Digital Signal Processing

Class 13
03/04/2025

ENGR 71

- Class Overview
  - Frequency Analysis of Discrete Signals
- Assignments
  - Reading:
    Chapter 4: Frequency Analysis of Signals

ENGR 71

- New Lab
  - Using frequency domain features for classification

Frequency Analysis for Discrete Signals

- Frequency analysis for discrete signals:
  - Three transforms to consider:
    - Discrete Time Fourier Transform - DTFT
      - Fourier transform of sampled signal
    - Discrete Time Fourier Series - DTFS
      - Fourier series of sampled periodic signal
    - Discrete Fourier Transform – DFT
      - Create periodic extension of finite sequence
      - Then find the Fourier series.
      - This is the transform that is most often used
      - Fast algorithm to compute: Fast Fourier Transform (FFT)

Discrete-time Fourier transform

- Discrete Time Fourier Transform (DTFT)
  - Fourier transform of sampled signal

$$x_s(t) = \sum_n x(nT_s)\delta\left(t - nT_s\right)$$

$$\mathcal{F}\left\{x_s(t)\right\} = \sum_n x(nT_s)\mathcal{F}\left\{\delta\left(t - nT_s\right)\right\} = \sum_n x(nT_s)e^{-jn\Omega T_s}$$

$$\left[\text{Using } \mathcal{F}\left\{\delta(t)\right\} = 1 \text{ and shift property } \mathcal{F}\left\{x(t-\tau)\right\} = X\left(\Omega\right)e^{-j\Omega\tau}\right]$$

Ω is the analog frequency variable

Note that: $\mathcal{F}\left\{x_s(t)\right\}$ is periodic:

$$\sum_n x(nT_s)e^{-jn\Omega T_s} = \sum_n x(nT_s)e^{-jn\left(\Omega + \frac{2\pi k}{T_s}\right)T_s}$$

So, the spectrum of a sampled signal is periodic

Discrete-time Fourier transform

– Discrete Time Fourier Transform (DTFT)

Define $\omega = \Omega T_s$ as the frequency of the discrete signal (in radians)

and define $x[n] = x(nT_s)$ as samples of the sampled signal

- Fourier transform of sampled signal

$$\boxed{\begin{aligned} X(\mathrm{e}^{j\omega}) &= \sum_n x[n]e^{-j\omega n} \qquad -\pi \le \omega < \pi \\ x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \end{aligned}}$$

- Note that this is a continuous function in the variable ω
- Measures frequency content of discrete signal
(Discrete frequency is in radians)
- DTFT is periodic in frequency ω

$$X(\mathrm{e}^{j(\omega+2\pi k)}) = \sum_n x[n]e^{-j(\omega+2\pi k)n} = \sum_n x[n]e^{-j\omega n}e^{-j2\pi k} = \sum_n x[n]e^{-j\omega n} = X(\mathrm{e}^{j\omega})$$

Discrete-time Fourier transform

– DTFT exists if sequence is absolutely summable

$$\left|X(\mathrm{e}^{j\omega})\right| \le \sum_n \left|x[n]\right| \left|e^{-j\omega n}\right| = \sum_n \left|x[n]\right| < \infty$$

– Relationship of z-transform to DTFT:

$$X(z)\Big|_{z=e^{j\omega}} = \sum_n x[n]\, z^{-n}\Big|_{e^{j\omega}} \quad \Rightarrow \quad \sum_n x[n] e^{-j\omega n} = X(\mathrm{e}^{j\omega})$$

– i.e. Z-transform computed on unit circle.
(Region of Convergence (ROC) must include unit circle.)

Discrete-time Fourier transform

- Eigenfunctions and the DTFT
  - Suppose input to system is $x[n]=e^{j\omega_o n}$
  - Output is

$$y[n]=\sum_k h[k]x[n-k]=\sum_k h[k]e^{j\omega_o(n-k)}$$

$$=e^{j\omega_o n}\sum_k h[k]e^{-j\omega_o k}=H\left(e^{j\omega_o}\right)e^{j\omega_o n}$$

  - Output is same as input multiplied by DTFT of the impulse response
  - That is to say, $x[n]=e^{j\omega_o n}$ are eigenvectors of systems with eigenvalues of $H\left(e^{j\omega_o}\right)$, the DTFT evaluated at ω_0

Discrete-time Fourier transform

- Since DTFT can be obtained from z-transform
  - Has same properties for time shifts, convolution, etc.
  - Expressed in terms of $e^{-j\omega}$ instead of z

Discrete-time Fourier transform

Discrete-time Fourier Transforms (DTFT)

