Mandatory exercise 1 MAT-MEK 9270 report

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September 2025

We consider the two dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{1}$$

and the second order finite difference scheme

$$\frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{i+1j}^n - 2u_{ij}^n + u_{i-1j}^n}{h^2} + \frac{u_{ij+1}^n - 2u_{ij}^n + u_{ij-1}^n}{h^2} \right).$$

This can be written as

$$u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1} = C^2 \big(u_{i+1j}^n - 2u_{ij}^n + u_{i-1j}^n + u_{ij+1}^n - 2u_{ij}^n + u_{ij-1}^n \big), \tag{2}$$

where $C = \frac{c\Delta t}{h}$ is the CFL number.

1.2.3. Exact solution

We want to show that the function

$$u(t, x, y) = e^{\imath(k_x x + k_y y - \omega t)} \tag{3}$$

satisfies the wave equation (1). We have that

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(-\imath\omega u) = (-\imath\omega)^2 u = -\omega^2 u.$$

Moreover,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\imath k_k u = (\imath k_x)^2 u = -k_x^2 u.$$

Furthermore, by symmetry we also have

$$\frac{\partial^2 u}{\partial y^2} = -k_y^2 u.$$

Inserting this into the equation (1) we get

$$-\omega^2 u = -c^2 (k_x^2 + k_y^2) u.$$

We see that this is satisfied if $\omega = c\sqrt{k_x^2 + k_y^2}$. Thus, the function (3) satisfies the wave equation (1).

1.2.4. Dispersion coefficient

We assume $m_x = m_y$ so that $k_x = k_y = k$ and consider the discrete version of the exact solution (3)

$$u_{ij}^n = e^{\imath(kh(i+j)-\tilde{\omega}n\Delta t}. \tag{4}$$

We want to show that if the CFL number is $C = 1/\sqrt{2}$, then $\tilde{\omega} = \omega$. Note that when $k_x = k_y = k$, $\omega = c\sqrt{k_x^2 + k_y^2} = \sqrt{2}ck$. We insert the mesh function (4) into the finite difference scheme (2).

For the left hand side we notice that $u_{ij}^{n+1} = u_{ij}^n e^{-\imath\tilde{\omega}\Delta t}$and $u_{ij}^{n-1} = u_{ij}^n e^{\imath\,\tilde{\omega}\,\Delta t}$. Hence,

$$\begin{aligned} u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1} &= u_{ij}^n\big(e^{\imath\tilde{\omega}\Delta t} + e^{-\imath\tilde{\omega}\Delta t} - 2\big) \\ &= u_{ij}^n\big(\cos(\tilde{\omega}\,\Delta t) - 1\big) \\ &= -4u_{ij}^n \sin^2\left(\frac{\tilde{\omega}\,\Delta t}{2}\right). \end{aligned} \tag{5}$$

Here we used the identity $e^{\imath x} + e^{-\imath x} = 2\cos(x)$ in the second line, and the identity $\cos(x) - 1 = -2\sin^2(\frac{x}{2})$ in the last line.

For the right hand side we notice that $u_{i+1j}^n = u_{ij}^n e^{\imath kh}$ and $u_{i-1j}^n = u_{ij}^n e^{-\imath kh}$. Hence, by similar computation as in (5)

$$u_{i+1j}^n - 2u_{ij}^n + u_{i-1j}^n = u_{ij}^n(e^{\imath kh} + e^{-\imath kh} - 2) = -4u_{ij}^n \sin^2\left(\frac{kh}{2}\right).$$

Moreover, by symmetry we also have

$$u_{ij+1}^n - 2u_{ij}^n + u_{ij-1}^n = -4u_{ij}^n \sin^2\left(\frac{kh}{2}\right).$$

Inserting this, and using $C = 1/\sqrt{2}$, into the equation (2) we get

$$-4u_{ij}^n \sin^2\left(\frac{\tilde{\omega}\,\Delta t}{2}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 \left(-4u_{ij}^n \sin^2\left(\frac{kh}{2}\right) - 4u_{ij}^n \sin^2\left(\frac{kh}{2}\right)\right).$$

This simplifies to

$$\sin^2\left(\frac{\tilde{\omega}\,\Delta t}{2}\right) = \sin^2\left(\frac{kh}{2}\right) \tag{6}$$

Hence, we have $\tilde{\omega}\,\Delta t = kh$, which can be written as $\tilde{\omega} = kh/\Delta t$. Recalling that $C = c\Delta t/h = 1/\sqrt{2}$, we have that $h/\Delta t = \sqrt{2}c$. Therefore,

$$\tilde{\omega} = \frac{kh}{\Delta t} = \sqrt{2}ck = \omega.$$