Digital Signal Processing

Class 1

01/21/2025

ENGR 71

- Class Overview
  - Syllabus, Class Policies
  - Introduction to DSP
- Assignments
  - Reading:
    - Chapter 1: Introduction
    - Chapter 2: Discrete-Time Signals and Systems
  - Homework 1: Due Jan. 26 (Sun.)

ENGR 71

- Homework Assignment:

  Homework 1 - Review of Complex Numbers

  (on Moodle page here)

  - Homework: Due Jan 26 (Sunday)
    - Please scan in as a pdf and put in dropbox

Class Information

- Class Information:
  - Office Hours

    Office Hours: Available after class
Wednesday 1:15 (during lab time)
By appointment

    Office: Singer Rm 346
Phone: (610) 328-8446
E-mail: amoser2@swarthmore.edu
  - When and Where?

    Class: Science Center 264 TTh 9:55 – 11:10
Lab: Nominally Self-scheduled
Time reserved: Wed. 1:15 – 4:00, Singer 246

Class Information

- Textbook (Required reading & problems):

  J.G. Proakis & D.G. Manolakis, *Digital Signal Processing*. Fifth Edition, Pearson, 2022. ISBN: 978-0137348244

  Digital edition available through TAP+ Program: here

- Reference useful for review of analog signal processing (not required)

  Chaparro, Luis F. (2014), Signals and Systems Using Matlab, 2nd Edition, Academic Press, ISBN: 978-0123948120.

- Other resources on class Moodle page

Grades

- Grades:
  - Homework: 20%
  - Exam 1: 25%
  - Exam 2: 25%
  - Labs: 15%
  - Final Project: 15%

- See Syllabus and Schedule on Moodle

Labs

- Labs:
  - Nominally self scheduled, but there is a time and room available: Wed. 1:15 – 4:00 PM, Singer 246
  - 2-3 students per lab group
  - Most labs will be computational
  - Possibly a few hardware labs

Topics

- Plan for class (order of topics we will cover)
  - Introduction to Digital Signal Processing (Chapter 1)
  - Review of Complex Variables
  - Review of Analog Signals & Systems
  - Sampling
  - Digital Signals and Systems (Chapter 2)
  - Z-Transform (Chapter 3)
  - Frequency Analysis of Signals (Chapter 4)
  - Frequency Analysis of LTI Systems (Chapter 5)
  - Discrete Fourier Transform (Chapter 7)
  - Design of Digital Filters (Chapter 10)
  - Special Topics if Time

Introduction to DSP

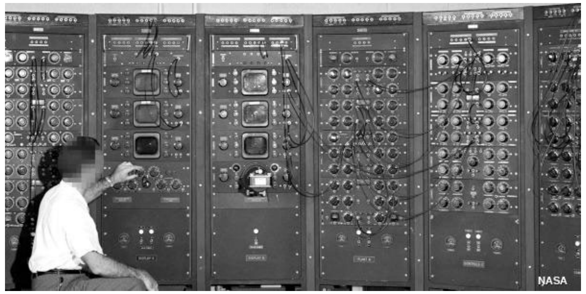
- Why is Digital Signal Processing Useful?
  - Practically every electronic device involves DSP
    - Delivery of content on internet
    - Speech recognition
      - Siri / Google Assistant
      - Irritating phone trees when you try to reach a human being
    - Audio processing / TV
      - Music recording/playback, video
    - Image processing
      - Enhancement, object recognition, compression
    - Telecommunications
      - Error correction, modulation/demodulation, software defined radio
    - Medical imaging
      - Almost all imaging modalities are now digital (CT, MRI, Ultrasound)

