a VISUAL Approach to CALCULUS problems

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Introduction

Calculus is a beautiful subject with a host of dazzling applications. As a teacher of calculus for more than fifty years and as an author of a couple of textbooks on the subject, I was stunned to learn that many *classical* problems in calculus can be easily solved by an innovative visual approach that makes no use of formulas. Here's a sample of four such (and more) problems:

Problem 1. Find the area of a parabolic segment.

Figure 1 shows a parabolic segment, the shaded region below the graph of the parabola y = x2 and above the interval from 0 to x. The area of the parabolic segment was first calculated by Archimedes more than 2000 years ago by a method that laid the foundations for integral calculus.

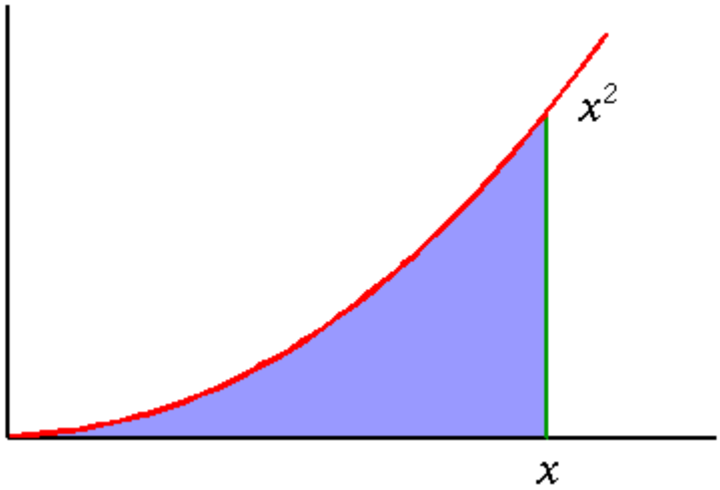


Figure 1. A parabolic segment

Problem 2. Find the area of the region under an exponential curve. Figure 2 shows the graph of the exponential function exp. We want the area of the shaded region below the graph to any point x.

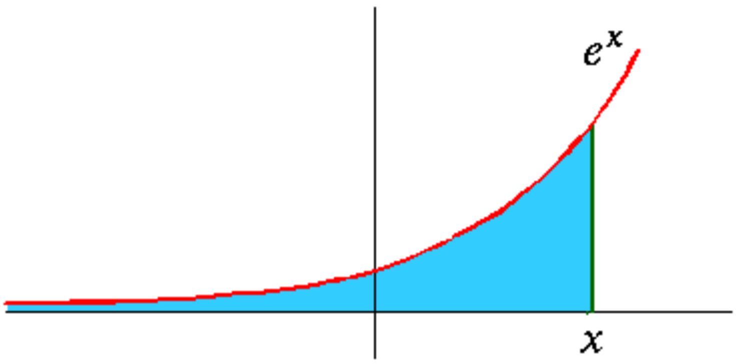


Figure 2. The region below an exponential curve

Problem 3. Find the area of the region under one arch of a cycloid.

A cycloid is the path traced out by a fixed point on the boundary of a circular disk that rolls along a horizontal line, and we want the area of the shaded region shown in Figure 3. This problem can also be done by calculus but it is more difficult than the first two. First, you have to find an equation for the cycloid, which is not exactly trivial. Then you have to integrate this to get the required area.

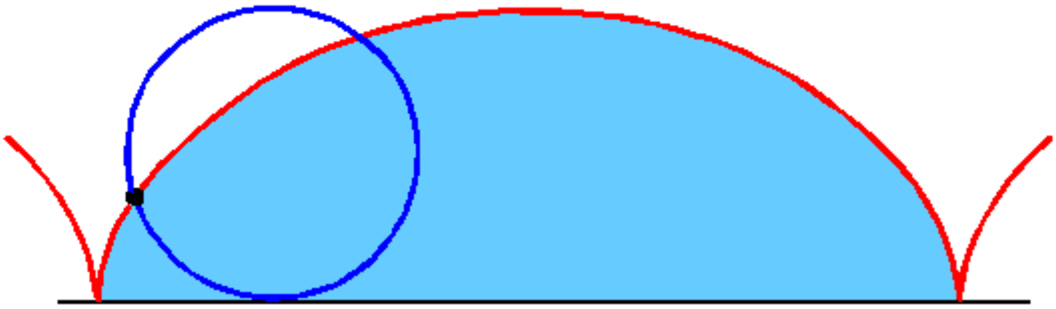


Figure 3. The region under one arch of a cycloid

Problem 4. Find the area of the region under a tractrix.

When a child drags a toy along the floor with a string of constant length, the toy traces out a tractrix as the child walks along the x axis all the way to infinity. (Figure 4.) We want to find the area of the region between the tractrix and the x axis. To solve this by the standard calculus method, you have to find the equation of the tractrix. This, in itself, is rather challenging&emdash;it requires solving a differential equation. Once you have the equation of the tractrix you have to integrate it to get the area. It can be done, but the calculation is somewhat demanding.

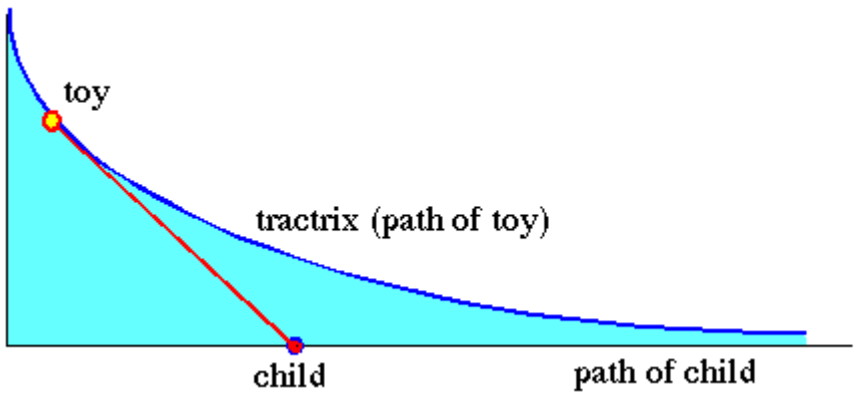


Figure 4. The region between a tractrix and the x axis.

All four of these classical calculus problems and many more can also be solved by a new method that relies on geometric intuition and is easily understood by very young students. Moreover, the new method also solves some problems *insoluable by calculus, and allows many increadible generalizations yet unknown in mathematics.*

For example, look at Figure 5, which shows the path traced out by the front wheel of a bicycle in motion. The rear wheel traces out another curve, and the problem is to find the area of the region between these two curves as the bicycle moves from an initial position to a final position. To do this with calculus you would need equations for the curves. But the problem can be solved with this new visual approach regardless of the shape of the bike's path. You don't need any equations!

