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Algorithms for unequal-arm Michelson interferometers

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Abstract

A method of data acquisition and data analysis is described in which the performance of Michelson-type interferometers with unequal arms can be made nearly the same as interferometers with equal arms. The method requires a separate readout of the relative phase in each arm, made by interfering the returning beam in each arm with a fraction of the outgoing beam. Instead of throwing away the information from a single arm by subtracting it from that from the other arm, the data in one arm is first used to estimate the laser phase noise and then correct for its effect in the normal differenced interferometer data.

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1. Background

The Michelson interferometer was devised as a method to make very precise relative distance measurements. In the common laboratory version of the instrument, a laser signal is divided by a beam splitter, the two divided beams are sent out along different paths, the beams are reflected back to the beam splitter, and the beams then interfere to produce a light fringe. The interferometer will detect changes in the difference in the lengths of the two arms by monitoring the intensity of the fringe.

The advantage of the interferometer over a system where a single arm is used and where the returning light interferes with a fraction of the outgoing light to form the fringes lies in the relative immunity of the interferometer to fluctuations in the phase of the laser. In a single arm, jitter in the laser phase over the round-trip light time would cause the interference pattern to fluctuate, mimicking a change in the path length. However, in an interferometer, the phase fluctuations are transmitted equally along the two arms and, when the return beams finally combine, the fluctuations will be the same in both signals and will cancel.

This scheme supposes, of course, that the lengths of the two arms are essentially equal. Indeed, if the two arms of length l_1 and l_2 are unequal by an amount $\Delta l = l_1 - l_2$, then the phase noise in the interference fringe will be given by (we adopt units in which the speed of light $c = 1$)

$$\Delta\phi(t) = p(t - 2l_1) - p(t - 2l_2) \approx \dot{p}(t - 2l_1)(2\Delta l),$$

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[redacted]

where $p(t)$ is the phase noise in the laser. The relative strain noise in the interferometer is therefore

$$\frac{\langle \Delta x \rangle}{l} = \langle \dot{p} \rangle \frac{2\Delta l}{\nu l},$$

where the brackets denote time average, and ν is the nominal laser frequency. Thus, the residual noise in the interferometer is a fraction $2\Delta l/l$ of the laser frequency noise.

In the laboratory, the armlength difference Δl may be initialized to near zero by minimizing this noise, and then this length may be stabilized by adjusting the path so as to maintain a constant intensity of the fringe, the path adjustment required giving the measure of the external influence on the armlengths. However, there are cases where the paths cannot be maintained at equal lengths. In a spaceborne gravitational wave experiment, for example, it is desired to very accurately measure the relative strain between free-flying spacecraft which, because of the solar system orbits on which they fly, cannot maintain equal distances between them. Even in laboratory laser interferometers it is sometimes impossible to assume that the arms can be made equal. That applies, for example, to delay-line interferometers, where the reentrance condition of the delay-line in each arm determines the necessary arm length. The Garching 30-m prototype in its old configuration is a prime example for this, where, indeed, a variant of the technique proposed here has been used for many years.

It is the purpose of this paper to describe a method of data processing that will achieve almost all the noise cancellation of an equal-arm interferometer, even in a case where the arms are rather badly unequal.

2. Unequal-arm interferometers

Two space missions have recently been proposed which consist of free-flying spacecraft that track each other with lasers [1,2]. In one of these [1], the spacecraft fly on heliocentric orbits that are non-circular to a few tenths of a percent. In the other, a geocentric mission [2], the deviations in the armlengths can be almost 2%. In order to be sure of detecting gravitational waves, these missions require a strain sensitivity of about $h \sim 10^{-20}\ \mathrm{Hz}^{-1/2}$. If one were to use normal interferometer techniques, the armlength difference would require the laser noise to be less than $h(l/2\Delta l) \approx 3 \times 10^{-18}\ \mathrm{Hz}^{-1/2}$ in the first case, and $3 \times 10^{-19}\ \mathrm{Hz}^{-1/2}$ in the second, geocentric, case. Neither of these laser phase stabilities are currently obtainable.

However, the equal-arm interferometer scheme described in the last section is only a particular case of a more general set of algorithms that can be used to analyze the data in the two arms. It is the case where a real-time subtraction is performed by applying the signal from one arm at one time to cancel the noise in the other arm at the same time, the times being the same because the times of flight in the two arms are identical. If the times of flight are different, then the information from the two arms may still be used to correct for the fluctuations of the laser, but the corrections would be applied at times consistent with the differences in armlengths.

