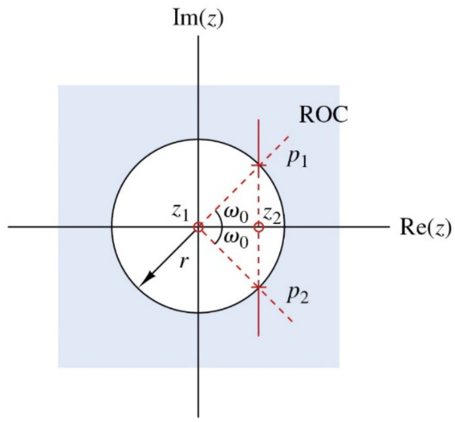
Find time-domain signal from pole-zero map

A pole zero-map is shown below:



The map shows zeros at $z_1 = 0$ and $z_2 = r\cos\omega_0$.

There are poles at the complex conjugate positions: $p_1 = re^{j\omega_0}$ and $p_2 = re^{-j\omega_0}$.

The rational polynomial expression from the pole-zero map is:

$$X(z) = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)} = \frac{z(z - r\cos\omega_0)}{(z - re^{j\omega_0})(z - re^{-j\omega_0})},$$ ROC $|z| > r$ possible with an overall scale factor not shown.

Expanding the denominator:

$$X(z) = \frac{z(z - r\cos\omega_0)}{(z - re^{j\omega_0})(z - re^{-j\omega_0})} = \frac{z(z - r\cos\omega_0)}{z^2 - rz(e^{j\omega_0} + e^{j\omega_0}) + r^2}$$

Using Euler's formula in the denominator:

$$X(z) = \frac{z(z - r\cos\omega_0)}{z^2 - 2rz\cos\omega_0 + r^2} = \frac{z^2}{z^2}\frac{1 - rz^{-1}\cos\omega_0}{1 - 2rz^{-1}\cos\omega_0 + r^2z^{-2}}$$

$$\boxed{X(z) = \frac{1 - rz^{-1}\cos\omega_0}{z^2 - 2rz^{-1}\cos\omega_0 + r^2z^{-2}}, \quad \text{ROC } |z| > r}$$

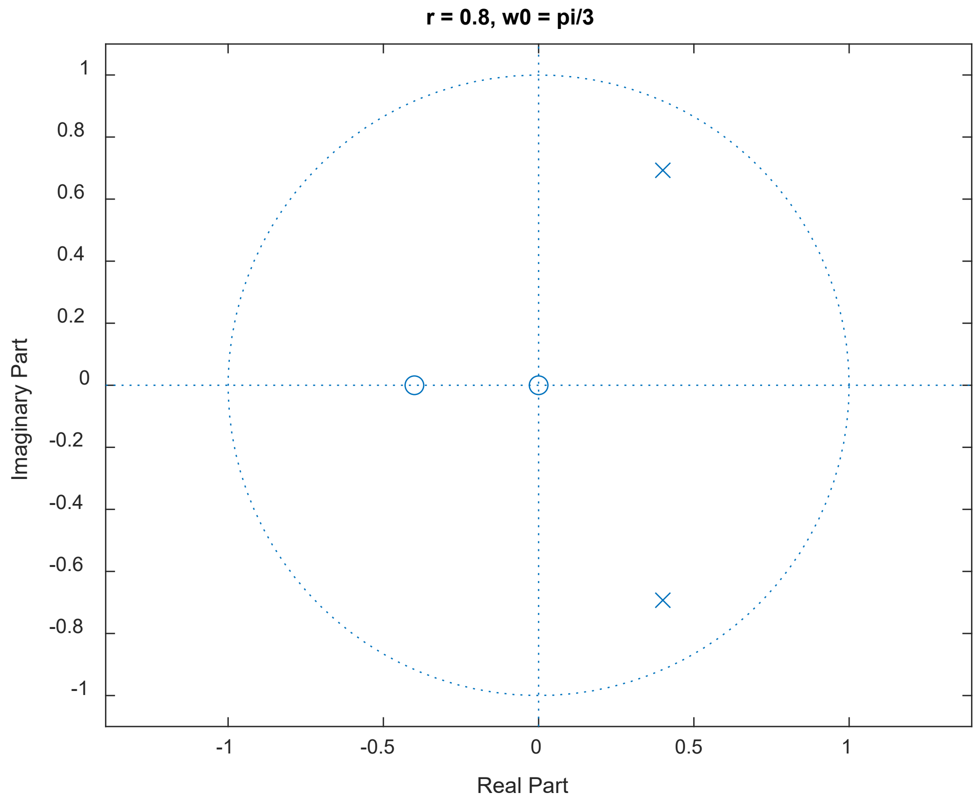
We see from Entry 9 in Table 3.3 of Proakis & Manolakis that the time-domain expression is:

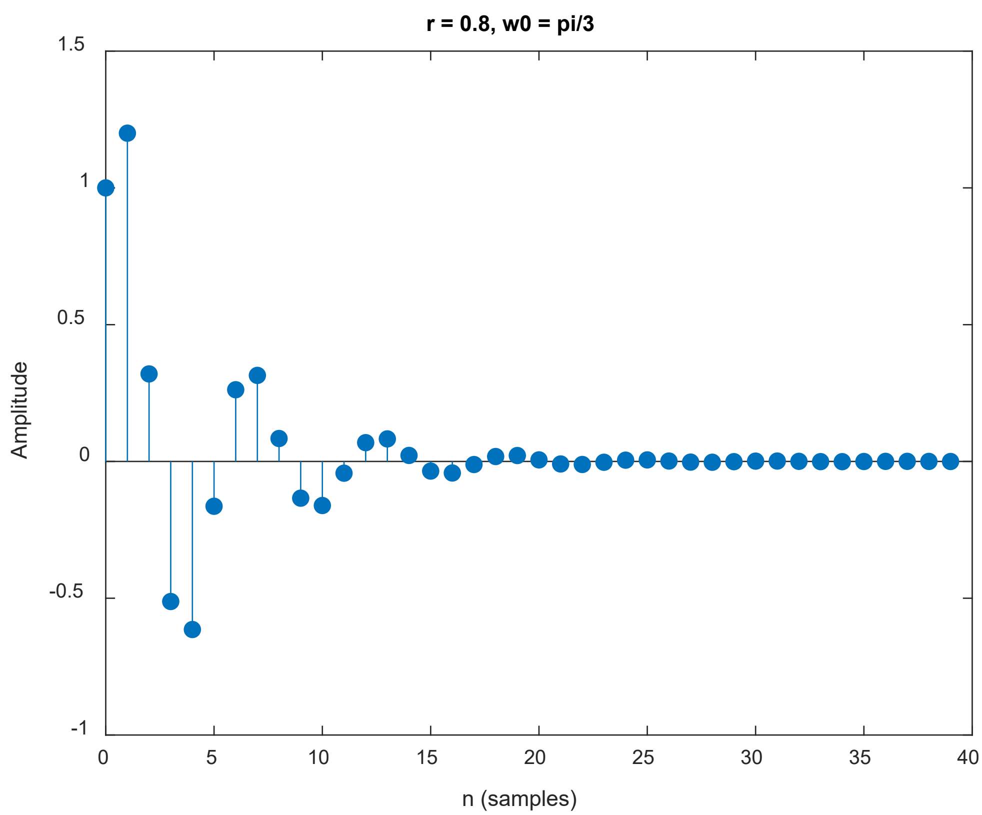
$\boxed{x(n) = r^n\cos(\omega_0 n)u(n)}$.

