Digital Signal Processing

Class 11
02/25/2025

ENGR 71

- Class Overview
  - Analysis of Linear Time-Invariant Systems in the z-Domain
- Assignments
  - Reading:

    Chapter 3: The z-Transform and its Applications to the Analysis of LTI

    Chapter 4: Frequency Analysis of Signals

ENGR 71

- Homework 4
  - Problems: 3.2 (b & f), 3.4(d), 3.12,
    3.14(b), 3.16, 3.31
    C3.3 (use Matlab)
    C3.5 (use Matlab)

  Due Mar. 2

Class Information

- Z-Transform Topics
  - The z-Transform
  - Properties of the z-Transform
  - Rational z-Transforms
  - Inversion of the z-Transform
  - Analysis of Linear Time-Invariant Systems in the z-Domain
  - The One-sided z-Transform

Review of z-transform topics covered

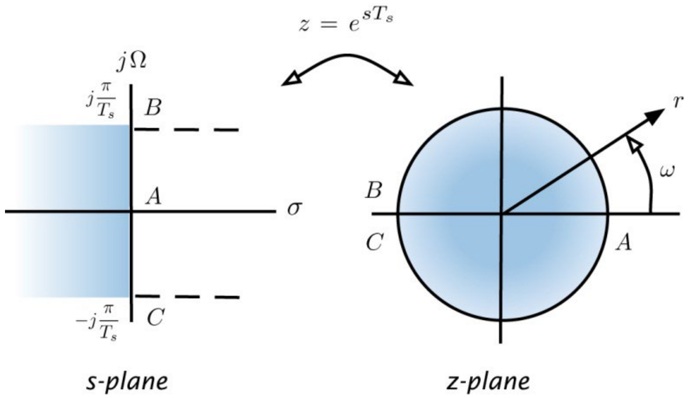
- Topics covered involving z-transform
  - Starting from system description as diagram
    - Determine difference equation
    - Find z-transform
    - Invert z-transform to find impulse or step response
  - Starting from system description as pole-zero map
    - Find z-transform
    - Invert z-transform to find impulse or step response
  - If you know impulse response for LTI system, you can find its response to any input through convolution.

Previous Class

- Worked through several z-transform examples
  - Ones discussed in class and additional examples can be found on Moodle page here:
    Time-domain signal from pole-zero map
    Z-Transform Examples
    Inverse z-transform with multiple-order pole

Laplace and z-Transforms

- Laplace and Z-transforms:
  - Laplace: $X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt \qquad x(t) = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{\gamma - jT}^{\gamma + jT} X(s) e^{st} ds$
  - z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$



  - Mainly concerned with causal signals and systems: $t \geq 0 \qquad n \geq 0$

$$x(t) = x(t)u(\mathrm{t})$$
$$x[n] = x[n]u[n]$$

    - Limits in sum and integral start at 0:

$$X(s) = \int_{0}^{+\infty} x(t) e^{-st} dt$$
$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Z-Transform

- Definition of z-transform:
  - Bilateral

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

  - Unilateral (causal signals & systems)

$$X(z) = \sum_{n=0}^{\infty} \mathrm{x}(n)\, z^{-n}$$

  - Inverse:

$$x(n) = \frac{1}{2\pi j} \oint X(z)\, z^{n-1} dz$$

    - Rarely use this, although this is common integral in complex variables math courses.
    - We compute forward & inverse by use of transform pairs and properties.
    - Can also find inverse by long division.

Z-Transform Inverse

- Complex (distinct) poles
  - Example: (Approach the same way as for real poles)

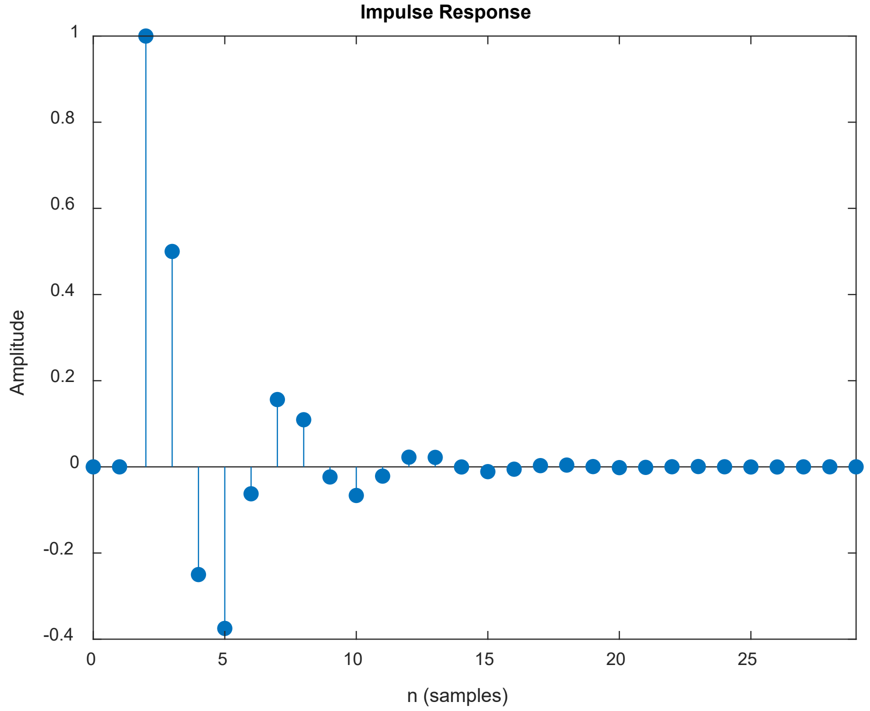
$$H(z) = \frac{z^{-2}}{1 - \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

