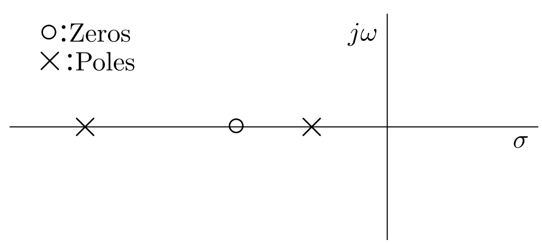
Lecture 2

Last time:

- Painted a picture of what this course is about, and where we are going with it.
- Started with a review of the Laplace Transform

$$L[f(t)] = \int_{0^-}^{\infty} e^{-st} f(t) dt$$

- Did an example to show that $L[1(t)] = 1/s$
- Also listed a few useful LT theorems:
  - $\mathscr{L}[e^{-at} f(t)] = F(s+a)$ where a is constant
  - $\mathscr{L}[tf(t)] = -\frac{d}{ds}F(s)$
  - $\mathscr{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$
  - $\mathscr{L}[af(t) + bg(t)] = aF(s) + bG(s)$ where a and b are constant
  - $\mathscr{L}[\frac{d}{dt}f(t)] = sF(s) - f(0^-)$
  - $\mathscr{L}[\frac{d^2}{dt^2}f(t)] = s^2F(s) - sf(0^-) - \frac{df}{dt}(0^-)$
  - $\mathscr{L}[\frac{d^n}{dt^n}f(t)] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1}f}{dt^{k-1}}(0^-)$
  - $\mathscr{L}[\int_{0^-}^{t} f(\tau)d\tau] = \frac{F(s)}{s}$
- We can express s-domain functions using a **pole-zero diagram**.



How would we transform the sine and cosine functions?

$$f(t) = \sin \omega t \; 1(t) \Rightarrow F(s) =$$

$$g(t) = \cos \omega t \; 1(t) \Rightarrow G(s) =$$

Let's use Euler's Identity, also called Euler's Theorem:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Add/subtract:

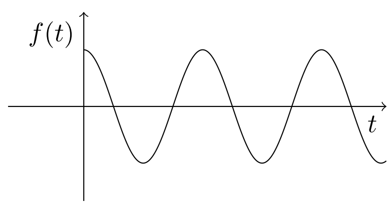
$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

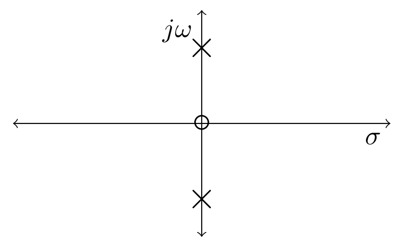
So,

$$\begin{aligned}
\mathscr{L}[\cos\omega t \ 1(t)] &= \mathscr{L}[\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})1(t)] \\
&= \frac{1}{2}\mathscr{L}[e^{j\omega t}1(t)] + \frac{1}{2}\mathscr{L}[e^{-j\omega t}1(t)] \\
&= \frac{1/2}{s - j\omega} + \frac{1/2}{s + j\omega} \\
&= \frac{s}{s^2 + \omega^2}
\end{aligned}$$

Hence,

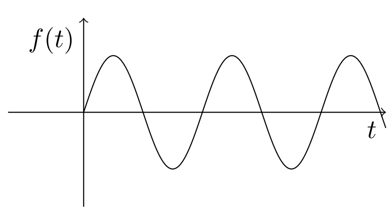
$$\mathscr{L}[\cos\omega t \ 1(t)] = \frac{s}{s^2 + \omega^2}$$

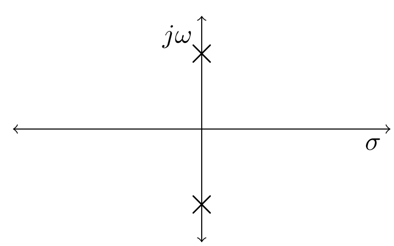




Similarly we can show that

$$\begin{aligned}
\mathscr{L}[\sin\omega t \ 1(t)] &= \mathscr{L}[\frac{1}{2j}(e^{j\theta} - e^{-j\theta})1(t)] \\
&= \frac{1}{2j}\mathscr{L}[e^{j\omega t}1(t)] - \frac{1}{2j}\mathscr{L}[e^{-j\omega t}1(t)] \\
&= \frac{1}{2j}\frac{1}{s - j\omega} + \frac{1}{2j}\frac{1}{s + j\omega} = \frac{1}{2j}\frac{s + j\omega - s + j\omega}{(s - j\omega)(s + j\omega)} \\
&= \frac{\omega}{s^2 + \omega^2}
\end{aligned}$$





We are now in a position to determine the Laplace Transform of most of the functions that are encountered in the study of LTI systems.

