Relating Classically Coupled Oscillators to Two-Level Quantum Systems

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by

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ABSTRACT

We study the analogy between a classical coupled oscillator system with a two-level quantum system by analysing its geometric phase. By exploiting the cyclic dynamics of the system, we establish and verify the analogy between the two regimes. Further, we investigate the effects of a mechanical dissipative agent on the coupled oscillator theoretically and experimentally. We model our coupled oscillator and verify the quality of fit by analysing the correlation matrix. In our findings, the fit parameters were poorly correlated implying good quality fit and the mechnical dissipative agent was found to be causing an exponential decay to the system.

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Chapter 1

Introduction

The study of classical and quantum mechanics is one of the most fascinating and essential topics in physics. The two disciplines have been the cornerstone of modern physics and have revolutionized our understanding of the natural world. Classical mechanics is the branch of physics that describes the behavior of macroscopic objects, while quantum mechanics deals with the behavior of subatomic particles. Throughout history, many great minds, such as Isaac Newton and Albert Einstein, have made significant contributions to classical mechanics. Similarly, quantum mechanics has been shaped by the works of notable physicists such as Max Planck, Erwin Schrödinger, and Werner Heisenberg. The classical and quantum analogies have played a pivotal role in bridging the gap between the two fields and have led to many exciting discoveries in the world of physics.

One of the interesting areas where classical and quantum analogies are commonly used is in the study of cyclical dynamical systems. In classical mechanics, a simple example of a cyclical system is a set of coupled oscillators. These oscillators exhibit a periodic behavior where their motions are linked and synchronized. In the realm of quantum mechanics, the equivalent system is a two-level quantum system, also known as a qubit. Qubits are the basic building blocks of quantum computing and exhibit similar periodic behaviors as coupled oscillators. The study of these systems has led to a better understanding of the behavior of complex systems and their underlying principles. Moreover, the classical-quantum analogy has allowed researchers to gain insights into quantum systems by studying classical analogs and vice versa. This has been an essential tool in the development of quantum technologies such as quantum computing, cryptography, and communication.

1.1 Objectives

The following objectives were formulated and achieved for this experiment:

1. To observe and analyse various fit parameters related to the dynamics of the coupled oscillator
2. To study the variation of geometric phase of the coupled oscillator
3. To analyse the correlation matrix to validate the quality of fit parameters
4. To study the effects of a mechanical damping agent on the coupled oscillator.

Chapter 2

Theory

In this chapter, the theoretical background will be detailed before we discuss the particulars of the experiment.

2.1 Dynamics of Cyclical Systems

The study of cyclical evolutions, where a system returns to its initial state at the end of the evolution, is a common topic in Physics. In quantum mechanics, the state vector of a system after a cyclical evolution is related to its initial state vector by a phase factor. This phase has a geometric component, which was first demonstrated by Pancharatnam in 1956 for cyclic evolution of the polarization state of light, and later by Berry in 1984 for adiabatic evolution of the Hamiltonian. The geometric phase for adiabatic evolution depends on the closed trajectory of the state ray in the Hilbert space and has been generalized to any cyclical evolution by Aharonov and Anandan. The geometric phase has been observed experimentally in various interference experiments, such as the Aharonov-Bohm effect, nuclear magnetic resonance, and laser optics. This concept has numerous applications, including the theory of insulators and quantum information.

This article focuses on a specific quantum mechanical system, a two-state atom interacting with a steadily oscillating electric field such as a laser, which exhibits Rabi oscillations and evolves cyclically. The article mathematically demonstrates that the motion of a system of two classical coupled oscillators is analogous to this quantum system. The periodic transfer of energy between the oscillators is similar to Rabi oscillations in the atom. This analogy is used to study the evolution of the oscillator as a trajectory on an equivalent Hilbert space and to obtain the associated geometric phase. The article also describes a simple experiment suitable for undergraduate students to demonstrate and measure the geometric phase, with the measured value found to be in good agreement with the expected phase. This analogy has been previously demonstrated in electrical RLC circuits to show phenomena such as electromagnetically induced transparency (EIT), double EIT, and Fano interference. The article concludes by noting that the use of analogies between classical and quantum mechanical systems is a topic of ongoing research and discussion.

2.2 Theoretical Framework

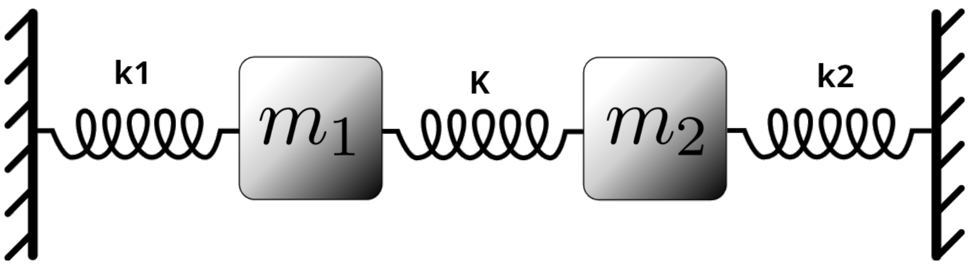

Figure 2.1: Schematic diagram of a system of coupled oscillators

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k\left(x_2 - x_1\right), \\ m_2\ddot{x}_2 &= k\left(x_1 - x_2\right) - k_2x_2. \end{aligned}$$

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{\xi} = \mathbf{A}\boldsymbol{\xi}$$

$$\boldsymbol{\xi} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} \omega_1^2 & -\Omega_1^2 \\ -\Omega_2^2 & \omega_2^2 \end{pmatrix},$$

$$\omega_\pm^2 = \frac{1}{2}\left(\omega_1^2 + \omega_2^2 \pm \left[4\Omega_1^2\Omega_2^2 + \left(\omega_1^2 - \omega_2^2\right)^2\right]^{1/2}\right).$$

$$\boldsymbol{\xi}(t) = F_+\boldsymbol{\xi}_+\exp\left(-\mathrm{i}\omega_+t - \mathrm{i}\phi_+\right) + F_-\boldsymbol{\xi}_-\exp\left(-\mathrm{i}\omega_-t - \mathrm{i}\phi_-\right),$$

The quantum two level system

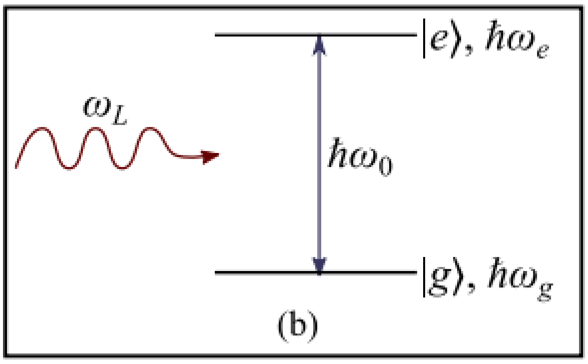

Figure 2.2: Level scheme of a two-state atom interacting with a single-mode laser as an example of a two-level quantum system.