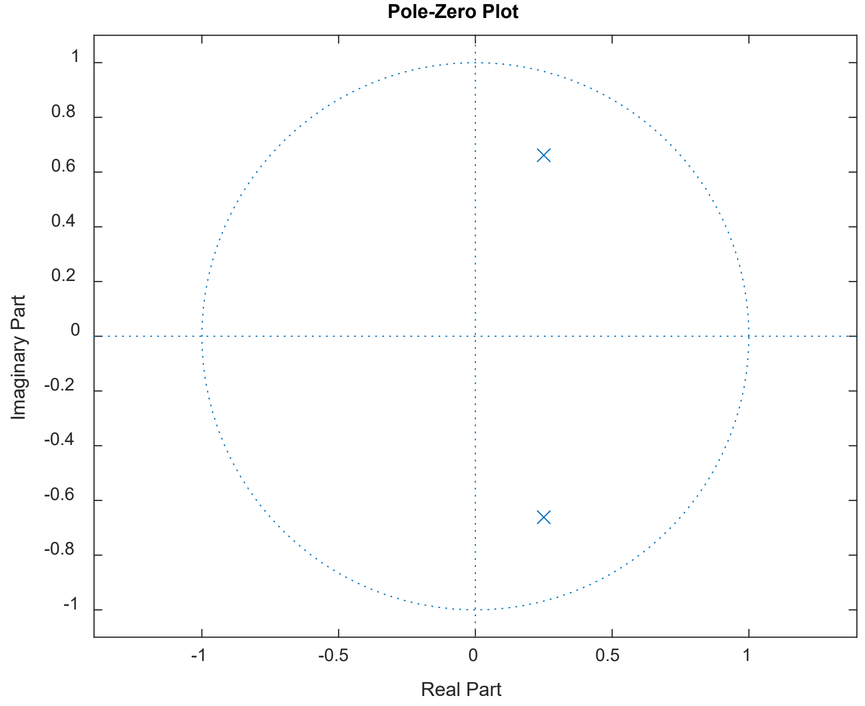
$$h(n) = 2\left[\delta(n) + \frac{2}{\sqrt{7}}\left(\frac{1}{\sqrt{2}}\right)^{(n-1)} \sin\left((n-1)\theta\right)u(n)\right]$$

Z-Transform Inverse

$$H(z)=\frac{z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

$$h(n)=2\left[\delta(n)+\frac{2}{\sqrt{7}}\left(\frac{1}{\sqrt{2}}\right)^{(n-1)}\sin\left((n-1)\theta\right)u(n)\right]$$





Complex conjugate pairs of poles

- If you have complex poles, they always occur in complex conjugate pairs
  - You can combine these to find that these poles give rise to oscillatory, sinusoidal terms

For complex conjugate pairs:

$$\frac{A}{1-pz^{-1}}+\frac{A^*}{1-p^*z^{-1}}$$

$$\mathcal{Z}^{-1}\left[\frac{A}{1-pz^{-1}}+\frac{A^*}{1-p^*z^{-1}}\right]=\left[A(p)^k+A^*\left(p^*\right)^k\right]u(n)$$

$$=\left[|A|e^{j\alpha}\left(|p|e^{j\beta}\right)^k+|A|e^{-j\alpha}\left(|p|e^{-j\beta}\right)^k\right]u(n)$$

$$=|A|r^k\left[e^{j(\beta k+\alpha)}+e^{-(\beta k+j\alpha)}\right]u(n)\quad\left(\text{where } r=|p|\right)$$

$$=2|A|r^k\cos\left(\beta k+\alpha\right)u(n)$$

Z-Transform Inverse

- Multiple-order poles
  - Example:

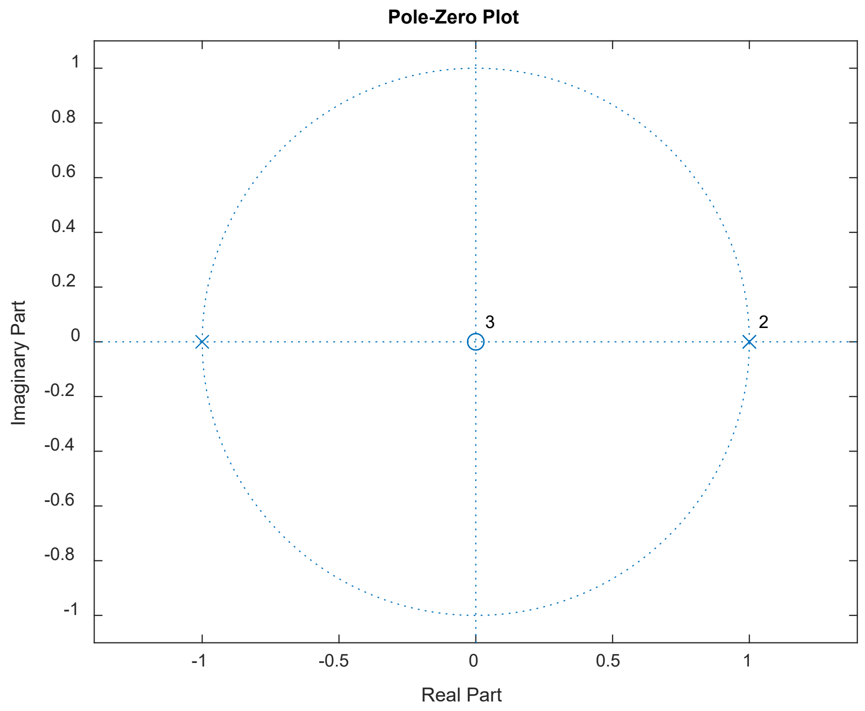
$$H(z) = \frac{1}{1 - z^{-1} - z^{-2} + z^{-3}}$$

