Digital Signal Processing

Class 26
04/24/2025

ENGR 71

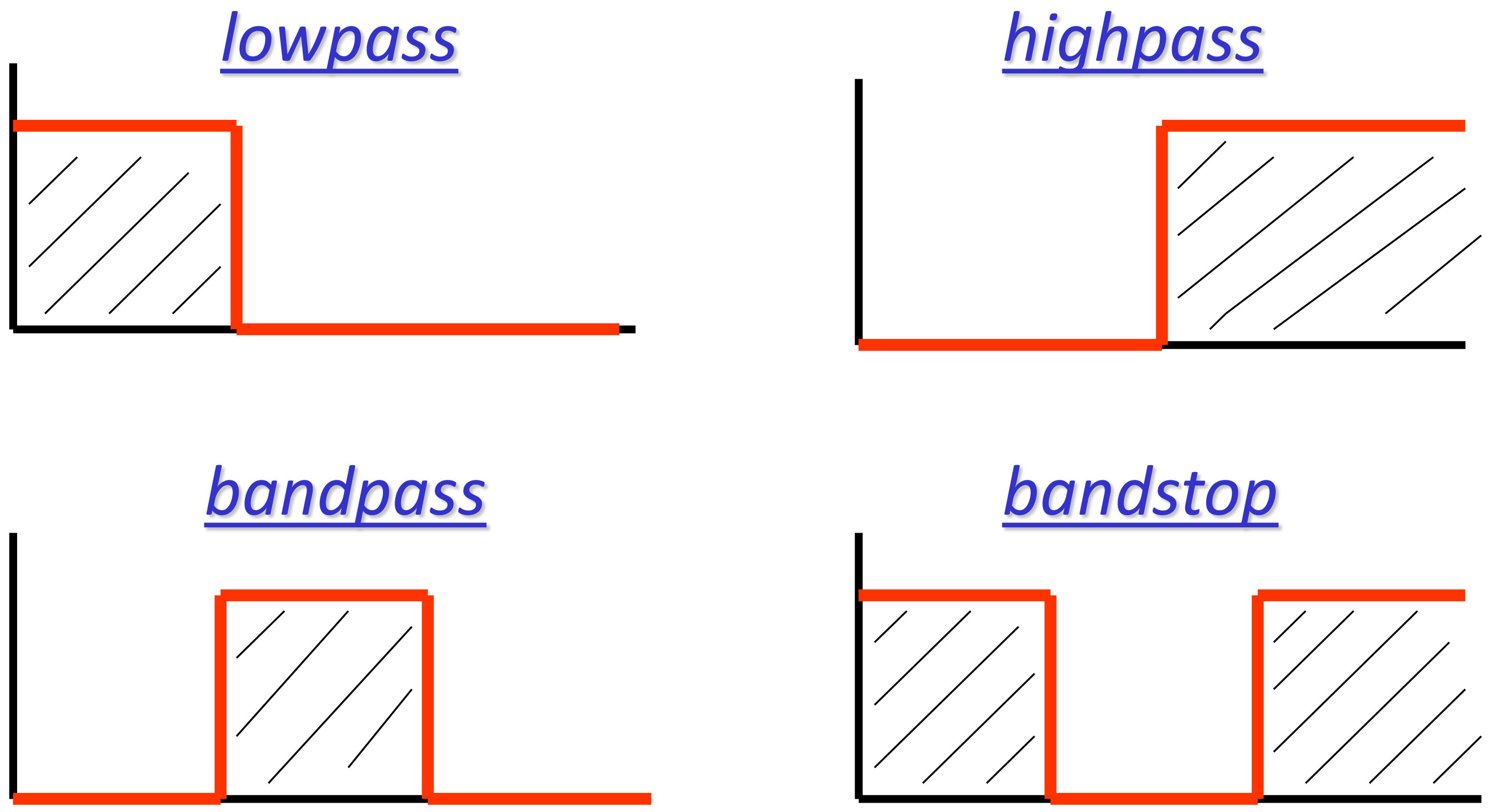
- Class Overview
  - Example for FIR frequency-sampling
  - Frequency transformations for IIR filters
    - Lowpass → highpass, bandpass, & bandstop filters
- Assignments
  - Reading:
    Chapter 10: Design of Digital Filters
    https://www.mathworks.com/help/signal/ug/fir-filter-design.html
  - Problems: Will be on Moodle page this afternoon
    - Due Date: None (will not need to be submitted)
  - Lab 3: “Fun with Filters”
    - Due May 4 (Sunday)

Project

- Projects
  - You can work in groups if you wish
  - Presentation at time reserved for final exam
    - Friday, May 9, 7:00-10:00 PM
    - Science Center 264
  - Submit a brief description of your proposed project and who you will be working with to the Project Description Dropbox
  - Submit slides from presentation to Project Dropbox
  - Submit written report to Project Dropbox by end of semester (May 15)