Proof of the Euler Identity

The subsection covers the proof of the Euler Identity, used in the previous example. Start with the Taylor series expansion of e^x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

So,

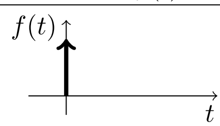
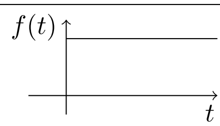
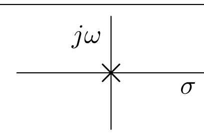
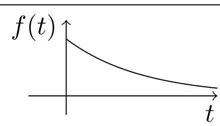
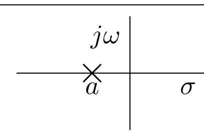
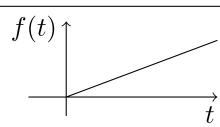
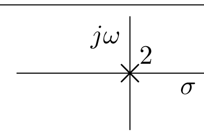
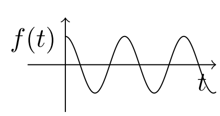
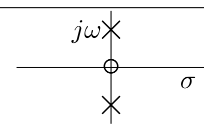
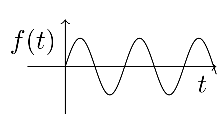
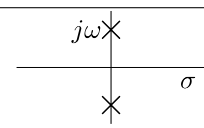
$$
\begin{aligned}
e^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \dots \\
&= 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \\
&= \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots}_{\cos\theta} + j\,\underbrace{j\theta - \frac{j\theta^3}{3!} + \frac{j\theta^5}{5!} - \dots}_{\sin\theta} \\
e^{j\theta} &= \cos\theta + j\sin\theta \\
e^{-j\theta} &= \cos\theta - j\sin\theta
\end{aligned}
$$

So,

$$
\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2}
$$

Summary So Far

Let's summarize our results so far into a table.

Function $f(t)$	Plot of $f(t)$	$F(s)$	Pole-Zero Diagram
$\delta(t)$		1	(no poles or zeros)
$1(t)$		$\frac{1}{s}$	
$e^{-at}1(t)$		$\frac{1}{s+a}$	
$t1(t)$		$\frac{1}{s^2}$	
$\cos\omega t1(t)$		$\frac{s}{s^2+\omega^2}$	
$\sin\omega t1(t)$		$\frac{\omega}{s^2+\omega^2}$	

We will now do some examples of the Laplace Transform.

Example

$$
f(t) = t^2 + e^{-2t}\sin 3t
$$

Find the Laplace transform of $f(t)$. $f(t)$ can be written formally (in causal form) as

$$f(t) = t^2 1(t) + e^{-2t} \sin 3t 1(t) \tag{1}$$

So,

$$\mathscr{L}[1(t)] = \frac{1}{s} \tag{2}$$

$$\mathscr{L}[t \cdot 1(t)] = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2} \tag{3}$$

$$\mathscr{L}[t \cdot t \cdot 1(t)] = -\frac{d}{ds}\frac{1}{s^2} = \frac{2}{s^3} \tag{4}$$

$$\mathscr{L}[\sin 3t \cdot 1(t)] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9} \tag{5}$$

$$\mathscr{L}[e^{-2t} \cdot \sin 3t \cdot 1(t)] = \frac{3}{(s+2)^2+3^2} = \frac{3}{(s+2)^2+9} \tag{6}$$

From (1), (4), and (6):

$$F(s) = \frac{2}{s^3} + \frac{3}{(s+2)^2+9}$$

Example

We want to solve:

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 4 \sin t \; 1(t)$$

where $y(t)$ is the output and $4 \sin t$ is the input, and where $y(0^-) = \dot{y}(0^-) = 0$ (zero initial conditions). In other words, we want the expression for $y(t)$. This is the procedure of the next step: Take the Laplace transform of the entire equation, i.e. equate the LT of the left-hand side to the LT of the right-hand side of the equation.

$$s^2 Y(s) + 5sY(s) + 6Y(s) = \frac{4}{s^2+1}$$

$$(s^2 + 5s + 6)Y(s) = \frac{4}{s^2+1}$$

$$\boxed{Y(s) = \frac{4}{(s^2+1)(s+2)(s+3)}}$$

We have found the Laplace transform of the solution! But how do we back out the solution from this? What is the time-function whose LT is $Y(s)$ above? We need an inverse Laplace transform.