The system illustrated in figure 1(b)[1] is that of an atom with a ground state $|g\rangle$ of energy $\hbar\omega_g$ and an excited state $|e\rangle$ of energy $\hbar\omega_e$, with resonant frequency $\omega_e - \omega_g = \omega_0$. It interacts with an oscillating electric field $\mathcal{E}(t) = \mathrm{Re}\left(\mathcal{E}_0 \mathrm{e}^{\mathrm{i}\omega_L t}\right)$ such as a single-mode laser. We define the dipole moment $\mu = e\langle g|\mathbf{r} \cdot \hat{\mathcal{E}}|e\rangle$, where e is the electronic charge and $\hat{\mathcal{E}}$ is the unit vector in the direction of the electric field. Then the Hamiltonian matrix of the system in the $\{|g\rangle, |e\rangle\}$ basis is [12]

$$\mathbf{H} = \begin{pmatrix} \hbar\omega_g & -\mu\mathcal{E}(t) \\ -\mu^*\mathcal{E}(t) & \hbar\omega_e \end{pmatrix}$$

If we transform the state vector using the unitary operator $U = |g\rangle \left\langle g \left| + \mathrm{e}^{-\mathrm{i}\omega_L t} \right| e \right\rangle \langle e|$, then the Hamiltonian matrix elements in the new basis $\left\{|\tilde{g}\rangle = U^\dagger|g\rangle = |g\rangle, |\tilde{e}\rangle = U^\dagger|e\rangle = \mathrm{e}^{\mathrm{i}\omega L t}|e\rangle\right\}$ become time independent under the rotating wave approximation,

$$\tilde{\mathbf{H}} = \hbar \begin{pmatrix} \omega_g & -\Omega/2 \\ -\Omega/2 & \omega_g + \Delta \end{pmatrix}$$

where we have defined $\Delta = \omega_0 - \omega_L$ (called the detuning) and $\Omega = \mu\mathcal{E}_0/\hbar$ (called the Rabi frequency). Ω is in general a complex quantity depending on the phase of the laser; here we choose the phase to be zero so that Ω is real. Physically, this may be interpreted as moving the laser source towards or away from the atom to ensure that

[1]Sharba Bhattacharjee et al 2018 Eur. J. Phys. 39 035404

Table 2.1: Relation between the quantities associated with the classical coupled oscillator and the equivalent quantum two-level system.

Coupled oscillator	**Two-level system**
Squares of amplitudes $\lvert\tilde{x}_1\rvert^2, \lvert\tilde{x}_2\rvert^2$	Probabilities $\lvert c_1\rvert^2, \lvert c_2\rvert^2$
First natural frequency ω_1	$\left(\omega_g^2 + \Omega^2/4\right)^{1/2}$
Second natural frequency ω_2	$\left[(\omega_g + \Delta)^2 + \Omega^2/4\right]^{1/2}$
Coupling $\Omega_1 = \Omega_2$	$\left[\Omega\left(2\omega_g + \Delta\right)/2\right]^{1/2}$

$\mu\mathcal{E}_0$ is real. Then the state $|\tilde{\psi}(t)\rangle$ of the system at a time t, with the initial condition $|\tilde{\psi}(0)\rangle = |\tilde{g}\rangle$, is given in the $\{|\tilde{g}\rangle, |\tilde{e}\rangle\}$ basis as [12]

$$|\tilde{\psi}(t)\rangle = c_1(t)|\tilde{g}\rangle + c_2(t)|\tilde{e}\rangle \equiv \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix},$$

where

$$\begin{aligned} c_1 =& \frac{1}{2}\Big[\left(\sqrt{\Delta^2+\Omega^2}-\Delta\right)\exp\left(-\mathrm{i}\left(\Delta+\sqrt{\Delta^2+\Omega^2}\right)t/2 - \mathrm{i}\omega_g t\right) \\ &+ \left(\sqrt{\Delta^2+\Omega^2}+\Delta\right)\exp\left(-\mathrm{i}\left(\Delta-\sqrt{\Delta^2+\Omega^2}\right)t/2 - \mathrm{i}\omega_g t\right)\Big]\left(\Delta^2+\Omega^2\right)^{-1/2} \\ c_2 =& \frac{1}{2}\Big[\Omega\exp\left(-\mathrm{i}\left(\Delta-\sqrt{\Delta^2+\Omega^2}\right)t/2 - \mathrm{i}\omega_g t\right) \\ &-\Omega\exp\left(-\mathrm{i}\left(\Delta+\sqrt{\Delta^2+\Omega^2}\right)t/2 - \mathrm{i}\omega_g t\right)\Big]\left(\Delta^2+\Omega^2\right)^{-1/2} \end{aligned}$$

The coefficients $c_1(t)$ and $c_2(t)$ are the probability amplitudes, so that at any given time t, $|c_1(t)|^2$ and $|c_2(t)|^2$ are the probabilities of the system collapsing to the states $|\tilde{g}\rangle$ and $|\tilde{e}\rangle$ respectively if its energy is measured.