Filters

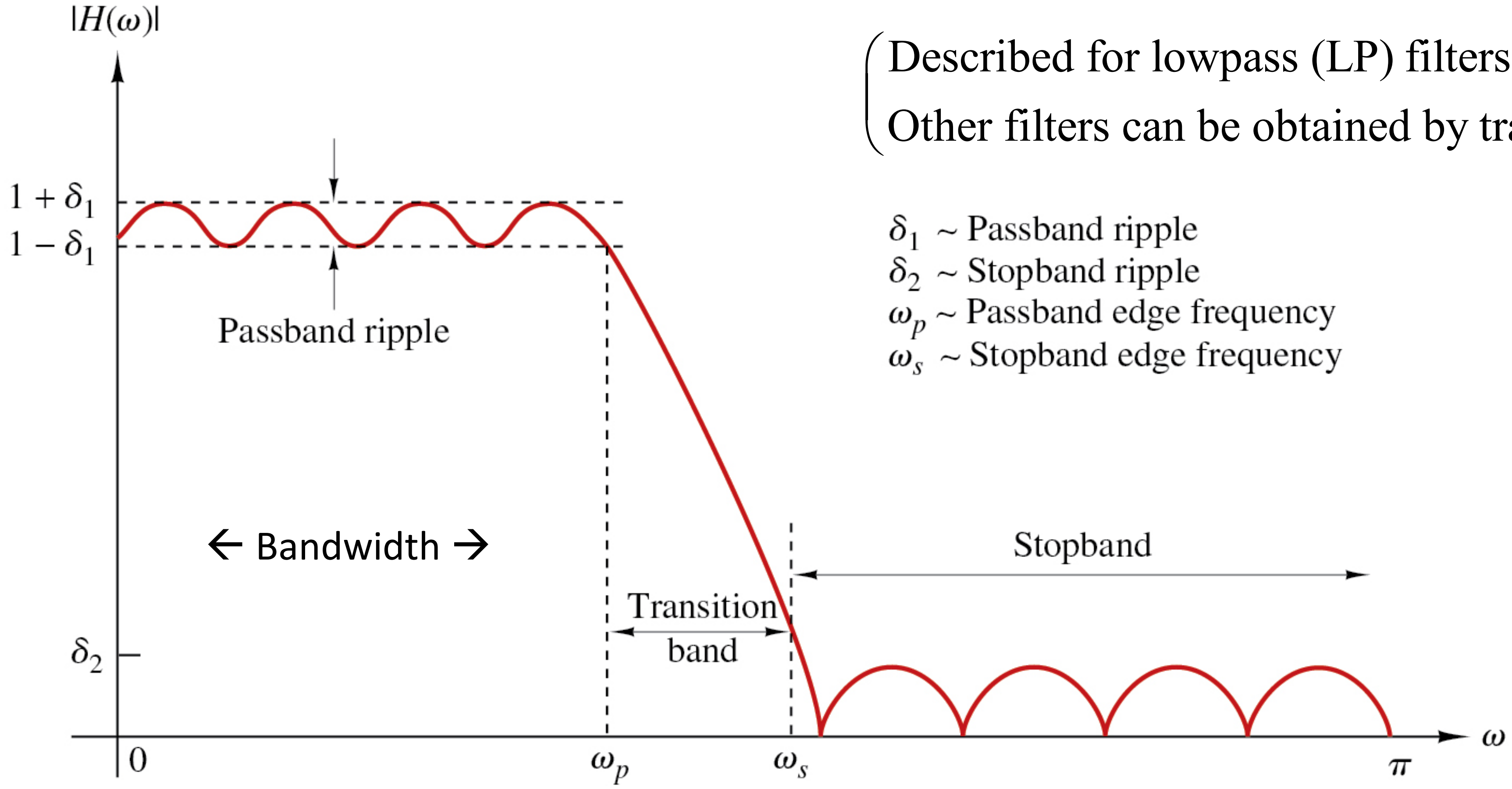
- Design of Digital Filters



Filter Design

- Specifications for physically realizable filters:

Described for lowpass (LP) filters.
Other filters can be obtained by transforming LP filters



Filters

- Design of Digital Filters
  - Finite Impulse Response (FIR)
    - Used when linear-phase required in passband
      - Constant time delay
      - No dispersion as a function of frequency
    - Three methods discussed
      - Windows, Frequency sampling, Iterative method for optimum equiripple filters
  - Infinite Impulse Response (IIR)
    - No requirement on linear-phase
    - Better characteristics for fewer parameters
    - Less memory
    - Lower computational complexity

Analog Filters

- Analog Filters
  - Four types of common analog filters
    - Butterworth
    - Chebyshev Type I
    - Chebyshev Type II
    - Elliptic

Matlab filter functions

Butterworth Filter

- Butterworth Filter
  - Magnitude is maximally flat at the origin and no ripples in either the passband or stopband
  - Magnitude changes monotonically with frequency
  - Compared to other types, has a slower roll-off
  - All pole filter
  - Frequency response of N'th order Butterworth filter

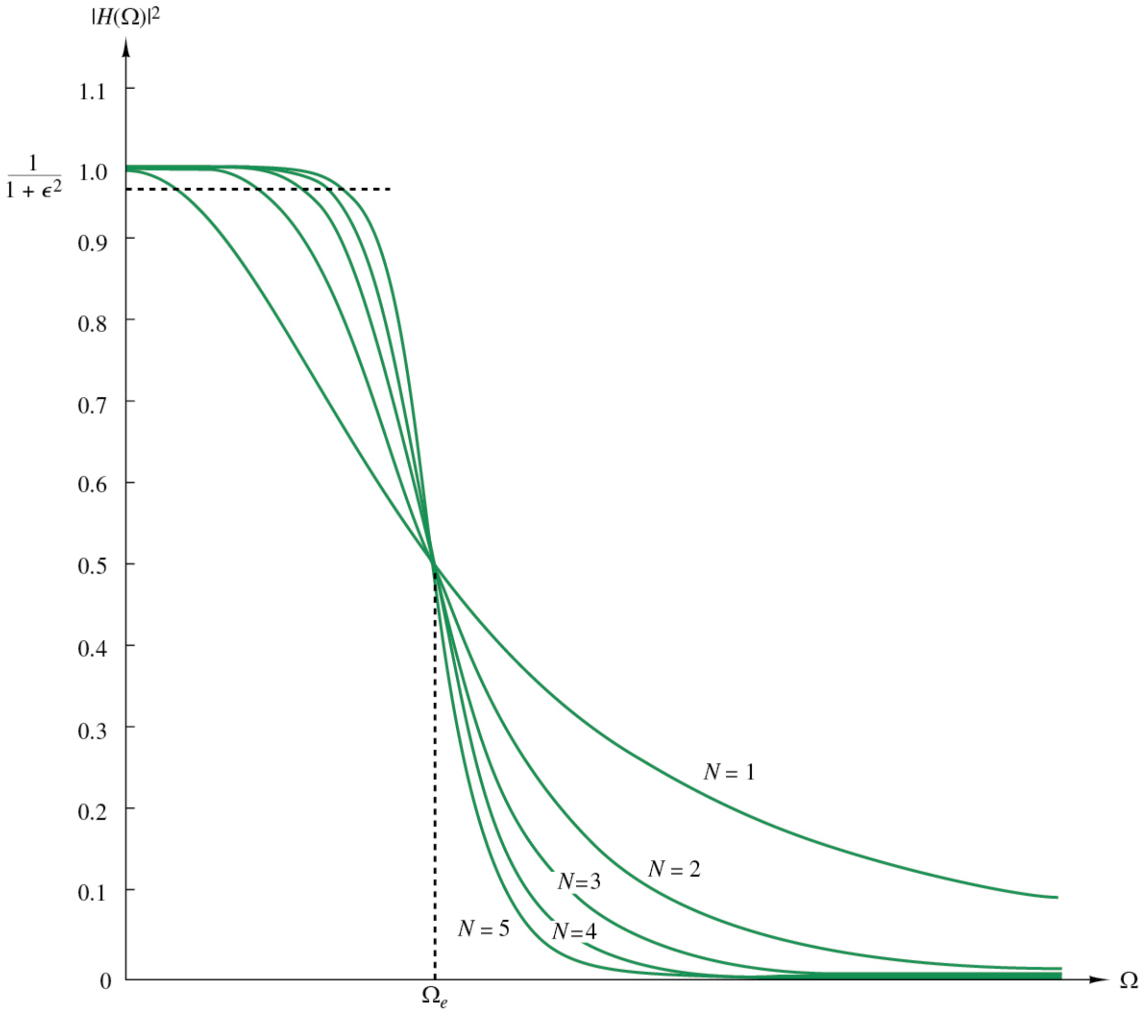
$$\left|H(\Omega)\right|^2 = \frac{1}{1+\left(\Omega/\Omega_c\right)^{2N}} = \frac{1}{1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}}$$

