The Well-ordering Theorem

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Introduction

This is an excerpt from my paper 'A Logical Alternative to Choice'. The well-ordering principle is accepted as an axiom and the well-ordering theorem is known to be choice equivalent. However, I show that both statements can be proven on totally ordered sets in first order logic independent of the axiom of choice.
This has implications for the mathematical representation of physical systems, since one assumes upon construction of a model that there exists a total ordering on the system's components. Since the well-ordering theorem implies that,

> Given any totally ordered set, one can define a well-ordering,

this provides further evidence that mathematics is an accurate representation of the natural world.

Addition and multiplication by 0 can be defined by

$$\begin{aligned}
\mathrm{n} + 0 &= |\mathrm{I_n} \sqcup \mathrm{I_0}| = |\mathrm{I_n} \sqcup \emptyset| = |\mathrm{I_n}| = \mathrm{n} \\
\mathrm{n} \times 0 &= \left| \bigsqcup_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{I_0} \right| = |\emptyset| = 0
\end{aligned} \tag{4}$$

Addition by zero is commutative, since $0 + \mathrm{n} = |\emptyset \sqcup \mathrm{n}| = |\mathrm{S_0(n)}| = \mathrm{n}$. However, left multiplication by zero is *poorly-defined*. This can be remedied by extending left multiplication *individually* for each n:

$$\mathrm{a} \times \mathrm{n} = \begin{cases} |\mathrm{I_0}| = 0 & \mathrm{a} = 0 \\ \left| \mathrm{I_0} \sqcup \left(\bigsqcup_{\mathrm{i}=1}^{\mathrm{a}} \mathrm{S_0(n)} \right) \right| & \mathrm{a} \in \mathbb{N} \end{cases}$$

5 Well-ordering

The Axiom of Choice was introduced by Ernst Zermelo in 1904[3] in order to prove his formulation of the well-ordering theorem, also known as Zermelo's theorem[5]:

Every set can be well-ordered.

A well-ordering refers to a set S with an order relation $\prec$ such that all subsets of S contain a least element under $\prec$. Given Zermelo's formulation, one has to assume the axiom of choice since sets are not implicitly ordered.
However, given a strict total ordering on a set, one can construct a well-ordering. This implies that the following statement can be deduced in first order logic independent of axioms:

Every set that can be totally ordered can be well-ordered.

5.1 Nested Sets

We begin with the well-defined order properties of nested sets.

Definition 5.1. *Let* S *be a set. Then* S *is a collection of nested sets if*

$$\forall\, \mathrm{x}, \mathrm{y} \in \mathrm{S},\ \mathrm{x} \neq \mathrm{y} \iff (\mathrm{x} \subset \mathrm{y}) \vee (\mathrm{y} \subset \mathrm{x})$$

Lemma 5.1. *Let* S *be a collection of nested sets,* $\mathrm{A}, \mathrm{B} \in \mathrm{S}$*. Then*

$$\mathrm{A} \subset \mathrm{B} \iff (\mathrm{A} \cap \mathrm{B} = \mathrm{A}) \wedge (\mathrm{A} \cup \mathrm{B} = \mathrm{B})$$

Lemma 5.2. *Let* S *be a collection of nested sets such that*

$$\exists\, \mathrm{z} \in \mathrm{S},\ (\mathrm{z} \neq \emptyset) \wedge (\forall\, \mathrm{x} \in \mathrm{S},\ \mathrm{z} \in \mathrm{x})$$

Then S *contains a least element.*

Proof. Since $\forall\, \mathrm{x} \in \mathrm{S},\ \mathrm{z} \in \mathrm{S}.$ the set

$$\mathrm{L} = \bigcap_{\mathrm{x} \in \mathrm{S}} \mathrm{x} \neq \emptyset$$

since $\mathrm{z} \in \mathrm{L}$. Furthermore, by the nested property of intersection,

$$\forall\, \mathrm{x} \in \mathrm{S},\ \mathrm{L} \subseteq \mathrm{x}$$

which implies that $\mathrm{L} \in \mathrm{S}$ and L is the least element in S. $\square$

5.2 The Well-ordering Principle

The order properties of nested sets can be generalized to any totally ordered set.

