Digital Signal Processing

Class 22
04/10/2025

ENGR 71

- Class Overview
  - Digital Filter Design
    - FIR filters
- Assignments
  - Reading:

    Chapter 10: Design of Digital Filters

    https://www.mathworks.com/help/signal/ug/fir-filter-design.html
  - Problems: 10.2, 10.3,10.6
    - Due April 20 (Sunday)

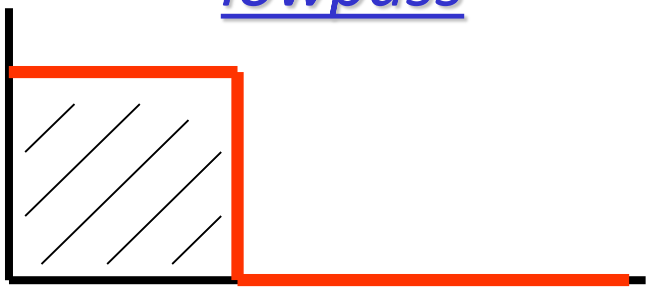
Project

- Projects
  - You can work in groups if you wish
  - Presentation at time reserved for final exam
    - Friday, May 9, 7:00-10:00 PM
    - Science Center 264
  - Submit a brief description of your proposed project and who you will be working with to the Project Description Dropbox
  - Submit slides from presentation to Project Dropbox
  - Submit written report to Project Dropbox by end of semester (May 15)

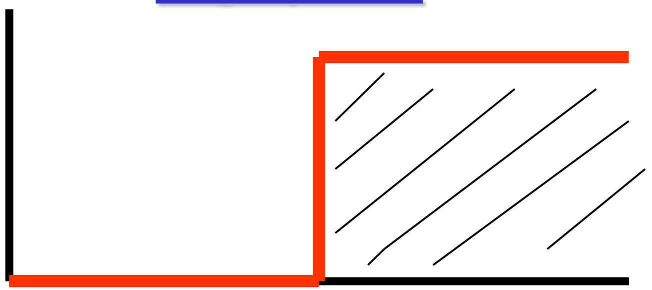
Filters

- Design of Digital Filters

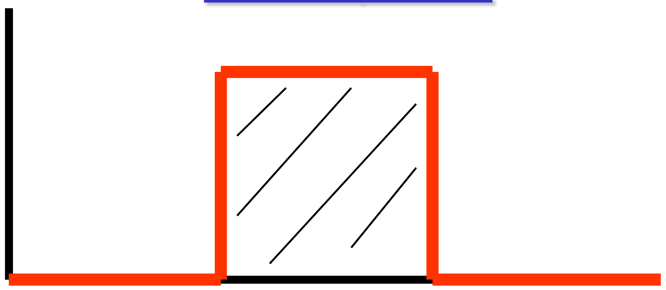
lowpass



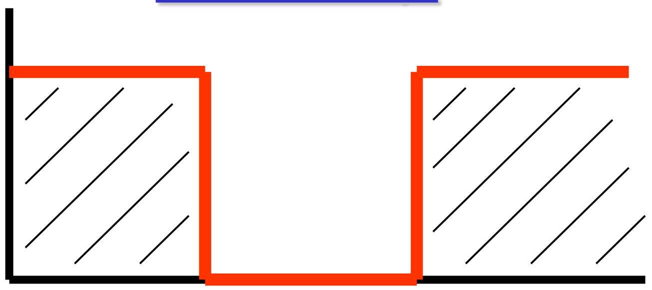
highpass



bandpass



bandstop



Filters

- Design of Digital Filters
  - Finite Impulse Response (FIR)
    - Used when linear-phase required in passband
      - Constant time delay
      - No dispersion as a function of frequency
  - Infinite Impulse Response (IIR)
    - No requirement on linear-phase
    - Better characteristics for fewer parameters
    - Less memory
    - Lower computational complexity

Filter Design

- Finite Impulse Response (FIR) Filters
  - In terms of impulse response

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \qquad \left(\quad b_k\text{'s are } h(k)\text{'s}\right)$$

  - In terms of transfer function:

$$H(z) = \sum_{k=0}^{M-1} h(k)z^{-k}$$

  - Always stable since finite impulse.

Filter Design

- Advantage of FIR filters is the can be designed to have linear phase in the passband
  - Only introduces time delay in the filtered signal
    - No dispersion (time delay dependent of frequency)
- Condition to guarantee linear phase:
  - For length *M* FIR filter: $h(n) = \pm h(M-1-n)$
  - Four cases:
    - Symmetric: $h(n) = +h(M-1-n)$
      - *M* even or *M* odd
    - Antisymmetric: $h(n) = -h(M-1-n)$
      - *M* even or *M* odd

Filter Design

- Frequency response for Type I FIR filter
  - Symmetric, *M* odd

$$h(n) = +h(M - 1 - n)$$

$$H(\omega) = \sum_{n=0}^{(M-1)/2} h(n)e^{-j\omega n}$$

$$H(\omega) = e^{-j\omega(M-1)/2} \sum_{n=0}^{(M-1)/2} a(n)\cos(\omega n)$$

where

$$a(0) = h\left(\frac{M-1}{2}\right)$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right), \quad n = 1,2,\ldots,\frac{M-1}{2}$$

Type I is the most versatile form.
Can be used for all low-pass, high-pass, band-pass, and band-stop filters

Filter Design

- Frequency response for Type II FIR filter
  - Symmetric, *M* even

$$h(n) = +h(M - 1 - n)$$

$$H(\omega) = \sum_{n=0}^{(M/2)-1} h(n) e^{-j\omega n}$$

$$H(\omega) = e^{-j\omega(M-1)/2} \sum_{n=1}^{M/2} b(n) \cos\left[\omega\left(n - \frac{1}{2}\right)\right]$$

where

$$b(n) = 2h\left(\frac{M}{2} - n\right), \quad n = 1, 2, \ldots, \frac{M}{2}$$

Type II is zero at $\omega=\pi$.
Cannot be used for high-pass filter

Filter Design

- Frequency response for Type III FIR filter
  - Antisymmetric, *M* odd

$$h(n) = -h(M - 1 - n)$$

$$H(\omega) = \sum_{n=0}^{(M-1)/2} h(n) e^{-j\omega n}$$

$$H(\omega) = j e^{-j\omega(M-1)/2} \sum_{n=1}^{(M-1)/2} a(n) \sin(\omega n)$$

where

$$a(n) = 2h\left(\frac{M-1}{2} - n\right), \quad n = 1, 2, \ldots, \frac{M-1}{2}$$

Type III is zero at $\omega=0$ and $\omega=\pi$. Cannot be used for low-pass or high-pass filter

Filter Design

– Frequency response for Type IV FIR filter
  - Antisymmetric, *M* even

$$h(n) = -h(M - 1 - n)$$

$$H(\omega) = \sum_{n=0}^{(M/2)-1} h(n) e^{-j\omega n}$$

$$H(\omega) = j e^{-j\omega(M-1)/2} \sum_{n=1}^{M/2} b(n) \sin\left[\omega\left(n - \frac{1}{2}\right)\right]$$

where

$$b(n) = 2h\left(\frac{M}{2} - n\right), \quad n = 1, 2, \ldots, \frac{M}{2}$$

Type IV is zero at ω=0.
Cannot be used for low-pass filter.