Ω_c is cut-off frequency

Ω_p is passband frequency

$1/\left(1+\varepsilon^2\right)$ is band-edge value frequency

Butterworth Filter



Chebyshev Filters

- Chebyshev Filters
  - Two types
    - Type I: all pole filter that has equiripple in passband, monotonic in stopband
    - Type II: poles & zeros. Monotonic in passband, equiripple in stopband

Chebysev Type I:

$$\left|H(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\Omega/\Omega_p\right)}$$

Chebysev Type II:

$$\left|H(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2\left[\dfrac{T_N^2\left(\Omega_s/\Omega_p\right)}{T_N^2\left(\Omega_s/\Omega\right)}\right]}$$

Ω_p is passband frequency

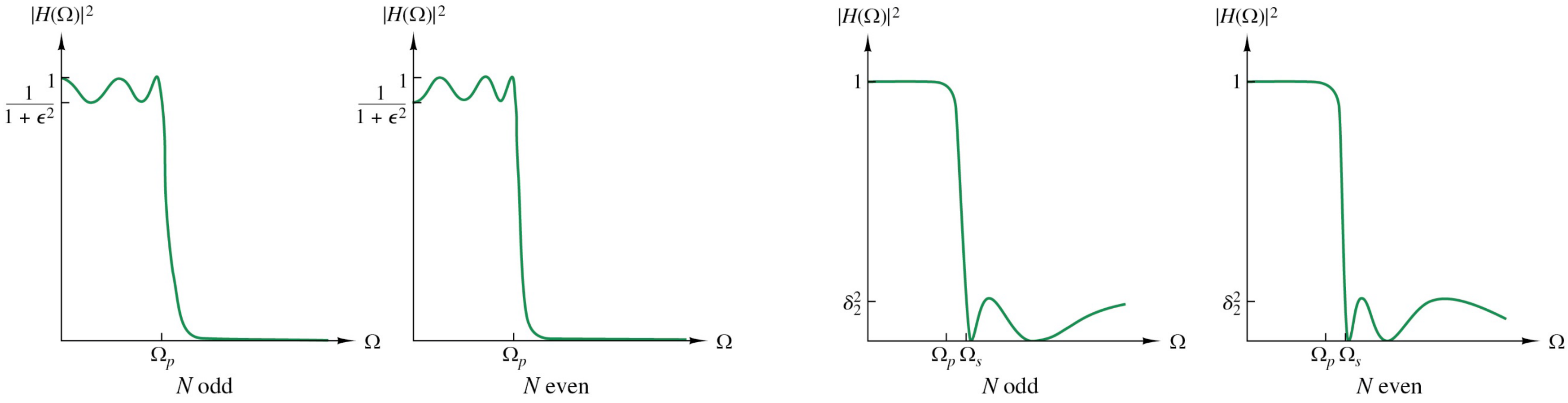
Ω_s is stopband frequency

ε is the ripple factor

T_N is a Chebyshev polynomial

Chebyshev Filters

- Chebyshev Filters



Type 1

Type 2

Elliptic Filters

- Elliptic Filters
  - Equiripple in pass and stop bands
  - Smallest order filter for given set of specifications
  - Smallest transition band
  - Phase is more nonlinear in passband than Butterworth and Chebyshev filters

$$\left|H(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 U_N\left(\Omega/\Omega_p\right)}$$

