KONERU LAKSHMAIAH EDUCATION FOUNDATION
(Deemed to be University, Estd. u/s. 3 of UGC Act 1956)

Department of Mathematics

DISCRETE MATHEMATICS

I/IV- B. Tech-(ODD Semester), Academic Year: 2025-2026

(Modeling and solving linear recurrence relations with constant coefficients)

Instructional Objective:

1. To Modal and solve **Linear recurrence relations with constant coefficients**

Learning Outcomes:

1. Able to Modal and solve **Linear recurrence relations with constant coefficients**

Introduction:

Recurrence Relations:

Many counting problems cannot be solved easily. To avoid the complexity of such problems recurrence relations play an important role.

A recursive definition of a sequence specifies one or more initial terms and a rule for determining subsequent terms from those that precede them. Also, recall that a rule of the latter sort (whether or not it is part of a recursive definition) is called a **recurrence relation** and that a sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Example1:

Determine f (1), f (2), f (3), and f (4) if f (n) is defined recursively by f (0) = 1 and for n = 0, 1, 2, . . .

f (n + 1) = f (n) + 2.

Solution:

f (0) = 1

Given that f (n + 1) = f (n) + 2, n = 0, 1, 2, . . .

Take n=0, f(1) = f(0) + 2 = 1 + 2 = 3

Take n=1, f(2) = f(1) + 2 = 3 + 2 = 5

Take n=2, f(3) = f(2) + 2 = 5 + 2 = 7.

Linear Recurrence Relation

A linear recurrence relation of degree k with constant coefficients is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

The recurrence relation in the definition is **linear** because the right-hand side is that in which a_{n-1}, a_{n-2} —- occurs in the first degree only and are not multiplied together.

Linear Homogeneous Recurrence Relation

If F(n)=0, then that recurrence relation is called **homogeneous**, otherwise is called **non-homogeneous**.

The **degree** is k because a_n is expressed in terms of the previous k terms of the sequence.

Examples

- The recurrence relation $P_n = 5P_{(n-1)}$ is a linear homogeneous recurrence relation of degree one.
- The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two.
- The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear.
- The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous.

Characteristic equation & Characteristic roots

The equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} --------c_k = 0$, is called **characteristic equation** of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{-2} + ------c_k a_{n-k}$ .The solutions of this equation are called the **characteristic roots** of the recurrence relation.

Example: $r^2 - 5r + 6 = 0$ is characteristic equation of recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ and characteristic roots are, $r_1 = 2$ and $r_2 = 3$.

Linear homogeneous Recurrence Relations with Constant Coefficients

Solution of recurrence relation corresponding to roots

Roots Of Ch. Equation	**Solution**
1. r_1, r_2(real and distinct roots)	**$c_1(r_1)^n + c_2(r_2)^n$**
2. r_1, r_1(2 real and equal roots)	**$[c_1 + c_2 n](r_1)^n$**
3. r_1, r_1, r_1, r_2 ----(3 real and equal roots)	**$[c_1 + c_2 n + c_3 n^2](r_1)^n + c_4(r_2)^n$+-----**

Example1: What is the solution of the recurrence relation $f_n = f_{n-1} + f_{n-2}$

Sol: The characteristic equation of the recurrence relation is $r^2 - r - 1 = 0$

Its roots are **$r_1 = [1+(5)^{(1/2)}]/2$ and $r_2 = [1-(5)^{(1/2)}]/2$**

Hence, the sequence $\{f_n\}$ is a solution to the recurrence relation if and only if

$f_n = \alpha_1 ([1+(5)^{(1/2)}]/2)^n + \alpha_{2(} [1-(5)^{(1/2)}]/2)^n$

Example 2: What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution: The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$.

Its roots are r = 2 and r = −1.

Hence, the sequence {an} is a solution to the recurrence relation if and only if $a_n = \alpha_1 2^n + \alpha_2(-1)^n$

From the initial conditions, it follows that $a_0 = 2 = \alpha_1 + \alpha_2$,

$a_1 = 7 = \alpha_1 \cdot 2^n + \alpha_2 \cdot (-1)^{n.}$

Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with $a_n = 3 \cdot 2^n - (-1)^n$

Non homogeneous Linear Recurrence Relations with Constant Coefficients

A recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{-2} + - - - - - c_k a_{n-k} + F(n)$ where $c_1, c_2, \ldots, c_k$ are real numbers and ***F(n)* is a function not identically zero depending only on *n***, is called linear non homogeneous recurrence relation with constant coefficients.

Note: Here, the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{-2} + - - - - - c_k a_{n-k}$ is called the associated homogeneous recurrence.

Note:
Every solution of non homogeneous linear recurrence relation with constant coefficients is of the form $\{\mathbf{a}_n^p + \mathbf{a}_n^h\}$, where $\{ a_n^h \}$ is a solution of the associated homogeneous recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{-2} + - - - - - c_k a_{n-k}$ and $\{a_n^p\}$ is a particular solution of the non homogeneous linear recurrence relation with constant coefficients.

Example:

Find all solutions of the recurrence relation. $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

Solution: This is a linear non homogeneous recurrence relation.

The solutions of its associated homogeneous recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ are

$a_n^h = \alpha_1.3^n + \alpha_2 2^n$ where α_1 and α_2 are constants

Because $F(n) = 7^n$, a reasonable trial solution is $a_n^p = C.7^n$ where C is a constant.

Substituting the terms of this sequence into the recurrence relation implies that

$C.7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$

Factoring out 7^{n-2}, this equation becomes $49C = 35C - 6C + 49$,
which implies that $20C = 49$, or that $C = 49/20$.

Hence, $a_n^p = \left[\frac{49}{20}\right].7^n$ **is a particular solution.**

Then, all solutions are of the form $a_n = \alpha_1.3^n + \alpha_2.2^n + 7^n(\frac{49}{20})$

Summary: In this session, it was discussed aboutRecurrence relations, solving Linear recurrence relations with constant coefficients.

Self-assessment questions:

1. Consider the recurrence relation $a_1=4$, $a_n=5n+a_{n-1}$. The value of a_{64} is __________

a) 10399
b) 23760
c) 75100
d) 53700
Answer: a

2. Determine the solution of the recurrence relation $F_n=20F_{n-1} - 25F_{n-2}$ where $F_0=4$ and $F_1=14$.

a) $a_n = 14*5^{n-1}$
b) $a_n = 7/2*2^n-1/2*6^n$
c) $a_n = 7/2*2^n-3/4*6^{n+1}$
d) $a_n = 3*2^n-1/2*3^n$
Answer: b

3. What is the recurrence relation for 1, 7, 31, 127, 499?

a) $b_{n+1}=5b_{n-1}+3$
b) $b_n=4b_n+7!$
c) $b_n=4b_{n-1}+3$
d) $b_n=b_{n-1}+1$
Answer: c

4.If $S_n=4S_{n-1}+12n$, where $S_0=6$ and $S_1=7$, find the solution for the recurrence relation.

a) $a_n=7(2^n)-29/6n6^n$
b) $a_n=6(6^n)+6/7n6^n$
c) $a_n=6(3^{n+1})-5n$
d) $a_n=nn-2/6n6^n$
Answer: b

5. Find the value of a_4 for the recurrence relation $a_n=2a_{n-1}+3$, with $a_0=6$.
a) 320
b) 221
c) 141
d) 65
Answer: c
6. The solution to the recurrence relation $a_n=a_{n-1}+2n$, with initial term $a_0=2$ are _________
a) 4n+7
b) 2(1+n)
c) $3n^2$
d) 5*(n+1)/2
Answer: b
7. Determine the solution for the recurrence relation $b_n=8b_{n-1}-12b_{n-2}$ with $b_0=3$ and $b_1=4$.
a) $7/2*2^n-1/2*6^n$
b) $2/3*7^n-5*4^n$
c) $4!*6^n$
d) $2/8^n$
Answer: a
8. What is the solution to the recurrence relation $a_n=5a_{n-1}+6a_{n-2}$?
a) $2n^2$
b) 6n
c) (3/2)n
d) n!*3

Answer: b
9. Determine the value of a2 for the recurrence relation $a_n = 17a_{n-1} + 30n$ with $a_0=3$.
a) 4387
b) 5484
c) 238
d) 1437
Answer: d
10. Determine the solution for the recurrence relation $a_n = 6a_{n-1}-8a_{n-2}$ provided initial conditions $a_0=3$ and $a_1=5$.
a) $a_n = 4 * 2^n - 3^n$
b) $a_n = 3 * 7^n - 5*3^n$
c) $a_n = 5 * 7^n$
d) $a_n = 3! * 5^n$
Answer: b

CLASS ROOM DELIVARY PROBLEMS

Session:

1. Describe the tower of Hanoi problem and model it as recurrence relations.
2. A young pair of rabbits (one of each gender) is placed on an island .A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Model it as a recurrence relation (assuming that rabbits never die).
3. You open a **recurring deposit account** in a bank with an **initial balance of ₹0**. You deposit **₹5,000 every month**, and the bank offers an interest rate of **0.6% per month** (compounded monthly). Write the difference equation and classify.

Session:

Solve the Homogeneous recurrence relation using characteristic roots

1) Solve the Fibonacci series using recurrence relation
2) Solve the recurrence relation $a_n = 4a_{n-1} + 21a_{n-2}$ for n≥2, with a_0=1 *and* a_1=-1
3) Solve the recurrence relation $a_n = 11a_{n-1} - 30a_{n-2}$ for n≥2, with a_0=0 *and* a_1=-1
4) Solve the recurrence relation $a_n = -2a_{n-1} + 15a_{n-2}$ for n ≥ 2 with a_0=1 and a_1= -1.
5) Solve the recurrence relation $a_n = a_{n-1} + 4a_{n-2} - 4a_{n-3}$ for $n \geq 3$ with a_0=0, a_1=1and a_2= -1

Session

Solve the Non Homogeneous recurrence relation using characteristic roots

1) Solve the recurrence relation $a_n = 2a_{n-1} + 2^n$, for $n \geq 1$ with $a_0 = 1$
2) Solve the recurrence relation $a_n - 2a_{n-1} + a_{n-2} = 3^n$ for n≥2, with a_0=2 *and* a_1=1.
3) Solve the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2} + 6$ for n≥2, with a_0=1 *and* a_1=2
4) Solve the recurrence relation $a_n = 4a_{n-1} + 12a_{n-2} + 4.5^n + 6$ for $n \geq 2$ with $a_0 = 0, a_1 = 1$
5) Solve the recurrence relation $a_n = -3a_{n-1} + 10a_{n-2} + 5.2^n$ with $a_0 = 0, a_1 = 1$
6) Solve the recurrence relation $a_n = 3a_{n-1} + 10a_{n-2} + 7.5^n$ with $a_0 = -1, a_1 = 1$
7) Solve Tower of Hanoi Problem.
8) Suppose you open a savings account with an initial deposit of ₹10,000. Each month, you deposit an additional ₹2,000. The bank offers an interest rate of **1% per month**, compounded monthly. Write down and classify the difference equation and calculate the amount the account after 5 months.

TUTORIAL PROBLEMS

1) A bank pays 6% (annual) interest on savings, compounding the interest monthly. If Bonnie deposits$1000 on the first day of May, how much will this deposit be worth a year later?
2) Obtain the solution to the recurrence relation
   $a_n = 7a_{n-2} - 6a_{n-3}$ for $n \geq 3$ with $a_0 = -1, a_1 = 0, a_2 = 3$
3) Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are -2, -2, -2, 7, 7, and 8 (that is, there are three roots, the root -2 with multiplicity three, the root 7 with multiplicity two, and the root 8 with multiplicity one). What is the form of the general solution?
4) Find all solutions of the recurrence relation
   $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$ for $n \geq 2$ with $a_0 = -1, a_1 = 0,$
5) Find all solutions of the recurrence relation
   $a_n = 3a_{n-1} + 10a_{n-2} + 5 + 8^n$ for $n \geq 2$ with $a_0 = -1, a_1 = 1,$
6)

HOME ASSIGNMENT PROBLEMS

1) Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 1, -2, -2, 4, 4,7, and 8 , What is the form of the general solution?
2) A plant is such that each of its seeds when one year old produces 8-fold and produces 18-fold when two year old or more. A seed is planted and as soon as a new seed is produced it is planted. Taking y_n to be the number of seeds produced at the end of the nth year, then model it into a recurrence relation in y_n and solve it.
3) A small town has a **population of 10,000** in the year 2020. The population increases by **2% per year**, and **300 people** move into the town each year due to migration. Form Difference equation and find population after 5 years.

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Department of Mathematics

DISCRETE STRUCTURES, 25MT1002

I/IV-B.Tech-(I st Sem), Academic Year: 2025-2026

Session-26 (GENERATING FUNCTIONS)

Instructional Objective:

1. To understand Generating function.
2. To determine generating function for sequence of real nos.

Learning Outcomes:

1. Able to determine the generating function for given sequence of real nos.
2. Able to determine the terms of the generating function..

Introduction:

Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series. Generating functions can be used to solve many types of counting problems, such as the number of ways to select or distribute objects of different kinds, subject to a variety of constraints, and the number of ways to make change for a dollar using coins of different denominations.

Explanation:

Generating functions can be used to solve recurrence relations by translating a recurrence relation for the terms of a sequence into an equation involving a generating function. This equation can then be solved to find a closed form for the generating function. Generating functions can be used to solve a wide variety of counting problems. In particular, they can be used to count the number of combinations of various types. Problems are equivalent to counting the solutions to equations of the form $e_1 + e_2 + \cdots + e_n = C$.

