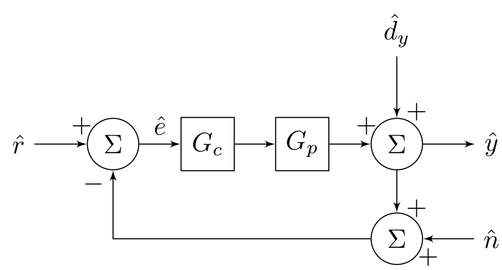
Lecture 15

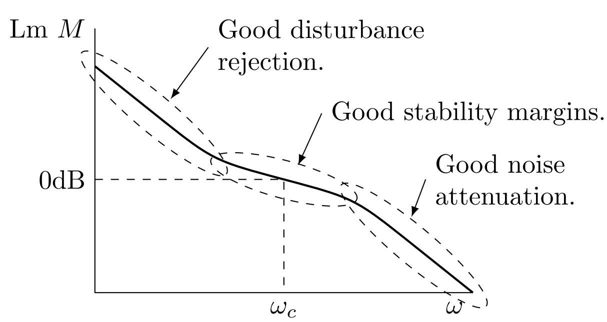
Last lecture:
Bode Plots

- Superposition
  - Lm $M_{total} = \sum$ Lm $M_{components}$
  - $\phi_{total} = \sum \phi_{components}$
- Poles/zeros have an effect for $\omega > pole/zero$
- Zeros add $90°$ of phase and add 20dB/decade of slope to Lm M
- Poles subtract $90°$ of phase and subtract 20dB/decade of slope to Lm M

Feedback



- The closed-loop system will be good at dealing with noise ($\hat{n}$) and disturbances ($\hat{d}_y$) if the Lm M plot of G_cG_p (the open-loop transfer function) has a good shape:



- Designing G_c to give the Bode plot of G_cG_p a good shape is call "loop-shaping".

Internal Stability

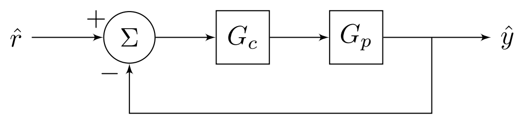
- Roots of $1 + G_cG_p$ strictly in LHP
- All hidden poles are strictly in LHP (G_c does not cancel unstable RHP poles in G_p)

This lecture:

- Closed-loop stability and robustness
- Transient performance

Closed-Loop Stability

How can we tell if a closed-loop feedback system is stable by looking at the loop gain?



$$T = \frac{G_c G_p}{1 + G_c G_p}$$

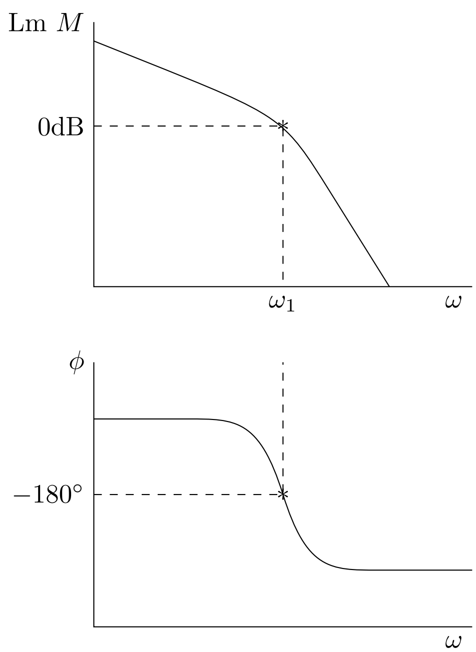
- If s_1 is a closed-loop pole then $1 + G_c(s_1)G_p(s_1) = 0$.
- If the closed-loop system is stable, then $s_1 = a + j\omega_1$ has $a < 0$.
- At the stability limit, $s = j\omega_1$ for at least one closed-loop pole.
- $1 + G_c(s_1)G_p(s_1) = 0 \quad \Rightarrow \quad 1 + G_c(j\omega_1)G_p(j\omega_1) = 0$
- We have already been plotting Lm and ϕ of $G_c(j\omega)G_p(j\omega)$. So, ω_1 will be a particular ω on our Bode plots.
- Let's look at this equation in terms of Lm and ϕ:

$$1 + G_c(j\omega_1)G_p(j\omega_1) = 0 \quad \Rightarrow \quad G_c(j\omega_1)G_p(j\omega_1) = -1$$

$$\text{Lm } M(\omega_1) = \text{Lm } G_c(j\omega_1)G_p(j\omega_1) = \text{Lm } (1) = 0dB$$

$$\phi(\omega_1) = \arg G_c(j\omega_1)G_p(j\omega_1) = \arg(-1) = \tan^{-1}\left(\frac{0}{-1}\right) = -180^\circ$$

