Coupled Oscillators -Analogy with two level quantum systems

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Abstract

In this report we discuss the equivalence of the classical equations of motion of a coupled oscillator to the Schrodingers equation of a two-level quantum system. A simple experimental demonstration of the analogy is presented. Phenomena usually observed in a two-level quantum system such as Rabi oscillation, energy level splitting and excited state transition probability due to coupling and Geometric phases are demonstrated in the classical oscillator, with suitable interpretation.

Introduction

Two-level quantum systems are arguably the simplest non-trivial quantum systems and they are said to deviate most dramatically from classical systems.However, there have been instances of demonstration of classical analogues of two-level quantum systems. In this report , we present a simple experimental demonstration of the analogy between the coupled oscillator and a two-level quantum system. Similar phenomenalike Rabi oscillation, quantum mechanical energy level repulsion and excited state transition probability due to couplingare demonstrated with a suitable interpretation in the context of coupled pendulum.

Analogy between Coupled oscillators and Two level quantum system

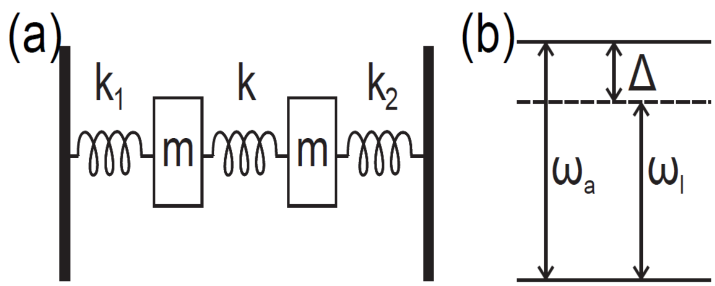

Figure 1: (a) A classical coupled oscillator (b) Representation of a two level atom with atomic transition at ω_a interacting with monochromatic light field of frequency ω_l.

Consider a pair of coupled oscillators as shown in Fig. 1, consisting of two equal masses of magnitude m each, attached to opposite walls with springs of spring constants k_1 and k_2, and attached to each other by a coupling spring of spring constant k. If the displacement of the masses from their equilibrium positions are denoted by q_1 and q_2, application of Newtons laws yields their equations of motion as

$$-\frac{d^2}{dt^2}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} V_1^2 & \Omega^2 \\ \Omega^2 & V_2^2 \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \text{...............(1)}$$

where $V_1 = \sqrt{\frac{k_1+k}{m}}$ and $V_2 = \sqrt{\frac{k_2+k}{m}}$ are the natural frequencies of each of the oscillators when the other mass is held fixed and $\Omega = \sqrt{\frac{k}{m}}$ is indicative of the coupling strength between the oscillators. We denote the square matrix on the right hand side of Eqn. (1) by A. The square roots of the two eigenvalues of A, denoted by ω_1 and ω_2, are

$$\omega_1 = \left(\frac{V_1^2+V_2^2}{2} + \frac{1}{2}\sqrt{(V_1^2-V_2^2)^2+4\Omega^4}\right)^{1/2} \text{............(2)}$$

$$\omega_2 = \left(\frac{V_1^2+V_2^2}{2} - \frac{1}{2}\sqrt{(V_1^2-V_2^2)^2+4\Omega^4}\right)^{1/2}$$

The motion of the two masses, in general, will be a superposition of the two normal modes of vibration, whose frequencies are given by ω_1 and ω_2. The general equations of motion are

$$q_1(t) = f_1 a_{11} e^{-i(\omega_1 t+\phi_1)} + f_2 a_{21} e^{-i(\omega_2 t+\phi_2)} \text{............(3)}$$

$$q_2(t) = f_1 a_{12} e^{-i(\omega_1 t+\phi_1)} + f_2 a_{22} e^{-i(\omega_2 t+\phi_2)}$$

where $\begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$ and $\begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}$ are the normalized eigen vectors of A corresponding to ω_1 and ω_2 respectively. Given the initial positions and velocities of the two oscillators f_1, f_2, ϕ_1 and ϕ_2 can be determined uniquely.