There are two ways to determine the inverse Laplace transform:

1. Formal method (definition)
2. Use of tables and partial fraction expansion

Inverse Laplace Transform

Go from $F(s)$ to $f(t)$ (complex integral).

$$f(t) = LT^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} e^{st} F(s) ds$$

This formula is rarely used. Most of the time, we do inverse Laplace transformation by the second method.

Inverse Laplace Transform of Rational Functions

We start with $Y(s)$ = some rational function; that is, a ratio of polynomials

$$Y(s) = \frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}$$

- If $m < n$, $Y(s)$ is said to be strictly proper
- If $m = n$, $Y(s)$ is proper
- If $m > n$, $Y(s)$ is improper

Strictly proper rational function

Step 1 Divide above and below by the coefficient a_n, and then factor out the denominator.

$$Y(s) = \frac{1}{a_n} \cdot \frac{b_m s^m + \ldots + b_1 s + b_0}{s^n + \ldots + \frac{a_1}{a_n} s + \frac{a_0}{a_n}} = \frac{\frac{1}{a_n}(b_m s^m + \ldots + b_1 s + b_0)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where the p_k's are the poles of $Y(s)$.

Step 2 Do a Heaviside Expansion of $Y(s)$ and determine the residuals.

- **Step 2 Case A:** $Y(s)$ is strictly proper and the p_k's are distinct. Then, the expansion has the form

$$Y(s) = \frac{R_1}{s+p_1} + \frac{R_2}{s+p_2} + \ldots + \frac{R_k}{s+p_k} + \ldots + \frac{R_n}{s+p_n} \tag{1}$$

R_k are called the residuals — they are constants. To determine R_k, multiply both sides of (1) by $(s+p_k)$ and evaluate $s = -p_k$:

$$(s+p_k)Y(s)\Big|_{s=-p_k} = \cancel{\frac{(s+p_k)R_1}{(s+p_1)}\Big|_{s=-p_k}} + \ldots + R_k + \ldots + \cancel{\frac{(s+p_k)R_n}{(s+p_n)}\Big|_{s=-p_k}}$$

$$\Rightarrow \quad R_k = (s+p_k)Y(s)\Big|_{s=-p_k}$$

Step 3 Use the linearity property of the Laplace transform to write

$$y(t) = R_1 e^{-p_1 t} + \ldots + R_k e^{-p_k t} + \ldots + R_n e^{-p_n t}$$

The problem is solved!

We will do some examples first, and then return to different cases for Step 2.

Example

$$Y(s) = \frac{s+1}{4s^2 + 15s + 9}$$

Find $y(t)$. $Y(s)$ is strictly proper. So, divide the numerator and denominator by 4.

$$Y(s) = \frac{\frac{s}{4} + \frac{1}{4}}{s^2 + \frac{15}{4}s + \frac{9}{4}} = \frac{\frac{1}{4}(s+1)}{\left(s+\frac{3}{4}\right)(s+3)}$$

Heaviside Expansion:

$$Y(s) = \frac{\frac{1}{4}(s+1)}{\left(s+\frac{3}{4}\right)(s+3)} = \frac{R_1}{s+\frac{3}{4}} + \frac{R_2}{s+3}$$

where

$$R_1 = \cancel{\left(s+\frac{3}{4}\right)} \frac{\frac{1}{4}(s+1)}{\cancel{\left(s+\frac{3}{4}\right)}(s+3)}\Big|_{s=-3/4} = \frac{\frac{1}{4}(s+1)}{s+3}\Big|_{s=-3/4} = \frac{\frac{1}{4}(-\frac{3}{4}+1)}{-\frac{3}{4}+3} = \frac{\frac{1}{4}(\frac{1}{4})}{\frac{9}{4}} = \frac{1}{36}$$

$$R_2 = \cancel{(s+3)} \frac{\frac{1}{4}(s+1)}{\left(s+\frac{3}{4}\right)\cancel{(s+3)}}\Big|_{s=-3} = \frac{\frac{1}{4}(s+1)}{s+\frac{3}{4}}\Big|_{s=-3} = \frac{\frac{1}{4}(-3+1)}{-3+\frac{3}{4}} = \frac{\frac{1}{4}(-2)}{-\frac{9}{4}} = \frac{2}{9}$$

