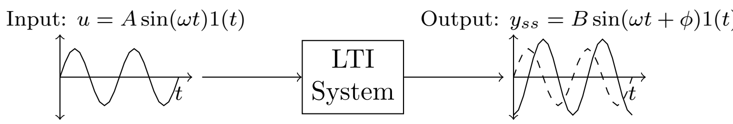
Lecture 13

Frequency Response

Designing controllers by frequency response is common in industry. We'll see it has some advantages over root locus design.



We've spent a lot of time focusing on the case where the input is a step function. Now let's look at the case where the input is a sinusoid. Assume:

$$u(t) = A\sin(\omega t)\cdot 1(t)$$

Then,

$$\mathscr{L}(u(t)) = U(s) = \frac{A\omega}{s^2+\omega^2} = \frac{A\omega}{(s+j\omega)(s-j\omega)}$$

$$Y(s) = GU = G\frac{A\omega}{(s+j\omega)(s-j\omega)}$$

$$Y = \frac{R_1}{s+j\omega} + \frac{R_2}{s-j\omega} + \text{extra terms from poles of } G(s)$$

$$y(t) = R_1 e^{-j\omega t} + R_2 e^{j\omega t} + \text{extra terms from poles of } G$$

Reminder about period:

- $\sin(t)$ is periodic with period $T = 2\pi$
- If we replace t with ωt, then: $\omega T = 2\pi$
- So, $\sin(\omega t)$ has a period of $T = \frac{2\pi}{\omega}$

If $G(s)$ is stable:

- All poles of $G(s)$ are in the LHP
- "extra terms" will decay to zero eventually

$$y_{\text{steady state}}(t) = R_1 e^{-j\omega t} + R_2 e^{j\omega t}$$

We can find R_1 and R_2:

$$R_1 = (s+j\omega)Y\big|_{s=-j\omega} = \left.\frac{G(s)A\omega}{s-j\omega}\right|_{s=-j\omega} = \frac{-AG(-j\omega)}{2j}$$

$$R_2 = (s-j\omega)Y\big|_{s=j\omega} = \left.\frac{G(s)A\omega}{s+j\omega}\right|_{s=j\omega} = \frac{AG(j\omega)}{2j}$$

$$\Rightarrow y_{ss}(t) = \frac{-AG(-j\omega)}{2j}e^{-j\omega t} + \frac{AG(j\omega)}{2j}e^{j\omega t}$$

What are $G(j\omega)$ and $G(-j\omega)$? We will use an example to show.

Example

Consider a system with transfer function $G(s) = 3 + s$. Find $G(j\omega)$ and $G(-j\omega)$ and evaluate at $\omega = 1, 2, 3$.

$$G(j\omega) = 3 + j\omega, \quad G(-j\omega) = 3 - j\omega$$

ω	$G(j\omega)$	$G(-j\omega)$
1	$3 + j$	$3 - j$
2	$3 + 2j$	$3 - 2j$
3	$3 + 3j$	$3 - 3j$

Example

Consider a system with transfer function $G(s) = (s + 2)(s + 4)$. Find $G(j\omega)$ and $G(-j\omega)$ and evaluate at $\omega = 1, 2, 3$.

$$G(s) = s^2 + 6s + 8$$

$$G(j\omega) = (j\omega)^2 + 6(j\omega) + 8 = -\omega^2 + j6\omega + 8 = (8 - \omega^2) + j(6\omega)$$

$$G(-j\omega) = (-j\omega)^2 - 6(j\omega) + 8 = -\omega^2 - j6\omega + 8 = (8 - \omega^2) - j(6\omega)$$

ω	$G(j\omega)$	$G(-j\omega)$
1	$7 + 6j$	$7 - 6j$
2	$4 + 12j$	$4 - 12j$
3	$-1 + 18j$	$-1 - 18j$

In general:

$$G(j\omega) = a(\omega) + jb(\omega)$$

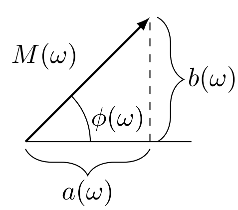
$$G(-j\omega) = a(\omega) - jb(\omega)$$

Recall that a complex number can be written in polar notation as:

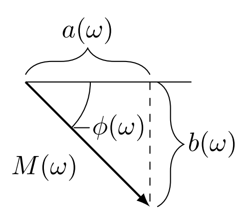
$$a + jb = Me^{j\phi}$$

So,

$$G(j\omega) = Me^{j\phi}$$
where $M(\omega) = \sqrt{(a(\omega)^2 + b(\omega)^2)}$
and $\phi = \tan^{-1}\left(\frac{b(\omega)}{a(\omega)}\right)$



$$G(-j\omega) = Me^{j(-\phi)}$$
where $M(\omega) = \sqrt{(a(\omega)^2 + b(\omega)^2)}$
and $\phi = -\tan^{-1}\left(\frac{b(\omega)}{a(\omega)}\right)$



Now, we will substitute these definitions of $G(j\omega)$ and $G(-j\omega)$ into the definition of $y_{ss}(t)$.

$$y_{ss}(t) = \frac{-AG(-j\omega)}{2j}e^{-j\omega t} + \frac{AG(j\omega)}{2j}e^{j\omega t}$$

$$y_{ss}(t) = \frac{-AMe^{-j\phi}}{2j}e^{-j\omega t} + \frac{AMe^{j\phi}}{2j}e^{j\omega t}$$

$$y_{ss}(t) = \frac{-AM}{2j} e^{-j(\omega t+\phi)} + \frac{AM}{2j} e^{j(\omega t+\phi)}$$

$$y_{ss}(t) = AM \left(\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} \right)$$

$$\boxed{y_{ss}(t) = AM(\omega) \sin(\omega t + \phi(\omega))}$$

- If the input is a sine wave, then the output at steady state is a sine wave.
- Frequency of input = frequency of output
- The output differs from the input in two ways:
  1. Amplitude of output = Amplitude of input $\times$ $M(\omega)$
  2. There is a phase shift of $\phi(\omega)$
- $M(\omega)$ is the gain or Bode magnitude
- $\phi(\omega)$ is the phase angle

Finding M and ϕ from a transfer function

Example

Find the expressions for $G(j\omega)$, $M(\omega)$ and $\phi(\omega)$ for $G(s) = s + 2$.

$$G(j\omega) = 2 + j\omega$$

$$M(\omega) = \sqrt{2^2 + \omega^2}$$

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega}{2}\right)$$

Example

Find the expressions for $G(j\omega)$, $M(\omega)$ and $\phi(\omega)$ for $G(s) = (s + 2)/(s + 3)$.

$$G(j\omega) = \frac{j\omega + 2}{j\omega + 3} = \frac{M_1 e^{j\phi_1}}{M_2 e^{j\phi_2}} = \frac{M_1}{M_2} e^{j(\phi_1 - \phi_2)}$$

$$M(\omega) = \frac{\sqrt{2^2 + \omega^2}}{\sqrt{3^2 + \omega^2}}$$

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{3}\right)$$

Graphical representations of frequency response

The frequency response is typically represented on a **Bode Diagram**, which includes two plots:

Plot 1: Magnitude plot: $20 \log_{10} M(\omega)$ in decibels vs. ω in rad/s, where ω is on a log scale.

Plot 2: Phase plot: ϕ in degrees vs. ω in rad/s, where ω is on a log scale.

A few notes on log magnitude:

- $20 \log_{10} M$ is often written as Lm M — "the log-magnitude of M".
- Multiplication/division in the linear scale is equivalent to addition/subtraction in the dB scale.

- Some common magnitudes in linear and log scales:

$$\begin{array}{rcl} M = 1 & \Rightarrow & \text{Lm } M = 20\log_{10}(1) = 0 \text{ dB} \\ M = 10 & \Rightarrow & \text{Lm } M = 20\log_{10}(10) = 20 \text{ dB} \\ M = 0.1 & \Rightarrow & \text{Lm } M = 20\log_{10}(0.1) = -20 \text{ dB} \\ M = 2 & \Rightarrow & \text{Lm } M = 20\log_{10}(2) = 6 \text{ dB} \\ M = \frac{1}{2} & \Rightarrow & \text{Lm } M = 20\log_{10}\left(\frac{1}{2}\right) = -6 \text{ dB} \\ M = \sqrt{2} & \Rightarrow & \text{Lm } M = 20\log_{10}\left(\sqrt{2}\right) = 3 \text{ dB} \end{array}$$