Equation (1) can be also written as

$$\left(i\frac{d}{dt}\right)^2 \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \left(\sqrt{A}\right)^2 \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

$$\Longrightarrow i\hbar\frac{d}{dt} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \hbar\sqrt{A} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \text{...........(4)}$$

It may be noted that Eqn. (4) is exactly of the form of Schrodingers equation governing a two level quantum system where $\hbar\sqrt{A}$ serves as the Hamiltonian of the system. Since A is a symmetric and positive definite matrix, it can be diagonalized by a similarity transformation, $P^{-1}AP$. The roots of the diagonal elements can be taken and the resulting matrix can be transformed back to obtain $\sqrt{A}$. The transformation matrix P is given by $\begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$ where $\theta = \frac{1}{2}tan^{-1}\left(\frac{2\Omega^2}{V_1^2 - V_2^2}\right)$. By following the above method, $\sqrt{A}$ can be determined and Eqn. (4) can be written as

$$\Longrightarrow i\hbar\frac{d}{dt} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} \omega_0 - \Delta & \Omega_0 \\ \Omega_0 & \omega_0 + \Delta \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \text{...........(5)}$$

where $\omega_0 = (\omega_1 + \omega_2)cos2\theta, \Delta = (\omega_1 + \omega_2)cos2\theta$ and $\Omega_0 = (\omega_1 - \omega_2)sin2\theta$

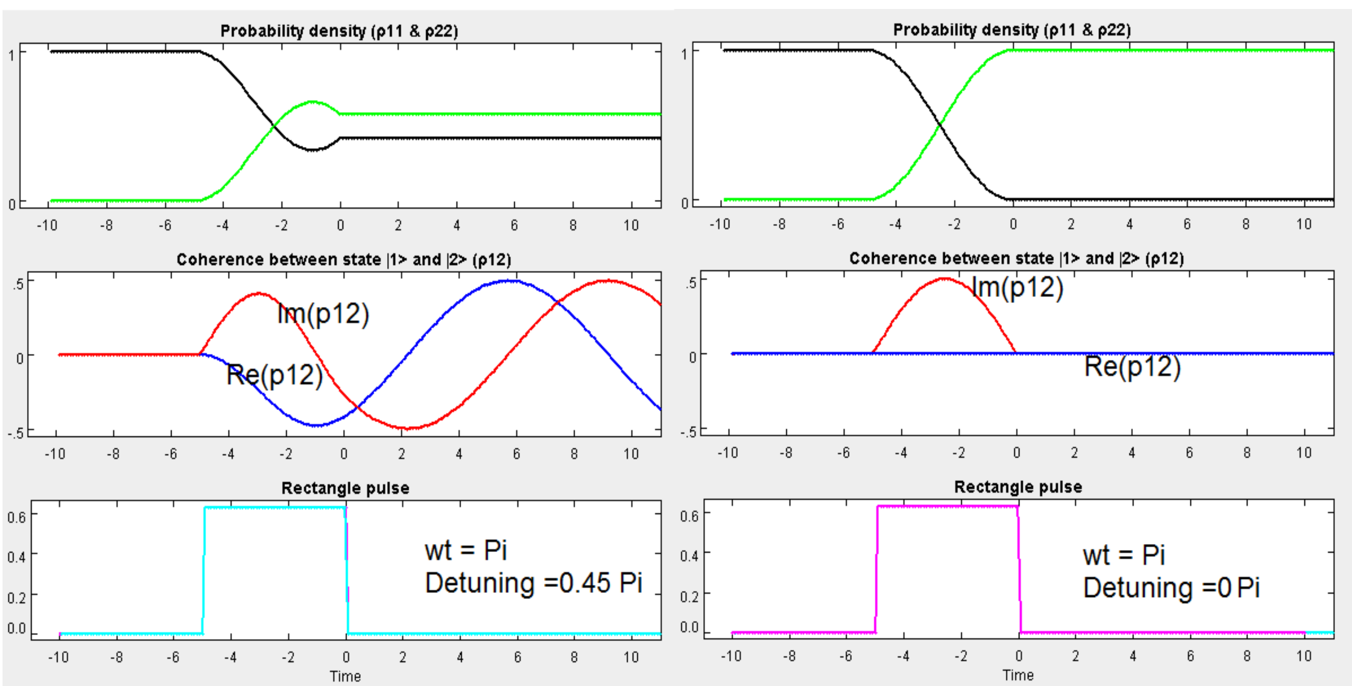

Figure 2: Probability densities for two level quantum system interacting with EM radiation with and with out detuning

