Z-Transform Example with Complex Poles

Example showing how to get impulse response for the more complicated transfer function we started in class.

Transfer function: $H(z)=\dfrac{z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2}}$

Find the partial fraction expansion of $H(z)$ in a form we can use to find the inverse z-transform:

$$H(z)=\frac{z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2}}=\frac{1}{z^{2}-\frac{1}{2}z+\frac{1}{2}}$$

$$\frac{H(z)}{z}=\frac{1}{z\left(z^{2}-\frac{1}{2}z+\frac{1}{2}\right)}=\frac{A}{z}+\frac{B}{z-p_1}+\frac{C}{z-p_2}$$

$$\text{Roots of } \; z^{2}-\tfrac{1}{2}z+\tfrac{1}{2}=0: \quad z=\frac{\tfrac{1}{2}\pm\sqrt{\left(\tfrac{1}{2}\right)^{2}-4\left(\tfrac{1}{2}\right)}}{2}=\frac{1\pm i\sqrt{7}}{4}$$

$$\text{so: } \mathrm{p}_1=\frac{1+i\sqrt{7}}{4},\; \mathrm{p}_2=\frac{1-i\sqrt{7}}{4} \quad ,\; \text{define: } p\equiv p_1=\frac{1+i\sqrt{7}}{4},\; p_2=p^{*}$$

$$p=|p|e^{i\theta}\;,\quad |p|=\frac{\sqrt{1+7}}{4}=\frac{1}{\sqrt{2}}\,,\quad \theta=\tan^{-1}\left(\sqrt{7}\right)$$

Find $A,\ B,$ and C :

$$\frac{A}{z}+\frac{B}{z-p}+\frac{C}{z-p^{*}}=\frac{A(z-p)\left(z-p^{*}\right)+Bz\left(z-p^{*}\right)+Cz(z-p)}{z(z-p)\left(z-p^{*}\right)}=\frac{1}{z(z-p)\left(z-p^{*}\right)}$$

$$\text{Set } z=0: A(0-p)\left(0-p^{*}\right)=A|p|^{2}=1 \quad\Rightarrow\quad A=\frac{1}{|p|^{2}}=\frac{4}{\left(1+i\sqrt{7}\right)}\frac{4}{\left(1-i\sqrt{7}\right)}=\frac{6}{8}=2$$

$$\text{Set } z=p:\; Bp\left(p-p^{*}\right)=2Bp\,\mathrm{Im}(p)=1 \quad\Rightarrow\quad B=\frac{16}{2\left(1+i\sqrt{7}\right)i\sqrt{7}}=\frac{\left(1-i\sqrt{7}\right)}{i\sqrt{7}}=-\left(1+\frac{i}{\sqrt{7}}\right)$$

$$B=-\sqrt{\frac{8}{7}}e^{i\phi}\;,\quad \phi=\tan^{-1}\left(\frac{1}{\sqrt{7}}\right)$$

$$\text{Set } z=p^{*}: Cp^{*}\left(p^{*}-p\right)=-2Cp^{*}\,\mathrm{Im}(p)\Rightarrow\quad C=\frac{-16}{2\left(1-i\sqrt{7}\right)i\sqrt{7}}=\frac{-\left(1+i\sqrt{7}\right)}{i\sqrt{7}}=-\left(1-\frac{i}{\sqrt{7}}\right)$$

$$C=-\sqrt{\frac{8}{7}}e^{i\varphi}\;,\quad \varphi=\tan^{-1}\left(\frac{-1}{\sqrt{7}}\right)=-\phi\Rightarrow\quad C=B^{*}$$

Find the inverse z-transform of $H(z)$:

$$H(z) = A + B\frac{z}{z-p} + C\frac{z}{z-p^*}$$

$$h[n] = A\delta[n] + \left[B\left(p\right)^n + C\left(p^*\right)^n\right]u(n)$$

$$h[n] = 2\delta[n] - \frac{2\sqrt{2}}{\sqrt{7}}\left[e^{i\phi}\left(\frac{1}{\sqrt{2}}\right)^n e^{in\theta} + e^{-i\phi}\left(\frac{1}{\sqrt{2}}\right)^n e^{-in\theta}\right]u(n)$$

$$h[n] = 2\delta[n] - \frac{2\sqrt{2}}{\sqrt{7}}\left(\frac{1}{\sqrt{2}}\right)^n 2\cos\left(n\theta + \phi\right)u[n]$$

