Digital Signal Processing

Class 20
04/03/2025

ENGR 71

- Class Overview
  - Digital Filter Design
    - FIR filters
- Assignments
  - Reading:

    Chapter 10: Design of Digital Filters

    https://www.mathworks.com/help/signal/ug/fir-filter-design.html
  - Problems:

    Chapter 7: 7.8, 7.9, 7.11(b), 7.14, 7.18, 7.25

    Pick one symmetry property from Table 7.1 and one property from Table 7.2 to prove. (Next class, say which ones.)

    Due: Friday, April 4

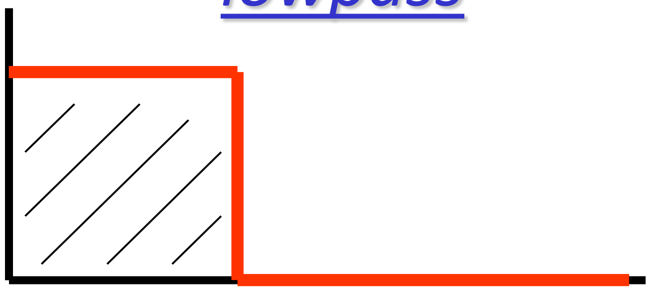
Project

- Projects
  - You can work in groups if you wish
  - Presentation at time reserved for final exam
    - Friday, May 9, 7:00-10:00 PM
    - Science Center 264
  - Submit slides from presentation to Project Dropbox
  - Submit written report to Project Dropbox by end of semester (May 15)

