Lecture 11

Last lecture: Complete seven rules for drawing a root locus.
Today: Continue discussion of root locus
- Closed-loop zeros
- Control system design via root locus

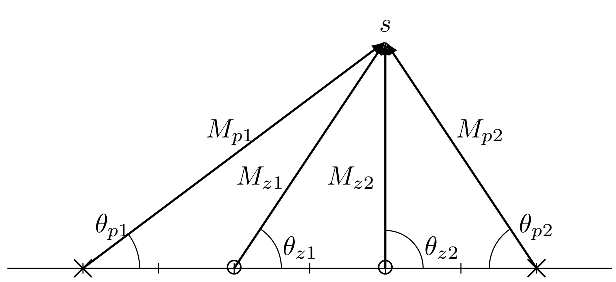
Angle and Magnitude Criterion

We will begin with an example of applying the angle and magnitude criteria. Recall,

$$\textbf{Angle Criterion:} \quad \sum^{m} \theta_{zi} - \sum^{n} \theta_{pi} = 180^\circ \pm \ell 360^\circ, \quad \ell = 0, 1, 2, \ldots \tag{1}$$

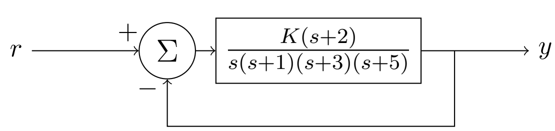
The angle criterion tells us if a point is on the root locus.

$$\textbf{Magnitude Criterion:} \quad \frac{\prod^n M_{pi}}{\prod^m M_{zi}} = K \tag{2}$$



Example

Consider the following system:

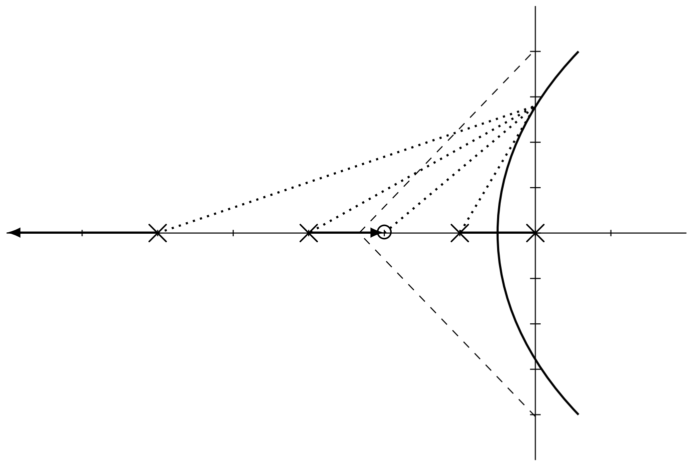


Then,

$$\sigma = \frac{0 + (-1) + (-3) + (-5) - (-2)}{4 - 1} = -\frac{7}{3}$$

$$\theta = \frac{180^\circ + \ell 360^\circ}{3 - 0} = 60^\circ + \ell 120^\circ, \ \ell = 0, 1, 2 \quad \Rightarrow \quad \theta = 60^\circ, 180^\circ, 300^\circ$$

At what gain does the system go unstable (the locus crosses into the right-hand plane)?



This occurs at approximate $s \approx \pm 2.8j$. Therefore,

$$K = \frac{\prod M_{pi}}{\prod M_{zi}} = \frac{(5.6)(4.1)(3.0)(2.8)}{3.5} = 55.1$$

We can confirm this in Matlab and find $K = 60.1$. So, our hand computation is fairly accurate.
At what gain does the locus break away from the real axis? First, we compute the break away point:

$$\sum^{m} \frac{1}{\sigma_b + z_i} = \sum^{n} \frac{1}{\sigma_b + p_i}$$

$$\frac{1}{\sigma_b} + \frac{1}{\sigma_b + 1} + \frac{1}{\sigma_b + 3} + \frac{1}{\sigma_b + 5} = \frac{1}{\sigma_b + 2}$$

