Digital Signal Processing

Class 14
03/05/2025

ENGR 71

- Class Overview
  - Frequency Analysis of Discrete Signals
  - Discussion of Lab 2
    - Classification using Frequency Domain Features
- Assignments
  - Reading:

    Chapter 4: Frequency Analysis of Signals

    Chapter 5: Frequency-Domain Analysis of LTI Systems

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- Exam
  - Take home exam
    - Will post Monday, March 17
    - Due March 24

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- Lab 2
  - Classification using Frequency Domain Features
  - Due Mar 22

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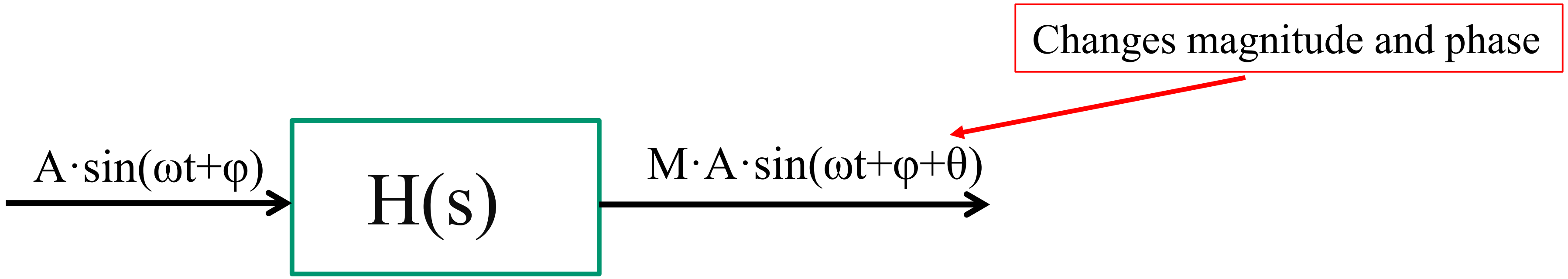
- Questions about HW4?

Frequency-Domain Analysis of LTI Systems

- Key concept behind the action of LTI systems on signals:
  - Signals can be decomposed into superposition of frequency components
  - Basis function for this decomposition are sines and cosines (and complex exponential)
  - **Frequency components of signal are unchanged when passed through Linear Time Invariant systems**
    - Only amplitude and phase change



Frequency-Domain Analysis of LTI Systems

- Frequency response of an LTI system
  - The response of an LTI system to any input is:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

  - Considering a complex exponential input $x(n) = Ae^{j\omega n}$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)Ae^{j\omega(n-k)} = A\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]e^{j\omega n}$$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$y(n) = AH(\omega)e^{j\omega n}$$

Frequency-Domain Analysis of LTI Systems

– This shows that complex exponentials are the eigenfunctions and $H(\omega)$ are the eigenvalues of an LTI system.

– Since any signal can be decomposed into complex exponentials, $H(\omega)$ completely characterizes the LTI system.

– Example of how the system modifies the amplitude and phase of a sinusoidal input but not the frequency:

- Impulse response of system is

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

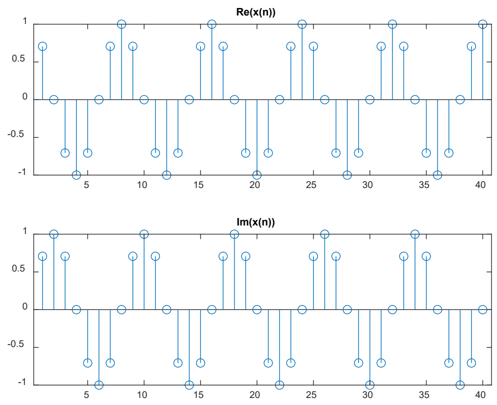
$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n)e^{-j\omega n} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} = \sum_{k=0}^{\infty} \left(\frac{1}{2}e^{-j\omega}\right)^n = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

Frequency-Domain Analysis of LTI Systems

– What does the system do to an complex exponential input (i.e. and input at some particular frequency)

- Consider an input with a frequency of $\pi/4$

$$x(n) = Ae^{j\omega n} = Ae^{jn\pi/4}$$



$$H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}, \quad H(\pi/4) = \frac{1}{1 - \frac{1}{2}e^{-\frac{j\pi}{4}}}$$

$$|H(\pi/4)| = 1.3572, \quad \phi = -28.68^{\circ}$$

- Example in book shows:

$$|H(\pi/2)| = 0.8944 \quad \phi = -26.6^{\circ}$$

$$|H(\pi)| = 0.6667 \quad \phi = 0^{\circ}$$

Frequency-Domain Analysis of LTI Systems

– If you want to see the affect of this LTI system on the amplitude and frequency at all frequencies, use Matlab `freqz`

$$H(w) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

Numerator and denominator coefficients are:

