ENGR 071
Digital Signal Processing

Class 04

01/30/2025

ENGR 71 Class 02

- Class Overview
  - Frequency Analysis
    - Fourier Series
    - Fourier Transform
  - Sampling

Laplace →Frequency

- Laplace transform and connection to frequency analysis
  - We have noted that the imaginary axis of s-plane corresponds to frequency
  - What is the connection to frequency
    - Simple pole in the left hand of the s-plane

For $X(s) = \dfrac{1}{s+a}$, Pole is at $s = -a$

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

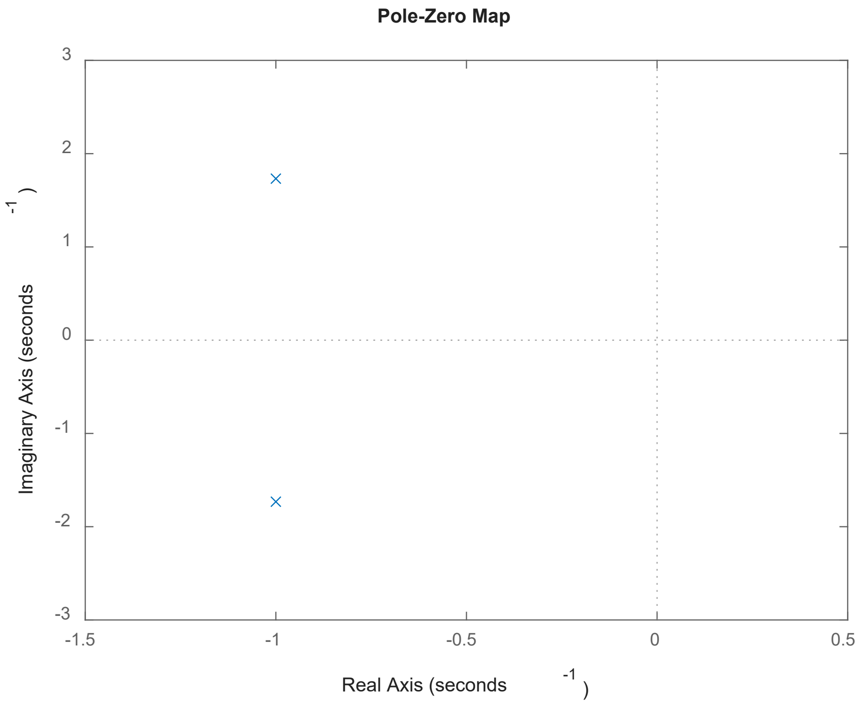
In the complex s-plane, the pole is complex: $a = \sigma + j\omega$

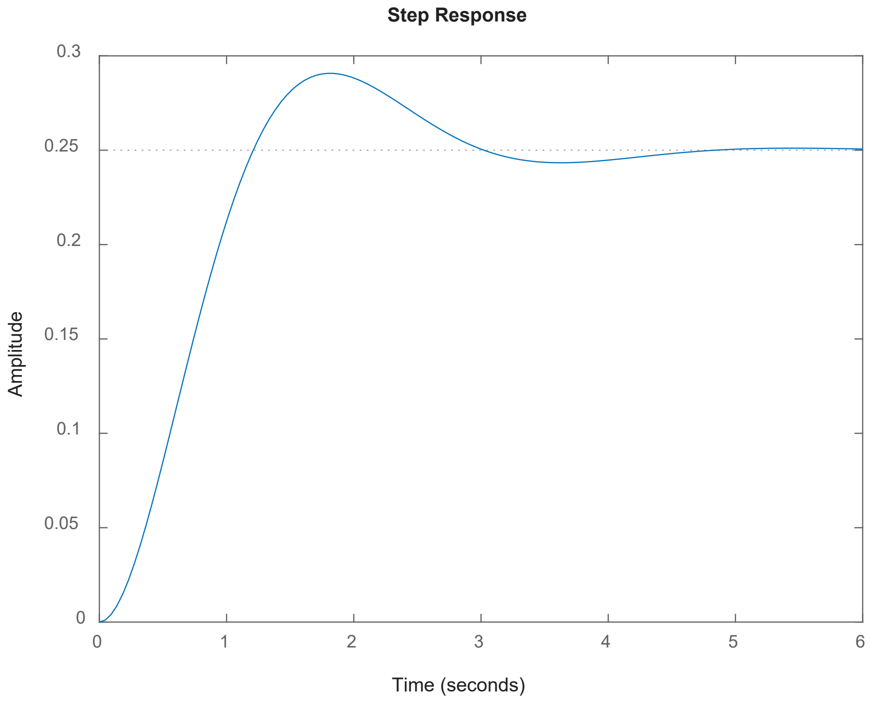
$$x(t) = e^{-\sigma t}\left[\cos \omega t + j \sin \omega t\right]$$

$$\mathrm{Re}\left(x(t)\right) = e^{-\sigma t} \cos \omega t$$

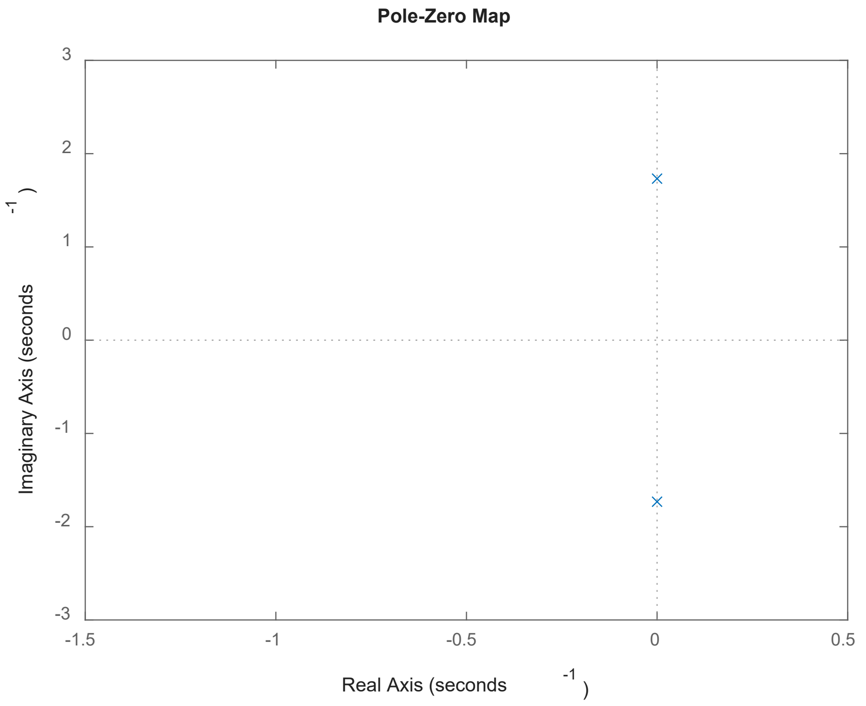
$e^{-\sigma t}$ dies off, it is transient response.

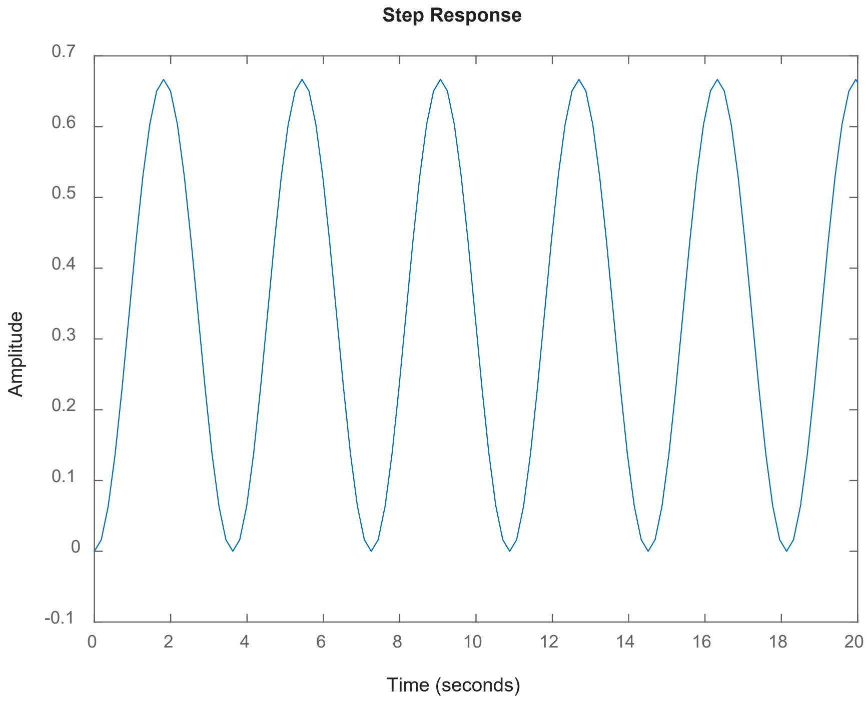
Laplace →Frequency





Laplace →Frequency





Laplace →Frequency

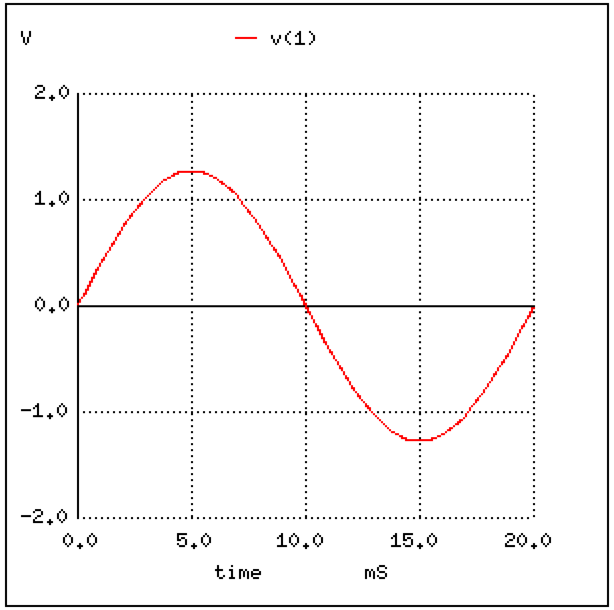
- Poles on the imaginary axis
  - signal looks like sinusoid.
- Can you start from the idea of decomposing a signal into frequency components and connect this with the Laplace transform?
  - Fourier Series
  - Fourier transform

Frequency Analysis

- Frequency Analysis
  - Why it's important for signal & image processing and communications applications:
    - Looking at a signal in terms of its frequency content lets you do things like:
      - Filter out noise
      - Enhance desirable characteristics
      - Compress data by removing frequencies that "won't be missed"
      - Feature recognition
      - Edge detection
      - Image reconstruction
      - Etc.

Frequency Analysis

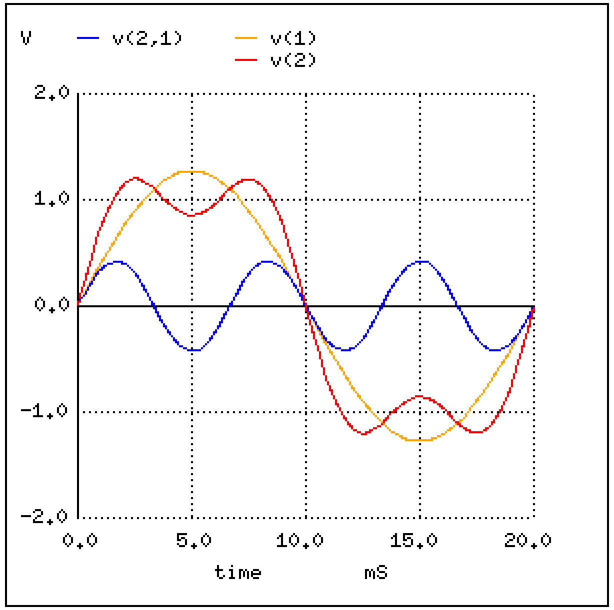
- Frequency Analysis
  - A periodic signal can be represented by a combination of sinusoidal signals.
  - Example: 50 Hz Square wave



1 50 Hz sine wave (1^{st} harmonic)

Frequency Analysis

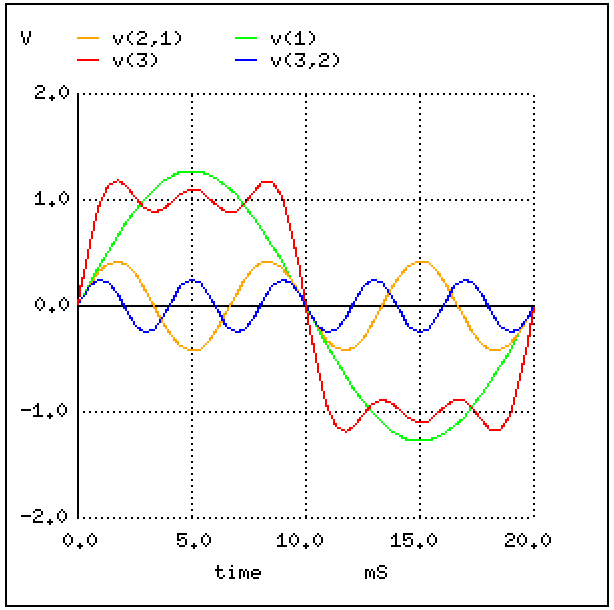
- Frequency Analysis
  - Any arbitrary signal can be represented by a combination of sinusoidal signals.
  - Example: 50 Hz Square wave




```
  1     50 Hz sine wave (1st harmonic)
+ 1/3 150 Hz sine wave (3rd harmonic)
```

Frequency Analysis

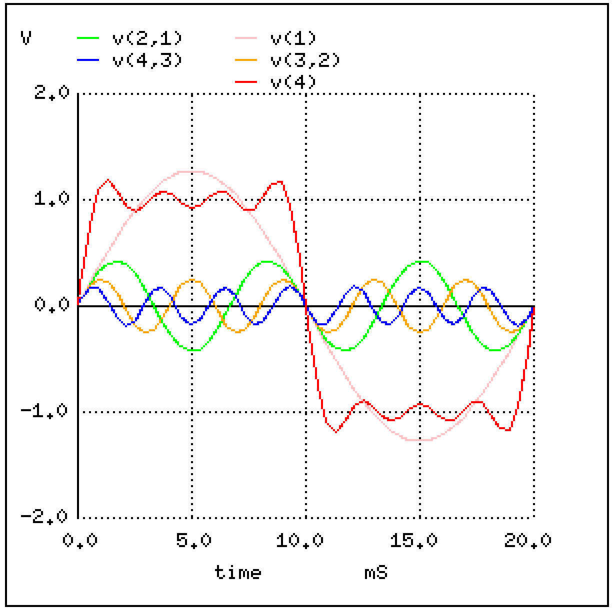
- Frequency Analysis
  - Any arbitrary signal can be represented by a combination of sinusoidal signals.
  - Example: 50 Hz Square wave




```
  1     50 Hz sine wave (1st harmonic)
+ 1/3  150 Hz sine wave (3rd harmonic)
+ 1/5  250 Hz sine wave (5th harmonic)
```

Frequency Analysis

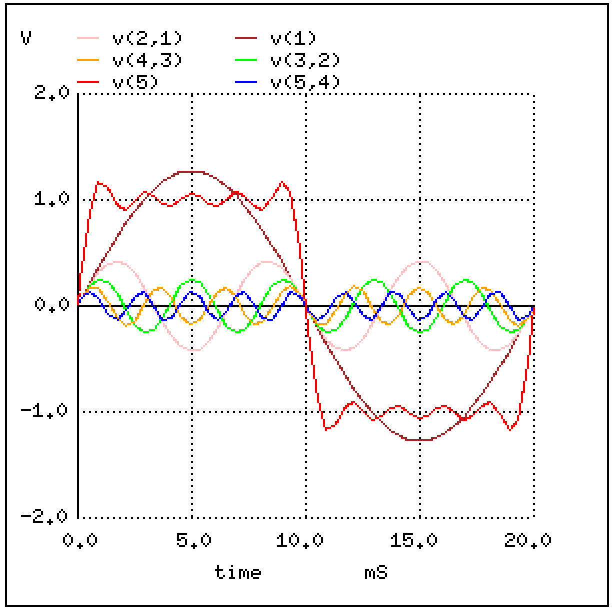
- Frequency Analysis
  - Any arbitrary signal can be represented by a combination of sinusoidal signals.
  - Example: 50 Hz Square wave



