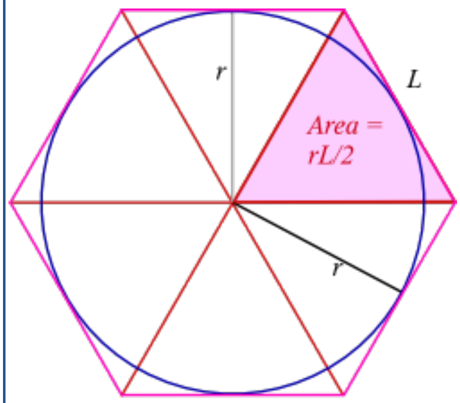
Area and Volume of a Circle and Sphere Using Simple Geometry

Previously, using simple geometry, we showed that the ratio of the circumference of a circle to its radius is independent of its size, and found a value for π accurate to about 10 decimal places. On this page, we'll carry that farther, and find the area of a circle and the area and volume of a sphere, using simple geometric arguments.

These formulas are, of course, trivial to find using calculus. However, the formulas were well known long before calculus was discovered; in fact, little more than the Pythagorean theorem is needed to derive them. Deriving them using calculus seems needlessly roundabout.

Area of a Circle

Figure 1: *Area of a polygon, with N sides and perimeter P, with inscribed circle:*



Perimeter = N·L
Total Area = N·rL/2 = r·P/2

As mentioned above, using the Pythagorean theorem, we've shown, here, that the ratio of the perimeter of a circle to its diameter is π. Given that, and using the same approach, we can easily find the area of a circle.

In figure 1 we've shown a circle inscribed in a polygon. (It happens to be a hexagon but that's not important to the argument.) It has ***N*** sides, and each side of the polygon has length ***L***. The perimeter of the polygon therefore has length ***N·L***. We can split the polygon into ***N*** triangles by cutting across from point to point; the area of one triangle is therefore:

$$\text{Area(one triangle)} = \frac{r \cdot L}{2}$$

Since there are ***N*** triangles in the polygon, the total area of the polygon must be

$$\begin{aligned}\text{Area(polygon)} &= r \cdot \frac{LN}{2} \\ &= \frac{r \cdot P}{2}\end{aligned}$$

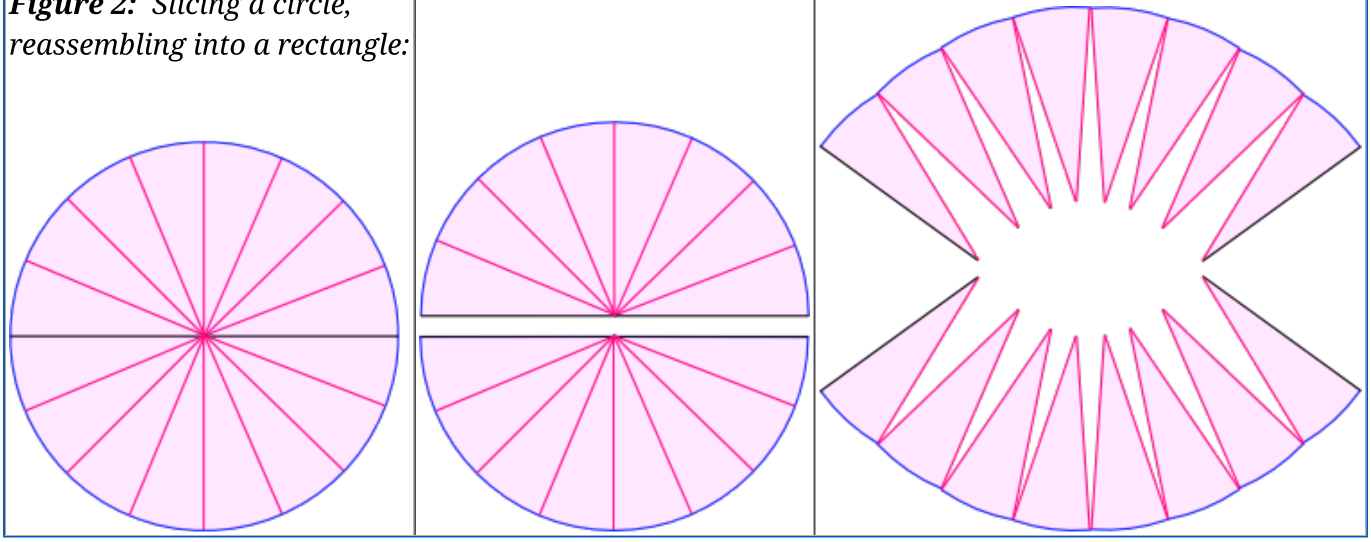
A circle is very much like a polygon with an arbitrarily large number of sides; as such, the area of a circle must also be the product of its radius and its perimeter, divided by 2. Since the perimeter of a circle is $P=2\pi r$, that means the area of the circle must be:

$$\boxed{\text{Area(circle)} = \pi r^2}$$

As stated, this argument isn't terribly rigorous. By inscribing a second polygon within the circle to provide a lower bound on the area, and using the formal definition of a limit rather than just saying a circle is *like* a polygon with an "*arbitrarily large*" number of sides, we could make it rigorous; however the added bulk would be substantial, with little additional clarity added for our pains, so we won't do that.

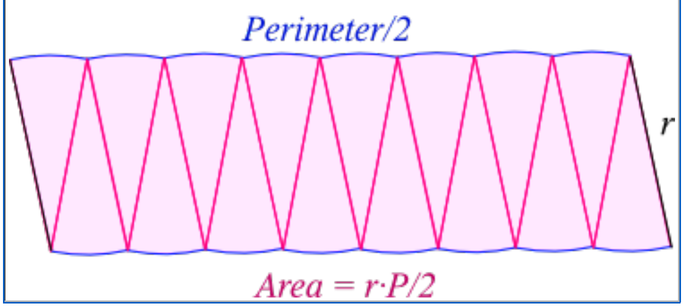
Alternatively, we can visualize the same argument (with even less rigor) by cutting a circle in two, and cutting each half in pie wedges, as we've shown in figure 2.

Figure 2: *Slicing a circle, reassembling into a rectangle:*



We then put the circle "back together", by interleaving the wedges from each side, as we've shown in figure 3. The result, at least if we use a large enough number of pie slices, will again be a rectangle, of area ***r·P/2***. (With a small number of pie slices, the total area is, of course, unchanged, but the result isn't so obvious, due the the "lumpy" nature of the border.)

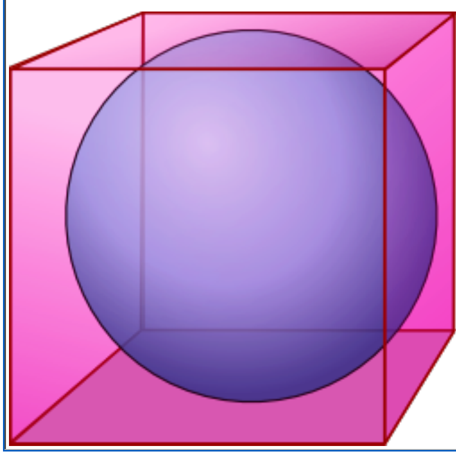
Figure 3: *Reassembled circle*



Volume of a Sphere

The general approach will be to find the values we want in terms of the area and volume of other shapes whose formulas we already know. This method is not new; it was used by Archimedes, thousands of years ago. There's more than one way we can apply it here; we can compare the volume of a sphere with cubes, cones, and cylinders in various ways. More specifically, we could show that the difference between the volume of a cylinder and the volume of a hemisphere is equal to the volume of a cone, or we could show directly that the volume of a hemisphere is equal to the volume of a cylinder *minus* the volume of a cone.

