# Digital Signal Processing

Class 17
03/25/2025

# ENGR 71

- Class Overview
  - Two Topics in Frequency Domain Analysis
    - Invertible systems
    - Minimum phase systems
  - Discrete Fourier Transform
- Assignments
  - Lab 2 due March 28
  - Reading:

    Chapter 7: The Discrete Fourier Transform

# Frequency-Domain Analysis of LTI Systems

- FIR systems are "all zero" systems"
  - There are no poles

FIR Transfer function : $H(z)=\sum_{k=0}^{M} b_k z^{-k} = \frac{B(z)}{1} = b_0 \prod_{k=1}^{M}\left(1-z_k z^{-1}\right)$

- IIR systems (with rational transfer function) have zeros and poles

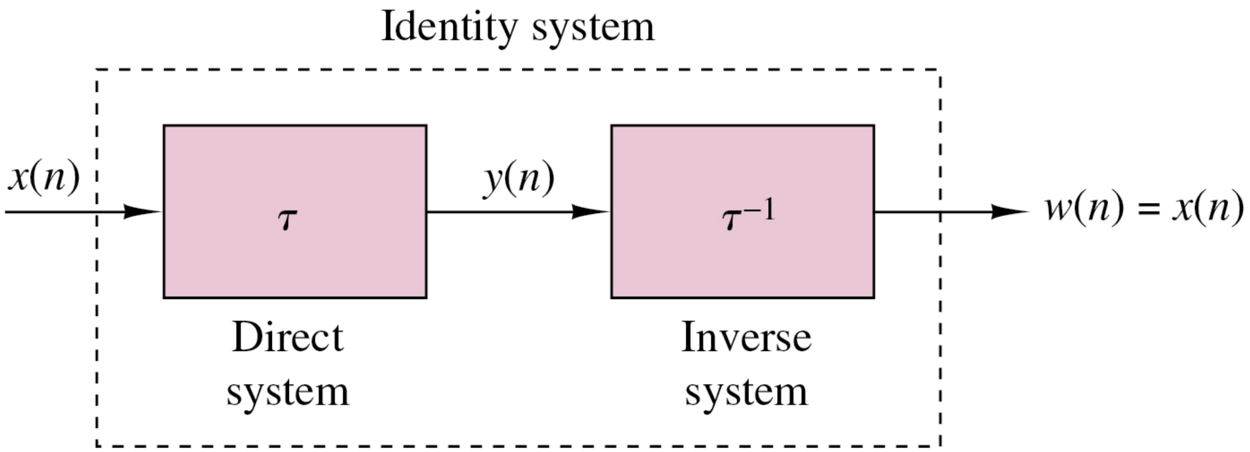
IIR Transfer function : $H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}\left(1-z_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1-p_k z^{-1}\right)}$

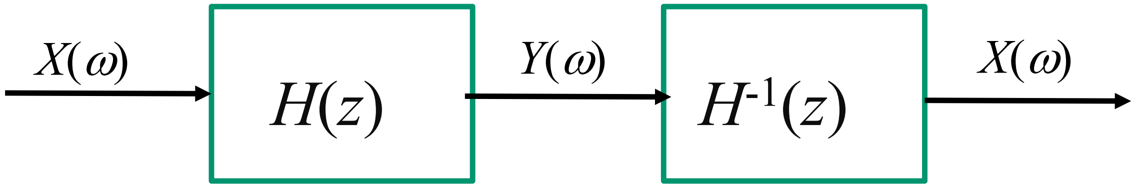
  - If they have no zeros, they are "all pole" systems

IIR all pole systems: $H(z)=\frac{1}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{1}{\prod_{k=1}^{N}\left(1-p_k z^{-1}\right)}$

# Frequency-Domain Analysis of LTI Systems

- Invertible systems




Question: Does inverse system exist, and is it stable?

# Minimum Phase Systems

- Minimum phase systems
  - When a signal passes through a system, how does its phase change?
  - Recall:
    - Phase change corresponds to a time delay for linear phase
    - Phase distortion for non-linear phase
      - Different time delays for different frequency components

# Filters

– Phase of filters
  - For an ideal filter:

$$Y(\omega) = H(\omega)X(\omega) \qquad \omega_1 < \omega < \omega_2$$

$$Y(\omega) = Ce^{-j\omega n_0} X(\omega)$$

  – It scales the magnitude of the input by C
    shifts the phase linearly with $\omega$
  – Linear phase filters are “good,”
    because they only introduce a time delay in the input signal
    » Time shift property of Fourier transform: $y(n) = Cx(n-n_0)$
  – What would be “bad” would be if phase of the input changed as a function of frequency, i.e., different frequency components would be delayed by different amounts.

# Filters

– Ideal filter : $H(\omega) = |H(\omega)| e^{j\Theta(\omega)} = Ce^{-j\omega n_0} \quad \omega_1 < \omega < \omega_2$

$$|H(\omega)| = C$$

$$\Theta(\omega) = -\omega n_0$$

– On previous slide:

- Delay is given by: $y(n) = Cy(n - n_0)$

$$\tau_g = -\frac{d\Theta(\omega)}{d\omega} = -\frac{d(-\omega n_0)}{d\omega} = n_0$$

– Generalize definition of "group delay" (or "envelope delay") for arbitrary phase:

$$\boxed{\text{Group delay: } \tau_g = -\frac{d\Theta(\omega)}{d\omega}}$$

Describes how delay depends on frequency

# Frequency-Domain Analysis of LTI Systems

- Minimum phase systems
  - Consider two all zero systems (FIR filters)

$$H_1(z) = 1 + \frac{1}{2}z^{-1} = z^{-1}\left(z + \frac{1}{2}\right) \quad \text{zero at } z = -\frac{1}{2}$$

$$H_2(z) = \frac{1}{2} + z^{-1} = z^{-1}\left(\frac{1}{2}z + 1\right) \quad \text{zero at } z = -2$$

- Same magnitude:

$$\left|H_1(\omega)\right| = \left|H_2(z)\right| = \sqrt{\frac{5}{4} + \cos\omega}$$

- Phase: $\Theta_1(\omega) = -\omega + \tan^{-1}\left(\dfrac{\sin\omega}{1/2 + \cos\omega}\right)$

$$\Theta_2(\omega) = -\omega + \tan^{-1}\left(\frac{\sin\omega}{2 + \cos\omega}\right)$$