1 50 Hz sine wave (1st harmonic)
+ 1/3 150 Hz sine wave (3rd harmonic)
+ 1/5 250 Hz sine wave (5th harmonic)
+ 1/7 350 Hz sine wave (7th harmonic)

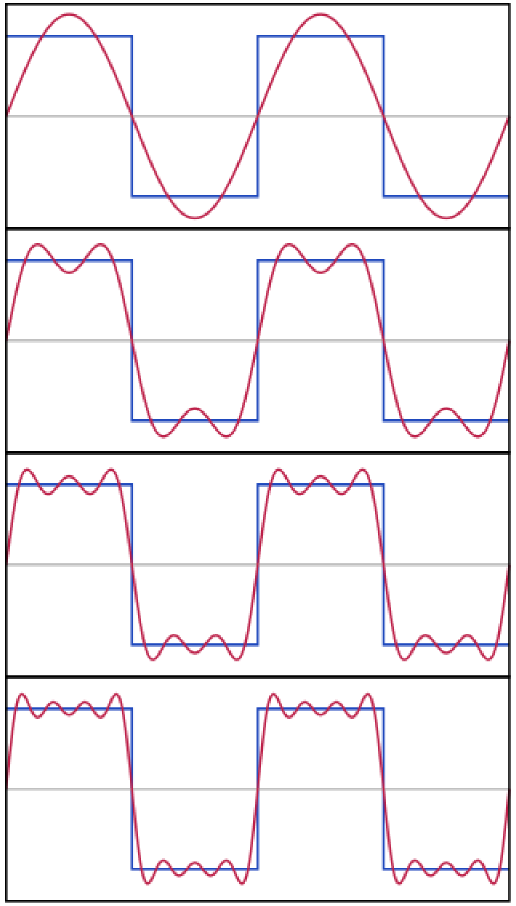
Frequency Analysis

- Frequency Analysis
  - Any arbitrary (periodic) signal can be represented by a combination of sinusoidal signals.
  - Example: 50 Hz Square wave




```
  1     50 Hz sine wave (1st harmonic)
+ 1/3 150 Hz sine wave (3rd harmonic)
+ 1/5 250 Hz sine wave (5th harmonic)
+ 1/7 350 Hz sine wave (7th harmonic)
+ 1/9 450 Hz sine wave (9th harmonic)
```

Frequency Analysis



Frequency Analysis

- Frequency Analysis:
  - Usual tools are:
    - Fourier Series for periodic functions
    - Fourier Transform for aperiodic functions
  - There are other (newer) tools that go beyond just decomposing signals into frequency components
    - Maintain time and frequency information about signals
      - e.g. Wavelet transform
    - We will get into this later…

Frequency Analysis

Who is Fourier?   Jean-Baptiste-Joseph Fourier

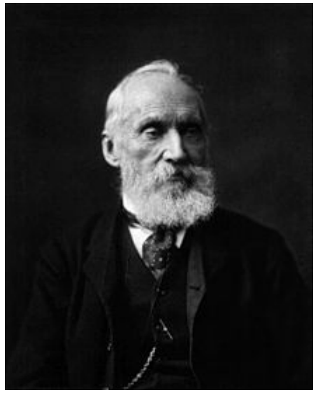
French mathematician/physicist (1768 – 1830)

Mainly interested in heat flow problems.

First published his theorem showing that any function can be represented as combination of pure frequency components in1822 in the book: “The Analytical Theory of Heat”

Lord Kelvin (William Thomson) (1824 – 1907) had this to say about Fourier’s theorem:

"Fourier's theorem is not only one of the most beautiful results of modern analysis, but it may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics."



Frequency Analysis for periodic signals

- The Fourier Series
  - Periodic functions: $x(t) = x(t + T_0)$

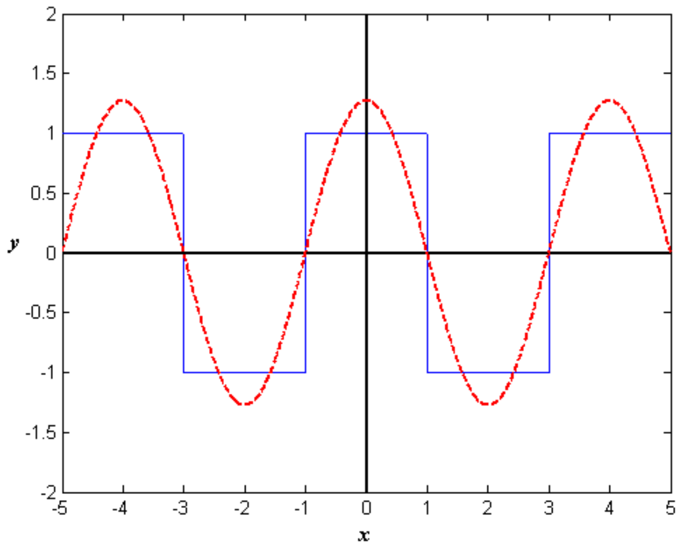
  Examples:

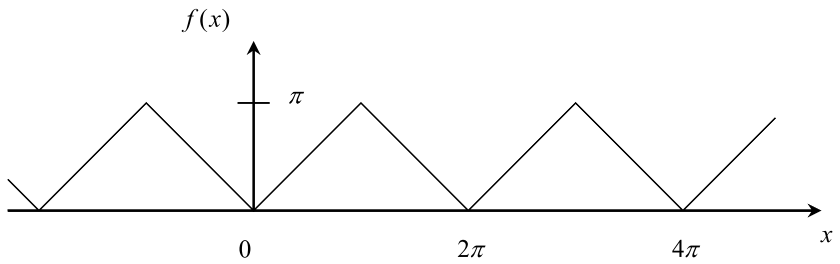
  Cosine function

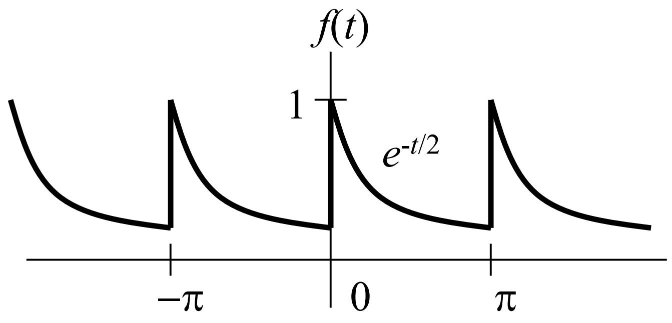
  Square wave pulse train

  Repeating triangular

  Repeating truncated exponential







Fourier Series

- The Fourier Series
  - Any periodic function can be represented as a sum of sines and cosines where the frequencies are multiples of the fundamental frequency of the periodic signal:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t)$$

Notice equation is periodic with period T_0

$$T_0 = \frac{1}{f_0}$$

Fourier Series

Note that the frequency f_o (cycles/sec or Hertz), period T_0 (secs), and angular frequency ω_0 (radians/sec) are related by:

$$f_0 = 1/T_0 \; ; \omega_0 = 2\pi f_0$$

So, you may see series in the forms:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T_0} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T_0} t\right)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Often, for simplicity, results and theorems are shown for signals that have period 2π, in which case:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$

Fourier Series

- Summary for Fourier Series representation of periodic function *x(t)* ("Trigonometric Form")

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \qquad \text{(Synthesis Eq.)}$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt \qquad \text{(Analysis Eqs.)}$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

– Remember: $\mathrm{x}(t) = x(t+T_0);\quad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

Fourier Series

- Few thoughts about the trigonometric form of Fourier Series
  - The a_0 coefficient is the mean value of the signal over one period.
    - D.C. level
  - For even functions, you only have cosine terms (because cosines are even functions and sines are odd functions)
  - For odd functions, you have only sine terms
  - For functions that are neither even or odd, you will have both sine and cosine terms

    $n = 1$ corresponds to the fundamental frequency

    $n = 2$, $n = 3$, etc. give you multiples of the fundamental frequency
    - overtones
  - The coefficients, a_n and b_n describe the strength of the frequency components
    - But, different strengths for the even and odd parts of the signals (sines & cosines)

Fourier Series

- Another form of Fourier series:
  Compact trigonometric series

$$a_n \cos\left(n\omega_0 t\right) + b_n \sin\left(n\omega_0 t\right) = C_n \cos\left(n\omega_0 t + \theta_n\right)$$

where $C_n = \sqrt{a_n^2 + b_n^2}$; $\theta_n = \tan^{-1}\left(\dfrac{-b_n}{a_n}\right)$

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(n\omega_0 t + \theta_n\right) \qquad C_0 = a_0$$

More convenient form for Fourier Series

- Exponential Form of Fourier Series
  - Signal is periodic:

$$x(t) = x(t+T_0) \qquad f_0 = \frac{1}{T_0} \qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t} \qquad \text{(Synthesis Eq.)}$$

$$X_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt \qquad \text{(Analysis Eq.)}$$

  - X_k's are complex numbers
  - To find the Fourier series, you need to know the period, T_0

(Use orthogonality of complex exponentials for analysis eq.)

Power in periodic signal

- Parseval's Theorem
  - Power in signal is: $P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 \, dt$ $\qquad P_x = \sum_{k=-\infty}^{+\infty} |X_k|^2$
    - Periodic signals have infinite energy but finite power

$$P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 \, dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \left(\sum_{k=-\infty}^{+\infty} X_k e^{j\omega_0 kt}\right)\left(\sum_{m=-\infty}^{\infty} X_m e^{j\omega_0 mt}\right)^* dt$$

$$P_x = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} X_k X_m^* \frac{1}{T_0}\int_{t_0}^{t_0+T_0} e^{j\omega_0(k-m)t} dt = 0 \text{ for } k \neq m$$

$$= T_0 \text{ for } k = m$$

$$P_x = \sum_{k=-\infty}^{+\infty} |X_k|^2 = X_0^2 + 2\sum_{k=1}^{+\infty} |X_k|^2 \quad \text{for real-valued signals}$$

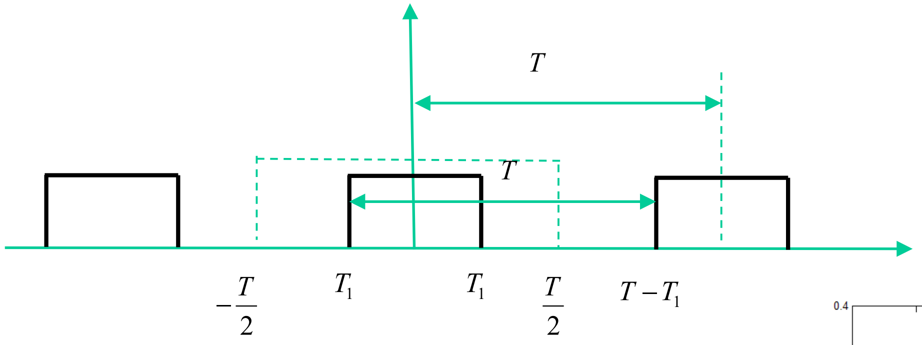
  - A plot of $|X_k|$ versus k shows the "power spectrum" of the signal

Fourier Series - Example

- Example:

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t}$$

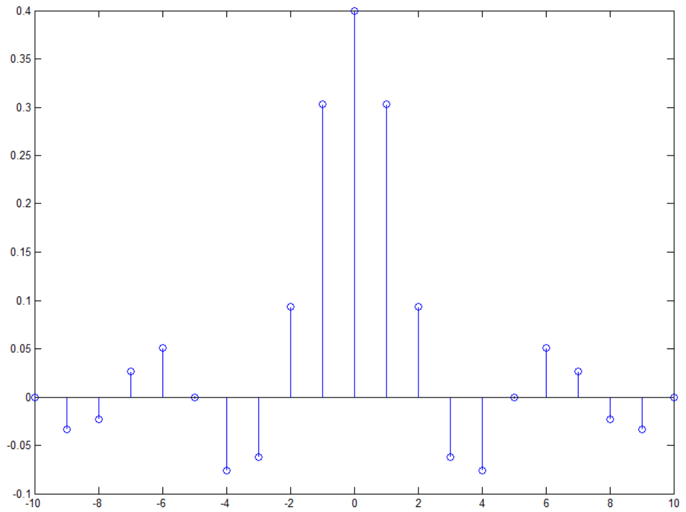
$$X_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$



$$X_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$X_k = \frac{\sin\left(k\omega_0 T_1\right)}{k\pi}$$

Power Spectrum



Fourier Series – Properties

Table 4.1 Basic Properties of Fourier Series

Basic Properties of Fourier Series		
	Time Domain	Frequency Domain
Signals and constants	$x(t)$, $y(t)$ periodic with period T_0, α, β	X_k, Y_k
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X_k + \beta Y_k$
Parseval's power relation	$P_x = \frac{1}{T_0}\int_{T_0} \lvert x(t)\rvert^2 dt$	$P_x = \sum_k \lvert X_k\rvert^2$
Differentiation	$\frac{dx(t)}{dt}$	$jk\Omega_0 X_k$
Integration	$\int_{-\infty}^{t} x(t')dt'$ only if $X_0 = 0$	$\frac{X_k}{jk\Omega_0}$ $k \neq 0$
Time shifting	$x(t-\alpha)$	$e^{-jk\alpha\Omega_0} X_k$
Frequency shifting	$e^{jM\Omega_0 t}x(t)$	X_{k-M}
Symmetry	$x(t)$ real	$\lvert X_k\rvert = \lvert X_{-k}\rvert$ even function of k $\angle X_k = -\angle X_{-k}$ odd function of k
Convolution in time	$z(t) = [x*y](t)$	$Z_k = X_k Y_k$

The Fourier Transform

- Fourier Transform
  - Most well-known frequency analysis tool
  - Decomposes signal into frequency components
    - Like Fourier series, but works on aperiodic signals
  - Relationship between Fourier Series and Transform
    - Obtain Fourier transform from Fourier series by letting period go to infinity and interval between harmonics go to zero.

Fourier Series → Fourier Transform

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t}$$

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt$$

$$\Big\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Define $X(\omega_k) \equiv T_0 X_k$ where $\omega_k = k\omega_0$

Also, $\omega_0 = \Delta\omega = \dfrac{2\pi}{T_0}$

ω_0 is the frequency between harmonics, i.e., $(k+1)\omega_0 - k\omega_0 = \omega_0$

Then

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} \frac{X(\omega_k)}{T_0} e^{j\omega_k t} = \sum_{k=-\infty}^{+\infty} X(\omega_k) e^{j\omega_k t}\frac{\Delta\omega}{2\pi}$$

As $T_0 \to \infty$, $\Delta\omega \to d\omega$, $\omega_k \to$ continuous variable, and sum $\to$ integral, so:

$$x(t) = \sum_{k=-\infty}^{+\infty} X(\omega_k) e^{j\omega_k t}\frac{\Delta\omega}{2\pi} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\,d\omega$$

$$X_k = \frac{X(\omega_k)}{T_0} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega_k t}\,dt \Rightarrow X(\omega) = \int_{-\infty}^{-\infty} x(t)e^{-j\omega t}\,dt$$

The Fourier Transform

- Fourier Transform
  - Equations:

$$X(\omega) = \mathscr{F}[x(t)] = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \qquad \text{(Analysis Equation)}$$

$$x(t) = \mathscr{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{+j\omega t}d\omega \qquad \text{(Synthesis Equation)}$$

  - The analysis equation “analyzes” signal for frequency components
  - The synthesis equation “synthesizes” the signal from its frequency components.