Figure 4: *Sphere inscribed in cube:*

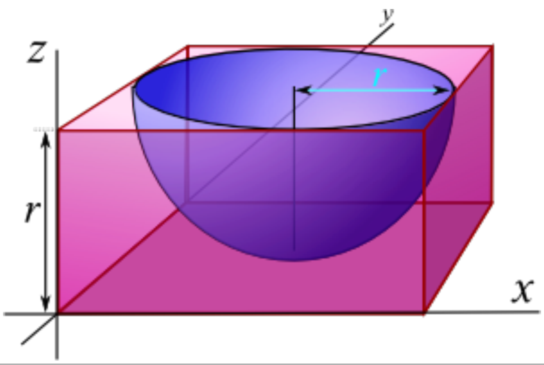


What we'll actually do here is find the difference between the volume of a cube and the volume of a sphere. We'll do this by *subtracting* the sphere from the cube. We'll first inscribe a sphere withing a cube (figure 4). We'll then cut the "hollow cube" in half, and look at a half-cube, *minus* a hemisphere (figure 5). Our approach will be to show that the volume of a half-cube, with an inscribed hemisphere cut out of it, is equal to the to the volume of a block *plus* the volume of a cone (figure 6, below).

To do this, we need an additional tool, which I will call:

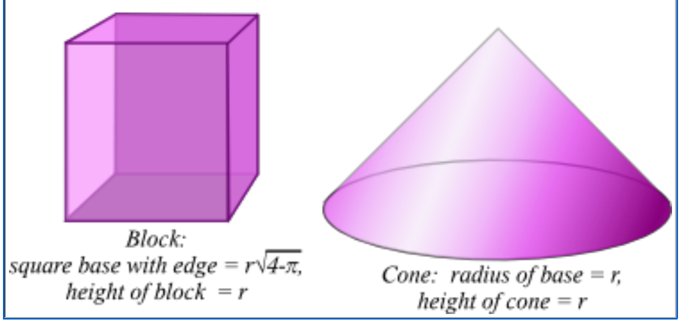
> ***The Slicing Principle:*** Take two objects, each the same length, each oriented parallel to the *z* axis. Pass a plane through both of them, perpendicular to the *z* axis. If the *area* of the cut is the same for each object, *no matter where the plane is located along the z axis*, then the two objects must have the same total volume.
>
> Let's make this concrete. Consider a pyramid. Cut it into thin horizontal "plates". The total volume of the pyramid is the sum of the volumes of all of the plates. If we allow the plates to slip across each other, so that the pyramid leans to one side or the other, the total volume is not affected. Similarly, if we change the *shape* of each plate *without* changing its thickness or volume (perhaps by making it circular rather than square), then the total volume of the pyramid -- and its height -- will not be affected, even though its new outline may be quite different. We can rephrase the condition above: Instead of slicing through each object with a plane at a particular location on the *z* axis, we shall cut each object into plates, with each plate perpendicular to the *z* axis. The volume of each object is the sum of the volumes of the plates it's been sliced into. If we *pair up* the plates between the two objects based on their location along the *z* axis, and if, in each pair of plates, the two plates have the same volume, then the two objects must have the same total volume.

Figure 5: *Half-sphere in half-cube:*



We will take this principle as "clearly true" (and I hope you will agree that it is), with no further effort to prove it rigorously. Rather, we shall now proceed to applying the slicing principle principle to find the volume of a sphere.