	Discrete-time signal	DTFT $X(e^{j\omega})$, periodic of period 2π
(1)	$\delta[n]$	$1, -\pi \leq \omega < \pi$
(2)	A	$2\pi A\delta(\omega), -\pi \leq \omega < \pi$
(3)	$e^{j\omega_0 n}$	$2\pi\delta(W - \omega_0), -\pi \leq \omega < \pi$
(4)	$\alpha^n u[n], \lvert\alpha\rvert < 1$	$\frac{1}{1-\alpha e^{-j\omega}}, -\pi \leq \omega < \pi$
(5)	$n\alpha^n u[n], \lvert\alpha\rvert < 1$	$\frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}, -\pi \leq \omega < \pi$
(6)	$\cos(\omega_0 n)u[n]$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)], -\pi \leq \omega < \pi$
(7)	$\sin(\omega_0 n)u[n]$	$-j\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)], -\pi \leq \omega < \pi$
(8)	$\alpha^{\lvert n\rvert}, \lvert\alpha\rvert < 1$	$\frac{1-\alpha^2}{1-2\alpha\cos(\omega)+\alpha^2}, -\pi \leq \omega < \pi$
(9)	$p[n] = u[n + N/2] - u[n - N/2]$	$\frac{\sin(\omega(N+1)/2)}{\sin(\omega/2)}, -\pi \leq \omega < \pi$
(10)	$\alpha^n\cos(\omega_0 n)u[n]$	$\frac{1-\alpha\cos(\omega_0)e^{-j\omega}}{1-2\alpha\cos(\omega_0)e^{-j\omega}+\alpha^2 e^{-2j\omega}}, -\pi \leq \omega < \pi$
(11)	$\alpha^n\sin(\omega_0 n)u[n]$	$\frac{\alpha\sin(\omega_0)e^{-j\omega}}{1-2\alpha\cos(\omega_0)e^{-j\omega}+\alpha^2 e^{-2j\omega}}, -\pi \leq \omega < \pi$

Discrete-time Fourier transform

Properties of the DTFT

Z-transform:	$x[n], X(z), \|z\| = 1 \in ROC$	$X(e^{j\omega}) = X(z)\|_{z=e^{j\omega}}$
Periodicity:	$x[n]$	$X(e^{j\omega}) = X(e^{j(\omega+2\pi k)}),\ k\ integer$
Linearity:	$\alpha x[n] + \beta y[n]$	$\alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
Time-shifting:	$x[n-N]$	$e^{-j\omega N} X(e^{j\omega})$
Frequency-shift:	$x[n]e^{j\omega_o n}$	$X(e^{j(\omega-\omega_0)})$
Convolution:	$(x * y)[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication:	$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Symmetry:	$x[n]$ real-valued	$\|X(e^{j\omega})\|$ even function of ω
		$\angle X(e^{j\omega})$ odd function of ω
Parseval's relation:	$\sum_{n=\infty}^{\infty} \|x[n]\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|X(e^{j\omega}\|^2 d\omega$	

Discrete-time Fourier series

- Consider the frequency representation of a periodic sequence where N is the period. $x[n+kN]=x[n]$
  - A periodic sequence can be represented in terms of a sum over basis functions:

$$\phi[k,n]=e^{j2\pi kn/N} \quad \left(\text{Different notation, but same as } s_k(n) \text{ in Proakis and Manolakis}\right)$$

  - These basis functions are periodic in k and n with period N
    - Easy to show. Substitute $k = k + rN$; substitute $n = n + rN$ where r is an integer
  - Basis functions are orthogonal over period N

$$\sum_{n=0}^{N-1}\phi[k,n]\times\phi^*[l,n]=\sum_{n=0}^{N-1}e^{j\frac{2\pi}{N}kn}\times e^{-j\frac{2\pi}{N}ln}=\begin{cases}N & k=l\\ 0 & k\neq l\end{cases}$$

Discrete-time Fourier series

– You can show orthogonality using our old friend, the geometric series, but not consider the finite geometric series:

$$\boxed{1+r+r^2+r^3+\cdots+r^{N-1}=\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r} \text{ for } r \neq 1}$$

$$\sum_{n=0}^{N-1} \phi[k,n]\times\phi^*[l,n]=\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn}\times e^{-j\frac{2\pi}{N}ln}=\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)n}$$

$$=\sum_{n=0}^{N-1}\left(e^{j\frac{2\pi(k-l)}{N}}\right)^n=\frac{1-e^{j\frac{2\pi(k-l)N}{N}}}{1-e^{j\frac{2\pi(k-l)}{N}}}=\frac{1-e^{j2\pi(k-l)}}{1-e^{j\frac{2\pi(k-l)}{N}}}=0 \text{ if } k\neq l$$

$$\boxed{\begin{array}{l}\text{If } k=l,\\ \displaystyle\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)n}=\sum_{n=0}^{N-1}\left(e^{j\frac{2\pi}{N}n}\right)^0=\sum_{n=0}^{N-1}1=N\end{array}}$$

Discrete-time Fourier series

– The orthogonality of $\phi[k,n] = e^{j2\pi kn/N}$ can be used to represent a periodic sequence (of period N) as:

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \quad \text{where} \quad X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

(The nomenclature in the book is different than that shown here. $c_k \equiv X[k]$)

which is the Fourier Series of $x[n]$.

