Circular and Hyperbolic Trig Functions Compared

This is mostly a "just for fun" page. It doesn't add much to what's already discussed elsewhere on the site. However, I found it interesting to go point by point through the circular and hyperbolic functions, so I decided to write it up.

We'll be starting with formulaic definitions of the functions and derivations of some of their properties, both to simplify what comes later and to give us a firm foundation for some of the derivations. We'll be developing the geometric properties of the functions as we go. Bear with us; there's going to be a rather large slab of algebra before we get to any pictures.

In an additional departure from what we've done elsewhere, we'll be defining the trig functions in terms of the exponential function, with which we'll start things off. This simplifies the derivations (a lot) and also makes the similarities between the circular and hyperbolic functions a little clearer (or at least I hope it does).

As a final heads-up, we'll be using x, y, u, v, t, and θ as real variables, and we'll be using z as a complex variable. We'll generally mention what each means as we go so the convention isn't critical.

1. The Exponential Function

We define the exponential function, e^z, as the function which has these two properties:

(1.1)
$$\begin{aligned} &\text{(a)} \quad e^{(z_0+z_1)} &&= e^{z_0} \cdot e^{z_1} \\ &\text{(b)} \quad \left.\frac{d}{dz}e^z\right|_{z=0} &&= 1 \end{aligned}$$

1.1 (a) says it behaves like an exponential function. (Lots of other functions behave that way, too, such as 10^x, for example.) 1.1 (b) says its derivative at zero is 1.

These two properties together fully define it. We'll need some additional (derived) properties as well; we've put their derivations, along with a brief discussion of complex numbers, over here to reduce the bloat on this page a bit.

The derivative of e^z is itself:

(1.2)
$$\frac{d}{dz}e^z = e^z$$

The magnitude of $e^{i\theta}$ is 1, for all real θ; consequently $e^{i\theta}$ maps the real line onto the *unit circle* in the complex plane:

(1.3)
$$\| e^{i\theta} \|^2 = e^{i\theta} \cdot \overline{e^{i\theta}} = e^{i\theta} e^{-i\theta} = 1$$

And finally,

(1.4) If we view the complex plane as an analog to $\mathbf{R}^2$, with real and imaginary parts forming the x and y components of the vectors, then the angle between the position vector of $e^{i\theta}$ and the x axis is θ.

In particular, property (1.4) implies that $e^{i\pi/2} = i$ and $e^{i\pi} = -1$.

2. Definitions of the Trig Functions

We'll be defining the sine and cosine directly, and the other functions in terms of them. For both the circular and hyperbolic functions, the tangent is the sine over the cosine, the cotangent is the inverse of the tangent, the secant is the inverse of the cosine, and the cosecant is the inverse of the sine.

2.1 The Circular Sine and Cosine

We define the sine and cosine as the imaginary and real parts of $e^{i\theta}$, respectively:

(2.1.1)
$$\begin{aligned} \sin(\theta) &\equiv \mathrm{Im}(e^{i\theta}) \\ \cos(\theta) &\equiv \mathrm{Re}(e^{i\theta}) \end{aligned}$$

The real and imaginary parts of a number can be found by adding and subtracting its complex conjugate in such a way that either the real or imaginary parts cancel, so we have the following formulas for sine and cosine:

$$\begin{aligned} \sin(\theta) &= \frac{e^{i\theta}-\overline{e^{i\theta}}}{2} = \frac{e^{i\theta}-e^{-i\theta}}{2} \\ \cos(\theta) &= \frac{e^{i\theta}+\overline{e^{i\theta}}}{2} = \frac{e^{i\theta}+e^{-i\theta}}{2} \end{aligned} \tag{2.1.2}$$

Since a complex number is the sum of its real part and *i* times its imaginary part, we have:

$$e^{i\theta} = \cos(\theta) + i\ \sin(\theta) \tag{2.1.3}$$

Since we already know that $e^{i\theta}$ carries **R** onto the unit circle (shown here), we know that the point $(\cos(\theta),\sin(\theta))$ must fall on the unit circle, and from (1.4) we know the point is at angle θ with the *x* axis. (Which is what we wanted to find, of course!)

