Digital Signal Processing

Class 24
04/17/2025

ENGR 71

- Class Overview
  - Wrapup FIR Filter Design
    - Review solution to problem 10.1
  - Digital Filter Design
    - IIR filters
- Assignments
  - Reading:
    Chapter 10: Design of Digital Filters
    https://www.mathworks.com/help/signal/ug/fir-filter-design.html
  - Problems: 10.2, 10.3,10.6
    - Due April 20 (Sunday)
  - Lab 3: “Fun with Filters”
    - Due May 4 (Sunday)

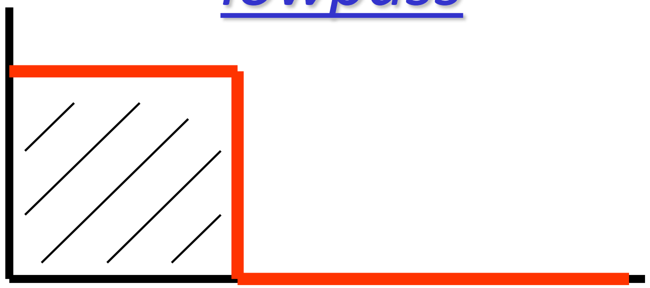
Project

- Projects
  - You can work in groups if you wish
  - Presentation at time reserved for final exam
    - Friday, May 9, 7:00-10:00 PM
    - Science Center 264
  - Submit a brief description of your proposed project and who you will be working with to the Project Description Dropbox
  - Submit slides from presentation to Project Dropbox
  - Submit written report to Project Dropbox by end of semester (May 15)

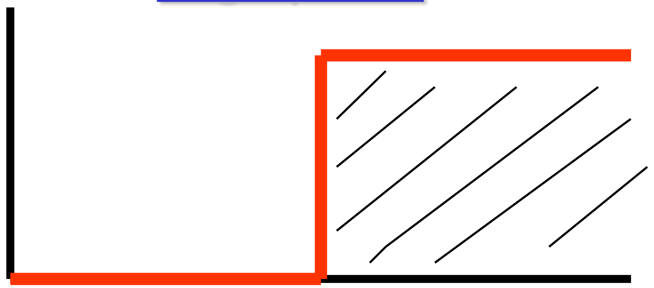
Filters

- Design of Digital Filters

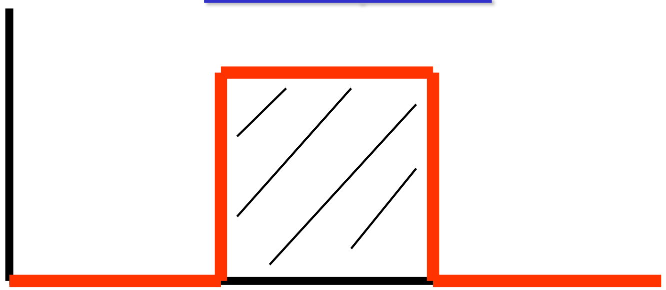
lowpass



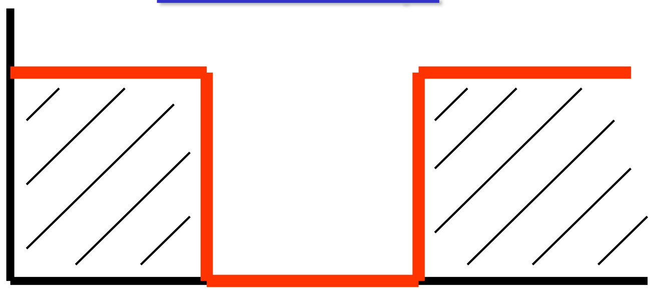
highpass



bandpass

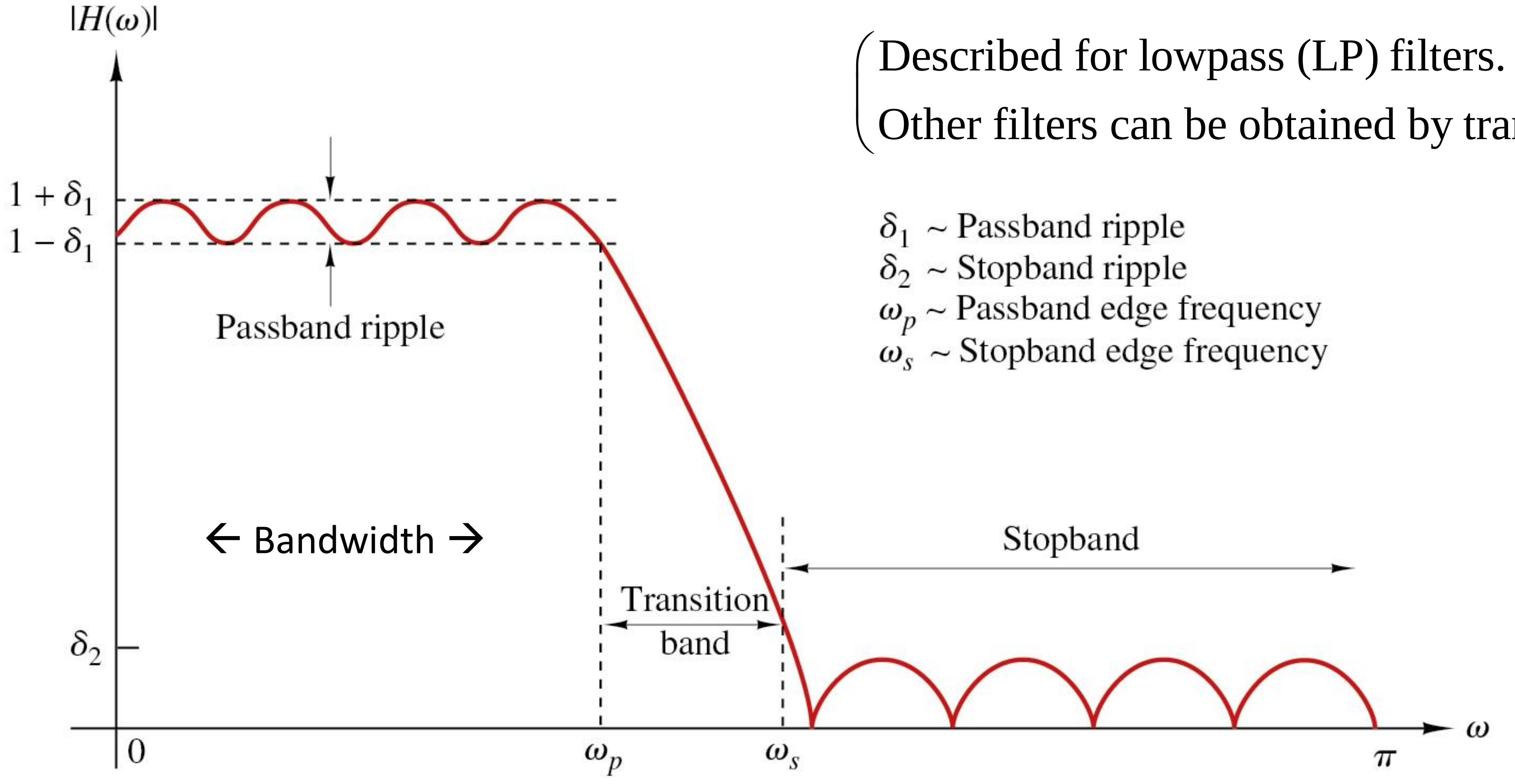


bandstop



Filter Design

- Specifications for physically realizable filters:



(Described for lowpass (LP) filters.
Other filters can be obtained by transforming LP filters)

δ_1 ~ Passband ripple
δ_2 ~ Stopband ripple
ω_p ~ Passband edge frequency
ω_s ~ Stopband edge frequency

Filters

- Design of Digital Filters
  - Finite Impulse Response (FIR)
    - Used when linear-phase required in passband
      - Constant time delay
      - No dispersion as a function of frequency
    - Three methods discussed
      - Windows, Frequency sampling, Iterative method for optimum equiripple filters
  - Infinite Impulse Response (IIR)
    - No requirement on linear-phase
    - Better characteristics for fewer parameters
    - Less memory
    - Lower computational complexity

FIR Filter Design

- Finite Impulse Response (FIR) Filters Wrap Up
  - We have only talked about low-pass filters
  - How do you construct high-pass, band-pass, and band-stop filters with the windowing method?
    - The approach is the same as for the low pass
      - You define the ideal transfer function $H_d(\omega)$
        - With linear phase based on filter length built in
      - Find the ideal impulse response, $h_d(n)$, using the inverse DTFT
      - Window the result to get the final impulse response: $h(n)=w(n)h_d(n)$
      - To find the transfer function for the modified impulse response, take the forward DTFT to get $H(\omega)$

