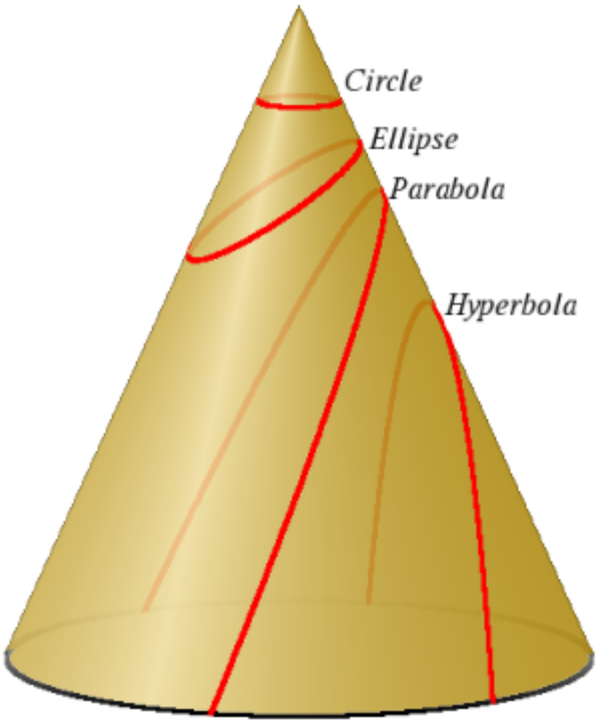
# Conic Sections

If we slice through a cone, depending on the angle of the cut, the edges will form a circle, ellipse, parabola, or hyperbola (figure 1). On this page, we'll discuss the shape each cut appears to have, simply from an inspection of the cone and the way the lines pass through it, and then we'll use a little algebra to prove that the sections really do have the claimed forms.

Note that each of the conic sections has its own "focus property", different for each. The focus properties are shown pictorially on our parabola focus, ellipse focus, and hyperbola focus pages, where we also derive equations for each of those figures starting with the definitions.

***Figure 1** -- Cone, with slices shown:*




## The Circle

This is really pretty obvious! If you cut through the cone parallel to the base the result is a circle; it must be true simply from the way we construct a cone.

## The Ellipse

This one may not be at all obvious! If we cut through the cone at an angle, certainly the section formed should have an *oval* shape -- but, since the cone is "fatter" farther from the point, one might expect the resulting figure to be egg-shaped, rather than a simple ellipse.

It can help to realize that, when one cuts through the cone this way, the "fattest" part of the cut is *not* on the centerline of the cone. The line of the cut is descending as it's crossing the cone. The cone can be viewed as being made up of stacked circles. As the cut crosses the centerline, it also crosses the widest chord of the circle through which it's cutting. But since that is exactly the widest point, moving an infinitesimal amount away from it to either side won't change the length of the chord -- this is just the principle on which finding maxima and minima is based. But as the cut crosses the centerline, it's descending at a nonzero (linear) rate, *and* the diameter of the cone is increasing at a nonzero (linear) rate. So, we would expect the diameter of the cut section to be *increasing* at the moment when the cut crosses the centerline. This is indeed the case, as we shall see later.

So, the section isn't egg-shaped -- but it also isn't symmetric about the centerline of the cone.

## *The Parabola*

If we cut exactly *parallel* to the side of the cone, we certainly won't get an ellipse. (In figure 1 the line labeled "parabola" should run along a cut which is exactly parallel to the left edge of the cone. However, it was drawn "free-hand" so it may not be *quite* parallel...)

The cut will remain at a fixed distance from the edge of the cone. If we imagine the cut as it goes far down the cone, we realize that there's no limit to how far apart the lines may become. But at the same time, it seems that they must be descending ever more rapidly down the cone's sides: as the cone becomes very very fat, the cut begins to look more and more like we're just peeling off a "strip of bark" from the cone, and the edges would seem to grow nearly parallel. Since they never cease to grow farther apart, however, they're certainly not approaching vertical asymptotes, like the curve of a tangent function. In fact, it appears that they never approach any asymptotes.

This certainly describes a parabola, and we'll see later that this is exactly what this curve is. (**para**bola == the cut which is **para**llel to the side of the cone.)