Remark

$(x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$

In general

$(x^n - 1)/(x - 1) = 1 + x + x^2 + \ldots\ldots + x^{n-1}$

Definition:

The generating function for the sequence $a_0, a_1 \ldots a_k, \ldots$ of real numbers is the infinite

Series, $G(x) = a_0 + a_1x + \cdots + a_kx^k + \cdots = \sum_{K=0}^{\infty} a_k x^k$

Example1

The generating functions for the sequences $\{a_k\}$ with $a_k = 3$, $a_k = k + 1$, and $a_k = 2k$

are $\sum_{K=0}^{\infty} 3x^k$, $\sum_{K=0}^{\infty} (k+1)x^k$, $\sum_{K=0}^{\infty} 2^k x^k$

Definition: Generating functions for finite sequences of real numbers by extending a finite sequence $a_0, a_1 \ldots a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on. The generating function G(x) of this infinite sequence $\{a_n\}$ is a polynomial of degree n , because no terms of the form $a_j\, x_j$ with $j > n$ occur, that is, $G(x) = a_0 + a_1x + \cdots + a_nx^n$.

Example2: What is the generating function for the sequence 1, 1, 1, 1, 1, 1?

Solution: The generating function of 1, 1, 1, 1, 1, 1 is $1 + x + x^2 + x^3 + x^4 + x^5$.

The function $f(x) = 1/(1 - x)$ is the generating function of the sequence $1, 1, 1, 1, \ldots,$ because $1/(1 - x) = 1 + x + x2 + \cdots$ for $|x| < 1$.

Example 3 : Find the number of solutions of $e_1 + e_2 + e_3 = 17$, where e1, e2, and e3 are nonnegative integers with $2 \le e1 \le 5$, $3 \le e2 \le 6$, and $4 \le e3 \le 7$.

Solution: The number of solutions with the indicated constraints is the coefficient of $x17$ in the expansion of $(x2 + x3 + x4 + x5)(x3 + x4 + x5 + x6)(x4 + x5 + x6 + x7)$. This follows because we obtain a term equal to x^{17} in the product by picking a term in the first sum x^{e1} , a term in the second sum x^{e_2} , and a term in the third sum x^{e3} , where the exponents e_1, e_2, and e_3 satisfy the equation $e1 + e2 + e3 = 17$ and the given constraints

Review questions:

1. The function $f(x) = 1/(1 - ax)$is the generating function for which sequence?
2. The generating function of the sequence 1, -1,1,-1,…

Summary:

In this session, we derived the terms of the generating function and evaluate of generating function for given sequence of real nos.

Self-assessment questions:

1. What is the sequence depicted by the generating series $4 + 15x^2 + 10x^3 + 25x^5 + 16x^6 + \cdots$?
   a) 10, 4, 0, 16, 25…
   b) 0, 4, 15, 10, 16, 25… [C]

c) 4, 0, 15, 10, 25, 16…
d) 4, 10, 15, 25…

2. What is the generating function for the sequence 1, 6, 16, 216…?
a) (1+6x)/x3
b) 1/(1−6x)
c) 1/(1−4x) [B]
d) $1-6x^2$

3. What is the generating function for generating series 1, 2, 3, 4, 5…?
a) 2/(1−3x)
b) 1/(1+x) [C]
c) $1/(1-x)^2$
d) $1/(1-x^2)$

4. What is the generating function for the generating sequence A = 1, 9, 25, 49…?
a) $1+(A-x^2)$
b) (1-A)-1/x
c) $(1-A)+1/x^2$ [B]
d) $(A-x)/x^3$

5. What will be the sequence generated by the generating function $4x/(1-x)^2$?
a) 12, 16, 20, 24,
b) 1, 3, 5, 7, 9,…
c) 0, 4, 8, 12, 16, 20,… [C]
d) 0, 1, 1, 3, 5, 8, 13,

Terminal questions

Class room problems

1. The generating functions for the sequences $\{a_k\}$ with $a_k = 3$, $a_k = k + 1$, and $a_k = 2k$ are $\sum_{K=0}^{\infty} 3x^k$, $\sum_{K=0}^{\infty} (k+1)x^k$, $\sum_{K=0}^{\infty} 2^k x^k$

2. Determine the generating function for the sequence 1, 1, 1, 1, 1, 1?
3. Find the generating function for the sequence 1, 2, 3, 4…
4. Obtain the generating function for the sequence 0, 1, -2, 3, -4…
5. Determine the coefficient of x^{27} in $x^4+x^5+x^6+$….

Tutorial problems

6. Determine Coefficient of x^k in the expansion of $(1+x)^n$

7. Determine the generating function for the sequence 1,1, 0, 1,1,1…

8. Find the generating function for the finite sequence 1, 4,16, 64, 256. Given sequence

1, 4, 16, 64, 256

9. For generating function, $(3x-4)^3$ provide a closed formula for the sequence it determines.
10. Determine the coefficient of x^0 in $(3x^2-2/x)^{15}$

Home assignment problems

10. Obtain the generating function for the sequence $1^2, 2^2, 3^2, 4^2$ …..
11. Determine the coefficient of x^{12} in $x^3(1-2x)^{10}$
12. Find a closed form for the generating function for each of these sequences. (For each sequence, use the most obvious choice of a sequence that follows the pattern of the initial terms listed. 0, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0
13. Find the generating function for the finite sequence 2, 2, 2, 2, 2, 2.
14. Obtain the generating function for the sequence 1, -1, 0, -1, 1, -1….
15. Obtain the generating function for the sequence $0^2, 1^2, 2^2, 3^2, 4^2$…..

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k$ $= 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k$ $= 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk}$ $= 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \leq n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$ $= 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$ $= 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k$ $= 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$

Introduction

There is an extremely powerful tool in discrete mathematics used to manipulate sequences called the generating function. A generating function is just a different way of writing a sequence of numbers.

The idea is this: instead of an infinite sequence (for example: 2,3,5,8,12 we look at a single function which encodes the sequence. But not a function which gives the nth term as output. Instead, a function whose power series (like from calculus) "displays" the terms of the sequence. So for example, we would look at the power series $2+3x+5x^2+8x^3+12x^4+\cdots$ which displays the sequence 2,3,5,8,12,…2,3,5,8,12,… as coefficients.

Session Description

Generating functions provide a powerful method for solving recurrence relations by transforming them into algebraic equations. The core idea is to represent the sequence defined by the recurrence relation as a power series, manipulate it algebraically, and then extract the solution by finding a closed-form expression for the generating function.

1. Define the Generating Function:

For a sequence $a_0, a_1, a_2, \ldots$, its generating function is defined as:

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n$$

2. Express the Recurrence Relation in Terms of the Generating Function:

Use the given recurrence relation to express A(x) in terms of itself, often involving multiplications by powers of x and additions/subtractions.

3. Solve for the Generating Function:

   Rearrange the equation obtained in the previous step to isolate A(x). This will result in an expression for A(x) as a function of x

4. Find the Closed-Form Solution:

   Expand the expression for A(x) (often using techniques like partial fractions or the binomial theorem) to obtain a power series representation.

   Identify the coefficient of x^n in the expanded form. This coefficient represents a_n, the solution to the recurrence relation.

   Example:

   Let's consider the recurrence relation: $a_n = 2a_{n-1} + 1$ with $a_0 = 1$.

   1. Generating function $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$
   2. Multiply the recurrence by x^n and sum from n=1 to infinity

   $$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (2a_{n-1} + 1) x^n$$

   Simplifying: $A(x) - 1 = 2xA(x) + \frac{x}{1-x}$

   3. Solve for the Generating Function:

   Thus: $A(x) = \frac{1}{(1-2x)(1-x)}$

   4. Closed-Form Solution:

Using partial fractions

$$A(x) = \frac{2}{1-2x} - \frac{1}{1-x}$$

Expanding using the geometric series formula:

$$A(x) = 2\sum_{n=0}^{\infty}(2x)^n - \sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}(2^{n+1}-1)x^n$$

- Therefore, $a_n = 2^{n+1} - 1$

Classroom delivery problems

1. Apply generating functions technic to solve $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 2$ with initial condition $a_0 = 3$ *and* $a_1 = 1$

Solution

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Consider the recurrence relation Type equation here.

$a_n = 2a_{n-1} + 3a_{n-2}$

Multiply with x^n on both sides and taking summation from n=2 to n=∞

$$\sum_{n=2}^{\infty} a_n x^n = 2\sum_{n=2}^{\infty} a_{n-1}x^n + 3\sum_{n=2}^{\infty} a_{n-2}x^n$$

$\rightarrow \sum_{n=2}^{\infty} a_n x^n = 2x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + 3\,x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2}$

$\rightarrow G(x) - a_0 - a_1x = 2x\,G(x) + 3\,x^2\,G(x)$

$\rightarrow G(x) - 3 - x = 2x\,G(x) + 3\,x^2\,G(x)$

$(3x^2 + 2x - 1)\,G(x) = -3 - x$

$\rightarrow \big(3x(x+1) - (x+1)\big)G(x) = -3-x$

$\rightarrow \big((3x-1)(x+1)\big)G(x) = -3 - x$

$\rightarrow G(x) = \frac{-3-x}{(3x-1)(x+1)} = \frac{10}{3}\frac{1}{1-3x} + \frac{1}{2}\frac{1}{x+1}$

$$\rightarrow G(x) = \frac{5}{2}\sum_{n=0}^{\infty} 3^n x^n - 2\sum_{n=0}^{\infty}(-1)^n x^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left(\frac{5}{2}3^n + \frac{1}{2}(-1)^n\right)x^n$$

Comparing coefficients of x^n on both sides

$a_n = \frac{5}{2}3^{n-1} + \frac{1}{2}(-1)^n$ Type equation here.is the solution

2. Solve the recurrence relations using generating function-: $Fn = Fn - 1 + Fn - 2$

$n \geq 2, \quad F1 = 1, \quad F0 = 0$

3. Solve $a_n = 3a_{n-1} + 2$ n≥1 with $a_0 = 1$ using generating functions

Solution

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Multiply with x^n on both sides and taking summation from n=1 to n=∞

$$\sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1}x^n + 2\sum_{n=1}^{\infty} x^n$$

$$\rightarrow G(x) - a_0 = 3x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + 2\sum_{n=1}^{\infty} x^n$$

$$\rightarrow G(x) - 1 = 3xG(x) + 2\frac{1}{1-x}$$

$$\rightarrow (1-3x)G(x) = 1 + \frac{2}{1-x}$$

$$\rightarrow \ G(x) = \frac{1}{1-3x} + \frac{3}{(1-3x)} - \frac{1}{(1-x)}$$

$$\rightarrow \ G(x) = \frac{4}{1-3x} - \frac{1}{1-x}$$

$$\rightarrow \sum_{n=1}^{\infty} a_n x^n = 4\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n$$

$$\rightarrow \sum_{n=1}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (43^n - 1)x^n$$

Therefore $a_n = 43^n - 1$

4. Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 4^n \,, n \geq 2 \; with \;\; a_0 = 2 \,, a_1 = 8$

Tutorial problems

5. Use generating functions to solve the recurrence relation $a_n = 7a_{n-1}$, with $a_0 = 5, for$ n = 1, 2, 3----

6. Use the method of generating function to solve the recurrence relation $a_n - 2a_{n-1} - 3a_{n-2} = 0 \,, n \geq 2 \; with \;\; a_0 = 3 \,, a_1 = 1$

7. Use generating functions to solve the recurrence relation
$a_n = a_{n-1} + 3^{n-1}$ for $n = 1, 2, 3, 4$----with initial condition $a_0 = 1$

8. Use generating functions to solve the recurrence relation
$a_n = 3a_{n-1} - 1$ for $n = 1, ,2, 3, 4$----with initial condition $a_0 = 2$

Home assignment problems

9. Use generating functions to solve the recurrence relation $a_n = 5a_{n-1}$-6 a_{n-2} for $n = 2, 3, 4$----with initial condition $a_0 = 0$ *and* $a_1 = 3$
10. Solve the recurrence relation $a_n = 58 - 16a_{n-2}$ for $n \geq 2$, with $a_0 = 16$ *and* $a_1 = 80$ Using generating functions
11. Suppose that a valid code word is an n-digit number in decimal notation containing an even number of 0s. Let an denote the number of valid code words of length n. Then, the corresponding recurrence relation is $a_n = 8a_{n-1} + 10^{n-1}$ for $n = 1, 2, 3, 4$----with initial condition $a_0 = 1$. Use generating functions to find an explicit formula for a_n
12. Solve the recurrence relation $a_n = a_{n-1} + 2(n-1),\ n \geq 1$ *with* $a_0 = 3$

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Department of Mathematics

Discrete Structures

25MT1002

I/IV-B.Tech-(I st Sem), Academic Year: 2025-2026

Session-25: (Introduction of Graphs, Graph terminology and Representation of Graphs in Matrices)

Instructional Objective:

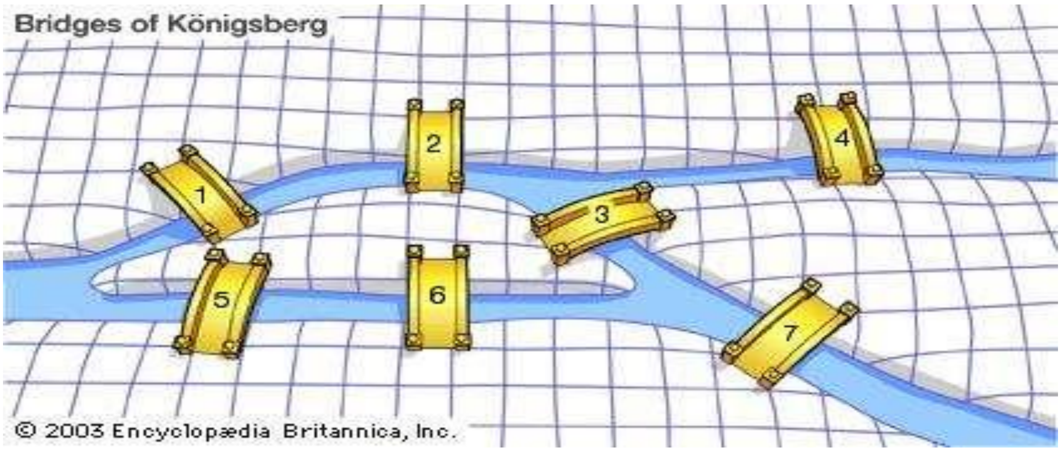
1. To represent the physical phenomena as a graph
2. To acquire the basic concepts of graphs
3. To understand the types of graphs

Learning Outcomes:

1. Able to draw physical phenomena into a graph
2. Able to identify different types of graphs
3. Able to distinguish about the special types of graphs.

