Digital Signal Processing

Class 9
02/17/2025

ENGR 71

- Class Overview
  - Z-Transform
- Assignments
  - Reading:

    Chapter 3: The z-Transform and its Applications to the Analysis of LTI

ENGR 71

- Homework 3
  - Problems: 2.9 (a), 2.17(a), 2.28(a & c),
    2.35, 2.46,
    C2.14(write your own code)
    C2.8 (use Matlab functions)

    Due Feb. 20

ENGR 71

- Lab 1-Aliasing Lab
  - Find a short piece of music to download
  - Subsample to demonstrate aliasing
  - Repeat the subsampling, but prior to subsampling, apply a low-pass anti-aliasing filter.
  - Compare the results
  - Mystery piece

Class Information

- Z-Transform Topics
  - The z-Transform
  - Properties of the z-Transform
  - Rational z-Transforms
  - Inversion of the z-Transform
  - Analysis of Linear Time-Invariant Systems in the z-Domain
  - The One-sided z-Transform

Z-Transform

- As with continuous systems LTI systems, there is an easier way to solve Discrete Linear Time (Shift) Invariant systems (LTI or LSI)
- The z-transform
  - Discrete version of Laplace transform
  - Many properties analogous to Laplace
    - Continuous Laplace: differential equations → algebraic equations
    - Discrete z-transform: difference equations → algebraic equations
    - Continuous Laplace: Stability determined by pole locations
      - Left half-plane
    - Discrete z-transform: Stability determined by pole locations
      - Inside unit circle
    - Other properties like convolution, time shift, initial & final values, etc.

Z-Transform

- Connection of Laplace and z-transform
  - Laplace transform of sampled signal

$$x(t) = \sum_n x(nT_s)\,\delta(t - nT_s)$$

$$\mathcal{L}\{x(t)\} = X(s) = \mathcal{L}\left\{\sum_n x(nT_s)\,\delta(t - nT_s)\right\}$$

$$X(s) = \sum_n x(nT_s)\,\mathcal{L}\{\delta(t - nT_s)\}$$

$$\mathcal{L}\{\delta(t)\} = 1 \quad ; \quad \mathcal{L}\{f(t - t_0) = F(s)e^{-st_0}$$

$$\text{so,} \quad \mathcal{L}\{\delta(t - nT_s)\} = 1e^{-snT_s} = \left(e^{-sT_s}\right)^n$$

$$X(s) = \sum_n x(nT_s)\left(e^{-sT_s}\right)^n \qquad z = e^{sT_s} \qquad x[n] = x(nT_s)$$

$$X(s) = \sum_n x[n]\,z^{-n} \qquad \boxed{\mathcal{Z}\{x[n]\} \equiv \sum_n x[n]\,z^{-n}}$$

Z-Transform

- The Laplace transform of sampled signal on the imaginary axis is periodic
  - In the Laplace domain the $s = j\Omega$ corresponds to frequency domain (Fourier transform)
  - On imaginary axis

$$X(\Omega) = \sum_n x(nT_s)\left(e^{-j\Omega T_s}\right)^n$$

This is periodic with period $2\pi/T_s$ (This is the sampling frequency)

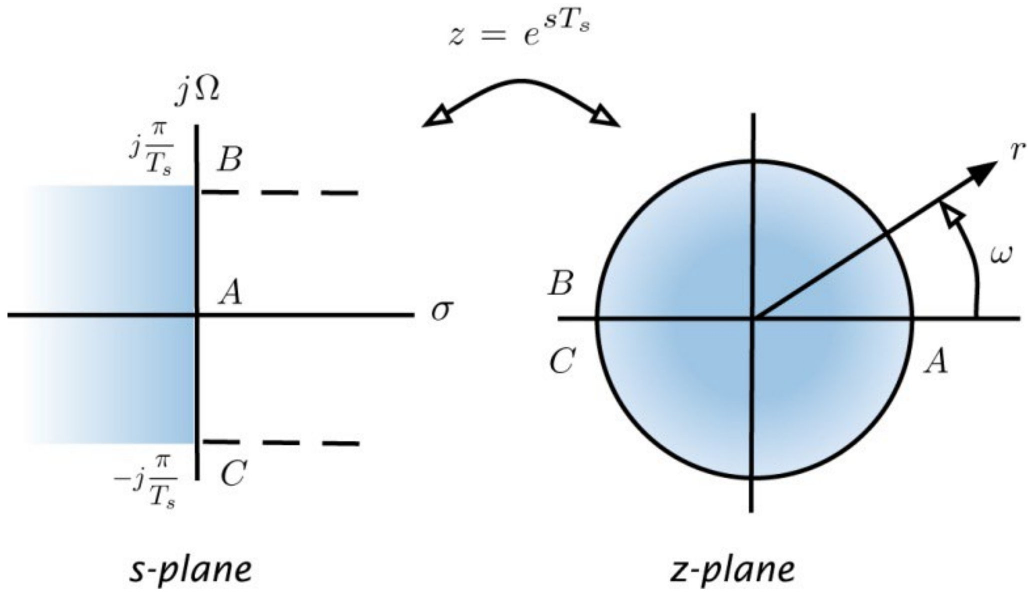
$$X(\Omega + 2\pi k/T_s) = X(\Omega)$$

$$z = e^{sT_s} = e^{(\sigma + j\Omega)T_s} = e^{\sigma T_s} e^{j\Omega T_s}$$

In polar form, defining: $r = e^{\sigma T_s}$ and $\omega = \Omega T_s$

Z-Transform

- Picture of the mapping s-plane to z-plane:



Notice that $j\pi/T_s$ is the Nyquist frequency
Left half-plane of strip maps into interior of unit circle
Imaginary axis maps to unit circle.
Right half-plane maps to exterior of unit circle

Z-Transform

- Computing the z-transform:
  - Important thing to remember: Geometric Series

$$\boxed{\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r} \quad \text{for } r \neq 1}$$

$$\boxed{\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad \text{for } |r| < 1}$$

Z-Transform

- Z-transform:
  - Laplace: $$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$
  - z-transform: $$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$
  - As in the case of the Laplace transform, we are mainly interested in causal signals and systems: $$x(t) = x(t)u(\mathrm{t})$$ $$x[n] = x[n]u[n]$$
  - Limits in sum and integral start at 0:

$$X(s) = \int_{0}^{+\infty} x(t) e^{-st} dt$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Z-Transform

- Z-transform:
  - Laplace: You can easily solve linear differential equations with constant coefficients in the Laplace domain.
    - These equations correspond to linear time-invariant systems
  - z-transform: Same function as Laplace, except for discrete time signals and systems.
    - LTI systems represented by difference equations.
    - You can solve for system response in the z-transform domain, and then use inverse z-transform to find response in sampled time domain.

Z-Transform

- Definition of z-transform:
  - Bilateral

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

  - Unilateral (causal signals & systems)

$$X(z) = \sum_{n=0}^{\infty} \mathrm{x}(n)\, z^{-n}$$

  - Inverse:

$$h(n) = \frac{1}{2\,\pi\, j} \oint H(z)\, z^{-n+1} dz$$

    - Rarely use this, although this is common integral in complex variables math courses.
    - We compute forward & inverse by use of transform pairs and properties.
    - Can also find inverse by long division.

Z-Transform

- Rational function

$$X(z)=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+\cdots+a_Nz^{-N}}=\frac{\sum_{k=0}^{M}b_kz^{-k}}{\sum_{k=0}^{N}a_kz^{-k}}$$

- Can also be written with positive powers of z

$$X(z)=\left(\frac{b_0z^{-M}}{a_0z^{-N}}\right)\frac{z^M+\left(b_1/b_0\right)z^{M-1}+\cdots+\left(b_M/b_0\right)}{z^N+\left(a_1/a_0\right)z^{N-1}+\cdots+\left(a_N/a_0\right)}$$

- The numerator and denominator can be factor into product of linear terms:

$$X(z)=\left(\frac{b_0}{a_0}\right)z^{N-M}\frac{\left(z-z_1\right)\left(z-z_2\right)\cdots\left(z-z_M\right)}{\left(z-p_1\right)\left(z-p_2\right)\cdots\left(z-p_M\right)}$$

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Z-Transform

- A few examples:
  - Impulse: $h(n) = \delta(n)$

$$H(z) = \sum_{n=0}^{\infty} \delta[n]\, z^{-n} = 1$$

Region of convergence is entire z-plane

  - Delayed impulse:

$$h(n) = \delta(n-1)$$

$$H(z) = \sum_{n=0}^{\infty} \delta[n-1]\, z^{-n} = z^{-1}$$

Region of convergence is entire z-plane

  - Causal exponential signal (geometric series)

$$x(n) = a^n u[n]$$

$$X(z) = \sum_{n=0}^{\infty} a^n\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{1}{1-\frac{a}{z}} \quad \text{if } \left|\frac{a}{z}\right| < 1$$