The Analog World

- In the good old days, life was analog



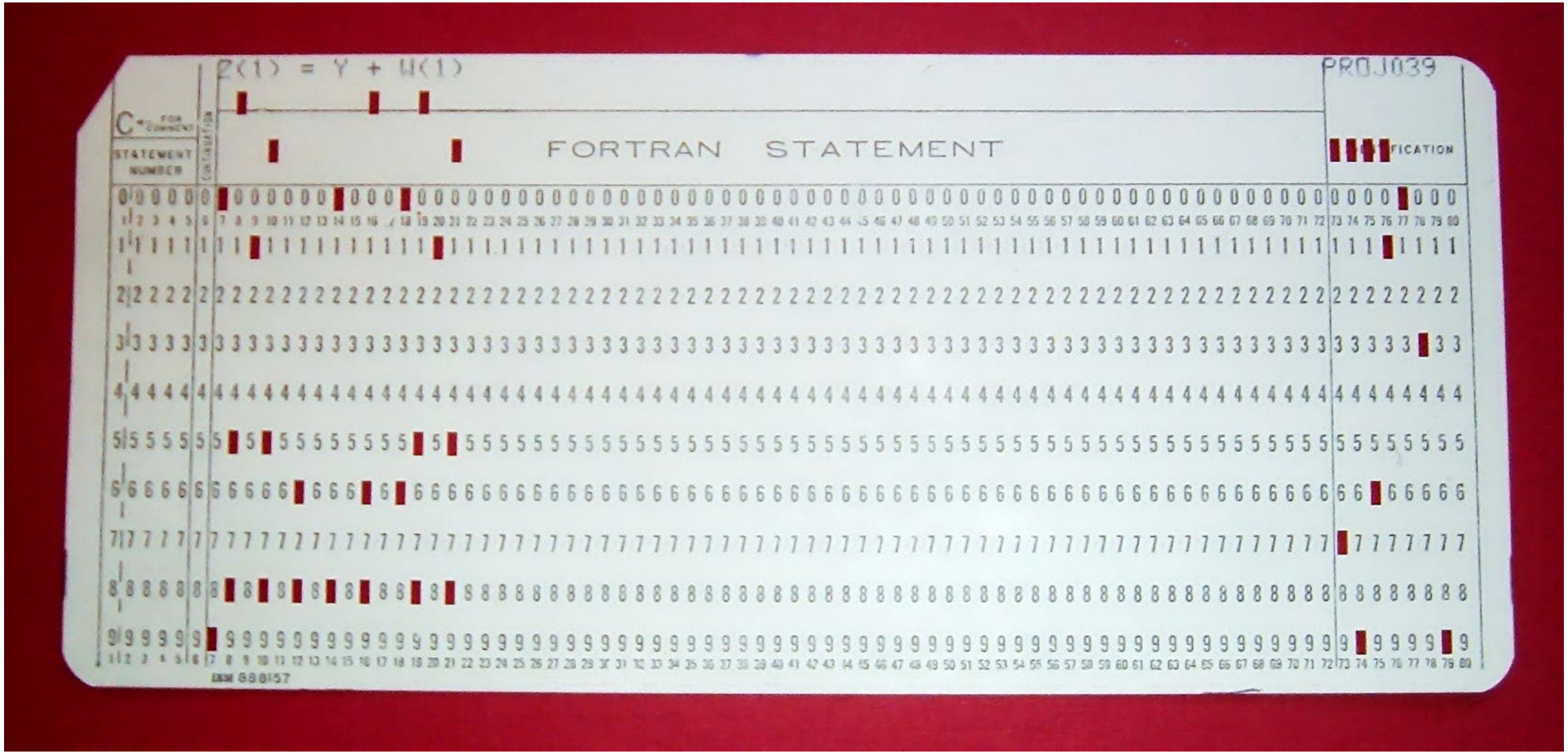


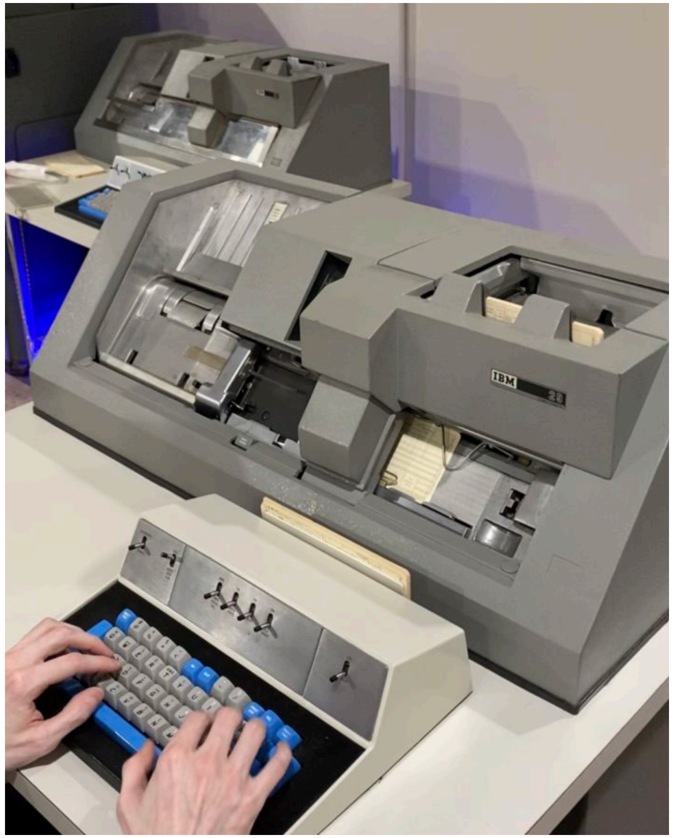
How did we get from analog to digital

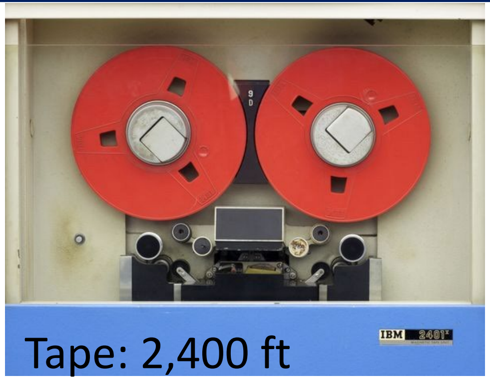
- How did we get here?
  - Conceptual advancements
    - Binary system (1679 - Leibniz)
    - Mathematical basis like Information Theory (1948 –Shannon)
  - Advancement in electronics
    - Transistors replaced vacuum tubes (Bardeen, Brattain, Shockley 1947)
    - Integrated circuits (Kilby - 1958)
    - Digital computers (1946 -ENIAC at Upenn)
  - Digital Storage
    - Punch cards and magnetic tape to disks→solid state drives→ cloud

How did we get from analog to digital



How did we get from analog to digital





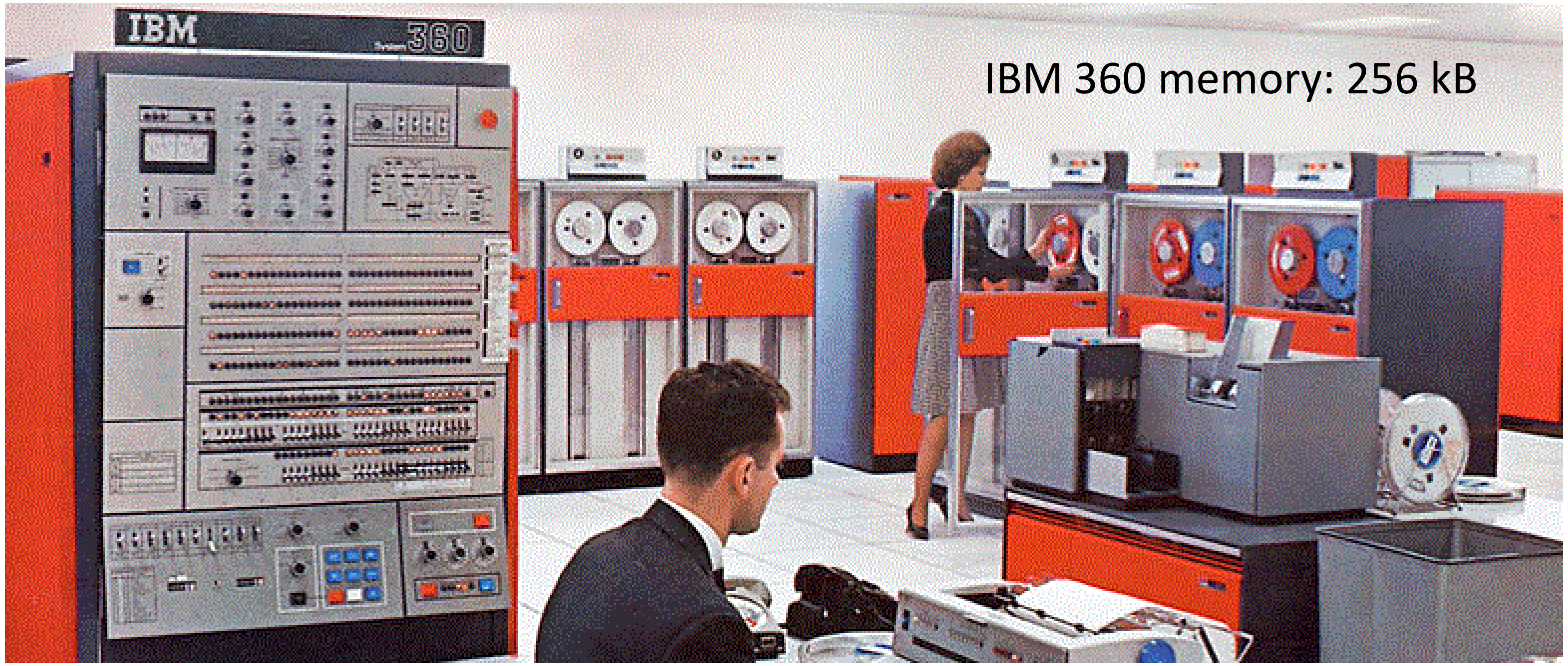
Tape: 2,400 ft

113 MB , 320 kB/sec.

washing machine size

Capacity: 7.25 MB

Transfer rate:
156 kB/sec.



IBM 360 memory: 256 kB

How did we get from analog to digital

- How did we get here?
  - Telecommunications
    - Analog to digital phone systems
    - Internet – relies on digital data transmission
  - Consumer electronics
    - Mobile phones analog to digital 2G, 3G ,…
    - Standards: Wi-Fi, Bluetooth
  - Digital economy

How did we get from analog to digital

- What drove the change
  - Efficiency
    - less susceptible to noise
  - Precision
    - error corrections
  - Economics
    - cost reduction associated with technology to mass produce integrated circuits and put more on a chip
  - Scalability
    - Easier to scale and upgrade

How did we get from analog to digital

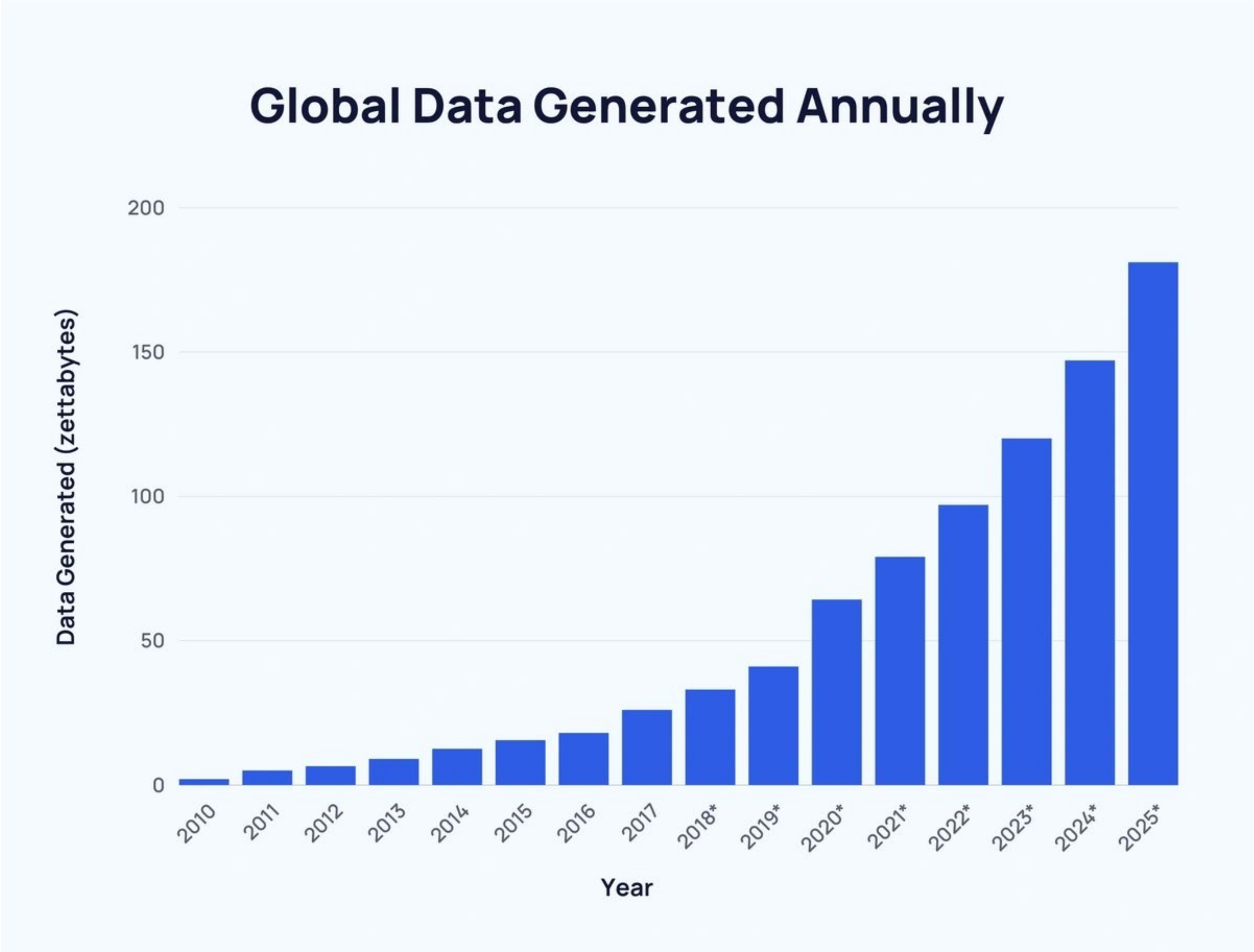
- What drove the change
  - Miniaturization
    - smaller device
    - portability
  - Global connectivity
    - Internet
  - Standardization
    - standardized formats allow devices to communicate and storage to be interpretable
    - Ease of duplication