The Fourier Transform

- Fourier Transform
  - Equations:

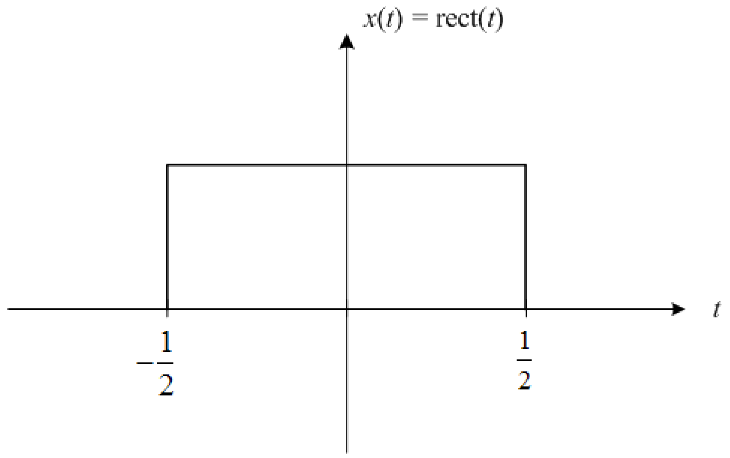
$$X(\omega) = \mathscr{F}[x(t)] = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \qquad \text{(Analysis Equation)}$$

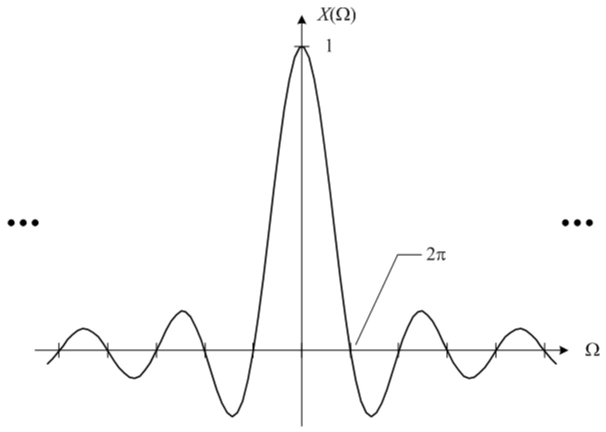
$$x(t) = \mathscr{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{+j\omega t}d\omega \qquad \text{(Synthesis Equation)}$$

  - Note that the factor of $1/2\pi$ in the synthesis equation is the usual convention for engineering.
    - In other fields $1/2\pi$ is in analysis equation or sometimes $1/\sqrt{2\pi}$ is in both equations (symmetric form).

The Fourier Transform

- Example:
  - Fourier transform of rectangular pulse (symmetric about 0)

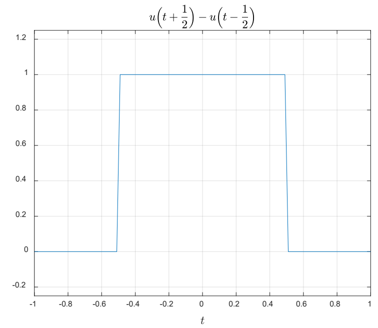




The Fourier Transform

- Example:
  - Fourier transform of rectangular pulse (symmetric about 0)

$$x(t) = \left[u(t+1/2) - u(t-1/2)\right]$$



$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = \int_{-1/2}^{1/2} e^{-j\omega t} dt$$

$$X(\omega) = \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-1/2}^{1/2} = \frac{e^{-j\frac{\omega}{2}} - e^{+j\frac{\omega}{2}}}{-j\omega} = \frac{2}{\omega}\left(\frac{e^{+j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{2j}\right)$$

$$X(\omega) = \frac{\sin\left(\omega/2\right)}{\left(\omega/2\right)} = \mathrm{sinc}\left(\omega/2\right)$$

(un-normalized form of sinc function)

The Fourier Transform

- When does Fourier transform exist?
  - Fourier Transform can be found if signal is absolutely integrable (and has finite number of discontinuities.)

$$\int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$$

[Note that although periodic signals are not absolutely integrable, you can find their Fourier transform using their Fourier series.]

  - Example of signal that does not have a Fourier transform

$$x(t) = e^{t} \ , \ 0 < t < +\infty$$

  - Examples of signals that do have Fourier transform:
    - Practically all physical signals since they will only exist for some finite time and have finite energy.
      - Music
      - Image data (using 2-D Fourier transform)

Fourier Transform from Laplace Transform

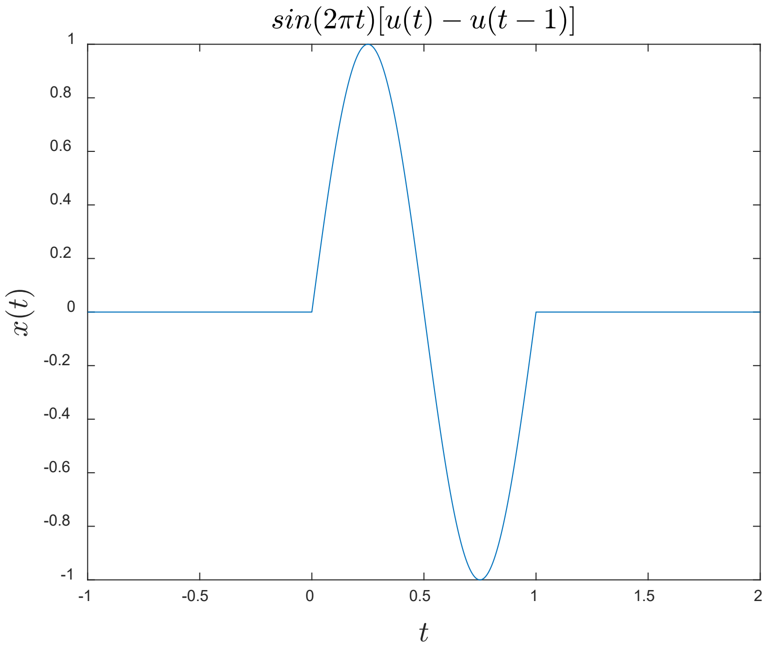
- Fourier Transform from Laplace Transform
  - Set Laplace transform variable $s = j\omega$

$$\mathscr{F}[x(t)] = \mathscr{L}[x(t)]\Big|_{s=j\omega} = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

  - Example:

$$x(t) = \sin\left(2\pi t\right)\left[u(t) - u(t-1)\right]$$

$$\mathscr{L}[x(t)] = X(s) = \frac{2\pi(1-e^{-s})}{s^2+4\pi^2}$$



Fourier Transform from Laplace Example

- Example:

$$\mathscr{F}[x(t)] = X(\omega) = X(s)\big|_{s=j\omega} = \frac{2\pi\left(1-e^{-j\omega}\right)}{\left(j\omega\right)^2 + 4\pi^2}$$

$$X(\omega) = \frac{2\pi e^{-j\omega/2}\left(e^{+j\omega/2} - e^{-j\omega/2}\right)}{4\pi^2 - \omega^2} = \frac{4j\pi e^{-j\omega/2}}{4\pi^2 - \omega^2}\frac{\left(e^{+j\omega/2} - e^{-j\omega/2}\right)}{2j}$$

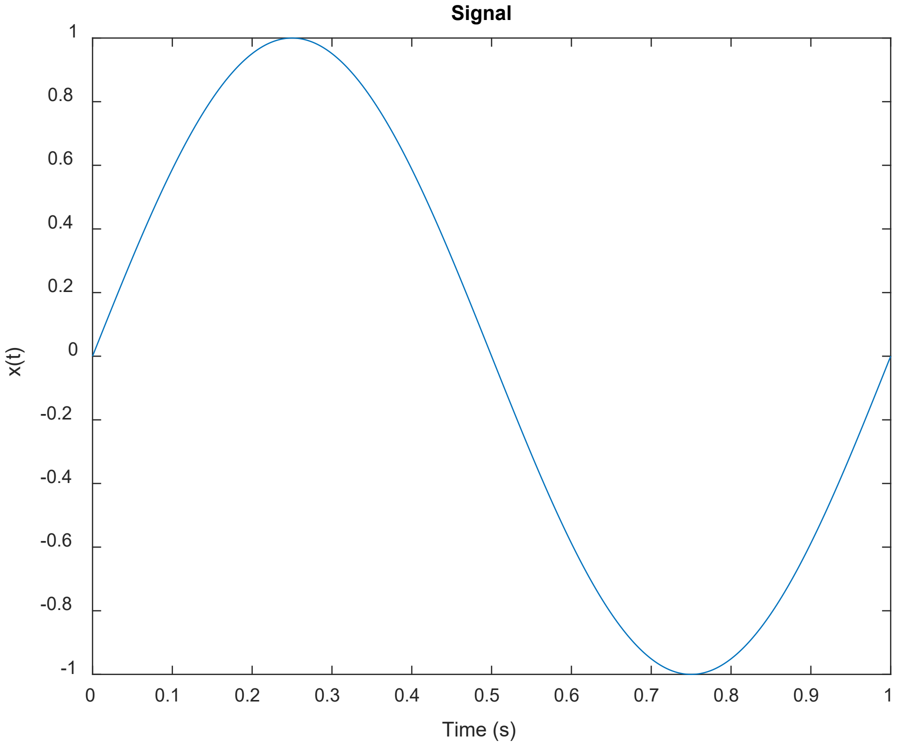
$$X(\omega) = \frac{4j\pi e^{-j\omega/2}\sin\left(\omega/2\right)}{4\pi^2 - \omega^2}$$

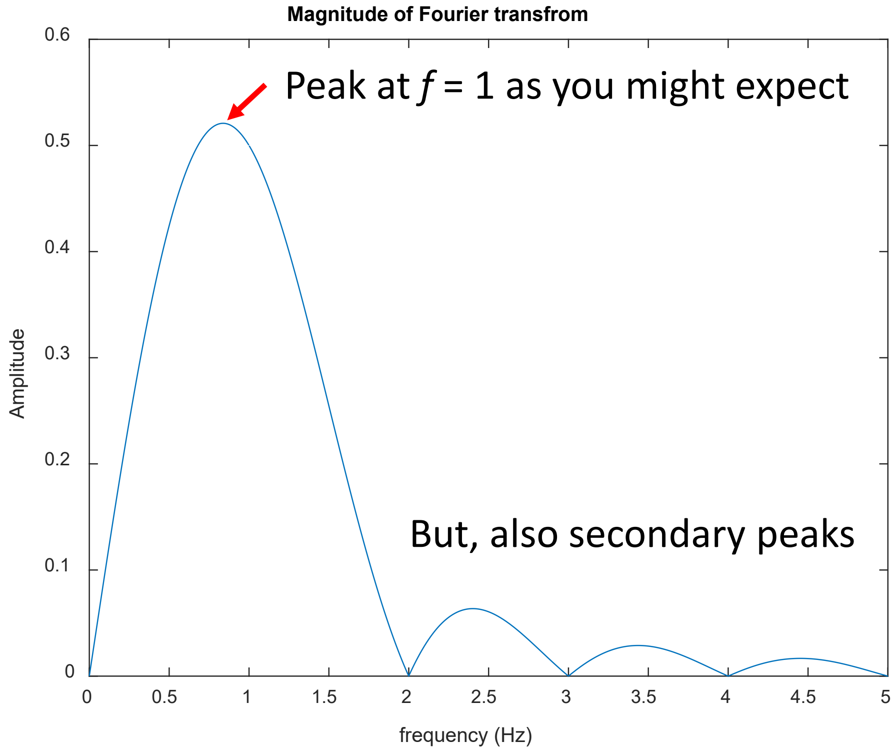
  – Note: $$\left|X(\omega)\right| = \left|\frac{4\pi\sin\left(\omega/2\right)}{4\pi^2 - \omega^2}\right|$$

    - What can we learn about spectral the spectral content of a signal like this?

Fourier Transform from Laplace Example

- Example





Fourier Series – From Laplace transform

You can also get the Fourier Series from the Laplace Transform

Fourier Series – From Laplace transform

- Fourier Series coefficients from Laplace transform:

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t}$$

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$$

  – Laplace transform of one period of periodic function:

$$x_1(t) = x(t)\left[u(t) - u(t-T_0)\right]$$

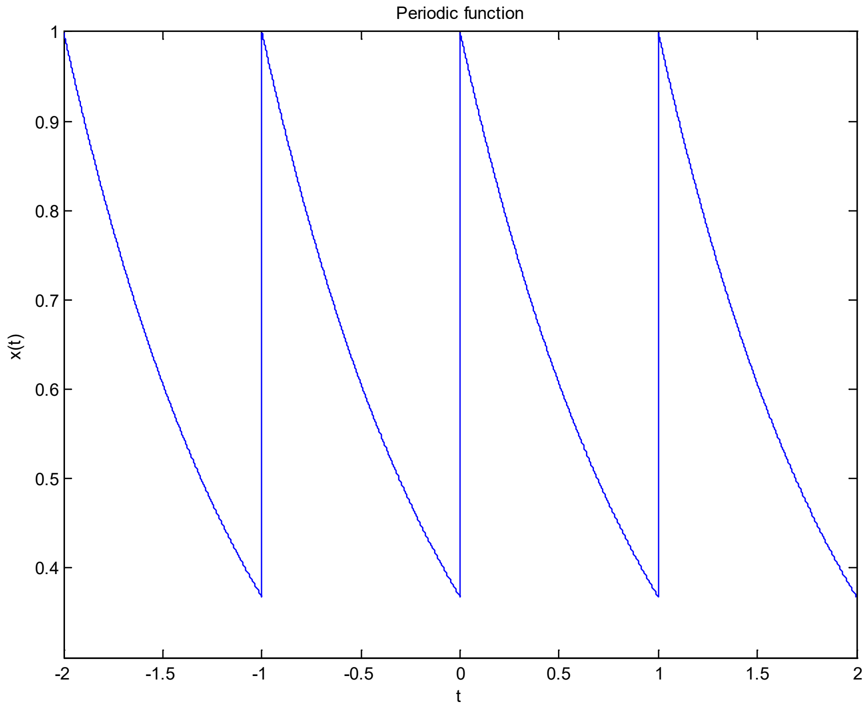
$$\mathcal{L}[x_1(t)] = X_1(s) = \int_{-\infty}^{+\infty} x(t)\left[u(t)-u(t-T_0)\right]e^{-st}dt = \int_0^{T_0} x(t)e^{-st}dt$$

  – Fourier Transform coefficients are found using Laplace as:

$$X_k = \frac{1}{T_0}\mathcal{L}\left[x_1(t)\right]_{s=jk\omega_0} \text{ where } \omega_0 = \frac{2\pi}{T_0}$$

Fourier Series – From Laplace transform

- Example:
  - Find the Fourier Series of: $x(t) = e^{-t}\left[u(t) - u(t-1)\right]$
    (one cycle of function from t =0 to t = 1)



Fourier Series – From Laplace transform

- Example:
  - Find the Fourier Series of: $x(t) = e^{-t}\left[u(t) - u(t-1)\right]$
    (one cycle of function from t =0 to t = 1)

$$X_k = \frac{1}{T_0}\mathcal{L}\left[x_1(t)\right]_{s=jk\omega_0} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T_0}$$

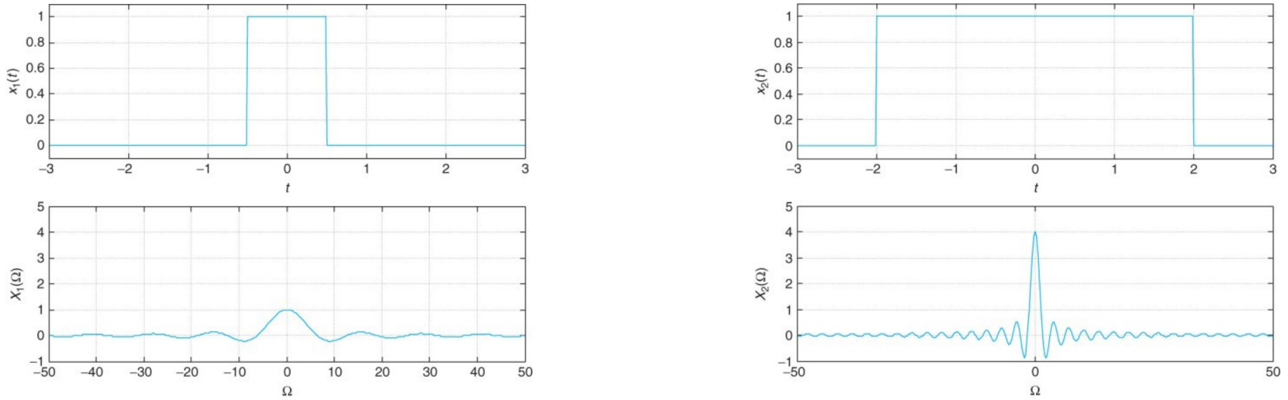
Use time and frequency shifting

$$x(t-t_0)u(t-t_0) \quad \Leftrightarrow \quad e^{-st_0}X(s) \quad \text{Frequency shift}$$

$$e^{s_0 t}x(t) \quad \Leftrightarrow \quad X(s-s_0) \quad \text{Time shift}$$

Properties of Fourier Transform

- Properties of the Fourier Transform
  - Very similar to those of Laplace transform and Fourier Series
    - Linearity: $\mathcal{F}[\alpha x(t)+\beta y(t)]=\alpha\mathcal{F}[x(t)]+\beta\mathcal{F}[y(t)]$
    - Inverse proportionality of Time and Frequency
      - If a signal is narrow in time, it is wide in frequency
      - If a signal is broad in time, it is narrow in frequency



Properties of Fourier Transform

- Duality: You can reverse the meaning of time and frequency
  Fourier transform pair:

$$x(t) \Leftrightarrow X(\omega)$$

  Has a dual Fourier transform pair:

$$\mathrm{X}(t) \Leftrightarrow 2\pi x(-\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\rho) e^{+j\rho t} d\rho$$

$$2\pi x(-\omega) = \int_{-\infty}^{+\infty} X(\rho) e^{-j\rho\omega} d\rho$$

$$2\pi x(-\omega) = \int_{-\infty}^{+\infty} X(t) e^{-j\omega t} dt = \mathcal{F}[X(t)]$$

– Example: How is this useful?

