Digital Signal Processing

Class 6
02/06/2025

ENGR 71

- Class Overview
  - Discrete-Time Signals and Systems
- Assignments
  - Reading:
    Chapter 2: Discrete-Time Signals and Systems
  - Lab 1 – Aliasing lab
    - Will be up on Moodle this afternoon

ENGR 71

- Lab 1-Aliasing Lab
  - Find a short piece of music to download
  - Subsample to demonstrate aliasing
  - Repeat the subsampling, but prior to subsampling, apply a low-pass anti-aliasing filter.
  - Compare the results
- More details and sample code will be placed on Moodle

Class Information

- Topics in Discrete-Time Signals and Systems
  - Discrete-Time Signals
  - Discrete-Time Systems
  - Analysis of Linear Time-Invariant Systems
  - Description of Systems by Difference Equations
  - Implementation of Discrete-Time Systems
  - Correlation of Discrete-Time Systems

Some aspects of these topics have already been discussed in the review of continuous systems

Discrete-Time Signals

- **Signals**

  (already covered in review of continuous signals)

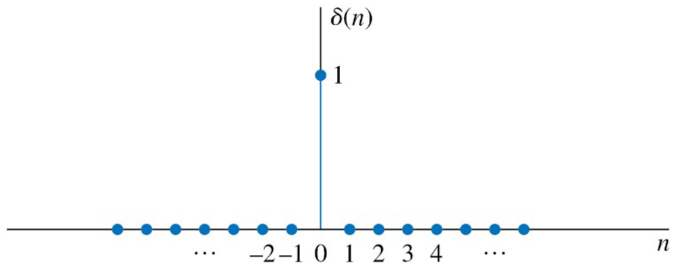
  – Discrete-signals are function of an integer index

  – Signal described by $x(n)$ where n is an integer indicating the sample number.

Discrete-Time Signals

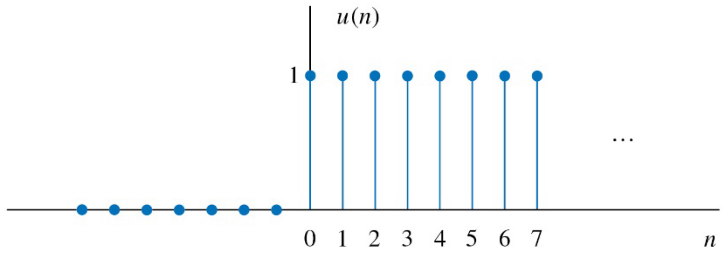
– Elementary discrete-time signals
- Unit sample sequence (impulse)

$$\delta(n) = 0, \quad n \neq 0$$
$$\delta(n) = 1, \quad n = 0$$



- Unit step signal

$$u(n) = 1, \quad n \geq 0$$
$$u(n) = 0, \quad n < 0$$

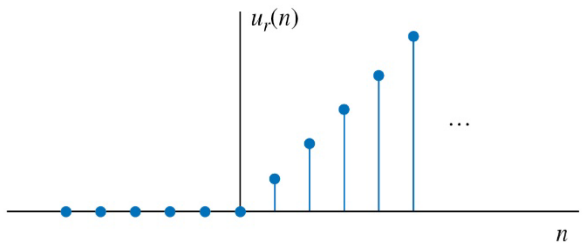


Discrete-Time Signals

– Elementary discrete-time signals

- Unit ramp

$$u_r(n) = n, \quad n \geq 0$$
$$u_r(n) = 0, \quad n < 0$$



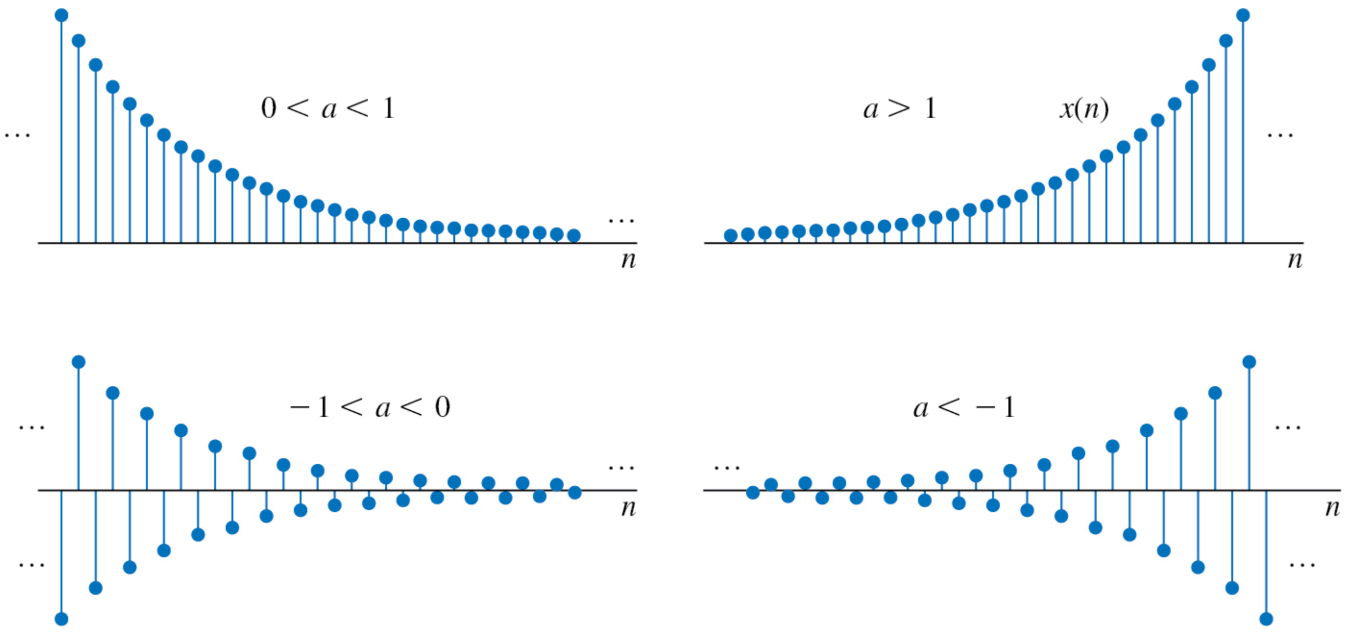
Discrete-Time Signals

– Elementary discrete-time signals

- Exponential signal (one we haven't discussed)

$$x(n) = a^n, \quad \text{for all } n$$

If a is real, $x(n)$ is real-valued signal.



