Lecture 14

Bode diagrams for more complex systems

We know how to sketch Bode diagrams for simple systems like s, $1/s$, $\tau s + 1$, and so on. If $G(s)$ is more complex, what do we do?

Imagine factoring $G(s)$ as:

$$G(s) = KG_1(s)G_2(s)\ldots G_N(s)$$

where each factored term ($G_i(s)$) is something we know how to sketch. Then in polar form we have:

$$G(j\omega) = Me^{j\phi} = KM_1e^{j\phi_1}M_2e^{j\phi_2}\ldots M_Ne^{j\phi_N}$$

$$G(j\omega) = (KM_1M_2\ldots M_N)e^{j(\phi_1+\phi_2+\ldots+\phi_N)}$$

So,

$$M = KM_1M_2\ldots M_N \Rightarrow \text{Lm } M = \text{Lm } K + \text{Lm } M_1 + \text{Lm } M_2 + \ldots \text{Lm } M_N$$

$$\phi = \phi_1 + \phi_2 + \ldots + \phi_N$$

We can use superposition for Lm M and ϕ — adding and together the effects of the different terms.

For superposition, we need to use a change mentality. For example:

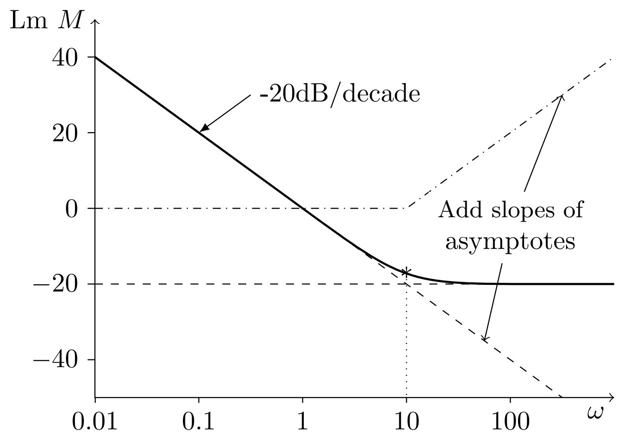
- **Don't think:**
  - Lm $(\tau s + 1)$ is a horizontal line at 0dB for $\omega < 1/\tau$
  - Lm $(\tau s + 1)$ is a line with 20dB/decade slope for $\omega > 1/\tau$.
- **Instead, think:**
  - Lm $(\tau s + 1)$ adds nothing for $\omega < 1/\tau$
  - Lm $(\tau s + 1)$ adds 20dB/decade slope for $\omega > 1/\tau$.

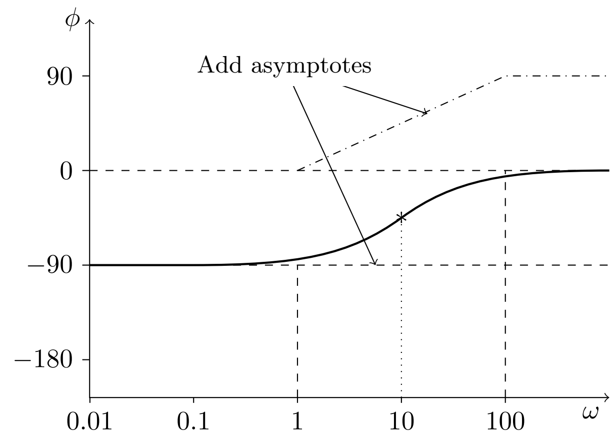
Example

Draw the Bode diagram for

$$G(s) = \frac{s/10+1}{s} = \frac{1}{s} \cdot (\frac{1}{10}s + 1)$$

$$\tau s + 1 \Rightarrow \tau = \frac{1}{10}, \text{ then } \frac{1}{\tau} = 10$$





- Start with −20dB/decade slope from integrator
- At $\omega = \frac{1}{\tau} = 10$rad/s, add 20dB/decade slope (net: 0dB/decade)