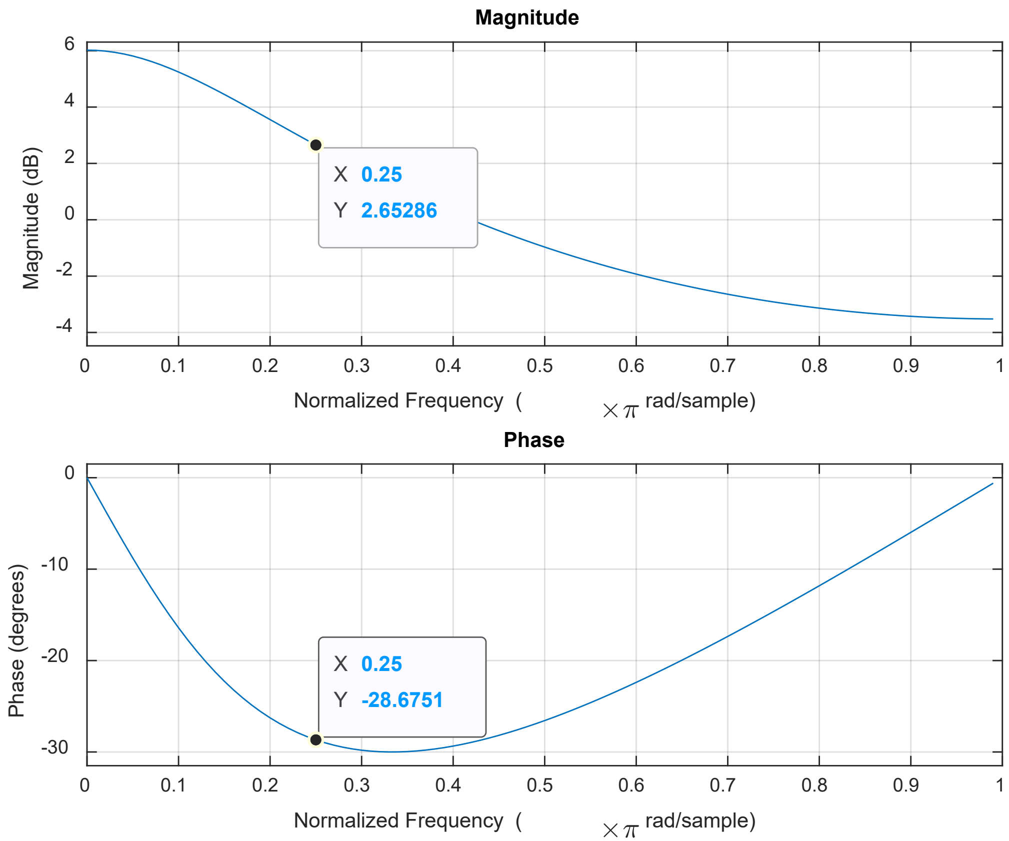
$$b = 1$$

$$a = [1, \quad -1/2]$$

```
freqz(b,a,n)  (where n is number of points to evaluate
                    frequency response)
```

Frequency-Domain Analysis of LTI Systems

```
freqz(1,[1,-1/2,100)
```



$$\left|H(\pi/4)\right| = 1.3572, \quad \phi = -28.68^{\circ}$$

$$\left|H(\pi/4)\right| = 20\log_{10}\left(1.3572\right) = 2.6529$$

Frequency-Domain Analysis of LTI Systems

– If an LTI system changes the magnitude and phase of an input
  - You can begin to see how filtering works
  - Consider what the LTI does to each frequency component

Example of a moving average filter:

$$y(n) = \frac{1}{M+1}\sum_{k=1}^{M} x(n-k)$$

Frequency response is (using the finite geometric series sum)

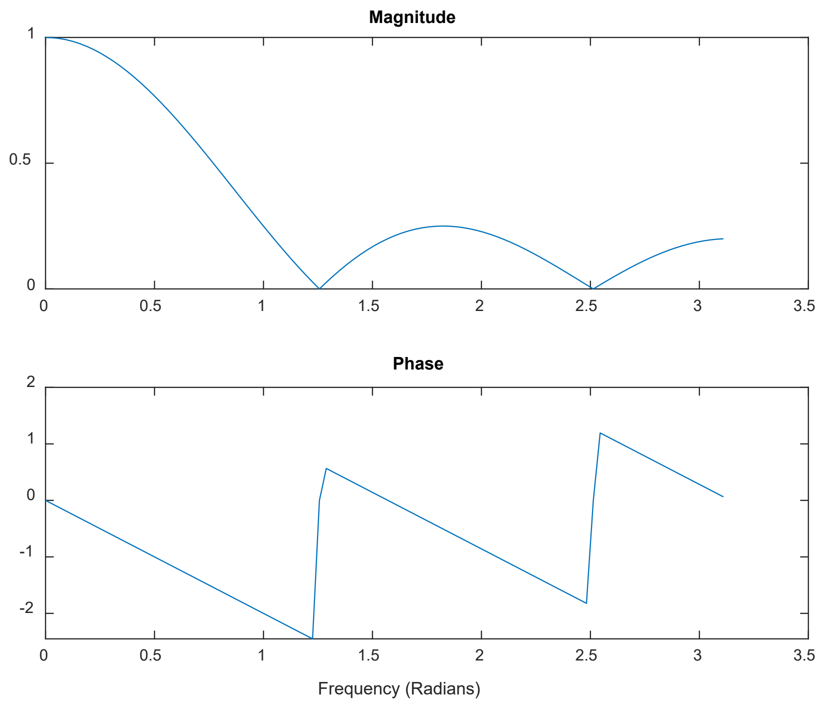
$$H(\omega) = \frac{1}{M+1}\sum_{k=0}^{M} e^{-j\omega k} = \frac{1}{M+1}\frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}}$$

$$H(\omega) = \frac{1}{M+1}\frac{\sin\left(\omega(M+1/2)\right)}{\sin\left(\omega/2\right)}e^{-j\omega/2}$$