The fundamental frequency is $\omega_o = 2\pi/N$

Notice that the frequency components for $X[k]$ are discrete

Both signal and Fourier series are discrete sequences.

(In contrast to Discrete Time Fourier Transform)

Discrete-time Fourier series

– Power spectrum

$$P_x = \frac{1}{N}\sum_{k=0}^{N-1}\left|x[n]\right|^2$$

Also

$$P_x = \frac{1}{N}\sum_{k=0}^{N-1}x[n]x^*[n] = \sum_{k=0}^{N-1}X^*[k]\left(\frac{1}{N}\sum_{k}^{N}x[n]e^{-j\frac{2\pi}{N}kn}\right)$$

$$P_x = \sum_{k=0}^{N-1}X^*[k]X[k]$$

$$P_x = \sum_{k=0}^{N-1}\left|X[k]\right|^2$$

$$\boxed{P_x = \frac{1}{N}\sum_{k=0}^{N-1}\left|x[n]\right|^2 = \sum_{k=0}^{N-1}\left|X[k]\right|^2}$$

Discrete-time Fourier series

– Energy spectrum

$$E_x = \sum_{k=0}^{N-1} \left| x[n] \right|^2$$

Also

$$E_x = \sum_{k=0}^{N-1} x[n] x^*[n] = \sum_{k=0}^{N-1} X^*[k] \left(\frac{N}{N} \sum_{k}^{N} x[n] e^{-j\frac{2\pi}{N}kn} \right)$$

$$E_x = N \sum_{k=0}^{N-1} X^*[k] X[k]$$

$$E_x = N \sum_{k=0}^{N-1} \left| X[k] \right|^2$$

$$\boxed{E_x = \sum_{k=0}^{N-1} \left| x[n] \right|^2 = \mathrm{N} \sum_{k=0}^{N-1} \left| X[k] \right|^2}$$

Discrete-time Fourier series

– Symmetry for real signals

$$X^*[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{+j\frac{2\pi}{N}kn} = X[-k]$$

$$\left|X[-k]\right| = \left|X[-k]\right|$$

$$-\measuredangle X[-k] = \measuredangle X[k]$$

$X[k]$ is also periodic

$$X[k+N] = X[k] \Rightarrow X[N-k] = X[-k]$$

$$\left|X[k]\right| = \left|X[N-k]\right|$$

$$\measuredangle X[k] = -\measuredangle X[N-k]$$

$$\left|X[0]\right| = \left|X[N]\right|$$

$$\left|X[1]\right| = \left|X[N-1]\right|$$

$$\left|X[N/2]\right| = \left|X[N/2]\right| \qquad N \text{ even}$$

$$\left|X[(N-1)/2]\right| = \left|X[(N+1)/2]\right| \qquad N \text{ odd}$$

$$\measuredangle X[0] = -\measuredangle X[N]$$

$$\measuredangle X[1] = -\measuredangle X[N-1]$$

$$\measuredangle X[N/2] = 0 \qquad N \text{ even}$$

$$\measuredangle X[(N-1)/2] = -\measuredangle X[(N+1)/2] \qquad N \text{ odd}$$

Discrete-time Fourier series

- Obtaining Fourier series coefficients for discrete sequences from the z-transform is similar to what you do for continuous signals from the Laplace transform.

For $x_1[n] = x[n]\big(u[n] - u[n-N]\big)$

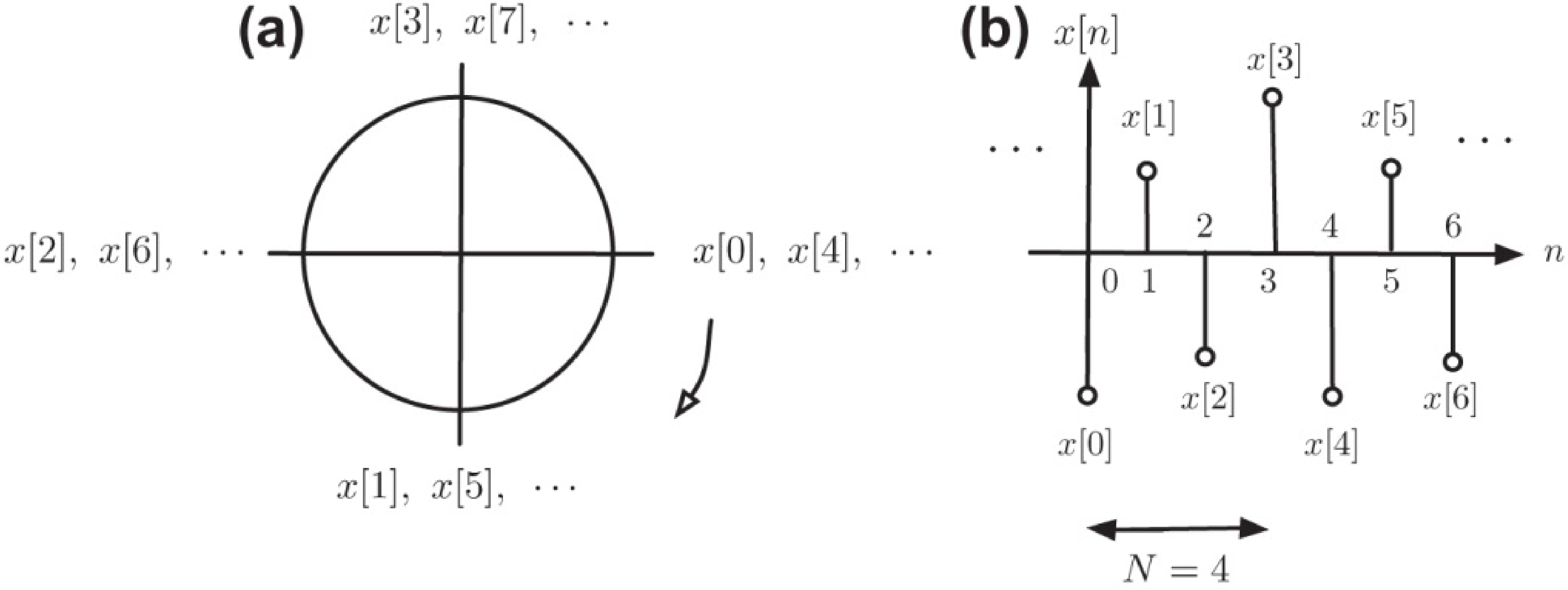
(i.e., one period of the periodic sequence $x[n]$)