To be specific, let us describe the following data analysis procedure [3] for a two-arm, unequal-arm interferometer formed by four spacecraft, as shown in Fig. 1. Each spacecraft is assumed to send a laser signal and to receive a signal from its counterpart. Each laser is assumed to have the same fundamental frequency ν. There is also assumed to be a two-way reference signal sent and received between the two spacecraft that are close to each other. The two central spacecraft correspond to the central beamsplitter of a laboratory Michelson interferometer. The single end spacecraft correspond to the end mirrors.

Let us then define:

- $p_m(t)$ as the phase noise (in cycles) of the laser in the mth spacecraft, so that the phase of the mth laser is $P_m = \nu t + p_m(t)$.
- $l_i(t)$ as the one-way light-time for the signal along the ith interferometer arm, including slow drift velocities from the orbits and faster changes produced by gravitational waves.

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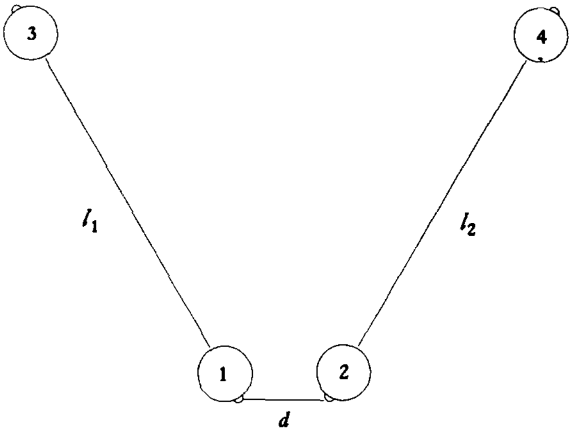

Fig. 1. Geometry of a spacecraft two-arm interferometer. Spacecraft 1 and 3 track each other, and spacecraft 2 and 4 track each other. Spacecraft 1 and 2 exchange a phase reference tracking signal.

The signal received by each spacecraft is allowed to interfere with a fraction of the local laser power being sent out. The phase of the beat signal read in the i^{th} spacecraft photodiode is then given by

$$\begin{aligned} s_i(t) &= P_k(t-l_i) - P_i(t) = -\nu l_i(t) + p_k(t-l_i) - p_i(t) \\ s_k(t) &= P_i(t-l_i) - P_k(t) = -\nu l_i(t) + p_i(t-l_i) - p_k(t), \end{aligned} \tag{1}$$

where i takes on the values $\{1,2\}$ and k takes on the appropriate value from the set $\{3,4\}$. The phase reference signal readout is similar:

$$\sigma_i(t) = -\nu d(t) + p_j(t-d) - p_i(t), \tag{2}$$

where d is the distance between the two close spacecraft and where $\{i,j\}$ are chosen from $\{1,2\}$.

If all signals $s_i(t)$ and $\sigma_i(t)$ are read out and telemetered from the individual spacecraft to one of the central spacecraft or to the ground, then combinations of these signals may be used to synthesize an interferometer in data analysis.

The differenced phase reference signal is given by

$$\sigma_2(t) - \sigma_1(t) = [p_1(t) + p_1(t-d)] - [p_2(t) + p_2(t-d)].$$

In the frequency domain [1], we use the transfer function for a signal differenced at the light time d to yield

$$\sigma_2(f) - \sigma_1(f) = [p_1(f) - p_2(f)]\left(1 + e^{2\pi i f d}\right), \tag{3}$$

so that, knowing the distance d, one can apply a linear filter and write the differenced phase reference signal as

$$\zeta(f) = p_1(f) - p_2(f) = \frac{\sigma_1(f) - \sigma_2(f)}{1 + e^{2\pi i f d}}. \tag{4}$$

The inverse Fourier transform $\zeta(t) = p_1(t) - p_2(t)$ will yield a time series that will tie the lasers in the two central spacecraft together as if they were beams from a single laser.

[1] We introduce here an estimate of the true spectrum, due to the discrete sampling and finite observing time. For simplicity, we neglect the errors associated to these estimates.

