Lecture 5

Last time:

- Reviewed Laplace Transform
- Transfer functions
- Time-domain $\leftrightarrow$ s-domain
- Initial Value, Final Value, Static (DC) Gain

We have seen that a system's behavior (i.e. output) depends on two things:

1. Its transfer function $G(s)$
2. The input $u(t)$

In general, we can write

$$G(s)\frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}$$

where $m \leq n$, so that $G(s)$ is a proper rational function.$u(t)$ can be anything — step, ramp, sinusoidal, etc... We will study a few simple cases. First, let's consider two cases for $G(S)$:

- 1st order transfer function
- 2nd order transfer function

(Here, the word "order" refers to the highest power of s in the denominator polynomial of $G(s)$ — in other words, the number of poles.) Next, let's consider two cases for $u(t)$:

- Unit step $\rightarrow U(s) = 1/s$ in the s-domain)
- Sinusoidal input $\rightarrow$ we will use the Bode diagram for this.

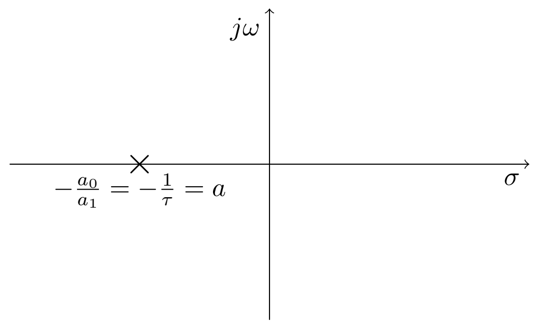
Nise, Chapter 4:

1st Order Systems

We'll start with a 1st-order system with no zeros (strictly proper): The denominator of $G(s)$ contains s to the 1st power only.

- $G(s) = \frac{b_0}{a_1 s + a_0}$ if $G(s)$ is strictly proper.
- $G(s) = \frac{K}{\tau s + 1}$ — Time constant form. Note that $G(0) = K =$ static gain.
- $G(s) = \frac{Ka}{s+a}$ — Pole-zero form, where $a = 1/\tau$:

$$G(s) = \frac{K}{\tau s + 1} = \frac{K/\tau}{s + 1/\tau} = \frac{Ka}{s+a}$$



The step response to this system is

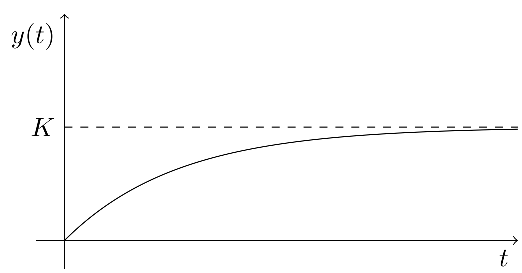
$$Y(s) = \frac{K}{s(\tau s + 1)} = \frac{\frac{K}{\tau}}{s(s + 1/\tau)} = \frac{R_1}{s} + \frac{R_2}{s + \frac{1}{\tau}}$$

$$R_1 = K, \quad R_2 = -K$$

and so,

$$Y(s) = \frac{K}{s} - \frac{K}{s + \frac{1}{\tau}}$$

$$y(t) = K\left[1 - e^{-t/\tau}\right] 1(t)$$



We can find the final/steady-state value as:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = K = y_{ss}$$

We can find the initial slope using the IVT or direct computation:

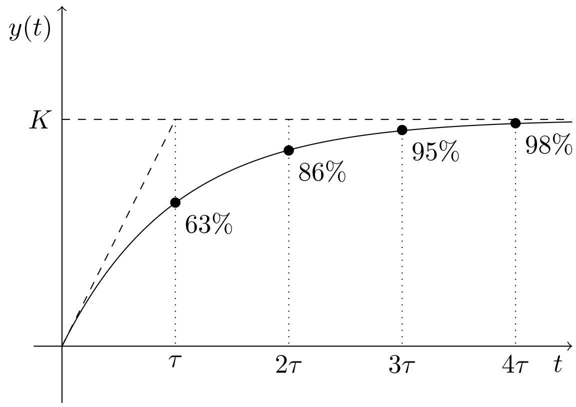
$$\dot{y}(0^+) = \lim_{s \to \infty} s^2 Y(s) = \lim_{s \to \infty} \frac{s\dfrac{K}{\tau}}{s + \dfrac{1}{\tau}} = \frac{K}{\tau}$$

Observe that

$$y(\tau) = K(1 - e^{-1}) = K(1 - 1/e) = 0.63K$$

τ is the **time constant** — the time taken by the response to reach 63% of it's steady state value. Other notable times:

- $2\tau \leftarrow 86\%$
- $3\tau \leftarrow 95\%$
- $4\tau \leftarrow 98\%$



τ is used as a measure of the system's speed of response. A large time constant means that the system is slow. 1st order systems are characterized by their time constant. This gives you a general idea of how fast the system responds to a step input. Overall, three quantities are used as performance characteristics for 1st-order systems.