Discrete-Time Signals

– Elementary discrete-time signals

- Exponential signal (one we haven't discussed)

If a is complex, $x(n)$ is complex-valued signal.

$$a = re^{j\theta}$$

$$x(n) = a^n = r^n e^{j\theta n} = r^n\left(\cos\theta n + j\sin\theta n\right)$$

$$x_R(n) = r^n \cos\theta n \quad \text{(Real part of signal)}$$

$$x_I(n) = r^n \sin\theta n \quad \text{(Imaginary part of signal)}$$

$$\left|x(n)\right| = A(n) = r^n \quad \text{(Magnitude of signal)}$$

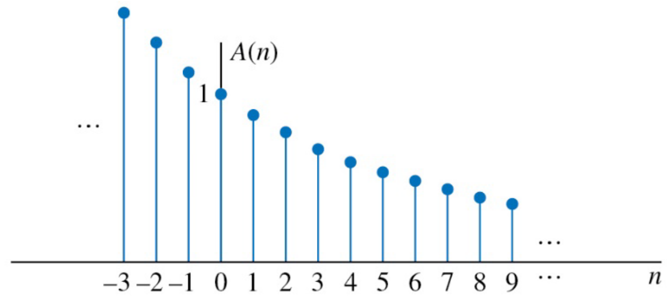
$$\angle x(n) = \phi(n) = \theta n \quad \text{(Phase of signal)}$$

Since it wraps, only consider range from $-\pi$ to π

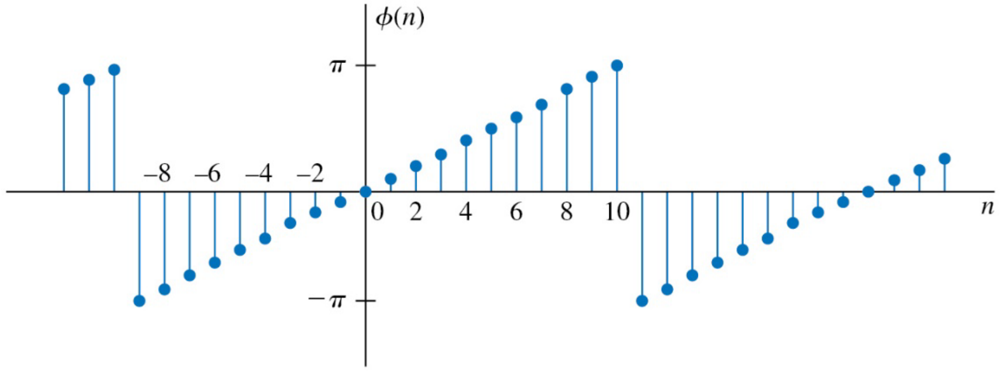
Discrete-Time Signals

– Elementary discrete-time signals

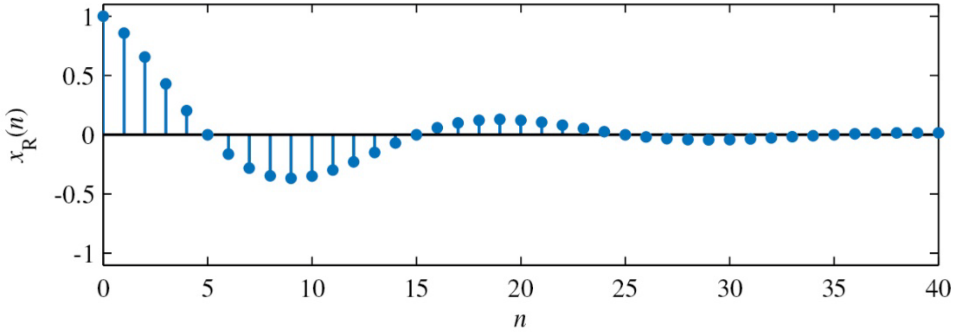
- Exponential signal (one we haven’t discussed)



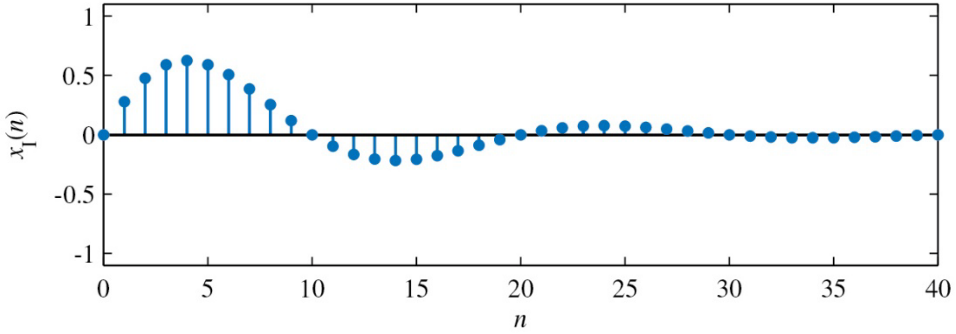
(a) Graph of $A(n) = r^n$, $r = 0.9$



(b) Graph of $\phi(n) = \frac{\pi}{10}n$, modulo 2π plotted in the range $(-\pi, \pi)$



(a)



(b)

Discrete-Time Signals

– Classification of signals

- Energy signals: Finite energy in signal

$$E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

- Power signals: Energy is infinite, but power is finite

$$P \equiv \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 < \infty$$

Discrete-Time Signals

– Classification of signals

- Power signal example:

$$x(n) = Ae^{j\omega_o n}$$

$$P \equiv \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|x(n)\right|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)x^*(n)$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} Ae^{j\omega_o n} Ae^{-j\omega_o n} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} A^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} A^2 \left(2N+1\right)$$

$$\boxed{P = A^2}$$

Discrete-Time Signals

- Classification of signals
  - Periodic or Aperiodic signals

$x(n+N) = x(n)$ for periodic signal

Infinite energy but finite power

(break sum up into sums over N samples, where N is the period)

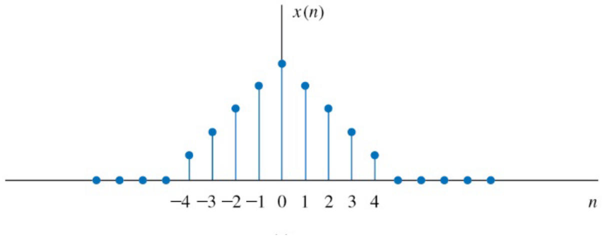
Power of periodic signal

$$P \equiv \frac{1}{N}\sum_{n=0}^{N-1}\left|x(n)\right|^2$$

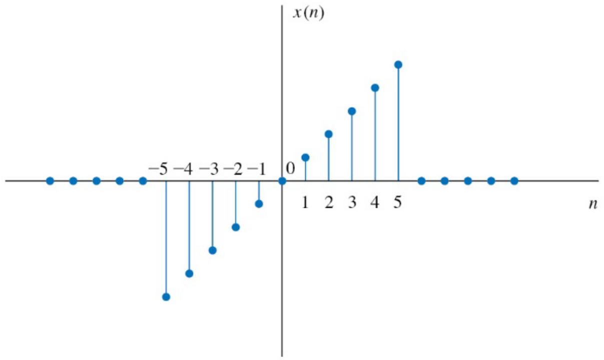
Discrete-Time Signals

– Classification of Signals
  - Even and Odd signals (Symmetric / Antisymmetric)

$x(-n) = x(n)$ for even (symmetric)



$x(-n) = -x(n)$ for odd (antisymmetric)