Frequency-Domain Analysis of LTI Systems

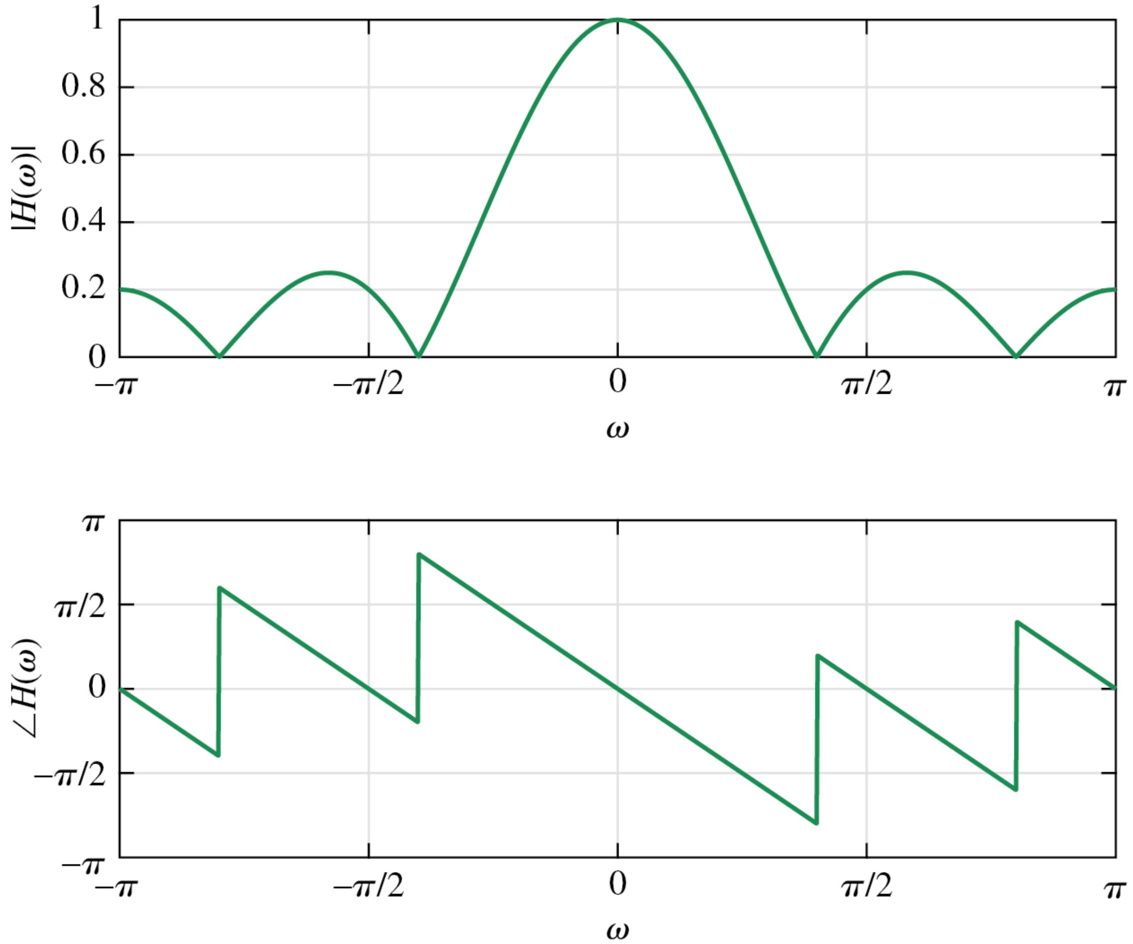
- Moving average (M=4):

$$H(z) = \frac{1}{M+1}\sum_{k=0}^{M} e^{-j\omega k} = \frac{1}{5}\left[1+z^{-1}+z^{-2}+z^{-3}+z^{-4}\right]$$




```
a =5;
b = [1,1,1,1,1];
[h,w] = freqz(b,a,100);
mag = abs(h);
phase  = angle(h);
figure(1)
subplot(2,1,1)
hold off
plot(w,mag)
title('Magnitude');
subplot(2,1,2)
plot(w,phase)
title('Phase');
xlabel('Frequency (Radians)')
```

Frequency-Domain Analysis of LTI Systems



This is a low-pass filter

M=4

Frequency-Domain Analysis of LTI Systems

– Example with Infinite impulse response

$$y(n) = ay(n-1) + bx(n)$$

We have found the impulse response for this system a few times:

$$H(z) = \frac{b}{1 - az^{-1}}$$

$$H(\omega) = \frac{b}{1 - ae^{-j\omega}}$$

$$\left|H(\omega)\right| = \frac{1-a}{\sqrt{1 - 2a\cos\omega + a^2}}$$

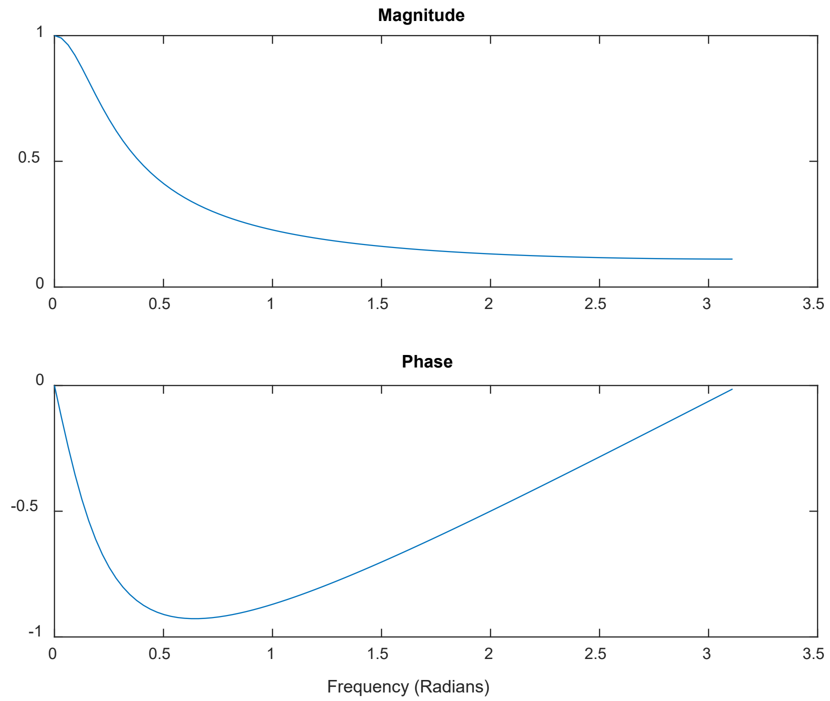
$$\phi = -\tan^{-1}\left(\frac{a\sin\omega}{1 - a\cos\omega}\right)$$

Frequency-Domain Analysis of LTI Systems

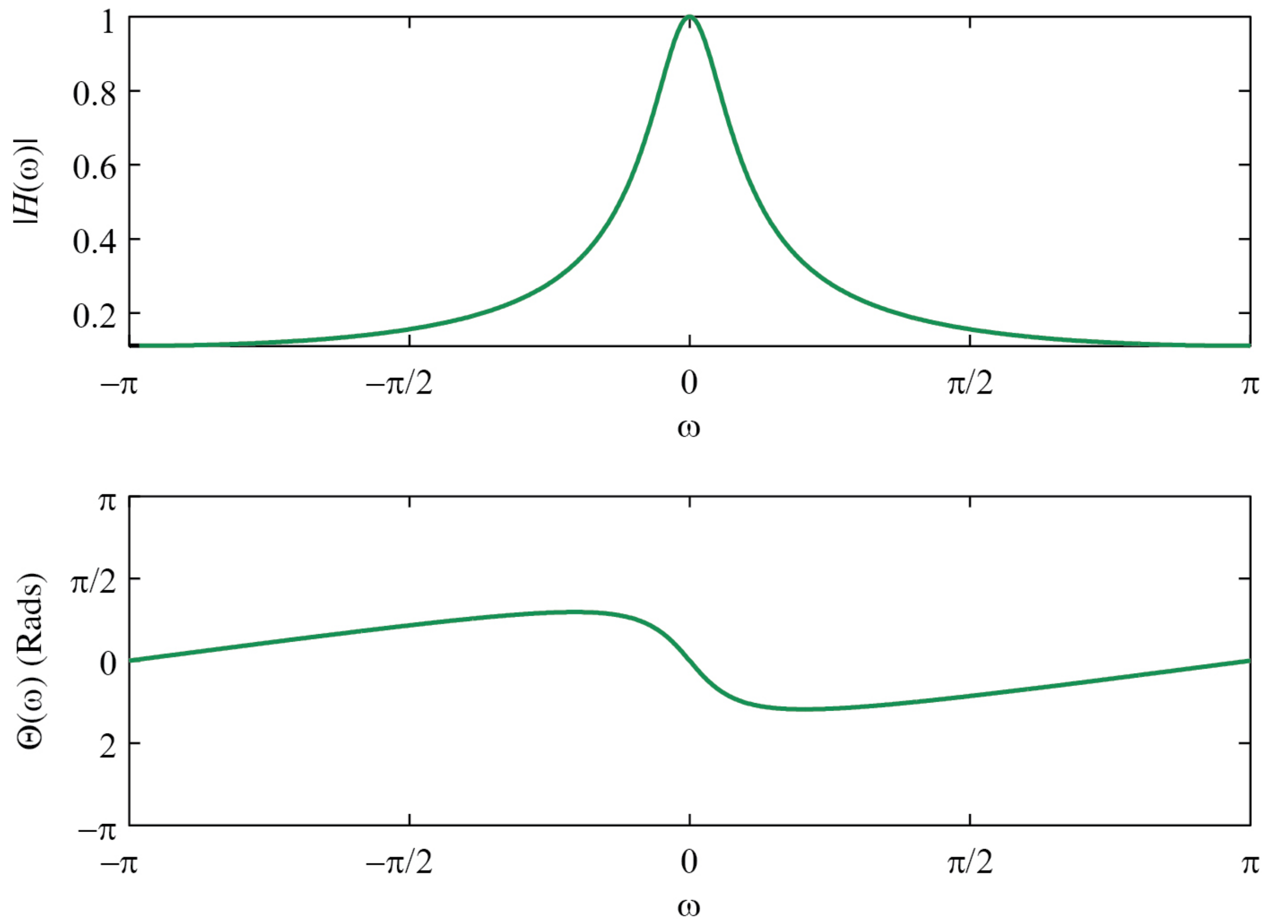
– Example with Infinite impulse response

$$y(n) = ay(n-1) + bx(n)$$

$$H(z) = \frac{1-a}{1-az^{-1}}, \quad a = 0.8$$



Frequency-Domain Analysis of LTI Systems



This is a also low-pass filter

a=0.8

Frequency-Domain Analysis of LTI Systems

- Transient and steady-state response of system
  - Example

$$y(n) = ay(n-1) + x(n), \quad y(-1)$$

$$y(0) = ay(-1) + x(0)$$

$$y(1) = a[ay(-1) + x(0)] + x(1)$$

$$y(2) = a\left[a[ay(-1) + x(0)] + x(1)\right] + x(2)$$

$$\vdots$$

$$y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k x(n-k)$$

Frequency-Domain Analysis of LTI Systems

- Transient and steady-state response of system
  - If the input is a complex exponential: $x(n) = Ae^{j\omega n}$

