VISUALISING THE FLOWS OF ORTHOGONAL POLYNOMIAL EXPANSIONS OF THE RIEMANN Ξ- FUNCTION

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MAY, 2024, VERSION 2.0

Abstract. In a comprehensive paper [1], Dan Romik derives new infinite series expansions for the Riemann $\Xi(t)$ function in three specific families of orthogonal polynomials; the Hermite polynomials, the symmetric Meixner-Pollaczek polynomials and the continuous Hahn polynomials. He also launched the idea to use the Poisson kernel from the theory of orthogonal polynomials to distort the Riemann $\Xi-$function and thereby induce a 'flow' in its zeros that might obey some dynamical evolution law. Building on Romik's work, we initially aimed at visualising only his three flows, however ended up embarking on a more ambitious journey to include all polynomials families in the Askey scheme. We discovered the visualised flows can be categorised into four types each with different attributes. As an important by-product of our study, we discovered specific direct links between the generating function(s) of the Laguerre, Continuous Dual Hahn and Wilson polynomials, and the Fourier cosine kernel of the Riemann $\Xi-$function. We will share does appendix B.

1. Introduction and preliminaries

When a conjecture about the zeros of function is hard to proof, a line of attack could be to use suitable approximations that are easier to handle. Another approach is to distort the function by a smartly chosen parameter and to study the evolution of its zeros with the hope to find possible patterns. During the last decades, these approaches have also been used to attack the Riemann Hypothesis. Early last century, Pólya started work in this direction and although his attempts were unsuccessful to prove the RH, his initial work has spurred the development of many new ideas by a variety of authors (e.g. Turán, De Bruijn, Newman; see chapters 1 of [1] and [2] for a comprehensive historic overview) and has culminated in the Polymath15 project in 2018 [2]. This project comprised of a combination of using existing numerical evidence about the RH, analytic proof and numerical computations. It managed to successfully reduce the range of validity of the RH when distorting the Riemann $Xi-$function with a special parameter and demonstrated how through further numerical verification of the RH, this range could be reduced further. The results make it clear that although we do get closer to the RH this way, a proof would remain out of reach. Both authors of the current paper have been deeply involved on the numerical/computational side of this project and the production of the associated graphs.

Shortly after the Polymath15 project had completed in 2019, Dan Romik published a paper in which he proposed an interesting new line of attack [1]. He started from the work of Turán who explored the more 'natural' idea of expanding the Riemann $\Xi-$function into a series involving orthogonal Hermite polynomials. Romik build on this idea and managed to derive new infinite series expansions for the Riemann Ξ-function, including rigorous asymptotics for their coefficients, in three specific families of orthogonal polynomials; the Hermite polynomials, the symmetric Meixner-Pollaczek polynomials and the continuous Hahn polynomials. He also introduced the provoking idea of using an intrinsic property from the theory of orthogonal polynomials to distort Ξ, the so-called 'Poisson kernel'. This turned out to be a quite natural way to distort the $\Xi-$function and thereby induce a 'Poisson flow' of its real and complex zeros thereby obeying some dynamical evolution law. To quote Romik p19, chapter 2 of [1]: "*(...) the point is that Poisson flows are a method of approximation that allows us considerable freedom in choosing a system of orthogonal polynomials to use, and it is conceivable that this might lead to new families of approximations with useful properties*".

Inspired by Romik's work, we started from the idea to use our learnings from Polymath15 project and to attempt to visualise his newly developed Poisson flows with the hope of gaining some new insights from them. Initially we followed the approach taken by Romik to directly link a generating function of a family of orthogonal polynomials, for a specific set of parameters, to the kernel of the Fourier cosine integral of the Rieman $\Xi-$function. However, we soon found out that this approach seems only viable for a relatively small set of families and only for a specific choice of parameters. The currently known families of orthogonal polynomials have been organised in the so-called Askey scheme, in which they are ranked by increasing complexity and the number of free parameters. Since each family has its own characteristics, we decided to become more ambitious and expanded our scope to visualise the Poisson flows for expansions of Ξ in all families in the Askey-scheme (but excluding the so-called $q-analogues$). Orthogonal polynomials come in two flavours, infinite and finite. We acknowledge that it feels more natural from the 'flow'-perspective, to only focus on the continuous/infinite types, especially since dynamical evolution laws (reflected in PDEs) seem mostly associated with those. However, we know from the Askey scheme that finite and infinite families are all interconnected through some limiting relation, hence we decided to keep the finite approximations in our scope.

Our more ambitious goal forces us to use the generic Fourier form that is independent of the function to be expanded. It therefore has the advantage that we could easily adjust use our code to expand any other sufficiently fast decaying entire function other than Ξ. An apparent disadvantage of this approach is that the computational intense Ξ-function will show up in each coefficient. However, a key insight from the Polymath15 project (p16, remark 4.1 of [2]) is that evaluating Ξ at smaller heights has become quite tractable when using modern software and hardware. A clear disadvantage of our generic expansion approach is that it doesn't open up an easy path towards establishing the asymptotics of the coefficients, nor does it pave the way for a rigorous proof of convergence in $\mathbb{C}$. This paper will therefore be more of an experimental nature and pays less attention to providing rigorous proof. The prime focus will be on visualising the Poisson flows in terms of all orthogonal polynomial families listed in the Askey scheme.

Having said that, during our investigations we have developed a significant amount of "by-products" that expand on Romik's initial work on the three families of orthogonal polynomials. We managed for instance to generalise his Meixner-Pollaczek and Continuous Hahn expansions towards a broader set of parameters than $\frac{4}{3}$ only. We also found similar expansions (i.e. with easy to compute coefficients) for the Laguerre, Continuous Dual Hahn and Wilson polynomial families. For readability purposes, we shared these results and their derivations in the appendices.

To keep our scope manageable, we will focus on visualising the evolution of the real zeros only. Pairs of complex zeros typically occur when two trajectories of real zeros 'collide' at a certain point in 'time'. We do have the tools to visualise the complex zeros as well (requiring a 3D-plot), but leave these out of scope for now. For our computations we require all relevant formulae associated with a specific family of orthogonal polynomials (hypergeometric function, weight, orthogonality relation, leading coefficients, etc.) and for this we will follow the 2010 standard work by Koekoek, Lesky and Swarttouw [5] that has been complemented by Koornwinder's 2022 paper containing new insights, additions and errata [6].

The more complex orthogonal polynomials in the Askey scheme offer an increasing number of free parameters and we found that the choice of parameters does impact the associated flow. It required a bit of trial and error to test how a chosen set of parameters would impact the produced flow. One of our key insights is that the Poisson flows, that are formally defined for the domain $0 < r < 1$, do most of the time remain valid for values of $r \geq 1$ and yield interesting new info. Another learning is that the Poisson flows could be summarised into four flavours that we will explain in more detail in this paper.

2. NAMING CONVENTIONS

The literature about orthogonal polynomials comes with its own historically evolved nomenclature that we will obviously strictly follow. However, some labels of the polynomials are not unique and for the unique labelling of the Poisson flows, we allowed ourselves a bit more freedom to ensure unambiguous and easy reference for the reader. We don't use any special mathematical symbols, but the Pochhammer symbol $(x)_n$ will appear quite often in the paper. This symbol reflects the rising factorial that could also be expressed as $\dfrac{\Gamma(x+n)}{\Gamma(x)}$.

3. Structure of the paper

In the next few chapters we will numerically and visually explore the Poisson flows of the real zeros of infinite series expansions for $\Xi(t)$ in a large number of different families of orthogonal polynomials. We will start in chapter 4 with a short introduction of the Riemann Ξ-function and how it has been distorted to induce the Pólya-De Bruijn flow. We will include a visual of this flow, since the vestige of its characteristic pattern shows up in different shape and forms for orthogonal polynomial expansions. We then introduce the Askey scheme in chapter 5 followed by the Poisson flow in chapter 6. After some computational considerations in chapter 7, we will present our results in a table, chapter 8 and picture format 9. We will conclude the paper with a short discussion and final remarks 10. For the readability of the paper, we have moved the many "by-products" of our investigations as well as their derivations, to the appendices A and B.

4. The Riemann Ξ-function and the Pólya-De Bruijn flow

The Riemann ξ-function:

$$\xi(s) = \frac{s\,(s-1)}{2}\,\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{1}$$

is an entire function that has been studied extensively. Let $\Xi(t)\colon \mathbb{C} \to \mathbb{C}$ denote the function:

$$\Xi(t) = \xi\left(\frac{1}{2} + it\right) \tag{2}$$

This is an entire even function with functional equations $\Xi(t) = \Xi(-t)$ and $\Xi(\bar{t}) = \overline{\Xi(t)}$, and the Riemann hypothesis is equivalent to the assertion that all the zeroes of $\Xi(t)$ are real. It has the well-known Fourier cosine transform:

$$\Xi(t) = \int_{-\infty}^{\infty} \Phi(x)\,\mathrm{e}^{i\,t\,x}\,dx \tag{3}$$

$$= 2\int_{0}^{\infty} \Phi(x)\,\cos(t\,x)\,dx \tag{4}$$

where Φ is the super-exponentially decaying even function:

$$\Phi(x) = 2\sum_{n=1}^{\infty}(2\,\pi^2\,n^4\,\mathrm{e}^{9x/2} - 3\,\pi\,n^2\,\mathrm{e}^{5x/2})\exp(-\pi\,n^2\mathrm{e}^{2x}) \qquad x \in \mathbb{R} \tag{5}$$

The distorting of $\Xi(t)$ by a real parameter λ is done by introducing an additional factor:

$$\Xi_\lambda(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\lambda\,x^2}\,\Phi(x)\,\mathrm{e}^{i\,t\,x}\,\mathrm{d}x \qquad t \in \mathbb{C}, \lambda \in \mathbb{R} \tag{6}$$

and in figure 1 the flow of the real zeros induces by this factor is visualised. It is known that the flows are determined by the time-reversed heat equation reflected in the PDE:

$$\frac{\partial^d}{\partial\lambda^d}\Xi_\lambda(t) = (-1)^d\,\frac{\partial^{2d}}{\partial t^{2d}}\Xi_\lambda(t)$$

with d = the d-th derivative. The graph clearly shows the collision points at a certain λ where they become a pair of complex zeros when going further back in "time".

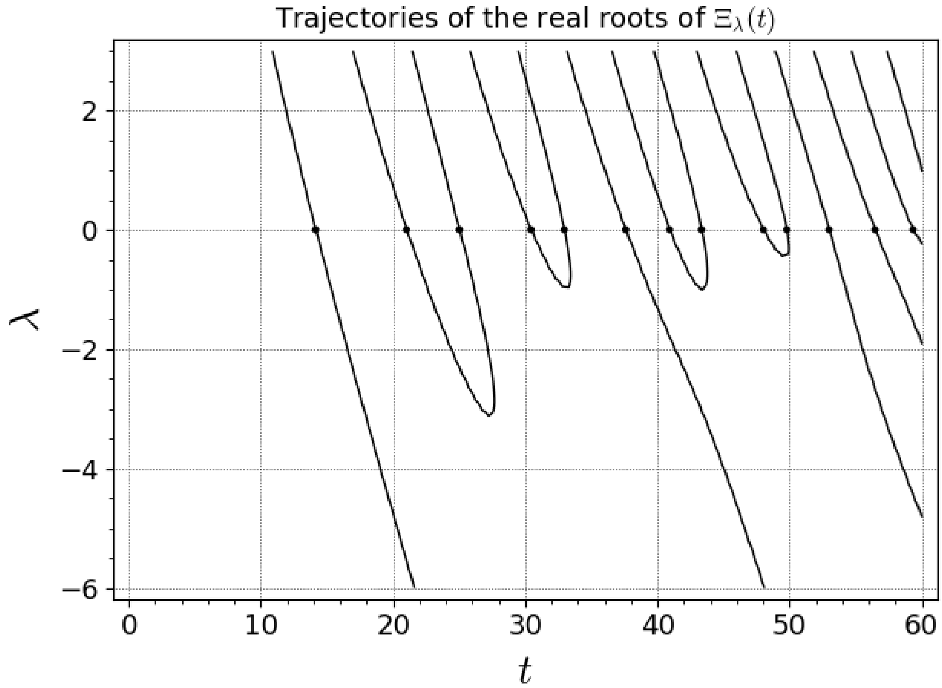

FIGURE 1. The black lines show the flow of the real roots of $\Xi_\lambda(t)$ when λ varies. Some "collisions" occur between the trajectories when $\lambda < 0$ where the zeros become complex pair. The RH is equivalent to no collisions occurring for $\lambda \geq 0$, The ordinates of the non-trivial zeros of $\xi(s)$ are shown as black dots. The concept of "hyperbolicity", i.e. the preservation of the reality of the zeros when time-parameter λ become more positive, is clearly visible in the flows.

- De Bruijn and Newman showed that there exists a constant Λ, now known as the De Bruijn-Newman constant, at which all zeros will be real and that this is also true for all $\lambda > \Lambda$. During the last decades, good progress has been made on the improvement of the lower bound of Λ, however only a minor reduction was scraped off the upper bound (< 0.5 instead of ≤ 0.5).
- In 2018 Tao and Rodgers [7] proved Newman's lower bound conjecture that $\Lambda \geq 0$, They used techniques from Random Matrix theory to predict the 'time to cool down' and compared this to known facts about the distribution of zeros at $\lambda = 0$. A student of Tao, Alex Dobner later found a much simpler proof [8] that $\Lambda \geq 0$ which avoids the heat equation approach. Dobner's approach instead relies on a Riemann-Siegel type approximation for $\Xi_\Lambda(t)$ in order to demonstrate the existence of zeros off the critical line.
- The objective of the 2018 Polymath15 project [2], was to lower the upper bound. This was successfully accomplished through a combination of analytic proof, existing numerical verification of the RH up to a certain height and advanced computer calculations. The new upper bound became $\Lambda = 0.2$. After further numerical verification of the RH by Platt et al [9], it has now been proven that the upper bound of the De Bruijn-Newman constant is $\lambda \leq 0.2$. Hence, a breach of the RH would imply that there exists some pair complex zeros at a certain T that collide in the region $0 \leq \Xi_\lambda(t) \leq 0.2$.
- For increasingly positive λ we know that the trajectories of zeros "cool down" into a deterministic arithmetic progression that is now well understood.
- Increasingly negative λ will induce more pairs of complex zeros that eventually will organise themselves on deterministic curves. A sparse, but regular set of real zeros (see the 1-st and 6-th trajectories in the graph) remains real 'forever' (probably because they are too far apart to be able to collide).

5. Askey scheme

The Askey scheme is a way of organizing orthogonal polynomials of hypergeometric or basic hypergeometric type into a hierarchy (see [5]).

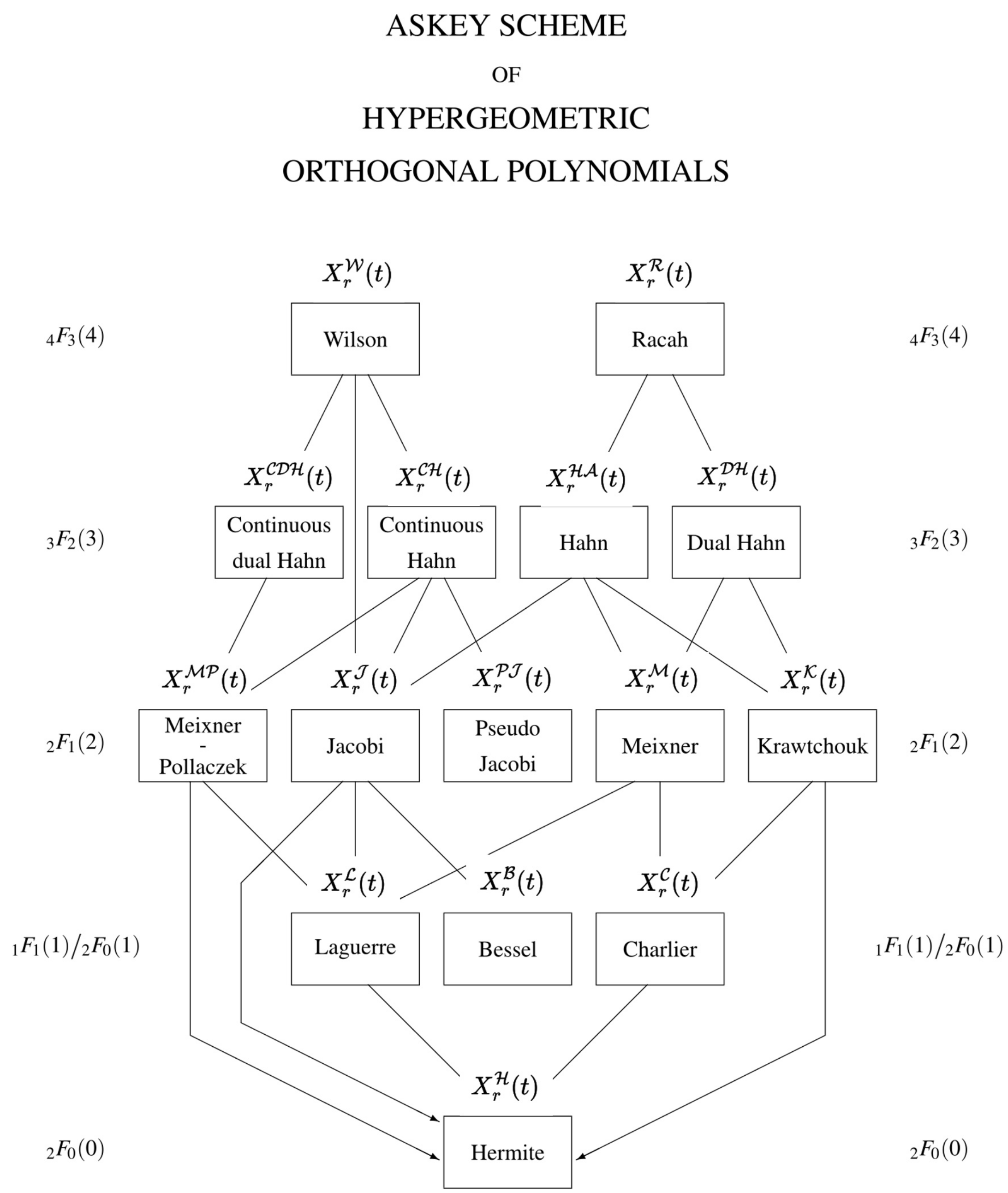

Figure 2. Askey scheme of orthogonal polynomials. The complexity of the polynomials increases from bottom to top which is reflected in the assocated hypergeometric function. The number between parenthesis reflects the number of free parameters. The arrows always flow from top to bottom and denote possible transformations (mostly based on infinite limits) from one family into another. We have added the labels $X_r^{\oplus}(t)$ as an easy and unique reference to the Poisson flow connected to a polynomial family.

