The Focus of an Ellipse

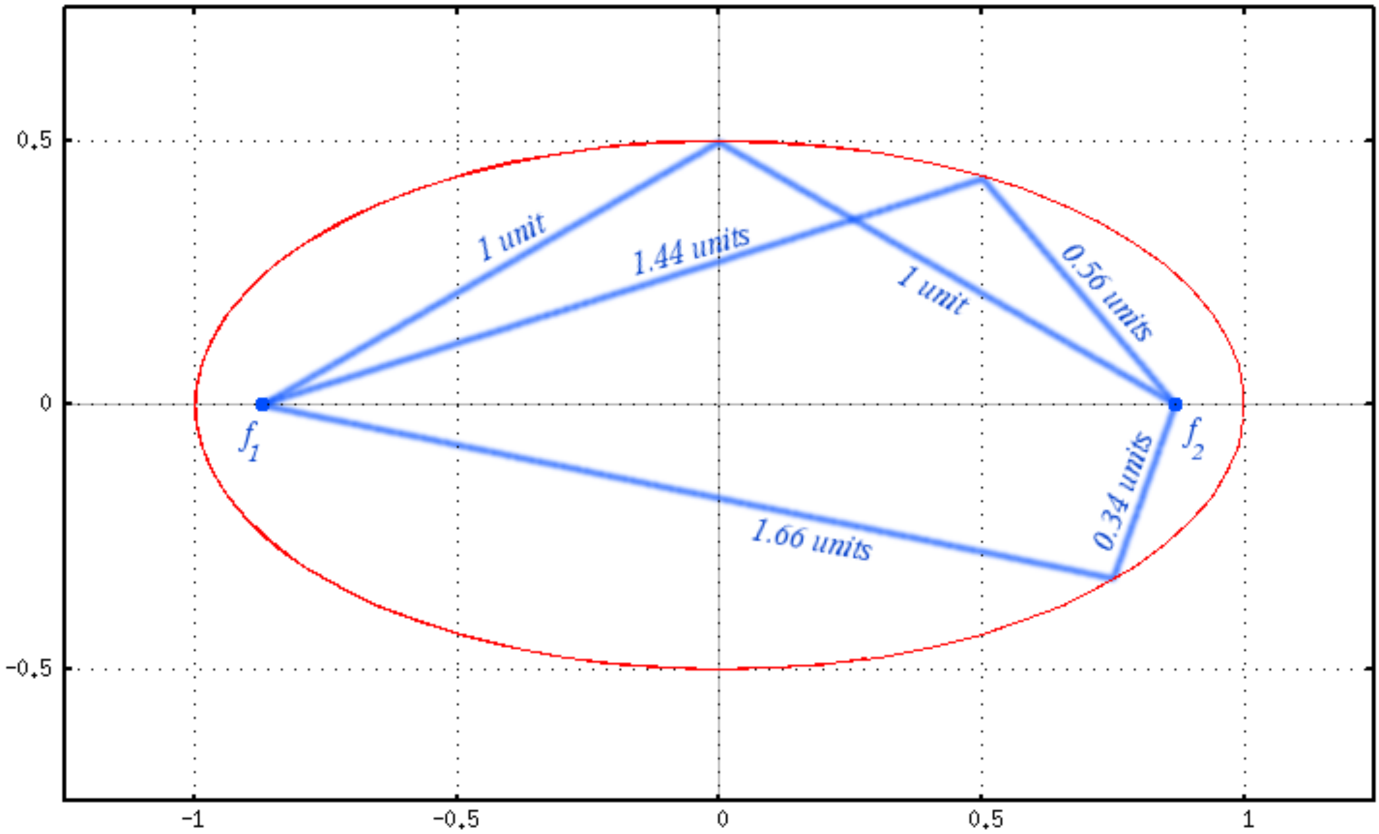
An ellipse has the property that any ray coming from one of its foci is reflected to the other focus. This is occasionally observed in elliptical rooms with hard walls, in which someone standing at one focus and whispering can be heard clearly by someone standing at the other focus, even though they're inaudible nearly everyplace else in the room.

