Digital Signal Processing

Class 8
02/13/2025

ENGR 71

- Class Overview
  - Correlation
  - z-Transform
- Assignments
  - Reading:

    Chapter 3: The z-Transform and its Applications to the Analysis of LTI
  - Lab 1 – Aliasing lab
    - Will be up on Moodle this afternoon

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- Lab 1-Aliasing Lab
  - Find a short piece of music to download
  - Subsample to demonstrate aliasing
  - Repeat the subsampling, but prior to subsampling, apply a low-pass anti-aliasing filter.
  - Compare the results
  - Mystery piece
- More details and sample code will be placed on Moodle: Lab 1

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- Homework 3
  - Problems: 2.9 (a), 2.17(a), 2.28(a & c),
    2.35, 2.46,
    C2.14(write your own code)
    C2.8 (use Matlab functions)

  Due Feb. 20

Class Information

- Topics in Discrete-Time Signals and Systems
  - Discrete-Time Signals
  - Discrete-Time Systems
  - Analysis of Linear Time-Invariant Systems
  - Description of Systems by Difference Equations
  - Implementation of Discrete-Time Systems
  - Correlation of Discrete-Time Systems

Correlation

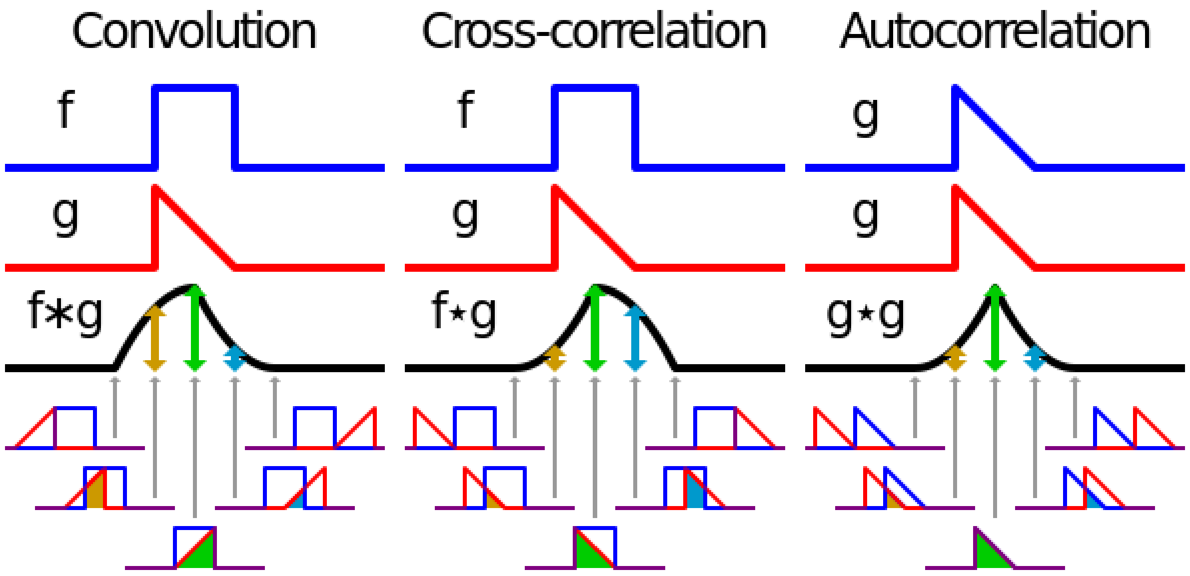
- Correlation
  - Objective of correlation is to determine similarity of signals.
  - Looks similar to convolution but an important difference
    - In convolution, one of the signals is folded
    - In correlation, both signals retain their respective orientations

Convolution: $$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k)$$ Signal y is folded: $y(k) \rightarrow y(-k)$

Correlation: $$r_{xy}(n) = \sum_{k=-\infty}^{\infty} x(k)y(k-n)$$

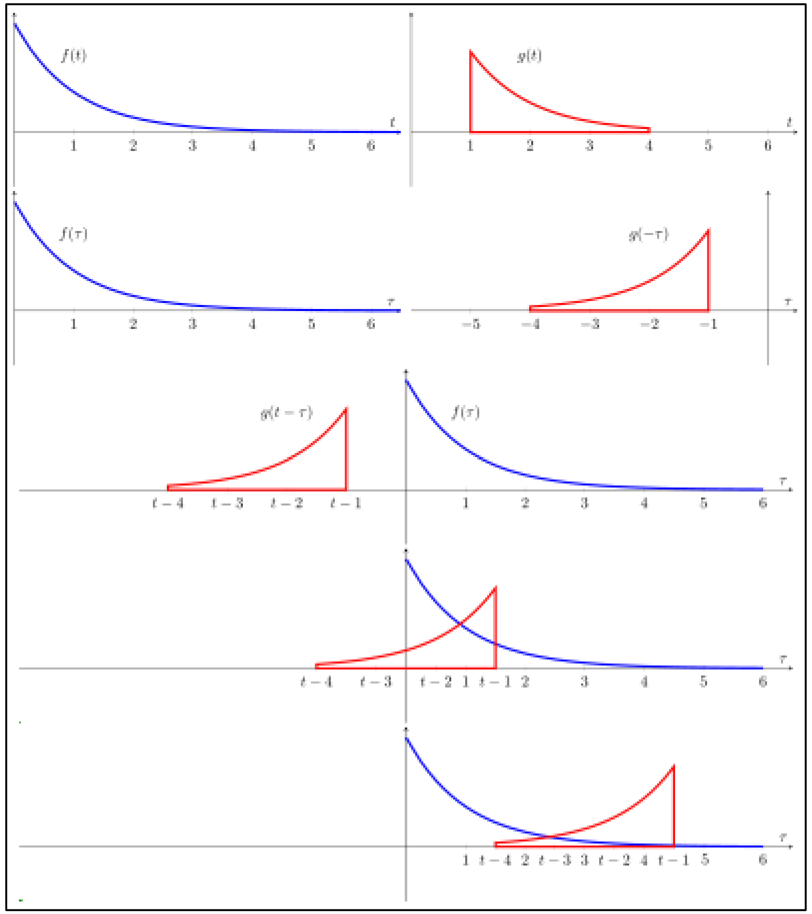
Relationship: $r_{xy}(n) = x(n) * y(-n)$

Correlation



https://lpsa.swarthmore.edu/Convolution/CI.html

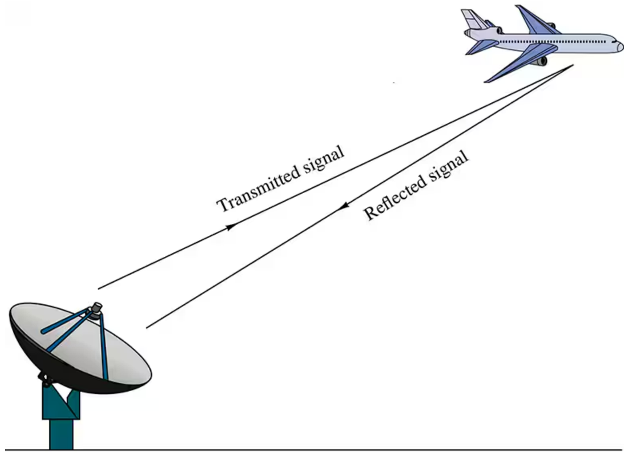
Correlation



https://en.wikipedia.org/wiki/Convolution

Correlation

- Convolution and correlation serve different purposes in signal processing
  - Convolution: Way of determining response of a LTI system to an input signal by convolving the impulse response with input
  - Correlation: Measure similarity of signal
- Example in book of correlation:



Transmitted signal: $x(n)$

Received signal: $y(n) = \alpha x(n-D) + w(n)$

Received signal is attenuated, α; delayed, D; and corrupted by noise, $w(n)$

Problem: Using cross-corrlation, determine if target is present, and if so, what is the delay.

Correlation

– Cross correlation:

- Notation is a bit different: the output is a function of the lag between signals

Expression for shifting y relative to x,

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l)$$

where y is shifted l units to the right for positive l, or to the left for negative l

Equivalent to $r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)y(n)$

Reversing order of x and y:

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)x(n-l) = \sum_{n=-\infty}^{\infty} y(n+l)x(n)$$

$$r_{yx}(l) = r_{xy}(-l)$$

Correlation

– Relationship between correlation and convolution

- From expression:

$$x(l) * y(l) = \sum_{n=-\infty}^{\infty} x(n)y(l-n)$$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l)$$

$$\Rightarrow \qquad r_{xy}(l) = x(l) * y(-l) \qquad \left(\text{but see note on next slide}\right)$$

– Autocorrelation

- Signal correlated with itself

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l) \quad \text{or} \quad \sum_{n=-\infty}^{\infty} x(n+l)x(n)$$

– Useful to determine if there are repeating patterns in signal.