Or, working backwards:

$$\begin{array}{rcl} \text{Lm } M = 0 \text{ dB} & \Rightarrow & M = 10^{0/20} = 1 \\ \text{Lm } M = 20 \text{ dB} & \Rightarrow & M = 10^{20/20} = 10 \\ \text{Lm } M = -20 \text{ dB} & \Rightarrow & M = 10^{-20/20} = 0.1 \\ \text{Lm } M = 6 \text{ dB} & \Rightarrow & M = 10^{6/20} = 2 \\ \text{Lm } M = -6 \text{ dB} & \Rightarrow & M = 10^{-6/20} = \frac{1}{2} \\ \text{Lm } M = 3 \text{ dB} & \Rightarrow & M = 10^{3/20} = \sqrt{2} \end{array}$$

Example

Evaluate the log-magnitude of $M = 20$.

$$\text{Lm } 20 = \text{Lm } (10 \cdot 2) = \text{Lm } 10 + \text{Lm } 2 = 20 + 6$$

$$\text{Lm } 20 = 26 \text{ dB}$$

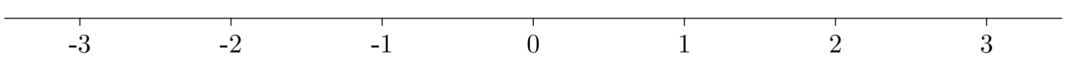
Evaluate the log-magnitude of $M = 5\sqrt{2}$.

$$\text{Lm } 5\sqrt{2} = \text{Lm } \left(10/\sqrt{2}\right) = \text{Lm } 10 - \text{Lm } \sqrt{2} = 20 - 3$$

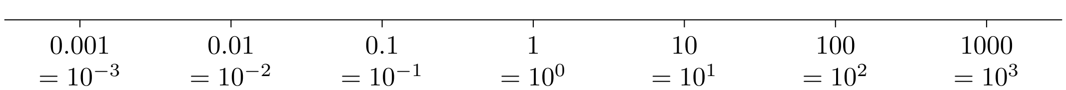
$$\text{Lm } 5\sqrt{2} = 17 \text{ dB}$$

Sketching Bode plots

Note: Before we begin, note that Bode plots are drawn on a **logarithmic** scale, not a linear one.



Linear Scale



Logarithmic Scale

Specifically, the x-axis is a logarithmic scale of the oscillation frequency ω. We refer to an increase of one power of 10 as a "decade". For instance, 10^0 to 10^1 is one decade; 10^2 to 10^3 is one decade; 10^{-2} to 10^0 is two decades, and so on.

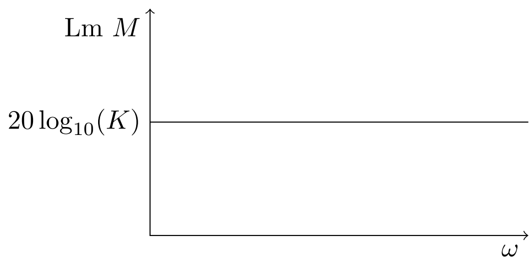
Drawing Bode Plots: If $G(s)$ is a rational function, then it consists of terms such as: K, $1/s$, s, $s+a$, $1/(s+a)$, $\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, etc... Let's look at how to sketch Bode plots for these simple terms.

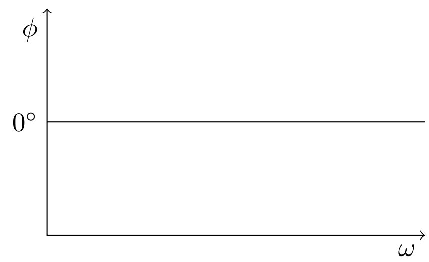
Gain Let $G(s) = K$. What is the Bode plot?

$$G(j\omega) = K + j(0)$$

$$M(\omega) = K$$

$$\phi(\omega) = \tan^{-1}\left(\frac{0}{K}\right) = 0$$





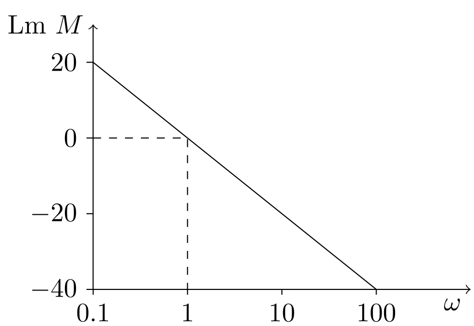
Integrator Let $G(s) = 1/s$. What is the Bode plot?