On this page, we'll show that this is true, by looking at several triangles in a magnified image of part of an ellipse. We'll also find an equation for an ellipse, and just for amusement we'll show that an "infinitely stretched" ellipse turns into a parabola.

Definition of an Ellipse

An ellipse has two foci. The sum of the distances from any point on the ellipse to the two foci is the same for every point on the ellipse.   In figure 1, we show an ellipse in which the foci are 1.7 units apart, and in which the sum of the distances to the two foci is 2 for every point on the ellipse.

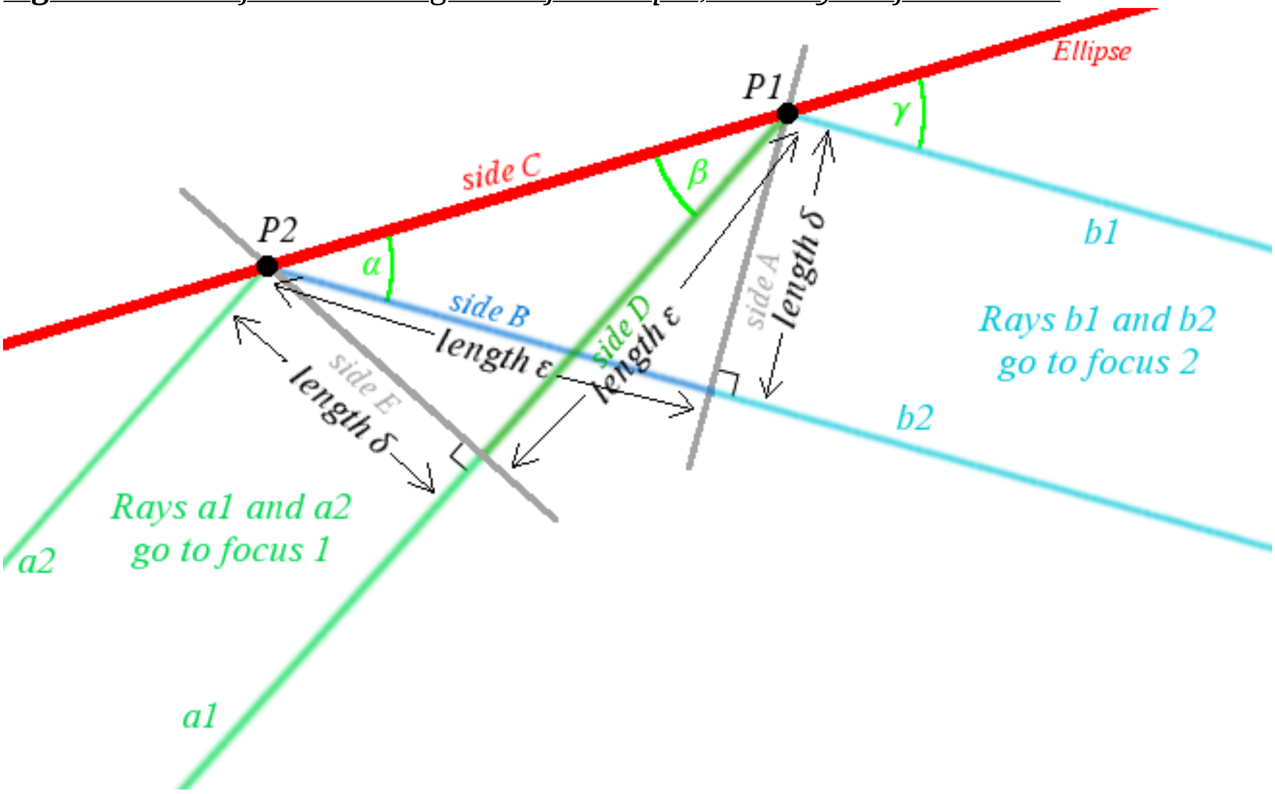
***Figure 1** -- An Ellipse, with foci at +/- 0.87, sum of radii = 2:*