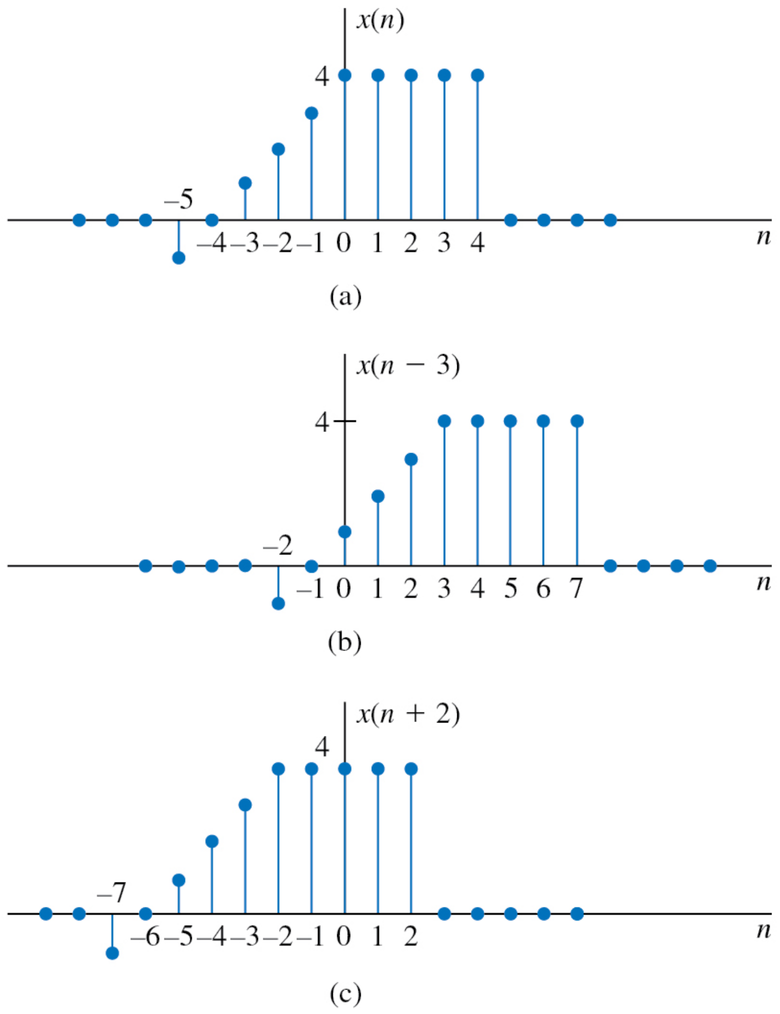
Discrete-Time Signals

- Operations on independent variable
  - Shifting (delay or advance) $y(n) = x(n \pm k)$
  - Folding (reflection) $y(n) \rightarrow x(-n)$
  - Down sampling $y(n) \rightarrow x(2n)$

Discrete-Time Signals

- Shifting

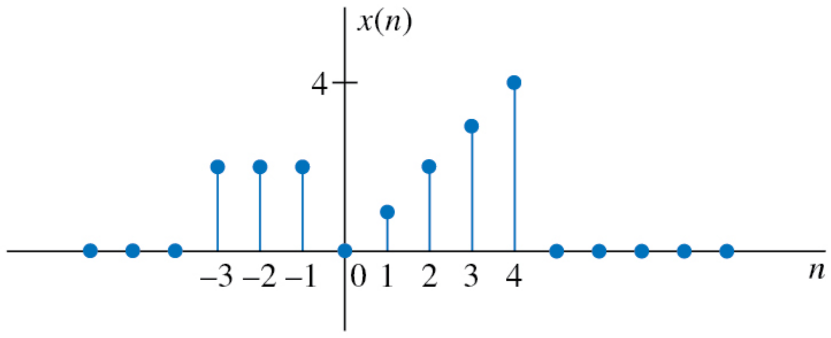
  $y(n) = x(n \pm k)$



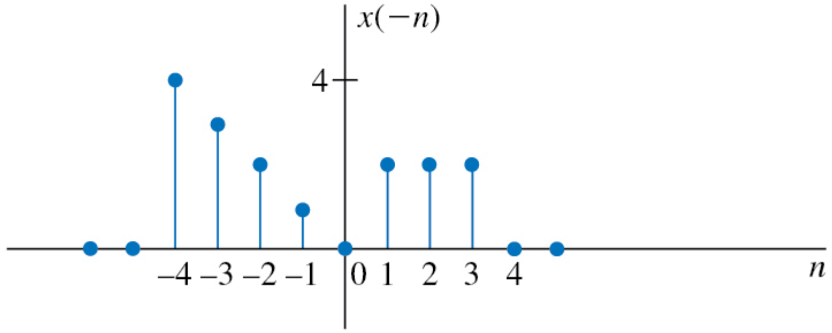
Discrete-Time Signals

- Folding (reflection)

$y(n) \rightarrow x(-n)$

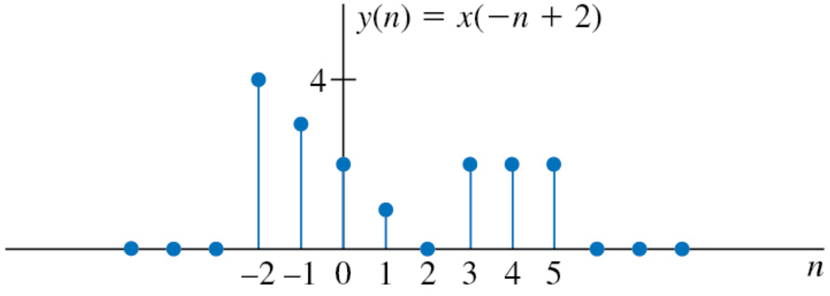


(a)



(b)

Fold and shift

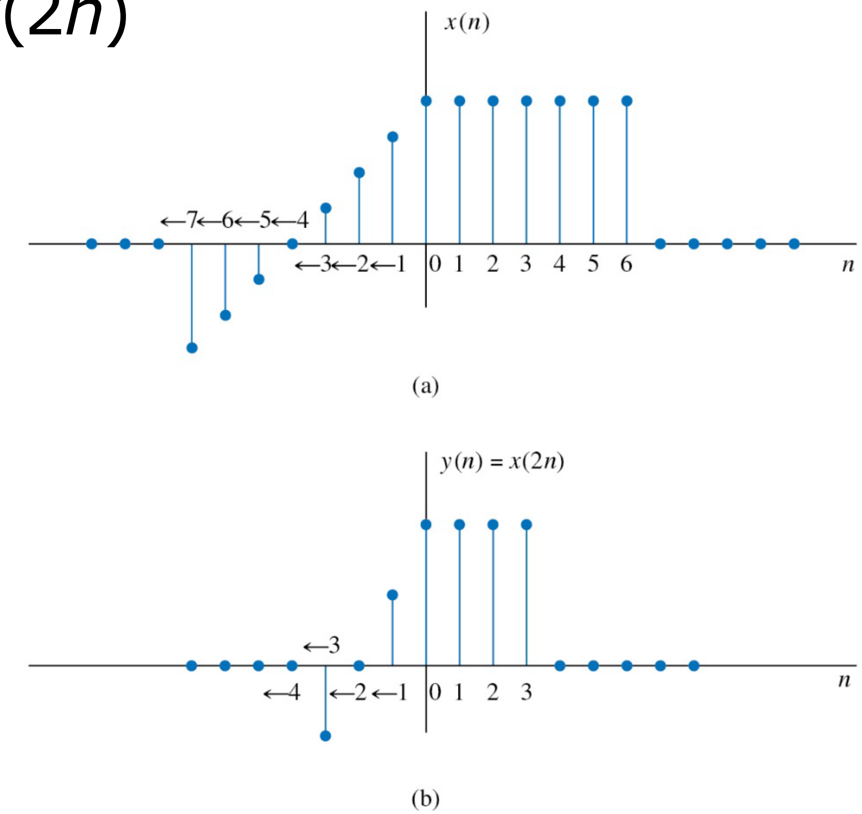


Discrete-Time Signals

- Down sampling

  If sampling, like dividing sampling rate by some integer value

  $y(n) \rightarrow x(2n)$



This is different than continuous time case, Since you can't divide samples or multiply by non-integer values.

