Senior Thesis in Physics

Influence Theory: Foundations, Applications, and an Extension

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Contents

Abstract

In this thesis, the mathematical foundations of Influence Theory — a novel approach to understanding physical phenomena that posits only that (i) events take place and (ii) that such events may influence each other — are explored. The potential advantage of this novel approach is demonstrated through the traditional, axiom-laden derivation of the Lorentz transformation. The elementary formalism of partially ordered sets and influence theory is next introduced, with an emphasis on quantifications of intervals of events. This power of this new approach is then tested through an application of the theory to the earlier example of special relativity, which closely follows the prior work of Bahreyni and Knuth. Lastly, a novel quantification scheme of subspace projection is developed, which allows for the quantification of *any* generalized interval in a partially ordered set with respect to a pair of coordinated chains.

Acknowledgements

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I would also like to thank my family for always finding time to talk to me on the phone. After four years away in Claremont, I return home this spring more convinced than ever that the relationships I share with my parents, siblings, and aunt are as dearly important as life itself.

Will Ballard
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Chapter 1

Introduction

Связь причин и последствий не имеет начала и не может иметь конца.

Л.Н. Толстой

The thread of cause and consequence has no beginning and can have no end.

L.N. Tolstoy

1.1 Spacetime as an Emergent Concept

Modern physics occupies a unique position in science due to its ability to investigate fundamental questions about the nature of space and time. Despite countless attempts to establish an understanding of these phenomena, it was not until the development of Newtonian mechanics in the 17th century that humanity arrived at a quantitative description. During the following three centuries, Newton's assumptions of absolute space and time guided his successors in their attempts to develop additional physical theories [1]. Nonetheless, humanity's conception of space and time underwent a further revolution during the *annus mirabilis* of 1905, when Einstein published his theory of special relativity [2]. In this momentous paper, Einstein postulated the unification of space and time into a four-dimensional non-Euclidean geometric structure known as spacetime. Among the most important consequences of Einstein's theory was his realization that spatial and temporal intervals measured between events are not absolute, but rather dependent on the relative motion of the observer. In a manner reminiscent of the dominance of Newton's views of space and time on his successors, Einstein's description of flat spacetime as a four-dimensional hyperbolic geometric structure has guided the development of subsequent theories, most notably relativistic quantum field theory. To this day, special relativity and Einstein's further generalization of the theory represent humanity's most successful description of spacetime – the stage upon which reality unfolds.

Nevertheless, there is growing speculation within the physics community that a new transformation of our understanding of spacetime may lie on the horizon [3]. In particular, certain physicists have begun to question whether spacetime and its geometry are emergent, rather than fundamental, concepts. As will be discussed below, the traditional (Einsteinian) derivation of special relativity presupposes certain characteristics of space and time, as well as an invariance of the laws of nature for all observers in sufficiently well-behaved reference frames. That is, the theory assumes both "the prior existence of spacetime" and some of its properties [4]. Conversely, advocates of the view that spacetime is an emergent concept argue that there are more primitive notions from which one can derive its overall structure and geometry. Examples of such assumptions include causality, quantum entanglement, and the concept of holography in string theory [5–7].

The objective of this thesis is to explore a particular abstraction of causality — known as influence theory — as a potential cornerstone for emergent spacetime. Unlike most approaches to the foundations of special relativity, this formalism abandons any notions of space or time and assumes only that events take place and have the ability to influence each other. My primary aim will be to extend previous work on the conception of spacetime provided by influence theory through a demonstration that familiar geometric concepts – such as the dot product and cross product – emerge from the requirement that observers quantify the ordering of events linked by influence in a consistent fashion.

1.2 Historical Background and Terminology

Before launching into a discussion about special relativity, it is important to define a few fundamental concepts. We begin with the definition of an *event*. In qualitative terms, an event is something that takes place at a given position in space and at a particular instant in time. As we often desire to describe when and where a specific event takes place, we must also introduce the concept of a *frame of reference*. This term denotes the coordinate system that an observer uses to assign spatial and temporal coordinates to an event. Restricting our attention to one time dimension and three spatial dimensions, we can therefore describe an event as an element of $\mathbb{R}^4$ characterized by four spacetime coordinates (t, x, y, z) [8]. Of particular interest are *inertial* frames of reference, which we define as a frame of reference in which a free object moves with constant velocity.

It is possible to view the events of the natural world from a wide range of reference frames. For instance, one can show that a frame that moves with a constant velocity relative to an inertial reference frame also provides an inertial coordinate system [9]. A fundamental question then arises: how do we relate the spacetime coordinates $\mathsf{x} = (t, x, y, z)$ that an observer assigns to an event in one inertial reference frame S, to the distinct spacetime coordinates $\mathsf{x}' = (t', x', y', z')$ that an observer in another inertial reference frame S' assigns to the same event? We do so by defining a transformation matrix A, which provides a bijective mapping from the four-dimensional spacetime $\mathbb{R}^4$ onto itself. That is, given the coordinates of some event x in an inertial frame S, we calculate its coordinates x' in another inertial frame S' as:

$$\mathsf{x}' = \mathsf{A}\,\mathsf{x} \tag{1.1}$$

The primary intention of special relativity is to determine the correct form of the transformation matrix A that relates the coordinates of a given event in distinct inertial frames. This topic was of considerable interest at the turn of the 19th century due to the discovery of a discord between the traditional Galilean transformations between inertial frames – which assume absolute spatial and temporal intervals – and Maxwell's equations of electromagnetism. To be more specific, it was found that Maxwell's equations violated the sacrosanct *principle of relativity* under a Galilean transformation. This principle, which has its origins in Galileo's 1632 publication *Dialogue Concerning the Two Chief World Systems*, states that the laws of physics should be the same in all inertial frames of reference [10]. Thus, to say that Maxwell's equations violated the principle of relativity is to say that the form of the equations changed under a Galilean transformation from one inertial frame to another [11].

As these revelations emerged, a series of physicists – most notably Lorentz and Poincaré – discovered that the form of Maxwell's equations was the same in all inertial reference frames if one adopts a different transformation matrix L [12]. The group of transformation matrices compatible with Maxwell's equations became known as the *group of Lorentz transformations*.

Let's take a closer look at one such Lorentz tranformation matrix. We imagine two inertial reference frames S and S', whose spatial Cartesian axes are parallel. Because the relative velocity between any pair of inertial reference frames is constant, we can assume that the origin of the S' spatial coordinate axis moves with constant velocity $\vec{\beta}$ relative to the origin of the S spatial coordinate system along the common x-axis direction. Lastly, assume that the spatial origins of the two coordinate systems coincide at the same spatial position at $t = t' = 0$. We define this particular orientation of two inertial frames as the *standard orientation*, a diagram of which can be found in Figure I. Now, define some event x which acquires spacetime coordinates $\mathsf{x} = (t, x, y, z)$ according to the S coordinate system. Then, the appropriate Lorentz transformation states that the spacetime coordinates $\mathsf{x}' = (t', x', y', z')$ of this event in the S' reference frame are given by the matrix equation [9]:

$$\mathsf{x}' = \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\frac{\gamma\beta}{c} & 0 & 0 \\ -\frac{\gamma\beta}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \mathsf{L}\,\mathsf{x} \tag{1.2}$$

where c is the speed of light and $\gamma = [1 - (\beta/c)^2]^{-1/2}$. We see that the appropriate transformation matrix L has the form:

$$\mathsf{L} = \begin{bmatrix} \gamma & -\frac{\gamma\beta}{c} & 0 & 0 \\ -\frac{\gamma\beta}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.3}$$

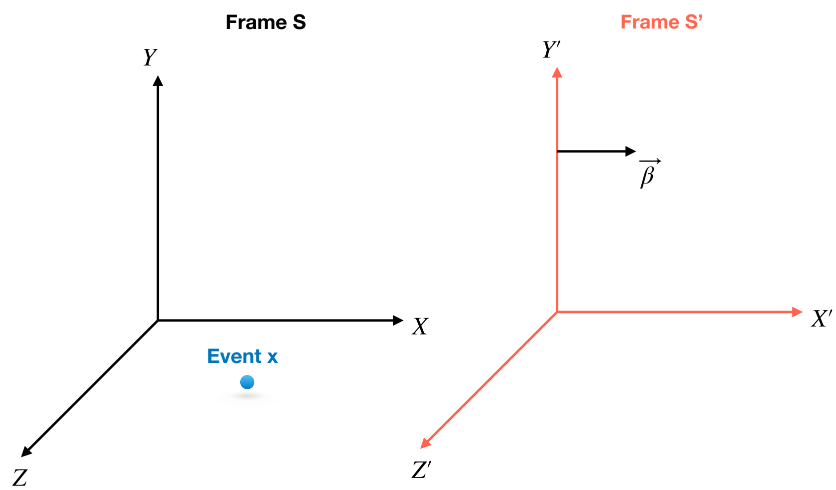


Figure 1.1: The standard orientation of two inertial reference frames at an instant of time t, as measured by the observer in S, in which occurs an event x.

As hinted at, the form of the Lorentz transformation was known to physicists before Einstein's publication of special relativity in 1905. In that case, one may ask what was so revolutionary about Einstein's theory? In short, Einstein's primary success was his revelation that the correct transformations between inertial reference frames must take the form of equation (1.2), provided that one accepts a set of simple axioms about the natural world.

1.3 Axiomatization of Physical Theories

Before proceeding to a derivation of the Lorentz transformation between inertial frames in standard orientation, a few words about the benefits of axiomatic analyses of physical theories are in order.

During his influential 1900 address to the International Congress of Mathematicians in Paris, David Hilbert identified the "mathematical treatment of the axioms of physics" as one of the most urgent, unsolved problems facing the mathematical sciences in the twentieth century [13]. There is little doubt that Hilbert's belief in the need of a detailed analysis of the foundations of physical theories stemmed from his successful axiomatization of Euclidean geometry one year prior. In particular, in his book entitled *The Foundations of Geometry*, Hilbert had established a complete set of independent, simple, and consistent axioms from which it was possible to deduce all of Euclidean geometry [14]. In Hilbert's opinion, the process of axiomatization represented a pivotal step in the transformation of geometry from a natural science into a pure mathematical science. For this reason, he firmly believed that the *axiomatic method* should play a similar role in the development of other scientific disciplines. In the words of Hilbert himself, "all other sciences ... should be treated according to the model set forth in geometry" [14].

The alluring simplicity and revolutionary consequences of special relativity give rise to a unique context in which to apply the axiomatic method. In his original paper (published just five years after Hilbert's address in Paris), Einstein purported to have derived the Lorentz transformations on the basis of two assumptions: (1) the principle of relativity and (2) the constancy of the speed of light [2]. Although Einstein's axiomatic system was unquestionably superior to the approaches of Poincaré and Lorentz, in 1910 Ignatowski noted the potential redundancy of Einstein's second postulate [15]. One year later, the team of Frank and Rothe demonstrated that it was in fact possible to derive the Lorentz transformations with only the principle of relativity [16]. In the century since, the axiomatic analysis of the foundations of special relativity has continued to garner significant attention. For instance, the conclusions of Frank and Rothe drew considerable skepticism from Pauli, who contended that Einstein's second postulate was necessary to derive the "physical content" of the conclusions of special relativity [17]. Additional approaches to the theory's axiomatic foundations appeared as well. Some researchers – such as Ueno in 1953 – emphasized the "physical role of observers", while others – such as Suppes in 1959 – gave priority to simplicity [18, 19]. More recently, Andréka et al. have adopted a logic-based approach to the axioms of special relativity [20].

Despite the variety of these approaches, all aspire towards the same objective: namely, the organization of a set of unambiguous logical propositions that allow for a clear understanding of the structure of special relativity [21]. The achievement of this goal would not only deepen our understanding of the insights of Einstein's theory, but also its possible extensions [15].

1.4 Derivation of the Lorentz Transformation

The purpose of the following is to illustrate the general tone of the Einsteinian approach to special relativity by deriving the Lorentz transformation between two inertial frames in standard orientation using traditional techniques. Although we have seen that there is a vibrant, ongoing debate about which assumptions are necessary to complete this derivation, we will assume a set of four uncontroversial and widely-accepted axioms.

1. The Homogeneity of Space and Time: This postulate states that the laws of physics should not privilege any particular location in space or moment in time.

2. The Isotropy of Space: This postulate asserts that the laws of physics are invariant under an arbitrary rotation of one's spatial coordinate axes.

3. The Principle of Relativity: This assumption states that the laws of physics take the same form in all inertial reference frames.

4. The Constancy of the Speed of Light: This assumption expresses the fact (verified by Michelson and Morley in 1887) that the speed of light c is the same in all inertial frames of reference [22].

In geometry, it is often helpful to think in terms of *invariants*, which are defined as quantities that do not change under a particular set of transformations. For instance, the conventional distance $d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ between two points in Euclidean two-space is an invariant under the set of all orthogonal transformations, which include rotations of a Cartesian coordinate system. By analogy, we should be able to define the Lorentz group as the set of all transformations that preserve an invariant "distance" between two arbitrary events in spacetime. Thus, the first step in our derivation will be to determine the form of that invariant interval between events, which we will later call the *spacetime interval*.

We begin by considering a very specific class of events: those which describe the emission and reception of a light signal. First, we define an inertial reference frame S, which utilizes Cartesian coordinate axes X, Y, and Z to assign spatial coordinates to events. Now, let event $P = (t_p, x_p, y_p, z_p)$ denote the emission of our photon from a specific location (x_p, y_p, z_p) in space at a given instant t_p in time, as measured in the S coordinate system. We then define event $Q = (t_q, x_q, y_q, z_q)$ as the reception of that same photon by a sensor at another location (x_q, y_q, z_q) at some time t_q, again as measured in S. Clearly, the photon in question travels a spatial distance Δd given by the Pythagorean theorem:

$$\Delta d = \sqrt{|x_p - x_q|^2 + |y_p - y_q|^2 + |z_p - z_q|^2} = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} \tag{1.4}$$

However, because the photon propagates with velocity c, we can also express this distance as $\Delta d = c|t_q - t_p| = c|\Delta t|$. Therefore, we can write that:

$$\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = c|\Delta t| \tag{1.5}$$

$$\implies \Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = 0, \tag{1.6}$$

where I have defined the interval Δs^2 between the events as $\Delta s^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$.

We now consider these same two events from the perspective of another inertial reference frame S'. We assume that the S and S' frames are in the aforementioned standard orientation[1], as depicted in Figure 1.1. Next, assume that events P and Q acquire spacetime coordinates (t'_p, x'_p, y'_p, z'_p) and (t'_q, x'_q, y'_q, z'_q) respectively in the S' reference frame. We can once again express the spatial distance $\Delta d'$ between these two events as measured by an observer in S' through the Pythagorean theorem:

$$\Delta d' = \sqrt{|x'_p - x'_q|^2 + |y'_p - y'_q|^2 + |z'_p - z'_q|^2} = \sqrt{(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2} \tag{1.7}$$

However, because the speed of light c is the same in all inertial reference frames, the distance $\Delta d'$ travelled by the photon in S' is also $c|t'_q - t'_p| = c|\Delta t'|$. From these observations arises the following result:

$$\Delta d' = c|\Delta t'| = \sqrt{(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2} \tag{1.8}$$

$$\implies (\Delta s')^2 = -c(\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 = 0 \tag{1.9}$$

This result demonstrates that the previously defined interval Δs^2 is identically zero in S'. However, recall that S' was an arbitrary inertial reference frame. This allows us to conclude that if the interval is zero in one inertial reference frame, then it must be identically zero in *all* inertial reference frames. That is, the interval Δs^2 between two events which correspond to the emission and reception of a light signal is an invariant under any transformation between inertial reference frames. From this point onward, we will refer to Δs^2 as the *spacetime interval* [23].

It remains to be shown that the the spacetime inverval Δs^2 between *any* pair of events is an invariant under a transformation between inertial frames. To see this, we once again return to our inertial frames S and S' in standard orientation and consider an arbitary pair of events. Assume that the observer in S measures the coordinate differences between these events to be $(\Delta t, \Delta x, \Delta y, \Delta z)$. Conversely, we assume that the observer in S' measures coordinate differences $(\Delta t', \Delta x', \Delta y', \Delta z')$. Next, we postulate that the equations that express the spacetime coordinates of an event measured by an observer in S' are linear in the spacetime coordinates that an observer in S assigns to the same event. We justify this assumption by the observation that non-linear terms would cause free objects to accelerate in S', which is impossible because S' is inertial.

Our assumption that the transformation is linear tells us that the coordinate differences $(\Delta t', \Delta x', \Delta y', \Delta z')$ are linear combinations of Δt, Δx, Δy, and Δz. Thus, our expression for the interval $(\Delta s')^2$ as measured by an observer in S' is a quadratic function of the coordinate differences measured in S. Our next step is to define the useful shorthand: $\Delta x^0 = \Delta t$, $\Delta x^1 = \Delta x$, $\Delta x^2 = \Delta y$, and $\Delta x^3 = \Delta z$. This allows us to write a general equation for $(\Delta s')^2$ in terms of

[1] Note that the homogeneity and isotropy of space together imply that I can rotate and translate the spatial axes of any inertial reference frame S' relative to S, such that they reach standard orientation.

the coordinate differences measured in S:

$$(\Delta s')^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \mathsf{M}_{\mu\nu}(\Delta x^\mu)(\Delta x^\nu) \tag{1.10}$$

where M is some 4×4 matrix that contains as its $\mathsf{M}_{\mu\nu}$ entry the correct coefficient of proportionality that arises from multiplying linear combinations of the coordinate differences. Note that we are free to assume that M is symmetric, since we can always express the summands $\mathsf{M}_{\mu\nu}(\Delta x^\mu)(\Delta x^\nu) + \mathsf{M}_{\nu\mu}(\Delta x^\nu)(\Delta x^\mu)$ as $(\mathsf{M}_{\mu\nu} + \mathsf{M}_{\nu\mu})(\Delta x^\mu)(\Delta x^\nu)$ if $\mu \neq \nu$.

It is important at this stage to ask what the functions represented by entries of the matrix M can depend on. The first of our set of axioms – the homogeneity of space and time – implies that there are no preferred positions in space or moments in time. Thus, there is no such thing as an absolute position or velocity – only *relative* velocities between frames have intrinsic physical meaning. Therefore, each entry $\mathsf{M}_{\mu\nu}$ must be a function only of the relative velocity $\vec{\beta}$ between the frames. However, we can go one step further by considering our second axiom – the isotropy of space. This postulate implies that there are no preferred directions in space. Therefore, the functions $\mathsf{M}_{\mu\nu}$ cannot depend on the direction of the relative velocity $\vec{\beta}$, but rather only the magnitude $|\vec{\beta}| = \beta$. We will revisit this important result after simplifying Equation (1.10).

