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HW 1 – CIS623
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A Logic Similar to Propositional Logic

Consider the following logic:
Syntax: A wff can be defined using following BNF rules.

$\phi ::= p \mid (\phi + \phi) \mid (\neg \phi)$

– p is propositional symbol representing a proposition from the set of propositions {p1, p2, p3,…}

Semantics:

– A proposition can map to “true” or “false”.

– “¬” is interpreted as “negation” (¬true=false ; ¬false=true).

– “+” is interpreted as “exclusive or”

true + true = false ; false + false = false
true + false = true ; false + true = true

Syntax Questions-1: $\phi ::= p \mid (\phi + \phi) \mid (\neg \phi)$

Which strings are wffs in this logic?

1. p1 - Yes
2. (p1+(¬p2)) - Yes
3. +p1 – No: needs two arguments
4. $(\neg \phi)$ - No: phi cannot be in a wff
5. p1+p2 – No: needs parenthesis
6. ¬ (p1) – No: negation needs to be inside parentheses
7. (¬ p1 – No: missing closing parentheses
8. (p1+(p2+p3)) - Yes

Syntax Questions-2: $\phi ::= p \mid (\phi + \phi) \mid (\neg \phi)$

- Let us assume that unary operator ¬ has higher precedence than + operator, and + is a left associative operator.
- What are fully parenthesized versions of the following wffs?

1. $\neg p1 + \neg p2$
2. $\neg \neg p1 + \neg p2 + p3$
3. $\neg p1 + p2 + \neg p3$

1. $((\neg p1) + (\neg p2))$
2. $(((\neg(\neg p1)) + (\neg p2)) + p3)$
3. $(((\neg p1) + p2) + (\neg p3))$

Proofs

Schema: Generic versions of formulas; instantiating symbols yields formulas.

Axioms: Formulas assumed to be true in the logic.

Inference Rules: Schema pairs saying what can be proved from what.

- If preconditions of an inference rule have been already proved, then the post-condition of that inference rule can be derived using that inference rule.

Proof: A sequence of applications of inference rules to axioms and premises

- Premises (wwfs that are assumed to be true for a specific proof)
- wffs (derived from and previously deducted wffs using inference rules)
- Conclusion (a wff that we want to derive from premises)

Proofs - Proving Results

- Notation: LHS $\vdash$ RHS, if there is a proof from LHS to RHS.
  - Wffs in LHS are assumed to be true.
  - We say that RHS is derived (deducted/proved) from RHS

  $p \wedge r, p \rightarrow q \vdash q$

  - We say that q is derived (deducted/proved) from $p \wedge r$ and $p \rightarrow q$

- Theorems: Formulas that can be proved, using inference rules applied to axioms, and theorems already proved.

  $\vdash \phi$ where ϕ is a theorem

Proof Questions-1: $\phi ::= p \mid (\phi + \phi) \mid (\neg \phi)$

Axioms

- Axioms are the formulas that are assumed to be true in the logic.
- Can you give some axioms in this logic? (using schemas - generic versions of formulas)

1. $(\phi + \neg\phi)$
2. $(\neg\phi + \phi)$
3. $\neg(\phi + \phi)$
4. $\neg(\neg\phi + \neg\phi)$

Proof Questions-2: $\phi ::= p \mid (\phi + \phi) \mid (\neg \phi)$

Inference Rules

- Can you give some inference rules in this logic? (using schemas – generic versions of formulas)

$\varphi + \psi$ φ --------- rule1 $\neg \psi$	$\varphi + \psi$ ψ --------- rule2	$\varphi + \psi$ $\neg \varphi$ --------- rule3 ψ
φ $\neg \psi$ --------- rule4 $(\varphi + \psi)$	$\neg \varphi$ ψ --------- rule5 $(\varphi + \psi)$	$\varphi + \psi$ $\neg \psi$ --------- rule6 (φ)
$\neg \neg \varphi$ --------- rule7 (φ)		

Proof Questions-3: $\phi ::= p \mid (\phi + \phi) \mid (\neg \phi)$

- Can you give a proof p3 from p1, p1+p2, p2+p3?

  p1, p1+p2, p2+p3 $\vdash$ p3 ???

1. p1 premise
2. p1 + p2 premise
3. p2 + p3 premise
4. ¬ p2 from 1 and 2 via rule 1
5. p3 from 3 and 4 via rule 3
6. p1, p1+p2, p2+3 |- p3

Semantics
Logical Consequence Relation

LHS ⊨ RHS

(read "LHS implies (logically implies) RHS")

if RHS is mapped to True under the conditions described by LHS.

Example:

- p ^ r, p → q ⊨ q
- This means that p ^ r, p → q logically implies q.
- Whenever p ^ r and p → q are True under the conditions, q is also True under those conditions.

Semantics Questions-1: Logical Consequence

$\phi ::= p \mid (\phi + \phi) \mid (\neg \phi)$

- Is ¬ p2 a logical consequence of p1, p1+p2 ?

p1, p1+p2 ⊨ ¬ p2 ???

1. p1 and p1+p2 are give as premises and are both assumed True
2. For p1+p2 to be True, one proposition must be True and the other False
3. Because p1 is already assumed True, p2 must be False
4. Therefore, ¬ p2 is a logical consequence of p1, p1+p2
5. p1, p1+p2 |= ¬ p2