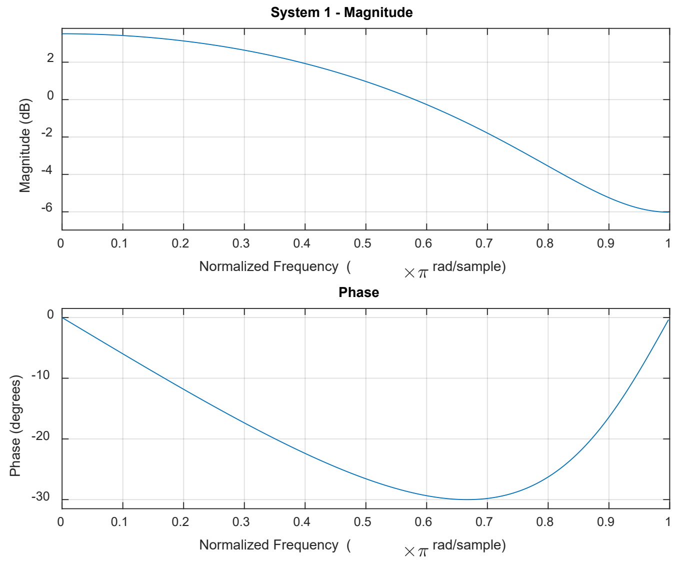
Different phase response

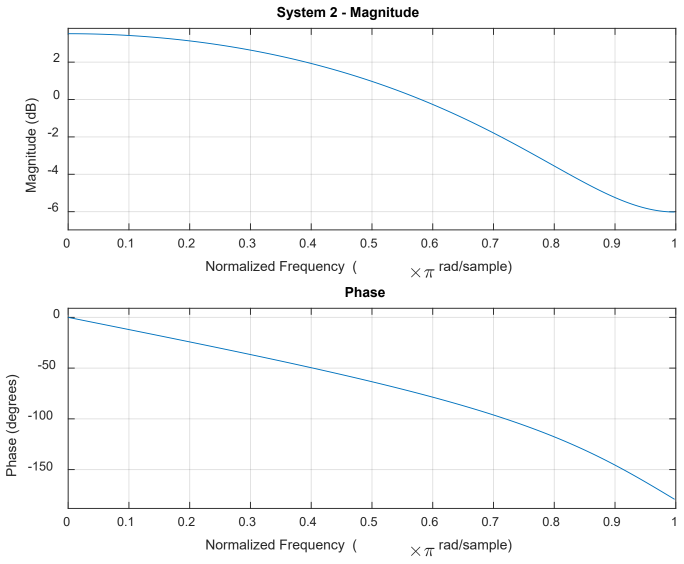
# Minimum Phase Systems

- Minimum and Maximum phase example

$$H_1(z) = 1 + \frac{1}{2} z^{-1} \quad \text{zero at } z = -\frac{1}{2}$$

$$H_2(z) = \frac{1}{2} + z^{-1} \quad \text{zero at } z = -2$$



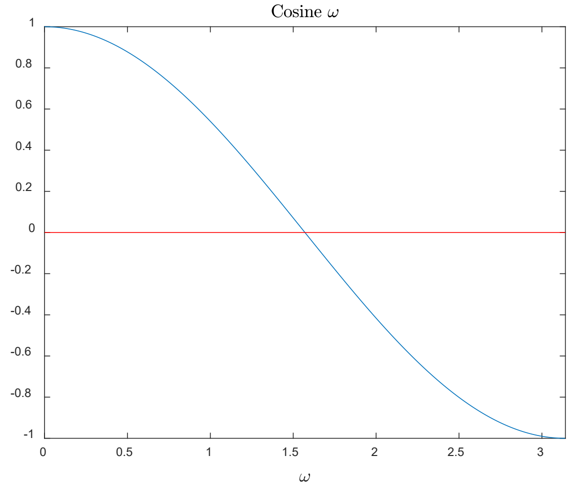



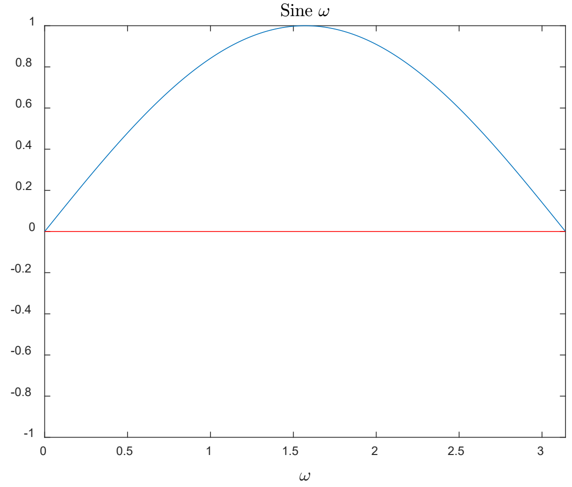
# Minimum Phase Systems

- Minimum phase systems
  - Phase:

$$\Theta_1(\omega) = -\omega + \tan^{-1}\left(\frac{\sin\omega}{1/2+\cos\omega}\right)$$

$$\Theta_2(\omega) = -\omega + \tan^{-1}\left(\frac{\sin\omega}{2+\cos\omega}\right)$$




cosine changes sign between 0 and $\pi$

sine does not change sign between 0 and $\pi$

# Minimum Phase Systems

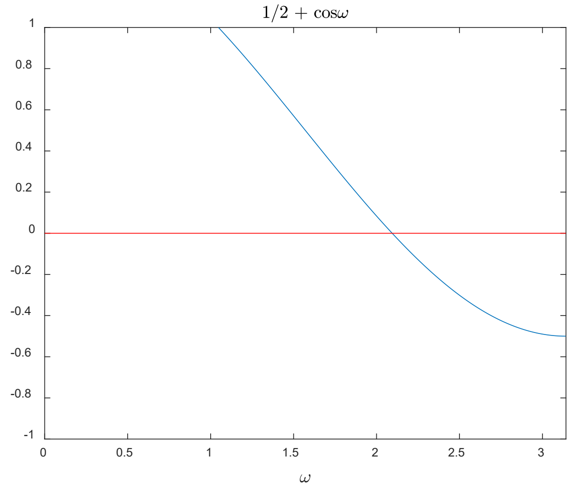
- Minimum phase systems
  - Phase:

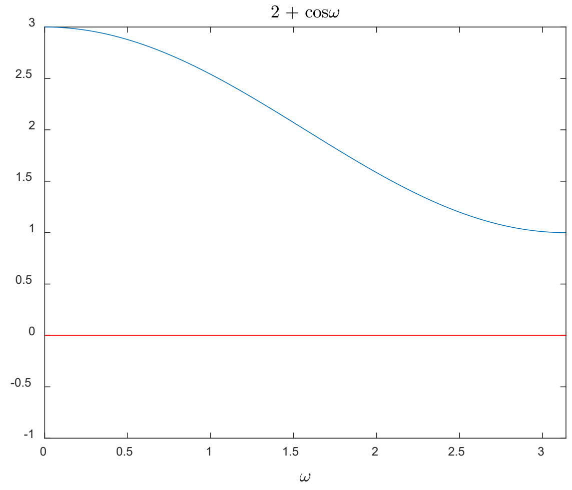
$$\Theta_1(\omega) = -\omega + \tan^{-1}\left(\frac{\sin\omega}{1/2+\cos\omega}\right)$$

$$\Theta_2(\omega) = -\omega + \tan^{-1}\left(\frac{\sin\omega}{2+\cos\omega}\right)$$