Introduction:

- Graph Theory is one of the most important tools in mathematics, starting from the problem of the Konigsberg bridge which is written by Leonhard Euler in 1736.



- Graph theory is becoming increasingly significant as it is applied to other areas of mathematics, science, and technology.
- Graph theory is rapidly moving into the mainstream of mathematics mainly because of its applications in diverse fields which include computer science (data mining, clustering, image capturing, networking), operations research (scheduling, traveling salesman problem), electrical engineering (communications networks and coding theory) and biochemistry.
- The important concept of graph coloring is utilized in resource allocation and scheduling. Paths walks, and circuits in graph theory are used in applications of the traveling salesman problem.
- Also, paths, walks and circuits in graph theory are used in tremendous applications say traveling salesman problem, database design concepts, resource networking.

- **Phenomena represented as Graphs:**

Now suppose that a network is made up of data centers and communication links between computers. We can represent the location of each data center by a point and each communications link by a line segment, as shown in Figure 1.

This computer network can be modeled using a graph in which the vertices of the graph represent the data centers and the edges represent communication links. In general, we visualize

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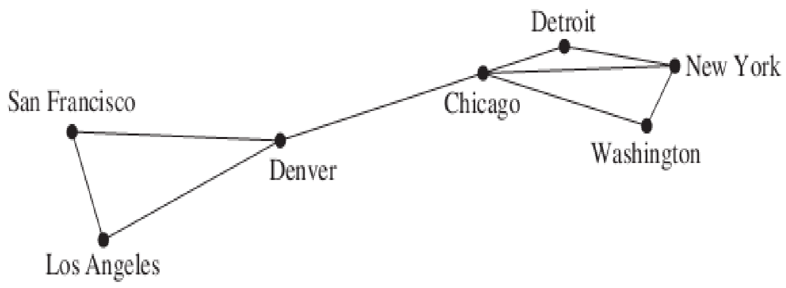
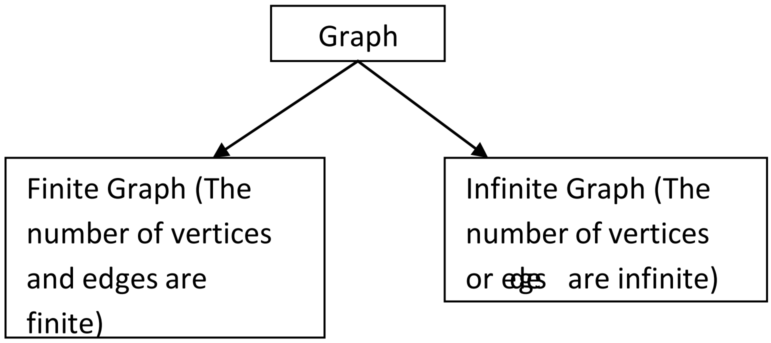


FIGURE 1 **A Computer Network.**

graphs by using points to represent vertices and line segments, possibly curved, to represent edges, where the endpoints of a line segment representing an edge are the points representing the endpoints of the edge. When we draw a graph, we generally try to draw edges so that they do not cross. However, this is not necessary because any depiction using points to represent vertices and any form of connection between vertices can be used. Indeed, there are some graphs that cannot be drawn in the plane without edges crossing (see Section 10.7). The key point is that the way we draw a graph is arbitrary, as long as the correct connections between vertices are depicted.

Definition of a Graph:

A graph G consists of a set V of vertices and a collection of edges (unordered pair of vertices)and is symbolically represented as G (V, E).



- The number of vertices is called the order
- The number of edges is called the degree

- An edge of a graph that joins a vertex to itself is called Loop.
- Two or more edges that join the same pair of distinct vertices are called multiple edges.
- Any two vertices connected by an edge are called adjacent vertices otherwise they are calledisolated vertices.
- The edge 'e' that joins the vertices u and v is said to be incident on each of its end points uand v.
- The sum of the degrees of vertices of a graph G is equal to twice the number of edges(Handshaking Theorem).

Types of Graphs:

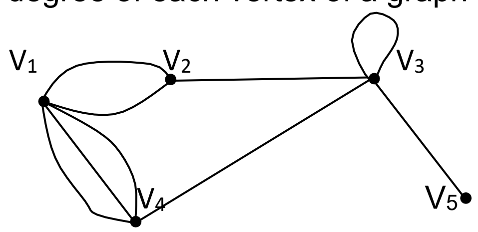
- **Trivial Graph** – A graph with only one vertex
- **Multigraph** – A graph with no loops
- **Simple Graph** –A graph which has neither loops nor multipleedges
- **Pseudograph** - A graph which has loops and multiple edges
- **Directed Graph** – A graph in which each edge has a direction
- **Regular Graph** – All vertices of a graph have the samedegree
- **Null Graph** - It contains only an isolated node (The edge set isempty)
- **Complete Graph:** Every vertex in a Graph G is adjacent toevery other vertex
- **Cycle Graph:** – It consists of n vertices V_1, V_2, V_3, …,V_n and edges {V_1, V_2}, { V_2, V_3}, …, { V_{n-1} , V_n} and { V_n , V_1} only.
- **Wheel Graph:**– when an additional vertex is added to the cycle and this new vertex is adjacent to each of the n vertices in cycle bythe new edges
- **Bipartite Graph** – If the vertex set V can be partitioned intotwo disjoint subsets V_1 and V_2 such that every edge in E connects a vertex in $V_1$and vertex in V_2

- **Complete Bipartite Graph** – A graph whose vertex set is partitioned into subsets V_1 and V_2 in which there is an edgebetween each pair of vertices.

Degree of a vertex:

- The degree of a vertex of an **undirected graph** is equal to the number of edges in G whichcontains the vertex and is denoted by deg (v)
  - A vertex of degree '0' is called an isolated vertex.
  - A degree of degree '0' is called an end vertex (A vertex is pendent iff it has a degree'1').

Example: Find the degree of each vertex of a graph



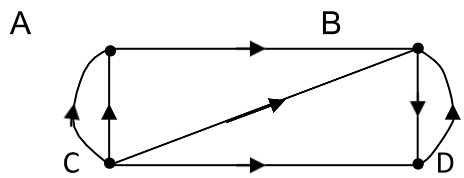
deg (V_1) = 5 ; deg (V_2) = 3 ; deg (V_3) = 5 ; deg (V_4) = 4 ; deg (V_5) = 1

- The degree of a **directed graph** is given by

$$\boxed{\text{Total deg } (V) = \text{Indeg } (V) + \text{outdeg } (V)}$$

  - The number of edges ending at V is called the in-degree of the vertex of a directedgraph and is denoted by Indeg (V) or deg^- (V).
  - The number of edges beginning at V is called the out-degree of the vertex of adirected graph and is denoted by outdeg (V) or deg^+ (V).
  - A vertex with zero indegree is called source.
  - A vertex with zero outdegree is called sink.

Example: Find the degree of each vertex of a digraph



Indeg (A) = 2,outdeg (A) = 1,Totaldeg (A) =

3 Indeg (B) = 3,outdeg (B) = 1,Totaldeg (B) =

4 Indeg (C) = 0,outdeg (C) = 3,Totaldeg (C)

= 3 Indeg (D) =2,outdeg (D) = 1,Totaldeg (D)

= 3

ADJACENCY and INCIDENCE MATRIX OF GRAPH:

If a graph has **n** number of vertices, then the adjacency matrix of that graph is **n x n**, and each entry of the matrix represents the number of edges from one vertex to another. An adjacency matrix is also called as **connection matrix**. Sometimes it is also called a **Vertex matrix**.

Adjacency Matrices

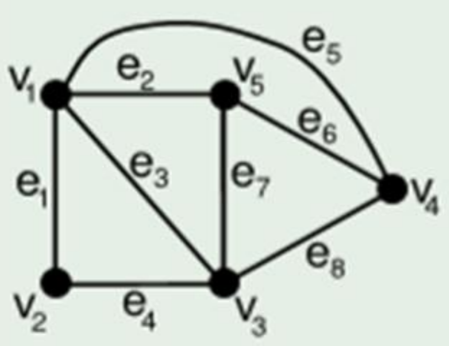
Def. $G=(V, E)$: simple graph, $V=\{v_1,v_2,\ldots,v_n\}$. A matrix A is called the adjacency matrix of G if $A=[a_{ij}]_{n\times n}$, where $a_{ij} =$ 1, if $\{v_i,v_j\} \in E,$
0, otherwise.

- **Def.** Let $G=(V, E)$: be an undirected graph. Suppose that $v_1,v_2,\ldots,v_n$ are the vertices and $e_1,e_2,\ldots,e_n$ are the edges of G . Then the incidence matrix with respect to this ordering of V and E is the n x m matrix $M=[m_{ij}]$, $where$

$$m_{ij} = \begin{cases} 0 \ if \ ej \ is \ not \ incident \ at \ vi \\ 1 \ if \ ej \ is \ incident \ at \ vi \\ 2 \ is \ ej \ is \ the \ loop \ at \ vi \end{cases}$$

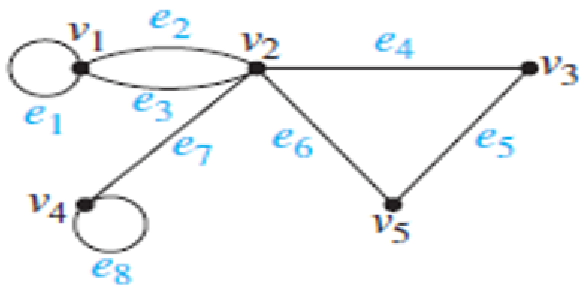
Adjacency matrix for the graph

example



	v_1	v_2	v_3	v_4	v_5
v_1	0	1	1	1	1
v_2	1	0	1	0	0
v_3	1	1	0	1	1
v_4	1	0	1	0	1
v_5	1	0	1	1	0

Example2:Obtain incidence matrix for the graph



$$\begin{array}{cc} & \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Review questions

1. What is a Graph?
2. What is the degree of a loop in a graph?
3. What is Adjacency Matrix?

Summary

In this session, the graph is introduced, and the types of graphs and the degree of the graph are explained with examples, adjacency matrices are introduced and explained with examples.

Self-assessment questions

1.A graph is a collection of...?

A) Row and columns
B) Vertices and edges
C) Equations
D) None of these

Answer: B

2.The degree of any vertex of graph is ...?

A) The number of edges incident with vertex
B) Number of vertices in a graph
C) Number of vertices adjacent to that vertex
D) Number of edges in a graph

Answer: A

3.If for some positive integer k, degree of vertex d(v)=k for every vertex v of the graph G, then G is called...?

a. K graph
b. K-regular graph
c. Empty graph
d. All of above

Answer: B

4.A graph with no edges is known as empty graph. Empty graph is also known as...?

a. Trivial graph
b. Regular graph
c. Bipartite graph
d. None of these

Answer: A

5.In a simple graph, the number of edges is equal to twice the sum of the degrees of the vertices.
a) True
b) False

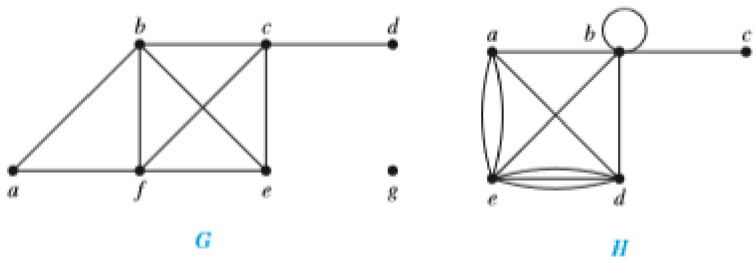
Answer: b

Terminal questions

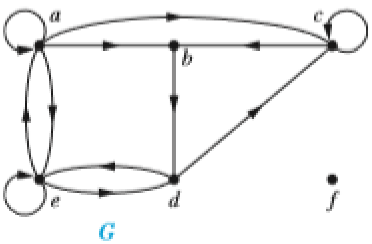
1. Draw a graph representing the problem of three houses and three utilities water, gas and electricity.

2. Identify the number of edges in a graph with 10 vertices each of degree 6.

3. What are the degrees and what are the neighborhoods of the vertices in the graphs G and H?



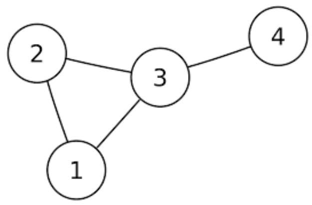
4. What are the in-degree and out-degree of each vertex in the directed graph G?



5.Draw the digraph G corresponding to adjacency matrix.

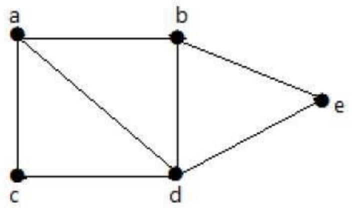
$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

6. Obtain the adjacency matrix for the given graph.



7.Draw a graph with the adjacency matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ with respect to the ordering of the vertices a, b, c, d.

