Design and Application of Allpass Filters with Equiripple Group Delay Errors

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***Abstract*—Allpass filters have found many applications in signal processing areas. In this paper, an iterative linear programming (LP) algorithm is first presented to solve the minimax phase error design of allpass filters. Then an iterative weighted minimax (WMM) method is presented to design allpass filters with equiripple group delay errors. Finally, the iterative WMM method is applied to the design of allpass-filter-based halfband filters with equiripple constant group delays. Design examples demonstrate that the new iterative LP algorithm is very efficient for the minimax allpass filter design, and the iterative WMM method is very effective for the equiripple constant group delay designs of allpass filters and allpass-filter-based halfband filters.**

Keywords - allpass filter; halfband filter; weighted minimax design; equiripple design

I. INTRODUCTION

Digital allpass filters have been used in many digital signal processing applications such as group delay and phase delay equalization and design and implementation of filter banks, halfband filters, notch filters, and variable fractional delay filters, etc. [2] [6] [7] [9] [12] [13] [14] Since an allpass filter has a unit magnitude across the whole frequency area [0, π], its design reduces to a pure phase approximation problem. Two widely used criterions for the approximation problem are the minimax [1] [11] [12] [13] [16] and the weighted least-squares [3] [8]. Between them, the minimax criterion is preferable because it minimizes the filter's maximum phase deviation from a desired one. Several algorithms for the minimax design can be found in the literature [1] [11] [12] [16]. All these algorithms obtained allpass filters with equiripple phase errors. By incorporating an iterative reweighting technique, the weighted least-squares methods can also be applied to the equiripple phase error design [8].

A minimax allpass filter has an equiripple phase error and the maximum phase error is the smallest. However, the filter's group delay deviation from its desired value is neither equiripple nor the smallest. In fact, the group delay deviation is much larger near the band edges than elsewhere in the frequency band of interest. If one would like to obtain an allpass filter with an equiripple group delay, the minimax phase error design does not work. To this end, some minimax or equiripple group delay error design methods had been proposed. Ref. [2] designs allpass variable fractional delay filters with minimax group delay errors using an iterative reweighted least-squares (IRLS) group delay error method. The program in [15] aims to obtain allpass filters with equiripple group delay errors using a least *p*th group delay error method. Since either the l_2 or the l_p norm of the group delay error is highly nonconvex, both methods are difficult to obtain true minimax group delay error filters.

Recently, a procedure was presented in [10] to design FIR filters with near equiripple group delay errors by iteratively updating the upper bound of phase error constraint. Inspired by [10], a new design method for allpass IIR filters with equiripple group delay errors is proposed in this paper. Instead of directly minimizing the group delay error, the method minimizes the weighted phase error of the filter. By iteratively updating the phase error weight using the envelope of the filter's group delay error, the method finally obtains allpass filters with equiripple group delay errors. The method is much different from the IRLS method in [2] though both of them use an iterative reweighting technique. While the IRLS method reweights the group delay error in an LS criterion, our method reweights the phase error in a minimax sense. We refer to our new method as an iterative weighted minimax (WMM) phase error method, since its iteration core is a WMM phase error problem. To solve the WMM phase error problem, an efficient iterative LP algorithm is also developed. Design examples of allpass filters and allpass-based halfband filters demonstrate the convergence and effectiveness of the iterative LP algorithm for WMM phase error problems and the iterative WMM phase error method for equiripple group delay error designs.

II. NEW MINIMAX PHASE ERROR DESIGN ALGORITHM

An *N*-th order digital allpass filter is described by the transfer function

$$A(z,\boldsymbol{a}) = z^{-N}\frac{1+a_1 z+\cdots+a_{N-1}z^{N-1}+a_N z^N}{1+a_1 z^{-1}+\cdots+a_{N-1}z^{-(N-1)}+a_N z^{-N}}, \quad (1)$$

