Coupled Oscillators -Analogy with two level quantum systems

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Abstract

In this report we focus on the mathematical equivalence between the Schrodinger equation of a two level quantum system and the equations of motion of a classical coupled oscillator. This analogy is experimentally demonstrated using two rulers connected with a rubber band. The results are compared with the dynamics of a two level atom interacting with a monochromatics light field.

1 Introduction

It is well known that there are many analogies between the formal structure of the classical hamiltonian dynamics and structure of quantum mechanics. These analogies can play a crucial role in formulating and understanding fundamentally new ideas and concepts, or in obtaining new results. In the history of physics, theories such as Black-hole thermodyanmics, formulation of wave mechanics by Schrodinger using the formal similarity between geometrical optics have been obtained using analogy as a basic methodological tool.

Two-level quantum systems deviate most from classical systems, but there are instances of demonstration of classical analogues to two-level quantum systems. According to the correspondence principle between classical and quantum mechanics, all objects obey the laws of quantum mechanics, and classical mechanics is just an approximation for large systems of objects (or a statistical quantum mechanics of a large collection of particles). P.A.M Dirac's work on drawing parallel relation between the schrodinger's equation and equations of motion of classical coupled oscillators, disallowed the existence of any direct classical counter parts. In this report the mathematical equivalence between the Schrodinger equation of a two level quantum system and the equations of motion of a classical coupled oscillators have been derived explicitly. This analogy is experiemntally demonstrated using a laser beam reflected from two rulers coupled with each other. Phenomena which is usually observed in a two-level quantum system such as Rabi oscillation is demonstrated in the classical oscillator, with suitable interpretation.

2 Analogy between Coupled oscillators and Two level quantum system

Consider a system of two objects of mass M. The two objects are attached to two springs with spring constants k_1 and k_2 (Figure 1).

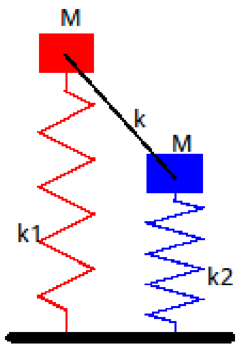

Figure 1: A classical coupled oscillator

The interaction force between the masses is represented by a third spring with spring constant k, which connects the two masses. We assume that the forces involved are spring-like forces (the magnitude of the force is proportional to the magnitude of the displacement from equilibrium).

Let x_1 ans x_2 denote the displacement of the masses from their equilibrium positions. Applying Newtons's law, the force on the left mass is equal to

$$F_1 = -k_1 x_1 + k(x_2 - x_1) = M\ddot{x}_1 \tag{1}$$

The force on the right mass is equal to

$$F_2 = -k_2 x_2 + k(x_2 - x_1) = M\ddot{x}_2 \tag{2}$$

The equations of motion are thus

$$M\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = 0 \tag{3}$$

$$M\ddot{x}_2 + (k_2 + k)x_2 - kx_1 = 0 \tag{4}$$

It is reasonable to assume that the resulting motion has an oscillatory behavior, we can consider following trial functions:

$$x_1(t) = B_1 e^{i\omega t} \tag{5}$$

$$x_2(t) = B_2 e^{i\omega t} \tag{6}$$

Substitution of $x_1(t)$ and $x_2(t)$ yields

$$
\begin{aligned}
(k_1 + k - M\omega^2)B_1 - kB_2 &= 0 && (7)\\
(k_2 + k - M\omega^2)B_2 - kB_1 &= 0 && (8)
\end{aligned}
$$

In order for a non-trivial solution to exist, the determinant of coefficients of B_1 and B_2 must vanish. This yields

$$
\left[k_1 + \left(k - M\omega^2\right)\right]\left[k_2 + \left(k - M\omega^2\right)\right] = k^2 \tag{9}
$$

from which we obtain

$$
\omega^2 = \frac{k_1 + k_2 + k}{2M} \pm \frac{1}{2M}\sqrt{(k_1 - k_2)^2 + 4k^2} \tag{10}
$$