Note that

$$\phi = \tfrac{\pi}{2} - \theta$$

$$\cos\left(n\theta - \theta + \tfrac{\pi}{2}\right) = -\sin\left((n-1)\theta\right)$$

$$h[n] = 2\delta[n] + \frac{4}{\sqrt{7}}\left(\frac{1}{\sqrt{2}}\right)^{(n-1)} \sin\left((n-1)\theta\right)u[n]$$

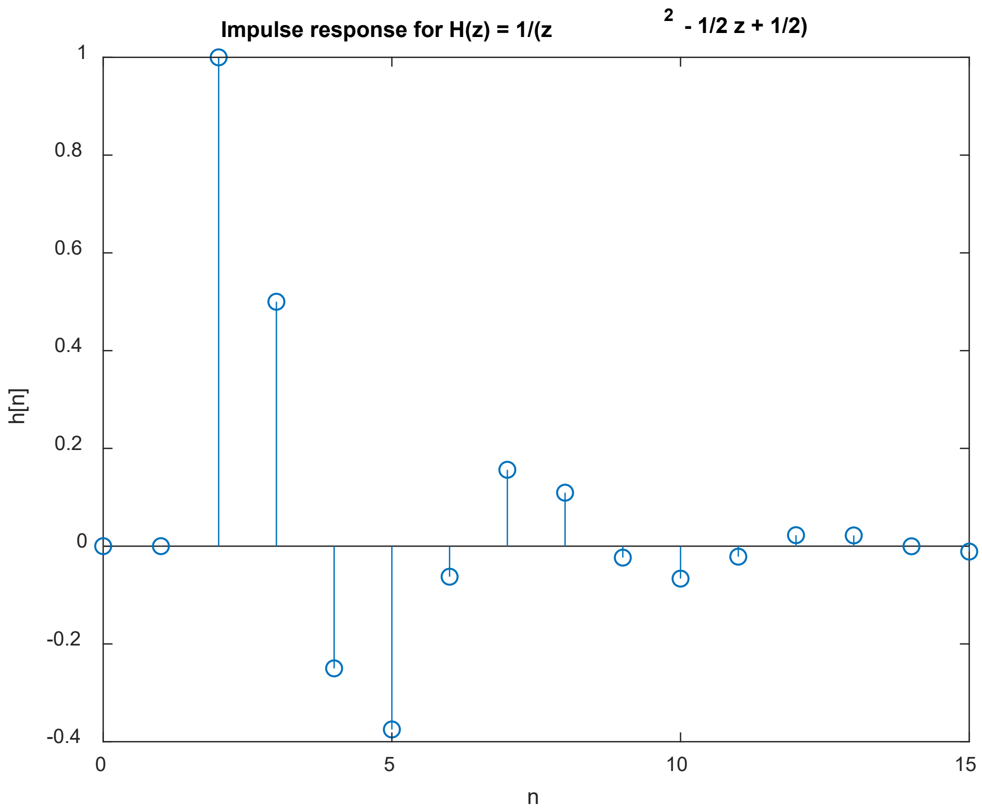
The final result is:

$$\boxed{\begin{array}{l} h[n] = 2\left[\delta[n] + \dfrac{2}{\sqrt{7}}\left(\dfrac{1}{\sqrt{2}}\right)^{(n-1)} \sin\left((n-1)\theta\right)u[n]\right], \\ \text{where} \\ \theta = \tan^{-1}\left(\sqrt{7}\right) \end{array}}$$

There is probably some way to get to this using the z-transform pair:

$$r^n \sin(\omega_0 n)u[n] \Leftrightarrow \frac{1 - r\sin(\omega_0 n)z^{-1}}{1 - 2r\cos(\omega_0 n)z^{-1} + r^2 z^{-2}}$$

but, as complicated as the method shown is, I think it's simpler.



Note the first two outputs are zero, which you expect from the difference equation for this system:

$$y[n]=\frac{1}{2}y[n-1]-\frac{1}{2}y[n-2]+x[n-2]$$