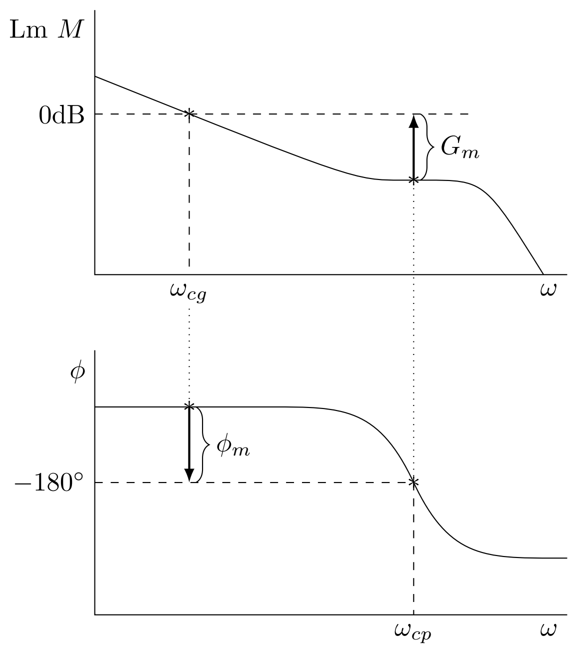
- Our system is marginally stable (which is BIBO unstable):



Gain and Phase Margin

We now introduce two new terms:

- Gain Margin G_m: How far below 0dB is the gain when $\phi = -180°$.
  - In other words, how much additional gain is needed to reach marginal stability?
  - The frequency at which the phase crosses $-180°$ is called the phase crossover frequency, ω_{cp}.
- Phase Margin ϕ_m: How far above $-180°$ is the phase when $\mathrm{Lm} = 0\mathrm{dB}$.
  - In other words, how much additional phase is needed to reach marginal stability?
  - The frequency at which the gain crosses 0dB is called the gain crossover frequency, ω_{cg} (or often just "crossover frequency", ω_c).



For stable opne-loop systems:

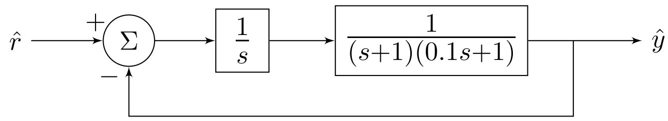
$$G_m > 0 \quad \text{and} \quad \phi_m > 0 \quad \Leftrightarrow \quad \text{Stable closed-loop system}$$

$$G_m \leq 0 \quad \text{and} \quad \phi_m \leq 0 \quad \Rightarrow \quad \textbf{Un}\text{stable closed-loop system}$$

- G_m, ϕ_m are kind of like safety factors telling us how close a system is to instability.
- Higher G_m, ϕ_m mean our closed-loop system is more robust (insensitive to modeling errors and parameter changes)
- According to ASME, we want $G_m \geq 6\mathrm{dB}$ and $\phi_m \geq 30°$ for robust stability.
- (The above rules assumes a stable open-loop system. If the open-loop system is unstable, these may not apply. However, this is a matter for another course.)