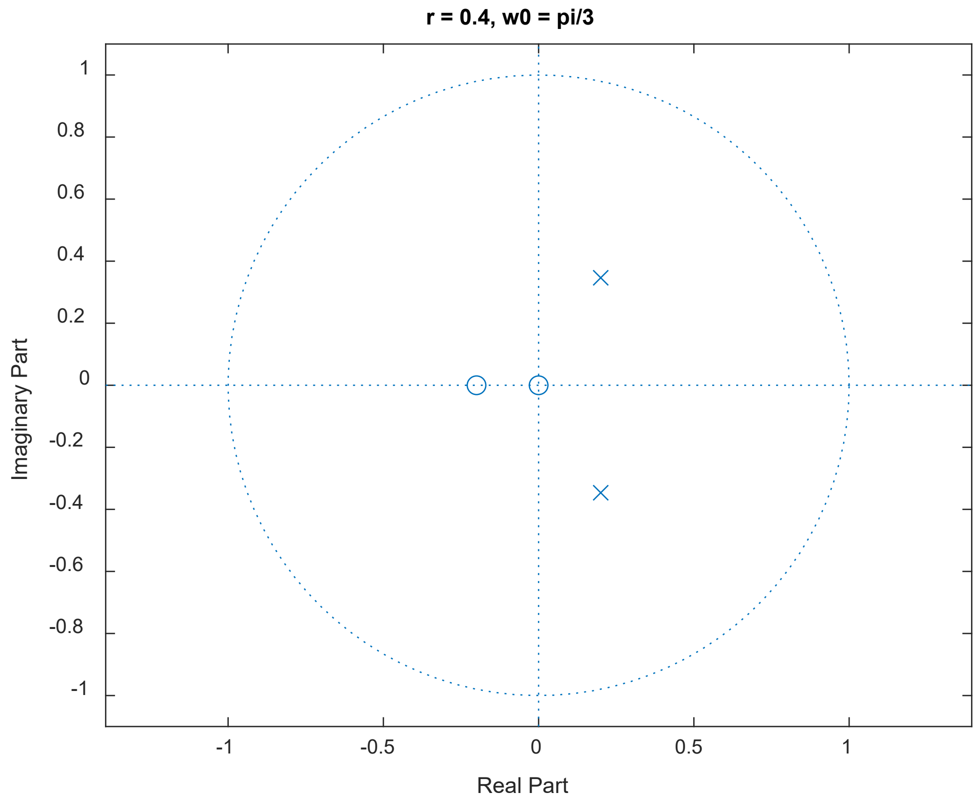
This $x(n)$ will be bounded as $n \to \infty$ if $r \le 1$.

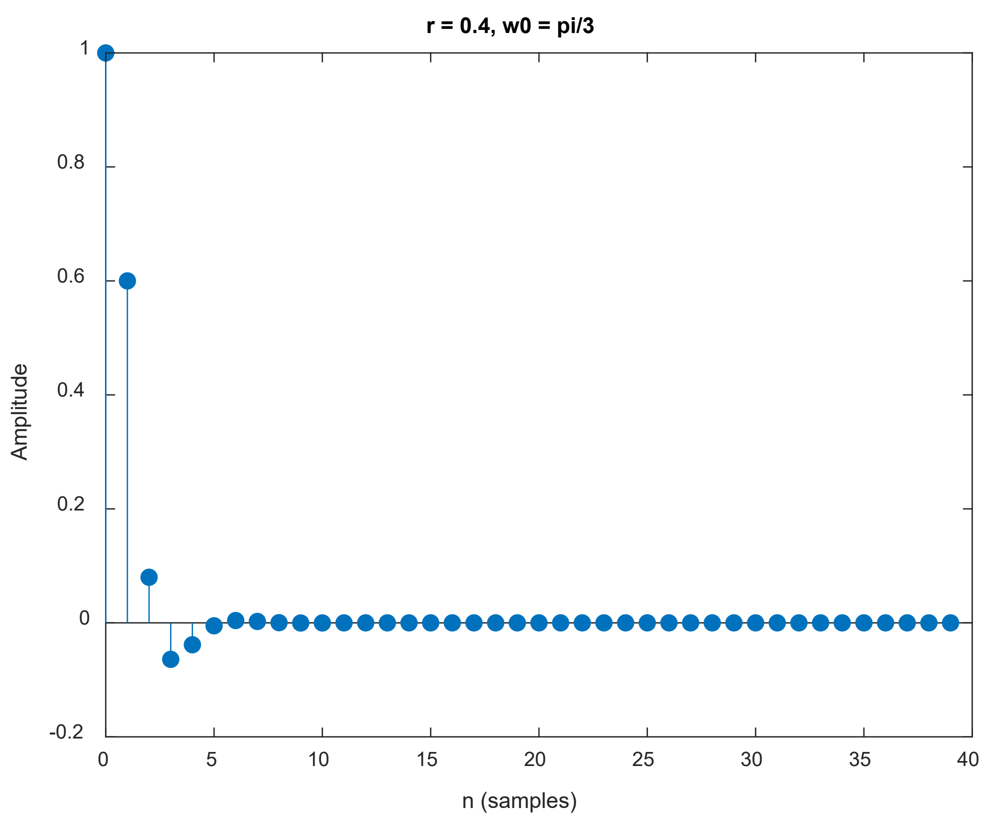
From the expression, we can see that $x(n)$ decreases more rapidly when r is small.