Correlation

– Note about complex signals

- We are only considering real-valued signals
- The actual definition of correlation including complex signals is:

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} \overline{x}(n) y(n-l)$$

where the overbar represent complex conjugation

Correlation

– Relationship between correlation and convolution

- From expression:

$$x(l) * y(l) = \sum_{n=-\infty}^{\infty} x(n)y(l-n)$$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l)$$

$$\Rightarrow \qquad r_{xy}(l) = x(l) * y(-l)$$

A note of caution:
We are only considering real-valued signals.
In the actual definition of correlation, you take the c

– Autocorrelation

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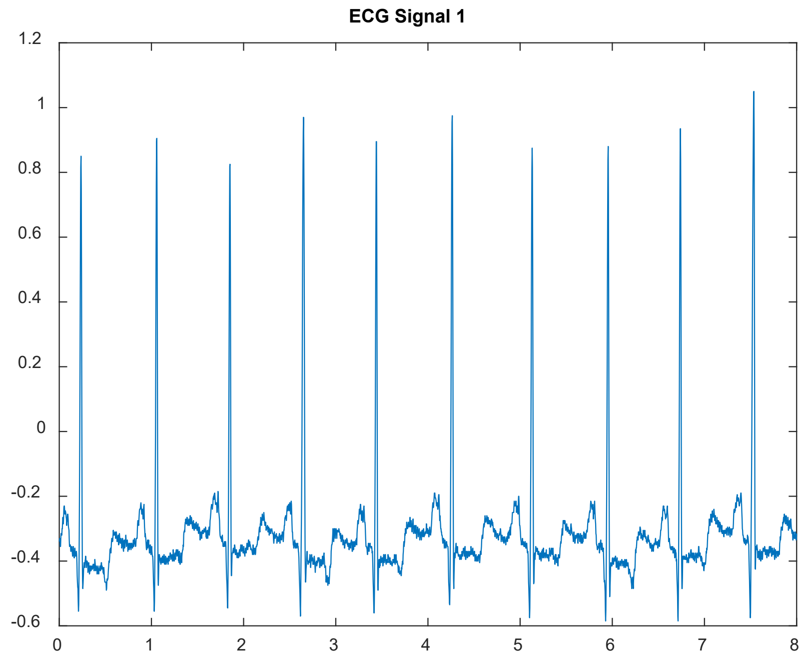
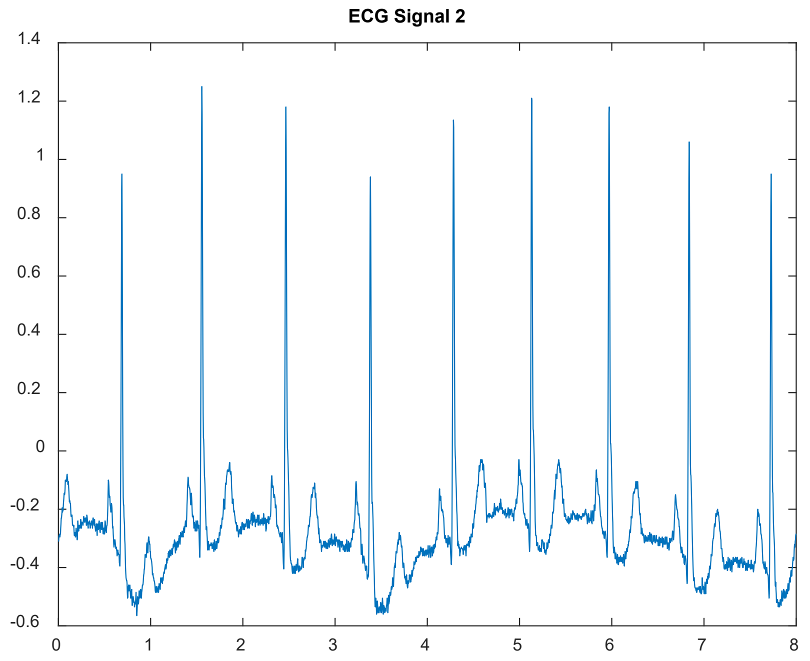
Correlation

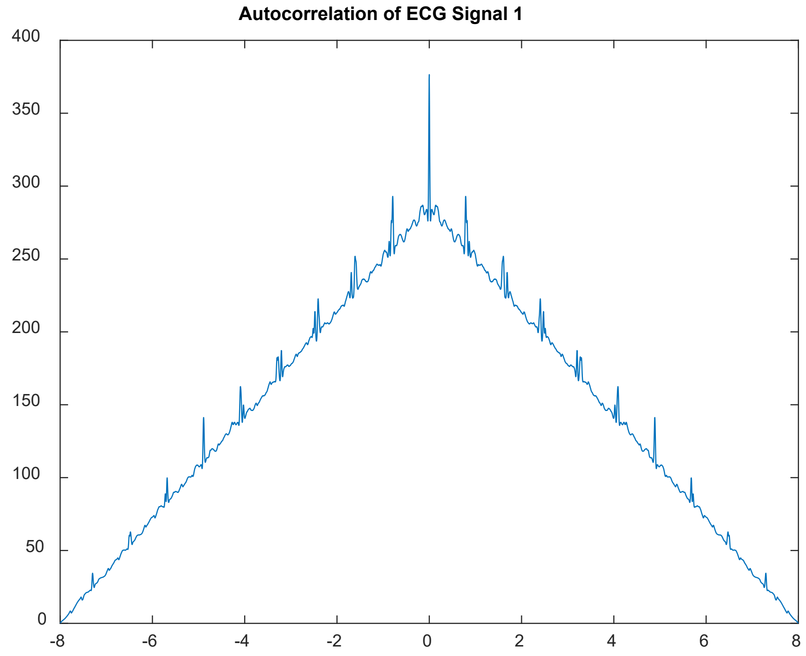
– Example of ECG signal

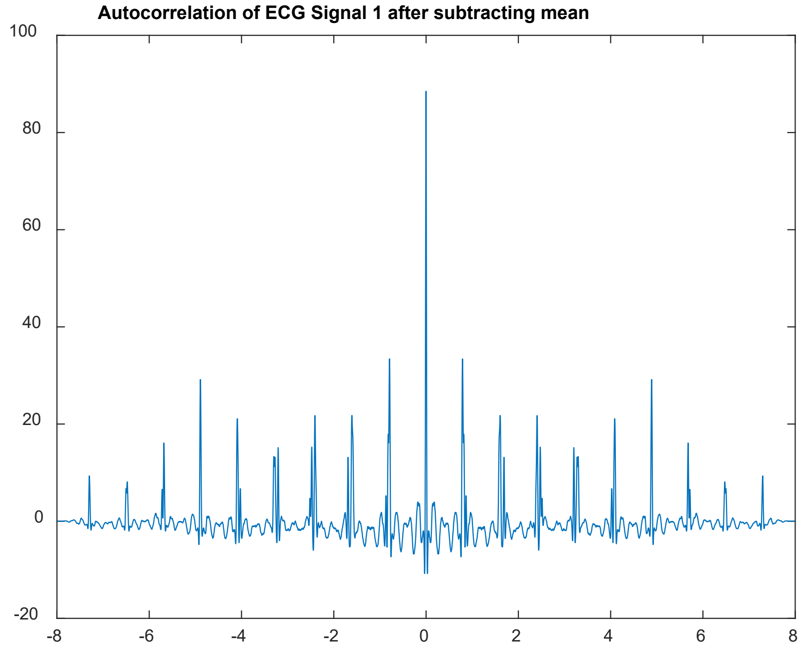




Correlation

– Example of ECG signal





Correlation

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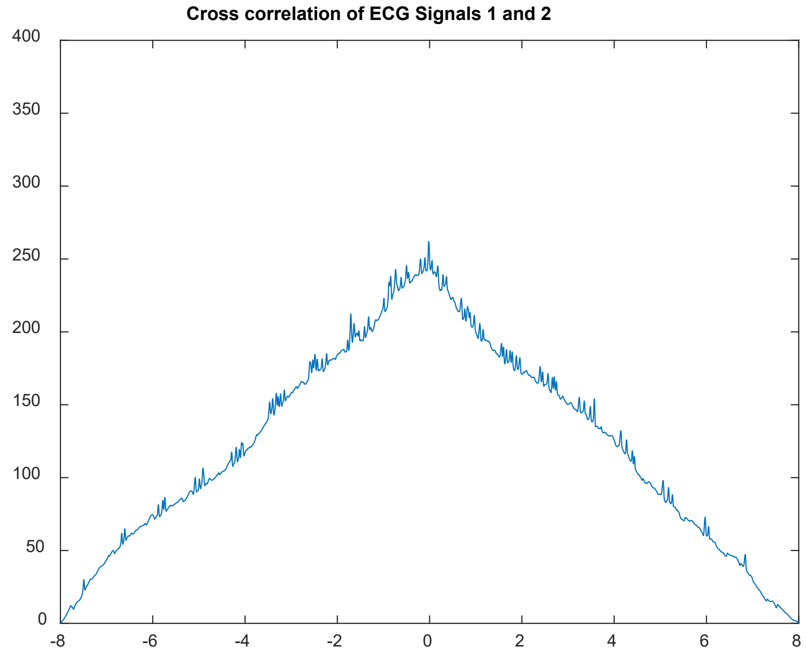