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k Ae^{j\omega(n-k)} = a^{n+1}y(-1) + A\left[\sum_{k=0}^{n} a^k e^{-j\omega k}\right]e^{j\omega n}$$

$$y(n) = a^{n+1}y(-1) + A\left[\sum_{k=0}^{n} \left(ae^{-j\omega}\right)^k\right]e^{j\omega n}$$

Using sum of finite geometric series

These die off as n increases

Steady-state

$$y(n) = a^{n+1}y(-1) + A\left[\frac{1-\left(ae^{-j\omega}\right)^{n+1}}{1-ae^{-j\omega}}\right]e^{j\omega n} = a^{n+1}y(-1) - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1-ae^{-j\omega}}e^{j\omega n} + \frac{A}{1-ae^{-j\omega}}e^{j\omega n}$$

Frequency-Domain Analysis of LTI Systems

- Steady-state for periodic input
  - Discrete Fourier Series of input:

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi k/N}$$

Output of each harmonic gets modified by: $H\left(\frac{2\pi k}{N}\right)$

$$y(n) = \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi k}{N}\right) e^{j2\pi k/N}$$

so also periodic with modified Fourier Series coefficients

Frequency-Domain Analysis of LTI Systems

- Steady-state for aperiodic input
  - Use convolution to find output:

$$Y(\omega) = H(\omega)X(\omega)$$

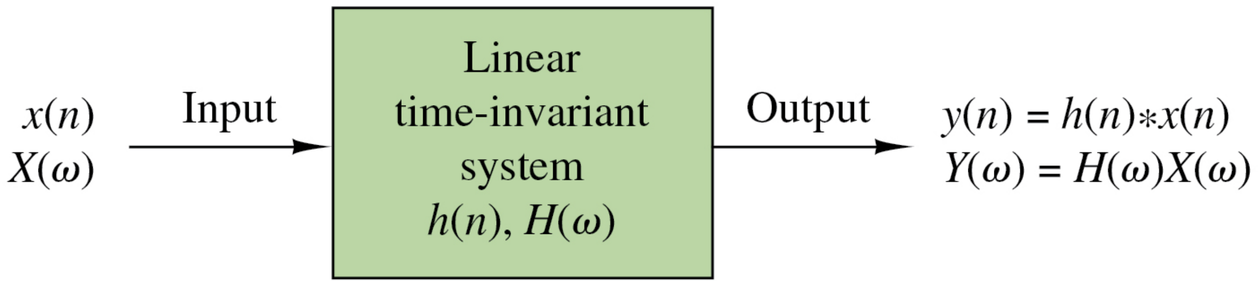
$$|Y(\omega)| = |H(\omega)||X(\omega)|$$

$$\measuredangle Y(\omega) = \measuredangle H(\omega) + \measuredangle X(\omega)$$

Energy Density:

$$|Y(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2$$

$$S_{yy} = |H(\omega)|^2 S_{xx}$$



Lab 2 -

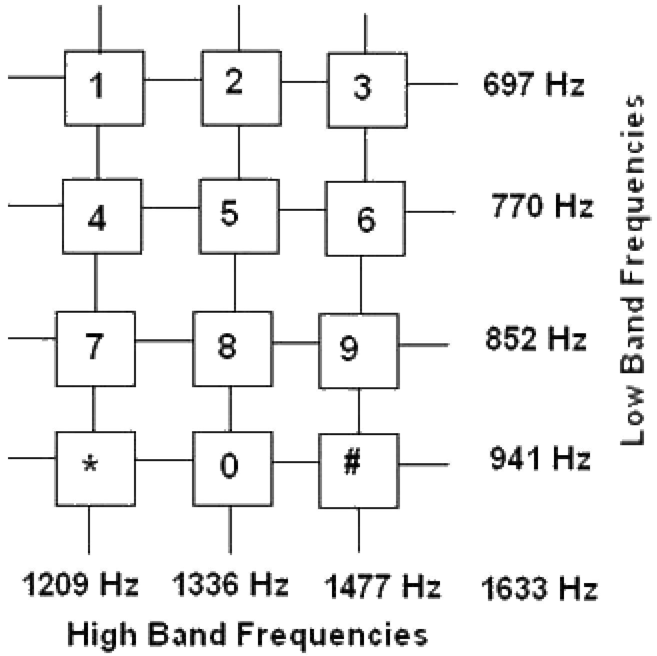
- Classification using Frequency Domain Features
  - Part A: Decoding phone numbers from touchtone frequencies
  - Part B: Classification of simple words from audio

Touchtone phone decoding

- Phone tones
  - Tools to use:
    - Fourier transform to see if you can identify discrete frequency that can be associated with numbers on the "dial"
    - Segment tones for numbers "dialed" and try to map to numbers

Touchtone phone decoding

- Phone tones



Touchtone phone decoding

- Part A: Decoding phone numbers from touchtone frequencies
  - Code to read files and plot time domain signals:

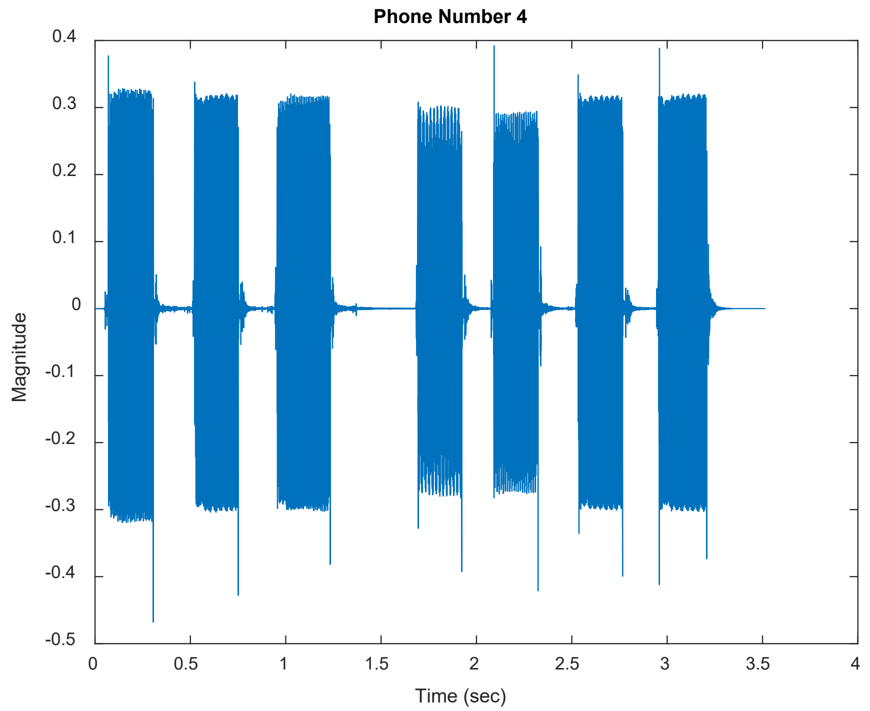
```
phone_number = 'phone_number_4.mp3';
[phn,fs] = audioread(phone_number);
phone_part1 = split(phone_number,'.');
phone_call = split(phone_part1{1},'_');
callnum = phone_call{3};
phn1 = phn(:,1);
nsamp = length(phn1);
tm = (1/fs)*[1:nsamp];
figure(1)
plot(tm,phn1);
xlabel('Time (sec)')
ylabel('Magnitude')
title(['Phone Number ',callnum])
```

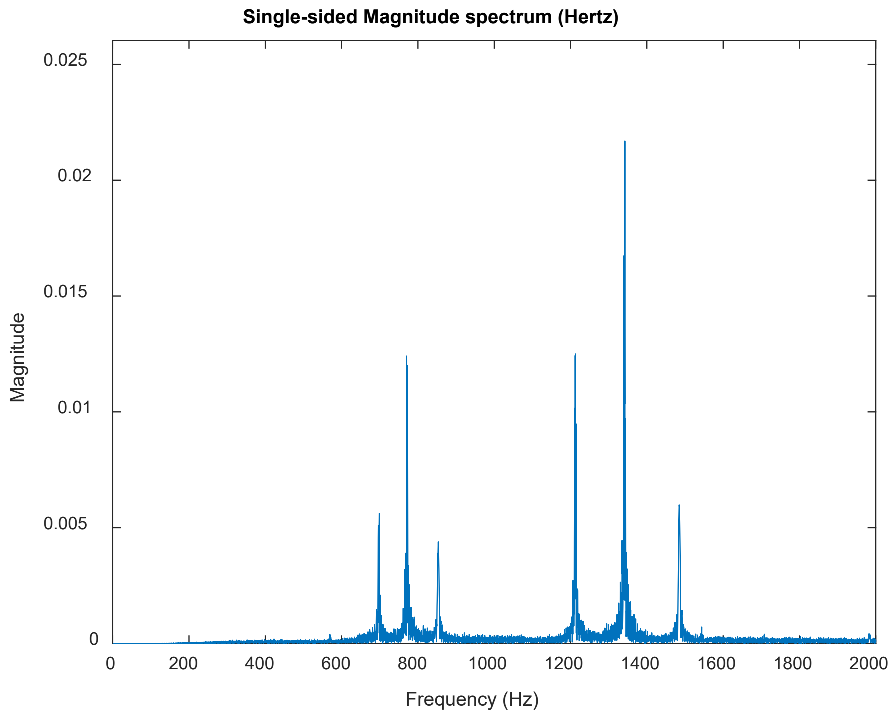
Touchtone phone decoding

- Code to plot magnitude of Fourier transform

```
fnyquist = fs/2;
x_mag = abs(fft(phn1))/nsamp;
bins = [0:nsamp-1];
freq_hz = bins*fs/nsamp;
% Plot only positive frequencies
n_2 = ceil(nsamp/2);
figure(2)
plot(freq_hz(1:n_2), x_mag(1:n_2))
max_mag = max(x_mag(1:n_2));
axis([0,2000,0,1.2*max_mag]);
xlabel('Frequency (Hz)')
ylabel('Magnitude');
title('Single-sided Magnitude spectrum (Hertz)');
```