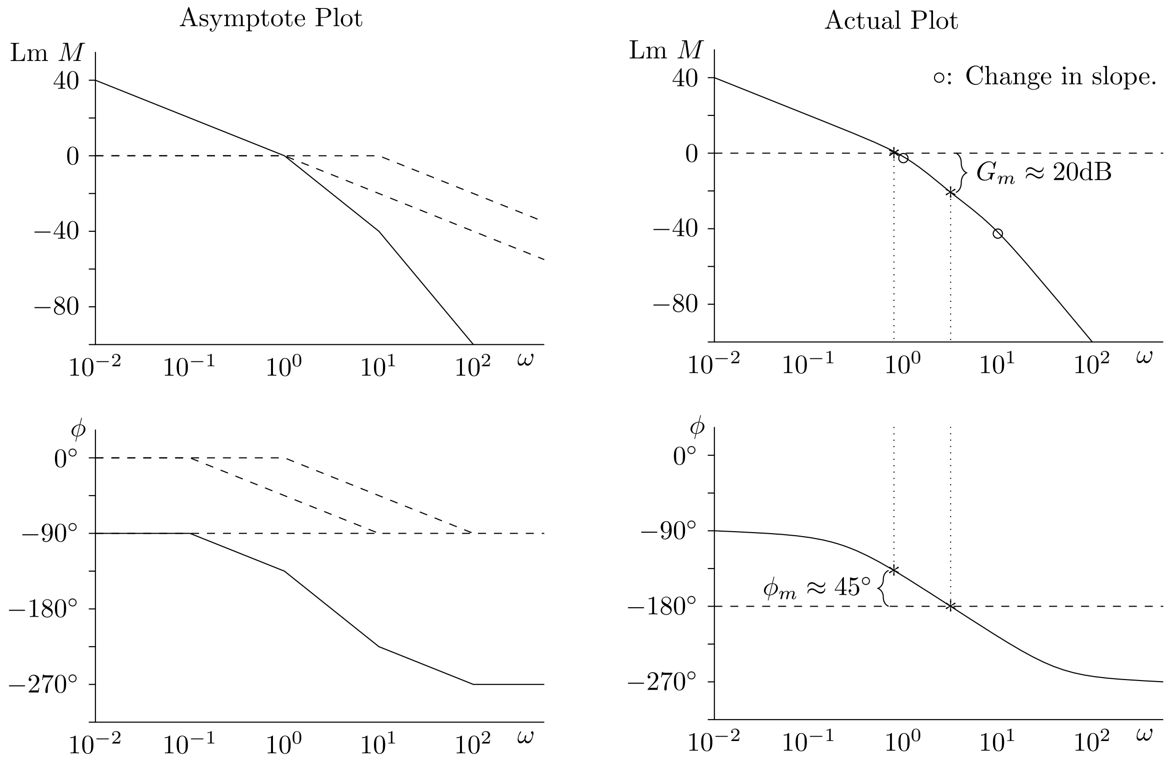
Example

Find the gain and phase margin for the following system.



$$L = G_c G_p = \frac{1}{s(s+1)(0.1s+1)}$$

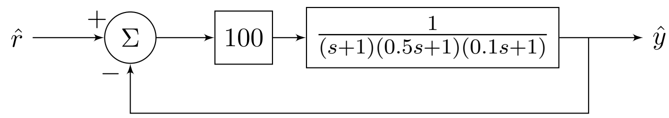
Use superposition to sketch Bode plots.



Both the gain and phase margin ar greater than zero, so the system is stable.

