Digital Signal Processing

Class 16
03/20/2025

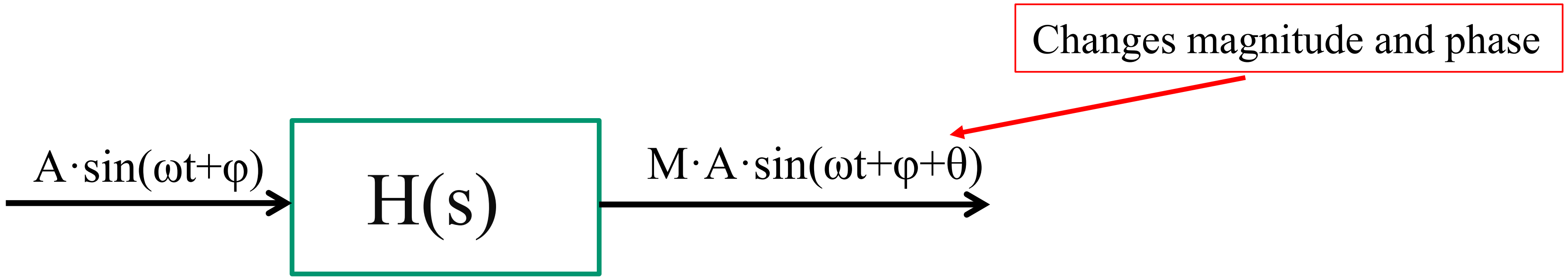
ENGR 71

- Class Overview
  - Frequency Analysis of Discrete Signals
- Assignments
  - Exam 1 due March 23
  - Lab 2 due March 28
  - Reading:

    Chapter 5: Frequency-Domain Analysis of LTI Systems

Frequency-Domain Analysis of LTI Systems

- Key concept behind the action of LTI systems on signals:
  - – Signals can be decomposed into superposition of frequency components
  - – Basis function for this decomposition are sines and cosines (and complex exponential)
  - – **Frequency components of signal are unchanged when passed through Linear Time Invariant systems**
    - Only amplitude and phase change



Frequency-Domain Analysis of LTI Systems

- Some interesting relationship for $H(\omega)$, $H^*(\omega)$, and $|H(\omega)|$
  - Section 5.2.1 of Proakis & Manolakis derives interesting relationships for the frequency response described by rational polynomials when the coefficients of the polynomials are real-valued numbers.
    - This is generally the case since discrete LTI systems are defined by difference equations like:

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad \text{where the } a\text{'s and } b\text{'s are weights of delayed inputs and outputs.}$$

$$\text{Transfer function : } H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$\text{Frequency response: } H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \frac{B(\omega)}{A(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$$

Frequency-Domain Analysis of LTI Systems

- Writing this in terms of poles and zeros

$$H(\omega) = b_0 \left. \frac{\prod_{k=1}^{M}\left(1 - z_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - p_k z^{-1}\right)} \right|_{z=e^{j\omega}} = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1 - p_k e^{-j\omega}\right)}$$

While the a's and b's are real-valued, the poles and zeros (z_k's and p_k's) can be real or complex, and, if complex, occur in complex conjugate pairs.

- What are the implications of this?

Skipping the details covered in the text:

$$H^*(\omega) = H(-\omega)$$

$$\left|H(\omega)\right|^2 = H(\omega)H(-\omega) = H(z)H(z^{-1})\Big|_{z=e^{j\omega w}}$$

One of the properties of the z-transform is that $H(z)H(z^{-1})$ is the z-transform of the autocorrelation of the impulse response.

Frequency-Domain Analysis of LTI Systems

- What are the implications (continued)?

One of the properties of the z-transform is that $H(z)H(z^{-1})$ is the z-transform of the autocorrelation of the impulse response.

Finally, since $\left|H(\omega)\right|^2 = H(z)H(z^{-1})$ and using the fact that the energy spectral density of a signal is the Fourier transform of the autocorrelation. (Wiener-Khintchine theorem)

$\boxed{\left|H(\omega)\right|^2 \text{ is the Fourier transform of the autocorrelation of the impulse response, } \left[r_{hh}(m)\right]}$

Frequency-Domain Analysis of LTI Systems

- Other results as a consequence of transfer function being rational polynomial with coefficients of polynomial real numbers and using autocorrelation relationship to z-transform:

Can show that $\left|H(\omega)\right|^2 = \dfrac{d_0 + 2\sum_{k=1}^{M} d_k \cos(k\omega)}{c_0 + 2\sum_{k=1}^{M} c_k \cos(k\omega)}$

where c's and d's are autocorrlation of a's and b's in polynomial

$$\mathrm{c}_l = \sum_{k=0}^{N-|l|} a_k a_{k+l} \quad -N \le l \le N, \quad c_l = c_{-l} \quad \text{since } a\text{'s and } b\text{'s are real}$$

$$d_l = \sum_{k=0}^{M-|l|} b_k b_{k+l} \quad -M \le l \le M, \quad d_l = d_{-l} \quad \text{since } a\text{'s and } b\text{'s are real}$$

From trigonometric identity $\cos(k\omega) = \sum_{m}^{k} \beta_m (\cos\omega)^m$

Frequency-Domain Analysis of LTI Systems

- The upshot of all this:
  The magnitude squared of the frequency response can always be written as a ratio of series expression of powers of cosines

$$\boxed{\left|H(\omega)\right|^2 = \frac{\beta_0 + \beta_1 \cos\omega + \beta_2 \cos^2\omega + \cdots}{\alpha_0 + \alpha_1 \cos\omega + \alpha_2 \cos^2\omega + \cdots}}$$

- Why is this worth knowing:

  Has significance for design of filters (and determining their stability).

  Also, computations involving cosines are numerically stable compared to complex exponential calculations

Frequency-Domain Analysis of LTI Systems

- Another thing to keep in mind about the relationship between $H(z)$ and $|H(\omega)|^2$
  - You can get $|H(\omega)|^2$ from $H(z)$, but you can’t get $H(z)$ from $|H(\omega)|^2$
  - You loose the phase information.

Frequency-Domain Analysis of LTI Systems

- Analysis of frequency response in terms of zero & pole locations
  - Writing the frequency response in terms of zeros and poles

$$H(\omega) = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1 - p_k e^{-j\omega}\right)} = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}\left(e^{j\omega} - z_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega} - p_k\right)}$$

Each factor can be written as a magnitude and phase:

$$\left(e^{j\omega} - z_k\right) = V_k(\omega) e^{j\Theta_k(\omega)} \quad \text{where } V_k(\omega) = \left|e^{j\omega} - z_k\right|$$

$$\left(e^{j\omega} - p_k\right) = U_k(\omega) e^{j\Phi_k(\omega)} \quad \text{where } U_k(\omega) = \left|e^{j\omega} - p_k\right|$$

$$\left|H(\omega)\right| = \left|b_0\right| \frac{V_1(\omega) V_2(\omega) \cdots V_M(\omega)}{U_1(\omega) U_2(\omega) \cdots U_n(\omega)},$$

$$\measuredangle H(\omega) = \omega(N-M) + \left[\Theta_1(\omega) + \Theta_2(\omega) + \cdots + \Theta_M(\omega)\right] - \left[\Phi_1(\omega) + \Phi_2(\omega) + \cdots + \Phi_N(\omega)\right]$$

