Z - transform Examples

(1) Moving average (FIR)

$$y[n] = \sum_{k=0}^{M-1} \alpha_k x[n-k]$$

example: $y[n] = \frac{1}{3} x[n] + \frac{1}{3} x[n-1] + \frac{1}{3} x[n-2]$

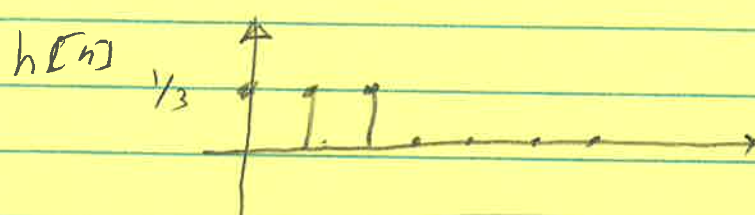
$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \quad \text{for causal signal}$$

$$Z\{x[n-k]\} = z^{-k} X(z) \quad \text{shift property}$$

$$Y(z) = \frac{1}{3} X(z) + \frac{1}{3} z^{-1} X(z) + \frac{1}{3} z^{-2} X(z)$$
$$= \frac{1}{3}(1 + z^{-1} + z^{-2}) X(z)$$

Transfer function: $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{3}(1 + z^{-1} + z^{-2})$

$$h[n] = Z^{-1}\{H(z)\} = \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2])$$



Summary: for $y[n] = \frac{1}{3} x[n] + \frac{1}{3} x[n-1] + \frac{1}{3} x[n-2]$

Transfer function: $H(z) = \frac{1}{3}(1 + z^{-1} + z^{-2})$

Impulse response: $h[n] = \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2])$

2 Simple ~~autoregressive~~ **First Order** filter (IIR)

$$y[n] = 0.5\,y[n-1] + x[n]$$

Impulse response: $x[n] = \delta[n]$

$y[0] = 0.5\,y[-1] + \delta[0] = 1$, $y[-1] = 0$ if causal

$y[1] = 0.5\,y[0] + \delta[1] = 0.5$

$y[2] = 0.5\,y[1] + \delta[2] = (0.5)(0.5)$

$y[3] = 0.5\,y[2] + \delta[3] = (0.5)^3$

$\vdots$

$y[n] = (0.5)^n \mu[n]$ ($\mu[n]$ to show 0, $n<0$)

$$Z\{y[n]\} = \sum_{n=0}^{\infty} (0.5)^n z^{-n} = \sum_{n=0}^{\infty} (0.5z^{-1})^n$$

$$= \frac{1}{1 - 0.5z^{-1}}$$ for $(0.5z^{-1}) < 1$ using geometric series

Impulse response; $h[n] = (0.5)^n \mu[n]$

Transfer function: $H(z) = \dfrac{1}{1-0.5z^{-1}} = \dfrac{z}{z-0.5}$

Region of convergence: $|0.5z^{-1}| < 1$

or $|z| > 0.5$

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Find transfer function and impulse response for original equation for system

$$y[n] = 0.5\,y[n-1] + x[n]$$

$$\mathcal{Z}\{y[n] - 0.5\,y[n-1] = x[n]\}$$

$$Y(z) - 0.5\,z^{-1}\,Y(z) = X(z)$$

$$(1 - 0.5\,z^{-1})\,Y(z) = X(z)$$

Transfer function: $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1-0.5\,z^{-1}} = \dfrac{z}{z-0.5}$

from z-transform table:

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-a}\right\} = a^n \mu[n] \quad , |z| > |a|$$

Impulse response: $h[n] = a^n \mu[n]$

Summary: For $y[n] = 0.5\,y[n-1] + x[n]$

Transfer function: $H(z) = \dfrac{1}{1-0.5\,z^{-1}} = \dfrac{z}{z-0.5}$, $|z| > 0.5$