Filter Design

– Task for FIR filter design:

• Determine the M coefficients, b_k, for

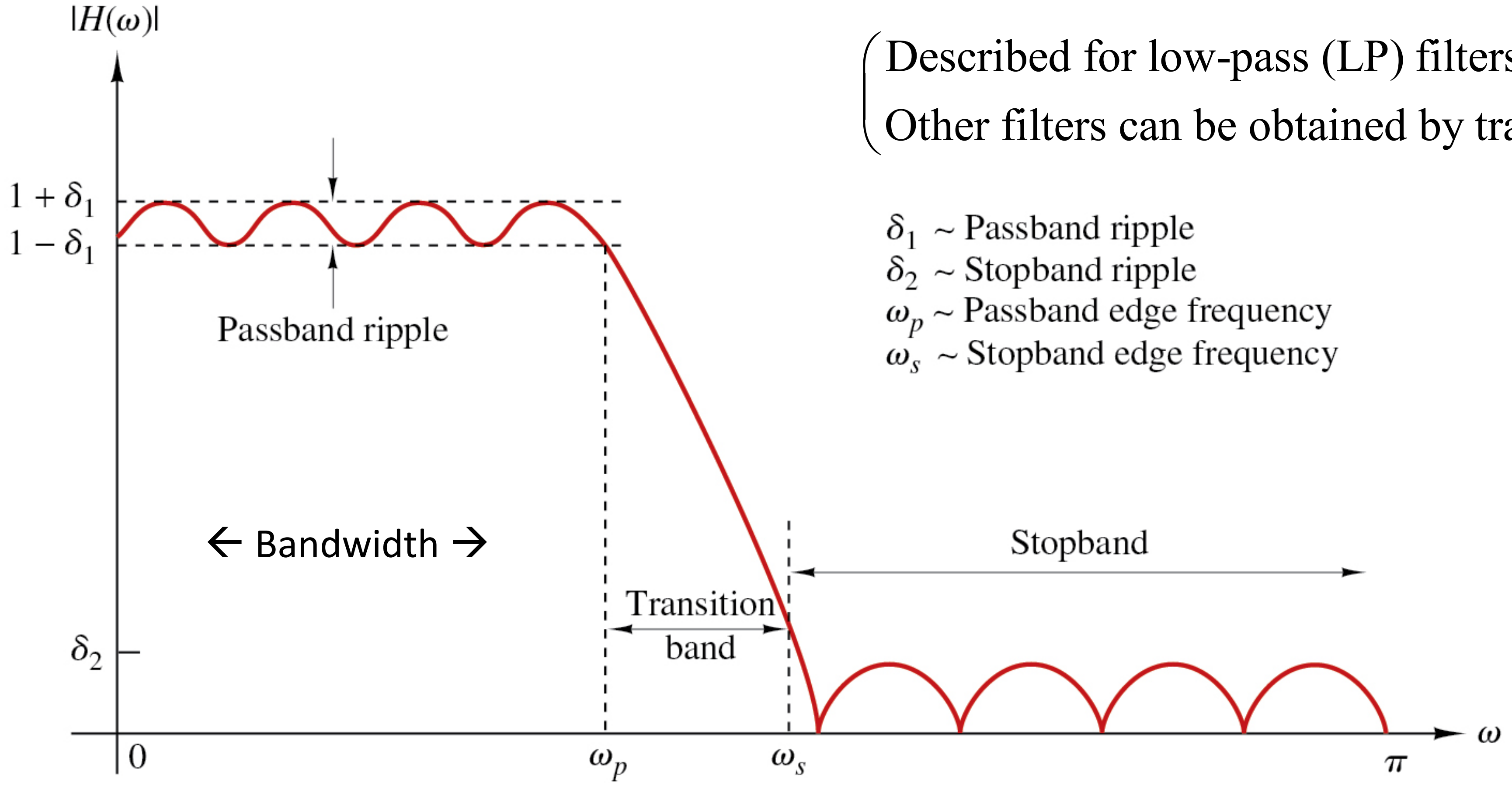
$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

that best match desired filter response, $H_d(\omega)$, in the frequency domain.

Filter Design

- Specifications for physically realizable filters:

(Described for low-pass (LP) filters.
Other filters can be obtained by transforming LP filters)



δ_1 ~ Passband ripple
δ_2 ~ Stopband ripple
ω_p ~ Passband edge frequency
ω_s ~ Stopband edge frequency

Linear-Phase FIR Filter Design

– Methods
  - Windowing impulse response
    - Specify desired response

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n) e^{-j\omega n}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

    - Cannot have an infinite impulse response, so window the $h_d(n)$
    - If truncate the series at some value of n, that is like a rectangular window
    - We will discuss different types of windowing functions

Linear-Phase FIR Filter Design

- Example: Low-pass filter
  - Ideal linear-phase low-pass filter:
    - Specify desired response

$$H_d(\omega) = 1e^{-j\omega(M-1)/2} \quad \text{for } 0 \le |\omega| \le \omega_c$$

Notice that a time delay of (M-1)/2 samples is "built in" with the linear phase.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

    - For concreteness, consider ω_c = 3π/16 (3/16 of Nyquist)
      Filter of length M=9 (odd)
      - For CD quality music, sampled at 44.1 kHz, This would correspond to ~ 8 kHz sampling which is what sampling rate is for digital phone.
      - This would correspond to a Nyquist frequency of ~4 kHz like listening to someone play a song over the phone

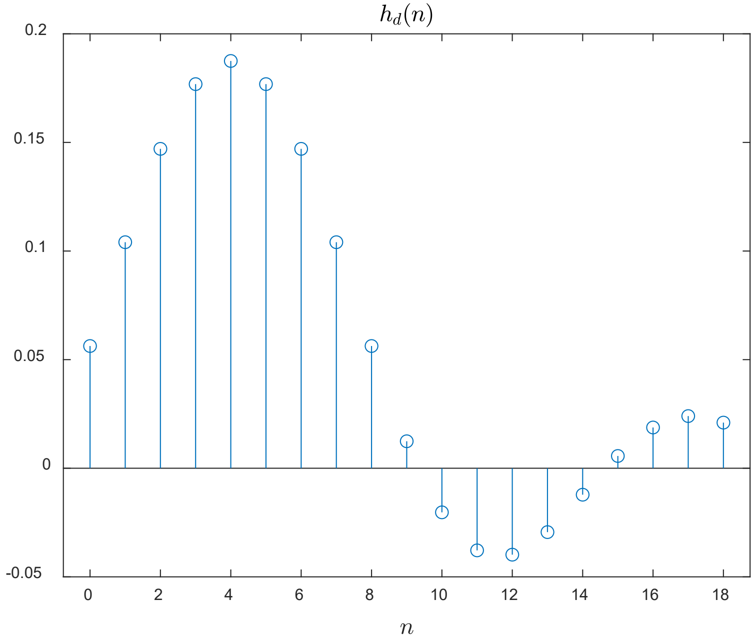
Linear-Phase FIR Filter Design

- Find time-domain impulse response

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H_d(\omega)e^{j\omega n}\,d\omega$$

$$\left(\text{On board}\right)$$

$$h_d(n) = \frac{1}{\pi}\frac{\sin\left[\omega_c\left(n-\frac{M-1}{2}\right)\right]}{\left(n-\frac{M-1}{2}\right)} = \frac{\omega_c}{\pi}\text{sinc}\left[\omega_c\left(n-\frac{M-1}{2}\right)\right]$$



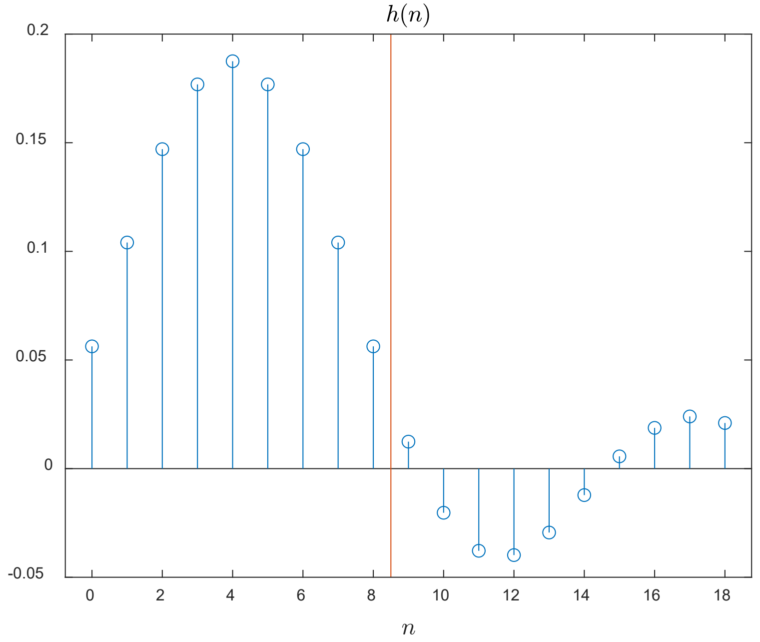
Caution! This is the unnormalized sinc function: sin(x)/x
You have to divide x by pi before calling
Matlab's sinc function