- Time constant τ: Time for the step response to rise to 63% of a final value.
- Settling time t_s: Time for a step response to settle to 98% of the final value.

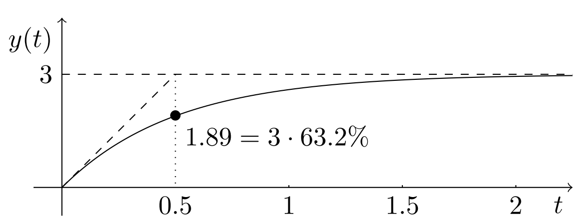
$$t_s = 4\tau$$

- Rise time t_r: Time taken by step response to go from 10% to 90% of the final value.

$$\begin{aligned}
y(t) &= K[1 - e^{-t/\tau}]1(t) \\
y(t_1) = 0.1\cancel{K} &= \cancel{K}[1 - e^{-t_1/\tau}] \\
e^{-t_1/\tau} &= 1 - 0.1 = 0.9 \\
-\frac{t_1}{\tau} &= \ln 0.9 = -0.1054 \\
\Rightarrow \quad t_1 &= 0.1054\tau \\
y(t_2) = 0.9\cancel{K} &= \cancel{K}[1 - e^{-t_2/\tau}] \\
e^{-t_2/\tau} &= 1 - 0.9 = 0.1 \\
-\frac{t_2}{\tau} &= \ln 0.1 = -2.3026 \\
\Rightarrow \quad t_2 &= 2.3026\tau \\
t_r &= t_2 - t_1 = (2.3026 - 0.1054)\tau \\
\Rightarrow \quad t_r &= 2.2\tau
\end{aligned}$$

So, rise time for a 1st order system is defined as $\mathbf{t_r = 2.2}\tau$.

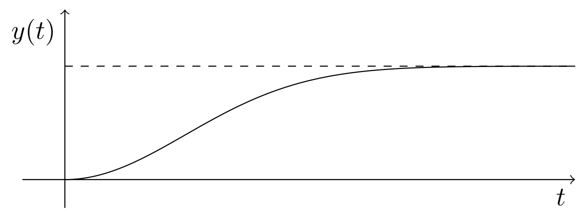
Note that for 1st-order systems, the slope will always be discontinuous at the origin. So, if you were given the following time-response:



You should be able to identify the transfer function as

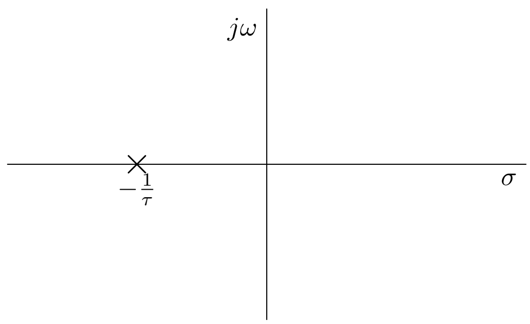
$$G(s) = \frac{3}{0.5s + 1}$$

However if you were given a response like



You know this is not the response of a 1st order system because of its initial slope.

Looking at a pole-zero diagram,



where τ is the time constant: the further away the pole is from the $j\omega$-axis, the faster the system response (to a step input).

2nd Order Systems

Let's now move to our second case for $G(s)$: second order systems. A strictly-proper second-order system has a transfer function

$$G(s) = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

Let's start by studying 2nd order transfer functions with no zeros.

$$G(s) = \frac{b_0}{a_2 s^2 + a_1 s + a_0}$$

This is usually written in 1 of 2 ways:

$$G(s) = \frac{K\gamma_0}{s^2 + \gamma_1 s + \gamma_0} \quad or \quad G(s) = \frac{K}{\alpha_2 s^2 + \alpha_1 s + 1}$$

In either case, K is the static gain $G(0) = K$ assuming both poles of $G(s)$ are in the LHP. The second-order system has the following response to a step input:

$$G(s) = \frac{K\gamma_0}{s^2 + \gamma_1 s + \gamma_0}, \quad U(s) = \frac{1}{s}$$

$$Y(s) = \frac{K\gamma_0}{s(s^2 + \gamma_1 s + \gamma_0)}$$

Initial value:

$$y(0^+) = \lim_{s\to\infty} sY(s) = 0$$

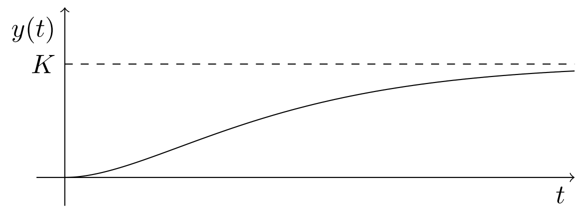
Initial slope:

$$\dot{y}(0^+) = \lim_{s \to \infty} s^2 Y(s) = 0$$

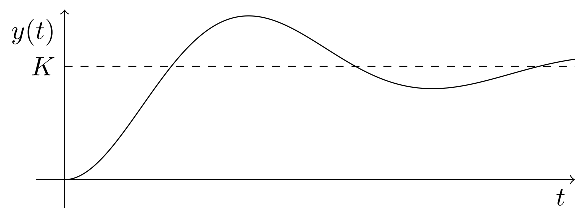
Assuming that the poles of $G(s)$ are strictly in the LHP, then the final value is:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = K$$

So, we have



or

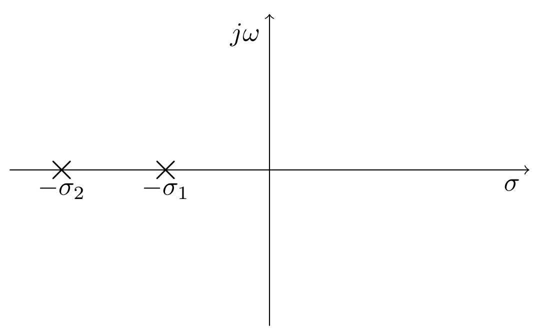


What happens here depends on the nature of the poles of $G(s)$.

Case 1: $G(s)$ has 2 real poles.

$$G(s) = \frac{K\gamma_0}{s^2 + \gamma_1 s + \gamma_0} = \frac{K\sigma_1\sigma_2}{(s+\sigma_1)(s+\sigma_2)}$$

(We need the $\sigma_1\sigma_2$ term so that K can continue to be the static gain.) Here, σ_1 and σ_2 are positive real numbers. Let $\sigma_2 > \sigma_1$.



Then the step response is

$$Y(s) = \frac{K\sigma_1\sigma_2}{s(s+\sigma_1)(s+\sigma_2)} = \frac{R_1}{s} + \frac{R_2}{s+\sigma_1} + \frac{R_3}{s+\sigma_2}$$

$$R_1 = sY(s)\big|_{s=0} = \frac{K\sigma_1\sigma_2}{\sigma_1\sigma_2} = K$$

$$R_2 = (s+\sigma_1)Y(s)\big|_{s=-\sigma_1} = \frac{K\cancel{\sigma_1}\sigma_2}{(-\cancel{\sigma_1})(-\sigma_1+\sigma_2)} = -\frac{K\sigma_2}{\sigma_2 - \sigma_1}$$

$$R_3 = (s+\sigma_2)Y(s)|_{s=-\sigma_2} = \frac{K\sigma_1\cancel{\sigma_2}}{(-\cancel{\sigma_2})(-\sigma_2+\sigma_1)} = -\frac{K\sigma_1}{\sigma_1-\sigma_2} = \frac{K\sigma_1}{\sigma_2-\sigma_1}$$

So,

$$Y(s) = \frac{K}{s} - \frac{K\sigma_2}{(\sigma_2-\sigma_1)(s+\sigma_1)} + \frac{K\sigma_1}{(\sigma_2-\sigma_1)(s+\sigma_2)}$$

$$y(t) = (K - \frac{K\sigma_2}{\sigma_2-\sigma_1}e^{-\sigma_1 t} + \frac{K\sigma_1}{\sigma_2-\sigma_1}e^{-\sigma_2 t})1(t)$$

We see indeed that at $t = 0^+$:

$$y(0^+) = K - \frac{K\sigma_2}{\sigma_2-\sigma_1} + \frac{K\sigma_1}{\sigma_2-\sigma_1} = K\left(1 - \frac{\sigma_2}{\sigma_2-\sigma_1} + \frac{\sigma_1}{\sigma_2-\sigma_1}\right) = 0$$

We can write this as

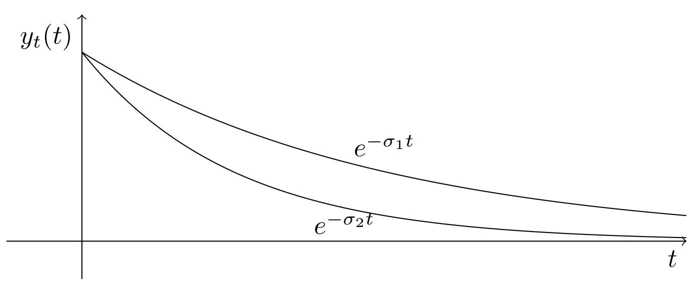
$$y(t) = (K - y_t(t))1(t)$$

where

$$y_t(t) = \left(\frac{K}{\sigma_2-\sigma_1}(\sigma_2 e^{-\sigma_1 t} - \sigma_1 e^{-\sigma_2 t})\right)1(t)$$