Hence, the square roots of the two eigenvalues (i,e ω_1 and ω_2) of matrix A represented by (7) and (8) can be rewritten in terms of natural frequencies of each oscillators $\nu_1 = \sqrt{\frac{k_1+k}{M}}$ and $\nu_2 = \sqrt{\frac{k_1+k}{M}}$ and coupling strength $\Omega = \sqrt{\frac{k}{M}}$ as

$$
\begin{aligned}
\omega_1 &= \left(\frac{{\nu_1}^2 + {\nu_2}^2}{2} + \frac{1}{2}\sqrt{\left({\nu_1}^2 - {\nu_2}^2\right)^2 + 4\Omega^4}\right)^{\frac{1}{2}} && (11)\\
\omega_2 &= \left(\frac{{\nu_1}^2 + {\nu_2}^2}{2} - \frac{1}{2}\sqrt{\left({\nu_1}^2 - {\nu_2}^2\right)^2 + 4\Omega^4}\right)^{\frac{1}{2}} && (12)
\end{aligned}
$$

The matrix A can be rewritten as

$$
\begin{pmatrix} {\nu_1}^2 & -\Omega^2 \\ -\Omega^2 & {\nu_2}^2 \end{pmatrix}
$$

Hence the displacement of two oscillators from their mean positions can written as

$$
\begin{aligned}
x_1(t) &= f_1 a_{11} e^{-i(\omega_1 t + \phi_1)} + f_2 a_{21} e^{-i(\omega_2 t + \phi_2)} && (13)\\
x_2(t) &= f_1 a_{12} e^{-i(\omega_1 t + \phi_1)} + f_2 a_{22} e^{-i(\omega_2 t + \phi_2)} && (14)
\end{aligned}
$$

Here $\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}$ and $\begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$ are the normalized eigen vectors of A. The constants f_1, f_2, ϕ_1, ϕ_2 are determined from the initial values of positions and velocities of the two oscillator.

The classical equations of motion (4) and (3) of the two oscillators can be written in the form of Schrodinger's equation as follows.

$$
i\hbar\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \hbar\sqrt{A}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{15}
$$

Equation 15 can be treated as a two-level quantum system and $\hbar\sqrt{A}$ as the hamiltonian of the system. Here the matrix A is symmetric. For every real non-zero coloumn vector $z = \begin{pmatrix} a \\ b \end{pmatrix}$, we have $z^T A z = \frac{a{k_1}^2}{m} + \frac{b{k_2}^2}{m} + \frac{k(a-b)^2}{m}$ which is always positive. Hence A is positive semidefinite. Therefore A can be diagonalized by a similarity transformation $T^{-1}AT$ where $T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\Omega^2}{\nu_1^2 - \nu_2^2}\right)$.

Equation 15 further reduces to

$$i\hbar\frac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=\frac{\hbar}{2}\begin{pmatrix}\omega_0-\Delta & \Omega_0\\ \Omega_0 & \omega_0+\Delta\end{pmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} \tag{16}$$

where $\omega_0=(\omega_1+\omega_2)\cos 2\theta$, $\Delta=(\omega_1-\omega_2)\cos 2\theta$ and $\Omega_0=(\omega_1-\omega_2)\sin 2\theta$.

2.1 Rabi Oscillation

It is the cyclic behaviour of a two-level quantum system in the presence of an oscillatory driving field. When an atom (or some other two-level system) is illuminated by a laser beam, it will cyclically absorb photons and re-emit them by stimulated emission. One such cycle is called a Rabi cycle and the inverse of its duration the Rabi frequency of the photon beam.

Consider two atomic levels with ground state $|g>$ and excited state $|e>$. The transition frequency $\omega_0=(E_e-E_g)/\hbar$. In the prescence of an electric field $E(t)=E_0\cos\omega t$, where $|\omega-\omega_0|<<\omega_0$. The two-level Hamiltonian of the system is

$$H=-\frac{\hbar\omega_0}{2}\sigma_z-A\cos(\omega t)\sigma_x \tag{17}$$

Here $A=<e|\hat{d}\cdot E_0|g>$, $\hat{d}$ is transistion dipole moment. The equation of state is given by