FIR Filter Design - Lowpass

- Lowpass

$$H_d(\omega) = \begin{cases} 1e^{-j\omega(M-1)/2}, & 0 \le |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1e^{-j\omega(M-1)/2} e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1e^{j\omega[n-(M-1)/2]} d\omega$$

$$= \frac{1}{2\pi} \left. \frac{e^{j\omega[n-(M-1)/2]}}{j[n-(M-1)/2]} \right|_{-\omega_c}^{\omega_c} = \frac{2}{2\pi}\left[\frac{e^{j\omega_c[n-(M-1)/2]} - e^{-j\omega_c[n-(M-1)/2]}}{2j[n-(M-1)/2]} \right]$$

$$h_d(n) = \frac{1}{\pi} \frac{\sin[\omega_c(n-(M-1)/2)]}{[n-(M-1)/2]} = \frac{\omega_c}{\pi} \operatorname{sinc}[\omega_c(n-(M-1)/2)]$$

(unnormalized sinc function)

Filter Design - Highpass

- Highpass

$$H_d(\omega)=\begin{cases}1e^{-j\omega(M-1)/2}, & \omega_c\le|\omega|\le\pi\\ 0, & \text{otherwise}\end{cases}$$

$$h_d(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d(\omega)e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{-\pi}^{-\omega_c}1e^{-j\omega(M-1)/2}e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{\omega_c}^{\pi}1e^{j\omega[n-(M-1)/2]}d\omega$$

$$=\frac{1}{2\pi}\frac{e^{j\omega[n-(M-1)/2]}}{j[n-(M-1)/2]}\Bigg|_{-\pi}^{-\omega_c}+\frac{1}{2\pi}\frac{e^{j\omega[n-(M-1)/2]}}{j[n-(M-1)/2]}\Bigg|_{\omega_c}^{\pi}=\frac{2}{2\pi}\left[\frac{e^{-j\omega_c[n-(M-1)/2]}-e^{-j\pi[n-(M-1)/2]}}{2j[n-(M-1)/2]}\right]+\frac{2}{2\pi}\left[\frac{e^{j\pi[n-(M-1)/2]}-e^{j\omega_c[n-(M-1)/2]}}{2j[n-(M-1)/2]}\right]$$

$$=\frac{2}{2\pi}\left[\frac{e^{j\pi[n-(M-1)/2]}-e^{j\pi[n-(M-1)/2]}}{2j[n-(M-1)/2]}\right]-\frac{2}{2\pi}\left[\frac{e^{j\omega_c[n-(M-1)/2]}-e^{-j\omega_c[n-(M-1)/2]}}{2j[n-(M-1)/2]}\right]$$

integer

$$h_d(n)==\left[\frac{\sin[\pi(n-(M-1)/2)]}{\pi[n-(M-1)/2]}\right]-\frac{\omega_c}{\pi}\left[\frac{\sin[\omega_c(n-(M-1)/2)]}{[n-(M-1)/2]}\right]=\mathrm{sinc}[\pi(n-(M-1)/2)]-\frac{\omega_c}{\pi}\mathrm{sinc}[\omega_c(n-(M-1)/2)]$$

$$\boxed{h_d(n)=\delta[n-(M-1)/2]-\mathrm{sinc}[\omega_c(n-(M-1)/2)]\quad \text{since } \sin(k)/(k)=0 \text{ except when } k=0\text{, in which case it is 1.}}$$

Filter Design - Bandpass

- Bandpass

$$H_d(\omega) = \begin{cases} 0, & \text{for } 0 \le |\omega| \le \omega_1 \\ 1e^{-j\omega(M-1)/2}, & \text{for } \omega_1 < |\omega| < \omega_2 \\ 0, & \text{for } \omega_2 \le |\omega| \le \pi \end{cases}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_2}^{-\omega_1} 1e^{-j\omega(M-1)/2}e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} 1e^{j\omega[n-(M-1)/2]}d\omega$$

$$= \frac{1}{2\pi}\frac{e^{j\omega[n-(M-1)/2]}}{j[n-(M-1)/2]}\Bigg|_{-\omega_2}^{-\omega_1} + \frac{1}{2\pi}\frac{e^{j\omega[n-(M-1)/2]}}{j[n-(M-1)/2]}\Bigg|_{\omega_1}^{\omega_2} = \frac{2}{2\pi}\left[\frac{e^{-j\omega_1[n-(M-1)/2]} - e^{-j\omega_2[n-(M-1)/2]}}{2j[n-(M-1)/2]}\right] + \frac{2}{2\pi}\left[\frac{e^{j\omega_2[n-(M-1)/2]} - e^{j\omega_1[n-(M-1)/2]}}{2j[n-(M-1)/2]}\right]$$

$$= \frac{2}{2\pi}\left[\frac{e^{j\omega_2[n-(M-1)/2]} - e^{j\omega_2[n-(M-1)/2]}}{2j[n-(M-1)/2]}\right] - \frac{2}{2\pi}\left[\frac{e^{j\omega_1[n-(M-1)/2]} - e^{-j\omega_1[n-(M-1)/2]}}{2j[n-(M-1)/2]}\right]$$

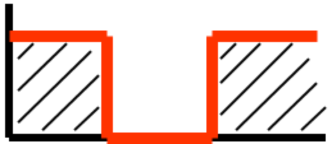
$$h_d(n) = \frac{\omega_2}{\pi}\frac{\sin[\omega_2(n-(M-1)/2)]}{[n-(M-1)/2]} - \frac{\omega_1}{\pi}\frac{\sin[\omega_1(n-(M-1)/2)]}{\omega_1[n-(M-1)/2]}$$

$$\boxed{h_d(n) = \frac{\omega_2}{\pi}\text{sinc}[\omega_2(n-(M-1)/2)] - \frac{\omega_1}{\pi}\text{sinc}[\omega_1(n-(M-1)/2)]}$$

Filter Design - Stopband

- Bandstop

$$H_d(\omega) = \begin{cases} 1e^{-j\omega(M-1)/2}, & \text{for } 0 \le |\omega| \le \omega_1 \\ 0, & \text{for } \omega_1 < |\omega| < \omega_2 \\ 1e^{-j\omega(M-1)/2}, & \text{for } \omega_2 \le |\omega| \le \pi \end{cases}$$



$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_2} 1e^{j\omega[n-(M-1)/2]}d\omega + \frac{1}{2\pi}\int_{-\omega_1}^{\omega_1} 1e^{-j\omega(M-1)/2}e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_2}^{\pi} 1e^{j\omega[n-(M-1)/2]}d\omega$$

$$= \frac{1}{2\pi}\left.\frac{e^{j\omega[n-(M-1)/2]}}{j[n-(M-1)/2]}\right|_{-\pi}^{-\omega_2} + \frac{1}{2\pi}\left.\frac{e^{j\omega[n-(M-1)/2]}}{j[n-(M-1)/2]}\right|_{-\omega_1}^{\omega_1} + \frac{1}{2\pi}\left.\frac{e^{j\omega[n-(M-1)/2]}}{j[n-(M-1)/2]}\right|_{\omega_2}^{\pi}$$

$$h_d(n) = \frac{\sin[\pi(n-(M-1)/2)]}{\pi[n-(M-1)/2]} - \frac{\omega_2}{\pi}\frac{\sin[\omega_2(n-(M-1)/2)]}{[n-(M-1)/2]} + \frac{\omega_1}{\pi}\frac{\sin[\omega_1(n-(M-1)/2)]}{\omega_1[n-(M-1)/2]}$$

$$\boxed{h_d(n) = \delta[n-(M-1)/2] - \left\{\frac{\omega_2}{\pi}\operatorname{sinc}[\omega_2(n-(M-1)/2)] - \frac{\omega_1}{\pi}\operatorname{sinc}[\omega_1(n-(M-1)/2)]\right\}}$$

FIR Filter Design

- Notice the relationship between lowpass & highpass and bandpass & stopband filters

Lowpass: $h_d^{LP}(n)$

Highpass: $h_d^{HP}(n) = \delta\left[n - \left(M - 1\right)/2\right] - h_d^{LP}(n);$

Bandpass: $h_d^{BP}(n)$

Stopband: $\delta\left[n - \left(M - 1\right)/2\right] - h_d^{BP}(n);$

- This is assuming you have a delay of $(M\text{-}1)/2$
  - With no delay it would be:

Lowpass: $h_d^{LP}(n)$

Highpass: $h_d^{HP}(n) = \delta\left(n\right) - h_d^{LP}(n);$

Bandpass: $h_d^{BP}(n)$

Stopband: $\delta\left(n\right) - h_d^{BP}(n);$

IIR Filters

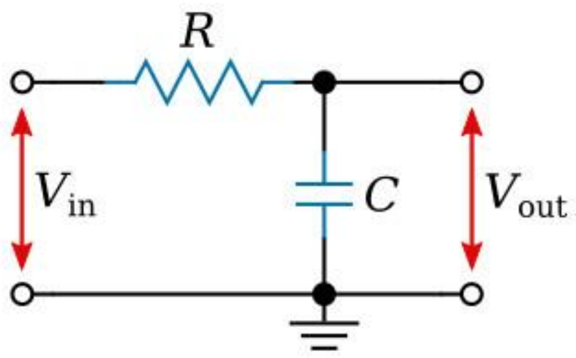
- Infinite Impulse Response Filters
  - Advantages
    - Usually require fewer coefficients to get similar response
    - Work faster
      - A consideration for hardware implementations
    - Require less memory
      - Again, probably on a consideration for hardware or firmware
  - Disadvantages
    - Nonlinear phase
      - Different frequency components have different delays
      - Causes distortion of signal's waveform shape