Linear-Phase FIR Filter Design

- Rectangular window:

$$w(n)=\begin{cases}1, & n=0,1,\ldots,M-1\\ 0, & \text{otherwise}\end{cases} \Rightarrow w(n)=\begin{cases}1, & n=0,1,\ldots,8\\ 0, & \text{otherwise}\end{cases}$$

$$h_d(n)=\begin{cases}\operatorname{sinc}\left[\omega_c\left(n-\dfrac{M-1}{2}\right)\right], & n=0,1,\ldots,M-1\\ 0, & \text{otherwise}\end{cases}$$



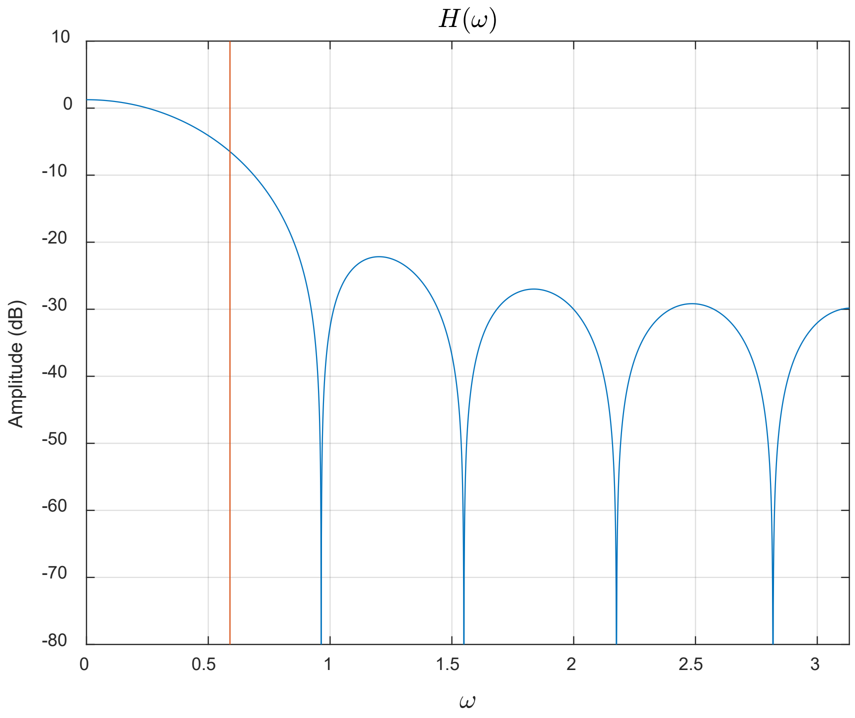
- General window:

$$h(n)=w(n)\cdot h_d(n)$$

$$\left(\text{For rectangular window: } h(n)=1\cdot h_d(n)\right)$$

Linear-Phase FIR Filter Design

- It would be nice to know what the frequency domain transfer function looks like after windowing
  - Can do it numerically in Matlab



Linear-Phase FIR Filter Design

- In general, you can see the effect of the window by convolving the window function in the frequency domain with the ideal filter transfer function.

$$H(\omega) = H_d(\omega) \otimes W\left(\omega\right)$$

For a window of lenght M

$$W\left(\omega\right) = \sum_{n=0}^{M-1} w(n) e^{-j\omega n}$$

For a rectangular window of length M $w(n)$ is 1 in the sum

$$W\left(\omega\right) = \sum_{n=0}^{M-1} e^{-j\omega n}$$

Linear-Phase FIR Filter Design

Use the expression for the finite geometric series:

$$S = \sum_{n=0}^{M-1} r^n = \frac{\left(1 - r^M\right)}{1 - r}$$

$$W(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{\left(1 - e^{-j\omega M}\right)}{1 - e^{-j\omega}}$$

(on board)

$$W(\omega) = e^{-j\omega(M-1)/2} \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

Magnitude: $\left|W(\omega)\right| = \frac{\sin(\omega M/2)}{\sin(\omega/2)}$ for $-\pi \le \omega \le \pi$

Phase: $\Theta(\omega) = \begin{cases} -\omega(M-1)/2 & \text{for } \sin(\omega M/2) \ge 0 \\ -\omega(M-1)/2 + \pi & \text{for } \sin(\omega M/2) < 0 \end{cases}$

Linear-Phase FIR Filter Design

The frequency domain transfer function is:

$$H(\omega) = \int_{-\pi}^{\pi} H_d(\nu) W(\omega - \nu) d\nu = \int_{-\pi}^{\pi} W(\omega) H_d(\omega - \nu) d\nu$$

For rectangular window:

$$H(\omega) = \int_{-\pi}^{\pi} W(\omega) H_d(\omega - \nu) d\nu = \int_{-\omega_c}^{\omega_c} e^{-j\nu(M-1)/2} \frac{\sin(\nu M/2)}{\sin(\nu/2)} 1 e^{-j(\omega-\nu)(M-1)/2} d\nu$$

$$H(\omega) = \int_{-\omega_c}^{\omega_c} e^{-j\nu(M-1)/2} e^{+j\nu(M-1)/2} e^{-j\omega(M-1)/2} \frac{\sin(\nu M/2)}{\sin(\nu/2)} d\nu$$

$$H(\omega) = e^{-j\omega(M-1)/2} \int_{-\omega_c}^{\omega_c} \frac{\sin(\nu M/2)}{\sin(\nu/2)} d\nu$$

Linear-Phase FIR Filter Design

```
% Set Filter Length
M=65;
wc = 3*pi/16;
n=[0:2*M];
x =wc*(n-(M-1)/2);
hd = (wc/pi)*sinc(x/pi);
% Plot impulse response for ideal filter
figure(1)
stem(n,hd)
title('$h_d(n)$','interpreter','Latex')
xlabel('$n$','interpreter','Latex')

% Plot showing truncated length M impulse
response
figure(2)
stem(n,hd)
title('$h(n)$','interpreter','Latex')
xlabel('$n$','interpreter','Latex')
hold on
plot([M-0.5,M-0.5],[-0.05,0.2])

% Select window type
type = 9;
alpha = 0.5;
[nn,win] = get_window(M,alpha,type);
% Plot Window function
figure(3)
stem(nn,win)
title('Window')
xlabel('$n$','interpreter','Latex')
```

```
hd_win = win.*hd(1:M);
% Plot windowed impulse response
figure(4)
% subplot(2,1,1)
stem(nn,hd(1:M))
% title('$h(n)$','interpreter','Latex')
% subplot(2,1,2)
hold on
stem(nn,hd_win)
title('$w(n)h(n)$','interpreter','Latex')
xlabel('$n$','interpreter','Latex')
legend('Ideal impulse','Modified impulse')
w = [-pi:0.001:pi];
hw = zeros(size(wc));
for n = 0:M-1
hw = hw+hd_win(n+1)*exp(-i*w*n);
end

% Plot transfer function for filter
figure(5)
plot(w,20*log10(abs(hw)))
axis([0,pi-0.01,-80,10])
grid on
title('$H(\omega)$','Interpreter','Latex')
ylabel('Amplitude (dB)')
xlabel('$\omega$','Interpreter','Latex')
hold on
plot([wc,wc],[-80,10])
```