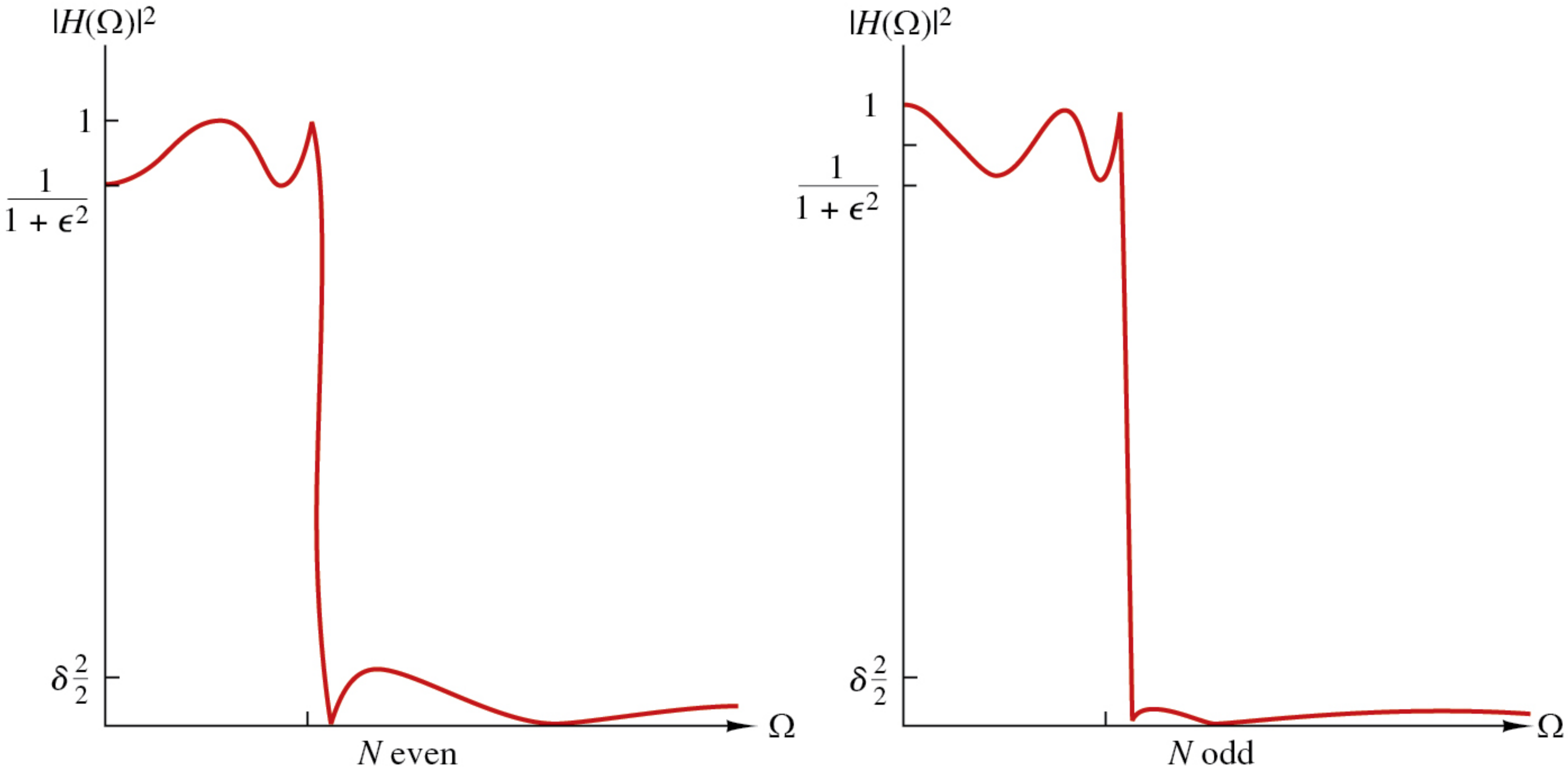
Ω_p is passband frequency

ε is the ripple factor

U_N is a Jacobian elliptic function of order N

Elliptic Filters

- Elliptic Filters



Bessel Filters

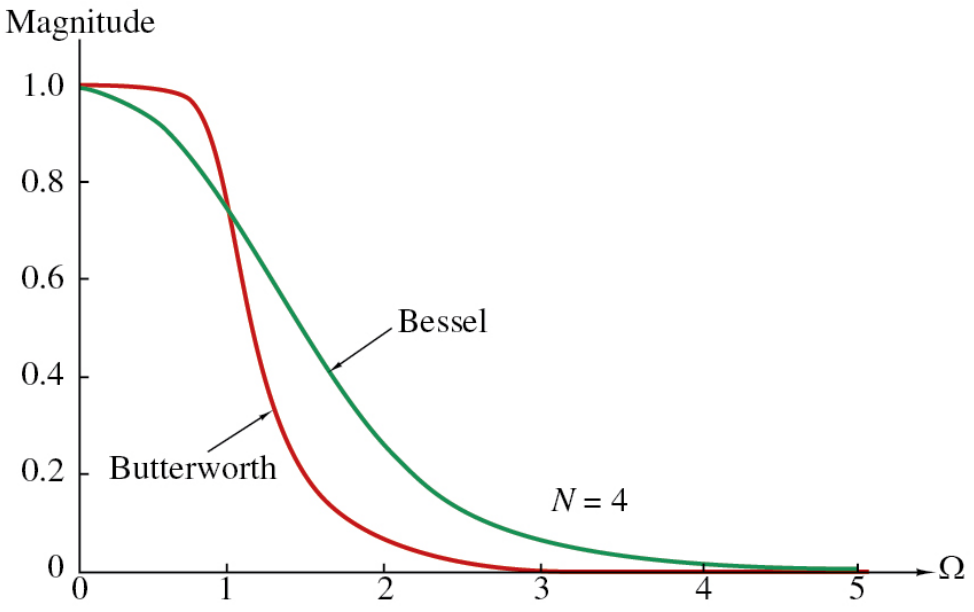
- Bessel Filters
  - All pole filters
  - Linear phase over passband
    - But when you transform it to digital filter, you lose that feature

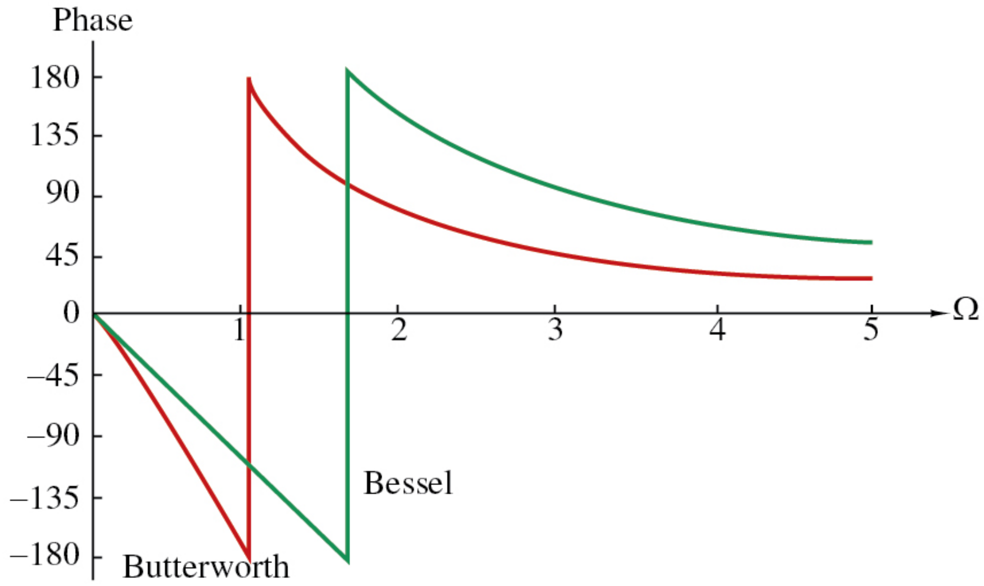
$$H(s) = \frac{1}{B_N(s)}$$

$B_N(s)$ is N'th order Bessel function

Bessel Filters

- Bessel Filters





Summary of Analog Filters

Analog Filter	Passband	Stopband	Transition Band	Specification
Butterworth	Monotonic	Monotonic	Broad	Pass/Stop band
Chebyshev-I	Equiripple	Monotonic	Narrow	Passband
Chebyshev-II	Monotonic	Equiripple	Narrow	Stopband
Elliptic	Equiripple	Equiripple	Very Narrow	Passband

