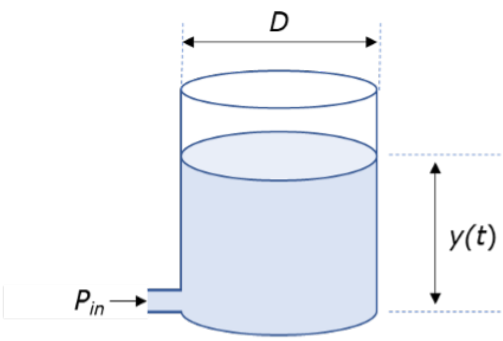
Laboratory 2: Runge-Kutta Methods

In this lab, you will study a family of iterative numerical algorithms known as the **Runge-Kutta** methods. In lecture, you will learn that the Runge-Kutta methods are useful for numerically approximating the solution to initial value problems; initial value problems occur frequently in physical linear systems, and Runge-Kutta are used when one cannot easily solve a linear system analytically. Unlike the other labs we will do this semester, this lab will *not* involve collecting data; you will submit a short formal report along with your code.



(a) **Tank diagram.**

(b) **Pipe diagram.**

Figure 1: **Fluid flow diagrams for Task 1.**

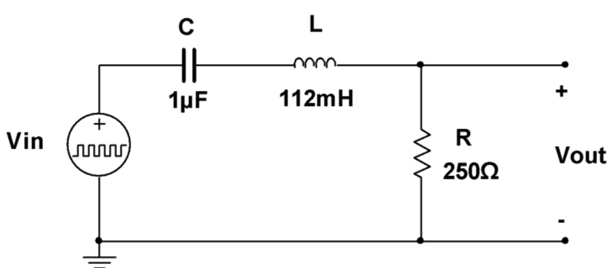


Figure 2: **RLC circuit used in Task 2.**

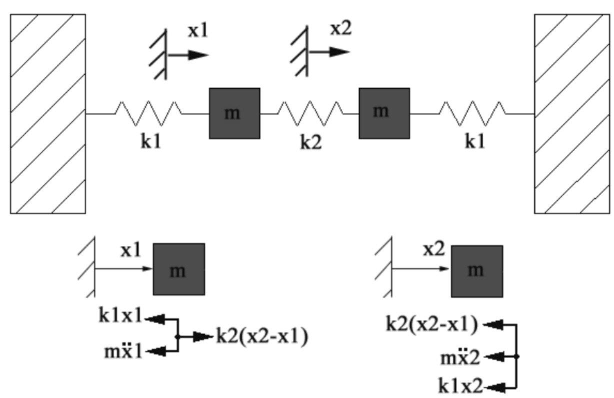


Figure 3: **Mechanical system used in Task 3.**

1 Lab instructions

This lab is divided into five tasks.

1.1 Task 0: Warm-up

Consider the following first-order ODE from the lecture slides:

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) = -2y(t) \tag{1.1}$$

with the initial condition $y(0) = 3$. The exact solution to this ODE is $y(t) = 3e^{-2t}$. Write a first-, second-, and fourth-order Runge-Kutta solver for this ODE.

1.2 Task 1: Fluid mechanics

Consider the fluid flow system shown in Figure 1. Assume that the *cylindrical* tank is empty at time $t = 0$, after which a pump at the entrance of the pipe instantaneously changes the pressure from 0 to P_{in} and fluid starts to flow through the pipe into the tank. The pump is run at 20 p.s.i. (pounds per square inch) with a flow rate of 4 g.p.m. (gallons per minute). In S.I. units, this translates to $P_{\text{in}} = 1.379 \times 10^5$ Pa and $K = 5.464 \times 10^8$ Pa$/(m^3/s)$. The diameter of the tank is 1 m, and the density of the fluid is $\rho = 997.0$ kg$/m^3$. The acceleration due to gravity is $g = 9.8\ m/s^2$.

To complete this task, you need to know two facts about fluid mechanics.

Fact 1: Pressure in a tank. In general, the pressure of the tank at a fixed depth h measured from the surface is described by $P(h) = \rho \cdot g \cdot h$, where g is the acceleration due to gravity. Since fluid enters at the bottom of the tank, the pressure depends on the depth $y(t)$ of the liquid, meaning that

$$P_{\text{bottom}}(t) = \rho \cdot g \cdot y(t). \tag{1.2}$$

where $P_{\text{bottom}}(t)$ denotes the pressure at the bottom of the tank. Note that $P_{\text{bottom}}(t)$ is expressed as a function of the time t since the height $y(t)$ of the fluid is changing over time.