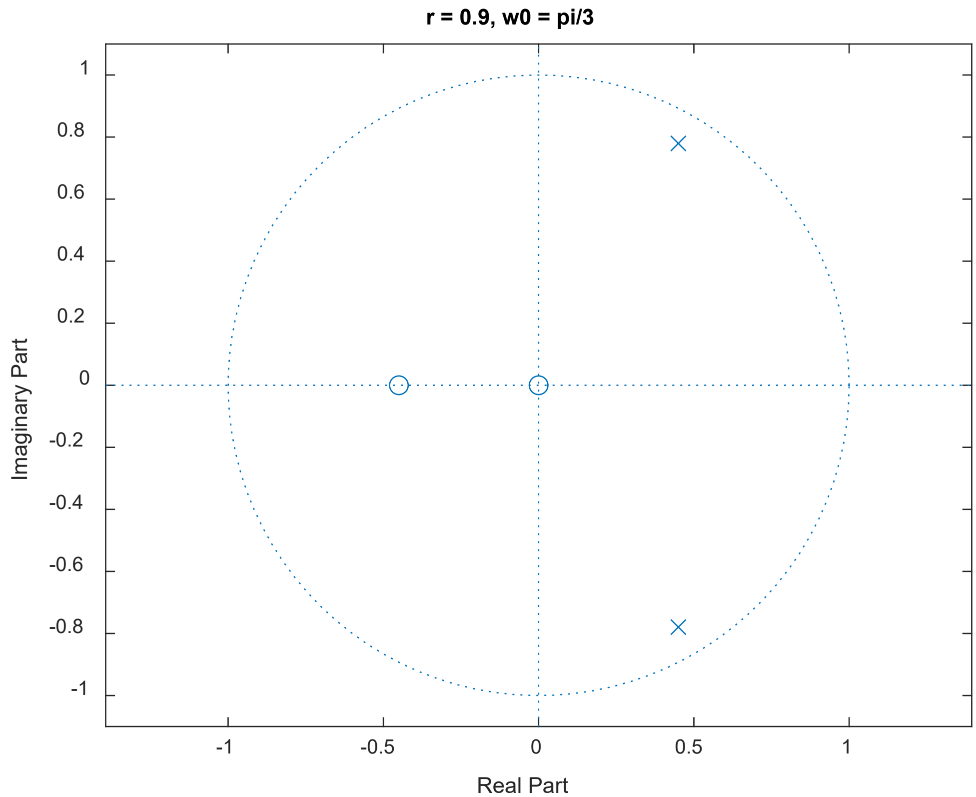
Also, $x(n)$ oscillates more rapidly when ω_0 is large. Examples are shown on the following pages.


r = 0.8, w0 = pi/3
Imaginary Part
Real Part
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1