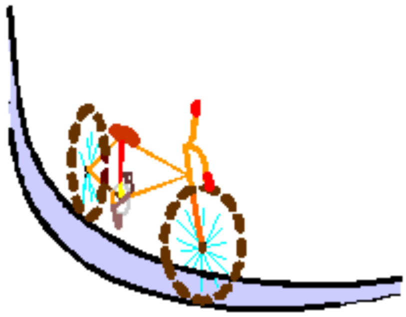


Figure 5. A region between the curves traced out by the rear and front wheels of a bicycle.

Historical background

Before describing this new method, some historical background is in order. The method was conceived in 1959 by a young undergraduate student at Yerevan University in Armenia named Mamikon A. Mnatsakanian. Mamikon, who is here today, will help illustrate this talk with some of his computer animation:

http://www.its.caltech.edu/~mamikon/calculus.html

Mamikon told me that when he showed his method to Soviet mathematicians they dismissed it out of hand and said "it can't be right&emdash;you can't solve calculus problems that easily." Mamikon went on to get a Ph.D. in physics; he pursued a full time career as professor of astrophysics at the University of Yerevan, and became an international expert in radiative transfer theory. He also continued to develop his powerful geometric methods, and eventually published a paper in 1981 outlining these methods, but the paper seems to have escaped notice, perhaps because it appeared in Russian in an Armenian journal with limited circulation (presented by Soviet Academician V.A.Ambartsumian):

Proceedings of the Armenian Academy of Sciences, vol.73, #2, pp.97-102, 1981

Mamikon came to California about a decade ago as part of an earthquake-preparedness program for Armenia. When the Soviet government collapsed, he was stranded in the US without a visa. With the help of a few mathematicians in Sacramento and at UC Davis he was granted status as an alien of extraordinary ability. He worked part time for the California Department of Education and at UC Davis. Here he further developed his methods in a form that can be used as a universal teaching tool to reach a wide audience, employing not only pictures, but also hands-on activities and computer-based manipulatives. He has taught these methods at UC Davis and in several elementary and high school classes in Northern California, ranging from Montessori elementary schools to inner-city public high schools. He has also demonstrated them at teacher conferences in Northern California. Response from both students and teachers has ranged from positive to enthusiastic, probably because the methods are vivid and dynamic and don't require the algebraic formalism of trigonometry or calculus.

About four years ago, Mamikon showed up at Project MATHEMATICS! headquarters and convinced me that his methods have the potential to make a significant impact on mathematics education, especially if they are combined with visualization tools of modern technology. Since then we have published several joint papers on innovative ideas in elementary mathematics and, if we can obtain adequate funding, we plan to produce a series of videotapes and workbooks to bring these powerful and exciting new geometric methods to a wide audience under the banner of Project MATHEMATICS!.

Mamikon's theorem for oval rings

Let's turn now to a description of Mamikon's method. Like *all great discoveries, it's based on a simple idea*. It started when young Mamikon was presented with a classical geometry problem, illustrated in Figure 6.

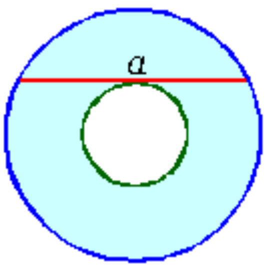


Figure 6. An annular ring bounded by two concentric circles.

This shows an annular ring and a chord of the outer circle tangent to the inner circle. The chord has given length a, and the problem is: Find the area of the annular ring.

As the late Paul Erdesh would have said, every baby can solve this problem. Look at the diagram in Figure 7. If the inner circle has radius r its area is ¼r2, and if the outer circle has

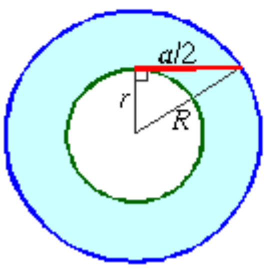


Figure 7. Calculating the area of the ring by the Pythagorean Theorem.

radius R its area is ¼R2, so the area of the ring is equal to ¼R2 &endash; ¼r2 = ¼(R2 &endash; r2). But the two radii and the tangent form a right triangle with legs r and a/2, and hypotenuse R, so by the Pythagorean Theorem, R2 &endash; r2 = (a/2)2, hence each ring has area ¼a2/4.

Note that the final answer depends only on a and not on the radii of the two circles!

If we knew in advance that the answer depends only on a, we could solve the problem another way. Shrink the inner circle to a point, and the annulus collapses to a disk of diameter a, with area equal to ¼a2/4.

Mamikon wondered if there was a way to see in advance why the answer depends only on the length of the chord. Then he thought of formulating the problem in a dynamic way. Take half the chord and think of it as a vector of length a/2 tangent to the inner circle. By moving this tangent vector around the inner circle, we see that it sweeps out the annular ring between the two circles.

But, for Mamikon it was obvious that the area is being swept due to pure rotation of the tangent segment

Now, translate each tangent vector parallel to itself so the point of tangency is brought to a common point. As the tangent vector moves around the inner circle, the translated vector rotates once around this common point and traces out a circular disk of radius a/2. So the tangent vectors sweep out a circular disk, as though they were all centered at the same point, as illustrated in Figure 8. And this disk has the same area as the ring.

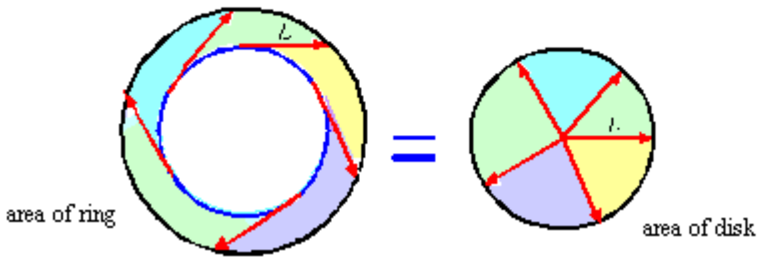
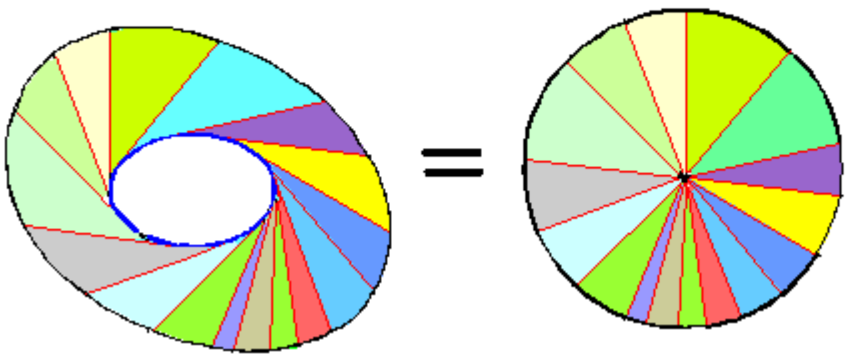


Figure 8. The area of the ring is equal to that of a circular disk.

Mamikon realized that this dynamic approach would also work if the inner circle is replaced by an arbitrary oval curve. Figure 9 shows the same idea applied to two different ellipses. As the tangent segment of constant length moves once around each ellipse, it sweeps out a more general annular shape that we call an oval ring.