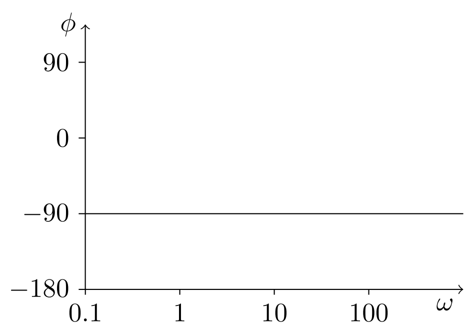
$$G(j\omega) = \frac{1}{j\omega}$$

$$M(\omega) = \frac{M_{num}}{M_{den}} = \frac{\sqrt{1^2 + 0^2}}{\sqrt{0^2 + \omega^2}} = \frac{1}{\omega}$$

$$\text{Lm } M(\omega) = 20\log_{10}\frac{1}{\omega} = 20\log_{10}(1) - 20\log_{10}(\omega) = -20\log_{10}\omega$$

$$\phi(\omega) = \phi_{num} - \phi_{den} = \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}\left(\frac{\omega}{0}\right) = 0 - 90° = -90°$$





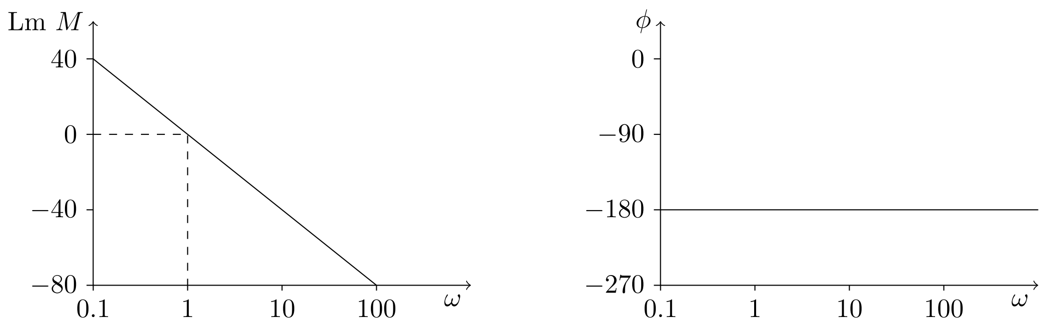
Double Integrator Let $G(s) = 1/s^2$. What is the Bode plot?

$$G(j\omega) = \frac{1}{(j\omega)^2} = \frac{1}{-\omega^2}$$

$$M(\omega) = \frac{M_{num}}{M_{den}} = \frac{\sqrt{1^2 + 0^2}}{\sqrt{0^2 + (-\omega^2)^2}} = \frac{1}{\omega^2}$$

$$\text{Lm } M(\omega) = 20\log_{10}\frac{1}{\omega^2} = 20\log_{10}(1) - 2 \times 20\log_{10}(\omega) = -40\log_{10}\omega$$

$$\phi(\omega) = \phi_{num} - \phi_{den} = \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}\left(\frac{0}{-\omega^2}\right) = 0 - 180° = -180°$$

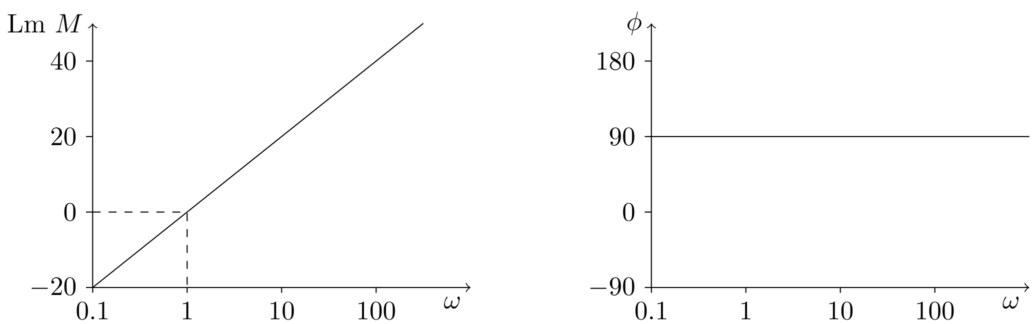


Differentiator Let $G(s) = s$. What is the Bode plot?