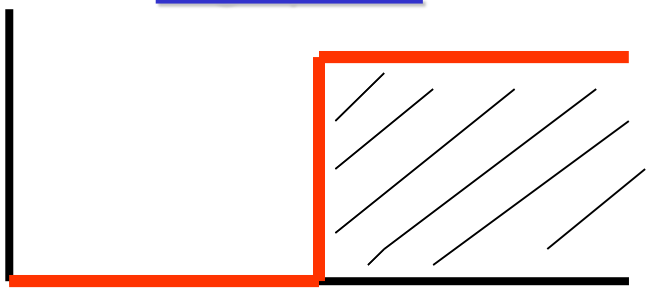
Filters

- Design of Digital Filters
  - Generally means frequency selective filters

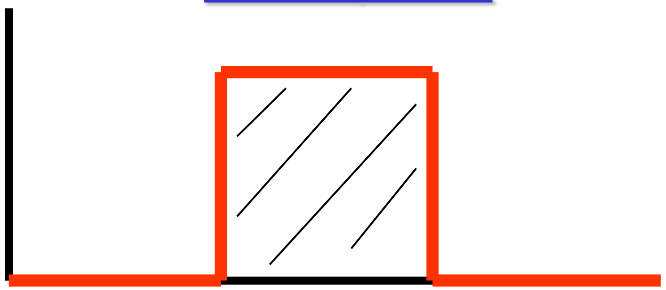
lowpass



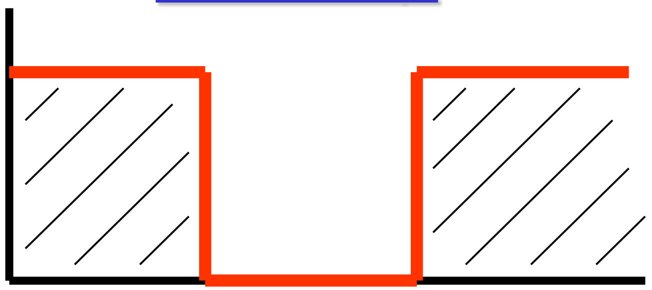
highpass



bandpass



bandstop

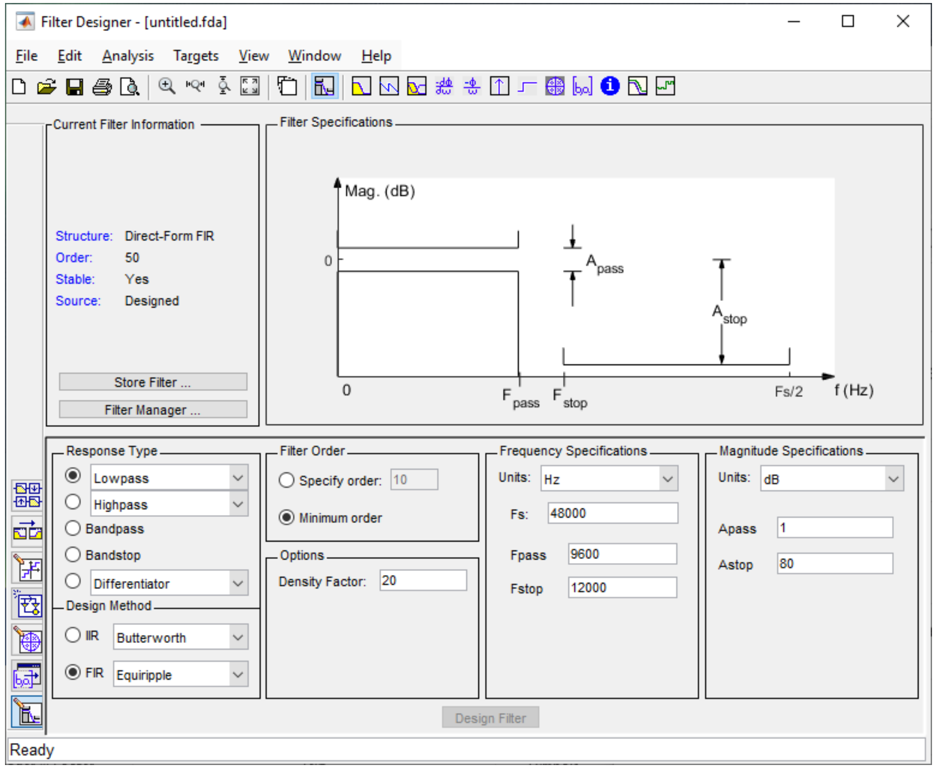


Filters

- Design of Digital Filters
  - Finite Impulse Response (FIR)
    - Used when linear-phase required in passband
      - Constant time delay
      - No dispersion as a function of frequency
  - Infinite Impulse Response (IIR)
    - No requirement on linear-phase
    - Better characteristics for fewer parameters
    - Less memory
    - Lower computational complexity

Filters

- Why learn about filter design?
  - Tools like Matlab's filterDesigner exist so why spend time learning about different methods?