$1/2+\cos\omega$ becomes negative since $\left|\frac{1}{2}\right| < 1$

$2+\cos\omega$ stays positive since $|2| > 1$



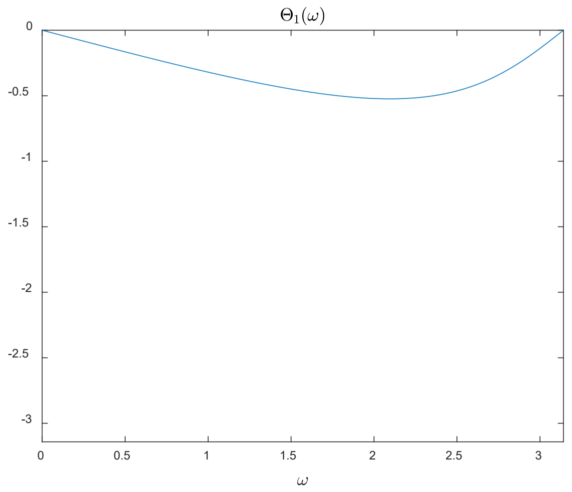



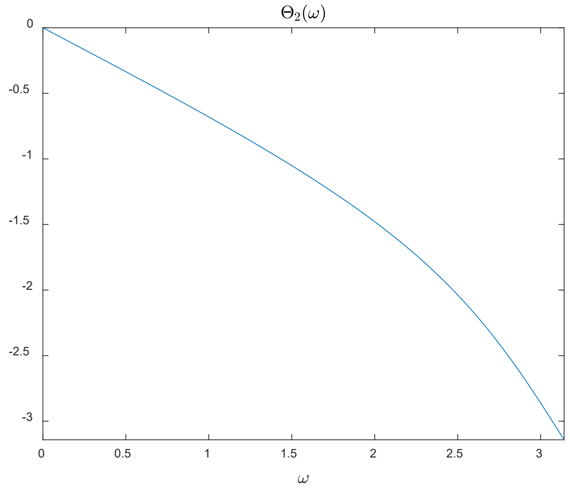
# Minimum Phase Systems

- Minimum phase systems
  - Phase:

$$\Theta_1(\omega) = -\omega + \tan^{-1}\left(\frac{\sin\omega}{1/2 + \cos\omega}\right)$$

$$\Theta_2(\omega) = -\omega + \tan^{-1}\left(\frac{\sin\omega}{2 + \cos\omega}\right)$$




Phase 1 starts at 0 and ends at 0: $-\pi + \pi = 0$

Phase 2 starts at 0 and ends at $-\pi$ : $-\pi + 0 = -\pi$

# Minimum Phase Systems

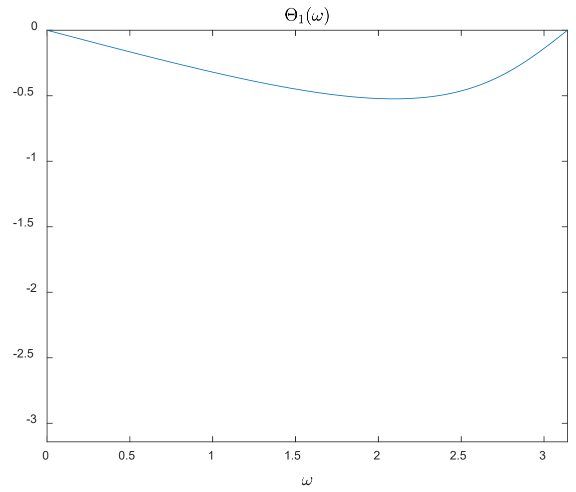
- Minimum phase systems
  - Phase:

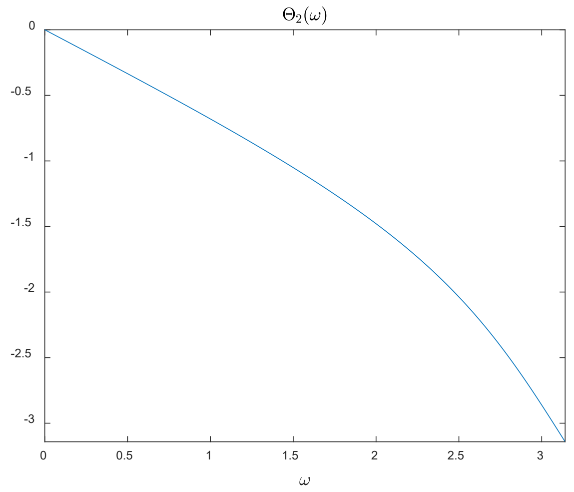
$$\Theta_1(\omega) = -\omega + \tan^{-1}\left(\frac{\sin\omega}{1/2 + \cos\omega}\right)$$

$$\Theta_2(\omega) = -\omega + \tan^{-1}\left(\frac{\sin\omega}{2 + \cos\omega}\right)$$

System 1 is a "minimum" phase system

System 2 is a "maximum" phase system



# Minimum Phase Systems

- Minimum phase systems
  - This type of behavior will be the same for any zeros whose magnitudes are either less than 1 or greater than 1 (inside or outside the unit circle)
  - It also holds for multiple zeros
    - If all zeros are inside unit circle, the phase change between $\omega = 0$ to $\omega = \pi$ will be minimum
    - If all zeros are outside unit circle, the phase change between $\omega = 0$ to $\omega = \pi$ will be maximum
    - If some are inside and some are outside, phase change is mixed
  - Minimum phase systems: All zeros inside unit circle
  - Maximum phase systems: All zeros outside unit circle
  - Mixed phase systems: Some zeros inside, some outside unit circle

# Minimum Phase Systems

- Minimum phase IIR systems
  - The same type of relationship holds for IIR systems
    - If all zeros are inside unit circle, the phase change between $\omega = 0$ to $\omega = \pi$ will be minimum
    - If all zeros are outside unit circle, the phase change between $\omega = 0$ to $\omega = \pi$ will be maximum
    - If some are inside and some are outside, phase change is mixed
  - Minimum phase systems: All zeros inside unit circle
  - Maximum phase systems: All zeros outside unit circle
  - Mixed phase systems: Some zeros inside, some outside unit circle

# Minimum Phase Systems

- Minimum phase IIR systems
  - For IIR systems to be stable, all poles must be inside the unit circle too.
  - Example:

Minimum phase:

$$H_{\min}(z)=\frac{1-\frac{13}{10}z^{-1}+\frac{2}{5}z^{-2}}{1-\frac{8}{5}z^{-1}+\frac{63}{100}z^{-2}}=\frac{\left(z-\frac{1}{2}\right)\left(z-\frac{4}{5}\right)}{\left(z-\frac{7}{10}\right)\left(z-\frac{9}{10}\right)}$$

Maximum phase:

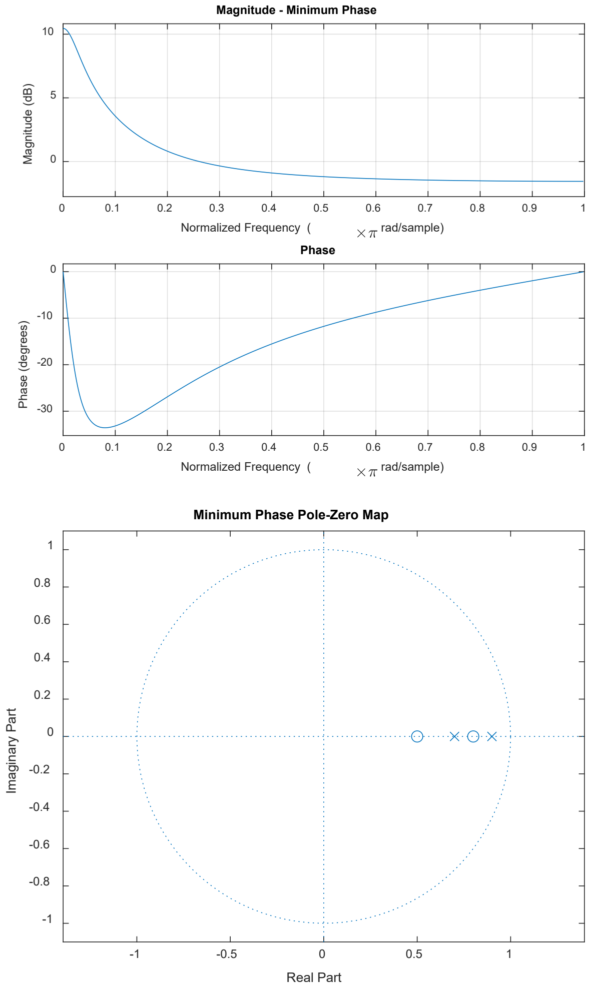
$$H_{\max}(z)=\frac{\left(\frac{2}{5}-\frac{13}{10}z^{-1}+z^{-2}\right)}{1-\frac{8}{5}z^{-1}+\frac{63}{100}z^{-2}}=\frac{\frac{2}{5}(z-2)\left(z-\frac{5}{4}\right)}{\left(z-\frac{7}{10}\right)\left(z-\frac{9}{10}\right)}$$

Mixed phase:

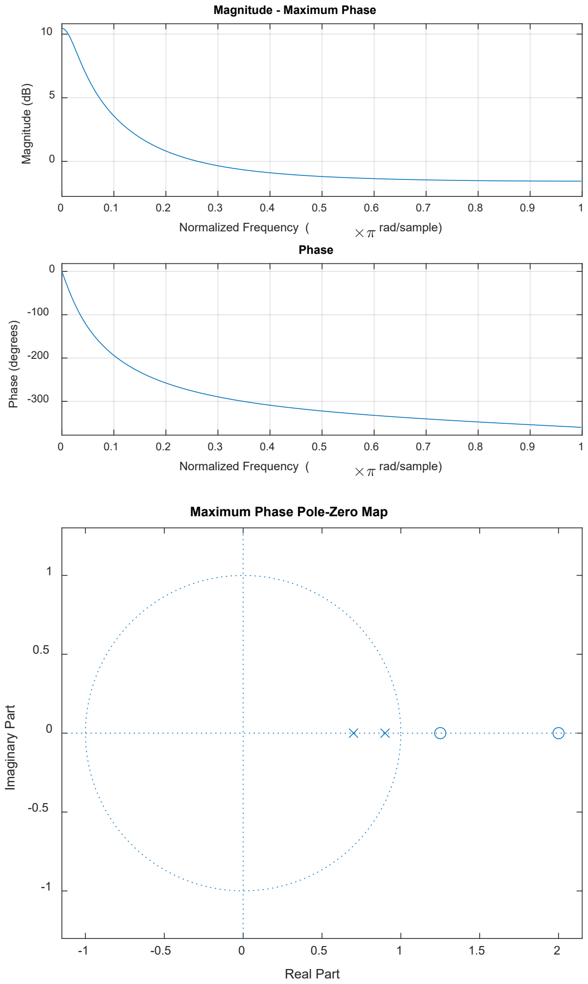
$$H_{\mathrm{mix}}(z)=\frac{\left(\frac{1}{5}-\frac{1}{2}z^{-1}+\frac{1}{5}z^{-2}\right)}{1-\frac{8}{5}z^{-1}+\frac{63}{100}z^{-2}}=\frac{\frac{1}{5}\left(z-\frac{1}{2}\right)(z-2)}{\left(z-\frac{7}{10}\right)\left(z-\frac{9}{10}\right)}$$

# Minimum Phase Systems

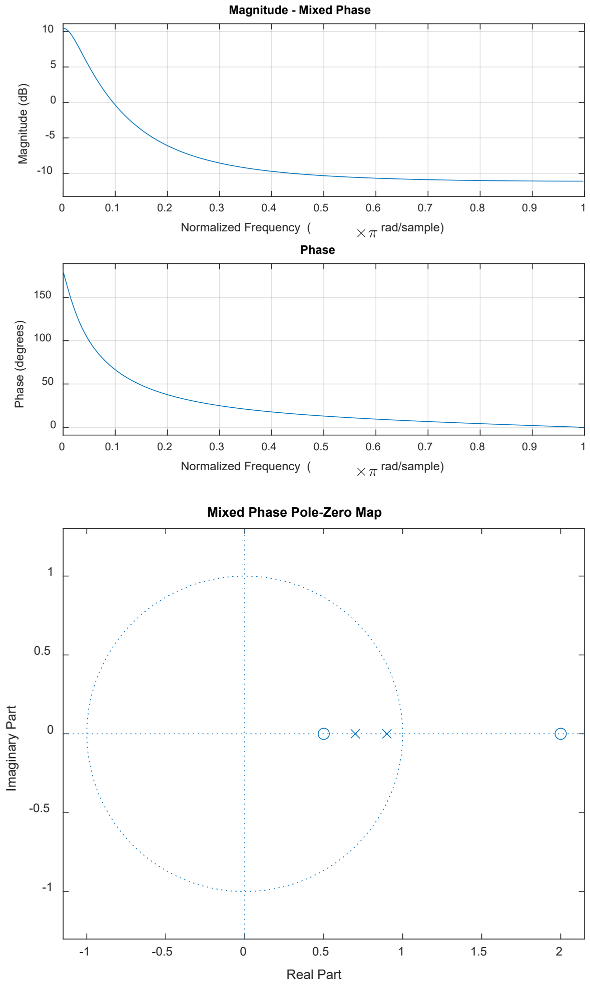
## Minimum phase:




## Maximum phase:

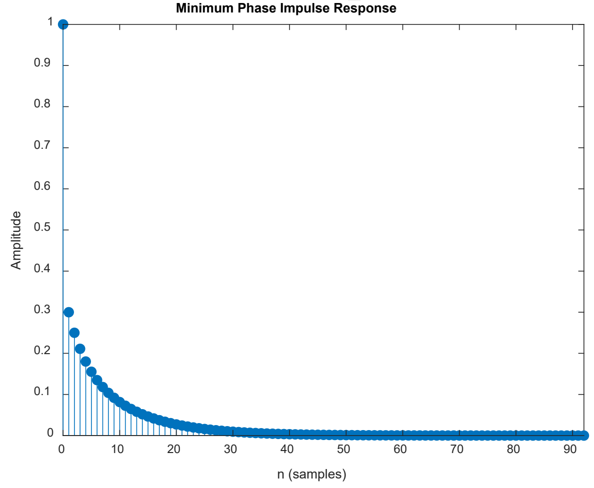


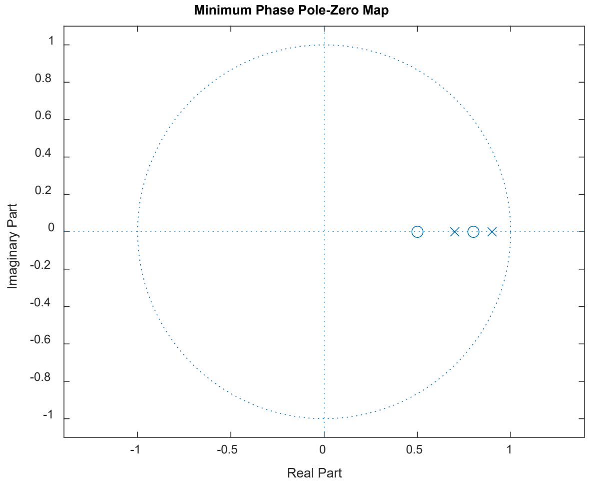

## Mixed phase:



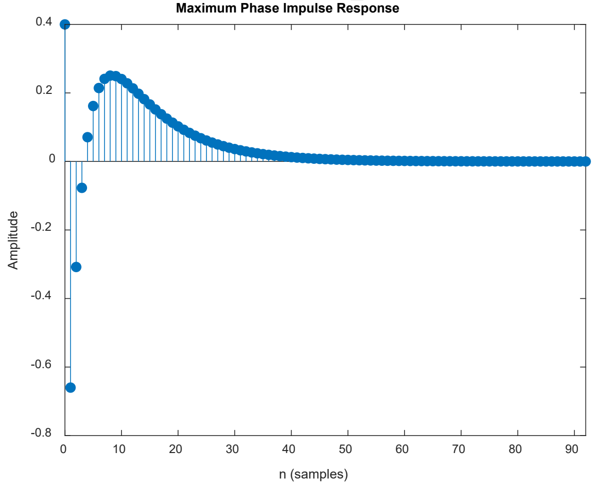
# Minimum Phase Systems

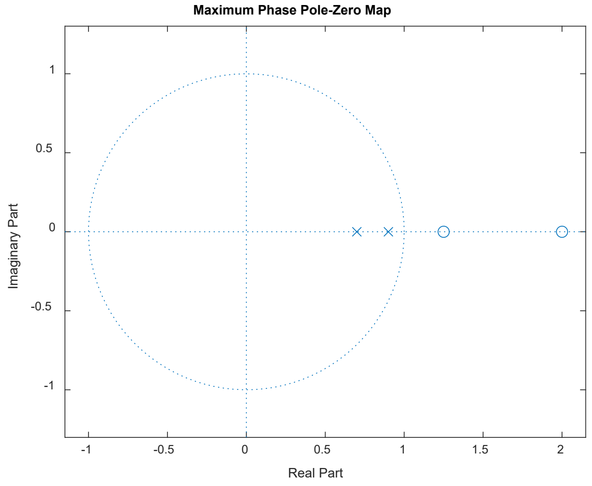
## Minimum phase:



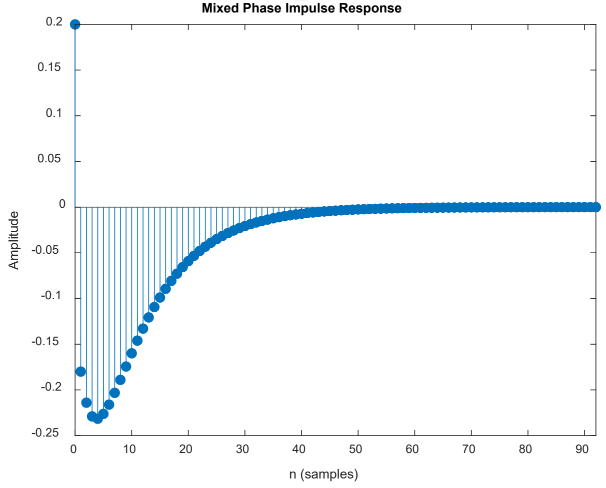



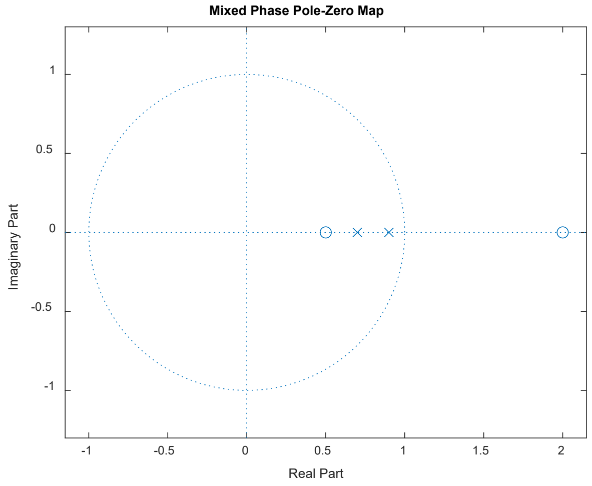

## Maximum phase:




## Mixed phase:



# Minimum Phase Systems

– Partial energy: $E(n) = \sum_{k=0}^{n} \left| h(k) \right|^2$

  - Among systems having the same magnitude response, the minimum phase system has the largest partial energy.

- Why is minimum phase important?
  - – Minimum phase distortion
  - – Smallest overall time delay
    - Notice that the impulse response dies off fastest for minimum phase

# Inversion and Minimum Phase

- Invertible systems and Minimum phase
  - If zeros and poles are all inside unit circle for stable system $H(z) = \frac{B(z)}{A(z)}$
  - Then since poles and zeros trade places for inverses system: $H^{-1}(z) = \frac{A(z)}{B(z)}$
  - Minimum phase delay systems have stable inverse, since all their poles (that used to be zeros) will be inside unit circle.
  - Inverse of FIR filters are all pole systems $H(z) = B(z) \Rightarrow H^{-1}(z) = \frac{1}{B(z)}$
  - Inverse of all pole systems are FIR filters $H(z) = \frac{1}{A(z)} \Rightarrow H^{-1}(z) = A(z)$