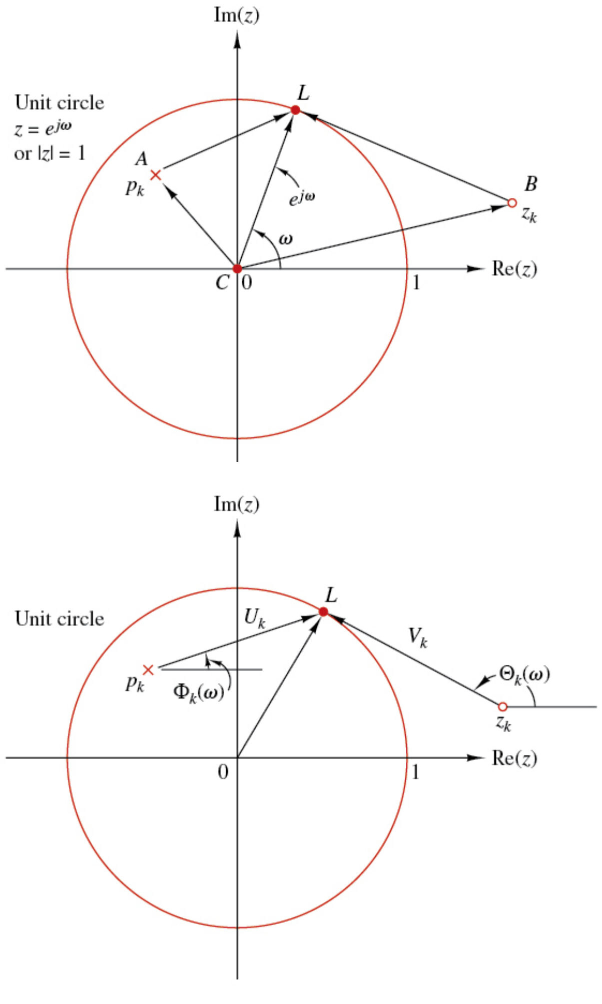
Frequency-Domain Analysis of LTI Systems

$e^{j\omega}$ is a point on the unit circle

Each $\left(e^{j\omega} - p_k\right)$ and $\left(e^{j\omega} - z_k\right)$ is a vector from the pole or zero to the $e^{j\omega}$ point on unit circle.

Magnitudes are given by U_k or V_k and angle determined from Φ_k or Θ_k

In terms of the V_k and U_k terms and associated phases:



(b)

Frequency-Domain Analysis of LTI Systems

- From the diagram you can see that the V_k and U_k correspond to distances from the zeros and poles (respectively) to the point on the unit circle associated with frequency ω

$$V_k(\omega) = \left| e^{j\omega} - z_k \right|$$

$$U_k(\omega) = \left| e^{j\omega} - p_k \right|$$

$$\left| H(\omega) \right| = \left| b_0 \right| \frac{V_1(\omega) V_2(\omega) \cdots V_M(\omega)}{U_1(\omega) U_2(\omega) \cdots U_n(\omega)},$$

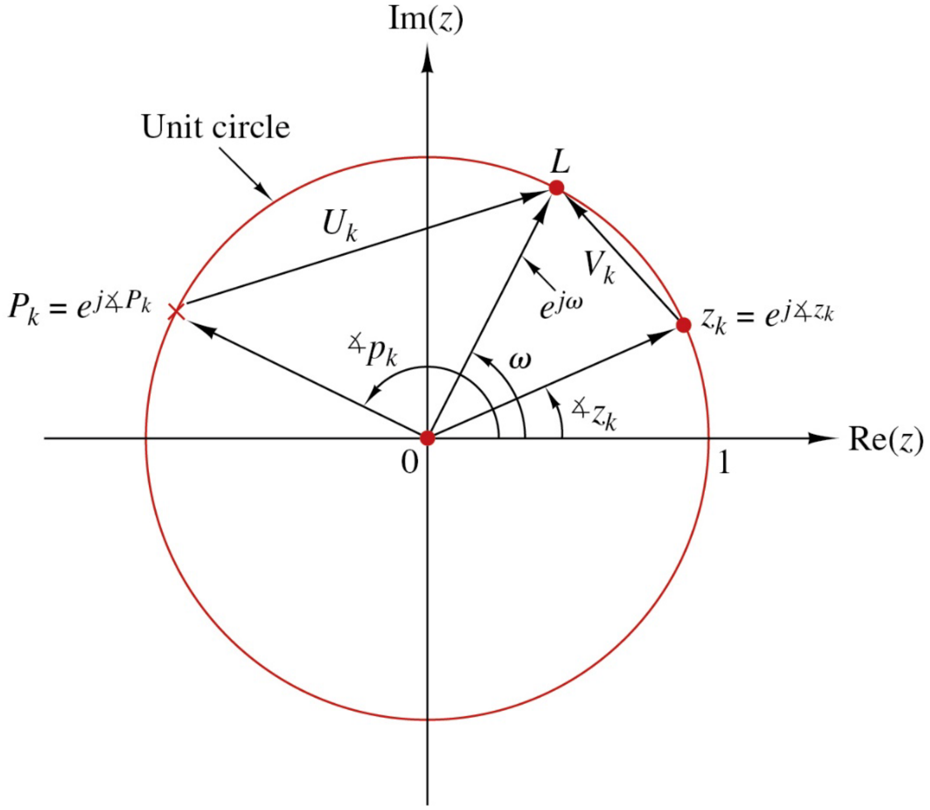
$$\begin{aligned} \measuredangle H(\omega) = \; & \omega(N - M) + \left[\Theta_1(\omega) + \Theta_2(\omega) + \cdots + \Theta_M(\omega) \right] \\ & - \left[\Phi_1(\omega) + \Phi_2(\omega) + \cdots + \Phi_N(\omega) \right] \end{aligned}$$

Frequency-Domain Analysis of LTI Systems

$$|H(\omega)| = |b_0| \frac{V_1(\omega)V_2(\omega)\cdots V_M(\omega)}{U_1(\omega)U_2(\omega)\cdots U_n(\omega)}$$

A zero close to a point at frequency ω on the unit circle causes the $|H(\omega)|$ to be small.

A pole close to a point at frequency ω on the unit circle causes $|H(\omega)|$ to be big.

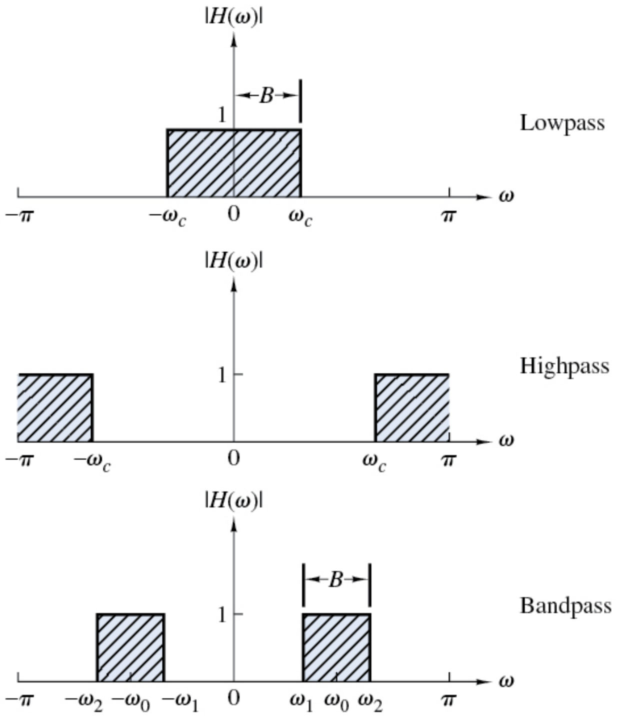


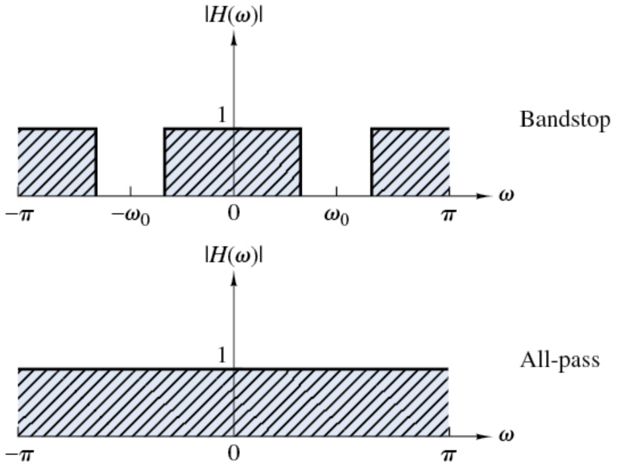
Filters

- Filters
  $Y(\omega) = H(\omega)X(\omega)$
  - The system function H acts on the input X to produce an output Y
    - Its effect on the input depends on the functional dependence of H on ω
  - Any LTI system can be thought of as a filter
    - It may:
      reduce high frequencies in the input, reduce low frequencies,
      enhance high frequencies, enhance low frequencies, … mess up all frequencies
    - Essentially, $H(\omega)$ can be designed to do almost anything you want to the frequency composition of the input to produce an output.