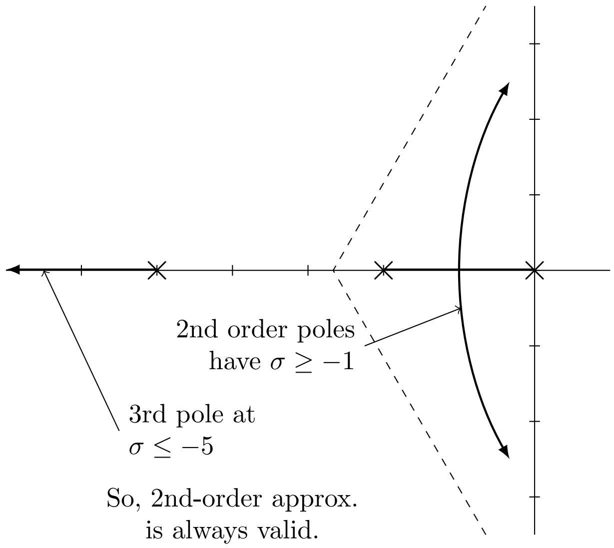
So, the breakaway is at $\sigma_b \approx -0.5$. Then,

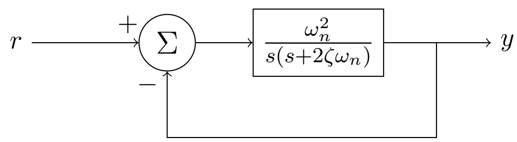
$$K = \frac{\prod M_{pi}}{\prod M_{zi}} = \frac{(4.5)(2.5)(0.5)(0.5)}{1.5} = 1.88$$

Is a second-order approximation valid for this system?

- The relative dominance of closed-loop poles is determined by the ratio of the real parts of the closed-loop poles as well as by the relative magnitudes of the residues evaluated at the closed-loop poles. The magnitude of the residues depend on both closed-loop poles and zeros.
- If the ratios of the real parts of the closed-loop poles exceed 5 and there are no zeros nearby, then the closed-loop poles nearest the $j\omega$-axis dominate the transient response behavior.
- Those closed-loop poles that have dominant effects on the transient response behavior are called *dominant closed loop poles.*

Need to always be careful about approximating higher order systems with its second order counterparts. We want the 2nd-order pair to be 5 times slower than the other poles.





A second order closed-loop transfer function comes from the following open-loop transfer function:

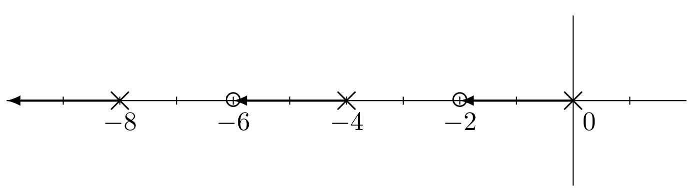
$$\frac{\hat{y}}{\hat{r}} = \frac{\frac{\omega_n^2}{s(s+2\zeta\omega_n)}}{1 + \frac{\omega_n^2}{s(s+2\zeta\omega_n)}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

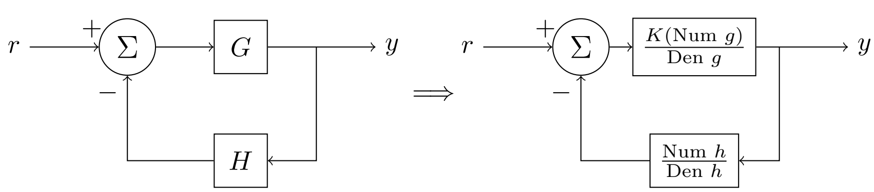
For instance, poles at $s = -1 \pm 2j$ give:

$$\frac{\hat{y}}{\hat{r}} = \frac{5}{s^2 + 2s + 5}$$

Closed-Loop Zeros

Does the root locus tell us about the closed-loop zeros? Consider this system:





We will solve for the closed-loop transfer function. What are the closed-loop zeros?

$$CLTF = \frac{\hat{y}}{\hat{r}} = \frac{\frac{K(\text{Num } g)}{\text{Den } g}}{1 + \frac{K(\text{Num } g)}{\text{Den } g} \cdot \frac{\text{Num } h}{\text{Den } h}} = \frac{K \cdot \text{Num } g \cdot \text{Den } h}{\text{Den } g \cdot \text{Den } h + K \cdot \text{Num } g \cdot \text{Num } h}$$

