# Digital Signal Processing

Class 18
03/27/2025

# ENGR 71

- Class Overview
  - Discrete Fourier Transform
- Assignments
  - Lab 2 due March 28
  - Reading:

    Chapter 7: The Discrete Fourier Transform
  - Problems:

    Chapter 7: 7.8, 7.9, 7.11(b), 7.14, 7.18, 7.25

    Pick one symmetry property from Table 7.1 and one property from Table 7.2 to prove. (Next class, say which ones.)

    Due: Friday, April 4

# Project Ideas

- Project Ideas
  - Speech recognition (more complex than Lab 2)
    - Classifier for multiple words
      - I can provide a dataset with multiple instances of several different words
  - Musical instrument tone recognition
    - Using recordings of musical instruments, determine note being played
    - Determine if instrument is in tune, sharp, or flat.
  - Identification of musical instruments
    - I have a dataset of recordings for several different instruments
  - Identification of music genre
    - From frequency characteristics, can you determine a type of music
      - Classical, rock, etc.

# Project ideas

- Filtering
  - Filtering to isolate sounds
  - Equalizer
- Noise reduction
- Audio effects processing
  - Reverb, echo, distortion
- Echo cancellation
- Several possibilities if you are interested in 2-D signal processing for image data

- Hardware projects
  - Link to site with collection of Arduino-based projects
- Theoretical research topics are also welcome
  - Paper on some interesting topic

# Discrete Fourier Transform

- **Fourier series for periodic signals:**

$$x(t) = x(t + T_0) \qquad f_0 = \frac{1}{T_0} \qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t} \qquad \text{(Synthesis Eq.)}$$

$$X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt \qquad \text{(Analysis Eq.)}$$

# Discrete Fourier Transform

- **Fourier transform aperiodic signals:**

$$X(\Omega) = \mathscr{F}[x(t)] = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt \qquad \text{(Analysis Equation)}$$

$$x(t) = \mathscr{F}^{-1}[X(\Omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\Omega)e^{+j\Omega t}d\omega \qquad \text{(Synthesis Equation)}$$

Time and frequency are continous variables

$-\infty < t < \infty$

$-\infty < \Omega < \infty$

Using $\Omega$ to distinguish it from dicrete time case where frequency is between -$\pi$ and $\pi$

# Discrete Fourier Transform

- **Discrete-time Fourier transform**

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad -\pi \le \omega < \pi \quad \text{(Analysis equation)}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \quad \text{(Synthesis equation)}$$

Time, labeled by the integer index *n*, is discrete $(t = nT_s)$

$-\infty < n < \infty$

$-\pi < \omega < \pi$

Limits on $\omega$ are imposed by the Nyquist condition

$\pi$ represents maximum positive frequency $f_{Nyquist} = \dfrac{f_s}{2} = \dfrac{1}{2T_s}$

(where $T_s$ is the sampling interval or alternatively, $f_s$ is the sampling frequency)

# Discrete Fourier Transform

- **Discrete Fourier series for a periodic sequence with period *N***

$$x_p[n+mN] = x_p[n] \quad m = \ldots, -1, 0, 1, \ldots$$

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

where

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

# Discrete Fourier Transform

- **Discrete Fourier Transform**

Discrete Fourier Transform (DFT) Analysis Equation

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j2\pi kn/N} \ , \quad k=0,1,2,\dots,N\text{ - }1$$

Inverse Discrete Fourier Transform (IDFT) Synthesis Equation

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j2\pi kn/N} \ , \quad n=0,1,2,\dots,N\text{ - }1$$

# Discrete Fourier Transform

- Discrete Fourier Series for periodic sequence $x_p\left[n+mN\right]=x_p\left[n\right]$

$$x_p\left[n\right]=\sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \quad , \quad n=0,1,2,\ldots,N\text{-}1$$

Fourier Coefficients

$$c_k=\frac{1}{N}\sum_{n=0}^{N-1} x_p\left[n\right]e^{-j2\pi kn/N} \quad , \quad k=0,1,2,\ldots,N\text{-}1$$

Identifying Fourier series coefficients as $c_k=\dfrac{1}{N}X\left(\dfrac{2\pi k}{N}\right)$

**Discrete Fourier series of periodic function is same as Discrete Fourier Transform of periodically extended finite sequence of length *N*.**

# Discrete Fourier Transform

- Example of DTFT, DFS, DFT

$$x[n] = \begin{cases} 1, & n = 0,1,2 \\ 0, & \text{otherwise} \end{cases}$$

  – DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = 1e^{0} + 1e^{-j\omega} + 1e^{-2j\omega} = e^{-j\omega}\left(e^{j\omega} + 1 + e^{-j\omega}\right) = e^{-j\omega}\left(1 + 2\cos\omega\right)$$