Region of convergence: $|z| > |a|$

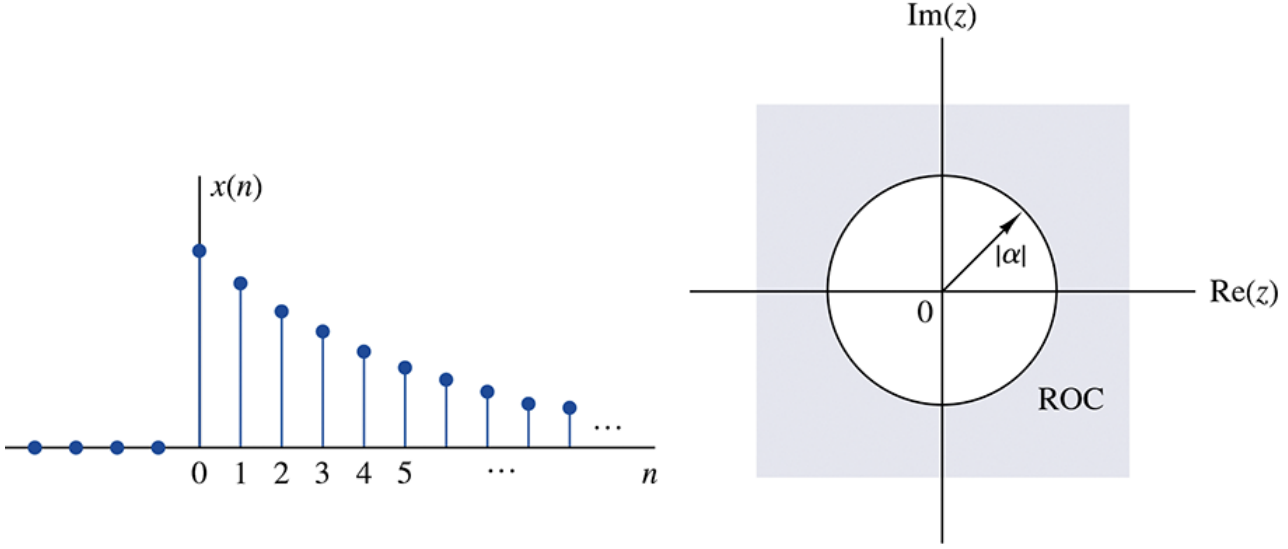
Z-Transform

Causal exponential signal (geometric series)

$$x(n) = a^n u[n] \qquad \text{(causal signal)}$$

$$X(z) = \sum_{n=0}^{\infty} a^n\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{1}{1-\frac{a}{z}} \quad \text{if } \left|\frac{a}{z}\right| < 1$$

Region of convergence: $|z| > |a|$



(a) (b)

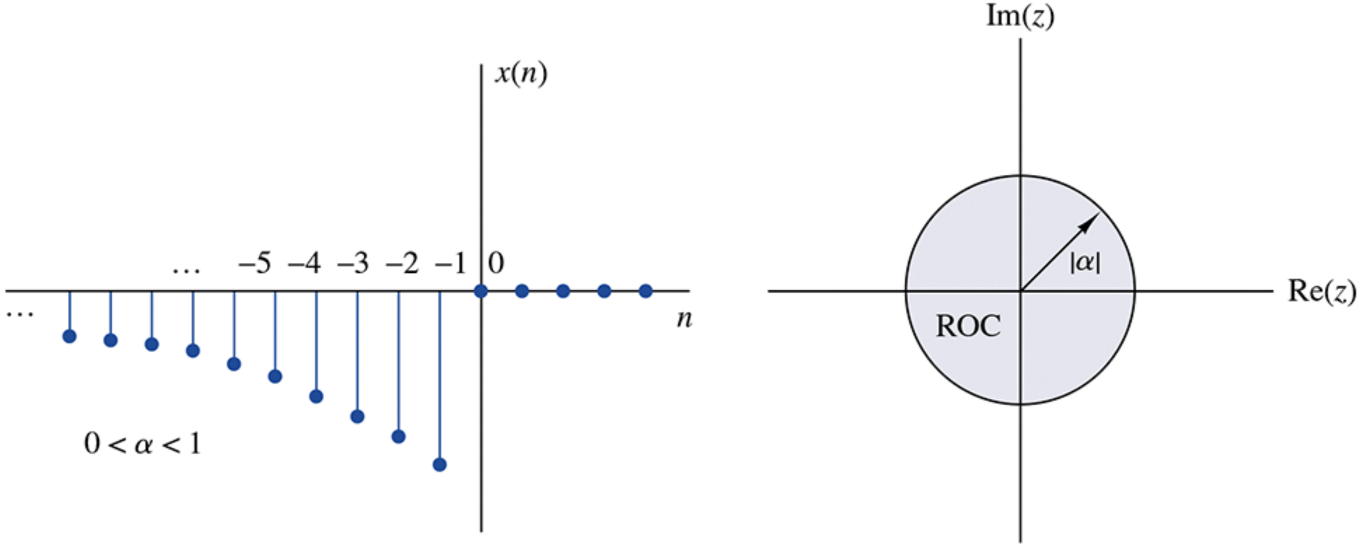
Z-Transform

$$x(n) = -a^n u[-n-1] \qquad \text{(anti-causal signal)}$$

$$X(z) = \sum_{n=-\infty}^{1} -a^n\ z^{-n} = \frac{1}{1-\frac{a}{z}} \quad \text{if} \left|\frac{z}{a}\right| < 1$$

Region of convergence: $|a| > |z|$

Same expression as before,
but region of convergence is different



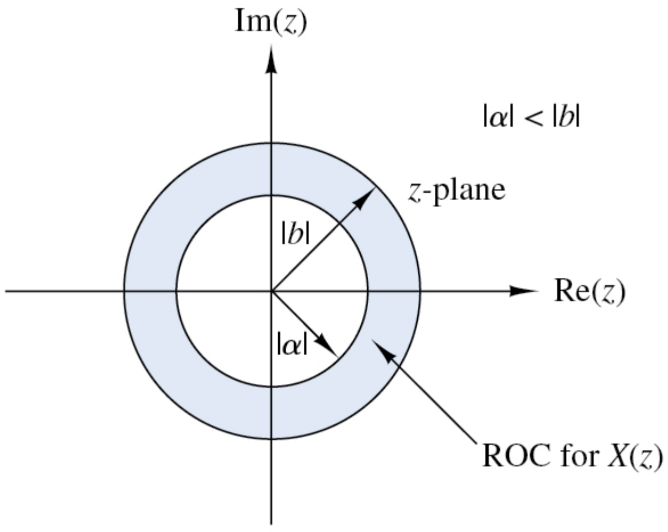
(a) (b)

Z-Transform

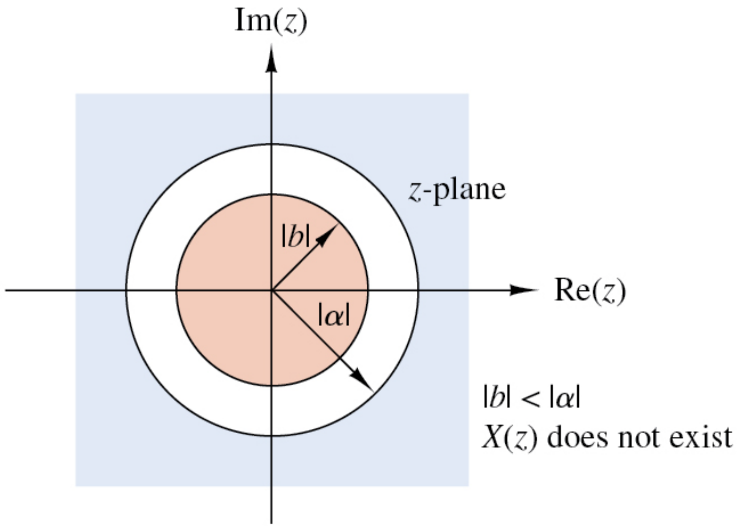
Neither caual nor anti-causal

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

$$H(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n + \sum_{n=1}^{\infty} \left(\frac{z}{b}\right)^n$$

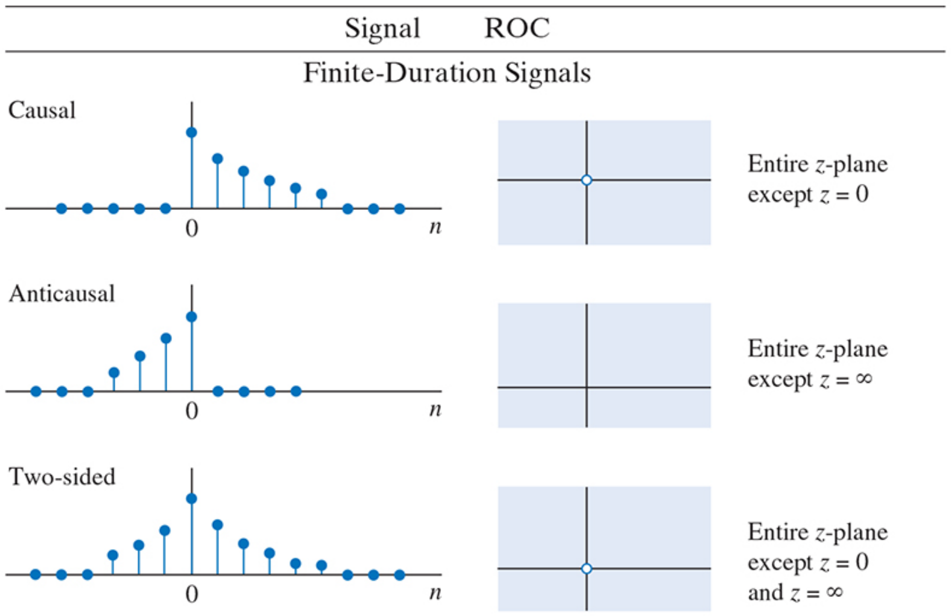


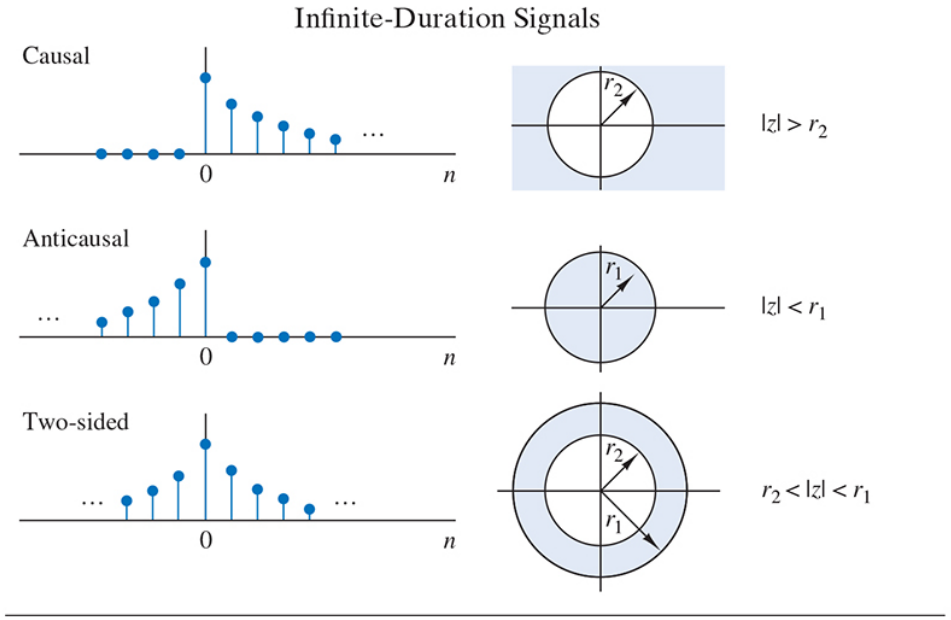
(b)



