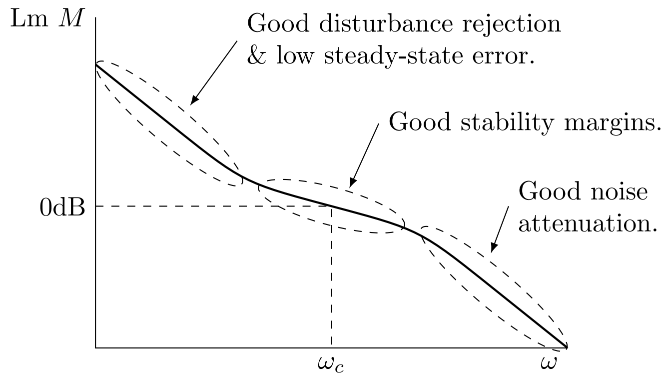
Lecture 16

Last lecture:

- Loop Shaping: Give our closed-loop system good performance by giving the Bod plot of our open-loop transfer function G_cG_p a good shape.
- The closed-loop system will reject disturbances and have low steady-state error if G_cG_p has large magnitude for low frequencies.
- The closed-loop system will attenuate noise if G_cG_p has small magnitude for high frequencies.
- The closed-loop system will be stable and robust if the slope of Lm G_cG_p is -20dB/decade at or near the 0dB crossover.

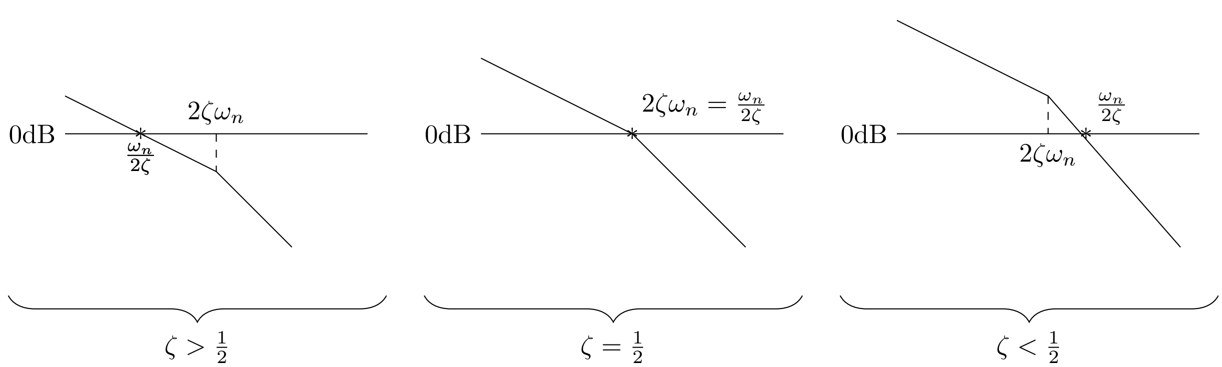


- If we want our closed-loop system to behave like a second-order system,

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

then our open-loop transfer function should look something like

$$G_cG_p = \frac{\frac{\omega_n}{2\zeta}}{s\left(\frac{s}{2\zeta\omega_n} + 1\right)}$$



This lecture: Controller design steps.
First, let's do some examples for what we learned last lecture.

Example

What do we want G_cG_p to look like if we want and overshoot percent and peak time such that $\zeta = \sqrt{2.5}$ and $\omega_n = 2\sqrt{2.5}$?

$$G_cG_p = \frac{\frac{2\sqrt{2.5}}{2(\sqrt{2.5})}}{s\left(\frac{s}{2(\sqrt{2.5})(2\sqrt{2.5})} + 1\right)}$$

$$G_cG_p = \frac{1}{s\left(\frac{s}{10} + 1\right)}$$

Example

What do we want G_cG_p to look like if we want and overshoot percent and peak time such that $\zeta = \frac{1}{2}\sqrt{10}$ and $\omega_n = 40\sqrt{10}$?