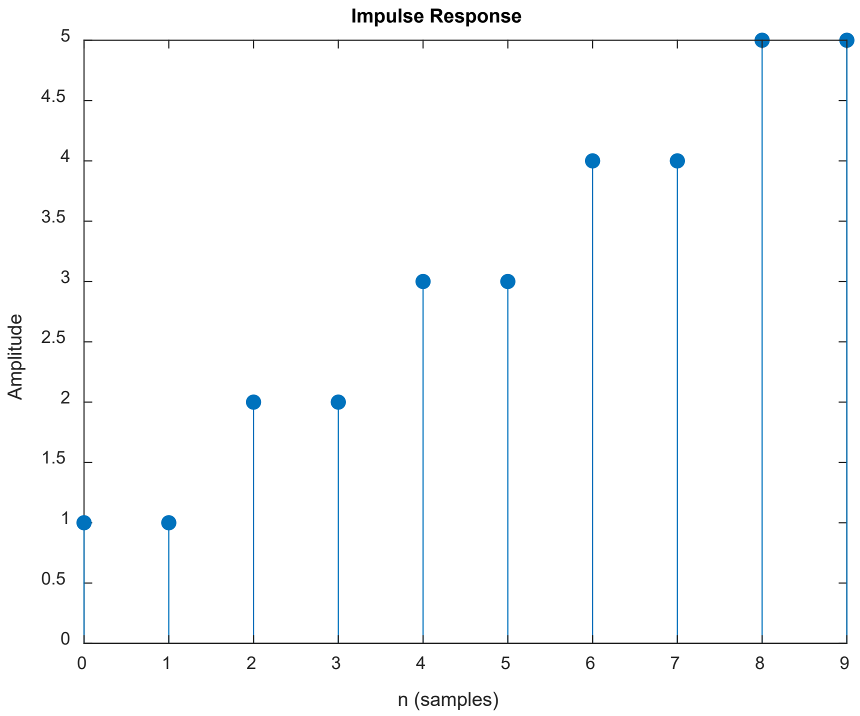
$$h(n) = \frac{1}{4}\left[\left(-1\right)^{n} + 2n + 3\right]u(n)$$

Z-Transform Inverse

$$H(z)=\frac{1}{1-z^{-1}-z^{-2}+z^{-3}}$$

$$h(n)=\frac{1}{4}\left[(-1)^{n}+2n+3\right]u(n)$$





Decomposition of Rational z-transform

- It is useful to decompose rational z-transforms into product of first-order and second-order terms:

$$H(z)=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}=b_0\frac{1+\left(b_1/b_0\right)z^{-1}+\cdots+\left(b_M/b_0\right)z^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}$$

$$H(z)=b_0\frac{\left(1-z_1z^{-1}\right)\left(1-z_2z^{-1}\right)\cdots\left(1-z_Mz^{-1}\right)}{\left(1-p_1z^{-1}\right)\left(1-p_2z^{-1}\right)\cdots\left(1-p_Mz^{-1}\right)}=b_0\frac{\prod_{k=1}^{M}\left(1-z_kz^{-1}\right)}{\prod_{k=1}^{N}\left(1-p_kz^{-1}\right)}$$

If $M>N$, do the usual division to get a sum of terms and a proper rational function

$$H(z)=\sum_{k=0}^{M-N}c_kz^{-k}+H_{pr}\left(z\right)$$

Decomposition of Rational z-transform

Break this out into real poles and complex conjugate pairs of poles

For complex conjugate pairs:

$$\frac{A}{1-pz^{-1}}+\frac{A^*}{1-p^*z^{-1}}=\frac{A\left(1-p^*z^{-1}\right)+A^*\left(1-pz^{-1}\right)}{\left(1-pz^{-1}\right)\left(1-p^*z^{-1}\right)}$$

$$\frac{A\left(1-p^*z^{-1}\right)+A^*\left(1-pz^{-1}\right)}{\left(1-pz^{-1}\right)\left(1-p^*z^{-1}\right)}=\frac{A-Ap^*z^{-1}+A^*-A^*pz^{-1}}{1-pz^{-1}-p^*z^{-1}+pp^*z^{-2}}$$

$$=\frac{b_0+b_1z^{-1}}{1+a_1z^{-1}+a_2z^{-2}}$$

(As shown, these second-order system give rise to sinusoidal terms in response)

$$b_0=2\operatorname{Re}(A)\quad,\quad a_1=-2\operatorname{Re}(p)$$

$$b_1=2\operatorname{Re}(A^*)\quad,\quad a_2=\left|p\right|^2$$

Decomposition of Rational z-transform

Now, write entire thing out in terms of real and complex poles (and delays if $M > N$)

$$H(z) = \sum_{k=0}^{M-N} c_k z^{-k} + \sum_{k=1}^{K_1} \frac{b_k}{1 + a_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{0k} + b_{1k} z^{-1}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

Decomposition of Rational z-transform

Recall that the overall z-transform in terms can be written terms of products of poles and zeros:

$$H(z) = b_0 \frac{\left(1 - z_1 z^{-1}\right)\left(1 - z_2 z^{-1}\right)\cdots\left(1 - z_M z^{-1}\right)}{\left(1 - p_1 z^{-1}\right)\left(1 - p_2 z^{-1}\right)\cdots\left(1 - p_M z^{-1}\right)} = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - p_k z^{-1}\right)}$$

You can collect the complex conjugate pairs in this form also.