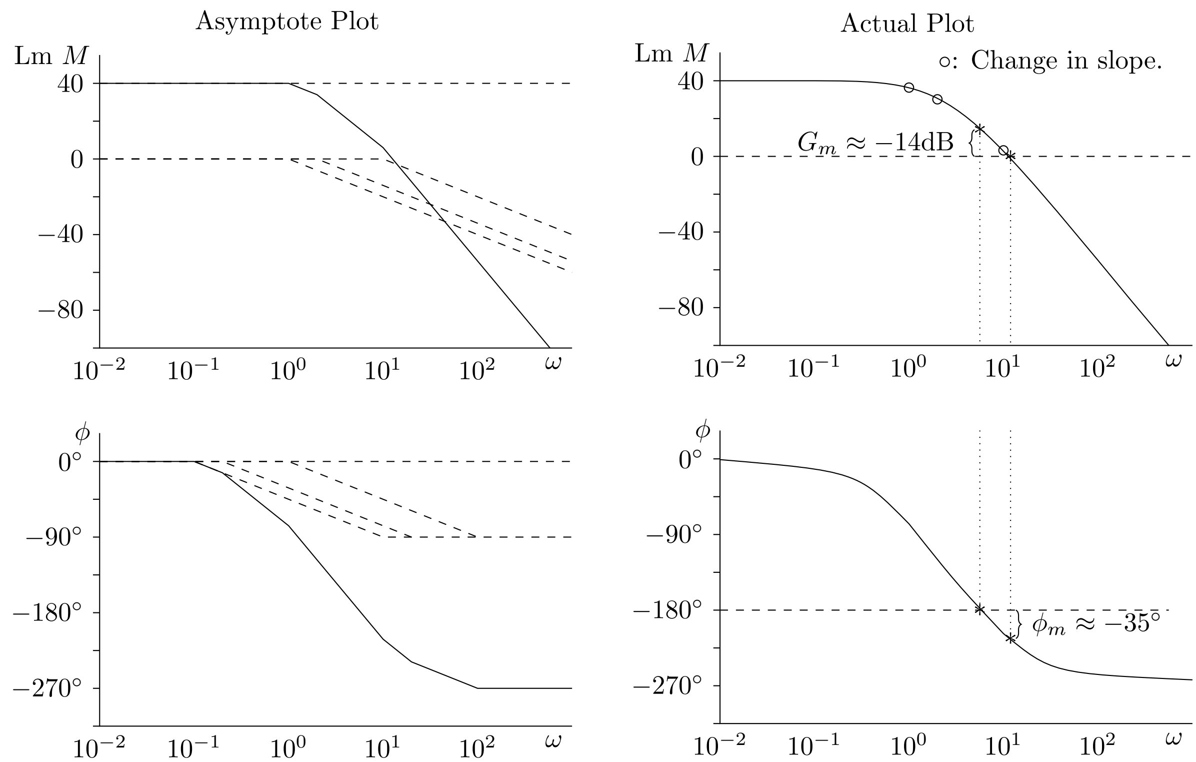
Example

Find the gain and phase margin for the following system.



$$L = G_c G_p = \frac{100}{(s+1)(0.5s+1)(0.1s+1)}$$

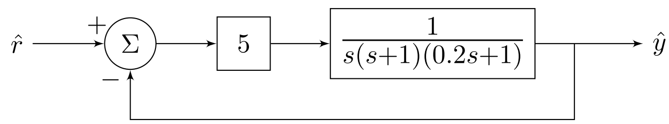
Use superposition to sketch Bode plots.



Both the gain and phase margin ar less than zero, so the system is **unstable**.

Example

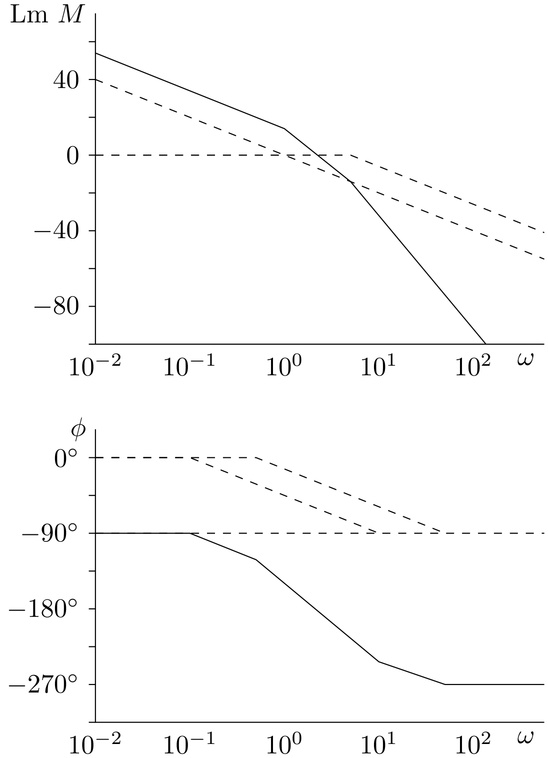
Find the gain and phase margin for the following system.