So,

$$Y(s) = \frac{1/36}{s+3/4} + \frac{2/9}{s+3}$$

Let's quickly check our work.

$$\frac{1/36}{s+3/4} + \frac{2/9}{s+3} = \frac{\frac{1}{36}(s+3) + \frac{2}{9}\left(s+\frac{3}{4}\right)}{\left(s+\frac{3}{4}\right)(s+3)} = \frac{s+3+8s+6}{36\left(s+\frac{15}{4}s+\frac{9}{4}\right)} = \frac{9s+9}{4\cdot 9\cdot\left(s+\frac{15}{4}s+\frac{9}{4}\right)} = \frac{s+1}{4s^2+15s+9} \quad \checkmark$$

Then,

$$Y(s) = \frac{1/36}{s+3/4} + \frac{2/9}{s+3}$$
$$y(t) = \left(\frac{1}{36}e^{-\frac{3}{4}t} + \frac{2}{9}e^{-3t}\right)1(t)$$
$$y(t) = \frac{1}{36}\left(e^{-\frac{3}{4}t} + 8e^{-3t}\right)1(t)$$

Example

$$\frac{dy}{dt} + 7y = 7\cdot 1(t), \quad y(0^-) = 0$$

Find $y(t)$. First, take the Laplace transform.

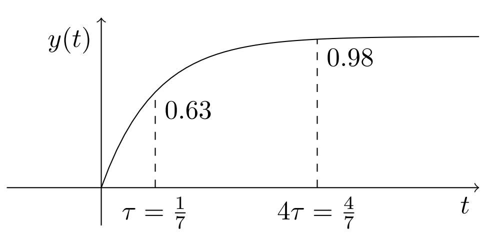
$$sY(s) + 7Y(s) = \frac{7}{s}$$

Then solve for $Y(s)$, expand, and find the residuals.

$$Y(s) = \frac{7}{s(s+7)} = \frac{R_1}{s} + \frac{R_2}{s+7}$$
$$R_1 = sY\Big|_{s=0} = 1$$
$$R_2 = (s+7)Y\Big|_{s=-7} = -1$$

So,

$$Y(s) = \frac{1}{s} - \frac{1}{s+7}$$
$$y(t) = (1-e^{-7t})\cdot 1(t)$$



Example

$$\frac{d^2y}{dt^2} + 4y = 8 \cdot 1(t), \quad y(0^-) = 0$$

Find $y(t)$. First, take the Laplace transform.

$$s^2Y(s) + 4Y(s) = \frac{8}{s}$$

Then solve for $Y(s)$, expand, and find the residuals.

$$Y(s) = \frac{8}{s(s^2+4)} = \frac{8}{s(s+2j)(s-2j)} = \frac{R_1}{s} + \frac{R_2}{s+2j} + \frac{R_3}{s-2j}$$

$$R_1 = sY\Big|_{s=0} = \frac{8}{4} = 2$$

$$R_2 = (s+2j)Y\Big|_{s=-2j} = \frac{8}{-2j(-4j)} = -1$$

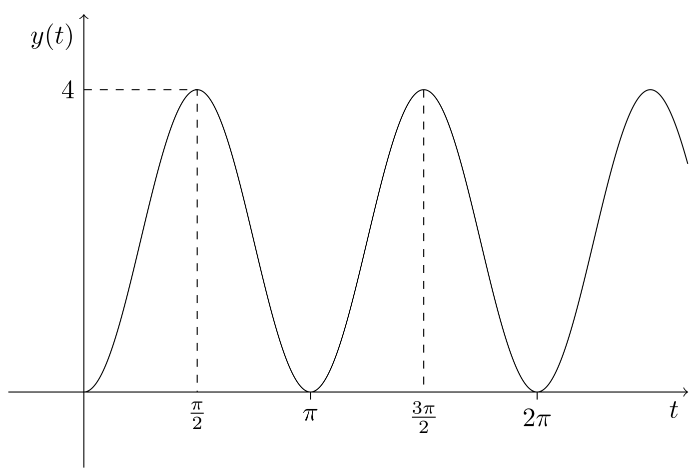
$$R_3 = (s-2j)Y\Big|_{s=+2j} = \frac{8}{+2j(+4j)} = -1$$

So,

$$Y(s) = \frac{2}{s} - \frac{1}{s+2j} - \frac{1}{s-2j}$$

$$y(t) = \left(2 - e^{2jt} - e^{-2jt}\right) 1(t) = 2\left(1 - \frac{e^{2jt} + e^{-2jt}}{2}\right) = 2\left(1 - \cos(2t)\right) 1(t)$$

$\omega = 2$, so therefore the vibration period is $T = \pi$.