Scale 1 is k_1 scale 2 is k_2 rubberband is K the extra rubberband we use changes k_2 but it also changes how quickly the system is damped

2.3 Modelling the coupled oscillator

The following equation was used to model the coupled oscillator:

$$f(t) = A\cos[\omega_+(t-d)+\phi] + B\cos[\omega_-(t-d)\phi] + C \tag{2.1}$$

The fitted values of A, B, ω_+ and ω_- are used to calculate the parameters ω_g, Δ and Ω of the Rabi model as

$$\begin{aligned}\Delta &= (\omega_+ - \omega_-)\frac{B-A}{B+A},\\ \Omega &= (\omega_+ - \omega_-)\left[1 - \left(\frac{B-A}{B+A}\right)^2\right]^{1/2},\\ \omega_g &= \frac{\omega_+ + \omega_- - \Delta}{2}\end{aligned} \tag{2.2}$$

Further we calculte the geometric phase as

$$\phi_G = \pi\left(1 - \frac{\Delta}{\sqrt{\Delta^2+\Omega^2}}\right) \tag{2.3}$$

2.4 The action of the coupling agent

When you stretch a rubber band there is considerable deformation to the polymer molecules in the rubber. As a result some of the work you do on the rubber band goes into exciting molecular vibrations i.e. heat. Some of the work you do is stored as elastic energy and some is dissipated as heat.

As the band is allowed to relax the elastic energy stored within it does work on you. However, as before some of this energy goes into molecular vibrations and some into heat.

In both cases the work is the integral of force with distance, i.e. the area under a force-distance graph, however, this graph will show some hysteresis due to the energy that is dissipated as heat[2]:

[2] Source:John Rennie, Net work done for rubber bands, https://physics.stackexchange.com/q/149122

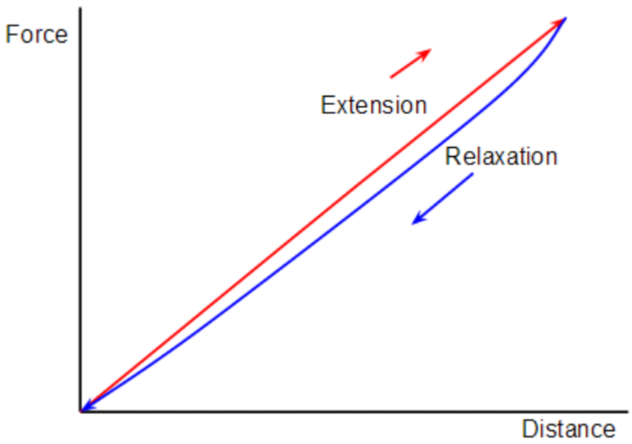

Figure 2.3: The dynamics of a rubber band

Because of the hysteresis you don't get as much work out as you put in and the net work is not zero.

In the case of permanent deformation the physics is basically similar, except that the difference between the extension and relaxation force-distance curves is greater so more work is lost as heat.

2.5 The Correlation Matrix

Suppose we have an equation with multiple parameters and we are trying to fit some dataset to this equation. In probability theory and statistics, it is essential to realise that there might be some correspondence between two of those parameters, even if they are independent in the physical sense. We define this property as *covariance*. It is the join variablity of two random variables. The greater two parameters correspond or correlate with each other, the higher will be the covariance.

We can construct a matrix that contains all the information about the covariance or correlation between all the pairs of parameters of an equation. A square matrix which can represent all the covariance values of all the pairs of aparameters of the equation is called the covariance or correlation matrix. So, if the entries of the column vector

$$\mathbf{X} = (X_1, X_2, \ldots, X_n,)^T \tag{2.4}$$

are random variables, each with finite variance and expected value, then the covariance matrix K_{XX} is the matrix whose (i, j) entry is the covariance

$$K_{X_iX_j} = \mathrm{cov}[X_i, X_j] = E[(X_i - E[X_i])(X_j - E[X_j])] \tag{2.5}$$

where E is an operator which denotes the expected value (mean) of its argument.

In our analysis, we used Python's `SciPy` library's `curve_fit` function. This function returns a tuple containing the two arrays. The first array contains the optimal parameters of the fittef function. The second array is a 2-D array, which represents the covariance matrix.

Note that the diagonal values of the covariance matrix give us the standard deviation in the parameters and the off-diagonal elements represents the covariance between (i, j) pairs. Therefore, if all the off-diagonal values are sufficently small, we can conclude it is a good quality fit.

Chapter 3

Experimental Set-up

Two metal rulers are fixed vertically at the bottom and are capable of vibrating with small amplitudes, acting as oscillators that are coupled by a rubber band, similar to a two-mass and three-spring system. The mass and elastic moduli of the rulers correspond to the mass and spring constant of an oscillator, with the first ruler behaving like a mass attached to a rigid wall with a spring constant in Figure 1, while the second ruler behaves like another mass with its own spring constant. The rubber band connecting the two rulers acts as a coupling spring. Assuming small amplitudes, the vibration of the rulers, including the angular displacement of any segment from the vertical, can be modelled as simple harmonic motion.

The displacements of the rulers are accurately measured using a He-Ne laser beam. The beam is split into two polarized beams using a polarizing cube beam splitter (PBS1), and the transmitted beam passes through a quarter-wave plate to make it circularly polarized. The beam is then reflected by a mirror attached to the ruler and passes through the quarter-wave plate again, making it linearly polarized with a phase shift of $\pi/2$ relative to the transmitted beam. The reflected beam is then reflected by PBS1 onto the quad photodetector D1 instead of being transmitted back to the laser source. The polarized beam reflected by PBS1 is then reflected by a second polarizing beam splitter (PBS2) and is again passed through the quarter-wave plate before being reflected by the plane mirror attached to the ruler. When the ruler undergoes small oscillations, the reflected beam makes an angular displacement along the normal line. The larger the displacement of the ruler, the greater the angular displacement of the reflected beam. The reflected beam falls onto the quad photodetector, and the signals detected by the upper and lower halves of the detector are separately amplified with the same gain and then subtracted. As the laser spot moves upward from the mean position, the upper two photo-detectors detect a larger signal than the lower two photo-detectors, and vice versa when the laser beam moves downward. Consequently, the difference between the output voltages of the two halves oscillates proportionally to the displacement of the oscillating ruler. This difference is then normalized to make an analogy with Rabi oscillation. The schematic setup is shown in Figure. (3)[1].