Proof that Rays from One Focus Travel to the Other Focus

Just looking at figure 1, this assertion certainly seems plausible; for a ray following any of the blue lines, the angle of incidence and the angle of reflection where it hits the ellipse look equal. That's what we'll now show is true. Figure 2 shows a highly magnified view of two points on the ellipse, **P1** and **P2**, which are extremely close together. They are assumed to be so close together, and the region in figure 2 is so small, that rays **b1** and **b2** which run to focus $\boldsymbol{f}_2$ are (nearly) *parallel*, and rays **a1** and **a2** which run to focus $\boldsymbol{f}_1$ are also (nearly) parallel. A careful examination of the picture demonstrates the proof; we shall now go over it in words.

***Figure 2** -- An infinitesimal segment of an ellipse, with rays to foci shown:*



Rays **a1** and **b1** run from point **P1** to the two foci, and rays **a2** and **b2** run from point **P2** to the two foci. By definition of the ellipse, the sum of the lengths of rays **a1** and **b1** must *equal* the sum of rays **a2** and **b2**. Ray **b2** is ε units longer than ray **b1**; therefore, if **a2**+**b2** is to equal **a1**+**b1**, it must be the case that ray **a2** is ε units *shorter* than ray **a1**. And that is the key to the proof!

Look at triangles **ABC** and **EDC**. For added clarity, these are shown highlighted in figure 3 and figure 4. They are each right triangles. Side **D** and side **B** are equal (each is ε units long), and side **C** is the hypotenuse of both triangles. We must, therefore, have the length of side **A** equal to the length of side **E**, as well; the triangles are mirror images of each other.

Since the two triangles are identical (save that one is flipped), we must have angle α equal to angle β. Now, since **b1** is parallel to **b2**, it must strike this tiny, *nearly straight* section of the ellipse at the same angle as **b2**, so we must also have angle α equal to angle γ. But then angle β must equal angle γ as well. But then, a beam coming from focus 2 which strikes the ellipse at **P1** will come in along ray **b1**; it will be reflected with an angle of reflection equal to the angle of incidence, which means it will have to be reflected along ray **a1**, which leads to focus 1. And that is what was to be shown.

