Lecture 7

Last time: Response of 2nd order systems:

- Underdamped Response
- Performance Metrics
  - Peak time t_p
  - Rise time t_r
  - Settling time t_s
  - Overshoot Percent $\%OS$
- 2nd order systems with a zero and 3rd and higher order systems
  - Same formulae can be used if the extra poles/zeros are far removed from the dominant 2nd order poles (far removed: 4 to 5 times further left of the $j\omega$-axis).

Note that we have not as of yet addressed the issue of accuracy of the final value of a system's output, in other words how close a system's final value is to the desired value. Another name for this is **steady-state error**. We will take care of a few other matters first, and then we will return to tack this problem later.

Stability

What is stability? Per the Webster Dictionary: "Stability is the property of a body that causes it when disturbed from a condition of equilibrium or steady motion to develop forces or moments that restore the original condition." This kind of stability is for small perturbations to the system.

In controls: A system is stable if all bounded inputs to the system produce bounded outputs; the system is unstable otherwise. This is called **Bounded-Input Bounded-Output** (BIBO) stability.

Let's look at things from the perspective of poles and zeros.

$$Y(s) = G(s)U(s)$$

So, the nature of the system output $y(t)$ depends on

- the poles of $G(s)$
- the poles of $U(s)$

Since $u(t)$ needs to be bounded, poles of $U(s)$ can include poles located:

- Strictly in the LHP

- At the origin (at most 1)
- On the $j\omega$-axis as a complex conjugate pair (there can be any number of these, so long as they are not repeated).

So, the requirements on poles of $G(s)$ for BIBO stability are:

- They must **all** be strictly in the LHP. Otherwise, it would be possible to find a bounded input that will result in an unbounded output.
  - Roots of denominator of $G(s)$ must all have strictly negative real parts.
- The system is BIBO unstable if $G(s)$ possesses poles in the RHP or on the imaginary axis.

So, we can look at the pole-zero pattern of a system's transfer function, and tell whether the system is BIBO stable or not.

Example

$$
\begin{aligned}
G(s) = \frac{1}{(s+1)(s+2)} \qquad & \longleftarrow \qquad \text{BIBO stable} \\
G(s) = \frac{1}{s} \qquad & \longleftarrow \qquad \text{BIBO unstable} \\
G(s) = \frac{\omega}{s^2+\omega^2} \qquad & \longleftarrow \qquad \text{BIBO unstable} \\
G(s) = \frac{2-s}{(s+1)(s+2)(s+3)} \qquad & \longleftarrow \qquad \text{BIBO stable} \\
G(s) = \frac{1}{(s+1)(s-2)(s+3)} \qquad & \longleftarrow \qquad \text{BIBO unstable}
\end{aligned}
$$

This is all easy if the denominator of $G(s)$ is given in factored form. But what if the denominator is not factored?

a) Use a calculator or Matlab to factor. This works only if you have numbers for the denominator coefficients.

b) Use a method that does not factor the denominator explicitly, but tells you how many of its roots are in the LHP, RHP and $j\omega$-axis (stability information). This is good when the coefficients of the denominator polynomials are either numbers **or** symbols.

Tests for Stability

Given the denominator polynomial, $p(s)$, of a transfer function

$$p(s) = a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

determine whether there are roots of $p(s)$ with positive real parts (RHP), or no real parts ($j\omega$-axis).

Test #1: This test comes from the mathematics of polynomials. Check the coefficients a_k. If any are missing or if any two a_k have opposite sign, then there is a zero of $p(s)$ on the imaginary axis or in the RHP. If this is the case, then the system is clearly BIBO unstable.

Example

$$p(s) = s - 4 \quad \Rightarrow \quad a_1 = 1,\ a_0 = -4. \text{ Sign change, therefore unstable.}$$

$$p(s) = s^2 + 9 \quad \Rightarrow \quad a_2 = 1,\ a_1 = 0,\ a_0 = 9.\ a_1 = 0, \text{ therefore unstable.}$$

This test is necessary to determine stability, but it is not sufficient. A system can satisfy this test and still be unstable!

