Determine the z-transforms, sketch the corresponding pole-zero patterns

2.2 b) $x(n) = (a^n + a^{-n})\,u(n)$ a real

$$X(z) = \sum_{n=0}^{\infty} (a^n + a^{-n})\,u(n)\,z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=0}^{\infty} a^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n + \sum \left(\left(\frac{1}{az}\right)\right)^n$$

$$= \frac{1}{1-\frac{a}{z}} + \frac{1}{1-\frac{1}{az}} = \frac{z}{z-a} + \frac{az}{az-1}$$

$$X(z) = \frac{az^2 - z + az^2 - a^2z}{(z-a)(az-1)} = \frac{2az^2 - a^2z - z}{(z-a)(az-1)}$$

Poles: $(z-a)(az-1) = 0$

$z = a$, $z = 1/a$

Zeros: $z(2az - a^2 - 1) = 0$ $2az = a^2 + 1$, $z = a^2+1/_{2a}$

$z = 0$, $z = a^2+1/_{2a}$

ROC:

$$\sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$\sum_{n=0}^{\infty} a^{-n} z^{-n} = \frac{1}{(1-\frac{1}{a}z^{-1})} \qquad \text{ROC: } |z| > \frac{1}{|a|}$$

for $X(z) = \max\left(|a|, \frac{1}{|a|}\right)$

3.2 f $x(n) = Ar^n \cos(\omega_0 n + \phi)\, u(n)$, $0 < r < 1$

$x(n) = A r^n \cos(\omega_n n) \cos\phi - Ar^n \sin(\omega_n n) \sin\phi$

$$X(z) = A\left(\frac{z(z - r\cos\omega_n)\cos\phi}{z^2 - 2rz\cos\omega_n + r^2} - \frac{rz \sin\omega_n \sin\phi}{z^2 - 2rz\cos\omega_n + r^2}\right)$$

$$X(z) = \frac{Az}{z^2 - 2rz\cos\omega_0 + r^2}\left[\cos\phi\,(z - r\cos\omega_0) - \sin\phi\,(r\sin\omega_0)\right]$$

Poles

$z^2 - 2rz\cos\omega_0 + r^2 = 0$

$$z = \frac{-2r\cos\omega_0 \pm \sqrt{4r^2\cos^2\omega_0 - 4r^2}}{2}$$

$$= \frac{-2r\cos\omega_0 \pm \sqrt{4r^2}\sqrt{\cos\omega_0^2 - 1}}{2}$$

Poles
$z_1 = r(-\cos\omega_0 + \sqrt{1 - \cos\omega_0^2}\; i)$
$z_2 = r(-\cos\omega_0 - \sqrt{1 - \cos\omega_0^2}\; i)$

Zeros

$Az = 0$

$z = 0$

$\cos\phi\,(z - r\cos\omega_0) = \sin\phi\,(r\sin\omega_0)$

$z = \tan\phi\, r\sin\omega_0 + r\cos\omega_0$

ROC $|z| > r$

3.4d) $x(n) = (-1)^n \left(\cos \frac{\pi}{3} n\right) u(n)$

$$X(z) = \frac{z\left(z + 1\cos(\pi/3)\right)}{z^2 + 2z\cos(\pi/3) + 1}$$

$$= \frac{z(z+0.5)}{z^2 + z + 1} = \frac{z^2 + 0.5z}{z^2 + z + 1}$$

$$= \frac{1 + 0.5z^{-1}}{1 + z^{-1} + z^{-2}}$$

ROC $|z| > 1$

3.12 Determine the causal signal $x(n)$ having the z-transform

$$\frac{X(z)}{z} = \frac{1}{(1-2z^{-1})(1-z^{-1})^2} = \frac{z^2}{(z-2)(z-1)^2}$$

$$\frac{z^2}{(z-2)(z-1)^2} = \frac{A}{z-2} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$z^2 = A(z-1)^2 + B(z-1)(z-2) + C(z-2)$$

$z=1$

$1 = C(-1)$, $C=-1$

$z=2$

$4 = A$, $A=4$

$z=3$

$9 = 4(2)^2 + B(2) - 1$

$9 - 16 + 1 = 2B$

$-6 = 2B$, $B=-3$

$$\frac{X(z)}{z} = \frac{4}{z-2} - \frac{3}{z-1} - \frac{1}{(z-1)^2}$$

$$X(z) = \frac{4z}{z-2} - \frac{3z}{z-1} - \frac{z}{(z-1)^2}$$

$$x(n) = [4(2^n) - 3(1)^n - n]u(n) = [4(2)^n - 3 - n]u(n)$$

3.14b) Determine the causal signal $x(n)$ if its $X(z)$ is given by

$$X(z) = \frac{1}{1 - z^{-1} + \frac{1}{2} z^{-2}} = \frac{1}{1 - \frac{1}{z} + \frac{1}{2z^2}}$$

$$X(z) = \frac{(1 - \frac{1}{2} z^{-1}) + \frac{1}{2} z^{-1}}{1 - z^{-1} + \frac{1}{2} z^{-2}} = \frac{1 - \frac{1}{2} z^{-1}}{1 - (1) z^{-1} + \frac{1}{2} z^{-2}} + \frac{\frac{1}{2} z^{-1}}{1 - z^{-1} + \frac{1}{2} z^{-2}}$$

form $a^k \cos bk$: $\frac{1 - a z^{-1} \cos b}{1 - 2a z \cos b\, z^{-1} + a^2 z^{-2}}$

$a^2 = \frac{1}{2}$

$\frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{2}} = 2$

$a = \frac{1}{\sqrt{2}}$ $\qquad 2a \cos b = 1$

$\frac{2}{\sqrt{2}} \cos b = 1$

$\cos b = \frac{\sqrt{2}}{2}$ $\quad \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

$$x(n) = \left[\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{4} n\right) + \left(\frac{1}{\sqrt{2}}\right)^n \sin\left(\frac{\pi}{4} n\right)\right] u(n)$$