Filters

- We have talked about low-pass, high-pass, band-pass, etc. filters
- Ideal ones (unrealizable in practice) look like these:





Notice that Bandwidth (B) only considers the width in the positive frequency half of the diagrams.

Filters

– Phase of filters

- For an ideal filter:

$$Y(\omega) = H(\omega)X(\omega) \qquad \omega_1 < \omega < \omega_2$$

$$Y(\omega) = Ce^{-j\omega n_0}X(\omega)$$

– It scales the magnitude of the input by C
shifts the phase linearly with ω

– Linear phase filters are "good,"
because they only introduce a time delay in the input signal

» Time shift property of Fourier transform: $y(n) = Cx(n\text{-}n_0)$

– What would be "bad" would be if phase of the input changed as a function of frequency, i.e., different frequency components would be delayed by different amounts.

Filters

– Ideal filter : $H(\omega) = |H(\omega)| e^{j\Theta(\omega)} = Ce^{-j\omega n_0} \quad \omega_1 < \omega < \omega_2$

$$|H(\omega)| = C$$

$$\Theta(\omega) = -\omega n_0$$

– On previous slide:

- Delay is given by: $y(n) = Cy(n - n_0)$

$$\tau_g = -\frac{d\Theta(\omega)}{d\omega} = -\frac{d(-\omega n_0)}{d\omega} = n_0$$

– Generalize definition of "group delay" (or "envelope delay") for arbitrary phase:

$$\boxed{\text{Group delay: } \tau_g = -\frac{d\Theta(\omega)}{d\omega}}$$

Filters

– Consider effect of phase on a sinusoidal input: $x(n) = \sin(\omega n)$

- Filter shifts phase by $\Theta(\omega)$: $y(n) = C\sin\left[\omega n + \Theta(\omega)\right]$
- Ideal filter:

$$y(n) = C\sin\left[\omega n + \Theta(\omega)\right] = C\sin\left[\omega\left(n - n_0\right)\right] = C\sin\left[\omega\left(n - \frac{-\Theta(\omega)}{\omega}\right)\right], \text{ for } \Theta(\omega) = -\omega n_0$$

Write this in terms of a delay term, $\tau_{pd} = -\dfrac{\Theta(\omega)}{\omega}$

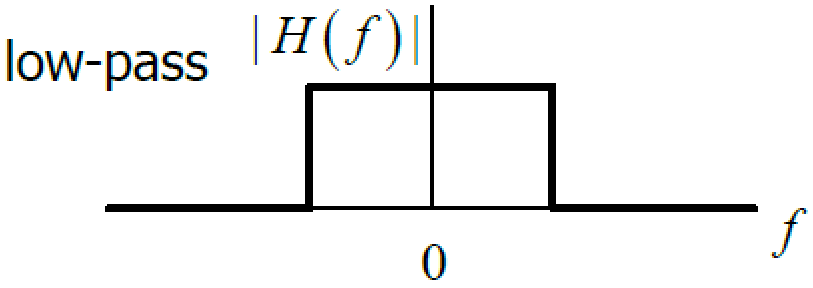
- For ideal filter: $y(n) = C\sin\left[\omega\left(n - \tau_{pd}\right)\right]$

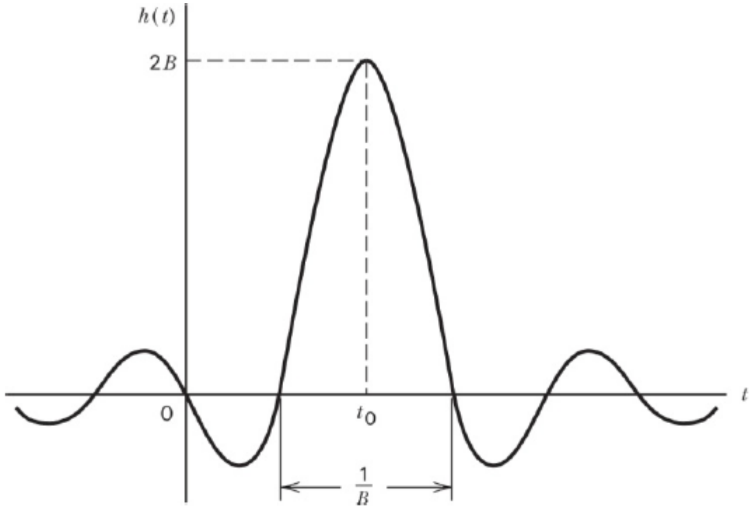
– Generalize definition of "phase delay" for filter phase:

$$\boxed{\text{Phase delay: } \tau_{pd} = -\frac{\Theta(\omega)}{\omega}}$$

Filters

- Why aren’t ideal filters realizable?
  - Ideal low-pass filter Filter is a rectangular pulse in the frequency domain:





In the time domain, there is some response from the filter before $t = 0$, so the ideal filter is non-casual.

Remember condition for casual system is that there is no response before time = 0

Filters

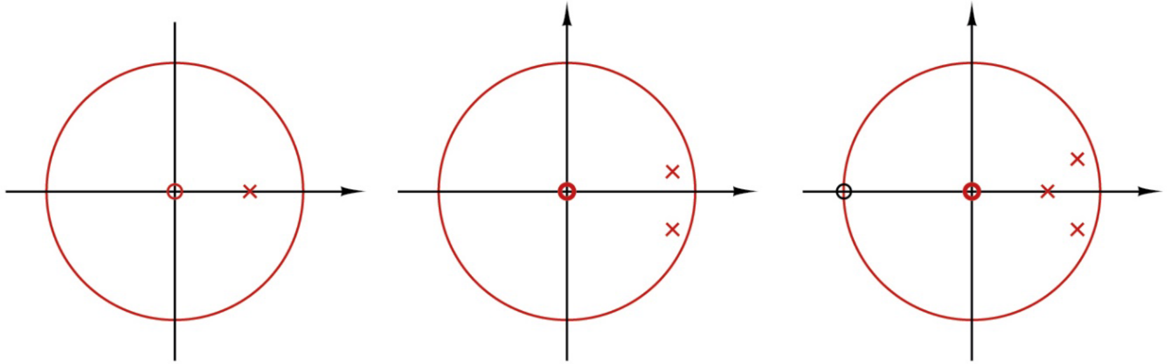
- Simple filter design by pole-zero placement
  - Place poles and zeros to affect the frequency response in the desired way
    - For stability, poles must be inside unit circle
    - Zeros can be anywhere
    - Complex poles and zeros must include complex conjugate pairs
    - Usually normalize frequency response so that the gain is 1 at specified frequency
      - Do this by setting b_0 so that $|H(\omega_0)| = 1$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=0}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1 - p_k e^{-j\omega}\right)}$$

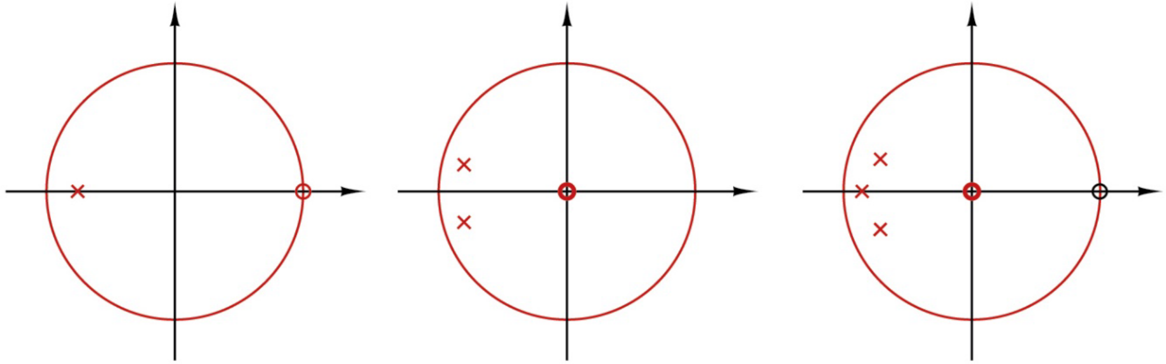
      - Usually have more poles than zeros