Discrete-Time Signals

– Arithmetic operations on signals

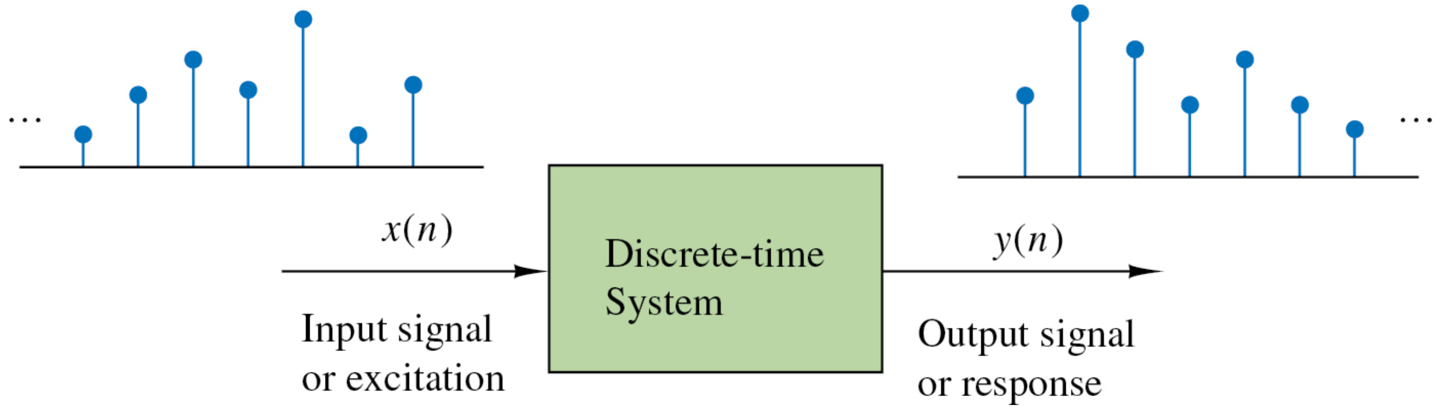
- Addition $y(n) = x_1(n) + x_2(n)$
- Multiplication $y(n) = x_1(n)x_2(n)$
- Scaling $y(n) = Ax_2(n)$

Discrete-Time Systems

- **Systems**
  (already covered in review of continuous signals)

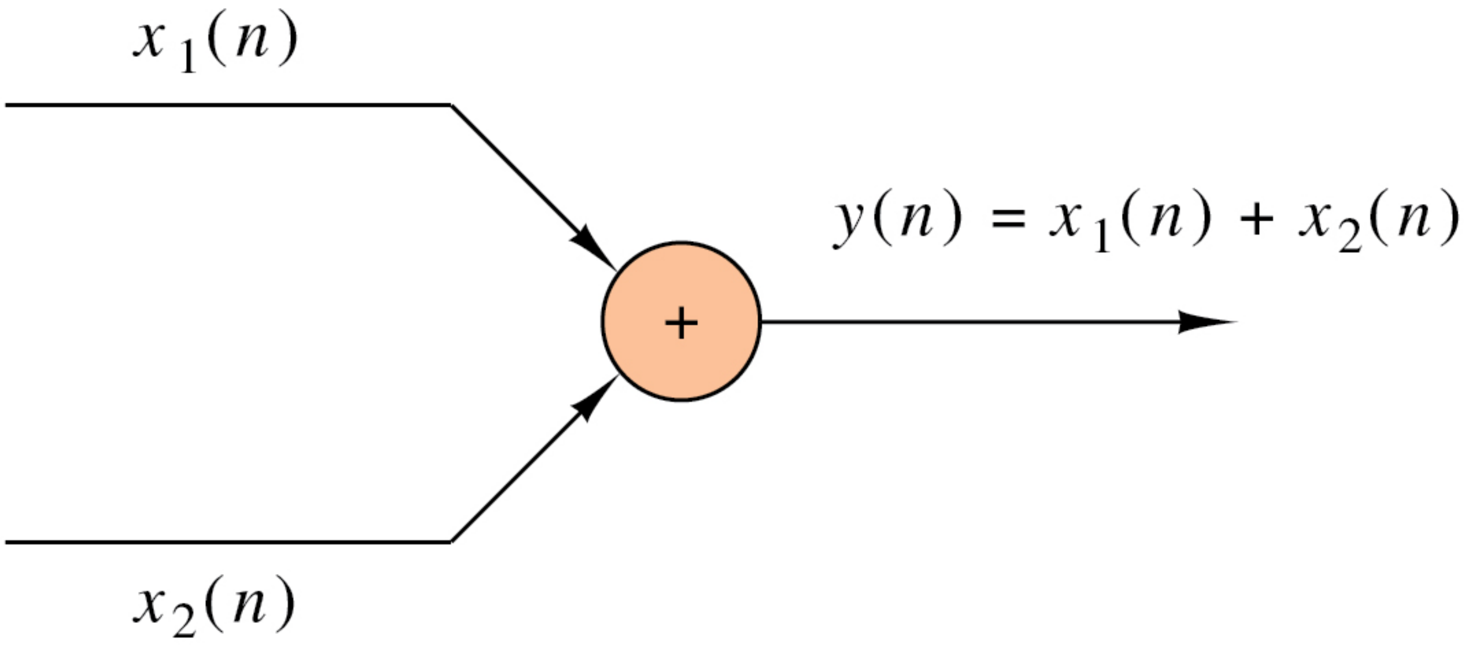
  System: Transforms input signal into output signal