[redacted]

The main signal is essentially an integrated Doppler measurement at the central point, formed by the combination

$$z_i(t) = s_i(t) + s_k(t - l_i) = p_i(t - 2l_i) - p_i(t) - 2\nu l_i(t). \tag{5}$$

By combining $z_1(t)$ and $z_2(t)$ from Eq. (5) and using the reference signal $\zeta(t)$ from Eq. (4), one can write the interferometer signal in terms of the noise in one laser only

$$\delta(t) \equiv z_1(t) - z_2(t) - \zeta(t - 2l_2) + \zeta(t) = p_1(t - 2l_1) - p_1(t - 2l_2) - 2\nu\Delta l(t). \tag{6}$$

The algorithm to be used in the case of unequal arms consists of a procedure to synthesize the laser phase noise in this signal so that its effect in Eq. (6) may be subtracted away. To do this, we first assume that the signal is dominated by laser phase noise in the bandwidth of interest, in which case the Fourier transform of $z_1(t)$ would be given in terms of the transform of $p_1(t)$ by

$$z_1(f) = p_1(f)\left(e^{4\pi i f l_1} - 1\right),$$

where the expression in parenthesis is the transfer function for differencing at the round-trip light-time. One may therefore use $z_1(t)$ to generate an estimate $\hat{p}_1(f)$ of $p_1(f)$:

$$\hat{p}_1(f) = \frac{z_1(f)}{e^{4\pi i f l_1} - 1} \tag{7}$$

Fourier reconstruction of the time series then gives estimates $\hat{p}_1(t)$ and $\hat{p}_2(t) = \hat{p}_1(t) - \zeta(t)$ of the phase noise of the lasers. These estimates can then be used to predict the effect of the laser noise in the interferometer via

$$\hat{z}_i(t) \equiv \hat{p}_i(t - 2l_i) - \hat{p}_i(t),$$

and the resulting estimate,

$$\hat{\delta}(t) \equiv \hat{z}_1(t) - \hat{z}_2(t),$$

of the differenced interferometer signal can then be subtracted from $\delta(t)$ to give a signal

$$\Delta(t) \equiv \delta(t) - \hat{\delta}(t),$$

which now does not contain the laser phase noise. This procedure will work as long as one remains far from the poles of Eq. (7), that is at frequencies well away from $f_n = n/2l_i$, where n is an integer. Of course, this procedure breaks down near $f = 0$ as well, and low frequency is the place where there is the most scientific interest. However, as one goes towards low frequencies, the noise in unequal-arm interferometers cancels anyway, and the noise still tends to zero at long periods. This will be shown explicitly in the next section.

3. Theoretical performance of the algorithm

The limitations on the procedure described above arrive from two sources – the random shot noise in the readout of the laser phase at each spacecraft and the error in the knowledge of the actual time-of-flight of the signals in the two arms. In this section, we will discuss the limitations that these errors place on the tolerances for the system.

We assume independent phase noise $n_i(t)$ in the readout of the ith arm ($i = 1,2$) and we explicitly write the armlength as a sum of slow drifts $l_i(t)$ outside the spectral band of interest and a gravitational wave signal within the band. This signal is added to one arm and subtracted from the other arm, a characteristic of

[redacted]

gravitational waves with a simple choice of wave polarization and propagation vectors. Its contribution to each arm can be shown [4] to be given by $\frac{1}{2}[h(t) - h(t-2l_i)]$. We use detection of the gravitational wave as our measure of sensitivity, but it is representative of any distance change that one wants to measure.

Our knowledge of the two armlengths l_1, l_2 is not exact, being limited by the errors we make in measuring the position of the central and end masses of the interferometer. Let δl_1, and δl_2 be such errors. We also assume, for the sake of simplicity, that the phase reference signal $\zeta(t)$ in Eq. (4) is null, i.e. that the two time series p_1 and p_2 are identical.

In the reconstruction of the laser phase noise, using the method described in the previous section and taking into account the contribution of the error δl_1, Eq. (7) is replaced by

$$\hat{p}_1(f) = \frac{z_1(f)}{\left[e^{4\pi i f(l_1+\delta l_1)} - 1\right]}. \tag{8}$$

Now $z_1(f)$ is equal to

$$z_1(f) = p_1(f)\left[e^{4\pi i f l_1} - 1\right] + n_1(f) + \nu h(f)\frac{\left[e^{4\pi i f l_1} - 1\right]}{2if},$$

where the coefficient of $h(f)$ comes from the Fourier transform of the gravitational wave response function. Therefore our estimate of the laser phase noise is given by

$$\hat{p}_1(f) = \left[p_1(f) + \frac{\nu h(f)}{2if}\right]\frac{\left[e^{4\pi i f l_1} - 1\right]}{\left[e^{4\pi i f(l_1+\delta l_1)} - 1\right]} + \frac{n_1(f)}{\left[e^{4\pi i f(l_1+\delta l_1)} - 1\right]}. \tag{9}$$