8.Obtain the incidence matrix of the following graph



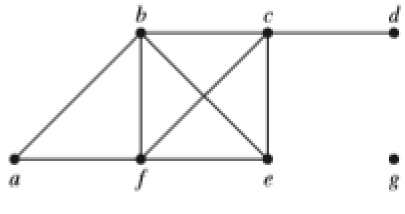
9.Draw an undirected graph represented by the given adjacency matrix $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Class Room Delivery problems:

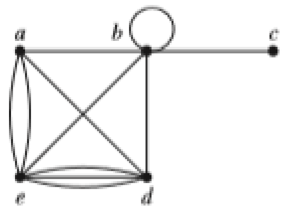
1.Draw a graph representing the problem of three houses and three utilities water, gas and electricity.

2. Identify the number of edges in a graph with 10 vertices each of degree 6.

3.What are the degrees and what are the neighborhoods of the vertices in the graphs G and H?

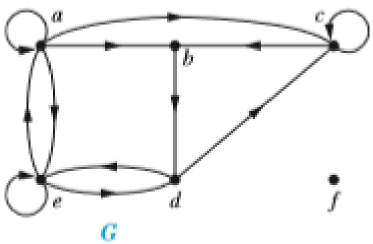


G



H

4.What are the in-degree and out-degree of each vertex in the directed graph G?

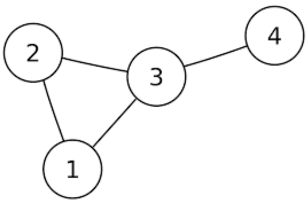


5.Draw a graph with the adjacency matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ with respect to the ordering of the vertices a, b, c, d.

TUTORIAL PROBLEMS

1. Identify the degree sequence in graphical 2,2,2,2.
2. Is there a simple graph with degree sequence (1,1,3,3,3,4,6,7)? Justify your answer.
3. Determine whether the following graph is a simple graph or multigraph, and obtain the degree of each vertex.

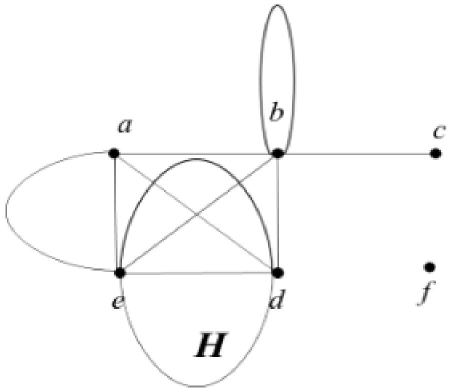
4. Obtain the adjacency matrix for the given graphs.



5. Draw a graph with the adjacency matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ with respect to the ordering of the vertices a, b, c, d.

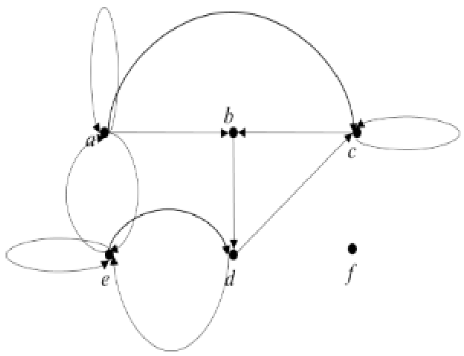
HOME ASSIGNMENT PROBLEMS

1. Determine whether the following graph is a simple graph, or multigraph and obtain the degree of each vertex.

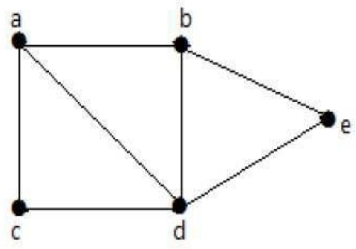


a.

2. Determine the in-degree and out-degree of the following graph and justify your Answer?



3. Obtain the incidence matrix of the following graphs.

i. 

4. Draw an undirected graph represented by the given adjacency matrix $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

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Department of Mathematics

Discrete Structures

23MT1001

Session-23: (Bipartite Graphs, Isomorphism of a graphs)

Instructional Objective:

1. To learn Special types of Graphs
2. To understand Concept of isomorphism

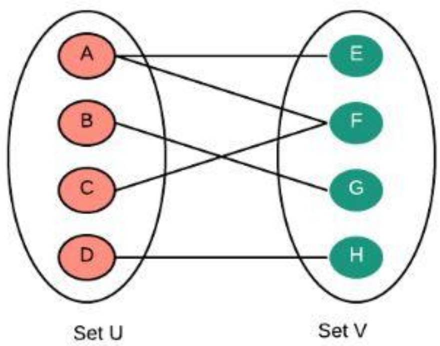
Learning Outcomes:

1. Able to identify Bipartite Graphs
2. Able to Check Graphs are isomorphic or not

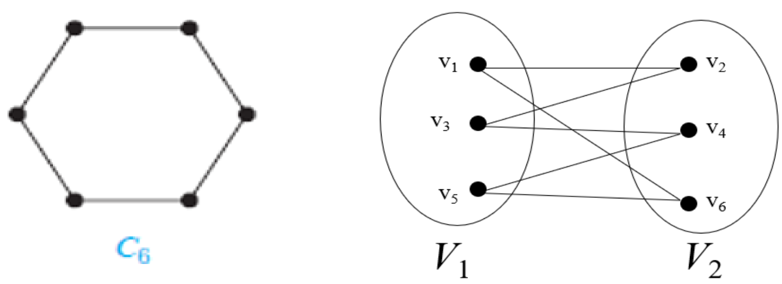
Introduction

BIPARTITE GRAPHS: A simple graph $G=(V,E)$ is called Bipartite if V can be partitioned into two vertices sets V_1 and V_2, $V_1 \cap V_2 = \emptyset$, such that every edge in the graph connect a vertex in V_1 and a vertex in V_2. (or simply)

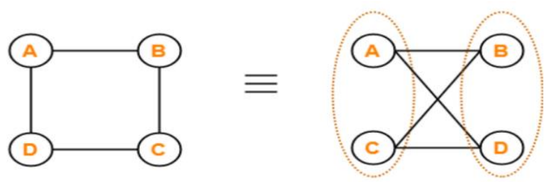
Bipartite Graphs is a graph whose vertices can be divided into two independent groups or sets so that for every edge in the graph, each end of the edge belongs to a separate group. There should not be any edge where both ends belong to the same set.



Example : C_6 is Bipartite Graph



Example : Verify the graph G is Bipartite or Not .



Example of Bipartite Graph

Here,

The vertices of the graph can be decomposed into two sets.

The two sets are X = {A, C} and Y = {B, D}.

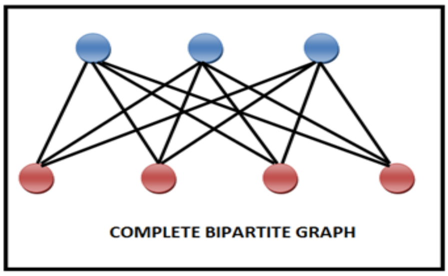
The vertices of set X join only with the vertices of set Y and vice-versa.

The vertices within the same set do not join.

Therefore, it is a bipartite graph.

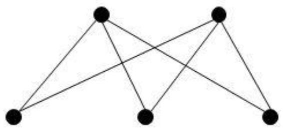
Complete Bipartite graph:

In the numerical area of graph hypothesis, a complete bipartite graph is a special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set.

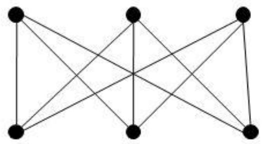


Example :

Complete Bipartite graphs ($K_{m,n}$):



$K_{2,3}$



$K_{3,3}$

$|V(K_{m,n})| = m+n$, $|E(K_{m,n})| = mn$, $K_{m,n}$ is regular if and only if $m=n$.

Isomorphism: A graph can exist in different forms having the same number of vertices, edges, and also the same edge connectivity. Such graphs are called isomorphic graphs. If two graphs G_1 and G_2 are said to be isomorphic then

- Their number of components (vertices and edges) is same.
- Their edge connectivity is retained.

Graph isomorphism is the area of pattern matching and widely used in various applications such as **image processing, protein structure, computer and information system, chemical bond structure, Social Networks**. Mainly, the application surveys in both various applications of graph isomorphism and their importance in the society.

Isomorphism of Graphs:

The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there is an one-to-one and onto function f from V_1 to V_2 with the property that $a\sim b$ in G_1 iff $f(a)\sim f(b)$ in G_2, $\forall a, b\in V_1$
f is called an isomorphism.

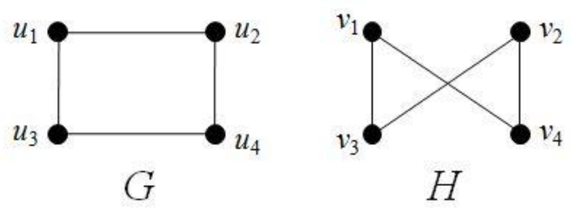
If $G_1 \cong G_2$ then G_1 & G_2 satisfies the following necessary conditions.

1. $|V(G_1)| = |V(G_2)|$.
2. $|E(G_1)| = |E(G_2)|$.
3. Degree sequences of G_1 and G_2 are same.
4. ($G_1 \cong G_2$) if and only if G_1 and G_2 are simple graphs.

All the above conditions are necessary for the graphs G_1 and G_2 to be isomorphic, but not sufficient to prove that the graphs are isomorphic.

- ($G_1 \cong G_2$) if the adjacency matrices of G_1 and G_2 are same.
- If the vertices $\{V_1, V_2, \ldots, V_k\}$ form a cycle of length K in G_1, then the vertices $\{f(V_1), f(V_2),\ldots, f(V_k)\}$ should form a cycle of length K in G_2.

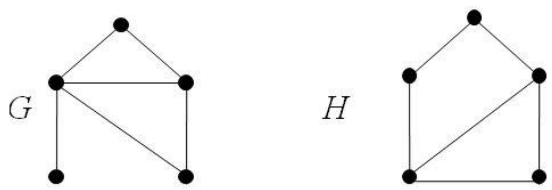
Example : Check G and H are isomorphic.



The above two graphs consists of

(1) The same number of vertices
(2) The same number of edges
(3) The same degree sequence
(4) G and H are simple graphs, also
(5) The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between $V(G)$ and $V(H)$.
Hence G and H are Isomorphic.

Example : Show that G and H are not isomorphic.



In the above graph G and H, G has a vertex of degree = 1, H don't have. Hence G and H are not isomorphic.

Review Questions

1. What is Bipartite Graph?
2. What is Graph Isomorphism?
3. Give any one application on Graph Isomorphism.

Summary

In this session, the concept of Bipartite graphs, complete Bipartite graphs and Graph Isomorphism is introduced and explained with examples. Some real-life applications are explained.

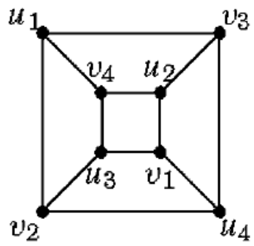
Self-Assessment Questions

1. Every Isomorphic graph must have_____________________ representation
   a) Cyclic
   b) Adjacency
   c) Tree
   d) Adjacency matrix

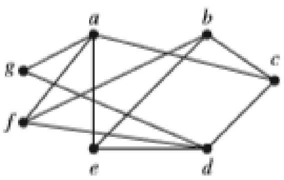
2. A cycle on n vertices is isomorphic to its complement. What is the value of n?
   a) 5
   b) 32
   c) 17
   d) 8

Terminal Questions

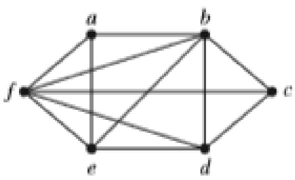
1. Test whether the graph is complete bipartite or not? Justify your answer.



2.Check whether the graphs G and H displayed are bipartite or not?

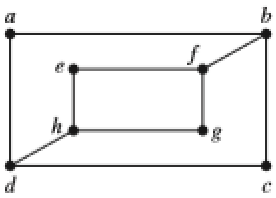


G

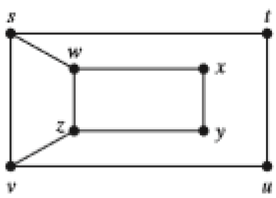


H

3.Determine whether the graphs G and H are isomorphic.

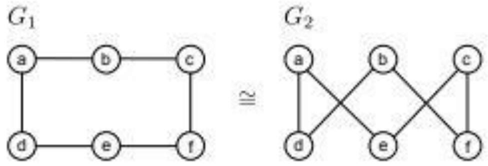


G

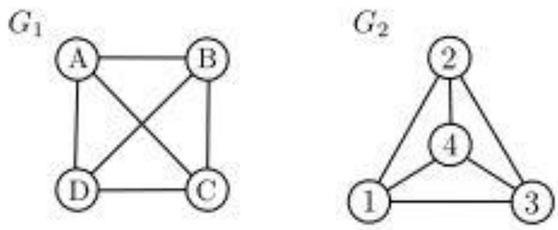


H

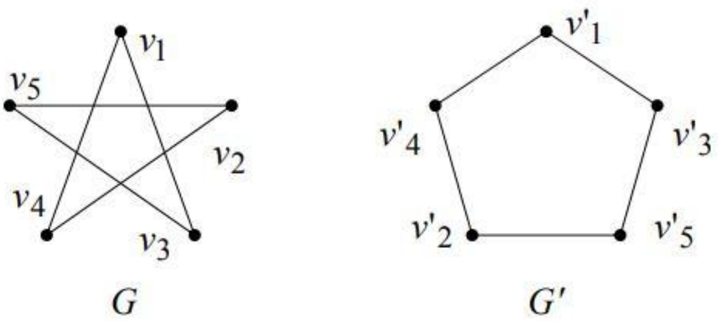
4.Determine whether the graphs G_1 and G_2 are isomorphic.