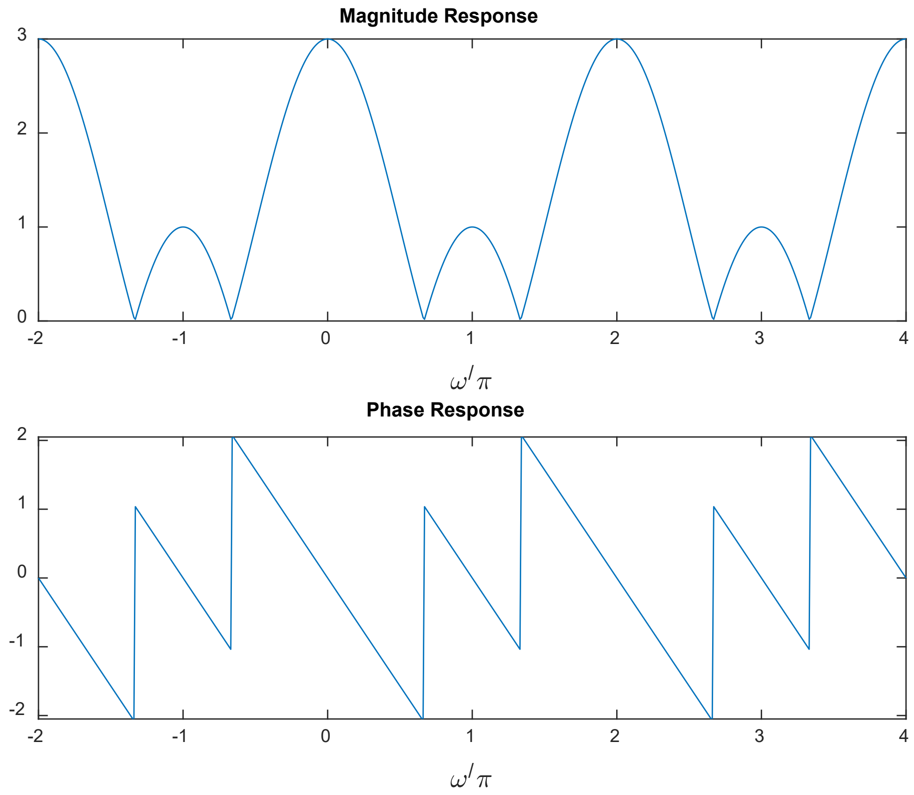
  – DFS & DFT

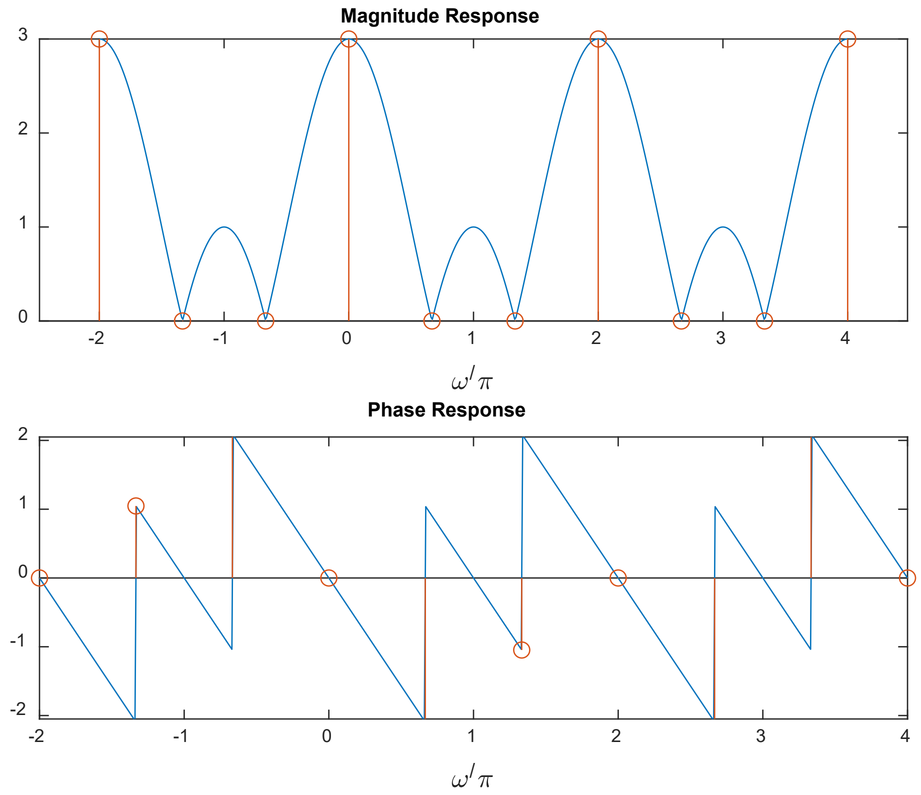
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$X[k] = 1e^{-j2\pi k0/N} + 1e^{-j2\pi k1/N} + 1e^{-j2\pi k2/N} = e^{-j2\pi k/N}\left(e^{j2\pi k1/N} + 1 + e^{-j2\pi k1/N}\right)$$

$$= e^{-j2\pi k/N}\left(1 + 2\cos\left(2\pi k/N\right)\right)$$

# Discrete Fourier Transform




DTFT

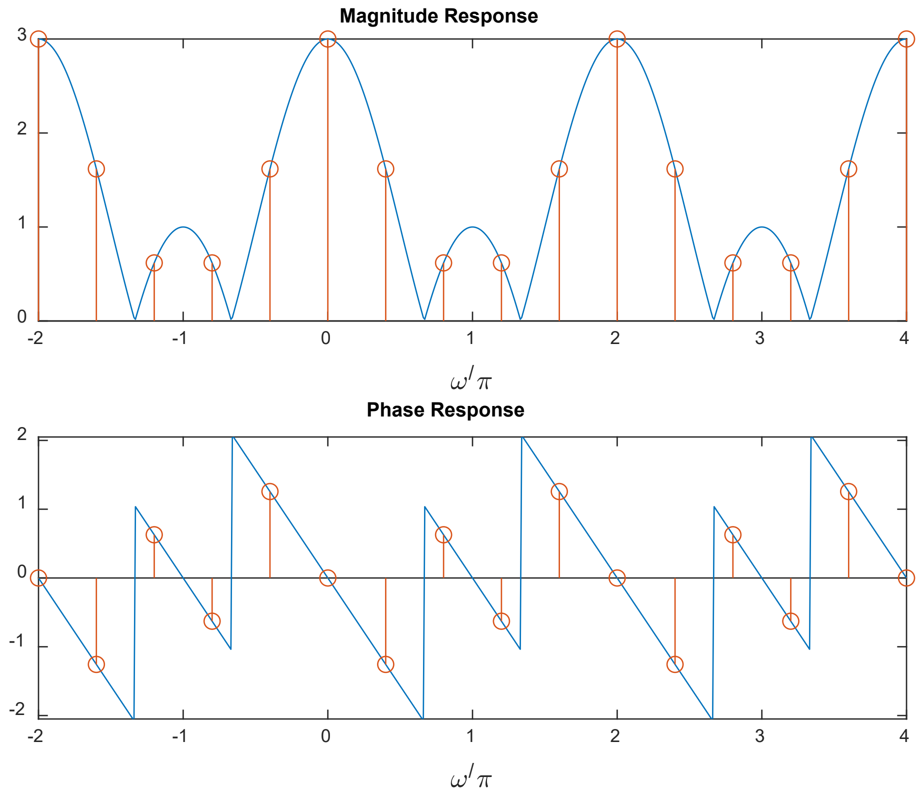
$$x[n] = \begin{cases} 1, & n = 0,1,2 \\ 0, & \text{otherwise} \end{cases}$$

