The Focus of a Parabola

A parabolic mirror focuses light. What does that mean? It means that all rays which run parallel to the parabola's axis which hit the face of the parabola will be reflected directly to the focus. But why does this happen?

On this page we provide a simple proof of this fact, just by examining an infinitesimal piece of a parabola and looking at some triangles.

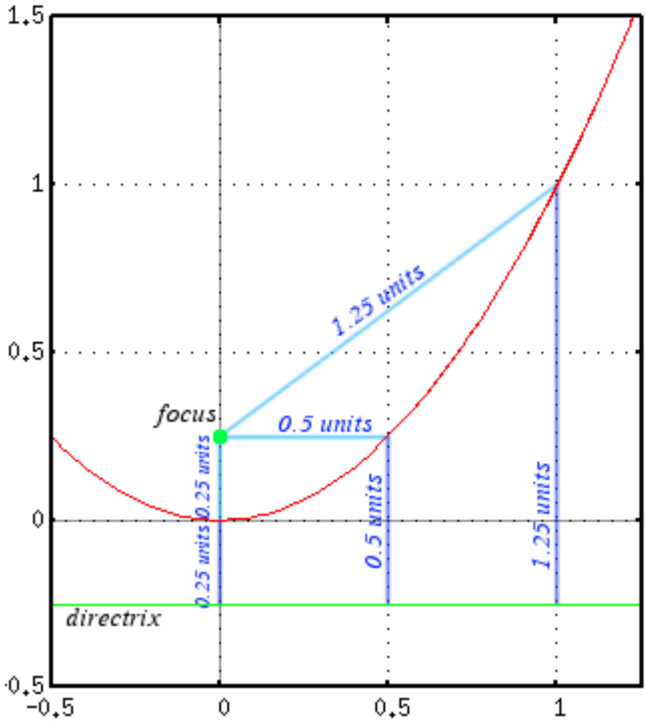
Definition of a Parabola

(Note -- A parabolic mirror is actually a "paraboloid", which just means that a vertical "slice" through the mirror's surface has the shape of a parabola. On this page I'm rather sloppily using the term "parabola" for the 2 dimensional curve which really is a parabola, and for the 3 dimensional surface of a mirror, which really is a paraboloid. Hopefully this won't cause any confusion.)

A "parabola" is the set of all points which are equidistant from a point, called the *focus*, and a line, called the *directrix*. Later on we'll show that this leads directly to the usual formula for a garden-variety parabola, $y=x^2$, but for now we're going to work directly with the definition. In figure 1 we've shown a portion of a parabola, with some distances marked off to illustrate this. This particular parabola has its focus located at (0,0.25), with its directrix running 1/4 unit below the X axis. We've shown the distances from the directrix and focus to three points on the parabola: the points at 0.25, 0.5, and 1.25 units from the focus and directrix.

(Note that a "paraboloid", such as the surface of a telescope mirror, is defined identically, save that the directrix is extended to a plane.)