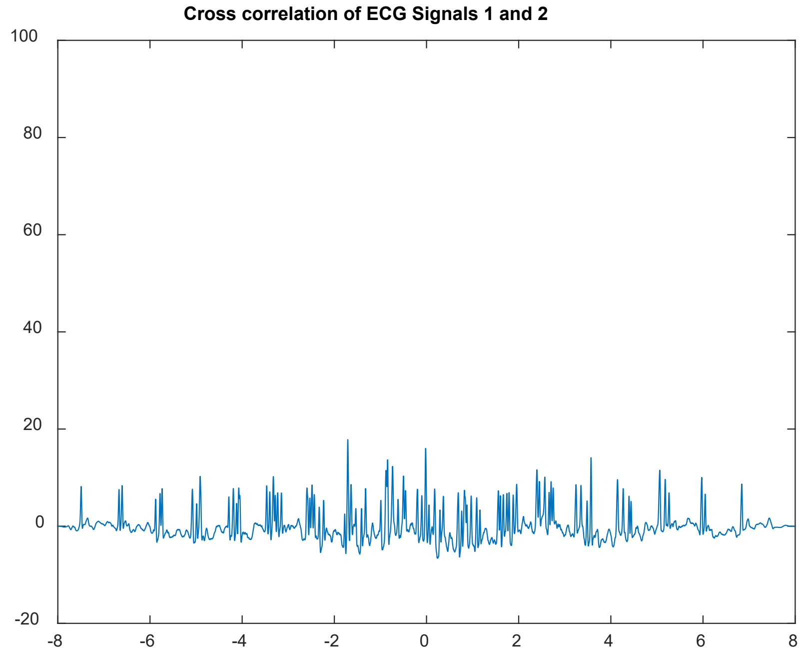

Correlation

– Properties:

- Energy of a signal is autocorrelation at zero lag:

$$E_x = \sum_{n=-\infty}^{\infty} x^2(n) = \sum_{n=-\infty}^{\infty} x(n)x(n-0) = r_{xx}(0)$$

- Inequality for cross-correlation in terms of energies

$$|r_{xy}(l)| \le \sqrt{E_x E_y} \qquad\qquad |r_{xx}(l)| \le r_{xx}(0) = E_x$$

I believe the book's proof of this makes an assumption which is equivalent to the result.

This depends on the Cauchy-Schwarz inequality:

$$\langle \mathbf{u},\mathbf{v} \rangle \le \|\mathbf{u}\| \cdot \|\mathbf{v}\| \quad \text{for vectors } \mathbf{u} \text{ and } \mathbf{v} \text{ in a vector space}$$

Correlation

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Correlation

– Normalized correlation

- Since the autocorrelation at zero lag is largest value:

$$\rho_{xx} \equiv \frac{r_{xx}(l)}{r_{xx}(0)} \qquad \rho_{xy} \equiv \frac{r_{xy}(l)}{\sqrt{r_{xx}(0)r_{yy}(0)}}$$

$$|\rho_{xx}| \leq 1 \qquad |\rho_{xy}| \leq 1$$

- Autocorrelation is an even function:

$$r_{xx}(l) = r_{xx}(-l)$$

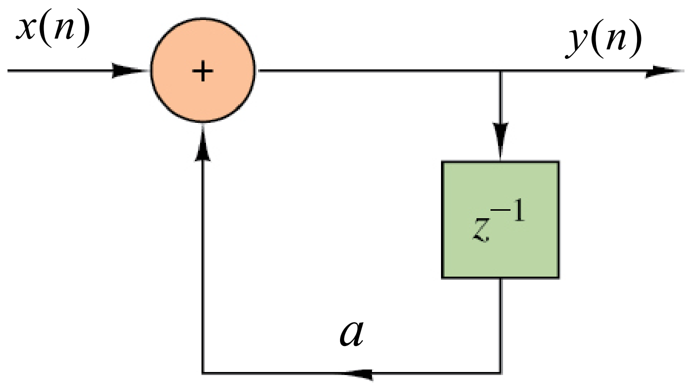
so you only have to compute it for $l \geq 0$

Correlation

- Example:
  - What is the autocorrelation for the impulse response for the first-order system shown below?
    - First-order model difference equation:

$$y(n) = ay(n-1) + x(n)$$

$$1 < a < 0 \quad \text{(i.e., } a = 0.8 \text{ in an example from last class)}$$



Impulse response:

$$h(0) = ah(-1) + \delta(0) = 1$$

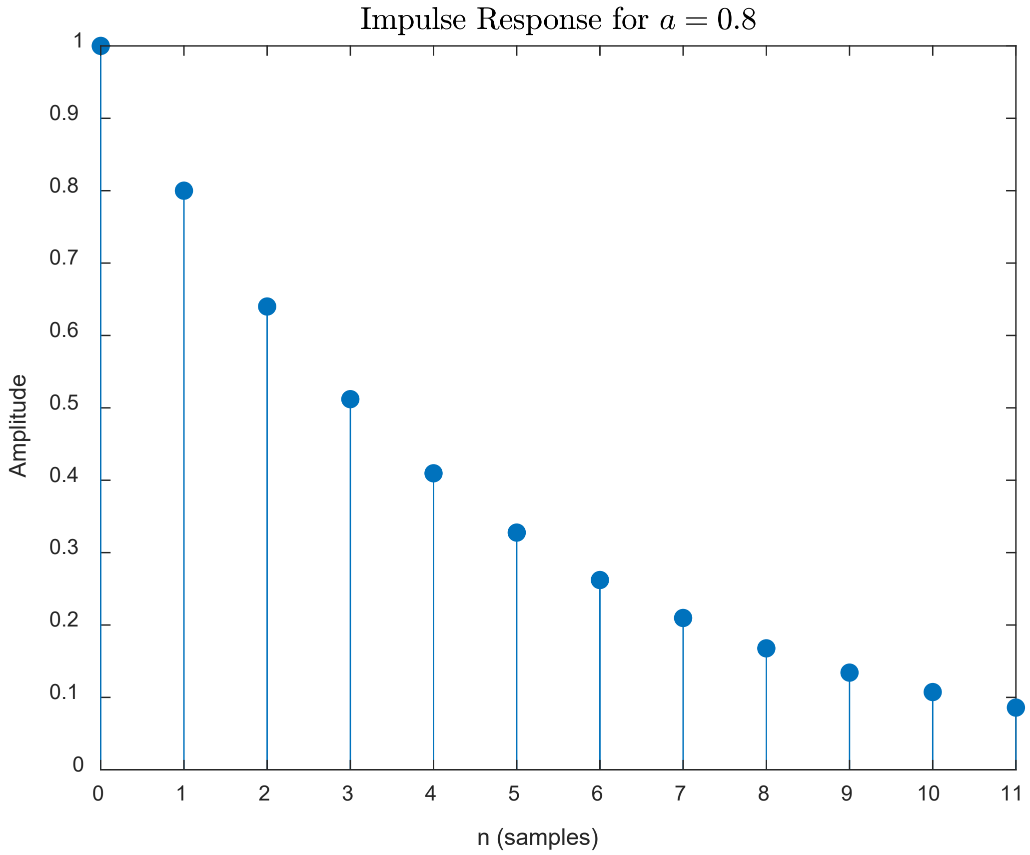
$$h(1) = ah(0) + \delta(1) = a \cdot 1 + 0$$

$$h(2) = ah(1) + \delta(2) = a^2$$

$$h(n) = a^n u(n)$$

Correlation

- Example:

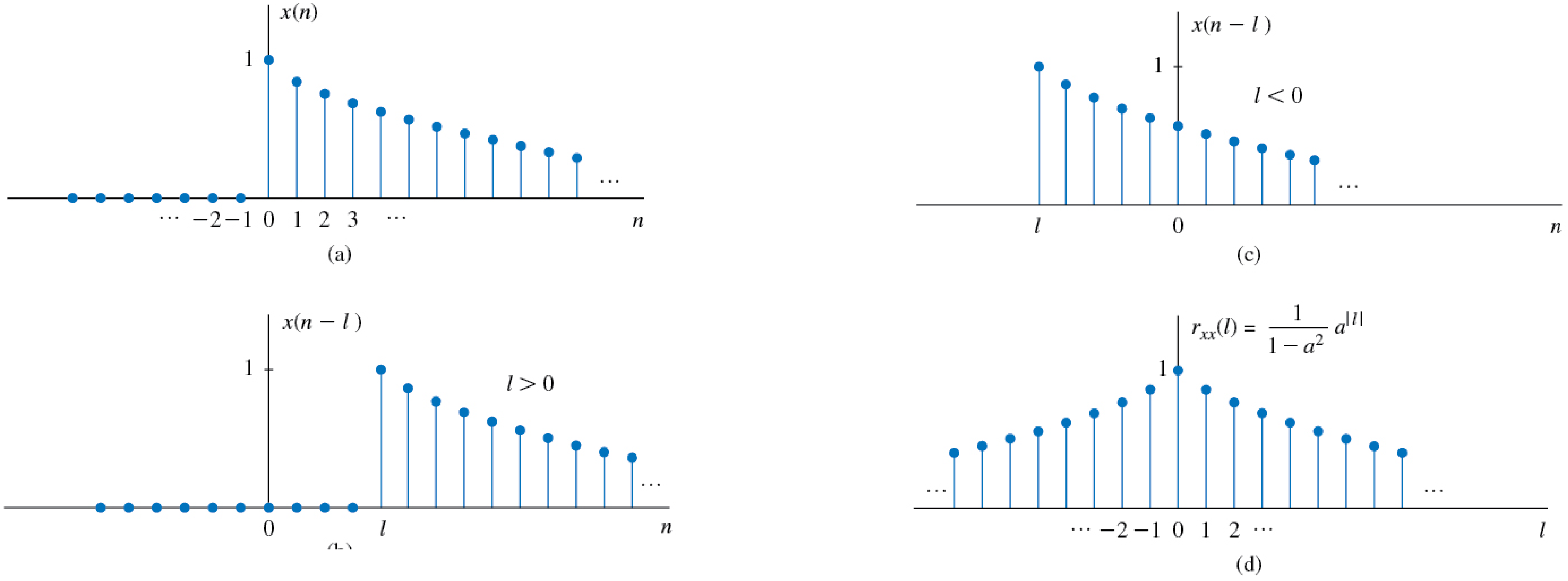


Correlation

- Example:

Correlation

- Example:



Correlation

– Correlation for periodic signals

- Definition needs to be modified since these are power signals, $E_x \to \infty$
- Cross-correlation and autocorrelation for power signals

$$r_{xy}(l) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n)y(n-l) \quad \text{Cross-correlation}$$

$$r_{xx}(l) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n)x(n-l) \quad \text{Autocorrelation}$$

- For periodic signals with period *N*

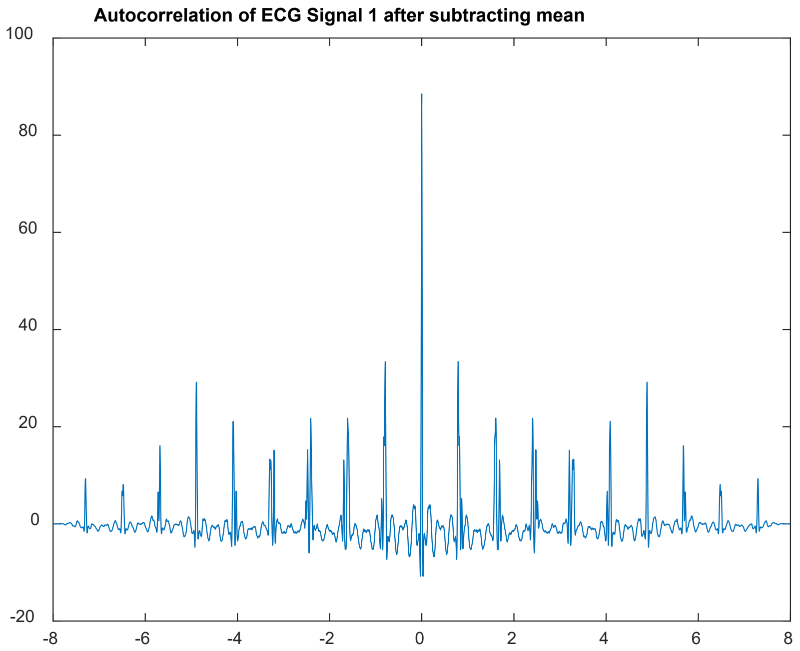
$$r_{xy}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)y(n-l) \quad \text{Cross-correlation}$$

$$r_{xx}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)x(n-l) \quad \text{Autocorrelation}$$

Correlation

– Correlation for periodic signals

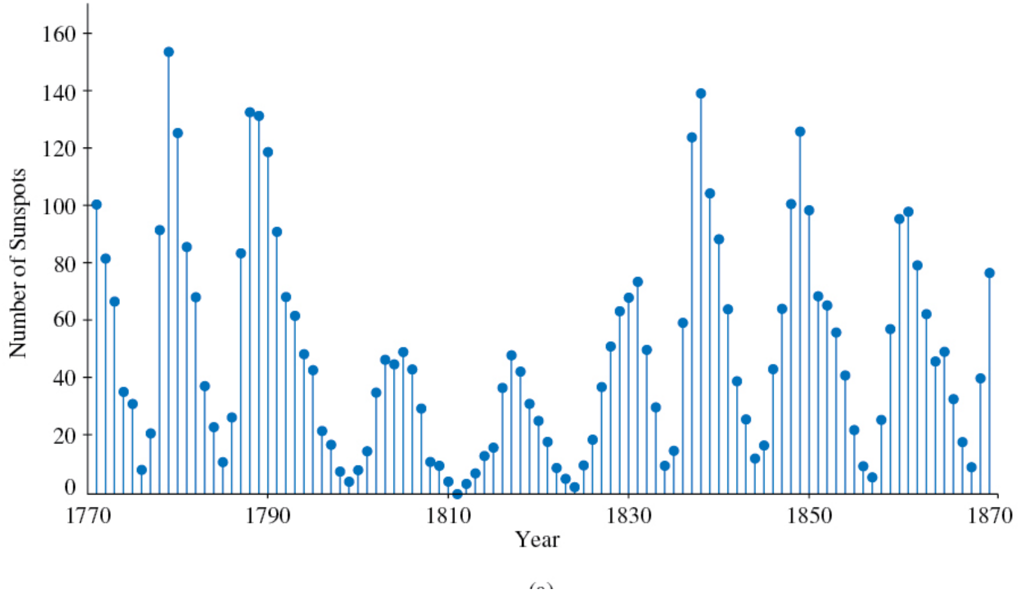
- Periodic signal will show peaks at lags corresponding to the period
- ECG signals are an example

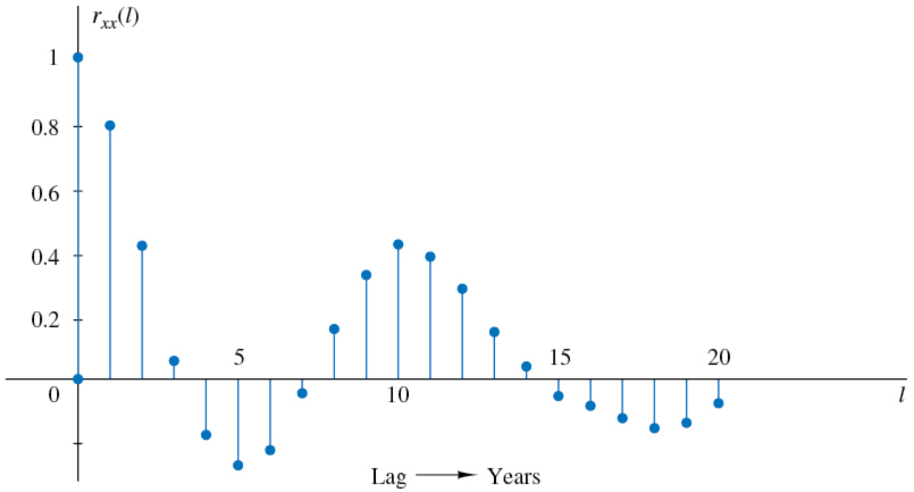


Correlation

– Correlation for periodic signals

- Example in book for 10 to 11 year periodic trend of sun spots is another





(b)

Correlation

– Relationship between input and output correlation for LTI systems

- This is an exercise in using the relationship between convolution and correlation
- For LTI system: $y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$

Using the relationship between correlation and convolution:

$$r_{yx}(l) = y(l) * x(-l)$$

$$r_{yx}(l) = \big(h(l) * x(l)\big) * x(-l)$$

$$r_{yx}(l) = h(l) * \big(x(l) * x(-l)\big)$$

$$r_{yx}(l) = h(l) * r_{xx}(l)$$

System response to autocorrelation of input signal *x* is cross correlation of signals *x* and *y*

Correlation

– Relationship between input and output correlation for LTI systems

Can also get interesting relationship between autocorrelation of input and autocorrelation of output:

$$r_{yy}(l) = y(l) * y(-l)$$

$$r_{yy}(l) = \big(h(l) * x(l)\big) * \big(h(-l) * x(-l)\big)$$

$$r_{yy}(l) = \big(h(l) * h(l)\big) * \big(x(l) * x(-l)\big)$$

$$r_{yy}(l) = r_{hh}(l) * r_{xx}(l)$$

Correlation

– Relationship between input and output correlation for LTI systems

In terms of the convolution sum:

$$r_{yy}(l) = \sum_{k=-\infty}^{\infty} r_{hh}(k) r_{xx}(l-k) \quad \text{, at } l = 0$$

$$r_{yy}(0) = E_y = \sum_{k=-\infty}^{\infty} r_{hh}(k) r_{xx}(k)$$

Energy of the output is the sum of the autocorrelations

of the impulse response and input signal summed over all lags.