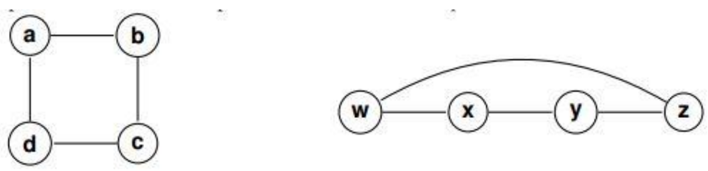
5.Verify graphs are isomorphic or not.



6.Determine whether the graphs G and G' are isomorphic.



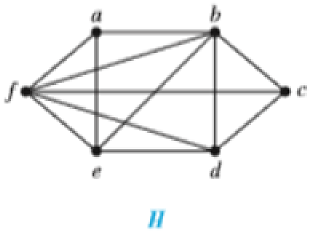
7.Verify graphs G and H are isomorphic or not.



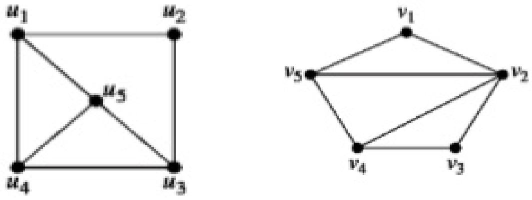
G H

TUTORIAL PROBLEMS

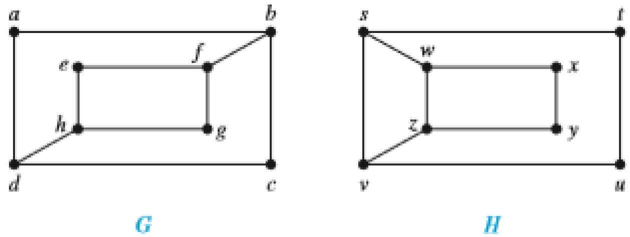
1. Check whether the graph H displayed are bipartite or not?



2. Check whether the graphs are isomorphic or not?

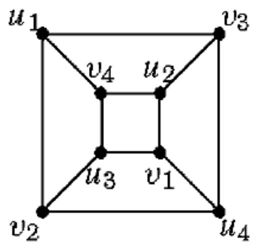


3. Is the graphs isomorphic or not?

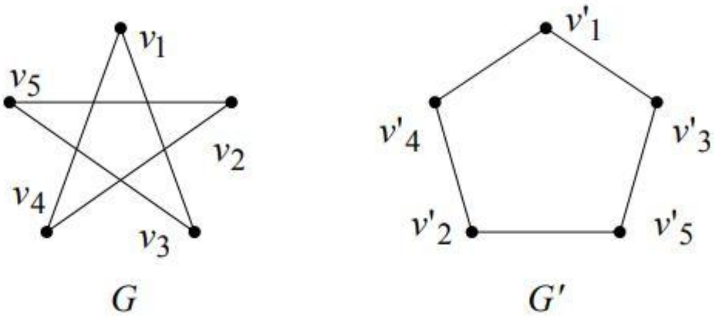


HOME ASSIGNMENT PROBLEMS

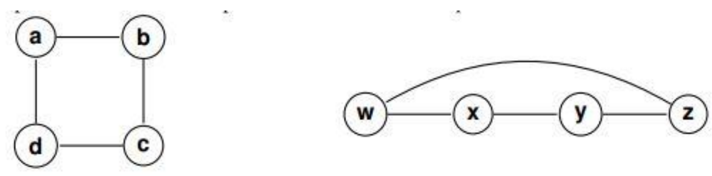
1. Test whether the graph is complete bipartite or not? Justify your answer.



2. Determine whether the graphs G and G' are isomorphic.

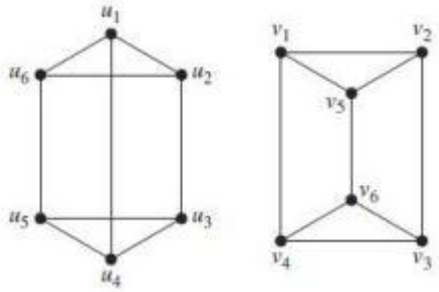


3. Verify graphs G and H are isomorphic or not?



G H

4. Check whether the graphs G and H are isomorphic or not?



G H

KONERU LAKSHMAIAH EDUCATION FOUNDATION
(DEEMED TO BE UNIVERSITY)
(Deemed to be University, Estd. u/s. 3 of UGC Act 1956)

Department of Mathematics

Discrete Structure

Course code: 23 MT1001

I/IV-B.Tech -(I st Sem), Academic Year: 2023-2024

Session-24 :Connectivity, Euler and Hamiltonian Path

APPLICATION OF EULER AND HAMILTONIAN PATHS

Instructional Objective:

1. To identify connectivity in a given graph.
2. To identify Euler paths in a given graph.
3. To identify Euler circuits in a given graph.
4. To identify Hamiltonian paths in a given graph
5. To identify Hamiltonian Circuits in a given graph

Learning Outcomes:

1. Able to known connectivity exists or not in a graph.
2. Able to known Euler paths exists or not in a given graph.
3. Able to known Euler circuits exists or not in a given graph.
3. Able to known Hamiltonian paths exists or not in a given graph.
4. Able to known Hamiltonian circuits exists or not in a given graph.

Introduction:

EULER PATHS AND CIRCUITS

The town of Konongsbrige, Prussia was devided in 4 sections by the branches of the Pregel River. These four sections included the two regions on the banks of the Pregel., Kneiphof Island and the region between the two branches of the Pregel. In the 18 th century 7 bridges connected these regions. The following graph depicts these regions and bridges.

The town people took long walks through town on Sunday. They wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice and return to the starting point.

Euler studied this problem using the multigraph obtained when the foue regions are represented by vertices and the bridges by edges. This multigraph is shown in below figure.

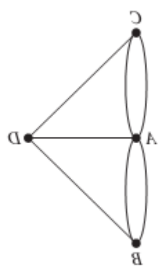


FIGURE 2 Multigraph Model of the Town of Königsberg.

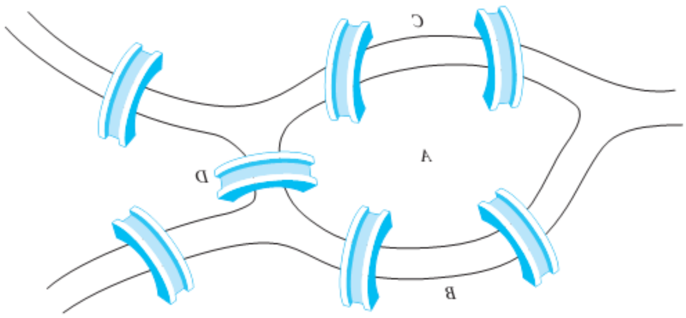


FIGURE 1 The Seven Bridges of Königsberg.

Connected graph*:* A graph G is called connected graph if there is a path in between every pair of vertices.

Euler circuit: An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

- An Euler path starts and ends at different vertices.
- An Euler circuit starts and ends at the same vertex

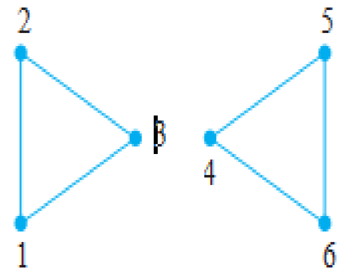
Theorem 1: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

Theorem 2: A connected multigraph has an Euler path not an Euler circuit if and only if it has exactly vertices of odd degree.

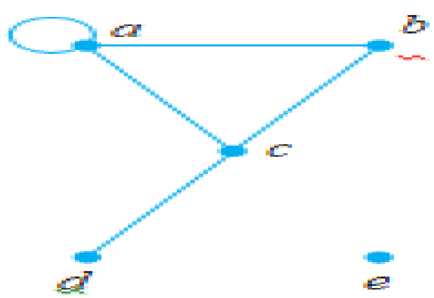
Theorem 3: (a) If a graph G has a vertex of odd degree, there can be no Euler circuit in G.
(b) If G is a connected graph and every vertex has even degree, then there is an Euler circuit in G.

Theorem 4: If a graph *G* has more than two vertices of odd degree, then there can be no Euler path in *G*.

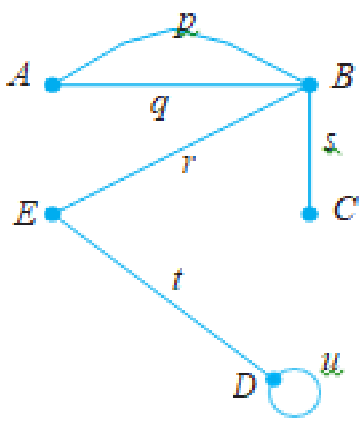
Example 1: Which of the following graphs are connected or disconnected graph.
Sol:



G1



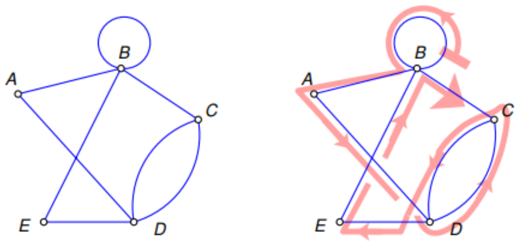
G2



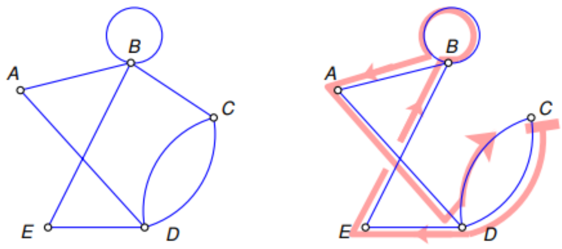
G3

Sol: Graph G1 and Graph G2 are disconnected graphGraph G3 is connected graph

Example 2: Consider the following graph and observe Euler path and Euler circuit.

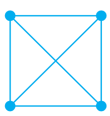


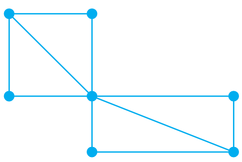
An Euler path: BBADCDEBC



Another Euler circuit: CDEBBADC

Example 3: Which of the graphs in Figures a, b, and c have an Euler circuit, an Euler path but not an Euler circuit, or neither?





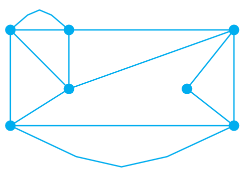


Figure a **Figure b** **Figure c**

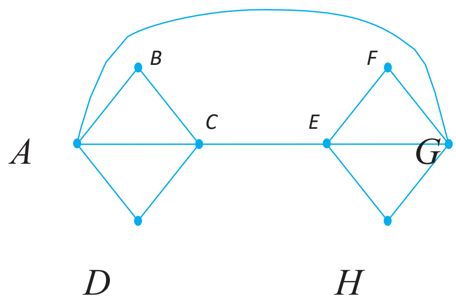
Solution

i) In Figure (a), each of the four vertices has degree 3; thus, by Theorems 1 and 2, there is neither an Euler circuit nor an Euler path.

ii) The graph in Figure (b) has exactly two vertices of odd degree. There is no Euler circuit, but there must be an Euler path.

iii)In Figure © , every vertex has even degree; thus the graph must have an Euler circuit.

Example 4: Construct an Euler circuit for the graph for following Figure



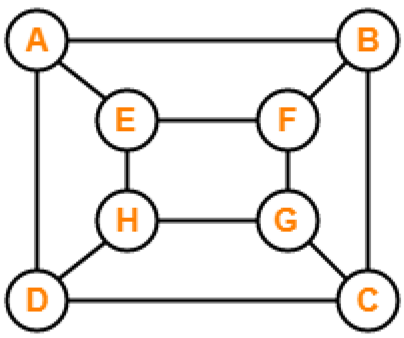
HAMILTONIAN PATHS AND HAMILTONIAN CIRCUITS

Definition: A simple path in graph G that passes through every vertex once is called Hamiltonian path, and a simple circuit in a graph G that passes through every vertex once is called Hamiltonian circuit.

Dirac's Theorem: If G is a simple graph with 'n' vertices with $n \geq 3$ such that the degree of every vertex in G is at least n/2, then G has Hamiltonian circuit.

Ore's Theorem: If G is a simple graph with 'n' vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of non adjacent vertices 'u' and 'v' in G then G has a Hamiltonian circuit.

Example 1: Does the following graph contains Hamiltonian circuit or Hamiltonian path?



Solution: The graph contains both a Hamiltonian path **ABCDHGFE** and Hamiltonian circuit **ABCDHGFEA**. The graph contains Hamiltonian circuit therefore it is a Hamiltonian graph.

Review questions

1) What is Euler Path?

2) What is Euler Circuit?

3) What is Hamiltonian Path?

4) What is Hamiltonian circuit?

Summary

In this session Euler paths, Euler circuits, Hamilton paths and Hamiltonian circuits were with were explained with examples and ***Dirac's Theorem*** statement ***Ore's Theorem*** statement are mentioned.