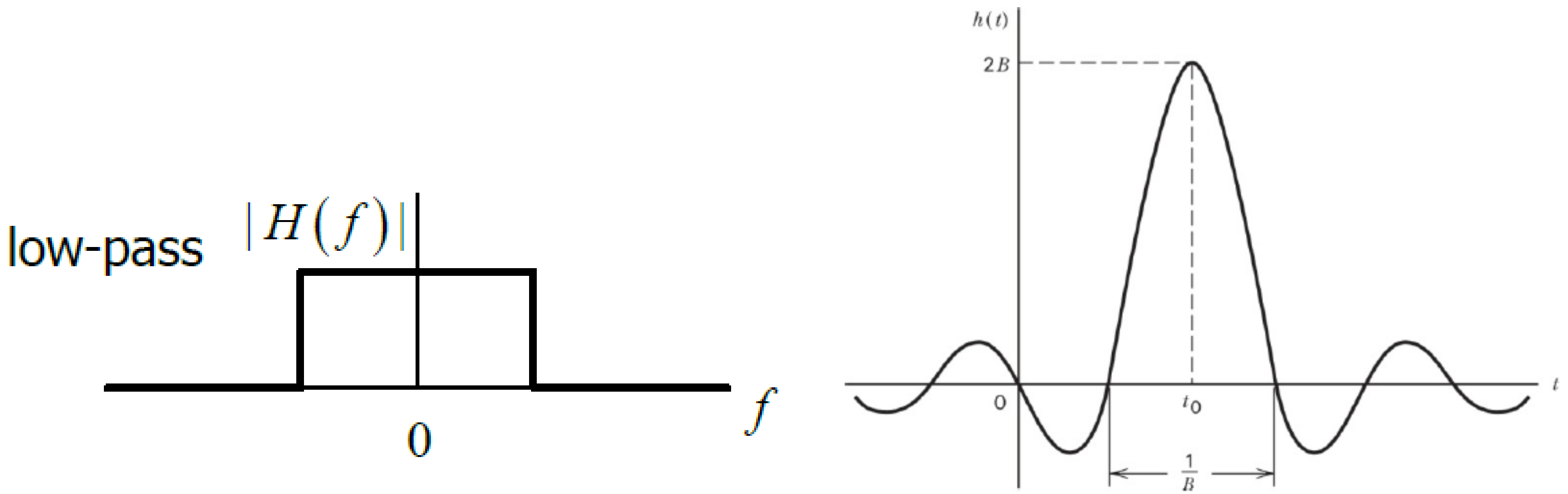
Filters

- Why learn about filter design?
  - Understanding methods …
    - For better design choices
    - To troubleshoot issues that may arise
    - To optimize for design constraints
    - To avoid “black box” thinking
  - Rounds out EE education
    - Something you are expected to understand

The intellectual satisfaction that comes from understanding how something works

Filters

- Why can't you have ideal filters?

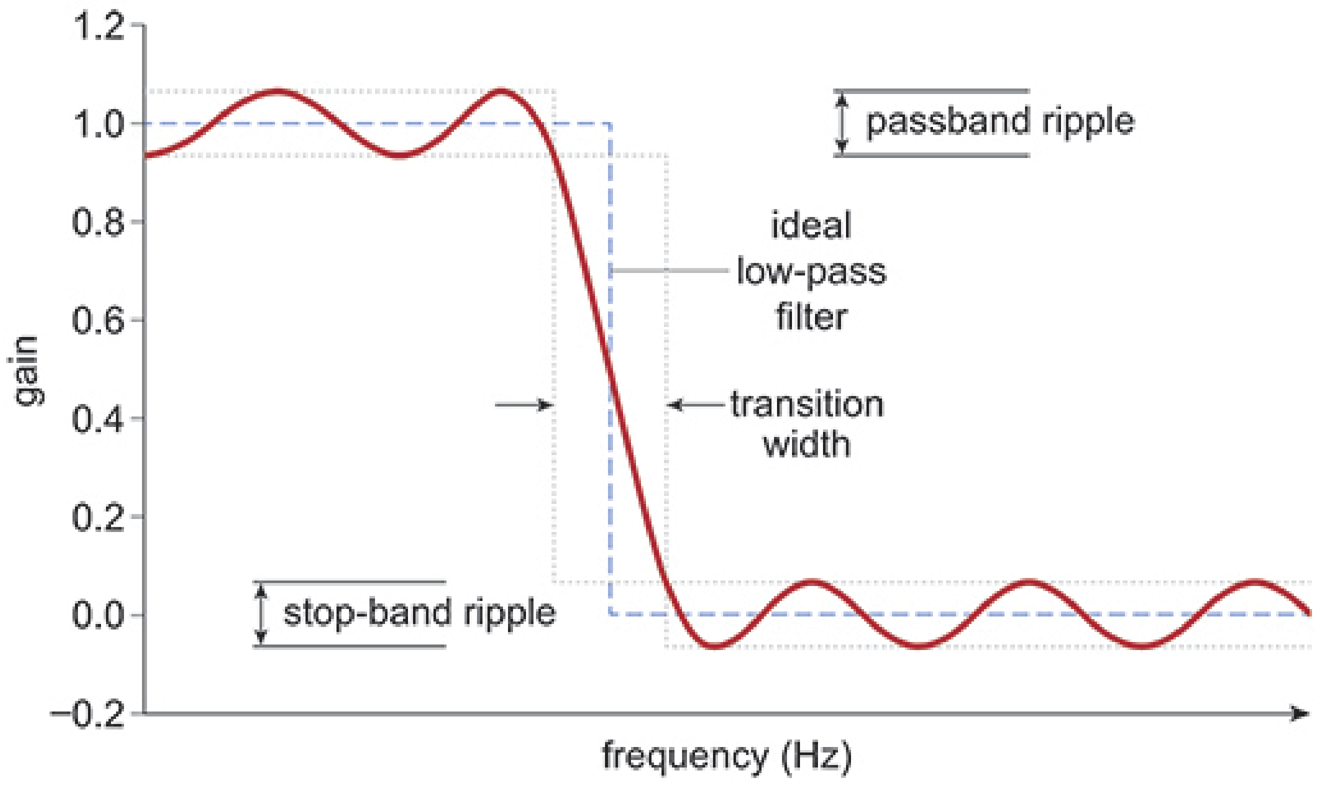


Condition for causal system: $h(t) = 0$ for all $t < 0$.