Determine the convolution of the following pairs by means of the z-transform

3.16 b) $x_1(n) = u(n)$; $x_2(n) = \delta(n) + \left(\frac{1}{2}\right)^n u(n)$

$$X_1(z) = \frac{z}{z-1} \qquad X_2(z) = 1 + \frac{z}{z-1/2}$$

$$Y(z) = \frac{z}{z-1}\left(1 + \frac{z}{z-1/2}\right)$$

$$Y(z) = \frac{z}{z-1} + \frac{z^2}{(z-1)(z-1/2)}$$

$$\frac{Y(z)}{z} = \frac{z(z-1/2) + z^2}{(z-1)(z-1/2)} = \frac{A}{z-1} + \frac{B}{z-1/2}$$

$$(z-1/2) + z = A(z-1/2) + B(z-1)$$

$z=1$

$$1/2 + 1 = A\frac{1}{2}$$

$$1.5 = \frac{A}{2} \qquad A=3$$

$$1/2 = -\frac{B}{2} \qquad B=-1$$

$$\frac{Y(z)}{z} = \frac{3}{z-1} - \frac{1}{z-1/2} \qquad Y(z) = \frac{3z}{z-1} - \frac{z}{z-1/2}$$

$$y(n) = \left[3(1)^n - \left(\frac{1}{2}\right)^n\right]u(n)$$

$$= \left[3 - (1/2)^n\right]u(n)$$

3.16c) $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$ $x_2(n) = \cos \pi n \ u(n)$

$$X_1(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \qquad X_2(z) = \frac{1 + z^{-1}}{1 + 2z^{-1} + z^{-2}}$$

$$Y(z) = \frac{1 + z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + 2z^{-1} + z^{-2})}$$

$$= \frac{A(1 + z^{-1})}{1 + 2z^{-1} + z^{-2}} + \frac{B}{1 - \frac{1}{2}z^{-1}}$$

$$A(1 + z^{-1})(1 - \tfrac{1}{2}z^{-1}) + B(1 + 2z^{-1} + z^{-2}) = 1 + z^{-1}$$

$z = \frac{1}{2}$ $z = 1$

$3 = 9B$

$B = \frac{1}{3}$

$$2(\tfrac{1}{2})A + \frac{4}{3} = 2$$

$$A = 2 - \tfrac{4}{3} = \tfrac{2}{3}$$

$$y(n) = \left[\frac{2}{3}\cos \pi n + \frac{1}{3}\left(\tfrac{1}{2}\right)^n\right] u(n)$$

3.31) Fibonacci using z transform techniques

$$y(n) = y(n-1) + y(n-2) + x(n)$$

$y(0) = 1$ $\quad x(n) = \delta(n)$

$$y(n) = y(n-1) + y(n-2) + \delta(n)$$

$$Y(z) = z^{-1} Y(z) + z^{-2} Y(z) + 1\, X(z)$$

$$(1 - z^{-1} - z^{-2}) Y(z) = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-1} - z^{-2}}$$

$$\frac{H(z)}{z} = \frac{z}{z^2 - z - 1}$$

$z^2 - z - 1$ roots: $r_1 = \frac{1+\sqrt{5}}{2}$ $\quad r_2 = \frac{1-\sqrt{5}}{2}$

$$\frac{H(z)}{z} = \frac{z}{\left(z - \left(\frac{1+\sqrt{5}}{2}\right)\right)\left(z - \left(\frac{1-\sqrt{5}}{2}\right)\right)} = \frac{A}{z - \frac{1+\sqrt{5}}{2}} + \frac{B}{z - \frac{1-\sqrt{5}}{2}}$$

$$A = \frac{\sqrt{5}+1}{2\sqrt{5}} \qquad B = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$y(n) = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{\sqrt{5}+1}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) u(n)$$

C3.3 Matlab

Difference Equation

$$Y(z) = H(z)\,X(z) = \frac{z^{-1} + z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}}\,X(z)$$

$$Y(z)\left(1 - 0.9z^{-1} + 0.81z^{-2}\right) = X(z)\left(z^{-1} + z^{-2}\right)$$

$$y(n) - 0.9y(n-1) + 0.81y(n-2) = \cancel{x(n)}\ x(n-1) + x(n-2)$$

$$y(n) = 0.9y(n-1) - 0.81y(n-2) + x(n-1) + x(n-2)$$

C3.5

$$Y(z) = Y(z)\left(z^{-1} + z^{-2}\right) + X(z)\left(2 + z^{-1}\right)$$

$$Y(z)\left(1 - z^{-1} - z^{-2}\right) = X(z)\left(2 + z^{-1}\right.$$

$$H(z) = \frac{2 + z^{-1}}{1 - z^{-1} + z^{-2}} = \frac{2z^2 + z}{z^2 - z - 1} = \frac{z(2z+1)}{z^2 - z - 1}$$

$$\cancel{h(n) = Z^{inv.}\left(\frac{2 + z^{-1}}{1 - z^{-1} - z^{-2}}\right)}$$

Diff eqn for impulse system

$$H(z) = \frac{2 + z^{-1}}{1 - z^{-1} - z^{-2}}$$

$$h(n) = h(n-1) + h(n-2) + 2\,\delta(n) + \delta(n-1)$$

$$\begin{cases} h(0) = 2 \quad h(1) = 3 \\ h(n) = h(n-1) + h(n-2) \quad n \geq 2 \end{cases}$$

d) We see the h(n) system grows, giving somewhat of a same difference equation.

~~Th~~ This agrees with what we got for part (c).