ECON 432 Homework 2

Due: Feb 5 (Wednesday) 6:00pm

1 Part I: Review Questions

1. Consider a fair casino roulette wheel with 37 slots: 18 red, 18 black, and 1 green (zero). Suppose you bet \$1 on black. If the ball lands on black, you receive \$2 (your original \$1 plus \$1 in winnings), and if it lands on red or zero, you receive \$0.

   (a) What are the expected return and Sharpe ratio (assuming a zero risk-free rate) of this bet?

   (b) Using the daily adjusted closing prices of the S&P500 from Jan 1, 2000 to Dec 31, 2024, estimate its daily Sharpe ratio (assuming a zero risk-free rate). Compare this with the Sharpe ratio calculated for the bet in part (**a**).

2. Suppose $r_t \sim \text{iid}(\mu, \sigma^2)$ where μ and σ^2 are finite. The ECDF, defined as

   $$\widehat{F}(x) = T^{-1} \sum_{t=1}^{T} I\{r_t \leq x\} \text{ for any } x \in \mathbb{R},$$

   is an estimator of the true CDF $F(x) = \mathrm{P}(r_t \leq x)$.

   (a) For any $x \in \mathbb{R}$, find the mean and variance of the binary random variable $Y_t = I\{r_t \leq x\}$.

   (b) Is $\widehat{F}(x)$ an unbiased estimator of $F(x)$?

   (c) Find the variance of $\widehat{F}(x)$.

   (d) Find the mean square error (MSE) of $\widehat{F}(x)$. Does the MSE go to zero as the sample size $T \to \infty$?

   (e) Do your answers in part (d) depend on the value of x?

3. Consider $X = U \cdot Z$, where $Z \sim N(0,1)$, U is independent of Z and satisfies $\mathrm{P}(U=1) = 0.8$ and $\mathrm{P}(U=5) = 0.2$.

   (a) What are the mean μ_X and σ_X^2 variance of X?

   (b) What's the probability of $|X| > 4(\mathrm{Var}(X))^{1/2}$? Compare it with the probability of $|Z| > 4(\mathrm{Var}(Z))^{1/2}$.

   (c) What's the kurtosis of X? Does X exhibit heavy tails or thin tails?

   (d) Prove that for any $x \in (-\infty, \infty)$,

$$F(x) = \mathrm{P}\left(X \leq x\right) = 0.8\Phi(x) + 0.2\Phi(x/5) \tag{1}$$

   where $\Phi(x)$ represents the standard normal CDF.

   (e) Derive the PDF of X from equation (1). Then, plot the PDF of X alongside the PDF of a normal distribution with mean μ_X and variance σ_X^2 using Python. Discuss your observations.

   (f) How can we generate a family of asymmetric and heavy-tailed distributions based on the distribution of X? Provide an example based on your proposed method and plot its PDF alongside the PDF of X.

4. Numerical differentiation is a valuable technique for approximating the derivative of complicated functions when obtaining an explicit derivative is difficult or impractical. The following problems will help in understanding this important concept.

   (a) Let $g(\theta)$ be a twice continuously differentiable function, and let θ be a scale. Applying L'Hôpital's rule, show that for any θ_0 in the domain of $g(\cdot)$:

$$\frac{\partial g(\theta_0)}{\partial \theta} = \lim_{\varepsilon \to 0} \frac{g(\theta_0 + \varepsilon) - g(\theta_0 - \varepsilon)}{2\varepsilon}, \tag{2}$$

$$\frac{\partial^2 g(\theta_0)}{\partial \theta^2} = \lim_{\varepsilon \to 0} \frac{g(\theta_0 + \varepsilon) + g(\theta_0 - \varepsilon) - 2g(\theta_0)}{\varepsilon^2}. \tag{3}$$

   Using (1) and (2), explain how to approximate the first- and second-order numerical derivatives with a small ε, say $\varepsilon = 10^{-5}$.

   (b) The PDF of student-t with degree of freedom v is

$$f(x; v) = \frac{\Gamma(\frac{v+1}{2})}{\sqrt{\pi v}\,\Gamma(\frac{v}{2})} \left(1 + \frac{x^2}{v}\right)^{-\frac{v+1}{2}}, \quad \text{for } x \in \mathbb{R},$$

where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\exp(-x)dx, \text{ for } \alpha > 0.$$

Using numerical differentiation with $\varepsilon = 10^{-5}$, compute:

$$\frac{\partial f(1; v_0)}{\partial v} \quad \text{and} \quad \frac{\partial^2 f(1; v_0)}{\partial v^2} \quad \text{for } v_0 = 2.$$

Hint: special.gamma (imported from scipy) is the gamma function in python.

(c) Plot $\partial f(x; v_0)/\partial v$ and $\partial^2 f(x; v_0)/\partial v^2$ ($v_0 = 2$) as functions x for $x \in (0, 10)$.

2 Part II. Data Exercises

1. Download the Microsoft daily price from Jan 2, 2022 to Dec 30, 2024 from *Yahoo* Finance, and perform the following data analysis:

   (a) Find the sample mean and sample variance of the daily log return (in percentage).

   (b) Estimate the CDF of the daily log return using ECDF. Plot the ECDF along with the normal CDF with mean and variance estimated in part **(a)**. Explain your findings.

   (c) Estimate the PDF of the daily log return using kernel estimation. Plot the kernel density estimator along with the normal PDF with mean and variance estimated in part **(a)**. Explain your findings.

   (d) Conduct the quantile v.s. quantile plot of the log return against the normal distribution. Explain your findings. Are the new findings consistent with those in parts **(b)** and **(c)**?

   (e) Calculate the sample skewness and the sample kurtosis. Test the null hypothesis that the skewness is zero, and the hypothesis that the kurtosis is less than or equal to 3. Are the estimators of skewness and kurtosis consistent with the normal assumption on the log return?

   (f) Perform the Jarque-Bera Test on the log return and explain the test result.

   (g) Now assume that the log return are i.i.d. with a Student-t distribution $t_v(\mu, \lambda)$. Find the maximum likelihood estimators and the standard errors of the unknown parameters μ, λ and v. Find the 95% asymptotic confidence interval for v. Test the null hypothesis H_0: $v \geq 10$. Explain the test result.