$$x(n) \xrightarrow{\mathscr{T}} y(n)$$



Discrete-Time Systems

- Block diagrams for systems
  - Adder



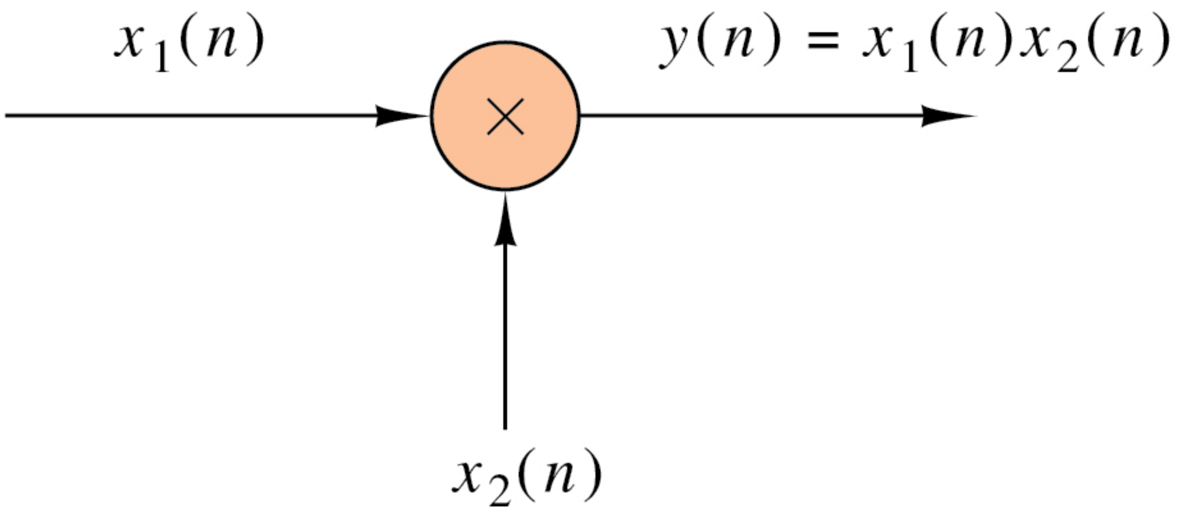
Discrete-Time Systems

- Block diagrams for systems
  - Constant multiplier

Discrete-Time Systems

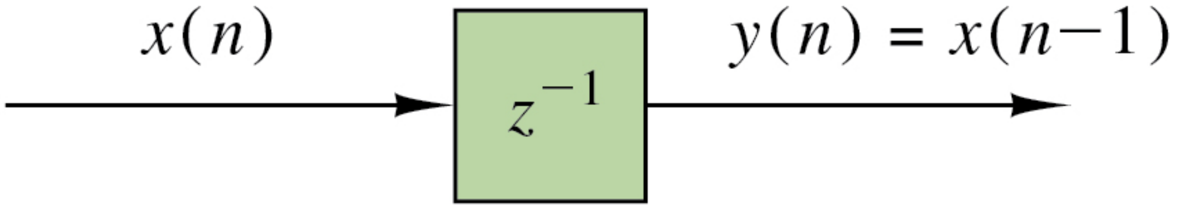
– Block diagrams for systems

- Signal Multiplier



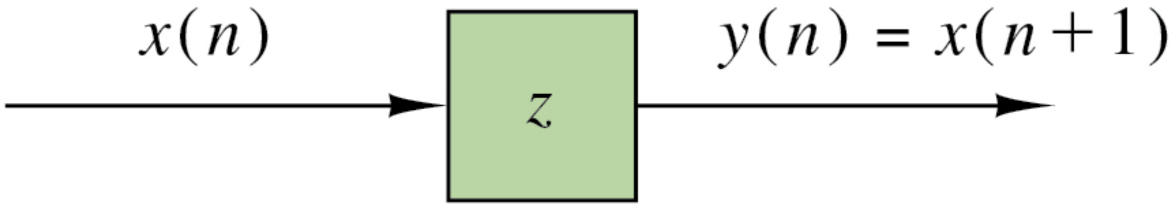
Discrete-Time Systems

– Block diagrams for systems
  - Unit delay element



Discrete-Time Systems

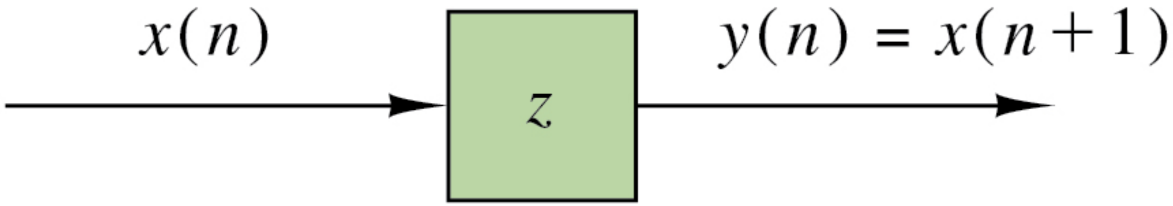
– Block diagrams for systems
  - Unit advance element



Discrete-Time Systems

– Block diagrams for systems

- Unit advance element

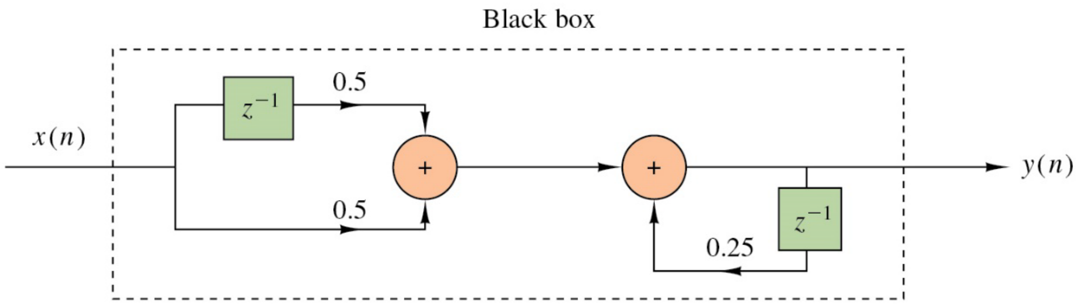


Discrete-Time Systems

– Building up a system from blocks

Realization 1 :

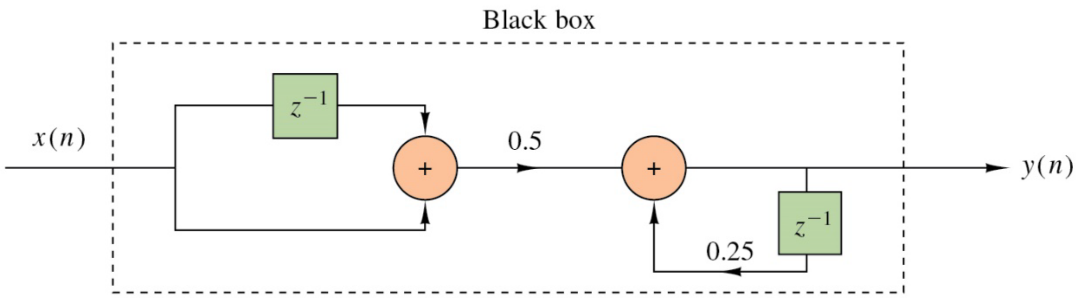
$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$$



(a)

Realization 2 :

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}\left[x(n) + x(n-1)\right]$$



(b)

Discrete-Time Systems

– Classification of Systems

(already covered in review of continuous systems)

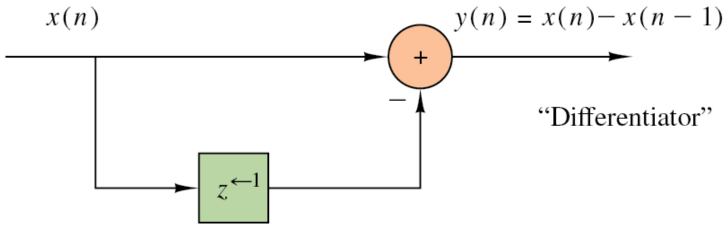
- Static versus Dynamic Systems
  - Static is memoryless: output only depends on current input
  - Dynamic has memory: output may depend on previous and current inputs
- Time Invariant (also known as Shift Invariant)
  - Input-output characteristics do not change with time (or sample #)
  - For "relaxed" system (i.e., no initial conditions)

$$x(n) \xrightarrow{\mathscr{T}} y(n) \quad \Rightarrow \quad x(n-k) \xrightarrow{\mathscr{T}} y(n-k)$$