For one pair of complex conjugate poles and zeros:

$$\frac{\left(1 - z_k z^{-1}\right)\left(1 - z_k^* z^{-1}\right)}{\left(1 - p_k z^{-1}\right)\left(1 - p_k^* z^{-1}\right)} = \frac{1 - z_k z^{-1} - z_k^* z^{-1} + z_k z_k^* z^{-2}}{1 - p_k z^{-1} - p_k^* z^{-1} + p_k p_k^* z^{-2}} = \frac{1 + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

where

$$b_{1k} = -2\,\mathrm{Re}\left(z_k\right) \quad , \quad a_{1k} = -2\,\mathrm{Re}\, p_k$$

$$b_{2k} = \left|z_k\right|^2 \quad , \quad \mathrm{a}_{2k} = \left|p_k\right|^2$$

Decomposition of Rational z-transform

Separating the parts with real and complex poles (and, assuming $M \leq \mathrm{N}$)

$$H(z) = b_0 \prod_{k=1}^{K_1} \frac{1 - b_k z^{-1}}{1 - a_k z^{-1}} \prod_{k=1}^{K_2} \frac{1 + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

Decompostion of systems into collections of first and secord order section:

$$H(z) = \sum_{k=1}^{K_1} \frac{b_k}{1 + a_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{0k} + b_{1k} z^{-1}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

$$H(z) = b_0 \prod_{k=1}^{K_1} \frac{1 - b_k z^{-1}}{1 - a_k z^{-1}} \prod_{k=1}^{K_2} \frac{1 + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

This will be useful with discussing filters

Analysis of LTI Systems in the z-Domain

- We’ve seen multiple ways of looking at systems:
  - System diagrams
  - Difference Equations in the time-domain
    - Impulse response
  - Transfer function in z-domain
    - Pole-zero maps

  (All are related)

Analysis of LTI Systems in the z-Domain

- Concentrate on systems described by linear difference equations with constant coefficients
  - Time domain:

$$y(n)+\sum_{k=1}^{N}a_k y(n-k)=\sum_{k=0}^{M}b_k x(n-k)$$

  - For any input, can find output response as convolution of unit impulse response with input:

$$y(n)=h(n)*x(n)$$

Analysis of LTI Systems in the z-Domain

- z-transform domain:

$$\left(1+a_1 z^{-1}+\cdots+a_N z^{-N}\right)Y(z)=\left(b_0+b_1 z^{-1}+\cdots+b_M z^{-M}\right)X(z)$$

  - For any input, can find output response as product of system response to unit impulse input:

$$Y(z)=H(z)X(z)$$

$$H(z)=\frac{b_0+b_1 z^{-1}+\cdots+b_M z^{-M}}{1+a_1 z^{-1}+\cdots+a_N z^{-N}}=\frac{B(z)}{A(z)}$$

Analysis of LTI Systems in the z-Domain

– Consider inputs that are also rational functions

$$X(z) = \frac{N(z)}{Q(z)}$$

– Example: sinusoidal input $x(n) = \cos\left(\omega_0 n\right) u(n) \;\Leftrightarrow\; X(z) = \dfrac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}$

- Output:

$$Y(z) = \frac{B(z)}{A(z)} \frac{N(z)}{Q(z)}$$

Analysis of LTI Systems in the z-Domain

– Writing transfer function and input in terms of poles and zeros:

  - assuming no initial conditions – This is zero-state response

(and no pole-zero cancellation)

$$Y(z)=\sum_{k=1}^{N}\frac{A_k}{1-p_k z^{-1}}+\sum_{k=1}^{L}\frac{Q_k}{1-q_k z^{-1}}$$

where p_k are poles of the system (transfer) function and q_k are poles of the input

A_k and Q_k have to be determined by partial fraction expansion

$\left(A_k\right.$ and Q_k are not to be confused with other use of A_k and Q_k in z-transform of $H(z)$ and $X(z)$.$\left.\right)$

In the time domain:

$$y(n)=\sum_{k=1}^{N}A_k\left(p_k\right)^n u(n)+\sum_{k=1}^{L}Q_k\left(q_k\right)^n u(n)$$

Analysis of LTI Systems in the z-Domain

In the time domain:

$$y(n) = \sum_{k=1}^{N} A_k \left(p_k\right)^n u(n) + \sum_{k=1}^{L} Q_k \left(q_k\right)^n u(n)$$

– Natural response:

$\sum_{k=1}^{N} A_k \left(p_k\right)^n u(n)$ (depends on input through A_k's)

– Forced response:

– $\sum_{k=1}^{L} Q_k \left(q_k\right)^n u(n)$ (depends on system through Q_k's)

Analysis of LTI Systems in the z-Domain

– Zero-state response has two parts:
  - Natural response
  - Forced response
– If all poles of system and input lie inside unit circle, $|p_k| < 1 \;\; \& \;\; |q_k| < 1$ response is transient (dies off as $n \to \infty$)
  – Poles of system should satisfy this condition if stable
  – Poles of input, not necessarily so
    Forced response goes on forever: steady-state response
    Poles of input are on unit circle:

$$\mathcal{Z}[\cos \omega_0 n] = \frac{1}{2}\left[\frac{1}{1 - e^{j\omega_0} z^{-1}} + \frac{1}{1 - e^{-j\omega_0} z^{-1}}\right]$$

Analysis of LTI Systems in the z-Domain

- Example:
- Find transient natural response and steady-state forced response of:

$$y(n) = \frac{1}{3} y(n-1) + x(n)$$
$$x(n) = \cos(\pi n / 6)$$

$$\left(\begin{aligned} \mathcal{Z}[\cos \omega_0 n] &= \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{1-e^{-j\omega_0}z^{-1}}\right] \\ \mathcal{Z}[\cos \omega_0 n] &= \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}} \end{aligned} \right)$$

Analysis of LTI Systems in the z-Domain

- Causality and Stability
  - Conditions for causal LTI system:

    $h(n) = 0, \quad \text{for } n < o$

    And, region of convergence (ROC) is exterior of some circle of radius r in the z-plane