Next, observe that Equation (1.10), which is a general equation for the spacetime interval measured in S', must hold in specific cases. One such case is our familiar example in which we consider the interval between events that correspond to the reception and emission of a light signal. In this case, we know from our conclusion below Equation (1.9) that $(\Delta s')^2 = \Delta s^2 = 0$. In addition, we also have that $c\Delta t = \Delta t = \Delta d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$, where I have followed the common convention of setting $c = 1$. This allows us to rewrite Equation (1.10) as:

$$(\Delta s')^2 = \mathsf{M}_{00}(\Delta d)^2 + 2\left[\sum_{i=1}^{3} \mathsf{M}_{0i}\Delta x^i\right]\Delta d + \sum_{i=1}^{3}\sum_{j=1}^{3} \mathsf{M}_{ij}(\Delta x^i)(\Delta x^j) = 0 \tag{1.11}$$

Although we have lost one degree of freedom by setting $\Delta t = \Delta d$, Equation (1.11) must hold for any arbitrary Δx, Δy, and Δz. For instance, the equation must hold if we set $\Delta x = -\Delta x$, $\Delta y = -\Delta y$, and $\Delta z = -\Delta z$:

$$\mathsf{M}_{00}(\Delta d)^2 + 2\left[\sum_{i=1}^{3} \mathsf{M}_{0i}(-\Delta x^i)\right]\Delta d + \sum_{i=1}^{3}\sum_{j=1}^{3} \mathsf{M}_{ij}(-\Delta x^i)(-\Delta x^j) = 0 \tag{1.12}$$

Subtracting Equation (1.12) from Equation (1.11) gives us the result:

$$4\left[\sum_{i=1}^{3} \mathsf{M}_{0i}\Delta x^i\right]\Delta d = 0 \tag{1.13}$$

The condition that Equation (1.13) be true for any arbitrary $\{x^i\}$ allows us to conclude that:

$$\mathsf{M}_{01} = \mathsf{M}_{02} = \mathsf{M}_{03} = 0 \tag{1.14}$$

By the symmetry of M, we conclude that $\mathsf{M}_{10} = \mathsf{M}_{20} = \mathsf{M}_{30} = 0$ as well. We continue by subjecting Equation (1.11) to additional specific cases. For instance, it must hold true if $\Delta x = 1$ and $\Delta y = \Delta z = 0$. Applying the result of Equation (1.14), we have that:

$$(\Delta s')^2 = \mathsf{M}_{00}(\Delta d)^2 + \mathsf{M}_{11}(\Delta x)^2 = 0 \tag{1.15}$$

Observing that $\Delta d = \Delta x$, we have that $-\mathsf{M}_{11} = \mathsf{M}_{00}$. The consideration of the cases in which we set (i) only $\Delta y = 1$ and (ii) only $\Delta z = 1$ lead to the results $-\mathsf{M}_{22} = \mathsf{M}_{00}$ and $-\mathsf{M}_{33} = \mathsf{M}_{00}$ respectively.

Lastly, consider the case in which $\Delta x = \Delta y = 1$ and $\Delta z = 0$. Then, Equation (1.11) becomes:

$$(\Delta s')^2 = \mathsf{M}_{00}\left[(\Delta d)^2 - (\Delta x)^2 - (\Delta y)^2\right] + 2\mathsf{M}_{12}(\Delta x)(\Delta y) = 0 \tag{1.16}$$

Recalling that $\Delta d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, equation (1.16) reduces to:

$$2\mathsf{M}_{12}(\Delta x)(\Delta y) = 0 \tag{1.17}$$

From which we conclude that $\mathsf{M}_{12} = \mathsf{M}_{21} = 0$. The consideration of the cases in which we set (i) $\Delta x = \Delta z = 1$ and $\Delta y = 0$ and (ii) $\Delta y = \Delta z = 1$ and $\Delta z = 0$ lead to the results $\mathsf{M}_{13} = \mathsf{M}_{31} = 0$ and $\mathsf{M}_{23} = \mathsf{M}_{32} = 0$ respectively. Thus, our matrix M must take the form:

$$\mathsf{M} = \begin{bmatrix} \mathsf{M}_{00} & 0 & 0 & 0 \\ 0 & -\mathsf{M}_{00} & 0 & 0 \\ 0 & 0 & -\mathsf{M}_{00} & 0 \\ 0 & 0 & 0 & -\mathsf{M}_{00} \end{bmatrix} \tag{1.18}$$

We can at last express Equation (1.10) as:

$$(\Delta s')^2 = -\mathsf{M}_{00}(\beta)\left[-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2\right] = -\mathsf{M}_{00}(\beta)\Delta s^2 \tag{1.19}$$

and thus:

$$(\Delta s')^2 = \phi(\beta)\Delta s^2 \tag{1.20}$$

where $\phi(\beta) = -\mathsf{M}_{00}(\beta)$ is an as yet undetermined function of the magnitude of the relative velocity between S and S' [24].

We now wish to determine the coefficient of proportionality $\phi(\beta)$ in Equation (1.20), which relates the spacetime interval $(\Delta s')^2$ between any arbitrary pair of events measured in S' in terms of the interval $(\Delta s)^2$ measured between those events in S. Recall that S and S' are in standard orientation (Figure 1.1), which requires that the spatial origin of S' move with velocity $+\vec{\beta}$ relative to S along the common x-axis. Now define a third frame S'' in standard orientation with S', whose spatial origin moves with velocity $-\vec{\beta}$ relative to the origin of S'

along the x-axis. Thus, we can use the result of Equation (1.20) to express the spacetime interval $(\Delta s'')^2$ measured in S'' in terms of $(\Delta s')^2$:

$$(\Delta s'')^2 = \phi(|-\vec{\beta}|)(\Delta s')^2 = \phi(\beta)(\Delta s')^2 \tag{1.21}$$

Because the frames S and S'' are identical, it follows that $\Delta s^2 = (\Delta s'')^2$. Therefore, we can substitute the result of Equation (1.21) into Equation (1.20) to arrive at:

$$(\Delta s')^2 = [\phi(\beta)]^2(\Delta s')^2 \implies \phi(\beta) = \pm 1 \tag{1.22}$$

We select the solution $\phi(\beta) = +1$ to preserve the unidirectionality of time [25]. Equation (1.20) thus becomes:

$$(\Delta s')^2 = (\Delta s)^2 \tag{1.23}$$

We therefore arrive at the important conclusion that *the spacetime interval between any two events is the same in all inertial reference frames.* This result, which is the mathematical consequence of all four axioms stated earlier, produces an important constraint on any proposed transformation between inertial frames. In particular, we now know that a transformation between inertial frames must preserve the spacetime interval between *any* pair of events. One can demonstrate that the aforementioned Galilean transformations do not satisfy this requirement [8].

Having learned that the spacetime interval must be invariant under a transformation between inertial reference frames, let us now inquire into what constraints this result imposes on the form of a transformation between inertial frames. In particular, we define our desired group of transformations as the set of all transformations $\mathsf{x} \longmapsto \mathsf{x}' = \mathsf{L}\,\mathsf{x}$ on $\mathbb{R}^4$, where L is a 4×4 matrix, that leave the spacetime interval Δs^2 invariant.

We first define a generic event x in $\mathbb{R}^4$ as the matrix $\mathsf{x}^\mathsf{T} = \begin{bmatrix} t & x & y & z \end{bmatrix}$. This allows us to express the spacetime interval between event x and the origin $O = (0,0,0,0)$ as the following matrix equation:

$$\Delta s^2 = \mathsf{x}^\mathsf{T}\,\mathsf{G}\,\mathsf{x} = \begin{bmatrix} t & x & y & z \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = -t^2 + x^2 + y^2 + z^2 \tag{1.24}$$

where

$$\mathsf{G} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.25}$$

is the Minkowski metric of flat spacetime.

One can express the requirement that the spacetime interval between x and the origin remain invariant under such a transformation via the following constraint:

$$\Delta s^2 = \mathsf{x}^\mathsf{T}\,\mathsf{G}\,\mathsf{x} = (\mathsf{x}')^\mathsf{T}\,\mathsf{G}\,(\mathsf{x}') = (\Delta s')^2 \tag{1.26}$$

Now, observe that our transformation matrix maps

$$\mathsf{x} \longmapsto \mathsf{x}' = \mathsf{L}\,\mathsf{x} \tag{1.27}$$

$$\text{and} \quad \mathsf{x}^{\mathsf{T}} \longmapsto (\mathsf{x}')^{\mathsf{T}} = \mathsf{x}^{\mathsf{T}}\,\mathsf{L}^{\mathsf{T}} \tag{1.28}$$

Therefore, we can express the middle equality of equation (1.13) as:

$$\mathsf{x}^{\mathsf{T}}\,\mathsf{G}\,\mathsf{x} = (\mathsf{x}^{\mathsf{T}}\,\mathsf{L}^{\mathsf{T}})\,\mathsf{G}\,(\mathsf{L}\,\mathsf{x}) \tag{1.29}$$

Because equation (1.16) must hold for *any* x, we require that our transformation L satisfy:

$$\mathsf{G} = \mathsf{L}^{\mathsf{T}}\,\mathsf{G}\,\mathsf{L} \tag{1.30}$$

If we assume once again that the frames S and S' are in standard orientation, then the transformation L should leave the coordinates y and z unchanged. That is, we require that $y' = y$ and $z' = z$. Moreover, we assume as before that the transformation L is linear to avoid the appearance of any fictious forces in the primed reference frame. We can therefore set the entries of L to arbitrary real values given by:

$$\mathsf{L} = \begin{bmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.31}$$

The matrix multiplication on the right-hand side of Equation (1.30) then supplies the following system of equations:

$$-A^2 + C^2 = -1 \tag{1.32}$$

$$-AB + CD = 0 \tag{1.33}$$

$$-B^2 + D^2 = 1 \tag{1.34}$$

From equation (1.32), we have that $A^2 = 1 + C^2 \geq 1$. If we assume that $A \geq 1$ to preserve the direction of time [26], then it is possible to define:

$$A = \cosh\Psi \tag{1.35}$$

for some real number Ψ. A useful hyperbolic trigonometric identity then tells us:

$$C^2 = A^2 - 1 \implies C = \sqrt{\cosh^2\Psi - 1} = \sinh\Psi \tag{1.36}$$

We can apply similar logic to equation (1.34), which leads to the following results:

$$D = \cosh\Omega \;\text{ and }\; B = \sinh\Omega \tag{1.37}$$

for some real number Ω. The substitution of these results into equation (1.33) gives us:

$$-\cosh\Psi\sinh\Omega + \sinh\Psi\cosh\Omega = \sinh(\Psi - \Omega) = 0 \implies \Psi = \Omega \tag{1.38}$$

Lastly, we let β be the unique real number defined by:

$$\tanh \Psi = -\beta \tag{1.39}$$

Then two more hyperbolic trigonometric identities lead us to the following results:

$$B = C = \sinh \Psi = \frac{\tanh \Psi}{\sqrt{1 - \tanh^2 \Psi}} = -\frac{\beta}{\sqrt{1-\beta^2}} = -\gamma\beta \tag{1.40}$$

$$A = D = \cosh \Psi = \frac{1}{\sqrt{1 - \tanh^2 \Psi}} = \frac{1}{\sqrt{1-\beta^2}} = \gamma \tag{1.41}$$

where $\gamma = [1-\beta^2]^{-1/2}$ [8].

We therefore conclude that the linear transformation $\mathsf{x} \mapsto \mathsf{x}' = \mathsf{L}\mathsf{x}$ that preserves the spacetime interval must have the form:

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \implies \mathsf{L} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.42}$$

This result is identical to equation (1.3) for the Lorentz transformation if we interpret β as the velocity of the spatial origin of the S' coordinate system relative to the spatial origin of the S coordinate system. To see that this interpretation must be correct, consider the equation for x' given by equation (1.42):

$$x' = \gamma\,(x - \beta t) \tag{1.43}$$

Observe that the spatial origin of the primed coordinate system ($x' = 0$) satisfies the equation $\frac{x}{t} = \beta$ in the unprimed system. Therefore, the spatial origin of S' does in fact move with velocity β relative to the spatial origin of S, as claimed [23]. We conclude that any coordinate transformation between inertial refrence frames in standard orientation must obey Equation (1.42).

The above derivation underlines how strongly Einstein's conclusions depend on assumptions concerning not only the constancy of the speed of light but also intrinsic properties of space and time. The derivation began with determining the form of the spacetime interval, which in many respects is the mathematical expression of the invariance of the speed of light [23]. Next, we demonstrated the invariance of the spacetime interval by invoking the homogeneity of space and time and the isotropy of space. Only then were we able to proceed to the form of the Lorentz transformation, from which follows the well-known phenomena of time dilation and length contraction. It is in this sense that the Einsteinian derivation of the Lorentz transformation presupposes the existence of spacetime and its characteristics. This realization leads to a compelling question: namely, is it possible to derive the same geometric properties of spacetime from a more primitive starting point, which assumes nothing about the invariance of a physical constant or the properties of space and time itself?

1.5 Introduction to Influence Theory

In the previous section, we demonstrated that one can define the group of Lorentz transformations as the set of all transformations that preserve the spacetime interval. This powerful constraint, which was intimately related to the constancy of the speed of light, led quite directly to the form of the transformation first derived by Poincaré and Lorentz. Despite the success of this approach, one may inquire, in the spirit of the axiomatic method, whether it is possible to arrive at the same spacetime interval and Lorentz transformation via another path. Ideally, this alternative approach would abandon the axioms listed earlier in favor of a set of more primitive assumptions about the natural world. As hinted at earlier, *influence theory*, which proposes influence as its core assumption, is one such alternative.

The general goal of influence theory is to derive the laws of physics from a very limited set of assumptions about influence. In contrast to most other approaches to the foundations of physics, this theory casts aside familiar notions such as positions in space and time, mass, energy, or momentum [27]. Rather, influence theory postulates only the existence of events and the capacity of certain events to influence others. Moreover, within this conception of reality there exist observers, who can quantify the ordering of events by assigning to them numerical valuations. In the context of special relativity, Knuth and Bahreyni have shown that requiring coordinated observers within this theoretical framework to quantify the ordering of events in a consistent manner "results in constraint equations that represent a discrete version of the Minkowski metric and Lorentz transformations" [5]. That is, they demonstrated that the geometric properties of spacetime – namely the spacetime interval and Lorentz transformation – emerge as a necessary consequence of more primitive assumptions regarding the ordering of events. We will see this for ourselves in Chapter 3. This contrasts sharply with the Einsteinian derivation of these concepts shown in the previous section, which relied heavily on powerful assumptions about space, time, and the laws of physics. It is in this sense that influence theory provides a potential foundation for emergent spacetime.

The conception of influence as a cornerstone for the emergence of spacetime is not unique to influence theory. In fact, causality-based derivations of the Lorentz transformation date back to 1964, when Zeeman demonstrated that one can regard the Lorentz transformations as the set of all mappings that preserve the causal ordering of events on Minkowski spacetime [28]. More recently, research into the relationship between causality and spacetime has proliferated due to the causal set theory approach to the problem of quantum gravity. This theory has its origins in a paper published in 1987, in which Sorkin and others proposed that the four-dimensional manifold of curved spacetime at the smallest scale in fact consists of discrete spacetime events, some of which are ordered by causality [29]. It is important to recognize that in most respects the methods and aims of the programs discussed above differ sharply from those of influence theory. Perhaps most important is the distinction that influence theory – unlike Zeeman's approach or causal set theory – assumes nothing about space and time.

Nevertheless, the tenet that causality represents a weaker axiom than those traditionally assumed in relativity theory is apparent in all three approaches. In the words of Fay Dowker, a leading researcher in the field of causal set theory: "causality is a more fundamental organizing principle even than space and time themselves" [30].

The remainder of this thesis will be dedicated to exploring the geometric structure of flat spacetime from the perspective of influence theory. In particular, I will review the past work of Knuth and Bahreyni, who demonstrated the emergence of the spacetime interval and Lorentz transformation from the constraint that coordinated observers quantify a partially-ordered set of events in a consistent manner. I then aim to extend this work by developing a method of *subspace projection*, which will allow for the quantification of *any* generalized interval that resides in a set of events, partially ordered by influence.

Chapter 2

Partially Ordered Sets and Influence Theory

2.1 Overview

In the previous chapter, I provided a derivation of the Lorentz transformation based on a set of four traditional axioms concerning properties of space, time, and the laws of physics. We now turn our attention towards *influence theory*, an alternative approach to this important derivation that postulates influence as its fundamental assumption. In particular, I aim to introduce the concept of partially ordered sets – the mathematical tool that we will use to describe the causal structure within a collection of events. This will allow a direct transition to a discussion of the premises at the foundation of influence theory, as well as the theory's most important results regarding the quantification of objects within a partially ordered set.

2.2 Introduction to Ordered Sets and Hasse Diagrams

We begin our discussion of ordered sets with a definition and set of axioms [31].

Definition 2.1. ***(Ordered Set):*** *An ordered set* $(S, \sim)$ *is the fusion of a set* S *together with a binary ordering relation* $\sim$.

Definition 2.2. ***(Binary Ordering Relation):*** *A binary ordering relation* $\sim$ *on a set* S *is any relation that satisfies the following three properties:*

1. *Reflexivity:* $a \sim a$ *for all* $a \in S$
2. *Antisymmetry: If* $a \sim b$ *and* $b \sim a$*, then* $a = b$ *for* $a, b \in S$.
3. *Transitivity: If* $a \sim b$ *and* $b \sim c$*, then* $a \sim c$ *for* $a, b, c \in S$.

Because the above axioms are rather abstract, it will help to consider a straightforward example.

Example 2.3. *Let* $M = \{1, 2, 3, 4, 5, 6\}$*. We may then define the following binary ordering relation on* M*: for* $a, b \in M$*, let* $a \sim b$ *if* $a \leq b$*. This ordering relation clearly satisfies the three axioms listed above:*

1. *Reflexivity:* $a \leq a$ *for all* $a \in M$*.*

2. *Antisymmetry: If* $a \leq b$ *and* $b \leq a$*, then* $a = b$ *for* $a, b \in M$*.*

3. *Transitivity: If* $a \leq b$ *and* $b \leq c$*, then* $a \leq c$ *for* $a, b, c \in M$*.*

We conclude that $\sim$ *does in fact define a binary ordering relation on* M *and therefore, that* $(M, \sim)$ *defines an ordered set. In fact, because for any* $a, b \in M$*, we can write that* $a \sim b$ *or* $b \sim a$ *(or both), we say that* $(M, \sim)$ *is a totally ordered set.*

Definition 2.4. ***(Totally Ordered Set):*** *If* $(S, \sim)$ *defines an ordered set and for any* $a, b \in S$*, we have that* $a \sim b$ *or* $b \sim a$ *(or both), then* $(S, \sim)$ *is a totally ordered set.*

It is important to observe that not all ordered sets are totally ordered. As an example, let us consider the same set M from Example 2.3, but with a different binary ordering relation.