IIR Filters

- Methods for designing IIR filters
  - Start with analog filter and convert to a digital filter
    - Specified in terms of $H(s)$, transfer function in Laplace domain
    - In Laplace domain, derivatives become powers of s
  - Three methods
    - Approximation of derivatives in analog filter description
    - Impulse invariance
      - Involves sampling the continuous impulse response
    - Bilinear transformation

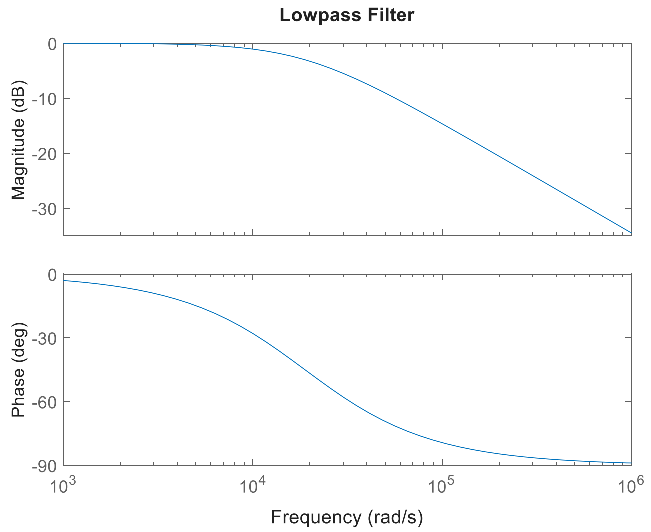
IIR Filters

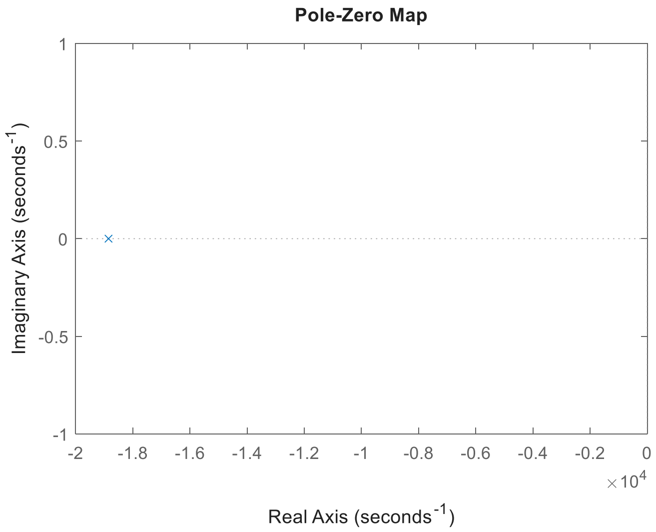
- Some simple analog filters
  - Lowpass



$$H(s) = \frac{1/RC}{s + (1/RC)} = \frac{\omega_c}{s + \omega_c}$$

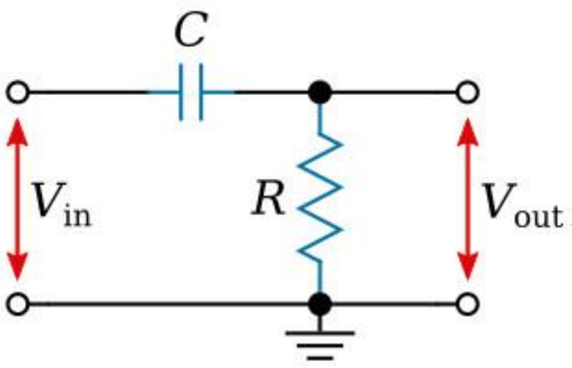
$$\omega_c = 1/RC \quad \text{(cutoff)}$$





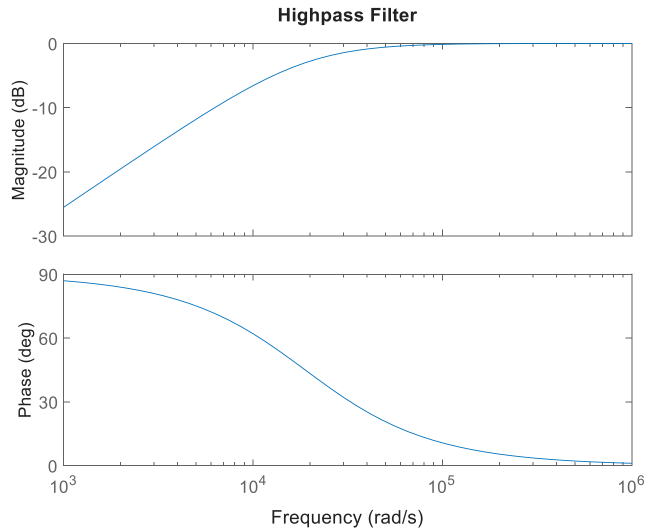
IIR Filters

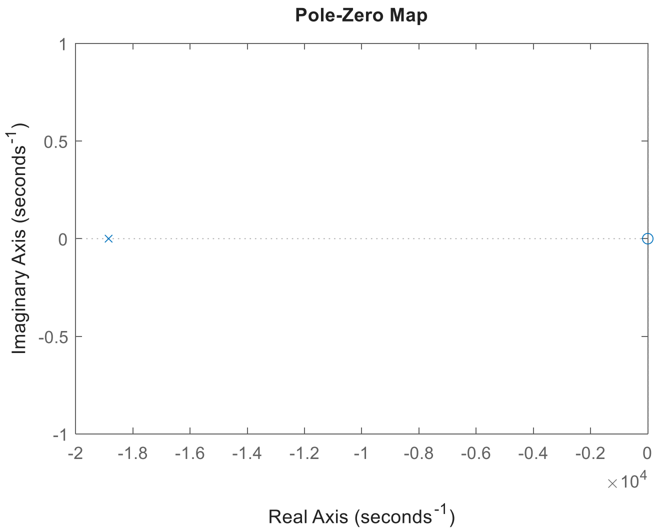
- Some simple analog filters
  - Highpass



$$H(s) = \frac{s}{s + (1/RC)} = \frac{s}{s + \omega_c}$$

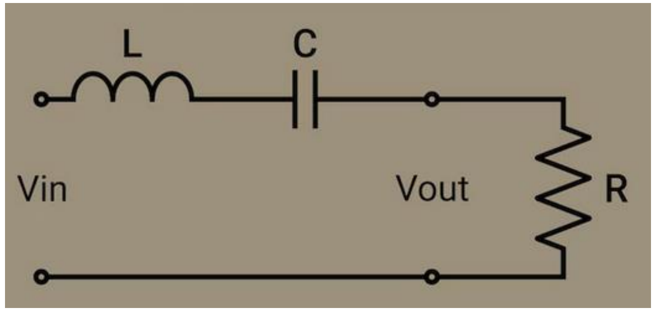
$$\omega_c = 1/RC \quad \text{(cutoff)}$$





IIR Filters

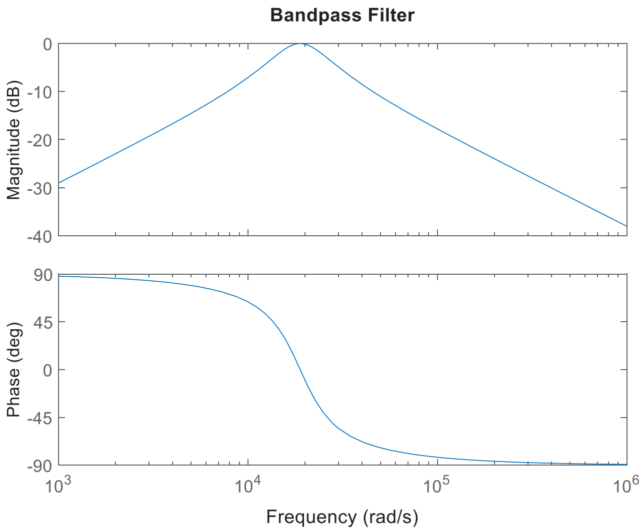
- Some simple analog filters
  - Bandpass:

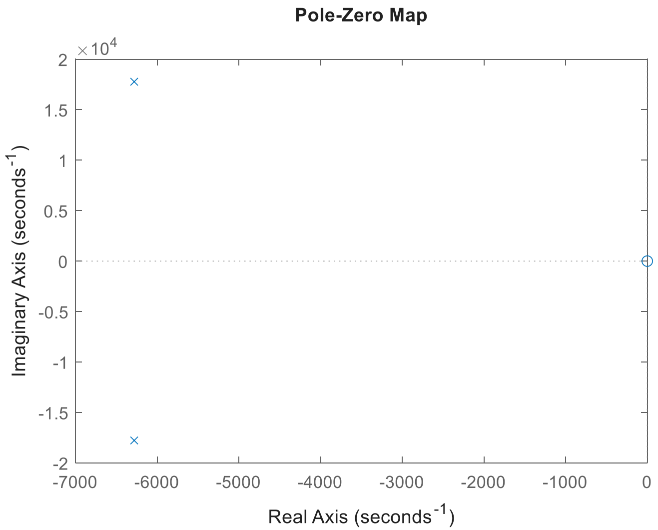


$$H(s)=\frac{(R/L)s}{s^2+(R/L)s+(1/LC)}=\frac{\beta s}{s^2+\beta s+\omega_0^2}$$

$$\omega_0=\sqrt{1/LC} \quad \text{(center frequency)}$$

$$\beta = R/L \quad \text{(bandwidth)}$$





IIR Filters-Derivative Approximation

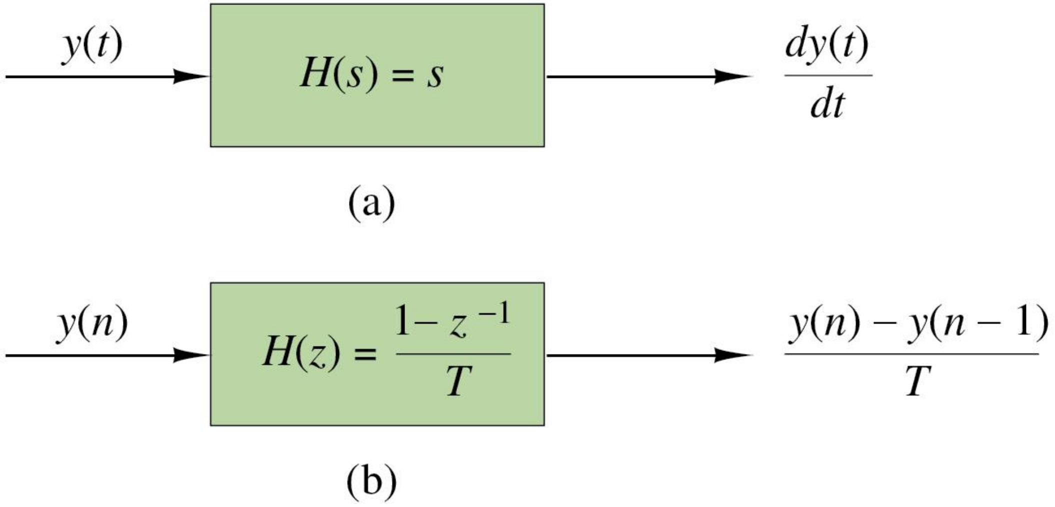
- Approximation of derivatives in analog filter
  - Convert Laplace domain system transfer function into discrete by first-order approximation of derivative

$$\frac{dy(t)}{dt} \approx \frac{y(nT) - y(nT - T)}{T} \rightarrow \frac{y(n) - y(n-1)}{T}$$

$$\mathcal{Z}\left[\frac{y(n) - y(n-1)}{T}\right] = \frac{1 - z^{-1}}{T} Y(z)$$

Replace s in analog transfer function with:

$$s \rightarrow \frac{1 - z^{-1}}{T}$$



IIR Filters-Derivative Approximation

Second order derivative is:

$$\frac{d^2y(t)}{dt^2} \approx \frac{\left[y(nT)-y(nT-T)\right]/T-\left[y(nT-T)-y(nT-2T)\right]/T}{T} \to \frac{y(n)-2y(n-1)+y(n-2)}{T}$$

$$\mathcal{Z}\left[\frac{y(n)-2y(n-1)+y(n-2)}{T^2}\right]=\frac{1-2z^{-1}-z^{-2}}{T^2}Y(z)=\left(\frac{1-z^{-1}}{T}\right)^2 Y(z)$$

Replace s^2 in analog transfer function with:

$$s^2 \to \left(\frac{1-z^{-1}}{T}\right)^2$$

In general

Replace s^k in analog transfer function with:

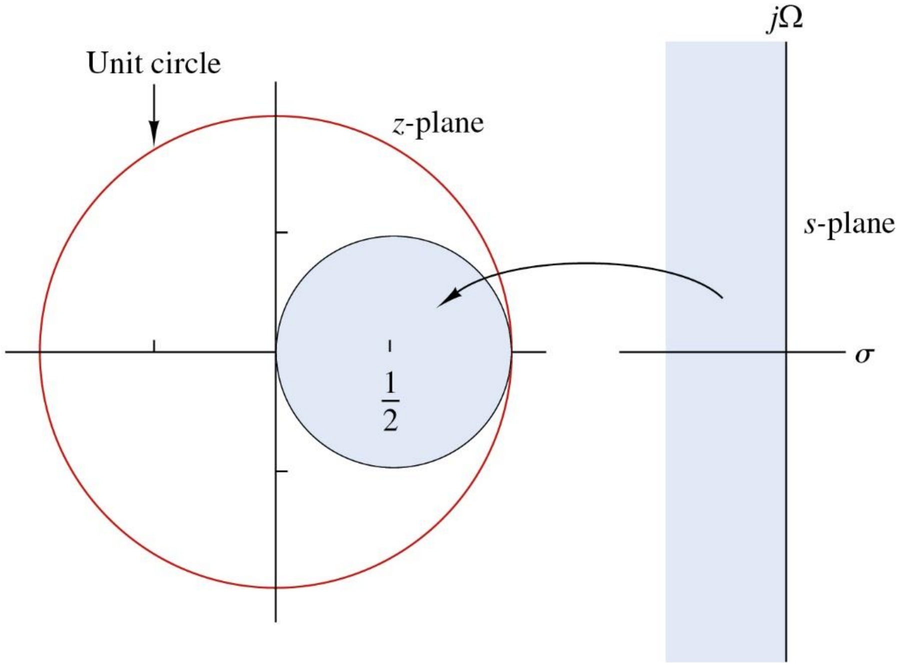
$$s^k \to \left(\frac{1-z^{-1}}{T}\right)^k \qquad \text{so} \qquad H(z)=H_a(s)\Big|_{s=\left(1-z^{-1}\right)/T}$$

IIR Filters-Derivative Approximation

- Approximation of derivatives in analog filter
  - Transformation:

$$s = \frac{1 - z^{-1}}{T} \quad ; \quad z = \frac{1}{1 - sT}$$

  - Maps negative half-plane of s into radius ½ circle centered at z = ½



IIR Filters-Derivative Approximation

- Example for lowpass filter

$$H(s) = \frac{1/RC}{s + (1/RC)} = \frac{\omega_c}{s + \omega_c}$$

$$H(z) = \frac{\omega_c}{(1 + z^{-1})/T + \omega_c} = \frac{\omega_c T}{1 + z^{-1} + \omega_c T}$$

$$H(z) = \frac{\omega_c T z}{(1 + \omega_c T)z + 1}$$

$$H(s) = \frac{1/RC}{s + (1/RC)} = \frac{\omega_c}{s + \omega_c}$$

$$H(z) = \frac{\omega_c}{(1 + z^{-1})/T + \omega_c} = \frac{\omega_c T}{1 + z^{-1} + \omega_c T}$$

$$H(z) = \frac{\omega_c T z}{(1 + \omega_c T)z + 1}$$

Set $f_s = 200000 \text{ Hz} \Rightarrow f_{NY} = 100000 \text{ Hz}$

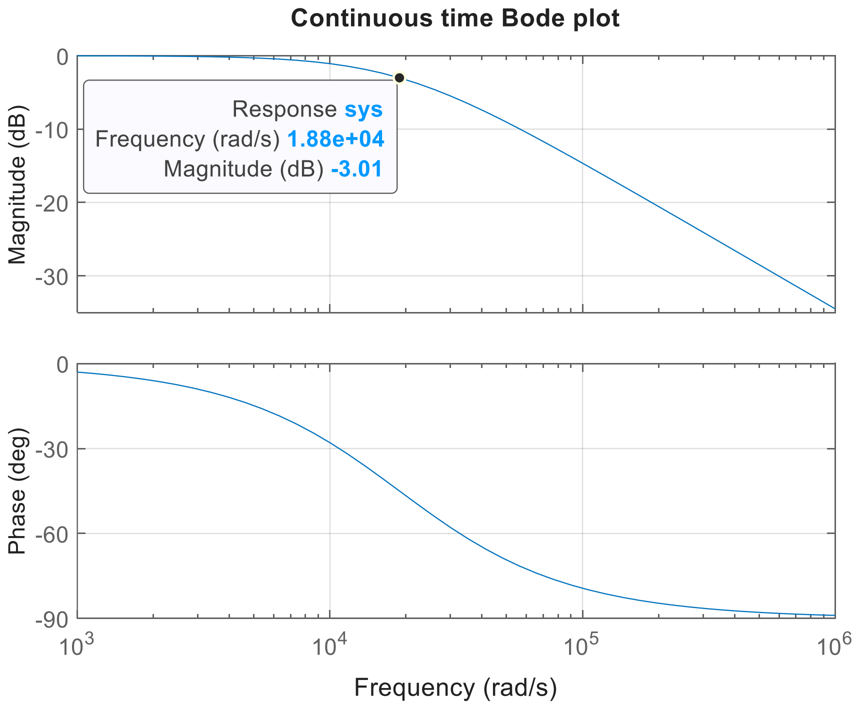
(This will be 1 on the normalized frequency scale in freqz)