Fact 2: Volume rate of flow through a pipe. In general, the volume rate of flow through a pipe is proportional to the pressure difference between the ends of the pipe. Assuming that a fluid is flowing from left to right (as in Figure 1b), the relationship between fluid flow and pressure can be written as

$$K \cdot F = P_1 - P_2 \tag{1.3}$$

where K is a proportionality constant related to the friction of the fluid flow through the pipe (due to the viscosity of the liquid and the roughness of the pipe interior), F is the volume flow rate, and P_1 and P_2 are the pressures at the left and right ends of the pipe respectively. Note that the volume flow rate F is defined to be the rate at which the volume of a liquid changes as a function of time, i.e.,

$$F = \frac{\mathrm{d}}{\mathrm{d}t}V(t) \tag{1.4}$$

where $V(t)$ denotes the volume of the liquid as a function of time.

Given these preliminaries, complete the following items.

1. *Differential equation.* Derive the first-order ODE describing the depth of the water in the tank.
2. *Solution.* Is the ODE analogous to any of the ODEs we saw in Lab 1? What are the analogous variables and constants? What is the solution and time constant τ?
3. *First-order Runge-Kutta.* Write a first-order Runge-Kutta solver to find the step response.
   - *Ground truth.* Let τ denote the time constant. Plot the exact solution at increments of $h = \tau/100$ until the system comes to equilibrium (which should be at around $t = 5 \cdot \tau$).
   - *Discrepancy definition.* Define a notion of discrepancy (or, if you like, error) between a numerical solution and the exact solution.
   - *Runge-Kutta.* Run the first-order Runge-Kutta algorithm with a step size h such that the discrepancy is no more than 5% (or a threshold of your choosing).
4. *Second-order Runge-Kutta.* Repeat step 3 with the second-order Runge-Kutta algorithm.
5. *Fourth-order Runge-Kutta.* Repeat step 3 with the fourth-order Runge-Kutta algorithm.

1.3 Task 2: Electrical system

Consider the RLC circuit shown in Figure 2.

1. *Differential equation.* Derive the second-order ODE describing the output voltage of the circuit.
2. *Solution.* Ensure that the exact solution for the impulse response of your ODE is as follows:

$$V_R(t) = -\frac{2\zeta\omega_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin\left(\sqrt{1-\zeta^2}\omega_0 t - \phi\right) \tag{1.5}$$

   where $\omega_0 = {}^1/\sqrt{LC}$, $\zeta = ({}^R/2)\sqrt{C/L}$, and $\phi = \arctan(\sqrt{1-\zeta^2}/\zeta)$.

3. *Runge-Kutta.* Repeat steps 3-5 of Task 1, assuming that the circuit is initially uncharged.

1.4 Task 3: Mechanical system

Consider the mechanical system shown in Figure 3. The state consists of two positions: x_1 and x_2. Both blocks have mass m and are coupled to the walls with two springs (with spring constants K_1) and to each each other with a third spring (with spring constant K_2).

1. *Differential equation.* The fourth-order ODE for this system is as follows:

$$m\ddot{x}_1 + (K_1 + K_2)x_1 - k_2 x_2 = 0 \tag{1.6}$$

$$m\ddot{x}_2 + (K_1 + K_2)x_2 - k_2 x_1 = 0 \tag{1.7}$$

2. *State space.* Use the state-space method to write the above ODE as a first-order ODE.
3. *Solution.* The exact solution is given by

$$x_1 = \frac{1}{2}\left(\cos\left(\omega_1 t\right) + \cos\left(\omega_2 t\right)\right) \tag{1.8}$$

$$x_2 = \frac{1}{2}\left(-\cos\left(\omega_1 t\right) + \cos\left(\omega_2 t\right)\right) \tag{1.9}$$

   where $\omega_1 = \sqrt{68}$ and $\omega_2 = 8$.

4. *Runge-Kutta.* Run fourth-order Runge-Kutta, assuming the following sets of initial conditions:
   (a) $x_1(0^+) = 1$ and $x_2(0^+) = -1$
   (b) $x_1(0^+) = 1$ and $x_2(0^+) = 1$

   Note that this system does not come to equilibrium, and so all of your plots should show a fixed time budget (e.g., 30 seconds).

5. *Questions.* Why do both sets of initial conditions yield simple harmonic motion? Why does one set of initial conditions result in a higher frequency of oscillation than the other?

1.5 Task 4: Simulink

Make a neat Simulink simulation diagram of the system from Task 3, and run it with all three sets of initial conditions. Since you have two symmetric equations, consider implementing one equation (i.e., for x_1) above the other (i.e., x_2) to ensure neatness. A primer on Simulink is provided on Moodle.

2 Your report

For this lab, each group will submit a single formal report. As a group, you are expected to share the work equally; every member of the lab group must read and approve of the content, formatting, and writing for each section of the lab report. The formatting requirements for this report are as follows.