The differenced "Doppler" signal, the Fourier transform of $\delta(t)$ from Eq. (6), has the following analytic form

$$\delta(f) = p_1(f)\left[e^{4\pi i f l_1} - e^{4\pi i f l_2}\right] + \left[n_1(f) - n_2(f)\right] + \nu h(f)\frac{\left[e^{4\pi i f l_1} + e^{4\pi i f l_2} - 2\right]}{2if} \tag{10}$$

The reconstructed contribution of laser phase noise to this phase difference can be written in terms of $\hat{p}_1(f)$

$$\hat{\delta}(f) \equiv \hat{z}_1(f) - \hat{z}_2(f) = \hat{p}_1(f)\left[e^{4\pi i f(l_1+\delta l_1)} - e^{4\pi i f(l_2+\delta l_2)}\right]. \tag{11}$$

After substituting Eq. (9) into Eq. (11) we get the following expression for the estimated phase difference

$$\hat{\delta}(f) = \left[p_1(f) + \frac{\nu h(f)}{2if}\right]\frac{\left[e^{4\pi i f l_1} - 1\right]\left[e^{4\pi i f(l_1+\delta l_1)} - e^{4\pi i f(l_2+\delta l_2)}\right]}{\left[e^{4\pi i f(l_1+\delta l_1)} - 1\right]} + n_1(f)\frac{\left[e^{4\pi i f(l_1+\delta l_1)} - e^{4\pi i f(l_2+\delta l_2)}\right]}{\left[e^{4\pi i f(l_1+\delta l_1)} - 1\right]}. \tag{12}$$

Finally, if we subtract the estimated phase difference due to the laser noise (Eq. (12)) from the actual phase difference (Eq. (10)), we get a signal, $\Delta(f)$, that has the following terms

$$\Delta(f) \equiv \delta(f) - \hat{\delta}(f) = P(f) + N(f) + H(f), \tag{13}$$

where $P(f)$, $N(f)$, and $H(f)$ are equal to

$$P(f) = 4\pi i f p_1(f)\left[\frac{\delta l_2\left(e^{4\pi i f l_1} - 1\right)e^{4\pi i f l_2} - \delta l_1\left(e^{4\pi i f l_2} - 1\right)e^{4\pi i f l_1}}{e^{4\pi i f l_1} - 1}\right], \tag{14a}$$

$$N(f) = \frac{n_1(f)\left[e^{4\pi i f l_2} - 1 + 4\pi i f\delta l_2\, e^{4\pi i f l_2}\right] - n_2(f)\left[e^{4\pi i f l_1} - 1 + 4\pi i f\delta l_1\, e^{4\pi i f l_1}\right]}{\left[e^{4\pi i f l_1} - 1 + 4\pi i f\delta l_1\, e^{4\pi i f l_1}\right]}, \tag{14b}$$

[redacted]

$$H(f) = 2\pi\nu h(f)\left[\frac{\delta l_2\left(e^{4\pi i f l_1}-1\right)e^{4\pi i f l_2} - \delta l_1\left(e^{4\pi i f l_2}-1\right)e^{4\pi i f l_1}}{e^{4\pi i f l_1}-1} + \frac{e^{4\pi i f l_2}-1}{2\pi i f}\right]. \tag{14c}$$

In the long wavelengths limit ($fl_1, fl_2 \ll 1$) Eqs. (14) simplify, and the equation for $\Delta(f)$ becomes

$$\Delta(f) \approx 4\pi i f p_1(f)\left[\frac{l_1\delta l_2 - l_2\delta l_1}{l_1}\right] + n_1(f)\left[\frac{l_2+\delta l_2 - l_2\delta l_1/l_1}{l_1}\right] - n_2(f) + 2\pi\nu h(f) l_2\left[2+\frac{\delta l_2}{l_2}-\frac{\delta l_1}{l_1}\right]. \tag{15}$$

Eq. (15) gives a requirement on the accuracy with which the armlengths must be determined in order for the data analysis algorithm correctly applied. From the relative size of noise in p_1 versus h, we derive the following requirement

$$\left|p_1(f)\left(\frac{f}{\nu}\right)\left[\frac{\delta l_2-\delta l_1}{l_2}+\frac{l_1-l_2}{l_1 l_2}\delta l_1\right]\right| \le |h(f)|. \tag{16}$$

As an example, let us assume that the relative laser frequency noise is about 5×10^{-13} Hz$^{-1/2}$. We further assume that the dominant frequency component of the gravitational wave we are trying to observe is 10^{-2} Hz, and that the gravitational wave amplitude is 10^{-20} Hz$^{-1/2}$. With these values, we find that the difference in armlength must be known to better than about 100 meters and that the individual armlength must be known absolutely to a factor $l/\Delta l$ worse than that (i.e., an error of ±5 kilometer for a $\Delta l/l$ of 2%).

