Lecture 4

Last time:

- Transfer Functions
- State-space forms

By this point, one should be familiar with the relationship between system representation in the complex plane (or s-plane poles and zeros) and it's representation in the time domain (time plots). Time domain:

$$u(t) \longrightarrow \boxed{\begin{array}{c}\text{Pysical}\\ \text{System}\end{array}} \longrightarrow y(t)$$

differential equations

Laplace domain:

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

transfer function

In general, we have

$$Y(s) = G(s)U(s)$$

where $G(s)$ is proper or strictly proper (never improper) and $U(s)$ is generally strictly proper. As a whole this means that $Y(s)$ is generally strictly proper but at "most" it is proper.

As an example, consider the case for strictly proper with distinct poles for $Y(s)$.

$$Y(s) = \frac{R_1}{s+p_1} + \frac{R_2}{s+p_2} + \ldots + \frac{R_n}{s+p_n}$$

$$y(t) = \sum_{j=1}^{n} \mathscr{L}^{-1}\left(\frac{R_j}{s+p_j}\right) = \sum_{j=1}^{n} R_j e^{-p_j t}$$

This is simply the sum of exponentials. Some of these may be complex, resulting in sines and cosines in the response. In general, the $y(t)$ above will be the sum of simple time functions.

1. If $Y(s)$ has a pole at the origin of the complex plane, $y(t)$ has a step function in the time domain.
2. If $Y(s)$ has a pole on the real axis, $y(t)$ has an exponential function in the time domain.
   (a) negative real axis: decaying exponential
   (b) positive real axis: growing exponential
3. If $Y(s)$ has a pair of poles on the imaginary axis, symmetrically placed with regards to the real axis (complex conjugates), then $y(t)$ has undamped oscillations of frequency ω (where ω is the imaginary part of the poles). Explanation:

   $$Y(s) = \frac{\omega}{s^2+\omega^2} \quad \Rightarrow \quad \text{poles are given by } s^2+\omega^2 = 0 \Rightarrow s^2 = -\omega^2 \Rightarrow s = \pm j\omega$$

   But, for this $Y(s)$, we know that $y(t) = \sin \omega t$ (undamped oscillation)

4. If $Y(s)$ has a pair of conjugate poles not on the imaginary axis, $y(t)$ will have decaying or growing oscillations. Explanation:

$$\text{Decaying Oscillation } (a > 0)\text{: } y(t) = e^{-at} \sin \omega t 1(t)$$

$$Y(s) = \mathscr{L}[y(t)] = ?$$

$$\mathscr{L}[\sin \omega t 1(t)] = \frac{\omega}{s^2 + \omega^2}$$

$$\mathscr{L}[e^{-at} \sin \omega t 1(t)] = \frac{\omega}{(s+a)^2 + \omega^2} = Y(s)$$

The poles of $Y(s)$ are $(s+a)^2 + \omega^2 = 0$. So,

$$(s+a)^2 = -\omega^2$$

$$s + a = \pm \omega$$

$$s = -a \pm j\omega$$

Complex conjugate poles in left-hand plane

$$\text{Growing Oscillation } (a > 0)\text{: } y(t) = e^{+at} \sin \omega t 1(t)$$

Following the same steps,

$$Y(s) = \frac{\omega}{(s-a)^2 + \omega^2}$$

$$s = a \pm j\omega$$

5. If $Y(s)$ has multiple poles at the same location on complex plane, the time-domain counterpart of a single pole is multiplied by t. For example, recall