Figure 6: *Cone and block, total volume equal to volume of half-cube minus volume of half-sphere:*



To start, note that the half-cube in figure 5 is oriented with the base lying on the *xy* plane, and the cube and cut sphere opening "up" along the *z* axis. We'll now pass a cutting plane through the half-cube, perpendicular to the *z* axis (figure 7). If the plane lies at offset ***z*** above the *xy* plane, then the area of the cut will be as in figure 8. The cross section of the cube is a square, of fixed area:

(1) $$\text{Area(cube slice)} = 4r^2$$

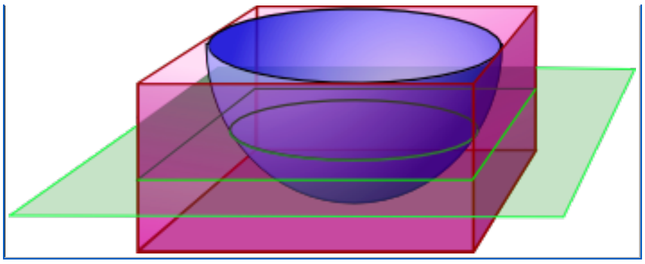
The cross section of the sphere is a circle, with radius which depends on ***z***:

(2) $$\begin{aligned}\text{Radius(sphere slice)} &= \sqrt{r^2 - (r-z)^2} \\ &= \sqrt{2rz - z^2}\end{aligned}$$

and area:

(3) $$\text{Area(sphere slice)} = \pi(2rz - z^2)$$

Figure 7: *Half-sphere in half-cube, sliced by plane, to obtain cross section:*



And the net area of the region *inside* the cube but *outside* the sphere is found by subtracting (3) from (1):

(4)
$$\text{Area(cube slice - sphere slice)} = 4r^2 + \pi z^2 - 2\pi r z$$

Next we'll slice through the block and cone in [figure 6](#) with the same plane, at the same height above the *xy* plane, with the block and cone both sitting on the *xy* plane. The result is as shown in [figure 9](#). The cross section of the block is, of course, fixed:

(5) $$\text{Area(block slice)} = (4 - \pi) \cdot r^2$$

The area of the cone slice depends on *z*, getting smaller toward the point:

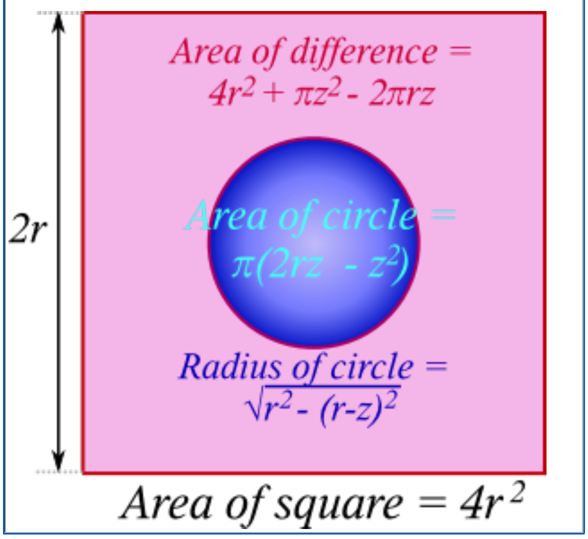
(6) $$\begin{aligned}\text{Area(cone slice)} &= \pi(r - z)^2 \\ &= \pi(r^2 + z^2) - 2\pi r z\end{aligned}$$

The sum of (5) and (6) is:

(7)
$$\text{Area(cone plus block slices)} = 4r^2 + \pi z^2 - 2\pi r z$$

This is equal to (4), the area of the slice through the cube, minus the area of the slice through the sphere, as shown in [figure 7](#). Consequently, we can conclude that the volume of the half-cube *minus* the volume of the hemisphere, as shown in [figure 5](#), must be equal to the volume of the block *plus* the volume of the cone, as shown in [figure 6](#).