Self-assessment questions

1. A path in a graph G is Euler path if ---------

   E) If the path contains every edge of G
   F) If the path contains every vertex of G
   G) If the path contains every vertex and every edge of G
   H) None of the above

   Answer: A

2. A circuit in a graph G is Euler circuit if ---------

A) If the circuit contains every vertex of G
B) If the circuit contains every edge of G
C) If the circuit contains every vertex and every edge of G
D) None of the above
Answer: B

3. A path in a graph G is Hamiltonian path if ---------

A) If the path contains every edge of G

B) If the path contains every vertex of G

C) If the path contains every vertex and every edge of G

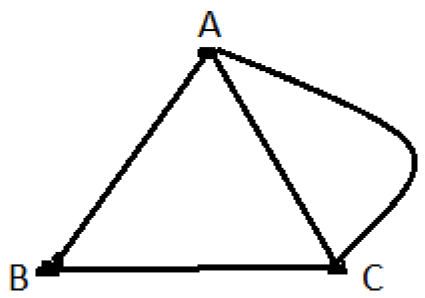
D) None of the above

Answer: B

4. A circuit in a graph G is Hamiltonian circuit if ---------

A) If the circuit contains every vertex of G
B) If the circuit contains every edge of G
C) If the circuit contains every vertex and every edge of G
D) None of the above
Answer: A

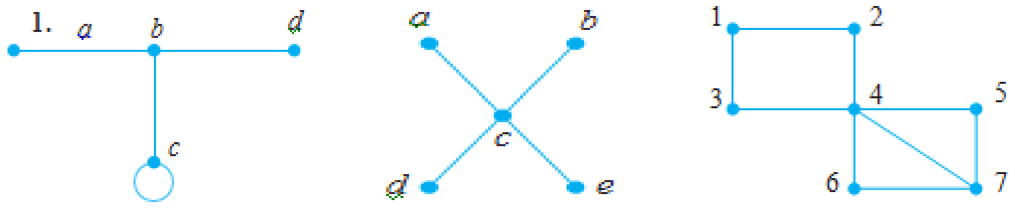
5. The following graph contains _______



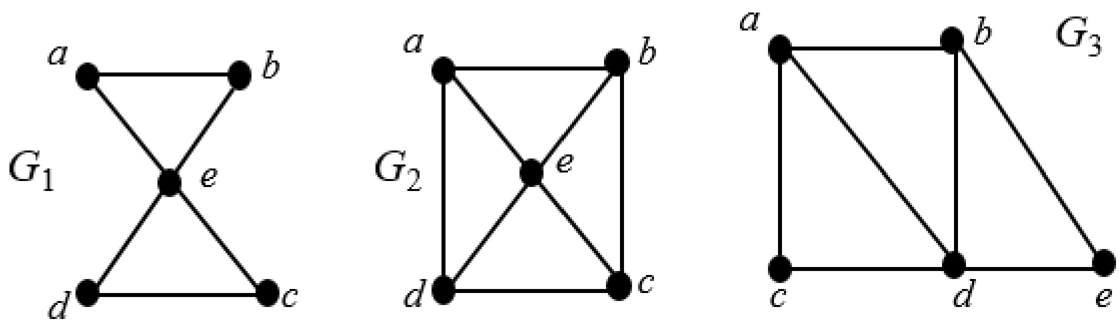
A) Euler path
B) Euler Circuit
C) Hamiltonian Path
D) Hamiltonian circuit
Answer: D

Class Room Delivery problems:

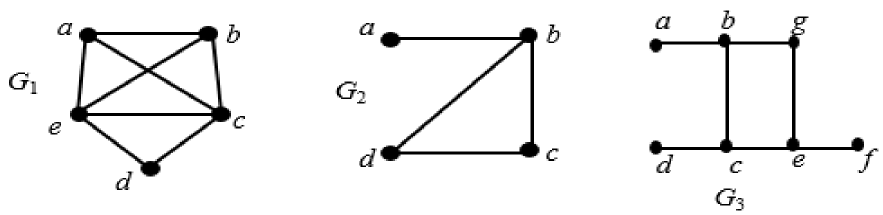
1. Which of the following graphs G_1, G_2, G_3 have Euler circuits or Euler Paths?



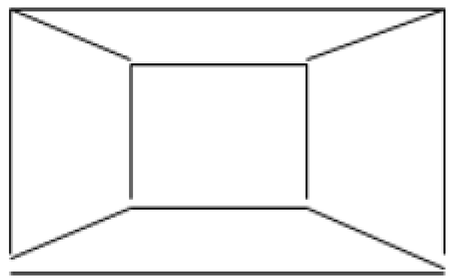
2. Which of the following graphs G_1, G_2, G_3 have Euler circuits or Euler Paths?



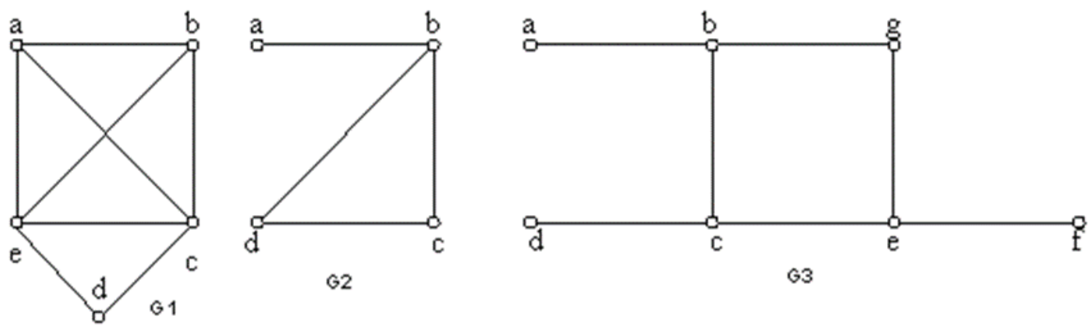
3. Which of the following graphs have a Hamilton circuit or a Hamilton path?



4). Is the following graph is Hamiltonian? If yes find the Hamiltonian cycle?

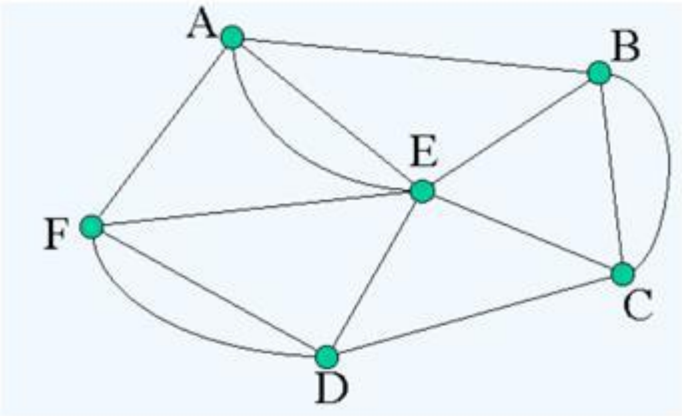


5) Which of the following graphs have Hamiltonian Circuits?

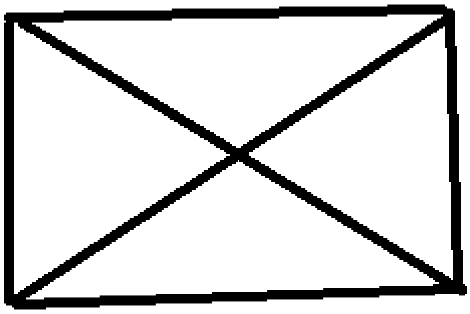


Practice Problems:

1) Does the following graph has Euler circuit or Hamilton circuit? if exist draw such graphs?

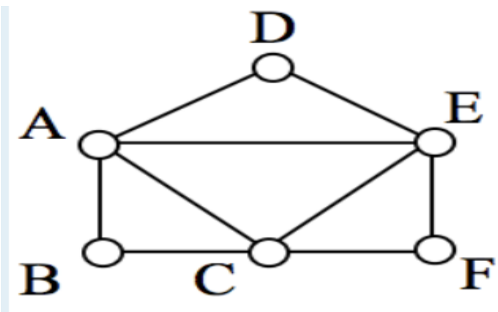


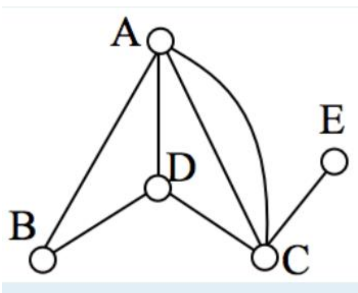
2) Determine three distinct Hamiltonian cycles in the following graph, if exists.



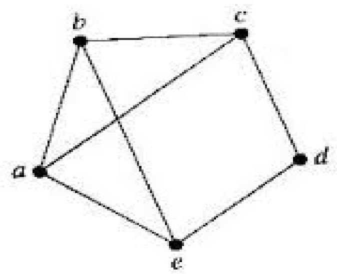
Tutorial Problems:

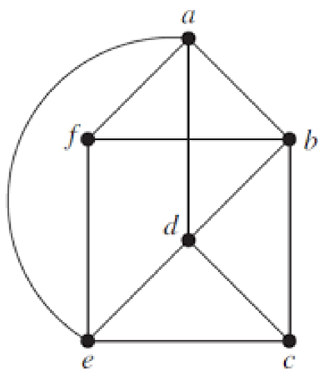
1) Which of the following graphs have Euler circuit or Euler path, if exist draw such graphs.

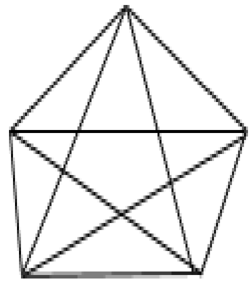




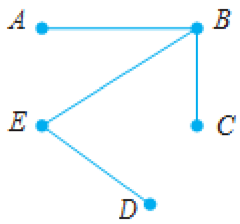
2). which of the following graphs have Euler circuit, then construct such graph if exist.

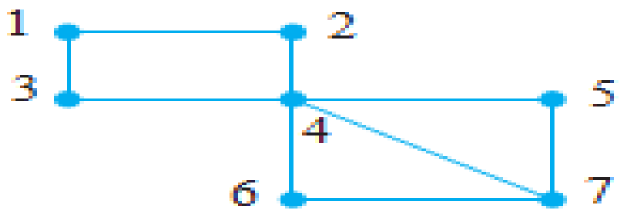




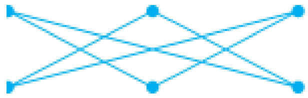


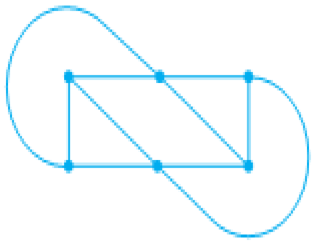
3). which of the following graphs have Hamilton circuit or Hamilton path, and then construct such graph if exist.





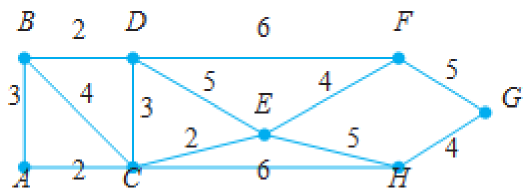
4) Which of the following graphs have Hamilton circuit or Hamilton path, then construct such graph if exist.

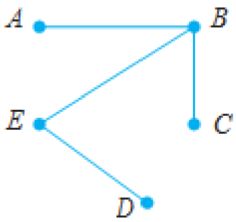




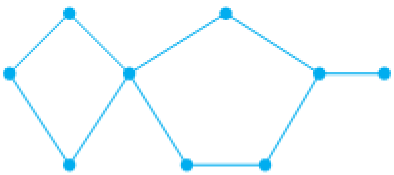
Home Assignment Problems:

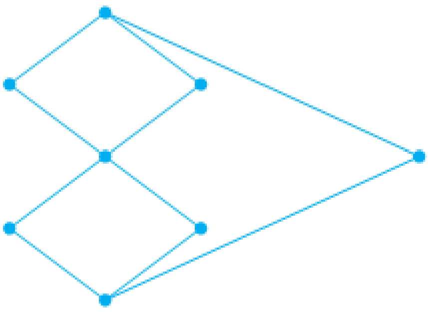
1) Which of the following graphs have Hamilton circuit or Hamilton path, and then construct such graph if exists





2) Which of the following graphs have Euler circuit or Euler path, then construct such graph if exists





Reference Books:
1.Joe L Mott, Abraham Kandel, Theodore P Baker, Discrete Mathematics for Computer Scientists and Mathematicians, Printice Hall of India, Second Edition, 2008.
2. Tremblay J P and Manohar R, Discrete Mathematical Structures with Applications to Computer Science, Tata McGraw Hill publishers, 1st edition, 2001,India.
Sites and Web links:
https://www.youtube.com/watch?v=KPx3Cyp__1s
https://www.youtube.com/watch?v=mok062lFajo

Department of Mathematics
Data Structures- 23MT1001
I/IV-B.Tech-(II Sem), Academic Year: 2022-2023

Session-25 (Shortest path Problems- Dijkstra's Algorithm, Planar graphs, Graph Coloring.)

Instructional Objective:

2. To understand the concepts of shortest path, planar graphs.
3. To understand the concept of graph coloring problems and chromatic number.

Learning Outcomes:

2. Able to apply shortest path problems in real world situations.
3. Able to apply graph coloring concepts in network systems.

Introduction:

Many problems can be modelled using graphs with weights assigned to their edges. As an illustration, consider how an airplane system can be modelled. We set up the basic graph model by representing cities by vertices and flights by edges.

- Problems involving distances can be modelled by assigning distances between cities to the edges.
- Problems involving flight time can be modelled by assigning flight times to edges.
- Problems involving fares can be modelled by assigning fares to the edges.

Graphs that have a number assigned to each edge are called **weighted graphs.**

- Weighted Graphs are used to model computer networks.
- Communication costs such as the monthly cost of leasing a telephone line, the response times over these lines or the distance between computers can all be studied using weighted graphs.

Determining a path of least length between vertices in a network is one such problem.

Explanation:

Shortest-Path Problems

Definition:

1. Graphs that have a number assigned to each edge are called *weighted graphs*.
2. The length of a path in a weighted graph is the sum of the weights of the edges of this path.

Shortest path Problem:
Determining the path of least sum of the weights between two vertices in a weighted graph

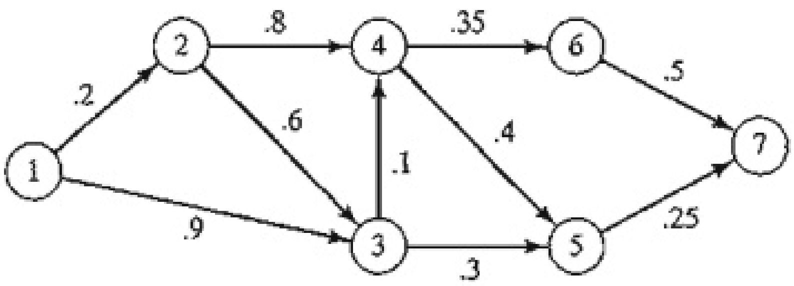
Use of the shortest path problems

- In graph theory, the **shortest path problem** is the problem of finding a path between two vertices (or nodes) in a graph such that the sum of the weights of its constituent edges is minimized.