Since $\sigma_2 > \sigma_1$ (each is positive) and

$$e^{-\sigma_2 t} < e^{-\sigma_1 t}$$



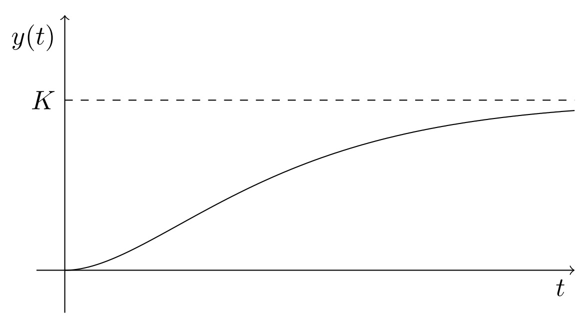
so then

$$\sigma_1 e^{-\sigma_2 t} < \sigma_2 e^{-\sigma_1 t}$$

Also given that $\frac{K}{\sigma_2-\sigma_1} > 0$, then

$$0 \leq y_t(t) \leq K$$

$y(t)$ is never greater than K, so no overshoot is possible. This system is said to be **overdamped**.



We will now consider a special case of Case 1: $\sigma_1 = \sigma_2 = \sigma$ (repeated poles). Then,

$$G(s) = \frac{K\sigma^2}{(s+\sigma)^2}$$

$$Y(s) = \frac{K\sigma^2}{s(s+\sigma)^2} = \frac{R_1}{s} + \frac{R_2}{(s+\sigma)^2} + \frac{R_3}{s+\sigma}$$

$$R_1 = K, \quad R_2 = (s+\sigma)^2 Y(s)\Big|_{s=-\sigma} = \frac{K\sigma^2}{-\sigma} = -K\sigma, \quad R_3 = \frac{d}{ds}\frac{K\sigma^2}{s}\Big|_{s=-\sigma} = \frac{-K\sigma^2}{\sigma^2} = -K$$

$$Y(s) = \frac{K}{s} - \frac{K\sigma}{(s+\sigma)^2} - \frac{K}{s+\sigma}$$

$$y(t) = K\left(1 - e^{-\sigma t} - \sigma t e^{-\sigma t}\right) 1(t)$$

Is $y(t)$ still bounded by 0 and K, in this case? Let

- At $t = 0^+$: $e^{-\sigma t} = 1$, $\sigma t e^{-\sigma t} = 0$, $\Rightarrow$ $y(0^+) = 0$.
- At $t \to \infty$: $e^{-\sigma t} = 0$, $\sigma t e^{-\sigma t} \to 0$, $\Rightarrow$ $\lim_{t\to\infty} y(t) = K$.
- Let $y_t(t) = -K\left(e^{-\sigma t} + \sigma t e^{-\sigma t}\right)$ and $y_{t_2}(t) = \sigma t e^{-\sigma t}$. Our final step is to check the peak location of $y_{t_2}(t)$. First, find the peak location:

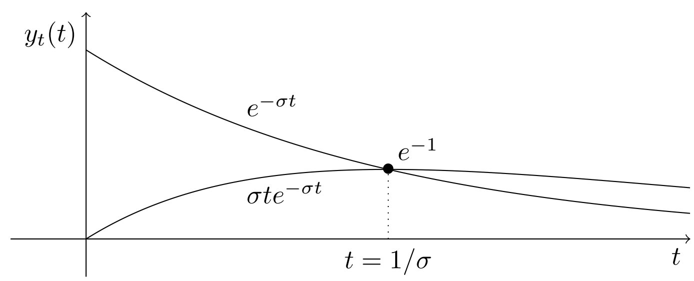
$$\dot{y}_{t_2} = \sigma\left(-t\sigma e^{-\sigma t} + e^{-\sigma t}\right) = 0$$

$$e^{-\sigma t}(1 - t\sigma) = 0 \quad \Rightarrow \quad t = \frac{1}{\sigma}$$

So,

$$y_{t_2}\left(\frac{1}{\sigma}\right) = \sigma\frac{1}{\sigma}e^{-\sigma\frac{1}{\sigma}} = e^{-1}$$

$$y\left(\frac{1}{\sigma}\right) = K\left(1 - e^{-1} - e^{-1}\right) = K - 2Ke^{-1}$$



So, $y(t)$ never exceeds K nor goes below 0. In fact, repeated poles normally have the fastest 2nd-order response possible without overshoot:

$$y_{\text{repeated}}(t) > y_{\text{distinct}}(t)$$

This system is said to be **critically damped**. In general, this response looks like a 1st-order response, except that there is no discontinuity in the slope at the origin.