Equation (5) can be compared to the quantum mechanical equations of motion of a two level quantum system, specifically to that in which a monochromatic laser beam is interacting with a two level atom close to resonance, as shown in Fig. 1. Δ would then denote the detuning, which is the difference between the laser frequency and the resonance frequency of the atomic transition line. Ω_0 would be the Rabi oscillation frequency, the frequency at which the population of the atomic states oscillate for laser frequency exactly at resonance. For non-zero detuning, the population oscillates with effective Rabi frequency, $\Omega_R = \sqrt{\Omega_0^2 + \Delta^2}$. Also, the transition probability to the excited state is Ω_0^2/Ω_R^2

In the case of the coupled oscillators, it can be shown using Eqn. (2) that $\omega_1 + \omega_2 \approx V_1 + V_2$ for $\Omega << \omega_1, \omega_2$. Using this approximation, it can be shown that detuning, $\Delta \approx V_1 - V_2$ and Rabi frequency, $\Omega_0 = \frac{2\Omega^2}{V_1+V_2}$. The effective Rabi frequency for finite detuning then becomes $\Omega_R = \omega_1 - \omega_2 = \sqrt{\Omega_0^2 + \Delta^2}$. Also, note that the square of the amplitudes of the oscillators normalized to the total energy of the system can be compared to the probability amplitudes of the population of the corresponding atomic states. In the case of laser atom interaction, Δ can be varied by changing the laser frequency and the Rabi frequency Ω_0 remains constant for a fixed amplitude of laser field. However, in the case of the coupled oscillators, Δ can only be varied by changing the spring constant of one of the oscillators. As a result, the Rabi frequency also changes, since it depends on the natural frequencies of the oscillators. However, the Δ dependence of the Rabi frequency can be suitably implemented in the analysis for the correct comparison of both the systems

Complete Quantum mechanical Wavefunction

Here Column matrix $q = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$ which we use can be considered as real part of quantum mechanical wave function. Taking Strocchi classical Hamiltonian, Eq. The Hamilton equations is given by symbolically writing $q = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$ and $p = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$ as column vectors and H as Hamiltonian matrix.Here p is momenta of oscillators

$$\dot{q} = Hp$$

and

$$\dot{p} = -Hq$$

$$\ddot{q} = H\dot{p} = -H^2 q$$

$$p = H^{-1}\dot{q}$$

A new set of complex amplitudes, the vector z, is constructed as

$$z = \frac{1}{\sqrt{2}}(q + ip).....(5)$$

From the above equations we can say that

$$\ddot{z} = -H^2 z$$

$$\implies i\dot{z} = Hz$$

From here we can say that z serves as complete wavefunction with complex amplitudes.

Experimental Demonstation of the Analogy

A schematic diagram of the experimental setup used to demonstrate the above analogy is shown in Fig. 2. Two rulers R_1 and R_2 are mounted vertically, side by side, on an optical table. When struck at the top end, the rulers vibrate with their natural frequencies, with the tensile property of their material accounting for the spring constant. A rubber band is put around them to couple the oscillators. The spring constant of the

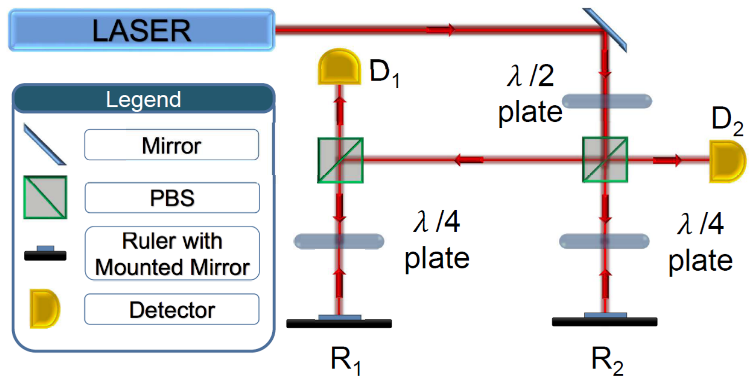

Figure 3: Schematic diagram of the experimental setup.

rubber band along with the distance of the rubber band from the base of the rulers decide the strength of the coupling and hence the Rabi frequency. In our experiment, we chose a suitable Rabi frequency by changing the position of the rubber band. The arrangement we used for quantitative measurement of the oscillations is also shown in Fig. 2. The aim here is to convert the motion of the scale to a measurable electrical signal. A motion sensor may be used for this purpose, but a simple setup was designed to accomplish this. The idea that the movement of the ruler would cause a difference of intensity of a beam of light reflected from the ruler and falling on a photodiode, and thus the produced voltage, is utilized.