Linear-Phase FIR Filter Design

- Comparison of FIR filters with different windows

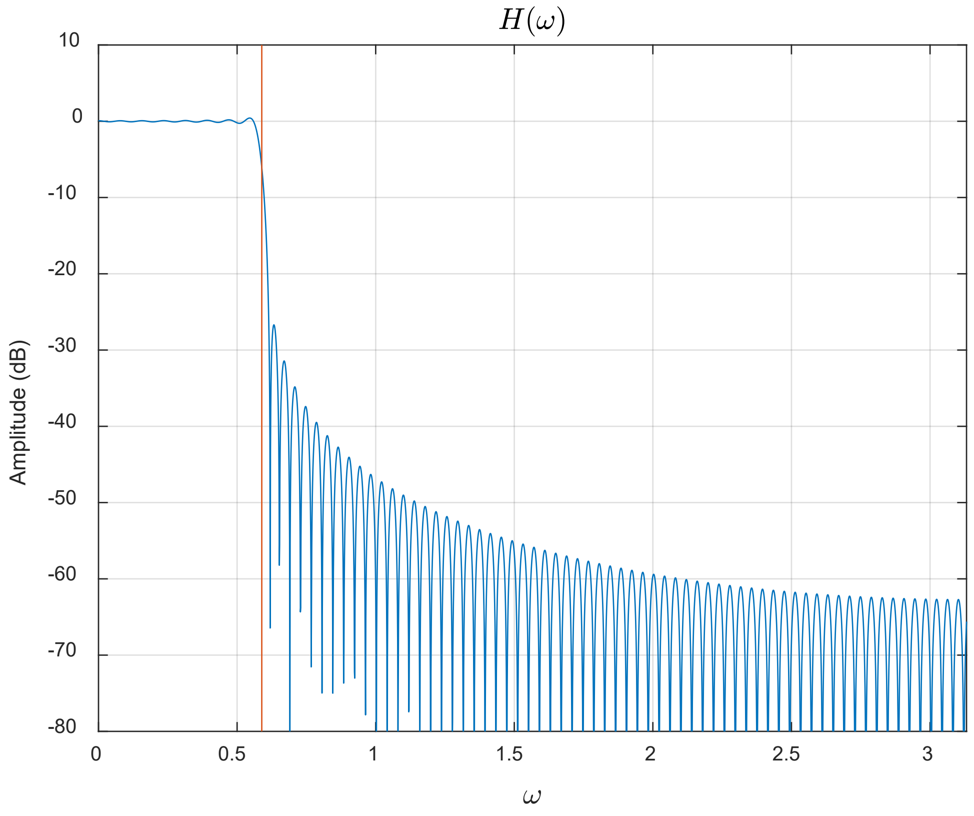
Type of window	Approximate transition width of main lobe	Peak sidelobe (dB)
Rectangular	$4\pi/M$	–13
Bartlett	$8\pi/M$	–25
Hanning	$8\pi/M$	–31
Hamming	$8\pi/M$	–41
Blackman	$12\pi/M$	–57

Wider main lobe – more smoothing and wider transition region
Larger sidelobes – more ripple
Making M larger makes transition narrower at expense of complexity and time delay introduced

Linear-Phase FIR Filter Design

- Gaussian window:
  - Specify desired response

$$w(n) = \exp\left[-\frac{1}{2\sigma^2}\left(\frac{n-(M-1)/2}{(M-1)/2}\right)^2\right]$$



Linear-Phase FIR Filter Design

- Ideal linear-phase band-pass filter:
  - Specify desired response

$$H_d^{BP}(\omega) = 1e^{-j\omega(M-1)/2} \quad \text{for} \quad -\omega_2 \le \omega \le -\omega_1 \quad \text{and} \quad \omega_1 \le \omega \le \omega_2$$

$$h_d^{BP}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega \Rightarrow \frac{1}{2\pi}\int_{-\omega_2}^{-\omega_1} H_d(\omega)e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} H_d(\omega)e^{j\omega n}d\omega$$

- Ideal linear-phase band-stop filter:
  - Design band-pass and subtract it from all pass:

$$h_d^{BS}(n) = \delta(n) - h_d^{BP}(n)$$

Linear-Phase FIR Filter Design

- Example:

10.1 Design an FIR linear-phase, digital filter approximating the ideal frequency response

$$H_d(\omega) = \begin{cases} 1, & \text{for } |\omega| \le \dfrac{\pi}{6} \\ 0, & \text{for } \dfrac{\pi}{6} < |\omega| \le \pi \end{cases}$$

a. Determine the coefficients of a 25-tap filter based on the window method with a rectangular window.

b. Determine and plot the magnitude and phase response of the filter.

c. Repeat parts (a) and (b) using the Hamming window.

d. Repeat parts (a) and (b) using a Bartlett window.

Bartlett (triangular)	$1 - \dfrac{2\left\lvert n - \dfrac{M-1}{2} \right\rvert}{M-1}$
Blackman	$0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}$
Hamming	$0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$

Frequency Sampling Method

- Frequency sampling method for FIR filter design
  - Conceptually straightforward
    - Specify desired response at set of equally spaced frequencies
    - Solve for *h*(*n*) from the specified response
    - Optimize frequency specification in transition band to reduce sidelobes
  - Select frequency points at

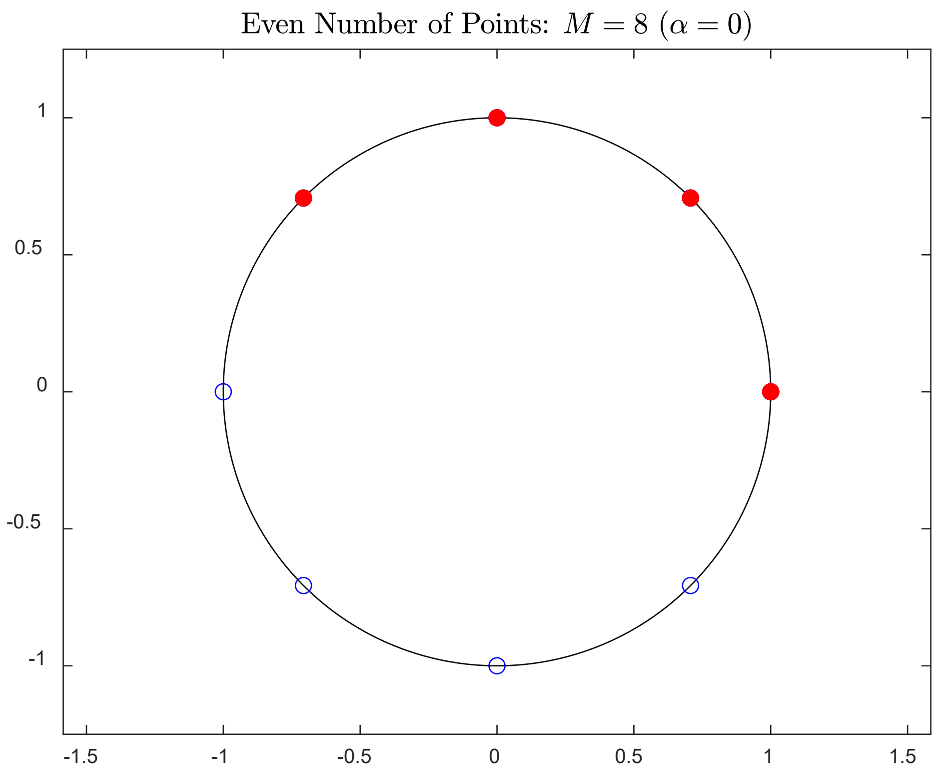
$$w_k = \frac{2\pi}{M}(k+\alpha), \quad k = 0,1,\ldots,\frac{M-1}{2} \text{ (for } M \text{ odd) or } k = 0,1,\ldots,\frac{M}{2}-1 \text{ (for } M \text{ even)}$$