Filters

- Note about definitions used for lowpass IIR filters
  - Two definitions are given for Butterworth filters.
    - In terms of passband edge frequency: Ω_p and cutoff frequency: Ω_c

$$\left|H(\Omega)\right|^2 = \frac{1}{1+\left(\Omega/\Omega_c\right)^{2N}} = \frac{1}{1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}} \quad \text{(for Butterworth filter, for example)}$$

    - At $\Omega = \Omega_c$

$$\left|H(\Omega_c)\right|^2 = \frac{1}{1+\left(\Omega_c/\Omega_c\right)^{2N}} = \frac{1}{2} \quad \left(\text{This is 1/2 power frequency, -3 dB}\right)$$

    - At $\Omega = \Omega_p$

$$\left|H(\Omega_p)\right|^2 = \frac{1}{1+\varepsilon^2\left(\Omega_p/\Omega_p\right)^{2N}} = \frac{1}{1+\varepsilon^2} \quad \left(\text{This is the power at the passband edge}\right)$$

Filters

- When you design LP filters (in Matlab for example)
  - Specify the attenuation at the edge of the passband in dB
    - How far below maximum value does the power drop
    - Usually something like -1 dB (~.794 of maximum $|H(\Omega)|^2$)

$$\mathrm{dB} = 10\log_{10}\left|H\left(\Omega\right)\right|^2 \quad \Rightarrow \quad -1\ \mathrm{dB} = 10^{-(1/10)} \approx .7943$$

  - Specify the attenuation at the edge of the stopband in dB
    - Again, how far below maximum value of $|H(\Omega)|^2$ in the stopband
    - Usually something like -80 dB (10^{-8} of maximum $|H(\Omega)|^2$)

$$-80\ \mathrm{dB} = 10^{-(80/10)} \approx 10^{-8}$$

Transformations of Lowpass Filters

- The goal is to find a way to design all filter types
  - Lowpass (LP)
  - Highpass (HP)
  - Bandpass (BP)
  - Bandstop (BS)

- We have concentrated on the design of lowpass filters

- Question is: How do you transform a lowpass filter into the other types?

FIR Filters

- FIR Filters
  - Lowpass, Highpass, Bandpass, and Bandstop filters
    - Can be created directly so there is no need to transform one form to another
  - Define ideal filter (with built in delay for linear phase)

    $H_d(\omega)$
  - Obtain ideal impulse response using inverse DTFT
  - Window impulse response to get physically realizable desired filter
    - Problem 10.2 is example of finding a bandstop filter

Transformations of IIR Filters

- Two approaches for IIR Filters:
  - Do the transformation in the analog domain
    - Transform that filter into the digital domain using one of the methods discussed
      - Derivative approximation
      - Impulse invariance
      - Bilinear Transform
    - Bilinear transform is preferred since it is the only one that can be used for HP, BP, and BS filters
  - Do the transformation in the digital domain
    - Transform analog lowpass filter to digital domain
    - Transform the digital lowpass filter into other forms (HP,BP,BS)

IIR Transformations: Analog Domain

- IIR filter transformations in the analog domain
  - Lowpass filters can be transformed into HP, BP, and BS filters through fairly straightford substitution of the Laplace variable, s
  - Transformation of lowpass to lowpass with different passband edge frequency
  - It is useful to first consider transformation of a prototype lowpass filter to a lowpass filter with desired cutoff frequency

IIR Transformations: Analog Domain

- Lowpass to Lowpass Transformation
  - Transformation of lowpass to lowpass with different passband edge frequency

$$s \to \frac{\Omega_p}{\Omega_{p'}} s$$

where Ω_p is original passband edge frequency

and $\Omega_{p'}$ is new passband edge frequency

$$H_l(s) = H_p\left(\frac{\Omega_p}{\Omega_{p'}} s\right)$$

$H_p(s)$ is the prototype lowpass filter

IIR Transformations: Analog Domain

- Example
  - Transform a 1st order Butterworth filter with passband edge frequency Ω_{p1} to one with passband edge frequency Ω_{p2}

$$H_{LP1}(s) = \frac{\Omega_{p1}}{s + \Omega_{p1}}$$

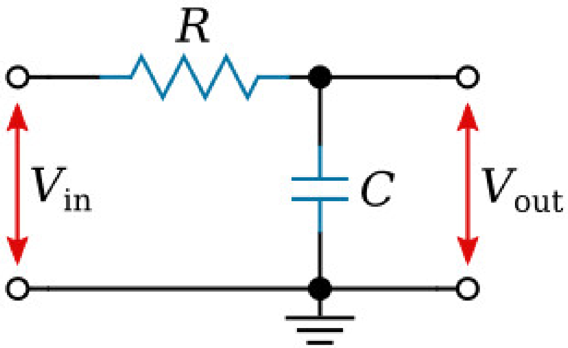
$$\mathrm{H}_{LP2}(s) = \frac{\Omega_{p1}}{\frac{\Omega_{p1}}{\Omega_{p2}} s + \Omega_{p1}} = \Omega_{p1}\left(\frac{1}{\frac{s}{\Omega_{p2}} + 1}\right) = \frac{\Omega_{p2}}{s + \Omega_{p2}}$$

$$H(s) = \frac{1}{s+1} \quad \text{(prototype analog filter for first-order Butterworth)}$$