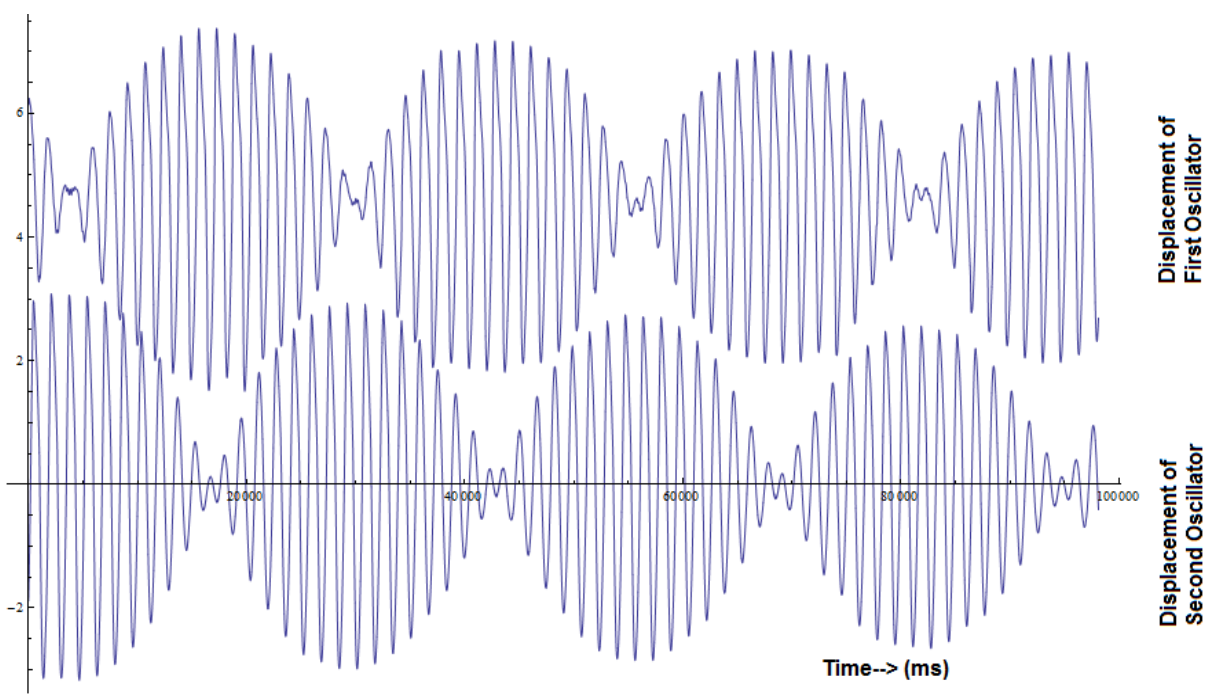

Figure 4: Displacement of coupled oscillators

A laser beam transmitted through a polarizing cube beam splitter (PBS) and a quarter wave plate is retro-reflected by a mirror mounted on the ruler R_2. The retro-reflected beam is reflected by the same cube beam splitter to fall on the quad-detector D_2. The reflected beam from the first cube beam splitter is again reflected by a second polarizing cube beam splitter and transmitted through a quarter wave plate. This beam is similarly retro-reflected by a mirror mounted on the ruler R_1 and transmitted through the second cube beam splitter to fall on the quad detector D_1. The quad detectors have four photo-diodes arranged as four quarters of a circle. When the rulers are at rest, these two reflected beams are made to fall at the center of the quad detectors. The vertical shift in the reflected beam due to the vibration of the ruler causes a difference in the intensity of light falling on either half of the photo diode. The movement of the rulers are traced by amplifying the potential difference between the two halves of the quad detectors and feeding it into an oscilloscope.

A typical signal at near resonant condition, that is $V_1 \approx V_2$ or $\Delta \approx 0$, is shown in Fig. 4. As in the case of Rabi oscillation of the population of a two level atom interacting with a laser beam, the oscillation amplitudes of the two oscillators are seen to alternate, one showing maximum oscillation amplitude while the other is showing minimum. This transfer of oscillation amplitude, which is equivalent to the transfer of energy, is observed to be maximum at $\Delta = 0$, where the transfer is nearly complete.