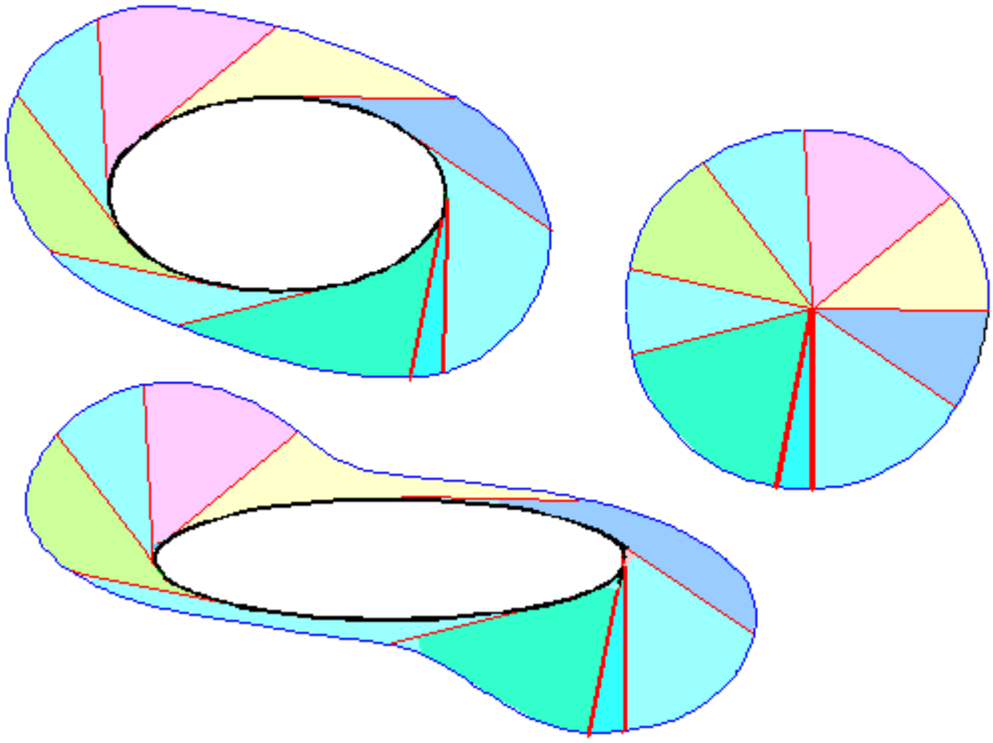
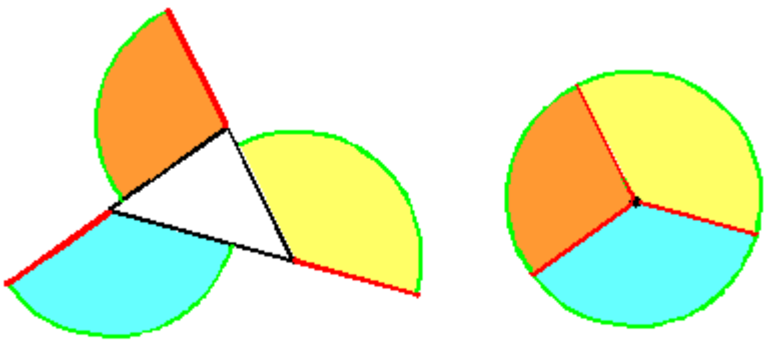


Figure 9. Oval rings swept by tangent segments of constant length moving around ellipses.

Again we can translate each tangent segment parallel to itself so the point of tangency is brought to a common point. As the tangent moves around the oval, the translated segments trace out a circular disk whose radius is that constant length. So, the area of the oval ring should be the area of the circular disk.

The Pythagorean Theorem *can not* help you find the areas for these oval rings. If the inner oval is an ellipse you can calculate the areas by integral calculus (which is not a trivial calculation); but if you do this calculation you find all these oval rings have equal areas depending only on the length of the tangent segment!

Is it possible that the same is true for any convex simple closed curve? Figure 10(a) illustrates the idea for a triangle.



(a) (b)

Figure 10. Region swept out as a tangent segment of constant length moves around a triangle.

As the tangent segment moves along an edge, it doesn't change direction so it doesn't sweep out any area. As it moves around a vertex from one edge to the next, it sweeps out part of a circular sector. And as it goes around the entire triangle it sweeps out three circular sectors that, together, fill out a circular disk, as shown in Figure 10(b).

The same is true for any convex polygon, as illustrated in Figure 11.

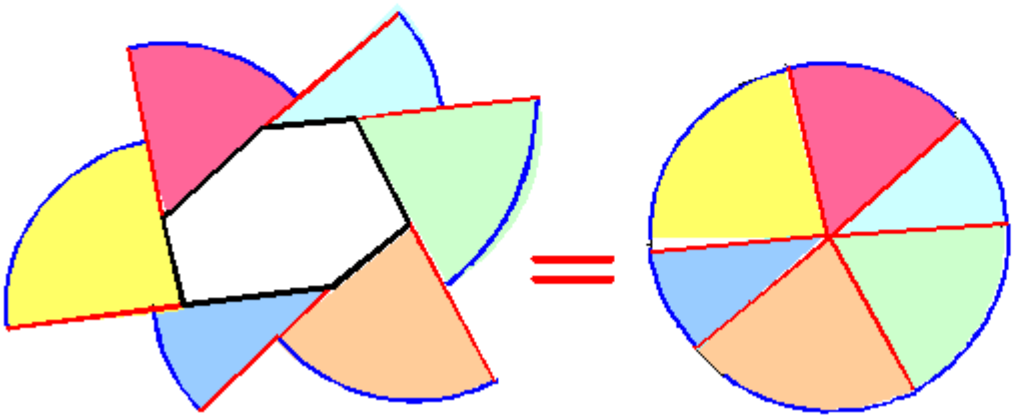


Figure 11. Region swept out as the tangent segment moves around a convex polygon.

The area of the region swept out by a tangent segment of given length moving around any convex polygon is equal to the area of a circular disk whose radius is that length. Therefore the same is true for any convex curve that is a limit of convex polygons. This leads us to:

MAMIKON'S THEOREM FOR OVAL RINGS

All oval rings swept out by a line segment of given length with one endpoint tangent to a smooth closed plane curve have equal areas, regardless of the size or shape of the inner curve. Moreover, the area depends only on the length L of the tangent segment and is equal to ¼L2, the area of a disk of radius L, as if the tangent segment was rotated about its endpoint.

Incidentally, Mamikon's Theorem for oval rings provides a new proof of the Pythagorean Theorem, as illustrated in Figure 12.