6. Another way to distort $\Xi(t)$: The Poisson flow

In paragraph 2.5 of [1], Romik introduces the interesting concept of a ”Poisson flow”, that is based on the idea that any series expansion of the Riemann Ξ-function in a system of orthogonal polynomials comes equipped with its own flow based on the standard construction of the so-called Poisson kernel from the theory of orthogonal polynomials.

We start by denoting $\phi = (\phi_n)_{n=0}^{\infty}$ as the family of polynomials that are orthogonal with respect the weight function $w(x)$. The Poisson flow for $\Xi(\tau)$ is now defined as follows:

$$X_r^{\phi}(t) = \begin{cases} \int_{-\infty}^{\infty} p_r(t,\tau)\,\Xi(\tau)\,w(\tau)\,\mathrm{d}\tau & \text{if } 0 < r < 1 \\ \\ \Xi(t) & \text{if } r = 1 \end{cases} \tag{7}$$

where $p_r(t,\tau)$ is the Poisson Kernel defined as:

$$p_r^{\phi}(x,y) = \sum_{n=0}^{\infty} \frac{r^n}{M_n}\,\phi_n(x)\,\phi_n(y) \tag{8}$$

with $M_n = \int_{\mathbb{R}} \phi_n(x)^2\,w(x)\,\mathrm{d}x$. The quantity M_n is the closed form of the orthogonality relation for a polynomial where the two indices are equal. Note that depending on the polynomial family, the integral could also be an infinite or even finite sum.

Swapping the summation and integration in the Poisson flow (assumming Fubini’s theorem applies), immediately yields the Fourier series expansion in terms of a family of orthogonal polynomials ϕ:

$$X_r^{\phi}(t) = \sum_{n=0}^{\infty} r^n\,\gamma_n\,\phi_n(t) \tag{9}$$

$$\gamma_n = \frac{1}{M_n} \int_{-\infty}^{\infty} \Xi(x)\,\phi_n(x)\,w(x)\,\mathrm{d}x \tag{10}$$

It is this generic structure that we decided to use for our computations, since it allows for the once-off calculation of the coefficients γ_n for a certain set of chosen parameters for the orthogonal polynomial under study. The big advantage is that all orthogonal polynomials in the Askey scheme can now be expressed into a hypergeometric function and a leading factor g that is only dependent on n and the free parameters. This allows us to optimise our pre-computations a bit further:

$$X_r^{\phi}(t) = \sum_{n=0}^{\infty} r^n\,\gamma_n\,g(n,\text{parms})\,{}_pF_q(n,t,\text{parms}) \tag{11}$$

$$\gamma_n = \frac{1}{M_n(\text{parms})} \int_{-\infty}^{\infty} \Xi(x)\,g(n,\text{parms})\,{}_pF_q(n,x,\text{parms})\,w(x)\,\mathrm{d}x \tag{12}$$

Collecting all terms that are independent of r and t, we will pre-compute γ_n as follows:

$$X_r^{\phi}(t) = \sum_{n=0}^{\infty} r^n\,\gamma_n\,{}_pF_q(n,t,\text{parms}) \tag{13}$$

$$\gamma_n = f(n,\text{parms}) \int_{-\infty}^{\infty} \Xi(x)\,{}_pF_q(n,x,\text{parms})\,w(x)\,\mathrm{d}x \tag{14}$$

with $f(n,\text{parms}) = \dfrac{g(n,\text{parms})^2}{M_n(\text{parms})}$

Remark 1: In appendix A, we have applied the above construct to each orthogonal polynomial in the Askey scheme. The resulting list of specific functions for each polynomial family is what we ended up encoding in our software. For Generalised Laguerre, Bessel-y, Jacobi and Pseudo-Jacobi families, we found that computing the $\frac{\partial^d}{\partial t^d} X_r^{\phi}(t,a)$, where d = the d-th partial t-derivative of the flow, could be done almost for "free". For these families we highlighted these extended equations for their Poisson flows.

Remark 2: Another benefit of our approach is that it is independent of the function to be distorted. It is a generic framework in which we in princple could plug in any entire and sufficiently fast decaying function without the need to change our code. We therefore decided to include another function than $\Xi(t)$ in our software and also computed its flows for comparative purposes. But what would be a useful function to compare $\Xi(t)$ to? We know that the non-trivial zeros ρ_n of $\xi(s)$ determine the oscillation (error) term in the prime counting function(s). Analogously, the equidistant zeros $\mu_n = 0 \pm 2\pi n i$ induce the oscillating term (sawtooth shaped) in the integer counting function. The entire function associated with the Hadamard product of these μ's is $\xi_N(s) = \frac{2}{s}\sinh\left(\frac{s}{2}\right)$. Analogously to $\Xi(t)$, we defined $\Xi^N(t) = \xi^N(it) = \frac{t}{2}\sin\left(\frac{t}{2}\right)$. Since the zeros are regularly spaced, all graphs showed parallel flows with the only exception for the Hahn polynomials where collisions also do occur (albeit regularly). These flows will not be included in this paper however could be viewed on our GitHub [12].

Remark 3: Our approach has the additional benefit that for most polynomial families, setting $t = 0$ induces the factor ${}_pF_q(n, t, \text{parms})$ to become 1. Hence, for $r = 1$, the sum of all coefficients γ_n simply becomes $X_r^{\phi}(0)$ and this can be used to ensure that computations of γ_n were done correctly and accurately. This doesn't work for all polynomial families, the Jacobi and Pseudo-Jacobi polynomials require $t = 1$ and $t = i$ respectively. More interestingly are the Meixner-Pollaczek, Continuous Hahn, Continuous Dual Hahn and Wilson polynomials that require $t = \pm ia$ to achieve the same effect. This provides us with a new free parameter to be varied and to induce a different type of flow. As an example, we share the resulting formula for the Meixner-Pollaczek family that uses the closed form of its generating function:

$$\hat{X}_r^{\mathcal{MP}}\left(\lambda, \frac{\pi}{2}\right) = \frac{2^{2\lambda-1}}{\pi\,\Gamma(2\lambda)\,(1-r^2)^{\lambda}} \int_{-\infty}^{\infty} \Xi(x)\,\Gamma(\lambda+ix)\,\Gamma(\lambda-ix)\left(\frac{1-r}{1+r}\right)^{ix} \mathrm{d}x \quad \mathrm{Re}\,(\lambda) > 0 \tag{15}$$

Remark 4: For some families of orthogonal polynomials, the Poisson Kernel $p_r(t,\tau)$ can be expressed into a simpler form. In the literature these are often referred to as expressions for "bilinear sums" or "bilinear generating functions" (see for instance [10] and [11]). Although we did not use these for our computations, we list the resulting expressions for the Hermite (leading to an alternative proof for its direct connection to the Pólya De Bruijn flow) and Laguerre flows as examples:

$$\Xi_\lambda(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Xi\left(t - 2\,i\,x\sqrt{\lambda}\right) \mathrm{e}^{-x^2} \mathrm{d}x \qquad t \in \mathbb{C}, \lambda \in \mathbb{R} \tag{16}$$

$$X_r^{\mathcal{H}}(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Xi\left(r\,t - i\,x\sqrt{r^2-1}\right) \mathrm{e}^{-x^2} \mathrm{d}x \quad \to X_r^{\mathcal{H}}(t) = \Xi_{\frac{r^2-1}{4}}(r\,t) \tag{17}$$

$$X_r^{\mathcal{L}}\left(t, -\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Xi\left(\left(\sqrt{r\,t} - i\,x\sqrt{r-1}\right)^2\right) \mathrm{e}^{-x^2} \mathrm{d}x \tag{18}$$

Remark 5: The disadvantage of our generic approach is that we need evaluate the Riemann $\Xi(t)$ function in the coefficient a significant number of times. It also is much more complicated to establish a good asymptotic for the coefficients of the expansion. A more attractive approach is to take (one of) the generating function(s) of a family of orthogonal polynomials and to link its closed form directly to the kernel of the cosine transform of $\Xi(t)$. Romik showed in his paper how this can be done for the Meixner-Pollaczek and Continuous Hahn families with the parameter(s) fixed at $\frac{3}{4}$. Following the same techniques, we managed to generalise his approach towards a broader set of parameters for these two families. We also found similar expressions for the coefficients of the Generalised Laguerre, Continuous Dual Hahn and Wilson polynomials, albeit for the latter two, the expansions are for $\Xi'(t)/t$ and/or $\Xi(0) \pm \Xi(t)$. Since this approach seems to only work for the "left-hand side" of the Askey-scheme and therefor doesn't enable realising our prime objective, we've captured our findings in appendix A.

7. Computational considerations

Visualising the Poisson flow of the Riemann Ξ-function requires a significant amount of evaluations of the function itself as well as the hypergeometric functions associated with the family of orthogonal polynomials. This might look like a 'daunting' task, however during the polymath15 project we discovered that with modern software and hardware it actually has become quite tractable at smaller heights (p16, remark 4.1 of [2]).

As explained in the previous section, the most intense task at hand is to pre-compute and store the coefficients (for each set of polynomial parameters) that are independent of t, r. This once-off pre-computation makes the evaluation of the flow $X_r^{\phi}(t)$ significantly faster, since only one hypergeometric function remains.

For both steps we have used the advanced FLINT/ARB C-library for arbitrary-precision ball arithmetic [3]. We have also considered using pari/gp and, although much easier to code, it proved to be slower and lacks some of the more advanced hypergeometric functions (${}_3F_2, {}_4F_3$ we need).

For the pre-computation step we require the full FLINT/ARB library and especially the multi-threaded function for the fast evaluation of the integral. This multi-threaded function seems not (yet) available in the 'wrapped', easier readable, version of FLINT/ARB under SageMath, hence we coded it all in "plain ARB".

For the flow visualisation step, we make use of the advanced plotting capabilities of Sagemath. In particular the *implicit plot* function that automatically plots where the contour of a function vanishes. Hence, there is no longer any need for dedicated root finding routines and this again speeds up computations considerably. To allow for easy integration between evaluating and plotting the Poisson flow and since in this step integration is no longer required, we made use of the "wrapped" version of FLINT/ARB under SageMath.

To keep a decent balance between gaining sufficient visual insights and the computations required, we decided to compute the flows up to $t = 60$ which comprises the flows of the first 13 ordinates of the non-trivial zeros at $r = 1$. We have set a target accuracy at $t = 60, r = 1$ of 20 digits, to allow for some contingency to 'peek' beyond that point if needed. The main tuning parameters to achieve this accuracy are the number of pre-computed coefficients, the precision settings in the software and the finite limits of the integrals/sums involved. Finding the right settings was a process of trial and error, but to ensure we got it right an automatic check was performed that $\Xi(60) - X_1^{\phi}(60)$ shows at least 20 zeros and that the sum of all pre-computed coefficients at $t = 0$ adds up to the correct value at the same accuracy.

Furthermore, we have verified the correct working our FLINT/ARB code by reproducing it for some random values in Maple™[4] and/or pari/gp.

For the hardware we have used an Apple Mac Studio, M2 ultra, 64Gb, using all its 24 cores to the max.

All documented software, including the accuracy settings that we have used to pre-compute the coefficients and to produce the visuals, is freely available from our GitHub [12]. All pre-computed coefficients are stored there as well.

8. Overview of the results

The visualisations of the Poisson flow for each family have been included in the next section. There appear to be four types of flows that can be loosely categorized as follows:

- *Type A:* Collisions seem to only occur for $r < 1$ (this resembles the Pólya-De Bruijn flow).
- *Type B:* Collisions seem to only occur for $r > 1$ (a "vertically flipped" Pólya-De Bruijn flow).
- *Type C:* Collisions don't occur at all, flows seems to remain parallel forever independent of r.
- *Type D:* After initial swings, the flow oscillates closer and closer around $r = 1$ when t increases.

In the table below we summarize which flows we observe for each family. We also include an observation about the behaviour of the coefficients (i.e. whether they are always positive/negative/alternating and/or increase/decrease monotone).

TABLE 1. Overview of our findings ordered by flow type

Polynomial	Flow	Type	Coefficients
Hermite	$X_r^{\mathcal{H}}(t)$	A	Odd index = 0, all positive real, monotone decreasing
Laguerre	$X_r^{\mathcal{L}}(t)$	A	Alternating with decreasing frequency, no pattern observed
Charlier	$X_r^{\mathcal{C}}(t)$	A	Alternating with decreasing frequency, no pattern observed
Meixner	$X_r^{\mathcal{M}}(t)$	A	Alternating with decreasing frequency, no pattern observed
Krawtchouk	$X_r^{\mathcal{K}}(t)$	A	Alternating without pattern, absolute value Gaussian shaped
Dual Hahn	$X_r^{\mathcal{DH}}(t)$	A	Alternating with decreasing frequency, absolute value Gaussian shaped
Meixner Pollaczek	$X_r^{\mathcal{MP}}(t)$	B	Odd index = 0, all positive real, monotone decreasing
Cont. Dual Hahn	$X_r^{\mathcal{CDH}}(t)$	B	All positive real, monotone decreasing
Cont. Hahn	$X_r^{\mathcal{CH}}(t)$	B	Odd index = 0, all positive real, monotone decreasing
Wilson	$X_r^{\mathcal{W}}(t)$	B,D	All positive real, monotone decreasing
Bessel-y	$X_r^{\mathcal{B}}(t)$	C	Alternating + + - -, monotone decreasing
Pseudo-Jacobi	$X_r^{\mathcal{PJ}}(t)$	C	All complex, real part positive, imaginary part alternating + -
Jacobi	$X_r^{\mathcal{J}}(t)$	D	Odd index = 0 when $\alpha = \beta$, alternating + -, monotone decreasing
Hahn	$X_r^{\mathcal{HA}}(t)$	D	Alternating with increasing frequency, no pattern observed
Racah	$X_r^{\mathcal{R}}(t)$	D	Alternating with decreasing frequency, absolute value Gaussian shaped

9. THE POISSON FLOWS VISUALISED

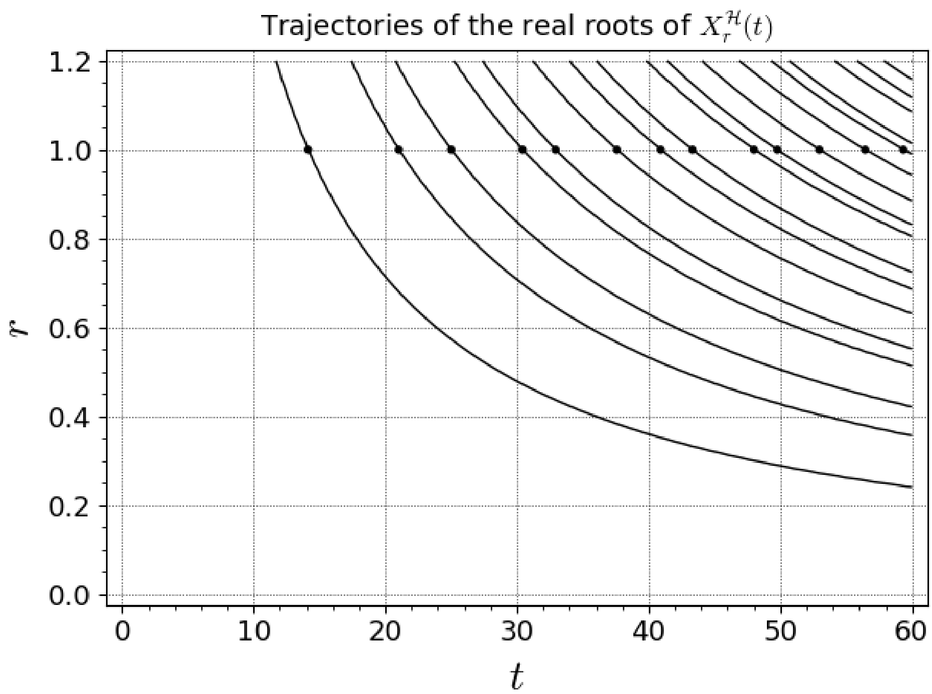

FIGURE 3. This graph shows the Poisson flow of the Hermite polynomial expansion. We know this flow is a simple transform of the Pólya De Bruijn flow and collisions between trajectories of real zeros will occur at higher t. We have done the actual computations to trace the 27th and 28th non-trivial zero (the first pair to collide in the Polya de Bruijn flow in the domain $-0.25 < \lambda < 0$) and indeed see their collision occurring around $t = 277, r = 0.34$.