$$G_cG_p = \frac{\frac{40\sqrt{10}}{2(\frac{1}{2}\sqrt{10})}}{s\left(\frac{s}{2(\frac{1}{2}\sqrt{10})(40\sqrt{10})} + 1\right)}$$

$$G_cG_p = \frac{40}{s\left(\frac{s}{400} + 1\right)}$$

Controller Design Steps

1. Transient requirements $\Longrightarrow \omega_n, \zeta$
2. Desired open-loop transfer function:

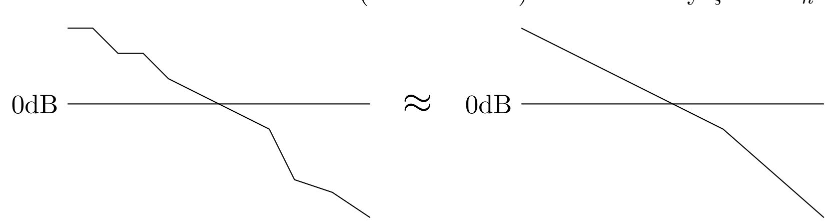
$$G_cG_p = OLTF = \frac{\frac{\omega_n}{2\zeta}}{s\left(\frac{s}{2\zeta\omega_n} + 1\right)}$$

3. $G_c =$ what we have to add to G_p in order to make G_cG_p look like OLTF (look like $\frac{\omega_n/2\zeta}{s\left(\frac{s}{2\zeta\omega_n}+1\right)}$)

This seems easy when G_p is really simple.

What do we do if G_p is more complicated and/or has extra poles/zeros?

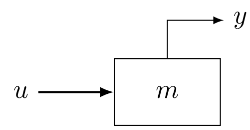
- Previously, we said our system would behave like a 2nd order system if extra poles/zeros are far enough left.
- For loop shaping, our closed-loop system will behave like a second-order system if G_cG_p has the right shape near the 0dB crossover
- Note: "near" means "within about 1 decade" ($\approx \pm 1$ decade) or dictated by ζ and ω_n.

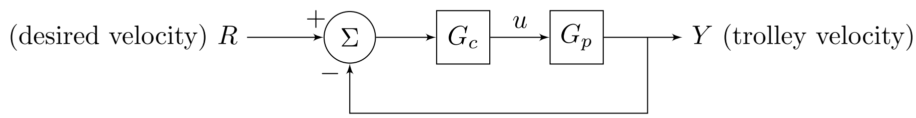


  - Still large at low freq. → rejects disturbances, low e_{ss}.
  - Still small at high freq. → attenuates noise
  - Similar shape near 0dB → stable, robust, ∼second order

Example

Velocity control of a trolley (mass $m = 1$kg) moving on a straight path, with actuator force u as the input u and velocity y as the output. For a step input, we want $OS\% < 20\%$, $t_s \le 12$.





$$G_p: \quad F = ma \quad \Rightarrow \quad u(t) = m\dot{y}(t)$$

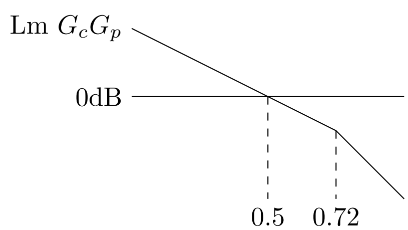
$$\Rightarrow U(s) = msY(s) \quad \Rightarrow \quad \frac{Y}{U} = G_p = \frac{1}{ms} = \frac{1}{s}$$

$$OS\% < 20\% \quad \Rightarrow \quad \zeta \ge 0.6$$

$$t_s \le 12 \quad \Rightarrow \quad \omega_n \ge 0.55$$

Let's pick $\zeta = 0.6$ and $\omega_n = 0.6$. Then, our desired G_cG_p is

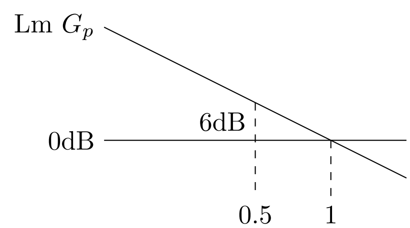
$$\text{Desired } G_cG_p = \frac{\frac{\omega_n}{2\zeta}}{s\left(\frac{s}{2\zeta\omega_n} + 1\right)} = \frac{\frac{0.6}{2\cdot 0.6}}{s\left(\frac{s}{2\cdot 0.6\cdot 0.6} + 1\right)} = \frac{0.5}{s\left(\frac{s}{0.72} + 1\right)}$$



Knowing that $G_p = 1/s$, we can quickly identify what we want for G_c from the desired G_cG_p:

$$\text{Desired } G_cG_p = \underbrace{\frac{1}{s}}_{G_p} \cdot \underbrace{\frac{0.5}{\frac{s}{0.72} + 1}}_{G_c} \quad \Rightarrow \quad G_c = \frac{0.5}{\frac{s}{0.72} + 1}$$

However, let's explore the design process in greater detail. Before designing G_c, what does the plant log magnitude look like?