  - Condition for stability of LTI

    $\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \Rightarrow \quad H(z)$ must contain unit circle within its ROC, since

    $$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \qquad \Rightarrow \qquad |\mathrm{H(z)}| \le \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}|$$

    evaluated on unit circle $(|z| = 1)$: $\quad |\mathrm{H(z)}| \le \sum_{n=-\infty}^{\infty} |h(n)| < \infty$

Analysis of LTI Systems in the z-Domain

- Causal and Stable
  - Causal, stable LTI converges for $|z|<1$
    so all poles must be inside the unit circle

Analysis of LTI Systems in the z-Domain

- Pole-zero cancellation
  - For system:

$$H(z) = b_0 \frac{\left(1 - z_1 z^{-1}\right)\left(1 - z_2 z^{-1}\right)\cdots\left(1 - z_M z^{-1}\right)}{\left(1 - p_1 z^{-1}\right)\left(1 - p_2 z^{-1}\right)\cdots\left(1 - p_M z^{-1}\right)}$$

  - If a pole is at the same location as zero (e.g.):

$$H(z) = b_0 \frac{\left(1 - z_1 z^{-1}\right)\cancel{\left(1 - p_2 z^{-1}\right)}\cdots\left(1 - z_M z^{-1}\right)}{\left(1 - p_1 z^{-1}\right)\cancel{\left(1 - p_2 z^{-1}\right)}\cdots\left(1 - p_M z^{-1}\right)}$$

reduces order of the system by 1

Analysis of LTI Systems in the z-Domain

- Pole-zero cancellation
  - A zero in the input could also cancel a pole in the transfer function and vice versa

$$H(z) = b_0 \frac{\left(1 - z_1 z^{-1}\right)\left(1 - z_2 z^{-1}\right)\cdots\left(1 - z_M z^{-1}\right)}{\left(1 - p_1 z^{-1}\right)\left(1 - p_2 z^{-1}\right)\cdots\left(1 - p_M z^{-1}\right)} \qquad X(z) = \frac{\left(1 - \alpha_1 z^{-1}\right)\left(1 - \alpha_2 z^{-1}\right)\cdots\left(1 - \infty_M z^{-1}\right)}{\left(1 - q_1 z^{-1}\right)\left(1 - q_2 z^{-1}\right)\cdots\left(1 - q_M z^{-1}\right)}$$

$$Y(z) = b_0 \frac{\left(1 - z_1 z^{-1}\right)\left(1 - z_2 z^{-1}\right)\cdots\left(1 - z_M z^{-1}\right)}{\left(1 - p_1 z^{-1}\right)\cancel{\left(1 - p_2 z^{-1}\right)}\cdots\left(1 - p_M z^{-1}\right)} \frac{\left(1 - \alpha_1 z^{-1}\right)\cancel{\left(1 - p_2 z^{-1}\right)}\cdots\left(1 - \infty_M z^{-1}\right)}{\left(1 - q_1 z^{-1}\right)\left(1 - q_2 z^{-1}\right)\cdots\left(1 - q_M z^{-1}\right)}$$

Analysis of LTI Systems in the z-Domain

- Pole-zero cancellation
  - If they almost cancel, zero suppresses response of system at that pole
  - We will use this later to construct notch filters

Analysis of LTI Systems in the z-Domain

- Multi-order poles
  - A single (complex conjugate pair) pole on the unit circle causes a steady-state response (i.e. sin or cos)
  - If you a double pole on the unit circle (whether from the system or combination of system and input) system will be unstable

Single poles have responses like $\left(\mathrm{p}_k\right)^n$

If $\left|\mathrm{p}_k\right| = 1$, response continues forever, but doesn't blow up

Double poles have responses like $n\left(\mathrm{p}_k\right)^n$

If $\left|\mathrm{p}_k\right| = 1$, response increases linearly with n (unstable)

Second order systems

- Detailed analysis of second-order systems
  - Second order system has two poles
  - Higher order systems are built using combination of second-order systems
  - General form of second order system:

$$y(n)+a_1 y(n-1)+a_2(n-2)=b_0 x(n) \quad \Rightarrow \quad \left(1+a_1 z^{-1}+a_2 z^{-2}\right)Y(z)=b_0 X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{b_0}{1+a_1 z^{-1}+a_2 z^{-2}}=\frac{b_0 z^2}{z^2+a_1 z+a_2}$$

Second order systems

$$H(z) = \frac{b_0 z^2}{z^2 + a_1 z + a_2}$$

Two zeros at $z = 0$

Two poles at:

$$p_1, p_2 = \frac{-a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_2}}{2}$$

Second-order system is stable if poles are inside unit circle:

$|p_1| < 1$ and $|p_2| < 1$

Second order systems

- You can find condition for stability from coefficients of difference equation

In terms of the poles:

$$a_1 = -(p_1 + p_2)2$$

$$a_2 = p_1 p_2$$

Condition for stability: $|\mathrm{p}_1| < 1 \quad |\mathrm{p}_2| < 1$

in terms of a's is:

$$|a_2| < 1$$

$$a_1 < 1 + a_2$$

Second order systems

- You can find condition for stability from coefficients of difference equation

In terms of the poles:

$$a_1 = -(p_1 + p_2)2$$
$$a_2 = p_1 p_2$$

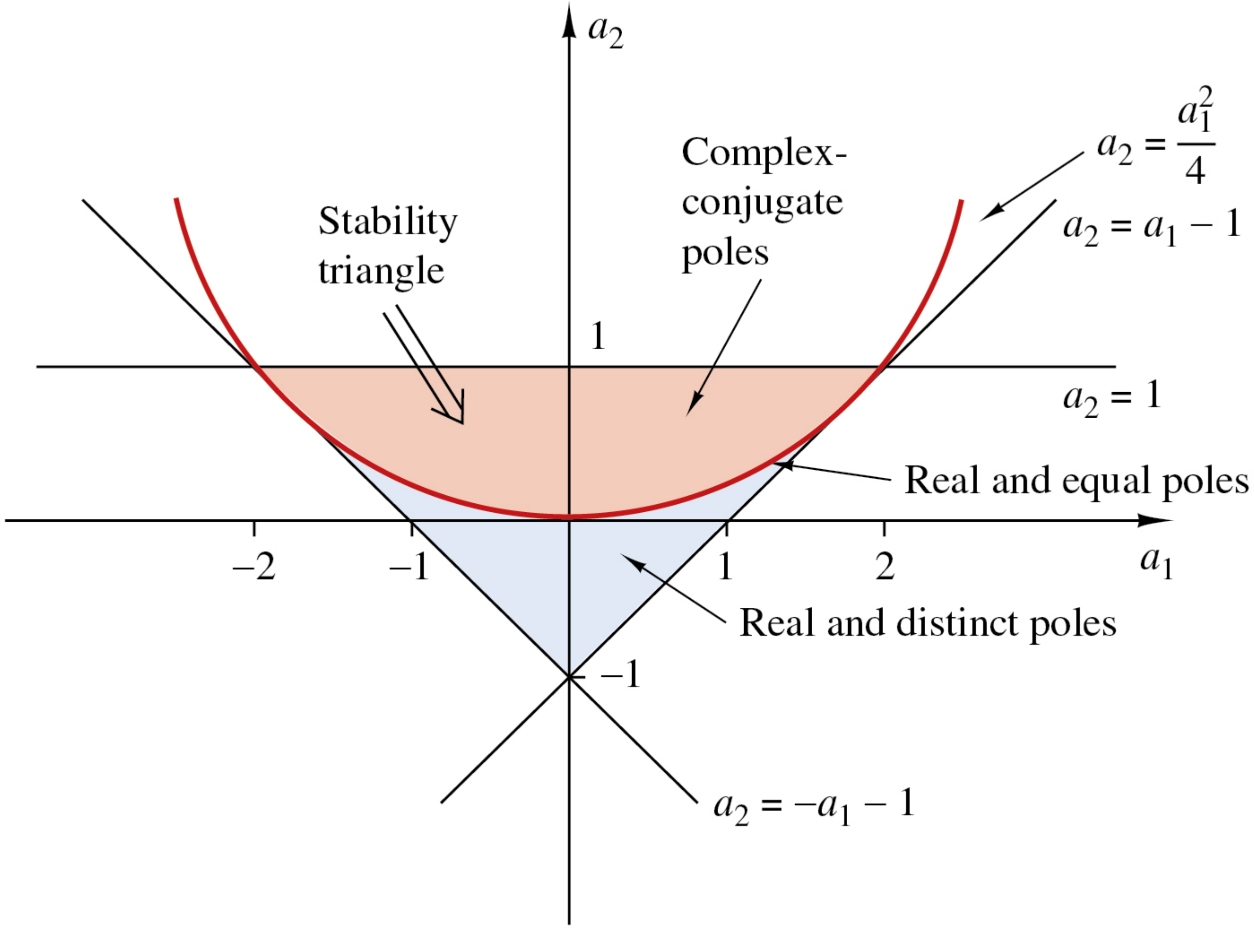
Condition for stability: $|\mathrm{p}_1| < 1 \quad |\mathrm{p}_2| < 1$

in terms of a's is:

$$|a_2| < 1$$
$$a_1 < 1 + a_2$$

Second order systems

– Triangle of stability



Second order systems

– Possibilities for poles depend on value of discriminant

From quadratic equation:

$$p_1, p_2 = \frac{-a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_2}}{2}$$