(a)

Z-Transform





Finite duration signals always converge

Infinite duration signals have regions have circular or annulus shaped regions of convergence

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Z-Transform Properties

- Linearity

$$a_1 x_1[n] + a_2 x_2[n] \Leftrightarrow a_1 X_1(z) + a_2 X_2(z)$$

– Example: Use linearity to find z-transform of

$$x(n) = (\cos \omega_0 n) u(n)$$
$$x(n) = (\sin \omega_0 n) u(n)$$

$$\mathcal{Z}[(\cos \omega_0 n) u(n)] = \frac{1 - z^{-1} \cos \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}} \quad \text{ROC } |z| > 1$$

$$\mathcal{Z}[(\sin \omega_0 n) u(n)] = \frac{z^{-1} \sin \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}} \quad \text{ROC } |z| > 1$$

Z-Transform Properties

- Time shift

For $x[n] \Leftrightarrow X(z)$

$$x[n-k] \Leftrightarrow z^{-k}X(z)$$

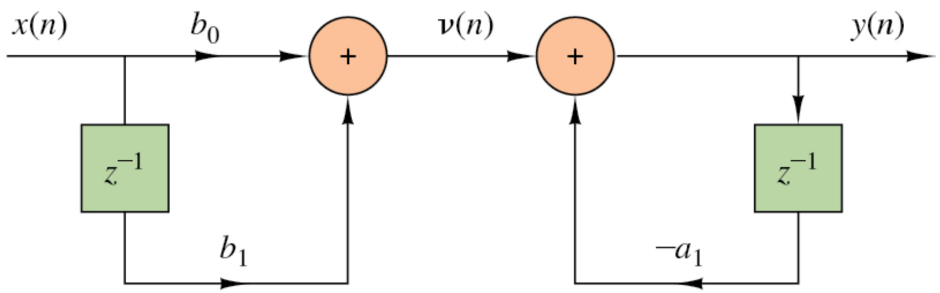
$$Z[x(n)] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$Z[x(n-k)] = \sum_{n=-\infty}^{\infty} x[n-k]z^{-n} = \sum_{n=-\infty}^{\infty} x[n-k]z^{-(n-k)}z^{-k} = z^{-k}\sum_{n=-\infty}^{\infty} x[n-k]z^{-(n-k)} = z^{-k}\sum_{m=-\infty}^{m} x[m]z^{-m}$$

$$Z[x(n-k)] = z^{-k}Z[x(n)]$$

Z-Transform Properties

- Time shift
  - We have seen how systems are described in terms of delays
  - In the z-transform domain, these delays become factors of z^{-1}



Z-Transform Properties

- Scaling

For $x[n] \Leftrightarrow X(z)$ ROC $r_1 < |z| < r_2$

$$\alpha^n x[n] \Leftrightarrow X\left(\frac{z}{\alpha}\right) \quad \text{ROC} \quad |\alpha| r_1 < |z| < |\alpha| r_2$$

Easy to prove from definition of z-transform

$$Z[x(n)] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$Z\left[\alpha^n x(n)\right] = \sum_{n=-\infty}^{\infty} x[n] \alpha^n z^{-n} = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{\alpha}\right)^{-n}$$

Z-Transform Properties

- Time reversal:

For $x[n] \Leftrightarrow X(z)$ ROC $r_1 < |z| < r_2$

$$x[-n] \Leftrightarrow X\left(\frac{1}{z}\right) \quad \text{ROC} \quad \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

Z-Transform Properties

- Convolution: (convolution in time is product in z-domain)

For $x[n] \Leftrightarrow X(z)$ & $y[n] \Leftrightarrow Y(z)$

$x[n] * y[n] \Leftrightarrow X(z)Y(z)$

ROC is at least the intersection of ROC's for x and y

Z-Transform Properties

- Note on convolution:
  - For LTI system, output of system is convolution of input with impulse response

$$y[n] = \sum_{k=0}^{\infty} x[k]h[n-k] = x[n]*h[n]$$

$$Z\{y[n]\} = Y(z) = Z\{x[n]*h[n]\} = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

  - *H(z)* is transfer function

Z-Transform Properties

- Correlation

For $x[n] \Leftrightarrow X(z)$ & $y[n] \Leftrightarrow Y(z)$

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(l)y(n-l) \Leftrightarrow X(z)Y(-z)$$

Using relationship between correlation and convolution:

$$r_{xy}(l) = x(l) * y(-l)$$

Z-Transform Properties

- Multiplication in time domain
  (Involves integration in complex space)

$$x(n)y^*(n) \Leftrightarrow \frac{1}{2\pi j}\oint X(v)Y^*\left(\frac{z^*}{v^*}\right)v^{-1}dv$$

  – Parseval's relation for energy in z-transform domain

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi j}\oint X(v)Y^*\left(\frac{1}{v^*}\right)v^{-1}dv$$

Z-Transform Properties

- Initial and final value theorem (for causal signals)
  - Initial value:

$$x(0) = \lim_{z \to \infty} X(z)$$

  - Final value

$$\lim_{n \to \infty} x(n) = \lim_{z \to 1} (z-1) X(z)$$

Z-Transform Properties

Z Transform Properties	
Property Name	**Illustration**
Linearity	$af_1[k]+bf_2[k] \xleftrightarrow{Z} aF_1(z)+bF_2(z)$
Left Shift by 1	$f[k+1] \xleftrightarrow{Z} zF(z)-zf[0]$
Left Shift by 2	$f[k+2] \xleftrightarrow{Z} z^2F(z)-z^2f[0]-zf[1]$
Left Shift by n	$f[k+n] \xleftrightarrow{Z} z^nF(z)-z^n\sum_{k=0}^{n-1} f[k]z^{-k}$ $= z^n\left(F(z)-\sum_{k=0}^{n-1} f[k]z^{-k}\right)$
Right Shift by n	$f[k-n] \xleftrightarrow{Z} z^{-n}F(z)$
Multiplication by time	$kf[k] \xleftrightarrow{Z} -z\frac{dF(z)}{dz}$
Scale in z	$a^kf[k] \xleftrightarrow{Z} F\left(\frac{z}{a}\right)$
Scale in time	$f\left[\frac{k}{n}\right] \xleftrightarrow{Z} F(z^n)$; n is an integer $n \geq 1$
Convolution	$f_1[k]*f_2[k] \xleftrightarrow{Z} F_1(z)F_2(z)$
Initial Value Theorem	$f[0]=\lim_{z\to\infty} F(z)$
Final Value Theorem (if final value exists)	$\lim_{k\to\infty} f[k]=\lim_{z\to 1}(z-1)F(z)$

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\text{Im}\{x(n)\}$	$\frac{1}{2}j[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\dfrac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}\,dv$	At least, $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

Z-Transform Rational Functions

- Rational function

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Can also be written with positive powers of z

$$X(z) = \left(\frac{b_0 z^{-M}}{a_0 z^{-N}}\right) \frac{z^M + (b_1/b_0) z^{M-1} + \cdots + (b_M/b_0)}{z^N + (a_1/a_0) z^{N-1} + \cdots + (a_N/a_0)}$$

- The numerator and denominator can be factor into product of linear terms:

$$X(z) = \left(\frac{b_0}{a_0}\right) z^{N-M} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_M)}$$

Z-Transform Rational Functions

- Zeros and Poles

$$\frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_M)}$$

$z_1, z_2, \ldots$ are the zeros

$p_1, p_2, \ldots$ are the poles

Z-Transform Rational Functions

- Zeros and Poles

$$\frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_M)}$$

$z_1, z_2, \ldots$ are the zeros

$p_1, p_2, \ldots$ are the poles

- Example:

$$x(n) = a^n u(n)$$

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad \text{ROC} \quad |z| > a$$

zero at $z = 0$,

pole at $z = a$

Z-Transform Rational Functions

- Example:

$$x(n) = a^n u(n)$$

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \quad \text{ROC } |z| > a$$

zero at $z = 0$,

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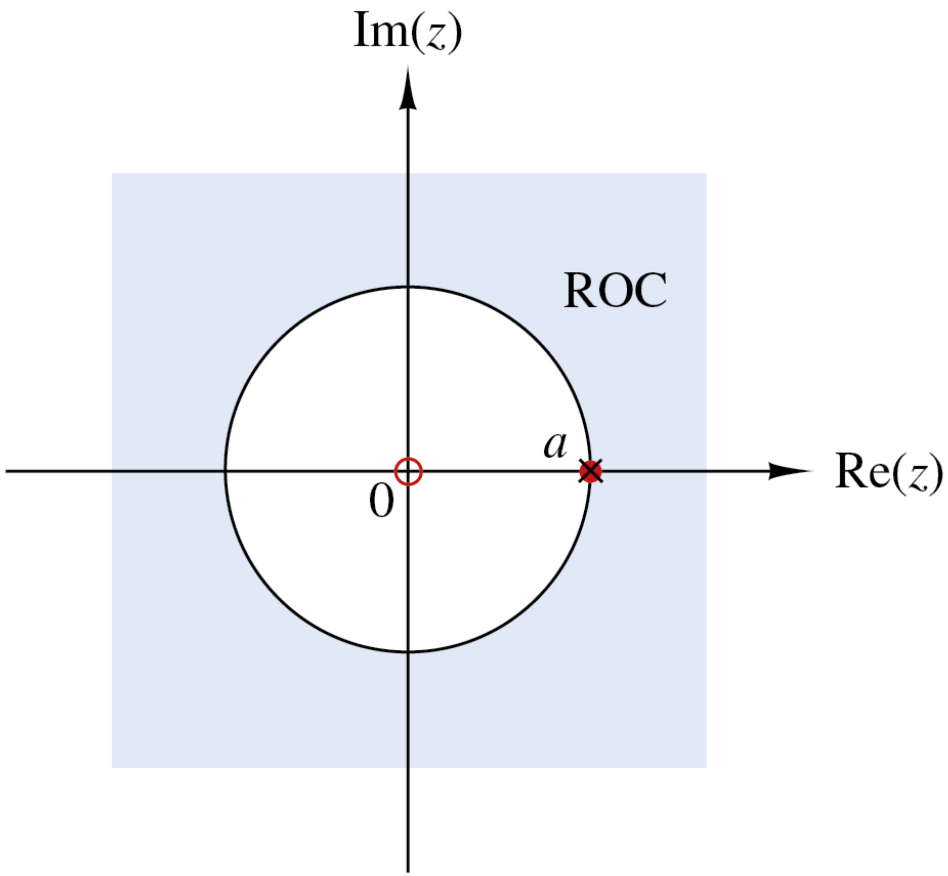
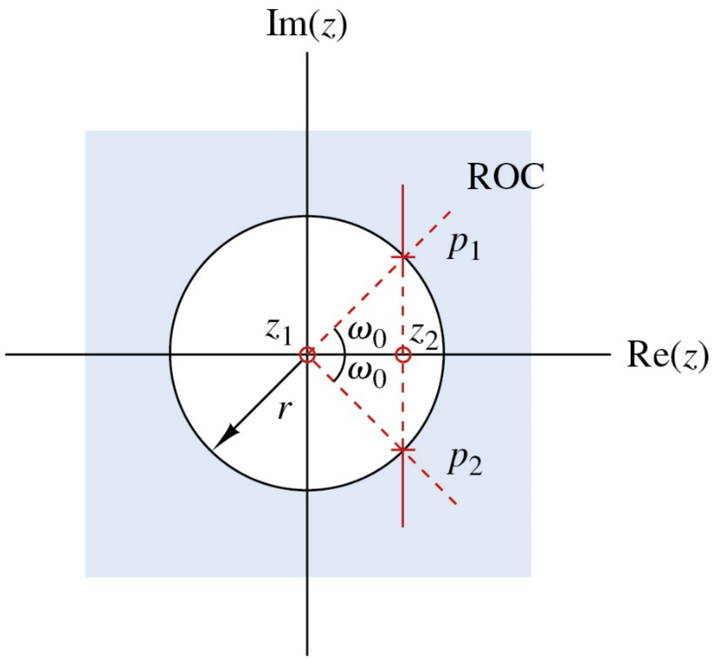


Z-Transform Rational Functions

- Example:

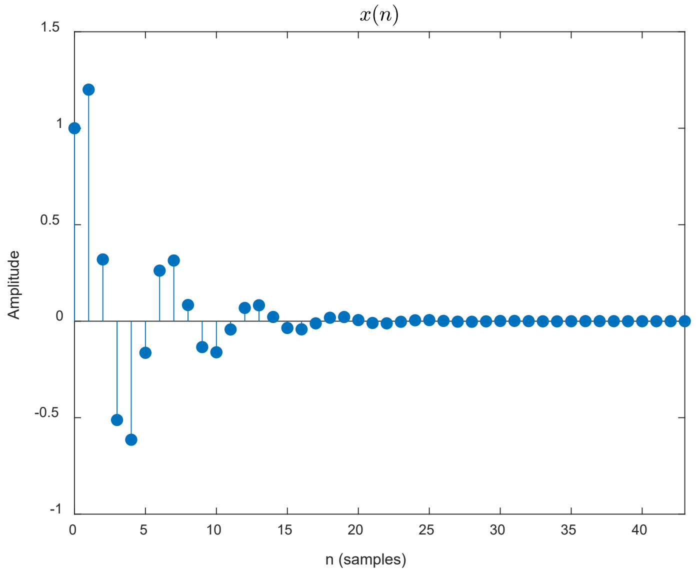


Zeros: $z_1 = 0,\, z_2 = r\cos\omega_0$

Poles: $p_1 = re^{j\omega_0},\, p_2 = re^{-j\omega_0}$ (complex conjugate pair)

$$X(z) = G\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = \frac{z(z-r\cos\omega_0)}{(z-re^{j\omega_0})(z-re^{-j\omega_0})} \qquad \text{ROC} \quad |z| > r$$

$$x(n) = r^n \cos(n\omega_0)u(n)$$



Z-Transform Rational Functions

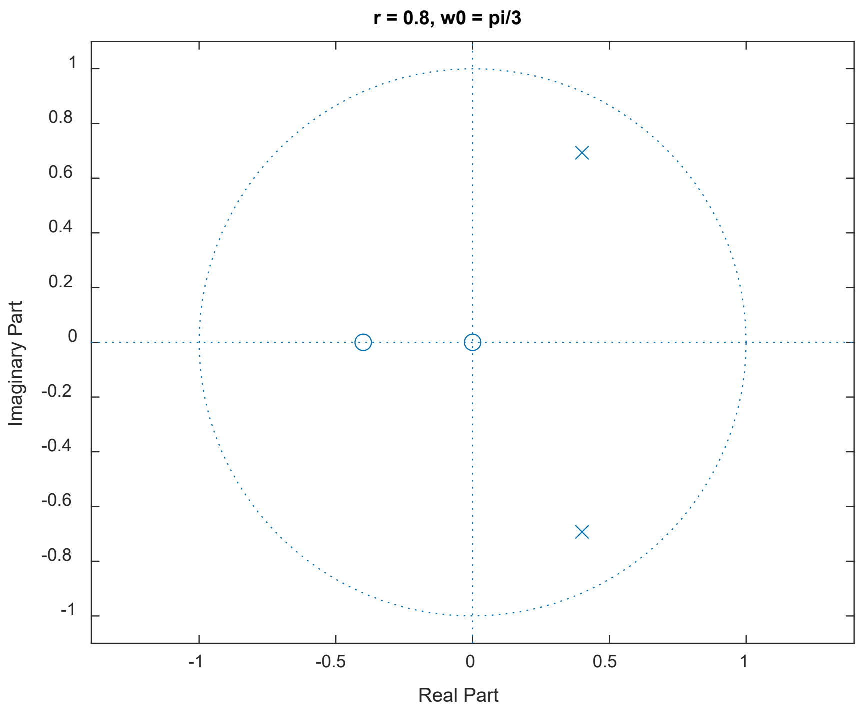
- Example:

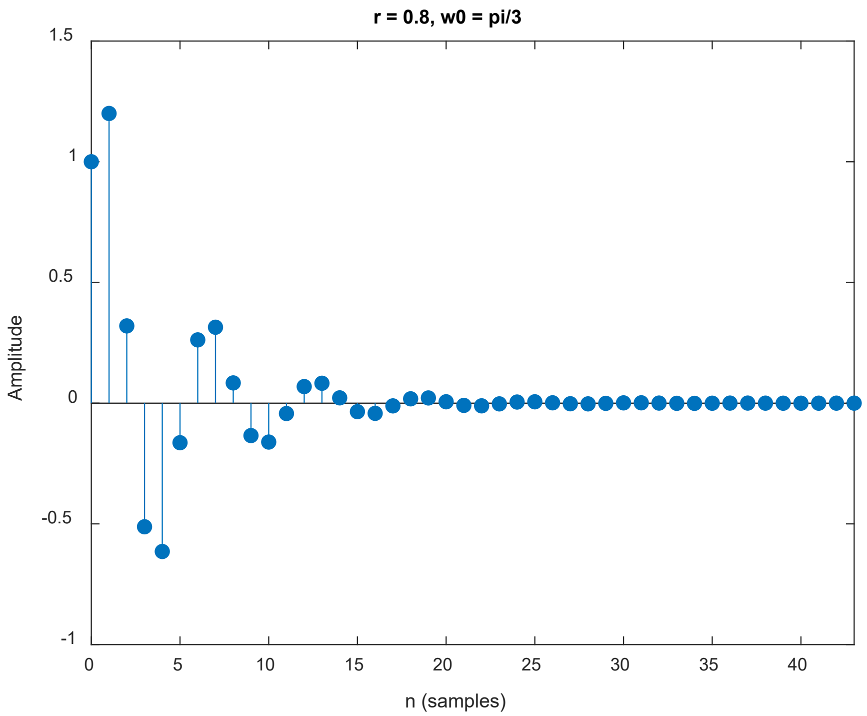
$$x(n) = r^n \cos(n\omega_0)u(n)$$

$$X(z) = \frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} \qquad \text{ROC} \quad |z| > r$$

