Prove that in a parabola the tangent at one end of a focal chord is parallel to the normal at the other end.

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> Prove that in a parabola the tangent at one end of a focal chord is parallel to the normal at the other end.

Now, I know prove this algebraically, and that's very easy, but I am not getting any visual picture of the above situation. It'd be great if someone could give a proof without (or with minimal) words for this one - these proofs are exciting!

EDIT: A quick-n-dirty working of what I call an algebraic proof:

WLOG, let the equation of the parabola be $y^2 = 4ax$. The coordinates of points on the focal chord and also on the parabola are: $P(at^2, 2at)$ and $Q(al^2, 2am)$. For these points to lie on a focal chord, $tm = -1$. The tangent at P is given as $y(2at) = 2a(x + at^2)$ so the slope is $1/t$. Similarly the slope of tangent at Q is $1/m$. So the slope of the normal at Q is $-m = 1/t =$ slope of tangent at P.

Now, I know I am using some 'shortcuts' here, but then this was just a fast-paced look at what I did. The point is, I know how to prove the statement using the regular 'equations approach'. I want to know if there are any visual proofs.

recreational-mathematics conic-sections alternative-proof

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edited Aug 25, 2013 at 9:49

asked Aug 25, 2013 at 9:27

Parth Thakkar
4,254 3 29 48

By algebraic, do you mean something like in.answers.yahoo.com/question/index?qid=20110919035218AAeNUxa ? – lab bhattacharjee Aug 25, 2013 at 9:32

Something similar. I'll just add a quick-n-dirty working of mine. – Parth Thakkar Aug 25, 2013 at 9:33

@labbhattacharjee, updated. – Parth Thakkar Aug 25, 2013 at 9:41

you shoud start with "WLOG we can assume the eqaution of the parabola to be $y^2 = 4ax$". What's meant by 'shortcuts'? – lab bhattacharjee Aug 25, 2013 at 9:45

I know we should *ideally* start with all those details...but that wasn't the point. And by shortcuts, I mean using the fact that *tm* = *-1*. In fact, using that fact is kind of 'cheating'. That fact (as it seems to me) comes from the theorem I am trying to prove. – Parth Thakkar Aug 25, 2013 at 9:48