Test #2: Routh-Hurwitz Criterion. Construct the **Routh Array** and examine the 1st column. The number of zeros of $p(s)$ in the RHP is the number of sign changes in the 1st column of the Routh array.

Basic rules for constructing the Routh array: Let $p(s)$ be:

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$$

Rules:

- The 1st two **rows** of the Routh array are constructed with the a_k coefficients. This means you will have $(n+1)/2$ (rounded up) **columns** in the Routh array.
- You will eventually have $n+1$ **rows** in the Routh array. To calculate an element of the array, form a 2×2 matrix, denoted A, using the preceding two rows' 1st column and column to the immediate right of the element you want to determine. The element is $= -(\det A)/A_{21}$.

Array:

a_n	a_{n-2}	a_{n-4}	$\cdots$
a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
b_1	b_2	b_3	$\cdots$
c_1	c_2	c_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$

Equations:

$$b_i = \frac{-\begin{vmatrix} a_n & a_{n-2i} \\ a_{n-1} & a_{n-(2i+1)} \end{vmatrix}}{a_{n-1}} = \frac{(a_{n-1} \times a_{n-2i}) - (a_n \times a_{n-(2i+1)})}{a_{n-1}}$$

$$c_i = \frac{-\begin{vmatrix} a_{n-1} & a_{n-(2i+1)} \\ b_1 & b_{i+1} \end{vmatrix}}{b_1} = \frac{(b_1 \times a_{n-(2i+1)}) - (a_{n-1} \times b_{i+1})}{b_1}$$

$$d_i = \frac{-\begin{vmatrix} b_1 & b_{i+1} \\ c_1 & c_{i+1} \end{vmatrix}}{b_1} = \frac{(c_1 \times b_{i+1}) - (c_{i+1} \times b_1)}{c_1}$$

etc...

When completed, the number of sign changes in the 1st column of the Routh array is the number of poles in the RHP.

For example,

$$p(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$$\begin{array}{cccc} & \text{Col. 1} & \text{Col. 2} & \text{Col. 3} \\ \text{Row 1} & a_4 & a_2 & a_0 \\ \text{Row 2} & a_3 & a_1 & \\ \text{Row 3} & b_1 & b_2 & \end{array} \qquad \begin{array}{cccc} & \text{Col. 1} & \text{Col. 2} & \text{Col. 3} \\ \text{Row 1} & a_4 & a_2 & a_0 \\ \text{Row 2} & a_3 & a_1 & \\ \text{Row 3} & b_1 & b_2 & \end{array}$$

$$b_1 = -\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = \frac{a_2 a_3 - a_4 a_1}{a_3} \qquad b_2 = -\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = \frac{a_0 a_3}{a_3}$$

Let's do some examples of using Routh-Hurwitz to determine stability. One way to keep track of the number of rows needed is to label the rows: s^n, s^{n-1}, ..., s^0. So, a third order polynomial will have rows labeled $s^3,\ s^2,\ s,\ s^0$ (4 rows).

Example

Consider a system with the following denominator polynomial function:

$$p(s) = s^3 + 4s^2 + 3s + 2$$

Determine stability using a Routh array.

$$\begin{array}{llll}
s^3: & 1 & 3 & 0 \\
s^2: & 4 & 2 & 0 \\
s^1: & -\frac{2-12}{4} = \frac{5}{2} & -\frac{1\cdot 0 - 0\cdot 4}{4} = 0 & \\
s^0: & -\frac{2}{5}\left(4\cdot 0 - 2\cdot \frac{5}{2}\right) = 2 & 0 &
\end{array}$$

There are no sign changes in the first column, so the system is stable.