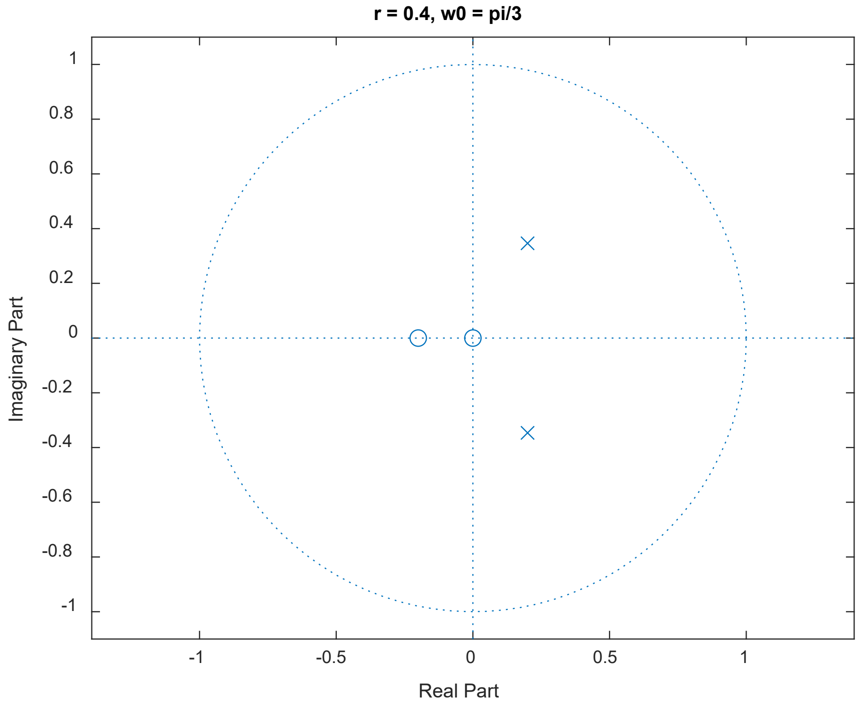
Try several values to see how rate of oscillation and decay vary with pole location