$$\mathrm{H}_2(s) = \frac{1}{s/\Omega_c + 1} = \frac{\Omega_c}{s + \Omega_c}$$

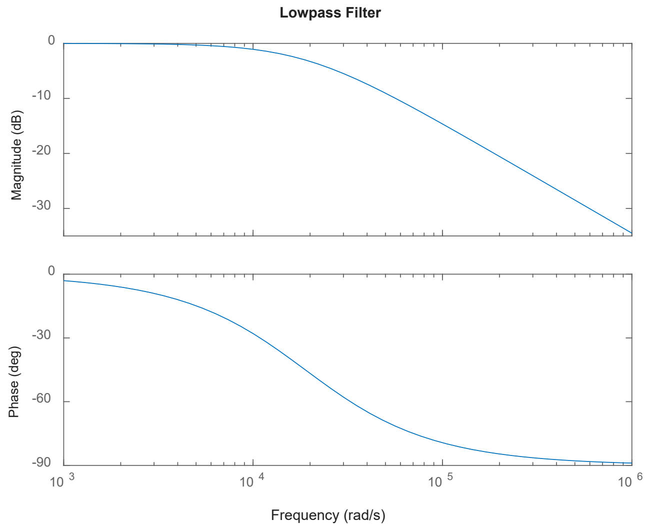
IIR Transformations: Analog Domain

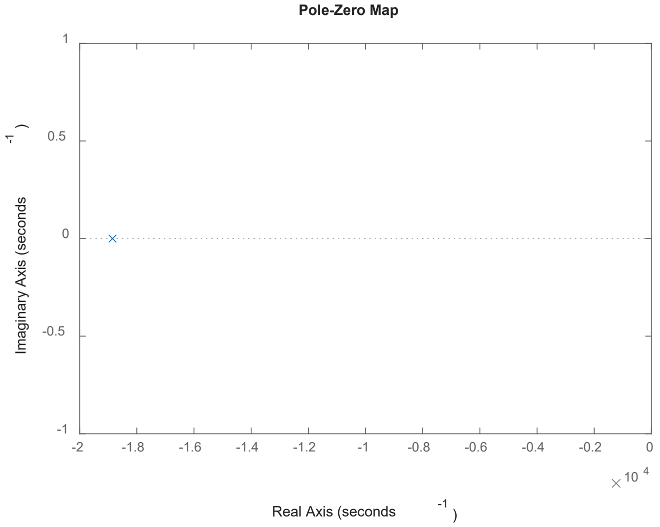
- Example prototype lowpass to lowpass



$$H(s) = \frac{1/RC}{s + (1/RC)} = \frac{\Omega_c}{s + \Omega_c}$$

$$\Omega_c = 2\pi/RC \quad \text{(cutoff)}$$





Filters

- Lowpass to Highpass Transformation

$$s \rightarrow \frac{\Omega_p \Omega_{p'}}{s}$$

where Ω_p is original passband edge frequency

and $\Omega_{p'}$ is new passband edge frequency

$$H_l(s) = H_p\left(\frac{\Omega_p \Omega_{p'}}{s}\right)$$

$H_p(s)$ is the prototype lowpass filter

Filters

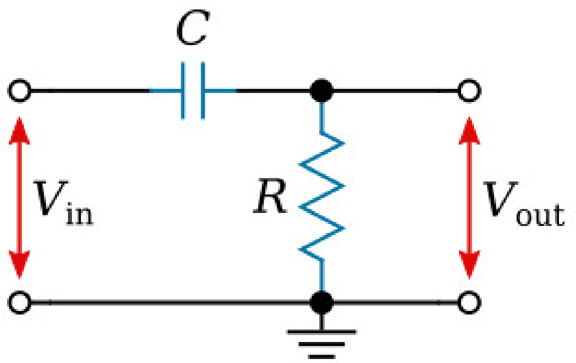
- Example: Lowpass to highpass
  - Transform a 1st order Butterworth filter with passband edge Ω_{p2} frequency Ω_{p1} to highpass with passband edge frequency

$$H_{LP1}(s) = \frac{\Omega_{p1}}{s + \Omega_{p1}}$$

$$\mathrm{H}_{HP2}(s) = \frac{\Omega_{p1}}{\frac{\Omega_{p1}\Omega_{p2}}{s} + \Omega_{p1}} = \Omega_{p1}\left(\frac{1}{\frac{\Omega_{p2}}{s} + 1}\right) = \frac{s}{s + \Omega_{p2}}$$

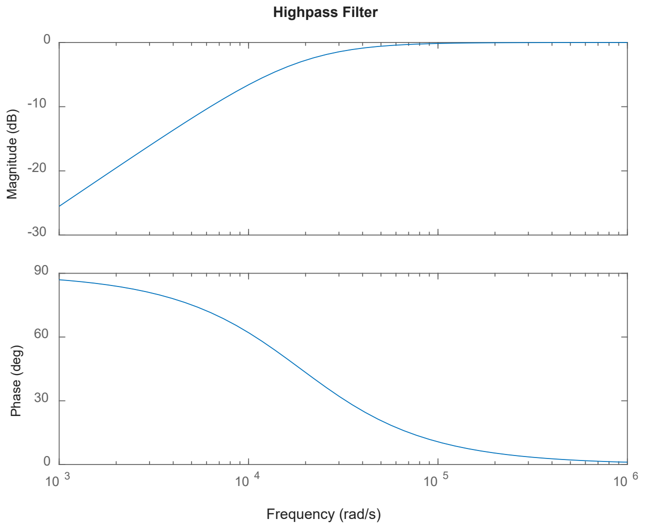
Filters

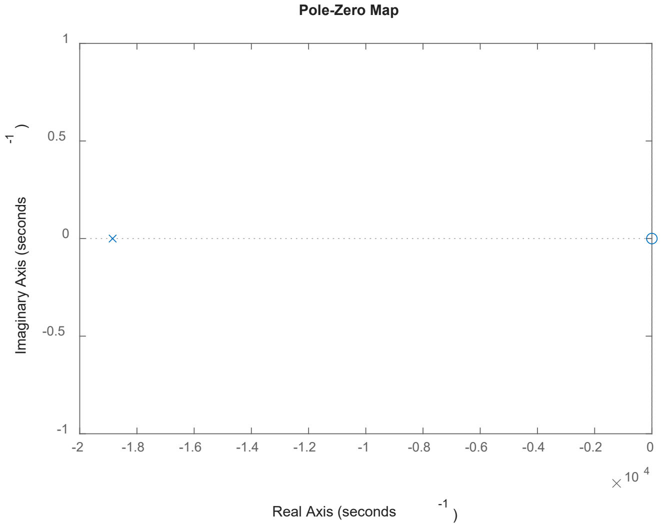
- Example: lowpass to highpass



$$H(s) = \frac{s}{s + (1/RC)} = \frac{s}{s + \Omega_c}$$

$$\Omega_c = 1/RC \quad \text{(cutoff)}$$





Filters

- Lowpass to Bandpass Transformation

$$s \rightarrow \Omega_p \frac{s^2 + \Omega_u \Omega_l}{s\left(\Omega_u - \Omega_l\right)}$$

where

Ω_p is prototype lowpass filter with passband edge Ω_p

Ω_l is lower band edge frequency

Ω_u is upper band edge frequency

$$H_b\left(s\right) = H_p\left(\Omega_p \frac{s^2 + \Omega_u \Omega_l}{s\left(\Omega_u - \Omega_l\right)}\right)$$

$H_p\left(s\right)$ is the prototype lowpass filter

Filters

- Example: Lowpass to bandpass
  - Transform a 1st order Butterworth filter with passband edge frequency $\Omega_{p1} = 1$ to one with bandpass filter with low and high band-edge frequencies Ω_l and Ω_u