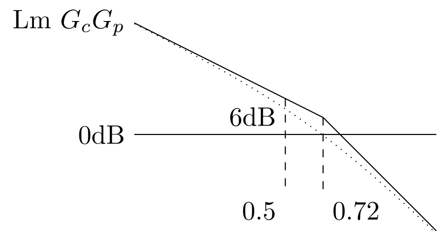


This meets some but not all of our requirements:

- Big at low frequencies
- Small at high frequencies
- Robust stability and performance

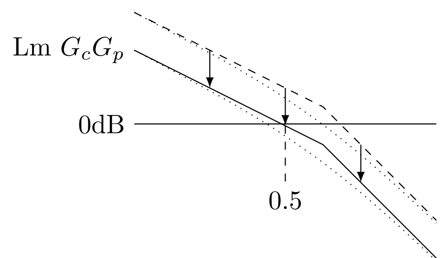
But, it does not have a second-order shape, so we will not meet our transient requirements. What is missing? We need a corner at $\omega = 0.72$. So, we add a pole to G_c:

$$\frac{1}{\frac{s}{0.72} + 1}$$



Next, we need the low-frequency asymptote to cross 0dB at $\omega = 0.5$.

- We use a gain to shift the Lm plot.
- The low-frequency asymptote is currently $\sim$ 6dB at $\omega = 0.5$, so we need G_c to provide -6dB of gain.



$$\text{Lm }(K) = 20\log_{10}(K) = -6$$

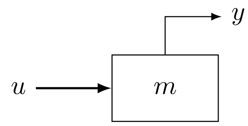
$$\log_{10}(K) = -0.3 \quad \rightarrow \quad K = 10^{-0.3} = 0.5$$

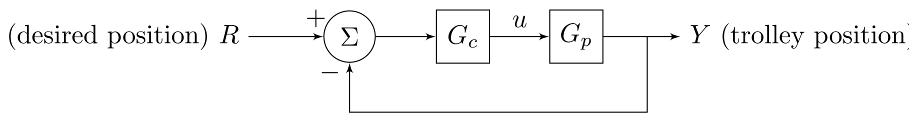
So,

$$\boxed{G_c = \frac{0.5}{\frac{s}{0.72} + 1}}$$

Example

Consider the above example again, except this the output $y(t)$ is the position rather than the velocity.





Then,

$$G_p : \quad F = ma \quad \Rightarrow \quad u(t) = m\ddot{y}(t)$$

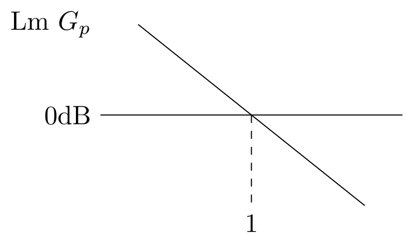
$$\Rightarrow U(s) = ms^2 Y(s) \quad \Rightarrow \quad \frac{Y}{U} = G_p = \frac{1}{ms^2} = \frac{1}{s^2}$$

For this example, we will use $\zeta = 0.6$ and $\omega = 2$, so the desired $G_c G_p$ will be

$$\frac{\frac{\omega_n}{2\zeta}}{s\left(\frac{s}{2\zeta\omega_n} + 1\right)} = \text{Desired } G_c G_p = \frac{\frac{5}{3}}{s\left(\frac{s}{2.4} + 1\right)}$$

This time, we have an extra integrator in G_p, so it will be more challenging to design G_c. $1/s$ is an unstable pole, so we cannot simply cancel the extra integrator with a zero at $s = 0$ (we would lose internal stability). In this example we will explore how to design G_c for the more complicated plant.

Before designing G_c, what does the open-loop log magnitude look like?



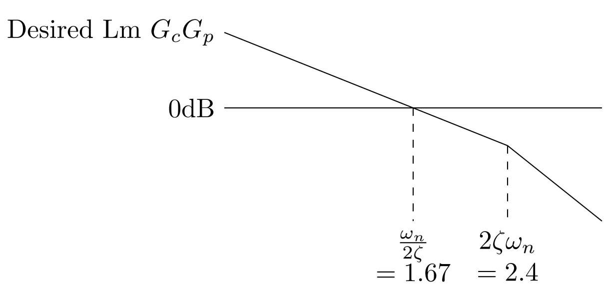
This meets two of our requirements:

- Big at low frequencies
- Small at high frequencies

However, it does not have -20dB/decade of slope near the 0dB crossover, so:

- Not stable and not robust
- No 2nd-order shape

What we really want is a shape like this:



How can we get our desired shape near the crossover?