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with a coefficient vector denoted by $\boldsymbol{a} = [a_1, a_2, \cdots, a_N]^T$. By substituting z with $e^{j\omega}$ in the transfer function, we obtain the frequency response of the filter as follows:

$$A(e^{j\omega}, \boldsymbol{a}) = e^{j\theta(\omega,\boldsymbol{a})}, \quad (2)$$

where $\theta(\omega, \boldsymbol{a})$ is the filter's phase response described by

$$\theta(\omega,\boldsymbol{a}) = -N\omega + 2\phi(\omega,\boldsymbol{a}), \quad (3)$$

$$\phi(\omega,\boldsymbol{a}) = \tan^{-1}\frac{S^T(\omega)\boldsymbol{a}}{1+C^T(\omega)\boldsymbol{a}}, \quad (4)$$

$$S(\omega) = [\sin\omega, \cdots, \sin(N-1)\omega, \sin(N\omega)]^T, \quad (5)$$

$$C(\omega) = [\cos\omega, \cdots, \cos(N-1)\omega, \cos(N\omega)]^T. \quad (6)$$

We assume the desired phase response denoted by $\theta_d(\omega)$ is defined on a close subset Ω of the interval $[0, \pi]$. Then the approximation error between $\theta(\omega, \boldsymbol{a})$ and $\theta_d(\omega)$ is

$$\theta_e(\omega,\boldsymbol{a}) = \theta(\omega,\boldsymbol{a}) - \theta_d(\omega). \quad (7)$$

From (3), we have

$$0.5\theta_e(\omega,\boldsymbol{a}) = \phi(\omega,\boldsymbol{a}) - \beta_d(\omega), \quad (8)$$

where

$$\beta_d(\omega) = 0.5[N\omega + \theta_d(\omega)]. \quad (9)$$

By taking the tangent of the two side of Eq. (8) and using (4), we get

$$\tan 0.5\theta_e(\omega,\boldsymbol{a}) = \frac{-\sin\beta_d(\omega) + \overline{S}^T(\omega)\boldsymbol{a}}{\cos\beta_d(\omega) + \overline{C}^T(\omega)\boldsymbol{a}}, \quad (10)$$

where

$$\overline{S}(\omega) = S(\omega)\cos\beta_d(\omega) - C(\omega)\sin\beta_d(\omega), \quad (11)$$

$$\overline{C}(\omega) = C(\omega)\cos\beta_d(\omega) + S(\omega)\sin\beta_d(\omega), \quad (12)$$

or, after some simple algebra using (5), (6) and (9),

$$\overline{S}(\omega) = \begin{bmatrix} \sin[\omega - 0.5N\omega - 0.5\theta_d(\omega)] \\ \sin[2\omega - 0.5N\omega - 0.5\theta_d(\omega)] \\ \vdots \\ \sin[N\omega - 0.5N\omega - 0.5\theta_d(\omega)] \end{bmatrix}, \quad (13)$$

$$\overline{C}(\omega) = \begin{bmatrix} \cos[\omega - 0.5N\omega - 0.5\theta_d(\omega)] \\ \cos[2\omega - 0.5N\omega - 0.5\theta_d(\omega)] \\ \vdots \\ \cos[N\omega - 0.5N\omega - 0.5\theta_d(\omega)] \end{bmatrix}. \quad (14)$$

Since the tangent function is monotonic in $[-0.5\pi, 0.5\pi]$ and $-0.5\pi \le \theta_e(\omega, \boldsymbol{a}) \le 0.5\pi$, imposing an upper bound $\Delta > 0$ on $|\theta_e(\omega, \boldsymbol{a})|$ is equivalent to impose an upper bound δ on $|\tan[0.5\theta_e(\omega, \boldsymbol{a})]|$ with $\delta = \tan(0.5\Delta)$. The minimax phase error design of the allpass filter $A(z, \boldsymbol{a})$ can then be described by

$$\min_{\delta,\, \boldsymbol{a}\in R(r)} \delta, \text{ s.t.: } \left|\frac{-\sin\beta_d(\omega) + \overline{S}^T(\omega)\boldsymbol{a}}{\cos\beta_d(\omega) + \overline{C}^T(\omega)\boldsymbol{a}}\right| \le \delta,\ \omega\in\Omega. \quad (15)$$

where $R(r)$ represents the stability domain of the filter with a maximum pole radius r.

The minimax problem (15) is nonconvex. We convert it into convex ones using the following iterative algorithm.

Step 1. Let $\boldsymbol{a}(0) = 0$ and $k = 0$.