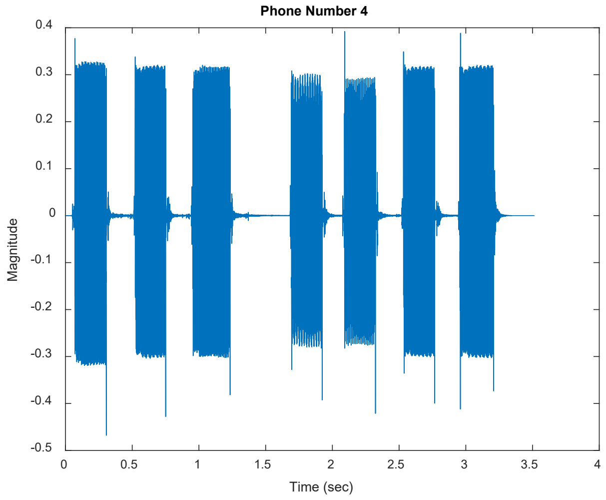
Touchtone phone decoding

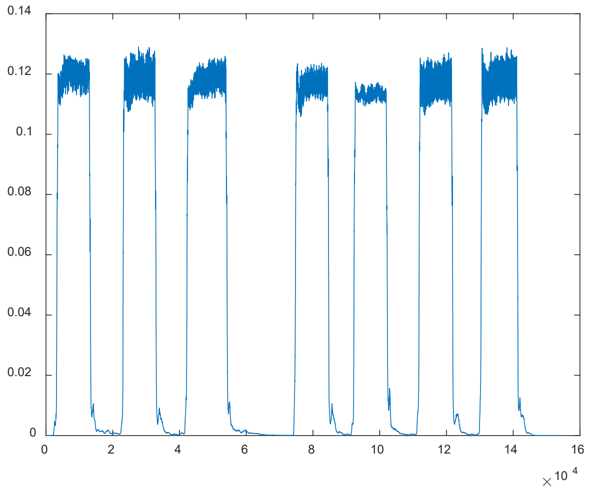




Touchtone phone decoding

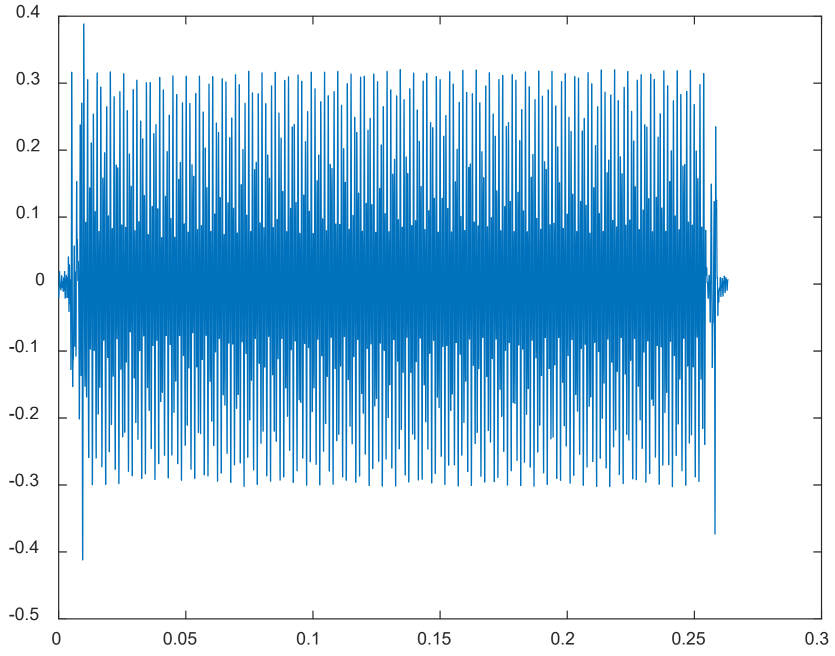
- How to partition key presses?
  - Ideas?
    - Moving average: movmean
    - Moving standard deviation: movstd
    - Moving variance: movvar