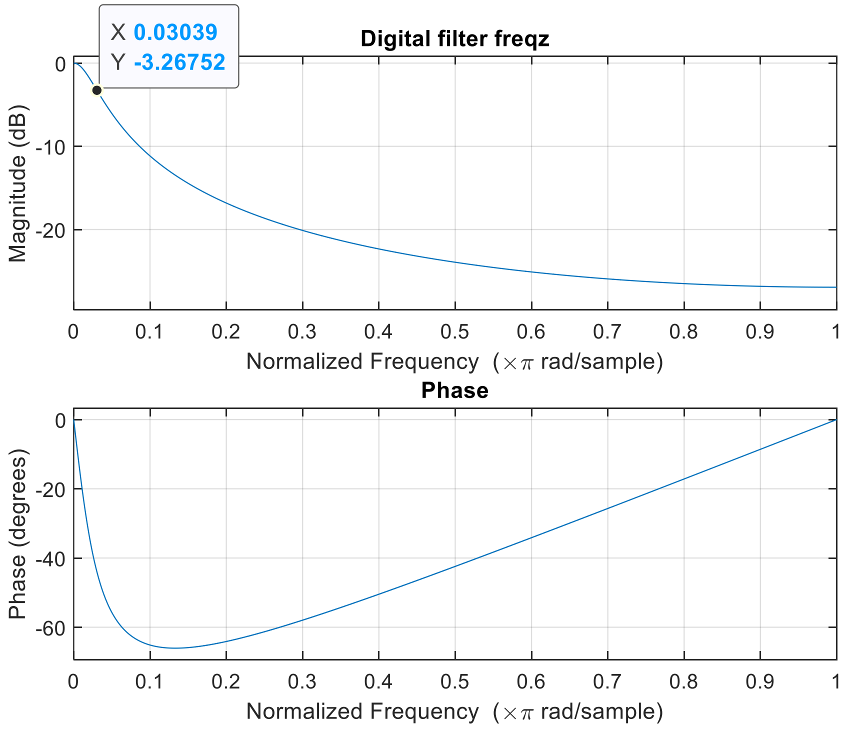
Set the cut-off frequency to 3000 Hz = 18850 rad/s

3000 is 0.03 of Nyquest, so the -3db point should be at normalized frequency 0.03

IIR Filters-Derivative Approximation

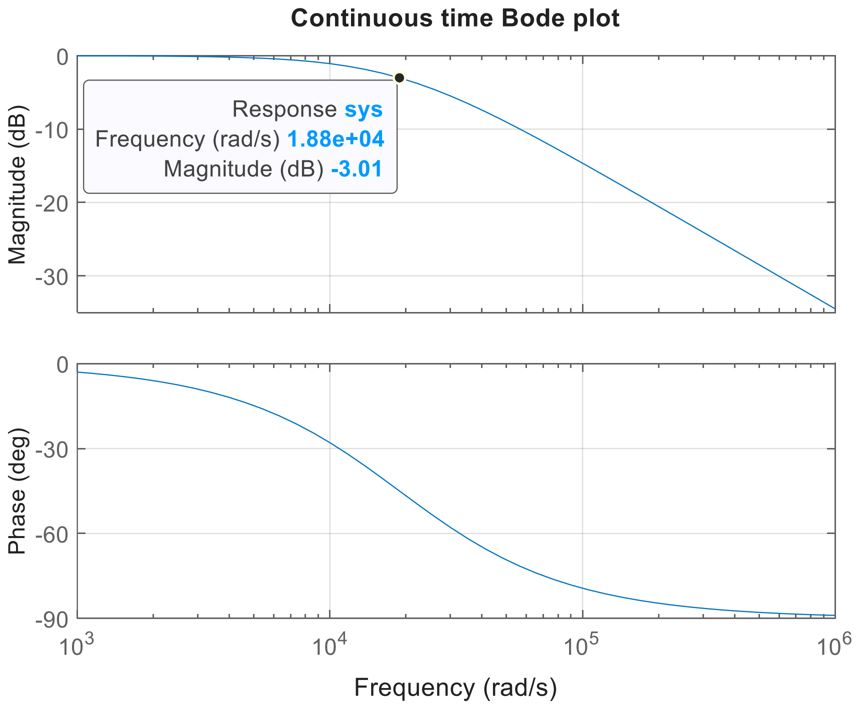
- Example for lowpass filter

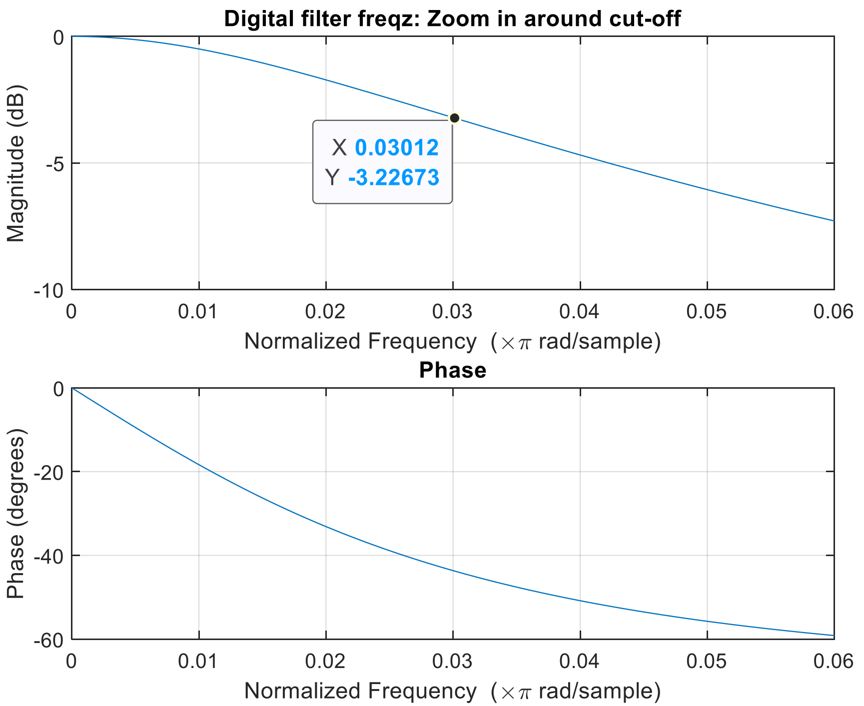




IIR Filters-Derivative Approximation

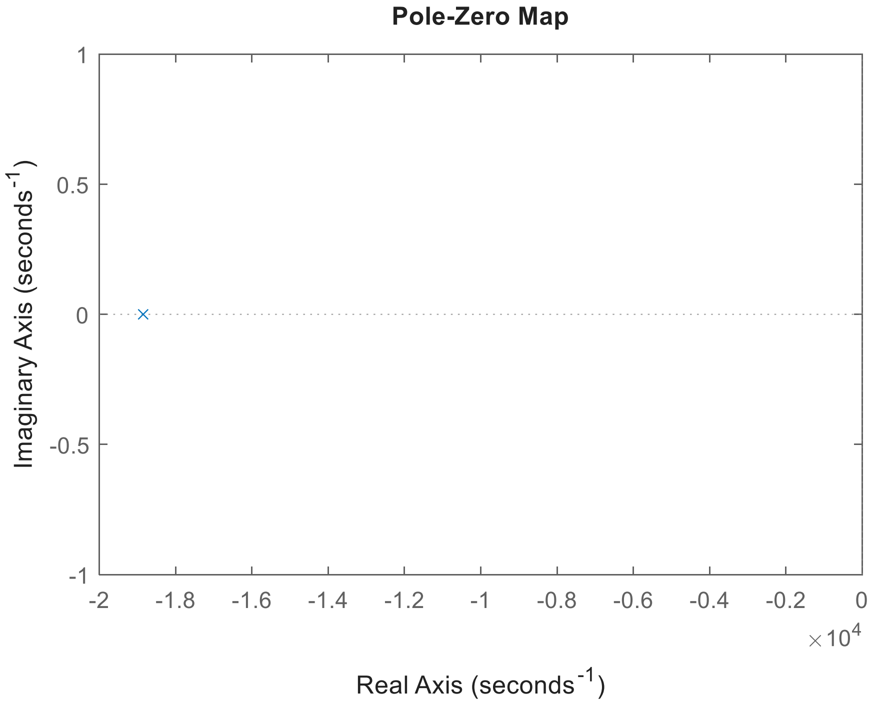
- Example for lowpass filter

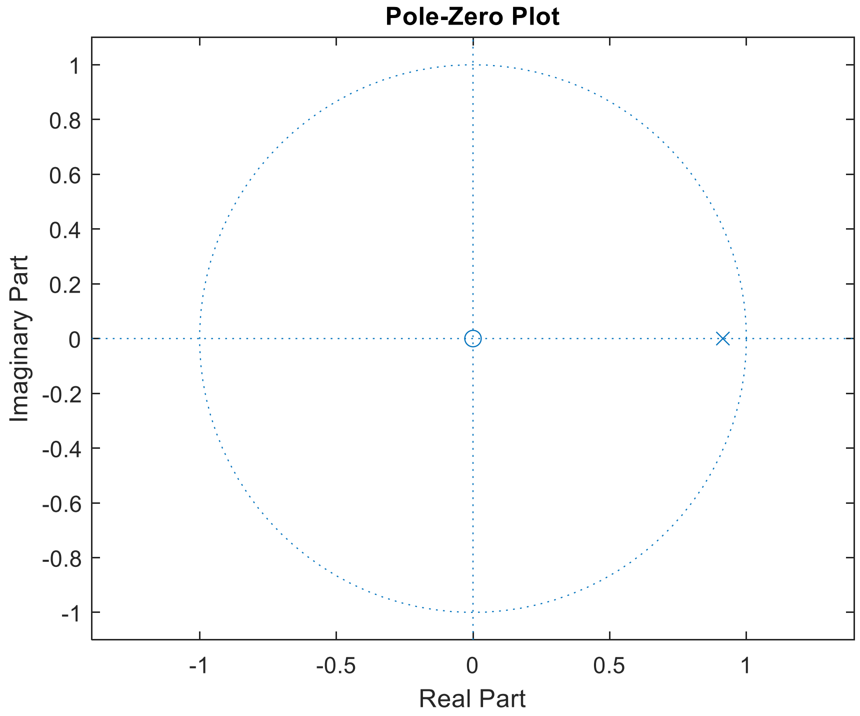




IIR Filters-Derivative Approximation

- Example for lowpass filter





IIR Filters-Derivative Approximation

- Example for highpass filter

$$H(s) = \frac{s}{s + (1/RC)} = \frac{s}{s + \omega_c}$$

$$H(z) = \frac{(1 + z^{-1})/T}{(1 + z^{-1})/T + \omega_c} = \frac{1 + z^{-1}}{1 + z^{-1} + \omega_c T}$$