- Add a zero and a pole
  - We want the pole at $\omega = 2\zeta\omega_n = 2.4$
  - We must place the zero before $\omega = \frac{\omega_n}{2\zeta} = 1.67$. We will say the zero is at $\omega = \omega_1$.
- Add a gain K to get the 0dB crossover we want
- Remember: we cannot add a zero at $s = 0$, canceling the $s = 0$ pole. This would create a "hidden instability".

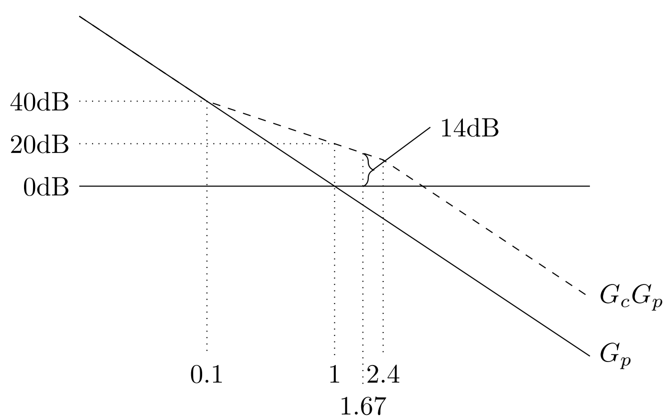
So, our controller will have the form

$$G_c = K \cdot \frac{\frac{s}{\omega_1} + 1}{\frac{s}{2.4} + 1}$$

We want ω_1 at least 1 decade away from the 0dB crossover. Let's pick $\omega_1 = 0.1$. Now, let's find K:

- The -20dB/decade asymptote should be 0dB at $\omega = \frac{\omega_n}{2\zeta} = 1.67$
- If $G_c = \frac{\frac{s}{0.1}+1}{\frac{s}{2.4}+1}$ then the magnitude of G_cG_p will be 14dB at $\omega = 1.67$.

$$\text{Lm } G_cG_p(1.67) = 20\log_{10}\left(\frac{\sqrt{\left(\frac{1.67^2}{0.1^2}\right)+1}}{1.67^2\sqrt{\left(\frac{1.67^2}{2.4^2}\right)+1}}\right) = 14\text{dB}$$



- So, add a gain that will shift the log magnitude by -14dB

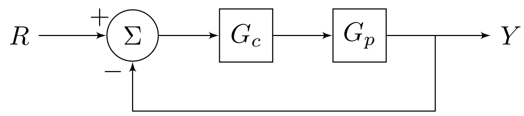
$$\text{Lm } (K) = 20\log_{10}(K) = -14$$

$$\log_{10}(K) = -0.7 \quad \rightarrow \quad K = 10^{-0.7} \approx 0.2$$

Therefore,

$$\boxed{G_c = \frac{0.2\left(\frac{s}{0.1}+1\right)}{\frac{s}{2.4}+1}}$$

Example

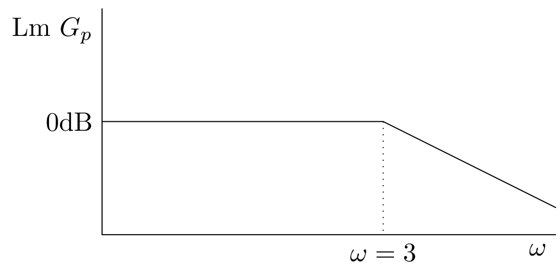


$$G_p = \frac{1}{\frac{s}{3}+1}$$

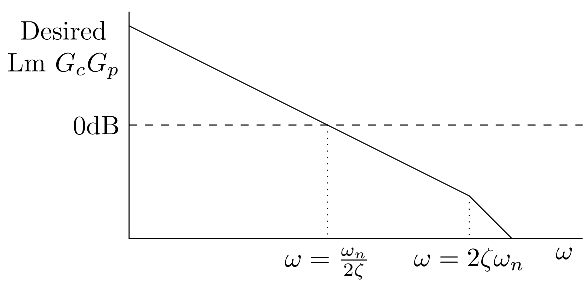
Find G_c such that:

- $t_s \leq 3$s
- $OS\% \leq 5\%$
- $e_{ss} \leq 5\%$

What we have:



What we want:



What should we do?