How did we get from analog to digital

- What drove the change
  - Consumer demand
    - convenience, content consumption, price reduction
  - Funding
    - Government and Military
- Innovation builds on itself
    - With devices came development of algorithms (DSP)

The world is digital now

Data, Data, Data



402.74 million terabytes created per day

Doubles about every 2 years

By 2030 there will be 660 zettabytes
zettabyte is 1,000,000 terrabytes
(zetta is 10^{21})

16 TB solid state drives use ~3.5 Watts
150 GW to store all this data
(about 150 nuclear)

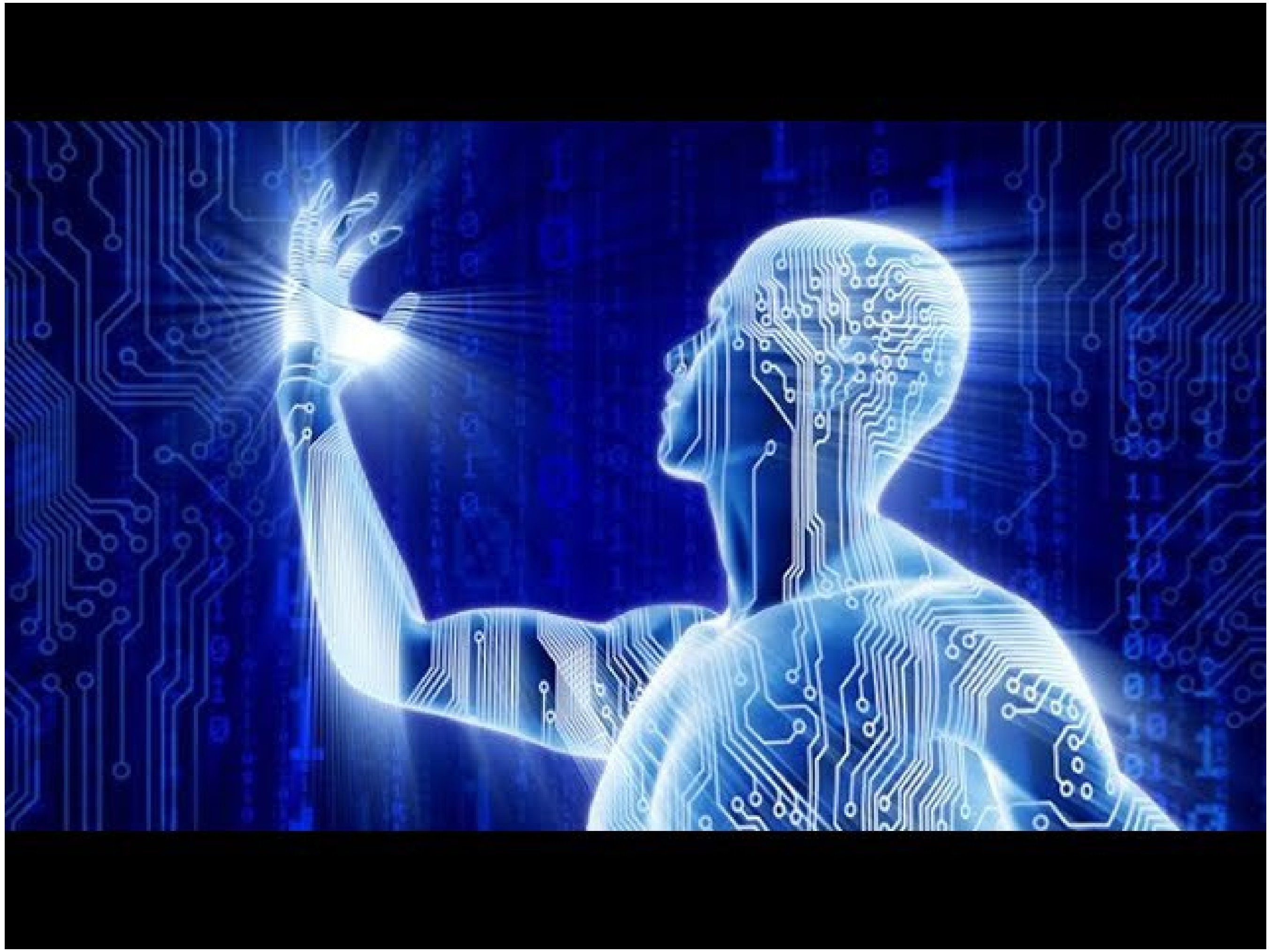
The next wave?

- Edge computing
  - More computation on low-level devices
    - Decentralized
    - low latency
    - bandwidth efficiency
    - security
- Internet of things
  - Everything will be connected
    - Smart homes
    - Supply chain
    - Optimized operations
- Decentralized technology
  - Blockchain
  - Digital currency

The next wave?

- AI and Machine Learning
  - Generative AI – music, text, … (these lectures?)
  - Autonomous systems
  - Personalization
- Quantum computing
  - Dramatic increase in computing power
    - New algorithms will have to be developed
- 5G and beyond
  - Faster connectivity, higher speeds
  - Metaverse – collection of shared virtual spaces
- Are we heading toward a technological singularity?
  - Technical growth becomes uncontrollable?
  - New hybrid/human species making analog humans obsolete?
  - Transhumanism?

Transhumanism

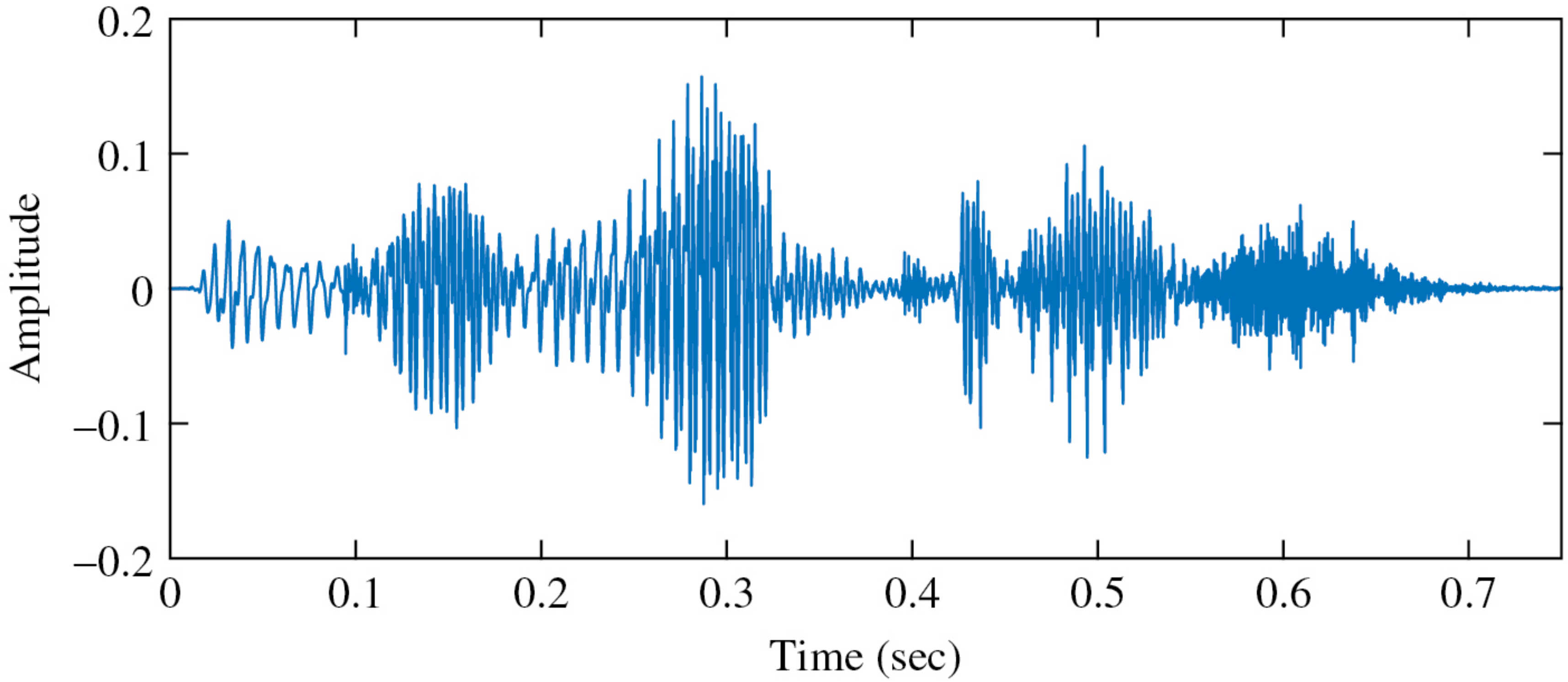


Signals

- **Signals**
  - Functions of 1 or more variables that carry useful information.
    - Typically, we think of 1-D signals as functions of time, but could be function of one spatial dimension
    - Example of 2-D signals are images
    - Higher dimensionality – multispectral data