So, the closed loop zeros satisfy:

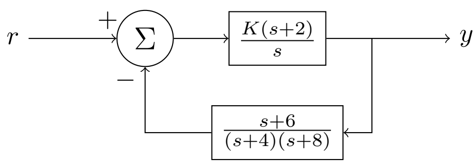
$$K \cdot \text{Num } g \cdot \text{Den } h = 0$$

$$\text{Num } g \cdot \text{Den } h = 0$$

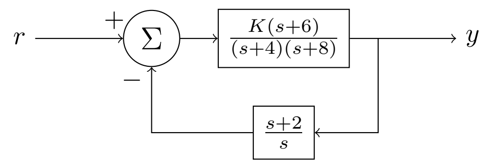
Therefore:

- The closed-loop zeros are the zeros of the forward path and the poles of the return path.
- The closed-loop zeros don't migrate with K.
- The closed-loop zeros are clearly not shown on the root locus.

For instance, these systems have the same root locus but different closed-loop zeros:



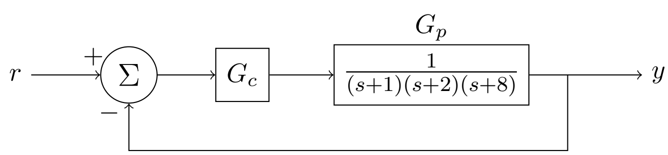
C.L. zeros: $-2, -4, -8$



C.L. zeros: $0, -6$

Control system design via root locus

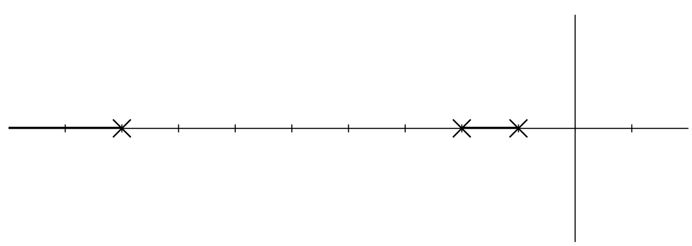
Consider the system shown:



Find G_c so that a unit step input at $r(t)$ leads to a $y(t)$ response that meets the following requirements:

1. overshoot $\leq 20\%$
2. peak time ≤ 2 seconds
3. steady-state error ≤ 0.4
4. system must remain stable

Let's start by trying a proportional controller $G_c = K_p$ and sketching a root locus. Proceeding through the first 4 rules, we have:



Next, we compute the asymptote center and angles.

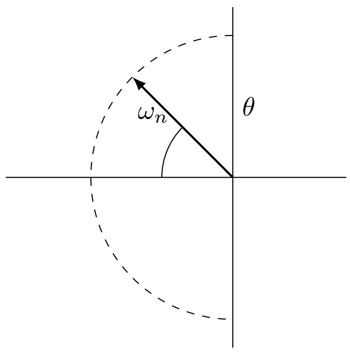
$$\sigma = \frac{-1 + (-2) + (-8)}{3 - 0} = -\frac{11}{3}$$

$$\theta = \frac{180^\circ + \ell 360^\circ}{3 - 0} = 60^\circ + \ell 120^\circ,\ \ell = 0, 1, 2 \quad \Rightarrow \quad \theta = 60^\circ, 180^\circ, 300^\circ$$

So, we can sketch the following root locus: *(next page)*

- For this system, we will have two poles near the imaginary axis and one pole much further left.
  - We might be able to treat this as a 2nd-order system
- No closed-loop zeros
- Let's review what we know about 2nd order systems without zeros.

From Lecture 6:

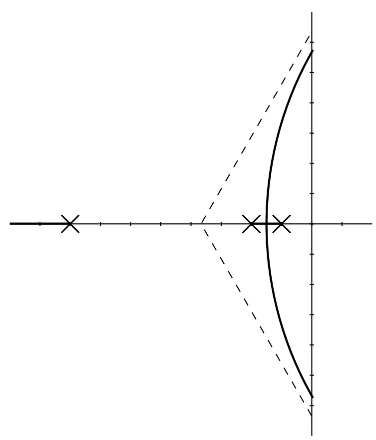


ω_n: undamped natural frequency
ζ: damping ratio, $\zeta = \cos\theta$

$$\zeta = \frac{-\ln\left(\frac{\%O.S.}{100}\right)}{\sqrt{\pi^2+\left(\ln\left(\frac{\%O.S.}{100}\right)\right)^2}}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

If the ratios of the real parts of the closed-loop poles exceed $> 5\times$ it can be ignored.