Equation (1.3) leads immediately to the the familiar identity,

$$\begin{aligned} 1 &= \|\cos(\theta) + i\cdot\sin(\theta)\|^2 \\ &= (\cos(\theta) + i\cdot\sin(\theta)\cdot(\cos(\theta) - i\cdot\sin(\theta)) \\ &= \cos^2(\theta) - i^2\cdot\sin^2(\theta) \\ &= \cos^2(\theta) + \sin^2(\theta) \end{aligned} \tag{2.1.4}$$

The derivatives of the sine and cosine also fall out of our definitions. We can just differentiate formulas (2.1.2) directly, and, by comparing the results with the definitions for sine and cosine, find their derivatives in about three lines.

Alternatively, we can differentiate $e^{i\theta}$. We have,

$$\frac{de^{i\theta}}{d\theta} = \cos'(\theta) + i\cdot\sin'(\theta) \tag{2.1.5}$$

But we know that e^z is its own derivative, so, using the chain rule, we also must have:

$$\begin{aligned} \frac{de^{i\theta}}{d\theta} &= i\cdot e^{i\theta} \\ &= i\,(\cos(\theta) + i\cdot\sin(\theta)) \\ &= i\cdot\cos(\theta) - \sin(\theta) \end{aligned} \tag{2.1.6}$$

Equating the corresponding real and imaginary parts from (2.1.5) and (2.1.6) we get where we were going, which is:

$$\begin{aligned} \sin'(\theta) &= \cos(\theta) \\ \cos'(\theta) &= -\sin(\theta) \end{aligned} \tag{2.1.7}$$

2.2 The Hyperbolic Sine and Cosine

We define the hyperbolic sine and cosine as the odd and even parts of e^θ, respectively, where θ is a real valued variable:

$$\begin{aligned} \sinh(\theta) &\equiv \mathrm{Odd}(e^\theta) \\ \cosh(\theta) &\equiv \mathrm{Even}(e^\theta) \end{aligned} \tag{2.2.1}$$

The breakdown of a function into odd and even parts is unique, and it goes like this:

$$\begin{aligned} \sinh(\theta) &= \frac{e^\theta-e^{-\theta}}{2} \\ \cosh(\theta) &= \frac{e^\theta+e^{-\theta}}{2} \end{aligned} \tag{2.2.2}$$

Since a function is the sum of its even and odd parts, we have:

$$e^\theta = \sinh(\theta) + \cosh(\theta) \tag{2.2.3}$$

Since *sinh* is odd and *cosh* is even, we also have:

$$e^{-\theta} = \sinh(-\theta) + \cosh(-\theta) = -\sinh(\theta) + \cosh(\theta) \tag{2.2.4}$$

If we multiply the left and right sides of (2.2.3) by $e^{-\theta}$, substituting in the right hand side from (2.2.4), we obtain the basic identity:

$$
\begin{aligned}
e^{\theta} \cdot e^{-\theta} &= (\sinh(\theta) + \cosh(\theta)) \cdot (\sinh(-\theta) + \cosh(-\theta)) \\
e^{\theta - \theta} &= (\sinh(\theta) + \cosh(\theta)) \cdot (-\sinh(\theta) + \cosh(\theta)) \\
1 &= \cosh^2\theta - \sinh^2\theta
\end{aligned} \tag{2.2.5}
$$

This tells us that the points (cosh *θ*, sinh *θ*) must lie *on the unit hyperbola* (which is the locus of the equation $x^2 - y^2 = 1$).

Finally, differentiating equations (2.2.2) leads to the derivatives for the hyperbolic functions:

$$
\begin{aligned}
\sinh'(\theta) &= \cosh(\theta) \\
\cosh'(\theta) &= \sinh(\theta)
\end{aligned} \tag{2.2.6}
$$

3. Sine and Cosine Interpreted Geometrically

We've shown the unit circle and the unit hyperbola, both centered on the origin. We've only drawn one branch of the unit hyperbola (there is another branch, symmetrically located on the left), and we haven't drawn the "co-hyperbola", which is the unit hyperbola rotated 90 degrees.