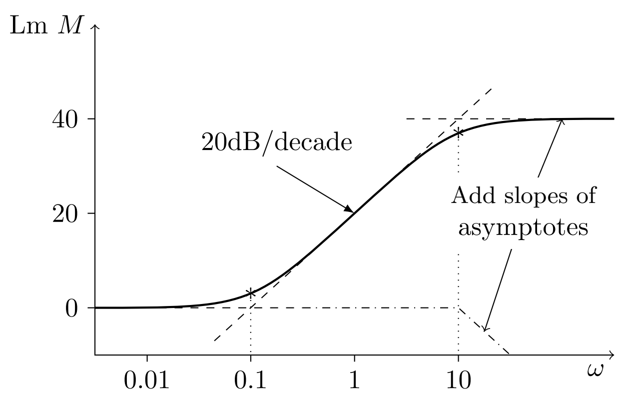
- Start with −90° phase from integrator
- Add 90° phase centered on $\omega = \frac{1}{\tau} = 10$rad/s (as $\omega \to \infty$: $\phi = 0°$)

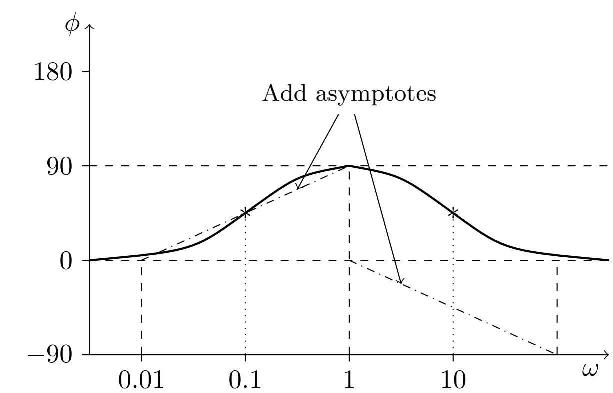
Example

Draw the Bode diagram for

$$G(s) = \frac{10s+1}{s/10+1}$$

$$\Rightarrow G(s) = 1 \cdot (10s+1) \cdot \left(\frac{1}{s/10+1}\right)$$





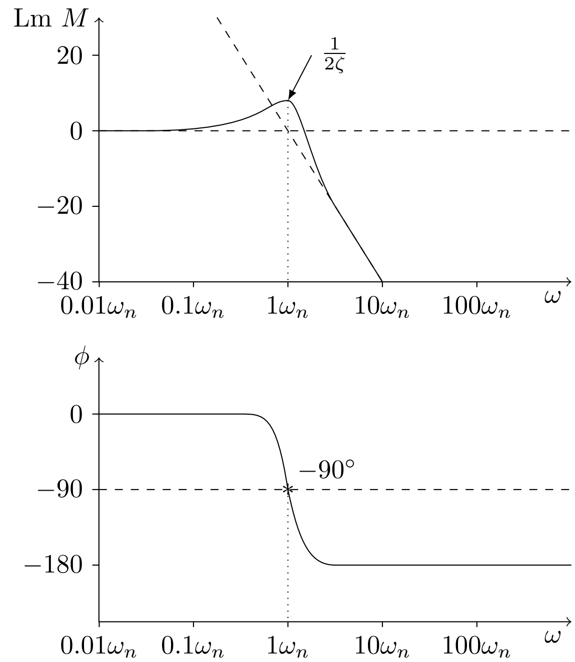
- Start with $K = 1$: Lm $M = 0$ and 0dB/decade slope
- At $\omega = \frac{1}{\tau_z} = 0.1$rad/s, add 20dB/decade slope (net: 20dB/decade)
- At $\omega = \frac{1}{\tau_p} = 10$rad/s, subtract 20dB/decade slope (net: 0dB/decade)

- Start with $K = 1$: 0° phase
- Add 90° phase centered on $\omega = \frac{1}{\tau_z} = 0.1$rad/s (net: $\omega = 90°$)
- Subtract 90° phase centered on $\omega = \frac{1}{\tau_p} = 10$rad/s (net: $\omega = 0°$)

Bode Plot Summary

- Use superposition to sketch more complex Bode plots
  - $\text{Lm } M_{total} = \sum \text{Lm } M_{components}$
  - $\phi_{total} = \sum \phi_{components}$
- Poles and zeros only have an effect for frequencies greater than the pole or zero ($\omega > p$ or $\omega > z$)
- Zeros add 90° of phase and add 20dB/decade of slope for $\omega > z$
- Poles subtract 90° of phase and subtract 20dB/decade of slope for $\omega > p$