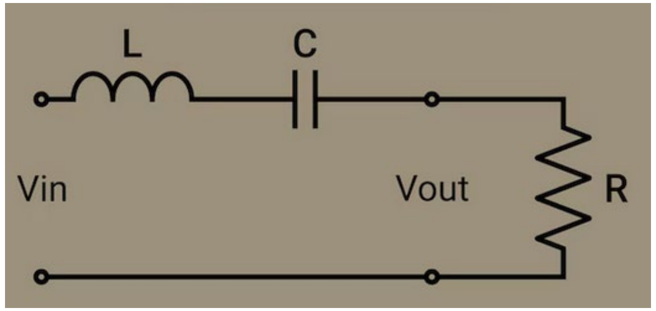
$$H_{LP}(s) = \frac{1}{s+1} \quad \text{(prototype analog filter for first-order Butterworth)}$$

$$\mathrm{H}_{BP}(s) = \frac{1}{\dfrac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} + 1} = \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u + s(\Omega_u - \Omega_l)}$$

$$\mathrm{H}_{BP}(s) = \frac{(\Omega_u - \Omega_l)s}{s^2 + s(\Omega_u - \Omega_l)s + \Omega_l \Omega_u}$$

Filters

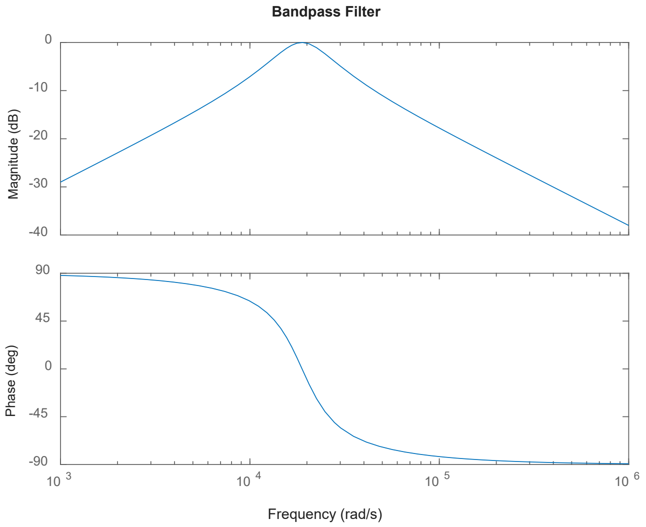
- Example: Lowpass to bandpass

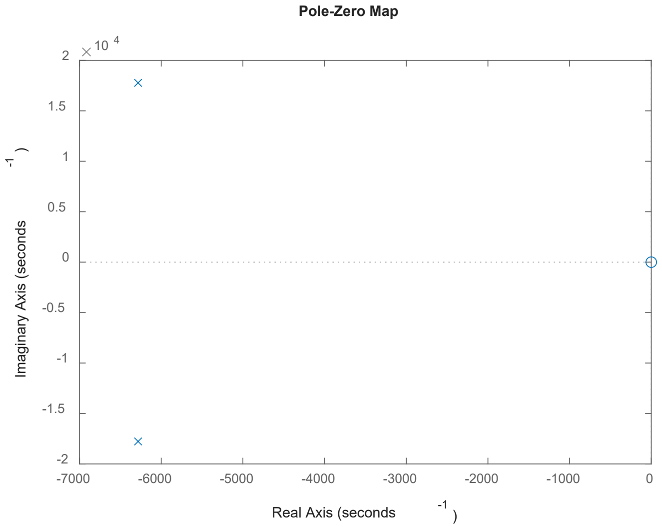


$$H(s) = \frac{(R/L)s}{s^2 + (R/L)s + (1/LC)} = \frac{\beta s}{s^2 + \beta s + \omega_0^2}$$

$$\omega_0 = \sqrt{1/LC} \quad \text{(center frequency)}$$

$$\beta = R/L \quad \text{(bandwidth)}$$





Filters

- Lowpass to Bandstop Transformation

$$s \to \Omega_p \frac{s\left(\Omega_u - \Omega_l\right)}{s^2 + \Omega_u \Omega_l}$$

where

Ω_p is prototype lowpass filter with passband edge Ω_p

Ω_l is lower band edge frequency

Ω_u is upper band edge frequency

$$H_{bs}(s) = H_p\left(\Omega_p \frac{s\left(\Omega_u - \Omega_l\right)}{s^2 + \Omega_u \Omega_l}\right)$$

$H_p(s)$ is the prototype lowpass filter

Filters

- Example: Lowpass to bandstop
  - Transform a 1st order Butterworth filter with passband edge frequency $\Omega_{p1} = 1$ to one with bandstop filter with low and high band-edge frequencies Ω_l and Ω_u

$$H_{LP}(s) = \frac{1}{s+1} \quad \text{(prototype analog filter for first-order Butterworth)}$$

$$\mathrm{H}_{BS}(s) = \frac{1}{\dfrac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u\Omega_l} + 1} = \frac{s^2 + \Omega_u\Omega_l}{s(\Omega_u - \Omega_l) + s^2 + \Omega_u\Omega_l}$$

$$\mathrm{H}_{BS}(s) = \frac{s^2 + \Omega_u\Omega_l}{s^2 + (\Omega_u - \Omega_l)s + \Omega_l\Omega_u}$$

Summary of Analog Transformations

Type of transformation	Transformation	Band edge frequencies of new filter
Lowpass	$s \longrightarrow \frac{\Omega_p}{\Omega'_p} s$	Ω'_p
Highpass	$s \longrightarrow \frac{\Omega_p \Omega'_p}{s}$	Ω'_p
Bandpass	$s \longrightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$	Ω_l, Ω_u
Bandstop	$s \longrightarrow \Omega_p \frac{s(\Omega_u - \Omega_c)}{s^2 + \Omega_u \Omega_l}$	Ω_l, Ω_u

Matlab routines for analog filters

- Matlab routines for analog filter prototypes:

```
buttap      - Butterworth
cheb1ap     - Chebyshev Type 1
cheb2ap     - Chebyshev Type 2
ellipap     - Elliptic  (equiripple)
besselap    - Bessel
```

- Matlab routines to convert prototypes to LP, HP, BP, and BS

```
lp2lp     - Lowpass to lowpass
lp2hp     - Lowpass to highpass
lp2bp     - Lowpass to bandpass
lp2bs     - Lowpass to bandstop pass
```