$$H(z) = \frac{z + 1}{(1 + \omega_c T)z + 1}$$

Set $f_s = 200000$ Hz $\Rightarrow f_{NY} = 100000$ Hz

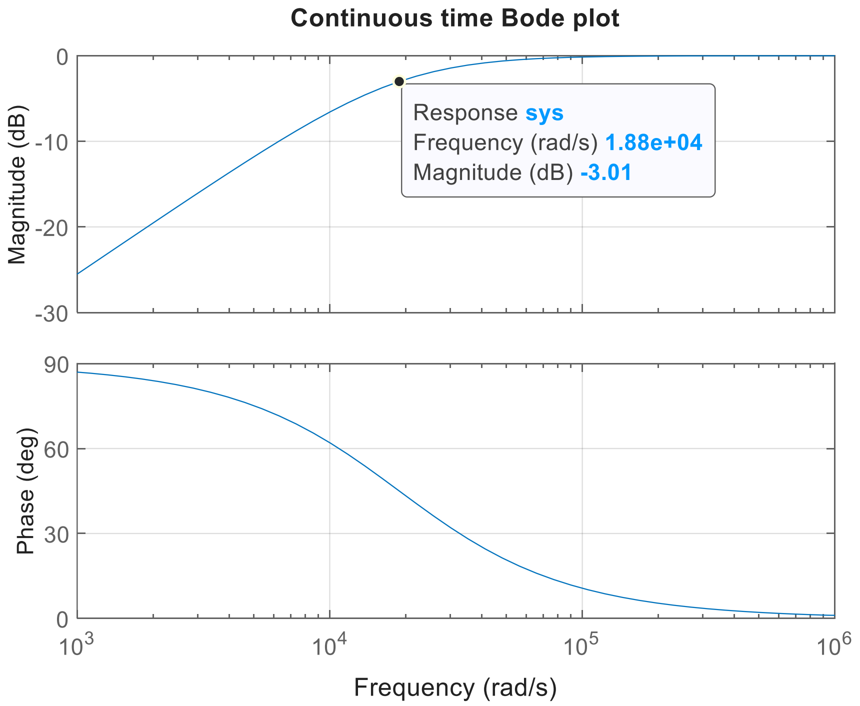
(This will be 1 on the normalized frequency scale in freqz)

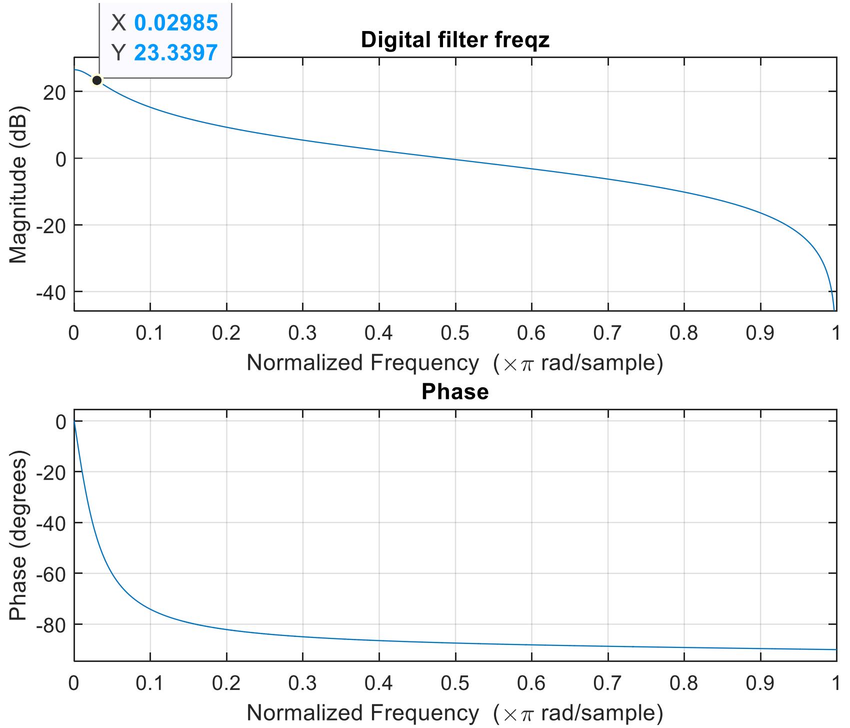
Set the cut-off frequency to 3000 Hz = 18850 rad/s

3000 is 0.03 of Nyquest, so the -3db point should be at normalized frequency 0.03

IIR Filters-Derivative Approximation

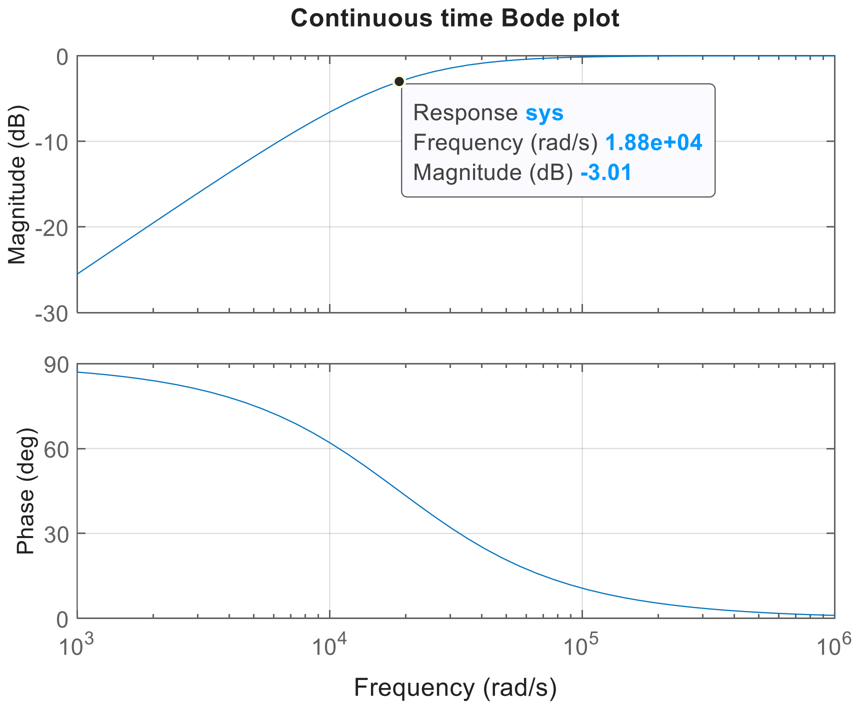
- Example for highpass filter   Does not work