The requirement on the precision of measuring the armlengths that we have deduced above can be easily achieved by computing the autocorrelation function of each phase difference $z_i(t)(i=1,2)$ [5]. The autocorrelation function of the laser noise has three maxima, at times zero and $\pm 2l_i$. Since the other noise sources have autocorrelation times smaller than $2l_i$, the armlength can be determined, within the error required, by searching for the position of the $2l_i$ peak.

4. Numerical simulation

In this section we will present a computer simulation of the signal processing for unequal-arm interferometers. We assume again that $\zeta(t)$ is null. We have simulated this single phase noise $p(t)$ of relative amplitude $\sim 5\times10^3$ rad using a gaussian random number generator. Shot noise $n_i(t) \sim 5\times10^{-4}$ rad, also with gaussian character, has been simulated for each of the interferometer channels ($i=1,2$). It has further been assumed that the end laser is perfectly phase-locked to its received signal, to simplify the analysis. Moreover, in order to approximate a realistic experiment, an error $\delta l_i = \pm10$ m in our knowledge of the two arm lengths has been introduced. The simulated experimental data has been assumed to be taken every second, for a total of $N = 2^{15} = 32768$ points.

To these noise records a simulated gravitational wave was added with amplitude $h = 10^{-20}$ and the data were analyzed to determine if the gravitational wave could be detected in the presence of the noise. Two cases were chosen. The first corresponds to the parameters for the geocentric mission, with its short round-trip light time but with the greater difference in the arms. This case thus tests the ability of the algorithm to perform with large discrepancies in armlengths. The round-trip light time for the two arms were taken to be $T_1 = 2l_1 = 7.2$ s and $T_2 = 2l_2 = 7.3$ s, and the simulated gravitational wave signal had a frequency of 10^{-2} Hz. The second case corresponds to the heliocentric mission. Here, the armlengths are greater and are relatively much closer to each other. The light time for the two arms was $T_1 = 16.70$ s and $T_2 = 16.73$ s. Because of the longer round-trip light time, part of the band of scientific interest will lie above the first pole of Eq. (7). To demonstrate the ability of the algorithm to perform in this range, a gravitational wave frequency of 10^{-1} Hz was chosen.

For all cases the phase readout in each arm is calculated using

[redacted]

$$z_i(t_k) = p(t_k - T_i) - p(t_k) + n_i(t_k) \pm \pi\nu \sum_{k'=0}^{k} [h(t_{k'} - T_i) - h(t_{k'})]\,\Delta t \qquad (k = 0, \ldots, N-1)$$

where $h(t) = h\cos(2\pi f_h t)$ is a pure sinusoidal gravitational wave signal of amplitude and frequency as stated above, and $\Delta t = 1$ s. The gravitational wave signal is added to arm 1 and subtracted from arm 2. Since T_i is not an integer number, the value of p at $t - T_i$ is not given. We have determined it by means of a linear fit between two successive points, i.e.

$$p(t_k - T_i) = \alpha p(t_k - \tau_i) + \beta p(t_k - \tau_i - 1),$$

where $\tau_i = \mathrm{Int}(T_i)$ and $\alpha + \beta = 1$. Since $p(t_k)$ is not defined for $t_k < 0$ and for $t_k > N$, we have minimized the boundary effect problem by closing the time series in a circular way.

Taking arm 1 as a reference, its phase readout signal is Fourier analyzed to give

$$z_1(f_n) = \sum_{k=1}^{N} z_1(t_k)\, e^{2\pi j f_n t_k},$$

where $f_n = n/N$. From $z_1(f)$ we get the estimate $\hat{p}(f)$ for $p(f)$ through Eq. (7), which now reads, taking into account the error δl and the discrete sampling,

$$\hat{p}(f_n) = \frac{z_1(f_n)}{1 - \alpha'\, e^{2\pi j \tau_i n/N} - \beta'\, e^{2\pi j(\tau_i+1)n/N}} \qquad n \geq 1 \tag{17}$$

In deriving Eq. (17) we have made use of the fact that the error δl does not change τ_i, but only the parameters α and β. The poles of Eq. (17) make it impossible to determine the zero frequency term, which we have taken to be zero, i.e. $\hat{p}_i(f = 0) = 0$. The estimate of $p(f)$ is then inverse transformed to give an estimate $\hat{p}(t)$ of $p(t)$.