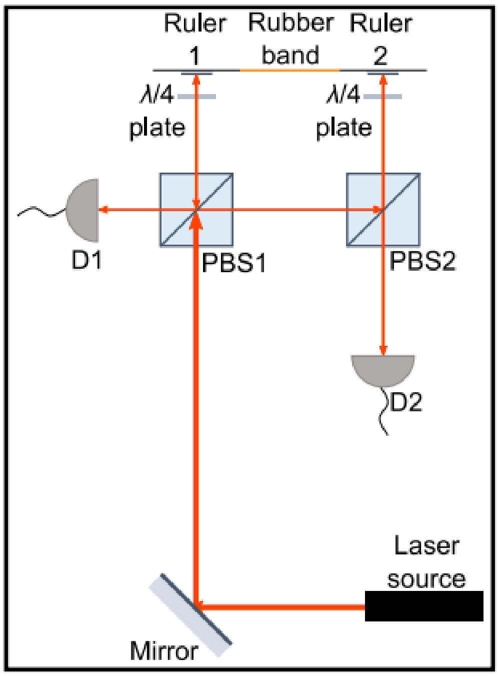

Figure 3.1: Schematic diagram of the experimental set-up (top view) showing the path taken by the laser beam. PBS1, PBS2:polarising cube beam splitters; D1, D2: quad photodetectors

[1]Sharba Bhattacharjee et al 2018 Eur. J. Phys. 39 035404

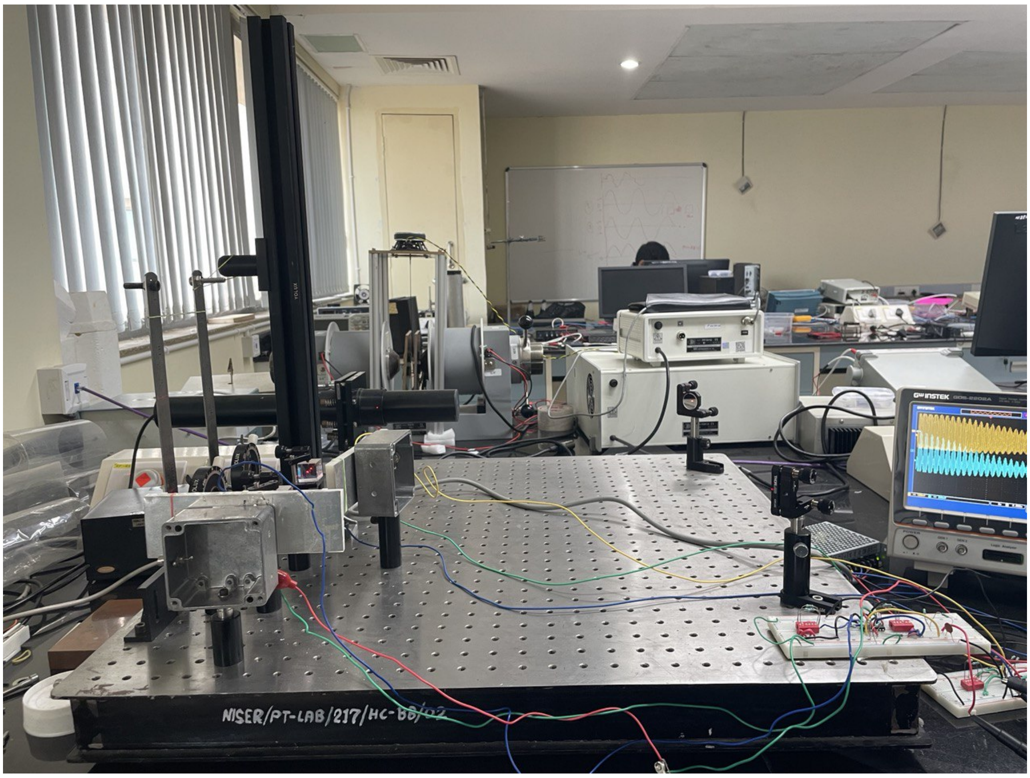

Figure 3.2: Overview of our experimental set-up. Notice the the black pillar we set-up for our innovation part. That beam was mounted and a rubber band was tied to this beam and one of the rulers. This extra rubber band induced the dampening.

The quad photo-detector comprises four photo-diodes arranged in a square cell, with the upper two detecting the upper signal of the moving laser spot and the lower two detecting the lower signal. To amplify the signals, we utilized the TL084 op-amp due to its superior slew rate and high-frequency response. The amplified signals are then subtracted using a differential amplifier (also TL084) to yield a signal voltage proportional to the displacement of the ruler. To eliminate high-frequency noise in our circuit, we incorporated an RC low pass filter. The signal is ultimately viewed through an oscilloscope, with the cut-off frequency of the RC filter set at 88.42Hz.

