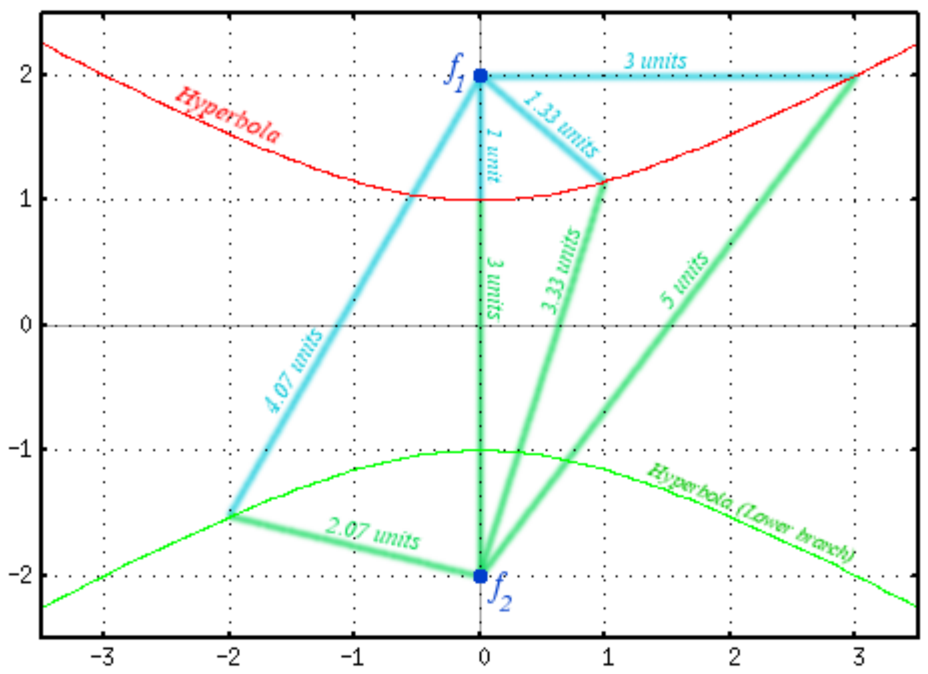
The Focus of a Hyperbola

A hyperbola can be considered as an ellipse turned inside out. Like the ellipse, it has two foci; however, the *difference* in the distances to the two foci is fixed for all points on the hyperbola. For an ellipse, of course, it's the sum of the distances which is fixed. If a hyperbola is "stretched" to the limit, it turns into a parabola, as does the ellipse; but for the hyperbola as we've drawn it here, it's the lower focus which goes to infinity (to form an upward-facing parabola), while for a similarly oriented ellipse, it's the upper focus (again, if you want form an upward-facing parabola). To turn an ellipse into a hyperbola, then, we can send its upper focus to infinity, and then retrieve a new lower focus from -infinity. At the boundary between the two, the figure is a parabola, which could be said to have three foci, two of which are infinitely distant (but we normally say, instead, that it has a directrix and just one focus).

In figure 1 we show a hyperbola with foci on the Y axis at +/- 2 units from the origin, for which the difference in the distances to the two foci is 2 units. We've shown both "branches" of the hyperbola, though on the rest of this page we'll be concerned only with the upper branch.

Like a parabola or ellipse, a hyperbola has its own "focus property": All incident rays which are directed at the lower focus and which hit the upper branch will be reflected to the upper focus, instead. Conversely, rays which come from the first focus and strike the hyperbola will be reflected along a path that makes them appear to have come from the *second* focus; in this sense, f_2 is an "image" of f_1. We shall prove that fact on this page, and then we'll show that an infinitely stretched hyperbola really does turn into a parabola, and go on to derive an equation for a hyperbola, and finally demonstrate that the familiar y=1/x is really a hyperbola.