Signals

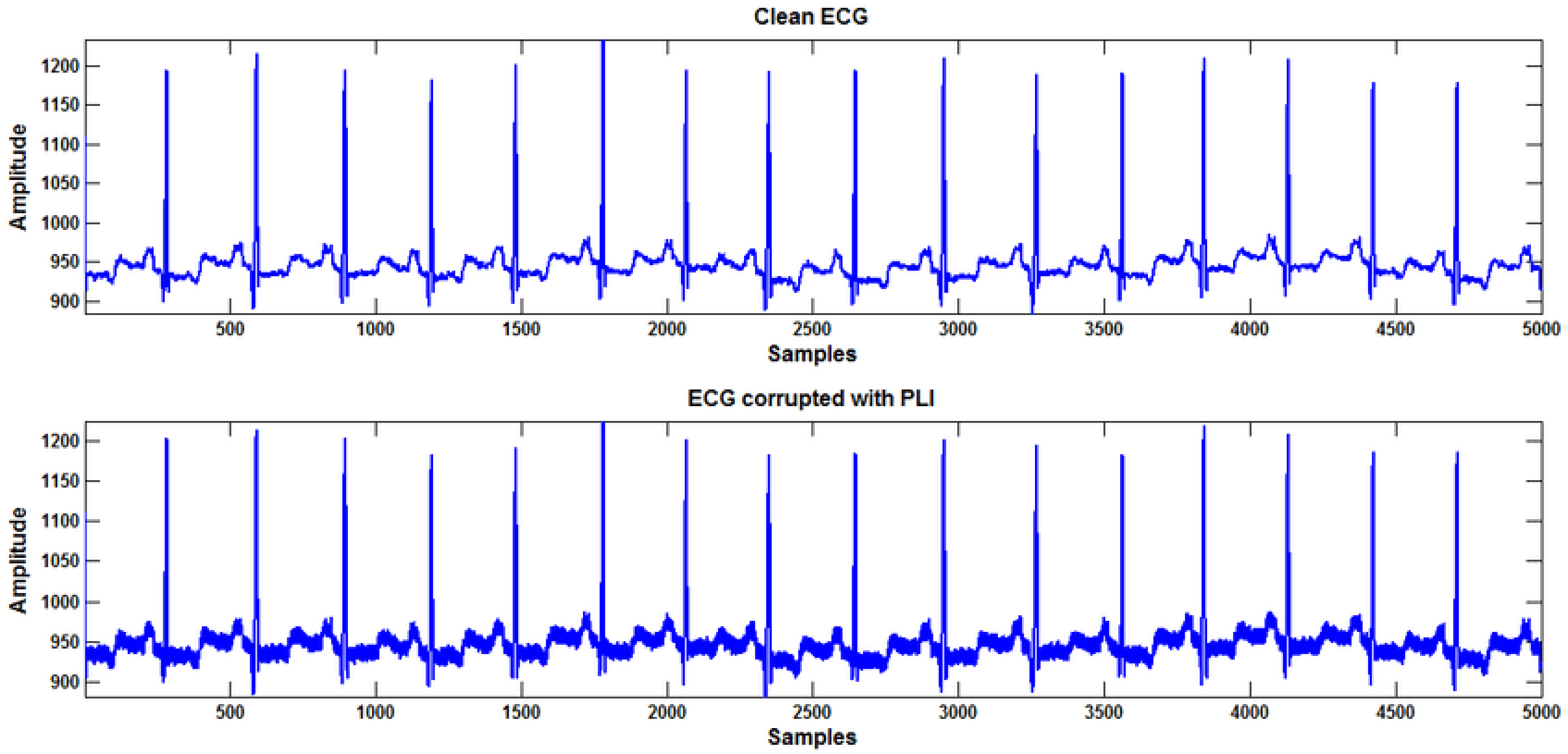
- Examples
  - Speech signal



Signals

– Examples

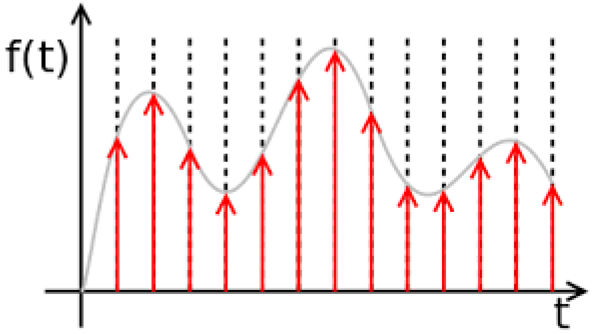
• Electrocardiogram



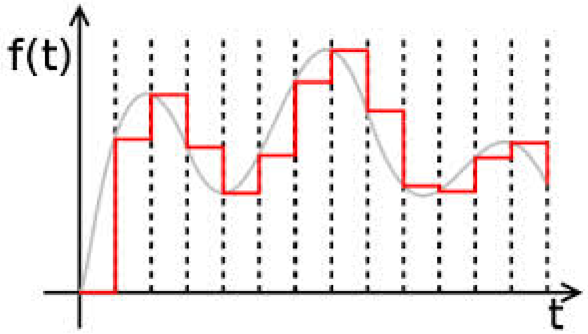
Signals

- What is a signal?
  - Ordered sequence of numbers (1-D)
    - e.g., sequence of amplitudes ordered by time
  - Ordered array of numbers
    - e.g., images (2 –D), volumes (3-D), higher dimensions
  - Types of signals (relevant to signal processing)
    - Continuous: time and amplitude are real numbers
    - Discrete: time sampled, amplitude is continuous
    - Digital: time sampled, amplitude quantized

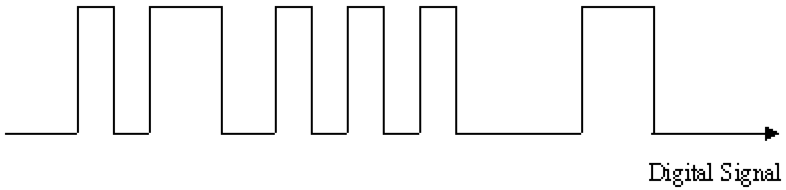
Signals



Continuous and Discrete signals



Digital signal



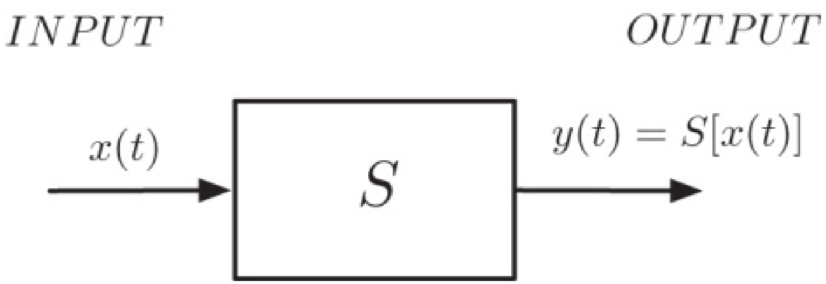
The term “digital signal”
is sometime used for encoded
sampled and quantized signals.