In the time domain, there is some response from the filter before $t = 0$, so the ideal filter is non-casual.

Causality and Its Implications

- Why can’t you have idea filters?



Causality and Its Implications

- Causality and Its Implications
  - Mathematical criterion for LTI causal system
    - Time domain (continuous systems):

      $h(t)$ has finite energy and $h(t) = 0$ for all $t < 0$.

    - Frequency domain: **Paley-Wiener Criterion**
      - In the frequency domain, the magnitude of the transfer function can be zero only at a discrete number of frequencies.
      - Mathematical description of Paley–Wiener criterion:
        For a realizable filter, necessary and sufficient condition for $|H(\Omega)|$ is

$$\int_{-\infty}^{\infty} \frac{\ln\left|H(\Omega)\right|}{1+\Omega^2} d\Omega < \infty$$

Causality and Its Implications

- Causality and Its Implications
  - Mathematical criterion for LTI causal system
    - Time domain (Discrete Systems):

      $h(n)$ has finite energy and $h(n) = 0$ for all $n < 0$.

    - Frequency domain: **Paley-Wiener Criterion**
      - In the frequency domain, the magnitude of the transfer function can be zero only at a discrete number of frequencies.
      - Mathematical description of Paley–Wiener criterion:
        For a realizable filter, necessary and sufficient condition for $|H(\omega)|$ is

$$\int_{-\pi}^{\pi} \left|\ln|H(\omega\right|\right| d\omega < \infty$$

Causality and Its Implications

– For causal systems the impulse response can be determined from just its even part
  (Or, its odd part plus the value at n=0)

$$h(n) = h_e(n) + h_o(n) \quad \text{where} \quad h_e(n) = \frac{1}{2}\big(h(n) + h(-n)\big) \quad \text{and} \quad h_o(n) = \frac{1}{2}\big(h(n) - h(-n)\big)$$

Since $h(n) = 0$ for $n < 0$

$$h(n) = 2h_e(n)u(n) - h_e(0)\delta(0)$$

and

$$h(n) = 2h_o(n)u(n) + h(0)\delta(0)$$

Causality and Its Implications

- Looking at this in the frequency domain:
  - Write the DTFT in terms of its real and imaginary components

$$H(\omega) = H_R(\omega) + jH_I(\omega)$$

  - Using the DTFT property

$$h_e(n) \leftrightarrow H_R(\omega) \text{ and } h_o(n) \leftrightarrow H_I(\omega)$$

    - Since $h(n)$ is completely determined by $h_e(n)$, $H(\omega)$ can be found from just $H_R(\omega)$. The same is true for $H_I(\omega)$.

  - The real and imaginary parts of the transfer function are interrelated for causal system

Causality and Its Implications

– You can get an explicit relationship between the real and imaginary part of the transfer function for causal systems

- Using:

$$h(n) = 2h_e(n)u(n) - h_e(0)\delta(0)$$

$$h_e(n) \leftrightarrow H_R(\omega)$$

$$H(\omega) = H_R(\omega) + jH_I(\omega)$$

Use the convolution theorem on the product of $h_e(n)u(n)$:

$$H_R(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_R(\lambda)U(\omega - \lambda)\,d\lambda$$

Use the DTFT of the unit step.

Combine all this together and you find:

Causality and Its Implications

– The Discrete Hilbert transform:

Relating the imaginary part of the transfer function to the real part:

$$H_I(\omega) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} H_R(\lambda)\cot\left(\frac{\omega-\lambda}{2}\right)d\lambda$$

A similar expression for the real part in terms of the imaginary part:

$$H_R(\omega) = h(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} H_I(\lambda)\cot\left(\frac{\omega-\lambda}{2}\right)d\lambda$$

(There is a more complete derivation of this in the book by Oppenheim & Schafer)

– The Hilbert transform is useful in digital communications for things like Single-Sideband modulation