Examples

1. The problem of finding the shortest path between two intersections on a road map may be modeled as a special case of the shortest path problem in graphs, where the vertices correspond to intersections and the edges correspond to road segments, each weighted by the length of the segment.
2. The Stagecoach Shipping Company transports oranges by six trucks from Los Angeles to six cities in the West and Midwest
3. Smart drives daily to work. Having just completed a course in network analysis, Smart is able to determine the shortest route to work. Unfortunately, the selected route is heavily patrolled by police, and with all the fines paid for speeding, the shortest route may not be the best choice. Smart has thus decided to choose a route that maximizes the probability of *not* being stopped by police.

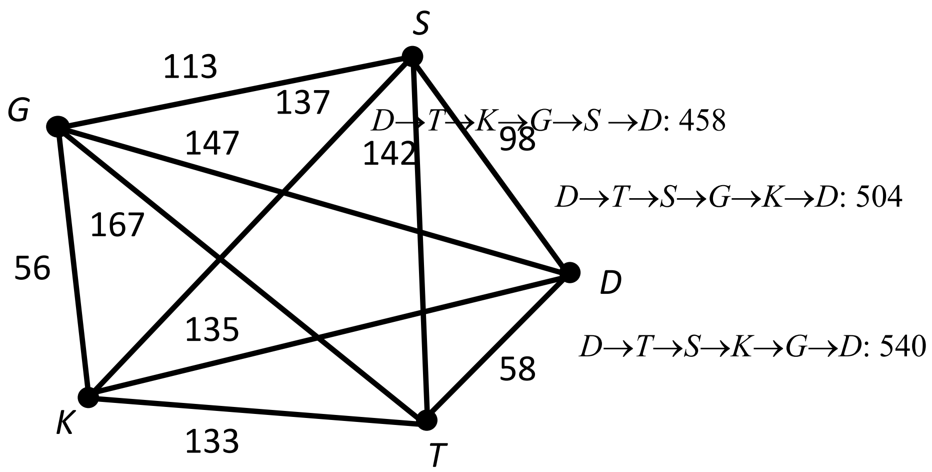
The network in the below figure shows the possible routes between home and work, and the associated probabilities of not being stopped on each segment. The probability of not beingstopped on a route is the product of the probabilities associated with its segments. For example, the probability of not receiving a fine on the route 1 -> 3 -> 5 -> 7 is .9 X .3 X .25 = .0675. Smart's objective is to select the route that *maximizes* the probability of not being fined.



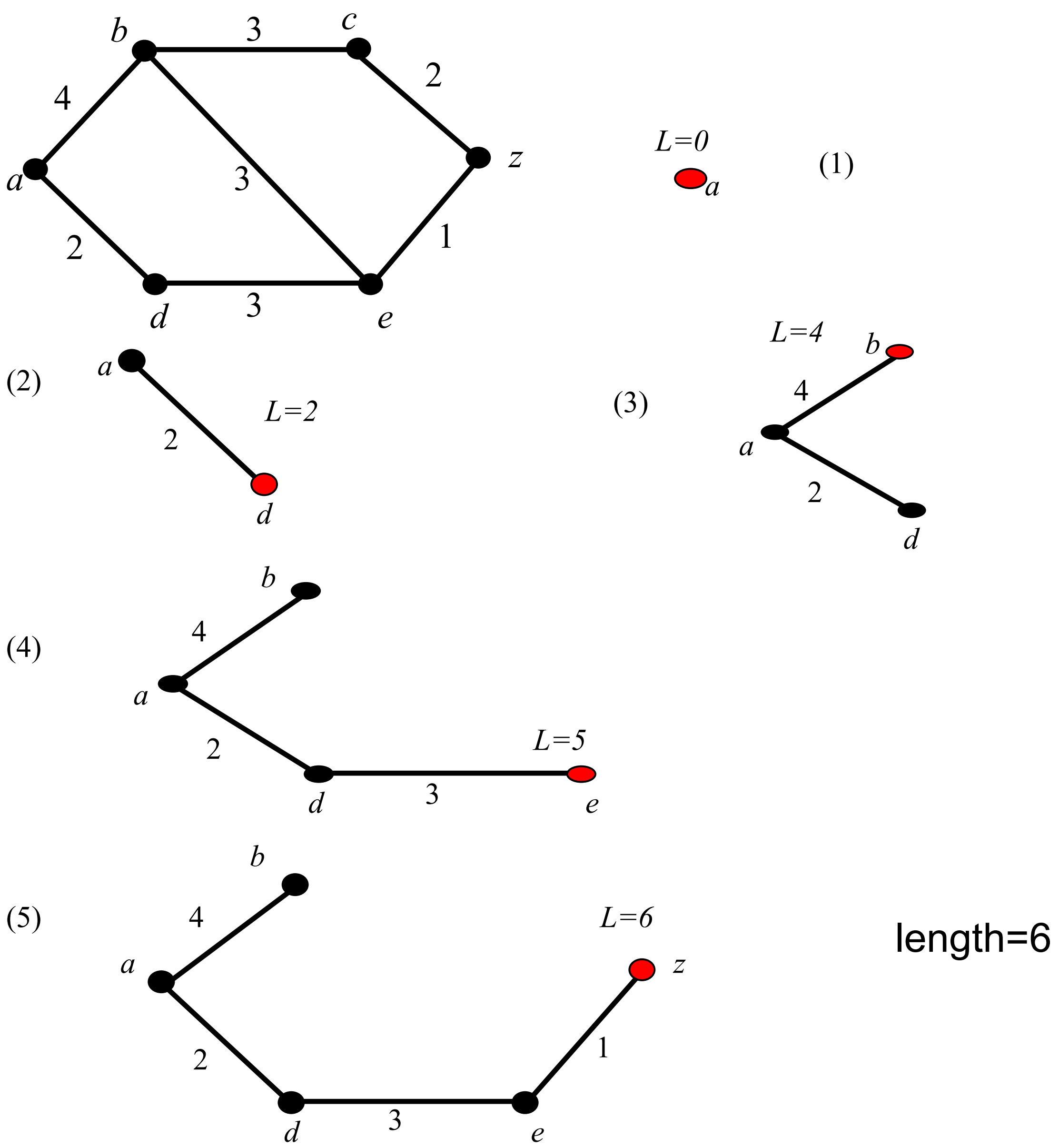
Example 4

The Traveling Salesman Problem:

A traveling salesman wants to visit each of *n* cities exactly once and return to his starting point. In which order should he visit these cities to travel the minimum total distance?



Example 5: What is the length of a shortest path between *a* and *z* in the weighted graph *G*?



Dijkstra's Algorithm (find the length of a shortest path from *a* to *z*)

It begins by labeling a with 0 and other vertices with ∞.

We use the notation $L_0(a) = 0$ and $L_0(v) = \infty$ for these labels before any iterations have taken place(subscript stands for 0^{th} iteration. These labels are the lengths of shortest paths from a to the vertices, where the paths contain only the vertex a. (Because no path from a to a vertex different from a exists, ∞ is the length of a shortest path between a and this vertex.

Let S_k denote this set after k iterations of the labelling procedure. We begin with $S_0 = \Phi$. The set S_k is formed from S_{k-1} by adding a vertex u not in S_{k-1} with the smallest label. Once u is added to S_k, we update the labels of all the labels not in S_k, so that $L_k(v)$, the label of the vertex v at the kth stage is the length of a shortest path from a to v that contains vertices only in S_k (vertices that were already in the distinguished set together with u)

Let v be a vertex not in S_k. To update the label of v note that L_k (v) is the length of a shortest path from a to v containing only vertices in S_k. The updating can be carried out efficiently when this observation is used: A shortest path from a to v containing only elements of S_k is either a shortest path from a to v that contains only elements of S_{k-1}. (the distinguished vertices not including u), or it is a shortest path from a to u at the (k-1) st stage with the edge (u,v) added. i.e.,

L_k (a,v)= min{ $L_{k-1,}$ (a,v), L_{k-1} (a,u) + w(u,v)}

This procedure is iterated successively , adding vertices to the distinguished set until z is added. When z is to the distinguished set , its label is the length of a shortest path from a to z.

Procedure *Dijkstra*(*G*: weighted connected simple graph, with all weights positive)

{*G* has vertices $a = v_0, v_1, \ldots, v_n = z$ and weights $w(v_i, v_j)$

where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in *G*}

for $i := 1$ **to** n

$L(v_i) := \infty$

$L(a) := 0$

$S := \varnothing$

while $z \notin S$

begin

$u :=$ a vertex not in S with $L(u)$ minimal

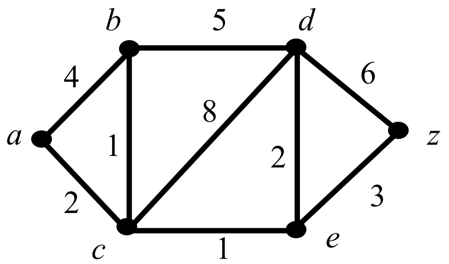
$S := S \cup \{u\}$

for all vertices v not in S

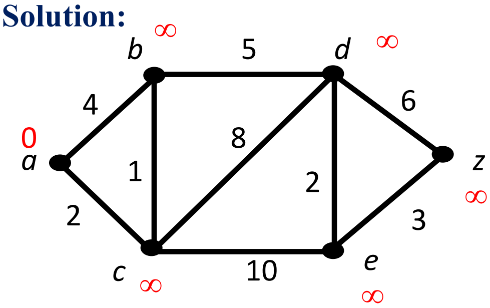
if $L(\mathrm{u}) + w(u, v) < L(v)$ **then** $L(v) := L(\mathrm{u}) + w(u, v)$

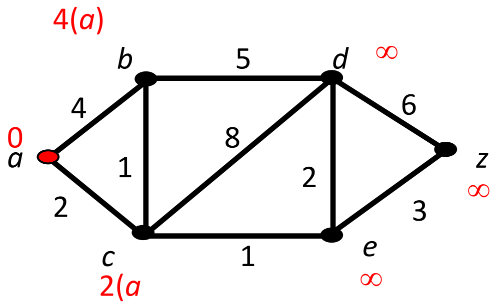
end {$L(z)$ = length of a shortest path from a to z}

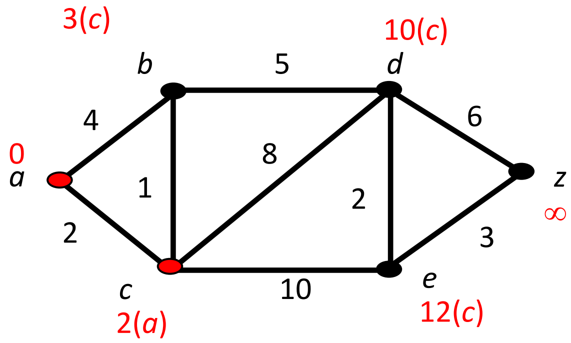
Problem: Use Dijkstra's algorithm to find the length of a shortest path between the vertices a and z in the weighted graph displayed in the following figure.

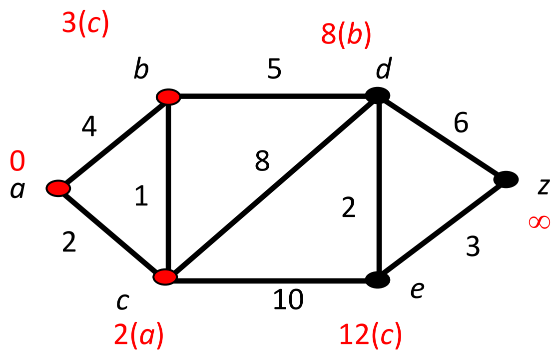


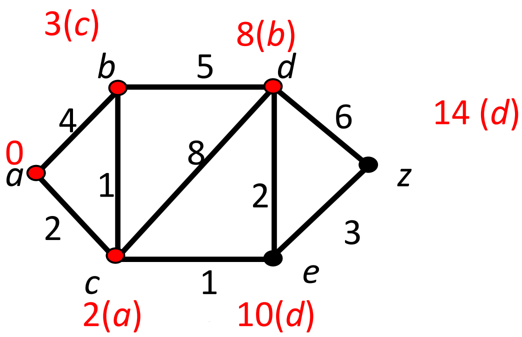
Solution:

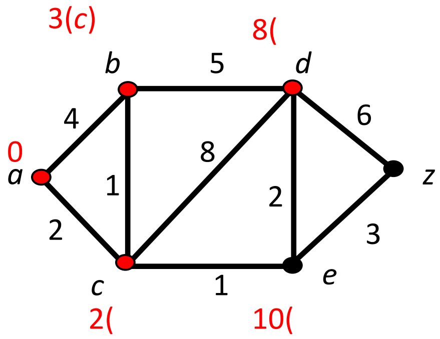


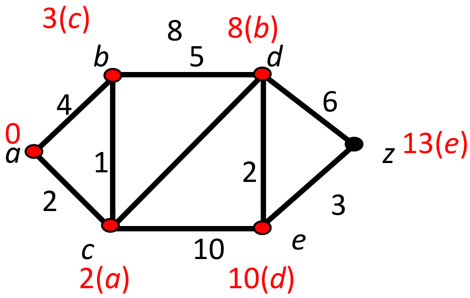


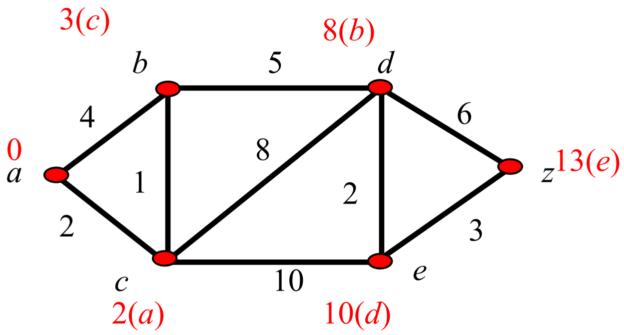












path: *a, c, b, d, e, z*   length: 13

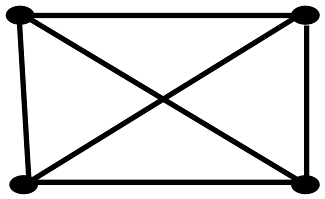
Note: 1. Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.