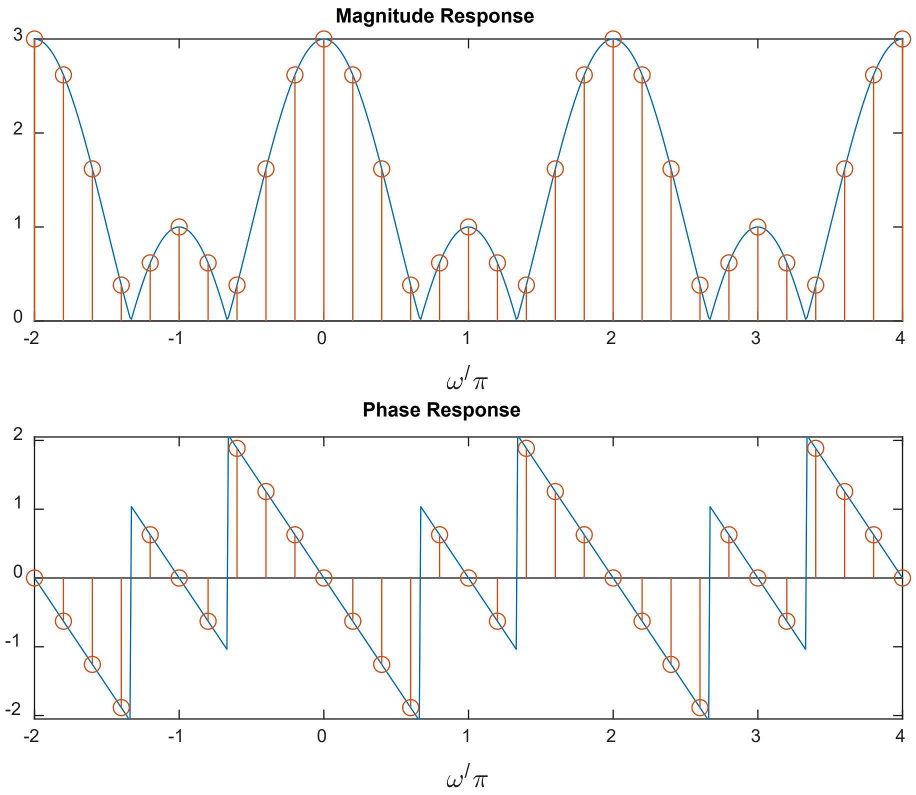
Comparison to DFS
(Plotted for 3 periods)