Systems

- What is a system?
  - Operates on signals.
    - May be realized in hardware or software or combination of the two
      - e.g. – Filtering to reduce noise
  - Systems have certain properties based on how they operate on signals
    - Linear
    - Non-linear systems

Systems

– System
  - Transforms input signal to output signal
  - Illustrated by "black box"



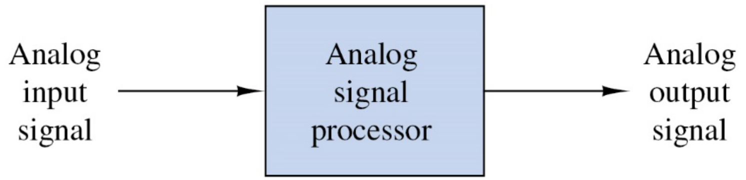
– Systems modeled as mathematical operations transforming input, $x(t)$, to output, $y(t)=S[x(t)]$

Systems

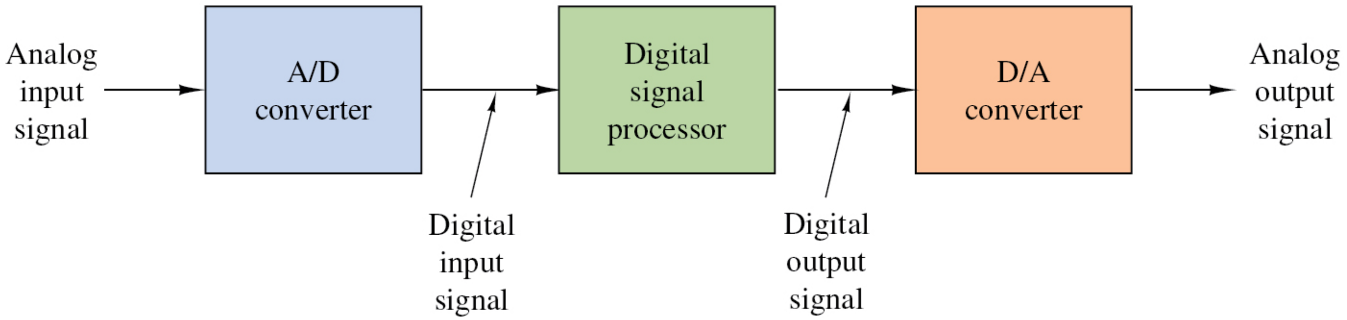
– System

- For digital signal processing, need to include conversion of analog signal to digital (and back again)

Analog signal processing:



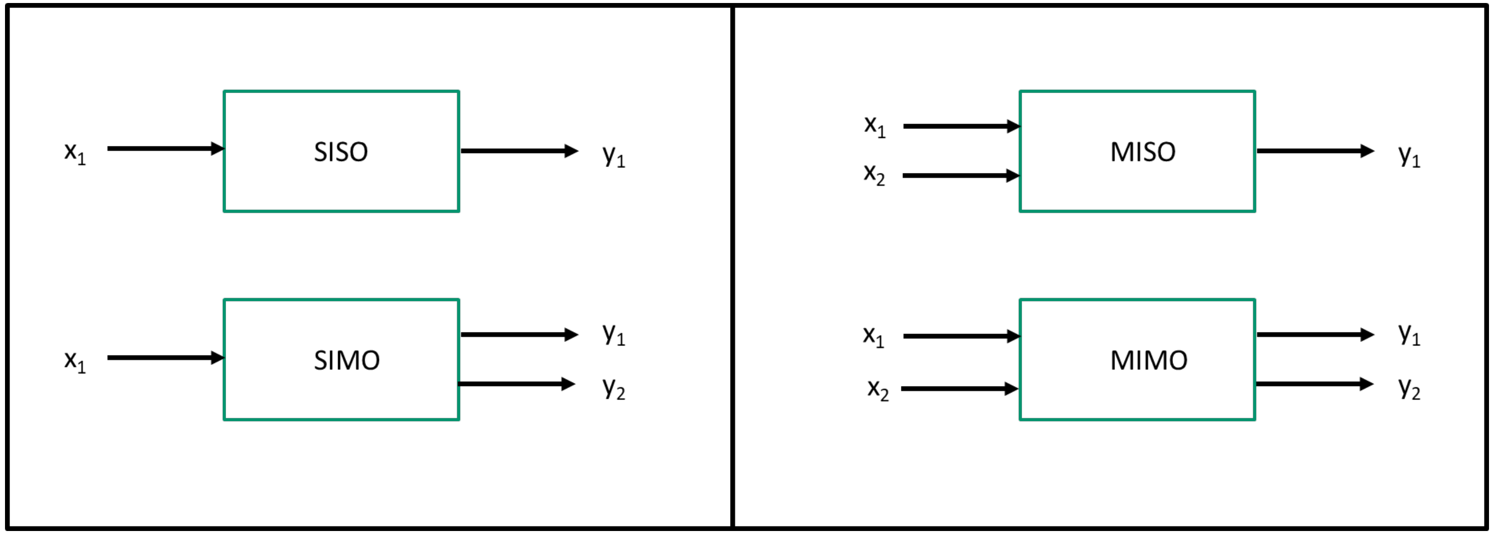
Digital signal processing:



Systems

– SISO and multivariate systems

- Single Input Single Output (SISO) is simplest
- Other possibilities include
  - Single Input Multiple Outputs (SIMO)
  - Multiple Input Single Output (MISO)
  - Multiple Input Multiple Outputs (MIMO)

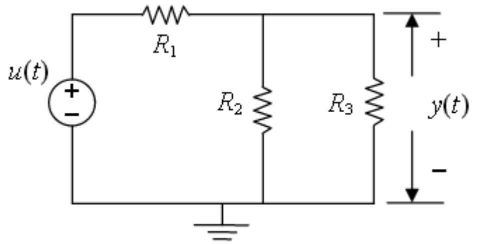


Classification of Systems

- Classification of systems
  - **Continuous-time**
    - Input and output signals are continuous time functions
  - **Discrete-time**
    - Input and output signals consist of sampled times
  - **Digital**
    - Inputs and outputs are discrete in time and amplitudes are quantized
  - **Hybrid**
    - Input and output signals can be mixed
    - Example Analog to Digital (A/D) converter

Classification of Systems

- **Static or Dynamic** (Also called **memoryless** or with **memory**)
  - Static system depend on the input at the present time
    - Example: resistive circuit excited by input voltage



$$y(t) = \frac{R_2 R_3}{R_1\left(R_2 + R_3\right) + R_2 R_3} u(t)$$

  - Dynamic system depends not only on the input at the current time, but also on the input at previous times.
    - Example would be circuits with capacitors and inductors

$$v_c(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau) d\tau$$

    - Another example would be a combination lock.
(i.e., needs to know two previous inputs plus present input to unlock.)

Classification of Systems

- **Causal Systems**
  - If output $y(t)$ at time t_0 only depends on input $x(t)$ for $t \leq t_0$, system is causal.
  - In other words, output can only be influenced by current input and what has happened before.
- **Non-Causal (or Acausal) Systems**
  - Can depend on current, previous, and <u>future</u> inputs
    - With buffers, you could include future inputs
      - Would not be real-time systems
    - For image data, non-causal systems are common
- **Anticausal Systems**
  - Depend only future inputs

Classification of Systems

- **Linear Systems**
  - If you scale the input to the system, the output scales by the same factor.
  - If you add two inputs and let the system operate on the inputs, the output is same as if you gave each input separately and summed the individual responses.
  - Mathematically:

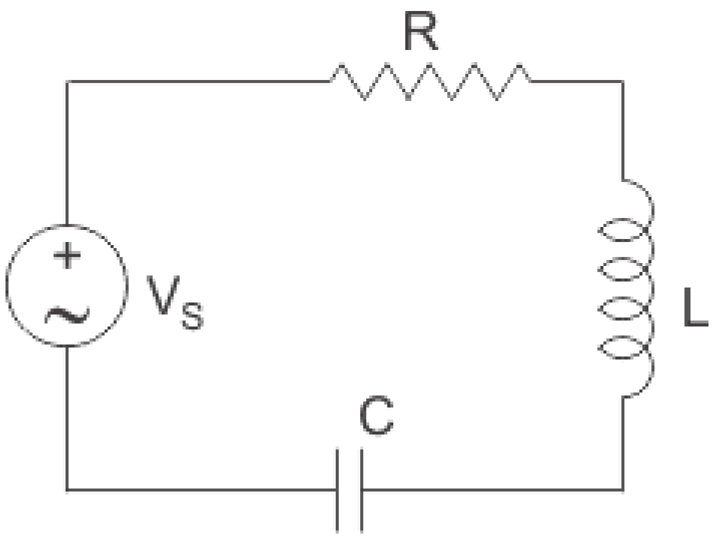
$$S[\alpha x(t) + \beta y(t)] = \alpha S[x(t)] + \beta S[y(t)]$$

  - If you superimpose two signals, output is superposition of two outputs.