Example

Consider a system with the following denominator polynomial function:

$$p(s) = s^3 + 4s^2 + 3s + 13$$

Determine stability using a Routh array.

$$\begin{array}{lll}
s^3: & 1 & 3 \\
s^2: & 4 & 13 \\
s^1: & -\frac{13-12}{4} = -\frac{1}{4} & \\
s^0: & \frac{-1}{-1/4}\left(4\cdot 0 + 13\cdot \frac{1}{4}\right) = 13 &
\end{array}$$

There are **two** sign changes in the first column, so the system has two RHP poles and is unstable.

Example

It is OK to multiply an entire row by a constant; this does not change what happens in the succeeding rows, and helps simplify the math. Consider a system with the following denominator polynomial function:

$$p(s) = s^4 + 10s^3 + 35s^2 + 50s + 24$$

Determine stability using a Routh array.

$$\begin{array}{lllll}
s^4: & 1 & 35 & 24 & \\
s^3: & \cancel{10}\ ^1 & \cancel{50}\ ^5 & 0 & \text{multiply row by } 1/10 \\
s^2: & \frac{35-5}{1} = \cancel{30}\ ^5 & \frac{24-0}{1} = \cancel{24}\ ^4 & 0 & \text{find } b_1 = 30 \text{ and } b_2 = 24\text{ , then multiply by } 1/6 \\
s^1: & \frac{25-4}{5} = \frac{21}{5} & 0 & & \\
s^0: & \frac{5}{21}\cdot\left(\frac{21}{5}\cdot 4 - 0\right) = 4 & 0 & &
\end{array}$$

There are no sign changes in the first column, so the system is stable.

Example

Consider a system with the following denominator polynomial function:

$$p(s) = s^4 + 3s^3 + 3s^2 + 12s + 8$$

Determine stability using a Routh array.

$$
\begin{array}{llll}
s^4: & 1 & 3 & 8 \\
s^3: & 3 & 12 & 0 \\
s^2: & -\frac{12-9}{3} = -1 & \frac{24-0}{3} = 8 & 0 \\
s^1: & -\frac{24+12}{-1} = 36 & 0 & \\
s^0: & \frac{36 \cdot 8 - 0}{36} = 8 & 0 &
\end{array}
$$

There are two sign changes in the first column, so the system is unstable with two poles in the RHP.

Variables in the Routh Array

Routh-Hurwitz is particularly useful for determining stability for systems with variables as coefficients.

Example

Consider a system with the following denominator polynomial function:

$$p(s) = s^3 + 2s^2 + 4s + (3 + K)$$

For what range of K will the system be stable?

$$
\begin{array}{llll}
s^3: & 1 & 4 & 0 \\
s^2: & 2 & 3+\mathrm{K} & 0 \\
s^1: & b_1 = \frac{8-3-K}{2} = \frac{5-K}{2} & 0 & 0 \\
s^0: & c_1 = \frac{2}{5-K} \cdot \left(\frac{5-K}{2} \cdot (3+K) - 0\right) = 3 + K & 0 &
\end{array}
$$

So, for the system to be stable we need

$$\frac{5-K}{2} > 0 \quad \Rightarrow \quad K < 5$$

$$3 + K > 0 \quad \Rightarrow \quad K > -3$$

Therefore the system is stable for $-3 < K < 5$

0-elements in the 1st Column of the Routh Array

How do we approach a system where we get 0 in the first column of the Routh array? This would require dividing by zero. The ϵ-method involves replacing the 0 with a value ϵ, proceeding with the Routh array, and then bring ϵ back to 0 once all elements of the Routh array have been determined.