Causality and Its Implications

- Summary of the implications of causality
  - Frequency response cannot be zero except at a finite set of points
  - The magnitude of $|H(w)|$ cannot be constant in any finite range of frequencies
  - The transition from passband to stopband cannot be infinitely sharp
  - The imaginary and real parts of the transfer function are interdependent and related by the Hilbert transform
    - The magnitude and phase of the transfer function cannot be chosen arbitrarily

Filter Design

- For LTI causal systems:

$$\boxed{y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M-1} b_k x(n-k)}$$

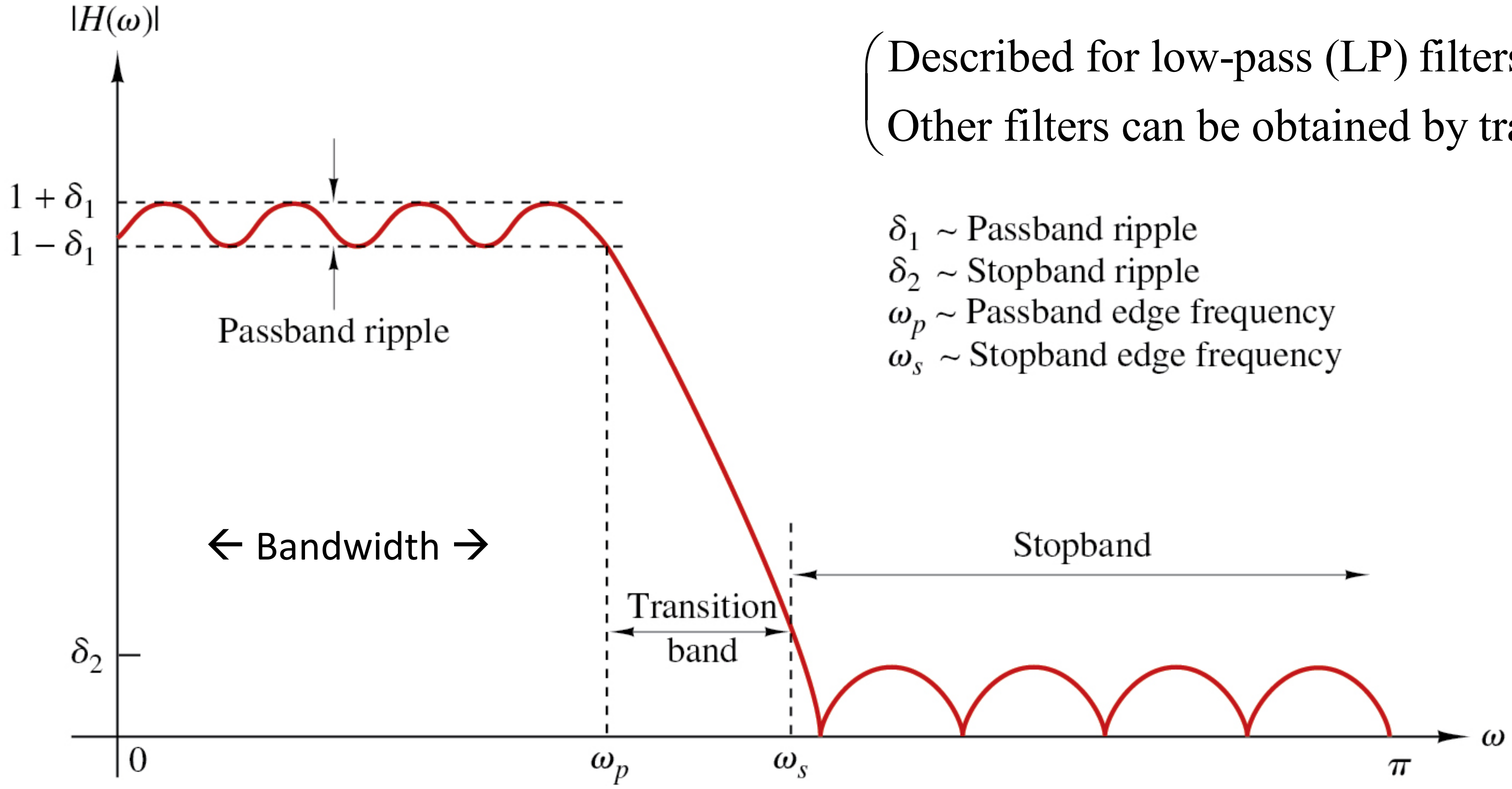
$$\boxed{H(\omega) = \frac{\sum_{k=0}^{M-1} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}}$$

  - For all the restrictions due to causality, we know that we cannot make an ideal filter, so the basic problem is:
  - Find $\{a_k\}$ and $\{b_k\}$ that give the best approximation to the desired filter specifications.