Principle of superposition

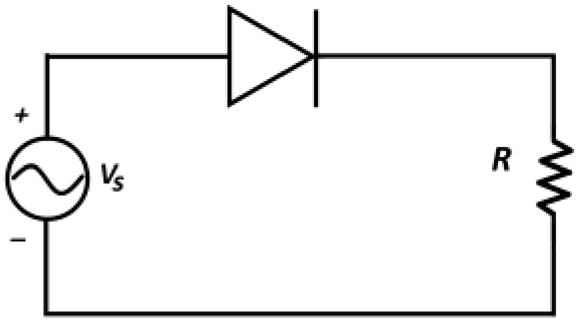
Classification of Systems

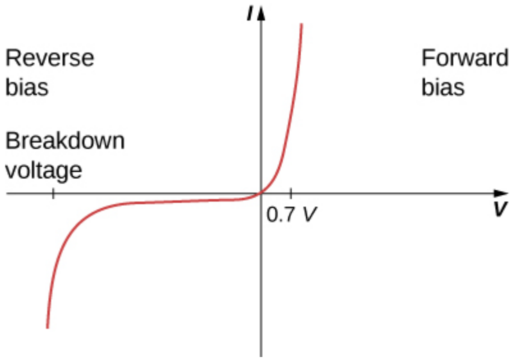
- Example of Linear and Non-linear Systems
  - RLC circuits (linear)



Classification of Systems

- Examples of Linear and Non-linear Systems
  - Diode circuits (non-linear)





Classification of Systems

- Examples of Linear and Non-linear Systems
  - RLC circuits (linear)
  - Diode circuits (non-linear)
  - Mass and Spring systems
    - linear if spring isn’t stretched too far
    - Non-linear if spring stretched beyond its elastic limit
  - No system is truly linear, but within some limited range of operation many systems can be treated as linear systems

Classification of Systems

- Mathematical examples of linear and non-linear systems

$$S[x(t)] = Ax(t)$$

$$S[x(t)] = x^2(t)$$

$$S[x(t)] = \sin\left(x(t)\right)$$

$$S[x(t)] = \frac{dx(t)}{dt} + ax(t)$$

Classification of Systems

- Mathematical examples of linear and non-linear systems

Linear

$$S[x(t)] = Ax(t)$$
$$S[ax(t) + by(t)] = A\alpha x(t) + A\beta y(t) = \alpha S[x(t)] + \beta S[y(t)]$$

Non-linear

$$S[x(t)] = x^2(t)$$
$$S[\alpha x(t) + \beta y(t)] = \left(\alpha x(t) + \beta y(t)\right)^2 = \alpha^2 x^2(t) + 2\alpha\beta x(t)y(t) + \beta^2 y^2(t) \neq \alpha S[x(t)] + \beta S[y(t)]$$

Non-linear

$$S[x(t)] = \sin\left(x(t)\right)$$
$$S[\alpha x(t) + \beta y(t)] = \sin\left(\alpha x(t) + \beta y(t)\right) = \sin\left(\alpha x(t)\right)\cos\left(\beta x(t)\right) + \cos\left(\alpha x(t)\right)\sin\left(\beta x(t)\right) \neq \alpha S[x(t)] + \beta S[y(t)]$$

Linear

$$S[x(t)] = \frac{dx(t)}{dt} + ax(t)$$
$$S[\alpha x(t) + \beta y(t)] = \frac{d\left(\alpha x(t) + \beta y(t)\right)}{dt} + a\left(\alpha x(t) + \beta y(t)\right)$$
$$= \frac{\alpha dx(t)}{dt} + \alpha a x(t) + \frac{\beta dy(t)}{dt} + \beta a y(t) = \alpha S[x(t)] + \beta S[y(t)]$$

Classification of Systems

- **Time Invariant**
  - Parameters of system do not change with time.
  - If you shift input time, output is shifted in same way
  - If input *x(t)* produces output *y(t)*,
    then for input at $x(t\text{-}t_0)$ the output produced would be $y(t\text{-}t_0)$
  - Examples
    - Capacitor is time invariant since:

$$v(t) = \frac{q(t)}{C} = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

If you consider input shifted by time t_0

$$v_{t_o}(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau - t_0)d\tau = \frac{1}{C}\int_{-\infty}^{t-t_0} i(\tau)d\tau = v(t - t_0)$$

- Example that is not time invariant:

$$y(t) = x(t) + \sin \omega t$$

Shifting input $x(t)$ by t_0:

$$y_{t_o}(t) = x(t - t_0) + \sin \omega t$$

but $y(t - t_0) \neq x(t - t_0) + \sin \omega(t - t_0)$

Classification of Systems

- **Stable Systems**
  - Any bounded input gives a bounded output.
- **Invertible Systems**
  - System is invertible if you can determine the input by observing the output.
  - An inverse system exists that could convert the output into the input.
- **Memoryless Systems**
  - Output at a given time depends only on input at that same time. (i.e., system doesn’t change with time)

Classification of Systems

- **Linear Time Invariant Systems**
  - Important class of systems
  - Can be represented by ordinary linear differential equation with constant coefficients.
  - Not all Linear D.E.'s with constant coefficients correspond to LTI systems
    - Must be causal and initially quiescent
  - What is so special about LTI systems?
    - LTI systems can be completely characterized by impulse response

Classification of Systems

High level classification of systems

- **Lumped or Distributed**
  - Lumped means elements of systems are localized and you need only consider the evolution of components in time.
    - Described by ordinary differential equations
    - Example is circuit with discrete elements (like R, L, C)
  - Distributed means system is distributed over space
    - Described by partial differential equations
    - Example is transmission lines
- **Passive or Active**
  - Passive systems can not deliver energy outside of system
    - Example: R-L-C circuits
  - Active systems can deliver energy outside of system
    - Example: Op Amp circuits

Review of Complex Numbers

Review of Complex Numbers

- What is i if $i^2 = -1$?
  - Definition for square root of -1: $i \triangleq \sqrt{-1}$ $\left(\text{or } j \triangleq \sqrt{-1}\right)$

    i is standard in mathematics, physics, most engineering fields

    j used in electrical engineering to avoid confusion with current
    - i is an **imaginary number**
  - **Complex numbers** (variables) have real and imaginary parts
    - **Rectangular form:**

      Complex number: $n = a + ib, \quad a = \text{Re}(n), \quad b = \text{Im}(n)$

      Complex variable: $z = x + iy, \; x = \text{Re}(z), \; y = \text{Im}(z) \quad [z \in \mathbb{C}, \quad x, y \in \mathbb{R}]$
    - **Polar form:**

      $z = r\angle\theta,$ magnitude: $|z| = r$, angle: θ

Argand Diagram

- Argand Diagram:
  - Complex number plotted as point in 2-D plane: **complex plane**
  - Conversion between rectangular and polar forms

$$z = x + iy \quad \Leftrightarrow \quad z = r\angle\theta$$

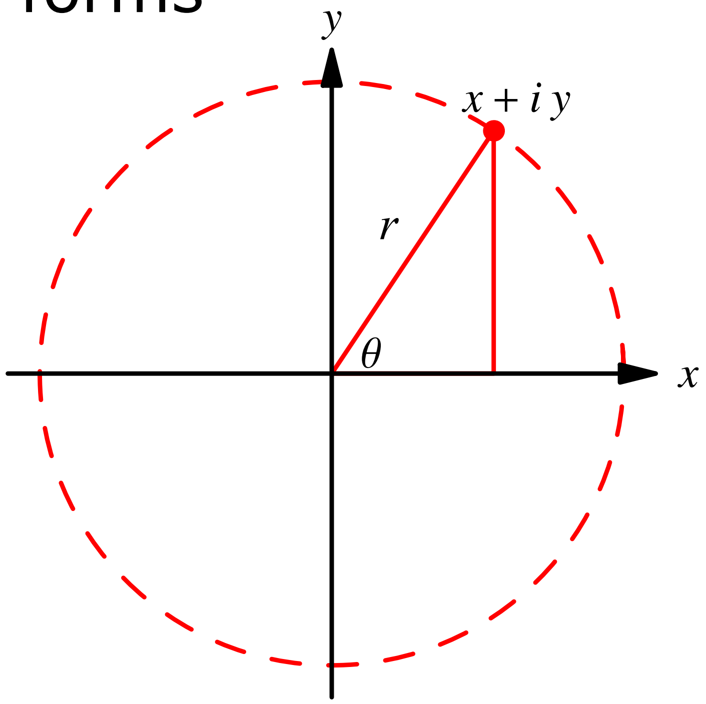
$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = r(\cos\theta + i\sin\theta)$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\frac{y}{x} \qquad (-\pi < \theta < \pi)$$



Examples

- Examples

$$z = 1 + i \quad \Leftrightarrow \quad \sqrt{2}\angle \pi/4$$

$$z = 1 + i\sqrt{3} \quad \Leftrightarrow \quad 2\angle \pi/3$$

$$z = 3 + i4 \quad \Leftrightarrow \quad 5\angle 0.9273$$

Complex Conjugation

- Complex Conjugation:
  - Reverse the sign of imaginary part.

    Complex conjugate of z: $z = x + iy$; $z = r\angle\theta$

    Denoted as $\bar{z}$ or z^*

    $z^* = x - iy$; $z^* = r\angle(-\theta)$
  - For functions, reverse sign of i everywhere in the function
    - Example:

      $f(t) = e^{it^2} + it$ (where t is a real-valued variable)

      $f^*(t) = e^{-it^2} - it$

Euler's Formula

- Euler's Formula:

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

  - There are several proofs of this formula

    Expand $e^{i\theta}$ in a Taylor series:

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \ldots$$

$$e^{i\theta} = \left[1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \ldots\right] + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^6}{6!} + \ldots\right] = \cos\theta + i\sin\theta$$

    recognizing that the first term in brackets is the Taylor series expansion for $\cos\theta$ and the second term is the series expansion for $\sin\theta$.