New Topic
The z-transform

Z-Transform

- As with continuous systems LTI systems, there is an easier way to solve Discrete Linear Time (Shift) Invariant systems (LTI or LSI)
- The z-transform
  - Discrete version of Laplace transform
  - Many properties analogous to Laplace
    - Continuous Laplace: differential equations → algebraic equations
    - Discrete z-transform: difference equations → algebraic equations
    - Continuous Laplace: Stability determined by pole locations
      - Left half-plane
    - Discrete z-transform: Stability determined by pole locations
      - Inside unit circle
    - Other properties like convolution, time shift, initial & final values, etc.

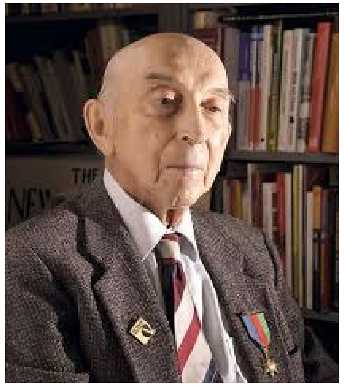
Z-Transform

- Z-transform is not named for anyone.
- Laplace had the concept, but who cared?
  No one was sampling signals at the time.

- (Re)-Introduced in 1947 by Wiltold Hurewicz (MIT)
  https://en.wikipedia.org/wiki/Witold_Hurewicz

Witold Hurewicz

- Further developed and denoted as z-transform
  in 1952 by control group at Columbia University
  http://www.ling.upenn.edu/courses/ling525/z.html



J. R. Ragazzini and L. A. Zadeh

Z-Transform

- Consider the Laplace transform of a sampled signal:

$$x(t) = \sum_n x\left(nT_s\right)\delta\left(t - nT_s\right)$$

$$\mathcal{L}\left\{x(t)\right\} = X(s) = \mathcal{L}\left\{\sum_n x\left(nT_s\right)\delta\left(t - nT_s\right)\right\}$$

$$X(s) = \sum_n x\left(nT_s\right)\mathcal{L}\left\{\delta\left(t - nT_s\right)\right\}$$

  - Use Laplace transform of delta function and time-shift property:

$$\mathcal{L}\left\{\delta(t)\right\} = 1 \quad ; \quad \mathcal{L}\{f(t - t_0) = F(s)e^{-st_0}$$

$$\text{so,} \quad \mathcal{L}\left\{\delta(t - nT_s)\right\} = 1e^{-snT_s} = \left(e^{-sT_s}\right)^n$$

  - Putting this all together:

$$X(s) = \sum_n x\left(nT_s\right)\left(e^{-sT_s}\right)^n$$

Z-Transform

- Consider the Laplace transform of a sampled signal (cont.)

$$X(s) = \sum_n x(nT_s)\left(e^{-sT_s}\right)^n$$

  - Denote the sampled values as: $x[n] = x(nT_s)$
  - Define z to be: $z = e^{sT_s}$
  - Then:

$$X(s) = \sum_n x[n] z^{-n}$$

  - The z-transform of a discrete signal, $x[n]$, is defined as:

$$\mathcal{Z}\{x[n]\} \equiv \sum_n x[n] z^{-n}$$

Z-Transform

- The Laplace transform on the imaginary axis is periodic
  - In the Laplace domain the $s = j\Omega$ corresponds to frequency domain (Fourier transform)
  - In z-transform domain:

$$X(\Omega) = \sum_n x(nT_s)\left(e^{-j\Omega T_s}\right)^n$$

This is periodic with period $2\pi / T_s$

$$X(\Omega + 2\pi k/T_s) = \sum_n x(nT_s)\,e^{-j(\Omega + 2\pi k/T_s)T_s n} = \sum_n x(nT_s)\,e^{-j(\Omega T_s + 2\pi k)n}$$

$$= \sum_n x(nT_s)\,e^{-j\Omega T_s n} e^{-j2\pi kn} = \sum_n x(nT_s)\,e^{-j\Omega T_s n}$$

$$X(\Omega + 2\pi k/T_s) = X(\Omega)$$

(I'm using Ω as the frequency for continuous signals)

Z-Transform

- What does it mean for this to be periodic with period $2\pi / T_{s?}$
  - Remember that the sampling frequency is: $\omega_s = 2\pi / T_{s?}$
  - So the z-transform taken on the unit circle: $z = e^{j\Omega T_s}$

    is periodic with the sampling frequency.
- Looking just at the z-transform variable z as a complex variable in polar form:

$$z = e^{sT_s} = e^{(\sigma + j\Omega)T_s} = e^{\sigma T_s} e^{j\Omega T_s}$$

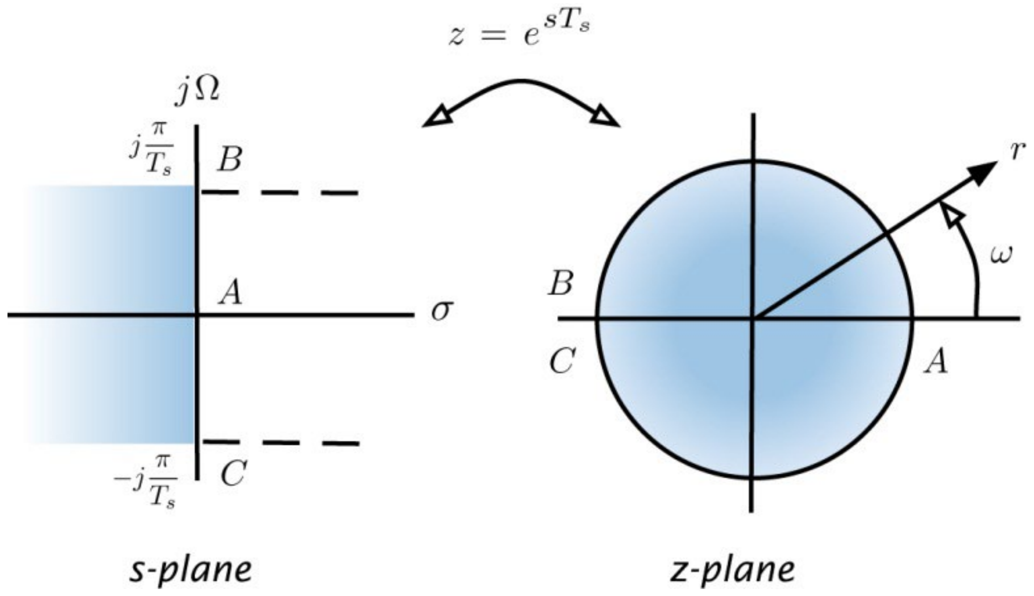
In polar form, defining: $r = e^{\sigma T_s}$ and $\omega = \Omega T_s$

  - The z-plane looks like circles of radius r with angle $-\pi \le \omega \le \pi$

  The imaginary axis in the Laplace domain becomes the unit circle in the z-transform domain.

Z-Transform

- Picture of the mapping s-plane to z-plane:



Notice that $j\pi/T_s$ is the Nyquist frequency
Left half-plane of strip maps into interior of unit circle
Imaginary axis maps to unit circle.
Right half-plane maps to exterior of unit circle

Z-Transform

- Computing the z-transform:
  - Most important thing to remember: Geometric Series

$$\boxed{S = 1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{1-r^n}{1-r} \quad \text{for } r \neq 1}$$

  - Easy proof: Multiply S by r and subtract S

$$S = 1 + r + r^2 + r^3 + \cdots + r^{n-1}$$

$$rS = r + r^2 + r^3 + r^4 + \cdots + r^{n-1} + r^n$$

$$rS - S = -1 + r^n \Rightarrow S = \frac{r^n - 1}{r - 1}$$

  - If $r < 1$, take limit as $n \rightarrow \infty$

$$\boxed{\sum_{n=0}^{\infty} r^n = \lim_{n \to \infty} \frac{1-r^n}{1-r} = \frac{1}{1-r} \quad \text{for } r < 1}$$

Z-Transform

- Z-transform:
  - Laplace: $$X(s)=\int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$
  - z-transform: $$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
  - As in the case of the Laplace transform, we are mainly interested in causal signals and systems: $$x(t)=x(t)u(\mathrm{t})$$ $$x[n]=x[n]u[n]$$
  - Limits in sum and integral start at 0:

$$X(s)=\int_{0}^{+\infty} x(t)e^{-st}\,dt$$

$$X(z)=\sum_{n=0}^{\infty} x[n]z^{-n}$$

Z-Transform

- Z-transform:
  - Laplace: You can easily solve linear differential equations with constant coefficients in the Laplace domain.
    - These equations correspond to linear time-invariant systems
  - z-transform: Same function as Laplace, except for discrete time signals and systems.
    - LTI systems represented by difference equations.
    - You can solve for system response in the z-transform domain, and then use inverse z-transform to find response in sampled time domain.