DFS

$$x[n] = [1,1,1]$$

# Discrete Fourier Transform




DFS

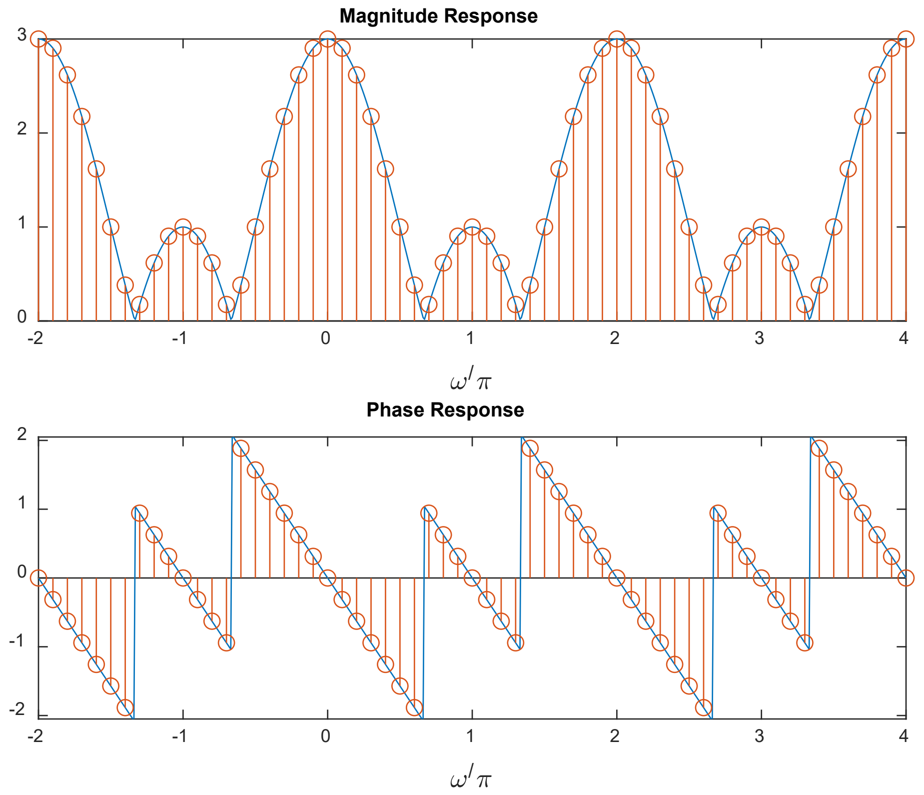
$x[n] = [1,1,1,0,0]$

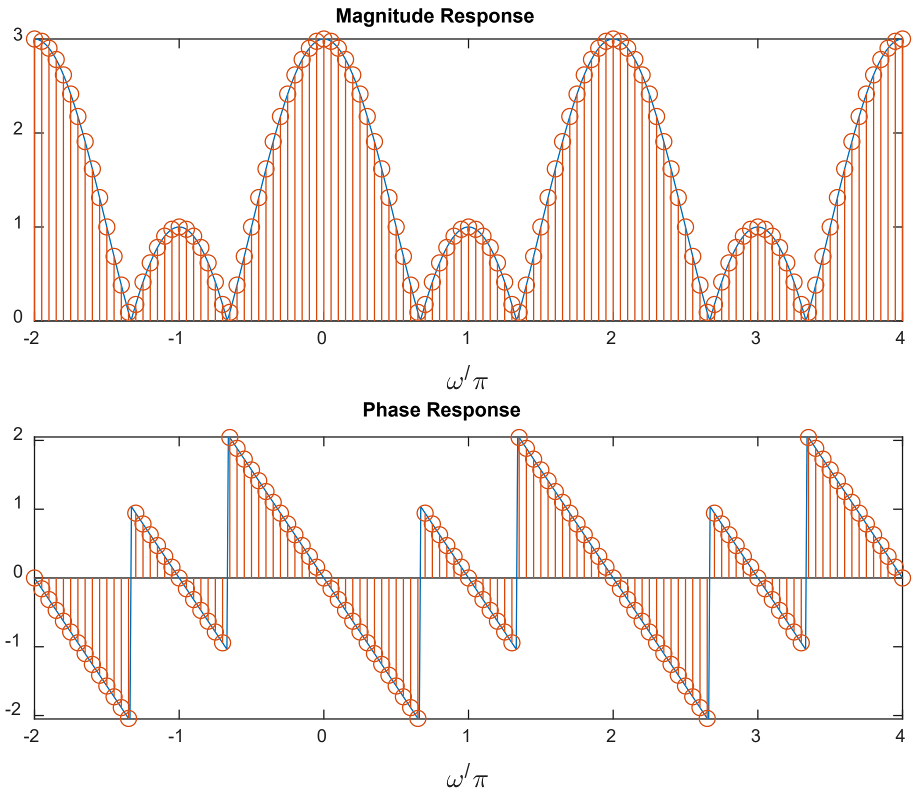
Comparison to DFS
(Plotted for 3 periods)