$$Z\left\{x_1[n]\right\} = \sum_{n=0}^{N-1} x[n] z^{-n}$$

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} Z\left\{x_1[n]\right\}\Big|_{z=e^{j\frac{2\pi}{N}k}}$$

Discrete-time Fourier series

– For periodic sequences, it is convenient to think of the sequence values as being on circle



Discrete-time Fourier series

- Periodic convolution
  - For periodic sequence, convolution is a bit different
    - The product of two periodic sequences is also periodic
  - Periodic convolution:

$$v[n] = \sum_{m=0}^{N-1} x[m]\,y[n-m] \quad \Leftrightarrow \quad V[k] = N X[k] Y[k]$$

$$w[n] = x[n]y[n] \quad \Leftrightarrow \quad W[k] = \sum_{m=0}^{N-1} X[m]\,Y[n-m]$$

All are periodic with period N

Discrete-time Fourier series

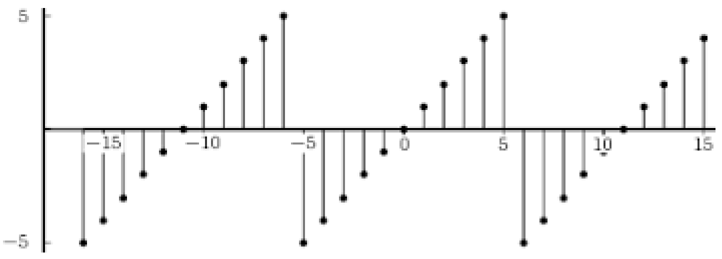
Fourier Series of Discrete-time Periodic signals		
	$x[n]$ periodic signal of period N	$X[k]$ periodic FS coefficients of period N
Z-transform	$x_1[n] = x[n](u[n] - u[n-N])$	$X[k] = \frac{1}{N}\mathcal{Z}(x_1[n])\big\vert_{z=e^{j2\pi k/N}}$
DTFT	$x[n] = \sum_k X[k]e^{j2\pi nk/N}$	$X(e^{j\omega}) = \sum_k 2\pi X[k]\delta(\omega - 2\pi k/N)$
LTI response	input $x[n] = \sum_k X[k]e^{j2\pi nk/N}$	output: $y[n] = \sum_k X[k]H(e^{jk\omega_0})e^{j2\pi nk/N}$ $H(e^{j\omega})$ (frequency response of system)
Time-shift (circular shift)	$x[n-M]$	$X[k]e^{-j2\pi kM/N}$
Modulation	$x[n]e^{j2\pi Mn/N}$	$X[k-M]$
Multiplication	$x[n]y[n]$	$\sum_{m=0}^{N-1} X[m]Y[k-m]$ periodic convolution
Periodic convolution	$\sum_{m=0}^{N-1} x[m]y[n-m]$	$NX[k]Y[n]$

Discrete Fourier Transform (DFT)

- The step from the Discrete Fourier Series to the Discrete Fourier Transform is a short one.
  - Consider a periodic sequence $x[n]$ (period N)
    - It has a Fourier series
  - Consider a finite length sequence $\mathrm{x}[n], \quad 0 \le n \le N-1$
  - One can think of making a periodic extension of this sequence and then take it's Fourier series.
    - This is essentially the Discrete Fourier transform
    - Except … traditionally, the 1/N goes with the sum over the DFT coefficients.