$$G(j\omega) = j\omega$$

$$M(\omega) = \sqrt{0^2 + \omega^2} = \omega$$

$$\phi(\omega) = \tan^{-1}\left(\frac{\omega}{0}\right) = 90°$$



The plot of s is a mirror image of the plot of $1/s$.

First-Order Zero Let $G(s) = \tau s + 1$. What is the Bode plot?

$$G(j\omega) = 1 + j(\tau\omega)$$

$$M(\omega) = \sqrt{1^2 + \tau^2\omega^2}$$

$$\phi(\omega) = \tan^{-1}\left(\frac{\tau\omega}{1}\right)$$

Neither the gain nor phase is a straight line. They will be harder to sketch than in previous cases. We will start by looking at the gain by finding:

- an approximation for M when $\tau\omega \ll 1$,
- an approximation for M when $\tau\omega \gg 1$, and
- a precise value of M when $\tau\omega = 1$

When $\tau\omega \ll 1$:

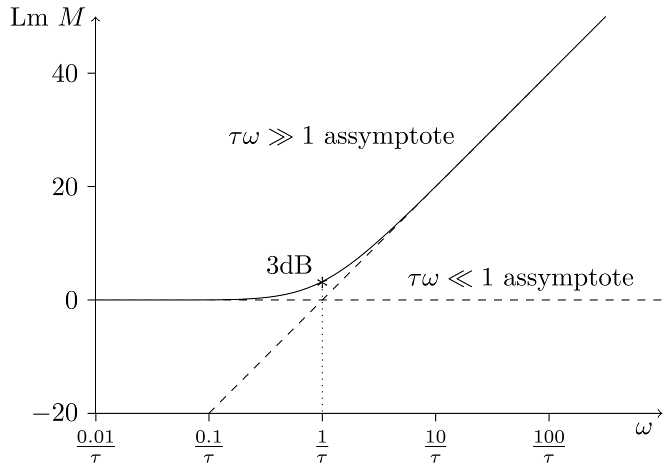
- $\tau\omega \approx 0$ so $M(\omega) \approx 1$.
- $20\log_{10}(1) = 0$

When $\tau\omega \gg 1$:

- $\tau\omega + 1 \approx \tau\omega$ so $M(\omega) \approx \tau\omega$.
- $20\log_{10}(\tau\omega)$ is a straight line with slope of 20dB/decade and is 0 at $\omega = 1/\tau$.

When $\tau\omega = 1$ precisely:

- $M(\omega) = \sqrt{1+1} = \sqrt{2}$.
- Lm $M(1/\tau) = 3$dB.



Next, we look at the phase by finding approximations for ϕ when::

- an approximation for M when $\tau\omega \ll 1$,
- an approximation for M when $\tau\omega \gg 1$, and
- a precise value of M when $\tau\omega = 1$

When $\tau\omega \ll 1$:

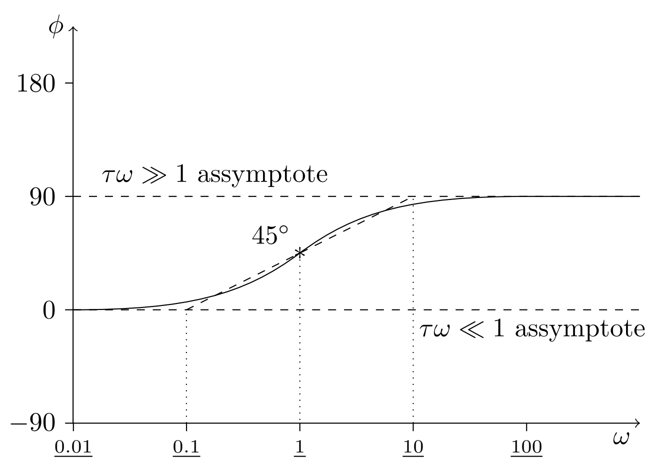
- $\tau\omega \approx 0$ so $\phi(\omega) \approx \tan^{-1}(0/1) = 0°$.