From $\hat{p}(t)$, the contributions $\hat{z}_i(t)$ of the laser phase noise to $z_i(t)$ were formed via

$$\hat{z}_i(t_k) = \hat{p}(t_k - 2(l_i + \delta l_i)) - \hat{p}(t_k).$$

The resulting estimate $\hat{\delta}(t_k) \equiv \hat{z}_1(t_k) - \hat{z}_2(t_k)$ was then subtracted from the true $\delta(t_k) \equiv z_1(t_k) - z_2(t_k)$, to give a signal $\Delta(t_k) \equiv \delta(t_k) - \hat{\delta}(t_k)$, whose power spectrum is given by Eq. (15). Apart from the remaining dependence on $p(f)$ due to the nonzero δl, $\Delta(t)$ contains only shot noise and gravitational wave signal. Its power spectrum is then analyzed to see if the pure sinusoidal $h(t)$ can be found against the background of the other noise.

The sequence of results of this data analysis is shown in Fig. 2. Figs. 2a and 2b represent a small portion of the time series $p(t)$ and the low frequency region ($f < 0.1$ Hz) of its power spectrum. Fig. 2c shows the same region of the power spectrum of $\delta(t)$. Finally, Figs. 2d and 2e show the final output of the data analysis, $\Delta(t)$, together with its power spectrum around the region $f \sim 0.01$ Hz. Fig. 2f displays the equivalent of 2(e) for the long baseline case with gravitational wave at 0.1 Hz. The counterparts for figures (a)–(d) for this case are indistinguishable from the shorter baseline case. In both cases, we notice that the signal in a bandwidth of $1/N$ Hz may be clearly seen above the shot noise background, with the expected signal-to-noise ratio of $\sqrt{N}$.

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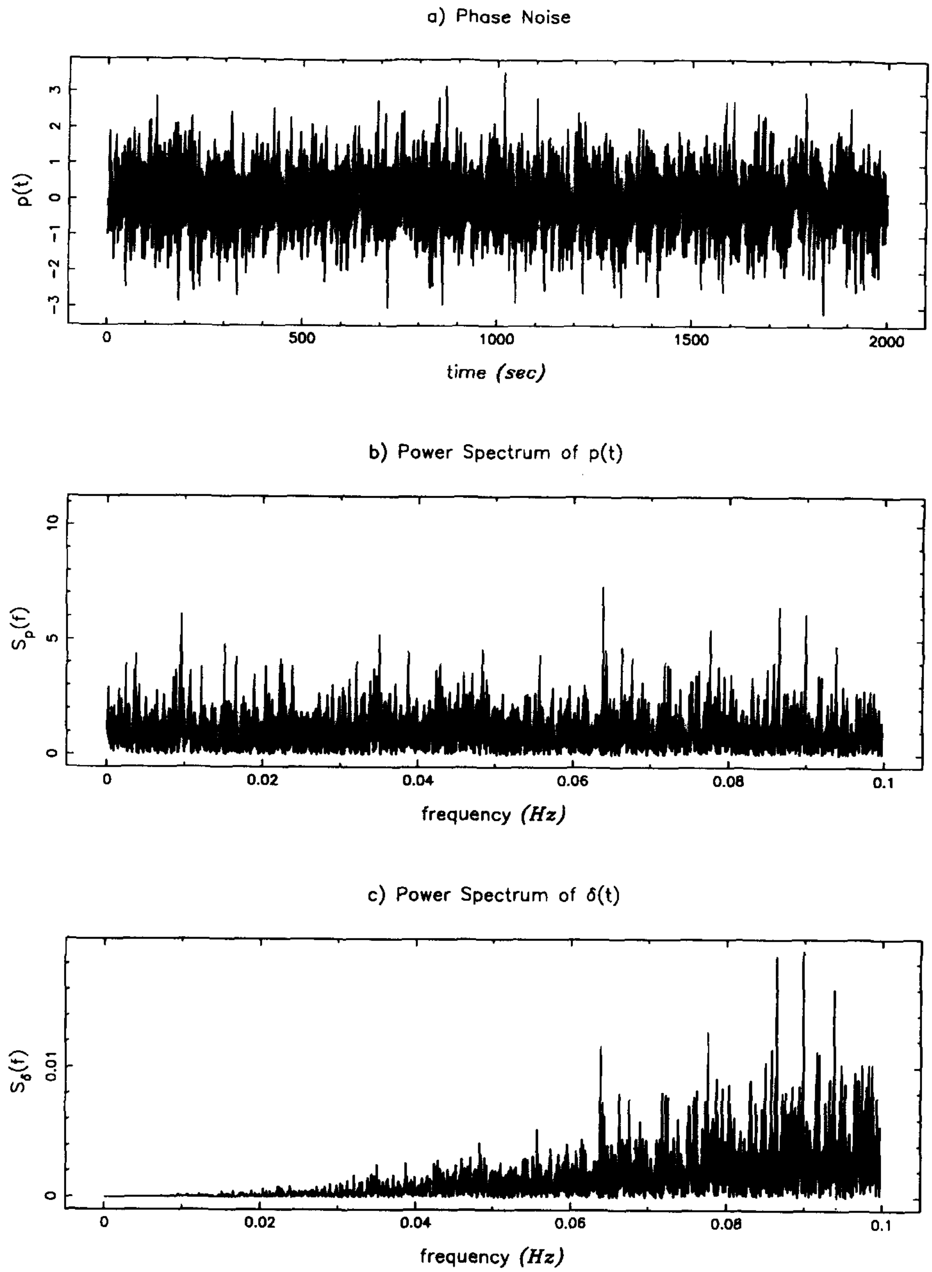