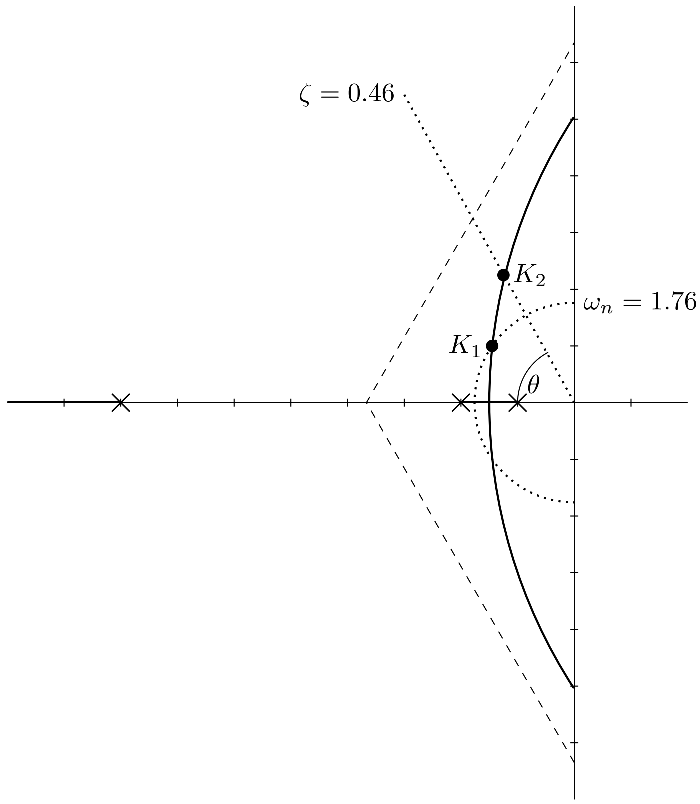


We can compute the damping ratio and natural frequency requirements for this system.

$$\zeta = \frac{-\ln(0.2)}{\sqrt{\pi^2 + (\ln 0.2)^2}} \approx 0.46, \quad \theta = \cos^{-1}\zeta = 63^\circ$$

$$2 = \frac{\pi}{\omega_n\sqrt{1-0.46^2}} \quad \Rightarrow \quad \omega_n = 1.76$$

We can plot the circle corresponding to $\omega_n = 1.76$ and line corresponding to $\zeta = 0.46$, identify where these intersect the root locus, and then use the Magnitude Criterion to find the associated gain.



Next, let's compute the breakaway point:

$$
\begin{aligned}
0 &= \frac{1}{\sigma_b+1} + \frac{1}{\sigma_b+2} + \frac{1}{\sigma_b+8} \\
0 &= 1 + \frac{\sigma_b+1}{\sigma_b+2} + \frac{\sigma_b+1}{\sigma_b+8} \\
0 &= (\sigma_b+2) + (\sigma_b+1) + \frac{(\sigma_b+1)(\sigma_b+2)}{\sigma_b+8} \\
0 &= (\sigma_b+2)(\sigma_b+8) + (\sigma_b+1)(\sigma_b+8) + (\sigma_b+1)(\sigma_b+2) \\
0 &= (\sigma_b^2+10s+16) + (\sigma_b^2+9s+8) + (\sigma_b^2+3s+2) \\
0 &= 3\sigma_b^2 + 22\sigma_b + 26 \\
\sigma_b &= -1.48,\ -5.85
\end{aligned}
$$

From Rule 4, we know that only $\sigma_b = -1.48$ is valid — the root locus is not on the real axis at $s = -5.85$. We are now going to show two methods of graphically computing a gain for this controller.