Example 2.5. *Let* $M = \{1, 2, 3, 4, 5, 6\}$ *and define the following ordering relation on* M*: for* $a, b \in M$*, let* $a \sim b$ *if* $a|b$*, where* $a|b$ *means that* b *is divisible by* a*. We can verify rather immediately that this ordering relation satisfies the three properties in Definition 2.2:*

1. *Reflexivity:* $a|a$ *for all* $a \in M$*. That is, every number in* M *is divisible by itself.*

2. *Antisymmetry: If* $a|b$ *and* $b|a$*, then* $a = b$ *for* $a, b \in M$*. In other words, if* a *is divisible by* b*, and* b *is divisible by* a*, then* a *and* b *must be the same number.*

3. *Transitivity: If* $a|b$ *and* $b|c$*, then* $a|c$ *for* $a, b, c \in M$*. That is,* c *is divisible by* b *and* b *is divisible by* a*, then* c *must also be divisible by* a*.*

Thus, we see that our newly defined ordering relation $\sim$ *does in fact define a binary ordering relation on* M *and conclude that* $(M, \sim)$ *defines an ordered set.*

However, note that $(M, \sim)$ in Example 2.5 does *not* satisfy the requirement stipulated in Definition 2.4 for a totally ordered set. To see this, consider the numbers 3 and 5, both of which are elements of M. We clearly can write neither $3 \sim 5$, nor $5 \sim 3$ because 3 and 5 are relatively prime – that is, they share no common divisor other than 1. Thus, $(M, \sim)$ is *not* a totally ordered set, which creates the need for a new mathematical object: a *partially* ordered set. That is, because $(M, \sim)$ is an ordered set that is not totally ordered, then we say that $(M, \sim)$ is a partially ordered set.

Definition 2.6. ***(Partially Ordered Set):*** *If $(S, \sim)$ defines an ordered set that is not totally ordered, then $(S, \sim)$ is a partially ordered set. Equivalently, we say that $(S, \sim)$ is a poset.*

We now shift our attention to Hasse diagrams (or equivalently: directed acyclic graphs), a class of drawings that will help us to visualize the ordering within an ordered set. For any ordered set $(S, \sim)$, the associated Hasse diagram has the following properties [32]:

1. The vertices of the Hasse diagram are the elements of S.

2. We draw an edge between two vertices $x, y \in S$ if: (a) $x \sim y$ and (b) there is no other element $z \in S$ such that $x \sim z \sim y$.

3. If $x \sim y$, then we place the vertex corresponding to y "above" the vertex corresponding to x.

We are now in a position to draw the Hasse diagrams that correspond to the ordered sets defined in Examples 2.3 and 2.5. These diagrams are shown in Figure 2.1.a and 2.1.b respectively.

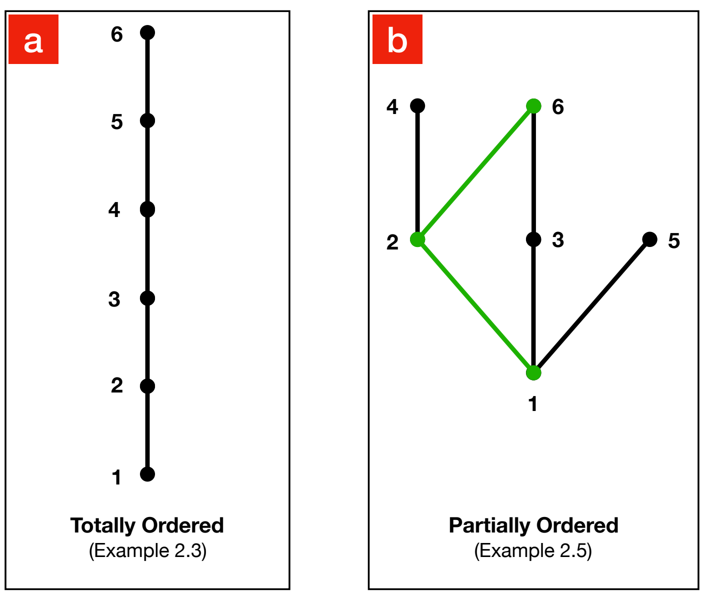


Figure 2.1: (a) The Hasse diagram that corresponds to the totally ordered set defined in Example 2.3. (b) The Hasse diagram that corresponds to the partially ordered set defined in Example 2.5.

The Hasse diagrams in Figures 2.1.a and 2.1.b illustrate well the distinction between total ordering and partial ordering. Whereas a diagram corresponding to a totally ordered set forms a chain of vertices (Figure 2.1.a), the same cannot be said of a diagram corresponding to a partially ordered set (Figure 2.1.b). That is to say, because there is no edge connecting the elements $\{3, 5\}$ or $\{5, 6\}$ in Figure 2.1.b, we know that the ordered set represented by this Hasse diagram is partially ordered.

We may now motivate the important concept of a *chain*. Observe that it is not necessarily true that every ordered subset of a partially ordered set is also partially ordered. That is, there may exist totally ordered sets within a partially ordered set. As an example, consider the set $W = \{1, 2, 6\}$, which is a subset of the set M defined in Example 2.5. If we retain the same divisibility ordering relation $\sim$ defined in that example, we see that $1 \sim 2$, $1 \sim 6$, and $2 \sim 6$. Thus, $(W, \sim)$ is a totally ordered set that lies within the partially ordered set $(M, \sim)$. The Hasse diagram in Figure 2.1.b helps to visualize this observation: we see that we can trace a "chain" (shown in green) that connects the vertices corresponding to the elements in $\{1, 2, 6\}$. We will refer to such a totally ordered subset as a *chain*, as the following definition specifies.

Definition 2.7. ***(Chain):*** *If $W \subseteq M$ and $\sim$ defines an ordering relation on M, then $(W, \sim)$ is an ordered subset of $(M, \sim)$. Furthermore, if $(W, \sim)$ is a totally ordered subset of $(M, \sim)$, then we refer to $(W, \sim)$ as a chain within M.*

2.3 Influence as a Binary Ordering Relation

Now that we are familiar with the class of objects known as ordered sets, we are prepared to considered the the physical notion of influence in a mathematically rigorous fashion. In particular, we will see that it is possible to regard a collection of events as a set with a partial order induced by influence. It is in this sense that this theory takes influence to be the fundamental organizing principle of reality.

We begin by postulating not only that events have the ability to influence one another, but also that this influence takes place in a directed, asymmetric fashion. That is to say, two events linked by influence can always be distinguished: one event represents the act of influencing, while the other represents the act of *being* influenced. This is a crucial assumption, since it permits us to assign an order to any pair of events linked through influence. We summarize these two observations in the following definitions:

Definition 2.8. *Influence refers to the conjunction of two events. We say that one of the events represents the act of influencing, while the other represents the act of being influenced.*

Definition 2.9. *If S is some non-empty collection of events, then for $x, y \in S$, we say $x \prec y$ if event x influences event y. Equivalently, we say that event y includes event x.*

Moreover, we posit that this ordering relation possesses the following three properties for some set S of events:

1. *Reflexivity*: $x \prec x$ for all $x \in S$.
2. *Antisymmetry*: If $x \prec y$ and $y \prec x$, then $x = y$ for $x, y \in S$.
3. *Transitivity*: If $x \prec y$ and $y \prec z$, then $x \prec z$ for $x, y, z \in S$.

It follows from Definitions 2.1 and 2.2 that $\prec$ defines a binary ordering relation on a set S of events and therefore that $(S, \prec)$ is an ordered set. However, we can go one step further: because in general there will exist events $x, y \in S$ such that we can write neither $x \prec y$ nor $y \prec x$, we conclude that $(S, \prec)$ is a partially ordered set. In other words, because there may be pairs of events in S that are *not* linked through influence, we conclude that $(S, \prec)$ is not totally ordered. Thus, it is a partially ordered set. For a pair of events $x, y \in S$ that are not related through influence, we say that x and y are *incomparable*.

Definition 2.10. ***(Incomparable Events):*** *Let $(S, \prec)$ be an ordered set of events, the order of which is induced by influence. If for some $x, y \in S$, we can write neither $x \prec y$ nor $y \prec x$, then we say that x and y are incomparable. Equivalently, we write that $x||y$.*

With the understanding that the influence relation $\prec$ defines a partial order on a given set of events, we can readily access the machinery of Hasse diagrams introduced in the previous section. For instance, we may now imagine that the Hasse diagrams shown in Figures 2.1.a and 2.1.b no longer represent partially ordered sets of integers, but rather sets of events ordered by influence. Dowker suggests that one imagine a Hasse diagram in this context as a sort of family tree, in which the ancestor of an event is always placed *below* the descendant [30]. For instance, in the case of the diagram in Figure 2.1.b, we see that event 3 is the descendant of event 1 (equivalently, event 1 includes event 3); conversely, events 5 and 6 are incomparable because there is no edge connecting their respective vertices.

In the domain of influence theory, chains within partially ordered sets of events take on particular importance, since they provide a means of quantifying other events in the partially ordered set. To achieve this, we begin by quantifying a chain of events within the poset with a function (valuation) that maps each event to a well-defined real number.

Bird's-Eye View: The main task of influence theory is to specify how one can use a valuation assigned to a chain within a poset to quantify arbitrary events in the poset with respect to that chain in a consistent fashion.

Our ultimate goal in this chapter is to consider how distinct chains can quantify the same interval of events within a poset in a consistent manner. It is this requirement of consistency that will lead to the emergence of the geometry of flat spacetime.

Thus, before we continue, we must specify a means by which to quantify the events along a chain in a poset, such as the one highlighted in green in Figure 1.b.

Definition 2.11. ***(Valuation):*** *Let $\boldsymbol{P}$ be a chain within some partially ordered set of events $(S, \prec)$ and $x, y \in \boldsymbol{P}$. Furthermore, let $v_{\mathbf{P}}$ be an injective functional $v_{\mathbf{P}} : S \to \mathbb{R}$. If $x \prec y \Rightarrow v(x) \leq v(y)$, then we say that $v_{\mathbf{P}}$ is a valuation assigned to the chain $\boldsymbol{P}$.*

In this section, we considered how influence defines a partial order on a set of events [27]. In addition, we specified the conditions necessary to define a valuation that we may use to quantify a chain of events within a partially ordered set. In the next section, we will seek to use such a valuation to quantify objects within the poset with respect to a given chain.

2.4 Quantification of Events and Intervals with Respect to One Chain

In this section, we will aim to determine how to quantify three objects within a poset with respect to a given chain: (1) an arbitrary event in a poset, (2) a closed interval (finite segment) along the specified chain, and (3) a generalized interval whose endpoints are arbitrary elements in the poset.

Events: We begin by rigorously defining a means of quantifying an arbitrary event x in a poset $(S, \prec)$ with respect to a particular chain $\mathbf{P}$ of events within that poset (recall that a chain, according to Definition 2.7, is a totally ordered subset of an ordered set). First, observe that it would be impossible to quantify x with respect to $\mathbf{P}$ if we have that $x || p$ for all $p \in \mathbf{P}$. That is to say, if x is incomparable with every event on the chain, then it impossible to quantify x with respect to that chain.

However, it is possible to quantify x with respect to $\mathbf{P}$ if there exists an event $p \in \mathbf{P}$ such that one of the following statements is true: (i) $x \prec p$ or (ii) $p \prec x$. If this is the case, then we may define two projection operators P and $\bar{P}$ which forward project and backward project x respectively onto the chain $\mathbf{P}$. In particular, we will define the backward projection $\bar{P}x$ of x as the event on $\mathbf{P}$ that is immediately "below" x (if it exists); similarly, we will define the forward projection Px of x as the event on $\mathbf{P}$ that is immediately "above" x (if it exists).

The Hasse diagrams shown in Figures 2.2.a and 2.2.b help to illustrate what we mean by the forward/backward projections of x onto the chain $\mathbf{P}$. In the case of Figure 2.2.a, we see that the event on chain $\mathbf{P}$ that is immediately below x is event p_{i+1}. Thus, the backward projection operator $\bar{P}$ sends x to the event p_{i+1}. Equivalently, we write that:

$$\bar{P}x = p_{i+1} \tag{2.1}$$

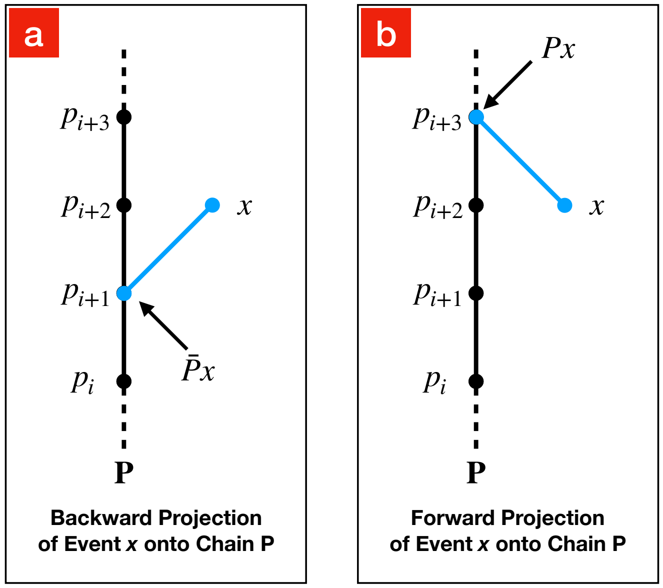


Figure 2.2: (a) A visualization of the backward projection of an event x onto the chain $\mathbf{P}$. (b) A visualization of the forward projection of an event x onto the chain $\mathbf{P}$.

In the case of Figure 2.2.b, we see that the event on chain $\mathbf{P}$ that is immediately above x is event p_{i+3}. Thus, the forward projection operator P sends x to the event p_{i+3}:

$$Px = p_{i+3} \tag{2.2}$$

It is important to underline that an arbitrary event x in the poset need not have a forward projection or a backward projection. In fact, as mentioned previously, if x is incomparable with every event on the chain $\mathbf{P}$, then it will have neither of the two. However, if the event x is related through influence with some event on the chain $\mathbf{P}$, then it is possible to quantify it with at least one real number with respect to that chain. In particular, if we have assigned some valuation $v_{\mathbf{P}} : \mathbf{P} \to \mathbb{R}$ to the chain $\mathbf{P}$, then we can quantify x with real numbers $x_v, \bar{x}_v$ that are equivalent to the valuations assigned to its forward and backward projections respectively (if they exist):

$$x_v = v(Px) \tag{2.3}$$

$$\bar{x}_v = v(\bar{P}x) \tag{2.4}$$

Thus, it is possible to quantify the arbitrary event x with a pair of real numbers $(x_v, \bar{x}_v)$ with respect to the chain $\mathbf{P}$, so long as it possesses both a forward and backward projection. This 2-tuple represents the consistent quantification of an arbirary event in a poset with respect a given chain.

Closed Interval: Now that we have developed a means of quantifying an arbitrary event with respect to a chain, we may consider the quantification of *intervals* of events. An interval is simply an object within the poset, which we define by its endpoints. The most straightforward example of an interval of events is a closed interval along a chain:

Definition 2.12. ***(Closed Interval):*** *Let $(S, \prec)$ be a partially ordered set of events, the order of which is induced by influence. Moreover, let $\mathbf{P}$ be a chain that lies within $(S, \prec)$. For any two events $p_i, p_j \in \mathbf{P}$, we define the corresponding closed interval $[p_i, p_j]_{\mathbf{P}}$ along $\boldsymbol{P}$ as:*

$$[p_i, p_j]_{\mathbf{P}} = \{p \in \mathbf{P} : p_i \prec p \prec p_j\} = \{p_i, p_{i+1}, \cdots, p_{j-1}, p_j\} \tag{2.5}$$

In less formal terms, a closed interval is simply a finite segment along a chain.

As before, we now seek to quantify a closed interval along a given chain using the valuation assigned to that chain. In their 2014 paper, Knuth and Bahreyni demonstrate that the only consistent means by which to quantify a closed interval along a chain is the following [5]:

Theorem 2.13. ***(Length of Closed Interval):*** *Let $\mathbf{P}$ be a chain that lies within some partially ordered set of events $(S, \prec)$, to which we have assigned a valuation $v_{\mathbf{P}} : \mathbf{P} \to \mathbb{R}$. Then the only consistent quantification of an arbitrary closed interval in $\mathbf{P}$ is given by the real number d:*

$$d([p_j, p_k]_{\mathbf{P}}) = v_{\mathbf{P}}(p_j) - v_{\mathbf{P}}(p_k) \tag{2.6}$$

where for some $p_j, p_k \in \mathbf{P}$, $[p_j, p_k]_{\mathbf{P}}$ denotes the corresponding closed interval along the chain $\mathbf{P}$.

Colloquially, we will refer to the quantification of a closed interval along a chain as the length of that interval.

Generalized Intervals: Our final objective is to specify a quantification for a generalized interval, the generalization of a closed interval. It turns out that there exist three different manners of quantifying a generalized interval with respect to a given chain: (1) a 4-tuple of valuations, (2) a 2-tuple of lengths (an interval pair), and (3) a scalar quantification (a "1-tuple").

We begin with the definition of a generalized interval. In short, a generalized interval is one whose endpoints are any arbitrary events within a poset. This concept contrasts with the notion of a closed interval, the end points of which must be located on a single chain. An example of a generalized interval $[a, b]$, whose endpoints are the events a and b, is shown in Figure 2.3. Moreover, as before, we will assume that we have assigned a valuation $v_{\mathbf{P}} : \mathbf{P} \to \mathbb{R}$ to the chain $\mathbf{P}$. It is now our goal to quantify the interval $[a, b]$ with respect to the chain $\mathbf{P}$ using that valuation.

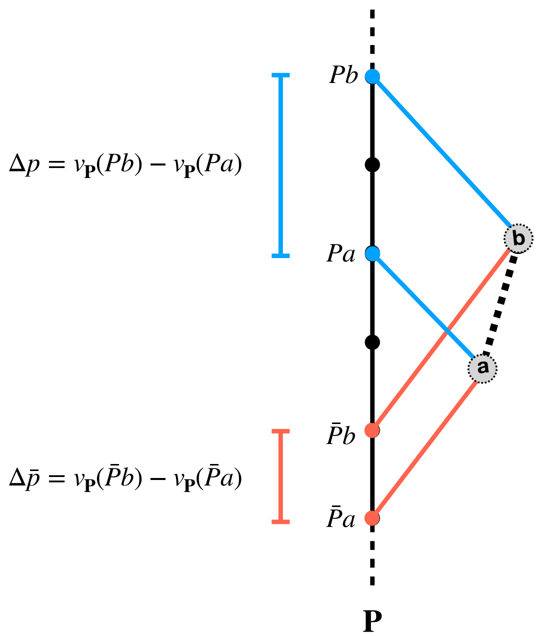


Figure 2.3: A generalized interval of events $[a, b]$ that we wish to quantify with respect to the chain $\mathbf{P}$.