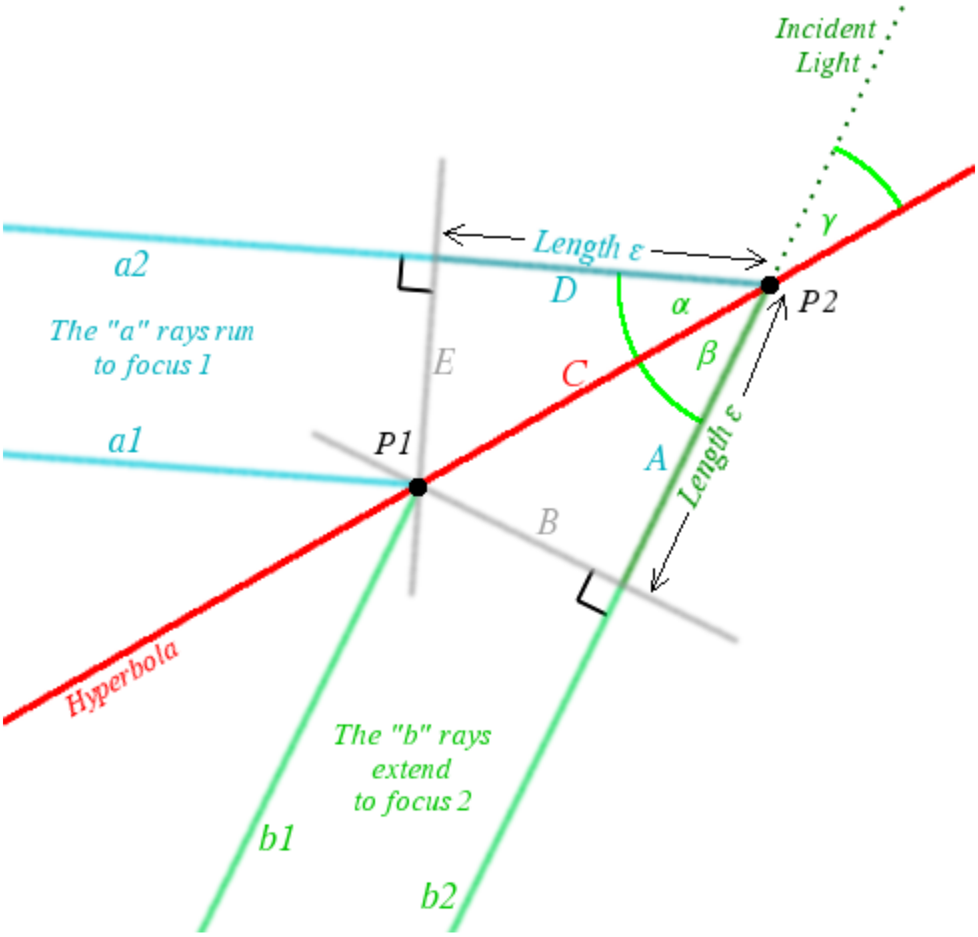
***Figure 1** -- Hyperbola, foci 4 units apart, difference=2*



Proof that Light Directed to f_2 is Reflected to f_1

In figure 2, we show two points on a hyperbola, **P1** and **P2**, which are infinitesimally close together, in a greatly magnified image. The proof is contained entirely in the picture; we will describe the details in words, below.

***Figure 2** -- An infinitesimal segment of a hyperbola, greatly magnified:*



From point **P1**, we show rays **a1** and **b1** which lead to focus 1 and focus 2, respectively. From point **P2**, we show rays **a2** and **b2** which lead to focus 1 and focus 2, respectively. **P1** and **P2** are assumed to be so close together that rays **a1** and **a2** are (nearly) parallel, and rays **b1** and **b2** are (nearly) parallel.

By the definition of a hyperbola, the difference in length between ray **a1** and ray **b1** must be equal to the difference in length between ray **a2** and ray **b2**. Ray **b2** is ε units longer than ray **b1**. Therefore, if the differences are to be equal, ray **a2** must also be ε units longer than ray **a1**. That is the key to the proof!