Discrete-Time Systems

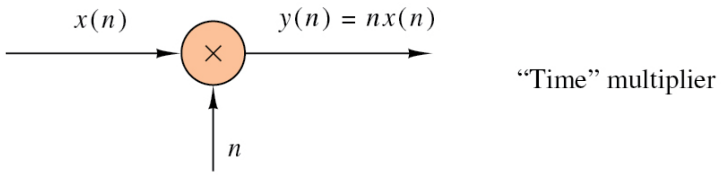
- Time Invariant/Variant examples

Time invariant



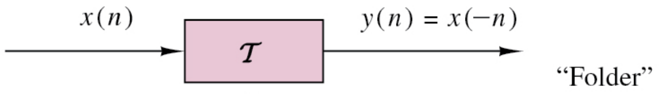
(a)

Time variant



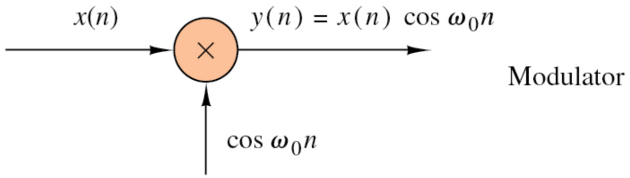
(b)

Time variant



(c)

Time variant

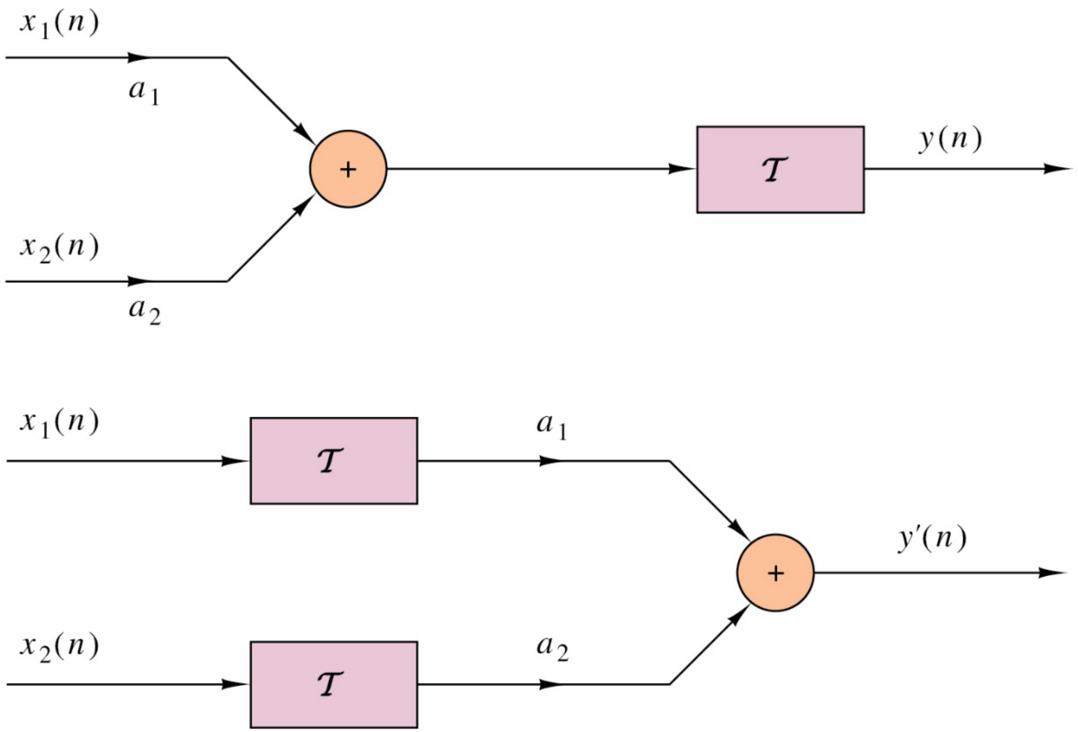


(d)

Discrete-Time Systems

- Linear versus Non-Linear Systems
  - Linear:

$$S[a_1x_1(t)+a_2x_2(t)]=a_1S[x_1(t)]+a_2S[x_2(t)]$$



Works for multiple sums

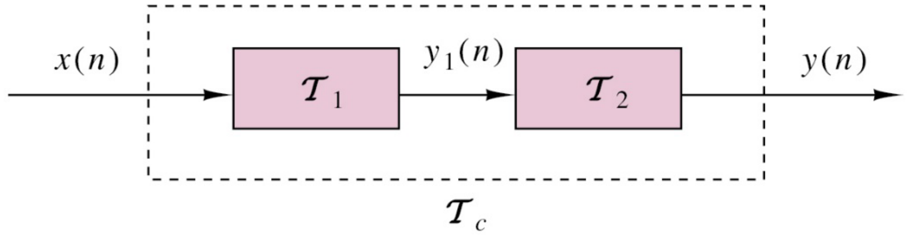
$$S\left[x(n)=\sum_{k=1}^{M-1}a_kx_k(t)\right]=y(n)=\sum_{k=1}^{M-1}a_ky_k(t)$$

$\boxed{\text{Principle of Superposition}}$

Discrete-Time Systems

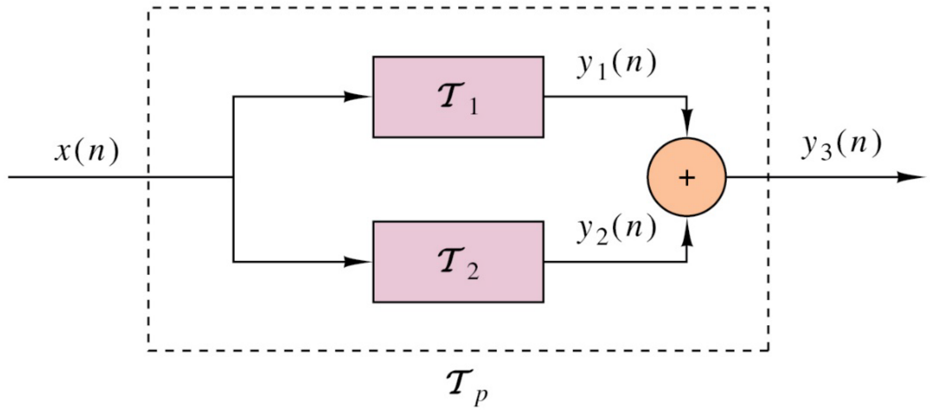
- Interconnections of systems

Series (cascade)



(a)

Parallel



(b)

Discrete-Time Systems

– Analysis of Linear Systems

- Two methods for analyzing linear systems
  - Direct method: (Think “time-domain”)
    - Solve equation describing system for $y(n)$:

$$y(n) = F\left[y(n-1), y(n-2), \ldots, y(n-N), x(n), x(n-1), \ldots, x(n-M)\right]$$

  - Transform method: (Think “frequency-domain”)
    - We will discuss z-transform and discrete Fourier transform later