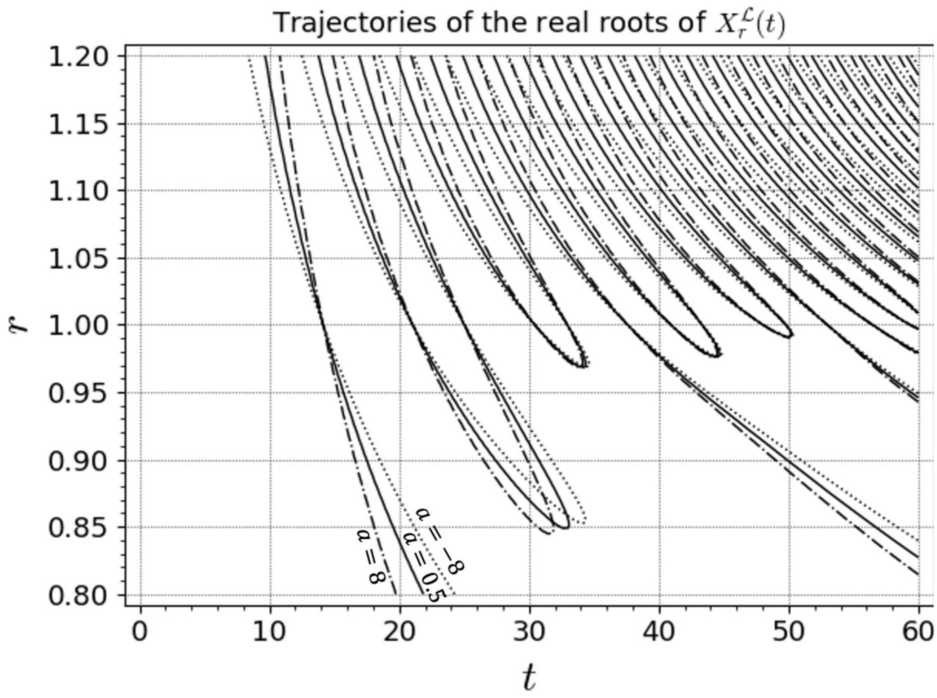

FIGURE 4. This graph shows the Poisson flow of the Generalised Laguerre polynomial expansion for the parameters $\alpha = -8, 0.5, 8$. We see similar patterns as in the Pólya De Bruijn flow, but these cannot be connected directly. The change of the parameter induces a rotation around $r = 1$. For $\alpha > 0$ the rotation is clock wise, $\alpha < 0$ goes counter clockwise. It would interesting to learn more about extreme rotations $\alpha \to \pm\infty$.

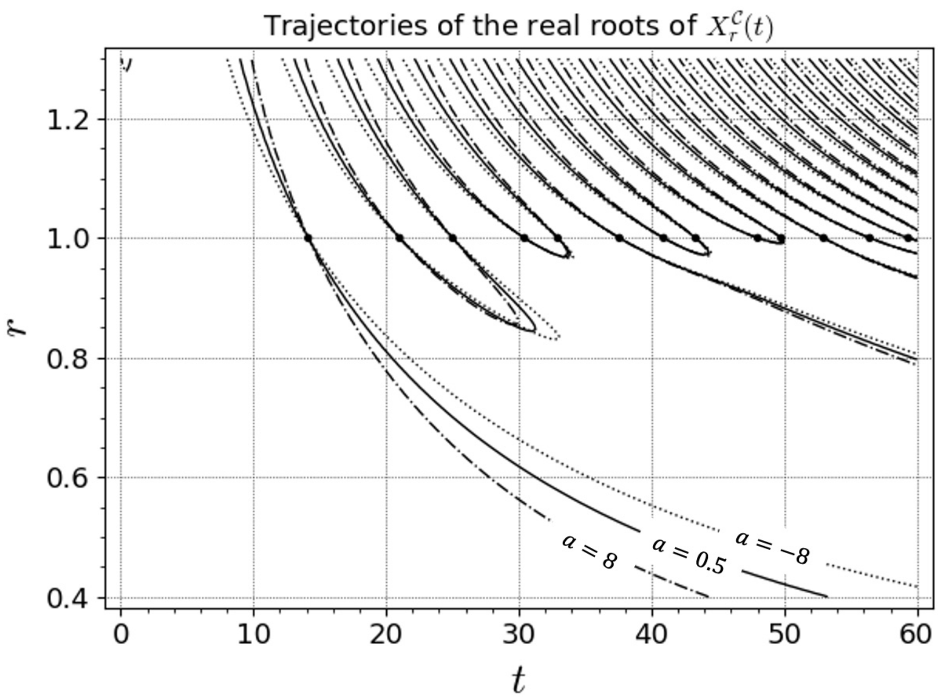

FIGURE 5. This graph shows the Poisson flow of the Charlier polynomial expansion for the parameters $\alpha = -8, 0.5, 8$. The flow looks very similar to the Laguerre flow including the rotations that are induced by changing the parameter α.

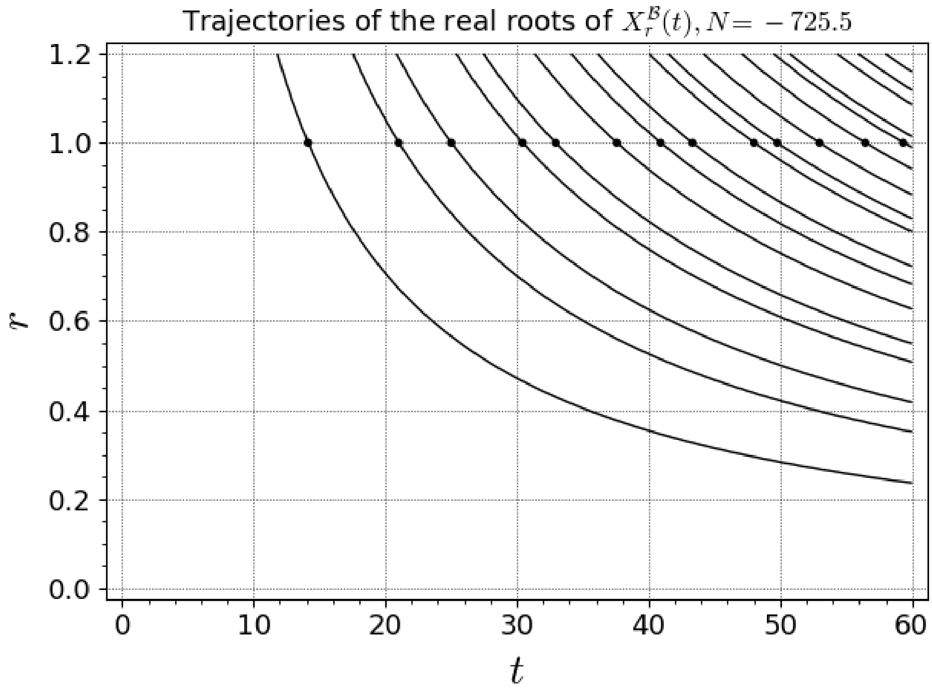

FIGURE 6. This graph shows the Poisson flow of the Bessel y polynomial expansion for a fixed set of parameters. The trajectories do not seem to collide neither in the r or the t directions. We computed both to considerable heights and continue to see parallel trajectories. This obviously doesn't mean collisions can't occur at greater heights.

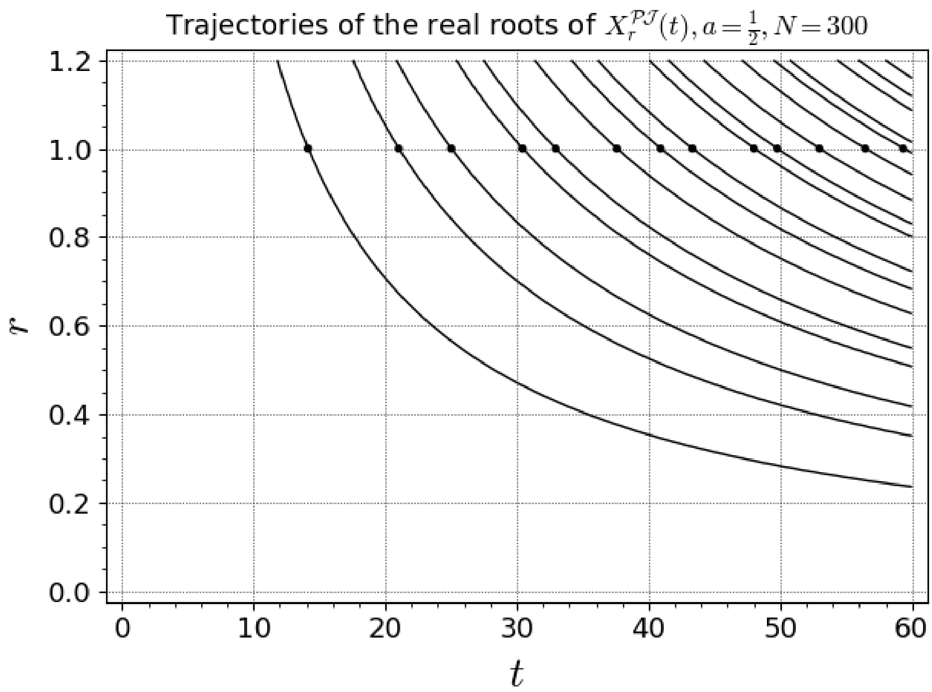

FIGURE 7. This graph shows the Poisson flow of the Pseudo Jacobi polynomial expansion for a fixed set of parameters. We observe a similar pattern without any collisions as the Bessel y expansion. We can't rule out collision do occur at some greater heights outside the reach of our computations.

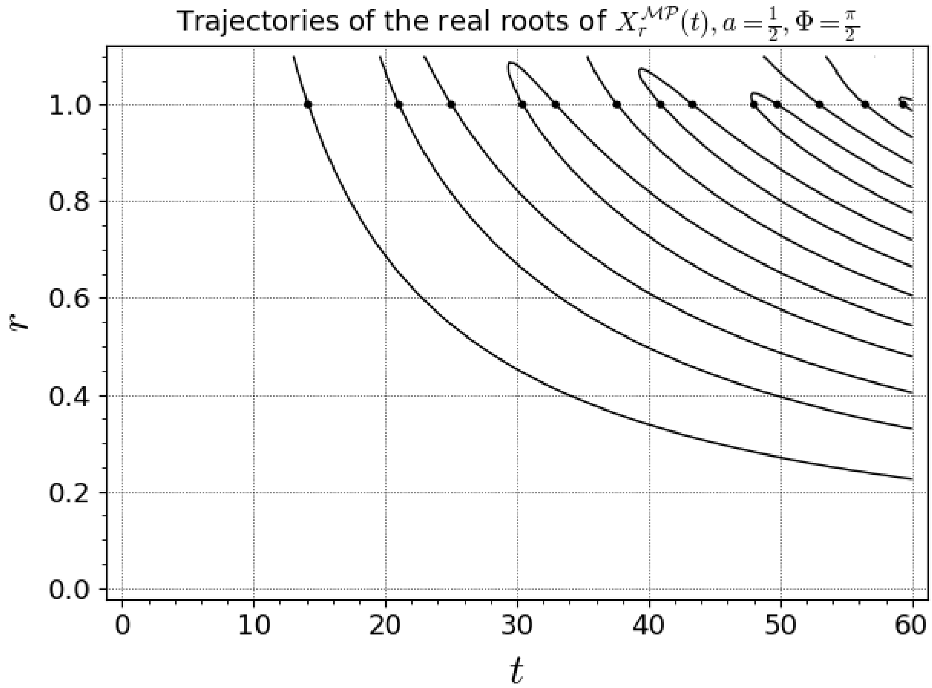

FIGURE 8. This graph shows the Poisson flow of the Meixner Pollaczek polynomial expansion for a fixed set of parameters. Collisions seem to only occur for $r > 1$ and their flows seem to be like a "vertically flipped" version of the Pólya De Bruijn flow.

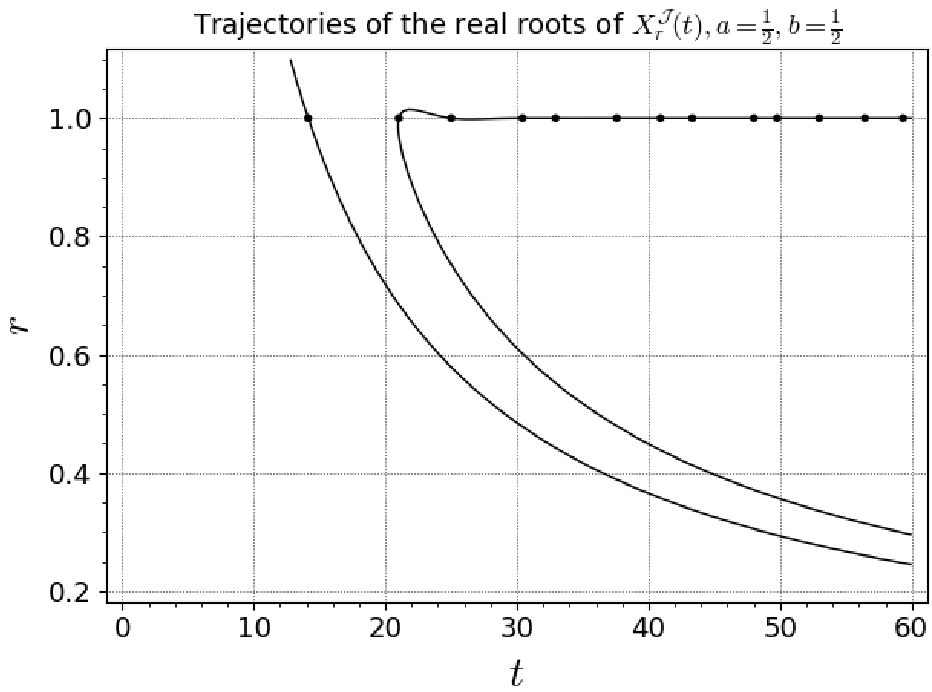

FIGURE 9. This graph shows the Poisson flow of the Jacobi polynomial expansion for a fixed set of parameters. This flow seems to oscillate with a rapidly decreasing amplitude around the line $r = 1$. We made several runs with different sets of parameters, including those for the Gegenbauer/Ultraspherical ($\alpha = \beta = \lambda - \frac{1}{2}$), Chebyshev ($\alpha = \beta = -\frac{1}{2}$) and Legendre/Spherical ($\alpha = \beta = 0$) polynomials, however without much difference.

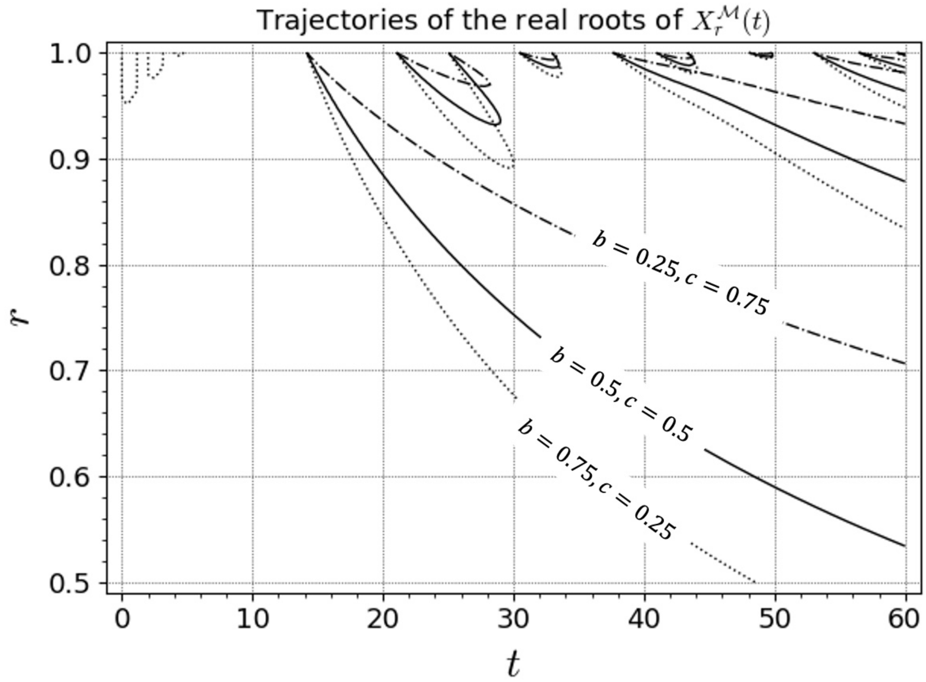

FIGURE 10. This graph shows the Poisson flow of the Meixner polynomial expansion for different sets of parameters. We again see the reflection of the Pólya De Bruijn flow. Changing the parameters induces a rotation clock and anti-clockwise. It would be interesting to better understand what happens when b, c go to $\pm\infty$