- Suppose you wanted to find the Fourier transform of the sinc function

Properties of Fourier Transform

- Example: Fourier transform of sinc function would be really hard to find from the definition:

$$x(t) = \text{sinc}\left(t/2\right) = \frac{\sin(t/2)}{t/2} \qquad X(\omega) = \int_{-\infty}^{+\infty} \frac{\sin\left(t/2\right)}{t/2} e^{-j\omega t} dt$$

But we know what the Fourier transform of the boxcar is:

Using:

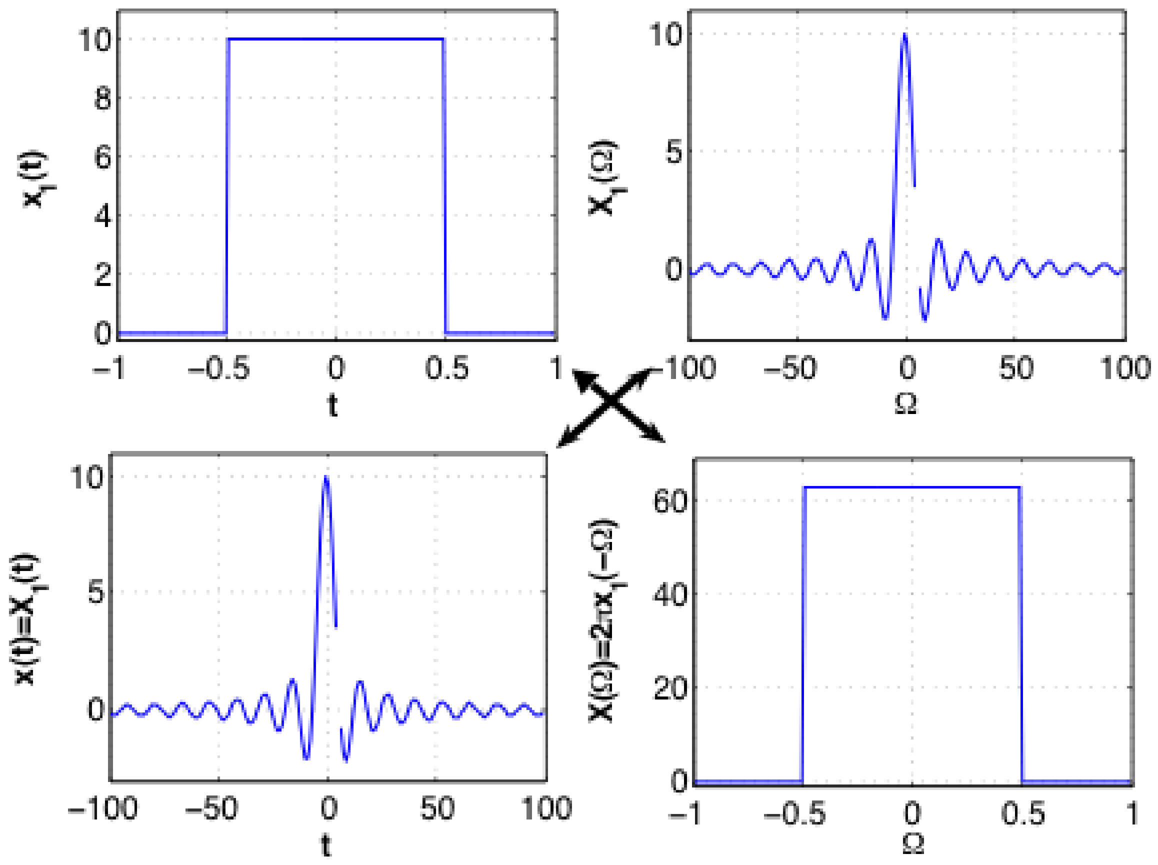
$$x(t) = \left[u(t+1/2) - u(t-1/2)\right] \qquad X(\omega) = \frac{\sin\left(\omega/2\right)}{\left(\omega/2\right)} = \text{sinc}\left(\omega/2\right)$$

$$x(t) \Leftrightarrow X(\omega) \quad \mathrm{X}(t) \Leftrightarrow 2\pi x(-\omega)$$

$$X(t) = \text{sinc}\left(t/2\right) \Leftrightarrow 2\pi x(-\omega) = 2\pi x\left(\omega\right) \quad \text{(boxcar is symmetric)}$$

$$\therefore \mathcal{F}\left[\text{sinc}\left(t/2\right)\right] = 2\pi\left[u\left(\omega + 1/2\right) - u\left(\omega - 1/2\right)\right]$$

Duality



Properties of Fourier Transform

– Frequency Shift property: (Signal Modulation)

If $\mathscr{F}[x(t)] = X(\omega)$

then $\mathscr{F}[x(t)e^{j\omega_0 t}] = X(\omega - \omega_0)$

$$\mathscr{F}[x(t)e^{j\omega_0 t}] = \int_{-\infty}^{+\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{+\infty} x(t)e^{-j(\omega-\omega_0)t}dt = X(\omega - \omega_0)$$

– Signal modulation

Instead of multiplying by $e^{j\omega_0 t}$, multiply by a $\cos(\omega_0 t)$

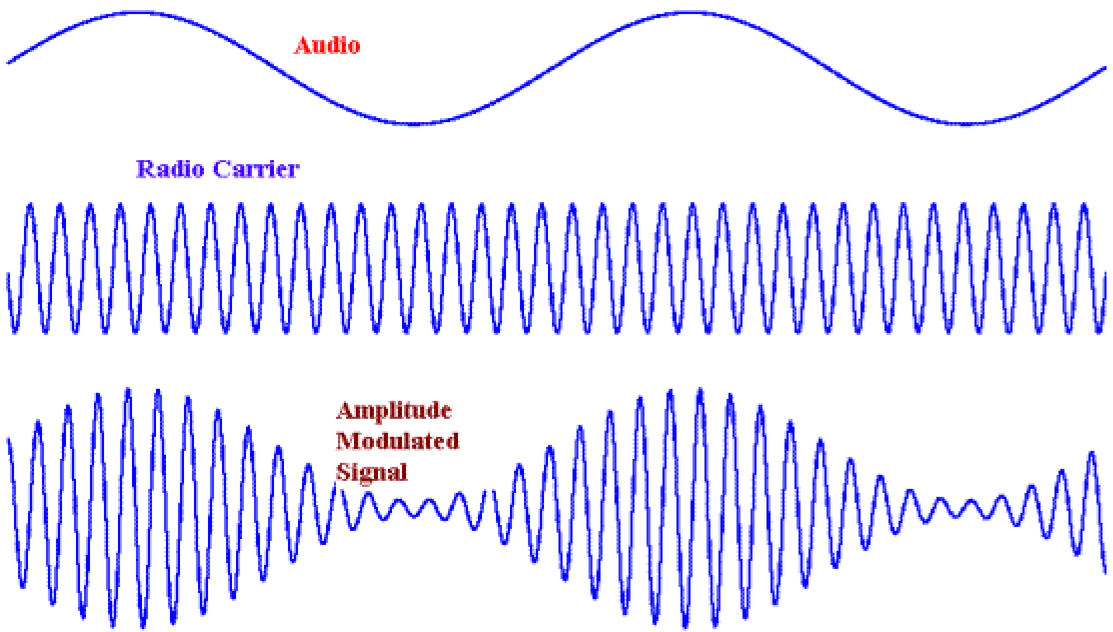
$$\cos(\omega_0 t) = 0.5\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$\mathscr{F}[x(t)\cos(\omega_0\,\mathrm{t})] = 0.5\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]$$

Modulation

– Why is modulation useful?

ω_o is the carrier frequency.



Modulation

– Why is modulation useful?

- We can hear sounds with frequencies ~20 Hz to ~20,000 Hz.
- For an antenna to radiate efficiently, it's roughly ¼ of the wavelength.

Relationship of velocity, frequency and wavelength:

$$c = \lambda \nu \quad \text{or} \quad \nu = {}^{c}\!/_{\lambda} \quad \text{or} \quad \lambda = {}^{c}\!/_{\nu}$$

Speed of electromagnetic radiation (light) is 3 x 10^8 m/s

so, for a 20 kHz signal, you need an antenna ~4 km (~2.4 miles)

That's a big antenna. (For 20 Hz, ~2,400 miles … BIG antenna!)

Put signal on a carrier wave on the order of 1000 kHz (AM radio)
(75 m. antenna)

Modulation also lets you assign different carrier frequencies to different radio stations.

Fourier transform of periodic signal

– Fourier Transform of periodic signal
  - Recall that we said if a signal is absolutely integrable, its Fourier transform exists
  - Note that this does <u>not</u> say, that non-absolutely integrable signals do not have Fourier transforms.
    – Periodic signals are a case in point.
  - Using the Fourier series representation of a periodic signal:

$$x(t) = \sum_k X_k e^{jk\omega_0 t} \qquad X_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$$

$$\mathcal{F}[x(t)] = \sum_k \mathcal{F}\left[X_k e^{jk\omega_0 t}\right]$$

Fourier transform of periodic signal

– Fourier Transform of periodic signal

- By the frequency shifting property:

$$\text{If} \quad \mathscr{F}[x(t)] = X(\omega)$$

$$\text{then} \quad \mathscr{F}[x(t)e^{j\omega_0 t}] = X(\omega - \omega_0)$$

- And, for a constant value, A, (in this case $A = X_k$)

$$\mathscr{F}[A] = 2\pi A\delta(\omega)$$

$$\mathscr{F}[x(t)] = \sum_k \mathscr{F}\left[X_k e^{jk\omega_0 t}\right] = 2\pi \sum_k X_k \delta(\omega - k\omega_0)$$

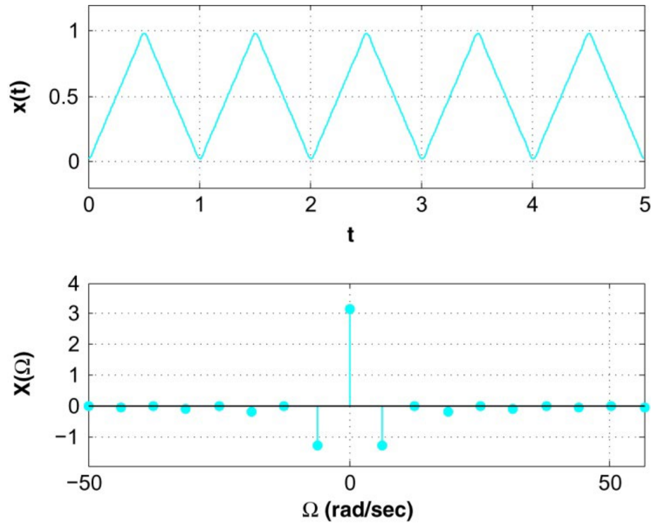
Fourier transform of periodic signal

– Fourier Transform of periodic signal

- Example:

$$x_1(t) = 2\left[tu\left(t\right) - 2(t-0.5)u(t-0.5) + (t-1)u(t-1)\right]$$

One period of $x(t)$ is $x_1(t)$ with fundamental frequency ω_0



Fourier Transform Pairs

Table 5.2 Fourier Transform Pairs

	Function of Time	Function of Ω
(1)	$\delta(t)$	1
(2)	$\delta(t-\tau)$	$e^{-j\Omega\tau}$
(3)	$u(t)$	$\frac{1}{j\Omega}+\pi\delta(\Omega)$
(4)	$u(-t)$	$\frac{-1}{j\Omega}+\pi\delta(\Omega)$
(5)	$\text{sign}(t)=2[u(t)-0.5]$	$\frac{2}{j\Omega}$
(6)	$A, -\infty<t<\infty$	$2\pi A\delta(\Omega)$
(7)	$Ae^{-at}u(t), a>0$	$\frac{A}{j\Omega+a}$
(8)	$Ate^{-at}u(t), a>0$	$\frac{A}{(j\Omega+a)^2}$
(9)	$e^{-a\|t\|}, a>0$	$\frac{2a}{a^2+\Omega^2}$
(10)	$\cos(\Omega_0 t), -\infty<t<\infty$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$
(11)	$\sin(\Omega_0 t), -\infty<t<\infty$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)]$
(12)	$p(t)=A[u(t+\tau)-u(t-\tau)], \tau>0$	$2A\tau\frac{\sin(\Omega\tau)}{\Omega\tau)}$
(13)	$\frac{\sin(\Omega_0 t)}{\pi t}$	$P(\Omega)=u(\Omega+\Omega_0)-u(\Omega-\Omega_0)$
(14)	$x(t)\cos(\Omega_0 t)$	$0.5[X(\Omega-\Omega_0)+X(\Omega+\Omega_0)]$

Fourier Transform Properties

Table 5.1 Basic Properties of Fourier Transform

	Time Domain	Frequency Domain
Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\Omega), Y(\Omega), Z(\Omega)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\Omega) + \beta Y(\Omega)$
Expansion/contraction in time	$x(\alpha t), \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Time differentiation	$\frac{d^n x(t)}{dt^n}, n \geq 1,$ integer	$(j\Omega)^n X(\Omega)$
Frequency differentiation	$-jtx(t)$	$\frac{dX(\Omega)}{d\Omega}$
Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Time shifting	$x(t - \alpha)$	$e^{-j\alpha\Omega} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(\Omega - \Omega_0)$
Modulation	$x(t)\cos(\Omega_c t)$	$0.5[X(\Omega - \Omega_c) + X(\Omega + \Omega_c)]$
Periodic signals	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\Omega)\rvert = \lvert X(-\Omega)\rvert$ $\angle X(\Omega) = -\angle X(-\Omega)$
Convolution in time	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$
Windowing/Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\Omega)$
Cosine transform	$x(t)$ even	$X(\Omega) = \int_{-\infty}^{\infty} x(t)\cos(\Omega t)dt,$ real
Sine transform	$x(t)$ odd	$X(\Omega) = -j\int_{-\infty}^{\infty} x(t)\sin(\Omega t)dt,$ imaginary