DFS

$x[n] = [1,1,1,0,0,0,0,0,0,0]$

# Discrete Fourier Transform




DFS

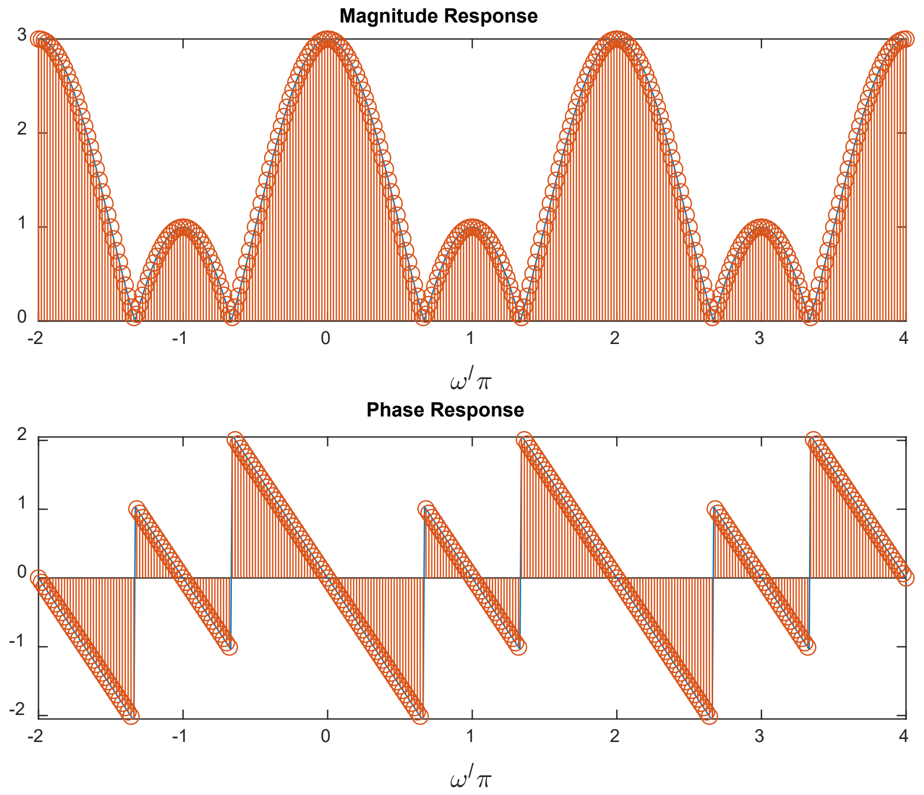
$x[n]=[1,1,1,0,...(17\ \ 0's)]$

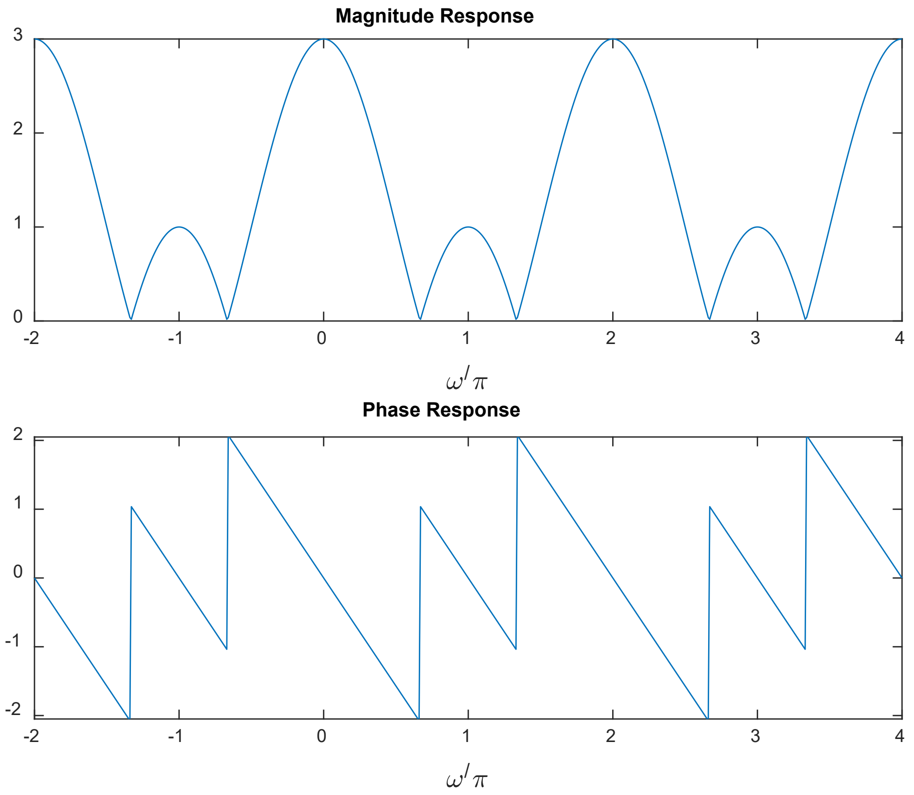
Comparison to DFS
(Plotted for 3 periods)