Discrete-Time Systems

– Analysis of Linear Systems

- Direct Method
  - The most general form for Linear Time Invariant (LTI) systems is a difference equation:

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=1}^{M} b_k x(n-k)$$

  - Can be solved by decomposing input into weighted sum of elementary signals, where the response to the elementary signals is known. The use superposition

$$x(n) = \sum_{k} c_k x_k(n) \qquad y_k(n) = S\left[x_k(n)\right] \qquad y = \sum_{k} c_k y_k(n)$$

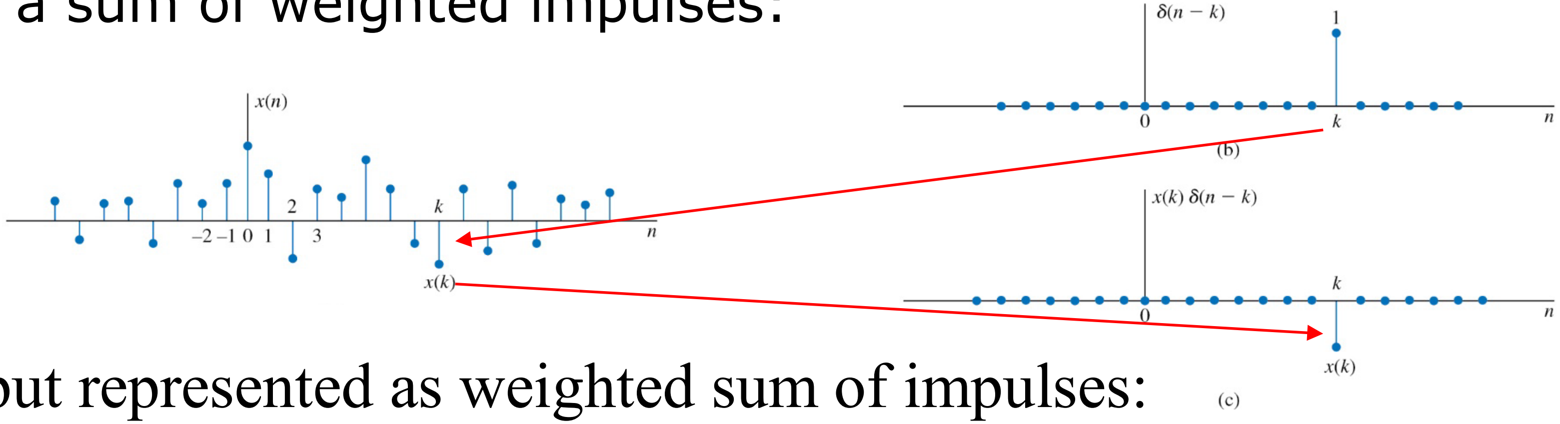
Discrete-Time Systems

– Impulse response:

- Select elementary functions to be delayed unit impulses

$$x_k(n) = \delta(n-k)$$

- As we've discussed before, any discrete signal can be decomposed into a sum of weighted impulses:



Input represented as weighted sum of impulses:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Discrete-Time Systems

- Impulse response for linear system:
  - Linearity:

$$y(n) = S\left[x(n)\right] = S\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k)S\left[\delta(n-k)\right]$$

  - Output for a system with no initial conditions ("relaxed")
  - In general $h(n,k) \equiv S\left[\delta(n-k)\right]$ could change with sample number is system is not time invariant.
  - For time invariant system:
- Linear Time Invariant System: $h(n-k) \equiv S\left[\delta(n-k)\right]$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad \text{where} \quad h(n) = S\left[\delta(n)\right] \text{ is impulse response}$$

Discrete Convolution

Discrete-Time Systems

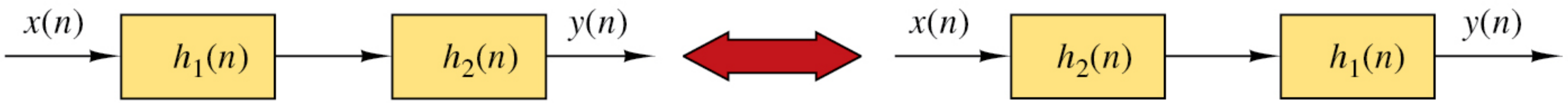
– Properties of Convolution:

- Commutivity: $x(n) * h(n) = h(n) * x(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

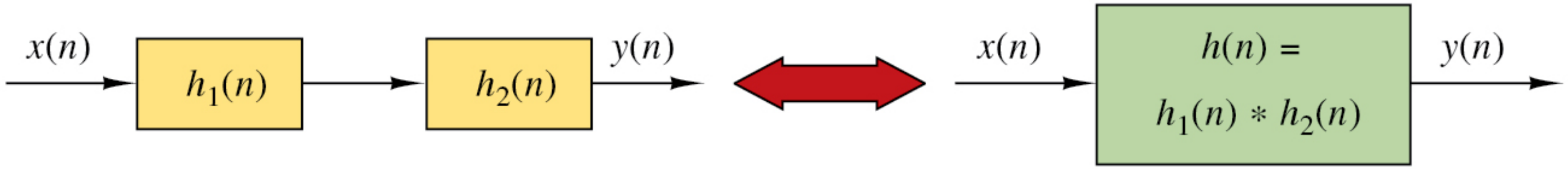
$$h_1(n) * h_2(n) = h_2(n) * h_1(n)$$



Discrete-Time Systems

- Properties of Convolution:
  - Associativity

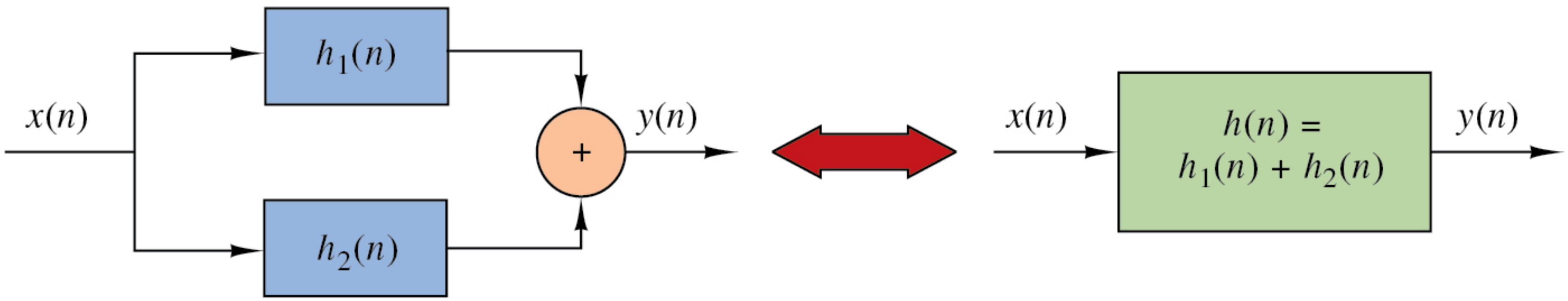
$$\left[x(n) * h_1(n)\right] * h_2(n) = x(n) * \left[h_1(n) * h_2(n)\right]$$