Euler's Formula

- Another proof:

Define the function: $f(\theta) = e^{-i\theta}\left(\cos\theta + i\sin\theta\right)$

$f'\left(\theta\right) = -ie^{-i\theta}\left(\cos\theta + i\sin\theta\right) + e^{-i\theta}\left(-\sin\theta + i\cos\theta\right)$

$f'\left(\theta\right) = \quad e^{-i\theta}\left(-i\cos\theta + \sin\theta - \sin\theta + i\cos\theta\right) = 0$

Since the derivative is zero, $f(\theta) = \text{constant}$.

Evaluating it at $\theta = 0$,

$f(0) = e^{0}\left(\cos 0 + i\sin 0\right) = 1$

so, $f(\theta) = \text{constant} = 1 = e^{-i\theta}\left(\cos\theta + i\sin\theta\right)$.

$$\boxed{\therefore\ \ e^{i\theta} = \cos\theta + i\sin\theta}$$

Euler's Identity

- Euler's Identity is found setting $\theta = \pi$

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$\boxed{e^{i\pi} + 1 = 0}$$

  - This is often called the most beautiful equation in mathematics
    - Links five fundamental mathematical constants: $e,\ i,\ \pi,\ 1,\ 0$
    - Also relates complex exponentiation with trigonometry

Polar Form of Complex Number

- Polar form of complex number using Euler's formula:

$$z = x + iy \quad \Leftrightarrow \quad z = r\angle\theta$$

$$x = r\cos\theta$$

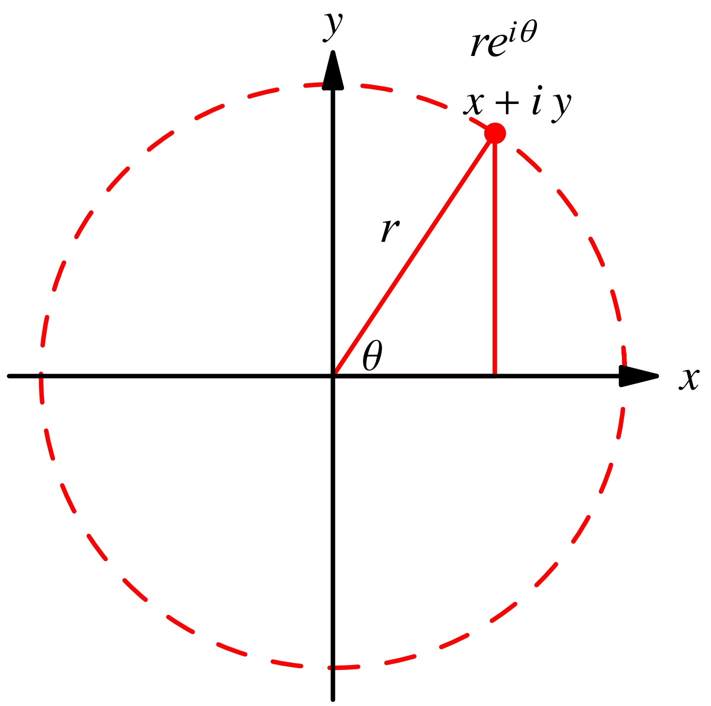
$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\frac{y}{x}$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = re^{i\theta}$$



Examples

- Example of rectangular to polar form

$$z = 1 - i \qquad \Leftrightarrow \qquad z = \sqrt{2}e^{-i\pi/4}$$

$$z = -1 + i\sqrt{3} \qquad \Leftrightarrow \qquad z = 2e^{i2\pi/3}$$

$$z = 3 + i4 \qquad \Leftrightarrow \qquad z = 5e^{i0.9273}$$

Examples

- Example of polar to rectangular form

$$z = 5e^{i\pi/3} \quad \Leftrightarrow \quad z = \frac{5}{2} + i\frac{5\sqrt{3}}{2}$$

$$z = e^{i3\pi/2} \quad \Leftrightarrow \quad z = 0 - i$$

$$z = e^{-i\pi/2} \quad \Leftrightarrow \quad z = 0 - i$$

$$z = e^{-i3\pi} \quad \Leftrightarrow \quad z = -1 + i0$$

Addition & Subtraction

- Addition (Subtraction)
  - Easiest to do addition (subtraction) in rectangular form:

$$z_1 = a_1 + ib_1 \quad \text{and} \quad z_2 = a_2 + ib_2$$

$$\begin{aligned} z_1 + z_2 &= \left(a_1 + ib_1\right) + \left(a_2 + ib_2\right) \\ &= \left(a_1 + a_2\right) + i\left(b_1 + b_2\right) \end{aligned}$$

Examples

- Examples

$$(1 - i) + (2 + i3) \quad \Leftrightarrow \quad 3 + i2$$

$$\left(1 + i\sqrt{3}\right) + \left(1 + i\sqrt{3}\right)^* \quad \Leftrightarrow \quad 2$$

$$\left(1 + i\sqrt{3}\right) - \left(1 + i\sqrt{3}\right)^* \quad \Leftrightarrow \quad i2\sqrt{3}$$

Multiplication & Division

- Multiplication $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$$

$$= a_1 a_2 + ib_1 a_2 + ia_1 b_2 + i^2 b_1 b_2$$

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

- Division

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)}{a_2 + ib_2}$$

$$= \frac{z_1}{z_2}\frac{z_2^*}{z_2^*} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{a_2^2 + b_2^2}$$

$$\frac{z_1}{z_2} = \frac{(a_1 a_2 + b_1 b_2) - i(a_1 b_2 - a_2 b_1)}{a_2^2 + b_2^2}$$

Examples

- Examples

$$(1-i)(1+i3) \quad \Leftrightarrow \quad 4+i2$$

$$(1+i)(1-i) \quad \Leftrightarrow \quad 2$$

$$\left(1+i\sqrt{3}\right)\left(1+i\sqrt{3}\right) \quad \Leftrightarrow \quad -2+i2\sqrt{3}$$

Examples

- Examples

$$\frac{1-i}{1+i3} \quad \Leftrightarrow \quad -\frac{1}{5} - i\frac{2}{5}$$

$$\frac{1+i}{1-i} \quad \Leftrightarrow \quad i$$

$$\frac{1+i2}{3-i4} \quad \Leftrightarrow \quad -\frac{1}{5} + i\frac{2}{5}$$

Multiplication & Division

- Easier to do multiplication and division in polar form

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Examples

- Examples

$$\left(1-i\right)\left(1+i\sqrt{3}\right) \qquad \sqrt{2}e^{-i\pi/4}2e^{i\pi/3} = 2\sqrt{2}e^{i\pi/12}$$

$$\left(1+i\right)\left(1-i\right) \qquad \sqrt{2}e^{i\pi/4}\sqrt{2}e^{-i\pi/4} = 2$$

$$\left(1+i\sqrt{3}\right)\left(1+i\sqrt{3}\right) \qquad 2e^{i\pi/3}2e^{i\pi/3} = 4e^{i2\pi/3} = 2\left(-1+i\sqrt{3}\right)$$

Examples

- Examples

$$\frac{1-i}{1+i\sqrt{3}} \qquad \frac{\sqrt{2}e^{-i\pi/4}}{2e^{i\pi/3}} = \frac{1}{\sqrt{2}}e^{-i7\pi/12}$$

$$\frac{1+i}{1-i} \qquad \frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{-i\pi/4}} = e^{-i\pi/2} = -i$$

$$\frac{1+i2}{3-i4} \qquad \frac{\sqrt{5}e^{i\tan^{-1}(2)}}{5e^{-i\tan^{-1}(4/3)}} = \frac{1}{\sqrt{5}}e^{i\left(\tan^{-1}(2)+\tan^{-1}(4/3)\right)} = \frac{1}{5}\left(-1+i2\right)$$

Power & DeMoivre's Formula

- Powers of complex numbers

$$z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$$

- DeMoivre's formula comes from the relationship between complex exponential and trigonometric function

$$e^{in\theta} = \left(\cos(\theta) + i\sin(\theta)\right)^n = \cos(n\theta) + i\sin(n\theta)$$

Examples

- Examples

$$(1-i)^3 \qquad \left[\sqrt{2}e^{-i\pi/4}\right]^3 = 2^{3/2}e^{-i3\pi/4} = -2(1+i)$$

$$(1-i)^4 \qquad \left[\sqrt{2}e^{-i\pi/4}\right]^4 = 2^2 e^{-i4\pi/4} = -4$$

$$i^5 \qquad \left[e^{-i\pi/2}\right]^5 = e^{-i5\pi/2} = i$$

Roots of Complex Numbers

- Roots of complex numbers $z^{\frac{1}{n}} = \left(re^{i\theta}\right)^{\frac{1}{n}}$
  - Must be careful, because there will be *n* roots
  - Note that $e^{i\theta} = e^{i(\theta+2k\pi)}$ for integer values of *k*