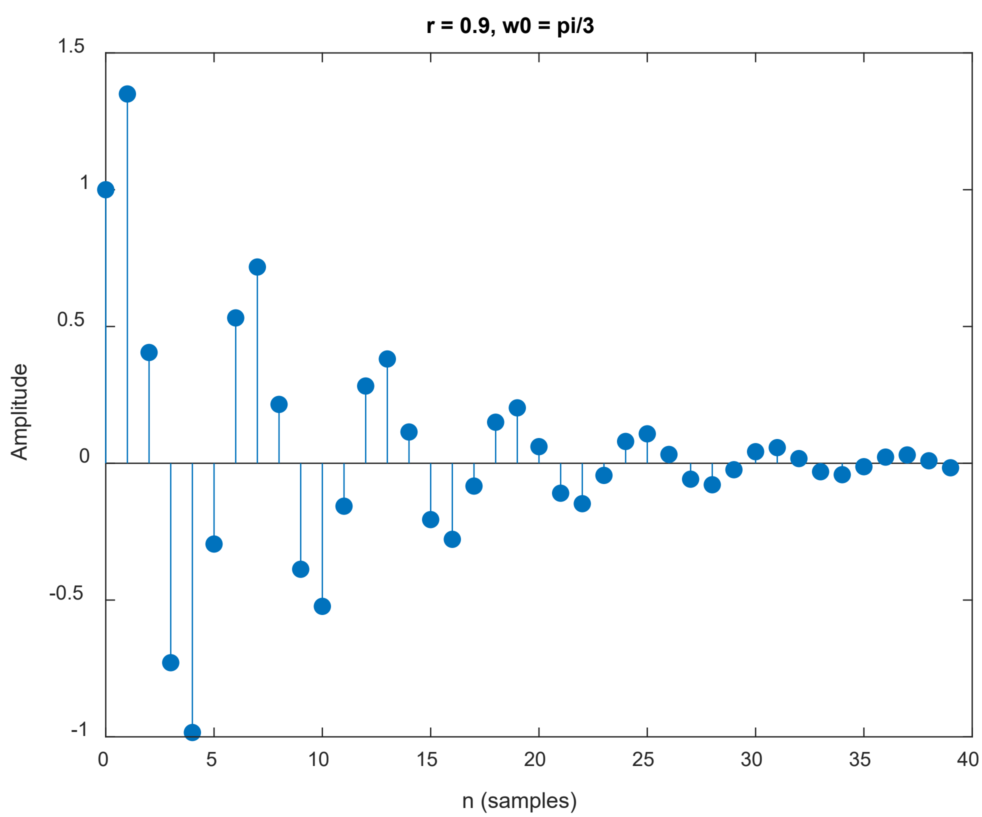

r = 0.8, w0 = pi/3
Amplitude
n (samples)
1.5
1
0.5
0
-0.5
-1
0
5
10
15
20
25
30
35
40


r = 0.4, w0 = pi/3
Imaginary Part
Real Part
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1

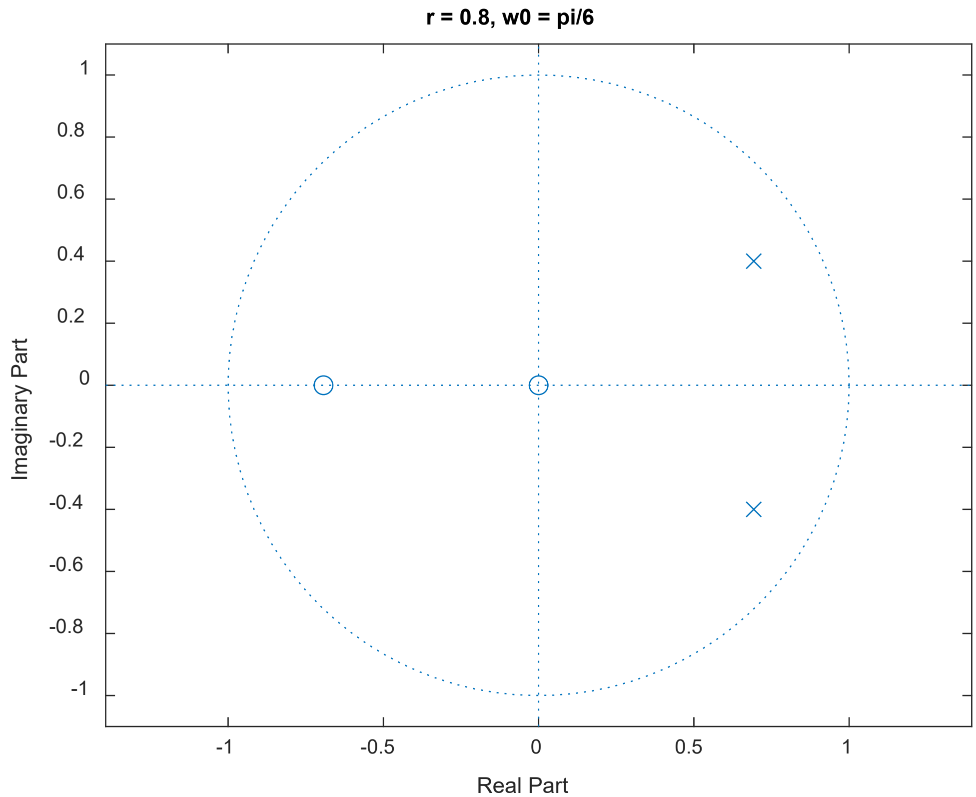

r = 0.4, w0 = pi/3
Amplitude
n (samples)
1
0.8
0.6
0.4
0.2
0
-0.2
0
5
10
15
20
25
30
35
40