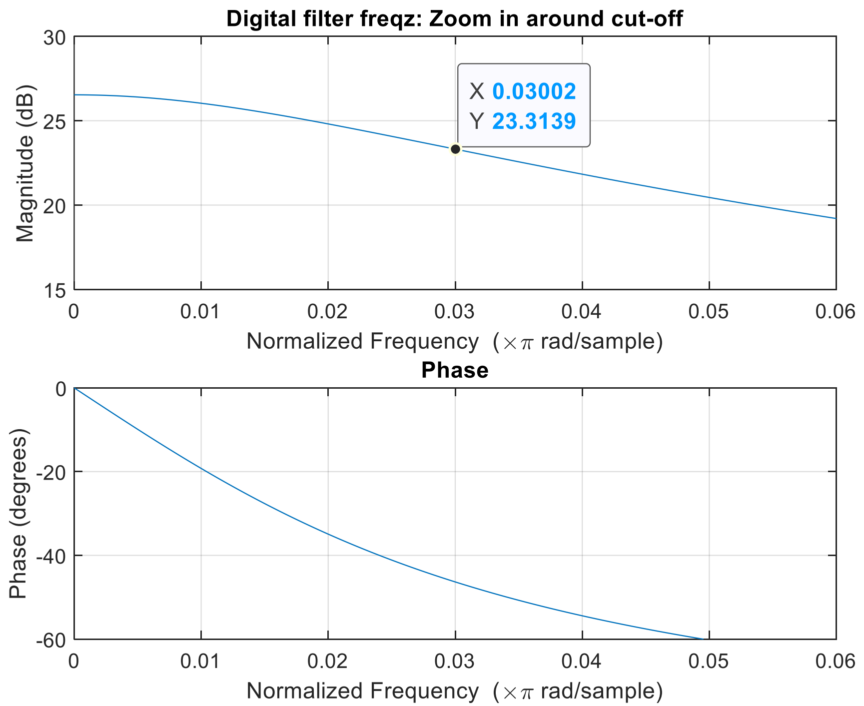




IIR Filters-Derivative Approximation

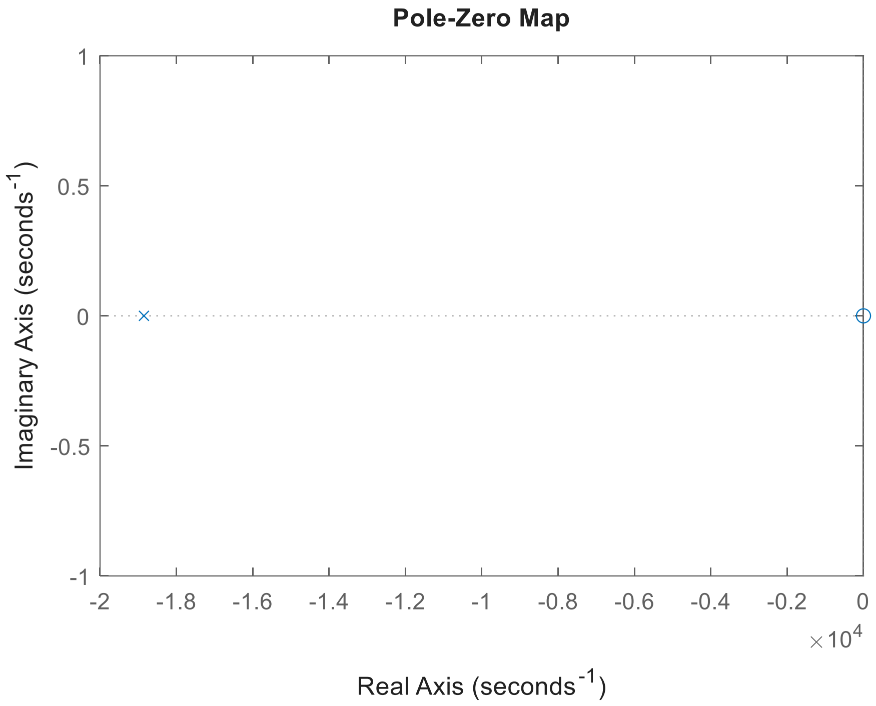
- Example for highpass filter

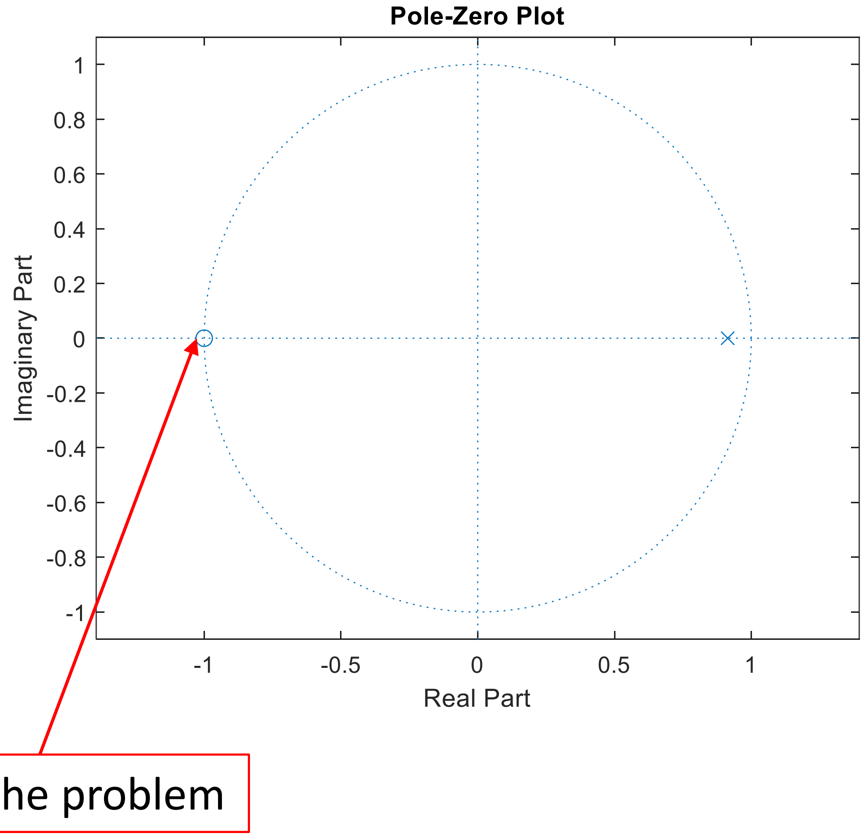




IIR Filters-Derivative Approximation

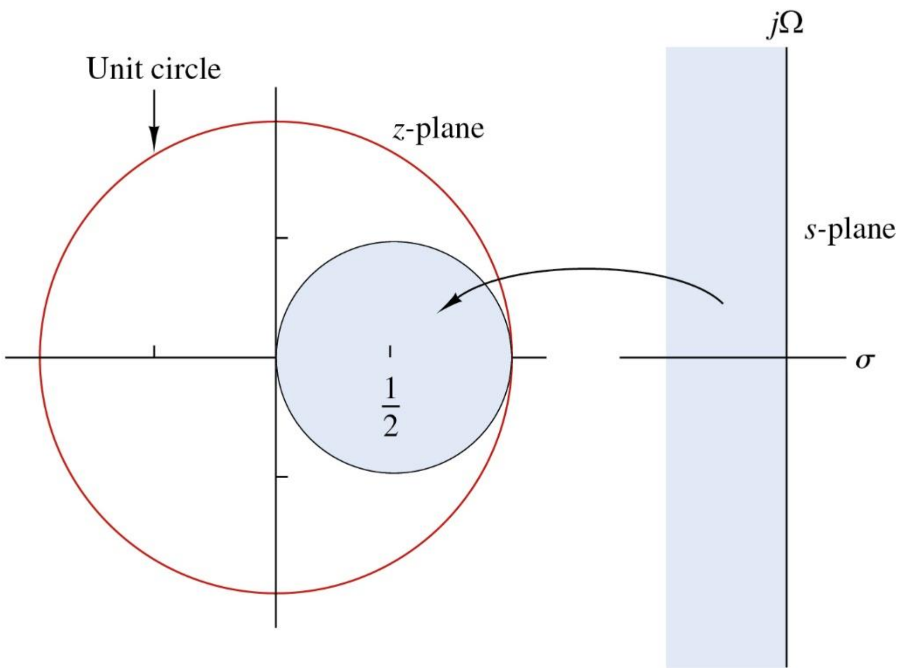
- Example for highpass filter





IIR Filters-Derivative Approximation

- Mapping of complex *s*-plane to *z*-plane
  - Approximation of derivatives



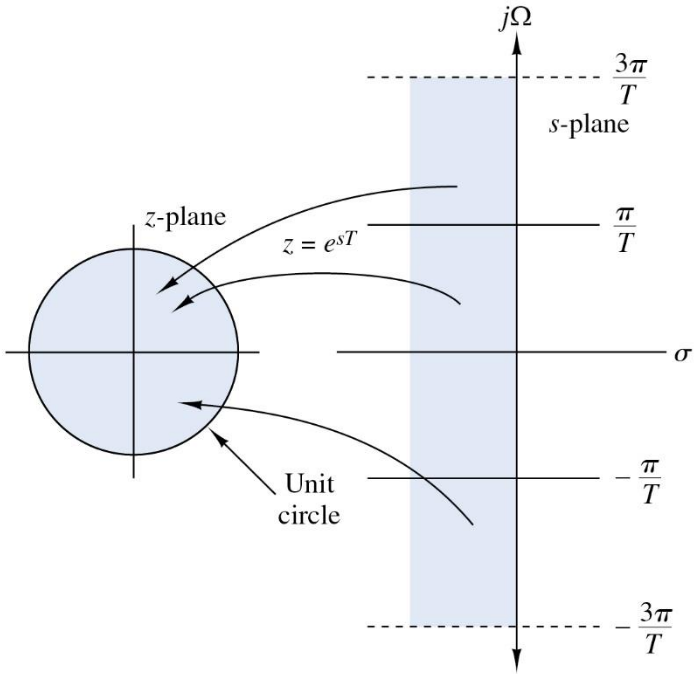
Only works for lowpass filters (or bandpass if poles and zeros map into the half circle shown)