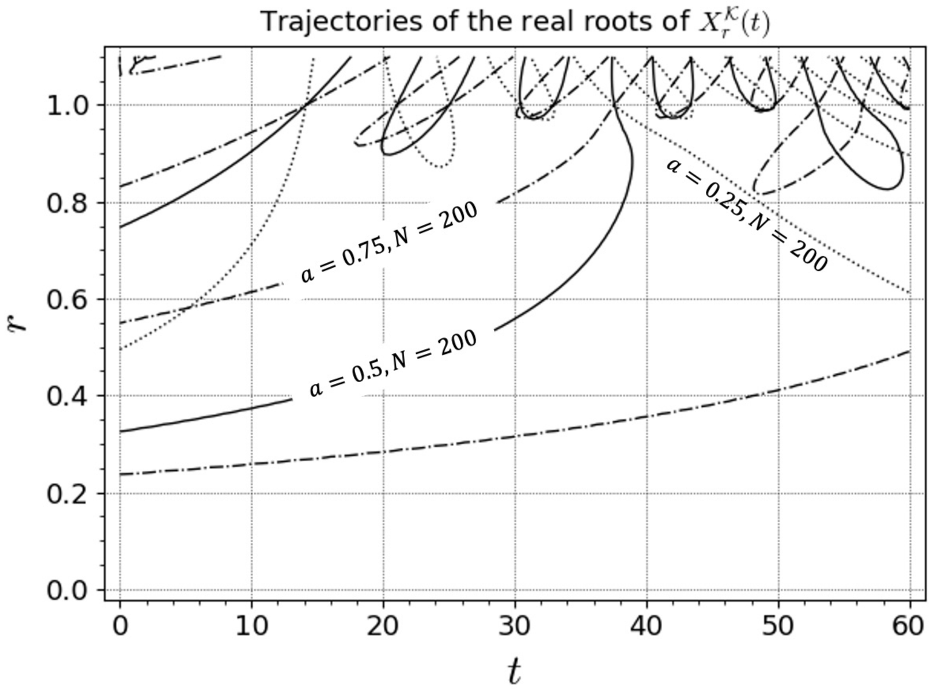

FIGURE 11. This graph shows the Poisson flow of the Krawtchouk polynomial expansion for different sets of parameters. We again see the vestige of the Pólya De Bruijn flow. Changing the parameters induces a rotation clock and anti-clockwise. It would be interesting to better understand what happens when a go to $\pm\infty$

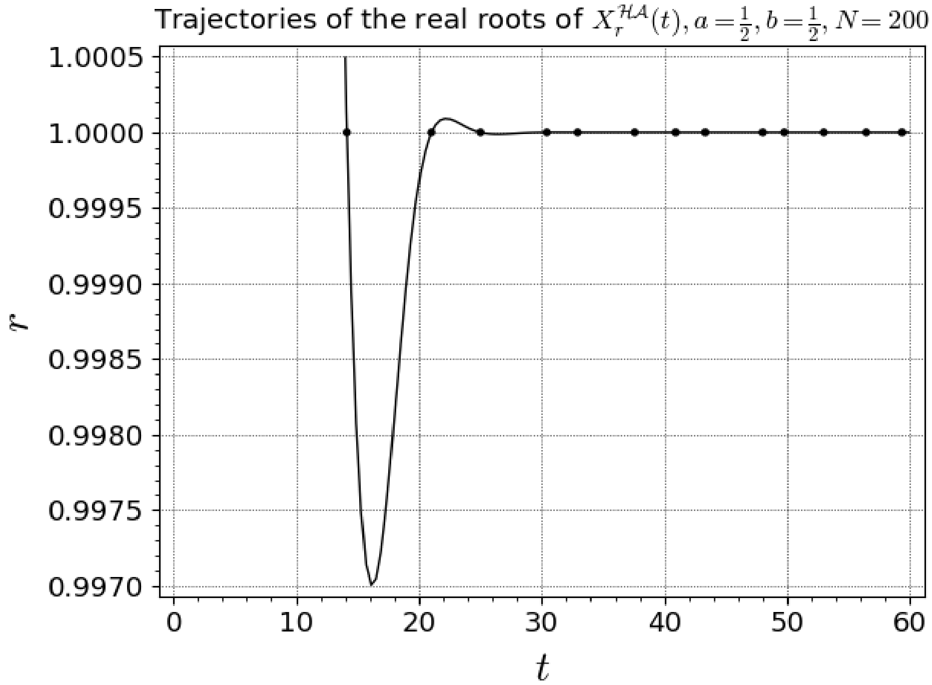

FIGURE 12. This graph shows the Poisson flow of the Hahn polynomial expansion for a fixed set of parameters. After an initial swing the flow starts to oscillate closer and closer around the line $r = 1$. Changing the parameter seems to only influence the amplitude of the initial swing.

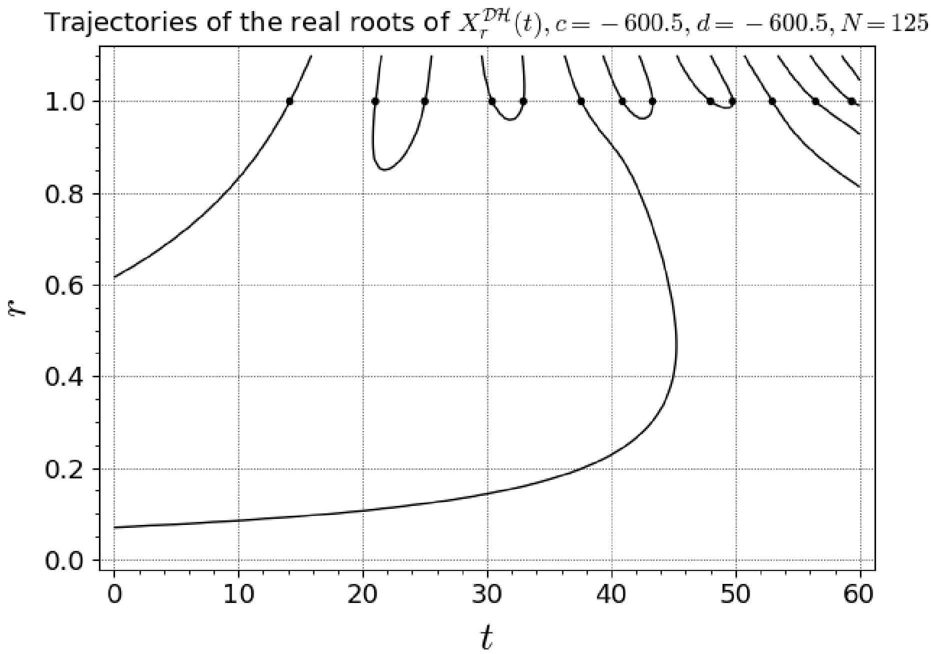

FIGURE 13. This graph shows the Poisson flow of the Dual Hahn polynomial expansion for a fixed set of parameters. We again see a similar flow pattern as the Pólya De Bruijn flow including the sparse set of trajectories that never collide.

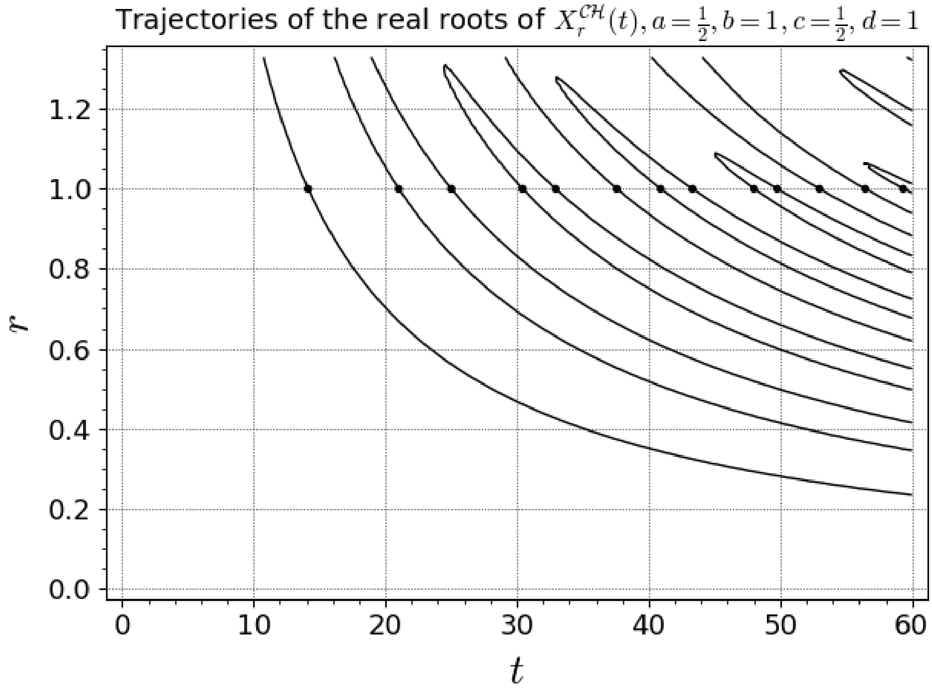

FIGURE 14. This graph shows the Poisson flow of the Continuous Hahn polynomial expansion for a fixed set of parameters. We see a "vertically flipped" Pólya De Bruijn flow. When t increases the flows line up vertically into an arithmetic progression. In theory this might be used to count the number of zeros between $0 < r < 1$ at point T, which should be equal to the number of zeros in the critical strip up till T.

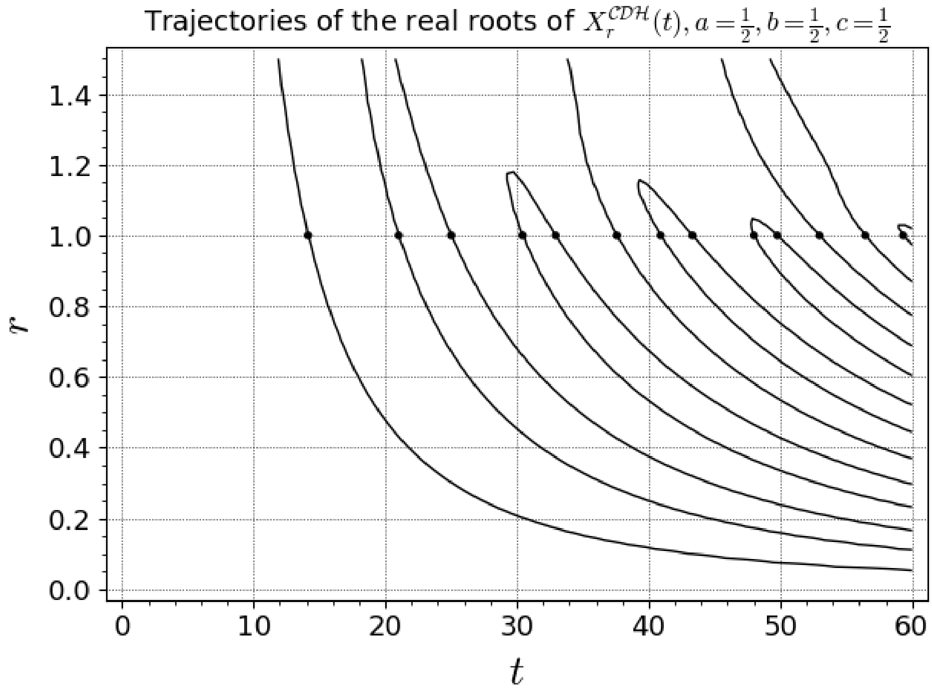

FIGURE 15. This graph shows the Poisson flow of the Continuous Dual Hahn polynomial expansion for a fixed set of parameters. We again see a "vertically flipped" Pólya De Bruijn flow. When t increases the flows line up vertically into an arithmetic progression. See also the comment under the Continuous Hahn flow.

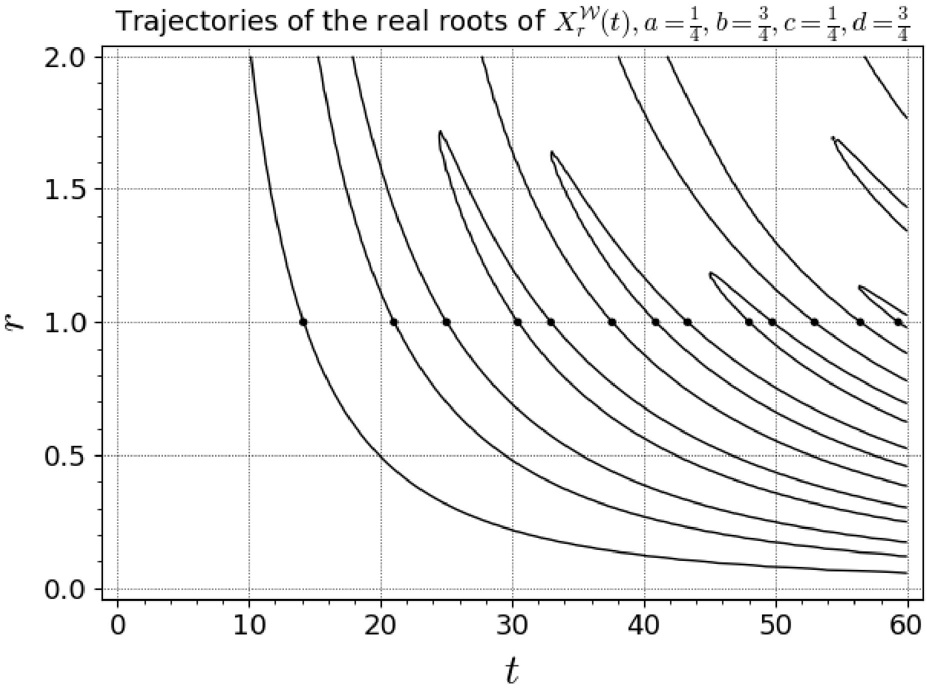

FIGURE 16. This graph shows the Poisson flow of the Wilson polynomial expansion for a fixed set of parameters. We once again see a "vertically flipped" Pólya De Bruijn flow, however it is very senstive to our choice of parameters. When we set those to f.i. $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$, the flow changes dramatically and, after a few initial swings, quickly starts to oscillate closer and closer around the line $r = 1$.

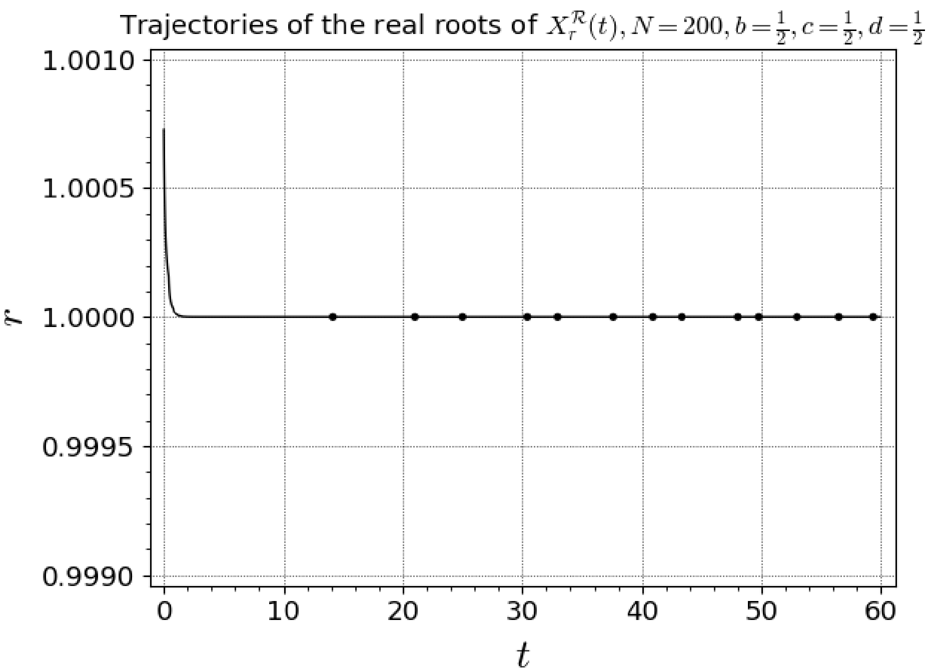

FIGURE 17. This graph shows the Poisson flow of the Racah polynomial expansion for a fixed set of parameters. This flow seems to oscillate extremely close around the line $r = 1$, so close that we can't even visualise the first wave (its amplitude must at least be smaller than 10^{-16}. Changing the parameters did not have any impact.

10. Discussion and concluding remarks

We have shown that visualising all Poisson flows in the Askey scheme is computationally tractable, albeit for limited heights of the $t-$ parameter. The visuals do provide an interesting new perspective on evolution of the real zeros of $\Xi(t)$ when expanded in a specific family of polynomials. We also learned more about the sensitivity of the observed flows to changes in parameters. Below we like summarise a few of our key insights.

- We start on a negative, but realistic note. The common characteristic of all Poisson flows is that they start off with a distortion/approximation of the original function $\Xi(t)$ at $r = 1$ where the complexity seems always highest. For each flow we observe that information seems to flow from this origin and could help explain what happens in the less complex 'distorted' domain ($r \neq 1$) where trajectories either collide or converge towards an arithmetic progression (or always stay parallel). Unfortunately, we do not (yet) see any information gained from this 'distorted' domain telling us more about the distribution of the zeros at $r = 1$.

- However, on a positive note, we find it really encouraging to see that the visualised flows for each expansion can be categorised into four very different types and this raises new questions. Why do we see a "vertically flipped" version of the Pólya De Bruijn flow for the Meixner-Pollaczek, Continuous (dual) Hahn and Wilson polynomials on the left side of the Askey scheme? Do the trajectories of the real zeros expanded in the Bessel y and Pseudo Jacobi polynomials indeed never collide in either the r or the t direction? What forces the Poisson flow to oscillate around the line $r = 1$ and fade out so quickly for the Jacobi, Hahn and Racah polynomials?

- Romik concludes in section 3.5 of [1], that the differential difference equation (DDE) that underpins the dynamical evolution of the real zeros of the Meixner Pollaczek expansion no longer preserves the property of "hyperbolicity" (the property of an entire function of having no non-real zeros) in the positive direction of the "time" (r) parameter. However, the visualised Poisson flow suggests that a form of "hyperbolicity" might well be preserved, but into the *negative* direction of "time". Would this be within the reach of a proof and an associated "De Bruijn Newman constant"? And could something similar be done for the Continuous Hahn flow?