# Deconvolution

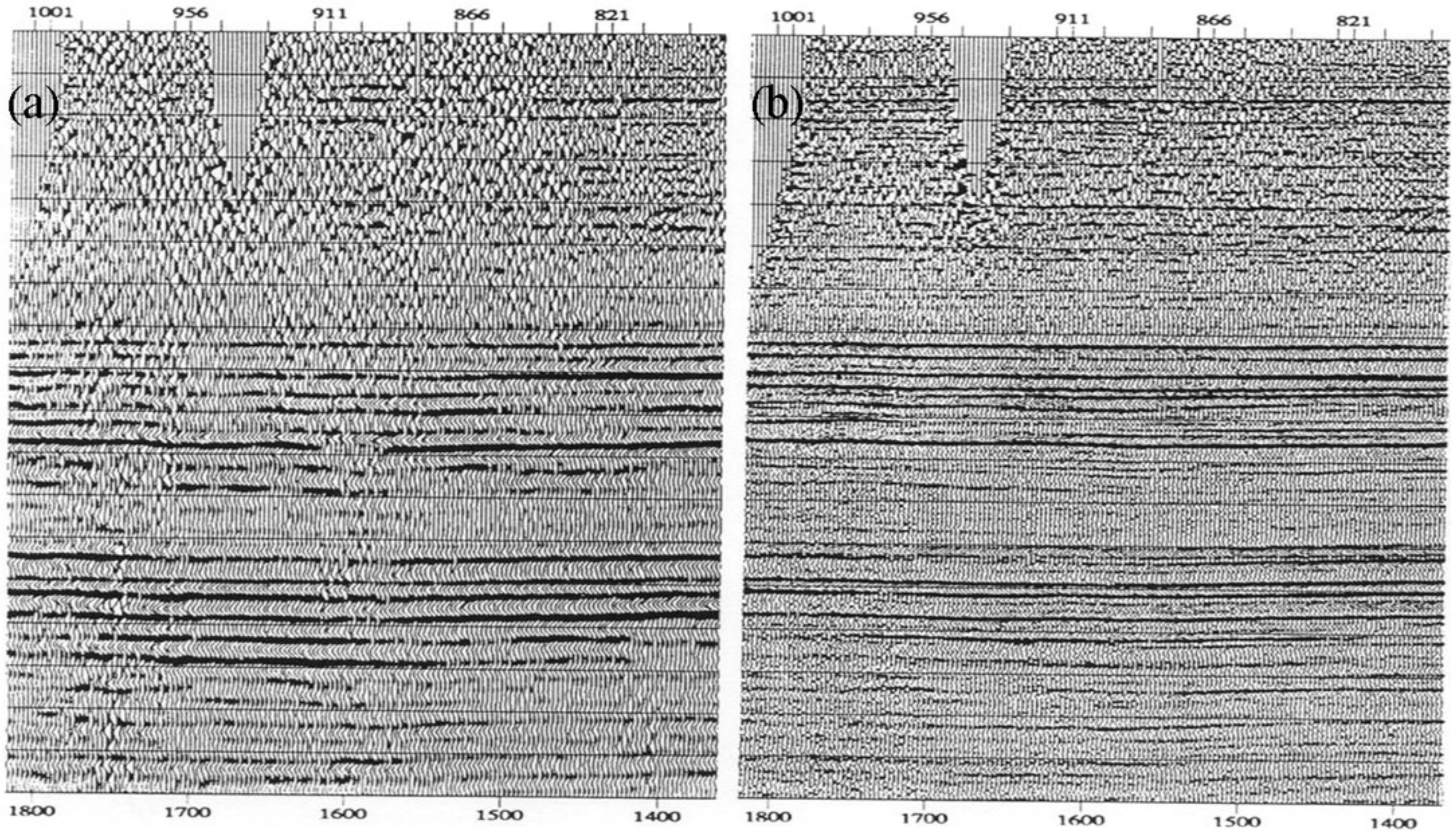
- Deconvolution
  - If a signal is modified by a rational polynomial system $Y(z) = \frac{B(z)}{A(z)} X(z)$

    and its inverse exists and is stable, you should be able to "undo" the filter with the inverse

    $$X(z) = H^{-1}(z)Y(z) = \frac{A(z)}{B(z)} Y(z)$$
  - Blurring or noise addition are often modeled by a transfer function
    - Theoretically, if you could figure out the transfer function, you could deconvolve the blurred signal to recover the original
    - Problem is, it is difficult to obtain the transfer function
    - Numerical methods are used, but they are sensitive to noise.

# Minimum Phase Systems

- Deconvolution of seismic data



# Discrete Fourier Transform

- Recall our Fourier transforms thus far:

**Fourier series for periodic signals:**

$$x(t) = x(t + T_0) \qquad f_0 = \frac{1}{T_0} \qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t} \qquad \text{(Synthesis Eq.)}$$

$$X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt \qquad \text{(Analysis Eq.)}$$

# Discrete Fourier Transform

– **Fourier transform aperiodic signals:**

$$X(\Omega) = \mathscr{F}[x(t)] = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt \qquad \text{(Analysis Equation)}$$

$$x(t) = \mathscr{F}^{-1}[X(\Omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\Omega)e^{+j\Omega t}d\omega \qquad \text{(Synthesis Equation)}$$

Time and frequency are continous variables

$-\infty < t < \infty$

$-\infty < \Omega < \infty$

Using $\Omega$ to distinguish it from dicrete time case where frequency is between -$\pi$ and $\pi$

# Discrete Fourier Transform

– **Discrete-time Fourier transform**

$$X(\omega)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad -\pi \le \omega < \pi \quad \text{(Analysis equation)}$$

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega \qquad \text{(Synthesis equation)}$$

Time, labeled by the integer index $n$, is discrete $(t = nT_s)$

$-\infty < n < \infty$

$-\pi < \omega < \pi$

Limits on $\omega$ are imposed by the Nyquist condition

$\pi$ represents maximum positive frequency $f_{Nyquist} = \dfrac{f_s}{2} = \dfrac{1}{2T_s}$

(where $T_s$ is the sampling interval or alternatively, $f_s$ is the sampling frequency)

# Discrete Fourier Transform

– **Discrete Fourier series for a periodic sequence with period *N***

$$x[n+mN]=x[n]$$

$$x[n]=\sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

where

$$c_k=\frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

# Discrete Fourier Transform

- For discrete systems, need a discrete frequency as well as discrete time
  - Over the range of frequencies, $-\pi \rightarrow \pi$, sample the frequency at $N$ points:

$$\omega = \frac{2\pi k}{N}, \quad 0 < k < N-1$$

$$X(\omega) \rightarrow X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi kn/N} \quad -\pi \le \omega < \pi$$

  - What kind of conditions does this impose on the time (or shift) signals we can have?

# Discrete Fourier Transform

– What kind of conditions does this impose on the time (or shift) signals?

– Break the sum over n into sections that are $N$-1 in length

$$X(\omega) \to X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi kn/N} = \cdots + \sum_{n=-N}^{-1} x[n]e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} + \sum_{n=N}^{2N-1} x[n]e^{-j2\pi kn/N} + \cdots$$

labeling each of these sums by $m$ :

$$X\left(\frac{2\pi k}{N}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=mN}^{mN+N-1} x[n]e^{-j2\pi kn/N}$$

change variables: $n = n' - mN$

$$X\left(\frac{2\pi k}{N}\right) = \sum_{m=-\infty}^{\infty} \sum_{n'=0}^{N-1} x[n' - mN]e^{-j2\pi k(n'-mN)/N} \qquad \left(e^{-j2\pi k(n'-mN)/N} = e^{-j2\pi kn'/N} e^{j2\pi km} = e^{-j2\pi kn'/N}\right)$$

# Discrete Fourier Transform

Replacing $n'$ with $n$ for notational convenience

$$X\left(\frac{2\pi k}{N}\right)=\sum_{m=-\infty}^{\infty}\sum_{n'=0}^{N-1}x[n-mN]e^{-j2\pi kn/N}$$

interchange the order of the summations:

$$X\left(\frac{2\pi k}{N}\right)=\sum_{n=0}^{N-1}\sum_{m=-\infty}^{\infty}x[n-mN]e^{-j2\pi kn/N}$$

Our (discretized in frequency) discrete-time Fourier transform

looks like a sum over our signal chopped up into segments of length $N$

Calling $\mathrm{x}_p[n]=\sum_{n'=0}^{N-1}x[n-mN]$

$$X\left(\frac{2\pi k}{N}\right)=\sum_{n=0}^{N-1}x_p[n]e^{-j2\pi kn/N}$$

# Discrete Fourier Transform

Consider the sum which is the periodic repetition of $x[n]$ every $N$ samples.