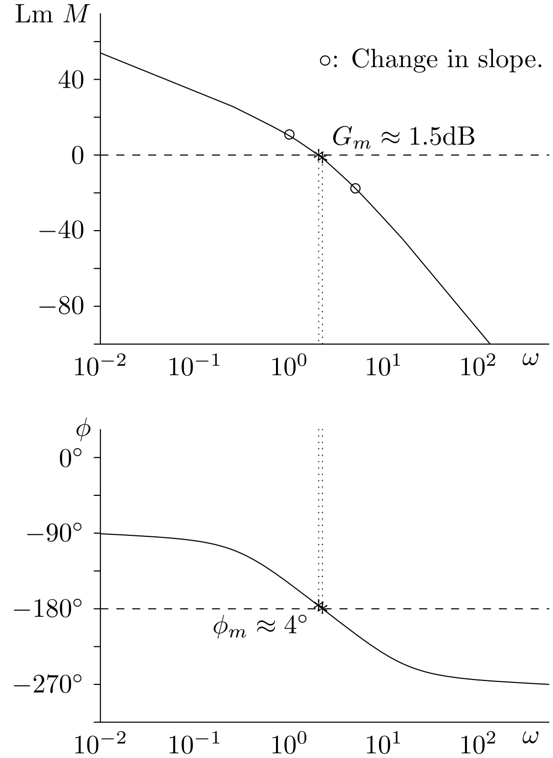
$$L = G_c G_p = \frac{1}{s(s+1)(0.2s+1)}$$

Use superposition to sketch Bode plots.

Asymptote Plot



Actual Plot



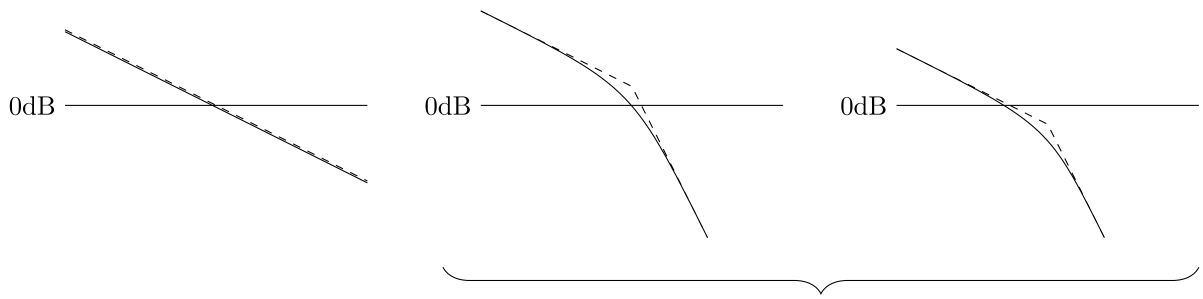
Both the gain and phase margin ar greater than zero, so the system is stable. However, it is **not robust**, because the gain and phase margins are too small. This would be a risky design.

Determining the Phase from the Gain Margins

- If $G_c G_p$ has no zeros or poles in the RHP, we call it a stable minimum phase system. In this case, if $G_m > 0$ then P_m is also > 0 — there will never be a case where one is > 0 while the other is < 0.
- This means that we can use only the gain plot to determine stability by using the following procedure:
- Since every pole adds −20dB/decade to slope of Lm and −90° to ϕ, we can estimate the phase at the crossover frequency by looking at the slope as the Lm crosses 0dB.

Slope of Lm asymptote at 0dB crossover	Approximate phase	Phase margin ϕ_m
−20dB/decade	−90°	90° (Stable)
−40dB/decade	−180°	0° (Marginal)
−60dB/decade	−270°	−90° (Unstable)

- This tells us that we want the slope of the asymptote of Lm G_cG_p to be -20dB/decade at or near the crossover.



$\phi = -90°$ $\qquad$ ϕ is in transition between $-90°$ and $-180°$

- Now we know how to achieve our primary goals (disturbance rejection, noise attenuation, stability robustness):
  - Lm G_cG_p is large for small ω (disturbance rejection)
  - Lm G_cG_p is small for large ω (noise attenuation)
  - **The slope of Lm G_cG_p is -20dB/decade at or near the 0dB crossover (robust stability)**
- Now let's look at how to meet our secondary goals.

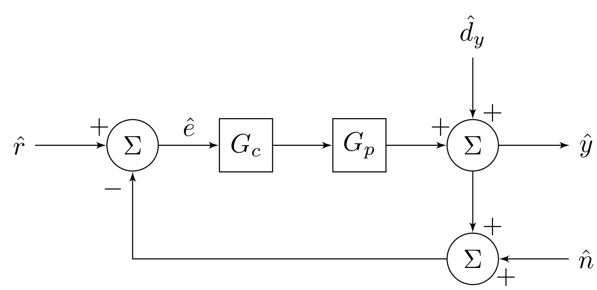
Time Response Considerations

Steady-State Error

- If Lm G_cG_p is large at small ω, e_{ss} will be small.
- Justification: for Lm G_cG_p to be large at small ω, G_cG_p must have a pole at/near 0 or a very high gain.
- So, we have already taken care of e_{ss} when we made Lm G_cG_p large at small ω for disturbance rejection.