DFS

$x[n]=[1,1,1,0,...(37\ \ 0's)]$

# Discrete Fourier Transform



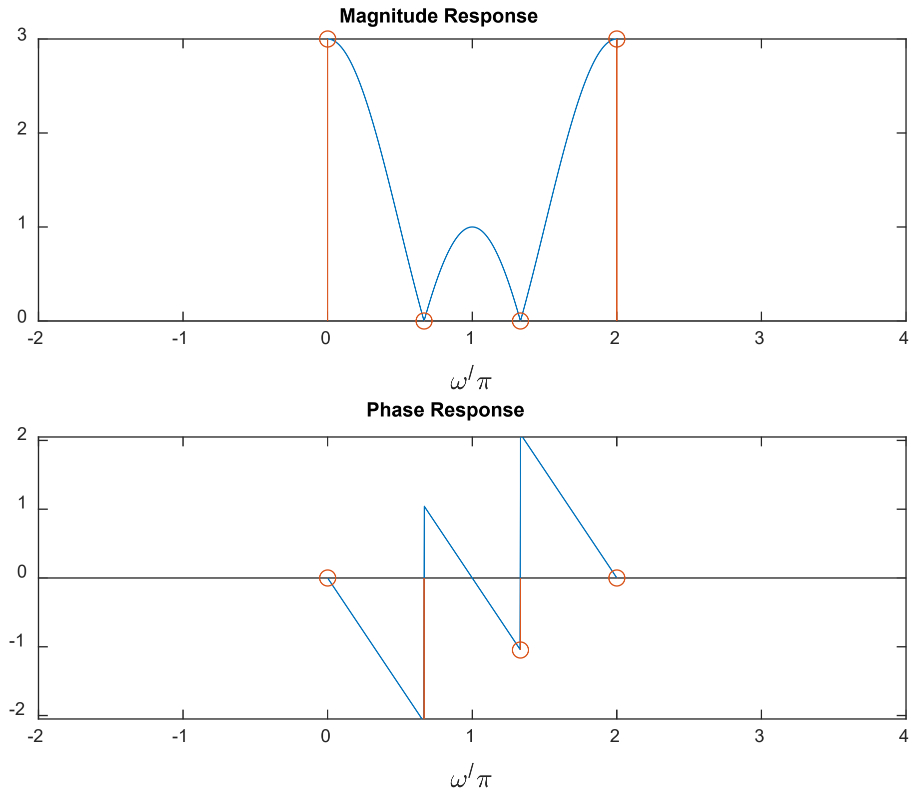

DFS

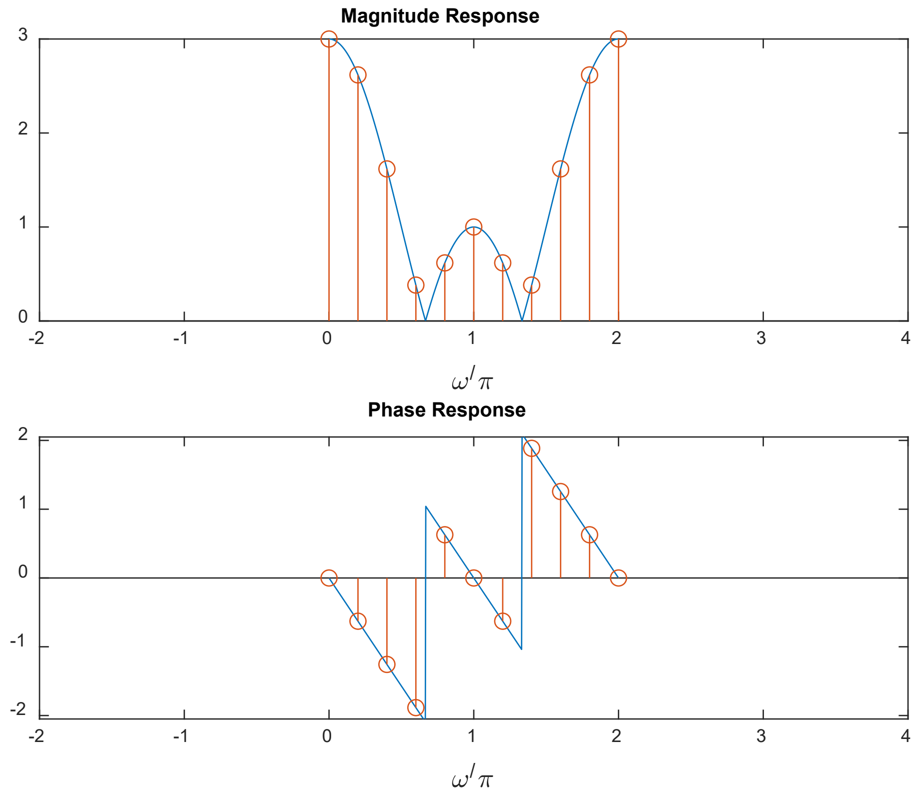
$x[n] = [1,1,1,0,...(97\ \ 0's)]$

Comparison to DFS
(Plotted for 3 periods)

DTFT

# Discrete Fourier Transform




DFT

$x[n]=[1,1,1]$

Comparison to DFT
(only 1 period)

DFT

$x[n]=[1,1,1,0,0,0,0,0,0,0]$

# Discrete Fourier Transform

- DFT as a vector-matrix multiplication

Define: $\mathrm{W}_N = e^{-j2\pi/N}$ (which is the $N$'th root of 1)

Then

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \ , \quad k = 0, 1, 2, \ldots, N\text{-}1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \ , \quad n = 0, 1, 2, \ldots, N\text{-}1$$

# Discrete Fourier Transform

- DFT as a vector-matrix multiplication
  - Define the vectors and matrix:

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \qquad \mathbf{W}_N = \begin{bmatrix} e^{-j2\pi 0\cdot 0/N} & e^{-j2\pi 0\cdot 1/N} & e^{-j2\pi 0\cdot 2/N} & \cdots & e^{-j2\pi 0\cdot(N-1)/N} \\ e^{-j2\pi 1\cdot 0/N} & e^{-j2\pi 1\cdot 1/N} & e^{-j2\pi 1\cdot 2/N} & \cdots & e^{-j2\pi 1\cdot(N-1)/N} \\ e^{-j2\pi 2\cdot 0/N} & e^{-j2\pi 2\cdot 1/N} & e^{-j2\pi 2\cdot 2/N} & \cdots & e^{-j2\pi 2\cdot(N-1)/N} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ e^{-j2\pi(N-1)\cdot 0/N} & e^{-j2\pi(N-1)\cdot 1/N} & e^{-j2\pi(N-1)\cdot 2/N} & \cdots & e^{-j2\pi(N-1)\cdot(N-1)/N} \end{bmatrix}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

# Discrete Fourier Transform

- DFT as a vector-matrix multiplication
  - The DFT and IDFT in vector-matrix notation is:

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$$

To invert this equation to find $\mathbf{x}_N$ :

$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N$ where $\mathbf{W}_N^{-1}$ is the matrix inverse of $\mathbf{W}_N$

Since $e^{-j2\pi/N} = \left(e^{j2\pi/N}\right)^*$

$$\mathbf{x}_N = \frac{1}{N} \mathbf{W}_N^* \mathbf{X}_N$$

so

$$\mathbf{W}_N^{-1} = \frac{1}{N} \mathbf{W}_N^* \Rightarrow \mathbf{W}_N \mathbf{W}_N^* = N\mathbf{I}$$

(Sort of "unitary" except for factor of N: $\mathbf{A}\mathbf{A}^* = \mathbf{I}$)

# Discrete Fourier Transform

- Relationship of DFT to z-transform and DTFT

Start with:

$$X(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

If sampled on unit circle, $z_k = e^{j2\pi k/N}$ ($N$ equally spaced points labeled by $k$)

$$X(\omega)\Big|_{\omega=2\pi k/N} = X(z)\Big|_{z=e^{j2\pi k/N}}$$

Discretized DTFT

If sequence is finite (with length $N$)

$$X(z)=\sum_{n=0}^{N-1} x(n)z^{-n}$$

Substitute IDFT for $x(n)$

$$X(z)=\sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}\right)z^{-n}$$

# Discrete Fourier Transform

- Relationship of DFT to $z$-transform and DTFT

Skipping a lot of steps, like interchanging order of summations and using the sum of a finite geometric series:

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j2\pi k/N} z^{-1}}$$

If you evaluate this on the unit circle, $z = e^{j\omega}$

$$X(\omega) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j(\omega - 2\pi k/N)}}$$

which is another interpolation formula for getting the DTFT from the DFT

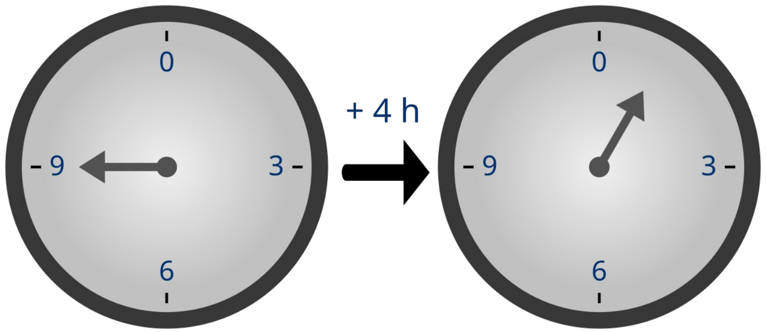
# Discrete Fourier Transform

- Indicate $x[n]$ and $X[k]$ are a DFT pair as: $x[n] \Leftrightarrow X[k]\left(\mathrm{I}\right)$