Definition 5.2 (Total order relation)**.** *Let* S *be a set with identity relation* $=$*. Then a binary relation* $<$ *is a total order relation on* S *if for all* $\mathrm{x}, \mathrm{y} \in \mathrm{S}$*:*

- $(\mathrm{x} < \mathrm{y}) \iff \neg(\mathrm{y} < \mathrm{x}) \wedge \neg(\mathrm{x} = \mathrm{y})$
- $(\mathrm{y} < \mathrm{x}) \iff \neg(\mathrm{x} < \mathrm{y}) \wedge \neg(\mathrm{x} = \mathrm{y})$
- $(\mathrm{x} = \mathrm{y}) \iff \neg(\mathrm{x} < \mathrm{y}) \wedge \neg(\mathrm{y} < \mathrm{x})$

Define $(\mathrm{S}, <)$ *to be the set* S *ordered under* $<$*.*

Definition 5.3 (Well-ordered)**.** $(\mathrm{S}, \prec)$ *is well-ordered if every subset* $\mathrm{X} \subseteq \mathrm{S}$ *contains a least element under* $\prec$*.*

Definition 5.4. *Let* $(\mathrm{A}, <)$ *and* $(\mathrm{B}, \ll)$ *be two totally ordered sets. Then a function* $\mathrm{f} : \mathrm{A} \longrightarrow \mathrm{B}$ *is order preserving if*

$$\forall\, \mathrm{x}, \mathrm{y} \in \mathrm{A},\ \mathrm{x} < \mathrm{y} \iff \mathrm{f}(\mathrm{x}) \ll \mathrm{f}(\mathrm{y})$$

One can construct an order-preserving bijection from any totally ordered set to a collection of nested sets. Define the partition of S induced by x to be the set $\{\mathrm{L_x}, \mathrm{x}, \mathrm{G_x}\}$ such that

1. $\mathrm{L_x} = \{\mathrm{s} \in \mathrm{S} | \mathrm{s} < \mathrm{x}\}$
2. $\mathrm{G_x} = \{\mathrm{s} \in \mathrm{S} | \mathrm{s} > \mathrm{x}\}$

Clearly, $\mathrm{S} = \mathrm{L_x} \sqcup \mathrm{x} \sqcup \mathrm{G_x}$.
Define a closed map F on S to be,

$$\mathrm{F(x)} = \mathrm{L_x} \cup \mathrm{x}$$

Define $\mathcal{L}[\mathrm{S}] = \{\mathrm{F(s)} | \mathrm{s} \in \mathrm{S}\}$.
As a totally ordered set, for each $\mathrm{x} \in \mathrm{S}$, F(x) clearly contains a maximal element. However, when considering the set F(x) as a member of $\mathcal{L}[\mathrm{S}]$, for all $\mathrm{y} \in \mathrm{S}$,

$$\mathrm{x} < \mathrm{y} \implies (\mathrm{L_x} \subset \mathrm{L_y}) \wedge (\mathrm{x} \in \mathrm{L_y}) \implies \mathrm{F(x)} \subseteq \mathrm{L_y} \subset \mathrm{F(y)}$$

Since F is an injectively defined function on S, it is trivially bijective. Hence, F is an order-preserving map such that

$$\mathrm{x} < \mathrm{y} \iff \mathrm{F(x)} \subset \mathrm{F(y)}$$

Theorem 5.3 (The well-ordering principle)**.** *Suppose* $(\mathrm{S}, <)$ *is totally ordered and contains a least element. Then every subset* $\mathrm{X} \subseteq \mathrm{S}$ *has a least element under* $<$.