Filters

- Picture of pole-zero placement for lowpass and highpass filters



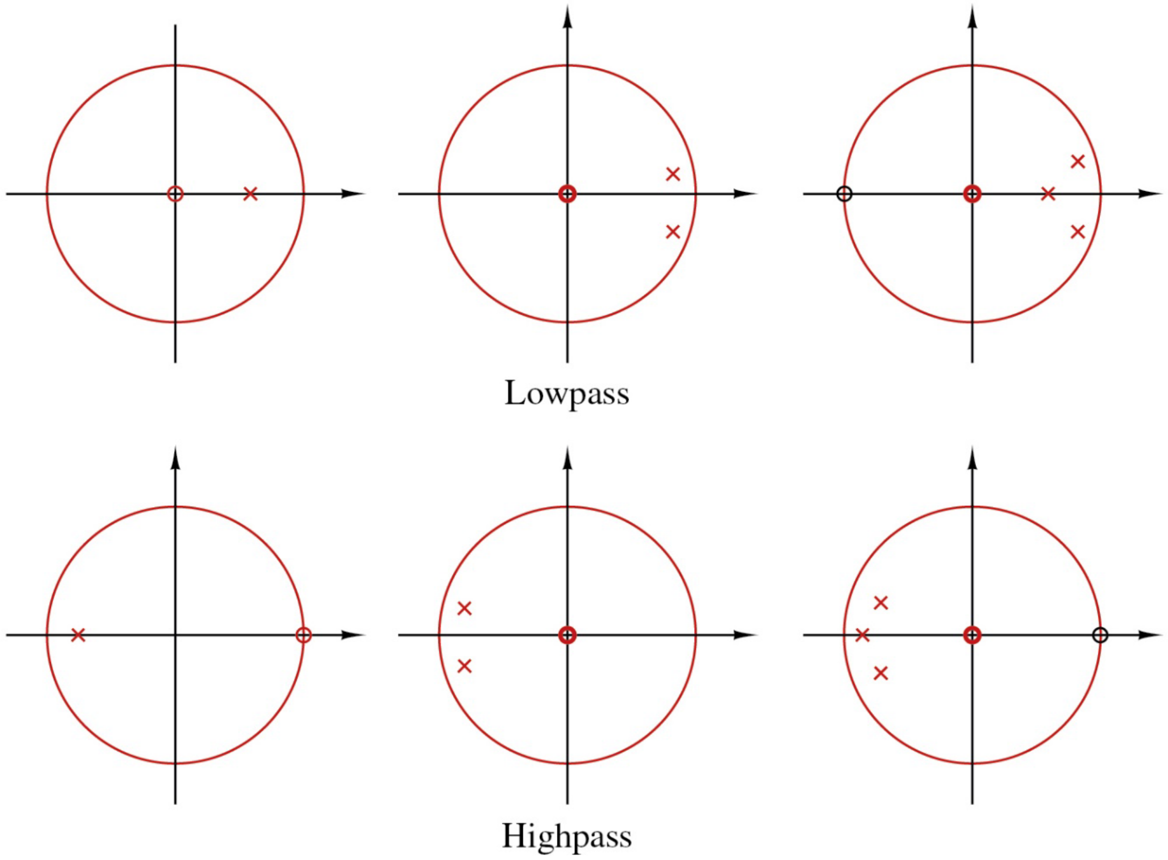
Lowpass



Highpass

Filters

- Picture of pole-zero placement for lowpass and highpass filters



Filters

- Single pole lowpass filter
  - Put pole close to unit circle at $\omega = 0$ (1 on real axis) to emphasize low frequencies
  - Put zero near $\omega = \pi$ (highest frequency, Nyquist) to suppress high frequencies
  - Normalize so $H(0) = 1$

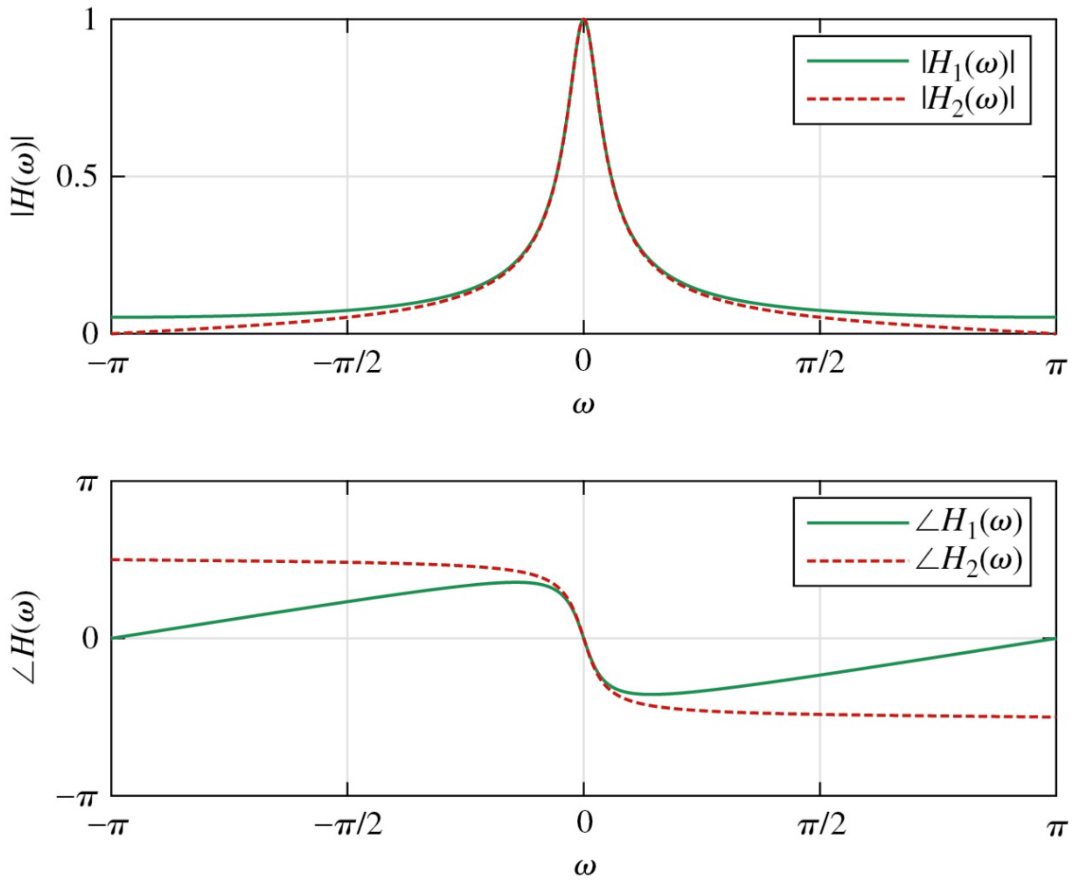
$$H_{LP}(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1-az^{-1}} \qquad H_{LP}(0) = \frac{1-a}{2}\frac{1+e^{-j0}}{1-ae^{-j0}} = \frac{1-a}{2}\frac{2}{1-a} = 1$$

$$H_{LP}(\omega) = \frac{1-a}{2}\frac{1+e^{-j\omega}}{1-ae^{-j\omega}} \qquad H_{LP}(\pi) = \frac{1-a}{2}\frac{1+e^{-j\pi}}{1-ae^{-j\pi}} = \frac{1-a}{2}\frac{1-1}{1+a} = 0$$