Figure 8: *Cross section through half-cube, with half-sphere cut out; cut* ***z*** *units from cube base:*



The volume of the block is, of course,

(8) $$\text{Volume(block)} = (4 - \pi) r^3$$

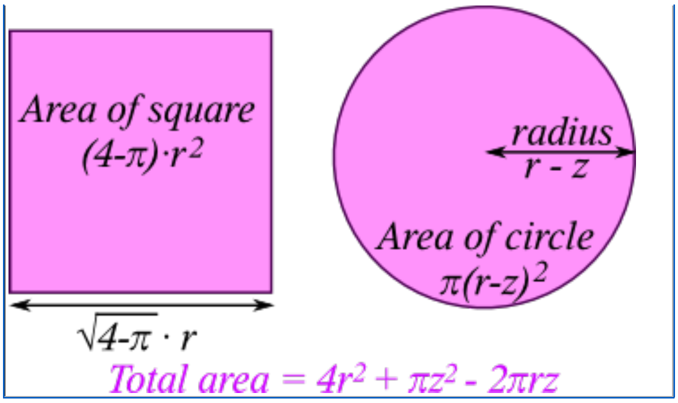
We determined, [**here**](#), that the volume of a cone with base area *A* and height *H* is (1/3)*A*·*H*. In fact, on that page we show a more general result for ***n*** dimensional hyperpyramids, but the result for 3 dimensions can be obtained by observing that we can pack exactly 3 pyramids into a cube, which shows that the volume of a regular pyramid is the area of the base times 1/3 the height. The same formula must, then, apply to cones as a consequence of the slicing principle given above: a cone is just the same as a pyramid, with its slices reshaped into circles rather than squares.

With that result in hand, the volume of the cone is:

(9) $$\text{Volume(cone)} = \frac{\pi}{3} r^3$$

Figure 9: *Cross sections through block and cone, showing total area through the two of them:*

And the difference between the volume of the half-cube and the volume of the hemisphere is:



(10)
$$\text{Volume(half-cube - hemisphere)} = (4 - \tfrac{2\pi}{3})r^3$$

But we know the volume of a half-cube, with height *r* and base *2r* on a side, is just $4r^3$. So, we can solve for the volume of the hemisphere:

(11)
$$\begin{aligned}\text{Volume(hemisphere)} &= 4r^3 - (4 - \tfrac{2\pi}{3})r^3 \\ &= \tfrac{2\pi}{3}r^3\end{aligned}$$

And the volume of the sphere is, of course, twice the volume of the hemisphere:

(12)
$$\boxed{\text{Volume(sphere)} = \tfrac{4}{3}\pi r^3}$$

Surface Area of a Sphere

The surface area of a sphere follows immediately from the volume, (12), and the fact that the volume of a pyramid (with any shape of base) is 1/3 the area of the base times the height. The argument is exactly analogous to the argument given above for the area of a circle.

We imagine the sphere inscribed in a polyhedron. We don't require the polyhedron be regular; consequently, we can have as many faces on it as we like. We imagine a polyhedron of many tiny faces. By drawing lines from each vertex to the center of the polyhedron, and then cutting up the polyhedron along the surfaces extending between each edge and the two lines drawn from that edge's endpoints to the center, we can divide it up into pyramids. Each pyramid is built on one face, and has its point at the center. (Of course, the pyramids are not regular 4-sided pyramids, but we don't care.)

The volume of each pyramid is r/3 times the area of the face on which it's built.

The total area of all the pyramid bases is equal to the total surface area of the polyhedron. The total volume of the polyhedron is the sum of the volumes of all the pyramids. Consequently, the total volume of the polyhedron must be

(13)
$$\text{Volume(polyhedron)} = \tfrac{1}{3}rA$$

where *A* is the surface area of the polyhedron.

By choosing a polyhedron with many facets, we can make the its volume as close as we like to the volume of the inscribed sphere. Similarly, its surface area can be made as close as we like to the surface area of the inscribed sphere. And so we must have, for the sphere as well as each polyhedron,

(14) $$\text{Volume(sphere)} = \frac{1}{3} r A$$

where *A* now represents the surface area of the sphere. But we just found the volume of the sphere, above. So, from (14) and (12), we have the surface area of the sphere:

(15) $$\boxed{\text{Surface(sphere)} = 4\pi r^2}$$

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