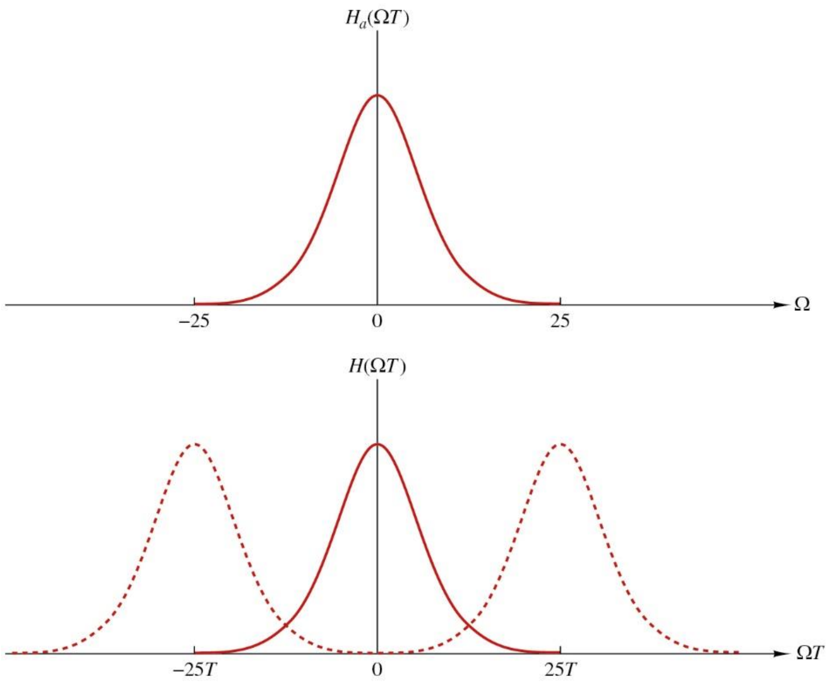
IIR Filters - Impulse Invariance

- Impulse invariance
  - Create sampled version of the continuous impulse response: $h(t) \rightarrow h(nT)$
  - When you sample in time, makes multiple copies of spectrum in frequency domain
  - Corresponds to mapping the s-plane to unit circle in z-plane multiple times.
  - If sample interval, T, is small enough, will not get aliasing for lowpass filter design
    - But, cannot be avoided for highpass or bandstop filters.

IIR Filters - Impulse Invariance

- Impulse invariance





Aliasing is the reason it's not useful for highpass or bandstop filters

IIR Filters – Bilinear Transform

- Bilinear transformation
  - Useful transformation for analog →digital filter design because it can be used for all filter types (LP,HP,BP,BS)
  - Bilinear transformation:

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \quad ; \quad z = \frac{2}{T}\left(\frac{1+sT/2}{1-sT/2}\right)$$

  - In complex variable theory this is a linear fractional transformation that has the general form:

$$w = \frac{az+b}{cz+d} \quad ; \quad z = \frac{-dw+b}{cw-a}$$

IIR Filters – Bilinear Transform

– In complex variable theory this is a linear fractional transformation that has the general form:

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For the bilinear transformation shown:

$$w = sT/2, \quad a = 1, \quad b = \text{-}1, \quad c = 1, \quad d = 1$$

- This is a conformal mapping
  - Maps each point in the *w* domain to a unique point in the *z* domain (except at $w = a/c$)
  - Derivative in nonzero and analytic
  - Preserves local angle preservation

IIR Filters – Bilinear Transform

- Motivation for bilinear transformation for DSP
  - Consider simple first-order system:

Differential equation: $y'(t) + ay(t) = bx(t) \Rightarrow y'(t) = -ay(t) + bx(t)$

System transfer function: $H(s) = \dfrac{b}{s+a}$

Integrate the differential equation: $y(t) = \displaystyle\int_{t_0}^{t} y'(\tau)d\tau + y(t_0)$

Approximating the integral by the trapezoidal rule at $t = nT$:

$$\begin{pmatrix} Area = (b-a)\cdot\frac{1}{2}\big(f(a)+f(b)\big) \\ t_0 = nT - T; \quad b-a = T; \quad f(a) = y(nT); \quad f(b) = y(nT-T) \end{pmatrix}$$

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T)$$

IIR Filters – Bilinear Transform

Substituting $y'(t) = -ay(t) + bx(t)$ at $t = nT$ into $y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T)$

$\left(\text{labeling } nT \text{ just by } n\right)$

$$y(nT) = \frac{T}{2}\left[\left(-ay(n) + bx(n)\right) + \left(-ay(n-1) + bx(n-1)\right)\right] + y(n-1)$$

Collect y on one side and x on the other:

$$y(n) + \frac{aT}{2}y(n) + \frac{aT}{2}y(n-1) - y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right]$$

$$\boxed{\left(1 + \frac{aT}{2}\right)y(n) - \left(1 - \frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right]}$$

IIR Filters – Bilinear Transform

Take the z-transform: $\mathcal{Z}\left\{\left(1+\frac{aT}{2}\right)y(n)-\left(1-\frac{aT}{2}\right)y(n-1)=\frac{bT}{2}\left[x(n)+x(n-1)\right]\right\}$

$$\left[\left(1+\frac{aT}{2}\right)-\left(1-\frac{aT}{2}\right)z^{-1}\right]Y(z)=\frac{bT}{2}\left[1+z^{-1}\right]X(z)$$

The system transfer function is:

$$H(z)=\frac{bT/2\left(1+z^{-1}\right)}{1+aT/2-\left(1-aT/2\right)z^{-1}}=\frac{b}{\dfrac{\left(1-z^{-1}\right)+aT/2\left(1+z^{-1}\right)}{T/2\left(1+z^{-1}\right)}}$$

The system transfer function is:

$$\boxed{H(z)=\frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}}$$

IIR Filters – Bilinear Transform

Compare: $H(z) = \dfrac{b}{\dfrac{2}{T}\dfrac{\left(1-z^{-1}\right)}{\left(1+z^{-1}\right)} + a}$ to continuous time transfer function: $H(s) = \dfrac{b}{s+a}$

The mapping from s to z plane is:

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

which is the bilinear transform

IIR Filters – Bilinear Transform

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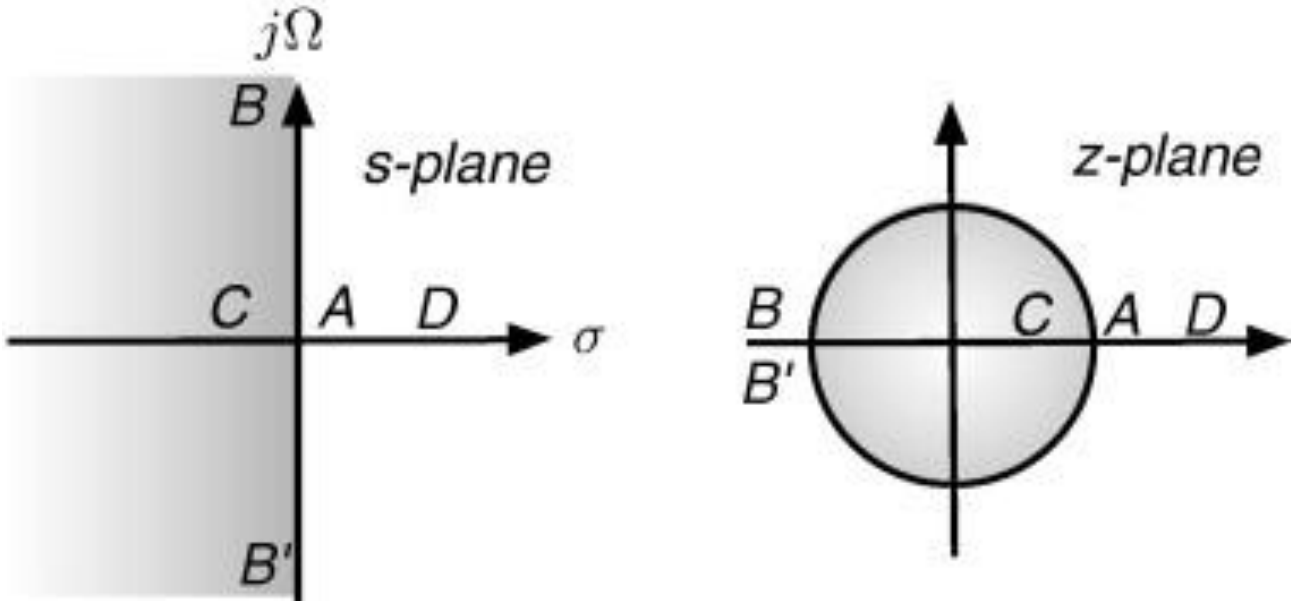
IIR Filters – Bilinear Transform

- Bilinear transform has some interesting properties



$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$s = \sigma + j\Omega; \quad z = re^{j\omega}$$

IIR Filters – Bilinear Transform

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \qquad s = \sigma + j\Omega; \quad z = re^{j\omega}$$

$r < 1 \Leftrightarrow \sigma < 0$ Stable system

$r > 1 \Leftrightarrow \sigma > 0$ Unstable system

$r = 1 \Leftrightarrow \sigma = 0$ $j\Omega$ axis in s-plane

$$\sigma = \frac{2}{T}\frac{r^2-1}{1+r^2+2r\cos\omega}$$

$$\Omega = \frac{2}{T}\frac{2r\sin\omega}{1+r^2+2r\cos\omega}$$

on the frequency axis (or unit circle)

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

very non-linear mapping of continuous to digital frequency

IIR Filters – Bilinear Transform

Frequency warping