Fig. 2. Sequence of results of the data analysis: (a) Small portion of the time series $p(t)$, (b) Low frequency region ($f < 0.1$ Hz) of the power spectrum of $p(t)$, (c) Low frequency region ($f < 0.1$ Hz) of the power spectrum of $\delta(t)$, (d) Final output of the data analysis, $\Delta(t)$, (e) Power spectrum of $\Delta(t)$ around the region $f \sim 0.01$ Hz, (f) Same as Fig. (e) for the long baseline case, with a gravitational wave at 0.1 Hz.

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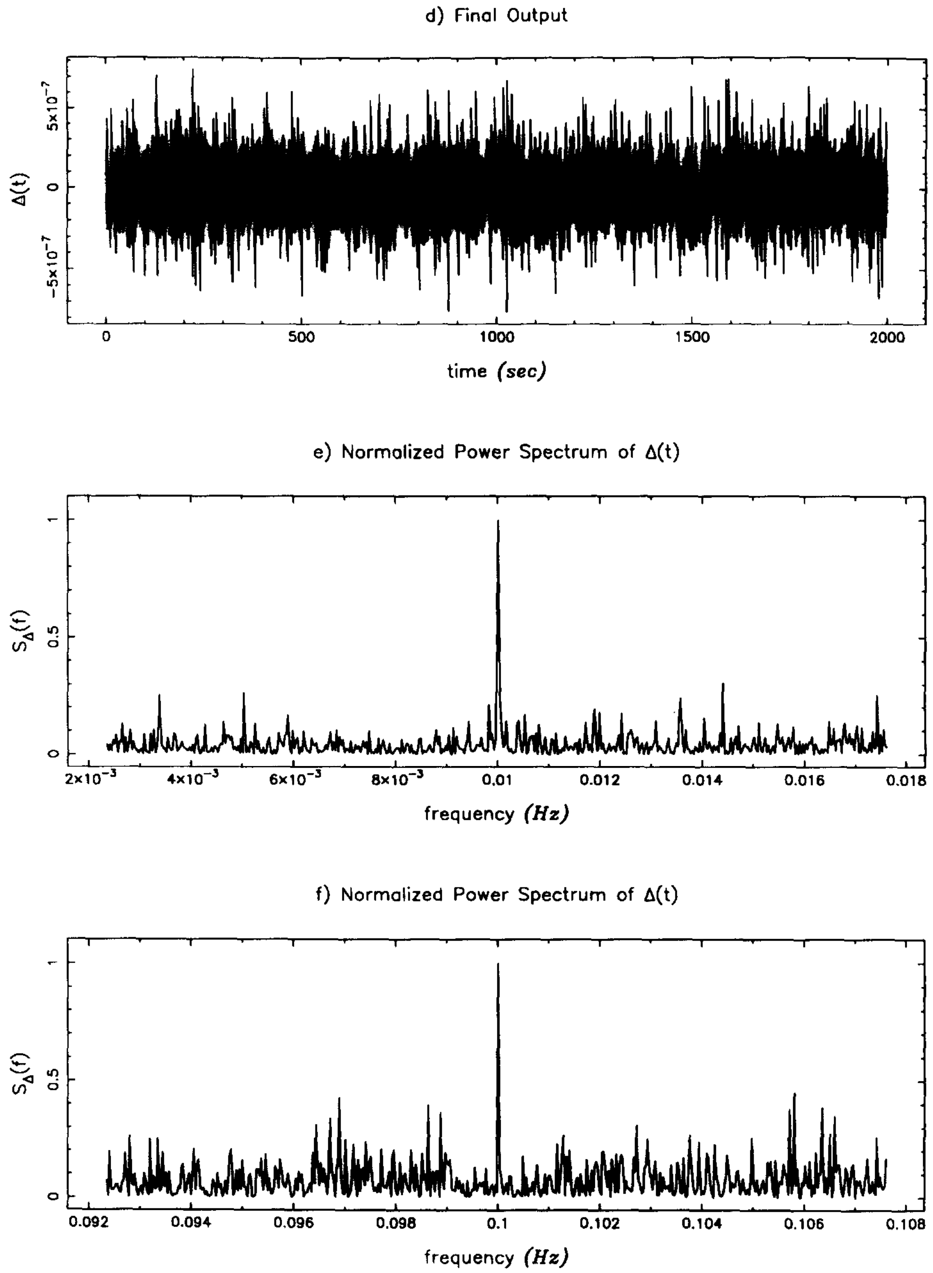