When $\tau\omega \gg 1$:

- $\tau\omega + 1 \approx \tau\omega$ and $\tau\omega \approx \infty$ so $\phi(\omega) \approx \tan^{-1}(\infty/1) = 90°$.

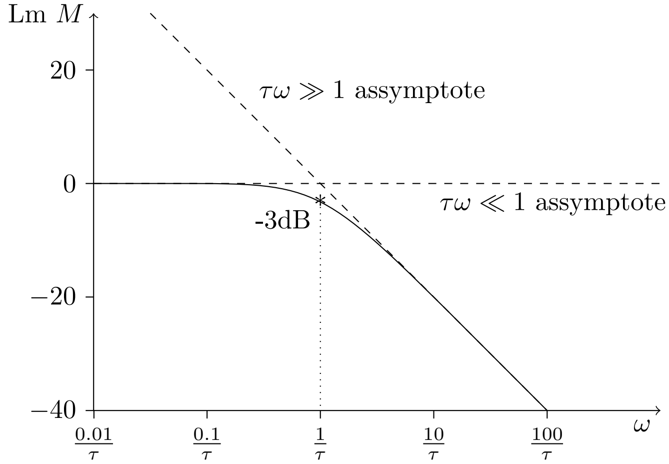
When $\tau\omega = 1$ precisely:

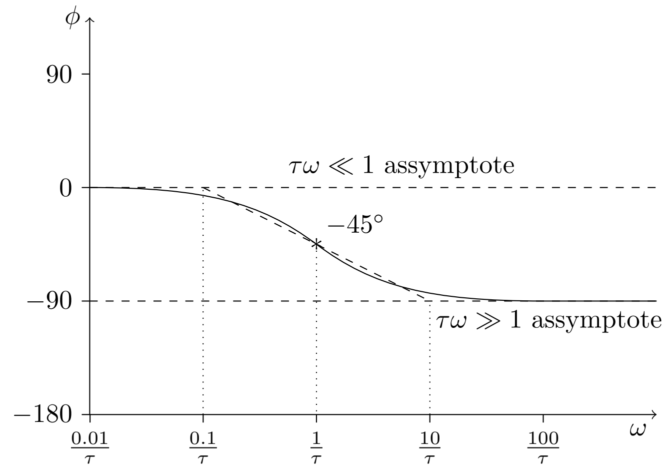
- $\phi(\omega) = \tan^{-1}(1/1) = 45°$.



A decent rule of thumb for drawing first-order phase plots is that the phase change takes place over roughly 2 decades, centered on the cutoff frequency (or $1/\tau$). This line is marked on the phase plot.

First-Order Pole The first-order pole $G(s) = 1/(\tau s + 1)$ is a mirror image of the first-order zero.





Second-Order Pole Let $G(s)$ be defined as

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

What is the Bode plot?

$$G(j\omega) = \frac{1}{\frac{-\omega^2}{\omega_n^2} + \frac{j(2\zeta\omega)}{\omega_n} + 1} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(2\zeta\frac{\omega}{\omega_n}\right)}$$

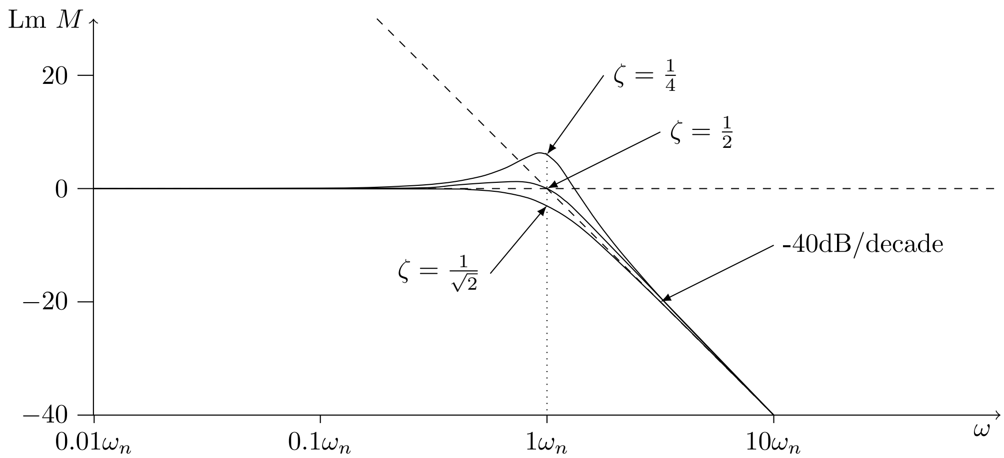
We will start by finding the gain as a function of ω.