- We found that the Pólya De Bruijn, Hermite, Meixner Pollaczek and Continuous Hahn flows share many similar properties. For this group we can directly link their generating functions to the kernel of the Fourier cosine integral and more importantly their odd indexed coefficients are all zero. In chapter 7 "Final remarks" of [1], Romik makes a similar observation and then considers some "toy" expansions for these polynomials for which all roots are known to be real. He indeed manages to show the (trivial) reality of the zeros, except for the Continuous Hahn polynomials. He raises the question whether anything could be said about the reality of the zeros of the g_n polynomial expansion (i.e. the Continuous Hahn polynomials with parameters $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}$). We found that with a slight change of parameters, it becomes quite easy to show that all zeros of the alternating generating function of the Continuous Hahn polynomial are real and regularly spaced for all $0 < r < 1$. To demonstrate this we start from definition [ref] and have:

$$
\begin{aligned}
(19)\qquad & \sum_{n=0}^{\infty}(-1)^n \frac{(2)_{2n}}{\left(\frac{3}{2}\right)_{2n}} \frac{r^{2n}}{(2n)!}\, p_{2n}\left(t, \frac{1}{2}, 1, \frac{1}{2}, 1\right) \\
&= \frac{1}{2}\left(\frac{1}{(1-r)^2}\, {}_2F_1\left(\left[1, \frac{1}{2} - it\right], [1], -\frac{4r}{(1-r)^2}\right) + \frac{1}{(1+r)^2}\, {}_2F_1\left(\left[1, \frac{1}{2} - it\right], [1], \frac{4r}{(1+r)^2}\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{(1-r)^2}\left(\left(\frac{r+1}{r-1}\right)^2\right)^{-\frac{1}{2}+t\mathrm{I}} + \frac{1}{(1+r)^2}\left(\left(\frac{r-1}{r+1}\right)^2\right)^{-\frac{1}{2}+t\mathrm{I}}\right)
\end{aligned}
$$

This results in the full list of "toy" expansions and the closed forms for their $n-$th zero (all real):

TABLE 2. "Toy" expansions and the reality of their zeros

Polynomial	Toy expansion for $0 < r < 1$	Formula for n-th zero, $n \in \mathbb{Z}$
"Polya De Bruijn"	$\sum_{n=0}^{\infty}(-1)^n \frac{r^{2n}}{(2n)!} t^{2n}$	$z_n(r) = \frac{\pi\left(n-\frac{1}{2}\right)}{r}$
Hermite	$\sum_{n=0}^{\infty}(-1)^n \frac{r^{2n}}{(2n)!} H_{2n}(t)$	$z_n(r) = \frac{\pi\left(n-\frac{1}{2}\right)}{2r}$
Meixner Pollaczek	$\sum_{n=0}^{\infty}(-1)^n r^{2n} P_{2n}^{\left(\frac{3}{4}\right)}\left(t, \frac{\pi}{2}\right)$	$z_n(r) = \frac{\pi\left(n-\frac{1}{2}\right)}{\log\left(\frac{1+r}{1-r}\right)}$
Continuous Hahn	$\sum_{n=0}^{\infty}(-1)^n \frac{2n+1}{\left(\frac{3}{2}\right)_{2n}} r^{2n} p_{2n}\left(t, \frac{1}{2}, 1, \frac{1}{2}, 1\right)$	$z_n(r) = \frac{\pi\left(n-\frac{1}{2}\right)}{\log\left(\frac{r+1}{r-1}\right) - \log\left(\frac{r-1}{r+1}\right)}$

- There appear to be only a limited set of orthogonal polynomials whose generating function(s) can be directly linked to the kernel of the Fourier cosine transform of $\Xi(t)$. We have included the derivation of those in appendix B. The table below is a simplified version of our findings in which we only show the integral part of the coefficients we found. It is intriguing to see how the expansions of $\Xi(t)$ seem to be become infeasible beyond the Continuous Hahn family (the Continuous Dual Hahn family is a bit of a "naughty" one). Why do the expansions of $\Xi'(t)/t$ and $\Xi(0) \pm \Xi(t)$ suddenly become feasible for the more complex polynomial families?

TABLE 3. Simplified coefficients for specific expansions

Polynomial	Integral part of the coefficient γ_n	Used to expand:
Hermite	$\int_{-\infty}^{\infty} \Phi(x)\, x^n\, \mathrm{e}^{-x^2/4}\, \mathrm{d}x$	$\Xi(t)$
Generalised Laguerre (α)	$\int_{-\infty}^{\infty} \Phi(x) \left(\frac{ix}{ix+1}\right)^n \left(\frac{1}{ix+1}\right)^{\alpha+1} \mathrm{d}x$	$\Xi(t)$
Meixner Pollaczek $\left(\lambda, \frac{\pi}{2}\right)$	$\int_{-\infty}^{\infty} \Phi(x) \tanh\left(\frac{x}{2}\right)^n \frac{\mathrm{e}^{\lambda x}}{(\mathrm{e}^x+1)^{2\lambda}}\, \mathrm{d}x$	$\Xi(t)$
Continuous Hahn $\left(\frac{1}{2}, 1, \frac{1}{2}, 1\right)$	$\int_{-\infty}^{\infty} \Phi(x) \tanh\left(\frac{x}{4}\right)^n \operatorname{sech}\left(\frac{x}{4}\right)^2 \mathrm{d}x$	$\Xi(t)$
Continuous Hahn (a, a, a, a)	$\int_{-\infty}^{\infty} \Phi(x) \tanh\left(\frac{x}{2}\right)^n \frac{\mathrm{e}^{ax}}{(\mathrm{e}^x+1)^{2a}} h_n(x,a)\, \mathrm{d}x$	$\Xi(t)$
Continuous Dual Hahn $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$	$\int_{0}^{\infty} \Phi(x) \tanh\left(\frac{x}{2}\right)^{2n} \operatorname{sech}\left(\frac{x}{2}\right) \mathrm{d}x$	$\Xi(t)$
Continuous Dual Hahn $\left(a, 1, \frac{1}{2}\right)$	$\int_{0}^{\infty} \Phi(x) \tanh\left(\frac{x}{2}\right)^{2n+1} \frac{x\,\mathrm{e}^{ax}}{(\mathrm{e}^x-1)^{2a}}\, \mathrm{d}x$	$\Xi'(t)/t$
Wilson $\left(0, \frac{1}{2}, \frac{1}{2}, 1\right)$	$\int_{0}^{\infty} \Phi(x) \tanh\left(\frac{x}{4}\right)^{2n} \frac{x}{\coth\left(\frac{x}{4}\right)} \mathrm{d}x$	$\Xi'(t)/t$
Wilson $\left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$	$\int_{0}^{\infty} \Phi(x) \tanh\left(\frac{x}{4}\right)^{2n} \mathrm{d}x$	$\Xi(0) + \Xi(t)$
Wilson $\left(\frac{1}{2}, 1, \frac{1}{2}, 1\right)$	$\int_{0}^{\infty} \Phi(x) \tanh\left(\frac{x}{4}\right)^{2n+2} \mathrm{d}x$	$(\Xi(0) - \Xi(t))/t^2$

where $h_n(x,a) = {}_2F_1\left(\left[a + \frac{n}{2}, a + \frac{n+1}{2}\right], \left[2a + n + \frac{1}{2}\right], \tanh\left(\frac{x}{2}\right)^2\right)$ and $\Phi(x)$ is defined in (5).

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Appendices

Appendix A. Equations used for computing the Poisson Flows

Legend:

$$\begin{aligned}
M_n(...) &= \text{the closed form of the orthogonality relation when } n = m.\\
w_x(...) &= \text{the weight function.}\\
f_n(...) &= \Big(\text{terms before the hypergeom of the polynomial}\Big)^2 / M_n(...)\\
X_r^{\phi}(t, ...) &= \text{the flow for polynomial family } \phi. \text{ Optional : } (d) = d\text{-th derivative.}\\
\gamma_n(...) &= \text{coefficients of the expansion of the flow.}
\end{aligned}$$

The Pólya-De Bruijn Flow

$$\Xi_\lambda(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Xi\left(t - 2\,i\,x\,\sqrt{\lambda}\right) \mathrm{e}^{-x^2} \mathrm{d}x \qquad t \in \mathbb{C}, \lambda \in \mathbb{R} \tag{A.1}$$

Poisson Flow for the Hermite polynomials

$$H_n(t) = i^n \, \frac{2^n\,\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}} \, {}_1F_1\left(-\frac{n}{2}, \frac{1}{2}, t^2\right) \tag{A.2}$$

$$M_n(a) = \sqrt{\pi}\, 2^n\, n! \tag{A.3}$$

$$w_x = \mathrm{e}^{-x^2} \tag{A.4}$$

$$f_n = \frac{2\,\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\pi\,\Gamma(\frac{n}{2} + 1)} \tag{A.5}$$

$$X_r^{\mathcal{H}}(t) = \sum_{n=0}^{\infty} \gamma_{2n}\, {}_1F_1\left(-n, \frac{1}{2}, t^2\right) r^{2n} \tag{A.6}$$

$$\gamma_n = f_n \int_0^{\infty} \Xi(x)\, w_x\, {}_1F_1\left(-\frac{n}{2}, \frac{1}{2}, x^2\right) \mathrm{d}x \tag{A.7}$$

$$\sum_{n=0}^{\infty} \gamma_n = \Xi(0) \tag{A.8}$$

Poisson Flow for the Generalised Laguerre polynomials

$$L_n^a(x) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1(-n, a+1, x) \tag{A.9}$$

$$M_n(a) = \frac{\Gamma(\alpha + 1 + n)}{\Gamma(n+1)} \tag{A.10}$$

$$w(x, a) = \mathrm{e}^{-x}\, x^a \tag{A.11}$$

$$f_n(a) = \frac{\Gamma\,(n + a + 1)}{\Gamma(a+1)^2\,\Gamma(n+1)} \tag{A.12}$$

$$X_r^{\mathcal{L}\,(d)}(t, a) = \frac{a}{(a)_{d+1}} \sum_{n=0}^{\infty} \gamma_n(a)\,(-n)_d\, {}_1F_1\,(d - n, a + d + 1, t)\; r^n \tag{A.13}$$

$$\gamma_n(a) = f_n(a) \int_{0^+}^{\infty} \Xi(x)\, w(x, a)\, {}_1F_1\,(-n, a+1, x)\; \mathrm{d}x \tag{A.14}$$

$$\sum_{n=0}^{\infty} \gamma_n(a) = \Xi(0) \tag{A.15}$$

Poisson Flow for the Bessel y_n polynomials

(A.16) $$y_n(x,a) = {}_2F_0\left(-n, n+a+1, [], -\frac{x}{2}\right)$$

(A.17) $$M_n(a) = \exp\left(\frac{1}{x}\right)\exp\left(\frac{i\sqrt{2xt-1}}{x}\right)$$

(A.18) $$w_x(a) = \mathrm{e}^{-\frac{2}{x}}\, x^a$$

(A.19) $$f_n(a) = -\frac{(2n+a+1)}{2^{a+1}\,\Gamma(-n-a)\,\Gamma(n+1)}$$

(A.20) $$X_r^{\mathcal{B}(d)}(t,a) = \frac{(-1)^d}{2^d}\sum_{n=0}^{\infty}\gamma(n,a)\,\frac{(n+a)_{d+1}\,(-n)_d}{n+a}\,{}_2F_0\left([d-n, n+a+d+1], [], -\frac{t}{2}\right) r^n$$

(A.21) $$\gamma(n,a) = f_n(a)\int_{0^+}^{\infty}\Xi(x)\,w_x(a)\,{}_2F_0\left([-n, n+a+1], [], \frac{-x}{2}\right)\mathrm{d}x$$

(A.22) $$\sum_{n=0}^{\infty}\gamma_n(a) = \Xi(0)$$

Poisson Flow for the Charlier polynomials

(A.23) $$C_n(x,a) = {}_2F_0\left([-n,-x], [], \frac{1}{a}\right)$$

(A.24) $$M_n(a) = a^{-n}\,\mathrm{e}^a\, n!$$

(A.25) $$w_x(a) = \frac{a^x}{\Gamma(x+1)}$$

(A.26) $$f_n(a) = \frac{a^n\,\mathrm{e}^{-a}}{n!}$$

(A.27) $$X_r^{C}(t,a) = \sum_{n=0}^{\infty}\gamma(n,a)\,{}_2F_0\left([-n,-t], [], -\frac{1}{a}\right) r^n$$

(A.28) $$\gamma_n(a) = f_n(a)\sum_{x=0}^{\infty}\Xi(x)\,w_x(a)\,{}_2F_0\left([-n,-x], [], -\frac{1}{a}\right)\mathrm{d}x$$

(A.29) $$\sum_{n=0}^{\infty}\gamma_n(a) = \Xi(0)$$

Poisson Flow for the Meixner Pollaczek polynomials with $\Phi = \frac{\pi}{2}$

(A.30) $$P_n^{(\lambda)}\left(x, \frac{\pi}{2}\right) = i^n\,\frac{(2\lambda)_n}{n!}\,2F_1\left([-n, \lambda + i\,x], [2\lambda], 2\right)$$

(A.31) $$M_n(\lambda) = \frac{2\,\pi\,\Gamma(n+2\lambda)}{2^{2\lambda}\,n!}$$

(A.32) $$w_x(\lambda) = \Gamma(\lambda+ix)\,\Gamma(\lambda-ix)$$

(A.33) $$f_n(\lambda) = \frac{4^{\lambda}\,\Gamma(n+2\lambda)}{2\,\pi\,\Gamma(2\lambda)^2\,\Gamma(n+1)}$$

(A.34) $$X_r^{\mathcal{MP}}(t,\lambda) = \sum_{n=0}^{\infty}\gamma_{2n}(\lambda)\,{}_2F_1\left([-2n, \lambda+it], [2\lambda], 2\right) r^{2n}$$

(A.35) $$\gamma_n(\lambda) = f_n(\lambda)\int_{-\infty}^{\infty}\Xi(x)\,w_x(\lambda)\,{}_2F_1\left([-n, \lambda+ix], [2\lambda], 2\right)\mathrm{d}x$$

(A.36) $$\sum_{n=0}^{\infty}\gamma_n(\lambda) = \Xi(i\lambda)$$

Poisson Flow for the Jacobi polynomials

$$P_n(\alpha,\beta,x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left([-n, n+\alpha+\beta+1], [\alpha+1], \frac{1-x}{2}\right) \tag{A.37}$$

$$M_n(\alpha,\beta) = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\,\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)\,n!} \tag{A.38}$$

$$w_x(\alpha,\beta) = (1-x)^{\alpha}\,(1+x)^{\beta} \tag{A.39}$$

$$f_n(\alpha,\beta) = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\alpha+\beta+1)\,(2n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\,\Gamma(\alpha+1)^2\,\Gamma(n+1)\,\Gamma(n+\beta+1)} \tag{A.40}$$

$$X_r^{\mathcal{J}\,(d)}(t,\alpha,\beta) = \frac{(-1)^d\,\alpha}{2^d\,(\alpha)_{d+1}}\sum_{n=0}^{\infty}\gamma_n(\alpha,\beta)\,\frac{(-n)_d\,(n+\alpha+\beta)_{d+1}}{n+\alpha+\beta} \tag{A.41}$$

$$\times\,{}_2F_1\left([d-n, n+\alpha+\beta+d+1], [\alpha+d+1], \frac{1-t}{2}\right) r^n \tag{A.42}$$

$$\gamma_n(\alpha,\beta) = f_n(\alpha,\beta)\int_{-1^+}^{1^-}\Xi(x)\,w_x(\alpha,\beta)\,{}_2F_1\left([-n, n+\alpha+\beta+1], [\alpha+d+1], \frac{1-x}{2}\right)\mathrm{d}x \tag{A.43}$$

$$\sum_{n=0}^{\infty}\gamma_n(\alpha,\beta) = \Xi(1) \tag{A.44}$$

Poisson Flow for the Pseudo Jacobi polynomials

$$P_n(x,\nu,N) = \frac{(-2i)^n\,(-N+i\nu)_n}{(n-2N-1)_n}\,{}_2F_1\left([-n, n-2N-1], [-N+i\nu], \frac{1-it}{2}\right) \tag{A.45}$$

$$M_n(\nu,N) = \frac{2\pi\,\Gamma(2N+1-2n)\,\Gamma(2N+2-2n)\,2^{2n-2N-1}\,n!}{\Gamma(2N+2-n)\,|\Gamma(N+1-n+i\nu)|^2} \tag{A.46}$$

$$w_x(\nu,N) = \left(1+x^2\right)^{-N-1}\,\mathrm{e}^{2a\,\arctan(x)} \tag{A.47}$$

$$f_n(\nu,N) = (-1)^n\,\frac{4^N\,(2N+1-2n)\,\Gamma(n-N+i\nu)^2\,|\Gamma(N+1-n+i\nu)|^2}{\pi\,n!\,\Gamma(2N+2-n)\,\Gamma(-N+i\nu)^2} \tag{A.48}$$

$$X_r^{\mathcal{PJ}\,(d)}(t,\nu,N) = \frac{(-i)^d}{2^d\,(-N+i\nu)_d}\sum_{n=0}^{\infty}\gamma_n(\nu,N)\,(-n)_d\,(n-2N-1)_d \tag{A.49}$$

$$\times\,{}_2F_1\left([d-n, n-2N-1+d], [d-N+i\nu], \frac{1-it}{2}\right) r^n \tag{A.50}$$

$$\gamma_n(\nu,N) = f_n(\nu,N)\int_{-\infty}^{\infty}\Xi(x)\,w_x(\nu,N)\,{}_2F_1\left([-n, n-2N-1], [-N+i\nu], \frac{1-ix}{2}\right)\mathrm{d}x \tag{A.51}$$

$$\sum_{n=0}^{\infty}\mathrm{Re}\,(\gamma_n(\nu,N)) = \Xi(i),\quad \sum_{n=0}^{\infty}\mathrm{Im}\,(\gamma_n(\nu,N)) = 0 \tag{A.52}$$