$$|\psi(t)>=C_g(t)e^{-iE_gt/\hbar}|g>+C_e(t)e^{-iE_et/\hbar}|e> \tag{18}$$

with $E_g=-\hbar\omega_0/2$ and $E_e=\hbar\omega_0/2$, we have

$$\dot{C}_g=\frac{i}{\hbar}A\cos(\omega t)e^{-i\omega_0 t}C_e \tag{19}$$

$$\dot{C}_e=\frac{i}{\hbar}A\cos(\omega t)e^{i\omega_0 t}C_g \tag{20}$$

Expanding the term $\cos(\omega t)=(e^{i\omega t}+e^{-i\omega t})/2$, we find the slowly rotating terms $e^{\pm(\omega-\omega_0)t}$ and the fast terms $e^{\pm(\omega+\omega_0)t}$. The time-evolution induced by the applied field is much slower than ω_0, we can neglect the quickly rotating terms,

$$\dot{C}_g=\frac{i}{2\hbar}Ae^{i(\omega-\omega_0)t}C_e \tag{21}$$

$$\dot{C}_e=\frac{i}{2\hbar}Ae^{i(\omega-\omega_0)t}C_g \tag{22}$$

We can eliminate C_g

$$\ddot{C}_e+i(\omega-\omega_0)\dot{C}_e+\frac{1}{4}\frac{A^2}{\hbar^2}C_e=0 \tag{23}$$

From the trial solution $C_e=e^{i\lambda t}$, we find two roots

$$\lambda_\pm=\frac{1}{2}\left(\Delta\pm\sqrt{\Delta^2+A^2/\hbar^2}\right) \tag{24}$$

where $\Delta = \omega_0 - \omega$ is the detuning. The general solution is

$$C_e(t) = C_+ e^{i\lambda_+ t} + C_- e^{i\lambda_- t} \tag{25}$$

and considering the initial conditions $C_g(0) = 1$ and $C_e(0) = 0$, the particular solution is

$$C_e(t) = i\frac{A}{\Omega_R \hbar} e^{i\Delta t/2} \sin(\Omega_R t/2) \tag{26}$$

$$C_e(t) = e^{-i\Delta t/2} i \left(\cos(\Omega_R t/2) + i\frac{\Delta}{\Omega_R} \sin(\Omega_R t/2) \right) \tag{27}$$

where $\Omega_R = \sqrt{\Delta^2 + A^2/\hbar^2}$ is Rabi frequency. The probability to find the atom in state $|e>$ is given by

$$P_e(t) = |C_e(t)|^2 = \frac{A^2}{\hbar^2 \Omega_R^2} \sin(\Omega_R t/2)^2 \tag{28}$$

Equation 16 can now be treated as the equations of motion of a two level atom close to resonance, interacting with a monochromatic laser beam. Δ corresponds to the difference between the resonance frequency of the atomic transistion line and laser frequency. Ω_0 is then Rabi Oscillation frequency . For $\Delta \neq 0$, the population of the atomic states oscillates with effective Rabi frequency. $\Omega_R = \sqrt{\Omega_0^2 + \Delta^2}$

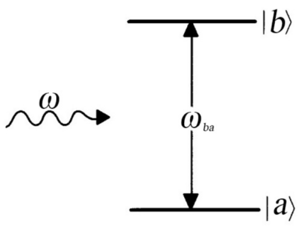

Figure 2: A two level atom with atomic transistion at ω_{ba} interacting with a monochromatic light of frequency ω_l

For $\Omega << \omega_1, \omega_2$, we have $\omega_1 + \omega_2 \approx \nu_1 + \nu_2$. From equations 11 and 12 we can find $\Delta \approx \nu_1 - \nu_2$ The Rabi frequency, $\Omega_0 = \frac{2\Omega^2}{\nu_1 + \nu_2}$ and $\Omega_R = \sqrt{\Delta^2 + \Omega_0^2}$

3 Experimental Demonstration

The experimental setup is shown in figure 3.