Figure 1: The Circular Functions

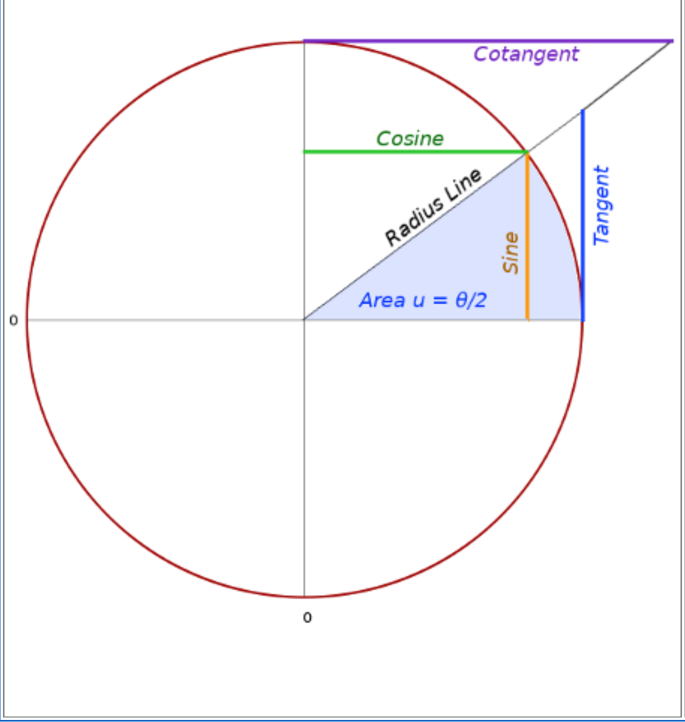
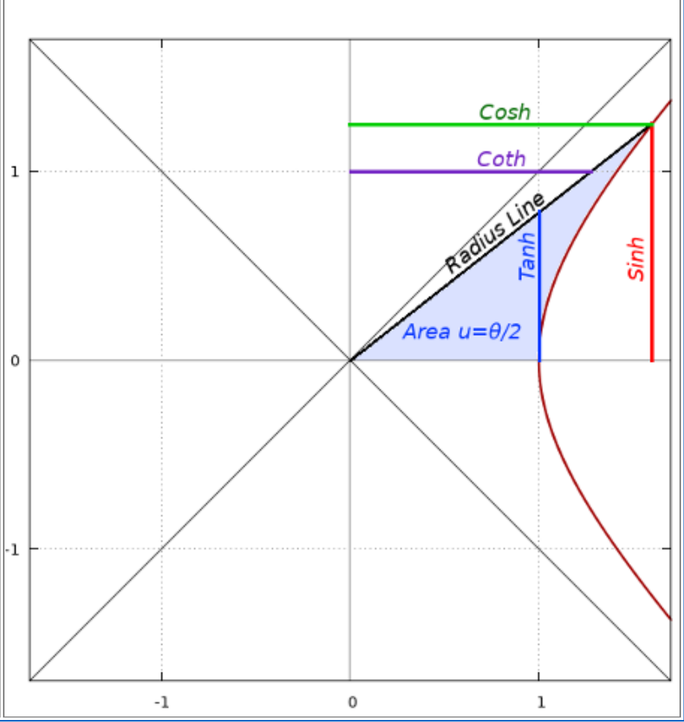


Figure 2: The Hyperbolic Functions



3.1 The Circular Functions

Draw a line from the origin out until it intersects the circle; we'll call this the "radius line". Draw a line segment from that intersection to the *x* axis. The length of that line segment is the sine (shown in orange in figure 1).

Draw a second line segment from the intersection to the *y* axis. The length of that segment is the cosine.

Put a tangent line on the circle, at the point where the circle crosses the *x* axis. Run that line segment up to where it hits the radius line. The length of that segment is the tangent.

Run another line from the *y* axis, also tangent to the circle, out until it intersects the radius line. The length of that line segment is the cotangent.