Discrete Fourier Transform

- Discrete Fourier Transform (DFT)
  - Signals may not be periodic, but are generally finite in length
    - In practice, all signals are finite.
    - If you are working with really long signals, you can always break it up into shorter length sections.
  - Although signal is not periodic, you can create a periodic extension of the signal by repeating the signal before and after real signal.
    - Create a periodic signal
  - You can then find the Discrete Time Fourier Series of the periodic extension of the signal



Discrete Fourier Transform

- Discrete Fourier Transform (DFT)
  - The DFT is usually written a little differently than the DTFS
  - For a finite length signal of length L, one often pads it out to a larger number of samples, N, that is L or greater:
  - The factor of 1/N is usually put with the "inverse" transform

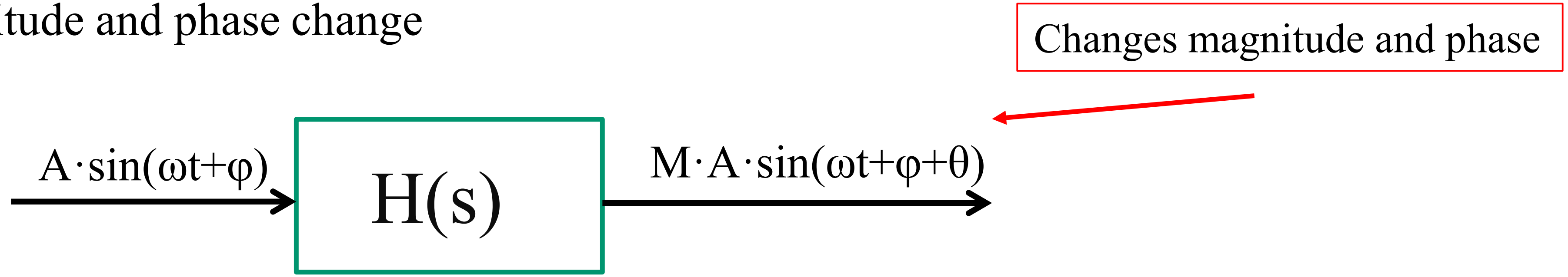
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi nk}{N}} \qquad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{\frac{j2\pi nk}{N}} \qquad 0 \le n \le N-1$$

We will discuss the Discrete Fourier Transform in more detail later

Frequency-Domain Analysis of LTI Systems

- Key concept behind frequency decomposition of signals:
  - Basis functions of sines and cosines (and complex exponential)
  - **Frequency components of signal are unchanged when passed through Linear Time Invariant systems**
    - Only amplitude and phase change



Frequency-Domain Analysis of LTI Systems

- Consider the Discrete-Time Fourier Transform of signals (Proakis & Manolakis refer to this just the Discrete Fourier transform in Chapter 5)
  - Frequency response completely characterizes LTI system
- We obtained the DTFT by taking the Fourier transform of sampled signal
- Previously, we considered the general expression for the DTFT for a signal, $x(n)$.

$$\boxed{\begin{aligned} &X(\mathrm{e}^{j\omega}) = \sum_{n} x(n) e^{-j\omega n} \qquad -\pi \le \omega < \pi \\ &x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \end{aligned}}$$

Frequency-Domain Analysis of LTI Systems

- Now, we concentrate on the frequency response of an LTI system
  - The response of an LTI system to any input is:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

  - Considering a complex exponential input $\quad x(n) = Ae^{j\omega n}$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)Ae^{j\omega(n-k)} = A\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]e^{j\omega n}$$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$y(n) = AH\left(\omega\right)e^{j\omega n}$$

Frequency-Domain Analysis of LTI Systems

– This shows that complex exponentials are the eigenfunctions and $H(\omega)$ are the eigenvalues of an LTI system.

– Since any signal can be decomposed into complex exponentials, $H(\omega)$ completely characterizes the LTI system.

– Example of how the system modifies the amplitude and phase of a sinusoidal input but not the frequency:

- Impulse response of system is

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

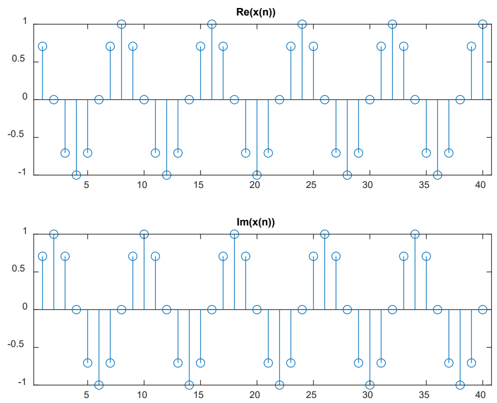
$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n) e^{-j\omega n} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} = \sum_{k=0}^{\infty} \left(\frac{1}{2} e^{-j\omega}\right)^n = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

Frequency-Domain Analysis of LTI Systems

– What does the system do to an complex exponential input (i.e. and input at some particular frequency)