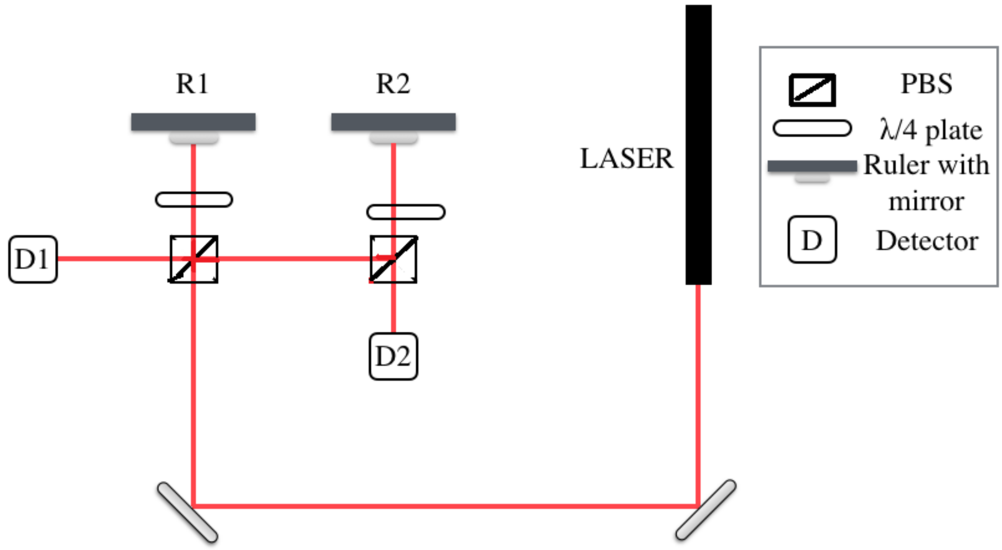

Figure 3: Schematic diagram of the experimental setup

R_1 and R_2 represents two identical rulers mounted vertically, side by side close to one another on an optical bench. The laser beam is transmitted by Polarising beam splitter (PBS1) and quarter wave plate and is retro-reflected by a mirror mounted on R1. The reflected beam passes through quater wave plate and (PBS1) and falls on detector (D1). A part of the laser beam which is reflected from PBS1 is further reflected by the second polarising beam splitter (PBS2) and is transmitted through quarter wave plate and falls on ruler R2. The beam reflected from a mirror mounted on R2 passes through QWP2 and PBS2 and falls on detector D2. When the rulers are at rest, the reflected beams are alligned to fall on the centre of quad detectors. A rubber band is put around both the rulers to couple them. The spring constant, mass and distance of the rubber from the rulers' base determine the strength of the coupling and Rabi frequency.

Here we convert the motion of the scale to measurable electrical signal. A four-quadrant detector was used to accomplish this. Figure 5 represents the circuit diagram of the detectors used. The quad detectors consists of four photodiodes arranged as a four quarters of a circle. In figure 5, U2 and U3 are the voltages produced at the upper and lower halves of the quad photodiodes due oscillating motion of the ruler. The signals are amplified through D1 and D2 amplifier. Here AD708 monolithic operational amplifiers have been used. The amplified signals are then fed to a differential amplifier. U1 is the amplified volatage proportional to the difference of U2 and U3. U1 is the final output of the detectors which is fed to the oscilloscope. Both D1 and D2 have same values of gain settings.