3 Answers

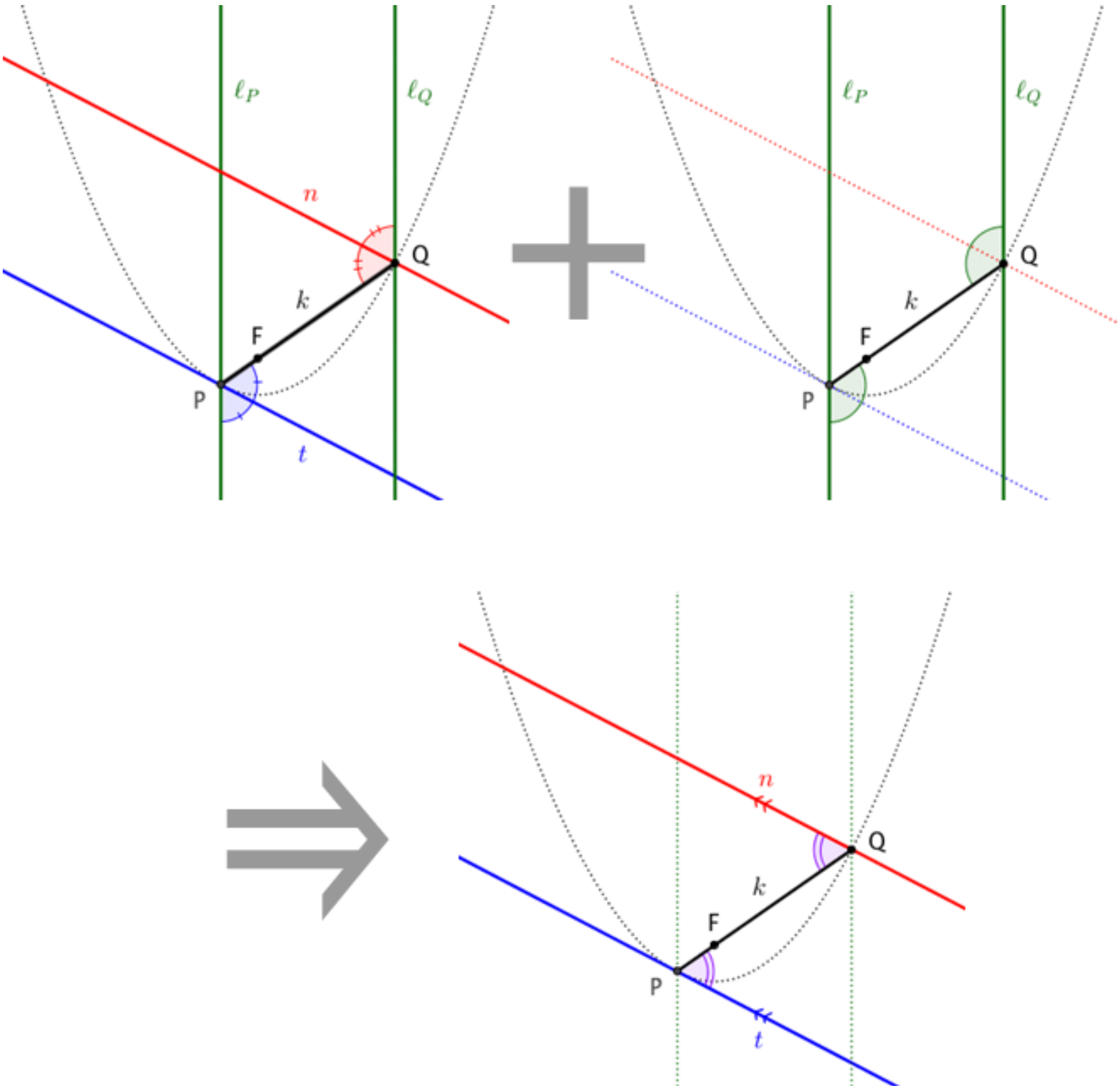
Sorted by: Highest score (default)

5

Let $k := \overleftrightarrow{PQ}$ be a line containing focal chord $\overline{PQ}$. Let lines ℓ_P and ℓ_Q, through P and Q, respectively, be parallel to the parabola's axis.

By the [Reflection Property of Parabolas](), the normal and tangent at P bisect angles made by k and ℓ_P; likewise, at Q. In particular, the tangent at P (call it t) and the normal at Q (call it n) bisect *alternate interior angles* formed by parallel lines ℓ_P and ℓ_Q cut by transversal k. These angles are congruent, so their half-angles are congruent. A pair of such half-angles comprise the alternate interior angles of t and n cut by k, whence $t \parallel n$: the tangent at P is parallel to the normal at Q.

Proof without words



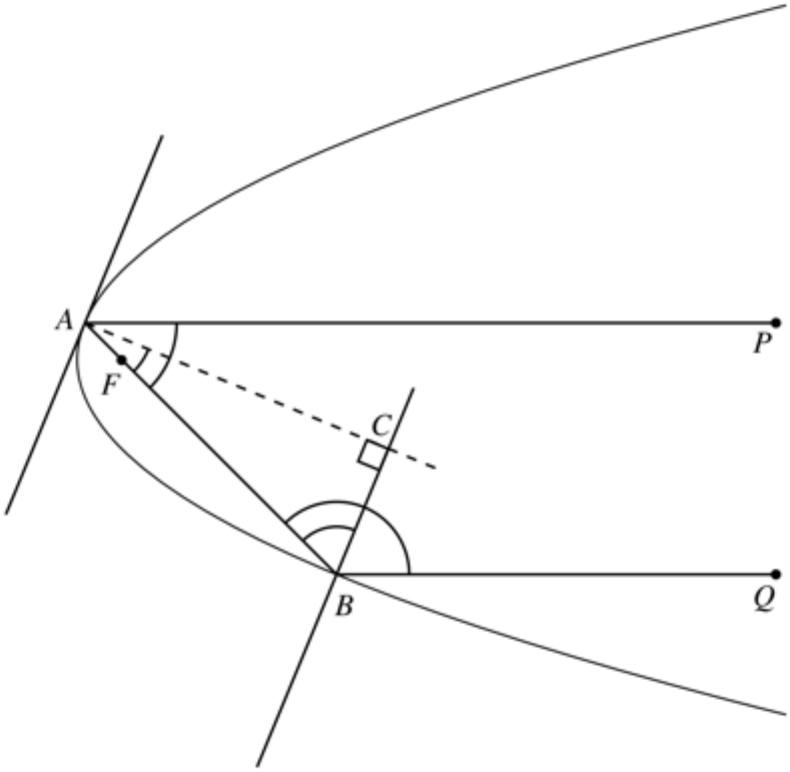
2 I fell for the proof without words. Off late I've started loving these things :D – Parth Thakkar Aug 25, 2013 at 18:12