$$\mathscr{L}[tf(t)] = -\frac{d}{ds} F(s)$$

So,

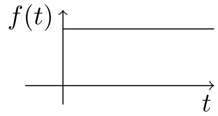
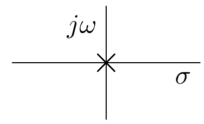
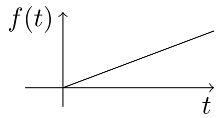
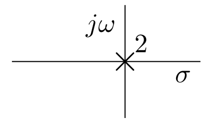
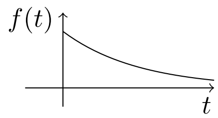
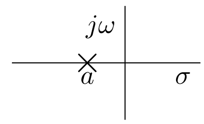
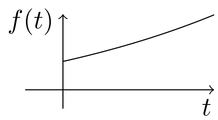
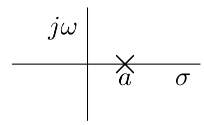
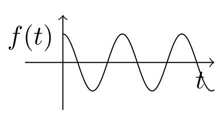
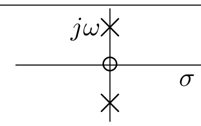
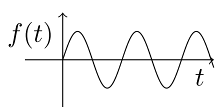
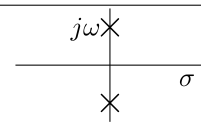
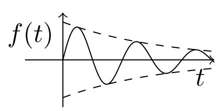
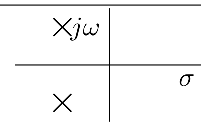
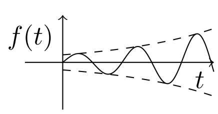
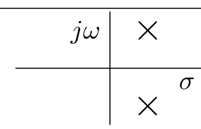
$$\mathscr{L}[t \cdot e^{-at} 1(t)] = -\frac{d}{ds} \frac{1}{s+a} = \frac{1}{(s+a)^2}$$

$$\mathscr{L}[t \cdot \sin \omega t 1(t)] = -\frac{d}{ds} \frac{\omega}{s^2 + \omega^2} = \frac{2\omega s}{(s^2 + \omega^2)^2}$$

and of course

$$\mathscr{L}[t 1(t)] = -\frac{d}{ds} \frac{1}{s} = \frac{1}{s^2}$$

Previously we looked at correspondences between the complex plane and the time domain.

Function $f(t)$	Plot of $f(t)$	$F(s)$	Pole-Zero Diagram
$1(t)$		$\frac{1}{s}$	
$t1(t)$		$\frac{1}{s^2}$	
$e^{-at}1(t)$		$\frac{1}{s+a}$	
$e^{at}1(t)$		$\frac{1}{s+a}$	
$\cos \omega t 1(t)$		$\frac{s}{s^2+\omega^2}$	
$\sin \omega t 1(t)$		$\frac{\omega}{s^2+\omega^2}$	
$e^{-at} \sin \omega t 1(t)$		$\frac{\omega}{(s+a)^2+\omega^2}$	
$e^{at} \sin \omega t 1(t)$		$\frac{\omega}{(s-a)^2+\omega^2}$	

- Poles in the LHP cause the time-domain equivalent to eventually decay to zero.
- Poles in the RHP cause the time domain equivalent to grow without bound.
- For distinct poles on the $j\omega$ axis, the time domain signal is bounded but does not decay.
  - Simple/single pole. This can only be at the origin and results in a step function in the time-domain.
  - Complex conjugate poles. This results in undamped oscillations.
  - Multiple poles at the same location on the $j\omega$-axis have the time function multiplied by t (e.g. $1(t) \to t1(t)$), causing them to grow without bound.

Some observations:

- The character of $y(t)$ principally depends on the poles of $Y(s)$ not on the zeros. For example:

$$Y(s) = \frac{\cdots}{s(s+a)(s^2+c^2)}$$

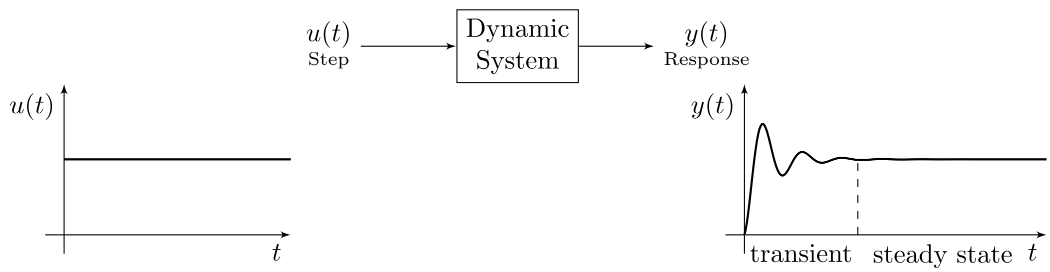
  Then $y(t)$ consists of step, exponential, and sine/cosine responses. The zeros help determine the residuals, in other words the "amount" that each pole contributes to the total response.

- The equation that gives the poles of $G(s)$ is often referred to as the characteristic equation.

$$\det(s\underline{I} - \underline{A}) = 0$$

More on system response: Final Values, initial values, and static gain

For a system S, some given input $u(t)$ gives rise to an output $y(t)$.