- Modulo arithmetic: $\left(m-n\right)\mathrm{mod}(N) = m - n + rN$

  where r is an integer choses such that $0 \leq n - m + rN \leq N - 1$

  - Book uses notation: $((m - n))_N$




Clock time is modulo 12
9 PM + 4 hours = 1 AM

$(9+4)\,\mathrm{mod}\,12 = 13 + r12, \quad r = -1$

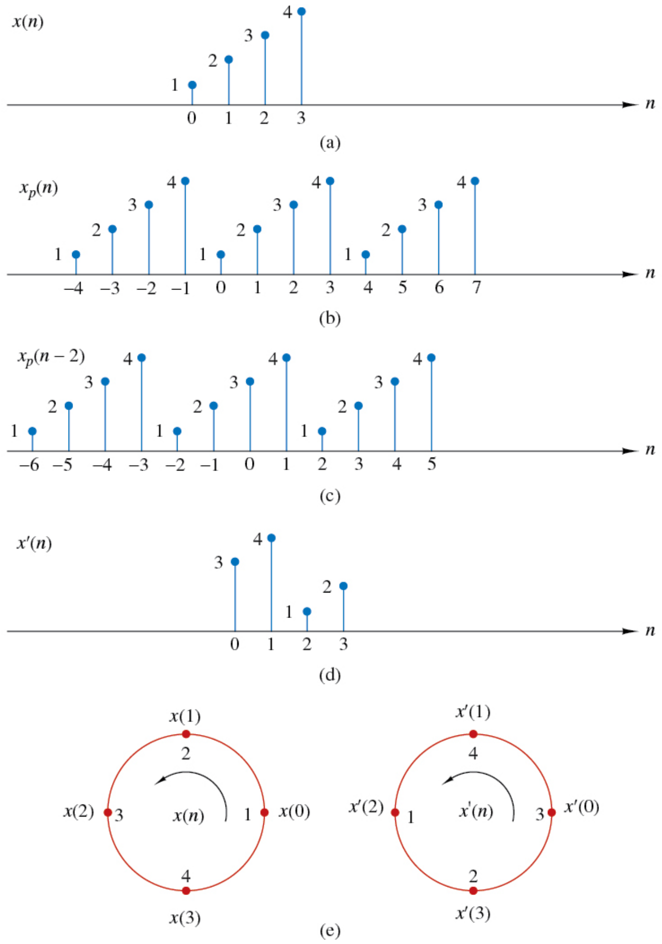
$(9+4)\,\mathrm{mod}\,12 = 1$

$\left(3-8\right)\mathrm{mod}\,12 = -5 + r12, \quad r = 1$

$\left(3-8\right)\mathrm{mod}\,12 = 7$

# Discrete Fourier Transform

$$\text{Example}: \; x[n] = \{1,2,3,4\}$$
$$((0-2))_4 = -2+1\cdot 4 = 2 \Rightarrow x[((0-2))_4] = x[2] = 3$$
$$((1-2))_4 = -1+1\cdot 4 = 3 \Rightarrow x[((1-2))_4] = x[3] = 4$$
$$((2-2))_4 = 0+0\cdot 4 = 0 \Rightarrow x[((2-2))_4] = x[0] = 1$$
$$((3-2))_4 = 1+0\cdot 4 = 1 \Rightarrow x[((3-2))_4] = x[1] = 2$$
$$x[((n-2))_4] = \{3,4,1,2\}$$



# Discrete Fourier Transform

- Some symmetries of sequence on a circle (with $N$ positions)
  - Circularly even if symmetric about point 0 on the circle

$$x[N-n]=x[n]$$

  - Circularly odd if antisymmetric about point 0 on the circle

$$x[N-n]=-x[n]$$

  - Time reversal:

$$x[((-n))_N]=x[N-n]$$

# Discrete Fourier Transform

- Symmetry properties of DFT

| $N$-Point Sequence $x(n)$, $0 \le n \le N-1$ | $N$-Point DFT |
|---|---|
| $x(n)$ | $X(k)$ |
| $x^*(n)$ | $X^*(N-k)$ |
| $x^*(N-n)$ | $X^*(k)$ |
| $x_R(n)$ | $X_{ce}(k) = \frac{1}{2}[X(k) + X^*(N-k)]$ |
| $jX_I(n)$ | $X_{co}(k) = \frac{1}{2}[X(k) - X^*(N-k)]$ |
| $x_{ce}(n) = \frac{1}{2}[x(n) + x^*(N-n)]$ | $X_R(k)$ |
| $x_{co}(n) = \frac{1}{2}[x(n) - x^*(N-n)]$ | $jX_I(k)$ |
| Real Signals | |
| Any real signal | $X(k) = X^*(N-k)$ |
| $x(n)$ | $X_R(k) = X_R(N-k)$ |
| | $X_I(k) = -X_I(N-k)$ |
| | $\lvert X(k)\rvert = \lvert X(N-k)\rvert$ |
| | $\angle X(k) = -\angle X(N-k)$ |
| $x_{ce}(n) = \frac{1}{2}[x(n) + x(N-n)]$ | $X_R(k)$ |
| $x_{co}(n) = \frac{1}{2}[x(n) - x(N-n)]$ | $jX_I(k)$ |

# Discrete Fourier Transform

- More Symmetry properties of DFT

  - Real-valued sequences: $X[N-k] = X^*[k] = X[-k]$

  - Real-valued even sequences: $x[n] = x[N-n]$

DFT $\quad X[k] = \sum_{n=0}^{N-1} x[n]\cos\left(\frac{2\pi kn}{N}\right), \quad 0 \le k \le N-1$