- It depends on if the pole of G_p is in a convenient place.
- Let's look at the transient requirements:
  - $OS\% \leq 5\%$: $\zeta \geq 0.69$. Let's use $\zeta = 0.7$.
  - $t_s = \frac{4}{\omega_n \zeta} \leq 3$: $\frac{4}{3} \leq \omega_n\zeta \;\rightarrow\; 2\zeta\omega_n \geq 2\frac{2}{3}$
- Therefore, the pole is at a convenient place ($2\zeta\omega_n = 3 \geq 2\frac{2}{3}$)
- So, we will give G_c a pole at the origin and a gain that will cause a good 0dB crossover.

$$2\zeta\omega_n = 3 \;\Rightarrow\; \omega_n = \frac{3}{2\zeta}$$

$$\text{crossover: } \frac{\omega_n}{2\zeta} = \frac{\frac{3}{2\zeta}}{2\zeta} = \frac{3}{4\zeta^2} \approx 1.53$$

Then,

$$(G_cG_p)_{desired} = \frac{1.53}{s(\frac{s}{3}+1)},\; G_p = \frac{1}{\frac{s}{3}+1} \quad \Rightarrow \quad G_c = \frac{1.53}{s}$$

- Because of the integrator in G_c, $e_{ss} = 0$.

Example

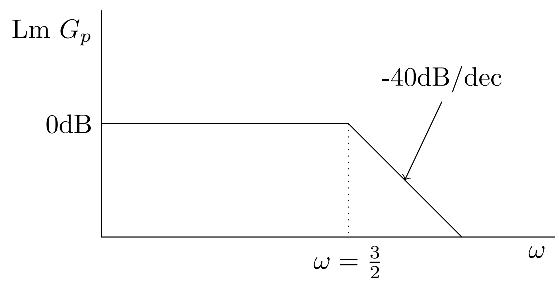
G_p is a second-order system with $\omega_n = \frac{3}{2}$ and $\zeta = \frac{2}{3}$.

$$G_p = \frac{1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1} = \frac{1}{\frac{4}{9}s^2 + \frac{8}{9}s + 1}$$

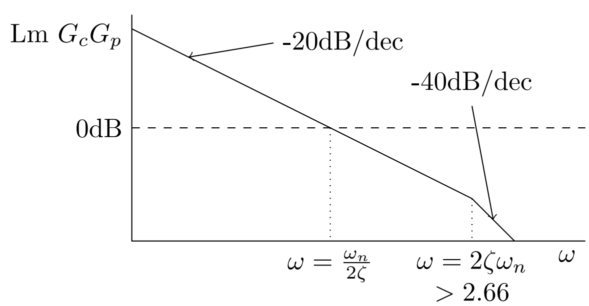
Find G_c such that:

- $t_s \leq 3$s
- $OS\% \leq 5\%$
- $e_{ss} \leq 5\%$

What we have:



What we want:



- We know from the previous example that $2\zeta\omega_n \geq 2\frac{2}{3}$
- What do we do?
  - Cancel the poles of G_p
  - Design a controller that undoes the dynamics of the plant and replaces them with better dynamics
  - If we want to give the system the same dynamics as the last example, then we have:

$$G_cG_p = \frac{\frac{\omega_n}{2\zeta}}{s\left(\frac{s}{2\zeta\omega_n} + 1\right)} \quad \Rightarrow \quad G_c = \frac{1}{G_p} \cdot \frac{\frac{\omega_n}{2\zeta}}{s\left(\frac{s}{2\zeta\omega_n} + 1\right)} = \frac{1.53\left(\frac{4}{9}s^2 + \frac{8}{9}s + 1\right)}{s(\frac{s}{3} + 1)}$$

$$G_cG_p = \frac{1.53\left(\frac{4}{9}s^2 + \frac{8}{9}s + 1\right)}{s(\frac{s}{3} + 1)\left(\frac{4}{9}s^2 + \frac{8}{9}s + 1\right)} = \frac{1.53}{s(\frac{s}{3} + 1)}$$

- This controller works by canceling out the dynamics of the plant. **We are able to do this here because we are only canceling stable poles.** If G_p had unstable poles, we would not be able to cancel them without losing internal stability.

Comments

- Only cancel LHP poles of G_p
- Unnecessarily canceling poles of G_p will make G_c unnecessarily complex
  - This can cause problems when implementing the controller on a real system (converting continuous-time to discrete-time control)
  - Depending how you implement your system, this make make G_c more expensive to implement