Bode Controller Design Examples

Assume unity feedback systems with the following plant transfer functions. Use loop shaping to design a controller (G_c) such that the closed loop system will meet the following requirements when subjected to a step input:

- overshoot $\leq 10\%$,
- settling time ≤ 3 seconds,
- steady state error $\leq 10\%$.

for the two plants

a) $G_p(s) = \dfrac{1}{\dfrac{s}{2}+1}$

b) $G_p(s) = \dfrac{1}{(s+1)(s+3)(s+15)}$

First, we find our constraints on ζ and ω_n from the given specifications.

$$\zeta \geq -\frac{\log\left(\dfrac{OS\%}{100}\right)}{\sqrt{\pi^2 - \log\left(\dfrac{OS\%}{100}\right)^2}} = 0.5912 \tag{1}$$

$$\omega_n \geq \frac{4}{t_s \zeta} = 2.2555 \tag{2}$$

We want to design our controller so that LM $G_c G_p$ is large at low frequencies, small at high frequencies, and has this desired form near the 0 dB crossing:

$$G_c G_p = \frac{\dfrac{\omega_n}{2\zeta}}{s\left(\dfrac{s}{2\zeta\omega_n}+1\right)} \tag{3}$$

a) Let's pick $\omega_n = 2.5$ and $\zeta = 0.6$. Then,

$$G_c G_p = \frac{\dfrac{2.5}{2\cdot 0.6}}{s\left(\dfrac{s}{2\cdot 0.6\cdot 2.5}+1\right)} = \frac{2.083}{s\left(\frac{s}{3}+1\right)} \tag{4}$$

Divide the desired $G_c G_p$ through by G_p to get G_c:

$$G_c = \frac{\dfrac{2.083}{s\left(\frac{s}{3}+1\right)}}{\dfrac{1}{\frac{s}{2}+1}} = \frac{2.083\left(\frac{s}{2}+1\right)}{s\left(\frac{s}{3}+1\right)} \tag{5}$$

We obtain strictly proper and therefore a valid controller.

b) This problem has a 3rd order transfer function, but our desired $G_c G_p$ is only 2nd order. We will look at two methods of approaching this, both using G_p inversion.

Method 1: Let's stick with $\omega_n = 2.5$ and $\zeta = 0.6$. Then, our desired $G_c G_p$ is still

$$G_c G_p = \frac{2.083}{s\left(\frac{s}{3}+1\right)} \tag{6}$$

Divide the desired G_cG_p through by G_p to get G_c:

$$\begin{aligned}
G_c &= \frac{\dfrac{2.083}{s\left(\frac{s}{3}+1\right)}}{\dfrac{1}{(s+1)(s+3)(s+15)}} \\
G_c &= \frac{2.083(s+1)(s+3)(s+15)}{s\left(\frac{s}{3}+1\right)} \\
G_c &= \frac{93.75(s+1)\left(\frac{s}{3}+1\right)\left(\frac{s}{15}+1\right)}{s\left(\frac{s}{3}+1\right)} \\
G_c &= \frac{93.75(s+1)\left(\frac{s}{15}+1\right)}{s}
\end{aligned}$$

However, this transfer function is **improper** (the degree of the numerator is larger than the degree of the denominator, in other words it has more zeros than poles). We can make it proper by adding a fast pole with no gain.

$$\Rightarrow G_c = \frac{93.75(s+1)\left(\frac{s}{15}+1\right)}{s\left(\frac{s}{100}+1\right)} \tag{7}$$

The exact location of this pole does not matter, so long as it is at least 4 to 5 times further left in the s-plane than the other poles and zeros. Placing it at $s=-100$ will suffice.

Method 2: Let's look at each of the poles of the plant:

(a) The pole at $s=-1$ is too low of frequency; we must cancel it.

(b) The pole at $s=-3$ is in a good spot. We will keep it.

(c) The pole at $s=-15$ doesn't help or hurt us. It is enough faster than the other poles and is far enough away that it doesn't affect slope or shape near the 0dB crossing. We will leave it alone.

Because the pole at $s=-15$ is far enough left, we could approximate the plant as

$$G_p \approx \frac{\frac{1}{15}}{(s+1)(s+3)} \tag{8}$$

Then, we can carry out the inversion using the approximate G_p. Let's stick with $\omega_n = 2.5$ and $\zeta = 0.6$.

$$\begin{aligned}
G_c &= \frac{\dfrac{2.083}{s\left(\frac{s}{3}+1\right)}}{\dfrac{1}{15(s+1)(s+3)}} \\
G_c &= \frac{2.083(s+1)(s+3)}{\frac{1}{15}s\left(\frac{s}{3}+1\right)} \\
G_c &= \frac{93.75(s+1)\left(\frac{s}{3}+1\right)}{s\left(\frac{s}{3}+1\right)} \\
\Rightarrow G_c &= \frac{93.75(s+1)}{s}
\end{aligned}$$

We could alternatively think of this as defining a desired G_cG_p that keeps the $s=-15$ pole:

$$G_cG_p = \frac{2.083}{s\left(\frac{s}{3}+1\right)\left(\frac{s}{15}+1\right)} \tag{9}$$

Then, divide through by the actual G_p:

$$
\begin{aligned}
G_c &= \frac{\dfrac{2.083}{s\left(\frac{s}{3}+1\right)\left(\frac{s}{15}+1\right)}}{\dfrac{1}{(s+1)(s+3)(s+15)}} \\
G_c &= \frac{93.75(s+1)\left(\frac{s}{3}+1\right)\left(\frac{s}{15}+1\right)}{s\left(\frac{s}{3}+1\right)\left(\frac{s}{15}+1\right)} \\
\Rightarrow G_c &= \frac{93.75(s+1)}{s}
\end{aligned}
$$