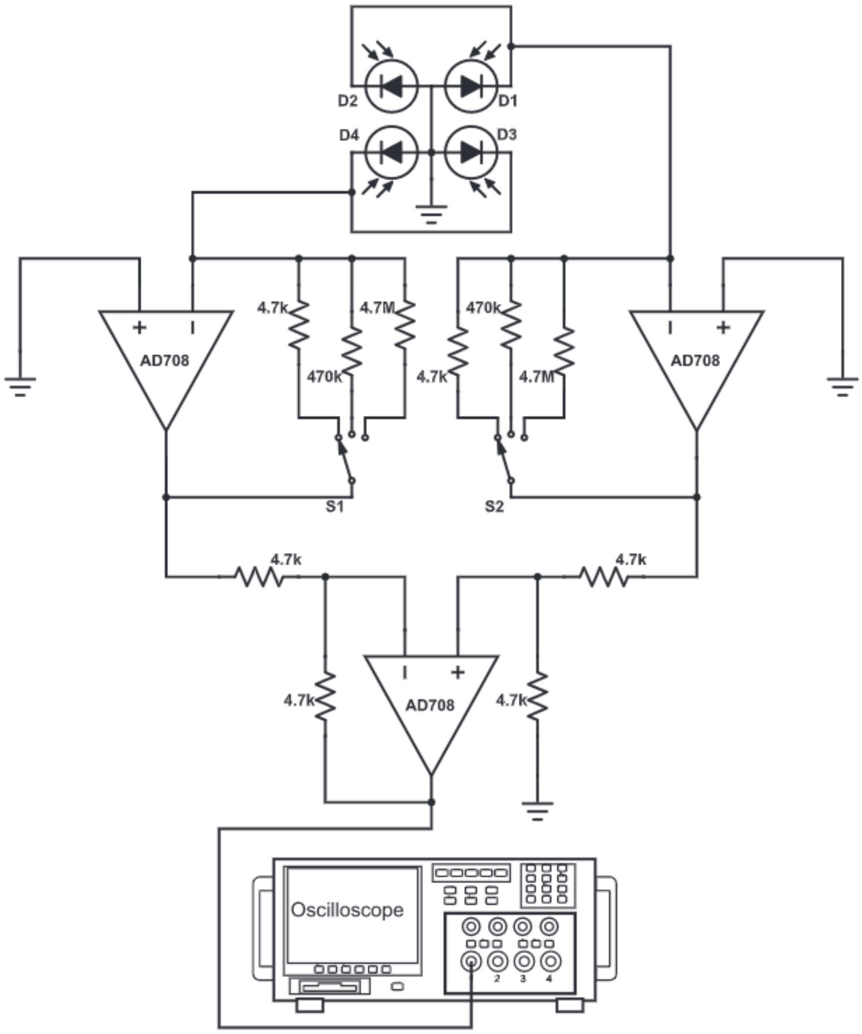

Figure 3.3: This is the TL084/AD708 circut diagram that was used to relay the voltage detcted by the photodetector.

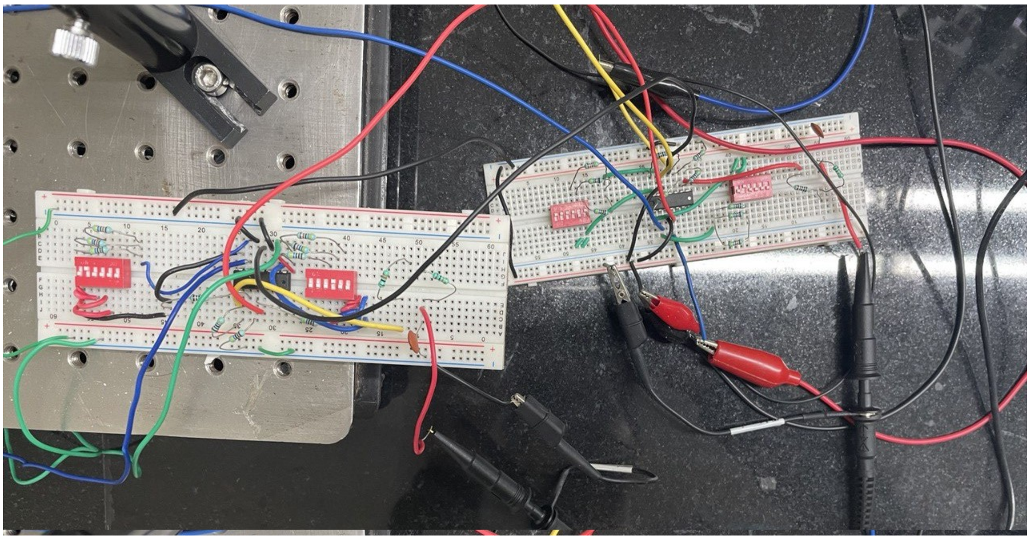

Figure 3.4: The circuitry of the amplifier (TL084) that was used to relay the data into our oscilloscope

Chapter 4

Data Analysis

The data analysis was performed using Python's `SciPy` library. We recorded the data for 20 combinations of masses attached to the oscillators and then for each file, the fitted parameters were found. Corresponding to each fit parameter, we found the the Rabi model parameters of *Delta* (detuning) and Ω (Rabi frequency). Then these parameters of the Rabi model were themselves fit according to their equations, relating them with geometric phase.

For error analysis, we used Python's `uncertainties` package, which preserved the significant digits when propagating errors. The complete code is on my GitHub. Some snippets are also attached with this report.

Here is how the code works:

- First we read the waveform from the file and import them into `NumPy` arrays.
- `SciPy`'s `curve_fit` function gives the covariance matrix. The diagonal elements of this matrix represent variance (which is the square of the standard deviation). Therefore, we take the diagonal elements and take their square root to get the standard deviation in parameters.
- Then we use the `uncertainties` package's `ufloat` object to store our nominal values with thier standard deviation.
- We convert all parameters as `ufloat` objects and then propagate through other formulae to find the error in them.

- We use `unumpy`, a superset of `NumPy` compatible with `uncertainties` package to finally calculate all the parameters with their associated errors.

In this section, we present our plots. In figure 4.1, we present the fitted curve for one of the data file (representing one mass combination for the coupled oscillator). The figure 4.5 represents the squares of the normalised amplitudes (basically the peaks of the 'carrier' wave).

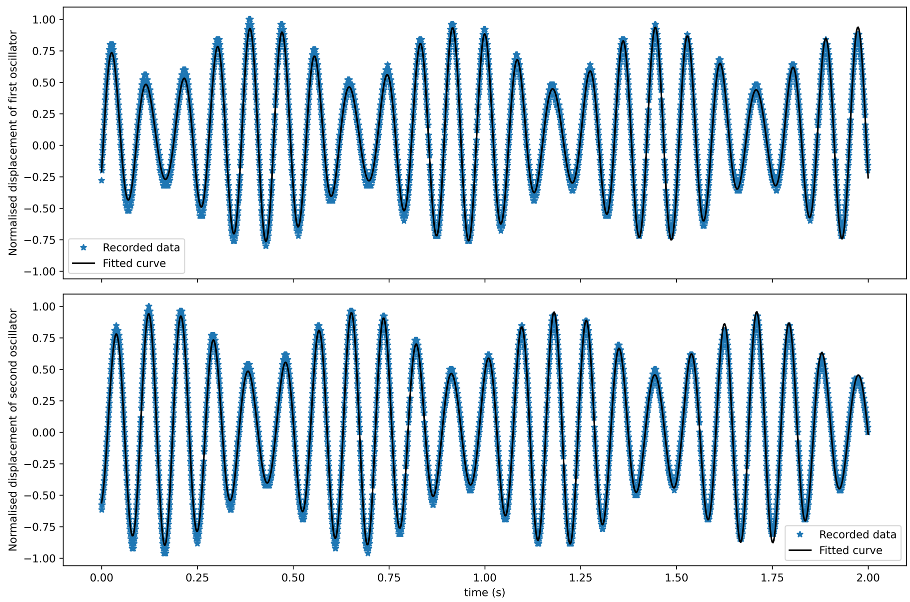

Figure 4.1: Normalised displacements of the two oscillators as functions of time t. The open circles are the experimental data, and the solid lines denote the fitted functions.