Filter Design

- Specifications for physically realizable filters:

(Described for low-pass (LP) filters.
Other filters can be obtained by transforming LP filters)



Filter Design

- Finite Impulse Response (FIR) Filters

$$y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_{M-1}(x - M + 1)$$

$$= \sum_{k=0}^{M-1} b_k x(n-k)$$

or in terms of impulse response:

$$y(n) = \sum_{k=0}^{M-1} h(k) x(n-k) \qquad \left(b_k\text{'s are } h(k)\text{'s} \right)$$

or in terms of transfer function:

$$H(z) = \sum_{k=0}^{M-1} h(k) z^{-k}$$

Filter Design

– Linear phase FIR filters

If $h(n) = \pm h(M-1-n)$ the phase will be linear

– Proof on board

Filter Design

– Linear phase FIR filters

- Book approaches this in a different way
  finding the z-transform for cases where length of the filter (*M*) is even or odd
- Result:

$$H(z)=\sum_{k=0}^{M-1} h(k)z^{-k} \quad \text{for } h(n) = \pm h(M-1-n)$$

$$H(z)=z^{-(M-1)/2}\left\{h\left(\frac{M-1}{2}\right)+\sum_{n=0}^{(M-1)/3} h(n)\left[z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2}\right]\right\}, \quad M \text{ odd}$$

$$=z^{-(M-1)/2}\left\{\sum_{n=0}^{(M/2)-1} h(n)\left[z^{(M-1-2n)/2} \pm z^{-(M-1-2n)/2}\right]\right\}, \quad M \text{ even}$$

Filter Design

– Linear phase FIR filters

Substituting z^{-1} for z

$$H(z^{-1}) = \sum_{k=0}^{M-1} h(k) z^{k} \quad \text{for } h(n) = \pm h(M-1-n)$$

and multipling both sides of equations by $z^{-(M-1)}$

result is

$$z^{-(M-1)} H(z^{-1}) = \pm H(z)$$

Filter Design

– Linear phase FIR filters

Roots of polynomial $H(z)$ are identical to roots of $H(z^{-1})$

so roots occur in reciprocal pairs.