Method 1: Graphically approximate the location for K_1 and K_2 (e.g., use a ruler).
$s_1 \quad at \quad K_1 \approx at(-1.4, 1),\ K_1 \approx (6.7) \cdot (1.1) \cdot (1.2) = 9$
$s_2 \quad at \quad K_2 \approx at(-1.16, 2.12),\ K_2 \approx (7.2) \cdot (2.1) \cdot (2.3) = 35$
$\therefore 9 \le K_p \le 35$

Method 2: Graphically estimate the real value for K_1 and K_2, and use geometry to find the imaginary value. For K_1, we note that the root locus is mostly vertical near the real axis. We can use assume the locus has the same real value as the breakaway point. So, assuming K_1 is at the point $(-1.48, \omega_1)$, then

$$1.76^2 = 1.48^2 + \omega_1^2 \quad \Rightarrow \quad \omega_1 = \sqrt{1.76^2 - 1.48^2} = 0.95$$

The point for K_2 is slightly right of the breakaway. Estimating K_2 is at the point $(-1.2, \omega_2)$, then

$$\zeta = \cos\theta \quad \Rightarrow \quad \theta = 62.6°$$

$$\tan\theta = \frac{opp}{adj} \quad \Rightarrow \quad \omega_2 = 1.2\tan(62.6°) = 2.315$$

Having an estimate of the location of K_1 and K_2, we can find the magnitude of each vector to from the open-loop poles to those points. Then, using the Magnitude Criterion,

$$\begin{aligned}
K_1 &\approx |(1.48 + 0.95j) - 1| \cdot |(1.48 + 0.95j) - 2| \cdot |(1.48 + 0.95j) - 8| \\
&\approx \sqrt{0.48^2 + 0.95^2} \cdot \sqrt{0.52^2 + 0.95^2} \cdot \sqrt{6.52^2 + 0.95^2} \\
&\approx (1.064)(1.083)(6.588) \\
&\approx 7.6
\end{aligned}$$

$$\begin{aligned}
K_2 &\approx |(1.2 + 2.315j) - 1| \cdot |(1.2 + 2.315j) - 2| \cdot |(1.2 + 2.315j) - 8| \\
&\approx \sqrt{0.2^2 + 2.315^2} \cdot \sqrt{0.8^2 + 2.315^2} \cdot \sqrt{6.8^2 + 2.315^2} \\
&\approx (2.32)(2.45)(7.18) \\
&\approx 40.8
\end{aligned}$$

So, $7.6 \leq K_p \leq 40.8$.

Next, we must check the steady-state error.

- Unity feedback
- Type 0 system
- Stable

For a step input:

$$e_{ss} = \frac{1}{1 + \lim_{s\to 0} G_c G_p}$$

$$e_{ss} = \frac{1}{1 + \lim_{s\to 0} \frac{K_p}{(s+1)(s+2)(s+8)}} = \frac{1}{1 + \frac{K_p}{16}} = \frac{16}{16 + K_p}$$

Alternatively, we can compute the steady-state error directly instead of using system types:

$$E(s) = \frac{1}{1 + G_c G_p} R(s) = \frac{1}{1 + \frac{K_p}{(s+1)(s+2)(s+8)}} \frac{1}{s}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{(s+1)(s+2)(s+8)}{(s+1)(s+2)(s+8) + K_p}$$

$$e_{ss} = \frac{(1)(2)(8)}{(1)(2)(8) + K_p} = \frac{16}{16 + K_p}$$

We get the same result with both methods. Then,

$$e_{ss} \leq 0.4 \quad \Rightarrow \quad K_p \geq 24$$

So, proportional control will work for $24 \leq K_p \leq 40.8$.

Finally, we can validate this with simulation in Matlab

- Define the transfer function:

```
s = tf('s');
Kp = 30; % or another valid value
Gc = Kp;
Gp = 1/((s+1)*(s+2)*(s+8)); % or using any other method
CLTF = Gc*Gp/(1+Gc*Gp); % or "CLTF = feedback(Gc*Gp,1)";
```

- Draw the root locus: `rlocus(Gp)`
- Simulate a step response: `step(CLTF)`

In actuality, the gain has an upper bound of $K_p = 38$ to meet the transient requirements.

Finally, we check if the 2nd-order approximation is valid.

- Complex poles 1.25 from the imaginary axis.
- Other pole is > 8 from the imaginary axis.

✓ Approximation is valid (ratio of other pole real part is $> 5\times$) farther left.