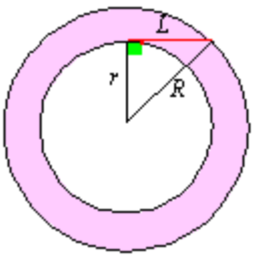


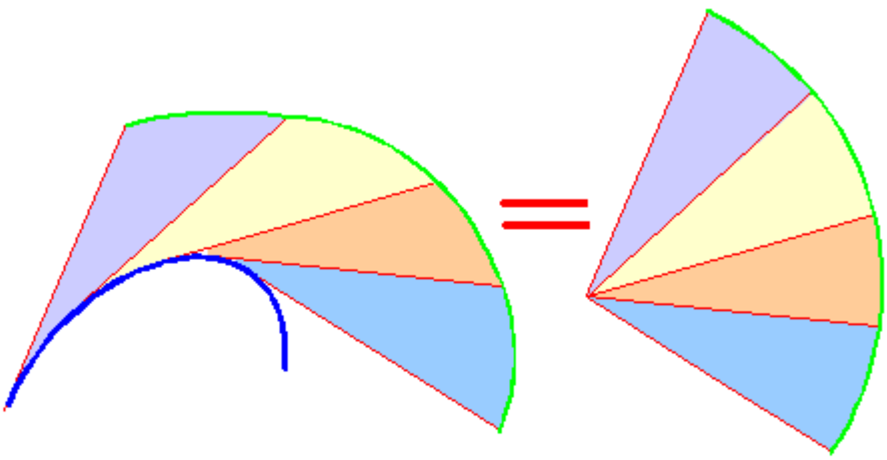
Figure 12. The Pythagorean Theorem deduced from Mamikon's Theorem on oval rings.

If the inner curve is a circle of radius r, the outer curve will also be a circle (of radius R, say) so the area of the oval ring is equal to the difference ¼R2 &endash; ¼r2. But by Mamikon's theorem, the area of the oval ring is also equal to ¼L2, where L is the constant length of the tangent segments. By equating areas we find R2 &endash; r2 = L2, from which we get R2 = r2 + L2, the Pythagorean Theorem *(for the right triangle RrL)*.

First generalization of Mamikon's Theorem

A generalized version of Mamikon's theorem is illustrated in Figure 13. The lower curve in Figure 13(a) is a more or less arbitrary smooth curve. Tangent segments to this curve of constant length sweep out a region, which is bounded by the lower curve and an upper curve traced out by the other extremity of each tangent segment. The exact shape of this region will depend on the lower curve and on the length of the tangent segments. We refer to this region as a tangent sweep.

When each tangent segment is translated parallel to itself so that each point of tangency is brought to a common point as shown in Figure 13(b), the set of translated segments is called the tangent cluster. (The tangent segments have been clustered together to emanate from a common point.) When the tangent segments have constant length as in this figure, the tangent cluster is a circular sector whose radius is that constant length.



(a) (b) (c)

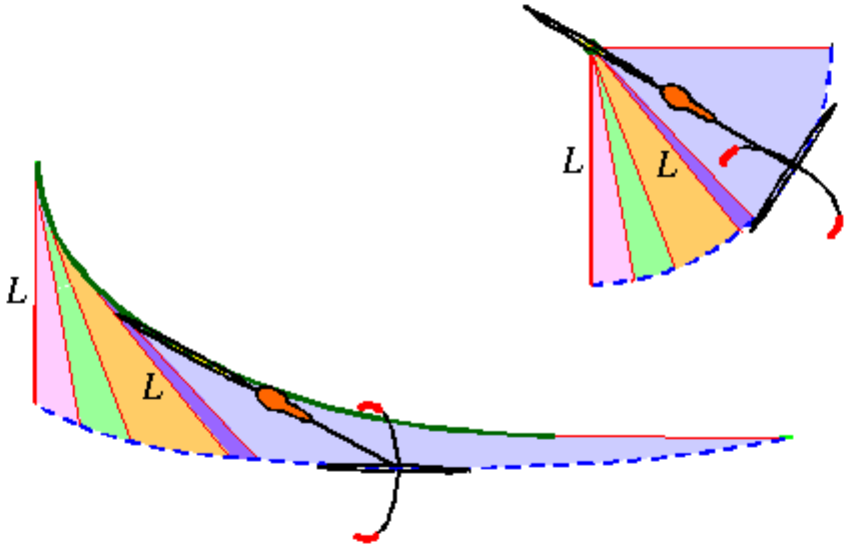
Figure 13. The tangent sweep and the tangent cluster for a general plane curve.

By the way, we could also translate the tangent segments so the other endpoints are brought to a common point, as in Figure 13(c). The resulting tangent cluster is a symmetric version of the cluster in (b). Now we can state:

MAMIKON'S THEOREM

The area of a tangent sweep is equal to the area of its *tangent cluster, regardless of the shape of the original curve.*

A physical example occurs when a bicycle's front wheel traces out one curve while the rear wheel (at constant distance from the front wheel) traces out another curve, as shown in Figure 14(a). The area of the tangent sweep is equal to the area of a circular sector depending only on the length of the bicycle and the change in angle from its initial position to its final position, as shown in Figure 14(b). The shape of the bike's path does not matter!

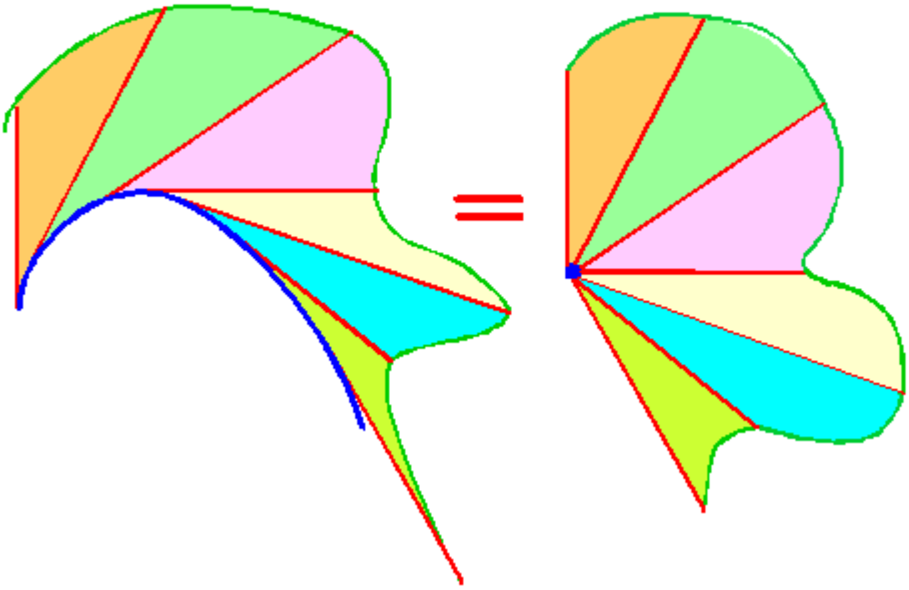


(a) (b)

Figure 14. Tangent sweep and tangent cluster generated by a moving bicycle.

Traxctrix and oval rings are particular cases of Bicyclix.