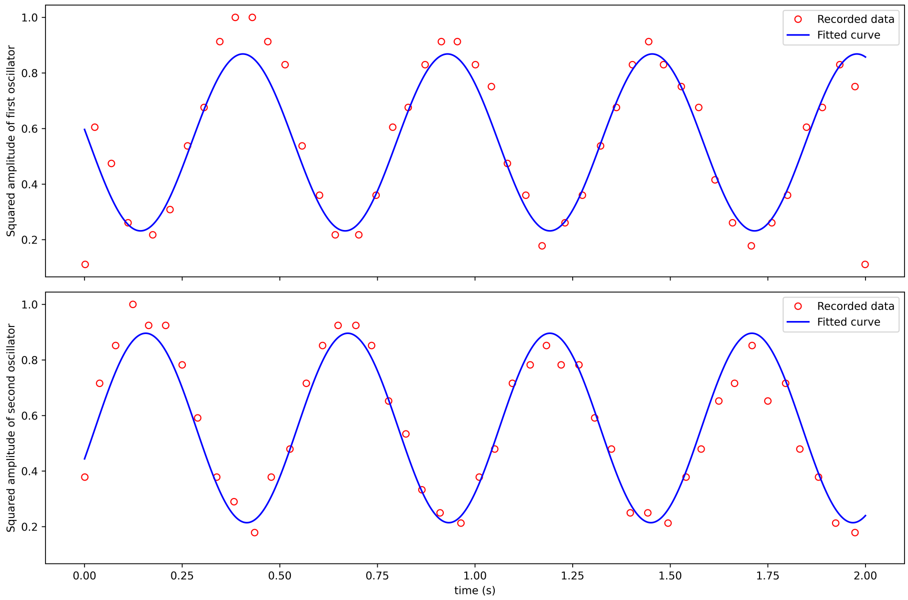

Figure 4.2: Squares of the fitted normalised amplitudes of the oscillators as functions of time. The open circles are the squares of the experimental amplitudes (obtained as the local extrema of the displacements), and the solid lines are the fitted functions

Similarly the other plots were obtained and that gave us the fitting parameters. In total we used 19 data files, each of which gave the Rabi model parameters we needed to study the dependence of geometric phase with detuning.

4.1 Calculation of Rabi model parameters and the Geometric Phase

Here we present the plots where we calculated the Rabi model parameters and the geometric phase and fitted thier functions.

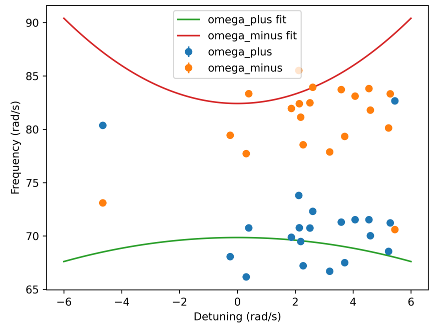

Figure 4.3: Normal mode frequencies ($\omega_{\pm}$) as functions of the detuning Δ. The circles are the experimental data, and the solid lines denote the fitted curves

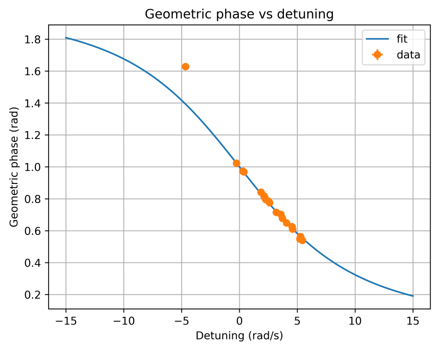

Figure 4.4: Geometric phase ϕ_G as a function of the detuning Δ. The circles are the experimental data, and the solid line denotes the fitted curve.

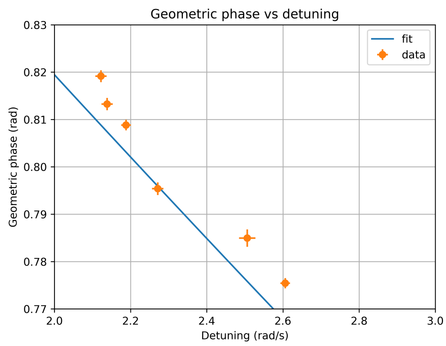

Figure 4.5: Zoomed in version of the geometric phase plot to show the error bars

4.2 Correlation Matrix Calculations

As stated earlier, the matrix was caluclated using Python's `SciPy`. Here, we will write the matrix for one data point. For the rest, please refer to the GitHub link.

$$\begin{pmatrix} 7.68580524e{-07} & 2.36824478e{-08} & 1.51761995e{-08} & -4.56947517e{-07} & -2.48776918e{-06} & -3.48929193e{-08} & -9.42783208e{-09} \\ 2.36824478e{-08} & 7.66811261e{-07} & 2.24914950e{-07} & 2.02813486e{-08} & -1.15823362e{-06} & -1.32197830e{-08} & -7.19123808e{-09} \\ 1.51761995e{-08} & 2.24914950e{-07} & 6.28335239e{-06} & 6.42283112e{-07} & -3.85895795e{-05} & -4.44334319e{-07} & 2.37039720e{-08} \\ -4.56947517e{-07} & 2.02813486e{-08} & 6.42283112e{-07} & 3.71393460e{-05} & 2.32215731e{-04} & 3.28217440e{-06} & 1.70175054e{-08} \\ -2.48776918e{-06} & -1.15823362e{-06} & -3.85895795e{-05} & 2.32215731e{-04} & 2.25269480e{-03} & 3.09135290e{-05} & -1.95585048e{-07} \\ -3.48929193e{-08} & -1.32197830e{-08} & -4.44334319e{-07} & 3.28217440e{-06} & 3.09135290e{-05} & 4.25921825e{-07} & -2.32283158e{-09} \\ -9.42783208e{-09} & -7.19123808e{-09} & 2.37039720e{-08} & 1.70175054e{-08} & -1.95585048e{-07} & -2.32283158e{-09} & 3.78885897e{-07} \end{pmatrix}$$