Find a transfer function from a Bode plot



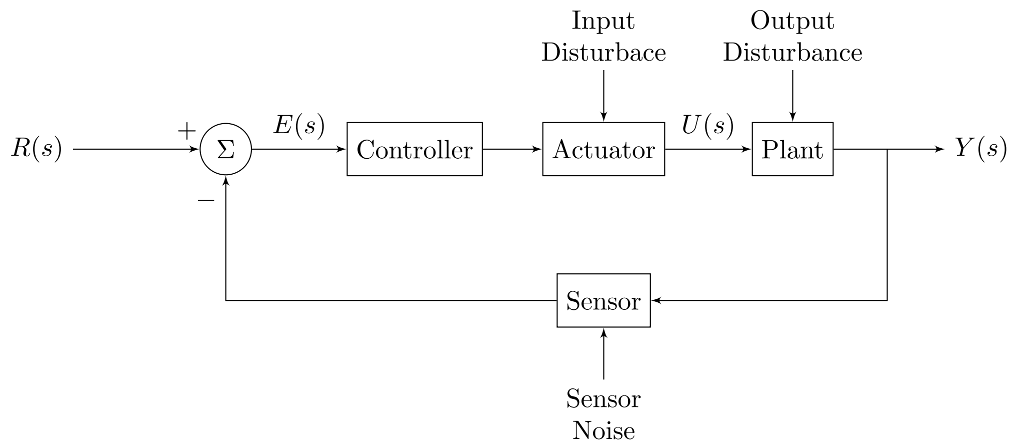
Given a Bode plot that is **known** to be 2nd-order, can we find the transfer function?

1. Find where the phase plot crosses 90°. This frequency is ω_n.
2. Find the magnitude at ω_n. This is $1/(2\zeta)$.

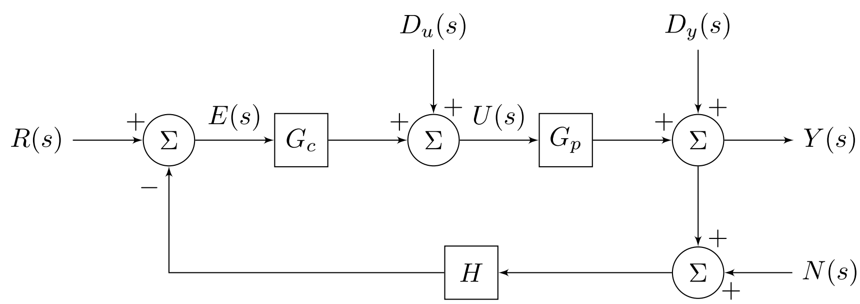
With ζ and ω_n known, the transfer function can be constructed.

Feedback Principles

Real systems have **disturbances** (uncontrolled inputs to the system).



Modeled in conventional block diagram form,



Typically:

- G_p is known
- H is given

Goal: Design G_c such that

- The feedback system is stable (internal stability)
- Transient behavior is good (minimal overshoot) — time domain, performance metric
- Steady-state behavior is good (minimal steady-state error) — time domain, performance metric

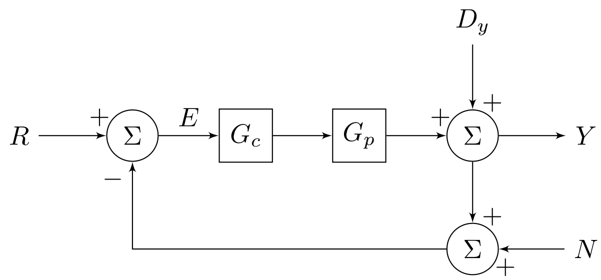
To simplify our closed-loop system, let's make some reasonable assumptions:

1. Ignore D_u ($D_u \approx 0$)
2. $H \approx 1$

How do we justify these assumptions?

- Typically, $D_u \ll D_y$
- We may have some control over D_u
- We want to use a sensor with negligible dynamics, otherwise it is not a good sensor.

So,



Sensitivity and Complementary Sensitivity

We will now introduce two new transfer functions used to describe the effect of disturbances on the system.