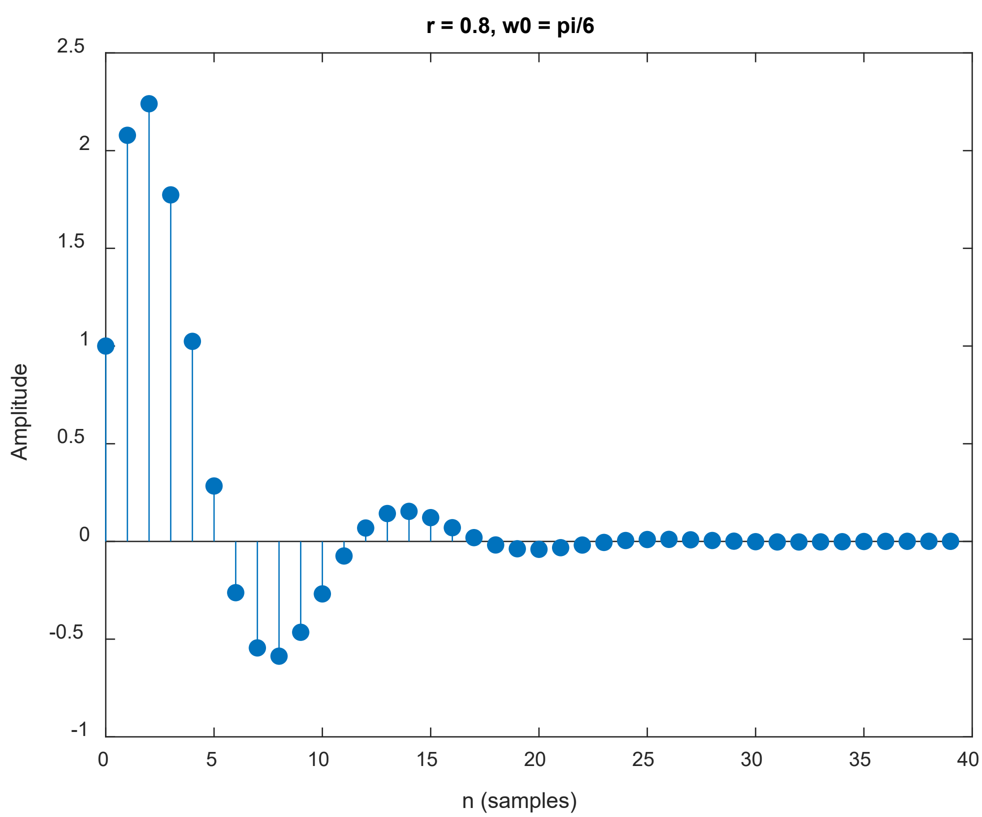
r = 0.9, w0 = pi/3
Imaginary Part
Real Part
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1



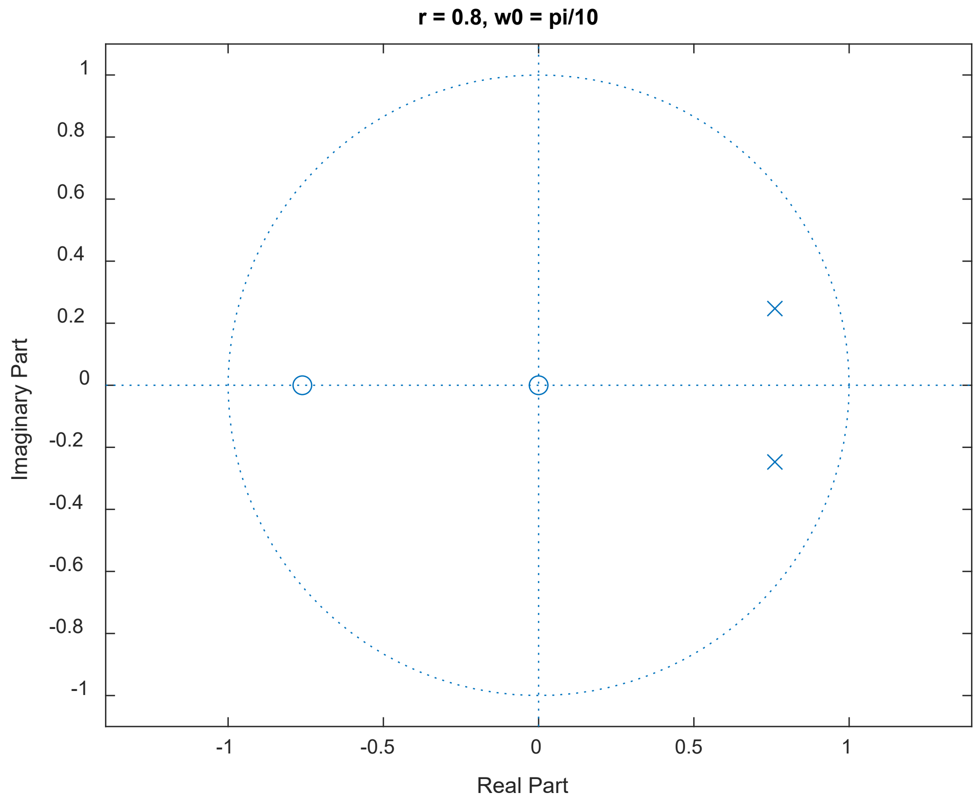
r = 0.9, w0 = pi/3
Amplitude
n (samples)
1.5
1
0.5
0
-0.5
-1
0
5
10
15
20
25
30
35
40