Step 2. Solve the following problem for $\boldsymbol{a}(k+1)$:

$$\boldsymbol{a}(k+1) = \arg\min_{\delta,\, \boldsymbol{a}\in R(r)} \delta, \quad (16)$$

$$\text{s.t.: } -\delta \le \frac{-\sin\beta_d(\omega) + \overline{S}^T(\omega)\boldsymbol{a}}{|\cos\beta_d(\omega) + \overline{C}^T(\omega)\boldsymbol{a}(k)|} \le \delta,\ \omega\in\Omega. \quad (17)$$

Step 3. If

$$\frac{\max_{\omega\in\Omega}|\theta_e(\omega,\boldsymbol{a}(k+1))| - \max_{\omega\in\Omega}|\theta_e(\omega,\boldsymbol{a}(k))|}{\max_{\omega\in\Omega}|\theta_e(\omega,\boldsymbol{a}(k))|} > \nu, \quad (18)$$

where the small number $\nu > 0$ is a relative tolerance of the maximum phase error, let $k = k + 1$ and go back to Step 2. Otherwise, terminate the algorithm.

In the problem (16) (17), the designed coefficient vector is constrained within $R(r)$ by some stability conditions. If the generalized positive realness condition in [4] is used, the constraint $\boldsymbol{a} \in R(r)$ can be rewritten as:

$$\mathrm{Re}\left\{e^{j\phi(\omega,\boldsymbol{a}(k))}[1+\psi^T(r,\omega)\boldsymbol{a}]\right\} > \varepsilon, \quad \forall\omega\in\Omega_0 = [0,\pi] \quad (19)$$

where $\psi(r,\omega) = [r^{-1}e^{-j\omega}, r^{-2}e^{-j2\omega}, \cdots, r^{-N}e^{-jN\omega}]^T$, $\varepsilon > 0$ is a small number. Since the cost in (16) and constraints (17) (19) are all linear in δ and $\boldsymbol{a}$, the problem (16) (17) is an LP one. To this end, we refer to the above algorithm an iterative LP algorithm.

Example 1. Design an 8th order minimax allpass filter with a desired linear phase response $\theta_d(\omega) = -7.0615\omega$ defined on $\Omega = [0, 0.8\pi]$.

In our designs, the continuous frequency intervals $\Omega_0 = [0, \pi]$ and $\Omega = [0, 0.8\pi]$ are replaced by the discrete frequency sets $\Omega_0 = \{k\pi/400 |\ k = 0, 1, \cdots, 400\}$ and $\Omega\cap\Omega_0$, respectively. The design parameters ν and ε are taken as 10^{-4} and 10^{-6}. The maximum pole radius is required to be smaller than $r = 0.98$.

The iterative LP algorithm converges very fast. It takes only 3 iterations, resulting in a filter with a maximum phase error of 3.274×10^{-5}. The maximum group delay deviation from its desired constant value 7.0615 is 2.676×10^{-3} samples.

The filter's phase error and group delay responses on Ω are shown by the dotted lines in Figs. 1 and 2, respectively. It is seen that the phase error is equiripple. But the group delay error is not. Instead, the group delay error near the high frequency edge of Ω is much larger than the error elsewhere.

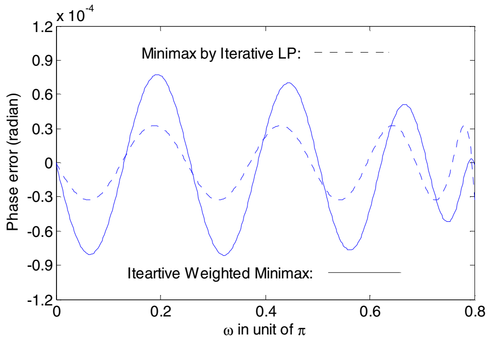

Fig. 1 Phase error responses of the allpass filters

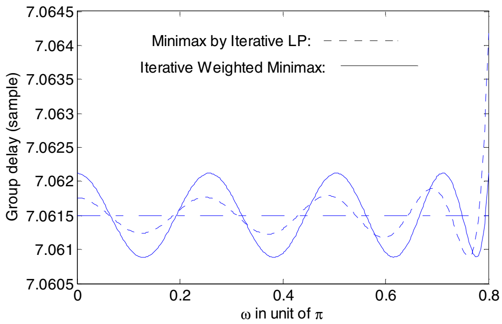

Fig. 2 Group delay responses of the allpass filters

III. ITERATIVE REWEIGHTED MINIMAX METHOD

In order to reduce the group delay error, we introduce a weight function $W(\omega) \geq 0$ in the minimax phase error problem (15), resulting in the following WMM phase error design:

$$\min_{\delta,\, \boldsymbol{a}\in R(r)} \delta\,,\ \text{s.t.: } W(\omega)\left|\frac{-\sin\beta_d(\omega)+\bar{S}^T(\omega)\boldsymbol{a}}{\cos\beta_d(\omega)+\bar{C}^T(\omega)\boldsymbol{a}}\right| \leq \delta,\ \omega\in\Omega\,, \quad (20)$$

which can be efficiently solved using the same iterative LP algorithm described in the previous section, except that Eq. (17) is replaced by

$$\frac{W(\omega)\left|-\sin\beta_d(\omega)+\bar{S}^T(\omega)\boldsymbol{a}\right|}{\left|\cos\beta_d(\omega)+\bar{C}^T(\omega)\boldsymbol{a}(k)\right|} \leq \delta, \quad \omega\in\Omega\,. \quad (21)$$

We then use the following iterative method to obtain the weight function under which the WMM problem (20) has a solution filter with an equiripple group delay error. We refer to the method an iterative WMM method.

Step 1. Let the initial weight function $W(\omega) = 1$ for $\omega\in\Omega$. Let E_g be a sufficiently large positive number.

Step 2. Solve for $\boldsymbol{a}_W$ the weighted minimax problem (20) using the iterative LP algorithm.

Step 3. Compute the group delay $\tau(\omega, \boldsymbol{a}_W)$ of the filter $A(z, \boldsymbol{a}_W)$. If

$$\left|E_g - \max_{\omega\in\Omega}\left|\tau(\omega,\boldsymbol{a}_W) - \frac{\partial}{\partial\omega}\theta_d(\omega)\right|\right| > \eta E_g \quad (22)$$

where $\eta > 0$ is a small relative tolerance of the maximum group delay error, we let

$$E_g = \max_{\omega\in\Omega}\left|\tau(\omega,\boldsymbol{a}_W) - \partial\theta_d(\omega)/\partial\omega\right|, \quad (23)$$

$$W(\omega) = W(\omega)\times\text{envelope}\left(\left|\tau(\omega,\boldsymbol{a}_W) - \partial\theta_d(\omega)/\partial\omega\right|\right), \quad (24)$$

and then go back to Step 2. Otherwise, terminate the algorithm.

Since the initial weight function is $W(\omega) = 1$ for $\omega\in\Omega$, the filter obtained in the first iteration of the above procedure is a minimax phase error filter.

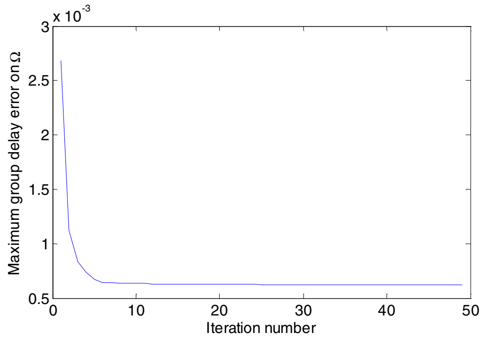

Fig. 3 Maximum group delay error versus iteration number

Example 2. Design an equiripple group delay allpass filter with the same specifications as Example 1. We assume a relative maximum group delay error tolerance of $\eta = 10^{-5}$. The iterative weighted minimax method converges in 49 iterations. Fig. 3 shows the convergence of the filter's maximum group delay error. The resultant filter has a maximum group delay error of 6.223×10^{-3} samples and a maximum phase error of 8.111×10^{-5}. The maximum group delay error is much smaller but the phase error is larger than those of the minimax filter obtained in Example 1. The solid lines in Figs. 1 and 2 draw the phase error and group delay responses of the filter.

IV. DESIGN OF HALFBAND FILTERS WITH EQUIRIPPLE GROUP DELAYS

It is known that if $A(z)$ is an Nth order allpass filter with a linear phase response $\theta_d(\omega) = -(N-0.5)\omega$ defined on $\Omega = [0, 2\omega_p]$, the filter obtained by $H(z) = 0.5[z^{-(2N-1)} + A(z^2)]$ is a linear phase halfband filter with a passband $[0, \omega_p]$.