$$x_p[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$

$x_p[n]$ is itself periodic with period N

$$x_p[n+N] = \sum_{m=-\infty}^{\infty} x[n + N - mN]$$

$$x_p[n+N] = \sum_{m=-\infty}^{\infty} x[n - (m-1)N]$$

$$x_p[n+N] = \sum_{m-1=-\infty}^{\infty} x[n - mN] = \sum_{m=-\infty}^{\infty} x[n - mN]$$

$$x_p[n+N] = x_p[n]$$

It's periodic in period $N$

# Discrete Fourier Transform

Since $x_p[n]$ is a periodic sequence with period $N$ it has a Discrete Fourier Series:

$$x_p[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

where

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n] e^{-j2\pi kn/N}$$

Going back a few slides:

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x_p[n] e^{-j2\pi kn/N}$$

So, the Fourier series coefficients are the discretized Fourier transform!

$$c_k = \frac{1}{N} X\left(\frac{2\pi k}{N}\right) = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n] e^{-j2\pi kn/N}$$

# Discrete Fourier Transform

– We now have our Discrete Fourier Transform analysis and synthesis equations:

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x_p[n] e^{-j2\pi kn/N} \qquad \text{Analysis equation}$$

$$x_p[n] = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{j2\pi kn/N} \qquad \text{Synthesis equation}$$

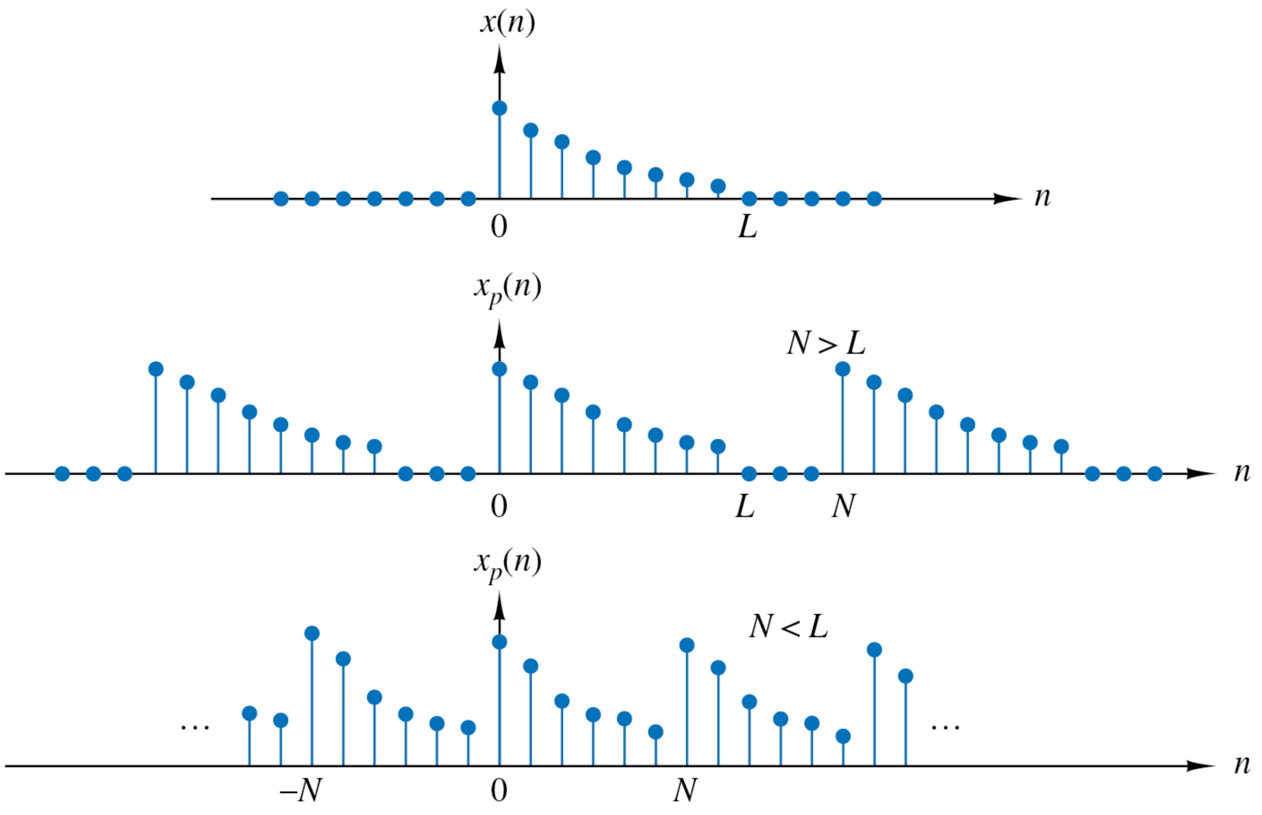
– Well, not quite, because is $x_p[n] = \sum_{m=-\infty}^{\infty} x[n-mN]$ the periodic extension of $x[n]$ the same as $x[n]$?

– If $x[n]$ is finite length, and the periodic extension doesn't overlap itself, this all works.

- If it overlapped, that would be "time" aliasing

# Discrete Fourier Transform

– Time aliasing



# Discrete Fourier Transform

– Bottom line for the Discrete Fourier Transform

- If signal is finite length, $L$, and $L < N$ (where $N$ is the number of frequency samples) then you can set

$$x[n] = \begin{cases} x_p[n], & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

- Now do have the analysis and synthesis equations:

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$ Analysis equation

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{j2\pi kn/N}$$ Synthesis equation

# Discrete Fourier Transform

- If this is all good, and there is no time aliasing, should be able to reconstruct the Discrete-time Fourier transform with a continuous variable $\omega$ with an interpolation formula
- Interpolation formula for $X(\omega)$

$$X(\omega)=\sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) P\left(\omega-\frac{2\pi k}{N}\right)$$

where

$$P(\omega)=\frac{1}{N}\frac{1-e^{j\omega N}}{1-e^{-j\omega}}$$

which by factoring out $e^{-j\omega N/2}$ in the numerator and $e^{-j\omega/2}$ in the denominator

can be written as:

$$P(\omega)=\frac{1}{N}\frac{\sin(\omega N/2)}{\sin(\omega/2)}e^{-j\omega(N-1)/2}$$

# Discrete Fourier Transform

– Summary of Discrete Fourier Transform

Discrete Fourier Transform (DFT) Analysis Equation

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \quad , \quad k = 0, 1, 2, \ldots, N\text{-}1$$

Inverse Discrete Fourier Transform (IDFT) Synthesis Equation

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N} \quad , \quad n = 0, 1, 2, \ldots, N\text{-}1$$

# Discrete Fourier Transform

– Padding a finite sequence:

- Good example in book:

$$x[n] = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases} \qquad X(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