$a = 0.9$ is close to 1

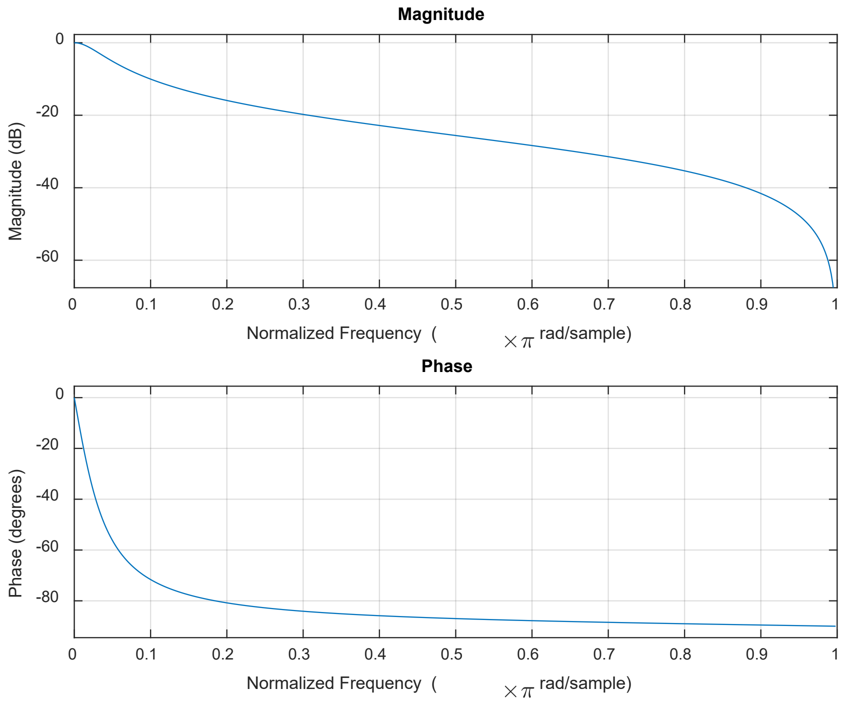
Filters

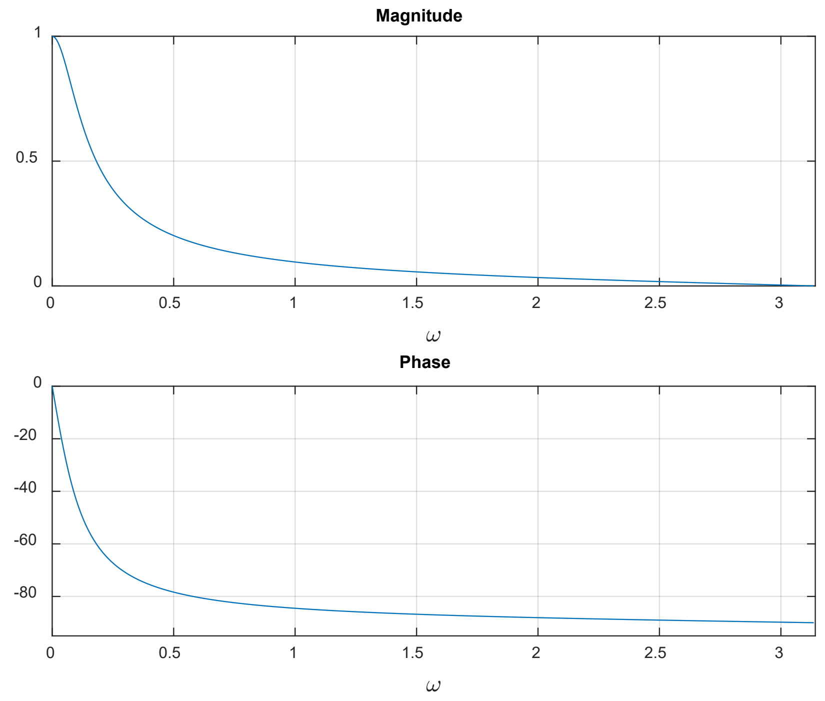
Book's picture, red line is this filter



Filters

Matlab (freqz db, freqz results linear)

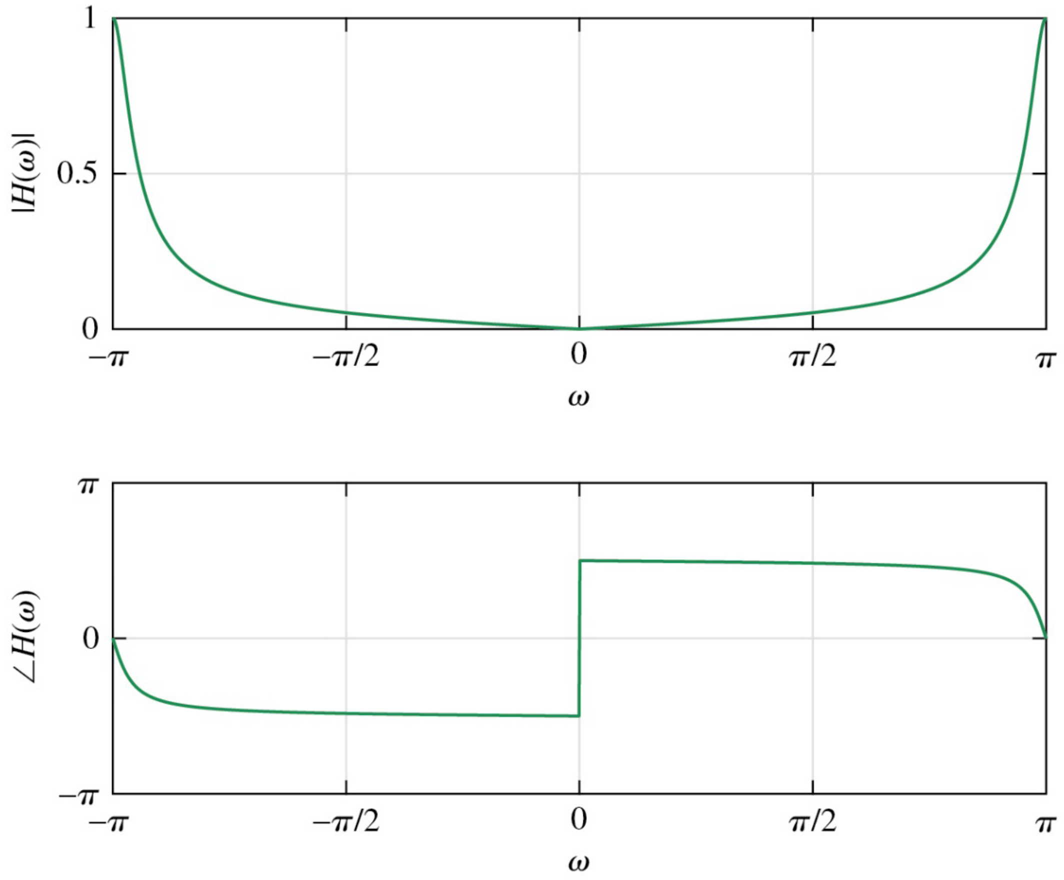




Filters

- Highpass filter by "folding" pole-zero locations :
  - Normalize so $H(0) = 1$

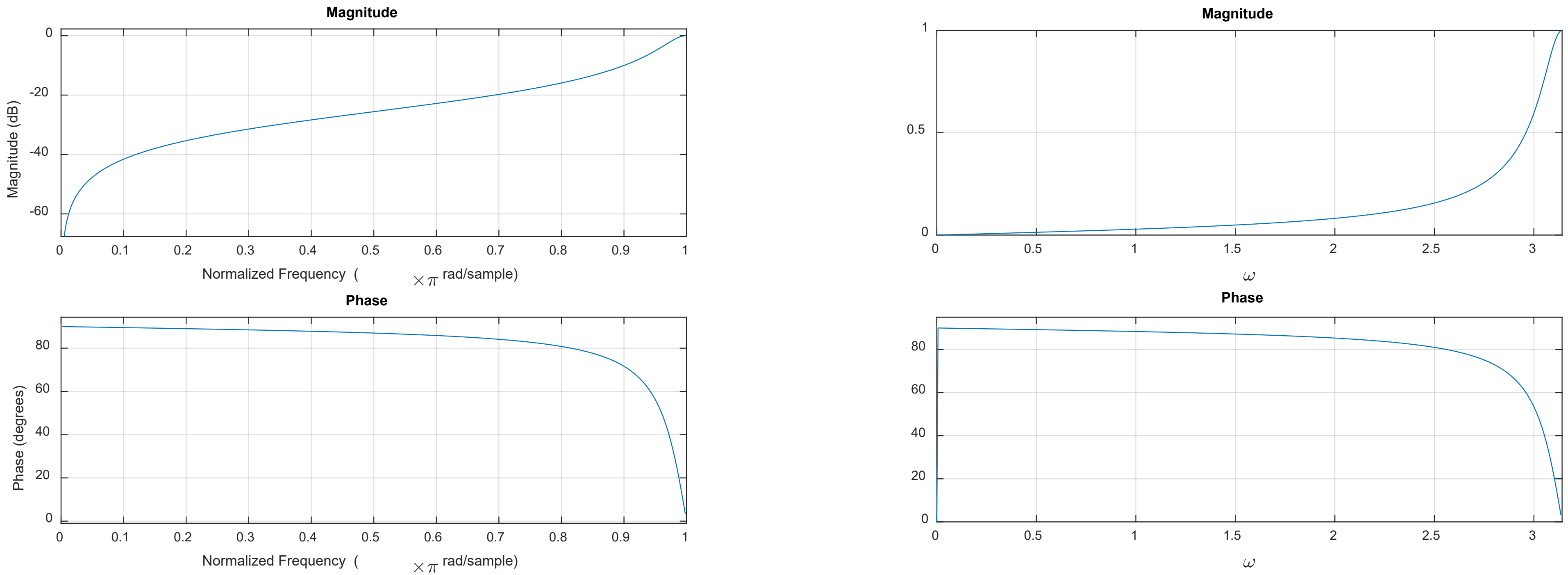
$$H_{HP}(z) = \frac{1-a}{2}\frac{1-z^{-1}}{1+az^{-1}}$$