The first possible quantification of this generalized interval with respect to the chain $\mathbf{P}$ consists simply of listing the valuations assigned to the forward and backward projections of the events a and b onto the chain $\mathbf{P}$ in a 4-tuple.

$$[a, b]\Big|_{\mathbf{P(4)}} = (v_{\mathbf{P}}(Pa), v_{\mathbf{P}}(Pb), v_{\mathbf{P}}(\bar{P}a), v_{\mathbf{P}}(\bar{P}b)) \tag{2.7}$$

The second option for quantification of the generalized interval $[a, b]$ with respect to the chain $\mathbf{P}$ takes advantage of the definition of the length of a closed interval in Theorem 2.13. In particular, note that the projections of events a, b onto $\mathbf{P}$ define closed intervals $[Pa, Pb]_{\mathbf{P}}$ and $[\bar{P}a, \bar{P}b]_{\mathbf{P}}$ along the chain $\mathbf{P}$ (shown in blue and red respectively in Figure 2.3). Therefore, we may reduce the above 4-tuple to an *interval pair*, which is a 2-tuple with entries equal to the lengths of the closed intervals $[Pa, Pb]_{\mathbf{P}}$ and $[\bar{P}a, \bar{P}b]_{\mathbf{P}}$:

$$[a, b]\Big|_{\mathbf{P(2)}} = \big(v_{\mathbf{P}}(Pb) - v_{\mathbf{P}}(Pa), v_{\mathbf{P}}(\bar{P}b) - v_{\mathbf{P}}(\bar{P}a)\big)_{\mathbf{P}} = (\Delta p, \Delta \bar{p})_{\mathbf{P}} \tag{2.8}$$

where we have defined:

$$\Delta p = d\left([Pa, Pb]_{\mathbf{P}}\right) = v_{\mathbf{P}}(Pb) - v_{\mathbf{P}}(Pa) \tag{2.9}$$

$$\Delta\bar{p} = d\left([\bar{P}a, \bar{P}b]_{\mathbf{P}}\right) = v_{\mathbf{P}}(\bar{P}b) - v_{\mathbf{P}}(\bar{P}a) \tag{2.10}$$

Digression: The notation Δp will be very important in later sections of this chapter. In general, if $\mathbf{P}$ is a chain within a poset, $v_{\mathbf{P}}$ is some valuation assigned to $\mathbf{P}$, and $[a, b]$ is a generalized interval that we wish to quantify, then Δp is the length of the closed interval along $\mathbf{P}$ whose endpoints are the forward projections of the events a and b onto $\mathbf{P}$. Equivalently:

$$\Delta p = d\left([Pa, Pb]_{\mathbf{P}}\right) = v_{\mathbf{P}}(Pb) - v_{\mathbf{P}}(Pa) \tag{2.11}$$

$\Delta\bar{p}$ is defined similarly.

The final option to quantify a generalized interval $[a, b]$ with respect to a chain $\mathbf{P}$ is to reduce the above 2-tuple to a single scalar value Δs^2. In their 2014 paper, Knuth and Bahreyni demonstrate that the product of the quantities Δp and $\Delta\bar{p}$ defined above furnishes a consistent scalar quantification of an arbitrary generalized interval [5]:

Theorem 2.14. ***(Scalar Quantification of Generalized Interval):*** *Let* $\mathbf{P}$ *be a chain that lies within some partially ordered set of events* $(S, \prec)$*, to which we have assigned a valuation* $v_{\mathbf{P}} : \mathbf{P} \to \mathbb{R}$*. Moreover, let* $a, b \in S$*. Then the only consistent quantification of an arbitrary generalized interval* $[a, b]$ *with respect to* $\boldsymbol{P}$ *is given by the real numer* Δs^2*:*

$$\Delta s^2\left([a, b]\right) = \Delta p \Delta\bar{p} \tag{2.12}$$

where as before $\Delta p = d\left([Pa, Pb]_{\mathbf{P}}\right)$ *and* $\Delta\bar{p} = d\left([\bar{P}a, \bar{P}b]_{\mathbf{P}}\right)$.

Figure 2.4 on the following page shows a summary of the quantifications of various objects in the poset detailed in this section.

2.5 Directionality

We now take a slight detour from the quantification of various objects within a poset to consider the structure induced by the partial order. Establishing this structure represents an essential step towards formulating a consistent means of quantifying intervals of events with respect to *multiple* chains – our ultimate objective for this chapter.

Though it may sound startling, the partially ordered sets discussed above in fact possess sufficient structure to allow for definitions of simple geometric concepts like directionality. To do so, we consider some arbitrary event x in the poset and its projections with respect to two distinct chains $\mathbf{P}$ and $\mathbf{Q}$ within the poset. Moreover, we will require that the event x possess both forward and backward projections onto the two chains. Of particular interest are instances in which these projections satisfy a condition described by Knuth and Bahreyni as *collinearity* [5].

Object we wish to quantify	How we quantify the object with respect to a chain P	
1 Arbitrary event $x \in S$ in the poset.	$(x_v, \bar{x}_v) = \left(v_{\mathbf{P}}(Px), v_{\mathbf{P}}(\bar{P}x)\right)$	**2-tuple** with entries equal to valuations assigned to forward and backward projections onto chain.
2 Closed interval $[p_j, p_k]_{\mathbf{P}}$ along chain.	$d([p_j, p_k]_{\mathbf{P}}) = v(p_j) - v(p_k)$	**Real number** equal to the difference of the valuations assigned to the endpoints of the interval.
3 Generalized interval $[a, b]$ with arbitrary events $a, b \in S$ as endpoints.	a $[a, b]\Big\vert_{\mathbf{P}(4)}$	**4-tuple** with entries equal to valuations assigned to forward and backward projections of endpoints onto chain.
	b $[a, b]\Big\vert_{\mathbf{P}(2)} = (\Delta p, \Delta \bar{p})_{\mathbf{P}}$	**2-tuple** with entries equal to lengths of closed intervals $[Pa, Pb]_{\mathbf{P}}$ and $[\bar{P}a, \bar{P}b]_{\mathbf{P}}$.
	c $\Delta s^2\left([a, b]\right) = \Delta p \Delta \bar{p}$	**Real number** equal to the product of lengths of closed intervals $[Pa, Pb]_{\mathbf{P}}$ and $[\bar{P}a, \bar{P}b]_{\mathbf{P}}$.

Figure 2.4: A summary of the various quantifications of objects within a partially ordered set with respect to a chain introduced in this section.

Definition 2.15. ***(Collinearity):*** *Let $(S, \prec)$ be a partially ordered set of events, the order of which is induced by influence. Next, let* $\mathbf{P}$ *and* $\mathbf{Q}$ *define two chains within S and rename our projection operators as: $P = P_1, \bar{P} = P_2, Q = Q_1, \bar{Q} = Q_2$. An event x in S is said to be collinear with respect to* $\mathbf{P}$ *and* $\mathbf{Q}$ *if and only if the following conditions are satisfied:*

1. *$Px = P_1x$, $\bar{P}x = P_2x$, $Qx = Q_1x$, $\bar{Q}x = Q_2x$ all exist.*
2. *$Px = P_j(Q_kx)$ for some $j, k \in \{1, 2\}$.*
3. *$\bar{P}x = P_m(Q_nx)$ for some $m, n \in \{1, 2\}$.*

Equivalently, we can follow Knuth and Bahreyni and say: "an element x that possesses both backward and forward projections onto two distinct finite chains ***P*** *and* ***Q*** *is said to be collinear with those projections if and only if each of its projections onto* ***P*** *can be found by first projecting onto* ***Q*** *and then onto* ***P****, and vice versa."*

As usual, we will rely on a pair of Hasse diagrams in Figures 2.5.a and 2.5.b to visualize what this definition says. I claim that both of these diagrams depict events that are collinear with respect to the pair of chains **P** and **Q**.

To verify this, we begin with the event x depicted in Figure 2.5.a. This event satisfies the first condition specified in Definition 2.15: namely, it has forward and backward projections onto both of the chains. As for the subsequent two conditions, we observe the following relationships:

$$Px = P_2(Q_1 x) \tag{2.13}$$

$$\bar{P}x = P_1(Q_2 x) \tag{2.14}$$

Thus, we conclude that event x is in fact collinear with respect to the chains **P** and **Q**. Similar relationships hold true for the event x' depicted in Figure 2.5.b:

$$Px' = P_1(Q_2 x') \tag{2.15}$$

$$\bar{P}x' = P_2(Q_1 x') \tag{2.16}$$

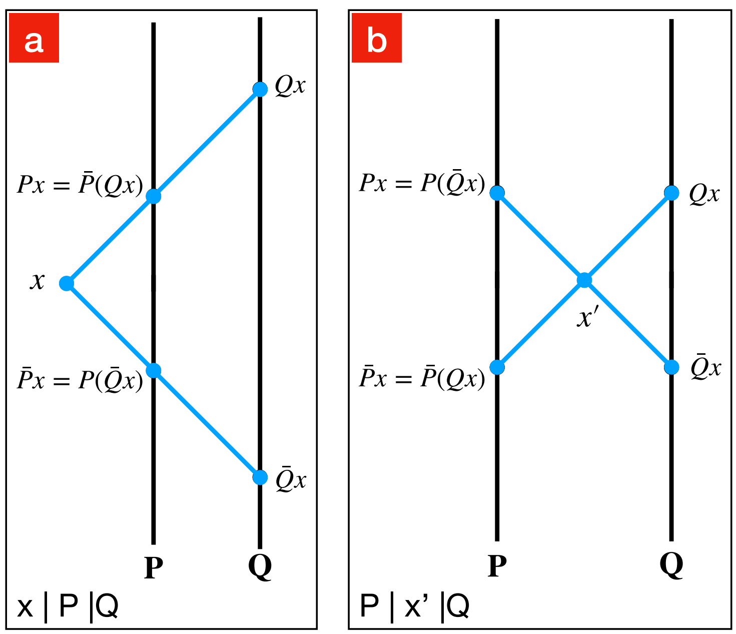


Figure 2.5: (a) An event x that is collinear with the chains **P** and **Q** and on the **P**-side of its projections onto these chains. (b) An event x' that is collinear with the chains **P** and **Q** and between its projections onto these chains.

Thus, Figures 2.5.a and 2.5.b both depict events that are collinear with respect to the chains **P** and **Q**. However, it is clear from these Hasse diagrams that collinearity can come in different flavors. In particular, we say that an event that exhibits the properties of x in Figure 2.5.a is on the **P**-side of its projections onto the $\mathbf{P}$ and $\mathbf{Q}$ chains, which we denote by $x|\mathbf{P}|\mathbf{Q}$. Similarly, we say that an event that exhibits the properties of x' in Figure 2.5.b is between its projections onto the $\mathbf{P}$ and $\mathbf{Q}$ chains, which we denote by $\mathbf{P}|x'|\mathbf{Q}$. One can imagine a final scenario in which an event x'' is on the $\mathbf{Q}$-side of its projections onto the $\mathbf{P}$ and $\mathbf{Q}$ chains, which we would denote by $\mathbf{P}|\mathbf{Q}|x''$ (See [5] for details).

2.6 Chain-Induced Subspaces and Coordination

As mentioned previously, our ultimate objective is to consider the quantification of a generalized interval with respect to multiple chains. In particular, we will consider generalized intervals whose endpoints are elements of the 1+1 subspace $\langle\mathbf{PQ}\rangle$ induced by a pair of chains $\mathbf{P}$ and $\mathbf{Q}$ that are *coordinated* with one another.

To unpack that sentence, we begin with the definition of the subspace $\langle\mathbf{PQ}\rangle$ induced by chains $\mathbf{P}$ and $\mathbf{Q}$. Observe first that the definition of collinearity in Definition 2.15 allows us to decompose any partially ordered set $(S, \sim)$ into two equivalence classes with respect to the chains $\mathbf{P}$ and $\mathbf{Q}$: the first class contains all events that are collinear with the chains $\mathbf{P}$ and $\mathbf{Q}$, while the second class contains all events that are not collinear with these chains.

We will refer to the first of these equivalence classes as the subspace $\langle\mathbf{PQ}\rangle$. That is, for any partially ordered set $(S, \sim)$ which contains chains $\mathbf{P}$ and $\mathbf{Q}$, we define:

$$\langle\mathbf{PQ}\rangle = \{x \in S : x \text{ is collinear with } \mathbf{P} \text{ and } \mathbf{Q}\} \tag{2.17}$$

As mentioned, we wish to consider the quantification of generalized intervals that reside in the subspace induced by two chains. However, before proceeding, we will impose a second condition on these chains: in particular, we require that the two chains be *coordinated*, as defined below.

Definition 2.16. ***(Coordinated Chains):*** *Let $(S, \prec)$ be a partially ordered set of events, the order of which is induced by influence. Moreover, let $\mathbf{P}$ and $\mathbf{Q}$ be two chains that lie within $(S, \prec)$. Next, define the closed intervals $[p_{min}, p_{max}]_{\mathbf{P}}$, $[\bar{p}_{min}, \bar{p}_{max}]_{\mathbf{P}}$ along* ***P*** *and $[q_{min}, q_{max}]_{\mathbf{Q}}$, $[\bar{q}_{min}, \bar{q}_{max}]_{\mathbf{Q}}$ along* ***Q****. The chains* ***P*** *and* ***Q*** *are said to be coordinated over these intervals if and only if three conditions are met.*

1. *The events in $[\bar{p}_{min}, \bar{p}_{max}]_{\mathbf{P}}$ forward project to the events in $[q_{min}, q_{max}]_{\mathbf{Q}}$ in a bijective fashion.*
2. *The events in $[p_{min}, p_{max}]_{\mathbf{P}}$ backward project to the events in $[\bar{q}_{min}, \bar{q}_{max}]_{\mathbf{Q}}$ in a bijective fashion.*
3. *The length (as given by Theorem 2.13) of any closed interval on* ***P*** *must equal the length of the image of its projection on* ***Q****, and vice-versa.*

A Hasse diagram of two coordinated chains is shown in Figure 2.6. In this diagram, we see that the three conditions for coordination are met: the mapping in blue between the closed intervals $[\bar{p}_{min}, \bar{p}_{max}]_{\mathbf{P}}$ and $[q_{min}, q_{max}]_{\mathbf{Q}}$ is

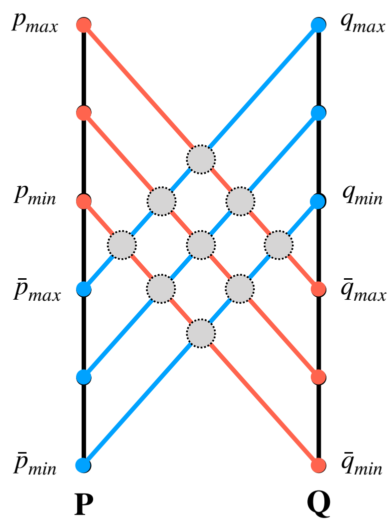


Figure 2.6: A pair of chains $\mathbf{P}$ and $\mathbf{Q}$ that are coordinated over the closed intervals $[p_{min}, p_{max}]_{\mathbf{P}}$, $[\bar{p}_{min}, \bar{p}_{max}]_{\mathbf{P}}$ along $\mathbf{P}$ and $[q_{min}, q_{max}]_{\mathbf{Q}}$, $[\bar{q}_{min}, \bar{q}_{max}]_{\mathbf{Q}}$ along $\mathbf{Q}$.

bijective, as is the mapping in orange between the intervals $[p_{min}, p_{max}]_{\mathbf{P}}$ and $[\bar{q}_{min}, \bar{q}_{max}]_{\mathbf{Q}}$. Moreover, the length of any closed interval on $\mathbf{P}$ is equal to the length of its image on $\mathbf{Q}$, and vice-versa (assuming that the length between any two consecutive events on either chain is equal to 1 unit).

In Section 2.4, we introduced a number of quantification schemes for various types of objects within a given poset. Now that we are familiar with the concept of coordinated chains, the question arises of how to consistently quantify an interval whose endpoints are *chains* rather than events. In particular, we would like to specify how to quantify an interval $[\mathbf{P}, \mathbf{Q}]$ whose endpoints are the coordinated chains $\mathbf{P}$ and $\mathbf{Q}$.

Theorem 2.17. ***(Distance between Two Chains)****: Let $(S, \prec)$ be a partially ordered set of events, the order of which is induced by influence. Moreover, let $\mathbf{P}$ and $\mathbf{Q}$ be a pair of coordinated chains that lie within $(S, \prec)$. Next, assign the valuations $v_{\mathbf{P}} : \mathbf{P} \to \mathbb{R}$ and $v_{\mathbf{Q}} : \mathbf{Q} \to \mathbb{R}$ to the chains $\mathbf{P}$ and $\mathbf{Q}$ respectively. Lastly, let $a \in \mathbf{P}$ and $b \in \mathbf{Q}$ be arbitrary events. It follows that the only consistent quantification of the interval $[\mathbf{P}, \mathbf{Q}]$, whose endpoints are the chains $\mathbf{P}$ and*

Q, *is given by the real number:*

$$d\left([\mathbf{P},\mathbf{Q}]\right) = \frac{\Delta p - \Delta q}{2} \tag{2.18}$$

where

$$\Delta p = d\left([Pa, Pb]_{\mathbf{P}}\right) = v_{\mathbf{P}}(Pb) - v_{\mathbf{P}}(Pa) \tag{2.19}$$

and

$$\Delta q = d\left([Qa, Qb]_{\mathbf{Q}}\right) = v_{\mathbf{Q}}(Qb) - v_{\mathbf{Q}}(Qa) \tag{2.20}$$

Colloquially, we will refer to the quantification of an interval of coordinate chains as the distance between those two chains.

Knuth and Bahreyni are careful to note that the distance between two coordinated chains is independent of the events chosen on either chain to compute this quantification [5].

2.7 Quantification of Intervals with Respect to Two Coordinated Chains

At last, we are prepared to consider the quantification of two classes objects with respect to a pair of coordinated chains: (1) closed intervals along one of the chains and (2) generalized intervals whose endpoints are located in the 1+1 subspace induced by the chains.

Closed Intervals: Equation 2.6 gives us a means of quantifying the length of a closed interval $[p_j, p_k]$ along a given chain $\mathbf{P}$ with respect to that chain. In particular, we saw that:

$$d([p_j, p_k]_{\mathbf{P}}) = v_{\mathbf{P}}(p_j) - v_{\mathbf{P}}(p_k) = v_{\mathbf{P}}(Pp_j) - v_{\mathbf{P}}(Pp_k) = \Delta p \tag{2.21}$$

where the last steps follow from the fact that the forward projection of an event $p \in \mathbf{P}$ onto the chain $\mathbf{P}$ is that same event p. However, we now let $\mathbf{Q}$ be another chain within the poset that is coordinated with $\mathbf{P}$. Then, we can also consider the length of the forward projection of our closed interval onto $\mathbf{Q}$:

$$d([Qp_j, Qp_k]_{\mathbf{Q}}) = v_{\mathbf{Q}}(Qp_j) - v_{\mathbf{Q}}(Qp_k) = \Delta q \tag{2.22}$$

By the third condition listed in the definition of coordination given in Definition 2.16, we require that $\Delta p = \Delta q$. Thus, it follows that we can rewrite the length of the closed interval $[p_j, p_k]$ given in equation 2.22 as:

$$d([p_j, p_k]_{\mathbf{P}}) = \Delta p = \frac{\Delta p + \Delta q}{2} \tag{2.23}$$

Thus, we now have a means of quantifying a closed interval along a chain with respect to a pair of coordinated chains. Importantly, this result differs by a sign from our equation for the distance between two coordinated chains given in equation 2.18.