Sensitivity (S) is the transfer function between the output disturbance D_y and the output Y.

$$\text{Sensitivity: } S = \frac{Y}{D_y} = \frac{1}{1 + G_c G_p}$$

Complementary Sensitivity (T) is the transfer function between the noise N and the output Y.

$$\text{Complementary Sensitivity: } T = \frac{Y}{N} = \frac{G_c G_p}{1 + G_c G_p}$$

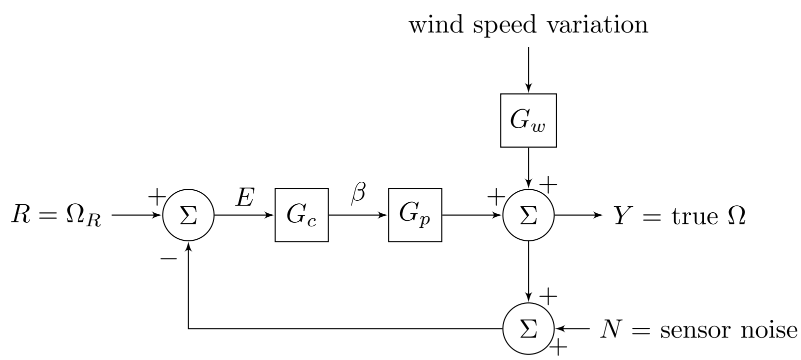
Controller Design

When designing a controller, the disturbances present a challenge:

- Input/Output disturbance
- Sensor noise
- Modeling error
- Variations in system parameters

Example

Consider a wind turbine where the controller goal is to maintain a constant rotational speed Ω_R by controlling the blade pitch angle β. What are some possible disturbances?



The wind turbine is subject to output disturbances (the impact of wind speed variation on rotational speed), sensor noise, modeling error (real dynamics are nonlinearl, but G_p is modeled as a linear system when designing G_c), and variations in system parameters (e.g. changes in air density will change the plant).

We want our system to have:

- Disturbance rejection — frequency domain, performance metric
- Noise attenuation — frequency domain, performance metric
- Robustness — frequency domain, performance **and** stability metric

Disturbance Rejection

$$\text{Sensitivity: } S = \frac{Y}{D_y} = \frac{1}{1 + G_c G_p}$$

If we don't want disturbances (D_y) to have much effect on the output (Y):

- What do we want the magnitude of S to be? **Small.**
- What do we want the magnitude of $G_c G_p$ to be for small S? **Big.**

$$d_y(t) = A\sin(\omega t) \longrightarrow \boxed{S} \longrightarrow y(t) = M_S A \sin(\omega t + \phi_S)$$

If M_S is small (magnitude of $G_c G_p$ is large), then disturbances will be rejected at Y.
Note: What is meant by "small"?

- Small compared to 1, i.e. $M_S \ll 1$.
- When $M_S = 1$, the magnitude of the disturbance = the magnitude of the output.

Noise Attenuation

$$\text{Complementary Sensitivity: } T = \frac{Y}{N} = \frac{G_c G_p}{1 + G_c G_p}$$

If we don't want disturbances (N) to have much effect on the output (Y):

- What do we want the magnitude of T to be? **Small.**
- What do we want the magnitude of $G_c G_p$ to be for small T? **Small.**

$$n(t) = A\sin(\omega t) \longrightarrow \boxed{T} \longrightarrow y(t) = M_T A \sin(\omega t + \phi_T)$$

If M_T is small (magnitude of $G_c G_p$ is small), then noise will be attenuated at Y.
Note: What is meant by "small"?

- Small compared to 1, i.e. $M_T \ll 1$.
- But, $\frac{Y}{R} = T$ and we want $R = Y$. Therefore, $M_T = 1$.

Loop Shaping

We want both S and T to be small! Unfortunately, this isn't possible.