Figure 15 illustrates the same ideas in a more general setting. The only difference is that the tangent segments to the lower curve need not have constant length. The tangent segments sweep out a region called the tangent sweep (shown in Figure 15(a)). The tangent cluster is the region obtained by translating each tangent segment parallel to itself so that each point of tangency is moved to a common point (Figure 15(b)).



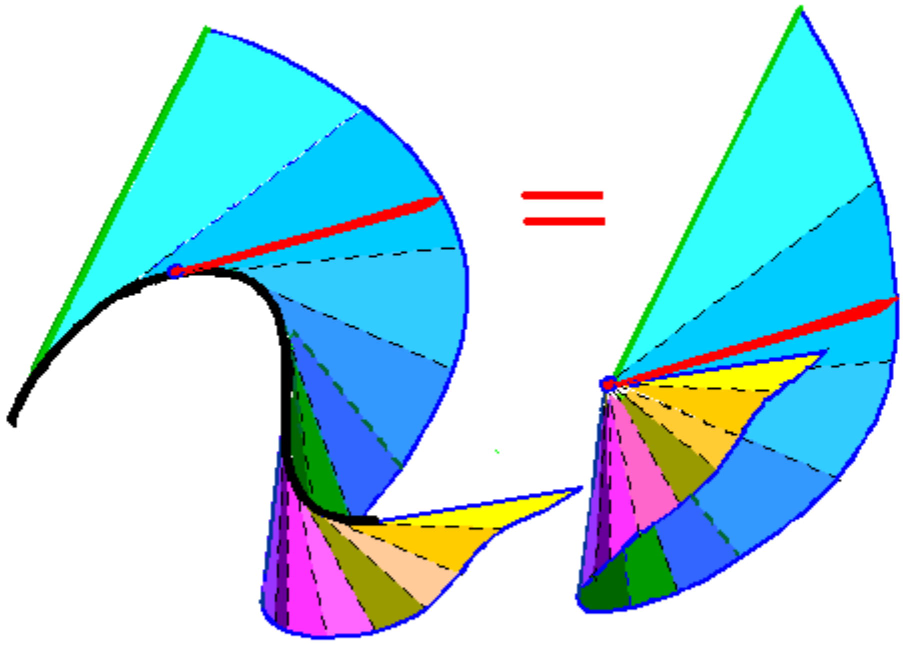
(a) (b)

Figure 15. The tangent sweep and the tangent cluster of a general plane curve.

Mamikon's theorem, which seems intuitively obvious by now, is that the area of the tangent cluster is equal to the area of the tangent sweep.

In the most general form of Mamikon's theorem the given curve need not lie in a plane. It can be any smooth curve in space, and the tangent segments can vary in length. The tangent sweep will lie on a developable surface. The shape of the tangent sweep depends on how the lengths and directions of the tangent segments change along the curve. When each tangent segment is translated parallel to itself so the point of tangency is brought to a common point, the set of translated segments is called the tangent cluster; it lies on a conical surface with vertex at this common point. Mamikon's general theorem equates the area of the tangent sweep with that of its tangent cluster.

GENERAL FORM OF MAMIKON'S THEOREM



The area of a tangent Sweep to a space curve is equal to the area of its conical Ikon.

This theorem, suggested by geometric intuition, can be proved also in a traditional manner, for example, by using differential geometry.

My first reaction to this theorem was "OK, that's a cool result in geometry. It must have some depth because it implies the Pythagorean Theorem. Can you use it to do anything else that's interesting?"

What you are about to see is a wide range of applications of this theorem. As already mentioned, curves swept out by tangent segments of constant length include oval rings and the tangent sweep of a bicycle. Another such example is the tractrix, the trajectory of a toy being pulled by a string of constant length by a child walking along a fixed straight line as shown in Figure 4.

All the examples with tangents of constant length reveal the striking property that the area of the tangent cluster can be expressed in terms of the area of a circular sector without using any of the formal machinery of traditional calculus. And what is more important, the animation shows why this happens.

But the most striking applications are to examples in which the tangent segments are not of constant length. These examples reveal the true power of Mamikon's method. The next example relates to exponential curves.

Exponential curves

Exponential functions are ubiquitous in the applications of mathematics. They occur in problems concerning population growth, radioactive decay, heat flow, and other physical situations where the rate of growth of a quantity is proportional to the amount present. Geometrically, this means that the slope of the tangent line at each point of an exponential curve is proportional to the height of the curve at that point. Exponential curves can also be described by their subtangents. The diagram in Figure 16 shows a general curve with a tangent line and the subtangent (the projection of the tangent on the x axis). The slope of the tangent is the height divided by the length of the subtangent. So, the slope is proportional to the height if and only if the subtangent is constant.

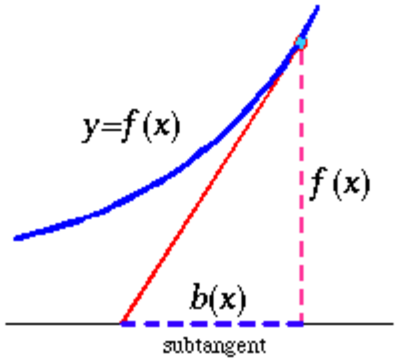


Figure 16. The slope of a curve is its height divided by the length of the subtangent.

The next diagram shows the graph of an exponential curve y = ex/b, where b is a positive constant. The only property of this curve that plays a role in this discussion is that the subtangent at any point has constant length b. This follows easily from differential calculus, but it can also be taken as the defining property of the exponential. In fact, exponential curves were first introduced in 1684 when Leibniz posed the problem of finding all curves with constant subtangents. The solutions are the exponential curves.

By exploiting the fact that exponential curves have constant subtangents, we can use Mamikon's theorem to find the area of the region under an exponential curve without using integral calculus. Figure 17 shows the graph of the exponential curve y = ex/b together with its tangent sweep as the tangent segments, cut off by the x axis, move to the left, from x all the way to _ ƒ. The corresponding tangent cluster is obtained by translating each tangent segment to the right so the endpoint on the x axis is brought to a common point, in this case, the lower vertex of the right triangle of base b and altitude ex/b. The resulting tangent cluster is the triangle of base b and altitude ex/b. Therefore the area of this region is equal to the area of this right triangle, so the area of the region between the exponential curve and the interval (_ ƒ, x] is equal to twice the area of this right triangle, which is its base times its altitude, or bex/b, the same result you would get by integration.