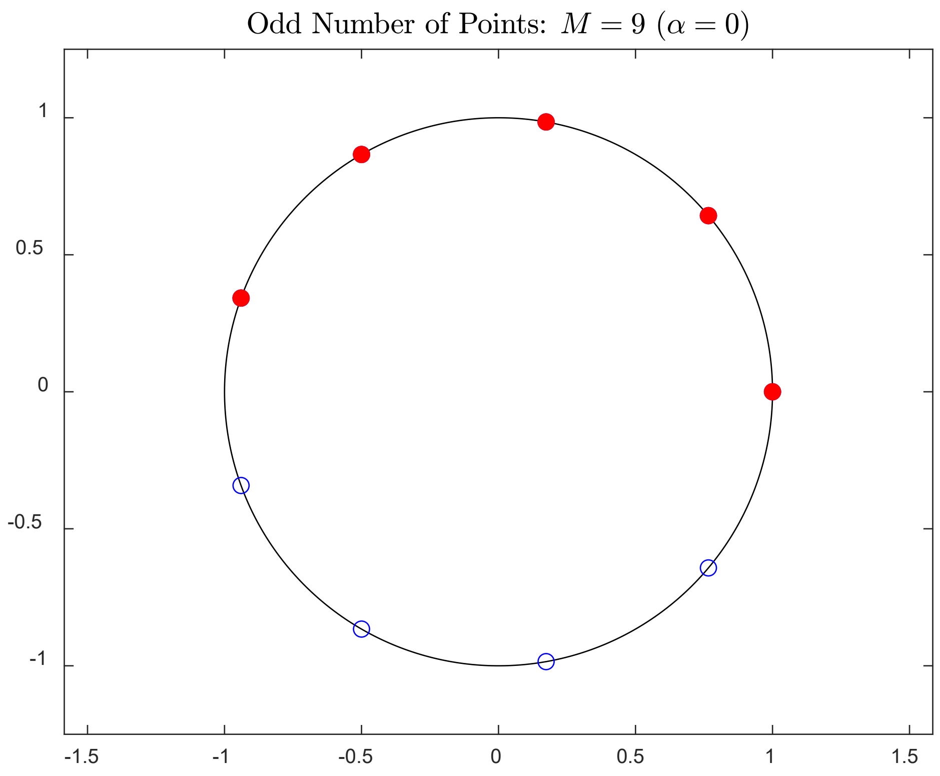
$$\alpha \text{ is either } 0 \; or \; 1/2$$

  - Frequency response at points: $H(k+\alpha) \equiv H\left(\frac{2\pi}{M}(k+\alpha)\right)$

Frequency Sampling Method

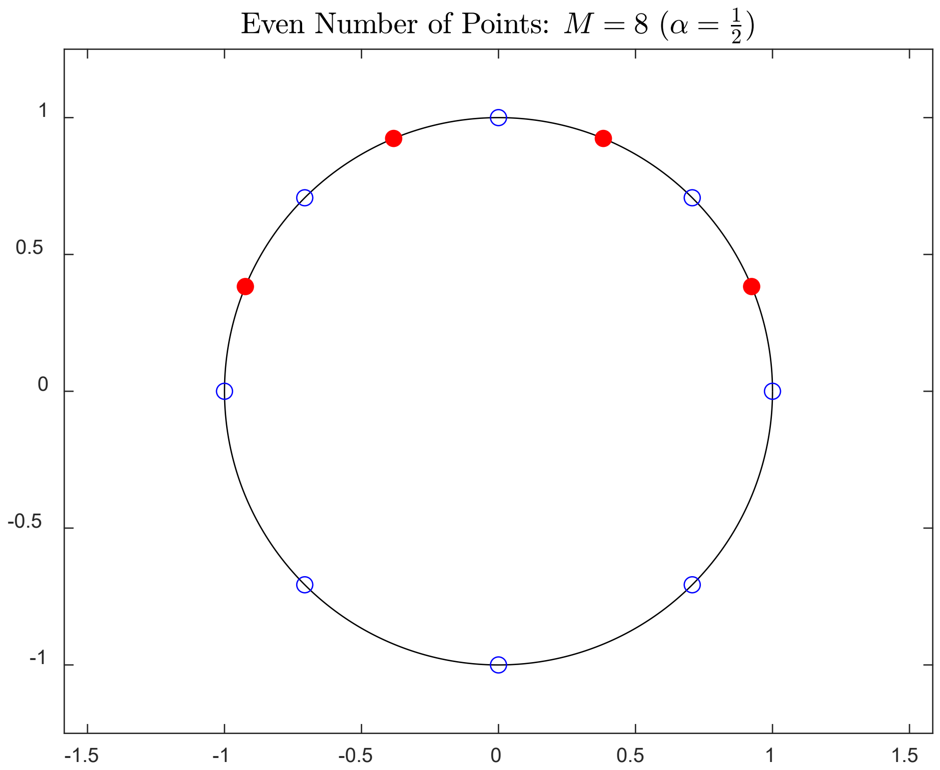
– Select frequency points: $\alpha = 0$

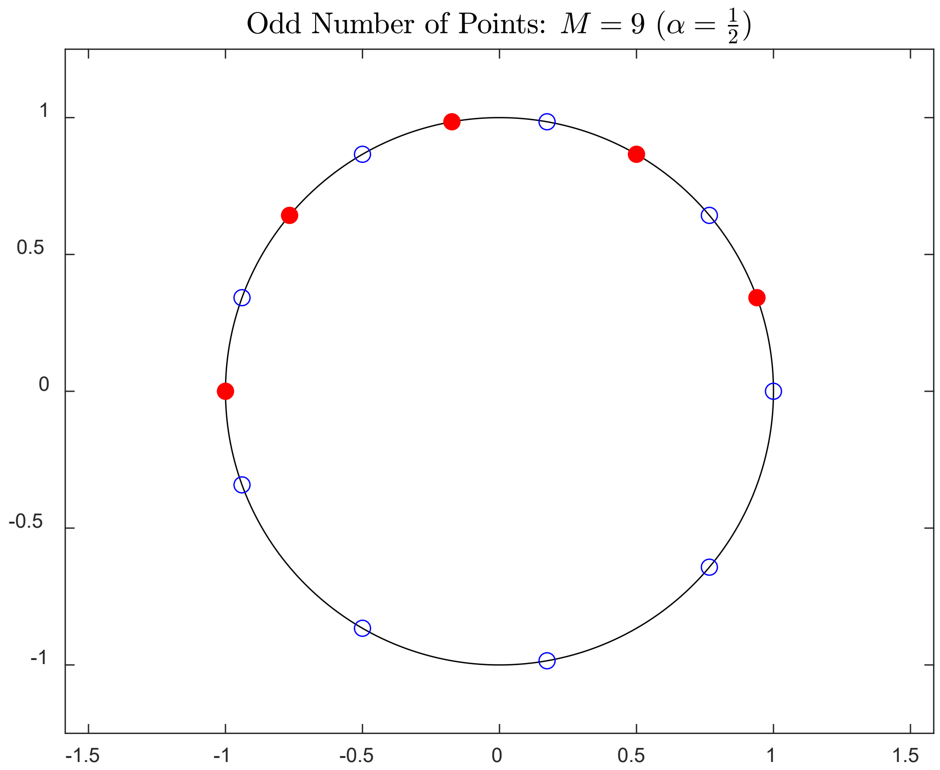




Frequency Sampling Method

– Select frequency points $\alpha = 1/2$





Frequency Sampling Method

- Frequency response at points is:

$$H\left(\frac{2\pi}{M}(k+\alpha)\right)$$

Simplify notation by defining: $H(k+\alpha) \equiv H\left(\frac{2\pi}{M}(k+\alpha)\right)$

- We need to find the filters impulse response *h*(*n*)
  - Use relationship to filter response and invert to find *h*(*n*)

$$H(k+\alpha) = \sum_{n=0}^{M-1} h(n)e^{-j\omega n} = \sum_{n=0}^{M-1} h(n)e^{-j(k+\alpha)n/M}$$

Multiply both sides by $\mathrm{e}^{+j2\pi km/M}$ and sum over k :

$$\sum_{k=0}^{M-1} H(k+\alpha)e^{2\pi km/M} = \sum_{k=0}^{M-1}\sum_{n=0}^{M-1} h(n)e^{-j2\pi(k+\alpha)n/M} e^{2\pi km/M}$$