Fig. 2. Continued.

Appendix A

In this Appendix we provide an alternative way of using the information from the two phase differences $z_1(t)$, $z_2(t)$ in order to remove the laser phase noise from an interferometer of unequal arms. This method is more direct, and, as we shall show below, its effectiveness is equal to the method we have described in the body of this paper.

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Let us consider the two phase differences $z_1(f)$, $z_2(f)$ in the Fourier domain

$$z_1(f) = \left[p(f) + \frac{\nu h(f)}{2if}\right]\left[e^{4\pi i f l_1} - 1\right] + n_1(f)\,,\; z_2(f) = \left[p(f) - \frac{\nu h(f)}{2if}\right]\left[e^{4\pi i f l_2} - 1\right] + n_2(f)\,.$$

If we divide $z_j(f)$ by the transfer function $e^{4\pi i f(l_j+\delta l_j)} - 1$ and then take the difference between the resulting two quantities we obtain the following expression

$$O(f) \equiv \frac{z_1(f)}{e^{4\pi i f(l_1+\delta l_1)} - 1} - \frac{z_2(f)}{e^{4\pi i f(l_2+\delta l_2)} - 1} = \mathcal{P}(f) + \mathcal{N}(f) + \mathcal{H}(f)\,,$$

where $\mathcal{P}$, $\mathcal{N}$, and $\mathcal{H}$ are

$$\mathcal{P}(f) = p(f)\left[\frac{e^{4\pi i f l_1} - 1}{e^{4\pi i f(l_1+\delta l_1)} - 1} - \frac{e^{4\pi i f l_2} - 1}{e^{4\pi i f(l_2+\delta l_2)} - 1}\right], \tag{A.1}$$

$$\mathcal{N}(f) = \frac{n_1(f)}{e^{4\pi i f(l_1+\delta l_1)} - 1} - \frac{n_2(f)}{e^{4\pi i f(l_2+\delta l_2)} - 1}, \tag{A.2}$$

$$\mathcal{H}(f) = \frac{\nu h(f)}{2if}\left[\frac{e^{4\pi i f l_1} - 1}{e^{4\pi i f(l_1+\delta l_1)} - 1} + \frac{e^{4\pi i f l_2} - 1}{e^{4\pi i f(l_2+\delta l_2)} - 1}\right]. \tag{A.3}$$

If we expand Eqs. (A.1)–(A.3) in the long wavelength limit ($f\delta l_i \ll f l_1, f l_1 \ll 1$) we deduce the following expression for $O(f)$

$$O(f) \approx p_1(f)\left[\frac{l_1\delta l_2 - l_2\delta l_1}{l_1 l_2}\right] + \frac{n_1(f)l_2 - n_2(f)l_1}{4\pi i f l_1 l_2} + \frac{\nu h(f)}{if}. \tag{A.4}$$

We note that Eq. (A.4) can be obtained from the corresponding expression deduced in section 3 if we divide Eq. (15) by $2ifl_2$, and neglect terms of order $O(h\,\delta l)$ and $O(n\,\delta l)$.

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