Poisson Flow for the Meixner polynomials

$$M_n(x,\beta,c) = {}_2F_1\left([-n,-x], [\beta], 1-\frac{1}{c}\right) \tag{A.53}$$

$$\hat{M}_n(\beta,c) = \frac{c^{-n}\,n!}{(\beta)_n\,(1-c)^{\beta}} \tag{A.54}$$

$$w_x(\beta,c) = \frac{\Gamma(x+\beta)}{\Gamma(\beta)\,\Gamma(x+1}\,c^x \tag{A.55}$$

$$f_n(\beta,c) = \frac{\Gamma(n+\beta)}{\Gamma(\beta)\,n!}\,(1-c)^b\,c^n \tag{A.56}$$

$$X_r^{\mathcal{M}}(t,\beta,c) = \sum_{n=0}^{\infty}\gamma_n(\beta,c)\,{}_2F_1\left([-n,-t], [\beta], 1-\frac{1}{c}\right) r^n \tag{A.57}$$

$$\gamma_n(\beta,c) = f_n(\beta,c)\sum_{x=0}^{\infty}\Xi(x)\,w_x(\beta,c)\,{}_2F_1\left([-n,-x], [\beta], 1-\frac{1}{c}\right)\mathrm{d}x \tag{A.58}$$

$$\sum_{n=0}^{\infty}\gamma_n(\beta,c) = \Xi(0) \tag{A.59}$$

Poisson Flow for the Krawtchouk polynomials

(A.60) $$K_n(x,p,N) = {}_2F_1\left([-n,-x],[-N],\frac{1}{p}\right)$$

(A.61) $$M_n(p,N) = \frac{(-1)^n\,n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n$$

(A.62) $$w_x(p,N) = \frac{\Gamma(N+1)}{\Gamma(x+1)\,\Gamma(N-x+1}\,(1-p)^{N-x}\,p^x$$

(A.63) $$f_n(p,N) = \frac{\Gamma(n-N)}{\Gamma(-N)\,\Gamma(n+1)}\,(1-p)^{-n}\,p^n$$

(A.64) $$X_r^{\mathcal{K}}(t,p,N) = \sum_{n=0}^{N}\gamma_n(p,N)\,{}_2F_1\left([-n,-t],[-N],\frac{1}{p}\right)\,r^n$$

(A.65) $$\gamma_n(p,N) = f_n(p,N)\sum_{x=0}^{N}\Xi(x)\,w_x(p,N)\,{}_2F_1\left([-n,-x],[-N],\frac{1}{p}\right)\,\mathrm{d}x$$

(A.66) $$\sum_{n=0}^{\infty}\gamma_n(p,N) = \Xi(0)$$

Poisson Flow for the Continuous Dual Hahn polynomials

(A.67) $$S_n(x^2;a,b,c) = (a+b)_n(a+c)_n\,{}_3F_2([-n,a+ix,a-ix],[a+b,a+c],1)$$

(A.68) $$M_n(a,b,c) = 2\pi\,n!\,\Gamma(n+a+b)\,\Gamma(n+a+c)\,\Gamma(n+b+c)$$

(A.69) $$w_x(a,b,c) = \left|\frac{\Gamma(a+ix)\,\Gamma(b+ix)\,\Gamma(c+ix)}{\Gamma(2ix)}\right|^2$$

(A.70) $$f_n(a,b,c) = \frac{\Gamma(n+a+b)\,\Gamma(n+a+c)}{2\,\pi\,\Gamma(a+b)^2\,\Gamma(a+c)^2\,\Gamma(n+1)\,\Gamma(n+b+c)}$$

(A.71) $$X_r^{\mathcal{CDH}}(t,a,b,c) = \sum_{n=0}^{\infty}\gamma_n(a,b,c)\,{}_3F_2\,([-n,a+it,a-it],[a+b,a+c],1)\;r^n$$

(A.72) $$\gamma(n,a,b,c) = f_n(a,b,c)\int_0^{\infty}\Xi(x)\,w_x(a,b,c)\,{}_3F_2\,([-n,a+ix,a-ix],[a+b,a+c],1)\;\mathrm{d}x$$

(A.73) $$\sum_{n=0}^{\infty}\gamma_n(a,b,c) = \Xi(\pm ia)$$

Poisson Flow for the Continuous Hahn polynomials

(A.74) $$p_n(t,a,b,c,d) = i^n\,\frac{(a+c)_n\,(a+d)_n}{n!}\,{}_3F_2\,([-n,n+a+b+c+d-1,a+it],[a+c,a+d],1)$$

(A.75) $$M_n(a,b,c,d) = \frac{\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)}{(2n+a+b+c+d-1)\,\Gamma(n+a+b+c+d-1)\,n!}$$

(A.76) $$w_x(a,b,c,d) = \Gamma(a+ix)\,\Gamma(b+ix)\,\Gamma(c-ix),\Gamma(d-ix)$$

(A.77) $$f_n(a,b,c,d) = \frac{\Gamma(n+a+c)\,\Gamma(n+a+d)\,\Gamma(n+a+b+c+d-1)\,(2n+a+b+c+d-1)}{2\,\pi\,\Gamma(a+c)^2,\Gamma(a+d)^2\,\Gamma(n+1)\,\Gamma(n+b+c)\,\Gamma(n+b+d)}$$

(A.78) $$X_r^{\mathcal{CH}}(t,a,b,c,d) = \sum_{n=0}^{\infty}\gamma_n(a,b,c,d)\,{}_3F_2\,([-n,n+a+b+c+d-1,a+it],[a+c,a+d],1)\;r^n$$

(A.79) $$\gamma(n,a,b,c,d) = f_n(a,b,c,d)\int_0^{\infty}\Xi(x)\,w_x(a,b,c,d)$$

(A.80) $$\times\,{}_3F_2\,([-n,n+a+b+c+d-1,a+ix],[a+c,a+d],1)\;\mathrm{d}x$$

(A.81) $$\sum_{n=0}^{\infty}\gamma_n(a,b,c,d) = \Xi(ia)$$

Poisson Flow for the Hahn polynomials

$$Q_n(t,\alpha,\beta,N) = {}_3F_2\left([-n, n+\alpha+\beta+1, -t], [\alpha+1, -N], 1\right) \tag{A.82}$$

$$M_n(\alpha,\beta,N) = \frac{(-1)^n\,(n+\alpha+\beta+1)_{N+1}\,(\beta+1)_n\,n!}{(2n+\alpha+\beta+1)\,(\alpha+1)_n\,(-N)_n\,N!} \tag{A.83}$$

$$w_x(\alpha,\beta,N) = \frac{\Gamma(x+\alpha+1)\,\Gamma(-x+\beta+N+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(x+1)\,\Gamma(N-x+1)} \tag{A.84}$$

$$f_n(\alpha,\beta,N) = \frac{(2n+\alpha+\beta+1)\,\Gamma(\beta+1)\,\Gamma(n+\alpha+1)\,\Gamma(n-N)\,\Gamma(n+\alpha+\beta+1)\,\Gamma(N+1)}{(-1)^n\,\Gamma(-N)\,\Gamma(\alpha+1)\,\Gamma(n+1)\,\Gamma(n+\beta+1)\,\Gamma(n+\alpha+\beta+N+2)} \tag{A.85}$$

$$X_r^{\mathcal{HA}}(\alpha,\beta,N) = \sum_{n=0}^{N} \gamma_n(\alpha,\beta,N)\,{}_3F_2\left([-n, n+\alpha+\beta+1, -t], [\alpha+1, -N], 1\right)\, r^n \tag{A.86}$$

$$\gamma_n(\alpha,\beta,N) = f_n(\alpha,\beta,N) \sum_{x=0}^{N} \Xi(x)\, w_x(\alpha,\beta,N)\,{}_3F_2\left([-n, n+\alpha+\beta+1, -x], [\alpha+1, -N], 1\right)\, \mathrm{d}x \tag{A.87}$$

$$\sum_{n=0}^{\infty} \gamma_n(\alpha,\beta,N) = \Xi(0) \tag{A.88}$$

Poisson Flow for the Dual Hahn polynomials

$$R_n(\lambda(x),\gamma,\delta,N) = {}_3F_2\left([-n, -x, x+\gamma+\delta+1], [\gamma+1, -N], 1\right) \tag{A.89}$$

$$\lambda(z) = z(z+\gamma+\delta+1)$$

$$M_n(\gamma,\delta,N) = \frac{\Gamma(\gamma+1)\,\Gamma(\delta+1)\,\Gamma(-n+N+1)\,n!}{\Gamma(n+\gamma+1)\,\Gamma(\delta-n+N+1)} \tag{A.90}$$

$$w_x(\gamma,\delta,N) = \frac{(2x+\gamma+\delta+1)\,\Gamma(\delta+1)\,\Gamma(x+\gamma+1)\,\Gamma(x-N)\,\Gamma(x+\gamma+\delta+1)\,\Gamma(N+1)}{(-1)^x\,\Gamma(-N)\,\Gamma(\gamma+1)\,\Gamma(x+1)\,\Gamma(x+\delta+1)\,\Gamma(x+\gamma+\delta+N+2)} \tag{A.91}$$

$$f_n(\gamma,\delta,N) = \frac{\Gamma(n+\gamma+1)\,\Gamma(-n+\delta+N+1)}{\Gamma(\gamma+1)\,\Gamma(\delta+1)\,\Gamma(n+1)\,\Gamma(N-n+1)} \tag{A.92}$$

$$X_r^{\mathcal{DH}}(t,\gamma,\delta,N) = \sum_{n=0}^{N} \gamma_n(\gamma,\delta,N)\, R_n(\lambda(t),n,\gamma,\delta,N)\, r^n \tag{A.93}$$

$$\gamma(n,\gamma,\delta,N) = f_n(\gamma,\delta,N) \sum_{x=0}^{N} \Xi(x)\, w_x(\gamma,\delta,N)\, R_n(\lambda(x),n,\gamma,\delta,N)\, \mathrm{d}x \tag{A.94}$$

$$\sum_{n=0}^{\infty} \gamma_n(\gamma,\delta,N) = \Xi(0) \tag{A.95}$$

Poisson Flow for the Wilson polynomials

$$\begin{aligned} W_n(y^2;a,b,c,d) &= (a+b)_n(a+c)_n(a+d)_n \\ &\quad \times {}_4F_3([-n, n+a+b+c+d-1, a+iy, a-iy], [a+b, a+c, a+d], 1) \end{aligned} \tag{A.96}$$

$$\begin{aligned} M_n(a,b,c,d) &= \frac{2\pi\,(n+a+b+c+d-1)_n n!\Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+a+d)}{\Gamma(2n+a+b+c+d)} \\ &\quad \times \Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d) \end{aligned} \tag{A.97}$$

$$w_x(a,b,c,d) = \left|\frac{\Gamma(a+ix)\,\Gamma(b+ix)\,\Gamma(c+ix)\;\Gamma(d+ix)}{\Gamma(2ix)}\right|^2 \tag{A.98}$$

$$\begin{aligned} f_n(a,b,c,d) &= \frac{\Gamma(n+a+b)\,\Gamma(n+a+c)\,\Gamma(n+a+d)\,\Gamma(n+a+b+c+d-1)}{2\pi\Gamma(a+b)^2\Gamma(a+c)^2\Gamma(a+d)^2\Gamma(n+1)\Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d)} \\ &\quad \times (2n+a+b+c+d-1) \end{aligned} \tag{A.99}$$

$$X_r^{\mathcal{W}}(t,a,b,c,d) = \sum_{n=0}^{\infty} \gamma_n(a,b,c,d)\,{}_4F_3\left(\left[-n, n+2, \frac{1}{2}+it, \frac{1}{2}-it\right], \left[\frac{3}{2}, 1, \frac{3}{2}\right], 1\right) r^n \tag{A.100}$$

$$\begin{aligned} \gamma(n,a,b,c,d) &= f_n(a,b,c,d) \int_0^{\infty} \Xi(x)\, w_x(a,b,c,d) \\ &\quad \times {}_4F_3\left(\left[-n, n+2, \frac{1}{2}+ix, \frac{1}{2}-ix\right], \left[\frac{3}{2}, 1, \frac{3}{2}\right], 1\right) \mathrm{d}x \end{aligned} \tag{A.101}$$

$$\sum_{n=0}^{\infty} \gamma_n(a,b,c,d) = \Xi(\pm ia) \tag{A.102}$$

Poisson Flow for the Racah polynomials

$$R_n(\lambda(z),\beta,\gamma,\delta,N) = {}_4F_3\left([-n, n-N-1+\beta+1, -z, z+\gamma+\delta+1], [-N, \gamma+1, \beta+\delta+1], 1\right) \tag{A.103}$$

$$\alpha+1 = -N, \quad \lambda(z) = z(z+\gamma+\delta+1)$$

$$M_n(\beta,\gamma,\delta,N) = \frac{(-\beta)_N\,(\gamma+\delta+2)_N\,(n-N+\beta)_n\,(-N+\beta-\gamma)_n\,(-N-\delta)_n\,(\beta+1)_n\,n!}{(\gamma-\beta+1)_N\,(\delta+1)_N\,(1-N+\beta)_{2n}\,(-N)_n\,(\beta+\delta+1)_n\,(\gamma+1)_n} \tag{A.104}$$

$$w_x(\beta,\gamma,\delta,N) = \frac{(2x+\gamma+\delta+1)\,(\gamma+1)_x\,(\beta+\delta+1)_x\,(\gamma+\delta+2)_x\,(-N)_x}{(x+\gamma+d+1)\,(N+\gamma+\delta+2)_x\,(\gamma-\beta+1)_x\,(\delta+1)_x\,\Gamma(x+1)} \tag{A.105}$$

$$f_n(\beta,\gamma,\delta,N) = \frac{(\gamma-\beta+1)_N\,(\delta+1)_N\,(-N+\beta+1)_{2n}\,(-N)_n\,(\beta+\delta+1)_n\,(\gamma+1)_n}{(-\beta)_N\,(\gamma+\delta+2)_N,(n-N+\beta)_n\,(-N+\beta-\gamma)_n\,(-N-\delta)_n\,(\beta+1)_n\,n!} \tag{A.106}$$

$$X_r^{\mathcal{R}}(t,\beta,\gamma,\delta,N) = \sum_{n=0}^{N} \gamma_n(\beta,\gamma,\delta,N)\,R_n(\lambda(t),\beta,\gamma,\delta,N)\,r^n \tag{A.107}$$

$$\gamma_n(\beta,\gamma,\delta,N) = f_n(\beta,\gamma,\delta,N) \sum_{x=0}^{N} \Xi(x)\,w_x(\beta,\gamma,\delta,N)\,R_n(\lambda(x),\beta,\gamma,\delta,N)\,\mathrm{d}x \tag{A.108}$$

$$\sum_{n=0}^{\infty} \gamma_n(\beta,\gamma,\delta,N) = \Xi(0) \tag{A.109}$$

(A.110)

Appendix B. A list of orthogonal polynomial expansions specific to the Riemann $\Xi-$function

B.1. **Derivation of an expansion of $\Xi(t)$ in terms of Generalised Laguerre polynomials.** We start from the definition of the generalised Laguerre polynomial family:

$$L_n^{\alpha}(t) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1(-n, \alpha+1, t) \tag{B.1}$$

Take its generating function and apply the mapping $x \mapsto \dfrac{ix}{1+ix}$:

$$\sum_{n=0}^{\infty} L_n^{\alpha}(t)\,x^n = \frac{1}{(1-x)^{\alpha+1}}\,\exp\left(-\frac{tx}{1-x}\right) \qquad |x|<1 \tag{B.2}$$

$$\sum_{n=0}^{\infty} L_n^{\alpha}(t)\left(\frac{ix}{1+ix}\right)^n = \mathrm{e}^{-itx}\left(\frac{1}{1+xi}\right)^{-\alpha-1} \qquad |x|\in\mathbb{R} \tag{B.3}$$

Our target integral representation of $\Xi(t) = \Xi(-t)$ is:

$$\Xi(t) = \int_{-\infty}^{\infty} \Phi(x)\,\mathrm{e}^{-itx}\,\mathrm{d}x \tag{B.4}$$

$$= \int_{-\infty}^{\infty} \Phi(x)\,\mathrm{e}^{-itx}\left(\frac{1}{1+xi}\right)^{-\alpha-1}\left(\frac{1}{1+xi}\right)^{\alpha+1}\mathrm{d}x \tag{B.5}$$

$$= \int_{-\infty}^{\infty} \Phi(x)\left(\sum_{n=0}^{\infty} L_n^{\alpha}(t)\left(\frac{ix}{1+ix}\right)^n\right)\left(\frac{1}{1+xi}\right)^{\alpha+1}\mathrm{d}x \tag{B.6}$$

$$= \sum_{n=0}^{\infty} L_n^{\alpha}(t)\int_{-\infty}^{\infty} \Phi(x)\left(\frac{ix}{1+ix}\right)^n\left(\frac{1}{1+xi}\right)^{\alpha+1}\mathrm{d}x \tag{B.7}$$

where $\Phi(x)$ is defined in equation (5).The Poisson flow becomes:

$$X_r^{\mathcal{L}}(t,\alpha) = \sum_{n=0}^{\infty} \gamma_n(\alpha)\, {}_1F_1\,(-n, \alpha+1, t)\; r^n \tag{B.8}$$

$$\gamma_n(\alpha) = \frac{(\alpha+1)_n}{n!} \int_{-\infty}^{\infty} \Phi(x) \left(\frac{ix}{ix+1}\right)^n \left(\frac{1}{ix+1}\right)^{a+1} \mathrm{d}x$$