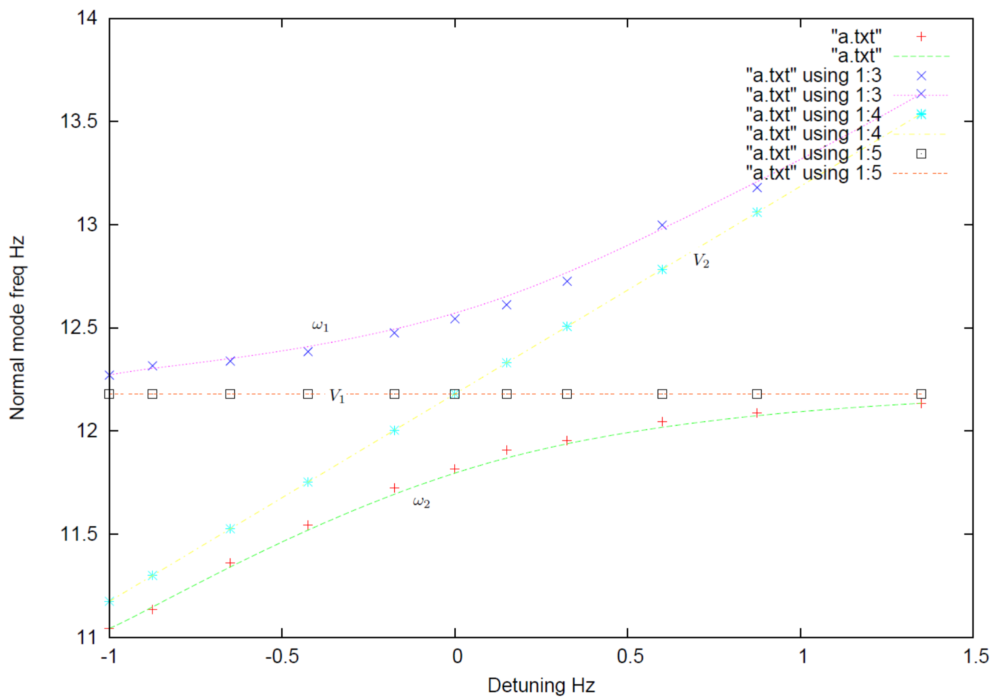

Figure 5: Normal mode frequency vs Detuning

For the quantitative measurements of the analogues of different features of the Rabi oscillation in these coupled oscillators, the following procedure was adopted. By preventing the motion of one ruler and only striking the other the natural frequencies of the oscillators, V1 and V2, were determined by fitting the resulting signals with the equation of a damped harmonic oscillator. The natural frequency V1 of oscillator R1 was changed by sticking small weights to it, while keeping V2 fixed. For each frequency V1, the coupled oscillations of the oscillators, like that in Fig. 5, were recorded and fitted with Eqns. (3) to determine the normal mode frequencies ω_1 and ω_2. Treating V1 as the independent parameter and V2 and Ω as the fitting parameters, the normal mode frequencies ω_1 and ω_2 were fitted with the expressions in Eqn. (2). The normal mode frequencies as a function of detuning Δ is shown in Fig. 6. The typical avoided crossing seen in atomic transition lines near resonance can be observed in the case of the classical coupled oscillators. Ω, which is related to the spring constant of the coupling spring, was determined from the fitting. Given a natural frequency and knowing Ω, the Rabi frequency Ω_0 can be determined using $\Omega_0 = \frac{2\Omega^2}{V_1+V_2}$. The