Generalized Intervals: Coordinated chains are of particular interest because they induce well-defined subspaces. For example, the subspace $\langle \mathbf{PQ} \rangle$ induced by the chains **P** and **Q** in Figure 2.6 is equivalent to the set of grey events located at the intersections of the projections. One can verify that all of these events are collinear with the chains **P** and **Q** and thus are elements of $\langle \mathbf{PQ} \rangle$ by checking the conditions in Definition 2.15. As a result, we may now focus our attention on a generalized interval $[a, b]$, the endpoints of which are elements of the subspace $\langle \mathbf{PQ} \rangle$ induced by the coordinated chains **P** and **Q**. Such a generalized interval is depicted in Figure 2.7.

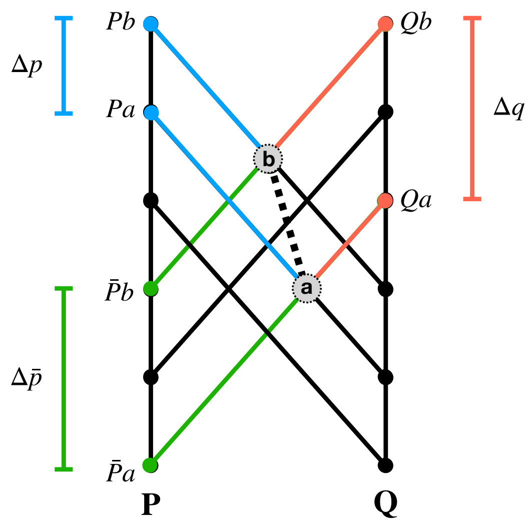


Figure 2.7: A generalized interval $[a, b]$ whose the endpoints are elements of the subspace $\langle \mathbf{PQ} \rangle$ induced by the coordinated chains **P** and **Q**, that we wish to quantify with respect to **P** and **Q**.

In Section 2.4, we learned how to quantify a generalized interval with respect to a given chain. As a result, we can use equation 2.8 to quantify the interval $[a, b]$ with respect to the chain **P** in the form of an interval pair:

$$[a, b]\Big|_{\mathbf{P(2)}} = (\Delta p, \Delta \bar{p})_{\mathbf{P}} = \big(v_{\mathbf{P}}(Pb) - v_{\mathbf{P}}(Pa), v_{\mathbf{P}}(\bar{P}b) - v_{\mathbf{P}}(\bar{P}a)\big)_{\mathbf{P}} \qquad (2.24)$$

where Δp and $\Delta \bar{p}$ are shown in blue and green respectively in Figure 2.7. Alternatively, we may use the scalar quantification of a generalized interval (Equation 2.12) to quantify $[a, b]$ with a single real number:

$$\Delta s^2\left([a, b]\right) = \Delta p \Delta \bar{p} \tag{2.25}$$

So far, we have simply done what we have done before and quantified $[a, b]$ with respect to the single chain $\mathbf{P}$. However, we may now take advantage of the coordination of the chains $\mathbf{P}$ and $\mathbf{Q}$ to quantify this interval with respect to both of these chains. In particular, recall that by the definition given in Definition 2.16, two chains are coordinated if and only if the length of any closed interval on $\mathbf{P}$ is equal to the length of the image of its projection onto the chain $\mathbf{Q}$. Because $[\bar{P}b, \bar{P}a]_{\mathbf{P}}$ is a closed interval on $\mathbf{P}$, it follows that its projection $[Qb, Qa]_{\mathbf{Q}}$ onto $\mathbf{Q}$ satisfies the following relationship:

$$\Delta \bar{p} = v_{\mathbf{P}}(\bar{P}b) - v_{\mathbf{P}}(\bar{P}a) = v_{\mathbf{Q}}(Qb) - v_{\mathbf{Q}}(Qa) = \Delta q \tag{2.26}$$

where Δq is depicted in orange in Figure 2.7.

It follows that we can rewrite the interval pair in equation 2.24 as the following quantification of $[a, b]$ with respect to the pair of coordinated chains $\mathbf{P}$ and $\mathbf{Q}$:

$$[a, b]\Big|_{\mathbf{P},\mathbf{Q(2)}} = (\Delta p, \Delta q)_{\mathbf{P},\mathbf{Q}} = \left(v_{\mathbf{P}}(Pb) - v_{\mathbf{P}}(Pa), v_{\mathbf{Q}}(Qb) - v_{\mathbf{Q}}(Qa)\right)_{\mathbf{P},\mathbf{Q}} \tag{2.27}$$

Similarly, we can rewrite the scalar quantification of the generalized interval $[a, b]$ in equation 2.25 as:

$$\Delta s^2\left([a, b]\right) = \Delta p \Delta q \tag{2.28}$$

Lastly, the results for the length of a closed interval along a given chain (equation 2.23) and the distance between two coordinated chains (equation 2.18) suggest that we rewrite the result for the scalar quantification of the interval $[a, b]$ in the following form:

$$\Delta s^2 = \Delta p \Delta q = \left(\frac{\Delta p + \Delta q}{2}\right)^2 - \left(\frac{\Delta p - \Delta q}{2}\right)^2 \tag{2.29}$$

Importantly, this representation of the scalar quantification of a generalized interval that resides in the subspace induced by a pair of coordinated chains is a function both of the length of a closed interval along one of the chains and the distance between our two coordinated chains. Moreover, the $(+, -)$ form of this scalar interval supplies us with the first hint of a metric with Lorentzian signature.

Chapter 3

Influence Theory and the Emergence of Minkowski Spacetime

3.1 Overview

In the previous chapter, we became familiar with the consistent quantification of sets of events that have acquired a partial ordering through influence. We placed particular emphasis on quantifying intervals of events with respect to multiple observer chains (totally ordered subsets within the larger poset). This crucially led us to the result in Equation 2.29, which provided a scalar quantification of a generalized interval that lies in the subspace induced by two coordinated observer chains with respect to those chains.

In this chapter, we wish to further investigate this equation, which bears striking resemblance to the Minkowski metric of flat spacetime discussed in Chapter 1 (see Equation 1.25). In particular, we will begin by considering the transformation of interval pairs from one observer chain to another. Then, following a helpful change of variables suggested by the form of Equation 2.29, we will arrive at the familiar Lorentz transformation derived in Chapter 1. Importantly, these results will serve to underline the upshot of the novel approach to special relativity offered by influence theory: namely, that one may view space and time as secondary concepts that emerge from a more fundamental notion – the consistent quantification of intervals.

3.2 Interval Pair Transformation

Recall that in order to quantify a generalized interval $[a, b]$ with respect to a chain $\mathbf{P}$, we may use the interval pair $(\Delta p, \Delta \bar{p})_{\mathbf{P}}$ (Equation 2.8). The entries of this 2-tuple are given by the lengths of the closed intervals $\Delta p = d([Pa, Pb]_{\mathbf{P}})$

and $\Delta\bar{p} = d([\bar{P}a, \bar{P}b]_{\mathbf{P}})$. Following our desire to consider the quantification of an interval with respect to multiple chains, we may now ask an important question: namely, how does one transform the entries of the interval pair written with respect to one chain to the entries of the interval pair written with respect to a different chain? In particular, we would like to move away from the rigid structure of coordinated chains and consider more general cases. One such case involves two chains $\mathbf{P}$ and $\mathbf{P'}$ that are linearly related. What does it mean for two chains to be linearly related? Figure 3.1 illustrates such a situation:

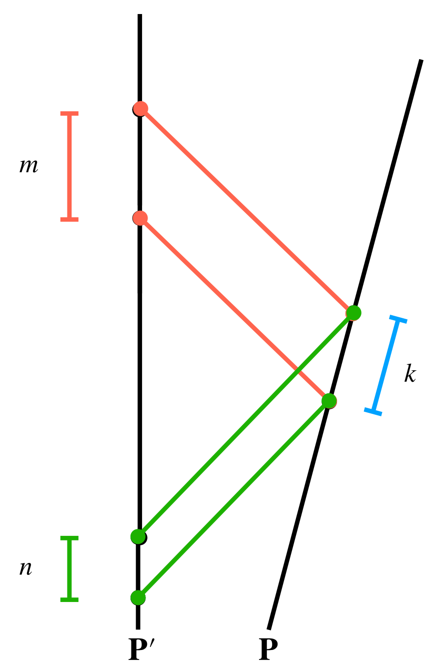


Figure 3.1: Two linearly related chains $\mathbf{P}$ and $\mathbf{P'}$, in which a closed interval of length k forward and backward projects onto $\mathbf{P'}$ with closed intervals of lengths m and n respectively.

The rationale for calling the chains in Figure 3.1 linearly related will become apparent later. For the moment, however, we first consider a closed interval along chain P that has fixed length k. Next, we assume that this interval forward projects onto a closed interval on $\mathbf{P'}$ with fixed length m, and backward projects onto a closed interval on $\mathbf{P'}$ with fixed length n. Thus, we can quantify the interval denoted in blue with respect to either $\mathbf{P}$ or $\mathbf{P'}$; these two quantifications are $(k, k)_{\mathbf{P}}$ and $(m, n)_{\mathbf{P'}}$.

We now wish to utilize this preliminary information to transform interval pairs between the two chains. In other words, given an interval $[a, b]$ and its interval pair $(\Delta p, \Delta\bar{p})_{\mathbf{P}}$ with respect to a chain $\mathbf{P}$, we desire a transformation

$L_{\mathbf{P}\to\mathbf{P}'}$ such that:

$$(\Delta p', \Delta\bar{p}')_{\mathbf{P}'} = L_{\mathbf{P}\to\mathbf{P}'}(\Delta p, \Delta\bar{p})_{\mathbf{P}} \tag{3.1}$$

where $(\Delta p', \Delta\bar{p}')_{\mathbf{P}'}$ is the interval pair of $[a, b]$ written with respect to $\mathbf{P}'$.

We consider such a situation in Figure 3.2, in which we wish to transform the interval pair of an interval $[a, b]$ written with respect to $\mathbf{P}$ to the interval pair of the same interval written with respect to $\mathbf{P}'$.

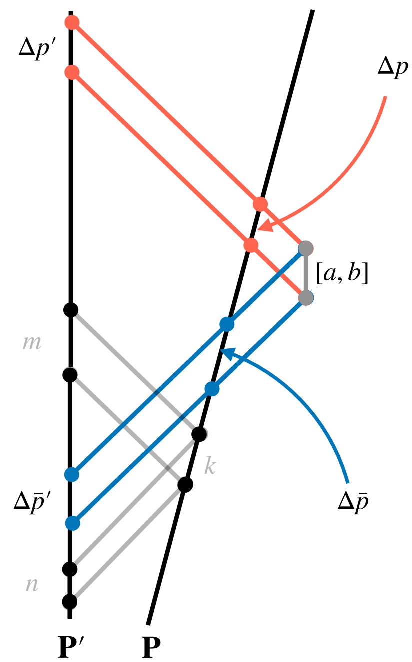


Figure 3.2: Two linearly related chains $\mathbf{P}$ and $\mathbf{P}'$, in which a closed interval of length k forward and backward projects onto $\mathbf{P}'$ with closed intervals of lengths m and n respectively.

In their 2014 paper, Knuth and Bahreyni show that if an arbitrary generalized interval $[a, b]$ is quantified with respect to $\mathbf{P}$ with an interval pair $(\Delta p, \Delta\bar{p})_{\mathbf{P}}$, then its interval pair with respect to $\mathbf{P}'$ is given by:

$$(\Delta p', \Delta\bar{p}')_{\mathbf{P}'} = L_{\mathbf{P}\to\mathbf{P}'}(\Delta p, \Delta\bar{p})_{\mathbf{P}} = (\Delta p\sqrt{\frac{m}{n}}, \Delta\bar{p}\sqrt{\frac{n}{m}})_{\mathbf{P}'} \tag{3.2}$$

which demonstrates that the pair transformation is in fact linear in Δp and $\Delta\bar{p}$, with scalars equal to $\sqrt{m/n}$ and $\sqrt{n/m}$, which were fixed earlier [5].

We next consider another special case, in which we have quantified the generalized interval $[a, b]$ with respect to two coordinated chains $\mathbf{P}$ and $\mathbf{Q}$. Recall that $\mathbf{P}$ and $\mathbf{Q}$ are coordinated if and only if (in addition to two other conditions) the length of any closed interval on $\mathbf{P}$ equals the length of the image of its projection on $\mathbf{Q}$, and vice-versa. Thus, we have that $\Delta\bar{p} = \Delta q$, and as seen in Equation 2.27, the interval pair for $[a, b]$ becomes $(\Delta p, \Delta q)_{\mathbf{P},\mathbf{Q}}$. We may then rewrite the result of Equation 3.2 in the form:

$$(\Delta p', \Delta q')_{\mathbf{P}',\mathbf{Q}'} = (\Delta p\sqrt{\frac{m}{n}}, \Delta q\sqrt{\frac{n}{m}})_{\mathbf{P}',\mathbf{Q}'} \tag{3.3}$$

3.3 A Helpful Change of Variables

Now that we have specified how to transform interval pairs from one chain to another linearly related chain, we return to the scalar quantification introduced in the previous chapter (Equation 2.29, repeated below).

$$\Delta s^2 = \Delta p \Delta q = \left(\frac{\Delta p + \Delta q}{2}\right)^2 - \left(\frac{\Delta p - \Delta q}{2}\right)^2 \tag{2.29r}$$

We initially observed that the $(+, -)$ form of this equation, written in terms of the length of a closed interval along a chain (Equation 2.23) and the distance between two coordinated chains (Equation 2.18), shares a Lorentzian signature with the Minkowski metric of flat spacetime (Equation 1.25).

This motivates the reinterpretation of the length of a closed interval as a time coordinate and the distance between two coordinated chains as a spatial coordinate:

$$\Delta t = \frac{\Delta p + \Delta q}{2} \tag{3.4}$$

$$\Delta x = \frac{\Delta p - \Delta q}{2} \tag{3.5}$$

This allows us to rewrite Equation 2.29 as:

$$\Delta s^2 = (\Delta t)^2 - (\Delta x)^2 \tag{3.6}$$

The result in Equation 3.6 is remarkable. In particular, we observe that the scalar quantification of a generalized interval of events in a partially ordered set of events is identical in form to the spacetime interval first introduced in Equation 1.6. Most starting is the realization that the result in Equation 3.6 has emerged independently of any preconceived notions about the nature of space or time; it merely represents a means for consistently quantifying intervals in a poset.

We then continue along this thread of reinterpretation of previous results by expressing the pair transformation in Equation 3.3 in a new form. In particular, we utilize the definitions in Equations 3.4 and 3.5 to solve for Δp and Δq:

$$\Delta p = \Delta t + \Delta x \tag{3.7}$$

$$\Delta q = \Delta t - \Delta x \tag{3.8}$$

such that Equation 3.3 becomes:

$$(\Delta t' + \Delta x', \Delta t' - \Delta x')_{\mathbf{P}',\mathbf{Q}'} = \left[\sqrt{\frac{m}{n}}(\Delta t + \Delta x), \sqrt{\frac{n}{m}}(\Delta t - \Delta x)\right]_{\mathbf{P}',\mathbf{Q}'} \tag{3.9}$$

We may now solve for the transformed coordinates $\Delta t'$ and $\Delta x'$

$$\Delta t' = \frac{\Delta t}{2}\left[\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}}\right] + \frac{\Delta x}{2}\left[\sqrt{\frac{m}{n}} - \sqrt{\frac{n}{m}}\right] \tag{3.10}$$

$$\implies \Delta t' = \frac{\Delta t}{2\sqrt{mn}}(m+n) + \frac{\Delta x}{2\sqrt{mn}}(m-n) \tag{3.11}$$

$$\Delta x' = \frac{\Delta t}{2}\left[\sqrt{\frac{m}{n}} - \sqrt{\frac{n}{m}}\right] + \frac{\Delta x}{2}\left[\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}}\right] \tag{3.12}$$

$$\implies \Delta x' = \frac{\Delta t}{2\sqrt{mn}}(m-n) + \frac{\Delta x}{2\sqrt{mn}}(m+n) \tag{3.13}$$

To simplify notation, we introduce the variable β, which quantifies the relationship between forward and backward projections of $[a, b]$ onto $\mathbf{P}'$:

$$\beta = \frac{n-m}{n+m} \tag{3.14}$$

such that

$$\implies \frac{1}{\sqrt{1-\beta^2}} = \left[\frac{(n+m)^2}{(n+m)^2} - \frac{(n-m)^2}{(n+m)^2}\right]^{-1/2} = \frac{n+m}{2\sqrt{mn}} \tag{3.15}$$

This allows us to rewrite Equations 3.11 and 3.13 in the form:

$$\Delta t' = \frac{\Delta t}{\sqrt{1-\beta^2}} - \frac{\beta \Delta x}{\sqrt{1-\beta^2}} \tag{3.16}$$

$$\Delta x' = -\frac{\beta \Delta t}{\sqrt{1-\beta^2}} + \frac{\Delta x}{\sqrt{1-\beta^2}} \tag{3.17}$$

Lastly, we define the variable: $\gamma = (1-\beta^2)^{-1/2}$, in which case the transformed coordinates become:

$$\Delta t' = \gamma \Delta t - \beta\gamma \Delta x \tag{3.18}$$

$$\Delta x' = -\beta\gamma \Delta t + \gamma \Delta x \tag{3.19}$$

Written in matrix form, the above transformations are equivalent to:

$$\begin{bmatrix} \Delta t' \\ \Delta x' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} \Delta t \\ \Delta x \end{bmatrix} \tag{3.20}$$

Like Equation 3.6, the result in Equation 3.20 is worthy of further scrutiny. In particular, it demonstrates that the pair transformation of Equation 3.3 – upon an appropriate change of variables – is identical to the Lorentz transformation first introduced in Equation 1.42. However, in contrast to the derivation of the Lorentz transformation in Chapter 1, our path to the result in Equation 3.20 did not rely on any prior assumptions about the nature of space or time. Instead, it merely represents a technique for transforming the quantification of a generalized interval of events from one chain to another [5].

3.4 The Significance of β

The introduction of the variable β in Equation 3.14 was critical to the above derivation. In our original discussion of the Lorentz transformation in Chapter 1, β quantified the relative velocity between two inertial reference frames. However, concepts such as space and time do not exist within the formalism of influence theory. As a result, we must consider the significance of β from a different perspective.