(returning to the procedure for inverse Laplace transform)

- **Case B:** $Y(s)$ is strictly proper but the p_k's are not distinct, i.e. there are some repeated roots. We'll demonstrate how to solve this with an example.

Example

$$Y(s) = \frac{3}{(s+1)(s+2)^2}$$

Expanded:

$$Y(s) = \frac{R_1}{s+1} + \frac{R_2}{(s+2)^2} + \frac{R_3}{s+2}$$

R_1 and R_2 are obtained the usual way.

$$R_1 = (s+1)Y(s)\Big|_{s=-1} = 3$$

$$R_2 = (s+2)^2 Y(s)\Big|_{s=-2} = -3$$

R_3 is found as follows

$$R_3 = \frac{d}{ds}\left[(s+2)^2 Y(s)\right]\Big|_{s=-2} = \left(\frac{d}{ds}\frac{3}{s+1}\right)\Big|_{s=-2} = \frac{-3}{(s+1)^2}\Big|_{s=-2} = -3$$

So,

$$Y(s) = \frac{3}{s+1} - \frac{3}{(s+2)^2} - \frac{3}{s+2} \overset{\text{check work:}}{=} \frac{3(s+2)^2 - 3(s+1) - 3(s+1)(s+2)}{(s+1)(s+2)^2} = \frac{3}{(s+1)(s+2)^2} \quad \checkmark$$

Then,

$$y(t) = \left(3e^{-t} + \mathscr{L}^{-1}\left[\frac{-3}{(s+2)^2}\right] - 3e^{-2t}\right)1(t)$$

What is the inverse Laplace of $\frac{-3}{(s+2)^2}$? Recall

$$\mathscr{L}[e^{-2t}\cdot 1(t)] = \frac{1}{s+2}$$

$$\mathscr{L}[t\cdot e^{-2t}\cdot 1(t)] = -\frac{d}{ds}\frac{1}{s+2} = \frac{1}{(s+2)^2}$$

So,

$$y(t) = \left(3e^{-t} - 3te^{-2t} - 3e^{-2t}\right)1(t)$$

Example

(Discussion Session)

$$Y(s) = \frac{2s+1}{s(s+1)^2}$$

Expanded:

$$Y(s) = \frac{R_1}{s} + \frac{R_2}{(s+1)^2} + \frac{R_3}{s+1}$$

Find the residuals:

$$R_1 = sY(s)\Big|_{s=0} = 1$$

$$R_2 = (s+1)^2 Y(s)\Big|_{s=-1} = 1$$

$$R_3 = \frac{d}{ds}\left[(s+1)^2 Y(s)\right]\Big|_{s=-1} = \frac{d}{ds}\frac{2s+1}{s}\Big|_{s=-1} = \frac{2s-(2s+1)}{s^2}\Big|_{s=-1} = \frac{-1}{s^2}\Big|_{s=-1} = -1$$

So,

$$Y(s) = \frac{1}{s} + \frac{1}{(s+1)^2} - \frac{1}{s+1}$$

Then,

$$y(t) = \left(1 + te^{-t} - e^{-t}\right)1(t)$$

Example

(Discussion Session)

$$Y(s) = \frac{3s+4}{s(s+1)^3}$$

Expanded:

$$Y(s) = \frac{R_1}{s} + \frac{R_2}{(s+1)^3} + \frac{R_3}{(s+1)^2} + \frac{R_4}{s+1}$$