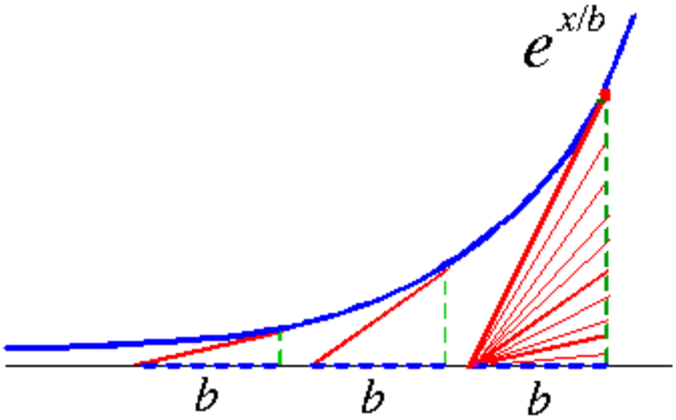
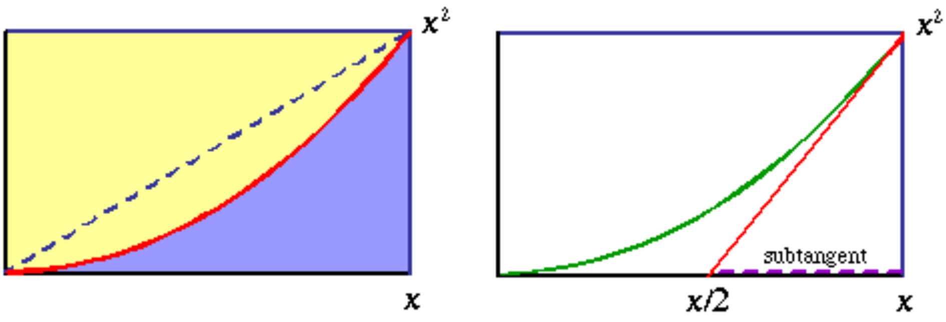


Figure 17. Finding the area of the region under an exponential curve by Mamikon's Method.

This yields the astonishing result that the area of the region under an exponential curve can be determined in an elementary geometric way without the formal machinery of integral calculus!

Area of a parabolic segment

We turn now to what is perhaps the oldest calculus problem in history&emdash;finding the area of a parabolic segment, the shaded region in Figure 18(a). The parabolic segment is inscribed in a rectangle of base x and altitude x2. The area of the rectangle is x3. From the figure we see that the area of the parabolic segment is less than half that of the rectangle in which it is inscribed. Archimedes made the stunning discovery that the area is exactly one-third that of the rectangle. Now we will use Mamikon's theorem to obtain the same result by a method that is not only simpler than the original Archimedes treatment but is also more powerful because it can be generalized to higher integer powers, and to arbitrary real powers as well.



(a) (b)

Figure 18. (a) A parabolic segment. (b) The subtangent to the parabola .

The parabola shown in Figure 18 has equation y = x2, but we shall not need this formula in our analysis. We use only the fact that the tangent line above any point x cuts off a subtangent of length x/2, as indicated in Figure 18(b). The slope of the tangent is x2 divided by x/2, or 2x.

To calculate the area of the parabolic segment we look at Figure 19 in which another parabola y = (2x)2 has been drawn, exactly half as wide as the given parabola. It is formed by bisecting each horizontal segment in the diagram. The two parabolas divide the rectangle into three regions, and our strategy is to show that all three regions have equal area. If we do this, then each has area one-third that of the circumscribing rectangle, as required.

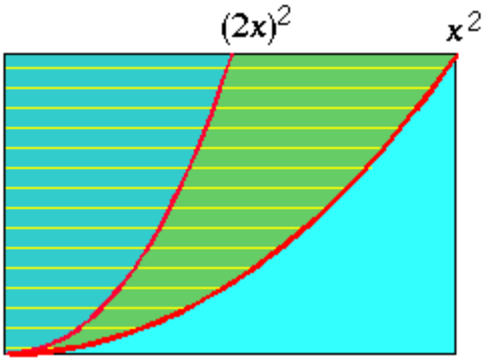


Figure 19. The two parabolas divide the rectangle into three regions of equal area.

The two shaded regions in Figure 19 formed by the bisecting parabola obviously have equal areas, so to complete the proof we need only show that region above the bisecting parabola has the same area as the parabolic segment below the original parabola. To do this we examine Figure 20. The two right triangles in this figure have equal area (they have the same altitude and equal bases). Therefore the problem reduces to showing that the two shaded regions in this diagram have equal areas. Here's where we use Mamikon's theorem.

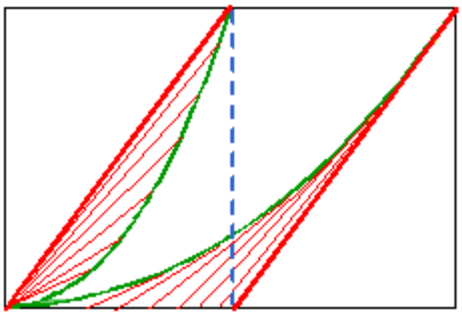


Figure 20. The tangent sweep of the lower curve is the tangent cluster above the upper one.

The shaded portion under the parabola y = x2 is the tangent sweep obtained by drawing all the tangent lines to the parabola and cutting them off at the x axis. And the other shaded portion is its tangent cluster, with each tangent segment translated so its point of intersection with the x axis is brought to a common point, the origin.

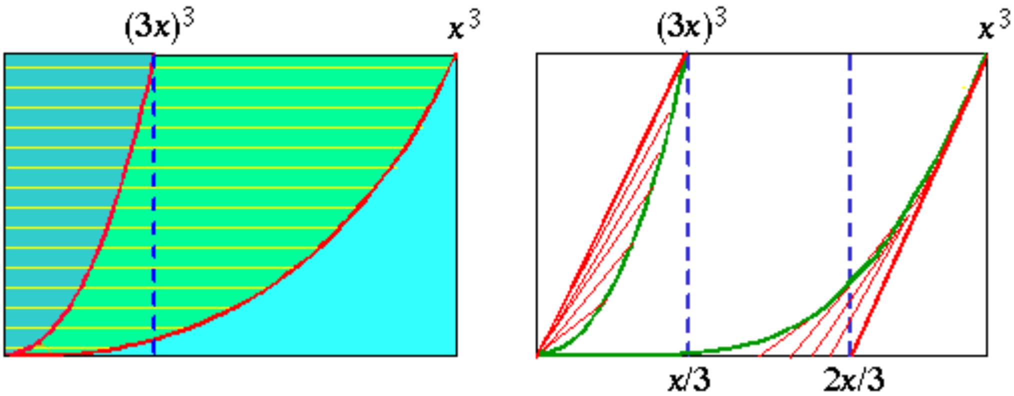
At a typical point (t, t2) on the lower parabola, the tangent intersects the x axis at t/2. Therefore, if the tangent segment from (t/2, 0) to (t, t2) is translated left by the amount t/2, the translated segment joins the origin and the point (t/2, t2) on the curve y = (2x)2. So the tangent cluster of the tangent sweep is the shaded region above the curve y = (2x)2, and by Mamikon's theorem the two shaded regions have equal areas, as required. So we have shown that the area of the parabolic segment is exactly one-third that of the circumscribing rectangle, the same result obtained by Archimedes.