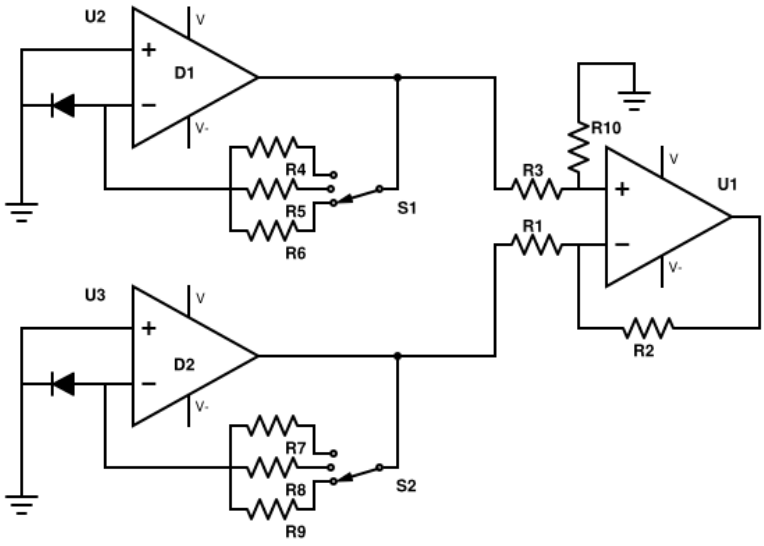

Figure 4: Circuit diagram of the detectors used in the experimental setup

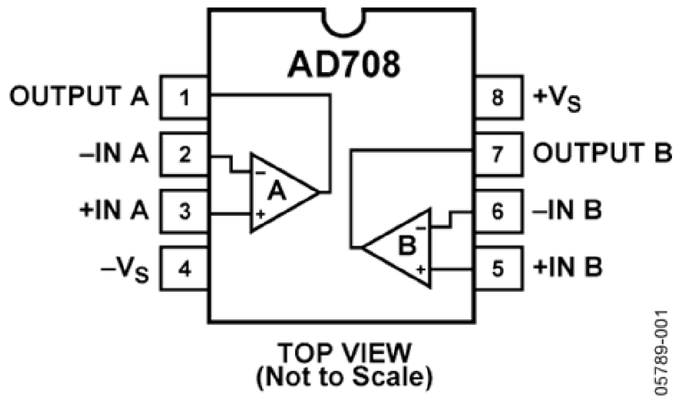

Figure 5: Pin configuration of AD708 operational amplifier

The vibration of the ruler when struck causes a difference of intensity of the reflected light beam falling on the photodiode. This is due to the vertical shift of reflected beam when the ruler is struck. This produces a voltage which is equal to the difference in voltages produced at the upper and lower end of the four-quadrant detector.

4 Observation and Ananlysis

In order to study the analogues of Rabi oscillation, one ruler was struck while the motion of one ruler was inhibited. The natural frequencies ν_1 and ν_2 of the oscillators are determined by fourier transformation of the signals generated from the oscilloscope. The natural frequency ν_1 of ruler was changed by attaching small masses to it. The natural frequency ν_2 of the second ruler was kept fixed. For each ν_1, displacement of both the rulers from their mean positions as a function of time was recorded. Corresponding ω_1 and ω_2 are determined using equations (13) and (14).

Figure 6 represents the signal obtained at resonance where, $\nu_1 = \nu_2$. The red dots are the experimental data and the blue solid line is the fitted curve. Real part of equations 13 and 14 is used for fitting. ϕ_1, ϕ_2, ω_1, ω_2 are determined from the fit.

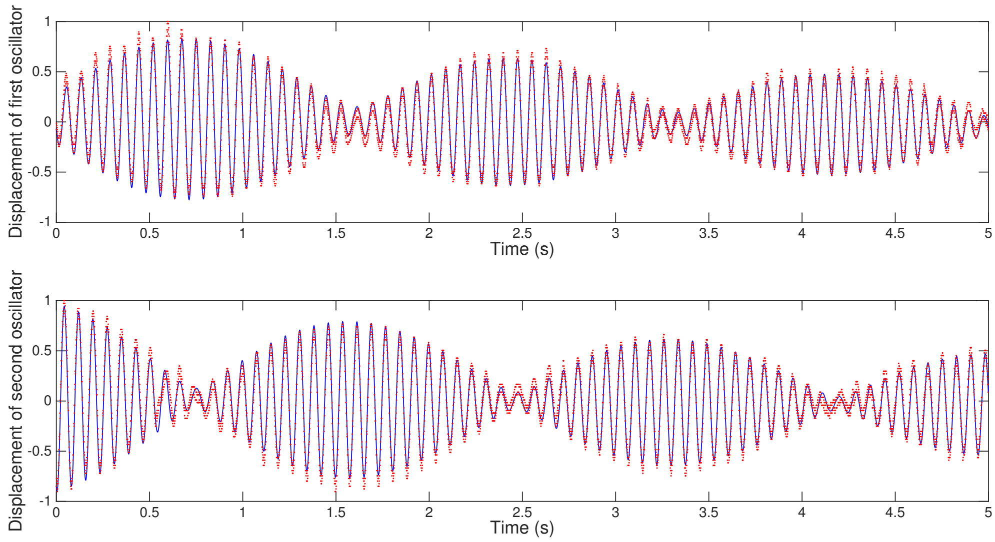

Figure 6: Normalised displacement of the rulers from their equilibrium positions as function of time

The above figure depicts the similarity with the Rabi oscillations.The oscillations amplitudes of the two rulers alternate. Both the signals deccay with time as the energy is dissipated with time.