Find the residuals:

$$R_1 = sY(s)\Big|_{s=0} = 4$$

$$R_2 = (s+1)^3 Y(s)\Big|_{s=-1} = -1$$

$$R_3 = \frac{d}{ds}\left[(s+1)^3 Y(s)\right]\Big|_{s=-1} = \frac{d}{ds}\frac{3s+4}{s}\Big|_{s=-1} = \frac{3s-(3s+4)}{s^2}\Big|_{s=-1} = \frac{-4}{s^2}\Big|_{s=-1} = -4$$

$$\begin{aligned} R_4 &= \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+1)^3 Y(s)\right]\Big|_{s=-1} = \frac{1}{2}\frac{d^2}{ds^2}\frac{3s+4}{s}\Big|_{s=-1} \\ &= \frac{d}{ds}\frac{-4}{2s^2}\Big|_{s=-1} = \frac{4}{s^3}\Big|_{s=-1} = -4 \end{aligned}$$

So,

$$Y(s) = \frac{4}{s} - \frac{1}{(s+1)^3} - \frac{4}{(s+1)^2} - \frac{4}{s+1}$$

Then,

$$y(t) = \left[4 - e^{-t}\left(\frac{1}{2}^2 t + 4t + 4\right)\right] 1(t)$$

How do we know

$$\mathscr{L}\left[\frac{1}{2}t^2 e^{-t} 1(t)\right] = \frac{1}{(s+1)^3} \ ?$$

We know

$$\mathscr{L}[te^{-t}] = \frac{1}{(s+1)^2}$$

So,

$$\mathscr{L}\left[t^2 e^{-t} 1(t)\right] = -\frac{d}{ds}\frac{1}{(s+1)^2} = \frac{2}{(s+1)^3}$$

$$\mathscr{L}\left[\frac{1}{2}t^2 e^{-t} 1(t)\right] = -\frac{d}{ds}\frac{1}{(s+1)^2} = \frac{1}{(s+1)^3}$$

$$g(t) = \left(4 - \frac{1}{2}t^2 e^{-t} - 4te^{-t} - 4e^{-t}\right) 1(t)$$

We will now look at one final case:

- **Case C:** $Y(s)$ is proper and the p_k's are distinct. We'll demonstrate how to solve this with an example.

Example

(Discussion Session)

$$Y(s) = \frac{s^2+1}{s^2+2s} = \frac{s^2+1}{s(s+2)}$$

This **won't work** if we try

$$Y(s) = \frac{R_1}{s} + \frac{R_2}{s+2}$$

If we put this over a common denominator:

$$Y(s) = \frac{(R_1+R_2)s + 2R_1}{s(s+1)}$$

We can see that the degree of the numerator is too low! The solution is a modification to the Heaviside expansion:

$$Y(s) = \frac{R_1}{s} + \frac{R_2}{s+2} + \mathbf{R_0}$$

Add something that will make the degree of the numerator ok. Adding a constant solves this problem. R_1 and R_2 are found the usual way.

$$R_1 = sY(s)\Big|_{s=0} = \frac{1}{2}$$

$$R_2 = (s+2)Y(s)\Big|_{s=-2} = -\frac{5}{2}$$

R_0 is found as follows:

$$R_0 = \lim_{s\to\infty} Y(s) = 1$$

So,

$$Y(s) = \frac{1}{2s} - \frac{5}{2(s+2)} + 1$$

$$y(t) = \left(\frac{1}{2} - \frac{5}{2}e^{-2t}\right)1(t) + \delta(t)$$

This process is in fact the same with repeated poles, so we will not consider that a 4th case. We will show this with an example for repeated poles.

Example

$$Y(s) = \frac{s^3+1}{s(s+2)^2} = \frac{s^3+1}{s^3+4s^2+4s} = \frac{R_1}{s} + \frac{R_2}{(s+2)^2} + \frac{R_3}{s+2} + R_0$$

Then,

$$R_1 = sY(s)\Big|_{s=0} = \frac{1}{4}$$

$$R_2 = (s+2)^2Y(s)\Big|_{s=-2} = \frac{7}{2}$$

$$R_3 = \frac{d}{ds}\left[(s+2)^2Y(s)\right]\Big|_{s=-2} = \frac{d}{ds}\frac{s^3+1}{s} = \frac{2s^3-1}{s^2} = -\frac{17}{4}$$

$$R_0 = \lim_{s\to\infty} Y(s) = \lim_{s\to\infty} \frac{s^3+1}{s^3+4s^2+4s} = 1$$

So,

$$Y(s) = \frac{1}{4s} + \frac{7}{2(s+2)^2} - \frac{17}{4(s+2)} + 1$$

$$y(t) = \left(\frac{1}{4} - \frac{7}{2}te^{-2t} - \frac{17}{4}e^{-2t}\right)1(t) + \delta(t)$$