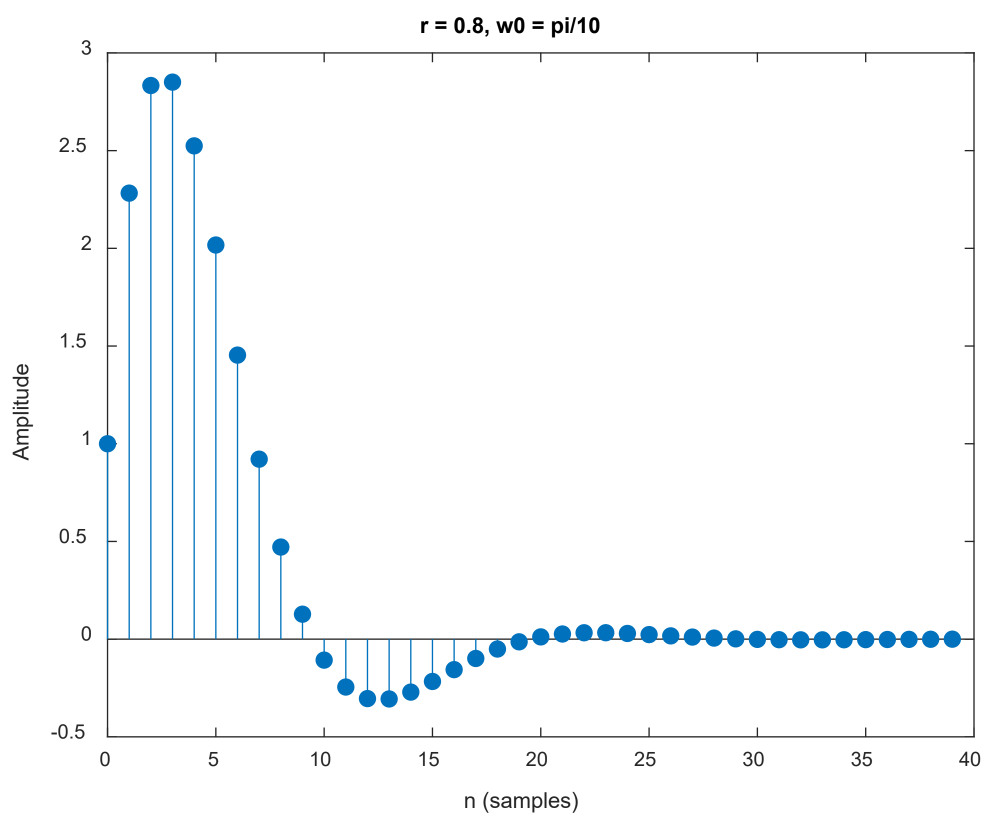


r = 0.8, w0 = pi/6
Imaginary Part
Real Part
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1



r = 0.8, w0 = pi/6
Amplitude
n (samples)
2.5
2
1.5
1
0.5
0
-0.5
-1
0
5
10
15
20
25
30
35
40


r = 0.8, w0 = pi/10
Imaginary Part
Real Part
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1


r = 0.8, w0 = pi/10
Amplitude
n (samples)
3
2.5
2
1.5
1
0.5
0
-0.5
0
5
10
15
20
25
30
35
40

Z