The most important observation about the quantity β is that its magnitude is bounded below by 0 and above by 1. The lower bound, in which $\beta = 0$, occurs when $m = n$. This situation is illustrated below in Figure 3.3. This figure illustrates two pairs of coordinated chains: $\mathbf{P}$ and $\mathbf{Q}$, as well as $\mathbf{P'}$ and $\mathbf{Q'}$. The diagram demonstrates that the case in which $\beta = 0$ occurs when the length of the forward projection of k onto chain $\mathbf{P'}$ from chain $\mathbf{P}$ equals the length of the forward projection of k from $\mathbf{Q}$ onto $\mathbf{Q'}$.

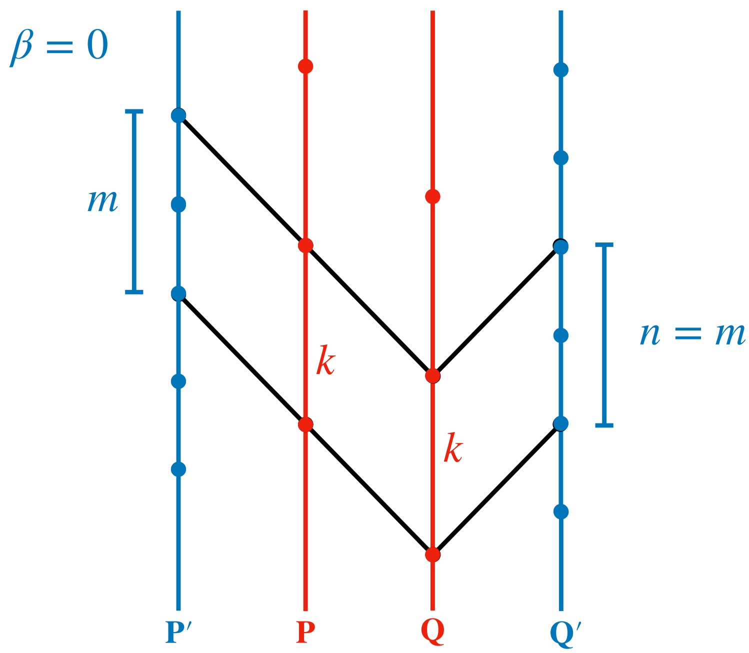


Figure 3.3: Two linearly related pairs of coordinated chains $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\mathbf{P'}, \mathbf{Q'}\}$ which yield $\beta = 0$. This is due to the fact that the interval k projects onto intervals of equal length m on chains $\mathbf{P'}$ and $\mathbf{Q'}$.

The second case, in which $|\beta| = 1$, requires that either $m = 0$ or $n = 0$. Figure 3.4 demonstrates an instance in which $n = 0$. Once again, we have two pairs of coordinated chains: $\mathbf{P}$ and $\mathbf{Q}$, as well as $\mathbf{P'}$ and $\mathbf{Q'}$. However, in contrast to Figure 3.3, we observe that while the interval k forward projects onto an interval of length m on chain $\mathbf{P'}$, the same interval k forward projects from chain $\mathbf{Q}$ onto a single event (an interval of zero length) on chain $\mathbf{Q'}$. As a result, the interval pair used to quantify the interval k with respect to the coordinated chains $\mathbf{P'}$ and $\mathbf{Q'}$ is $(m, 0)_{\mathbf{P'},\mathbf{Q'}}$ which yields a $\Delta s^2 = m \cdot 0 = 0$ scalar quantification.

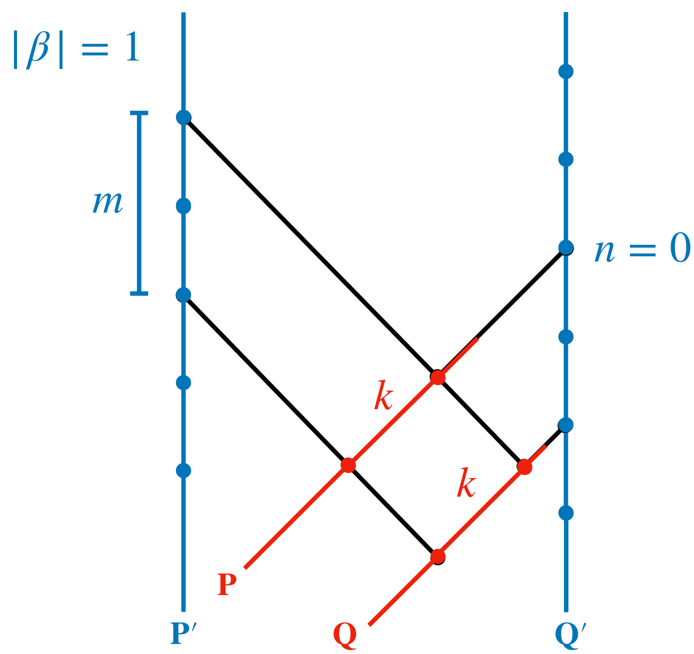


Figure 3.4: Two linearly related pairs of coordinated chains $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\mathbf{P'},\mathbf{Q'}\}$ which yield $\beta = 1$. This is due to the fact that the interval k forward projects onto intervals of zero length on the chain $\mathbf{Q'}$.

Now that we have assessed the behavior of β, we are prepared to contrast our conclusions with the significance of β in traditional special relativity. There is an obvious analogy with the well-known result from special relativity that no massive object may travel faster than the speed of light. That is, in both the traditional space-time picture and the perspective suggested by influence theory, the magnitude of β may not exceed 1. There is a further similarity between the behavior of the scalar quantification in influence theory and the spacetime interval – its analogue in the traditional formalism of special relativity. We have just seen in the paragraph above that the scalar quantification of an interval along chain $\mathbf{P}$ with respect to coordinated chains $\mathbf{P'}$ and $\mathbf{Q'}$ is 0 if $\{\mathbf{P'}, \mathbf{Q'}\}$ are related to $\{\mathbf{P}, \mathbf{Q}\}$ such that $|\beta| = 1$. This is analogous to a result first introduced in Equation 1.9: namely, that the spacetime interval between two events that correspond to the emission and reception of a light signal is 0 [5].

3.5 Summary

The above results mark an opportune moment to return to the discussion with which we began our exploration of special relativity in Chapter 1. There, we inquired into the assumptions behind the traditional derivation of the Lorentz transformation. These presuppositions included the homogeneity of space and time, the isotropy of space, the principle of relativity, and the constancy of the speed of light. Although these assumptions provided one path to the final form of the Lorentz transformation, a pair of questions arose. First, we desired to know whether one could arrive at the same physical law of the Lorentz transformation on the basis of fewer postulates about the natural world. Second, we speculated as to whether or not one could demonstrate spacetime to be an emergent concept. This would be achieved by means of a derivation of the geometric structure of flat spacetime that relies on zero preconceptions regarding the nature of space and time itself.

As we have seen in the prior two chapters, influence theory presents a compelling case for answering in the affirmative to the two questions above. The whole edifice of influence theory rests upon a pair of very straightforward assumptions regarding the natural world: (1) events happen and (2) it is possible to order events through influence. Absent are any notions related to space, time, or relative motion. Adopting these two assumptions as a starting point, we then proceeded to examine the structure that arises within a partially ordered set of events. At first, this structure appeared abstract and wholly removed from physical reality. However, an advantageous change of variables revealed a startling connection between the tools developed to consistently quantify intervals of events within a partially ordered set and the geometric structure of flat spacetime. In particular, we saw that the scalar quantification (Equation 2.29) is analogous to the spacetime interval, while the pair transformation (Equation 3.3) parallels the Lorentz transformation. It thus appears that the equations that characterize the relationship between the spacetime coordinate intervals corresponding to pairs of events materialize independently of any assumptions regarding space, time, and the laws of physics. It is in this sense that Influence theory suggests that spacetime itself is an emergent concept and thereby breaks with a centuries-old tradition that regards space and time as fundamental building blocks of reality.

Chapter 4

Geometry from the Perspective of Influence Theory

4.1 Overview

At the end of Chapter 2, we arrived at the essential result in Equation 2.29 that provides a scalar quantification of a generalized interval that lies in the subspace induced by a pair of coordinated chains with respect to those chains. The aim of this chapter is to extend the work of Knuth and Bahreyni by deriving a method of subspace projection, which will permit the quantification of *any* generalized interval with respect to a pair of coordinated chains. In particular, we will develop a means of projecting an arbitrary interval onto the subspace induced by a pair of coordinated chains. In order to accomplish this goal, we will rely on concepts introduced in Chapter 2 – such as betweenness and collinearity – to revisit familiar geometric notions such as orthogonality and the Pythagorean theorem from the perspective of influence theory. Mindful of the connections between geometry and the formalism of influence theory, we will ultimately find in the method of subspace projection an analogue to the traditional scalar product of Euclidean geometry.

4.2 A Special Case to Motivate Orthogonality

In Chapter 2, we developed various methods for quantifying an interval that lies in the subspace induced by a pair of coordinated chains. Our ultimate objective for this chapter is to formalize the quantification of intervals in a more general setting. In particular, we aim to define a method of *subspace projection*, that will permit the quantification of all intervals in a poset with respect to a pair of coordinated chains — including those intervals that do not lie within the induced subspace — so long as they have projections onto the chains in question.

Nevertheless, we cannot make the jump to this general calculation directly. Instead, it is necessary to derive an intermediate result that will provide us with the tools necessary to make sense of the more general case. This intermediate step relies on the notion of orthogonal subspaces. Although the rigorous definition and treatment of orthogonal subspaces is beyond the scope of this work, we will consider a special case, as described below.

For the first time, we consider a system of *two* pairs of coordinated chains. The first pair consists of the coordinated chains $\mathbf{P}$ and $\mathbf{Q}$, and together they induce the subspace $\langle\mathbf{PQ}\rangle$. The second pair consists of the chains $\mathbf{R}$ and $\mathbf{S}$, which together induce the subspace $\langle\mathbf{RS}\rangle$. The special case of interest to us occurs when a class of intervals known as *pure antichain-like* intervals in the subspace $\langle\mathbf{PQ}\rangle$ project onto the chains $\mathbf{R}$ and $\mathbf{S}$ in a specific manner. We will define the notion of pure antichain-like intervals below. In particular, we require that the two events which define the endpoints of the pure antichain-like interval in $\langle\mathbf{PQ}\rangle$ (i) forward project onto the same event on $\mathbf{R}$, (ii) backward project onto the same event on $\mathbf{R}$, (iii) forward project onto the same event on $\mathbf{S}$, and (iv) backward project to the same event on $\mathbf{S}$. If the projections of all pure antichain-like intervals in $\langle\mathbf{PQ}\rangle$ onto the chains $\mathbf{R}$ and $\mathbf{S}$ behave in this manner, we will consider the two subspaces to be orthogonal. Make careful note, however, that we are *not* defining this behavior of projection of pure antichain-like intervals to be equivalent to orthogonality. Rather, these conditions merely represent a special case of orthogonal subspaces, a concept that deserves a fuller and more rigorous treatment in the future.

In light of the special case described above, a pair of questions arise: what exactly are pure antichain-like intervals and why are they relevant to motivating the concept of orthogonal subspaces? We answer the first question with a definition:

Definition 4.1. ***(Antichain-Like Intervals):*** *Let $(S,\prec)$ be a partially ordered set of events, the order of which is induced by influence. Moreover, let $\mathbf{P}$ and $\mathbf{Q}$ form a pair of coordinated chains that lie within $(S,\prec)$. As usual, we assign valuations $v_{\mathbf{P}} : \mathbf{P} \to \mathbb{R}$ to the chain $\mathbf{P}$ and $v_{\mathbf{Q}} : \mathbf{Q} \to \mathbb{R}$ to the chain $\mathbf{Q}$. Lastly, let events $a, b \in \langle\mathbf{PQ}\rangle$. Then the interval $[a,b]$ forms an antichain-like interval if:*

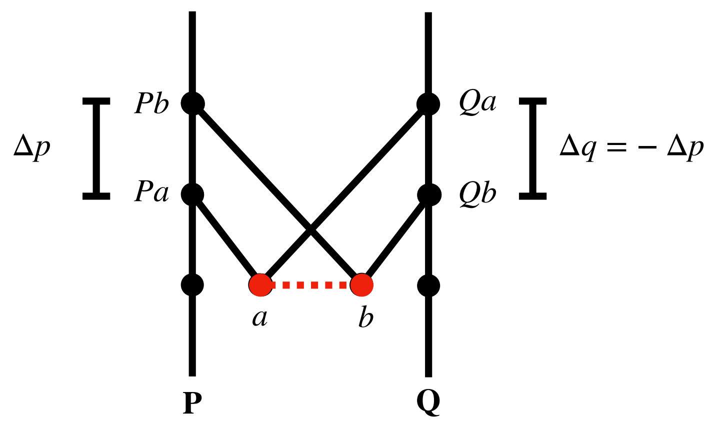
$$\Delta p \Delta q = \left[v_{\mathbf{P}}(Pb) - v_{\mathbf{P}}(Pa)\right]\left[v_{\mathbf{Q}}(Qb) - v_{\mathbf{Q}}(Qa)\right] < 0.$$

Equivalently, the interval is antichain-like if Δp and Δq are both non-zero and of opposite sign.

As a special case, we say the interval $[a,b]$ is purely antichain-like if, in addition to the above condition, $|\Delta p| = |\Delta q|$. If the interval is purely antichain-like, then we may always write its interval pair quantification with respect to the coordinated chains $\mathbf{P}$ and $\mathbf{Q}$ as:

$$[p,q]\Big|_{\mathbf{P},\mathbf{Q}(\mathbf{2})} = (\Delta p, \Delta q)_{\mathbf{P},\mathbf{Q}} = (\Delta p, -\Delta p)_{\mathbf{P},\mathbf{Q}}.$$

As always, it is helpful to consider an example to clarify the meaning of the above definition. Figure 4.1 demonstrates a pure antichain-like interval $[a, b]$ that resides in the subspace $\langle \mathbf{PQ} \rangle$. As described in Definition 4.1, $[a, b]$ is antichain-like because Δp and Δq are of opposite sign. To see this, recall that valuations assigned to chains are always monotonic (Definition 2.11); hence $\Delta p = v_{\mathbf{P}}(Pb) - v_{\mathbf{P}}(Pa) > 0$ (because Pa influences Pb), but $\Delta q = v_{\mathbf{Q}}(Qb) - v_{\mathbf{Q}}(Qa) < 0$ (because Qa is influenced by Qb). It is pure, since $|\Delta p| = |\Delta q|$. Hence, the interval pair quantification of $[a, b]$ with respect to the coordinated chains $\mathbf{P}$ and $\mathbf{Q}$ is indeed $(\Delta p, -\Delta p)_{\mathbf{P},\mathbf{Q}}$ — that is, the interval $[a, b]$ is pure antichain-like, as claimed.



$$[p, q]\Big|_{\mathbf{P},\mathbf{Q}(\mathbf{2})} = (\Delta p, \Delta q)_{\mathbf{P},\mathbf{Q}} = (\Delta p, -\Delta p)_{\mathbf{P},\mathbf{Q}}$$

Figure 4.1: A pure antichain-like interval $[a, b]$ that resides in the subspace $\langle \mathbf{PQ} \rangle$ induced by the coordinated chains $\mathbf{P}$ and $\mathbf{Q}$.

Next, recall that in the special case of orthogonality under consideration, the two events which define the endpoints of *any* pure antichain-like interval in $\langle \mathbf{PQ} \rangle$ must (i) forward project onto the same event on $\mathbf{R}$, (ii) backward project onto the same event on $\mathbf{R}$, (iii) forward project onto the same event on $\mathbf{S}$, and (iv) backward project to the same event on $\mathbf{S}$. Once again, we use a Hasse diagram (Figure 4.2) to elucidate what these conditions mean. In this case, we define the events $p \in \mathbf{P}$ and $q \in \mathbf{Q}$ such that $[p, q]$ is a pure antichain-like interval in $\langle \mathbf{PQ} \rangle$. Moreover, we have introduced the chains $\mathbf{R}$ and $\mathbf{S}$, and assign to them respectively the monotonic valuations $v_{\mathbf{R}} : \mathbf{R} \to \mathbb{R}$ and $v_{\mathbf{S}} : \mathbf{S} \to \mathbb{R}$.

The projections of p and q onto the chains $\mathbf{P}$ and $\mathbf{Q}$ are depicted with black, dotted lines. The confirmation that $[p, q]$ is pure antichain-like is similar to the discussion regarding Figure 4.1, so we do not repeat that same analysis here. However, we do verify that the four conditions regarding the projections of p and q onto the chains $\mathbf{R}$ and $\mathbf{S}$ (depicted with dotted, blue lines) are met; we see that (i) $Rp = Rq$, (ii) $\bar{R}p = \bar{R}q$, (iii) $Sp = Sq$, and (iv) $\bar{S}p = \bar{S}q$. So long as every pure antichain-like interval in $\langle\mathbf{PQ}\rangle$ projects onto the chains $\mathbf{R}$ and $\mathbf{S}$ in this manner, the special case of orthogonality under consideration is met.

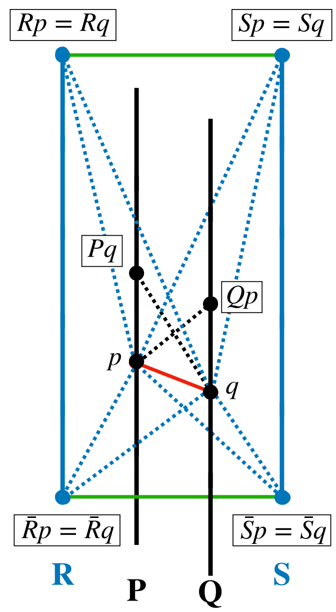


Figure 4.2: A system of two pairs of coordinated chains $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\mathbf{R}, \mathbf{S}\}$, in which the pure antichain-like interval $[p, q]$ in $\langle\mathbf{PQ}\rangle$ projects onto $\mathbf{R}$ and $\mathbf{S}$ as desired; the pure antichain-like intervals $[\bar{R}p, \bar{S}p]$ and $[Rp, Sp]$ in $\langle\mathbf{RS}\rangle$ projects onto $\mathbf{P}$ and $\mathbf{Q}$ similarly.

This leads to the second question posed earlier: why are pure antichain-like intervals essential to motivating the concept of orthogonal subspaces? The answer lies in the fact that pure antichain-like intervals represent the only type of intervals in $\langle\mathbf{PQ}\rangle$ that can always meet the four projection-related conditions described above. Moreover, requiring that all pure antichain-like intervals in $\langle\mathbf{PQ}\rangle$ project onto the chains $\mathbf{R}$ and $\mathbf{S}$ in the aformentioned manner guarantees that the same class of intervals in $\langle\mathbf{RS}\rangle$ will project onto the chains $\mathbf{P}$ and $\mathbf{Q}$ in the same way. Although by no means a conclusive proof, we can use Figure 4.2 as a first test of this consequence.