    You can see this from

    $$e^{i(\theta+2k\pi)} = \cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi) = \cos(\theta) + i\sin(\theta)$$

    $$z^{\frac{1}{n}} = \left(re^{i\theta}\right)^{\frac{1}{n}} = \left(re^{i(\theta+2k\pi)}\right)^{\frac{1}{n}} = r^{\frac{1}{n}}e^{i\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)}$$

    n roots since this yields unique values for $k = 0, 1, \ldots, n-1$

Examples

- Examples

$$\sqrt[3]{1-i} \qquad \left[\sqrt{2}e^{-i(\pi/4+k2\pi)}\right]^{1/3} = 2^{1/6}e^{i7\pi/12}\ , \quad 2^{1/6}e^{i15\pi/12}\ , \quad 2^{1/6}e^{i23\pi/12}$$

$$i^{1/4} \qquad \left[e^{i(\pi/2+k2\pi)}\right]^{1/4} = e^{i\pi/8}\ , \quad e^{i5\pi/8}\ , \quad e^{i9\pi/8}\ , \quad e^{i13\pi/8}$$

$$\sqrt{1} \qquad \left[e^{i(0+k2\pi)}\right]^{1/2} = e^{0}\ , \quad e^{i\pi} = 1\ , \quad -1$$

$$1^{1/3} \qquad \left[e^{i(0+k2\pi)}\right]^{1/3} = e^{0}\ , \quad e^{i2\pi/3}\ , \quad e^{i4\pi/3}$$

Matlab Commands

- Matlab commands for complex values

abs	Absolute value and complex magnitude
angle	Phase angle
complex	Create complex array
conj	Complex conjugate
cplxpair	Sort complex numbers into complex conjugate pairs
i	Imaginary unit
imag	Imaginary part of complex number
isreal	Determine whether array uses complex storage
j	Imaginary unit
real	Real part of complex number
sign	Sign function (signum function)
unwrap	Shift phase angles

A few odds and ends to remember

- A few odds and ends to remember:

$$i^2 = -1$$

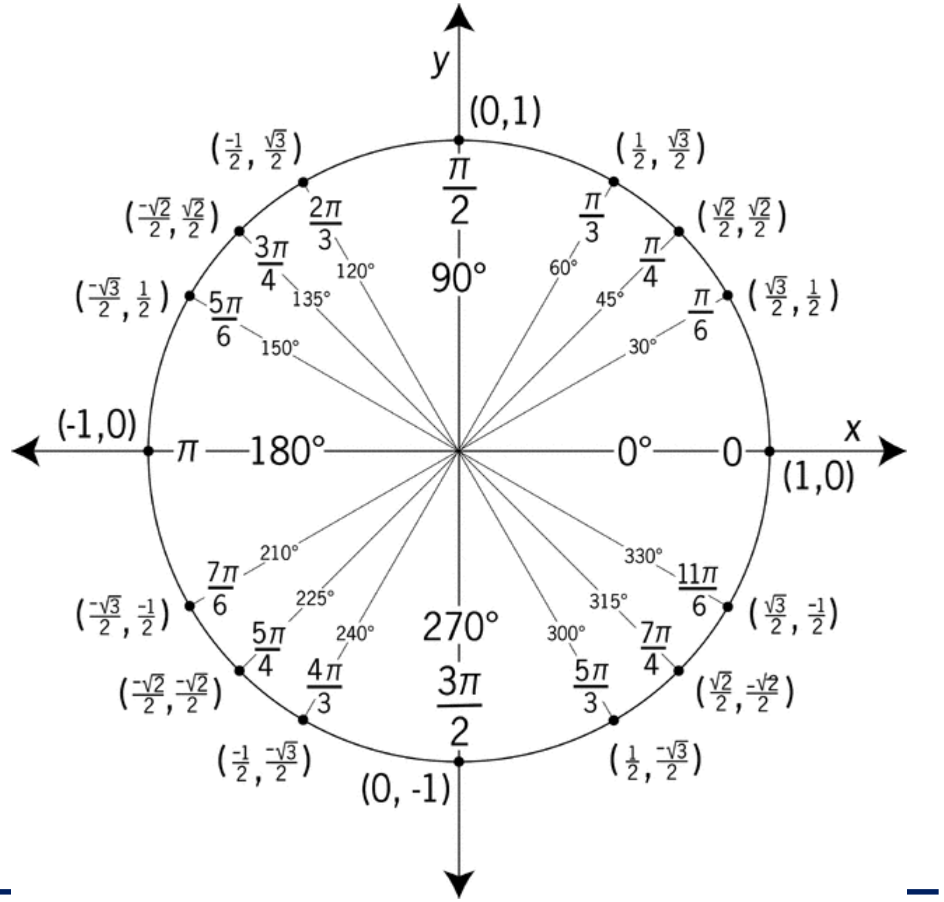
$$e^{\pm i\pi} = -1$$

$$e^{i2\pi k} = 1 \quad \text{(for integer } k\text{)}$$

$$e^{\pm i\frac{\pi}{2}} = \pm i$$

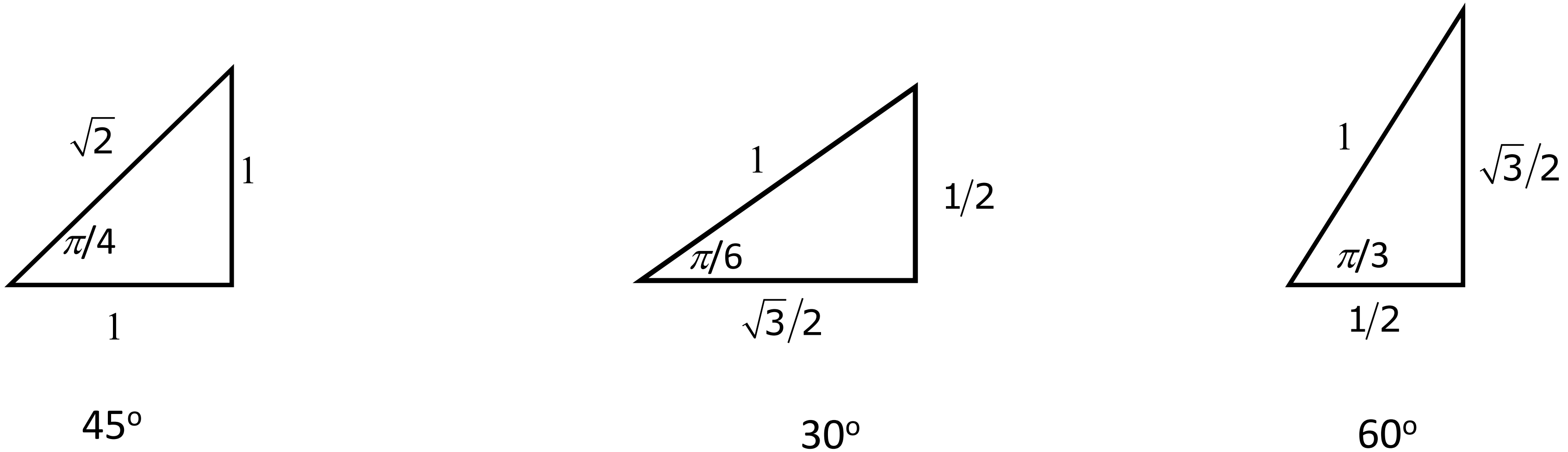
$$-i = \frac{1}{i}$$

Points on the unit circle in complex plane



A few odds and ends to remember

- Convenient triangles:



A few odds and ends to remember

- Geometric Series:

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

$$\sum_{k=1}^{n} r^k = \frac{r\left(1 - r^n\right)}{1 - r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}$$

for $|r| < 1$

$$\sum_{k=1}^{\infty} r^k = \frac{r}{1 - r}$$

for $|r| < 1$

Examples

- Example using geometric series:
  a) If you drop a ball from 20 meters, and it recovers ¾ of its height at each bounce (coefficient of restitution), how far has the ball travelled after 10 bounces?

$$T_n = H + 2Hr\sum_{k=0}^{n-1} r^k = 20 + 2 \bullet 20 \bullet (3/4)\frac{1-(3/4)^n}{1-(3/4)}, \quad T_{10} = 133.2424$$

  b) If you let it bounce forever (and it is a perfect ball), how far will it travel?

$$T_\infty = H + 2Hr\sum_{k=0}^{\infty} r^k = 20 + 2 \bullet 20 \bullet (3/4)\frac{1}{1-(3/4)}, \quad T_\infty = 140$$