Z-Transform

- Definition of z-transform:
  - Bilateral

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

  - Unilateral (causal signals & systems)

$$H(z) = \sum_{n=0}^{\infty} h[n]\, z^{-n}$$

  - Inverse:

$$h[n] = \frac{1}{2\pi j} \oint_R H(z)\, z^{-n+1} dz$$

    - Rarely use this, although this is common integral in complex variables math courses.
    - We compute forward & inverse by use of transform pairs and properties.
    - Can also find inverse by long division.

Z-Transform

- A few examples:
  - Impulse: $h(n) = \delta(n)$

$$H(z) = \sum_{n=0}^{\infty} \delta[n]\, z^{-n} = 1$$

Region of convergence is entire z-plane

  - Delayed impulse:

$$h(n) = \delta(n-1)$$

$$H(z) = \sum_{n=0}^{\infty} \delta[n-1]\, z^{-n} = z^{-1}$$

Region of convergence is entire z-plane

  - Exponential signal (geometric series)

$$h(n) = a^n u[n]$$

$$H(z) = \sum_{n=0}^{\infty} a^n\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{1}{1-\frac{a}{z}} \quad \text{if } \left|\frac{a}{z}\right| < 1$$

Region of convergence: $|z| > |a|$

Z-Transform

$$h(n) = -a^n u[-n-1]$$

$$H(z) = \sum_{n=-\infty}^{1} -a^n \; z^{-n}$$

Let $m = -n$

$$H(z) = -\sum_{m=1}^{\infty} a^{-m} \; z^m = -\sum_{m=1}^{\infty} \left(z/a\right)^m$$

$$H(z) = -\frac{z/a}{1-z/a} \quad \text{for} \quad \left|\frac{z}{a}\right| < 1$$

$$H(z) = -\left(\frac{a/z}{a/z}\right)\left(\frac{z/a}{1-z/a}\right) = -\frac{1}{a/z-1} = \frac{1}{1-a/z} \text{ for } \left|\frac{z}{a}\right| < 1$$

$$\boxed{H(z) = \frac{1}{1-\frac{a}{z}} \quad \text{if} \left|\frac{z}{a}\right| < 1}$$

Region of convergence: $|a| > |z|$

Same expression as before,
but region of convergence is different

Z-Transform

Z Transform Pairs		
Time Domain *	**Z Domain**	
	z	**z^{-1}**
$\delta[k]$ (unit impulse)	1	1
$\gamma[k]$ † (unit step)	$\Gamma(z)=\frac{z}{z-1}$	$\Gamma(z)=\frac{1}{1-z^{-1}}$
a^k	$\frac{z}{z-a}$	$\frac{1}{1-z^{-1}a}$
e^{-bTk}	$\frac{z}{z-e^{-bT}}$	$\frac{1}{1-z^{-1}e^{-bT}}$
k	$\frac{z}{(z-1)^2}$	$\frac{z^{-1}}{(1-z^{-1})^2}$
$\sin(bk)$	$\frac{z\sin(b)}{z^2-2z\cos(b)+1}$	$\frac{z^{-1}\sin(b)}{1-2z^{-1}\cos(b)+z^{-2}}$
$\cos(bk)$	$\frac{z(z-\cos(b))}{z^2-2z\cos(b)+1}$	$\frac{1-z^{-1}\cos(b)}{1-2z^{-1}\cos(b)+z^{-2}}$
$a^k\sin(bk)$	$\frac{az\sin(b)}{z^2-2az\cos(b)+a^2}$	$\frac{az^{-1}\sin(b)}{1-2az^{-1}\cos(b)+a^2z^{-2}}$
$a^k\cos(bk)$	$\frac{z(z-a\cos(b))}{z^2-2az\cos(b)+a^2}$	$\frac{1-az^{-1}\cos(b)}{1-2az^{-1}\cos(b)+a^2z^{-2}}$

Z-Transform

- Bounded-Input Bounded-Output Stability:
  - Rule #1: Poles inside unit circle (causal signals)
  - Rule #2: Unit circle in region of convergence
    - Analogy in continuous-time: imaginary axis would be in region of convergence of Laplace transform
- Example:

$$a^n u[n] \overset{Z}{\leftrightarrow} \frac{1}{1 - a\, z^{-1}} \quad \text{for} \quad |z| > |a|$$

BIBO stable if $|a| < 1$ by rule #1

BIBO stable if $|z| > |a|$ includes unit circle;

hence, $|a| < 1$ by rule #2

Z-Transform

- Z-transform Properties
  - Linearity: $a_1 x_1[n] + a_2 x_2[n] \Leftrightarrow a_1 X_1(z) + a_2 X_2(z)$
  - Right shift (Delay)

    For $x[n] \Leftrightarrow X(z)$ $\quad x[n-1]\, u[n-1] \Leftrightarrow z^{-1} X(z)$

    - In general

$$x[n-m]u[n-m] \Leftrightarrow z^{-m} X(z)$$

    - Also a form of this used for solving difference equations with initial conditions, where start of time is not also delayed:

$$x[n-1]\, u[n] \Leftrightarrow z^{-1} X(z) + z^{-1} x[-1]$$

$$x[n-m]u[n] \Leftrightarrow z^{-m} X(z) + z^{-m}\left(\sum_{l=1}^{m} x[-l] z^{l}\right)$$

Z-Transform

- Z-transform Properties
  - Left shift (Advance)

$$\text{For} \quad x[n] \Leftrightarrow X(z)$$

$$x[n+1]\ u[n] \Leftrightarrow zX(z) - zx[0]$$

$$x[n+m]\ u[n] \Leftrightarrow z^m X(z) - z^m \left(\sum_{k=0}^{m-1} x[k]\ z^{-k} \right)$$

  - Convolution: (convolution in time is product in z-domain)

$$\text{For} \quad x[n] \Leftrightarrow X(z) \quad \& \quad y[n] \Leftrightarrow Y(z)$$

$$x[n] * y[n] \Leftrightarrow X(z)Y(z)$$

Z-Transform

- Z-transform Properties
  - Multiplication in time becomes scaling in z-domain

$$\text{For} \quad x[n] \Leftrightarrow X(z)$$

$$\alpha^n x[n]u[n] \Leftrightarrow X\left(\frac{z}{\alpha}\right)$$

  - Time reversal:

$$\text{For} \quad x[n] \Leftrightarrow X(z)$$

$$x[-n] \Leftrightarrow X\left(\frac{1}{z}\right)$$

  - Initial value: $x[0] = \lim_{z\to\infty} X(z)$
  - Final value: $\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$

    if poles of $(z-1)X(z)$ are inside unit circle

Z-Transform

- Note on convolution:
  - For LTI system, output of system is convolution of input with impulse response

$$y[n] = \sum_{k=0}^{\infty} x[k]h[n-k] = x[n]*h[n]$$

$$Z\{y[n]\} = Y(z) = Z\{x[n]*h[n]\} = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

  - *H(z)* is transfer function

Z-Transform

Z Transform Properties	
Property Name	**Illustration**
Linearity	$af_1[k] + bf_2[k] \xleftrightarrow{Z} aF_1(z) + bF_2(z)$
Left Shift by 1	$f[k+1] \xleftrightarrow{Z} zF(z) - zf[0]$
Left Shift by 2	$f[k+2] \xleftrightarrow{Z} z^2F(z) - z^2f[0] - zf[1]$
Left Shift by n	$f[k+n] \xleftrightarrow{Z} z^nF(z) - z^n\sum_{k=0}^{n-1} f[k]z^{-k}$ $= z^n\left(F(z) - \sum_{k=0}^{n-1} f[k]z^{-k}\right)$
Right Shift by n	$f[k-n] \xleftrightarrow{Z} z^{-n}F(z)$
Multiplication by time	$kf[k] \xleftrightarrow{Z} -z\frac{dF(z)}{dz}$
Scale in z	$a^kf[k] \xleftrightarrow{Z} F\left(\frac{z}{a}\right)$
Scale in time	$f\left[\frac{k}{n}\right] \xleftrightarrow{Z} F(z^n);$ n is an integer $n \geq 1$
Convolution	$f_1[k] * f_2[k] \xleftrightarrow{Z} F_1(z)F_2(z)$
Initial Value Theorem	$f[0] = \lim_{z\to\infty} F(z)$
Final Value Theorem (if final value exists)	$\lim_{k\to\infty} f[k] = \lim_{z\to 1}(z-1)F(z)$