Example

Consider a system with the following denominator polynomial function:

$$p(s) = s^4 + 2s^3 + s^2 + 2s + 5$$

$$
\begin{array}{lccll}
s^4: & 1 & 1 & 5 & \\
s^3: & 2 & 2 & 0 & \\
s^2: & -\frac{2-2}{2} = \cancel{0}\ \epsilon & -\frac{0-10}{2} = 5 & 0 & \text{Imagine that the zero is some small value } \epsilon \\
s^1: & \frac{2\epsilon-10}{\epsilon} & 0 & & \\
s^0: & \frac{10\epsilon-50}{\epsilon} \cdot \frac{\epsilon}{2\epsilon-10} = 5 & & &
\end{array}
$$

So, the first column has elements

$$1,\ 2,\ \epsilon,\ \frac{2\epsilon-10}{\epsilon},\ 5$$

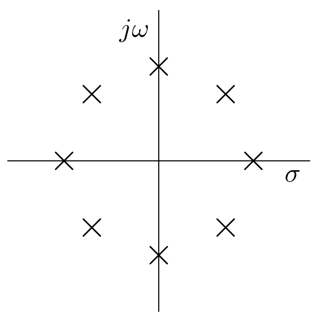
Stability depends on the ϵ and $\frac{2\epsilon-10}{\epsilon}$ terms. What happens as we bring ϵ back to 0? If we start with negative ϵ, then we know the system is unstable because ϵ is an element of the first column. If we start with positive ϵ and approach zero from the right, then

$$\lim_{\epsilon\to 0^+} \frac{2\epsilon-10}{\epsilon} = -\infty$$

So, the system is unstable and has two unstable poles.

Row of 0-elements in the Routh Array

How do we approach a system where an entire row is filled with zeros? This implies that an even number of poles of the system are arranged symmetrically around the origin. **This is in fact the only way to have poles on the $j\omega$ axis**.



This implies that either there are poles in the RHP and/or there are poles on the $j\omega$-axis. **Therefore, the system is BIBO unstable**. This is the only situation that will yield poles on the imaginary axis. There is a way to determine how many of the poles (roots) are on the imaginary axis and how many are in the RHP. We will demonstrate this with an example.

Example

Consider a system with the following transfer function:

$$G(s) = \frac{507s}{s^5 + 3s^4 + 10s^3 + 30s^2 + 169s + 507}$$

And therefore the following denominator polynomial function:

$$p(s) = s^5 + 3s^4 + 10s^3 + 30s^2 + 169s + 507$$

$$\begin{array}{lllll}
s^5: & 1 & 10 & 169 & \\
s^4: & \cancel{3}\ ^{1} & \cancel{30}\ ^{10} & \cancel{507}\ ^{169} & \text{Divide row by 3.} \\
s^3: & \frac{10-10}{1} = 0 & \frac{169-169}{10} = 0 & 0 & \text{Entire row is zeros!} \\
s^1: & \cdots & & &
\end{array}$$

Given the row of zeros, we cannot proceed. The approach around this is to use an auxiliary polynomial, formed from the coefficients of the prior row.

$$P_{aux}(s) = s^4 + 10s^2 + 169$$

We differentiate this polynomial with respect to s and obtain

$$\frac{dP_{aux}}{ds}(s) = 4s^3 + 20s + 0$$

Then, we replace the row of zeros with the coefficients of this new polynomial (we will divide this row through by 4 to simplify the later math). We can then continue with the standard procedure.

$$\begin{array}{llll}
s^5: & 1 & 10 & 169 \\
s^4: & 1 & 10 & 169 \\
s^3: & \cancel{0}\ 1 & \cancel{0}\ 5 & 0 \\
s^2: & \frac{10-5}{1} = 5 & \frac{169-0}{1} = 169 & \\
s^1: & \frac{25-169}{5} = -\frac{144}{5} & 0 & \\
s^0: & 169 & 0 &
\end{array}$$

We can see that this system is unstable, with 2 poles in the RHP. When we have a row of zeros, this tells us that there are an even number of poles symmetrically arranged around the origin. So, the remaining 3 poles must by in the LHP with 2 of them as mirror images of the poles in the RHP. There can be no poles on the jw-axis, because otherwise a symmetric arrangement would be impossible.

In fact, we can check the pole locations with Matlab. The poles are at:

$$s = -3,\ -2 \pm 3i,\ +2 \pm 3i$$

This confirms what we deduced from the Routh-Hurwitz approach.