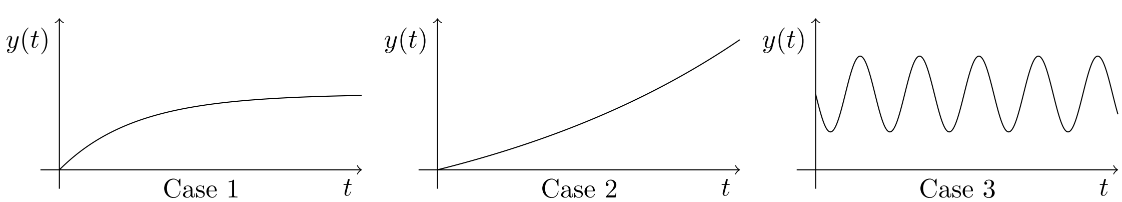
The output $y(t)$ has both a transient response (the immediate reaction to $u(t)$) and a steady-state response (the long-term reaction to $u(t)$). This depends on the the input as well as the system transfer function.

Final Value The final value of $y(t)$ is its value after a very long time

$$F.V. \triangleq \lim_{t\to\infty} y(t)$$

There are three possibilities:

1. $y(t)$ has a final value (the limit exists)
2. $y(t)$ is unbounded (it has no final value)
3. $y(t)$ is bounded by has no final value (limit undefined)



Is it possible to look at the expression of $Y(s)$ and tell which of these situations we have?

Yes!

A final value exists if and only if all poles of $Y(s)$ are strictly in the LHP, except for a single pole at the origin.

- If $Y(s)$ has any poles in the RHP, $y(t)$ is unbounded.
- If $Y(s)$ has a pair of complex conjugate poles on the imaginary axis, the final values is undefined.

If a final value exists, it can be found using the **final value theorem**. If $\mathscr{L}[y] = Y$ and poles of sY lie strictly in the LHP, then the Final Value Theorem states

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

Proof.

$$\begin{aligned}
\mathscr{L}\left[\frac{dy}{dt}\right] &= sY - y(0^-) \\
sY &= y(0^-) + \mathscr{L}\left[\frac{dy}{dt}\right] \\
&= y(0^-) + \int_{0^-}^{\infty} e^{-st}\frac{dy}{dt}(t)dt \\
\lim_{s\to 0} sY(s) &= y(0^-) + \int_{0^-}^{\infty} \frac{dy}{dt}(t)dt \\
&= y(0^-) + \lim_{t\to\infty} y(t) - y(0^-) \\
&= \lim_{t\to\infty} y(t)
\end{aligned}$$

□

Example

Consider

$$G(s) = \frac{3}{(s+4)(s+3)}$$

What is the final value for a unit step, $U(s) = 1/s$?

$$Y(s) = G(s)U(s) = \frac{3}{s(s+4)(s+3)}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{3}{(s+4)(s+3)}$$

$$\lim_{t\to\infty} y(t) = \frac{1}{4}$$

Now consider

$$G(s) = \frac{3}{(s-4)(s+3)}$$

What is the final value for a unit step, $U(s) = 1/s$? We must be careful! If we apply the FVT blindly, we get

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{3}{(s-4)(s+3)}$$

$$\lim_{t\to\infty} y(t) = -\frac{1}{4}$$

But, $y(t)$ is unbounded because there is a pole in the RHP! So, the FVT is only usable if all the poles of $Y(s)$ are **strictly** in the LHP (except for a simple pole at the origin). Consider another case where

$$G(s) = \frac{3}{(s^2+4)}$$

What is the final value for a unit step, $U(s) = 1/s$? The FVT would say that:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{3}{(s^2+4)}$$

$$\lim_{t\to\infty} y(t) = \frac{3}{4}$$

but again, this would be incorrect because of the poles on the imaginary axis. $y(t)$ has no final value.