Figure 7 shows the normal mode frequencies as a function of detuning Δ. The dots represents the experimental data and the solid lines are the fitted curve. Equations 11 and 12 are used for the fitting. Both these equations are explicitily expressed as a function of Δ. The natural frequency ν_2 is kept to be constant. ν_1 is observed to be a linearl function of Δ The dotted lines are the natural frequencies of both the oscillators. At resonance i,e $\Delta = 0$ we can see a avoided crossing for the classical oscillators which is similar to what is observed in atomic transition line near resonance. Ω is determined from the fitting.

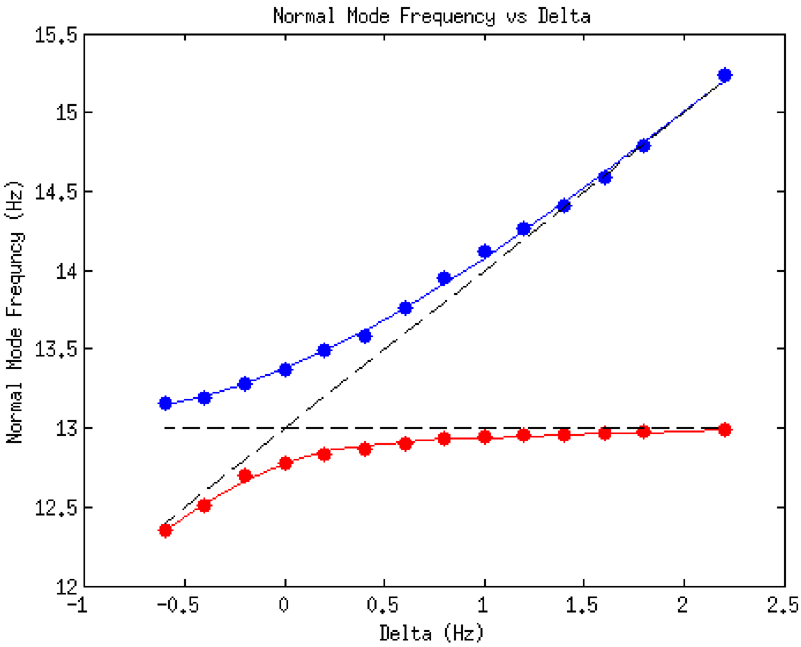

Figure 7: Normal mode frequencies as a function of detuning Δ

For a given natural frequency and Ω as determined from above, the Rabi frequency can be calculated as $\Omega_0 = \frac{2\Omega^2}{\nu_1+\nu_2}$. Further the effective Rabi frequency Ω_R is deterimed using $\Omega_R = \sqrt{\Omega_0^2 + \Delta^2}$. Figure shows the variation of Ω_R as a function of Δ