Power in aperiodic signal

- Parseval's Theorem for aperiodic signal
  - Energy in signal

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt = \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)\,\mathrm{e}^{-j\omega t}\,d\omega\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)\left[\int_{-\infty}^{+\infty} x(t)\,\mathrm{e}^{-j\omega t}\,dt\right]d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(\omega)X(\omega)d\omega$$

$$\boxed{E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega}$$

  - Describes how energy is distributed among signal's harmonic components

Spectrum of a signal

- Parseval's Power and Energy relations
  - Periodic signals (Fourier Series)

$$\boxed{P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = \sum_{k=-\infty}^{+\infty} |X_k|^2}$$

Parseval (1799) came up with this for series … stated it, didn't prove it

  - Aperiodic signals (Fourier transform)

$$\boxed{E_x = \int_{-\infty}^{+\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2\, d\omega}$$

This is actually Plancherel's theorem (1910)

  - Describes how power or energy are distributed among frequency components

Spectrum of a signal

– Magnitude spectrum:

$$\left|X(\omega)\right| \text{ vs } \omega \quad (\text{or } \left|X_k\right| \text{ vs } k)$$

– Phase Spectrum

$$\angle X(\omega) \text{ vs } \omega \quad (\angle X_k \text{ vs } k)$$

– Energy/Power Spectral Density

$$\left|X(\omega)\right|^2 \text{ vs } \omega \quad (\text{or } \left|X_k\right|^2 \text{ vs } k)$$

Spectrum of a signal

- Relation of Spectral Density to Autocorrelation
  - Autocorrelation (measures self-similarity of signal)

$$\varphi_x(\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t-\tau)dt$$

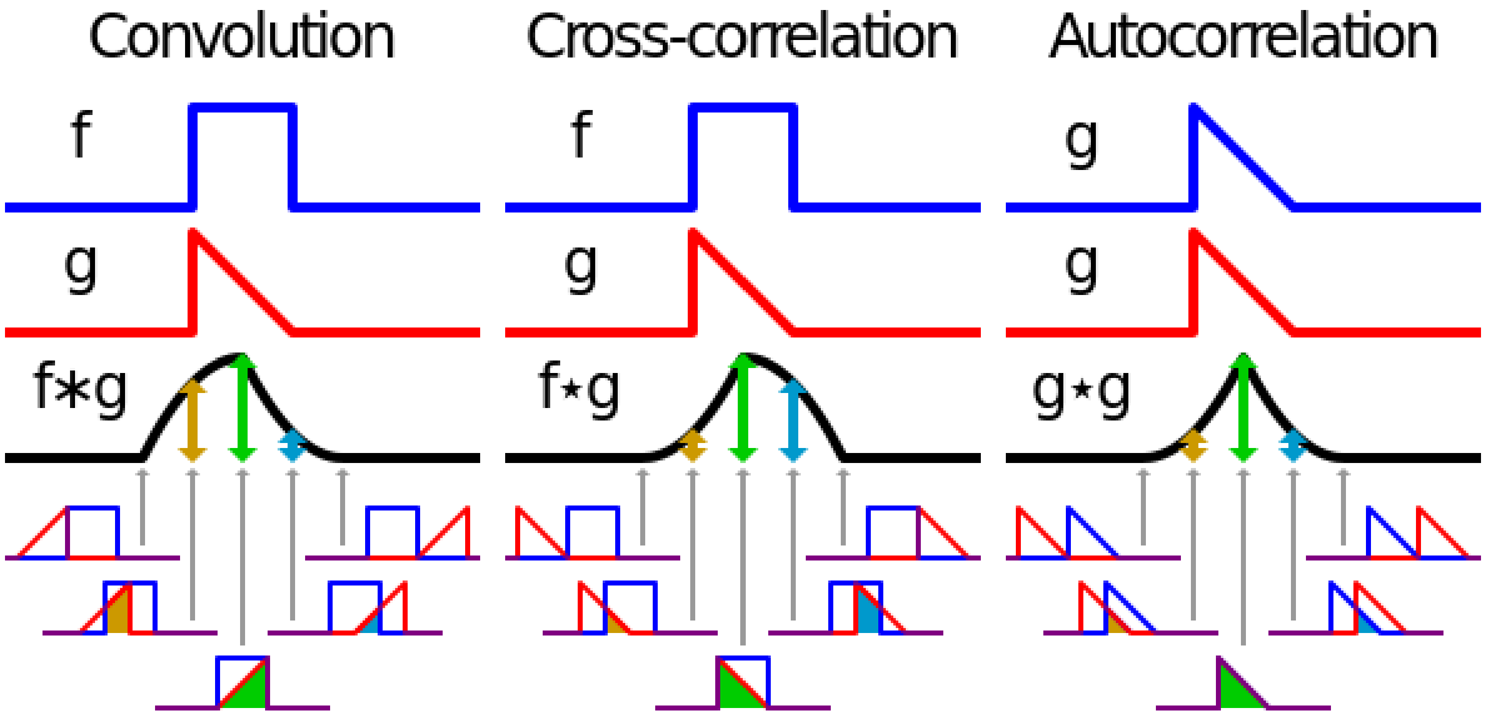
Note that this is different than convolution of signal with itself

$$x(\tau) * x(\tau) = \int_{-\infty}^{+\infty} x(t)x(\tau - t)dt$$

  - Inverse Fourier transform of $|X(\omega)|^2$ is the autocorrelation
  - Energy spectral density and autocorrelation are Fourier transform pairs

Convolution, Cross-correlation, Auto-correlation

$$[x*y](t)=\int_{-\infty}^{+\infty}x(\tau)y(t-\tau)d\tau \qquad \varphi_{xy}(\tau)=\int_{-\infty}^{+\infty}x(t)y^*(t-\tau)dt \qquad \varphi_x(\tau)=\int_{-\infty}^{+\infty}x(t)x^*(t-\tau)dt$$



Signal transformations – Summary

- Why signal transformations?
  - Laplace transform represents a signal in terms of complex exponentials.

$$X(s) = \mathcal{L}[x(t)] = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \quad \text{where } s \text{ is a complex variable } (s = \sigma + j\omega)$$

  - You can think of this as a decomposition of the signal into components on the basis vectors e^{-st}
    - The complex, continuous variable s parameterizes the basis vectors.
  - In the Laplace domain, you can examine the stability of a system and determine both the steady-state and transient behavior of the system.
    - Useful for control applications.

Signal transformations – Summary

- Fourier
  - Frequency analysis of systems and signals looks only at the steady-state behavior (not transients).
    - This is useful for applications in signal processing, communications and image processing
  - The Fourier series and Fourier transform are the tools for frequency analysis.

$$X(\omega) = \mathscr{F}[x(t)] = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$ Fourier Transform for aperiodic signals

- This is a special case of the Laplace transform where you are just looking at the imaginary component of $s = \sigma + j\omega$
  - There are other mathematical differences having to do with convergence.

Signal transformations – Summary

– Frequency Analysis

- Useful in communications, signal & image processing
- **Fourier Series – For periodic signals**

$$x\left(t\right)=\sum_{k=-\infty}^{+\infty}X_n e^{jk\omega_0 t}$$

$$X_k=\frac{1}{T_0}\int_{T_0}x(t)e^{-jk\omega_0 t}dt$$

- **Fourier Transform – For aperiodic signals**

$$X(\omega)=\mathscr{F}[x(t)]=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt$$

$$x(t)=\mathscr{F}^{-1}[X(\omega)]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\omega)e^{+j\omega t}d\omega$$

SAMPLING CONTINUOUS → DISCRETE SIGNALS

Sampling for Discrete Signals

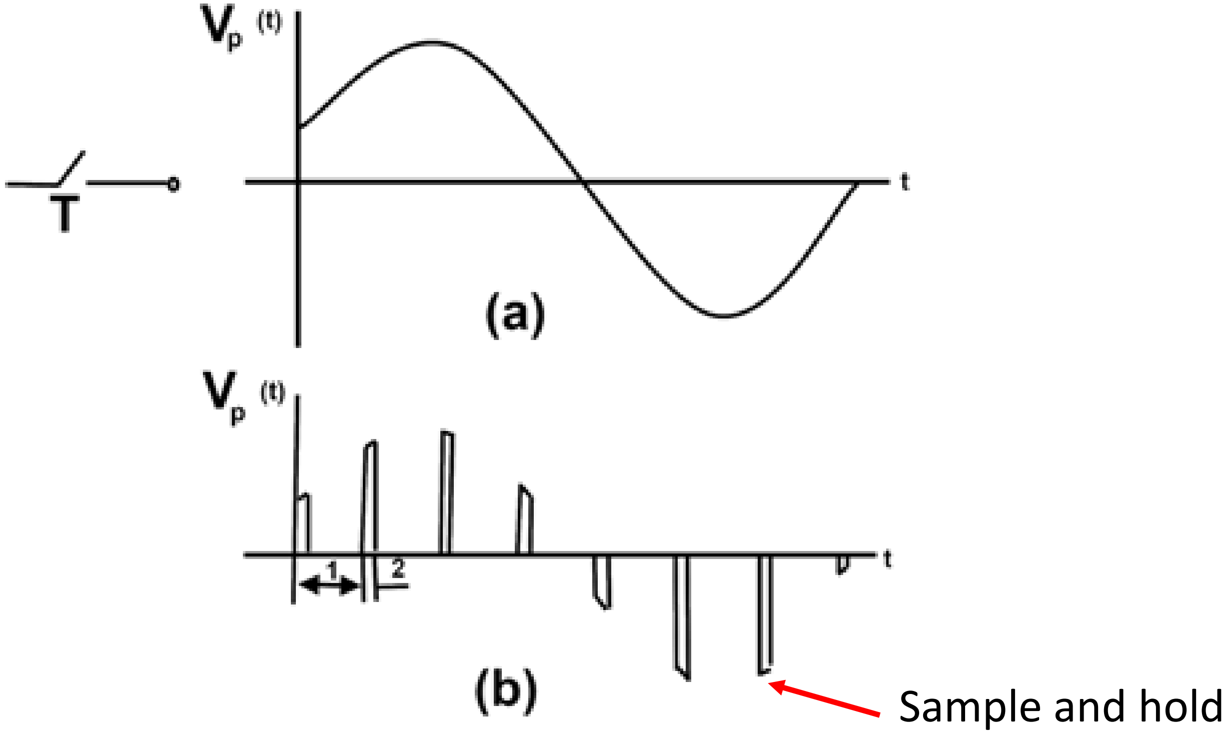
- Discrete Signals
  - Sampled in time
    - Simplest case is uniform sampling
  - Amplitude is continuous
- Digital Signals
  - Discrete signals
  - Amplitude is quantized
    - Could be uniform
    - Often more efficient to have non-uniform levels that are data dependent
- Binary Signals
  - Digital signals that have been coded in binary.

Sampling for Discrete Signals

- Discrete Signals
  - Fundamental issue: How do you pick the sampling interval?
    - Time steps too small – redundant data
    - Time steps too large – data loss
    - You can figure out the optimal step size based on the frequency content of the data.
  - How do you get a discrete signal?
    - Take samples of continuous signal at fixed time-steps
    - Sampling pulse must have some width, but usually, we neglect this width

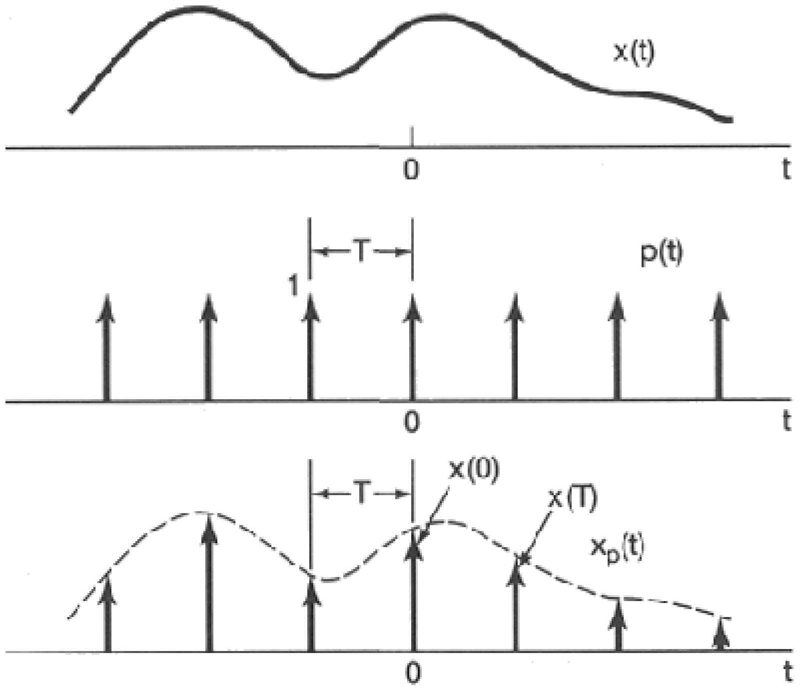
Sampling for Discrete Signals

- Sampling (showing width of sampling pulse)



Sampling for Discrete Signals

- Sampling (ignore width of sampling pulse)



Sampling for Discrete Signals

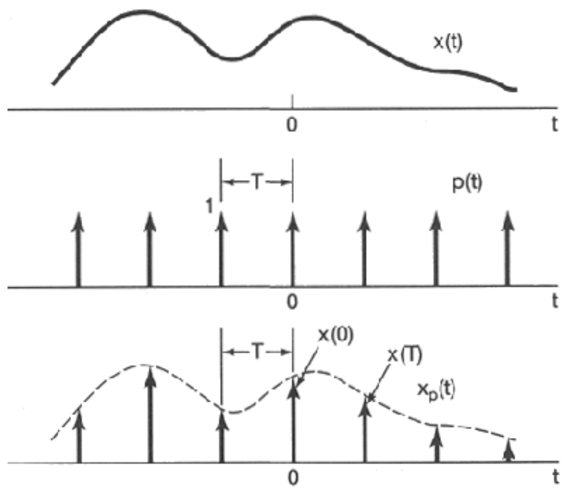
- Sampling a signal at sample interval T_s
  - Signal: $x(t)$
  - Sampled signal: $x_s(t)$
  - Sampling function: $\delta_{T_s}(t) = \sum_n \delta(t - nT_s)$

    $\delta(t)$ is unit impulse
  - Multiply signal by the sampling function

$$x_s(t) = x(t)\delta_{T_s}(t) = \sum_n x(t)\delta(t - nT_s) = \sum_n x(nT_s)\delta(t - nT_s)$$

    - The last sum is true since only the values at $t = nT_s$ matters.
    - **Notice both sampled signal and original signal are shown as continuous functions of time.**
    - The discrete signal could be represented as a sequence of numbers:

$$\{x_k\} = \{\cdots x(-kT_s), \cdots, x(-2T_s), x(-1T_s), x(0T_s), x(1T_s), x(2T_s), \cdots, x(kT_s), \cdots\}$$



Sampling for Discrete Signals

- Aliasing and the Nyquist-Shannon sampling theorem:
  - The sampling function is periodic,

    $\delta_{Ts}(t)$ periodic with a period of T_s
  - Since it is periodic, we can find its Fourier series:

$$f(t) \equiv \delta_{Ts}(t) \ \text{ with period } T_s \qquad \left(\text{Fundamental frequency } \ \Omega_s = \frac{2\pi}{T_s} \right)$$

(Here, I will use a capital Omega to represent the sampling frequency)

Definition of Fourier Series

$$f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t}$$

$$D_n = \frac{1}{T_0} \int_{T_0} f(t) e^{-jn\omega_0 t} dt$$

Sampling for Discrete Signals

$$f(t) \equiv \delta_{Ts}(t) \text{ with period } T_s \qquad \left(\text{Fundamental frequency } \Omega_s = \frac{2\pi}{T_s} \right)$$

$$D_n = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jn\,\Omega_s\,t} dt = \frac{1}{T_s} e^{-jn\Omega_s 0} = \frac{1}{T_s}$$

$$f(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jn\omega_0 t} \quad \Rightarrow \quad \delta_{T_s}(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\Omega_s t} = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} e^{jn\Omega_s t}$$

– Recall that the sampled signal is: $x_s(t) = x(t)\delta_{T_s}(t)$

$$x_s(t) = x(t)\delta_{T_s}(t) = x(t)\frac{1}{T_s}\sum_{n=-\infty}^{+\infty} e^{jn\Omega_s t} = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} x(t) e^{jn\Omega_s t}$$

Sampling for Discrete Signals

– Big question: What is the frequency content of the sampled signal and how does it compare to the frequency content of the original signal?
– Take the Fourier transform of both.