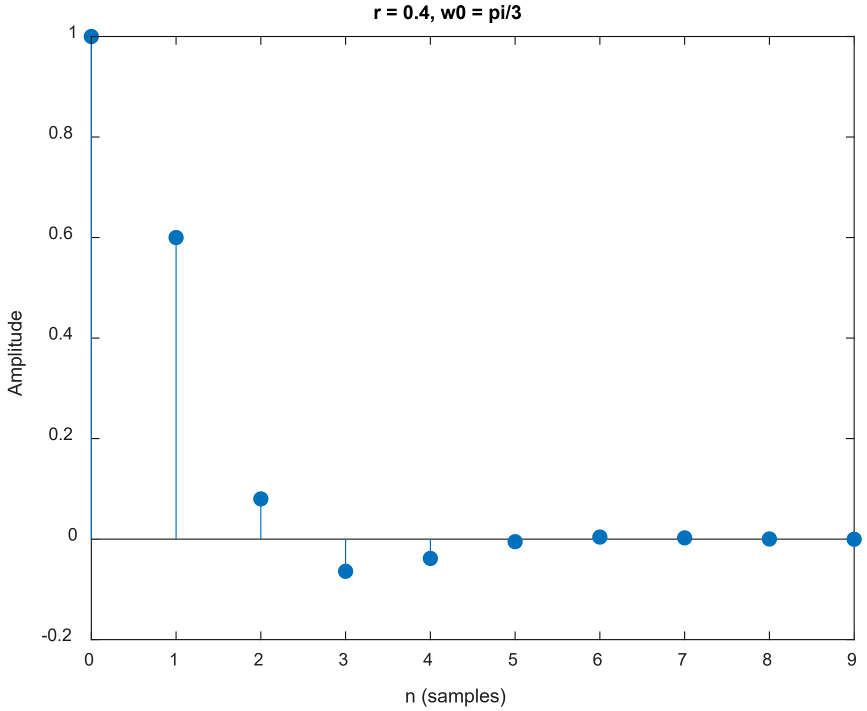
Z-Transform Rational Functions



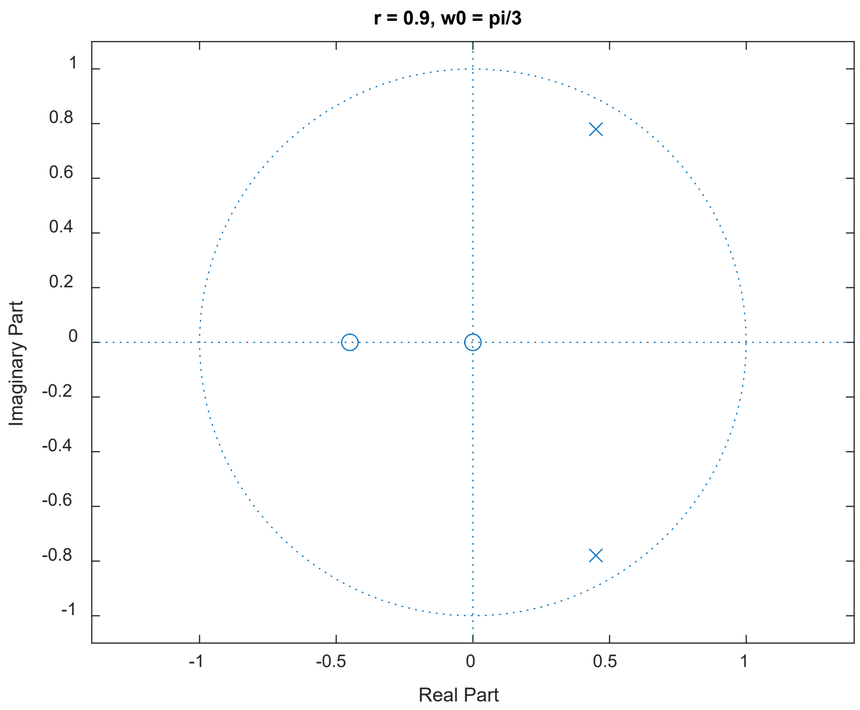


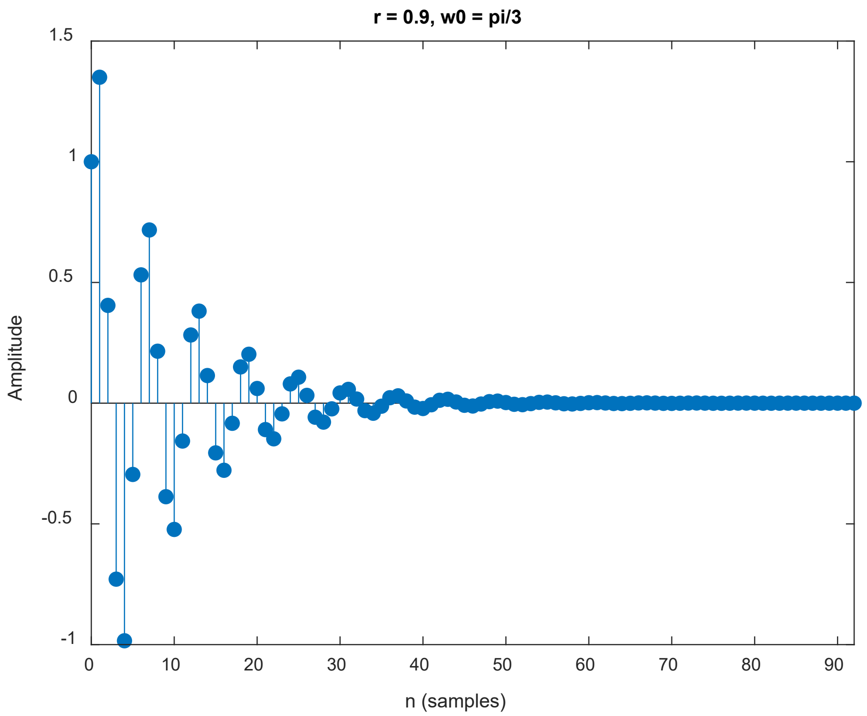
Z-Transform Rational Functions



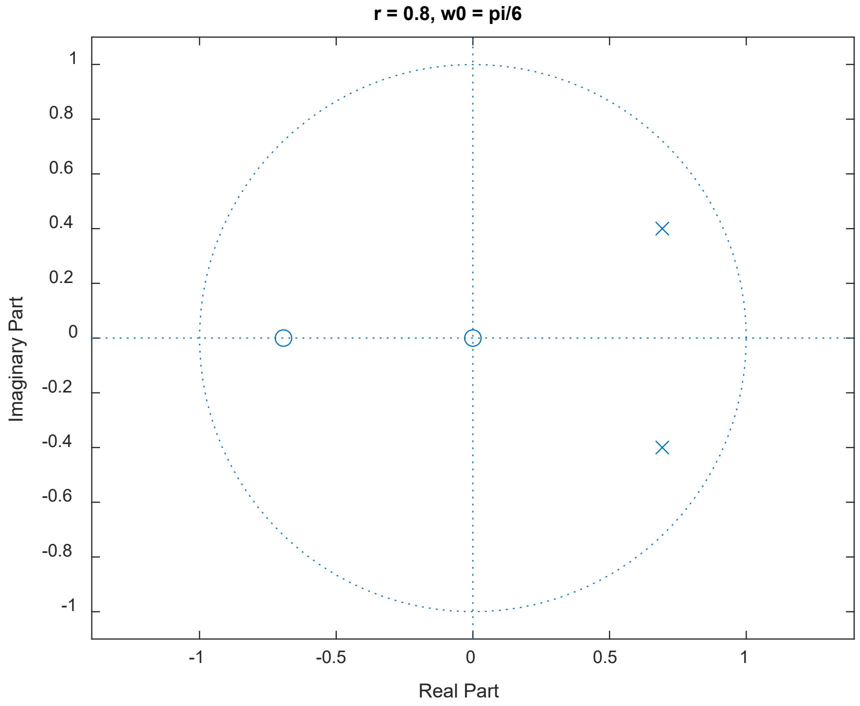


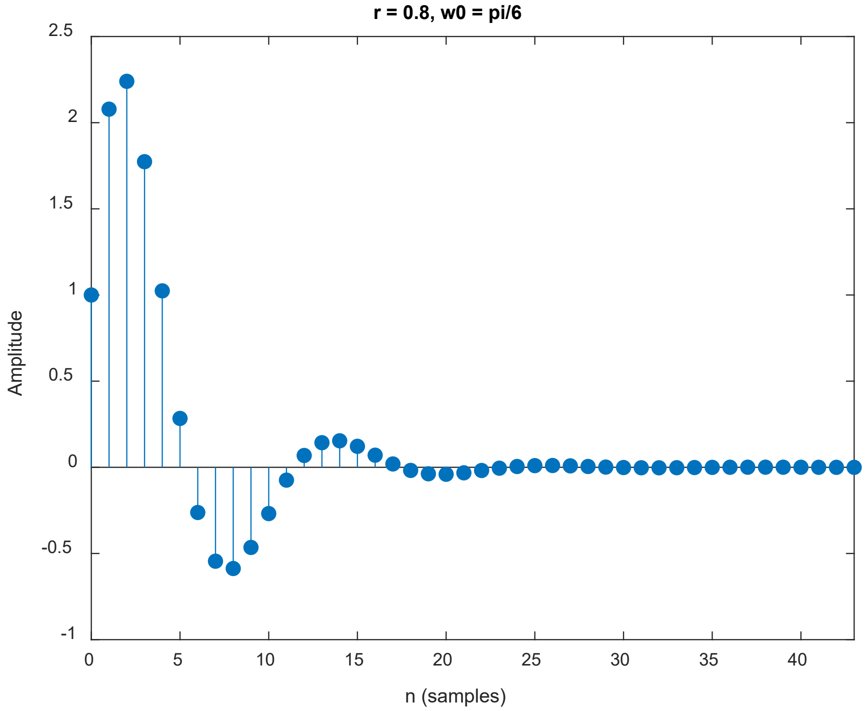
Z-Transform Rational Functions



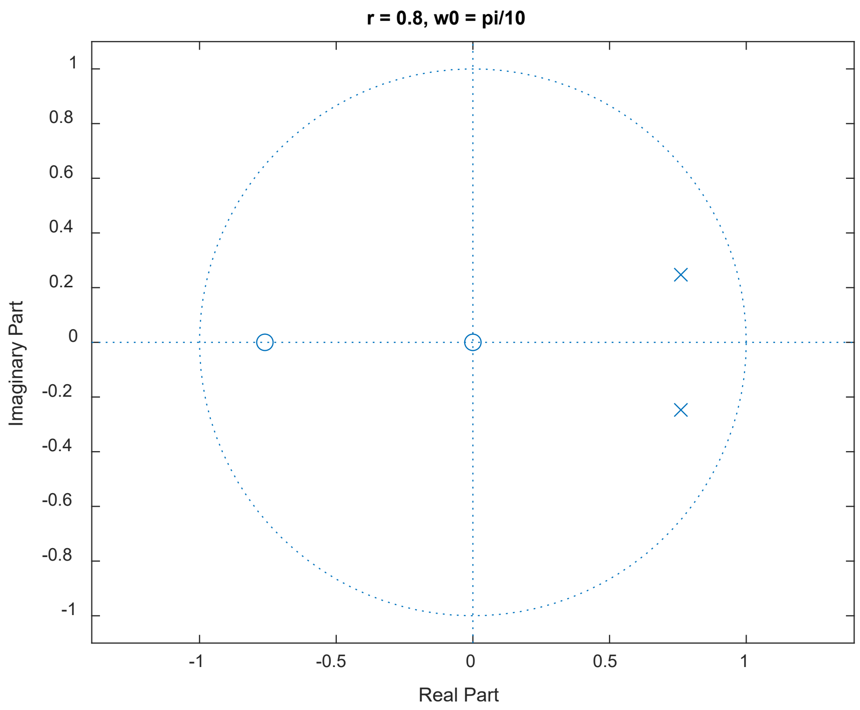


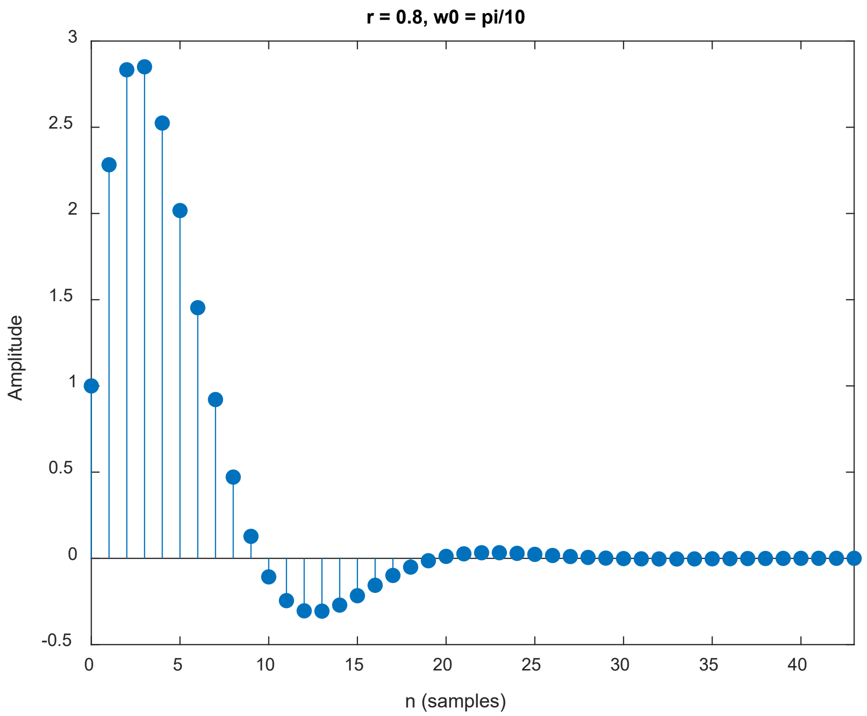
Z-Transform Rational Functions





Z-Transform Rational Functions





Z-Transform Inverse

- Three ways to find inverse
  - Contour integration in complex plane using Cauchy residue theorem (which we will not do)

$$h(n) = \frac{1}{2\pi j} \oint H(z)\, z^{-n+1} dz$$

  - Power series expansion of function (which we will not do)

$$H(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n}$$

$$x(n) = c_n$$

Z-Transform Inverse

- Three ways to find inverse
  - Partial fraction expansion and then beat it into shape we recognize from table (which we will do)

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

To be a proper rational expression,
order of numerator must be less than that
of the denominator

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

Z-Transform of rational functions

- Detailed examples of how to find inverse z-transform:

$$H_1(z) = \frac{1 + 2z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

Easier to work with:

$$H_1(z) = \left(\frac{z^2}{z^2}\right)\frac{1 + 2z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = \frac{z^2 + 2z + 2}{z^2 - 3z \ + 2}$$

– As in the case of the Laplace, do partial fractions expansion

- However, to have proper rational polynomial expression, do partial fractions of

$$\frac{H(z)}{z} = \frac{z^2 + 2z + 2}{z\left(z^2 - 3z + 2\right)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

Z-Transform

$$\frac{H(z)}{z} = \frac{z^2 + 2z + 2}{z\left(z^2 - 3z + 2\right)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$$A = 1 \quad B = -5 \quad C = 5$$

$$h[n] = \delta[n] - 5u[n] + 5 \cdot 2^n u[n]$$

Z-Transform

- A few Matlab tools:

zplane(b,a) plots poles and zeros in z-plane

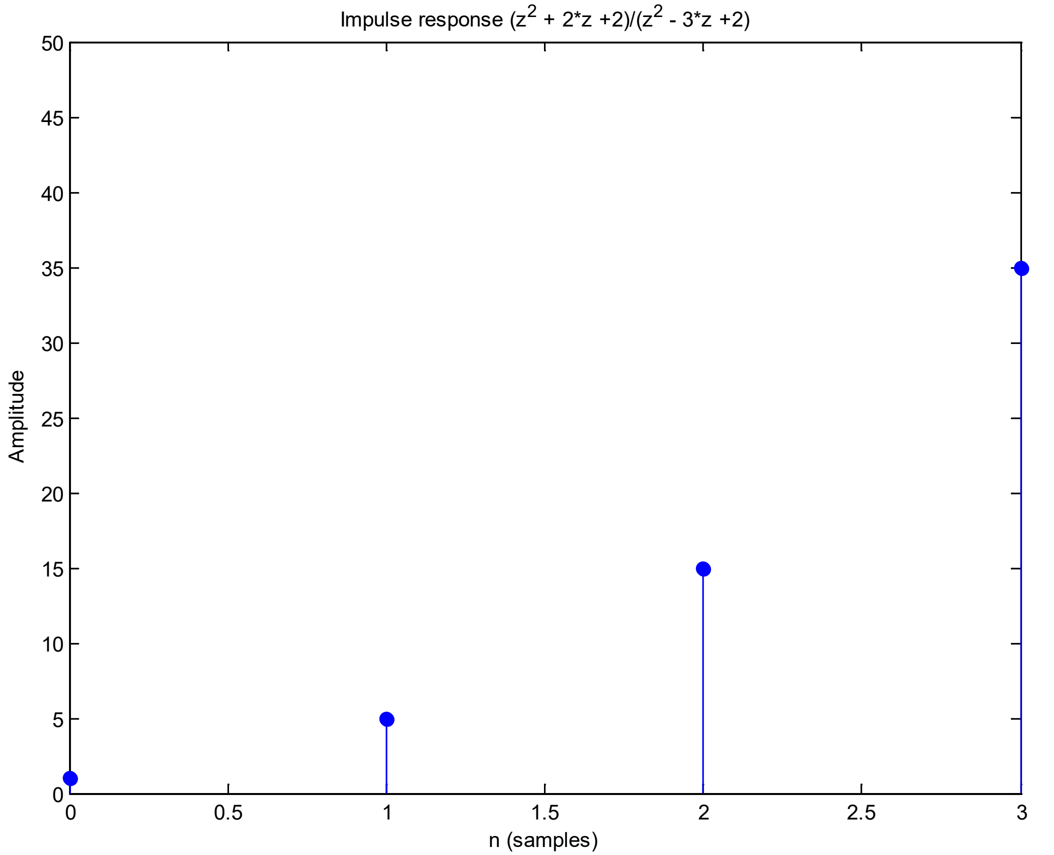
$$H_1(z) = \frac{1 + 2z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

b = [1 2 2]; a = [1 -3 2];

impz(b,a)