Frequency Sampling Method

$$\sum_{k=0}^{M-1} H(k+\alpha)e^{j2\pi km/M} = \sum_{k=0}^{M-1}\sum_{n=0}^{M-1} h(n)e^{-j2\pi(k+\alpha)n/M} e^{j2\pi km/M} = \sum_{k=0}^{M-1} h(n)e^{-j2\pi\alpha n/M} \sum_{n=0}^{M-1} e^{j2\pi k(m-n)/M}$$

$$\left(\text{Show on board that } \sum_{n=0}^{M-1} e^{j2\pi k(m-n)/M} = M\delta_{mn}\right)$$

$$\boxed{h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha)e^{j2\pi(k+\alpha)n/M} = \frac{1}{M}\sum_{k=0}^{M-1} H\left(\frac{2\pi}{M}(k+\alpha)\right)e^{j2\pi(k+\alpha)n/M}}$$

Note that for case $\alpha = 0$ this is the DFT

$$H(k) = \sum_{n=0}^{M-1} h(n)e^{-j2\pi kn/M}$$

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k)e^{j2\pi kn/M}$$

Frequency Sampling Method

Since $h(n)$ is real. There is creates a symmetry condition on $H\left(k+\alpha\right)$:

$$H\left(k+\alpha\right)=H^{*}\left(M-k-\alpha\right)$$

Along with the symmetry condition on $h(n)$ for linear phase: $h(n)=\pm h(M-1-n)$

you can show that frequency response values are need at :

- $\dfrac{\left(M+1\right)}{2}$ values for M odd
- $\dfrac{M}{2}$ values for M even

Frequency Sampling Method

Additionally, you can write H(k+α) terms of a real part and a phase, where the phase is known:

$$H(k+\alpha) = H_r\left(\frac{2\pi}{M}(k+\alpha)\right)\exp\left(j\left[\frac{\beta\pi}{2} - \frac{2\pi(k+\alpha)(M-1)}{2M}\right]\right)$$

where: β=0 for the symmetric case: $h(n) = +h(M-1-n)$

β=1 for the antisymmetric case: $h(n) = +h(M-1-n)$ $\left(\text{changes the phase by } 90^{\circ}\right)$

Frequency Sampling Method

Convenient to define a new real-valued function

$$G(k+\alpha) = (-1)^k H_r\left(\frac{2\pi}{M}(k+\alpha)\right)$$

Then

$$H(k+\alpha) = G(k+\alpha)e^{j\pi k}\exp\left(j\left[\frac{\beta\pi}{2} - \frac{2\pi(k+\alpha)(M-1)}{2M}\right]\right)$$

Frequency Sampling Method

Subsituting

$$H(k+\alpha) = G(k+\alpha)e^{j\pi k} \exp\left(j\left[\frac{\beta\pi}{2} - \frac{2\pi(k+\alpha)(M-1)}{2M} \right] \right)$$

into

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha)e^{j2\pi(k+\alpha)n/M}$$

leads to simplier expression to find the impulse response looking at the four cases:

Symmetric $h(n)$: β=0

$\beta = 0, \quad \alpha = 0$

$\beta = 0, \alpha = 1/2$

Antisymmetric $h(n)$: β=1

$\beta = 1, \quad \alpha = 0$

$\beta = 1, \alpha = 1/2$

Frequency Sampling Method

	Symmetric
$\alpha = 0$	$H(k) = G(k)e^{j\pi k/M}, \qquad k = 0, 1 \ldots, M-1$ $G(k) = (-1)^k H_r\left(\frac{2\pi k}{M}\right), \qquad G(k) = -G(M-k)$ $h(n) = \frac{1}{M}\left\{G(0) + 2\sum_{k=1}^{U} G(k)\cos\frac{2\pi k}{M}\left(n+\frac{1}{2}\right)\right\}$ $U = \begin{cases} \frac{M-1}{2}, & M \text{ odd} \\ \frac{M}{2} - 1, & M \text{ even} \end{cases}$
$\alpha = \frac{1}{2}$	$H\left(k+\frac{1}{2}\right) = G\left(k+\frac{1}{2}\right)e^{-j\pi/2}e^{j\pi(2k+1)/2M}$ $G\left(k+\frac{1}{2}\right) = (-1)^k H_r\left[\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\right]$ $G\left(k+\frac{1}{2}\right) = G\left(M-k-\frac{1}{2}\right)$ $h(n) = \frac{2}{M}\sum_{k=0}^{U} G\left(k+\frac{1}{2}\right)\sin\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)$

Frequency Sampling Method

Antisymmetric	
	$H(k) = G(k)e^{j\pi/2}e^{j\pi k/M}, \qquad k = 0, 1, \ldots, M-1$
$\alpha = 0$	$G(k) = (-1)^k H_r\left(\frac{2\pi k}{M}\right), \qquad G(k) = G(M-k)$
	$h(n) = -\frac{2}{M}\sum_{k=1}^{(M-1)/2} G(k)\sin\frac{2\pi k}{M}\left(n+\frac{1}{2}\right), \qquad M \text{ odd}$
	$h(n) = \frac{1}{M}\left\{(-1)^{n+1}G(M/2) - 2\sum_{k=1}^{(M/2)-1} G(k)\sin\frac{2\pi}{M}k\left(n+\frac{1}{2}\right)\right\}, \quad M \text{ even}$
	$H\left(k+\frac{1}{2}\right) = G\left(k+\frac{1}{2}\right)e^{j\pi(2k+1)/2M}$
$\alpha = \frac{1}{2}$	$G\left(k+\frac{1}{2}\right) = (-1)^k H_r\left[\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\right]$
	$G\left(k+\frac{1}{2}\right) = -G\left(M-k-\frac{1}{2}\right); \qquad G(M/2) = 0 \text{ for } M \text{ odd}$
	$h(n) = \frac{2}{M}\sum_{k=0}^{V} G\left(k+\frac{1}{2}\right)\cos\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)$
	$V = \begin{cases} \frac{M-3}{2}, & M \text{ odd} \\ \frac{M}{2}-1, & M \text{ even} \end{cases}$

Frequency Sampling Method

- Usefulness of frequency sampling method
  - You can define the values of the $H(k)$ to be
    - 1 in the pass-band
    - 0 in the stop-band
  - You define the value (or values) in the transition band to get the fall-off the way you want it

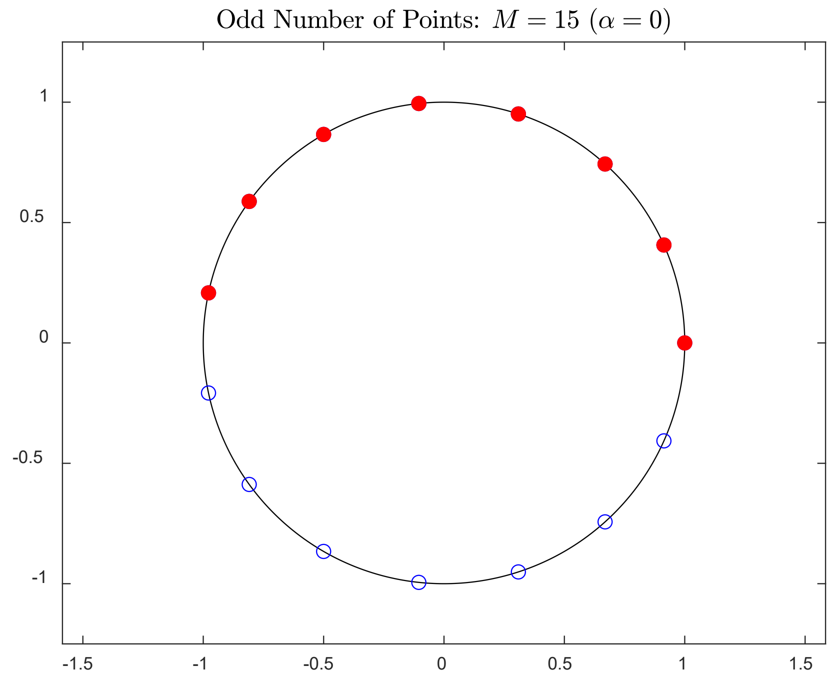
Frequency Sampling Method

- Example: M=15, α=0 symmetric case (β=0)

Subsituting

$$H_r\left(\frac{2\pi k}{15}\right)=\begin{cases}1.0 & k=0,1,2,3\\ 0.4 & k=4\\ 0.0 & k=5,6,7\end{cases}$$

If sampling at 3000 Hz, Nyquist at 1500 Hz
Specify frequency points every 100 Hz
Pass band ends at 300 Hz
Stop band starts at 500 Hz
Transition point specified at 400 Hz



Frequency Sampling Method

- There is a way of determining optimum frequency in the transition band
  - See Appendix B in book or get the original paper: Rabiner, Gold, McGonegal. (1970) An approach to the approximation problem for nonrecursive digital filters.