If you choose $N = L$

$$X[k] = \begin{cases} L, & k = 0 \\ 0, & k = 1, 2, \ldots, N-1 \end{cases}$$

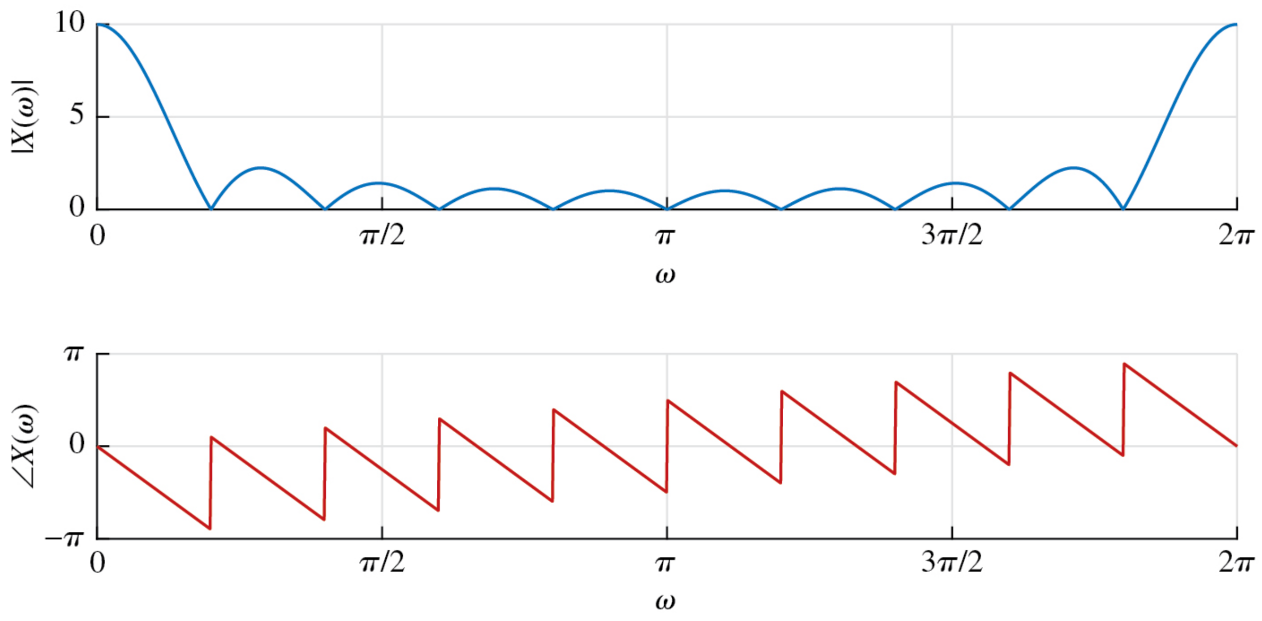
– Other choices for $N$, padding signal by 0's for $N$-$L$

$N$=50

$N$=100

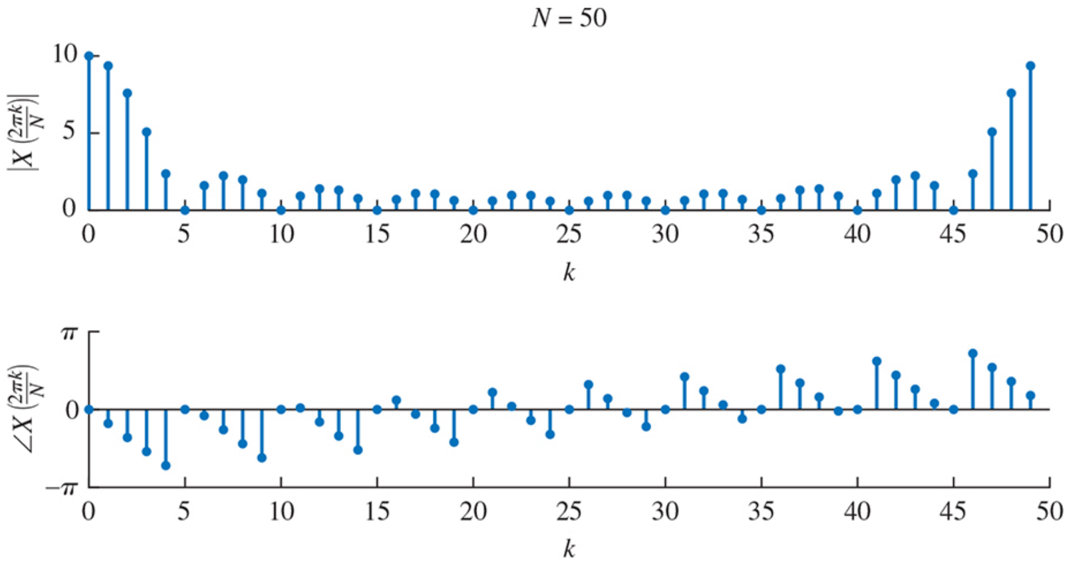
# Discrete Fourier Transform

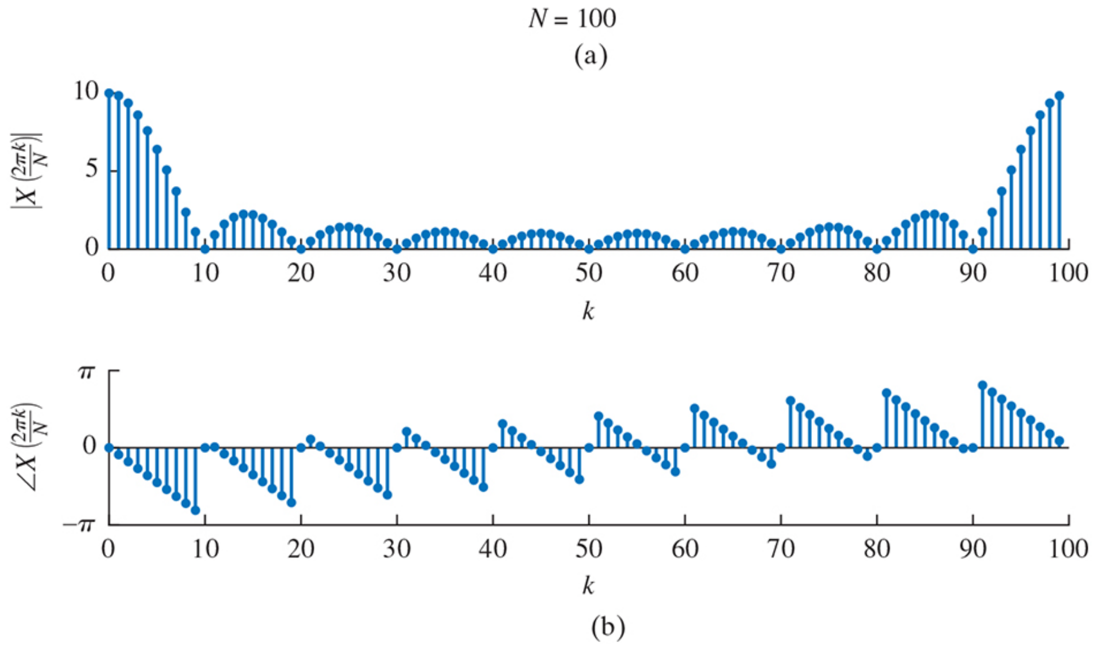
– Magnitude and phase of $X(\omega)$



# Discrete Fourier Transform

– Magnitude and phase of $X[k]$ for $N$=50 and $N$ = 100



# Discrete Fourier Transform

Magnitude and phase of $X(\omega)$