Filters

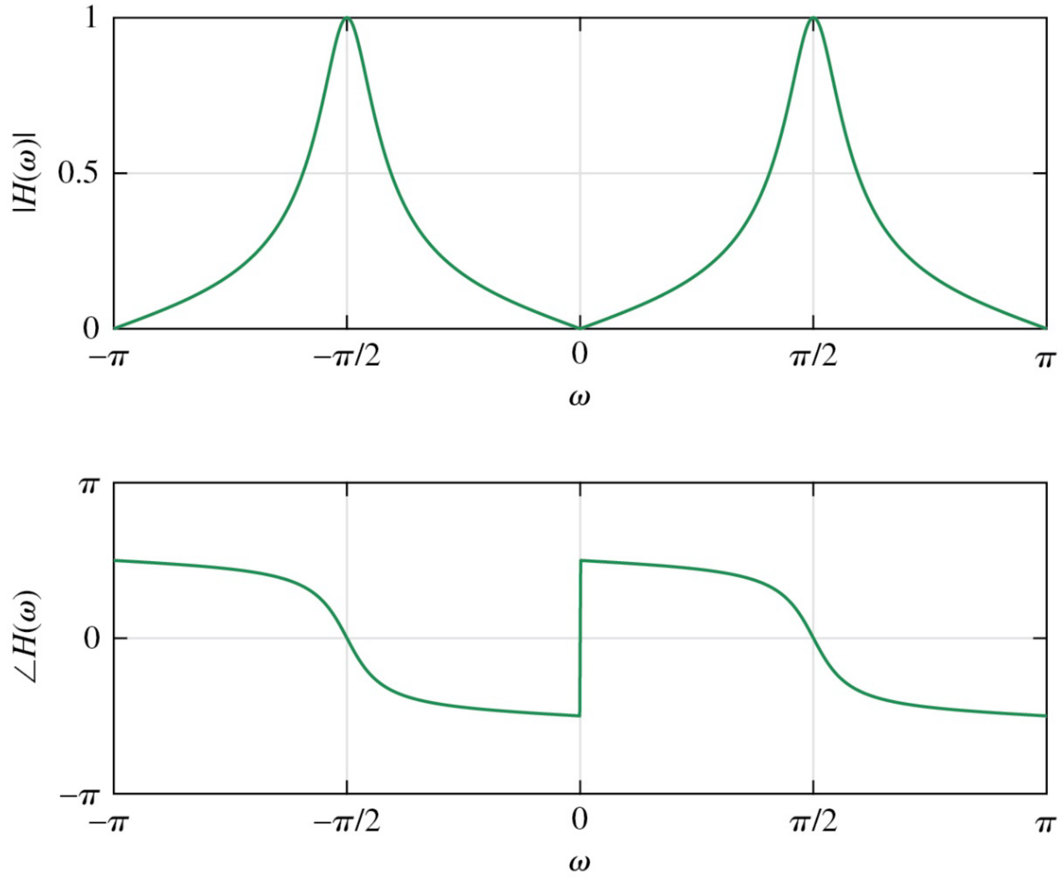
- Highpass filter by "folding" pole-zero locations :
  - Normalize so $H(0) = 1$

$$H_{HP}(z) = \frac{1-a}{2}\frac{1-z^{-1}}{1+az^{-1}}$$



Filters

- See examples in book for
  - Two-pole bandpass filter
    centered at $\omega = \pi/2 \quad H(\pi/2) = 1,$
    $H(0) = H(\pi) = 0, \quad H(4\pi/9) = 1/\sqrt{2}$



Filters

- See examples in book for
  - Two-pole lowpass filter
  - Two-pole bandpass filter with pass band centered at
  - p/2, zero at 0 and pi, and

centered at $\omega = \pi/2 \quad H(\pi/2) = 1, \quad H(0) = H(\pi) = 0, \quad H(4\pi/9) = 1/\sqrt{2}$

Filters

- Simple method to convert lowpass to highpass

$$H_{HP}(\omega) = H_{LP}(\omega - \pi)$$

Lowpass :

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Frequency domain:

$$H_{LP}(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$$

Highpass:

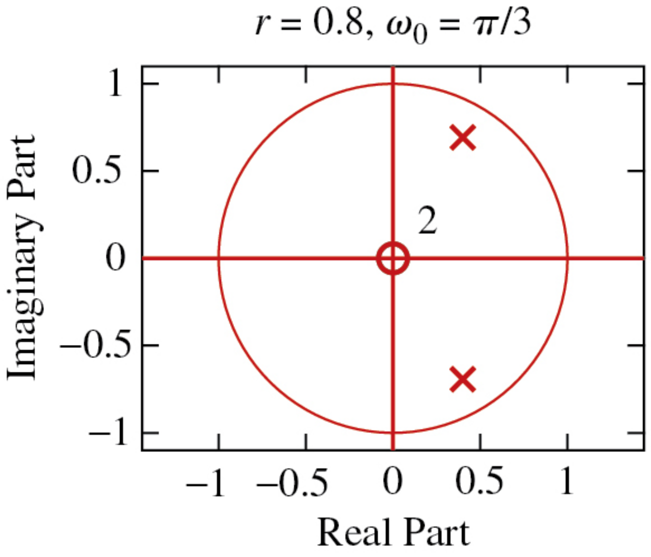
$$y(n) = -\sum_{k=1}^{N} (-1)^k a_k y(n-k) + \sum_{k=0}^{M} (-1)^k b_k x(n-k)$$

Frequency domain:

$$H_{HP}(\omega) = \frac{\sum_{k=0}^{M} (-1)^k b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} (-1)^k a_k e^{-j\omega k}}$$