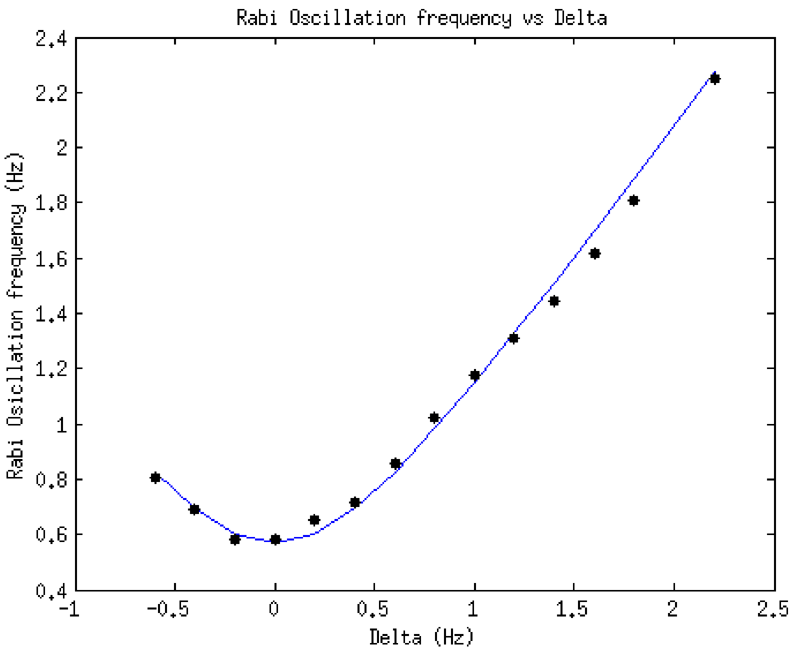

Figure 8: Effective Rabi frequency as a function of Δ

The energy of ruler R1 is proportional to $|x_1|^2$. From equation 13 and 14 we get,

$$|x_1|^2 = |f_1 a_{11}|^2 + |f_2 a_{21}|^2 + f_1 a_{11} f_2 a_{21} \cos(\Omega_R t + (\phi_1 - \phi_2)) \quad (29)$$
$$|x_2|^2 = |f_1 a_{12}|^2 + |f_2 a_{22}|^2 + f_1 a_{12} f_2 a_{22} \cos(\Omega_R t + (\phi_1 - \phi_2)) \quad (30)$$

At time $t = 0$ ruler R1 is struck, therefore it has maximum amplitude $\implies$ maximum energy. At time $t = \pi/\Omega_R$, R1 has minimum amplitude and hence minimum energy. The percentage of energy transferred from R1 to R2 is proportional to $(|x_1|_{max}{}^2 - (|x_1|_{min}{}^2)/|x_1|_{max}{}^2$. The normalised energy transformed is shown in figure 9.

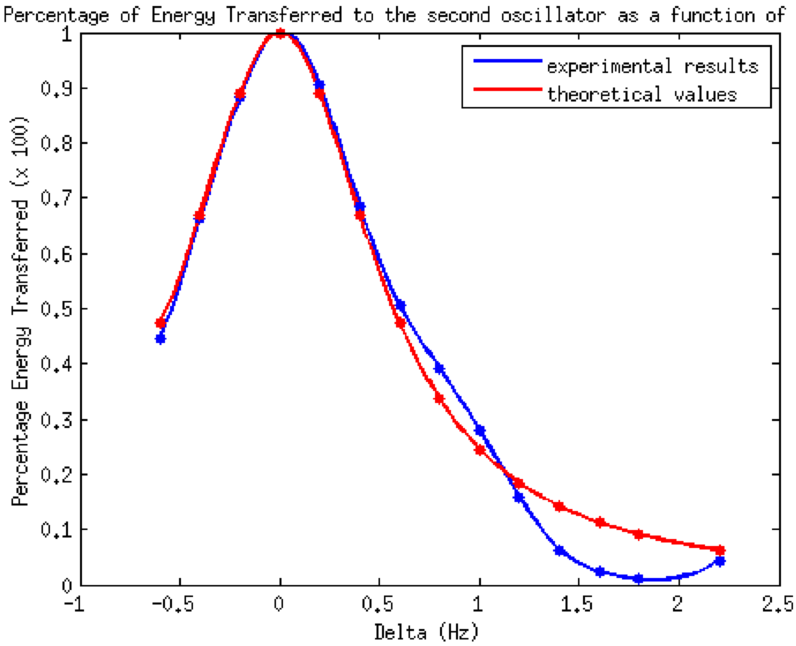

Figure 9: Normalised energy tranformed from R_1 to R_2

From figure 9 we can see that there is a maximum energy transfer at resonance as expected from above calculations and this energy transfer decreases as we go out of resonance. Theoretically the minimum ampltiude of oscillations of both the rulers should be equal to zero at resonance, but a non-zero minimum amplitude was observed. This can be attributed due to non-perfect intial conditions or due to other elements in the setup.

5 Conclusion

The mathematical equivalence between the classical coupled oscillation and rabi oscillation is consistent with the experiment. Therefore a periodical transistion of a two level quantum system is equilvalent to a pair of classical coupled oscillator. As a result of the analogy, many basic phenomena observed in the quantum system can be simulated in the coupled classical oscillator. This experiment can also be used to study geometric phase.

6 References

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