Discrete-Time Systems

- Properties of Convolution:
  - Distributivity

$$x(n) * \left[h_1(n) + h_2(n)\right] = x(n) * h_1(n) + x(n) * h_2(n)$$



Discrete-Time Systems

- Causal LTI Systems:

$$h(n) = 0 \quad \text{for} \quad n < 0$$

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

Discrete-Time Systems

- Stable LTI systems
  - For bounded input to yield bounded output

$$|x(n)| \le M_x$$

$$|y(n)| \le M_y$$

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

Also, impulse response must got to zero as $n \to \infty$

Discrete-Time Systems

- Stable LTI systems
  - Example of conditions for stability
    - Consider the following causal impulse response:

$$h(n) = a^n u(n)$$

$$\sum_{k=-\infty}^{\infty} \left|h(k)\right| < \infty \quad \Rightarrow \quad \sum_{k=-\infty}^{\infty} \left|a^n u(n)\right| = \sum_{k=0}^{\infty} \left|a^n\right| < \infty$$

$$\text{Geometric series:} \quad \sum_{k=0}^{\infty} \left|a^n\right| = \frac{1}{1-\left|a\right|} \quad \text{if } \left|a\right| < 1$$

$$\text{If } \left|a\right| \geq 1, \quad \sum_{k=0}^{\infty} \left|a^n\right| \to \infty$$

$$\therefore \text{ System with } h(n) = a^n u(n) \text{ is stable for } \left|a\right| < 1\text{, unstable for } \left|a\right| \geq 1$$

Discrete-Time Systems

- Finite Impulse Response systems (FIR)
  - For causal system,

    $$h(n) = 0, \quad n < 0 \text{ and } n \geq M$$

    is an FIR system

    $$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

  - This is a weighted sum of input values

    $$x(n), x(n-1), \ldots, x(n-M+2), x(n-M+1)$$

  - Finite memory length

Discrete-Time Systems

- Infinite Impulse Response systems (IIR)
  - For causal system, $h(n)$ persists for n going to infinity

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

  - Has infinite memory. Response depends on all previous inputs
  - Results from system with recursion where current output depends on previous outputs.

Discrete-Time Systems

– LTI systems characterized by constant-coefficient difference equations

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

N'th order difference equation

N'th order system

Discrete-Time Systems

– Compare this to continuous system

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_1\frac{dy}{dt} + a_0 y =$$

$$b_m\frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1} x}{dt^{m-1}} + b_{m-2}\frac{d^{m-2} x}{dt^{m-2}} + \cdots + b_1\frac{dx}{dt} + b_0 x$$

Discrete-Time Systems

– If output only depends on previous and current input

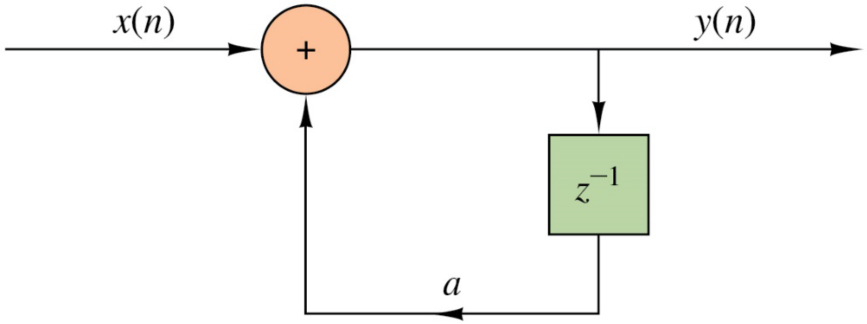
$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

System will be FIR

Discrete-Time Systems

– Example of a simple recursive difference equation
  - First order autoregressive model

$$y(n) = ay(n-1) + x(n)$$



Discrete-Time Systems

- First order autoregressive model

$$y(n) = ay(n-1) + x(n)$$

$$y(0) = ay(-1) + x(0)$$

$$y(1) = ay(0) + x(1) = a\left(ay(-1) + x(0)\right) + x(1) = a^2 y(-1) + ax(0) + x(1)$$

$$y(2) = ay(1) + x(2) = a\left(a^2 y(-1) + ax(0) + x(1)\right) + x(2) = a^3 y(-1) + a^2 x(0) + ax(1) + x(2)$$

$$\vdots$$

$$y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k x(n-k)$$

Depends on initial condition *y*(-1)

Discrete-Time Systems

- Same drill as before
  - If the system is relaxed (no initial condition)

    Zero-state solution

$$y_{ZS}(n) = \sum_{k=0}^{n} a^k x(n-k)$$

  - If the system has no input
    Zero-input solution

$$y_{ZI}(n) = a^{n+1} y(-1)$$

Discrete-Time Systems

- Notice the impulse response of this system

$$h(0) = ay(-1) + \delta(0) = 1 \quad \text{(with no initial condition)}$$

$$h(1) = ay(0) + \delta(1) = a$$

$$h(2) = ay(1) + \delta(2) = a^2$$

$$\vdots$$

$$h(n) = a^n u(n)$$

$$y_{ZS}(n) = \sum_{k=0}^{n} a^k x(n-k) = \sum_{k=0}^{n} h(k)x(n-k)$$

Zero-state response is convolution of input with impulse response

Discrete-Time Systems

– Don’t forget about geometric series

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

$$\sum_{k=1}^{n} r^k = \frac{r\left(1 - r^n\right)}{1 - r}$$

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}$$

for $|r| < 1$

$$\sum_{k=1}^{\infty} r^k = \frac{r}{1 - r}$$

for $|r| < 1$

Discrete-Time Systems

– Difference equations

Can be solved in similar manner as differential equations

$$y(n) - ay(n-1) = x(n)$$

Guess solution of homogeneous equation is of the form: λ^n

$$y(n) - ay(n-1) = 0$$

$$\lambda^n - a\lambda^{n-1} = 0$$

$$\lambda - a = 0 \quad \text{(Characteristic equation)}$$

$$\lambda = a$$

$$y_h(n) = Ca^n$$

Find C from initial condition for impulse: $x(n) = \delta(n)$

$$y(0) = 1 \Rightarrow 1 = Ca^0 \Rightarrow C = 1$$

Impulse response is:

$$h(n) = a^n$$

Step response is:

$$s(n) = \sum_{k=-\infty}^{\infty} h(k)u(n-k)$$

$$s(n) = \sum_{k=0}^{n} h(k)u(n-k)$$

$$s(n) = \sum_{k=0}^{n} h(k) = \sum_{k=0}^{n} a^n = \frac{1-a^{n+1}}{1-a}$$

Discrete-Time Systems

– Difference equations

Can be solved in similar manner as differential equations

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = x(n)$$

Guess solution of homogeneous equation is of the form: λ^n

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} = 0$$

$$\lambda^2 + a_1 \lambda + a_2 = 0 \quad \text{(Characteristic equation)}$$

$$\lambda_{(1,2)} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$y_h(n) = C_1 \lambda_1^n + C_2 \lambda_2^n$$

Find C_1 and C_2 from initial conditions for impulse: $x(n) = \delta(n)$

Solve for impulse response, solve for step response.

Discrete-Time Systems

– Example:
  - Determine the impulse and step response for the following second-order discrete system:

$$y(n) = 0.6\,y(n-1) - 0.08\,y(n-2) + x(n)$$