Area of a generalized parabolic segment

The argument used to derive the area of a parabolic segment extends to generalized parabolic segments, in which x2 is replaced by higher powers. Figure 21(a) shows the graphs of y = x3 and y = (3x)3, which divide the rectangle of area x4 into three regions. The curve y = (3x)3 trisects each horizontal segment in the figure, hence the area of the region above this cubic is half that of the region between the two cubic curves. In this case we will show that the area of the region above the trisecting cubic is equal to that below the original cubic, which means that each region has area one-fourth that of the circumscribing rectangle.

To do this we use the fact that the subtangent to the cubic is one-third the length of the base, as shown in Figure 21(b). One shaded region in Figure 21(b) is the tangent sweep of the original cubic, and the other is the corresponding tangent cluster, so they have equal areas.

The two right triangles are congruent, so they have equal areas. Therefore the region above the trisecting cubic has the same area as the cubic segment below the curve y = x3, and each is one-fourth that of the rectangle, or x4/4.



(a) (b)

Figure 21. Mamikon's method used to find the area of a cubic segment.

In the quartic case we use the two curves y = x4 and y = (4x)4 to divide the rectangle of area x5 into three regions. Using the fact that the subtangent to the quartic at x has length x/4, we can use the same argument to show that the area of the region between the two quartics is three times that of each of the other two pieces, so the quartic segment below y = x4 has area one-fifth that of the rectangle, or x5/5. The argument also extends to all higher powers, a property not shared by the Archimedes treatment of the parabolic segment. For the curve y = xn we use the fact the subtangent at x has length x/n.

Cycloid

We turn next to the cycloid, the curve traced out by a point on the perimeter of a circular disk that rolls without slipping along a horizontal line. A classical problem is to show that the area of the region between one arch of the cycloid and the horizontal line is three times the area of the rolling disk, as suggested by Figure 22.

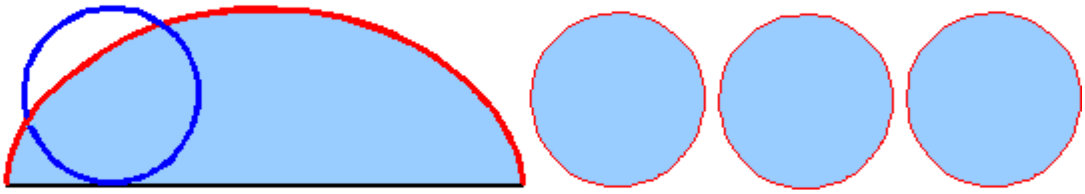


Figure 22. The area of the region under one arch of a cycloid is three times that of the rotating circular disk.

The standard calculus method of solving this problem is to first determine parametric equations for the cycloid, then calculate the area by integration. The same result can be obtained from Mamikon's theorem without the need to find parametric equations or to perform any integration.

Figure 23 shows a cycloidal arch inscribed inside a rectangle whose altitude is the diameter d of the rolling disk and whose base is the circumference of the disk, ¼d. The area of the circumscribing rectangle is ¼d2, which is four times the area of the disk. So it suffices to show that the unshaded region above the arch and inside the rectangle has area equal to that of the disk.

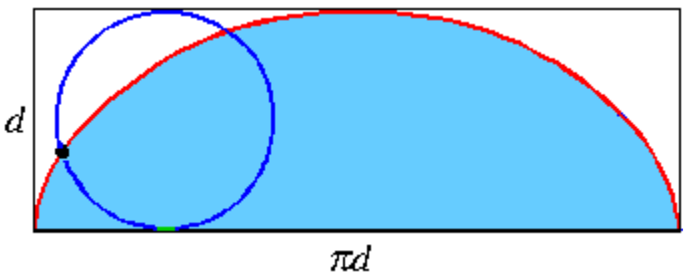


Figure 23. The unshaded region above the arch and inside the rectangle has area of the rolling circle.

To do this we show that the unshaded region is the tangent sweep of the cycloid, and the corresponding tangent cluster is a circular disk of diameter d. By Mamikon's theorem, this disk has the same area as the tangent sweep. Because the area of the disk is one-fourth the area of the rectangle, the area of the region below the arch must be three-fourths that of the rectangle, or three times that of the rolling disk.

It remains to shows that the tangent cluster of the unshaded region is a circular disk as asserted. As the disk rolls along the base it is always tangent to the upper and lower boundaries of the circumscribing rectangle. Denote the upper point of tangency by P and the lower point of tangency by P0, as in Figure 24.

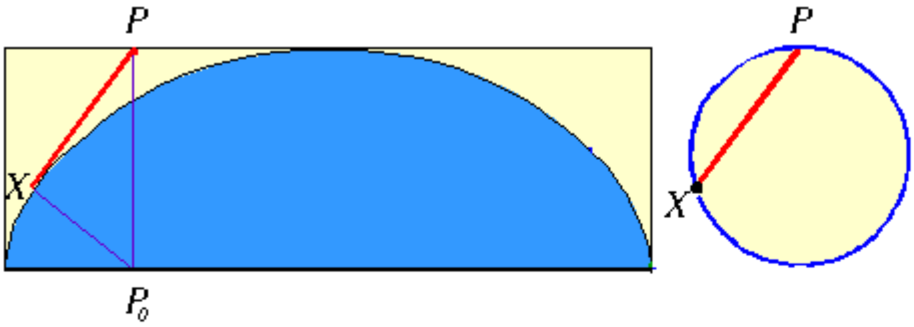


Figure 24. Proof that the tangent segment to the cycloid is a chord of the rolling disk.

The diameter PP0 divides the rolling circle into two semicircles, and any triangle inscribed in these semicircles must be a right triangle. The disk undergoes instantaneous rotation about P0, so the tangent to the cycloid at any point X is perpendicular to the instantaneous radius of rotation and therefore must be the vertex of a right triangle inscribed in the semicircle with diameter PP0. Consequently, the chord XP of the rolling disk is always tangent to the cycloid.

Extend the upper boundary of the circumscribing rectangle beyond the arch and choose a fixed point O on this extended boundary. Translate each chord parallel to itself so point P is moved horizontally to the fixed point O. Then the other extremity X moves to a point Y such that segment OY is equal in length and parallel to PX. Consequently, Y traces out the boundary of a circular disk of the same diameter, with OY being a chord equal in length and parallel to chord PX. Therefore the tangent cluster is a circular disk of the same diameter as the rolling disk, and Mamikon's theorem tells us that its area is equal to that of the disk.

New results

In the time that remains, I would like to mention a new discovery made as a result of Mamikon's investigations. Recall two of the examples mentioned earlier: the tractrix (which has constant tangents) and the exponential (which has constant subtangents). The tractrix and exponential have been studied for centuries, but apparently no one realized that they are related to one another. Mamikon has discovered that they are part of a new family of curves that we will describe presently.

Figure 25 shows an arbitrary curve together with a given base line (shown here horizontally as the x axis). At a general point P of this curve a tangent segment of length t cuts off a subtangent of length s along the base line.