Take under consideration the events $\bar{R}p \in \mathbf{R}$ and $\bar{S}p \in \mathbf{S}$. These events define the endpoints of the interval $[\bar{R}p, \bar{S}p] \in \langle\mathbf{RS}\rangle$ (shown in green in Figure 4.2). I first claim that this interval is pure antichain-like with respect to the chains $\mathbf{R}$ and $\mathbf{S}$. To verify this, we examine the conditions in Definition 4.1. We first calculate:

$$\Delta r = v_{\mathbf{R}}(R\bar{S}p) - v_{\mathbf{R}}(R\bar{R}p) = v_{\mathbf{R}}(Rp) - v_{\mathbf{R}}(\bar{R}p) > 0, \tag{4.1}$$

which is positive since $R\bar{R}p = \bar{R}p$ influences $R\bar{S}p = Rp$ (remember that valuations are always monotonic). In the case of Δs, we find that:

$$\Delta s = v_{\mathbf{S}}(S\bar{S}p) - v_{\mathbf{S}}(S\bar{R}p) = v_{\mathbf{S}}(\bar{S}p) - v_{\mathbf{S}}(Sp) < 0, \tag{4.2}$$

which is negative because $S\bar{S}p = \bar{S}p$ influences $S\bar{R}p = Sp$. Hence, $\Delta r \Delta s < 0$ and the interval $[\bar{R}p, \bar{S}p]$ is antichain-like with respect to the chains $\mathbf{R}$ and $\mathbf{S}$. It is a *pure* antichain-like interval because:

$$|\Delta r| = |v_{\mathbf{R}}(Rp) - v_{\mathbf{R}}(\bar{R}p)| = |v_{\mathbf{S}}(\bar{S}p) - v_{\mathbf{S}}(Sp)| = |\Delta s|, \tag{4.3}$$

which confirms that we may write the interval pair quantification of $[\bar{R}p, \bar{S}p]$ with respect to the chains $\mathbf{R}$ and $\mathbf{S}$ in the form $(\Delta r, -\Delta r)_{\mathbf{R},\mathbf{S})}$. Hence, $[\bar{R}p, \bar{S}p]$ is indeed a pure antichain-like interval in $\langle\mathbf{RS}\rangle$.

I next claim that the interval $[\bar{R}p, \bar{S}p]$ *forward* projects onto the chains $\mathbf{P}$ and $\mathbf{Q}$ in the desired manner. Observe that the events $\bar{R}p$ and $\bar{S}p$ both forward project onto the same event on chain $\mathbf{P}$: $P\bar{R}p = p = P\bar{S}p$. The forward projections of these events onto chain $\mathbf{Q}$ are likewise equal: $Q\bar{R}p = q = Q\bar{S}p$. Hence, the endpoints of the pure antichain-like interval $[\bar{R}p, \bar{S}p]$ forward project onto the chains $\mathbf{P}$ and $\mathbf{Q}$ as promised. The limitations of Figure 4.2 are such that we cannot consider how the endpoints $\bar{R}p$ and $\bar{S}p$ backward project onto $\mathbf{P}$ and $\mathbf{Q}$, but the calculation would be similar. Moreover, the backward projections of the pure antichain-like interval $[Rp, Sp]$ behave in a similar manner – the calculation is nearly identical to the discussion above of $[\bar{R}p, \bar{S}p]$. Hence, these examples give credence to the claim that requiring pure antichain-like intervals $\langle\mathbf{PQ}\rangle$ to project onto the chains $\mathbf{R}$ and $\mathbf{S}$ in the manner described implies that pure antichain-like intervals $\langle\mathbf{RS}\rangle$ to project onto the chains $\mathbf{P}$ and $\mathbf{Q}$ in the same way. For these reasons, we are comfortable in saying that the subspaces $\langle\mathbf{PQ}\rangle$ and $\langle\mathbf{RS}\rangle$ are orthogonal.

4.3 Hasse Diagrams from a New Perspective

The special case of orthogonality considered in Section 4.2 is valuable because it permits us to construct new diagrams that shed considerable light on the relationships between multiple pairs of coordinated chains. In particular, it is of great benefit to transition from the traditional "aerial" view of Hasse diagrams — in which we depict chains as vertical lines — to a geometric view — in which we depict the same chains as simple points. An example of this transition is shown in Figures 4.3.a and 4.3.b: the left frame depicts the traditional Hasse diagram of the system of coordinated chains $\{\mathbf{P}, \mathbf{Q}\}$ and $\{\mathbf{R}, \mathbf{S}\}$ that we considered in detail in Section 4.2, while the right frame portrays the geometric view of this same system of orthogonal subspaces.

In order to construct the geometric view depicted in Figure 4.3.b, we arrange the chains in accordance with the betweenness relations first considered in Section 2.5. In particular, an examination of the projections of the interval $[p, q]$ onto the chains $\mathbf{R}$ and $\mathbf{S}$ in Figure 4.3.a shows that the chain $\mathbf{R}$ lies on the $\mathbf{P}$-side of the coordinated chains $\{\mathbf{P}, \mathbf{Q}\}$, while the chain $\mathbf{S}$ lies on the $\mathbf{Q}$-side of the coordinated chains $\{\mathbf{P}, \mathbf{Q}\}$. This permits us to conclude that the interval $[p, q]$ lies *between* the chains $\mathbf{R}$ and $\mathbf{S}$ — the case of directionality shown in Figure 2.5.b. We may also apply the same analysis to the intervals $[\bar{R}p, \bar{S}p]$ and $[Rp, Sp]$, which are depicted in green in Figure 4.3.a. An examination of the projections of these intervals onto the chains $\mathbf{P}$ and $\mathbf{Q}$ reveal that these intervals lie between the chains $\mathbf{P}$ and $\mathbf{Q}$. Hence, Figure 4.3.b indeed represents the only manner by which we can position the four chains in accordance with the betweenness relations: the interval $[p, q]$ (shown in red) must lie between the chains $\mathbf{R}$ and $\mathbf{S}$, while the intervals $[\bar{R}p, \bar{S}p]$ and $[Rp, Sp]$ (shown in green) must lie between the chains $\mathbf{P}$ and $\mathbf{Q}$.

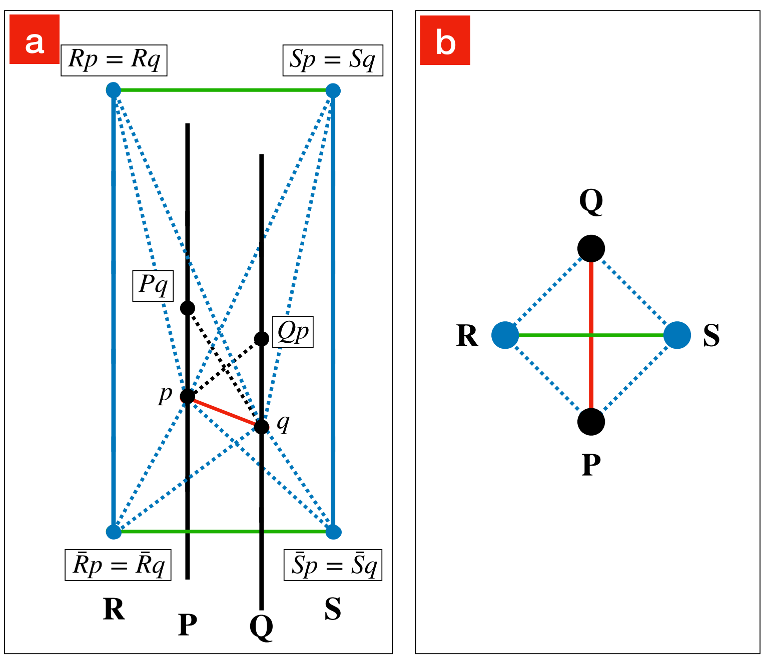


Figure 4.3: A comparison of the traditional Hasse diagram of the system of coordinated chains in Figure 4.2 with a geometric view which represents chains as dots and arranges them in accordance with betweenness relations.

4.4 An Analogue to the Pythagorean Theorem

Before continuing to a derivation of the central result of the chapter — a means of projecting any generalized interval in a poset onto the subspace induced by a pair of coordinated chains — we first review a crucial lemma from Knuth and Bahreyni regarding the scalar quantification of intervals that reside in orthogonal subspaces. In their original 2014 paper, Knuth and Bahreyni considered an extension of the special case of orthogonality described in Section 4.3. They begin with the same four chains $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ from Section 4.3 and then define a fifth chain $\mathbf{O}$ that is collinear with the pair of coordinated chains $\{\mathbf{P}, \mathbf{Q}\}$ as well as the pair $\{\mathbf{R}, \mathbf{S}\}$. This ensures that the pairs $\{\mathbf{P}, \mathbf{O}\}$ and $\{\mathbf{O}, \mathbf{R}\}$ are coordinated. Moreover, this implies that the subspaces $\langle \mathbf{OP} \rangle$ and $\langle \mathbf{RO} \rangle$ satisfy the special case of orthogonality from earlier. They suppose that the chains $\mathbf{R}$ and $\mathbf{P}$ are coordinated as well. As usual, we assign to each chain a valuation; for instance, the chain $\mathbf{P}$ has valuation $v_{\mathbf{P}}$. Figure 4.4 depicts this new configuration.

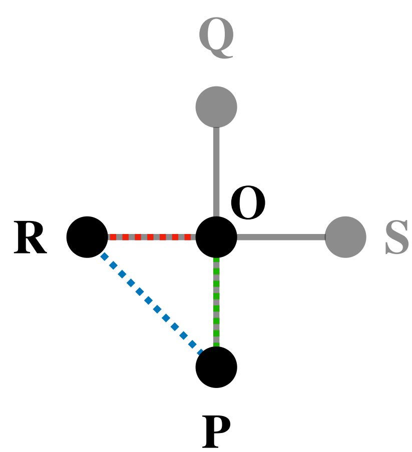


Figure 4.4: The same system of chains from Figure 4.3 with the introduction of a new chain $\mathbf{O}$ so that the chains $\mathbf{O}$, $\mathbf{P}$, and $\mathbf{R}$ are pairwise coordinated and the subspaces $\langle \mathbf{RO} \rangle$ and $\langle \mathbf{OP} \rangle$ are orthognal in the sense of Section 4.2. The intervals $[r, o]$ (red), $[o, p]$ (green), and $[p, r]$ (blue) are all antichain-like.

Subsequently, Knuth and Bahreyni define three events $r \in \mathbf{R}$, $o \in \mathbf{O}$, and

$p \in \mathbf{P}$ so that these events define the endpoints of three pure antichain-like intervals: the first is $[r, o]$ — depicted in red in the diagram — and resides in the subspace $\langle \mathbf{RO} \rangle$. The second interval is $[o, p]$ — depicted in green — and resides in the subspace $\langle \mathbf{OP} \rangle$. The third interval is $[p, r]$ and is depicted in blue.

We summarize the interval pair quantification of these pure antichain-like intervals below:

$$\begin{aligned} [r,o]\Big|_{\mathbf{R},\mathbf{O}(\mathbf{2})} &= [v_{\mathbf{R}}(r) - v_{\mathbf{R}}(Ro), -(v_{\mathbf{R}}(r) - v_{\mathbf{R}}(Ro))]_{\mathbf{R},\mathbf{O}} \\ [o,p]\Big|_{\mathbf{O},\mathbf{P}(\mathbf{2})} &= [v_{\mathbf{O}}(o) - v_{\mathbf{O}}(Op), -(v_{\mathbf{O}}(o) - v_{\mathbf{O}}(Op))]_{\mathbf{O},\mathbf{P}} \\ [p,r]\Big|_{\mathbf{P},\mathbf{R}(\mathbf{2})} &= [v_{\mathbf{P}}(p) - v_{\mathbf{P}}(Pr), -(v_{\mathbf{P}}(p) - v_{\mathbf{P}}(Pr))]_{\mathbf{P},\mathbf{R}} \end{aligned} \tag{4.4}$$

Next, if we define $\Delta a = v_{\mathbf{R}}(r) - v_{\mathbf{R}}(Ro)$, $\Delta b = v_{\mathbf{O}}(o) - v_{\mathbf{O}}(Op)$, and $\Delta c = v_{\mathbf{P}}(p) - v_{\mathbf{P}}(Pr)$, then we may rewrite the expressions in Equation 4.4 as:

$$\begin{aligned} [r,o]\Big|_{\mathbf{R},\mathbf{O}(\mathbf{2})} &= [\Delta a, -\Delta a]_{\mathbf{R},\mathbf{O}} \\ [o,p]\Big|_{\mathbf{O},\mathbf{P}(\mathbf{2})} &= [\Delta b, -\Delta b]_{\mathbf{O},\mathbf{P}} \\ [p,r]\Big|_{\mathbf{P},\mathbf{R}(\mathbf{2})} &= [\Delta c, -\Delta c]_{\mathbf{P},\mathbf{R}} \end{aligned} \tag{4.5}$$

As one might expect, it would valuable to relate the quantification of the interval $[p, r]$ to to the quantifications of $[r, o]$ and $[o, p]$ — which share the common event $o \in \mathbf{O}$ as an endpoint. Unfortunately, there is no relation for expressing the interval pair quantification of $[p, r]$ in terms of the interval pair quantifications of $[r, o]$ and $[o, p]$. In particular, the interval-pair is *not* additive in the special case of orthogonality under consideration:

$$[p,r]\Big|_{\mathbf{P},\mathbf{R}(\mathbf{2})} \neq [r,o]\Big|_{\mathbf{R},\mathbf{O}(\mathbf{2})} + [o,p]\Big|_{\mathbf{O},\mathbf{P}(\mathbf{2})} \tag{4.6}$$

However, as Knuth and Bahreyni show, it turns out that one can express the *scalar* quantification of $[p, r]$ in terms of the scalar quantifications of $[r, o]$ and $[o, p]$. In particular, in the case that the chains $\mathbf{O}$, $\mathbf{P}$, and $\mathbf{R}$ are pairwise coordinated, the subspaces $\langle \mathbf{PO} \rangle$ and $\langle \mathbf{RO} \rangle$ are orthogonal in the sense described in Section 4.2, and the three relevant intervals are pure antichain-like, then the interval-scalar is additive. In particular, we have that:

$$\underbrace{\Delta s^2([p,r])}_{\Delta c^2} = \underbrace{\Delta s^2([r,o])}_{\Delta a^2} + \underbrace{\Delta s^2([o,p])}_{\Delta b^2}, \tag{4.7}$$

where the values of the scalar quantifications of each interval are given by the product of the components of the respective interval-pair in Equation 4.5. Hence, we have that:

$$\Delta c^2 = \Delta a^2 + \Delta b^2. \tag{4.8}$$

This result provides a compelling analogue to the Pythagorean theorem in traditional Euclidean geometry that will be essential in the upcoming derivation.

4.5 Subspace Projection

We are at last prepared to shift our attention to the central result of this thesis: a derivation of the method of subspace projection. As we have seen through the course of this work, the majority of results in influence theory center around how we can quantify objects that lie within the subspace induced by a pair of coordinated chains. The purpose of this section is to derive a means of quantifying *any* generalized interval in a poset — including those that do not inhabit an induced subspace — with respect to a pair of coordinated chains.

We begin by defining the concept of distance between an event and a chain. This contrasts with the result of Theorem 2.17, which allowed us to quantify the distance between two coordinated chains. To achieve our goal of quantifying the distance between an event x and chain $\mathbf{P}$ (to which we have assigned the valuation $v_{\mathbf{P}}$), we let $p \in \mathbf{P}$ be some event on the chain. Hence, we may consider the interval $[x,p]$. The interval pair quantification of $[x,p]$ with respect to the chain $\mathbf{P}$ is:

$$\begin{aligned} [x,p]\Big|_{\mathbf{P(2)}} &= \big(v_{\mathbf{P}}(Pp) - v_{\mathbf{P}}(Px), v_{\mathbf{P}}(\bar{P}p) - v_{\mathbf{P}}(\bar{P}x)\big)_{\mathbf{P}} \\ &= \big(v_{\mathbf{P}}(p) - v_{\mathbf{P}}(Px), v_{\mathbf{P}}(p) - v_{\mathbf{P}}(\bar{P}x)\big)_{\mathbf{P}} \\ &= (\Delta p, \Delta\bar{p})_{\mathbf{P}}. \end{aligned} \tag{4.9}$$

The next step is to consider the scalar quantification of this interval. By taking the product of the entries of the interval pair, we find:

$$\begin{aligned} \Delta s^2([x,p]) &= \Delta p \Delta\bar{p} \\ &= \underbrace{\left(\frac{\Delta p + \Delta\bar{p}}{2}\right)^2}_{\text{Length along Chain}} - \underbrace{\left(\frac{\Delta p - \Delta\bar{p}}{2}\right)^2}_{\text{Dist. between Chains}}, \end{aligned} \tag{4.10}$$

where in the final step, we have made use of the symmetric/anti-symmetric decomposition of the scalar quantification first seen in Equation 2.29[1]. There, we saw that the scalar quantification can be seen as a function both of the length of a closed interval along one of the chains and the distance between the two coordinated chains. In particular, we saw in that instance that the second squared quantity $(\Delta p - \Delta\bar{p})/2$ of the decomposition was the distance between the two chains with respect to which we quantified the interval. Although in the present case, we are considering only a single chain, we may still interpret this quantity in parentheses of Equation 4.10 as a distance between the event x and the chain $\mathbf{P}$. Hence, we define the distance $d([x,\mathbf{P}])$ between the event x

[1]Note that in Chapter 2, we were concerned with the scalar quantification of intervals that reside in the subspace induced by a pair of coordinated chains. Nonetheless, we may use the same decomposition in the present case.

and the chain $\mathbf{P}$ as:

$$\begin{aligned} d([x,\mathbf{P}]) &= \frac{\Delta p - \Delta \bar{p}}{2} \\ &= \frac{v_{\mathbf{P}}(p) - v_{\mathbf{P}}(Px) - [v_{\mathbf{P}}(p) - v_{\mathbf{P}}(\bar{P}x)]}{2} \end{aligned} \tag{4.11}$$

We are now ready to begin our derivation of a means of subspace projection. Let $(S, \prec)$ be an ordered set of events, the order of which is induced by influence. Moreover, we let $\mathbf{P}$ and $\mathbf{Q}$ make up a pair of coordinated chains within S, to which we have assigned the valuations $v_{\mathbf{P}}$ and $v_{\mathbf{Q}}$ respectively. Lastly, we let the events $x, y \in S$ define the endpoints the interval $[x, y]$. Our objective is to deduce a consistent means of quantifying this interval $[x, y]$ — which need not reside in the subspace $\langle \mathbf{PQ} \rangle$ — with respect to the coordinated chains $\mathbf{P}$ and $\mathbf{Q}$. We will call this scalar quantification $\Pi_{\langle \mathbf{PQ} \rangle}[x, y]$.