B.2. **Derivation of an expansion of $\Xi(t)$ in terms of Meixner Pollaczek polynomials.** We start from the definition of the Meixner Pollaczek polynomials:

$$P_n^{(\lambda)}\left(t, \frac{\pi}{2}\right) = i^n \frac{(2\lambda)_n}{n!}\, 2F_1\,([-n, \lambda + i\,t], [2\lambda], 2) \tag{B.9}$$

Take its generating function and apply the mapping: $x \mapsto i\,\dfrac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}$:

$$\sum_{n=0}^{\infty} P_n^{(\lambda)}\left(t, \frac{\pi}{2}\right) x^n = (1 - i\,x)^{-\lambda + i\,t}\,(1 + i\,x)^{-\lambda - i\,t} \qquad |x| < 1 \tag{B.10}$$

$$\sum_{n=0}^{\infty} P_n^{(\lambda)}\left(t, \frac{\pi}{2}\right) \left(i\,\frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\right)^n = \mathrm{e}^{-itx}\, 4^{-\lambda}\, \mathrm{e}^{-\lambda\,x}\,(\mathrm{e}^x + 1)^{2\lambda} \qquad |x| \in \mathbb{R} \tag{B.11}$$

Our target integral representation of $\Xi(t) = \Xi(-t)$ is:

$$\Xi(t) = \int_{-\infty}^{\infty} \Phi(x)\, \mathrm{e}^{-itx}\, \mathrm{d}x \tag{B.12}$$

$$= 4^{\lambda} \int_{-\infty}^{\infty} \Phi(x)\, \mathrm{e}^{-itx}\, 4^{-\lambda}\, \mathrm{e}^{-\lambda\,x}\,(\mathrm{e}^x + 1)^{2\lambda}\, \frac{\mathrm{e}^{\lambda\,x}}{(\mathrm{e}^x + 1)^{2\lambda}}\, \mathrm{d}x \tag{B.13}$$

$$= 4^{\lambda} \int_{-\infty}^{\infty} \Phi(x) \left(\sum_{n=0}^{\infty} i^n\, P_n^{(\lambda)}\left(t, \frac{\pi}{2}\right) \left(\frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\right)^n\right) \frac{\mathrm{e}^{\lambda\,x}}{(\mathrm{e}^x + 1)^{2\lambda}}\, \mathrm{d}x \tag{B.14}$$

$$= 4^{\lambda} \sum_{n=0}^{\infty} i^n\, P_n^{(\lambda)}\left(t, \frac{\pi}{2}\right) \int_{-\infty}^{\infty} \Phi(x)\, \tanh\left(\frac{x}{2}\right)^n \frac{\mathrm{e}^{\lambda\,x}}{(\mathrm{e}^x + 1)^{2\lambda}}\, \mathrm{d}x \tag{B.15}$$

where $\Phi(x)$ is defined in equation (5). The Poisson flow becomes:

$$X_r^{\mathcal{MP}}(t,\lambda) = \sum_{n=0}^{\infty} \gamma_{2n}(\lambda)\, {}_2F_1\,([-2n, \lambda + it], [2\lambda], 2)\; r^{2n} \tag{B.16}$$

$$\gamma_n(\lambda) = 4^{\lambda}\, \frac{(2\lambda)_n}{n!} \int_{-\infty}^{\infty} \Phi(x)\, \tanh\left(\frac{x}{2}\right)^n \frac{\mathrm{e}^{\lambda\,x}}{(\mathrm{e}^x + 1)^{2\lambda}}\, \mathrm{d}x$$

B.3. **Derivation of an expansion of $\Xi(t)$ in terms of Continuous Hahn polynomials with parameters $a = \frac{1}{2}, b = 1, c = \frac{1}{2}, d = 1$.** We start from the definition of the Continuous Hahn family of polynomials:

$$p_n(t; a, b, c, d) = \frac{(a+b)_n (a+d)_n}{n!}\, {}_3F_2([-n, n+a+b+c+d-1, a+it], [a+c, a+d], 1)$$

We take one of its generating functions valid for $|x| < 1$ and as long as $a = c$ and $b = d$ simpler forms can be found. When we set $a = \frac{1}{2}, b = 1, c = \frac{1}{2}, d = 1$ it becomes:

$$\sum_{n=0}^{\infty} \frac{(a+b+c+d-1)_n\, p_n(t; a, b, c, d)}{(a+c)_n (a+d)_n\, i^n}\, x^n = \frac{{}_2F_1\left(\left[\frac{a+b+c+d-1}{2}, a+it\right], [a+c], -\frac{4x}{(1-x)^2}\right)}{(1-t)^{a+b+c+d-1}} \tag{B.17}$$

$$= \frac{1}{(1-x)^2} \left(\frac{(x+1)^2}{(x-1)^2}\right)^{-\frac{1}{2} - it} \tag{B.18}$$

We apply the mapping $x \mapsto \left(\frac{e^{x/2}-1}{e^{x/2}+1}\right)$ to make it converge for $x \in \mathbb{R}$ and obtain:

$$\sum_{n=0}^{\infty} \frac{(2)_n\, p_n\left(t; \frac{1}{2}, 1, \frac{1}{2}, 1\right)}{n!\, \left(\frac{3}{2}\right)_n\, i^n} \left(\frac{e^{x/2}-1}{e^{x/2}+1}\right)^n = e^{-ixt}\, \frac{\left(e^{x/2}+1\right)^2}{4\, e^{x/2}} \tag{B.19}$$

Our target integral representation of $\Xi(t) = \Xi(-t)$ is:

$$\Xi(t) = \int_{-\infty}^{\infty} \Phi(x)\, e^{-ixt}\, dx \tag{B.20}$$

$$= \int_{-\infty}^{\infty} \Phi(x)\, e^{-ixt}\, \frac{\left(e^{x/2}+1\right)^2}{4\, e^{x/2}}\, \frac{4e^{x/2}}{(e^{x/2}+1)^2}\, dx \tag{B.21}$$

$$= \int_{-\infty}^{\infty} \Phi(x) \left(\sum_{n=0}^{\infty} \frac{(2)_n\, p_n\left(t; \frac{1}{2}, 1, \frac{1}{2}, 1\right)}{n!\, \left(\frac{3}{2}\right)_n\, i^n} \left(\frac{e^{x/2}-1}{e^{x/2}+1}\right)^n\right) \frac{4e^{x/2}}{(e^{x/2}+1)^2}\, dx \tag{B.22}$$

$$= 4 \sum_{n=0}^{\infty} \frac{(2)_n\, p_n\left(t; \frac{1}{2}, 1, \frac{1}{2}, 1\right)}{n!\, \left(\frac{3}{2}\right)_n\, i^n} \int_{-\infty}^{\infty} \Phi(x) \left(\frac{e^{x/2}-1}{e^{x/2}+1}\right)^n \frac{e^{x/2}}{(e^{x/2}+1)^2}\, dx \tag{B.23}$$

where $\Phi(x)$ is defined in equation (5). We can now define the Poisson flow for $\Xi(t)$:

$$X_r^{\mathcal{CH}}\left(t, \frac{1}{2}, 1, \frac{1}{2}, 1\right) = \sum_{n=0}^{\infty} \gamma_{2n}\, {}_3F_2\left(\left[-2n, 2n+2, \frac{1}{2}+it\right], \left[1, \frac{3}{2}\right], 1\right) r^{2n} \tag{B.24}$$

$$\gamma_n = (n+1) \int_{-\infty}^{\infty} \Phi(x)\, \tanh\left(\frac{x}{4}\right)^n\, \operatorname{sech}\left(\frac{x}{4}\right)^2\, dx$$

B.4. **Derivation of an expansion of $\Xi(t)$ in terms of Continuous Hahn polynomials with all parameters equal.** We start from some finite sum definitions of the Meixner-Pollaczek and the Continuous Hahn polynomials that are obtained by just writing the hypergeometric function as a finite sum:

$$f_n(x,a) = P_n^{(a)}\left(t, \frac{\pi}{2}\right) = i^n\, \frac{(2a)_n}{n!}\, 2F_1\left([-n, a+i\,t], [2a], 2\right) \tag{B.25}$$

$$= (-i)^n \sum_{m=0}^{n} \frac{2^m\, \Gamma(2a+n)}{(n-m)!\, \Gamma(2a+m)} \binom{-a+ix}{m} \tag{B.26}$$

$$g_n(x,a) = p_n(t; a, a, a, a) = \frac{(2a)_n (2a)_n}{n!}\, {}_3F_2([-n, n+4a-1, a+it], [2a, 2a], 1) \tag{B.27}$$

$$= (-i)^n \sum_{m=0}^{n} \frac{(4a-1)_{n+m}}{(n-m)!\, (2a)_m^2} \binom{-a+ix}{m} \tag{B.28}$$

Next, we express $f_n(x,a)$ into $g_n(x,a)$ and vice versa:

$$g_n(x,a) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^{n-2k} \left(2a-\frac{1}{2}\right)_{n-k}}{k!\, (2a)_{n-2k}}\, f_{n-2k}(x,a) \tag{B.29}$$

$$f_n(x,a) = \frac{(2a)_n}{2^n \left(2a-\frac{1}{2}\right)_{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left(n-2k+2a-\frac{1}{2}\right) \binom{n+2a-\frac{1}{2}}{k} g_{n-2k}(x,a) \tag{B.30}$$

The proof of the above relations follows the same steps as the proof for the special case $a = \frac{3}{4}$ in appendix (A.6) of Romik's extended paper [1]. The steps of the proof are not repeated here, since the only difference is that in equation (A.45) we have to start with $p = n - m$ and $q = m + 2a - \frac{3}{2}$ and in equation (A.49) we have to use $N = n + 2a - \frac{1}{2}, m = m + 2a - \frac{3}{2}$ (note there is typo in (A.49), the last factor in the RHS should be 2^{n-m-1} instead of 2^{n-m+1}).

The second equation above allows us to equate the generating functions of $f_n(x, a)$ and $g_n(x, a)$ as follows:

$$\sum_{n=0}^{\infty} f_n(t,a)\, x^n = \sum_{n=0}^{\infty} \frac{(2a)_n}{2^n \left(2a - \frac{1}{2}\right)_n} \, {}_2F_1\left(\left[a + \frac{n}{2}, a + \frac{n+1}{2}\right], \left[2a + n + \frac{1}{2}\right], -x^2\right) g_n(t,a)\, x^n \tag{B.31}$$

$$= (1 - ix)^{-a+it}\,(1 + ix)^{-a-it}) \tag{B.32}$$

where the derivation towards the hypergeometric function again follows the exact same steps as in Romik's paper (Lemma 5.7. and its proof on p50).

We apply the mapping $x \mapsto \left(i\,\frac{e^x - 1}{e^x + 1}\right)$ to make it converge for $x \in \mathbb{R}$ and set

$$h(x,n,a) = {}_2F_1\left(\left[a + \frac{n}{2}, a + \frac{n+1}{2}\right], \left[2a + n + \frac{1}{2}\right], \left(\frac{e^x - 1}{e^x + 1}\right)^2\right)$$

We obtain:

$$\sum_{n=0}^{\infty} \frac{(2a)_n}{2^n \left(2a - \frac{1}{2}\right)_n} h(x,n,a)\, g_n(t,a) \left(i\,\frac{e^x - 1}{e^x + 1}\right)^n = 4^{-a} e^{itx} e^{-ax}\,(e^x - 1)^{2a} \tag{B.33}$$

Our target integral representation of $\Xi(t) = \Xi(-t)$ is:

$$\Xi(t) = \int_{-\infty}^{\infty} \Phi(x)\, e^{-ixt}\, dx \tag{B.34}$$

$$= 4^a \int_{-\infty}^{\infty} \Phi(x)\, 4^{-a} e^{itx}\, e^{-ax}\, (e^x - 1)^{2a}\, \frac{e^{ax}}{(e^x - 1)^{2a}} dx \tag{B.35}$$

$$= 4^a \int_{-\infty}^{\infty} \Phi(x) \left(\sum_{n=0}^{\infty} \frac{(2a)_n}{2^n \left(2a - \frac{1}{2}\right)_n} h(x,n,a)\, g_n(t,a) \left(i\,\frac{e^x - 1}{e^x + 1}\right)^n\right) \frac{e^{ax}}{(e^x - 1)^{2a}} dx \tag{B.36}$$

$$= 4^a \sum_{n=0}^{\infty} i^n\, \frac{(2a)_n\, g_n(t,a)}{2^n \left(2a - \frac{1}{2}\right)_n} \int_{-\infty}^{\infty} \Phi(x)\; h(x,n,a) \left(\frac{e^x - 1}{e^x + 1}\right)^n \frac{e^{ax}}{(e^x - 1)^{2a}} dx \tag{B.37}$$

We can now define the Poisson flow for $\Xi(t)$ valid for $a \in \mathbb{R}, a \neq \frac{1}{4}$:

$$X_r^{\mathcal{CH}}(t,a,a,a,a) = \sum_{n=0}^{\infty} \gamma_{2n}\, {}_3F_2\left([-2n, 2n + 4a - 1, a + it], [2a, 2a], 1\right) r^{2n} \tag{B.38}$$

$$\gamma_n = \frac{(-1)^n\, 2^{2a-n}\, (2a)_n\, (4a-1)_n}{n!\, \left(2a - \frac{1}{2}\right)_n} \int_{-\infty}^{\infty} \Phi(x)\, \tanh\left(\frac{x}{2}\right)^n \frac{e^{ax}}{(e^x + 1)^{2a}}$$

$$\times\, {}_2F_1\left(\left[a + \frac{n}{2}, a + \frac{n+1}{2}\right], \left[2a + n + \frac{1}{2}\right], \tanh\left(\frac{x}{2}\right)^2\right) dx$$

B.5. **Derivation of an expansion of $\Xi(t)$ in terms of Continuous Dual Hahn polynomials for parameters $a = \frac{1}{2}, 0, \frac{1}{2}$.** We start from the definition of the Continuous Dual Hahn family of polynomials:

$$S_n(t^2; a, b, c) = (a+b)_n (a+c)_n\, {}_3F_2([-n, a + it, a - it], [a+b, a+c], 1) \tag{B.39}$$

and take one of its generating functions and set $a = \frac{1}{2}, b = 0, c = \frac{1}{2}$ (note that formally all parameters should be larger than 0, however putting one to 0 does work numerically):

$$\sum_{n=0}^{\infty} \frac{(1)_n\, S_n(t^2;a,b,c)}{(a+b)_n\,(a+c)_n\, n!}\, x^n = \frac{1}{1-x}\, {}_3F_2\left([1, a+it, a-it], [a+b, a+c], \frac{x}{x-1}\right) \tag{B.40}$$

$$= \frac{i}{2}\, \frac{\mathrm{e}^{\pi t}\,(1+\sqrt{x})^{-2it}\,(x-1)^{ti} + \mathrm{e}^{-\pi t}\,(1+\sqrt{x})^{2it}\,(x-1)^{-ti}}{\sqrt{x-1}} \tag{B.41}$$

We now apply the mapping $x \mapsto \left(\frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\right)^2$ which yields:

$$\sum_{n=0}^{\infty} \frac{S_n\left(t^2; \frac{1}{2}, 0, \frac{1}{2}\right)}{\left(\frac{1}{2}\right)_n\, n!} \left(\frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\right)^{2n} = \cos(tx)\, \cosh\left(\frac{x}{2}\right) \tag{B.42}$$

Our target integral representation of $\Xi(t) = \Xi(-t)$ is:

$$\Xi(t) = 2\int_0^{\infty} \Phi(x) \cos(tx)\, \mathrm{d}x \tag{B.43}$$

$$= 2\int_0^{\infty} \Phi(x)\, \cos(tx)\, \cosh\left(\frac{x}{2}\right)\, \mathrm{sech}\left(\frac{x}{2}\right)\, \mathrm{d}x\mathrm{d}x \tag{B.44}$$

$$= 2\int_0^{\infty} \Phi(x) \left(\sum_{n=0}^{\infty} \frac{S_n\left(t^2; \frac{1}{2}, 0, \frac{1}{2}\right)}{\left(\frac{1}{2}\right)_n\, n!} \left(\frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\right)^{2n}\right) \mathrm{sech}\left(\frac{x}{2}\right)\, \mathrm{d}x\mathrm{d}x \tag{B.45}$$

$$= 2\sum_{n=0}^{\infty} \frac{S_n\left(t^2; \frac{1}{2}, 0, \frac{1}{2}\right)}{\left(\frac{1}{2}\right)_n\, n!} \int_0^{\infty} \Phi(x)\, \tanh\left(\frac{x}{2}\right)^{2n}\, \mathrm{sech}\left(\frac{x}{2}\right)\, \mathrm{d}x \tag{B.46}$$

where $\Phi(x)$ is defined in equation (5). The Poisson flow for $\Xi(t)$ becomes:

$$X_r^{\mathcal{CDH}}\left(t, \frac{1}{2}, 0, \frac{1}{2}\right) = \sum_{n=0}^{\infty} \gamma_n\, {}_3F_2\left(\left[-n, \frac{1}{2}+it, \frac{1}{2}-it\right], \left[\frac{1}{2}, 1\right], 1\right) r^n \tag{B.47}$$

$$\gamma_n = 2\int_0^{\infty} \Phi(x)\, \tanh\left(\frac{x}{2}\right)^{2n}\, \mathrm{sech}\left(\frac{x}{2}\right)\, \mathrm{d}x$$

B.6. **Derivation of an expansion of $\Xi'(t)/t$ in terms of Continuous Dual Hahn polynomials.** We start from the definition of the Continuous Dual Hahn family of polynomials:

$$S_n(t^2; a, b, c) = (a+b)_n (a+c)_n\, {}_3F_2([-n, a+it, a-it], [a+b, a+c], 1) \tag{B.48}$$

and take one of its generating functions and set $b = 1/2, c = 1$ and keep a as a free parameter:

$$\sum_{n=0}^{\infty} \frac{S_n(t^2;a,b,c)}{(b+c)_n\, n!}\, x^n = (1-x)^{-a+it}\, {}_2F_1\left([b+it, c+it], [b+c]; x\right) \tag{B.49}$$

$$= (1-x)^{-a+it} \left(-i\, \frac{(1-\sqrt{x})^{-2it} - (1+\sqrt{x})^{-2it}}{4\, t\, \sqrt{x}}\right) \tag{B.50}$$