## *The Hyperbola*

If we cut even more steeply, so that the slice is not parallel to the side, we form a hyperbola. (**hyper** => more; a hyperbola is cut more steeply than a parabola.)

A primary trait of a hyperbola is that the sides approach asymptotes. For any hyperbola, we can find two lines (the "asymptotes") such that, if we go far enough out on the legs, the sides of the curve approach arbitrarily closely to them. We can see that the edges of any cut which is steeper than the parabola will approach asymptotes:

First, if we cut straight down, as the hyperbola in figure 1 appears to be, this property is obvious: as we go arbitrarily far down the cone, the cone swells out on either side of the cut, and the cut is left traveling almost "dead center". The radius of curvature of a circle about the cone increases arbitrarily as we descend, and the distance between the two sides of the cut becomes almost the same as the diameter of the circle; eventually the hyperbola will look almost as though we cut straight down through the point ... which would certainly just produce two straight lines.

If we cut down from the *point* at an angle, then the fraction of the way from the center of the cone to its edge at which the cut is located would be fixed, all the way down. The edges would be straight lines, which would separate from each other at a smaller angle than the "sides" of the cone. If, however, we start our cut some distance to one side of the point, it will still *eventually* find itself about the same fraction of the distance from the center of the circle to the edge as if we had cut from the *point* ... and as we descend, the "initial aberration" we provided by starting the cut to one side of the point will make less and less difference. In short, the edges will again approach asymptotes.

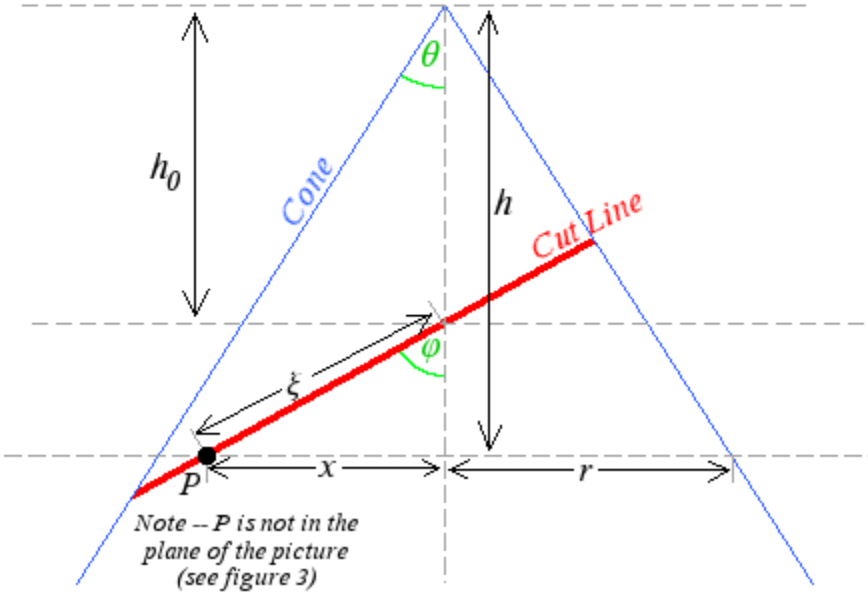
We'll prove later that this curve really is a hyperbola.

## *Proof, Part I -- Preliminaries, and Proof that a Parallel Cut yields a Parabola*

In figure 2, we show a flattened view of the cone from the side, with the "cut" line passing through it. Note that the "cut *line*" is really a cut *plane*, viewed edge-on. We have marked the included angle $\theta$ between the edge of the cone and the centerline, and the angle $\varphi$ between the cut line and the centerline.

We have coordinates $\xi$ and $\boldsymbol{y}$ in the plane of the cut. Given a particular point, $\boldsymbol{P}$, on the edge of the cut, $\xi$ is its distance from the centerline of the *cone*, measured along the centerline of the *cut*, and $\boldsymbol{y}$ is the perpendicular distance from the centerline of the *cut* to point $\boldsymbol{P}$. Note that $\boldsymbol{P}$ is at the edge of the cone -- *not* in the plane of the picture in figure 2. $\boldsymbol{P}$ has coordinates $(\xi,\boldsymbol{y})$ in the cut plane. $\boldsymbol{x}$ is the perpendicular distance from the centerline of the *cone* to the point $(\xi,\mathbf{0})$ in the plane of the cut. We wish to find an equation relating $\xi$ and $\boldsymbol{y}$.