Let $A(e^{j\omega}) = e^{j\theta(\omega)}$. Then, $H(e^{j\omega}) = 0.5[e^{-j(2N-1)\omega} + e^{j\theta(2\omega)}] = e^{-j[(N-0.5)\omega-0.5\theta(2\omega)]}\cos[(N-0.5)\omega+0.5\theta(2\omega)]$, and the group delay response of $H(e^{j\omega})$ is $(N-0.5)+\tau(2\omega)$ where $\tau(\omega)$ is the group delay response of the allpass filter $A(z)$, i.e., $\tau(\omega) = -\partial\theta(\omega)/\partial\omega$. In order to obtain a halfband filter $H(z)$ with an equiripple group delay, we may first design an equiripple group delay allpass filter $A(z)$ by the iterative weighted minimax method, and then let $H(z) = 0.5[z^{-(2N-1)} + A(z^2)]$.

Table 1 Maximum error measurements of the halfband filters

Method	MMEP	MMES	MPE	MGDE
Iterative Weighted Minimax	1.74×10^{-7}	5.90×10^{-4}	5.90×10^{-4}	9.23×10^{-3}
Minimax by SOCP [1]	2.92×10^{-8}	2.38×10^{-4}	2.36×10^{-4}	$>3\times10^{-2}$

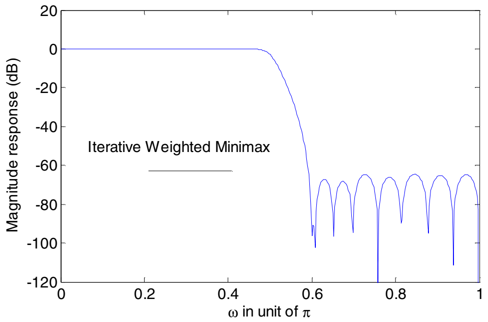

Fig. 4 Magnitude response of the halfband filter

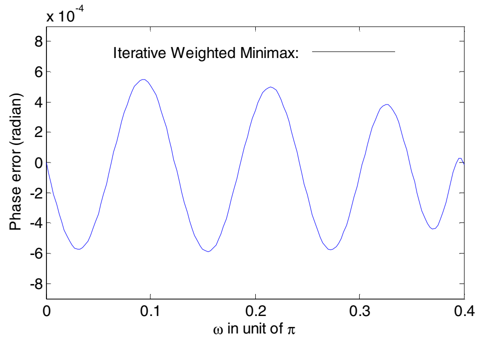

Fig. 5 Phase error response of the halfband filter

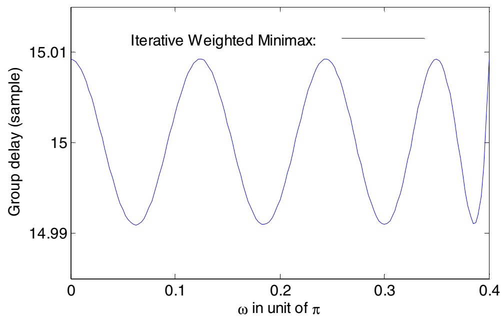

Fig. 6 Group delay response of the halfband filter

Example 3. Design a halfband filter with a denominator order 16 and a passband [0, 0.4π]. The corresponding allpass filter is of 8th order and has a desired phase response $\theta_d(\omega) = -7.5\omega$ defined on $\Omega = [0, 0.8\pi]$. Using the iterative weighted minimax method, we obtain an allpass filter with a maximum phase error of 1.18×10^{-3} and a maximum group delay error of 9.23×10^{-3} samples. The maximum passband magnitude error (MMEP), maximum stopband magnitude error (MMES), maximum phase error (MPE), and maximum group delay error (MGDE) of the halfband filter based on the resultant allpass filter are listed in the first row of Table 1. Corresponding measurements of the halfband filter obtained by the second-order conic programming (SOCP) based minimax design in [1] are also listed in Table 1, where the MMEP, MMES, and MPE are directly computed from the data in Table II of [1], and the MGDE are estimated from Fig. 1(b) of [1]. It is seen that the MGDE by the iterative WMM method is much smaller than that in [1]. The magnitude response, passband phase error and group delay of the halfband filter are shown in Figs. 4 - 6.

V. Conclusion

A novel iterative LP algorithm for the minimax phase error design of allpass filters has been presented. It only takes a few iterations to obtain the minimax filter. In order to obtain allpass filters with equiripple group delays, an iterative weighted minimax method has been also presented. The iterative method converges fast and has been effectively applied in the design of halfband filters with equiripple group delays.

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