Filters

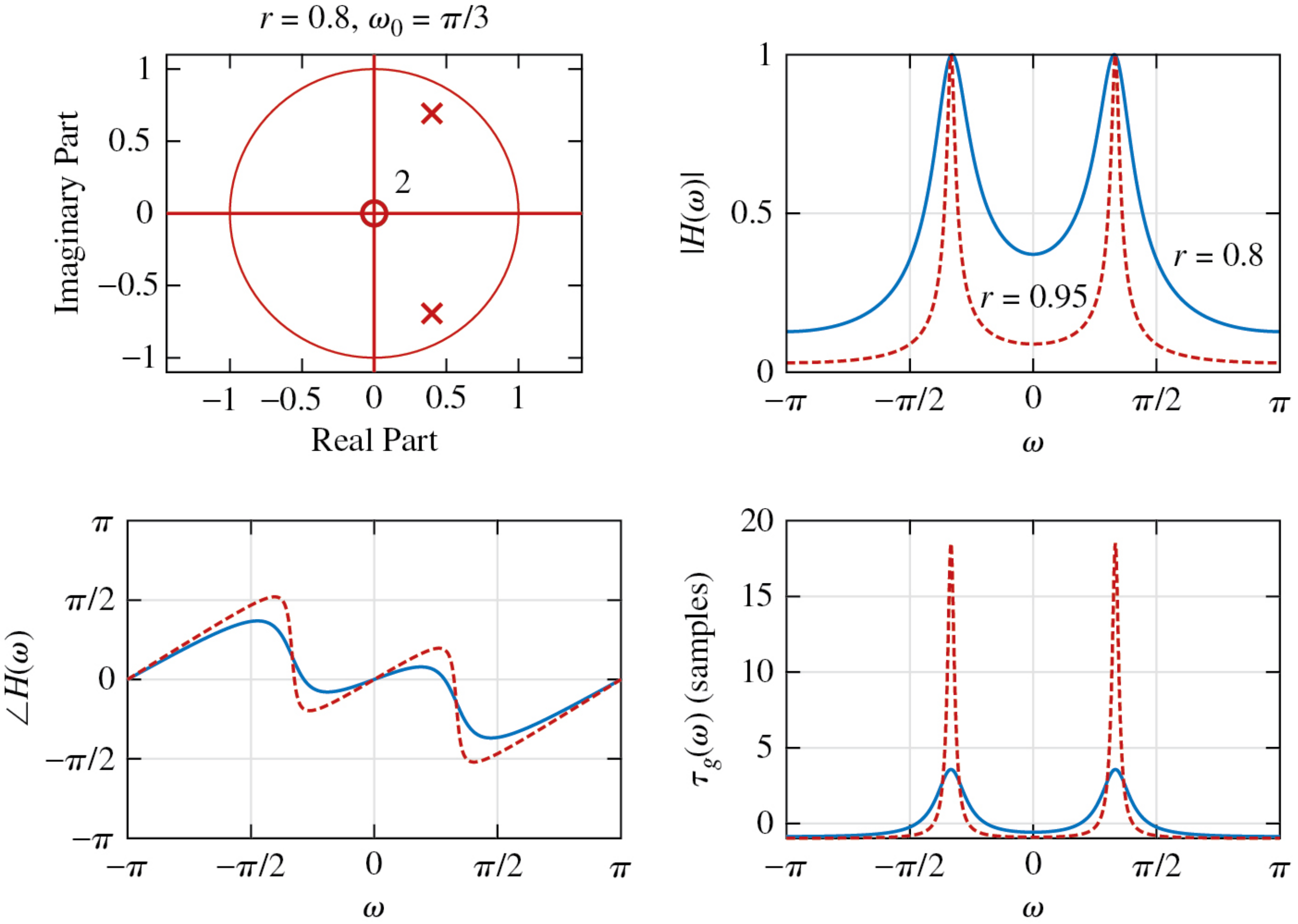
- Proakis & Manolakis give several examples of simple filters designed by pole placement:
  - Resonators:
    - Pole placement of complex conjugate pair near unit circle
      - Enhances frequency corresponding to location of poles so "resonates" at that frequency



$$p_{1,2} = re^{\pm j\omega_0}$$

Filters

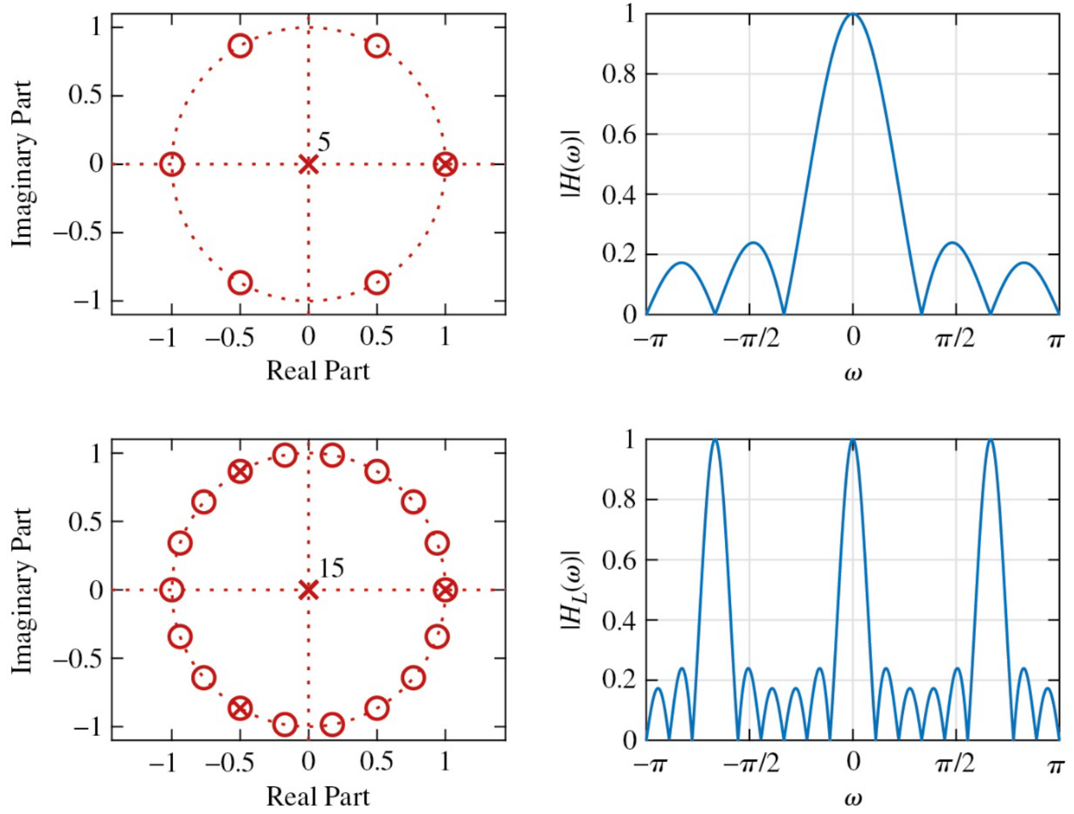
- Resonators:



Notice group delay is negative of the derivative of the phase.

Filters

- Comb filter: Multiple, narrow passbands
  - Good for good for getting rid of noise with repeating harmonics



Filters

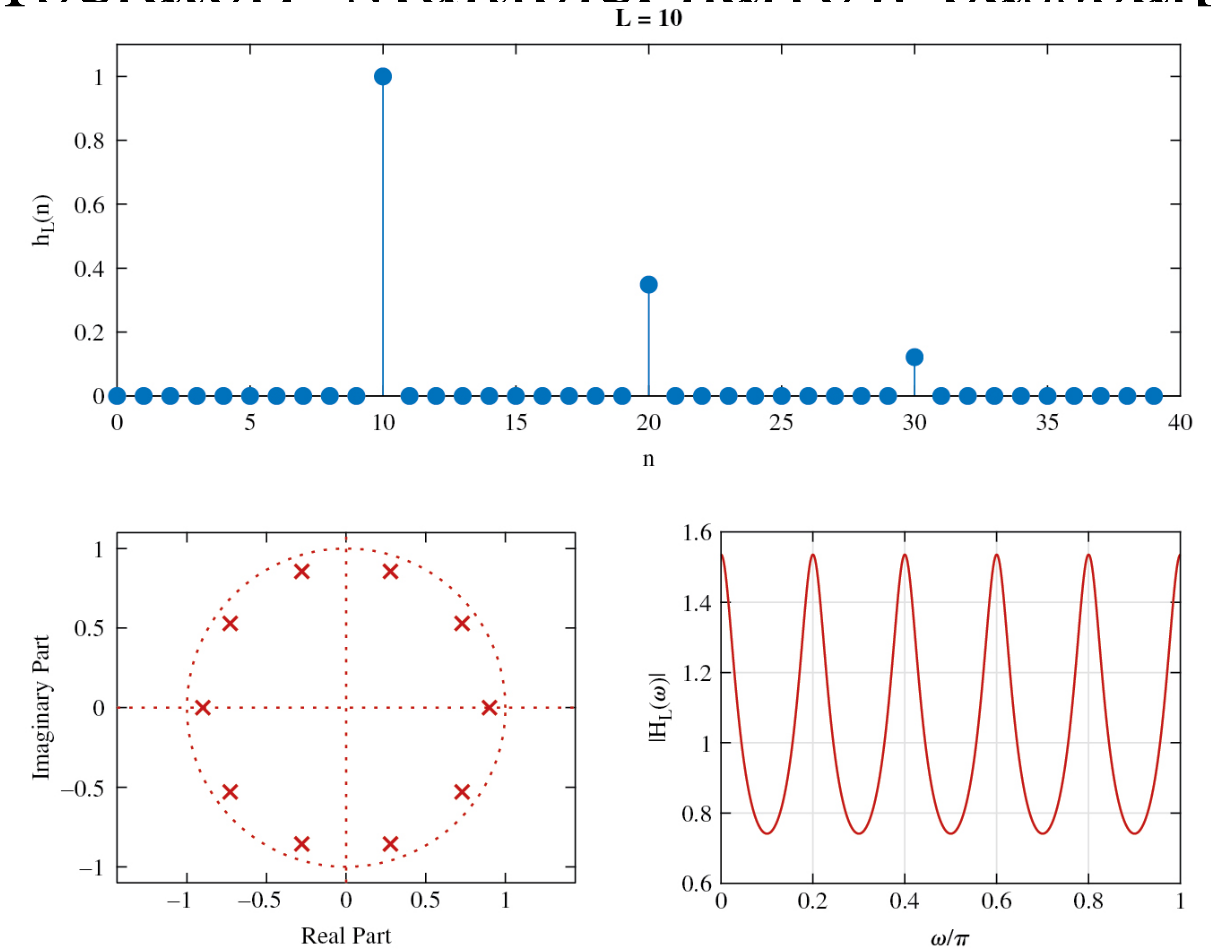
- Reverberator: Multiple, narrow passbands
  - Example: Music in a live performance bounces off walls of auditorium
    music in studio sounds "dry" so add some reverb