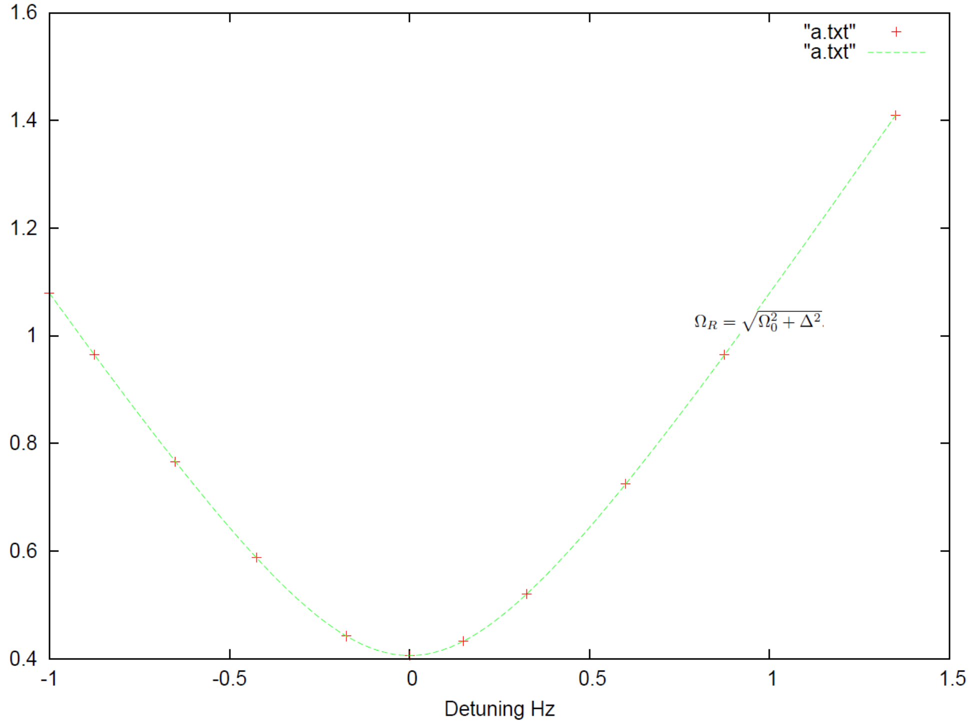

Figure 6: Normal mode frequency vs Detuning

effective Rabi frequency , which is the rate at which energy is transferred between the oscillators, is plotted as a function of Δ in the inset of Fig. 6.

The experimental data compares with the theoretical plot of $\Omega_0^2/\sqrt{\Omega_0^2+\Delta^2}$as shown in green line in Fig 7. A systematic shift of the experimental data (Red line)from the theoretical model can be seen. We infer this systematic shift to be due to the additional masses added to R1 to change its natural frequency. Due to the additional mass δm, the natural frequency V1 changes as $\sqrt{(k1+k)/(m+\delta m)}$ and at the same time, the coupling strength Ω changes as $\sqrt{k/(m+\delta m)}$