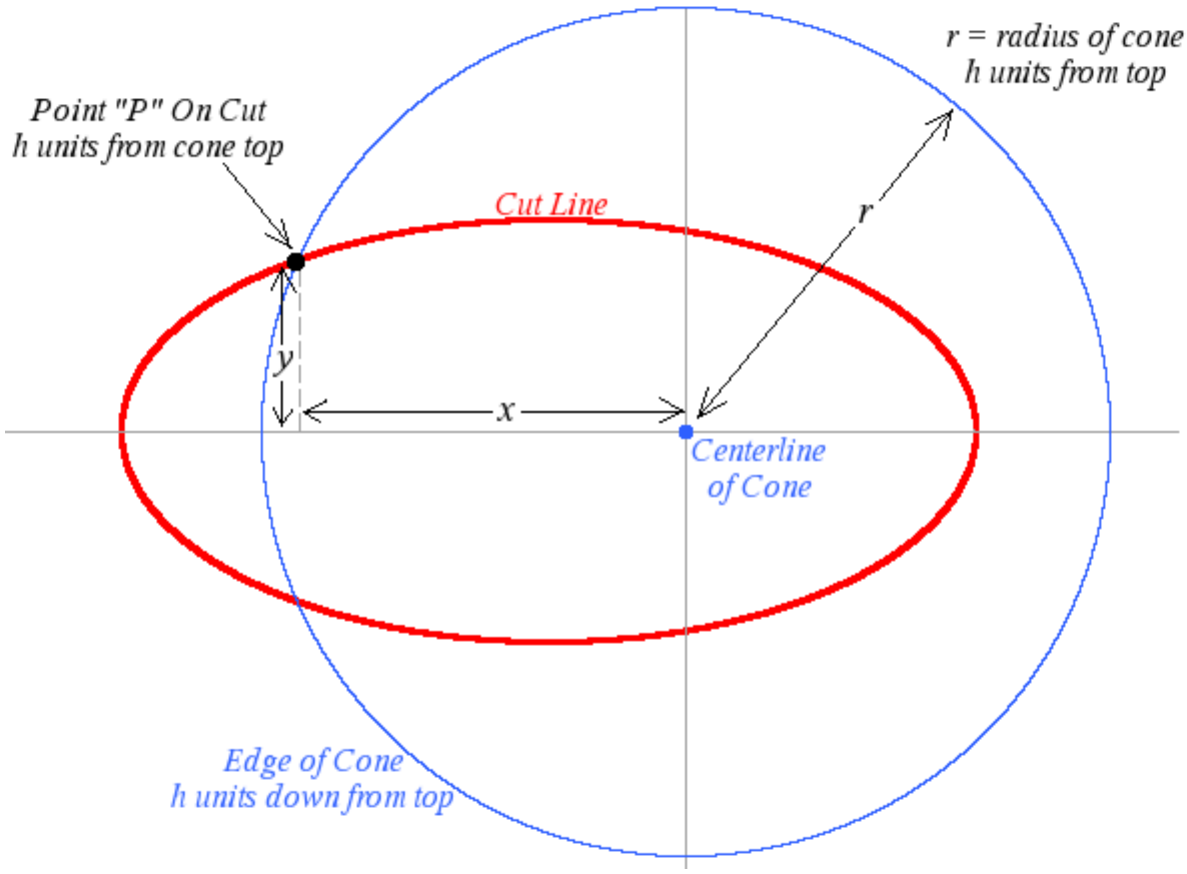
***Figure 2** -- Cone with cut, flattened, seen from the side:*




This should be clearer in figure 3 below, which shows the cone from the *top*; the distances *x*, *y*, and *r*, shown in figure 3, relate points all of which lie in a plane ***h*** units from the top of the cone.

The cut crosses the centerline of the cone a distance $\boldsymbol{h}_0$ from the top of the cone, and the point ***P*** at (ξ,***y***) on the cut edge lies distance ***h*** from the top of the cone.  In the plane of figure 2, the point ***P*** may be seen to lie ***x*** units from the centerline of the cone, and figure 3 we can see it lies ***y*** units from the centerline of the cut.  ***h*** units from the top, the cone has radius ***r***.  Since the cone's edge ***h*** units from the top is a circle, we can see immediately that $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{r}^2$.