Figure 1 *-- A Parabola*

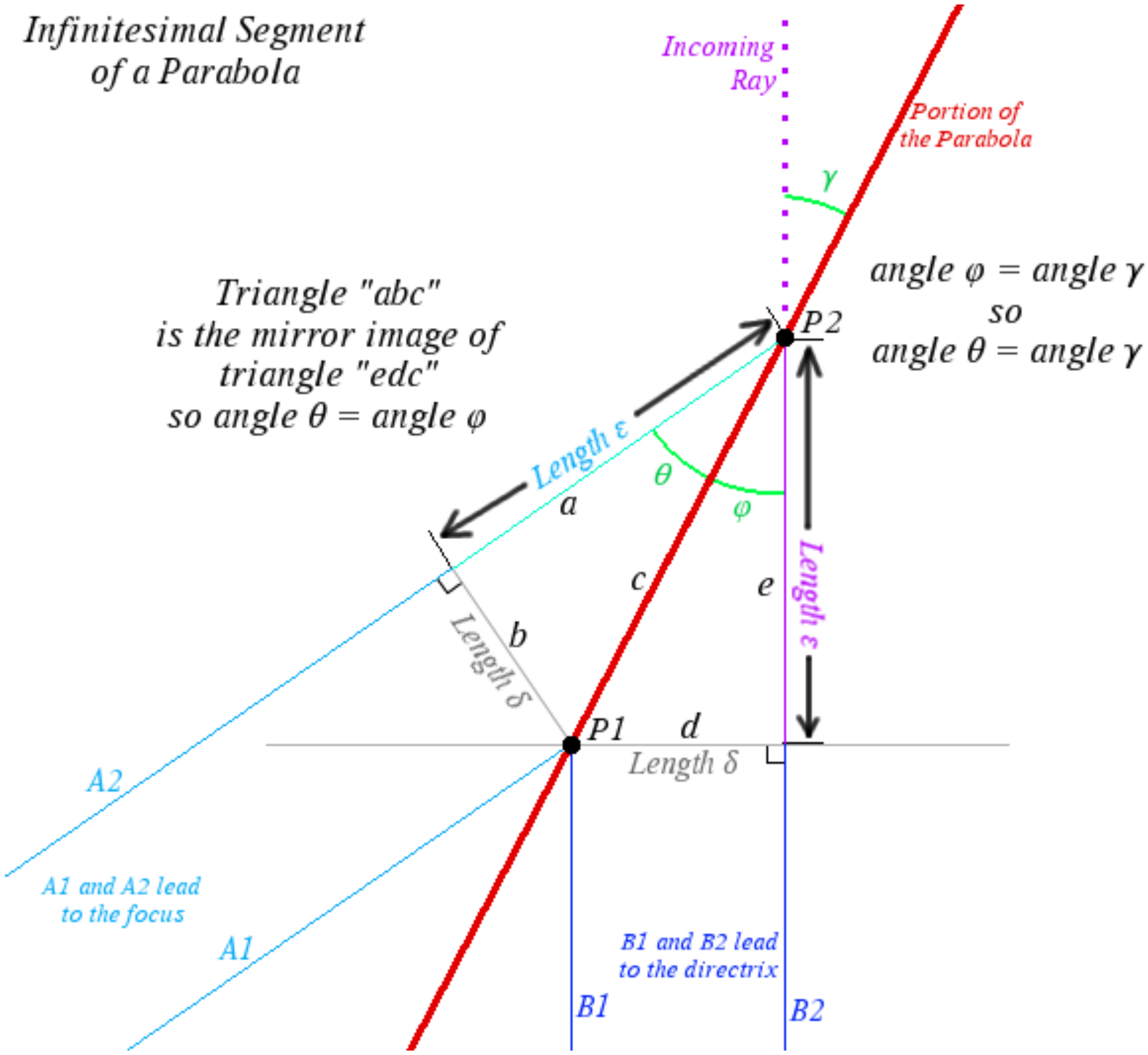


Proof that the Parabola Brings Light to a Focus

We'll now look at an infinitesimal segment of the parabola. In figure 2 we've shown a highly magnified view of two points on the parabola, marked **P1** and **P2**, along with the lines leading from the focus and directrix to them. The segment shown in the picture is so small, and **P1** and **P2** are so close together, that the lines from the focus to **P1** and **P2** are (almost exactly) *parallel*. (The lines perpendicular to the directrix which lead to **P1** and **P2** are, of course, necessarily parallel.)

The proof is contained entirely in the picture. We will, however, discuss it in a bit more detail, below.

Figure 2:



Lines **A1** and **B1** lead from point **P1** to the focus and directrix, respectively. Since **P1** is on the parabola, lines **A1** and **B1** must be the *same length*.

Just a little farther along the parabola we have marked point **P2**. We've drawn line **A2** from **P2** to the focus, and we've drawn line **B2** straight down to the directrix. As mentioned, **P1** and **P2** are actually so close together that **A1** and **A2** are (essentially) parallel.

Segment ***b*** runs perpendicularly from the end of **A1** to **A2**. **A2** is longer than **A1** by the piece extending past segment ***b***, marked "***a***"; it is ε units long.

Segment ***d*** runs perpendicularly from the end of **B1** to **B2**, and ***e*** is the "extension" of **B2** versus **B1**. Since **P2** must also be equidistant from the directrix and the focus, segment ***e*** must also be ε units long. That is the key to the proof!

Triangles **abc** and **edc** are right triangles and two of their sides (***a*** and ***e***, and the hypotenuse, which is ***c*** for each of them) are certainly the same length. So, the third pair of sides, ***b*** and ***d***, must also be the same length. So, if lines **A1** and **A2** are δ units apart, then **B2** and **B2** must also be δ units apart.

So, triangles **abc** and **edc** are just mirror images, so angles θ and φ must be identical. Angles φ and γ are also identical, since they're opposite angles of two intersecting lines. But then angles θ and γ must also be identical.