- Consider an input with a frequency of π/4

$$x(n) = Ae^{j\omega n} = Ae^{jn\pi/4}$$



$$H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}, \quad H(\pi/4) = \frac{1}{1 - \frac{1}{2}e^{-\frac{j\pi}{4}}}$$

$$|H(\pi/4)| = 1.3572, \quad \phi = -28.68^{\circ}$$

- Example in book shows:

$$|H(\pi/2)| = 0.8944 \quad \phi = -26.6^{\circ}$$

$$|H(\pi)| = 0.6667 \quad \phi = 0^{\circ}$$

Frequency-Domain Analysis of LTI Systems

– If an LTI system changes the magnitude and phase of an input
  - You can begin to see how filtering works
  - Consider what the LTI does to each frequency component

Example of a moving average filter:

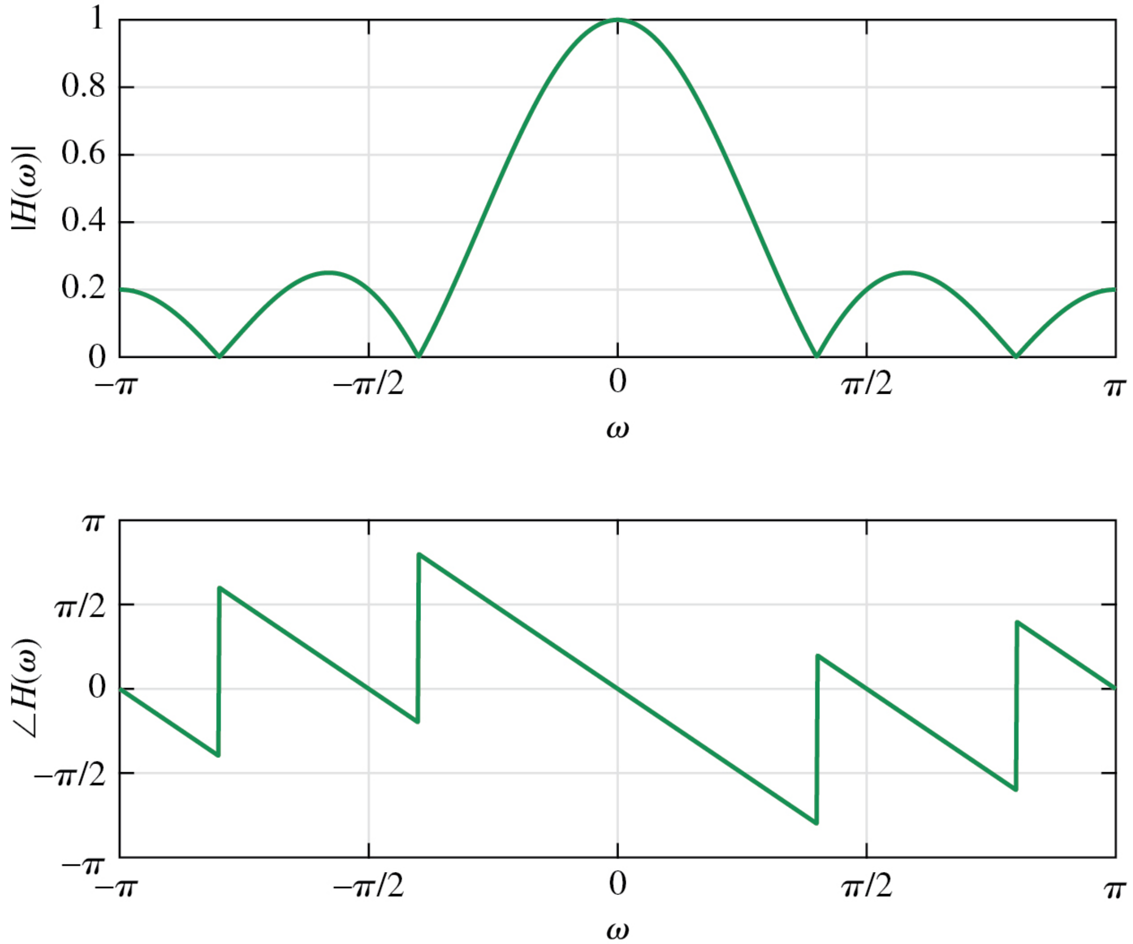
$$y(n) = \frac{1}{M+1} \sum_{k=1}^{M} x(n-k)$$

Frequency response is (using the finite geometric series sum)

$$H(\omega) = \frac{1}{M+1} \sum_{k=0}^{M} e^{-j\omega k} = \frac{1}{M+1} \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}}$$

$$H(\omega) = \frac{1}{M+1} \frac{\sin\left(\omega(M+1/2)\right)}{\sin\left(\omega/2\right)} e^{-j\omega/2}$$

Frequency-Domain Analysis of LTI Systems



This is a low-pass filter

M=4

Frequency-Domain Analysis of LTI Systems

– Example with Infinite impulse response

$$y(n) = ay(n-1) + bx(n)$$

We have found the impulse response for this system a few times:

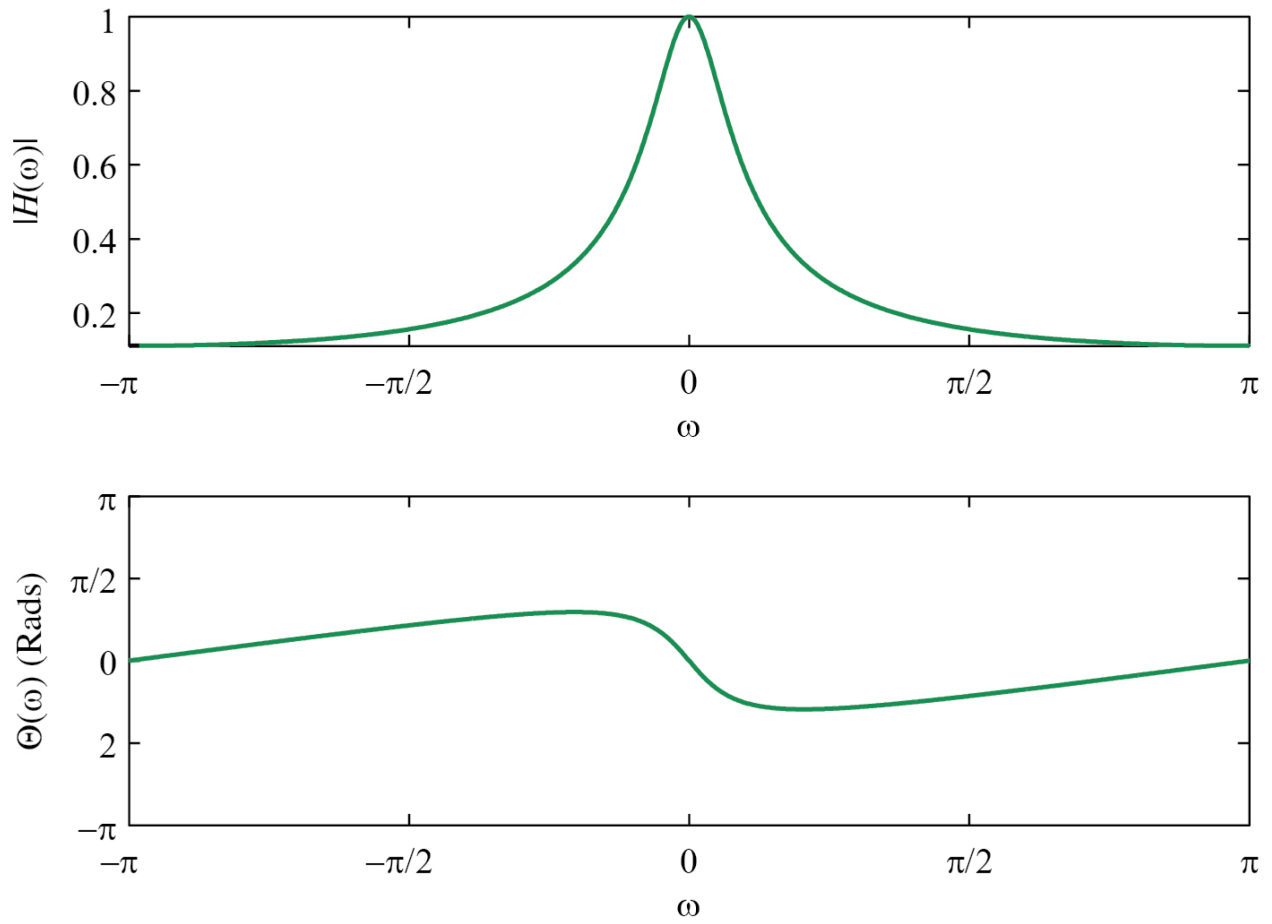
$$H(z) = \frac{b}{1 - az^{-1}}$$

$$H(\omega) = \frac{b}{1 - ae^{-j\omega}}$$

$$|H(\omega)| = \frac{1-a}{\sqrt{1 - 2a\cos\omega + a^2}}$$

$$\phi = -\tan^{-1}\left(\frac{a\sin\omega}{1 - a\cos\omega}\right)$$

Frequency-Domain Analysis of LTI Systems



This is a also low-pass filter

a=0.8

Frequency-Domain Analysis of LTI Systems

- Transient and steady-state response of system
  - Example

$$y(n) = ay(n-1) + x(n), \quad y(-1)$$

$$y(0) = ay(-1) + x(0)$$

$$y(1) = a[ay(-1) + x(0)] + x(1)$$

$$y(2) = a\big[a[ay(-1) + x(0)] + x(1)\big] + x(2)$$

$$\vdots$$

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k x(n-k)$$

Frequency-Domain Analysis of LTI Systems

- Transient and steady-state response of system
  - If the input is a complex exponential: $x(n) = Ae^{j\omega n}$

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k A e^{j\omega(n-k)} = a^{n+1}y(-1) + A\left[\sum_{k=0}^{n} a^k e^{-j\omega k}\right]e^{j\omega n}$$

$$y(n) = a^{n+1}y(-1) + A\left[\sum_{k=0}^{n}\left(ae^{-j\omega}\right)^k\right]e^{j\omega n}$$