- Matlab routines to convert analog to digital filters

```
bilinear  - Bilinear method
impinvar  - Impulse invariance method
```

https://www.mathworks.com/help/signal/analog-filters.html?s_tid=CRUX_lftnav

IIR Transformations: Digital Domain

- Transforming IIR lowpass digital filters
  - Goal is to come up with a mapping of z^{-1}

$$\text{Mapping: } z^{-1} \rightarrow g\left(z^{-1}\right)$$

  - Requirements
    - Must map points inside unit circle to points inside unit circle
      - Stability
    - Unit circle must be mapped into itself
      - Preserve interpretation of unit circle as frequency response
      - Mapped frequency response should also be frequency response

IIR Transformations: Digital Domain

- Implications of requirements:
  - If the unit has to map into itself:

$$\text{Mapping: } z^{-1} \rightarrow g\left(z^{-1}\right)$$

$$\left|z^{-1}\right| = 1 \;\Rightarrow \left|\, g\left(z^{-1}\right)\right| = 1$$

$$z = re^{j\omega} \qquad \text{When } r \;=\; 1, \quad z^{-1} = e^{-j\omega}$$

$$g\left(e^{-j\omega}\right) \equiv g\left(\omega\right) = \left|g\left(\omega\right)\right| e^{j\arg\left(g\left(\omega\right)\right)}$$

where

$$\left|g\left(\omega\right)\right| = 1 \;\text{ for all } \omega$$

IIR Transformations: Digital Domain

$$g\left(e^{-j\omega}\right) \equiv g(\omega) = \left|g(\omega)\right| e^{j\arg(g(\omega))}$$

where

$$\left|g(\omega)\right| = 1 \quad \text{for all } \omega$$

$g(\omega)$ must be all-pass, all frequencies mapped to unit circle

What function has this property?

$$g\left(z^{-1}\right) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

IIR Transformations: Digital Domain

What function has this property $\left|g\left(e^{-j\omega}\right)\right| = 1$

Try

$$g\left(z^{-1}\right) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

$$g\left(e^{-j\omega}\right) = \frac{e^{-j\omega} - \alpha}{1 - \alpha e^{-j\omega}}$$

$$\left|g\left(e^{-j\omega}\right)\right|^2 = \left(\frac{e^{-j\omega} - \alpha}{1 - \alpha e^{-j\omega}}\right)\left(\frac{e^{j\omega} - \alpha}{1 - \alpha e^{j\omega}}\right) = \frac{1 - \alpha e^{j\omega} - \alpha e^{-j\omega} + \alpha^2}{1 - \alpha e^{-j\omega} - \alpha e^{j\omega} + \alpha^2} = 1$$

IIR Transformations: Digital Domain

A product of terms like this will also have magnitude 1 for $z = e^{-j\omega}$ since each term has magnitude 1.

General form of transformation of z^{-1}

$$g\left(z^{-1}\right) = \pm \prod_{k=1}^{n} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}}$$

Notice that poles and zeros are related:

zeros: $z = 1/a_k$

poles: $z = a_k$

IIR Transformations: Digital Domain

If want to map passband edge frequency, ω_{p1} into ω_{p2}

$$\omega_{p2} = g\left(\omega_{p1}\right)$$

Also would like simplest function.

If only one parameter (LP &HP), would expect one factor

If two parameters (BP &BS), would expect two factors

$\mathrm{LP} \to \mathrm{LP}$ and $\mathrm{LP} \to \mathrm{HP}$

Form is $g\left(z^{-1}\right) = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$

$\mathrm{LP} \to \mathrm{BP}$ and $\mathrm{LP} \to \mathrm{BS}$

Form is $g\left(z^{-1}\right) = \left(\dfrac{z^{-1} - \alpha_1}{1 - \alpha_1 z^{-1}}\right)\left(\dfrac{z^{-1} - \alpha_2}{1 - \alpha_2 z^{-1}}\right) = \dfrac{z^{-2} - \left(\alpha_1 + \alpha_2\right) z^{-1} + \alpha_1 \alpha_2}{1 - \left(\alpha_1 + \alpha_2\right) z^{-1} + \alpha_1 \alpha_2 z^{-2}}$

Note that α's are different
For the different transforms

IIR Transformations: Digital Domain

LP $\rightarrow$ LP and LP $\rightarrow$ HP have one parameter (different for LP $\rightarrow$ LP and LP $\rightarrow$ HP)

Form is $g\left(z^{-1}\right) = \dfrac{z^{-1} - a}{1 - az^{-1}}$

parameter is determined by mapping z in prototype LP filter such that $\omega_{p'}$ is the cutoff frequency

LP $\rightarrow$ BP and LP $\rightarrow$ BS have 2 parameters (different for LP $\rightarrow$ BP and LP $\rightarrow$ BS)

Form is $g\left(z^{-1}\right) = \dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$

parameters are determined by mapping prototype filter such that you have

upper and lower passband edges.

Summary of Digital Transformations

Type of transformation	Transformation	Parameters
Lowpass	$z^{-1} \longrightarrow \dfrac{z^{-1} - a}{1 - az^{-1}}$	$\omega'_p =$ band edge frequency new filter $a = \dfrac{\sin[(\omega_p - \omega'_p)/2]}{\sin[(\omega_p + \omega'_p)/2]}$
Highpass	$z^{-1} \longrightarrow -\dfrac{z^{-1} + a}{1 + az^{-1}}$	$\omega'_p =$ band edge frequency new filter $a = -\dfrac{\cos[(\omega_p + \omega'_p)/2]}{\cos[(\omega_p - \omega'_p)/2]}$
		$\omega_l =$ lower band edge frequency $\omega_u =$ upper band edge frequency $a_1 = 2\alpha K/(K + 1)$

Summary of Digital Transformations

Bandpass

$$z^{-1} \longrightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$$

$$a_2 = (K-1)/(K+1)$$

$$\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$$

$$K = \cot \frac{\omega_u - \omega_l}{2} \tan \frac{\omega_p}{2}$$

ω_l = lower band edge frequency

ω_u = upper band edge frequency

Bandstop

$$z^{-1} \longrightarrow \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-1} - a_1 z^{-1} + 1}$$

$$a_1 = 2\alpha/(K+1)$$

$$a_2 = (1-K)/(1+K)$$

$$\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$$

$$K = \tan \frac{\omega_u - \omega_l}{2} \tan \frac{\omega_p}{2}$$

IIR Transformations

If you do the mapping in the analog domain and use the bilinear transform to get the digital filter you should get same result as mapping the LP analog filter to digital filter and doing the transformation in the digital domain.