The Fourier transform of the original signal is:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Sampling for Discrete Signals

– The Fourier transform of the sampled signal is:

$$X_s(\omega) = \int_{-\infty}^{\infty} x_s(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} x(t) e^{jn\Omega_s t} e^{-j\omega t} dt$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) e^{jn\Omega_s t} e^{-j\omega t} dt = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) e^{-j(\omega - n\Omega_s)t} dt$$

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\Omega_s)$$

Sampling for Discrete Signals

- Interesting!
- Original signal has frequency "content"

$$X(\omega)$$

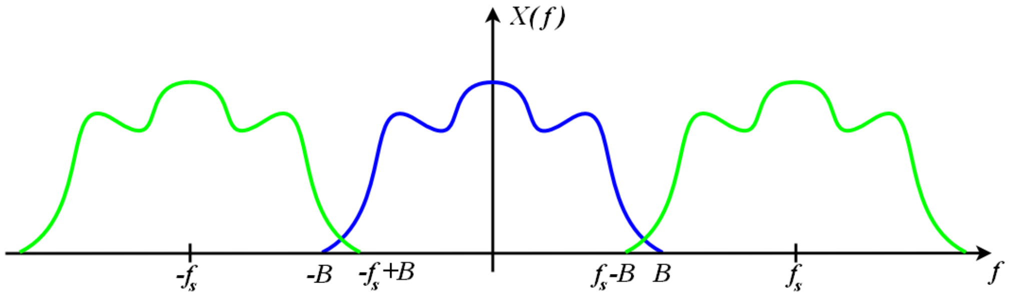
- Sampled signal has frequency "content"

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\Omega_s)$$

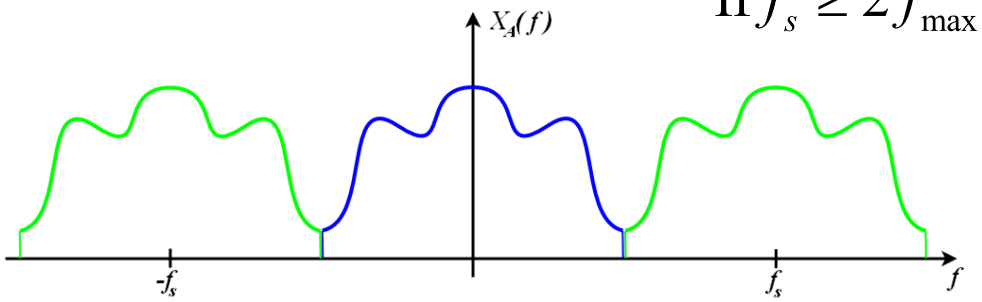
  - Sampled signal has repeated copies of original frequency spectrum offset by $n\Omega_s$
- What does this mean?
  - Sampled signal will have frequencies that original doesn't have!
  - High frequency components of signal may be overlapped by low frequency components of sampled signal

Sampling for Discrete Signals

- Aliasing: Frequencies that are in sampled signal but not in original are called "aliased"



If $f_s \geq 2 f_{max}$, spectra do not overlap



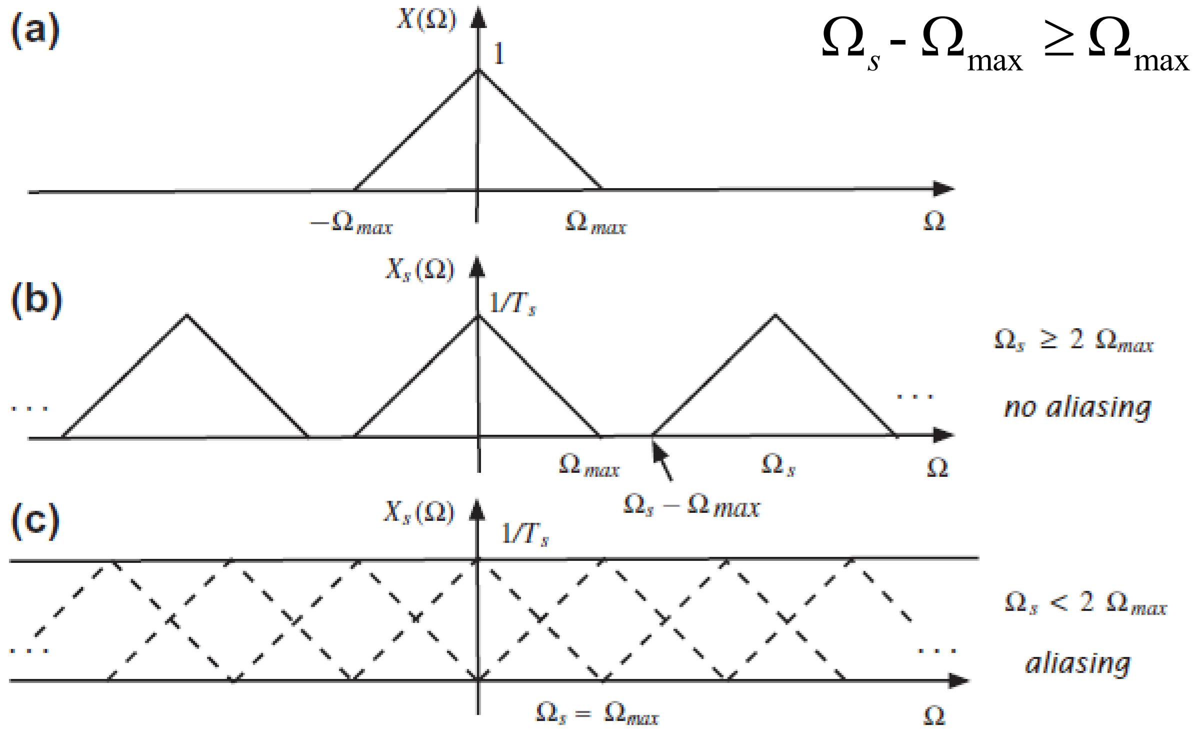
Sampling for Discrete Signals

- Aliasing: Frequencies that are in sampled signal but not in original are called "aliased"

spectra do not overlap if

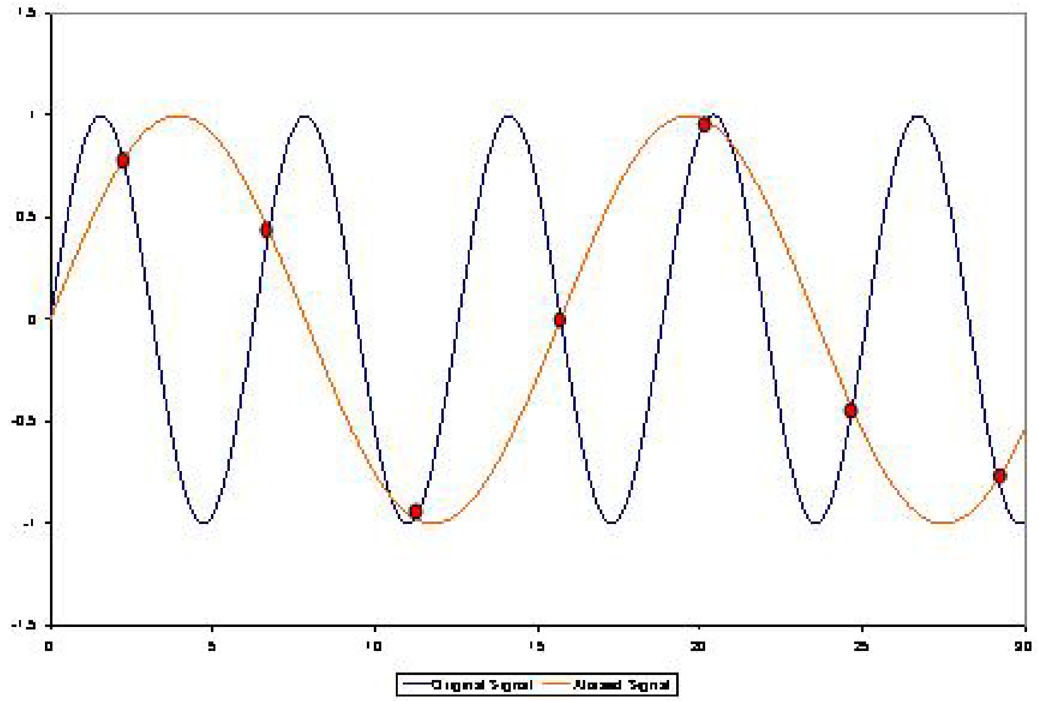
$$\Omega_s - \Omega_{\max} \geq \Omega_{\max} \quad \Rightarrow \quad \Omega_s \geq 2\Omega_{\max}$$

$$f_s \geq 2 f_{\max}$$



Sampling for Discrete Signals

- Not that complicated:

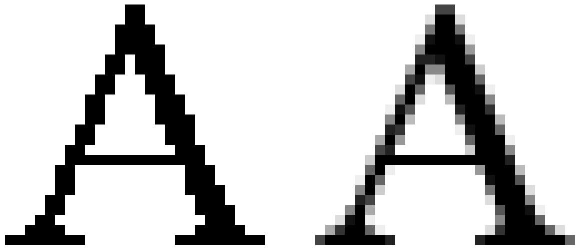


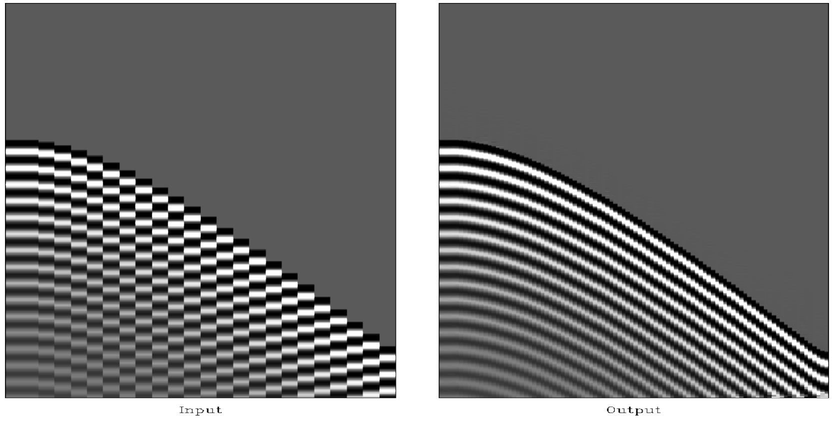
Sampling for Discrete Signals

- Lots of interesting examples
  - A strobe light "samples" motion,
    jerky movements captured when light flashes
  - Fan blades that appear to turn backwards
  - Moire patterns are another example

Sampling for Discrete Signals

– "jaggies"





Sampling for Discrete Signals

- Nyquist-Shannon sampling theorem:
  - For a band-limited signal, there will be no overlapping frequencies if you sample at a rate twice the maximum frequency in the signal.

$$\Omega_s \geq 2\omega_{\max} \quad \text{or} \quad F_s \geq 2f_{\max} \quad \text{or} \quad T_s \leq \frac{1}{2f_{\max}}$$

  - Consider digital music (CD's, Pandora, Spotify, …)
    - Sampling rate is 44.1 kHz
    - Good for frequencies up to 22.05 kHz which is about the upper limit of normal human hearing. (64 – 23,000 Hz)
    - Not so great for dogs (67-45,000) and cats (45-64,000)
    - Rotten for bats, (110 kHz), beluga whales (123 kHz) and porpoises (150 kHz)

Sampling for Discrete Signals

- Reconstruction of original signal from sampled signal
  - If the original signal is band-limited, and $-\omega_{\max} \le \omega \le \omega_{\max}$
  - If sampled signal has sampling frequency $\Omega_s \ge 2\omega_{\max}$
  - Then, you should be able to exactly recover the original signal from the sampled signal

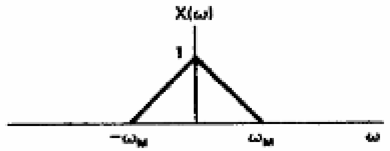
$$X(\omega)$$

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\Omega_s)$$

Sampling for Discrete Signals

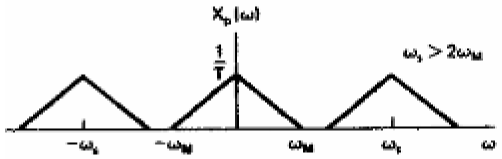
- Reconstruction of original signal from sampled signal

Original signal



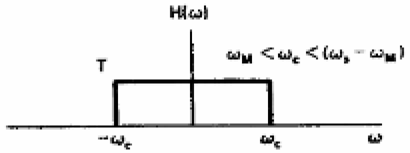
$$X(\omega)$$

Sampled signal
$\omega_s > 2\,\omega_m$



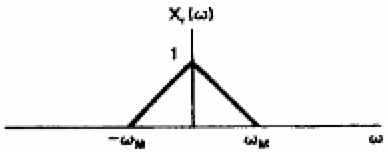
$$X_s(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n\Omega_s)$$

Ideal
reconstruction filter
(low-pass)



$$H_{\text{rect}}(\omega) = T_s \quad -\Omega_s/2 \le \omega \le \Omega_s/2$$

Reconstructed signal
(=Original signal)



$$X_r(\omega) = X_s(\omega) H_{\text{rect}}(\omega)$$

Sampling for Discrete Signals

- Reconstruction of original signal from sampled signal

$$X_s(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n\Omega_s)$$

$$H_{\text{rect}}(\omega) = T_s \quad -\Omega_s/2 \le \omega \le \Omega_s/2$$

$$X_r(\omega) = X_s(\omega) H_{\text{rect}}(\omega)$$

- In the time domain:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$h_{\text{sinc}}(t) = \frac{\sin(\pi t/T_s)}{(\pi t/T_s)}$$

$$x_r(t) = [x_s * h_{\text{sinc}}](t) = \int_{-\infty}^{\infty} x_s(\tau) h_{\text{sinc}}(t-\tau)\, d\tau$$

Sampling for Discrete Signals

- Reconstruction of original signal from sampled signal

$$x_r(t) = [x_s * h_{\text{sinc}}](t) = \int_{-\infty}^{\infty} x_s(\tau) h_{\text{sinc}}(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\delta(\tau - nT_s) h_{\text{sinc}}(t-\tau) d\tau$$

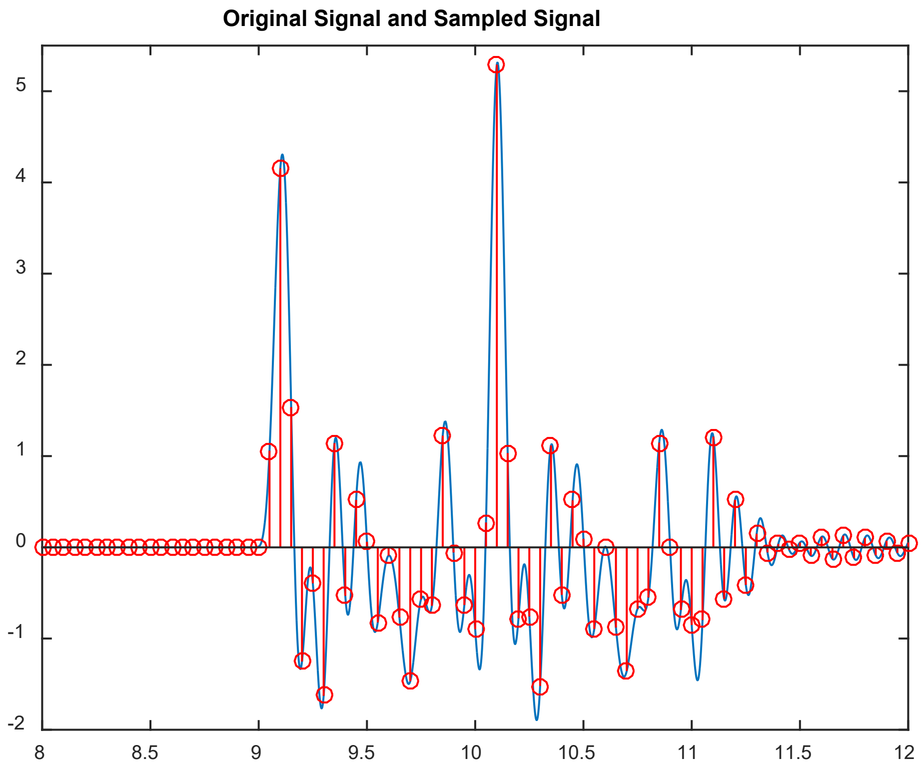
$$= \sum_{n=-\infty}^{\infty} x(nT_s) \int_{-\infty}^{\infty} \delta(\tau - nT_s) h_{\text{sinc}}(t-\tau) d\tau$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) h_{\text{sinc}}(t - nT_s)$$

$$h_{\text{sinc}}(t) = \frac{\sin(\pi t/T_s)}{(\pi t/T_s)}$$

$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left(\pi\left(t-nT_s\right)/T_s\right)}{\left(\pi\left(t-nT_s\right)/T_s\right)}}$$