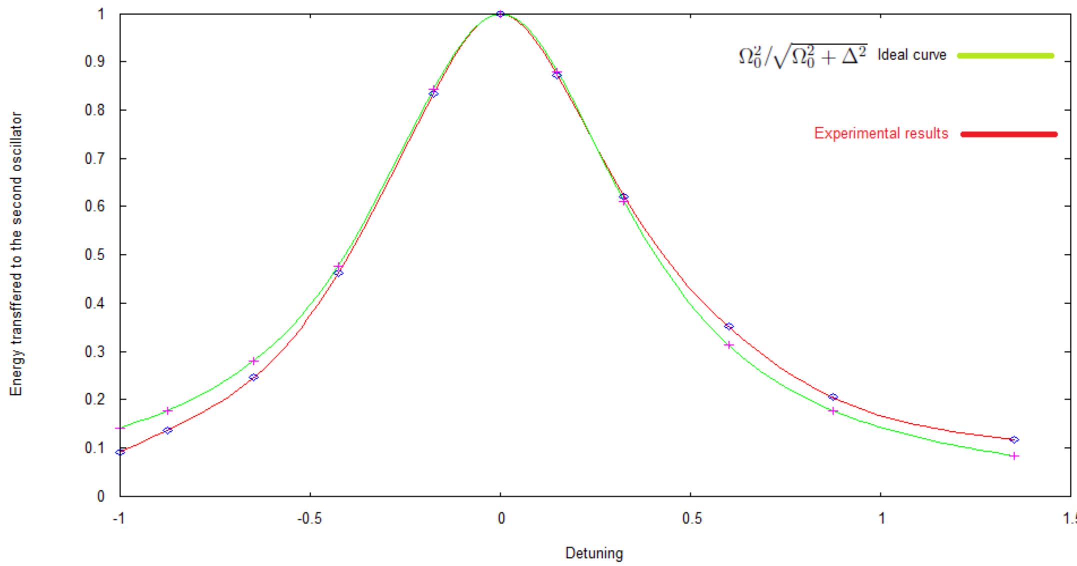

Figure 7: Percentage of energy transferred from ruler R1 to R2

Geometric Phases

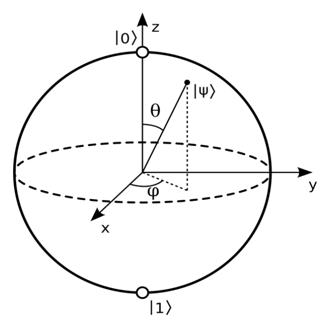

Figure 8: Bloch sphere representation of two level system

A wave vector representing a two level quantum system can be represented on Bloch Sphere as

$$|\psi\rangle = cos(\theta/2) + e^{i\phi}sin(\theta/2)$$

Here$|0\rangle$ can be viewed as a state where complete energy of system is present in first oscillator ,and $|1\rangle$ can be viewed as a state where complete energy of system is present in second oscillator. For a cyclic Evolution of Hamiltonian ,Wave function acquired a phase depending on its evolution on Bloch sphere. This phase is called Geometric Phase because its purely geometric in nature.It is given by half of the solid angle covered by the path taken by wave function on Bloch sphere.

In Resonance condition for one Rabi oscillation bloch vector traces a path along a great circle. So the solid angle covered is given by 2π.So π phase is acquired.That means

$$|\psi(t+\frac{1}{\Omega_0})\rangle = -|\psi(t)\rangle$$

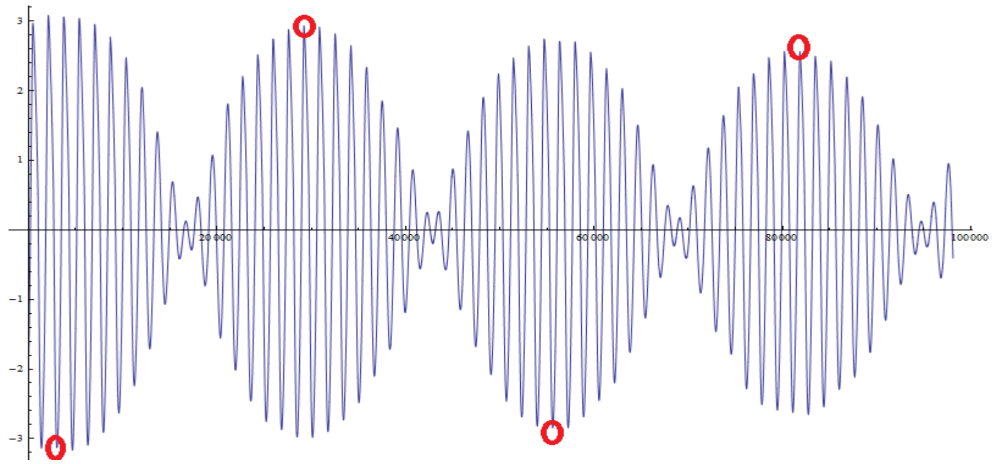

Figure 9: Points corresponding to max Amplitude(Energy) in first oscillator after each rabi cycle

Maximum amplitude for wave vector with complex amplitude is given by

$$z_{max} = max(\sqrt{q_1^2 + p_1^2})$$

q_1 represents position and p_1 represents the momentum given by Eqn(5.) But calculation of momentum was not possible because of noise present in calculation of $\dot{q}$. So we were not able to measure the geometric phase throught this process.

Conclusion

The equivalence between a pair of classical coupled oscillators and two level quantum system has been explicitly worked out. Some aspects of the equivalence were demonstrated using a simple experimental setup suitable for undergraduates. Such mathematical exercises and experimental demonstrations could aid students in the development of intuition in the non-intuitive subject of quantum mechanics.

The mathematical equivalence seems to suggest that this classical system is capable of reproducing all phenomenon observable in its quantum counterpart. Innovative ideas to construct a simple pair of coupled oscillators, whose natural frequencies can be adjusted without changing their masses, would be an improvement to the setup and would enable one to dispense with the mass correction that had to be performed in this case. Extension of this experiment like two dimetional coupled oscillators could also be used to demonstrate more advanced ideas like geometric phase which we were not able to do by this setup and and phenomena associated with it like relative phase between $|0\rangle$ and $|1\rangle$.

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