Initial Value The initial value $y(0^+)$ is the value of the response $y(t)$ at the instance the control input is applied. In a similar manner to the FVT, we can find the initial value $y(0^+)$. The **initial value theorem** states that

$$\lim_{s\to\infty} sY(s) \triangleq y(0^+)$$

Proof.

$$\begin{aligned}
\mathscr{L}\left[\frac{dy}{dt}\right] &= sY - y(0^-) \\
sY &= y(0^-) + \int_{0^-}^{\infty} e^{-st}\frac{dy}{dt}(t)dt \\
&= y(0^-) + \int_{0^-}^{0^+} \underbrace{e^{-st}}_{=e^{-0t}=1} \frac{dy}{dt}(t)dt + \int_{0^+}^{\infty} e^{-st}\frac{dy}{dt}(t)dt \\
&= \cancel{y(0^-)} + \left(y(0^+) - \cancel{y(0^-)}\right) + \int_{0^+}^{\infty} e^{-st}\frac{dy}{dt}(t)dt \\
&\text{Now, take the limit of both sides.} \\
\lim_{s\to\infty} sY(s) &= y(0^+) + \int_{0^-}^{\infty} 0\cdot\frac{dy}{dt}(t)dt \\
&= y(0^+) + 0 \\
\lim_{s\to\infty} sY(s) &= y(0^+)
\end{aligned}$$

□

Example

$$Y(s) = \frac{3}{s(s^2+4)}$$

$$y(0^+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty}\frac{3}{s^2+4} = 0$$

Static Gain The **static gain** tells you how well a system responds to a step command in the steady-state. The static gain is defined for a step input with magnitude a. Then,

$$\text{Static Gain} \triangleq \frac{\lim_{t\to\infty} y(t)}{a}$$

Let's look at this using the FVT.

$$Y(s) = G(s)U(s)$$

So,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0}[sG(s)U(s)]$$

Recall that $U(s) = a/s$. So,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0}\left[\cancel{s}G(s)\frac{a}{\cancel{s}}\right] = \lim_{s\to 0}[aG(s)] = aG(0)$$

Therefore

$$\text{Static Gain} = G(0)$$

This is also called the DC Gain in some texts. So, take a system's transfer function and set $s = 0$ — that is the system's static gain. Note that this concept only applies if all the poles of $G(s)$ are strictly in the LHP.

Another way to define static gain: It is the value of $y(t)$ at steady-state when $u(t) = 1(t)$. This will be equal to $G(0)$.

Example

$$G(s) = \frac{10(s+7)}{(s+10)(s+20)}, \quad u(t) = 1(t)$$

Find the initial value, final value, and initial slope of $y(t)$.

$$Y(s) = \frac{1}{s} \cdot \frac{10(s+7)}{(s+10)(s+20)}$$

Initial value:

$$y(0^+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \left[\frac{10(s+7)}{(s+10)(s+20)}\right]$$

$$y(0^+) = \frac{10\left(\frac{1}{s} + \frac{7}{s^2}\right)}{\left(1+\frac{10}{s}\right)\left(1+\frac{20}{s}\right)} = 0$$

Final value: First, we can tell the final value exists because all the poles of $Y(s)$ are strictly in the LHP with just one at the origin. Hence, we can use the FVT.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{10(s+7)}{(s+10)(s+20)}$$

$$\lim_{t\to\infty} y(t) = \frac{10(7)}{(10)(20)} = \frac{7}{20}$$

Given that this is a step input, this also means that the Static Gain= $\frac{7}{20}$.
Initial slope: In other words, find $\dot{y}(0^+)$ (initial value of the derivative of $y(t)$).

$$\mathscr{L}[y(t)] = Y(s) = \frac{10(s+7)}{s(s+10)(s+20)}$$

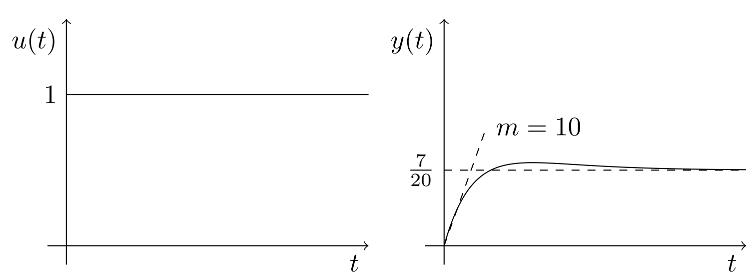
$$\mathscr{L}[\dot{y}(t)] = sY(s) = \frac{10(s+7)}{(s+10)(s+20)}$$