Chapter 5

Extensions and Innovation

The system we have considered is an ideal system. Classically, what we have modelled is a dissipationless system that will continue its motion without a loss in energy. But all real experiments have a dissipative element in the system and in this case, it is the damping due to the springs, which are not ideal. Upon observing the readings, it was obvious to us that there is a damping present due to the rubber band connecting the two scales and due to the nature of the scales which will also lose energy over time. But this dissipation is not obvious enough to be observable in the dataset that we collect from the oscilloscope. Its observation takes at least 10 seconds using the oscilloscope under normal conditions. To study the damping effect of our 'springs', we attached one more rubber band to one of the scales using a rigid support. The system essentially remains the same as before with just the effective spring constant changing for one of the springs. But now, the damping is significant enough to be taken into account. The modified equations for the coupled oscillator with damping become

$$F_1 = -k_1 x_1 + K(x_2 - x_1) = m_1 \ddot{x}_1$$

$$F_1 = -(k_2 + k_3)x_2 + K(x_2 - x_1) - k_v \frac{d^\alpha x_2}{dt^\alpha} = m_2 \ddot{x}_2$$

α and k_v are parameters that cannot be determined theoretically. An approximate analytic solution for a single mass oscillator exists only for small $\frac{k_v}{m_2}$. It can give us an idea of the form of damping caused due to the rubber band[1].

$$m \frac{d^2 x}{dt^2} + k_{\mathrm{v}} \frac{d^\alpha x}{dt^\alpha} + kx = 0$$

[1] Yan Huang et al 2020 IOP Conf. Ser.: Mater. Sci. Eng. 782 022093

$$\frac{d^2x}{dt^2}+2\beta\frac{d^\alpha x}{dt^\alpha}+\omega_0^2x=C$$

Where, $2\beta=\frac{\mathrm{k_v}}{\mathrm{m}},\omega_0^2=\frac{\mathrm{k}}{\mathrm{m}}$, and the initial condition is set to be $x(0)=x_0,\dot{x}(0)=0$. For the case of $0<\beta<1$, when β can be regarded as a small parameter, the approximate analytical solution of the equation(3) can be obtained by means of the average method as follows

$$x(t)=x_0e^{-\beta\omega_0^{\alpha-1}\sin\frac{\alpha\pi}{2}t}\cos\left[\left(\omega_0+\beta\omega_0^{\alpha-1}\cos\frac{\alpha\pi}{2}\right)t\right]$$

The corresponding solution of equation (1) is

$$x(t)=x_0e^{-\frac{1}{2m}k_v\left(\sqrt{\frac{k}{m}}\right)^{\alpha-1}\sin\frac{\alpha\pi}{2}t}\cos\left\{\left[\sqrt{\frac{k}{m}}+\frac{1}{2m}k_v\left(\sqrt{\frac{k}{m}}\right)^{\alpha-1}\cos\frac{\alpha\pi}{2}\mid t\right\}\right.$$

$$\Delta=\frac{1}{2m}k_{\mathrm{v}}\left(\sqrt{\frac{k}{m}}\right)^{\alpha-1}\sin\frac{\alpha\pi}{2}$$

$$\omega_{\mathrm{v}}=\frac{1}{2m}k_{\mathrm{v}}\left(\sqrt{\frac{k}{m}}\right)^{\alpha-1}\cos\frac{\alpha\tau}{2}$$

The equation (5) can be changed into

$$x(t)=x_0e^{-\Delta t}\cos\left[\left(\omega_0+\omega_{\mathrm{v}}\right)t\right]$$

This equation is a cosine simple harmonic oscillator term multiplied by an exponential damping term. The system that we are studying in our experiment is a coupled oscillator and it does not have a simple cosine dependence for the harmonic oscillation part and instead is a linear combination of two normal modes hence, beats can emerge because the two oscillators are connected and their natural oscillating frequencies are of the same order of magnitude. Since these beats have a more complicated solution, we cannot directly say how the damped solution will look. But, we know that what the rubber band is essentially doing is dissipating energy. We know that the energy of an oscillator is directly proportional to the square of the amplitude.

Hence, we know that if energy is dissipated, the amplitude will decrease and we saw that for a single oscillator. For the coupled oscillator, we propose that we will see an exponential decay in the amplitude of oscillation in the beats and a part of our experiment works towards finding out this decay term. What we have to do is to find the amplitude in every single beat unit and plot it as a function of time. If this turns out to be an exponential decay then we have verified our hypothesis. This is because the total energy of the system is directly proportional to the sum of squares of individual oscillator amplitudes. Since beats have the form

$$\cos(\omega_1 t) + \cos(\omega_2 t) = 2\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)$$

We expect our final curve to look like

$$a(t) = e^{-\Delta t}\left\{2\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\right\}$$

Experimentally, we only need to obtain the maximum values of the cosine part to get an exponential decaying term.

5.1 Observations

Here we have tabulated the decay constants we found for different masses after doing curve-fitting, along with the plots.

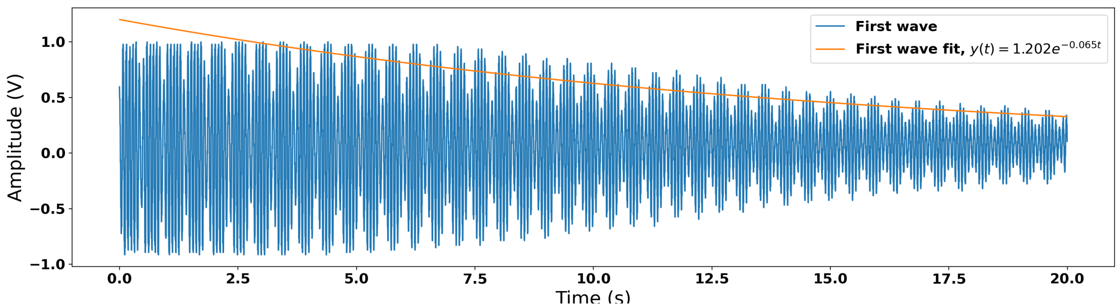

Figure 5.1: Decaying amplitude due to daming by the rubber band attached to the mounted aluminium beam, fitted with the it's decay law