- Section 1: **Title (5 points)**
  - ☐ Title of the lab.
  - ☐ Date of submission.
  - ☐ Names of the members of your lab group.
  - ☐ Emails of the members of your lab group.
- Section 2: **Abstract (5 points)**
  - ☐ A short paragraph describing the main question(s) considered in the lab, the experiments you conducted, and the main results. The abstract should be no more than 5-6 sentences.
- Section 3: **Introduction (5 points)**
  - ☐ Motivation: Explain why Runge-Kutta methods are important and what they are used for, ensuring that you cite any sources that you use.
  - ☐ Purpose: Explain the purpose of this lab and the main question(s) under consideration.
  - ☐ Experiments: Explain (at a high level) the experiments you performed and why you believe they address the main question(s) considered in this lab.
  - ☐ Results: Explain (at a high level) your main results and whether they agree with your theoretical expectation.
- Section 4: **Preliminaries (5 points)**
  - ☐ Derive the ODEs for each task in this lab. State each of the exact solutions, and answers to the questions in each task description above.
- Section 5: **Methods (5 points)**

- ☐ Describe the steps you followed to produce your results. You may refer to the lab handout (which you should properly cite) and report any deviations from the specified procedure.
- ☐ Motivate and describe the definition of discrepancy you defined in Task 1.
- ☐ Include a screenshot and a brief description of your Simulink model for Task 4.

- Section 6: **Experimental results (35 points)**
  - ☐ Tasks 0, 1, and 2.
    - ☐ Well-commented code for each Runge-Kutta method (Appendix B).
    - ☐ A table showing the largest value of h that gave a "good" discrepancy.
    - ☐ A graph with three subplots: (1) the exact solution, (2) a simulation with "good" discrepancy, and (3) a simulation with "bad" discrepancy. For each plot, you should show the individual points as well as a line linking these points.
  - ☐ Task 3.
    - ☐ Well-commented code for the simulation of the exact solution with initial conditions $x_1(0^+) = 1$ and $x_2(0^+) = 0$ (Appendix B).
    - ☐ A labeled plot of the exact solution which shows both variables.
    - ☐ A table showing the largest value of h that gave a "good" discrepancy.
    - ☐ A graph showing the results of your Runge-Kutta simulation for the initial conditions $x_1(0^+) = 1$ and $x_2(0^+) = -1$. There is no need to include code.
    - ☐ A graph showing the results of your Runge-Kutta simulation for the initial conditions $x_1(0^+) = 1$ and $x_2(0^+) = 1$. There is no need to include code.
  - ☐ Task 4
    - ☐ Plot your Simulink simulations for each set of initial conditions.
- **Section 7: Discussion (5 points)**
  - ☐ For each task, discuss how the step size h was chosen, and the relative difficulty of finding a "good" or "bad" discrepancy.
  - ☐ Qualitatively compare the simulations from Task 3 and Task 4.
- Section 8: **Conclusion (5 points)**
  - ☐ Summarize your report—the motivation, experiments, and main results—in one or two short paragraphs.
- Section 9: **Acknowledgments (5 points)**
  - ☐ If you discussed this lab with anyone other than the members of your lab group, acknowledge them here. Include a brief description of what was discussed with each person.
- Section 10: **References (5 points)**
  - ☐ Properly cite any sources (including this document) that you used when for this lab.
- Section 11: **Appendices (5 points)**
  - ☐ Appendix A: Include the statement of each lab member's contribution(s) to this lab.
  - ☐ Appendix B: Include the code used to fit the data in each task.

In addition to including the aforementioned sections, a portion of your grade for this lab will depend on the writing quality and formatting of this report.

- **Formatting (5 points)**
  - ☐ *Figures:* Each figure is labeled and is referenced in the main text. All figures include a caption directly below the figure which describes the main idea in 1-2 short sentences.
  - ☐ *Plots:* All plots include a title and, if appropriate, a legend. The vertical and horizontal axes are labeled and units are included (if appropriate).
  - ☐ *Sections:* The paper is broken up into clearly defined and numbered sections (e.g., for the introduction, experiments, conclusion, etc.).
- **Grammar, spelling, and conciseness (5 points)**
  - ☐ *Grammar/Spelling:* The report is generally free of grammatical and spelling mistakes.
  - ☐ *Conciseness:* The report is concise; the writing is clear and matter of fact.
- **Audience & reproducibility (5 points)**
  - ☐ *Audience:* Readers with technical knowledge of the subject, but who otherwise have no knowledge of the lab, should be able to understand your report.
  - ☐ *Reproducibility:* The report is written such that a knowledgeable reader could reproduce your results after reading your report and the references therein.