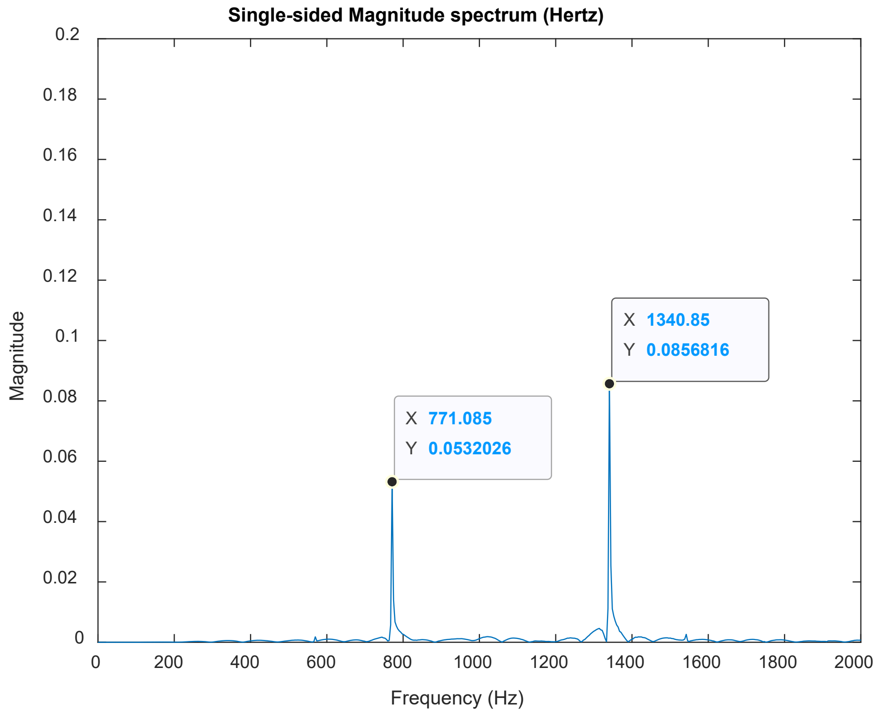




Touchtone phone decoding

- Once you've obtained segments, find peaks





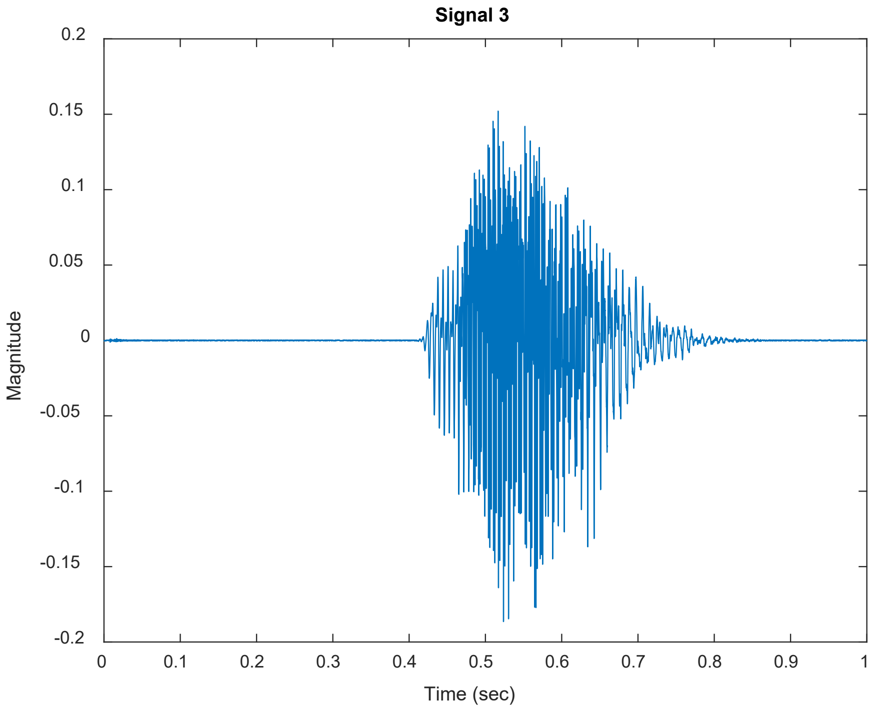
#5: 770 Hz & 1336 Hz

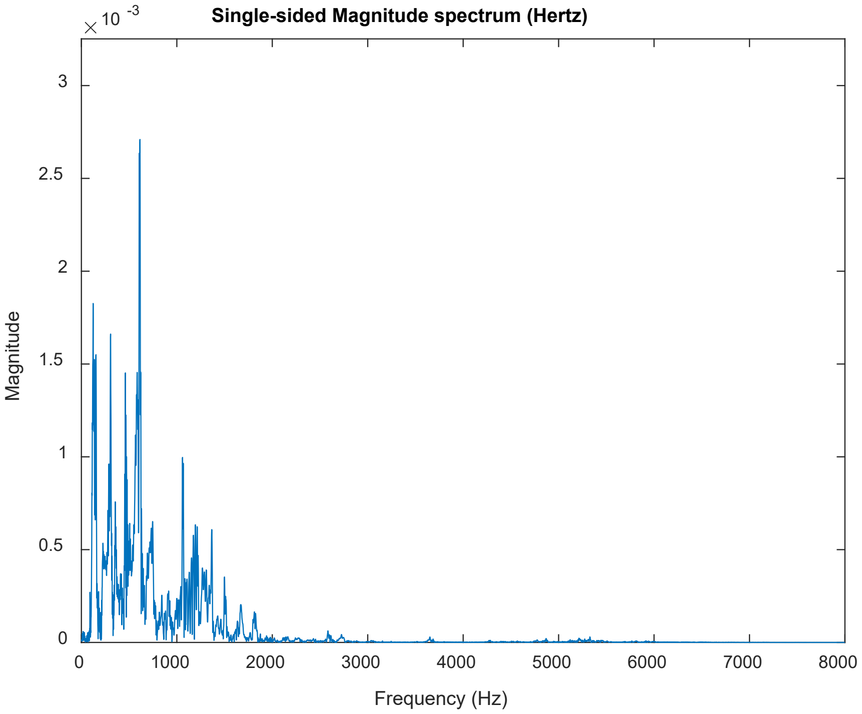
Speech recognition

- Speech recognition
  - There are some very sophisticated methods of speech recognition, which actually seem to work some time.
  - We won't be using these.
  - Dataset with single words from google.
  - We will just try to distinguish two words, like "yes" and "no"
  - This will involve obtaining attributes in the frequency domain and using them in a classifier
    - A very neat tool in Matlab called classificationLearner
  - A good start for features is finding the power in some set of frequency bands
    - Need to normalize by total power

Speech recognition

- Read data:





Speech recognition

- Get features:
  - You could try looking at spectra for a few yes’s and a few no’s to see if anything jumps out.
    - Perhaps a frequency band that is strong in one and weak in another
    - Try finding energy in set of frequency bands
      - How many?
      - What are the boundaries?
      - Normalize by total power?
      - Take ratios of bands that look most different?
      - Do some on-line research for good features in speech recognition

Speech recognition

- How to use classificationLearner