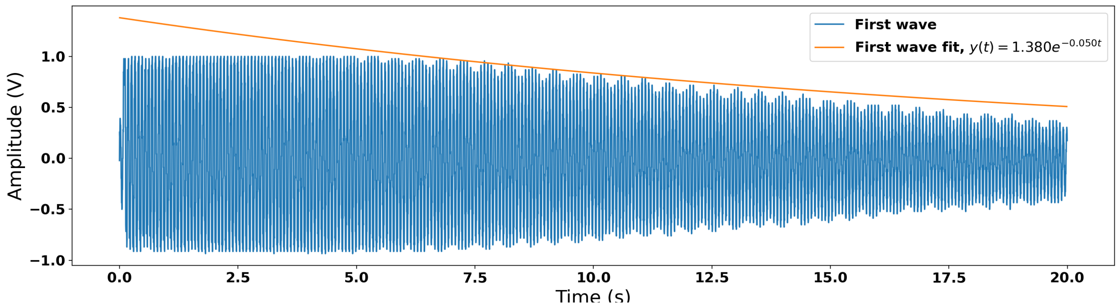

Figure 5.2: Decaying amplitude due to daming by the rubber band attached to the mounted aluminium beam, fitted with the it's decay law

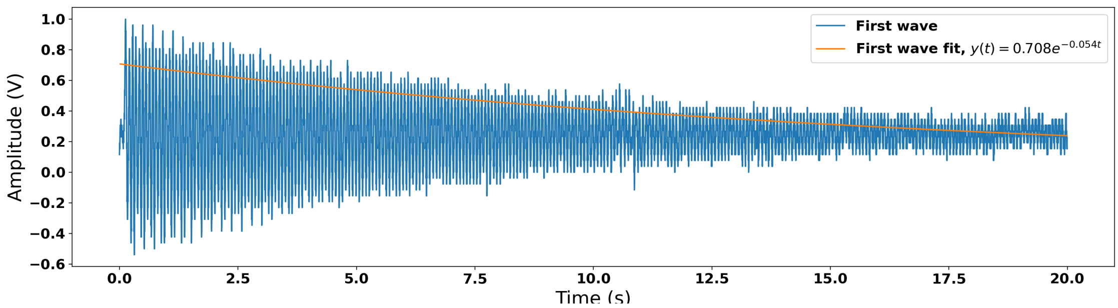

Figure 5.3: Decaying amplitude due to daming by the rubber band attached to the mounted aluminium beam, fitted with the it's decay law

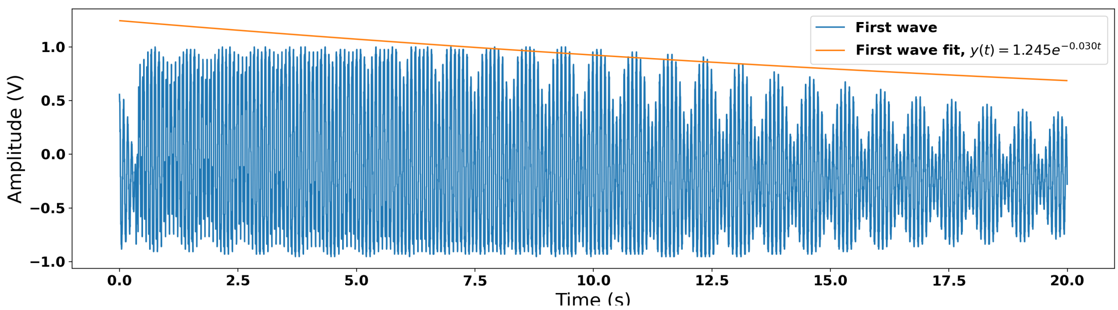

Figure 5.4: Decaying amplitude due to daming by the rubber band attached to the mounted aluminium beam, fitted with the it's decay law

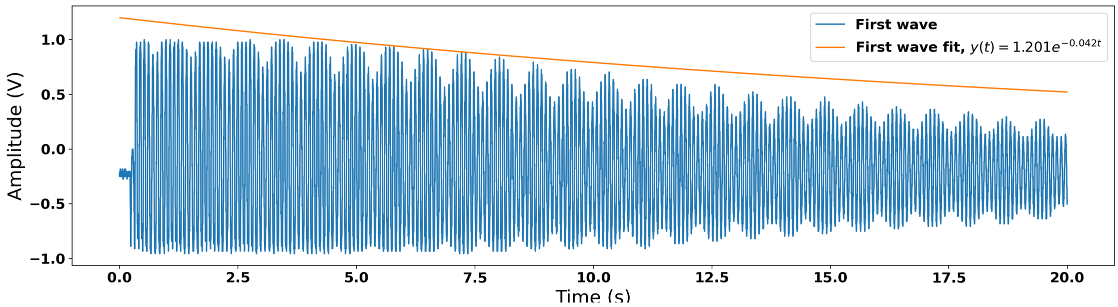

Figure 5.5: Decaying amplitude due to daming by the rubber band attached to the mounted aluminium beam, fitted with the it's decay law

Chapter 6

Results and Discussion

1. We analysed the coupled oscillator and the analogies with the classical coupled oscillator were verified.

2. It was found that the phase change of its displacement after one Rabi period is roughly equal to the dynamic phase of the coupled oscillator.

3. The geometric phase in the coupled oscillator as measured from the experiment is found to be in very good agreement with the expected phase based on theory of a two-state system.

4. In our innovation part, we first theoretically derived how the damping will vary and then experimentally found the parameters of the decay law. We observed that it was in agreement with the theory (exponential decay of the form we derived).

5. The decay law/parameters depend on the mass on the oscillators to as it is apparent that a massive body will dampens slowly as compared to a lighter object due to inertia. We could have recorded the masses for better validation.

Chapter 7

Summary and Conclusions

1. We have experimentally demonstrated that the classical system of coupled oscillators evolves in a manner similar to a two-state atom interacting with a laser.

2. This analogy may be used to simulate in classical coupled oscillators various phenomena associated with the dynamics of quantum mechanical systems

3. The Rabi model parameters of the equivalent two-level quantum mechanical system have been calculated, and the geometric phases associated with the cyclical evolution have been estimated and found to be in very good agreement with the expected values

4. In the future, this experiment can be expanded to other ways of modelling damping/decay observed in two-level quantum systems as we did mechanically here. One way to do it is to do it electromagnetically by exploiting the use of eddy currents.