Z-Transform

- Detailed examples of how to find inverse z-transform:

$$H_1(z) = \frac{1+2z^{-1}+2z^{-2}}{1-3z^{-1}+2z^{-2}}$$

Easier to work with:

$$H_1(z) = \left(\frac{z^2}{z^2}\right)\frac{1+2z^{-1}+2z^{-2}}{1-3z^{-1}+2z^{-2}} = \frac{z^2+2z+2}{z^2-3z+2}$$

– As in the case of the Laplace, do partial fractions expansion

  - However, for reasons that will become clear, do partial fractions of

$$\frac{H(z)}{z} = \frac{z^2+2z+2}{z\left(z^2-3z+2\right)} = \frac{A}{z}+\frac{B}{z-1}+\frac{C}{z-2}$$

Z-Transform

$$\frac{H(z)}{z} = \frac{z^2 + 2z + 2}{z\left(z^2 - 3z + 2\right)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$$A = 1 \quad B = -5 \quad C = 5$$

$$h[n] = \delta[n] - 5u[n] + 5 \cdot 2^n u[n]$$

Z-Transform

- A few Matlab tools:

  zplane(b,a) plots poles and zeros in z-plane

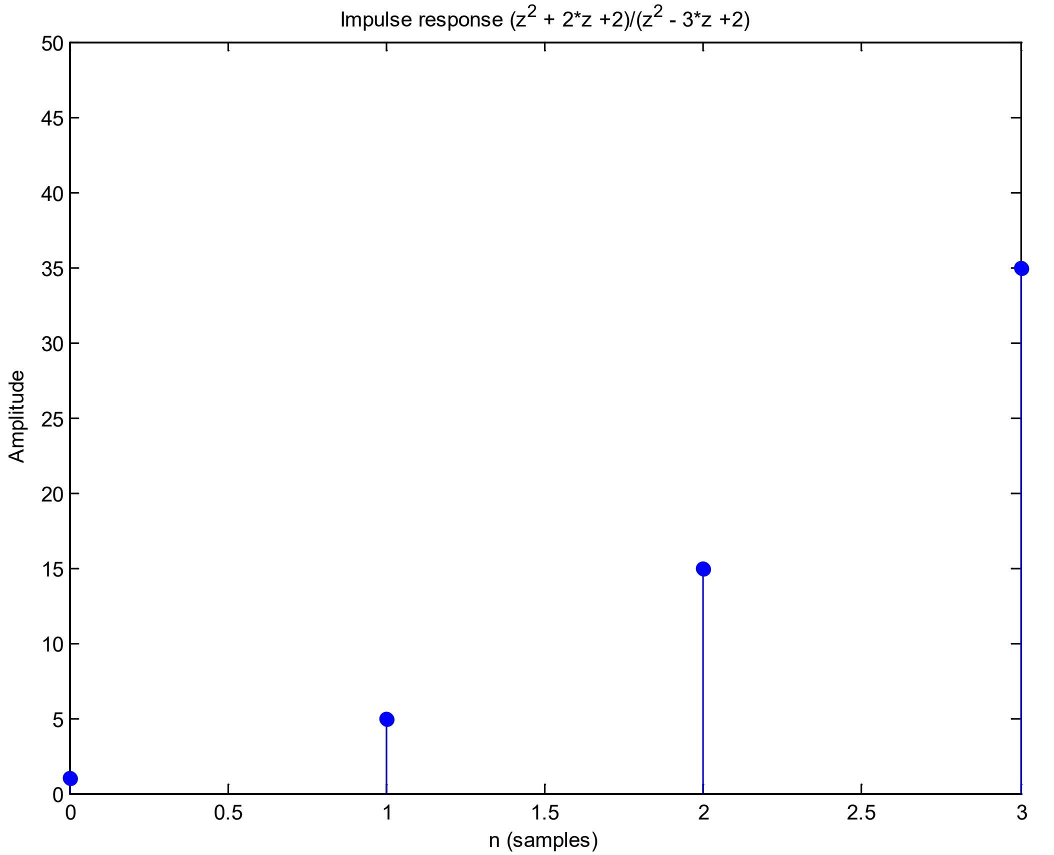
$$H_1(z) = \frac{1 + 2z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

  b = [1 2 2]; a = [1 -3 2];

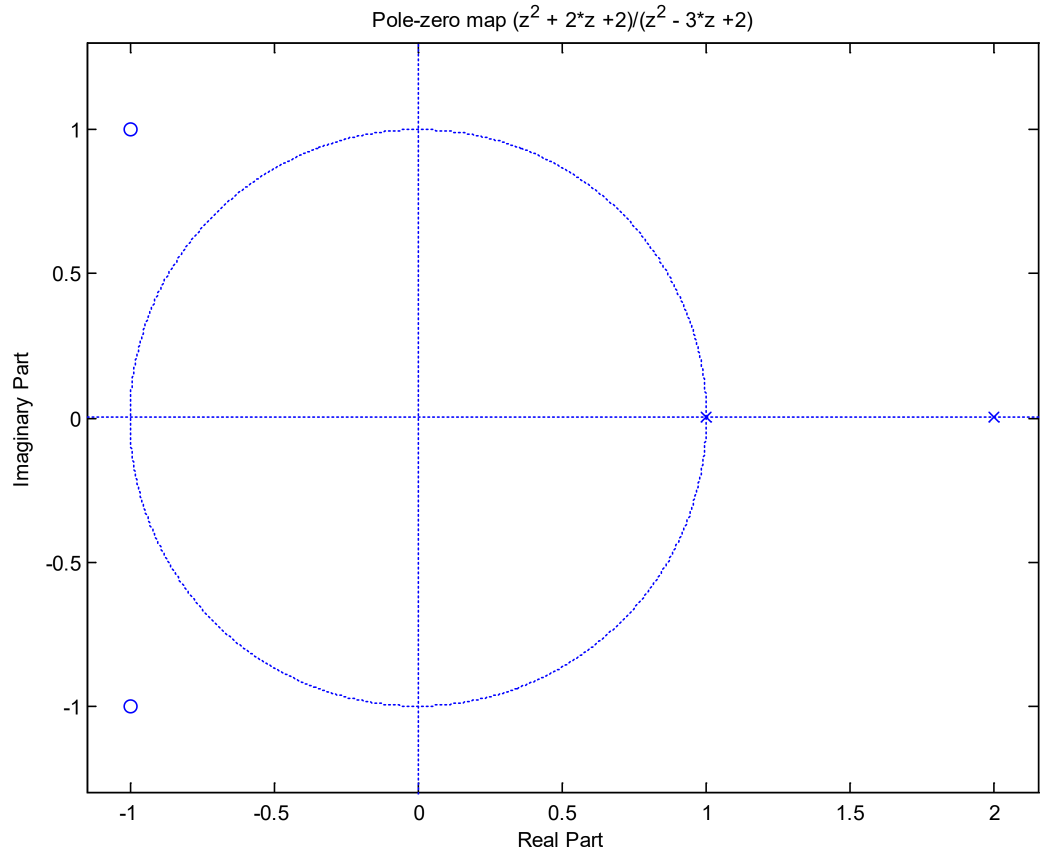
  impz(b,a)

  zplane(b,a)

Z-Transform


Impulse response (z^2 + 2*z +2)/(z^2 - 3*z +2)
Amplitude
n (samples)
0
5
10
15
20
25
30
35
40
45
50
0
0.5
1
1.5
2
2.5
3

Z-Transform



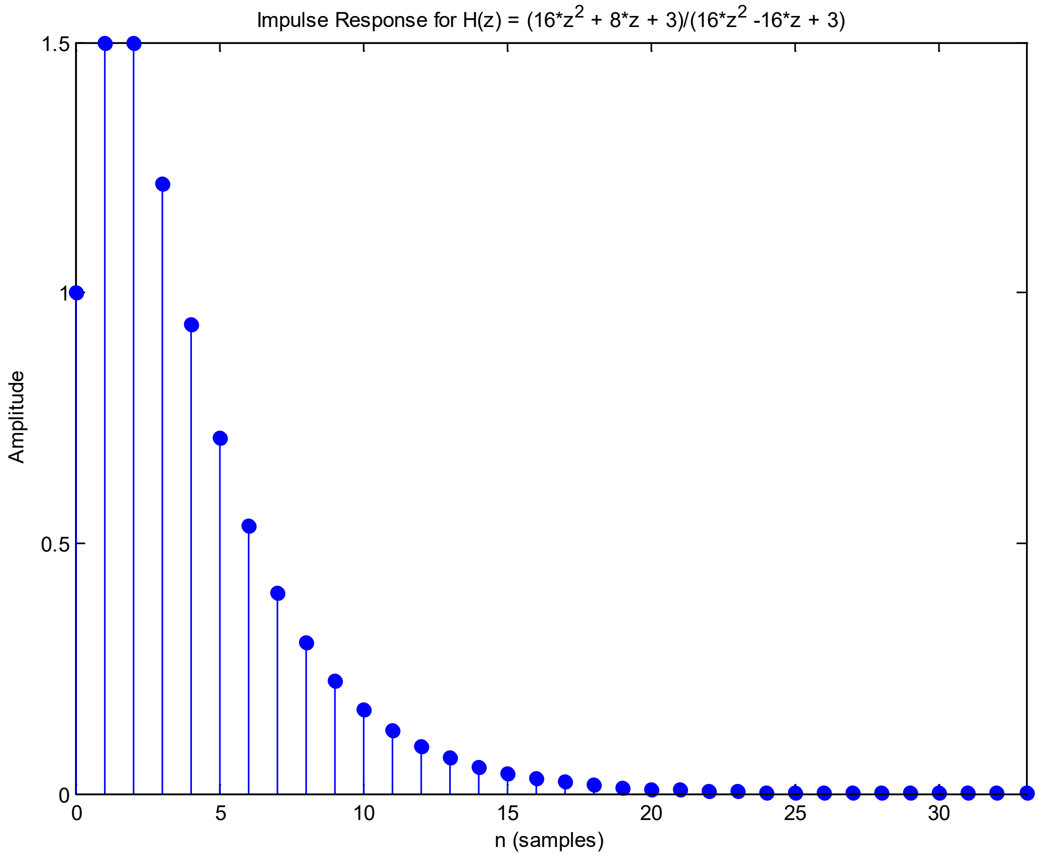
Pole-zero map (z2 + 2*z +2)/(z2 - 3*z +2)
Imaginary Part
Real Part
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1.5
2