$$Disc = a_1^2 - 4a_2$$

Discriminant can be

Positive: $a_1^2 > 4a_2$ two real distinct roots

Negative: $a_1^2 < 4a_2$ complex conjugate pair of roots

Zero: $a_1^2 = 4a_2$ multiple pole at $a_1/2$

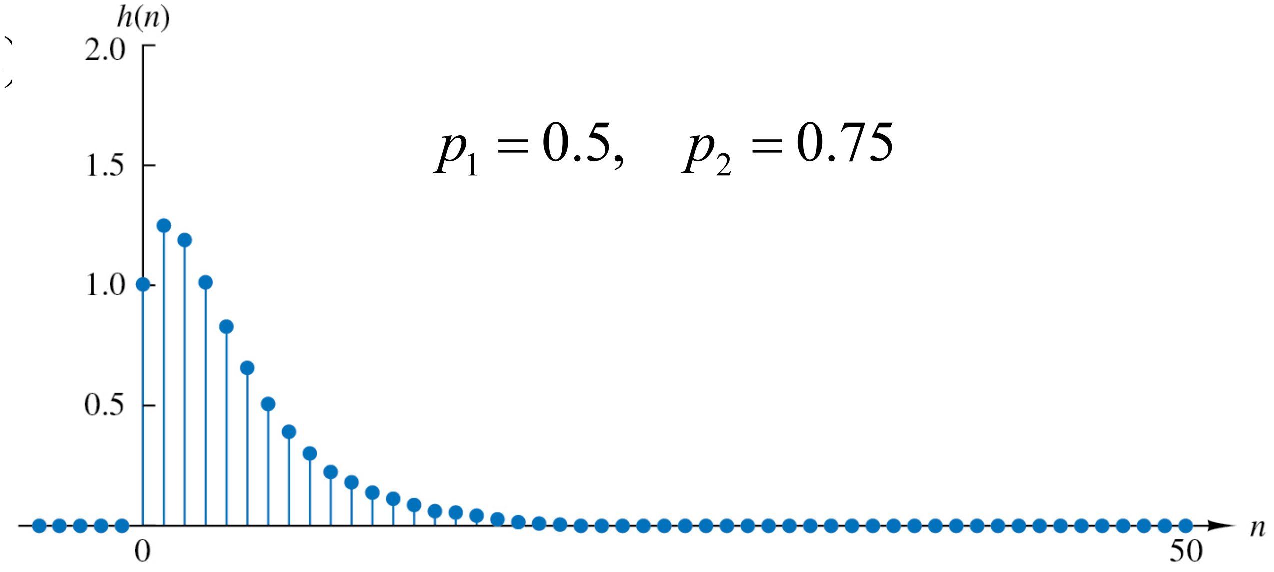
Second order systems

– Real distinct poles $a_1^2 > 4a_2$

$$H(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}}$$

$$A_1 = \frac{b_0 p_1}{p_1 - p_2}, \quad A_2 = \frac{-b_0 p_2}{p_1 - p_2}$$

$$h(n) = \frac{b_0}{p_1 - p_2}\left(p_1^{n+1} - p_2^{n+1}\right)u(n)$$



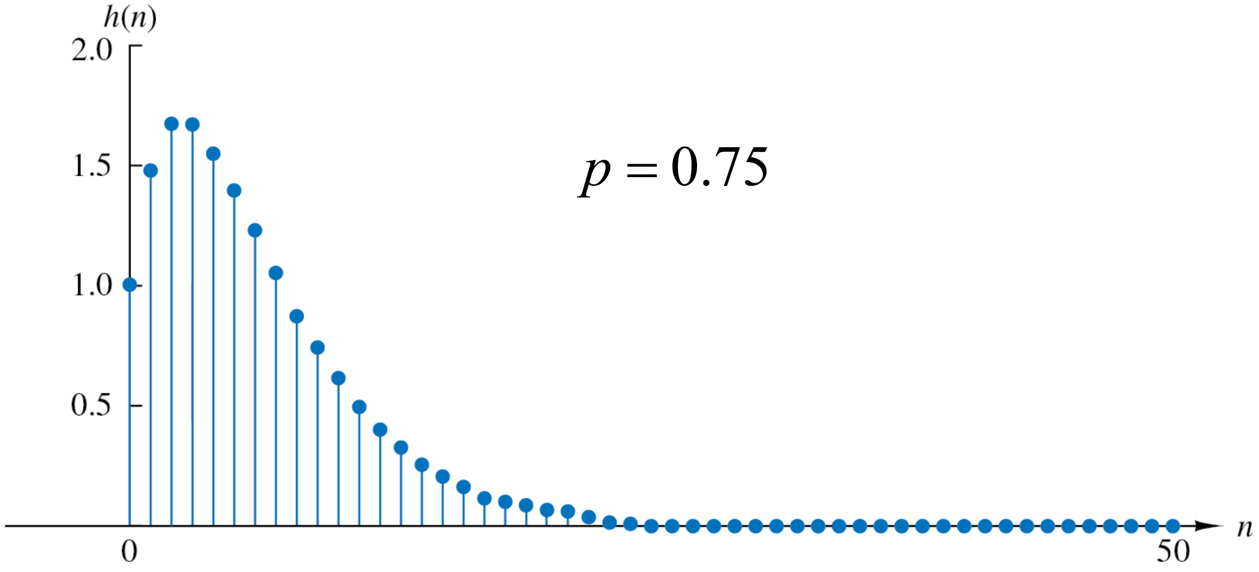
$p_1 = 0.5, \quad p_2 = 0.75$

Second order systems

– Real equal poles $a_1^2 = 4a_2$

$$H(z) = \frac{b_0}{\left(1 - pz^{-1}\right)^2}$$

$$h(n) = b_0\left(n+1\right)p^n u(n)$$



$p = 0.75$

Second order systems

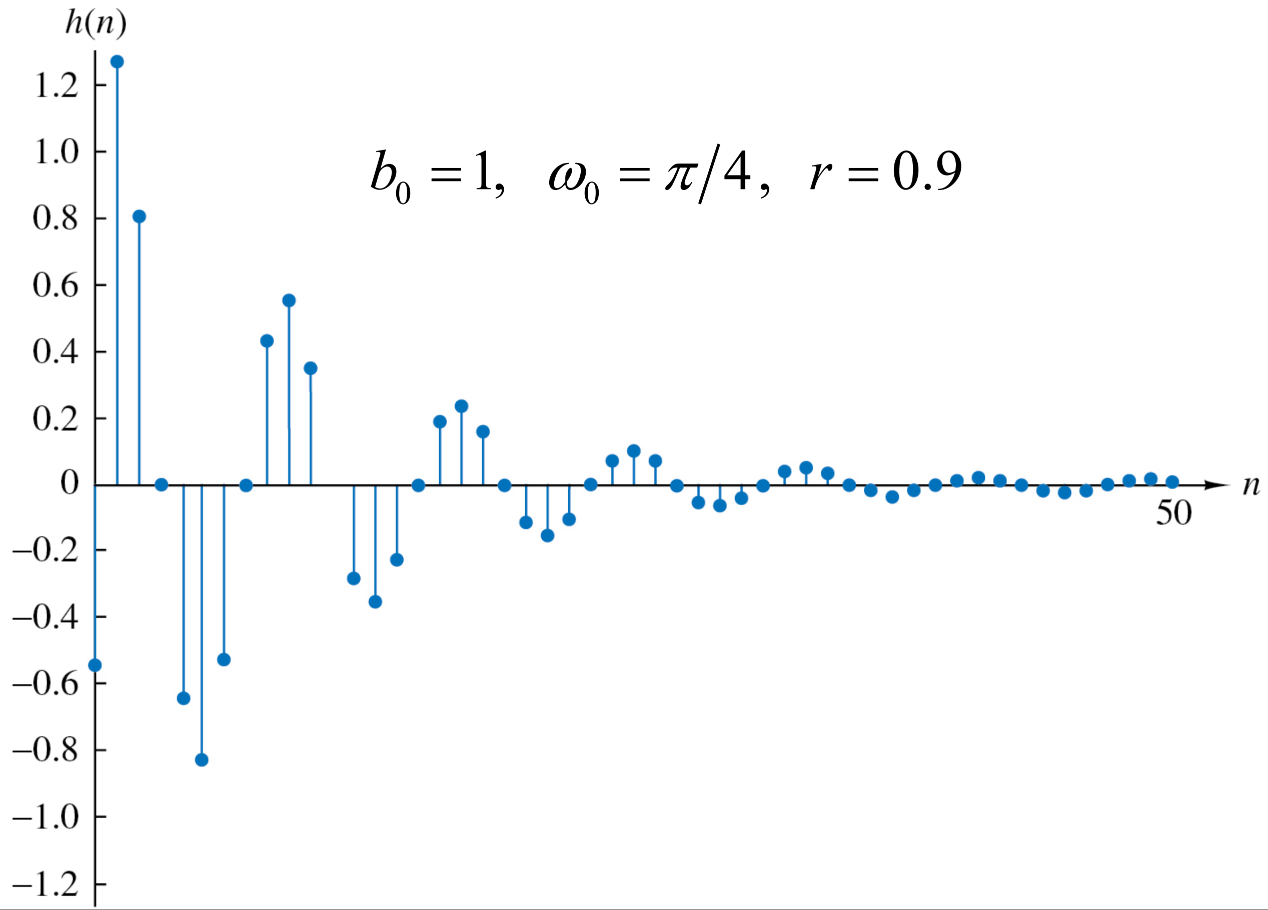
– Complex conjugate poles $a_1^2 < 4a_2$