Then, apply the IVT:

$$\dot{y}(0^+) = \lim_{s\to\infty}(s[sY(s)]) = \lim_{s\to\infty} \frac{10s(s+7)}{(s+10)(s+20)}$$

$$\dot{y}(0^+) = \frac{10\left(1+\frac{7}{s}\right)}{\left(1+\frac{10}{s}\right)\left(1+\frac{20}{s}\right)} = \frac{10}{1\cdot 1} = 10$$

Sketch of $u(t)$ and $y(t)$:



Example

$$G(s) = \frac{s-2}{(s+1)(s+4)}$$

Sketch the output to a unit ramp input.

$$U(s) = \frac{1}{s^2} \quad \Rightarrow \quad Y(s) = \frac{s-2}{s^2(s+1)(s+4)}$$

$Y(s)$ has two poles at the origin — therefore, there is no final value! Let's look at the initial value and initial slope: Initial value:

$$y(0^+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \left[\frac{s-2}{s(s+1)(s+4)}\right] = 0$$

Initial slope:

$$\mathscr{L}[y(t)] = Y(s) = \frac{s-2}{s^2(s+1)(s+4)}$$

$$\mathscr{L}[\dot{y}(t)] = sY(s) = \frac{s-2}{s(s+1)(s+4)}$$

Then, apply the IVT:

$$\dot{y}(0^+) = \lim_{s\to\infty} (s[sY(s)]) = \lim_{s\to\infty} \frac{s-2}{(s+1)(s+4)} = 0$$

So, we have an initial value and initial slope of zero, and we cannot obtain a final value. Note, however, that slope transfer function

$$\mathscr{L}[\dot{y}(t)] = \frac{s-2}{s(s+1)(s+4)}$$

admits a final value. So, we can find a final slope.

$$\lim_{t\to\infty} \dot{y}(t) = \lim_{s\to 0} s^2 Y(s) = \lim_{s\to 0} \frac{s-2}{(s+1)(s+4)} = \frac{-2}{4} = -\frac{1}{2}$$

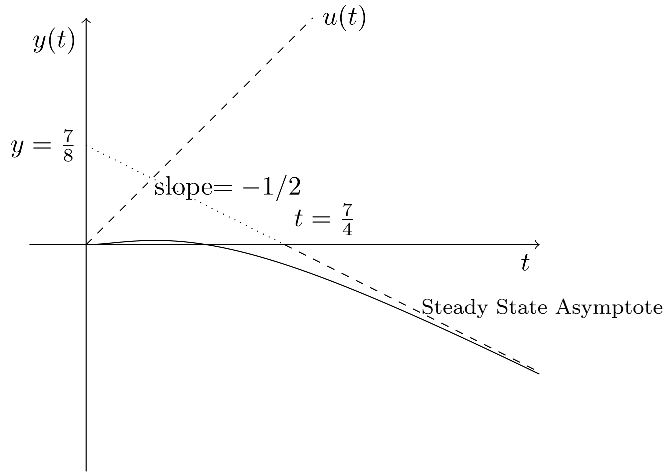
Although we will not go over the steps here, it can be found that

$$y(t) = \left(\frac{7}{8} - e^{-t} + \frac{1}{8}e^{-4t} - \frac{1}{2}t\right) 1(t)$$

This confirms our finding:

- Initial value $y(0^+) = \frac{7}{8} - 1 + \frac{1}{8} - 0 = 0$
- Initial slope $\dot{y}(0^+) = 0 + 1 - \frac{4}{8} - \frac{1}{2} = 0$
- No final value (as the $-\frac{1}{2}t$ term will continue growing)
- Final slope $\lim_{t\to\infty} \dot{y}(t) = 0 - 0 + 0 - \frac{1}{2} = -\frac{1}{2}$

Additionally, we can see that as t becomes large, $y(t) \approx \dfrac{7}{8} - \dfrac{1}{2}t$.



Example

$$G(s) = \frac{20}{s^2 + 6s + 144}$$

What is the Static Gain? (The final value of output for input $1(t)$.) First, what are the poles of $G(s)$?

$$s^2 + 6s + 144 \quad \Rightarrow \quad s = -3 \pm \frac{\sqrt{36 - 4 \cdot 144}}{2} = -3 \pm j\sqrt{135}$$

So, the static gain exists. Then,

$$\text{Static Gain} = G(0) = \frac{20}{144} \approx 0.14$$