Z-Transform

$$H_2(z) = \frac{1-\left(1/2\right)z^{-1} + \left(3/16\right)z^{-2}}{1-z^{-1}+\left(3/16\right)z^{-2}} = 1 - \frac{3z}{z-1/4} + \frac{3z}{z-3/4}$$

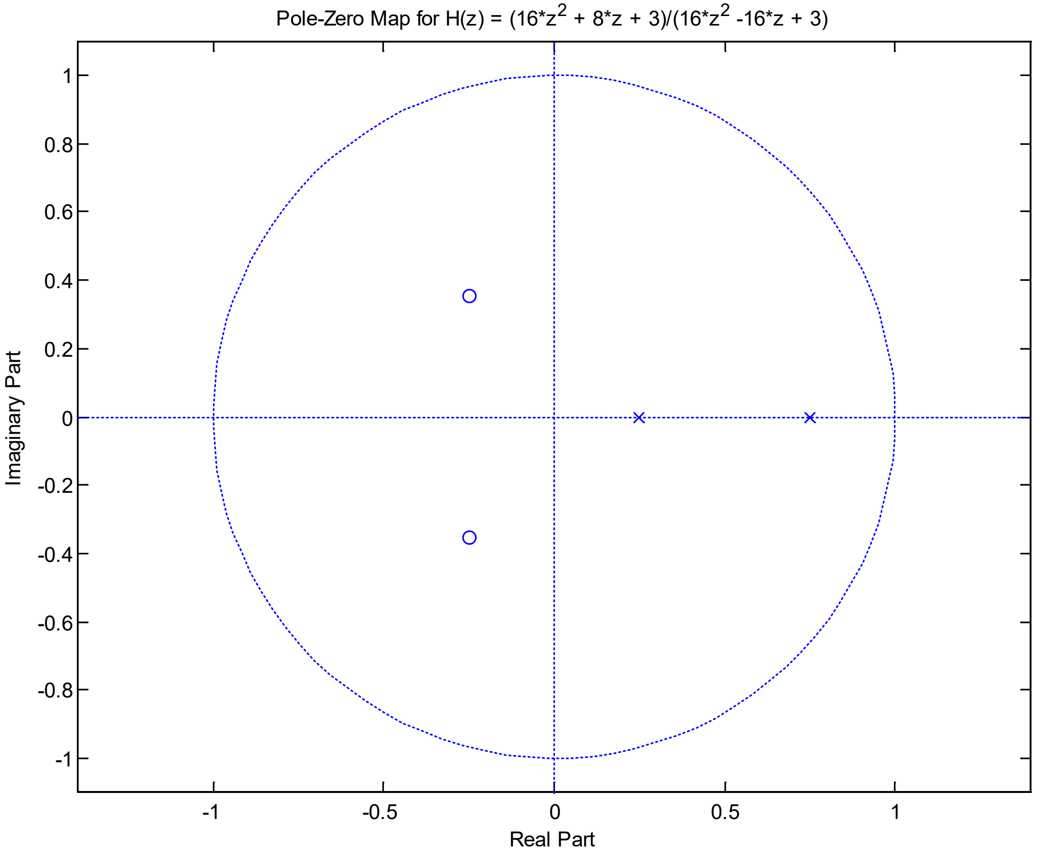
$$h_2[n] = \delta[n] - 3\cdot\left(\tfrac{1}{4}\right)^n u[n] + 3\cdot\left(\tfrac{3}{4}\right)^n u[n]$$

Z-Transform



Impulse Response for H(z) = (16*z^2 + 8*z + 3)/(16*z^2 -16*z + 3)
1.5
1
0.5
0
Amplitude
0
5
10
15
20
25
30
n (samples)

Z-Transform



Pole-Zero Map for H(z) = (16*z^2 + 8*z + 3)/(16*z^2 -16*z + 3)
Imaginary Part
Real Part
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1

Z-Transform

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n u[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n u[n]$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(n+1)\,\alpha^n u[n]$	$\dfrac{1}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos \omega_o n)\, u[n]$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$(r^n \sin \omega_o n)\, [n]$	$\dfrac{1-(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$

Z-Transform

TABLE 5.1 (Unilateral) z-Transform Pairs

No.	$x[n]$	$X[z]$
1	$\delta[n-k]$	z^{-k}
2	$u[n]$	$\frac{z}{z-1}$
3	$nu[n]$	$\frac{z}{(z-1)^2}$
4	$n^2u[n]$	$\frac{z(z+1)}{(z-1)^3}$
5	$n^3u[n]$	$\frac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^n u[n]$	$\frac{z}{z-\gamma}$
7	$\gamma^{n-1}u[n-1]$	$\frac{1}{z-\gamma}$

Z-Transform

8	$n\gamma^n u[n]$	$\dfrac{\gamma z}{(z-\gamma)^2}$
9	$n^2\gamma^n u[n]$	$\dfrac{\gamma z(z+\gamma)}{(z-\gamma)^3}$
10	$\dfrac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$	$\dfrac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^n \cos \beta n\, u[n]$	$\dfrac{z(z-\lvert\gamma\rvert\cos\beta)}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^n \sin \beta n\, u[n]$	$\dfrac{z\lvert\gamma\rvert\sin\beta}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\dfrac{rz[z\cos\theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n] \qquad \gamma=\lvert\gamma\rvert e^{j\beta}$	$\dfrac{(0.5re^{j\theta})z}{z-\gamma}+\dfrac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\dfrac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$

$$r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}}$$

$$\beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert}$$

$$\theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$$

Z-Transform

Z- Transform Operations

Operation	$f[k]$	$F[z]$
Addition	$f_1[k] + f_2[k]$	$F_1[z] + F_2[z]$
Scalar multiplication	$af[k]$	$aF[z]$
Right-shift	$f[k-m]u[k-m]$	$\frac{1}{z^m}F[z]$
	$f[k-m]u[k]$	$\frac{1}{z^m}F[z] + \frac{1}{z^m}\sum_{k=1}^{m} f[-k]z^k$
	$f[k-1]u[k]$	$\frac{1}{z}F[z] + f[-1]$
	$f[k-2]u[k]$	$\frac{1}{z^2}F[z] + \frac{1}{z}f[-1] + f[-2]$
	$f[k-3]u[k]$	$\frac{1}{z^3}F[z] + \frac{1}{z^2}f[-1] + \frac{1}{z}f[-2] + f[-3]$
Left-shift	$f[k+m]u[k]$	$z^m F[z] - z^m \sum_{k=0}^{m-1} f[k]z^{-k}$
	$f[k+1]u[k]$	$zF[z] - zf[0]$
	$f[k+2]u[k]$	$z^2F[z] - z^2f[0] - zf[1]$
	$f[k+3]u[k]$	$z^3F[z] - z^3f[0] - z^2f[1] - zf[2]$

Z-Transform

Multiplication by γ^k	$\gamma^k f[k]u[k]$	$F\left[\frac{z}{\gamma}\right]$
Multiplication by k	$kf[k]u[k]$	$-z\frac{d}{dz}F[z]$
Time Convolution	$f_1[k] * f_2[k]$	$F_1[z]F_2[z]$
Frequency Convolution	$f_1[k]f_2[k]$	$\frac{1}{2\pi j}\oint F_1[u]F_2\left[\frac{z}{u}\right]u^{-1}\,du$
Initial value	$f[0]$	$\lim_{z\to\infty} F[z]$
Final value	$\lim_{N\to\infty} f[N]$	$\lim_{z\to 1}(z-1)F[z]$ poles of $(z-1)F[z]$ inside the unit circle.