$$\text{Small } G_c G_p \rightarrow \text{Small } T$$

$$\text{Large } G_c G_p \rightarrow \text{Small } S$$

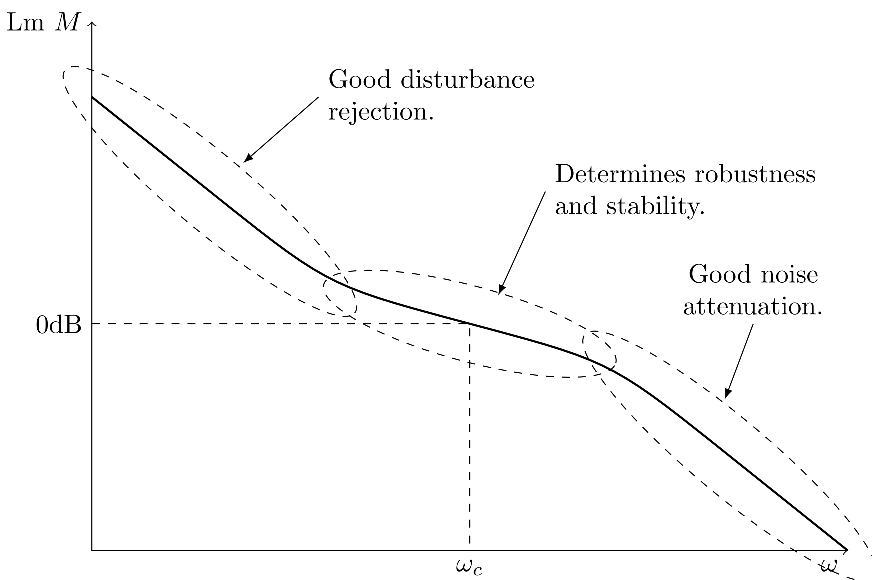
$$S + T = \frac{1}{1 + G_c G_p} + \frac{G_c G_p}{1 + G_c G_p}$$

$$\boxed{S + T = 1}$$

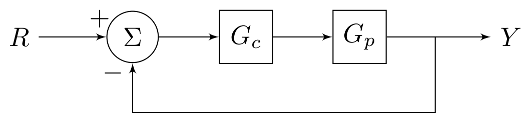
If S and T add up to 1, can S or T be greater than 1? YES! S and T are complex numbers, so $S + T = 1$ is like vector addition.

Improving disturbance rejection makes noise attenuation worse, and vice versa. What do we do?

- What if I told you that disturbances are typically low-frequency signals and noise is a typically high-frequency signal?
  - Make $M_S \ll 1$ at low frequency
    ($M_T = 1$ at low frequency)
  - Make $M_T \ll 1$ at high frequency
    ($M_S = 1$ at high frequency)
- In terms of $G_c G_p$:
  - Make $G_c G_p$ large for low frequency and small for high frequency
- A good shape for $G_c G_p$ is something like:



Stability Revisited

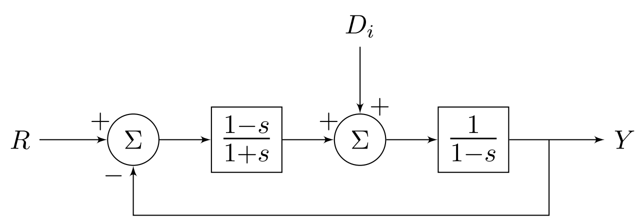


- The definition of stability relies on the location of the closed-loop poles:

$$den = 1 + G_c G_p = 0 \quad \left(\text{where } \frac{Y}{R} = \frac{num}{den}\right)$$

- But, this might not tell the entire story. There might be "hidden poles" not captured by that equation.
- For instance, we can design G_c to cancel out poles of G_p.
- Why do we care about these hidden poles?
  - **Because they can resurface when we don't expect it.**

Example



First, consider the closed-loop transfer function:

$$\frac{Y}{R} = \frac{\frac{1-s}{1+s}\frac{1}{1-s}}{1 + \frac{1-s}{1+s}\frac{1}{1-s}} = \frac{\frac{1}{s+1}}{1 + \frac{1}{s+1}} = \frac{1}{s+2}$$

So, the closed-loop TF has a single, stable pole at $s = -2$. Now let's look at the effect of the input disturbance:

$$\frac{Y}{D_i} = \frac{\frac{1}{1-s}}{1 + \frac{1-s}{1+s}\frac{1}{1-s}} = \frac{\frac{s+1}{s-1}}{1+s+1} = \frac{s+1}{(s+2)(1-s)}$$

Even though the unstable pole was canceled out of the closed-loop TF, it is still present in the system's response to an input disturbance.

- We *thought* we had a stable system, but the "hidden" pole came back to haunt us.
- A system is called **internally stable** if every closed-loop transfer function (not just R to Y) is stable.
  - This system is not internally stable.
- In general, do not cancel unstable poles of the plant.