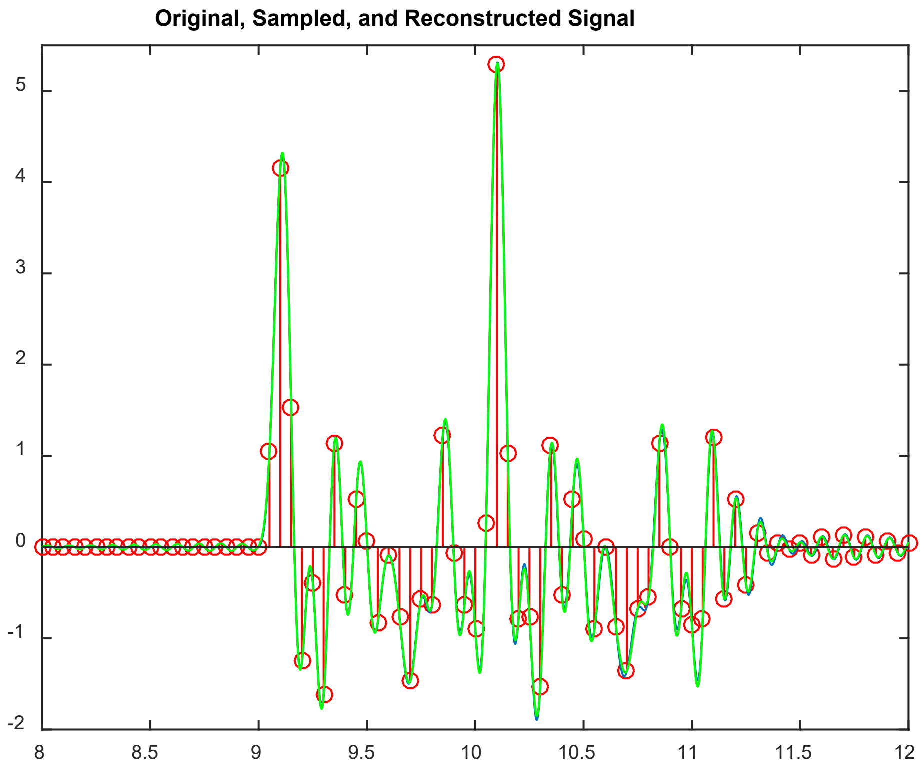
Sampling for Discrete Signals



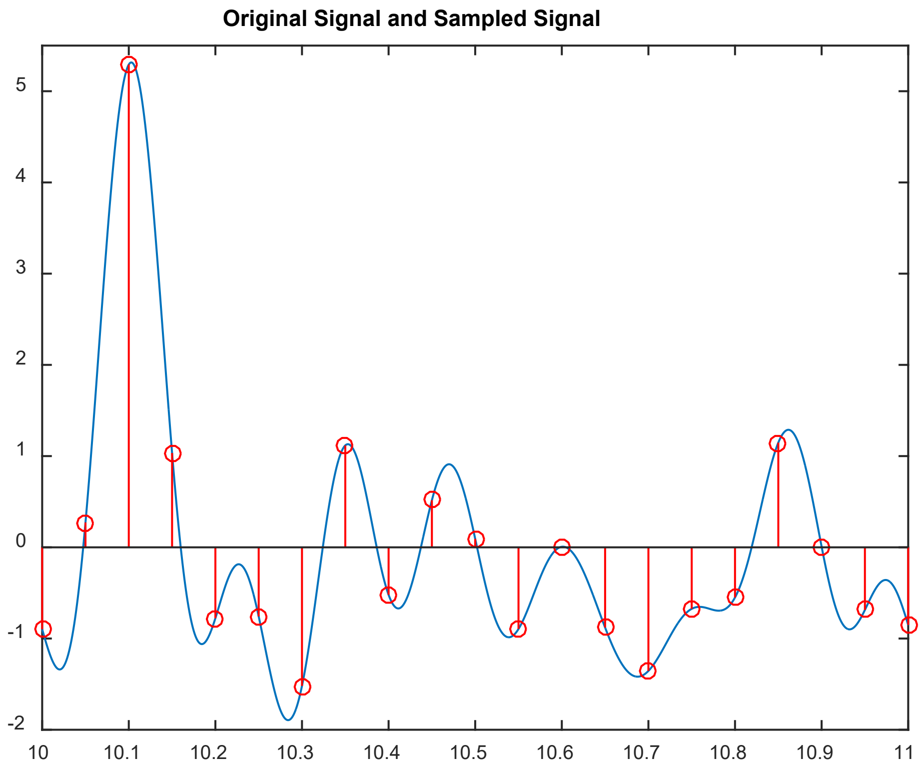
Sampled at 20 Hz

Signal constructed from 10 freq. components (1 to 10 Hz)