$$M = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

From here, we will find M when $\frac{\omega}{\omega_n} = 1$ precisely and find approximations for M when $\frac{\omega}{\omega_n} \ll 1$ and when $\frac{\omega}{\omega_n} \gg 1$.

- When $\frac{\omega}{\omega_n} \ll 1$, $M \approx 1/\sqrt{1^2 - 0^1} = 1$
    - Lm $(1) = 0$dB

- When $\dfrac{\omega}{\omega_n} \gg 1$,
$$M \approx \frac{1}{\sqrt{\frac{\omega^4}{\omega_n^4} + 2\zeta\frac{\omega^2}{\omega_n^2}}} \approx \frac{1}{\frac{\omega^2}{\omega_n^2}}$$
  - Lm $M =$ Lm $\left(\frac{1}{\frac{\omega^2}{\omega_n^2}}\right) =$ Lm $(1) -$ Lm $\left(\frac{\omega}{\omega_n}\right)^2 = 0 - 40\log_{10}\left(\frac{\omega}{\omega_n}\right)$
  - Straight line with slope of -40dB/decade and 0dB at $\omega = \omega_n$
- When $\omega = \omega_n$ precisely,
$$M = \frac{1}{\sqrt{(1-1^2)^2 + (2\zeta 1)^2}} = \frac{1}{2\zeta}$$
  - If $\zeta = 1/\sqrt{(2)}$, $M = 1/\sqrt{2}$ and Lm $M = -3$dB.
  - If $\zeta = 1/2$, $M = 1$ and Lm $M = 0$dB.
  - If $\zeta = 1/4$, $M = 2$ and Lm $M = 6$dB.
  - **As ζ goes down, the magnitude at $\omega = \omega_n$ goes up.**



Next, we find the phase as a function of ω.
$$\phi = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right)$$
From here, we will find ϕ when $\frac{\omega}{\omega_n} = 1$ precisely and find approximations for ϕ when $\frac{\omega}{\omega_n} \ll 1$ and when $\frac{\omega}{\omega_n} \gg 1$.

- When $\frac{\omega}{\omega_n} \ll 1$ (meaning $\omega \to 0$), $\phi \approx -\tan^{-1}(\frac{0}{1}) = 0°$
- When $\omega = \omega_n$, $\phi \approx -\tan^{-1}(\frac{2\zeta}{0}) = -90°$
- When $\frac{\omega}{\omega_n} \gg 1$ (meaning $\omega \to \infty$), $\phi \approx -\tan^{-1}(\frac{2\zeta}{-\infty}) = -180°$

Decreasing ζ decreases the range of frequencies over which the phase shift occurs — that is, the phase shift is "sharper", and the slope at the center ($\omega = \omega_n$) is closed to vertical. When drawing a 2nd-order phase plot by hand, the phase shift can be approximated by estimating that the phase shift starts at
$$\omega_{start} = \omega_n \cdot 10^{-\zeta}$$
and ends at
$$\omega_{end} = \omega_n \cdot 10^{\zeta}$$
More simply, this means that the phase change takes place over 2ζ decades. For example,

- If $\zeta = 1/2$, the phase shift happens over 1 decade — half a decade for $0°$ to $-90°$, and half a decade for $-90°$ to $-180°$.
- If $\zeta = 1/4$, the phase shift happens over half a decade — one quarter of a decade for $0°$ to $-90°$, and one quarter of a decade for $-90°$ to $-180°$.