Optimum Equiripple FIR Filters

- Optimum Equiripple Linear-Phase FIR Filters
  - Windowing and frequency selection methods
    - Lack good control over the pass-band and stop-band frequencies
  - Optimum equiripple spreads approximation error is spread evenly across pass-band and stop-band.
  - Remember our 4 types of FIR filters?
    - Type I FIR – symmetric, M odd, most versatile form
    - Type II FIR –symmetric, M even, cannot be used for high-pass filters
    - Type III FIR –antisymmetric, M odd, cannot be used for LP or HP filters
    - Type IV FIR - antisymmetric, M even, cannot be used for LP filters
  - Chapter 10.2.4 has a very general description of optimization considering a

Optimum Equiripple FIR Filters

- Chapter 10.2.4 has a very general description of optimization considering all 4 types of FIR filters simultaneously
- We will just consider Type I

$$H(\omega) = \sum_{n=0}^{(M-1)/2} h(n) e^{-j\omega n}$$

$$H(\omega) = e^{-j\omega(M-1)/2} \sum_{n=0}^{(M-1)/2} a(n) \cos(\omega n)$$

where

$$a(0) = h\left(\frac{M-1}{2}\right)$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right), \quad n = 1, 2, \ldots, \frac{M-1}{2}$$

Optimum Equiripple FIR Filters

- Magnitude of frequency response is given by:

Type 1 FIR filter response:

$$H_r(\omega) \equiv \left|H(\omega)\right| = \sum_{n=0}^{(M-1)/2} a(n)\cos(\omega n)$$

- When you design a filter you have a desired response which is also real-valued, since we know the phase
  - Required it to be linear

  Desired frequency response of filter: $H_d(\omega)$

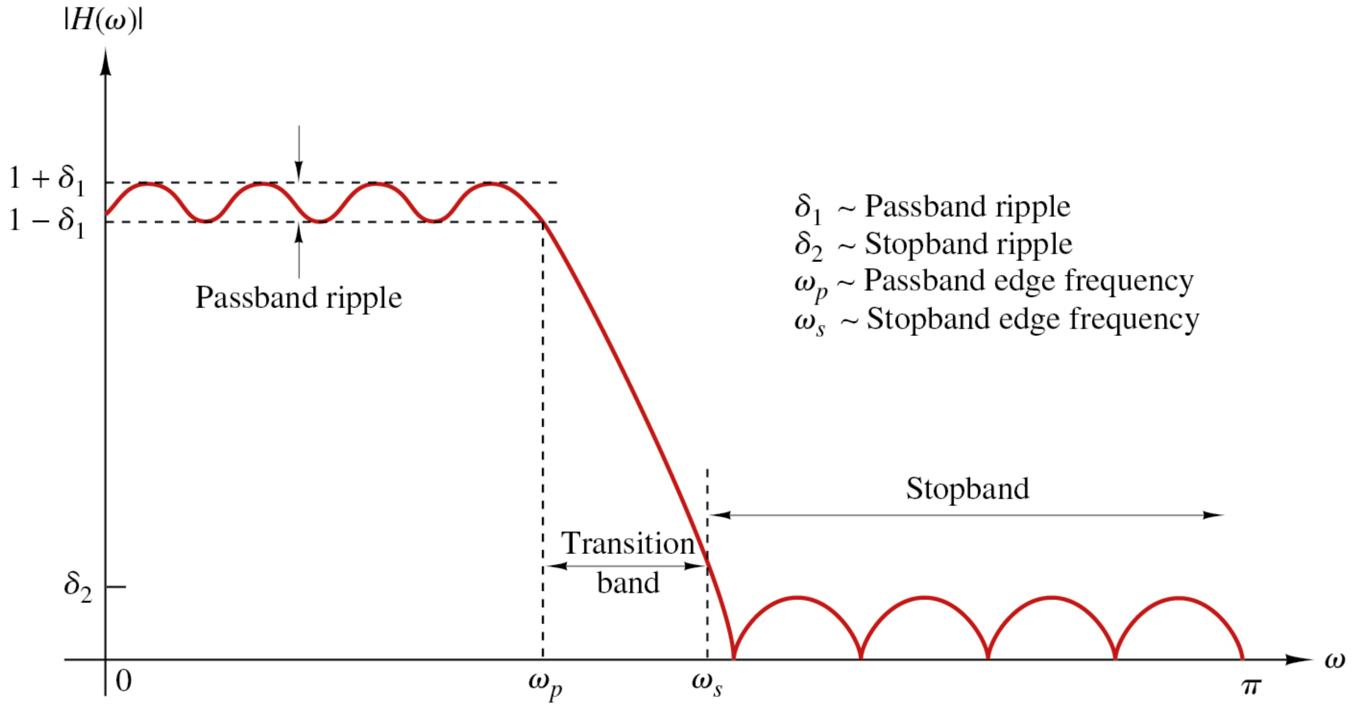
  - If we want to spread the error out in some way, we can define a weighting function

Optimum Equiripple FIR Filters

– If we want to spread the error out in some way, we can define a weighting function $W(\omega)$

Weighting function

$$W(\omega)=\begin{cases} 1 & \omega \text{ in Pass-band} \\ \delta_2/\delta_1 & \omega \text{ in Stop-band} \end{cases}$$



Optimum Equiripple FIR Filters

- The optimization problem:
  - The error between the desired filter $H_{dr}(\omega)$, the analytic description of the filter we end up with by adjusting parameters

$$E(\omega) = W(\omega)\left[H_{dr}(\omega) - H_r(\omega)\right] \qquad \text{where } H_r(\omega) = \sum_{n=0}^{(M-1)/2} a(n)\cos(\omega n)$$

  - Mathematical description of problem:

$$\min_{\text{over}\{a(k)\}} \left[\max_{\text{over}\,\omega \in S} \left| W(\omega) H_{dr}(\omega) - H_r(\omega) \right| \right]$$

    - *S* is the set of frequency bands.
      For low-pass filter, 2 elements: pass-band and stop-band
    - This is called the Chebyshev optimization problem
  - Find parameters, $a(k)$, that minimize the maximum error in the pass-band and stop-band

Optimum Equiripple FIR Filters

– This optimization problem was solved in a paper by Parks & McClellan (1972)

Parks, T. W., and McClellan, J. H. 1972a. "Chebyshev-Approximation for Nonrecursive DigitalFilters with Linear Phase," IEEE Trans. Circuit Theory, Vol. CT-19, pp. 189–194.

- Alternation Theorem:

Solve: $\dfrac{dE(\omega)}{d\omega} = -\dfrac{dH_r(\omega)}{d\omega} = 0$ which has a solution for a set of $L-1$ frequencies

that are peaks in the error you are trying to minimize.