$$T = \frac{G_cG_p}{1 + G_cG_p} \quad \Rightarrow \quad G_cG_p \gg 1 \text{ at low fre.} \quad \Rightarrow \quad T = 1 \quad \Rightarrow \quad Y = R$$

Transient Behavior:



- Transient behavior of the system is governed by

$$\frac{\hat{y}}{\hat{r}} = \frac{G_cG_p}{1 + G_cG_p} = T$$

- If we want to set t_{rise}, t_{settle}, t_{peak}, $\%OS$, etc... then CLTF must be similar to a 2nd order system.

$$T \approx \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where ζ and ω_n are chosen based on designed transient performance.

- What does that mean for G_cG_p?
  - Find an expression of G_cG_p in terms of T

$$T = \frac{G_cG_p}{1 + G_cG_p} \quad \rightarrow \quad T(1 + G_cG_p) = G_cG_p$$

$$T = G_cG_p - G_cG_p(T)$$

$$\boxed{G_cG_p = \frac{T}{1-T}}$$

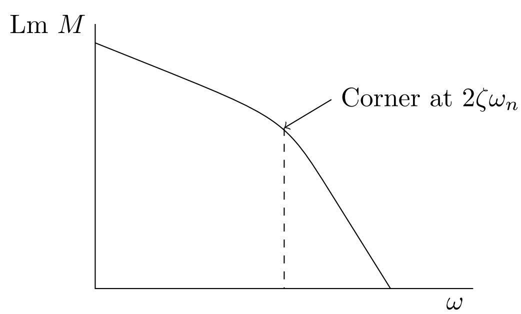
  - Find expression for G_cG_p in terms of ζ and ω_n

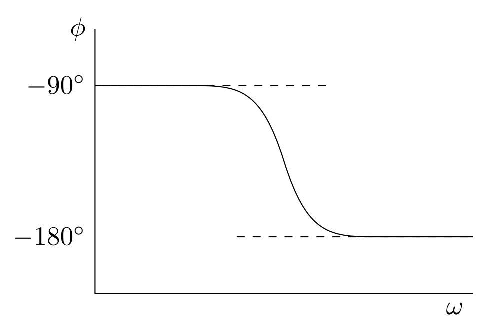
$$G_cG_p = \frac{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}{1 - \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2 - \omega_n^2}$$

$$= \frac{\omega_n^2}{s^2 + 2\omega_n\zeta s} = \frac{\omega_n^2}{s(s + 2\omega_n\zeta)}$$

$$\boxed{G_cG_p = \frac{\frac{\omega_n}{2\zeta}}{s\left(\frac{s}{2\zeta\omega_n} + 1\right)}}$$

- The Bode diagram of G_cG_p will look like:





- Where is the 0dB line? Let's use the low-frequency asymptote to estimate it.

$$G_cG_p(j\omega) = \frac{\omega_n}{2\zeta(j\omega)\left(\frac{j\omega}{2\zeta\omega_n}+1\right)}$$

$$|G_cG_p(j\omega)| = \frac{\omega_n}{2\zeta\omega\sqrt{\frac{\omega^2}{4\zeta^2\omega_n^2}+1}}$$

Approximate $|G_cG_p(j\omega)|$ when ω is small ($\omega \ll \omega_n$)

$$|G_cG_p(j\omega)| \approx \frac{\omega_n}{2\zeta\omega}$$

- Our low-frequency asymptote will have slope of -20dB/decade and intercept 0dB when

$$\frac{\omega_n}{2\zeta\omega} = 1 \quad \Rightarrow \quad \omega = \frac{\omega_n}{2\zeta}$$

- To meet transient requirements Lm $|G_cG_p|$ must have:
  - A low-ω asymptote with -20dB/decade slope and a 0dB intercept at $\omega = \omega_n/2\zeta$.
  - A "corner" decreasing the slope to -40dB/decade at $\omega = 2\zeta\omega_n$.
- Note that if $\zeta < \frac{1}{2}$ the corner will be before the low-ω asymptote gets to 0dB.