Look at triangles **ABC** and **DEC** in figure 1. They're right triangles, they each have side C as the hypotenuse, and sides A and D are equal (each is ε units long). Therefore the two triangles are identical (save that one's flipped over). Therefore, angles α and β must be equal.

The line marked "Incident Light" is the extension of ray **b2** through the hyperbola, and the angle it forms with the hyperbola must, of course, be equal to the angle **b2** forms with the hyperbola. So, angles β and γ must be equal.

But then angles α and γ must also be equal. Light coming to point **P2** which was directly at focus f_2 must come in along the "Incident Light" line, with incident angle γ. Its angle of reflection must equal its angle of incidence; but then, it must be reflected along ray **a2**. Ray **a2** leads to focus f_1, which is what was to be shown.

Note that the argument given here was identical to the argument we used on the parabola focus page. The only significant difference is that figure 2 on the parabola page was shown rotated, so that lines **B1** and **B2** run straight down to the directrix. They could just as well be running down to a lower focal point which was located very far away; we discuss this at a bit more length in the next section.

Stretching a Hyperbola Makes a Parabola

Move focus f_2 down to -infinity. Now, lines which are directed toward f_2 must all go *straight down*. So, all lines arriving parallel to the axis will be reflected to focus f_1, and the figure must be a parabola, as we saw here on the parabola focus page. Q.E.D.

That may have been too concise. So here's a version with a few more steps, which proves the claim directly from the definitions:

Shift everything in figure 1 down a bit so that the upper branch of the hyperbola touches the origin. Place f_1 δ units above the X axis, and fix the difference in the distance to the two foci at **2(f**-δ) units, where the distance between the foci is 2**f**. Now, send the lower focus arbitrarily far away. The hyperbola will still just touch the origin, but what happens at nearby points on the hyperbola?

As we move away from the origin, and consequently farther from f_1, we must also move farther from f_2 in order to keep the difference in the distances fixed. But since f_2 is arbitrarily far away, all lines to f_2 run (almost exactly) *straight down*. Draw a line parallel to the X axis, and δ units below the origin; call it the *directrix*. Lines leading to f_2 are all (almost exactly) perpendicular to the directrix. A point on the hyperbola which is ε units farther from f_1, and consequently ε units farther from f_2, must also be ε units farther from the directrix. So, the difference in distances between a point on the hyperbola and f_1 and the directrix must also be the same for all points on the hyperbola. By the difference in the distances for the point at the origin is zero (it's δ units from f_1 and from the directrix). So, all points on the hyperbola are equidistant from f_1 and the directrix, which is the definition of a parabola. And that is what was to be shown.

This proof would benefit enormously from a picture, but I haven't drawn one for it (as yet). A little farther down the page we'll do this over symbolically.

An Equation for a Hyperbola

So far we've just worked directly with the definition of a hyperbola. Let's find an equation for one.

Given a particular point on the hyperbola, we define the following:

$$
(1)\quad \begin{aligned}
l_1 &\equiv \mathit{distance\ to}\ f_1 \\
l_2 &\equiv \mathit{distance\ to}\ f_2 \\
r &\equiv l_2 - l_1 \\
l_1 &= \sqrt{(y-f)^2 + x^2} \\
l_2 &= \sqrt{(y+f)^2 + x^2}
\end{aligned}
$$

We then have:

$$(2)\quad \sqrt{(y+f)^2 + x^2} - \sqrt{(y-f)^2 + x^2} = r$$

Moving the second root to the right, squaring, and eliminating common terms, we obtain:

$$(3)\quad (y+f)^2 = r^2 + (y-f)^2 + 2r\sqrt{(y-f)^2 + x^2}$$

Multiplying out the squares, we obtain:

$$(4)\quad 4yf = r^2 + 2r\sqrt{(y-f)^2 + x^2}$$

Moving r^2 to the other side and squaring again, we obtain:

$$(5)\quad (4yf - r^2)^2 = 4r^2\left((y-f)^2 + x^2\right)$$

Multiplying everything out, eliminating common terms, and collecting what's left, we get:

$$(6)\quad (16f^2 - 4r^2)y^2 - 4r^2x^2 + (r^4 - 4r^2f^2) = 0$$

This is still pretty ugly. Observing that we must always have r < 2f, we make the following substitution:

$$(7)\quad \gamma \equiv \frac{2f}{r}$$
$$1 < \gamma < \infty$$

which, with a little rearrangement, leads to:

$$\boxed{(8)\quad y^2 - \left(\frac{1}{\gamma^2-1}\right)x^2 = \frac{1}{4}r^2}$$

A Symbolic Proof that a Stretched Hyperbola is a Parabola

The form of (8) isn't useful for taking a limit as the lower focus goes to infinity, because the hyperbola heads off to infinity, too. We want a formula which keeps the upper branch of the hyperbola touching the X axis as we "stretch" it. To obtain an easily recognizable form, we'd also like the "end" of the hyperbola to be touching the origin.

To that end, we'll first define the following:

$$(9)\quad h \equiv f - \frac{l_2 - l_1}{2}$$

where l_1 and l_2 are the distances to f_1 and f_2, as above. We'll then shift the axes up a bit, so that the foci fall as:

$$(10)\quad f_1 = (0, h)$$
$$f_2 = (0, -2f + h)$$

We can then immediately see that:

$$(11)\quad l_1 = \sqrt{(y-h)^2 + x^2}$$
$$l_2 = \sqrt{(y + 2f - h)^2 + x^2}$$
$$l_2 - l_1 = 2(f - h)$$

and:

$$(12) \quad 2(f-h) = \sqrt{(y+2f-h)^2 + x^2} - \sqrt{(y-h)^2 + x^2}$$

Squaring both sides and dividing by 2, we obtain the following mess:

$$(13) \quad 2(f-h)^2 = y^2 + h^2 + 2f^2 + 2fy - 2fh - 2hy + x^2 \\ - \sqrt{[(y+2f-h)^2 + x^2][(y-h)^2 + x^2]}$$

Multiplying out and canceling common terms, we obtain:

$$(14) \quad h^2 - 2fh = y^2 + 2fy - 2hy + x^2 \\ - \sqrt{[(y+2f-h)^2 + x^2][(y-h)^2 + x^2]}$$

Now we divide through by ***f*** and take the limit as ***f*** goes to infinity:

$$(15) \quad -2h = 2y - \sqrt{4[(y-h)^2 + x^2]}$$

Collecting all but the square root on the left, we square it again:

$$(16) \quad (y+h)^2 = (y-h)^2 + x^2$$

Multiply out the squares:

$$(17) \quad 4yh = x^2$$

And we're done:

$$\boxed{(18) \quad y = \frac{1}{4h}x^2}$$

(Compare with the formula for a parabola we found on the parabola focus page.)

What About y=1/x ?

It's also a hyperbola; it's just rotated relative to the hyperbolas we've been looking at. We'll start with the usual formula:

$$(19) \quad y = \frac{1}{x}$$

and we'll rotate the axes to the right 45 degrees, which is an angle of $-\frac{\pi}{4}$. The rotation matrix is:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{20}$$

We obtain the following substitutions:

$$\begin{aligned} \zeta &= \frac{x-y}{\sqrt{2}} \\ \xi &= \frac{x+y}{\sqrt{2}} \\ x &= \frac{\zeta+\xi}{\sqrt{2}} \\ y &= \frac{-\zeta+\xi}{\sqrt{2}} \end{aligned} \tag{21}$$

Plugging them into (19) we obtain:

$$\frac{1}{\sqrt{2}}(-\zeta+\xi) = \frac{\sqrt{2}}{\zeta+\xi} \tag{22}$$

or:

$$\xi^2 - \zeta^2 = 2 \tag{23}$$

Looking back at equation (8), we see that it matches equation (23) with the following values:

$$\begin{aligned} r &= 2\sqrt{2} \\ \gamma &= \sqrt{2} \\ f &= 2 \end{aligned} \tag{24}$$

and this is, indeed, a hyperbola.

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