ω_p and ω_s are also extreme points as well as $\omega = 0$ and $\omega = \pi$. The frequence are a set $\{\omega_n\}$

$$W(\omega_n)\left[H_{dr}(\omega_n) - H_r(\omega_n)\right] = (-1)^n \delta, \quad n = 0,1,\ldots,L+2$$

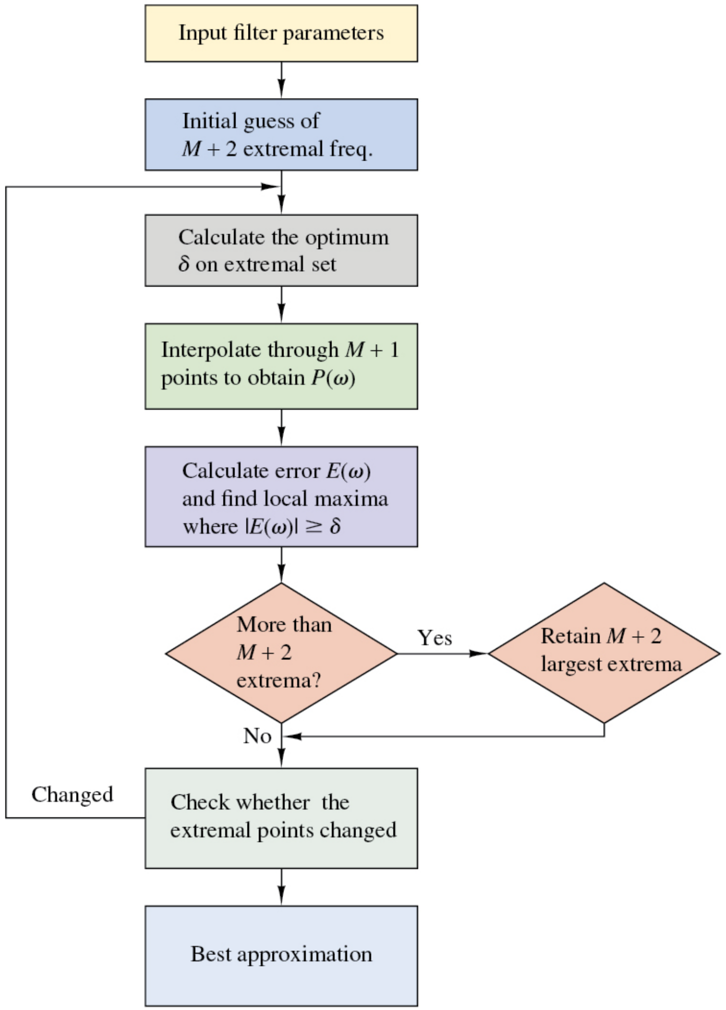
δ is the maximum value of the error function. (For LP filter, δ=δ_2)

Optimum Equiripple FIR Filters

- Alternation Theorem says there is a unique solution.
- To find solution (*a*'s that minimize the error) an iterative approach is used:
  The Remez exchange algorithm

Optimum Equiripple FIR Filters

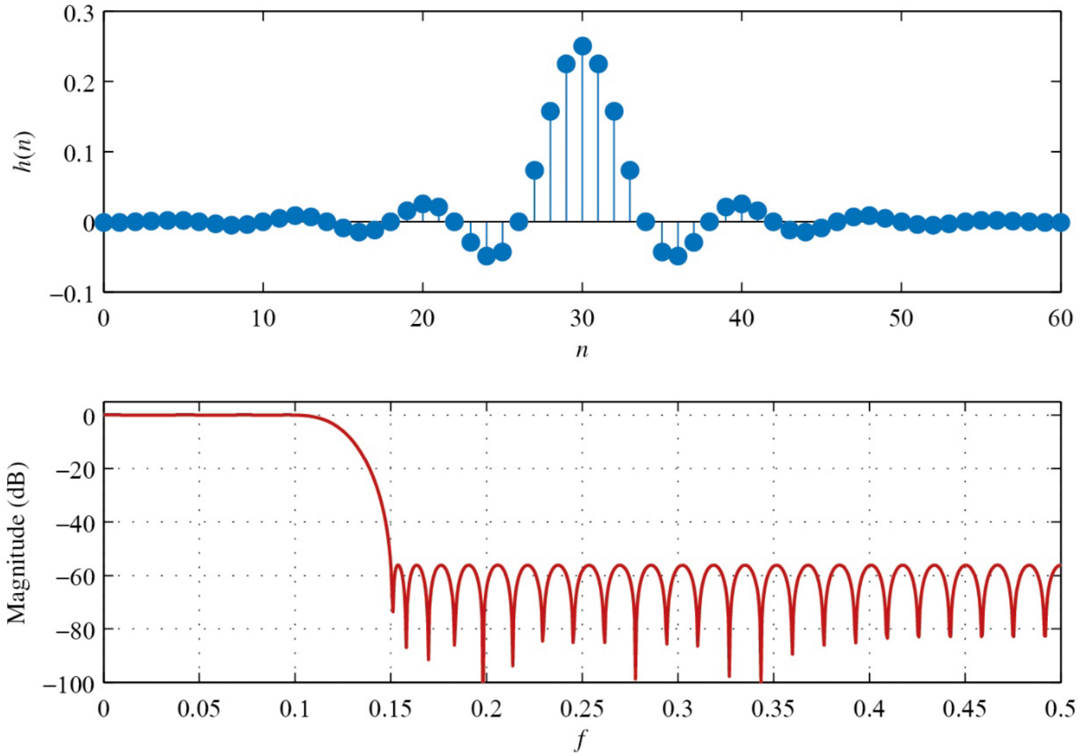
- The Remez exchange algorithm



Optimum Equiripple FIR Filters

- Example of equiripple filter:

Design a lowpass filter of length M=61 with a passband edge frequency f_p = 0.1 and a stopband edge frequency f_s=0.15



Comparison of FIR Design Methods

- Comparison of methods for linear-phase FIR filters
  - Window method was usual method up until 1970's
  - In 70's frequency sampling and Chebyshev approximation where developed
  - Window method
    - Advantages: Easiest to do if you don't have Matlab
    - Disadvantages: Poor control of pass-band & stop-band frequencies
  - Frequency sampling is improvement over window method
    - Pick frequencies where pass-band frequency response is 1 and stop-band frequency response is 0
    - Problem is picking response in transition region

Comparison of FIR Design Methods

- Comparison of methods for linear-phase FIR filters
  - Chebyshev method gives you control over filter parameters and allowed deviation from ideal filter.

    $\omega_s,\ \omega_p,\ \delta_2/\delta_1$
  - You do need to set the length of the filter.
  - What you would like would be to set $\omega_s,\ \omega_p,\ \delta_1,\ \delta_2$ and figure out how long the filter has to be to meet the criteria.
    - Two formulas: one from Kaiser, another (more accurate one) from paper by Rabiner et al.

Comparison of FIR Design Methods

Kaiser formula for filter length for given criteria:

$$\hat{M} = \frac{-20\log_{10}\left(\sqrt{\delta_1\delta_2}\right)-13}{14.6\left(\omega_s-\omega_p\right)/2\pi}$$

Rabiner:

$$\hat{M} = \frac{D_\infty\left(\delta_1,\delta_2\right)-f\left(\delta_1,\delta_2\right)\left(\Delta f\right)^2}{\Delta f}+1, \text{ where } \Delta f = \left(\omega_s-\omega_p\right)/2\pi \text{ and}$$

$$\begin{aligned} D_\infty(\delta_1,\delta_2) &= [0.005309(\log_{10}\delta_1)^2+0.07114(\log_{10}\delta_1)-0.4761](\log_{10}\delta_2) \\ &\quad -[0.00266(\log_{10}\delta_1)^2+0.5941\ \log_{10}\delta_1+0.4278] \\ f(\delta_1,\delta_2) &= 11.012+0.51244(\log_{10}\delta_1-\log_{10}\delta_2) \end{aligned}$$

Comparison of FIR Design Methods

- Frequency sampling method for FIR filter design
  - Conceptually straightforward
    - Specify desired response at set of equally spaced frequencies
    - Solve for $h(n)$ from the specified response
    - Optimize frequency specification in transition band to reduce sidelobes
  - Optimal Equiripple linear-phase FIR filters
    - Enable more precise control of pass and stop-band critical frequencies