***Figure 3** -- Ellipse segment, with triangle ABC highlighted:*

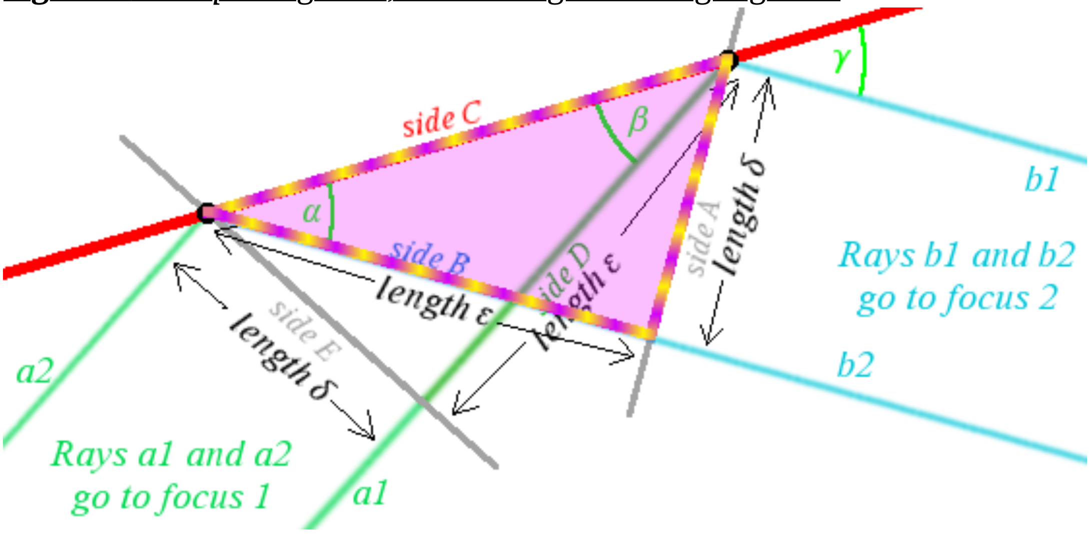
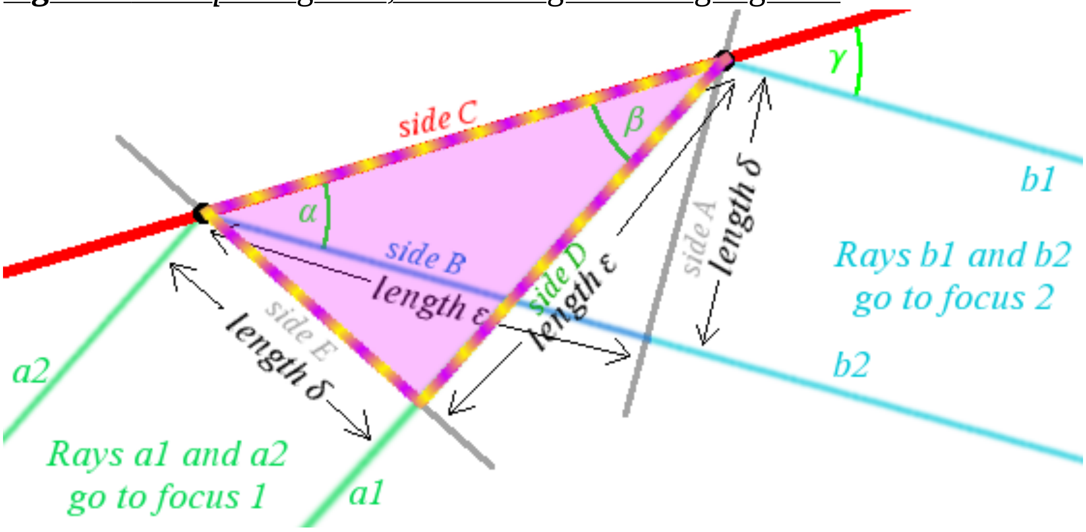


***Figure 4** -- Ellipse segment, with triangle EDC highlighted:*



The Limit of a Stretched Ellipse is a Parabola

Rotate the ellipse so it's vertical, with focus f_2 above, and focus f_1 below (this just makes this argument a little easier to state). Send focus f_2 up to infinity. Now, lines which come from f_2 must all go *straight down*. So, all lines arriving parallel to the axis will be reflected to focus f_1, and the figure must be a parabola, as we saw here on the parabola focus page. Q.E.D.

That may have been a little too concise, so here's a longer version that argues directly from the definition of a parabola:

Suppose we make the foci of an ellipse arbitrarily far apart. At the same time, suppose we keep the sum of the distances from points on the ellipse to the foci just slightly longer than the distance between the foci. If the distance between the foci is $2f$, let's set the sum of the distances to the foci from any point on the ellipse to $2(f+h)$. And now, let's shift the ellipse so that one end of it just touches the origin, with its long axis extending to the right along the X axis. What will it look like?

The "close" focus, f_1, must be h units from the origin. Let's draw a line, parallel to the Y axis, h units from the origin, on the *opposite* side of the Y axis from f_1. Call it the *directrix*. Now, the origin, which lies on the ellipse, is h units from the directrix, and h units from f_1. As we move along the ellipse to points that are farther from f_1, they must grow *closer* to f_2, which we have assumed is arbitrarily far away. Because f_2 is so far away, all lines to it will appear parallel. So, a point on the ellipse which is $h+\delta$ units from f_1 must be δ units farther to the right -- or δ units from the Y axis, or $h+\delta$ units from the directrix. That means the figure is, by definition, a parabola.