Proof. Let z be the least element of S. For each $\mathrm{x} \in \mathrm{S}$, define

$$\mathrm{F(x)} = \mathrm{L_x} \sqcup \mathrm{x}$$

By definition of least element, $\mathrm{F(z)} = \emptyset \sqcup \mathrm{z} = \mathrm{z}$. Since

$$\forall \mathrm{x}, \mathrm{y} \in \mathrm{S},\ \mathrm{x} < \mathrm{y} \implies \mathrm{F(x)} \subset \mathrm{F(y)}$$

F is an order-preserving bijection. In addition,

$$\forall \mathrm{x} \in \mathrm{S},\ \mathrm{z} = \mathrm{F(z)} \subseteq \mathrm{F(x)}$$

thus, $\mathcal{L}[\mathrm{S}] = \{\mathrm{F(x)} | \mathrm{x} \in \mathrm{S}\}$ is totally ordered under $\subset$ with least element z.

Let $X \subseteq S$. Define

$$\mathcal{L}[X] = \{F(x) | x \in X\}$$

Then $\forall F(x) \in \mathcal{L}[X]$, $z \subseteq F(x)$ which implies that

$$F_0 := \bigcap_{F(x) \in \mathcal{L}[X]} F(x) \neq \emptyset$$

Thus, $F_0 \in \mathcal{L}[X]$ by the properties of intersection and

$$(\forall x \in X)(F_0 \subseteq F(x))$$

Since $\mathcal{L}[X]$ is bijective and $F_0 \in \mathcal{L}[X]$,

$$\exists!\, x_0 \in X,\ F(x_0) = F_0$$

Thus,

$$(\forall x \in X)(F_0 \subseteq F(x)) \implies (\forall x \in X)(x_0 \leq x)$$

and $x_0 \in X$ is the least element of X. □

5.3 The Well-ordering Theorem

Lemma 5.4 (Existence of a well-ordering)**.** *Let* $(S, <)$ *be a totally ordered set. Then there exists a strict total ordering* $\prec$ *such that* $(S, \prec)$ *has a least element.*

Proof. Let $z \in S$ such that

$$\begin{aligned} L_z &:= \{x \in S | x < z\} \neq \emptyset \\ G_z &:= \{x \in S | z < x\} \neq \emptyset \end{aligned} \tag{5}$$

such that $S = L_z \sqcup z \sqcup G_z$. Define the order relation $\prec$ such that

$$z \prec \{x_1 | (\forall x_1 \in L_z)\} \prec \{x_2 | (\forall x_2 \in G_z)\}$$

and $\forall x_1, x_2 \in S$,

$$x_1 \prec x_2 \iff \neg(x_1 = x_2) \wedge \begin{cases} ((x_1 \in L_z) \wedge (x_2 \in G_z)) \\ \vee((x_1 < x_2) \wedge (x_1 \in L_z) \wedge (x_2 \in L_z)) \\ \vee((x_1 < x_2) \wedge (x_1 \in G_z) \wedge (x_2 \in G_z)) \\ \vee(x_1 = z) \end{cases}$$

Then z is the least element of $(S, \prec)$. □

Theorem 5.5 (Well-ordering theorem)**.** *Suppose* $(\mathrm{S}, <)$ *is totally ordered. Then* S *can be well-ordered.*

Proof. If S has 2 elements, then it is trivially well-ordered. Thus, suppose $(\mathrm{S}, <)$ is a total ordering on more than 2 elements. By Lemma 5.4, there exists a strict total order relation $\prec$ such that $(\mathrm{S}, \prec)$ contains a least element. By the well-ordering principle, for all subsets $\mathrm{X} \subseteq \mathrm{S}$, $(\mathrm{X}, \prec)$ has a least element. Thus, every set $\mathrm{X} \subseteq \mathrm{S}$ has a least element under $\prec$ and S is well-ordered. $\square$

References

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