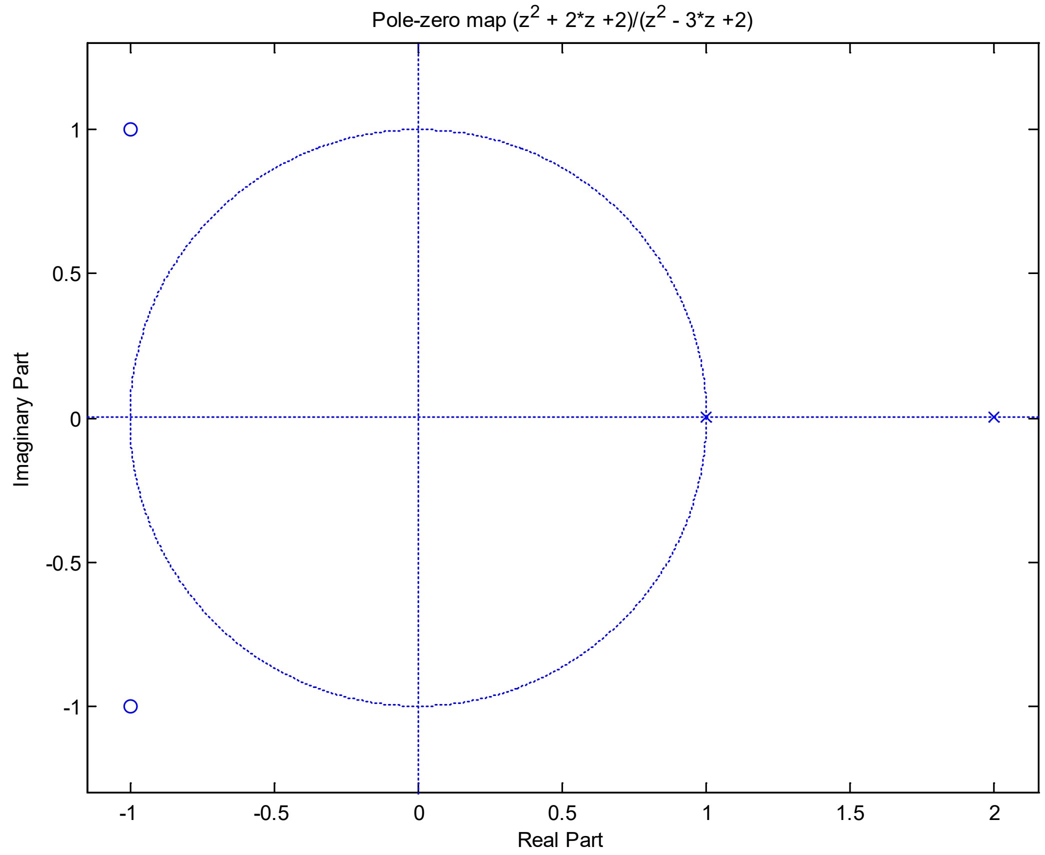
zplane(b,a)

Z-Transform



Impulse response (z^2 + 2*z +2)/(z^2 - 3*z +2)
Amplitude
n (samples)
50
45
40
35
30
25
20
15
10
5
0
0
0.5
1
1.5
2
2.5
3

Z-Transform



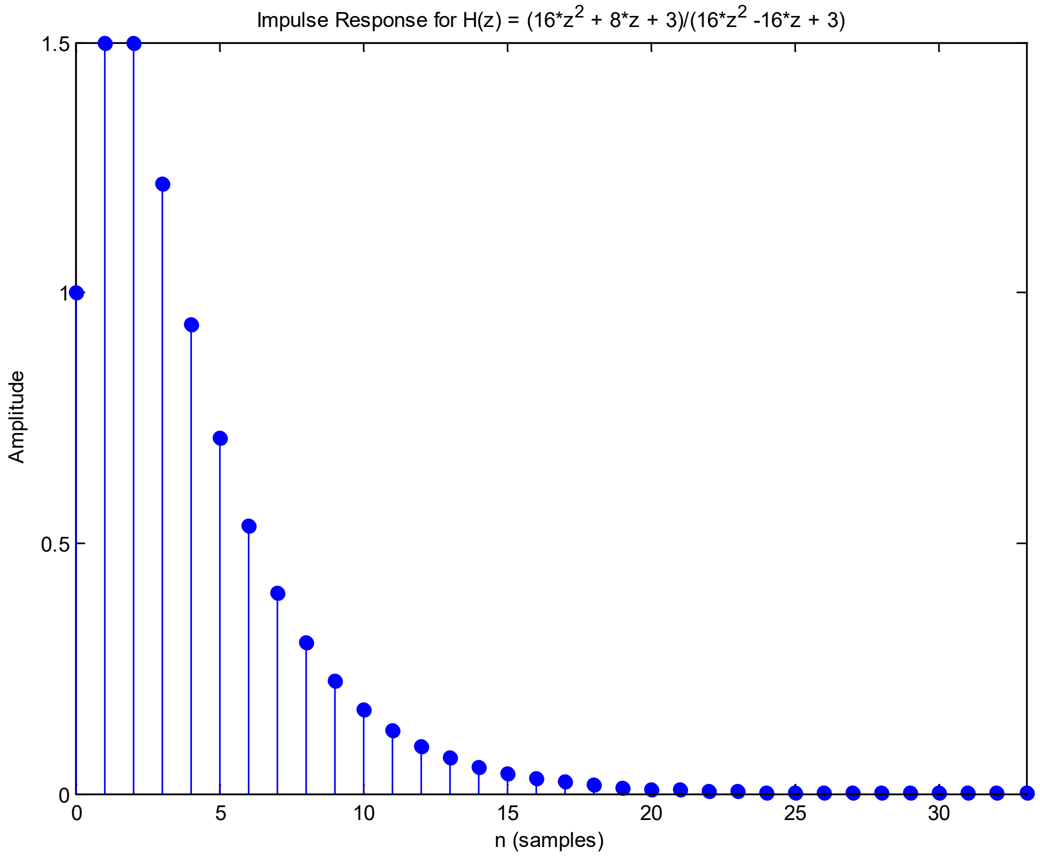
Pole-zero map (z^2 + 2*z +2)/(z^2 - 3*z +2)
1
0.5
0
-0.5
-1
Imaginary Part
-1
-0.5
0
0.5
1
1.5
2
Real Part

Z-Transform

$$H_2(z) = \frac{1-(1/2)z^{-1}+(3/16)z^{-2}}{1-z^{-1}+(3/16)z^{-2}} = 1 - \frac{3z}{z-1/4} + \frac{3z}{z-3/4}$$

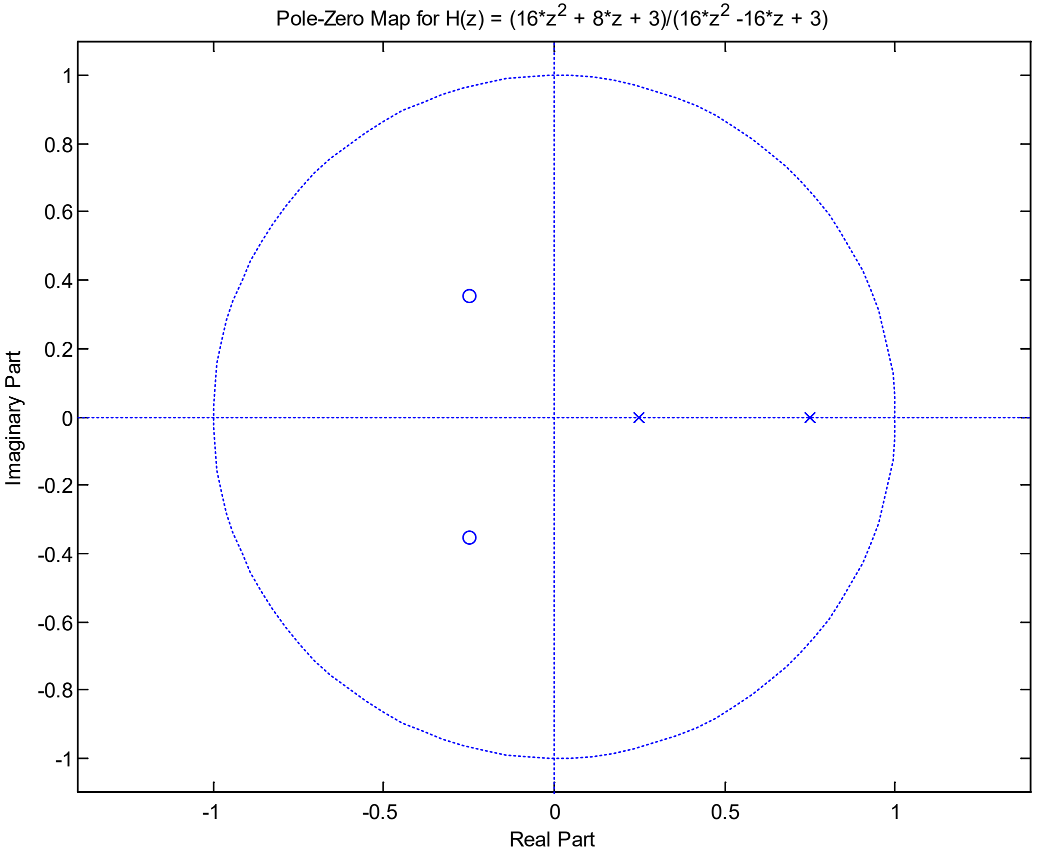
$$h_2[n] = \delta[n] - 3\cdot\left(\tfrac{1}{4}\right)^n u[n] + 3\cdot\left(\tfrac{3}{4}\right)^n u[n]$$

Z-Transform



Impulse Response for H(z) = (16*z^2 + 8*z + 3)/(16*z^2 -16*z + 3)
1.5
1
0.5
0
0
5
10
15
20
25
30
Amplitude
n (samples)