2.Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph with n vertices.

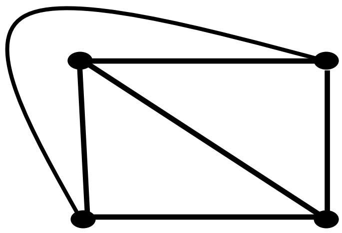
Planar Graphs

Definition:
A graph is called *planar* if it can be drawn in the plane without any edge crossing. Such a drawing is called a *planar representation* of the graph

Example: Is K_4 planar?



K_4

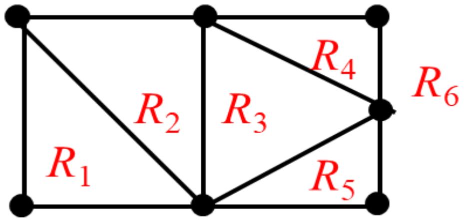


There is no edge crossing. K_4 is planar

Euler's Formula

A planar representation of a graph splits the plane into regions, including an unbounded region.

Example : How many regions are there in the following graph?



Total regions : 6

PROPERTIES OF PLANAR GRAPHS:

- If a connected simple planar graph G has e edges and r regions then r≤ 2e/3
- If a connected simple planar graph G has e edges and n(≥3) vertices, then e ≤ 3n-6.
- A complete graph K_n is planar if n< 5.
- A complete bipartite graph $K_{m,n}$ is planar if and only if m<3 or n>3.

Remarks:

In planar graphs, the following properties hold good

1. In a planar graph with 'n' vertices, sum of degrees of all the vertices is $n \sum deg(V_i) = 2|E|$

2. According to Sum of Degrees of Regions Theorem, in a planar graph with 'n' regions, Sum of degrees of regions is $n \sum deg(r_i) = 2|E|$

Where, $|V|$ is the number of vertices, $|E|$ is the number of edges, and $|R|$ is the number of regions.

Example: Suppose that a connected planar graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Sol.

$v = 20$, $2e = 3\times20 = 60$, $e = 30$

$r = e-v+2 = 30-20+2 = 12$

Example: A connected planar graph has 10 vertices each of degree 3. Into how many regions does a representation of this planar graph split the plan.
Solution: Here n= 10, and degree of each vertex is 3. So sum of all degrees of all vertices = 3x 10= 30 = 2e (by hand shaking rule). So e(number of edges)= 15. Again by euler's formula we have n-e + r= 2. Hence 10-15+r =2. Therefore, the number of regions, r = 7.

GRAPH COLORING:

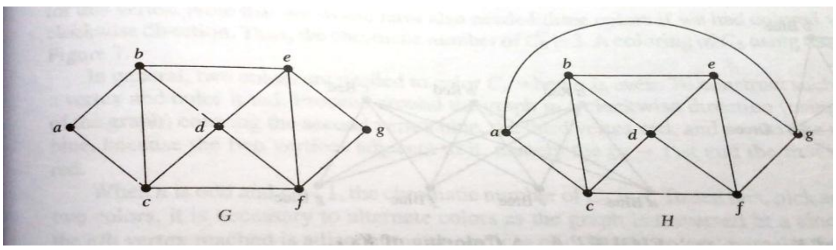
When a map is colored two regions with a common border are assigned different colours. One way to ensure different that two adjacent regions never have the same colour is to use a different colour for each region. However, this is inefficient and on maps with many regions it would be hard to distinguish similar colours. Instead a small number of colours should be used whenever possible.

Consider the problem of determining the least number of colours that can be used to colour a map so that adjacent regions never have the same colour.

Definition: A colouring of a simple graph is the assignment of a colour to each vertex of the graph so that no two adjacent vertices are assigned the same colour.

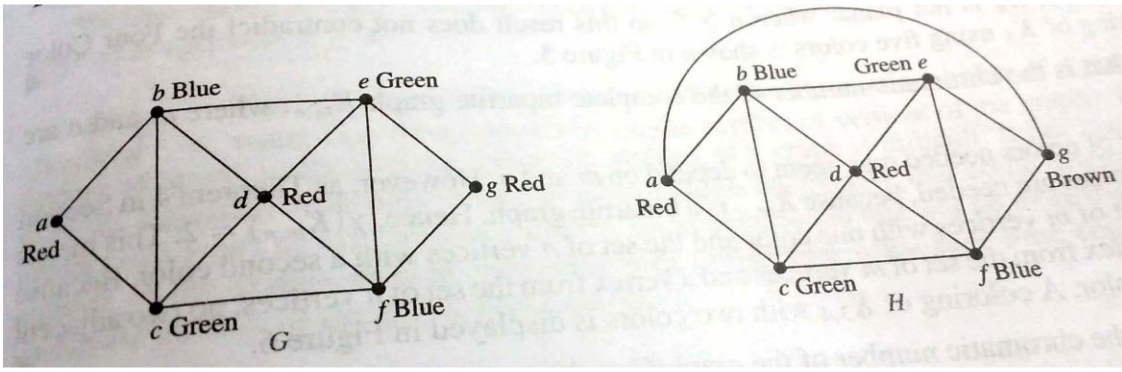
Definition: the chromatic number of a graph is the least number of colours needed for colouring of this graph. It is denoted by X(G).

1) Determine the chromatic numbers of the graphs G and H shown below.



The chromatic number of G is at least three because the vertices a, b, c must be assigned different colours. To set if G can be coloured with three colours assign red to a, blue to b and green to c. Then d can be coloured red because it is adjacent to b and c. further e must be coloured green because it is adjacent only to vertices coloured red and blue. And f must be coloured blue because it is adjacent to red and green, finally g must be coloured red because it is adjacent to blue and green. This produces a colouring of G using exactly three colours.

The graph H is made up of G with an edge connecting a and g. Any attempt to colour H using three colours must follow the same reasoning as that used to colour G, except at the last stage, when all the vertices other than g have been coloured. Then because g is adjacent to vertices coloured red , blue and green, a fourth colour say brown needs to be used. Hence H has a chromatic number 4. A colouring of G and H are as shown below.



Review questions:

1. What is a planar graph?
2. Define Euler's formula?
3. What is Chromatic number?

Summary:

In this session, it was discussed about Shortest path problems - Dijkstra's Algorithm, planar graphs and Graph colouring.

Self-assessment questions:

1. Dijkstra's Algorithm is used to solve _____________ problems.
   a) All pair shortest path
   b) Single source shortest path
   c) Network flow
   d) Sorting
   Answer: b
2. For a connected planar simple graph G=(V, E) with e=|E|=16 and v=|V|=9, then find the number of regions that are created when drawing a planar representation of the graph?
   a) 321
   b) 9
   c) 1024
   d) 596
   Answer: b
3. What is the number of edges of the greatest planar subgraph of $K_{3,2}$ where m,n≤3?
   a) 18
   b) 6

c) 128

d) 702

Answer: b

4. A connected planar simple graph G=(V, E) with e=|E|=16 and v=|V|=9, then find the number of regions that are created when drawing a planar representation of the graph?
a) 21
b) 94
c) 1024
d) 9
Answer: d

5. The chromatic number of a graph is the property of ____________
a) graph coloring
b) graph ordering
c) group ordering
d) group coloring
Answer: a

6. Is K_4 is planar?
a) Planar
b) Non planar
c) Disconnected
d) None of these
Answer : a

7. What will be the chromatic number for an bipartite graph having n vertices?
a) 0
b) 1
c) 2
d) n
Answer : C

8. What will be the chromatic number for a complete graph having n vertices?
a) 0
b) 1
c) n
d) n!
Answer: C

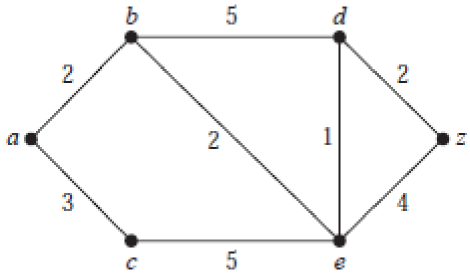
9. Dijkstra's Algorithm cannot be applied on ______________
a) Directed and weighted graphs
b) Graphs having negative weight function
c) Unweighted graphs
d) Undirected and unweighted graphs
Answer: b

10. A non-planar graph can have ____________
a) complete graph
b) subgraph
c) line graph
d) bar graph

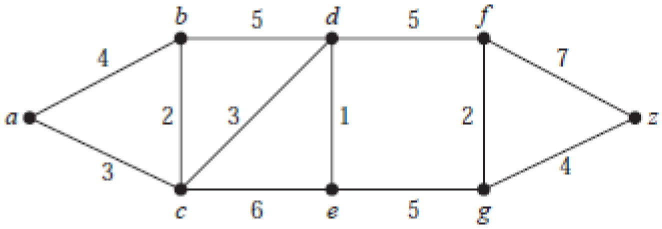
Answer: b

CLASSROOM DELIVARY PROBLEMS

1. Determine the length of a shortest path between *a* and *z* in the given weighted graph.



2. Apply Dijkstra's Algorithm to find the length of a shortest path between *a* and *z* in the given weighted graph

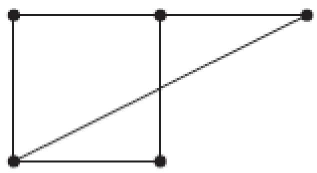
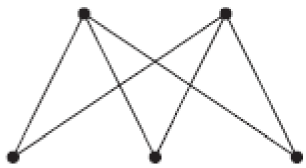


2. Prove that complete graph K_4 is planar whereas complete graph K_5 is not planar.

3. Is $K_{3,3}$ is a planar? Justify?

4. Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

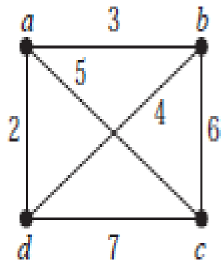
5. Draw the given planar graph without any crossings.

a.  b. 

6. Find the chromatic number of Kn?

TUTORIAL PROBLEMS

1.Solve the traveling salesperson problem for this graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.

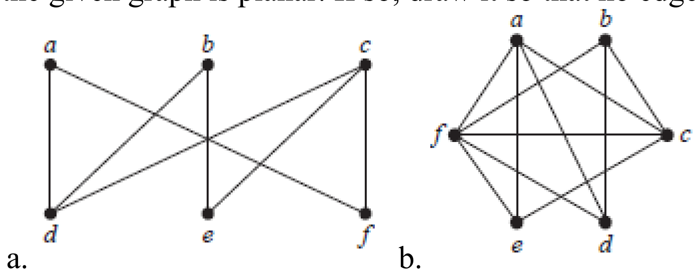


2.Which of these non-planar graphs have the property that the removal of any vertex and all edges incident with that vertex produces with planar graph?

a) K_6 b) $K_{3,3}$

3.Suppose that a connected planar graph has six vertices, each of degree four. Into how many regions is the plane divided by a planar representation of this graph?

4.Determine whether the given graph is planar. If so, draw it so that no edges cross.

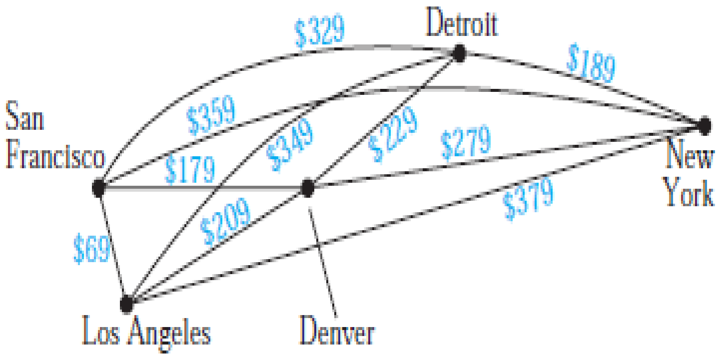


5.A connected planar graph has 10 vertices each of degree 3. Into how many regions does a representation of this planar graph split the plane.

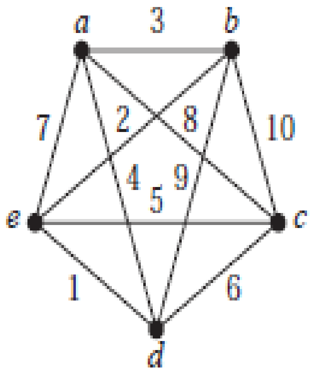
6. Find the chromatic number of the complete bipartite graph Km,n, where m and n are positive integers?

HOME ASSIGNMENT PROBLEMS

1. Find a route with the least total airfare that visits each of the cities in this graph, where the weight on an edge is the least price available for a flight between the two cities.



2. Solve the traveling salesperson problem for this graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



3. Which of these non-planar graphs have the property that the removal of any vertex and all edges incident with that vertex produces with planar graph?
   a) K_5 b) $K_{3, 4}$
4. Can five houses be connected to two utilities without connections crossing?
5. Suppose that a planar graph has k connected components, e edges, and v vertices. Also suppose that the plane is divided into r regions by a planar representation of the graph. Find a formula for r in terms of e, v and k.
6. Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?