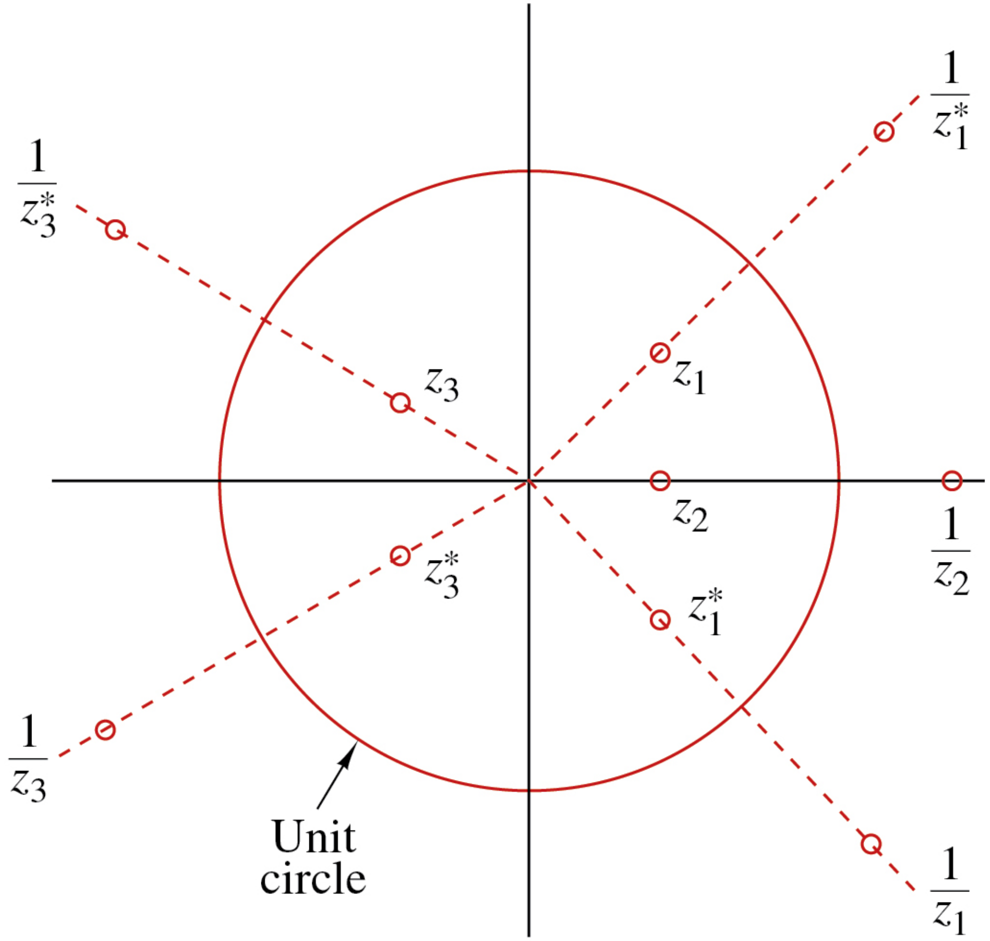
Furthermore, if $h(n)$ is real, roots occur in complex conjugate pairs.

If z_1 is a root, so is $1/z_1$, z_1^*, and $1/z_1^*$

– This makes for an interesting requirement for location of zeros for linear phase FIR filters

Filter Design

– Linear phase FIR filters



Filter Design

– Looking at frequency response ($z = e^{j\omega}$)

For $h(n) = h(M-1-n)$: $\quad H(\omega) = H_r(\omega) e^{-j\omega(M-1)/2}$

Where $H_r(\omega)$ is real function of ω:

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\left[\omega\left(\frac{M-1}{2} - n\right)\right], \quad M \text{ odd}$$

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\cos\left[\omega\left(\frac{M-1}{2} - n\right)\right], \quad M \text{ even}$$

Phase for M both even and odd is

$$\Theta(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right), & \text{if } H_r(\omega) > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi, & \text{if } H_r(\omega) < 0 \end{cases}$$

Filter Design

For $h(n) = -h(M - 1 - n)$ (anti-symmetric case) center point is at $n = (M-1)/2$ so $h\left(\frac{M-1}{2}\right) = 0$

$$H(\omega) = H_r(\omega) e^{-j\left[-\omega(M-1)/2 + \pi/2\right]}$$

$$H_r(\omega) = 2\sum_{n=0}^{(M-3)/2} h(n)\sin\left[\omega\left(\frac{M-1}{2} - n\right)\right], \quad M \text{ odd}$$

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\sin\left[\omega\left(\frac{M-1}{2} - n\right)\right], \quad M \text{ even}$$

Phase for M both even and odd is

$$\Theta(\omega) = \begin{cases} \frac{\pi}{2} - \omega\left(\frac{M-1}{2}\right), & \text{if } H_r(\omega) > 0 \\ \frac{3\pi}{2} - \omega\left(\frac{M-1}{2}\right) + \pi, & \text{if } H_r(\omega) < 0 \end{cases}$$

Linear-Phase FIR Filter Design

– Methods

- Windowing impulse response
  - Specify desired response

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n) e^{-j\omega n}$$

$$h_d(n) = \int_{-\pi}^{\pi} H_d(\omega) e^{-j\omega n} d\omega$$

  - Cannot have an infinite impulse response, so window the $h_d(n)$
  - If you just truncate the series at some value of n, that is like a rectangular window
  - We will discuss different types of windowing functions

Linear-Phase FIR Filter Design

- Methods
  - Frequency sampling methods
    - Specify desired response at set of equally spaced frequencies
    - Solve for $h(n)$ from the specified response
  - Optimal Equiripple linear-phase FIR filters
    - Enable more precise control of pass and stop-band critical frequencies