A light ray coming straight down from the top of the page (parallel to the parabola's axis), along line **B2**, strikes the parabola at angle γ. It's naturally reflected at *the same angle* -- so, since γ = θ, the ray will head down line **A2**, straight to the focus, as was to be shown.

If it Brings Parallel Rays to a Focus, Must it Be a Parabola?

Yes, absolutely ... *if* it's continuous.

Pick a focus, and pick a point on an arbitrary curve. If, at that point, incoming light is reflected to the focus, then the curve must have the same slope as a parabola at that point. That is, its derivative must match that of a parabola which passes through that point. If its derivative matches that of a parabola at *every* point, then, since a function is the integral of its derivative, it must *be* a parabola.

There's a caveat here, though, which is that this argument only works if the curve is continuous. A discontinuous curve could take a finite number of "jumps", and still bring incoming rays to a focus nearly everywhere. That's the principle on which Fresnel lenses and mirrors are based.

On the other hand, camera lenses which do not behave as ideal parabolic lenses also don't "really" focus light to a *point*. That is, parallel rays are brought together into a tiny disk at the film plane (the "blur disk"), but not a single point. It's a trade-off: Parabolic mirrors (and lenses) focus on-axis rays perfectly (within the limits set by diffraction), but do a poor job of focusing items which are off-axis. Camera lenses are optimized to bring rays from many angles to a "pretty good" focus.

The Parabola as a Limit

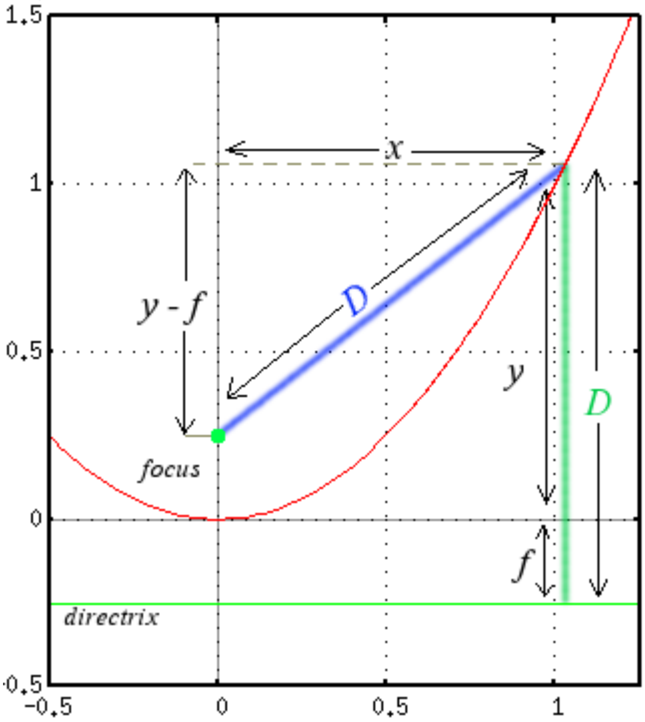
As we can see on the ellipse focus and hyperbola focus pages, a parabola can be considered as the limit of a "stretched ellipse", or as the limit of a "stretched hyperbola". It can also be considered as the figure which stands midway between an ellipse and hyperbola. It has a single focal point and a focal *line*, but the directrix could also be considered to represent either an "ellipse-like" focal point at $+\infty$ or a "hyperbola-like" focal point at $-\infty$, or perhaps both.

The Equation of a Parabola

Just for completeness, we'll now go ahead and derive the simplest of the parabola equations, to show that the definition we're talking about is really the usual $y=x^2$ thing.

In figure 3, we've shown the parabola again, with a number of distances marked. We'll use those to find the equation which describes the parabola.

Figure 3 *-- Parabola with distances marked:*



Since, by definition, the distance from a point on the parabola to the focus equals the distance from that point to the directrix, the two distances marked "*D*" must be identical. We can identify a right triangle in the upper left of figure 3 bounded by y-f, x, and D, and we can see on the green line that y+f must be D. So, we can read off:

$$(1) \quad (y - f)^2 + x^2 = D^2$$

$$(2) \quad y + f = D$$

Squaring (2) and equating the D^2 terms, we obtain

$$(3) \quad (y-f)^2 + x^2 = (y+f)^2$$

Multiplying it out and collecting terms we see,

$$\boxed{(4) \quad y = \frac{1}{4f}x^2}$$

and, when f = 1/4, this is just the familiar $y=x^2$.

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