***Figure 3** -- Cone with cut, seen from the **top**:*




I hope I haven't confused you totally with these attempts at describing what we see in the images. The situation is simple enough in 3 dimensions; the difficulty is in producing a clear picture using only *two* dimensions! But let us proceed.

From figures 1 and 2 we can read off the following:

(1)
$$\begin{aligned} x &= \xi \sin\phi \\ r &= h \tan\theta \\ h &= h_0 + \xi \cos\phi \end{aligned}$$

and as already noted we can see that:

(2) $$y^2 + x^2 = r^2$$

Expanding (2) with the formulas from (1) for ***x*** and ***h*** we obtain:

(3) $$y^2 + \xi^2 \sin^2\phi = (h_0 + \xi\cos\phi)^2 \tan^2\theta$$

Rearranging a little,

(4) $$y^2 + (\sin^2\phi - \cos^2\phi\tan^2\theta)\xi^2 - (2h_0\cos\phi\tan^2\theta)\xi = h_0^2\tan^2\theta$$

And now we need to stop and take a good look at the second term in (4) before we do anything more to it. If, similar to the way we have drawn figure 1, φ > θ, then we also have $\tan\phi > \tan\theta$, and so, $\cos\phi\tan\theta < \sin\phi$, and so the second term must be *positive.*

If $\varphi = \pi/2$, then $\cos(\varphi)=0$, $\sin(\varphi)=1$, and we obtain a circle, just as we would expect.

If, on the other hand, we have $\varphi = \theta$ then the second term in (4) will be *zero*. In that case we can replace $\varphi$ with $\theta$ everywhere to obtain,

$$\boxed{(5) \quad \xi = \frac{1}{2h_0\sin\theta\tan\theta}(y^2 - h_0^2\tan^2\theta)}$$

which is certainly the equation for a parabola, as we saw here, on the parabola focus page. So, if the cut is parallel to the side of the cone, the edge does, indeed, form a parabola.

## *Proof that $\varphi > \theta$ => we obtain an Ellipse*

When $\varphi > \theta$, the second term in (4) is positive. To reduce the clutter we make the following substitutions:

$$(6) \quad \begin{aligned} k_1 &= h_0\cos\phi\tan\theta \\ k_2 &= \sin^2\phi - \cos^2\phi\tan^2\theta \end{aligned}$$

Note that $k_1$ and $k_2$ are both positive, and, further, $k_2 < 1$. We pull out the coefficient on $\xi^2$,

$$(7) \quad y^2 + k_2(\xi^2 - 2\frac{k_1}{k_2}\xi) = {h_0}^2\tan\theta$$

complete the square,

$$(8) \quad y^2 + k_2(\xi - \frac{k_1}{k_2})^2 - \frac{{k_1}^2}{k_2} = {h_0}^2\tan\theta$$

shift the origin by making the following substitution,

$$(9) \quad \zeta = \xi - \frac{k_1}{k_2}$$

and finally obtain:

$$\boxed{(10) \quad y^2 + k_2\zeta^2 = {h_0}^2\tan^2\theta + \frac{{k_2}^2}{k_1}}$$

Looking back at (6), we see that that the coefficient on $\zeta^2$ is between 0 and 1, and the term on the right is positive. Comparing this with equation 8 on the ellipse focus page, we see that this is, indeed, the equation for an ellipse.

Note that the substitution in (9) is necessary exactly because the minor axis of the ellipse does not touch the centerline of the cone.