$$y(n) = ax(n-L) + a^2x(n-2L) + a^3x(n-3L) + \cdots$$

$0 < a < 1$ is strength of bounce

Filters

- Reverberator: Multiple, narrow passbands



Filters

- Notch filters: (This one I'll go through in more detail)

DC notch filter:

$$H(z) = \frac{z-1}{z-\lambda}$$

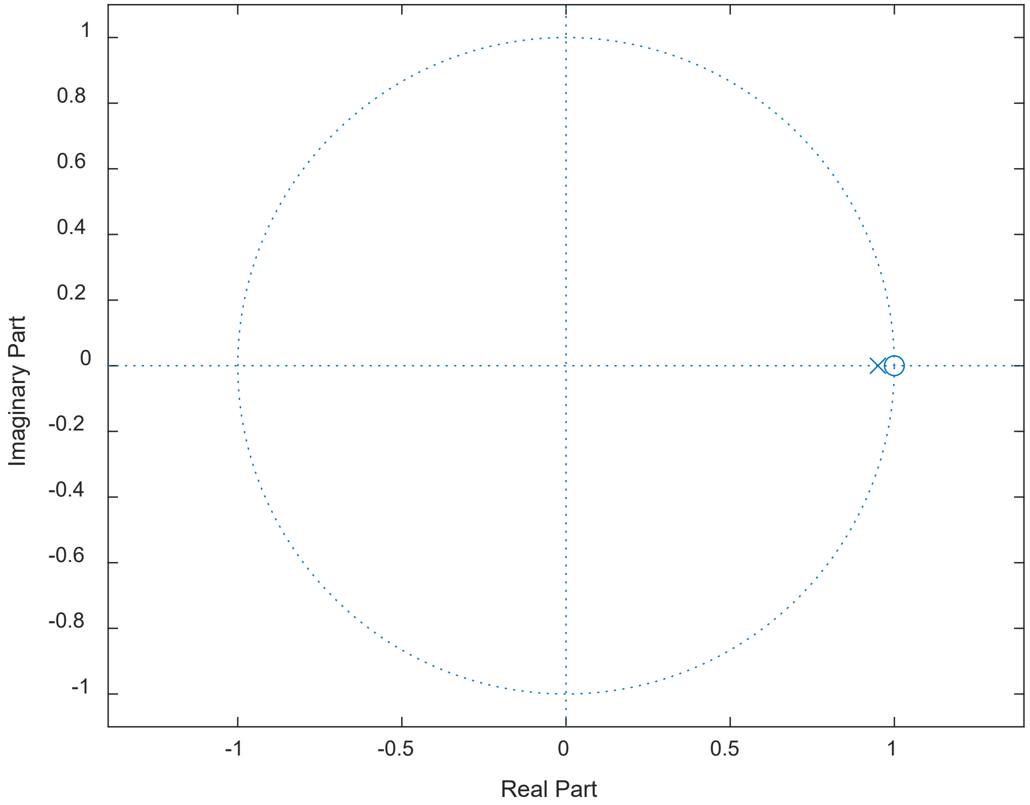
Zero in the numerator makes magnitude of impulse response zero at $z = 1$

If you put a pole at λ very close to 1, it almost cancels out the zero in the numerator.

You need to normalize the filter so you have unit gain.

$$H(z) = \left(\frac{1+\lambda}{2}\right)\frac{z-1}{z-\lambda}$$

Try this for different values of λ



Show demo for DC notch filter: notch_signal.m

Filters

- Notch filters at other frequencies

$$H(e^{j\omega}) = \frac{\left(z - e^{j\omega_0}\right)\left(z - e^{-j\omega_0}\right)}{\left(z - \alpha e^{j\omega_0}\right)\left(z - \alpha e^{-j\omega_0}\right)} = \frac{\left(z^2 - 2\cos\omega_0 + 1\right)}{\left(z^2 - 2\alpha\cos\omega_0 + \alpha^2\right)}$$

Zero in the numerator makes magnitude of impulse response zero at $z = e^{\pm j\omega_0}$

If you put a pole very close to zero, it almost cancels it out.

You need to normalize the filter so you have unit gain at DC.

$$H(z) = \left(\frac{1 + \alpha^2 - 2\alpha\cos\omega_0}{2 - 2\cos\omega_0}\right)\frac{\left(z^2 - 2\cos\omega_0 + 1\right)}{\left(z^2 - 2\alpha\cos\omega_0 + \alpha^2\right)}$$

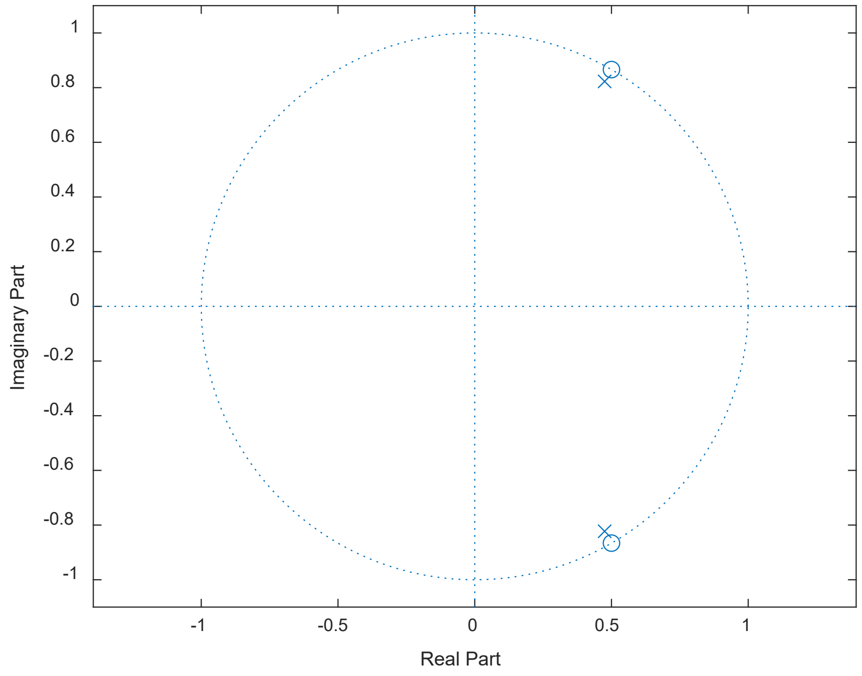
Filters

- Notch filter at 60 Hz

$$\omega_0 = 2\pi f / F_s$$

for 60 Hz with $F_s = 360\text{Hz}$

$$\omega_0 = \pi / 3$$



Show demo for DC notch filter: filter_60_120Hz.m