IDFT $\quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\cos\left(\frac{2\pi kn}{N}\right), \quad 0 \le n \le N-1$

  - Real-valued odd sequences: $x[n] = x[N-n]$

DFT $\quad X[k] = -j\sum_{n=0}^{N-1} x[n]\sin\left(\frac{2\pi kn}{N}\right), \quad 0 \le k \le N-1$

IDFT $\quad x[n] = j\frac{1}{N}\sum_{k=0}^{N-1} X[k]\sin\left(\frac{2\pi kn}{N}\right), \quad 0 \le n \le N-1$

# Discrete Fourier Transform

- More Symmetry properties of DFT

  - Duality:

  If x[n] $\Leftrightarrow$ X[k], then $X[n] \Leftrightarrow x[((-k))_N]$

# Discrete Fourier Transform

- Circular Convolution:
  - Start with assumption that product of two DFT's is going to be something in terms of the time-domain sequences

$$X_3[k] = X_1[k]X_2[k]$$

$$X_1[k] = \sum_{n=0}^{N-1} x_1[n] e^{-j2\pi kn/N} \quad , \quad k = 0, 1, 2, \ldots, N\text{-}1$$

$$X_2[k] = \sum_{n=0}^{N-1} x_2[n] e^{-j2\pi kn/N} \quad , \quad k = 0, 1, 2, \ldots, N\text{-}1$$

Inverse of $X_3[k]$:

$$x_3[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_3[k] e^{j2\pi kn/N}$$

# Discrete Fourier Transform

Inverse of $X_3[k]$:

$$x_3[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_3[k] e^{j2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k] X_2[k] e^{j2\pi kn/N}$$

$$x_3[n] = \frac{1}{N}\sum_{k=0}^{N-1} \left(\sum_{n=0}^{N-1} x_1[n] e^{-j2\pi kn/N}\right)\left(\sum_{l=0}^{N-1} x_2[l] e^{-j2\pi kl/N}\right) e^{j2\pi kn/N}$$

(Many steps skipped … interchange order of summation, use sum of finite geometric series, recognize modulo arithmetic …)

$x_3$ is circular convolution of $x_1$ and $x_2$

$$x_3[m] = \sum_{n=0}^{N-1} x_1[n] x_2[((m-n))_N]$$

$$x_1[n] \odot x_2[n] \equiv \sum_{n=0}^{N-1} x_1[n] x_2[((m-n))_N]$$

# Discrete Fourier Transform

- Circular Shift of Sequence

If $x[n] \Leftrightarrow X[k]$,

$$x[\left(n-m\right)\mathrm{mod}(N)] \Leftrightarrow W_N^{km} X[k] = e^{j2\pi km/N} X[k]$$

or

$$x[((n-m))_N] \Leftrightarrow W_N^{km} X[k] = e^{j2\pi km/N} X[k]$$

# Discrete Fourier Transform

- Properties of DFT:
  - – Linearity
  - – Circular convolution
  - – Time reversal
  - – Circular shift of sequence
  - – Circular frequency shift
  - – Complex conjugate properties
  - – Circular correlation
  - – Multiplication of sequences
  - – Parseval’s Theorem

# Discrete Fourier Transform

- Properties of DFT:

| Property | Time Domain | Frequency Domain |
|---|---|---|
| Notation | $x(n), y(n)$ | $X(k), Y(k)$ |
| Periodicity | $x(n) = x(n+N)$ | $X(k) = X(k+N)$ |
| Linearity | $a_1x_1(n) + a_2x_2(n)$ | $a_1X_1(k) + a_2X_2(k)$ |
| Time reversal | $x(N-n)$ | $X(N-k)$ |
| Circular time shift | $x((n-l))_N$ | $X(k)e^{-j2\pi kl/N}$ |
| Circular frequency shift | $x(n)e^{j2\pi ln/N}$ | $X((k-l))_N$ |
| Complex conjugate | $x^*(n)$ | $X^*(N-k)$ |
| Circular convolution | $x_1(n) \circledN x_2(n)$ | $X_1(k)X_2(k)$ |
| Circular correlation | $x(n) \circledN y^*(-n)$ | $X(k)Y^*(k)$ |
| Multiplication of two sequences | $x_1(n)x_2(n)$ | $\frac{1}{N}X_1(k) \circledN X_2(k)$ |
| Parseval's theorem | $\sum_{n=0}^{N-1} x(n)y^*(n)$ | $\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$ |

# Discrete Fourier Transform

- Examples:

  Problem 7.11 (a)

**7.11** Given the eight-point DFT of the sequence

$$x(n) = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & 4 \le n \le 7 \end{cases}$$

compute the DFT of the sequences

**a.** $x_1(n) = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le 4 \\ 1, & 5 \le n \le 7 \end{cases}$

**b.** $x_2(n) = \begin{cases} 0, & 0 \le n \le 1 \\ 1, & 2 \le n \le 5 \\ 0, & 6 \le n \le 7 \end{cases}$

# Discrete Fourier Transform

- Examples:

  Circular Convolution example

**Example 7.2.1**

Perform the circular convolution of the following two sequences:

$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

**Solution**

Each sequence consists of four nonzero points. For the purposes of illustrating the operations involved in circular convolution, it is desirable to graph each sequence as points on a circle. Thus the sequences $x_1(n)$ and $x_2(n)$ are graphed as illustrated in Fig. 7.2.2(a). We note that the sequences are graphed in a counterclockwise direction on a circle. This establishes the reference direction in rotating one of the sequences relative to the other.

# Discrete Fourier Transform

- Circulant Matrix