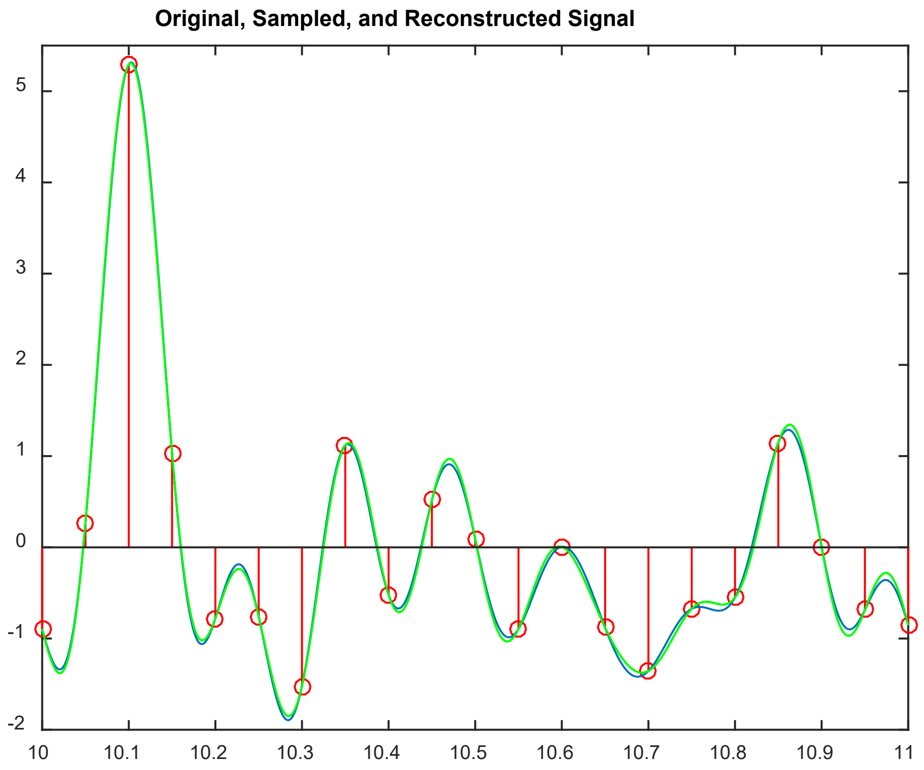
Sampling for Discrete Signals



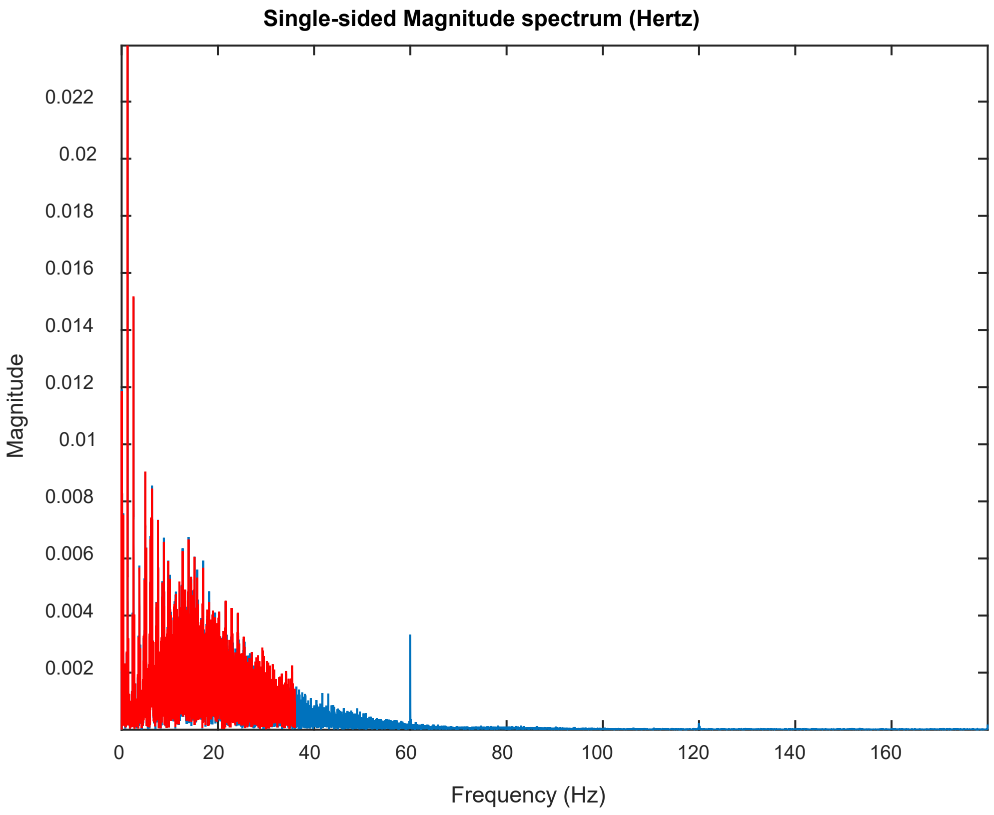
Sampling for Discrete Signals



Sampling for Discrete Signals



Sampling for Discrete Signals

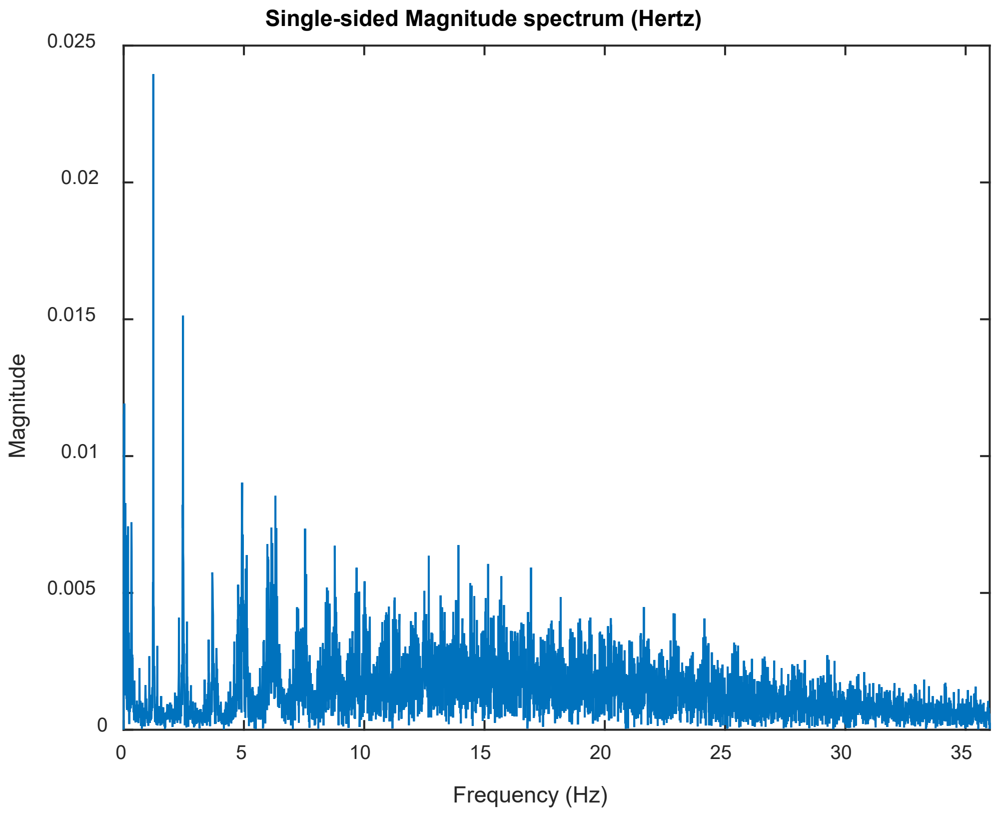


ECG signal

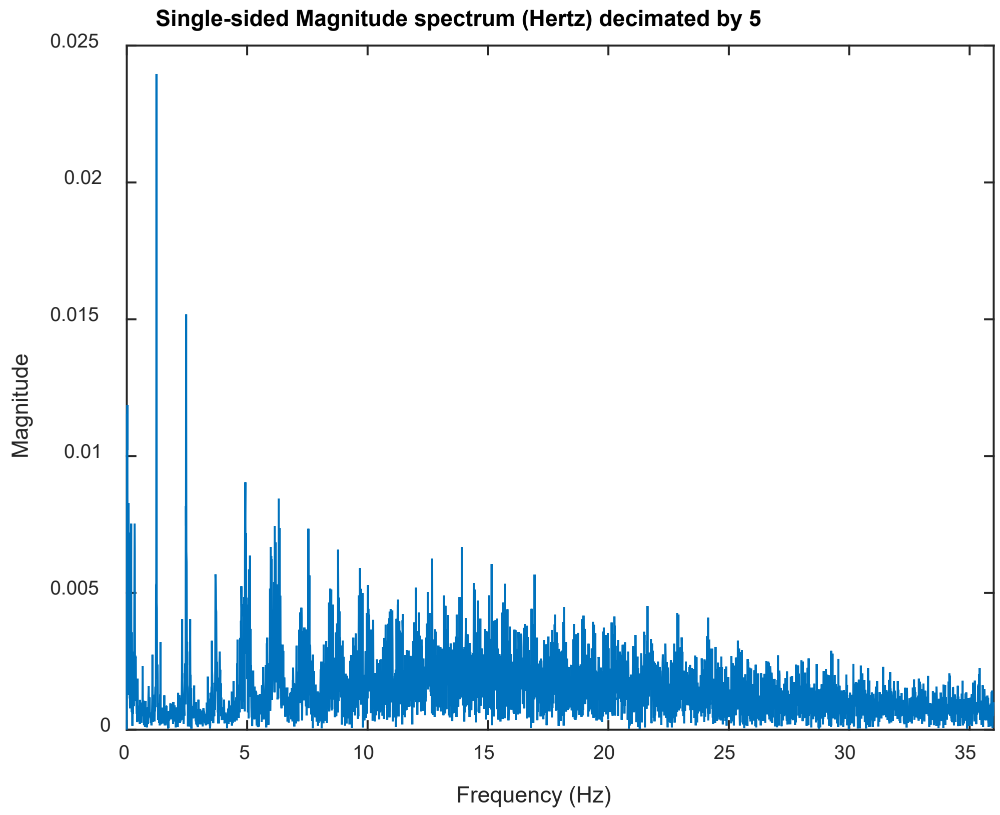
Red - Decimated by 5

Like sampling at 360/5 = 72 Hz
Max freq is 36 Hz

Sampling for Discrete Signals

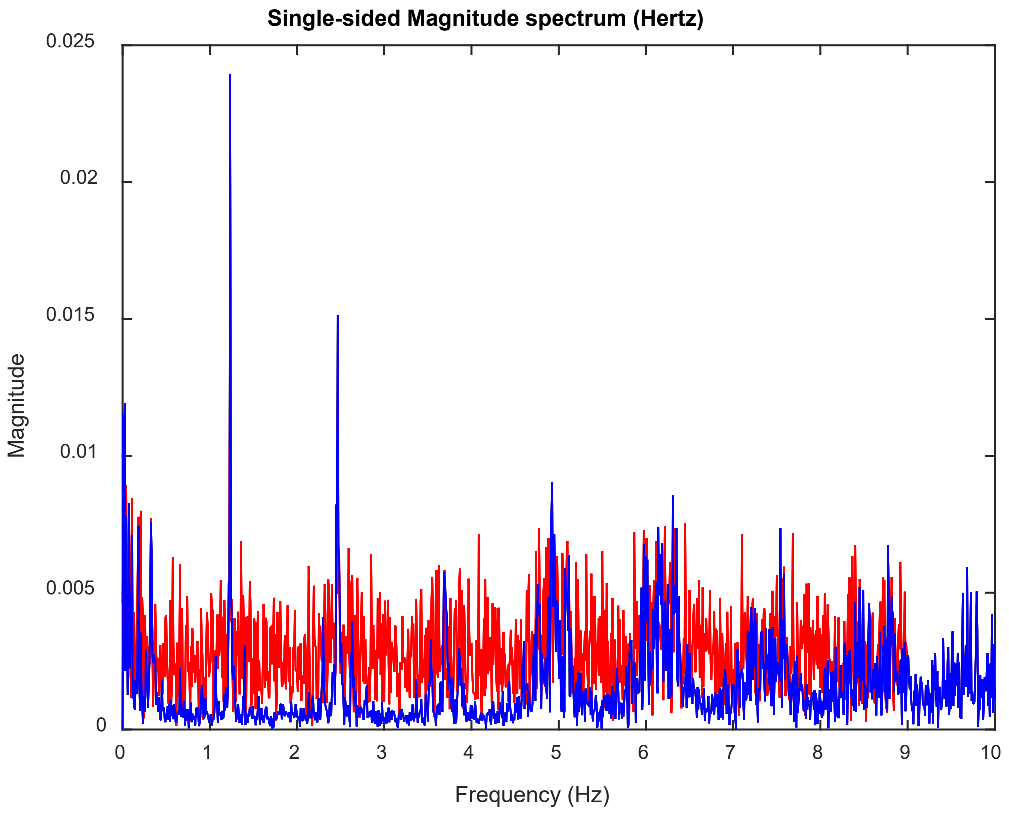


Sampling for Discrete Signals



Decimated by 5

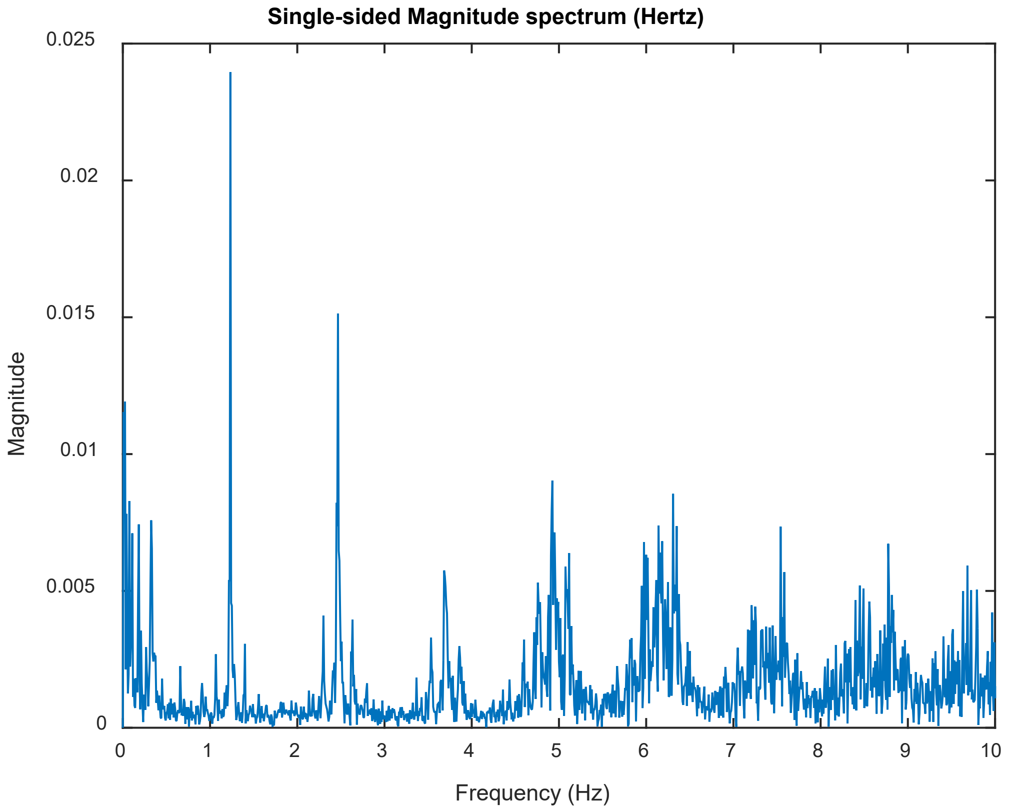
Sampling for Discrete Signals



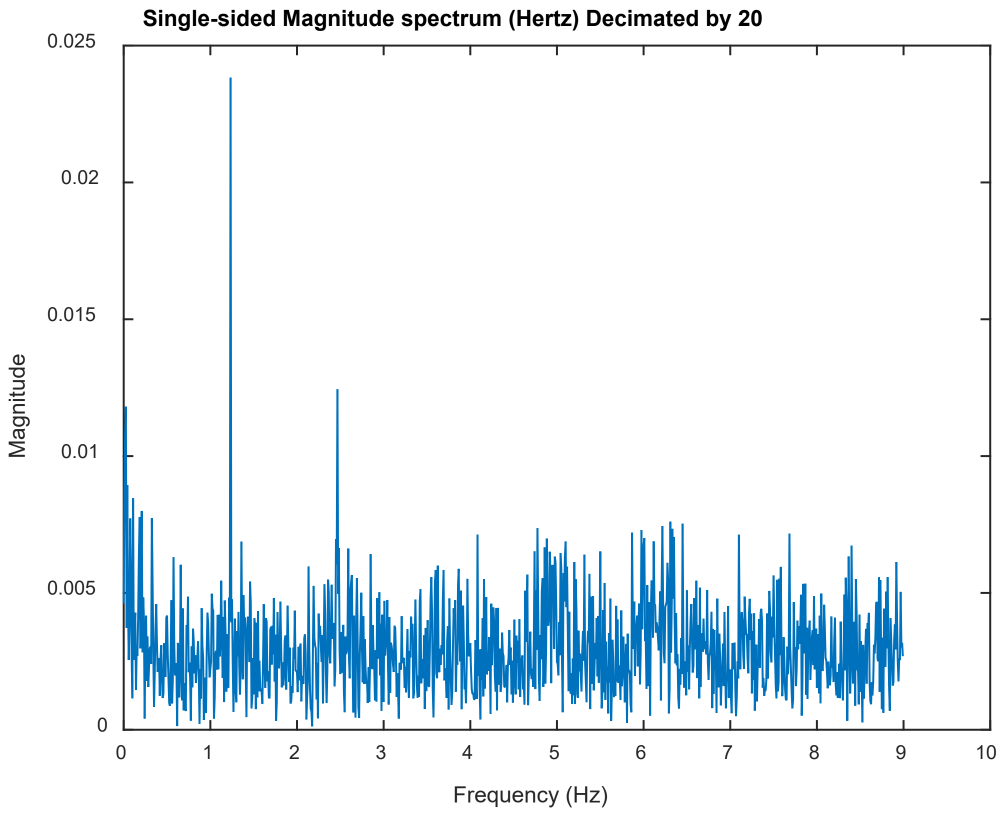
Red - Decimated by 20

Like sampling at 360/20 = 18 Hz
Max. freq is 9 Hz

Sampling for Discrete Signals

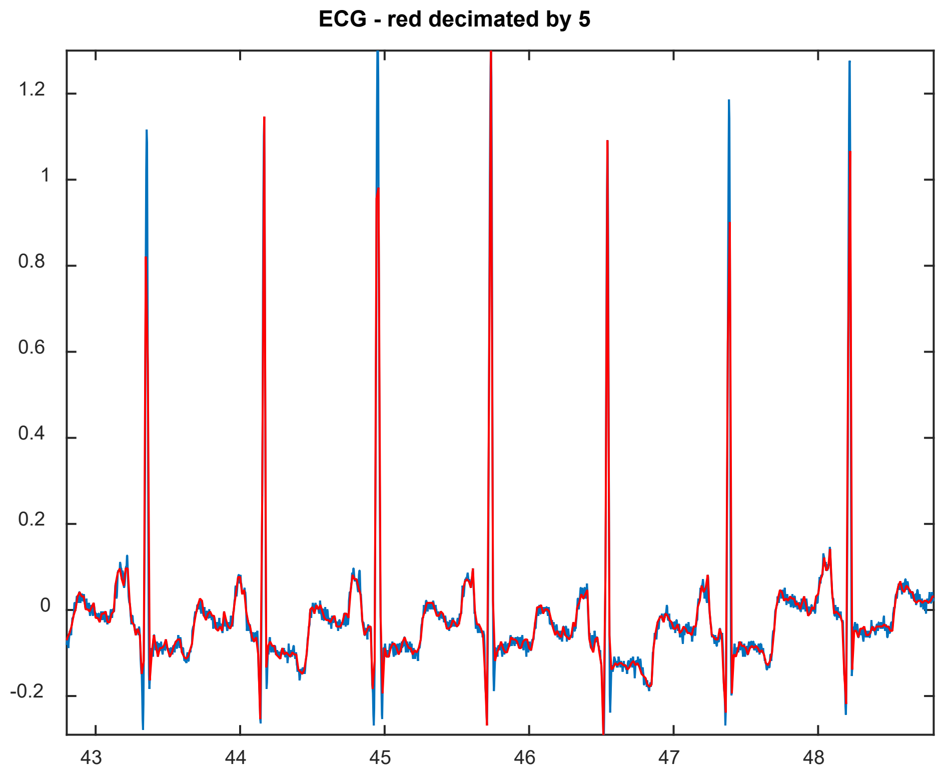


Sampling for Discrete Signals



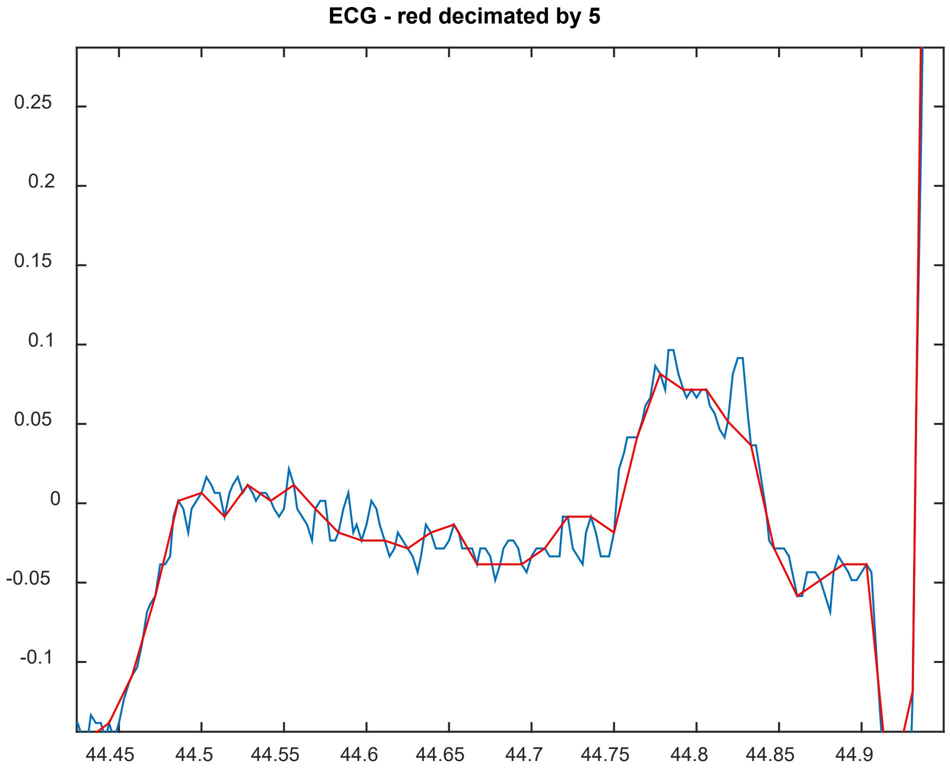
Decimated by 20

Sampling for Discrete Signals



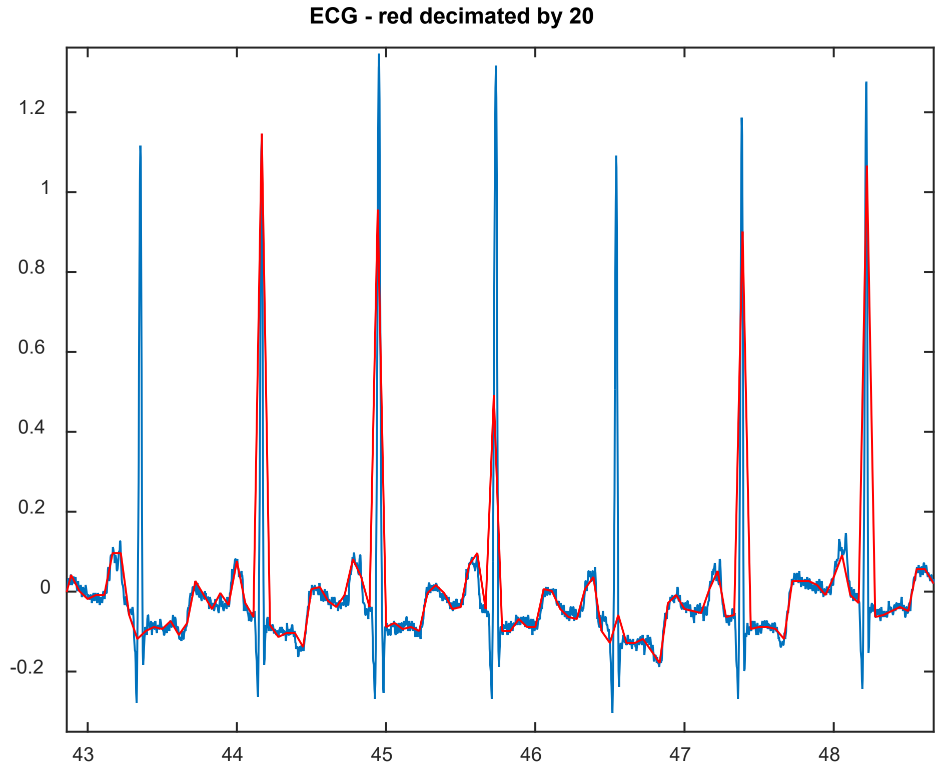
Decimated by 5

Sampling for Discrete Signals



Decimated by 5

Sampling for Discrete Signals



Decimated by 20

Sampling for Discrete Signals

- What to do when signals are not band-limited?
  - Apply a low-pass filter to eliminate frequencies above the range of interest.
    - Anti-aliasing filter
  - Then sample with sampling rate that matches the band-width of the filter.