$$H(z) = \frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}}$$

$$= \frac{A}{1 - re^{j\omega_0} z^{-1}} + \frac{A^*}{1 - re^{-j\omega_0} z^{-1}}$$

$$A = \frac{b_0 p}{p - p^*} = \frac{b_0 r e^{j\omega_0}}{r\left(e^{j\omega_0} - e^{-j\omega_0}\right)} = \frac{b_0 e^{j\omega_0}}{2j \sin \omega_0}$$

$$h(n) = \frac{b_0 r^n}{\sin \omega_0} \sin\left((n+1)\omega_0\right) u(n)$$



One-sided z-transform

- One-sided z-transform
  - For causal signals and systems, the same as bilateral

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

  - Useful for solving difference equations with initial conditions

One-sided z-transform

- Shift Property for one-sided z-transform

$x(n) \Leftrightarrow X^{+}(z)$

$x(n-k) \Leftrightarrow ?$

From definition for one-sided transform: $\sum_{n=0}^{\infty} x(n-k)z^{-n}$

Let $m = n - k; \;\; n = m+k; \;\; n=0 \Rightarrow m=-k$

$$\sum_{m=-k}^{\infty} x(m)z^{-(m+k)} = z^{-k}\left[\sum_{m=-k}^{-1} x(m)z^{-m} + \sum_{m=0}^{\infty} x(m)z^{-m}\right] = z^{-k}\left[\sum_{m=1}^{k} x(-m)z^{-m} + \sum_{m=0}^{\infty} x(m)z^{m}\right]$$

$$\boxed{\begin{array}{l} x(n) \Leftrightarrow X^{+}(z) \\ x(n-k) \Leftrightarrow z^{-k}\left[X^{+}(z) + \sum_{n=1}^{k} x(-n)z^{n}\right] \end{array}}$$

Something similar for positive shifts

$x(n) \Leftrightarrow X^{+}(z)$

$x(n+k) \Leftrightarrow z^{k}\left[X^{+}(z) - \sum_{n=0}^{k-1} x(n)z^{-n}\right]$

Difference Equations

– Why is this useful for difference equations with initial conditions

$$y(n)+a_1y(n-1)+a_2y(n)=b_0x(n)+b_1x(n-1)$$

For causal system with initial conditions: $y(-1)$ & $y(-2)$

$$\mathcal{Z}^+\left[y(n)+a_1y(n-1)+a_2y(n)=b_0x(n)+b_1x(n-1)\right]$$

$$Y^+(z)+a_1z^{-1}\left[Y^+(z)+y(-1)z\right]+a_2z^{-2}\left[Y^+(z)+y(-1)z+y(-2)z^2\right]=(b_0+b_1z^{-1})X(z) \quad \text{(input causal)}$$

$$Y^+(z)+a_1z^{-1}Y^+(z)+a_2z^{-2}Y^+(z)+\left[a_1y(-1)+a_2y(-1)z^{-1}+a_2y(-2)\right]=(b_0+b_1z^{-1})X(z)$$

$$\boxed{\left(1+a_1z^{-1}+a_2z^{-2}\right)Y^+(z)=\left(b_0+b_1z^{-1}\right)X(z)-\left[a_1y(-1)+a_2y(-1)z^{-1}+a_2y(-2)\right]}$$

Difference Equations

- Solution of difference equations with initial conditions

$$Y^+(z) = \frac{\sum_{k=0}^{\infty} b_k z^{-k}}{1+\sum_{k=1}^{\infty} a_k z^{-k}} X(z) + \frac{\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^{n}}{1+\sum_{k=1}^{\infty} a_k z^{-k}}$$

$$Y^+(z) = H(z)X(z) + \frac{N_0(z)}{A(z)}$$

$H(z)X(z)$ is the zero-state response (no initial conditions)

$\frac{N_0(z)}{A(z)}$ is the zero-input response (due to initial conditions)

$$y(n) = y_{zs}(n) + y_{zi}(n)$$