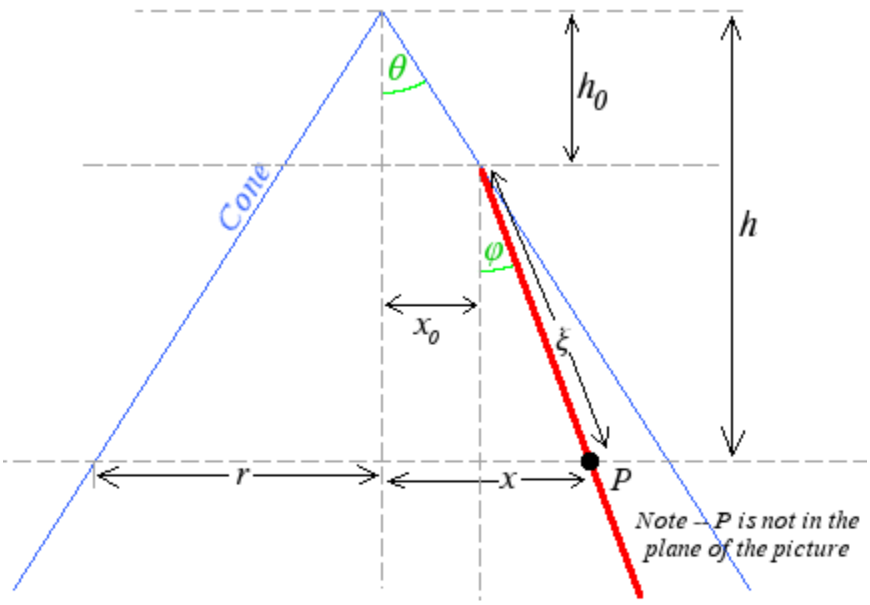
### *Proof that φ < θ => we obtain a Hyperbola*

Looking back at (6) we see that, if $\varphi < \theta$, then $k_2$ must be between 0 and $-\infty$. The right hand side of (10), on the other hand, is positive, of any magnitude. Comparing this with equation 8 on our hyperbola focus page, we see that in this case (10) is, indeed, the equation for a hyperbola.

### *Cuts which Don't Cross the Centerline of the Cone*

As it happens, the cut labeled "Hyperbola" in figure 1 appears to go straight down, and never cross the centerline at all. We have not yet proven that such cuts, nor cuts which angle "outward" and hence also avoid the centerline, also produce hyperbolas. We need to change our analysis slightly, and we need a new picture. In figure 4, we've turned the cut so it never crosses the center line. We've kept the same meanings for the symbols used in figure 2 and figure 3, but we've added a new one: $x_0$ is the horizontal distance to the start of the cut.

***Figure 4*** *-- Cone, side view, with cut that doesn't cross the center line:*




From figure 4, we can read off:

$$
\begin{aligned}
x &= x_0 + \xi \sin\phi \\
x_0 &= h_0 \tan\theta \qquad (11) \\
r &= (h_0 + \xi\cos\phi)\tan\theta
\end{aligned}
$$

Plugging the values for ***r*** and ***x*** from (11) into (2) we obtain:

$$(12)\quad y^2 + (h_0 \tan\theta + \xi \sin\phi)^2 = (h_0 + \xi \cos\phi)^2 \tan^2\theta$$

Multiplying out and collecting terms we obtain:

$$(13)\quad y^2 + (\sin^2\phi - \cos^2\phi \tan^2\theta)\xi^2 + 2h_0(\tan\theta \sin\phi - \cos\phi \tan^2\theta)\xi = 0$$

Note that we must have φ < θ (or else the "cut" misses the cone entirely). By inspection, then, we see that both coefficients in (13) are *negative*. To reduce clutter we'll make the following substitutions:

$$(14)\quad \begin{aligned} k_3 &= -(\sin^2\phi - \cos^2\phi \tan^2\theta) \\ k_4 &= -h_0 \tan\theta \cdot (\sin\phi - \cos\phi \tan\theta) \end{aligned}$$

Note that both $k_3$ and $k_4$ are positive and unbounded.

We'll now complete the square and collect terms to obtain:

$$(15)\quad y^2 - k_3(\xi + \tfrac{k_4}{k_3})^2 - \tfrac{{k_4}^2}{k_3} = 0$$

We change variables to shift the origin:

$$(16)\quad \zeta = \xi + \tfrac{k_4}{k_3}$$

and finally arrive at:

$$\boxed{(17)\quad y^2 - k_3\,\zeta^2 = \tfrac{{k_4}^2}{k_3}}$$

The coefficients are positive and unbounded. Comparing this with equation 8 on our hyperbola focus page, we see that it is, indeed, the equation for a hyperbola.



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