These die off as n increases

Using sum of finite geometric series

Steady-state

$$y(n) = a^{n+1}y(-1) + A\left[\frac{1-\left(ae^{-j\omega}\right)^{n+1}}{1-ae^{-j\omega}}\right]e^{j\omega n} = a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1-ae^{-j\omega}}e^{j\omega n} + \frac{A}{1-ae^{-j\omega}}e^{j\omega n}$$

Frequency-Domain Analysis of LTI Systems

- Steady-state for periodic input
  - Discrete Fourier Series of input:

$$x(n)=\sum_{k=0}^{N-1} c_k e^{j2\pi k/N}$$

Output of each harmonic gets modified by: $H\left(\frac{2\pi k}{N}\right)$

$$y(n)=\sum_{k=0}^{N-1} c_k H\left(\frac{2\pi k}{N}\right) e^{j2\pi k/N}$$

so also periodic with modified Fourier Series coefficients

Frequency-Domain Analysis of LTI Systems

- Steady-state for aperiodic input
  - Use convolution to find output:

$$Y(\omega) = H(\omega)X(\omega)$$

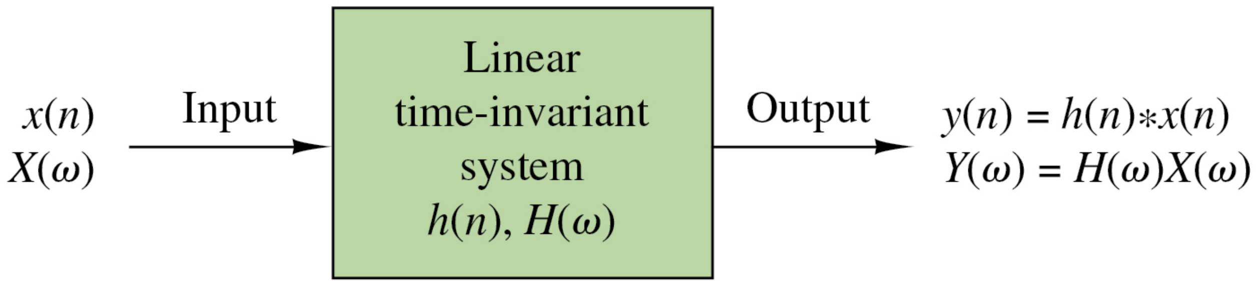
$$|Y(\omega)| = |H(\omega)||X(\omega)|$$

$$\measuredangle Y(\omega) = \measuredangle H(\omega) + \measuredangle X(\omega)$$

Energy Density:

$$|Y(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2$$

$$S_{yy} = |H(\omega)|^2 S_{xx}$$



Time for some applications rather than theory

- Two tasks presented which will be subject of the next lab
  1. Phone tones
  2. Speech recognition

Time for some applications rather than theory

- Phone tones
  - Can you figure out what numbers are being "dialed" by the tones they produce?
  - Tools to use:
    - Fourier transform to see if you can identify discrete frequency that can be associated with numbers on the "dial"
    - Segment tones for numbers "dialed" and try to map to numbers

Time for some applications rather than theory

- I have a set of *.mp3 files for numbers being dialed (show some code)

```
phone_number = 'phone_number_2.mp3';
[phn,fs] = audioread(phone_number);
phone_part1 = split(phone_number,'.');
phone_call = split(phone_part1{1},'_');
callnum = phone_call{3};
phn1 = phn(:,1);
Normalize amplitudes to have maximum value of 1
phn1 = phn1/max(abs(phn1));
tm = (1/fs)*[1:length(phn1)];
figure(1)
plot(tm,phn1);
xlabel('Time (sec)')
ylabel('Magnitude')
title(['Phone Number ',callnum])
```

```
segment = phn1(startseg(k):endseg(k));
tmseg = (1/fs)*[1:length(segment)];
figure
plot(tmseg,segment)
nsamp = length(segment);
fnyquist = fs/2;
x_mag = abs(fft(segment))/nsamp;
bins = [0:nsamp-1];
freq_hz = bins*fs/nsamp;
% Plot only positive frequencies
n_2 = ceil(nsamp/2);
figure()
plot(freq_hz(1:n_2), x_mag(1:n_2))
xlabel('Frequency (Hz)')
ylabel('Magnitude');
title('Single-sided Magnitude spectrum (Hertz)');
axis([0,2000,0,0.2])
```

Time for some applications rather than theory

- Speech recognition
  - There are some very sophisticated methods of speech recognition, which actually seem to work some time.
  - We won’t be using these.
  - Dataset with single words from google.
  - We will just try to distinguish two words, like “yes” and “no”
  - This will involve obtaining attributes in the frequency domain and using them in a classifier
    - A very neat tool in Matlab called classificationLearner
  - A good start for features is finding the power in some set of frequency bands
    - Need to normalize by total power