This visual argument could be shown more clearly with a picture but I haven't drawn one for it (as yet). I'll also show this symbolically, below.

An Equation for an Ellipse

We've done everything so far just using the definition of an ellipse. Let's find an equation for one.

Referring back to figure 1, if each focus is f units from the origin, then the distance from a point on the ellipse to focus f_1 must be

$$(1) \quad l_1 = \sqrt{(x+f)^2 + y^2}$$

and the distance from a point on the ellipse to focus f_2 must be

$$(2) \quad l_2 = \sqrt{(x-f)^2 + y^2}$$

If we let the sum of the distances to the foci total $2r$, then we have

$$(3) \quad \sqrt{(x+f)^2 + y^2} + \sqrt{(x-f)^2 + y^2} = 2r$$

or

$$(4) \quad \sqrt{(x+f)^2 + y^2} = 2r - \sqrt{(x-f)^2 + y^2}$$

Squaring both sides, expanding $(x+f)^2$ and $(x-f)^2$ on the left and right respectively, and canceling common terms, we obtain,

$$(5) \quad 4xf - 4r^2 = -4r\sqrt{(x-f)^2 + y^2}$$

Dividing through by 4 and then squaring yet again, we obtain,

$$(6) \quad x^2f^2 + r^4 - 2xfr^2 = r^2[(x-f)^2 + y^2]$$

We expand $(x-f)^2$ and cancel common terms,

$$(7) \quad x^2f^2 + r^4 = r^2[x^2 + f^2 + y^2]$$

Dividing through by r^2 and collecting terms,

$$\boxed{(8) \quad (1 - \tfrac{f^2}{r^2})x^2 + y^2 = r^2 - f^2}$$

Since we must have r > f, we see that the coefficient on x^2 must be between zero and one, and the term on the right can be any positive value.

Limit of a Stretched Ellipse

Above we gave a visual argument (sans picture) to show a stretched ellipse turns into a parabola. Here we'll do it symbolically.

Let's change coordinates so that the left end of the ellipse just touches the Y axis; then we'll stretch it to the right. So, let:

$$(9) \quad \begin{aligned} h &= x + r \\ x &= h - r \end{aligned}$$

Substituting into (8) we get:

$$(10) \quad (1 - \tfrac{f^2}{r^2})(h-r)^2 + y^2 = r^2 - f^2$$

Multiplying through by $r^2/(r^2-f^2)$ and multiplying out the square:

$$(11) \quad h^2 - 2hr + r^2 + \frac{r^2y^2}{r^2-f^2} = r^2$$

Collecting terms:

$$(12) \quad h = \frac{h^2}{2r} + \frac{ry^2}{2(r^2-f^2)}$$

Now we're going to set:

$$(13) \quad r = f + \delta$$

If we keep δ fixed, then focus $\boldsymbol{f}_1$ will remain fixed δ units to the right of the Y axis even as we vary ***f***, and the left end of the ellipse will remain at the Y axis even as we vary ***f***. Substituting (13) into (12), we obtain:

$$(14) \quad h = \frac{h^2}{2(f+\delta)} + \frac{(f+\delta)y^2}{2(f+\delta+f)(f+\delta-f)}$$

$$= \frac{h^2}{2(f+\delta)} + \frac{(1+\frac{\delta}{f})y^2}{2\delta(2+\frac{\delta}{f})}$$

and, in the limit as ***f*** goes to infinity,

$$\boxed{(15) \quad h = \frac{y^2}{4\delta}}$$

which was to be shown (compare with the parabola equation we found previously).

Back to...
Proof by Picture

Basic Stuff | Home | Physics

Page created on 11/6/06, and last updated on 11/15/06