math.stackexchange.com/questions/1756391/… – Beautifully irrational Aug 5, 2021 at 8:09

@Blue could you pls see this post thanks. – Beautifully irrational Aug 5, 2021 at 8:10

2

In the following diagram, the axis of the parabola is horizontal and the focus F is at the dot. The line through the focus intersects the parabola at A and B.



Note that $\angle PAB$ and $\angle ABQ$ are supplementary (they sum to π) since $\overline{AP}$ and $\overline{BQ}$ are both parallel to the axis. By the reflection property for parabolas, $\overline{AC}$, being perpendicular to the tangent at A, bisects $\angle PAB$, and $\overline{BC}$, being normal to the parabola at B, bisects $\angle ABQ$. Thus, $\angle CAB$ and $\angle CBA$ are complementary (they sum to $\pi/2$), and therefore, $\triangle ACB$ is a right triangle. Since $\overline{AC}$ is perpendicular to the tangent at A and perpendicular to the normal at B, the tangent at A and the normal at B are parallel.

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edited Aug 26, 2013 at 15:57

answered Aug 25, 2013 at 18:47

robjohn ♦
328k 34 424 802

I guess if you know the reflection property for parabolas, the diagram above would be a proof without words. – robjohn ♦ Aug 25, 2013 at 19:09

1

Here's a geometric proof, based on the fact that a line (thought of as a light ray) going through the focus of a parabola reflects to a line parallel to the axis of the parabola. This is sometimes called the reflective property of the parabola. Call the focus F, and have the parabola arranged with its axis the y axis. Pick the chord BA through F, so that A lies to the right of F and B to the left.

We need to name some reference points: Pick a point A_L to the left of A on the tangent line T_A, and another point A_R to its right. Similarly pick the points B_L, B_R to the left and right of B on the tangent line T_B. Also pick a point A' above A on the vertical through A and another point A'' below A on that vertical; similarly pick points B' and B'' above and below B on the vertical through B.

Now the reflective property of the parabola means in this notation that the angles $A_L AF$ and $A' AA_R$ are equal. Call that common angle α, and note that by the vertical angle theorem (opposite angles of intersecting lines are equal) we also have α equal to the angle $A_L AA''$.

Similarly we have the three equal angles, call each β, namely angles $B_R BF$ and $B_L BB'$ from the reflective property and the further equal angle in this triple $B_R BB''$ again from the vertical angle theorem.

Now because the segment BA may be extended to a line transverse to the two parallel verticals through A and B, we have that angle $B'' BB_R$ is equal to angle BAA'. A diagram shows that the first of these is 2β, while the second is $\pi - 2\alpha$. This brings us almost to the end of the argument, since we now have $\alpha + \beta = \pi$. So if we let the two tangent lines $T_A,\ T_B$ meet at the point P, we see (again referring to a sketch) that triangle BPA is a right triangle with its right angle at P. But this means the two tangent lines $T_A,\ T_B$ are perpendicular, so we may conclude finally that the normal line N_B through B is parallel to the tangent line T_A at A.