As before, we can form the tangent cluster by translating each tangent segment of length t parallel to itself so the point of tangency is brought to a common point O, as in Figure 25. Let C denote the other endpoint of the tangent. As P moves along the given curve, point C traces out the curve defining the tangent cluster. We can also translate the subtangent of length s. These subtangents will be parallel to the given base line (shown horizontal in Figure 25). One endpoint of the translated subtangent is at C. When point P moves along a tractrix, t is constant and C moves along a circle. When point P moves along an exponential, s is constant and C moves along a line.

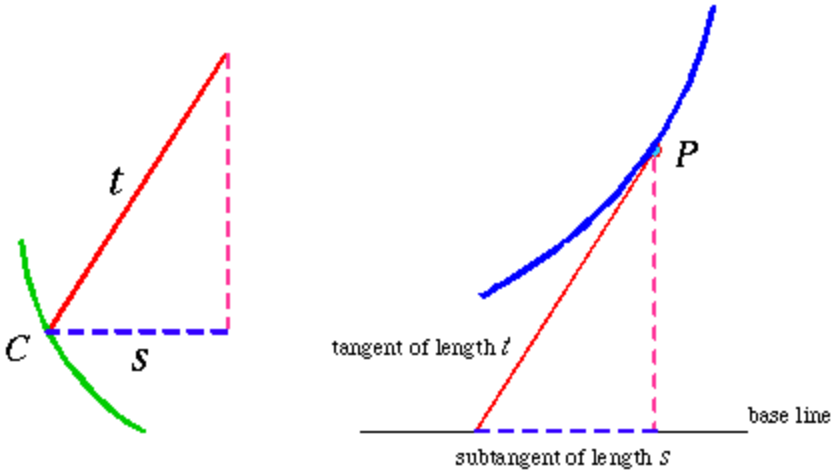


Figure 25. The tangent and subtangent to a general curve translated by the same amount.

Now suppose the original curve has the property that some linear combination of t and s is constant, say

at + bs= constant for some choice of a and b, with a „ 0 and b „ 0. What can we say about the path of C ?

When b = 0, the tangent t is constant and C lies on a circle. When a= 0 the subtangent s is constant and C lies on a straight line. Now we can easily show that for general a and b, C always lies on a conic section.

Let's see why this is true. If _ , 0, divide by _ and rewrite the equation as t + _s = constant, where _ = _/_

or t = constant &endash; _s = _(d &endash; s), where d is another constant. To show that C lies on a conic we refer to Figure 26.

Use point O as a focus and take as directrix a line perpendicular to the subtangents at distance d from the focus. The quantity (d &endash; s) is the distance of C from the directrix, and t is the distance of C from the focus. The equation t = _(d &endash; s) states that the distance of C from the focus is _ times its distance from the directrix. Therefore C lies on a conic section with eccentricity _. The conic is an ellipse, parabola, or hyperbola according as 0 < _ < 1, _ = 1, or _ > 1. The limiting cases _ = 0 and ƒ give a circle and straight line. So in this family of curves, the area swept out by the tangent segment is a portion of the area of a conic section.

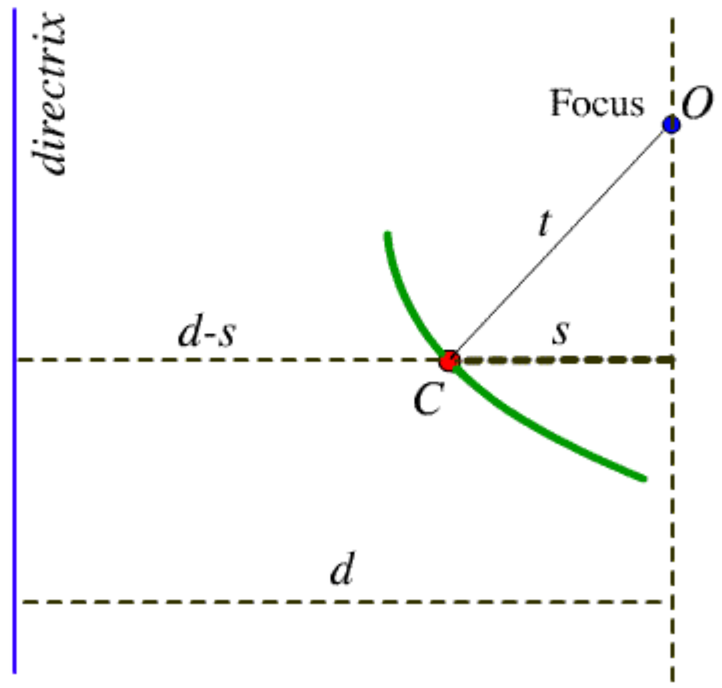


Figure 26. If _t + _s is constant, point C lies on a conic with eccentricity _/_.

Thus, we have learned something new. Both the tractrix and the exponential, which have been studied for centuries, turn out to be special cases of a family of curves characterized by the equation _t + _s = constant. The tangent cluster of each member of this family is bounded by a portion of a conic section.

Summary

The foregoing examples display a wide canvas of geometric ideas that can be treated with Mamikon's methods. Mamikon and I believe that video is the ideal medium for communicating these ideas. The examples discussed in this talk will form the core of a pilot videotape, the first of a series of contemplated videotapes we hope to produce under the umbrella of Project MATHEMATICS!. Like all videotapes produced by Project MATHEMATICS!, the emphasis will be on dynamic visual images presented with the use of motion, color, and special effects that employ the full power of television to convey important geometric ideas with a minimal use of formulas.

The animated sequences will be designed by Mamikon himself, using Flash Animation or Java Applets, which can easily be placed on the Internet and accessed from the Project's web site: http://www.projectmathematics.com. Professional animators will be used to render the Flash Animation to a format suitable for broadcast quality television.

These animated sequences will reveal in a dynamic way how tangent sweeps are generated by moving tangent segments, and how the tangent segments can be translated to form tangent clusters. They will also reveal that many classical curves can be generated in a natural way by their intrinsic geometric and mechanical properties.

It should be pointed out that Mamikon's methods are also applicable to many plane curves not mentioned above. We plan to treat these in subsequent videotapes. For example, the following figures have been successfully treated by this method: ellipse, hyperbola, catenary, logarithm, cardioid, epi-cycloid, hypo-cycloids, involutes, evolutes, Archimedean spiral, Bernoulli lemniscate, sine and cosine.

The methods also apply to finding volumes of three-dimensional figures such as the ellipsoid, paraboloid, three types of hyperboloid, catenoid, pseudosphere, torus, and other solids of revolution.

I'd like to conclude with a small philosophical remark about calculus. Newton and Leibniz are generally regarded as the discoverers of integral calculus. Their great contribution was to unify work done by many other pioneers and to relate the process of integration with the process of differentiation. If you analyze Mamikon's method you see that it has some of the same ingredients, because it relates moving tangent segments with the areas of the regions swept out by these tangent segments. So the relation between differentiation and integration is naturally imbedded in Mamikon's method (The construction of the tangent cluster is based on the knowledge of the slope of the initial curve).