We now apply the mapping $x \mapsto \left(\frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\right)^2$ which yields:

$$\sum_{n=0}^{\infty} \frac{S_n\left(t^2; a, \frac{1}{2}, 1\right)}{\left(\frac{3}{2}\right)_n\, n!} \left(\frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\right)^{2n} = \frac{\mathrm{e}^{-ax}\,(\mathrm{e}^x - 1)^{2a}}{-2\, 4^a\, t} \left(\frac{\mathrm{e}^x + 1}{\mathrm{e}^x - 1}\right) \sin(tx) \tag{B.51}$$

Our target integral representation of $\Xi'(t) = \Xi'(-t)$ is:

$$\Xi'(t) = -2i \int_0^{\infty} \Phi(x)\, x \sin(tx)\, \mathrm{d}x \tag{B.52}$$

$$= -4^{a+1}\, t \int_0^{\infty} \Phi(x)\, \frac{\mathrm{e}^{-ax}\, (\mathrm{e}^x - 1)^{2a}}{-2\, 4^a\, t} \left(\frac{\mathrm{e}^x + 1}{\mathrm{e}^x - 1}\right) \sin(tx)\, \frac{x\, \mathrm{e}^{ax}}{(\mathrm{e}^x - 1)^{2a}} \left(\frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\right) \mathrm{d}x \tag{B.53}$$

$$= -4^{a+1}\, t \int_0^{\infty} \Phi(x) \left(\sum_{n=0}^{\infty} \frac{S_n\left(t^2; a, \frac{1}{2}, 1\right)}{\left(\frac{3}{2}\right)_n\, n!} \left(\frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\right)^{2n} \right) \frac{x\, \mathrm{e}^{ax}}{(\mathrm{e}^x - 1)^{2a}} \left(\frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\right) \mathrm{d}x \tag{B.54}$$

$$= -4^{a+1}\, t \sum_{n=0}^{\infty} \frac{S_n\left(t^2; a, \frac{1}{2}, 1\right)}{\left(\frac{3}{2}\right)_n\, n!} \int_0^{\infty} \Phi(x)\, \tanh\left(\frac{x}{2}\right)^{2n+1} \frac{x\, \mathrm{e}^{ax}}{(\mathrm{e}^x - 1)^{2a}}\, \mathrm{d}x \tag{B.55}$$

where $\Phi(x)$ is defined in equation (5). The Poisson flow for $\Xi'(t)/t$ becomes:

$$X_r^{\mathcal{CDH}}(t, a) = \sum_{n=0}^{\infty} r^n\, \gamma_n(a)\, {}_3F_2\left([-n, a + it, a - it], \left[a + \frac{1}{2}, a + 1\right], 1\right) \tag{B.56}$$

$$\gamma_n(a) = -4^{a+1}\, \frac{\left(a + \frac{1}{2}\right)_n\, (a + 1)_n}{\left(\frac{3}{2}\right)_n\, n!} \int_0^{\infty} \Phi(x)\, \tanh\left(\frac{x}{2}\right)^{2n+1} \frac{x\, \mathrm{e}^{ax}}{(\mathrm{e}^x - 1)^{2a}}\, \mathrm{d}x$$

or

$$\gamma_n(a) = \frac{\left(a + \frac{1}{2}\right)_n^2\, (a + 1)_n^2}{2\pi\, n!\, \Gamma(n + a + \frac{1}{2})\, \Gamma(n + a + 1)\, \Gamma(n + 3/2)} \times \int_{0^+}^{\infty} \frac{\Xi'(x)}{x}\, w_x\left(a, \frac{1}{2}, 1\right) {}_3F_2\left([-n, a + ix, a - ix], \left[a + \frac{1}{2}, a + 1\right], 1\right) \mathrm{d}x$$

B.7. **Derivation of an expansion of $\frac{\Xi'(t)}{t}$ in terms of Wilson polynomials for parameters** $0, \frac{1}{2}, \frac{1}{2}, 1$. We start from the definition of the Wilson family of polynomials:

$$W_n(t^2; a, b, c, d) = (a + b)_n (a + c)_n (a + d)_n \times {}_4F_3([-n, n + a + b + c + d - 1, a + it, a - it], [a + b, a + c, a + d], 1) \tag{B.57}$$

Its generating function valid for $|x| < 1$. We set $a = 0, b = \frac{1}{2}, c = \frac{1}{2}, d = 1$ and obtain a simpler form (note that formally all parameters should be larger than 0, however putting one to 0 does work numerically):

$$\sum_{n=0}^{\infty} \frac{W_n(t^2; a, b, c, d)}{(a + d)_n (b + c)_n\, n!}\, x^n = {}_2F_1\left([a + it, d + it], [a + d]; x\right) {}_2F_1\left(b - it, c - it,], [b + c], x\right) \tag{B.58}$$

$$= i \left(\frac{(1 - \sqrt{x})^{4it} - (1 + \sqrt{x})^{4it}}{8\, t \sqrt{x}} \right) (1 - t)^{-2it} \tag{B.59}$$

We apply the mapping $x \mapsto \left(\frac{\mathrm{e}^{x/2} - 1}{\mathrm{e}^{x/2} + 1}\right)^2$ to make it converge for $x \in \mathbb{R}$ and obtain:

$$\sum_{n=0}^{\infty} \frac{W_n\left(t^2; 0, \frac{1}{2}, \frac{1}{2}, 1\right)}{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n\, n!} \left(\frac{\mathrm{e}^{x/2} - 1}{\mathrm{e}^{x/2} + 1}\right)^{2n} = \frac{\coth\left(\frac{x}{4}\right) \sin(tx)}{4t} \tag{B.60}$$

Our target integral representation of $\Xi(t) = \Xi(-t)$ is:

$$\Xi(t) = -2\int_0^\infty \Phi(x)\, x \sin(t\,x)\,\mathrm{d}x \tag{B.61}$$

$$= -\int_0^\infty \Phi(x)\,\frac{\coth\left(\frac{x}{4}\right)\,\sin(tx)}{4t}\,\frac{8tx}{\coth\left(\frac{x}{4}\right)}\mathrm{d}x \tag{B.62}$$

$$= -\int_0^\infty \Phi(x)\left(\sum_{n=0}^\infty \frac{W_n\left(t^2;0,\frac{1}{2},\frac{1}{2},1\right)}{\left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n\,n!}\left(\frac{\mathrm{e}^{x/2}-1}{\mathrm{e}^{x/2}+1}\right)^{2n}\right)\frac{8tx}{\coth\left(\frac{x}{4}\right)}\mathrm{d}x \tag{B.63}$$

$$= -8t\left(\sum_{n=0}^\infty \frac{W_n\left(t^2;0,\frac{1}{2},\frac{1}{2},1\right)}{\left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n\,n!}\int_0^\infty \Phi(x)\,\tanh\left(\frac{x}{4}\right)^{2n}\right)\frac{8tx}{\coth\left(\frac{x}{4}\right)}\mathrm{d}x \tag{B.64}$$

where $\Phi(x)$ is defined in equation (5). We can now define the Poisson flow for $\frac{\Xi'(t)}{t}$:

$$X_r^{\mathcal{W}}(t) = \sum_{n=0}^\infty \gamma_n\,{}_4F_3\left([-n,n+1,it,-it]\,,\left[\frac{1}{2},\frac{1}{2},1\right],1\right)\,r^n \tag{B.65}$$

$$\gamma_n = -\frac{8}{2n+1}\int_0^\infty \Phi(x)\,\tanh\left(\frac{x}{4}\right)^{2n}\,\frac{x}{\coth\left(\frac{x}{4}\right)}\mathrm{d}x$$

or

$$\gamma_n = \frac{2}{\pi^3\,(2n+1)}\int_{0^+}^\infty \frac{\Xi'(x)}{x}\,w_x\left(0,\frac{1}{2},\frac{1}{2},1\right){}_4F_3\left([-n,n+1,ix,-ix]\,,\left[\frac{1}{2},\frac{1}{2},1\right],1\right)\mathrm{d}x$$

B.8. **Derivation of an expansion of** $(\Xi(0)+\Xi(t))$ **in terms of Wilson polynomials for parameters** $\frac{1}{2},0,\frac{1}{2},0$**.** We start from the definition of the Wilson family of polynomials:

$$W_n(t^2;a,b,c,d) = (a+b)_n(a+c)_n(a+d)_n \times {}_4F_3([-n,n+a+b+c+d-1,a+it,a-it],[a+b,a+c,a+d],1) \tag{B.66}$$

Its generating function valid for $|x|<1$. We set $a=\frac{1}{2}, b=0, c=\frac{1}{2}, d=0$ and obtain a simpler form (note that formally all parameters should be larger than 0, however putting two to 0 does work numerically):

$$\sum_{n=0}^\infty \frac{W_n(t^2;a,b,c,d)}{(a+d)_n(b+c)_n\,n!}\,x^n = {}_2F_1\left([a+it,d+it],[a+d];x\right)\,{}_2F_1\left(b-it,c-it,],[b+c],x\right) \tag{B.67}$$

$$= \frac{1}{4}\left((1-\sqrt{x})^{-2it}-(1+\sqrt{x})^{-2it}\right)\left((1-\sqrt{x})^{2it}-(1+\sqrt{x})^{2it}\right) \tag{B.68}$$

We apply the mapping $x \mapsto \left(\frac{\mathrm{e}^{\frac{x}{2}}-1}{\mathrm{e}^{\frac{x}{2}}+1}\right)^2$ to make it converge for $x\in\mathbb{R}$ and obtain:

$$\sum_{n=0}^\infty \frac{W_n\left(t^2;\frac{1}{2},0,\frac{1}{2},0\right)}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\,n!}\left(\frac{\mathrm{e}^{\frac{x}{2}}-1}{\mathrm{e}^{\frac{x}{2}}+1}\right)^{2n} = \frac{\cos(tx)+1}{2} \tag{B.69}$$

Our target integral representation of $\Xi(t) = \Xi(-t)$ is:

$$\Xi(t) = 2\int_0^\infty \Phi(x)\,\cos(t\,x)\,\mathrm{d}x \tag{B.70}$$

$$= -\Xi(0 + 4\int_0^\infty \Phi(x)\,\frac{\cos(tx)+1}{2}\mathrm{d}x \tag{B.71}$$

$$= -\Xi(0) + 4\int_0^\infty \Phi(x)\left(\sum_{n=0}^{\infty} \frac{W_n\left(t^2;\frac{1}{2},0,\frac{1}{2},0\right)}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\,n!}\left(\frac{\mathrm{e}^{\frac{x}{2}}-1}{\mathrm{e}^{\frac{x}{2}}+1}\right)^{2n}\right)\mathrm{d}x \tag{B.72}$$

$$= -\Xi(0) + 4\sum_{n=0}^{\infty} \frac{W_n\left(t^2;\frac{1}{2},0,\frac{1}{2},0\right)}{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\,n!}\int_0^\infty \Phi(x)\,\tanh\left(\frac{x}{2}\right)^{2n}\mathrm{d}x \tag{B.73}$$

where $\Phi(x)$ is defined in equation (5). We can now define the Poisson flow for $\Xi(0) + \Xi(t)$:

$$X_r^W\left(t,\frac{1}{2},0,\frac{1}{2},0\right) = \sum_{n=0}^{\infty} \gamma_n\,{}_4F_3\left(\left[-n,n,\frac{1}{2}+it,\frac{1}{2}-it\right],\left[\frac{1}{2},1,\frac{1}{2}\right],1\right)r^n \tag{B.74}$$

$$\gamma_n = 4\int_0^\infty \Phi(x)\,\tanh\left(\frac{x}{4}\right)^{2n}\,\mathrm{d}x$$

B.9. **Derivation of an expansion of** $(\Xi(0) - \Xi(t)/\,t^2)$ **in terms of Wilson polynomials for parameters** $\frac{1}{2},1,\frac{1}{2},1$**.** We start from the definition of the Wilson family of polynomials:

$$W_n(t^2;a,b,c,d) = (a+b)_n(a+c)_n(a+d)_n \times {}_4F_3([-n,n+a+b+c+d-1,a+it,a-it],[a+b,a+c,a+d],1) \tag{B.75}$$

Its generating function is valid for $|x| < 1$. We set $a = \frac{1}{2}, b = 1, c = \frac{1}{2}, d = 1$ and obtain a simpler form:

$$\sum_{n=0}^{\infty} \frac{W_n(t^2;a,b,c,d)}{(a+d)_n(b+c)_n\,n!}\,x^n = {}_2F_1\,([a+it,d+it],[a+d];x)\;{}_2F_1\,(b-it,c-it,],[b+c],x) \tag{B.76}$$

$$= \left(\frac{(1-\sqrt{x})^{-2it}-(1+\sqrt{x})^{-2it}}{16\,t\,\sqrt{x}}\right)\left(\frac{(1-\sqrt{x})^{2it}-(1+\sqrt{x})^{2it}}{16\,t\,\sqrt{x}}\right) \tag{B.77}$$

We apply the mapping $x \mapsto \left(\frac{\mathrm{e}^{x/2}-1}{\mathrm{e}^{x/2}+1}\right)^2$ to make it converge for $x \in \mathbb{R}$ and obtain:

$$\sum_{n=0}^{\infty} \frac{W_n\left(t^2;1,\frac{1}{2},1,\frac{1}{2}\right)}{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n\,n!}\left(\frac{\mathrm{e}^{x/2}-1}{\mathrm{e}^{x/2}+1}\right)^{2n} = \frac{(1-\cos(tx))}{8\,t^2}\left(\frac{\mathrm{e}^{x/2}+1)}{\mathrm{e}^{x/2}-1}\right)^2 \tag{B.78}$$

Our target integral representation of $\Xi(t) = \Xi(-t)$ is:

$$\Xi(t) = 2\int_0^\infty \Phi(x)\,\cos(t\,x)\,\mathrm{d}x \tag{B.79}$$

$$= \Xi(0) - 16\,t^2\int_0^\infty \Phi(x)\,\frac{(1-\cos(tx))}{8\,t^2}\left(\frac{\mathrm{e}^{x/2}+1)}{\mathrm{e}^{x/2}-1}\right)^2\left(\frac{\mathrm{e}^{x/2}-1)}{\mathrm{e}^{x/2}+1}\right)^2\mathrm{d}x \tag{B.80}$$

$$= \Xi(0) - 16\,t^2\int_0^\infty \Phi(x)\left(\sum_{n=0}^{\infty} \frac{W_n\left(t^2;1,\frac{1}{2},1,\frac{1}{2}\right)}{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n\,n!}\left(\frac{\mathrm{e}^{x/2}-1}{\mathrm{e}^{x/2}+1}\right)^{2n}\right)\left(\frac{\mathrm{e}^{x/2}-1)}{\mathrm{e}^{x/2}+1}\right)^2\mathrm{d}x \tag{B.81}$$

$$= \Xi(0) - 16\,t^2\sum_{n=0}^{\infty} \frac{W_n\left(t^2;1,\frac{1}{2},1,\frac{1}{2}\right)}{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n\,n!}\int_0^\infty \Phi(x)\,\tanh\left(\frac{x}{4}\right)^{2n+2}\mathrm{d}x \tag{B.82}$$

where $\Phi(x)$ is defined in equation (5). We can now define the Poisson flow for $\frac{\Xi(0)-\Xi(t)}{t^2}$:

(B.83)

$$X_r^{\mathcal{W}}(t) = \sum_{n=0}^{\infty} \gamma_n \, {}_4F_3\left(\left[-n, n+2, \frac{1}{2}+it, \frac{1}{2}-it\right], \left[\frac{3}{2}, 1, \frac{3}{2}\right], 1\right) r^n$$

$$\gamma_n = 16 \int_0^{\infty} \Phi(x) \tanh\left(\frac{x}{4}\right)^{2n+2} \mathrm{d}x$$

or

$$\gamma_n = \frac{16\,(n+1)}{\pi^3} \int_{0^+}^{\infty} \frac{\Xi(0)-\Xi(x)}{x^2} w_x\left(\frac{1}{2}, 1, \frac{1}{2}, 1\right) {}_4F_3\left(\left[-n, n+2, \frac{1}{2}+ix, \frac{1}{2}-ix\right], \left[\frac{3}{2}, 1, \frac{3}{2}\right], 1\right) \mathrm{d}x$$

Note: The RH would follow when (for $r = 1$) only real t solutions exist of:

$$t^2 \sum_{n=0}^{\infty} \gamma_n \, {}_4F_3\left(\left[-n, n+2, \frac{1}{2}+ix, \frac{1}{2}-ix\right], \left[\frac{3}{2}, 1, \frac{3}{2}\right], 1\right) = \Xi(0) \tag{B.84}$$

An infinite number of solutions for t real must exist, however for solutions for t complex (i.e. zeros off the critical line) things are much more complicated. The factor t^2 will always be complex for $t = a + bi, a, b \neq 0$, hence the factor $\sum_{n=0}^{\infty} \gamma_n \, {}_4F_3\left(\left[-n, n+2, \frac{1}{2}+ix, \frac{1}{2}-ix\right], \left[\frac{3}{2}, 1, \frac{3}{2}\right], 1\right)$ must be the exact conjugate of t^2 to become real and equal to $\Xi(0)$. The existence of such exact conjugate is equivalent to the RH and obviously very hard to prove.

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