We know that the quantification $\Pi_{\langle \mathbf{PQ} \rangle}[x, y]$ depends only on four quantities: the scalar distances between x and each of the chains $\mathbf{P}$ and $\mathbf{Q}$ — $d([x,\mathbf{P}])^2$ and $d([x,\mathbf{Q}])^2$ — as well as the scalar distances between y and each of the chains $\mathbf{P}$ and $\mathbf{Q}$ — $d([y,\mathbf{P}])^2$ and $d([y,\mathbf{Q}])^2$. Recall that there are arbitrary elements $p \in \mathbf{P}$ and $q \in \mathbf{Q}$ implicit in the definitions of these distances. Hence, we may begin by writing:

$$\Pi_{\langle \mathbf{PQ} \rangle}[x, y] = f[d([x,\mathbf{P}])^2, d([x,\mathbf{Q}])^2, d([y,\mathbf{P}])^2, d([y,\mathbf{Q}])^2]. \tag{4.12}$$

We also know that $\Pi_{\langle \mathbf{PQ} \rangle}[x, y]$ depends linearly on these quantities. This is logical, as non-linear terms would lead to an expression of the distance $\Pi_{\langle \mathbf{PQ} \rangle}[x, y]$ written in terms of quantities that themselves are not distances. Given this piece of information, we may then write:

$$\Pi_{\langle \mathbf{PQ} \rangle}[x, y] = \alpha d([x,\mathbf{P}])^2 + \beta d([x,\mathbf{Q}])^2 + \gamma d([y,\mathbf{P}])^2 + \delta d([y,\mathbf{Q}])^2, \tag{4.13}$$

where α, β, γ, and δ are the unknown scalar constants of the linear combination that we must determine through the consideration of three special cases:

Case #1: We first consider the case in which $x = y$. In this instance, we would expect the coordinated chains $\mathbf{P}$ and $\mathbf{Q}$ to quantify the zero interval $[x, y]$ with zero: $\Pi_{\langle \mathbf{PQ} \rangle}[x, y] = 0$. Since the condition that $x = y$ implies that $d([x,\mathbf{P}]) = d([y,\mathbf{P}])$ and $d([x,\mathbf{Q}]) = d([y,\mathbf{Q}])$, we may rewrite Equation 4.13 as:

$$0 = d([x,\mathbf{P}])^2(\alpha + \gamma) + d([x,\mathbf{Q}])^2(\beta + \delta). \tag{4.14}$$

Since the above equation must hold for all possible values of $d([x,\mathbf{P}])^2$ and $d([x,\mathbf{Q}])^2$, we conclude that:

$$\begin{aligned} \gamma &= -\alpha \\ \delta &= -\beta, \end{aligned} \tag{4.15}$$

which allows the following new expression for Equation 4.13:

$$\Pi_{\langle \mathbf{PQ} \rangle}[x, y] = \alpha \left(d([x,\mathbf{P}])^2 - d([y,\mathbf{P}])^2 \right) + \beta \left(d([x,\mathbf{Q}])^2 - d([y,\mathbf{Q}])^2 \right) \tag{4.16}$$

Case #2: In the second special case, we consider a situation in which $x \in \langle \mathbf{PQ} \rangle$ and $y \in S$ such that the interval $[x, y]$ is orthogonal to the subspace $\langle \mathbf{PQ} \rangle$ in the sense of orthogonality considered in Section 4.2. See Figure 4.5 for a visualization of these conditions. This permits us to use the additive property of the scalar quantification of orthogonal intervals.

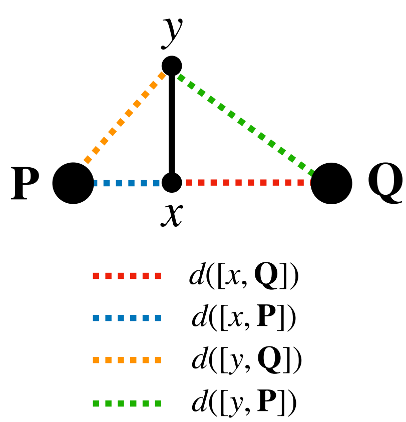


Figure 4.5: A geometric view of the conditions under consideration in Case # 2 of the derivation of the method of subspace projection. Event x resides in the subspace $\langle \mathbf{PQ} \rangle$ while Event y is such that the interval $[x, y]$ is orthogonal to $\langle \mathbf{PQ} \rangle$ in the sense of Section 4.2.

In particular, if we let $\Delta s^2([x, y])$ be the scalar quantification of the interval $[x, y]$, then we may use the orthogonality of the intervals $[p, x]$ and and $[x, y]$ to write (following Equation 4.7):

$$\begin{aligned} \Delta s^2([p, y]) &= \Delta s^2([p, x]) + \Delta s^2([x, y]) \\ d([y, \mathbf{P}])^2 &= d([x, \mathbf{P}])^2 + \Delta s^2([x, y]) \\ \implies \Delta s^2([x, y]) &= d([y, \mathbf{P}])^2 - d([x, \mathbf{P}])^2 \end{aligned} \tag{4.17}$$

Similarly, we may use the orthogonality of the intervals $[q, x]$ and and $[x, y]$ to write (following Equation 4.7):

$$\begin{aligned} \Delta s^2([q,y]) &= \Delta s^2([q,x]) + \Delta s^2([x,y]) \\ d([y,\mathbf{Q}])^2 &= d([x,\mathbf{Q}])^2 + \Delta s^2([x,y]) \\ \implies \Delta s^2([x,y]) &= d([y,\mathbf{Q}])^2 - d([x,\mathbf{Q}])^2 \end{aligned} \tag{4.18}$$

The results of Equations 4.17 and 4.18 imply that:

$$d([y,\mathbf{P}])^2 - d([x,\mathbf{P}])^2 = d([y,\mathbf{Q}])^2 - d([x,\mathbf{Q}])^2. \tag{4.19}$$

Hence, we may substitute the result of Equation 4.19 into Equation 4.16 (recalling that the quantification of $[x,y]$ with respect to the chains $\mathbf{P}$ and $\mathbf{Q}$ should be zero, since $[x,y]$ is orthogonal to $\langle\mathbf{PQ}\rangle$):

$$\begin{aligned} \alpha\left(d([x,\mathbf{P}])^2 - d([y,\mathbf{P}])^2\right) + \beta\left(d([x,\mathbf{Q}])^2 - d([y,\mathbf{Q}])^2\right) &= 0 \\ \alpha\left(d([x,\mathbf{P}])^2 - d([y,\mathbf{P}])^2\right) + \beta\left(d([x,\mathbf{P}])^2 - d([y,\mathbf{P}])^2\right) &= 0 \\ (\alpha+\beta)\left[d([x,\mathbf{P}])^2 - d([y,\mathbf{P}])^2\right] &= 0. \end{aligned} \tag{4.20}$$

Since the above result must hold for all possible configurations of x and y, we conclude that $\beta = -\alpha$. Hence, we may update Equation 4.16 as:

$$\begin{aligned} \Pi_{\langle\mathbf{PQ}\rangle}[x,y] &= \alpha\left[\left(d([x,\mathbf{P}])^2 - d([y,\mathbf{P}])^2\right) - \left(d([x,\mathbf{Q}])^2 - d([y,\mathbf{Q}])^2\right)\right] \\ &= \alpha\left[d([x,\mathbf{P}])^2 - d([x,\mathbf{Q}])^2 + d([y,\mathbf{Q}])^2 - d([y,\mathbf{P}])^2\right] \end{aligned} \tag{4.21}$$

We require one final special case to determine the constant of normalization α.

Case #3: The final special case to consider occurs when we assign one event to each chain. Without loss of generality, assume that $x \in \mathbf{P}$ and $y \in \mathbf{Q}$. In that case, $d([x,\mathbf{P}]) = d([y,\mathbf{Q}]) = 0$ and $d([x,\mathbf{Q}]) = d([y,\mathbf{P}]) = d([\mathbf{P},\mathbf{Q}])$, where $d([\mathbf{P},\mathbf{Q}])$ is the distance between the coordinated chains $\mathbf{P}$ and $\mathbf{Q}$ as given in Theorem 2.17. Moreover, since $x \in \mathbf{P}$ and $y \in \mathbf{Q}$ in this case, we would expect the projection of the interval $[x,y]$ onto the subspace $\langle\mathbf{PQ}\rangle$ to equal $d([\mathbf{P},\mathbf{Q}])$. Hence, Equation 4.21 reduces to:

$$d([\mathbf{P},\mathbf{Q}]) = 2\alpha d([\mathbf{P},\mathbf{Q}])^2, \tag{4.22}$$

from which we conclude that:

$$\alpha = \frac{1}{2d([\mathbf{P},\mathbf{Q}])}. \tag{4.23}$$

Substituting this result into Equation 4.21, we arrive at the equation's final form:

$$\Pi_{\langle\mathbf{PQ}\rangle}[x,y] = \frac{d([x,\mathbf{P}])^2 - d([x,\mathbf{Q}])^2 + d([y,\mathbf{Q}])^2 - d([y,\mathbf{P}])^2}{2d([\mathbf{P},\mathbf{Q}])}. \tag{4.24}$$

As desired, Equation 4.24 yields the consistent quantification of the interval $[x,y]$ with respect to the coordinated chains $\mathbf{P}$ and $\mathbf{Q}$. Equivalently, we may think of Equation 4.24 as the projection of an arbitrary generalized interval $[x,y]$ onto the subspace induced by a pair of coordinated chains $\mathbf{P}$ and $\mathbf{Q}$.

Chapter 5

Conclusion

In this thesis, we have explored the mathematical foundations and physical applications of *Influence Theory*, an approach to understanding the order of the universe solely in terms of events and the interactions of influence between such events.

In Chapter 1, we motivated our project by considering the specific example of Einstein's theory of special relativity. In particular, we performed a careful analysis of the assumptions at the core of the traditional derivation of the Lorentz transformation: (i) the homogeneity of space and time, (ii) the isotropy of space, (iii) the principle of relativity, and (iv) the constancy of the speed of light for all inertial observers. Our derivation, which revealed the importance of the four aforementioned assumptions at various stages, ultimately led us to inquire: is there a more fundamental set of assumptions about nature from which one may derive Einstein's theory? This led us to the alternative approach of Influence Theory, which posits that we view the world simply as an arena in which events take place and influence one another. Influence Theory offers clear advantages over a traditional conception of spacetime: in particular, it removes the necessity of presupposing the existence of space and time and their properties (e.g. isotropy, homogeneity). Convinced of the potential fruits of the perspective that Influence Theory had to offer, we then moved to acquaint ourselves with its mathematical foundations.

In Chapter 2, we conducted a brief survey of the mathematics of partially ordered sets, which lie at the heart of the formalism used by Influence Theory to investigate the natural world. After familiarizing ourselves with partially ordered sets, we next reviewed some of the core results of influence theory already derived by Bahreyni and Knuth in their 2014 paper [5]. We paid close attention to the various ways in which one may consistently quantify events, chains, and intervals within a partially ordered set. We also considered intrinsic geometric properties of partially ordered sets (such as directionality and chain-coordination) that emerged from the initial set of axioms of influence theory. Together, the system of quantifications and geometric concepts led us to a powerful system of scalar quantification of particular intervals of events within a partially ordered set. It was the form of this quantification (Equation 2.29) that first hinted at the emergence of Minkowski spacetime from our primitive assumptions about the nature of events and influence.

In Chapter 3, we investigated in greater detail the scalar quantification of a given interval of events offered by Equation 2.29, following closely the work of Bahreyni in [5]. In particular, we reflected on the possibility of transforming interval pair quantifications between various chains, before performing a helpful change of variables to uncover the familiar forms of the Minkowski metric of flat spacetime and Lorentz transformation. These startling results demonstrated that these foundational results of modern physical theory emerged out of the formalism of influence theory — which we originally developed simply as a means to quantify various objects within a partially ordered set of events. Importantly, these results were attained without any prior assumptions about the nature of space or time, an achievement that contrasts sharply with the traditional approach to special relativity considered in Chapter 1. Another remarkable and familiar result that emerged from the framework of influence theory was the maximal property of the speed of light.

Chapter 4 represented the principal contribution of this thesis to the influence theory formalism. In prior chapters, we had seen the importance of coordinated chains to the success of our attempts to quantify the various objects of a partially ordered set. Our primary objective in Chapter 4 was the development of a method of *subspace projection*. Such a result would allow for the quantification of *any* generalized interval with respect to a pair of coordinated chains — including those generalized intervals that do not reside in the subspace induced by the relevant coordinated chains. To achieve this goal, we reviewed and built on some of the geometric concepts introduced in earlier chapters such as directionality. Of particular importance was our development of a special case of orthogonality of two pairs of coordinated chains and the multiplicative property of the scalar interval under those conditions. Once equipped with these geometric tools, we were ready to tackle the problem of subspace projection. We proceeded by deducing a general form for the desired function and then determined the precise form of the function by considering three special cases. This derivation culminated in the result of Equation 4.24 which gives an expression for the projection of an arbitrary generalized interval onto the subspace induced by a pair of coordinated chains.

It is hoped this thesis has served its desired role as (i) an introduction to the fascinating approach to physics provided by influence theory and (ii) a resource for future work in this area of research. Not only have we seen an application of the theory to special relativity, one of the pillars of modern physics; we have also explored the boundaries of the theory's current domain of knowledge and derived a method of subspace projection that might serve to further develop the formalism at a later date. Of particular interest would be a method of quantifying intervals of events that mirrors the traditional cross product, in the same way that the method of subspace projection derived in these pages mirrors the traditional dot product.

Bibliography

[1]R. Rynasiewicz, "Newton's Views on Space, Time, and Motion", in *The Stanford Encyclopedia of Philosophy*, edited by E. N. Zalta, Summer 2014 (Metaphysics Research Lab, Stanford University, 2014).

[2]A. Einstein, "Zur Elektrodynamik bewegter Körper", en, Annalen der Physik **322**, 891–921 (1905).

[3]L. Boi, "Theories of Space-Time in Modern Physics", en, Synthese **139**, 429–489 (2004).

[4]H. S. Yang, "Emergent Spacetime: Reality or Illusion?", arXiv:1504.00464 [gr-qc, physics:hep-ph, physics:hep-th], arXiv: 1504.00464 (2015).

[5]K. H. Knuth and N. Bahreyni, "A potential foundation for emergent spacetime", English, Journal of Mathematical Physics **55**, WOS:000345643100018, 112501 (2014).

[6]M. Van Raamsdonk, "Building up spacetime with quantum entanglement", en, General Relativity and Gravitation **42**, 2323–2329 (2010).

[7]M. Atiyah, R. Dijkgraaf, and N. Hitchin, "Geometry and physics", Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences **368**, 913–926 (2010).

[8]P. Szekeres, *A Course in Modern Mathematical Physics: Groups, Hilbert Space and Differential Geometry* (Cambridge University Press, 2004).

[9]T. Moore, *A General Relativity Workbook* (University Science Books, 2012).

[10]G. Galilei and S. Drake, *Dialogue concerning the two chief world systems, Ptolemaic & Copernican*, English (University of California Press, Berkeley, 1953).

[11]G. Kar, S. Kunkri, and S. K. Choudhary, "Special relativity, causality and quantum mechanics — 1", en, Resonance **11**, 41–52 (2006).

[12]H. Poincaré, "La dynamique de l'électron", Revue générale des sciences pures et appliquées **19**, 386–402 (1908).

[13]D. Hilbert, "Mathematical problems", en, Bulletin of the American Mathematical Society **8**, 437–479 (1902).

[14] L. Corry, “The Origin of Hilbert’s Axiomatic Method1”, en, in *The Genesis of General Relativity*, edited by M. Janssen, J. D. Norton, J. Renn, T. Sauer, and J. Stachel, Boston Studies in the Philosophy of Science (Springer Netherlands, Dordrecht, 2007), pp. 1680–1777.

[15] S. S. Stepanov, “On simplied axiomatic foundations of special relativity”, en, 9.

[16] P. Frank and H. Rothe, “Über die Transformation der Raumzeitkoordinaten von ruhenden auf bewegte Systeme”, en, Annalen der Physik **339**, 825–855 (1911).

[17] W. Pauli, *Theory of Relativity*, English (Pergamon Press, B.I. Publications, New York, Bombay, 1958).

[18] Y. Ueno, “On the Equivalency for Observers in the Special Theory of Relativity”, en, Progress of Theoretical Physics **9**, 74–84 (1953).

[19] P. Suppes, “Axioms for Relativistic Kinematics with or without Parity”, en, in *Studies in Logic and the Foundations of Mathematics*, Vol. 27, edited by L. Henkin, P. Suppes, and A. Tarski, The Axiomatic Method (Elsevier, Jan. 1959), pp. 291–307.

[20] H. Andréka, J. X. Madarász, and I. Németi, “Logic of Space-Time and Relativity Theory”, en, in *Handbook of Spatial Logics*, edited by M. Aiello, I. Pratt-Hartmann, and J. Van Benthem (Springer Netherlands, Dordrecht, 2007), pp. 607–711.

[21] K. Stiegler, “The axiomatic foundations of the theory of special relativity”, en, International Journal of Theoretical Physics **5**, 403–419 (1972).

[22] A. A. Michelson and E. W. Morley, “On the relative motion of the Earth and the luminiferous ether”, en, American Journal of Science **Series 3 Vol. 34**, 333–345 (1887).

[23] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed., Published: Paperback (Butterworth-Heinemann, Jan. 1980).

[24] B. Schutz, *A First Course in General Relativity*, 2nd ed. (Cambridge University Press, 2009).

[25] F. Cooperstock and S. Tieu, *Einstein’s Relativity: The Ultimate Key to the Cosmos*, SpringerLink : Bücher (Springer Berlin Heidelberg, 2012).

[26] M. Srednicki, *Quantum Field Theory* (Cambridge Univ. Press, Cambridge, 2007).

[27] K. H. Knuth and J. L. Walsh, “An Introduction to Influence Theory: Kinematics and Dynamics”, en, Annalen der Physik **531**, 1800091 (2019).

[28] E. C. Zeeman, “Causality Implies the Lorentz Group”, Journal of Mathematical Physics **5**, 490–493 (1964).

[29] L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, “Space-time as a causal set”, Physical Review Letters **59**, 521–524 (1987).

[30]F. Dowker, “Causal sets as discrete spacetime”, Contemporary Physics **47**, 1–9 (2006).

[31]B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order* (Cambridge University Press, Cambridge, UK, 2010).

[32]S. Shahriari, “The Lattice of Subgroups”, in *Algebra in Action: a Course in Groups, Rings, and Fields*, Vol. 27, Pure and Applied Undergraduate Texts (American Mathematical Society, Providence, Rhode Island, 2017), pp. 167–185.