And finally, we consider the *angle* -- i.e., the argument to the trig functions. Consider the region "Area u" shown in blue. The area of the entire unit circle is π, and the "angle" of the entire circle is 2π. So, if the angle made by the radius line with the *x* axis is θ, then the area of *u* must be (θ/2π)·π = θ/2.

In other words, the "angle" (which is the argument to the trig functions) is twice the area of *u*.

3.2 The Hyperbolic Functions

Draw a line from the origin out until it intersects the hyperbola; we'll call this the "radius line". Draw a line segment from that intersection to the *x* axis. The length of that line segment is the hyperbolic sine (shown in red in figure 2).

Draw a second line segment from the intersection to the *y* axis. The length of that segment is the hyperbolic cosine.

Put a tangent line on the hyperbola, at point (1,0), which is where it crosses the *x* axis. Run that line segment up to where it hits the radius line. The length of that segment is the hyperbolic tangent.

Run another line from the *y* axis, at point (0,1), out until it intersects the radius line. The length of that line segment is the hyperbolic cotangent. (If we also showed the "co-hyperbola", which is rotated 90 degrees, the cotangent line would be tangent to it.)

And finally, we consider the *angle* -- i.e., the argument to the trig functions. Consider the region "Area u" shown in blue. It's a bit harder to prove than the case for the circular functions, but in fact the area of *u* is θ/2. We prove that (via straightforward integration) here.

In other words, the "angle" (which is the argument to the trig functions) is twice the area of *u*.

4. Bonus Section: Sum of Angles Formulas

Defining the functions as parts of the exponential function makes deriving the sum formulas easy, as well. It also shows the similarities between the circular and hyperbolic functions.

4.1 Circular Trig Functions of a Sum of Angles

We'll find the sine angle-sum formula and cosine angle-sum formula simultaneously. Basic property (1.1(a)) of the exponential function tells us that,

(4.1.1) $$e^{i\theta+i\phi} = e^{i\theta}\cdot e^{i\phi}$$

And from that and equation (2.1.3), we're almost done:

(4.1.2) $$\begin{aligned}\cos(\theta+\phi)+i\cdot\sin(\theta+\phi) &= e^{i(\theta+\phi)}\\ &= e^{i\theta}\cdot e^{i\phi}\\ &= (\cos\theta+i\sin\theta)\cdot(\cos\phi+i\sin\phi)\\ &= \cos\theta\cos\phi-\sin\theta\sin\phi+i(\cos\theta\sin\phi+\cos\phi\sin\theta)\end{aligned}$$

Equating the real and imaginary parts of the first and last formula, we obtain:

(4.1.3) $$\begin{aligned}\cos(\theta+\phi) &= \cos\theta\cos\phi-\sin\theta\sin\phi\\ \sin(\theta+\phi) &= \cos\theta\sin\phi+\cos\phi\sin\theta\end{aligned}$$

4.2 Hyperbolic Trig Functions of a Sum of Angles

We'll proceed as we did in the previous section. Basic property (1.1(a)) of the exponential function tells us that,

(4.2.1) $$e^{\theta+\phi} = e^{\theta}\cdot e^{\phi}$$

And with that and equation (2.2.3), we're almost done:

(4.2.2) $$\begin{aligned}\cosh(\theta+\phi)+\sinh(\theta+\phi) &= e^{\theta+\phi}\\ &= e^{\theta}\cdot e^{\phi}\\ &= (\cosh\theta+\sinh\theta)\cdot(\cosh\phi+\sinh\phi)\\ &= (\cosh\theta\cosh\phi+\sinh\theta\sinh\phi)+(\cosh\theta\sinh\phi+\cosh\phi\sinh\theta)\end{aligned}$$

The product of two even functions is even, and the product of two odd functions is even.

The product of an odd function with an even function is odd.

So, we can equate the even part of the last expression in (4.2.2) with the even part of the first expression, and equate the odd parts, and we obtain:

$$\begin{aligned}\cosh(\theta+\phi) &= \cosh\theta\cosh\phi+\sinh\theta\sinh\phi \\ \sinh(\theta+\phi) &= \cosh\theta\sinh\phi+\cosh\phi\sinh\theta\end{aligned} \tag{4.2.3}$$

and we're done.

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