Z-Transform



Pole-Zero Map for H(z) = (16*z^2 + 8*z + 3)/(16*z^2 -16*z + 3)
Imaginary Part
Real Part
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1

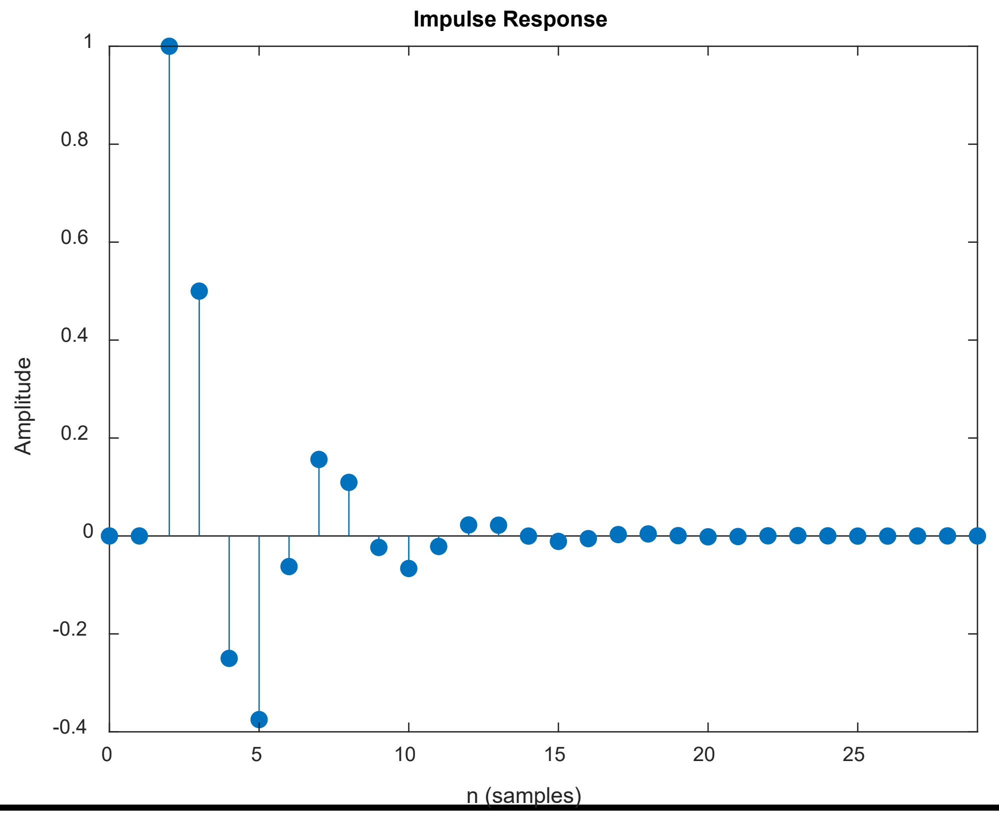
Z-Transform

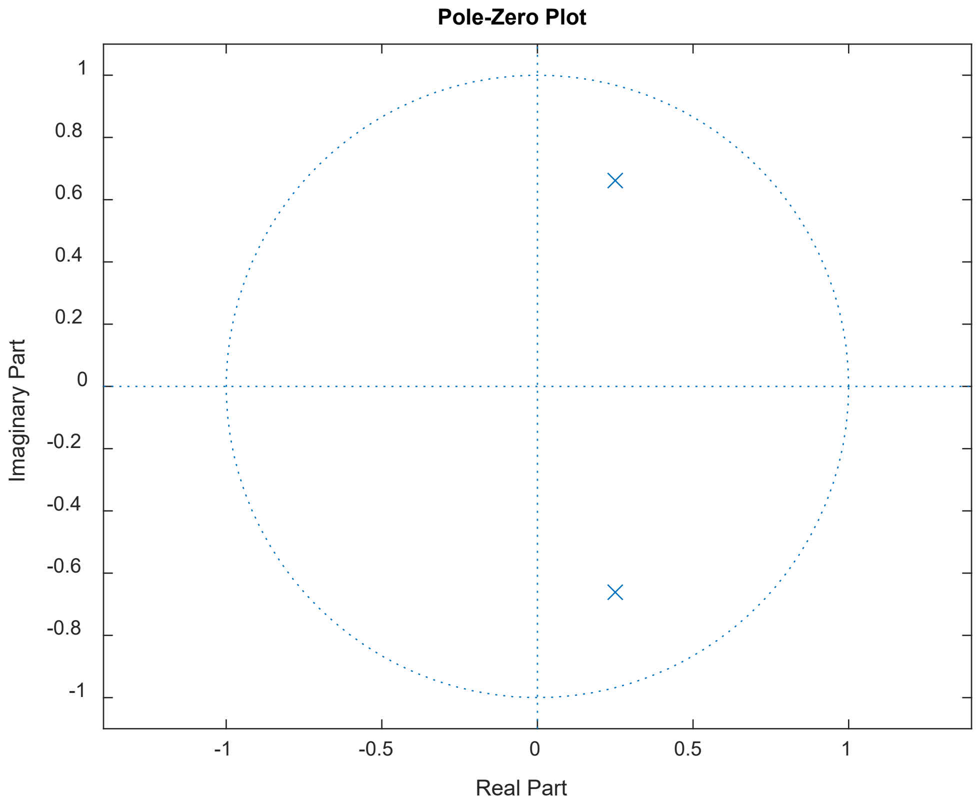
$$H_2(z) = \frac{z^{-2}}{1-(1/2)z^{-1}+(1/2)z^{-2}} = \frac{0+0z^{-1}+z^{-2}}{1-(1/2)z^{-1}+(1/2)z^{-2}}$$

$$h_2[n] = 2\left[\delta[n] + \frac{2}{\sqrt{7}}\cdot\left(\frac{1}{\sqrt{2}}\right)^{n-1}\sin\left((n-1)\theta\right)u(n)\right] \text{ where } \theta = \tan^{-1}\left(\sqrt{7}\right)$$

Z-Transform

$$H(z)=\frac{z^{-2}}{1-(1/2)z^{-1}+(1/2)z^{-2}}=\frac{0+0z^{-1}+z^{-2}}{1-(1/2)z^{-1}+(1/2)z^{-2}}$$





Z-Transform

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(n+1)\ \alpha^n u[n]$	$\frac{1}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos \omega_o n)\ u[n]$	$\frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$(r^n \sin \omega_o n)\ [n]$	$\frac{1-(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$

Z-Transform

TABLE 5.1 (Unilateral) z-Transform Pairs

No.	$x[n]$	$X[z]$
1	$\delta[n-k]$	z^{-k}
2	$u[n]$	$\dfrac{z}{z-1}$
3	$nu[n]$	$\dfrac{z}{(z-1)^2}$
4	$n^2u[n]$	$\dfrac{z(z+1)}{(z-1)^3}$
5	$n^3u[n]$	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^n u[n]$	$\dfrac{z}{z-\gamma}$
7	$\gamma^{n-1}u[n-1]$	$\dfrac{1}{z-\gamma}$

Z-Transform

8	$n\gamma^n u[n]$	$\dfrac{\gamma z}{(z-\gamma)^2}$
9	$n^2\gamma^n u[n]$	$\dfrac{\gamma z(z+\gamma)}{(z-\gamma)^3}$
10	$\dfrac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$	$\dfrac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^n \cos\beta n\, u[n]$	$\dfrac{z(z-\lvert\gamma\rvert\cos\beta)}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^n \sin\beta n\, u[n]$	$\dfrac{z\lvert\gamma\rvert\sin\beta}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\dfrac{rz[z\cos\theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n] \quad \gamma=\lvert\gamma\rvert e^{j\beta}$	$\dfrac{(0.5re^{j\theta})z}{z-\gamma}+\dfrac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\dfrac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$

$$r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}}$$

$$\beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert}$$

$$\theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$$

Z-Transform

$\mathcal{Z}$- Transform Operations

Operation	$f[k]$	$F[z]$
Addition	$f_1[k] + f_2[k]$	$F_1[z] + F_2[z]$
Scalar multiplication	$af[k]$	$aF[z]$
Right-shift	$f[k-m]u[k-m]$	$\frac{1}{z^m}F[z]$
	$f[k-m]u[k]$	$\frac{1}{z^m}F[z] + \frac{1}{z^m}\sum_{k=1}^{m} f[-k]z^k$
	$f[k-1]u[k]$	$\frac{1}{z}F[z] + f[-1]$
	$f[k-2]u[k]$	$\frac{1}{z^2}F[z] + \frac{1}{z}f[-1] + f[-2]$
	$f[k-3]u[k]$	$\frac{1}{z^3}F[z] + \frac{1}{z^2}f[-1] + \frac{1}{z}f[-2] + f[-3]$
Left-shift	$f[k+m]u[k]$	$z^mF[z] - z^m\sum_{k=0}^{m-1} f[k]z^{-k}$
	$f[k+1]u[k]$	$zF[z] - zf[0]$
	$f[k+2]u[k]$	$z^2F[z] - z^2f[0] - zf[1]$
	$f[k+3]u[k]$	$z^3F[z] - z^3f[0] - z^2f[1] - zf[2]$

Z-Transform

Multiplication by γ^k	$\gamma^k f[k]u[k]$	$F\left[\frac{z}{\gamma}\right]$
Multiplication by k	$kf[k]u[k]$	$-z\frac{d}{dz}F[z]$
Time Convolution	$f_1[k] * f_2[k]$	$F_1[z]F_2[z]$
Frequency Convolution	$f_1[k]f_2[k]$	$\frac{1}{2\pi j}\oint F_1[u]F_2\left[\frac{z}{u}\right]u^{-1}\,du$
Initial value	$f[0]$	$\lim_{z\to\infty} F[z]$
Final value	$\lim_{N\to\infty} f[N]$	$\lim_{z\to 1}(z-1)F[z]$ poles of $(z-1)F[z]$ inside the unit circle.