Impulse response: $h[n] = a^n \mu[n]$

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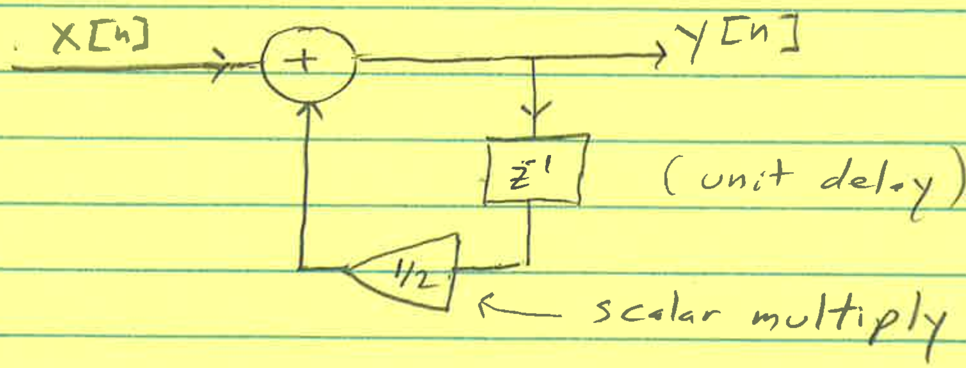
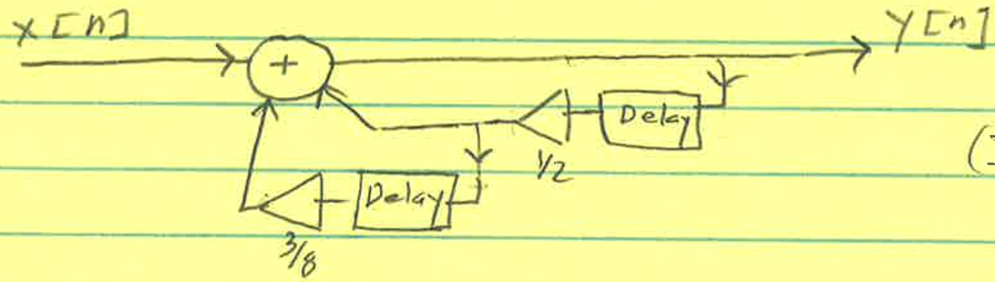
Diagram for $y[n] = 0.5\,y[n-1] + x[n]$



3 System with 2 delays: Find difference equation



(IIR)

$$y[n] = \frac{1}{2} y[n-1] + \left(\frac{1}{2}\right)\left(\frac{3}{8}\right) y[n-2] + x[n]$$

a)
$$\boxed{y[n] - \frac{1}{2} y[n-1] - \frac{3}{16} y[n-2] = x[n]}$$

b) Find transfer function:

$$\mathcal{Z}\left\{ y[n] - \frac{1}{2} y[n-1] - \frac{3}{16} y[n-2] \right\} = \mathcal{Z}\left\{ x[n] \right\}$$

$$Y(z) - \frac{1}{2} z^{-1} Y(z) - \frac{3}{16} z^{-2} Y(z) = X(z)$$

$$\left(1 - \frac{1}{2} z^{-1} - \frac{3}{16} z^{-2}\right) Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1} - \frac{3}{16} z^{-2}}$$

or

$$H(z) = \frac{z^2}{z^2 - \frac{1}{2} z - 3/16}$$

More useful to factor denominator

$(z^2 - \frac{1}{2}z - \frac{3}{16})$

Poles at: $z = \dfrac{\frac{1}{2} \pm \sqrt{(\frac{1}{2})^2 - 4(-\frac{3}{16})}}{2}$

$z_p = \dfrac{\frac{1}{2} \pm \sqrt{\frac{4}{16} + \frac{12}{16}}}{2} = \dfrac{\frac{1}{2} \pm \sqrt{\frac{16}{16}}}{2}$

$z_p = 3/4$ and $-1/4$

$$\boxed{H(z) = \frac{z^2}{(z-\frac{3}{4})(z+\frac{1}{4})}}$$

largest pole at $z = 3/4$ inside unit circle, so stable.

ROC: $|z| > |3/4|$

for causal system

c) Find impulse response

$$\frac{H(z)}{z} = \frac{z}{(z-\frac{3}{4})(z+\frac{1}{4})}$$

(divide by z so we can do partial fractions and look up inverse)

$$\frac{z}{(z-\frac{3}{4})(z+\frac{1}{4})} = \frac{A}{(z-\frac{3}{4})} + \frac{B}{(z+\frac{1}{4})}$$

$$\frac{z}{(z-\frac{3}{4})(z+\frac{1}{4})} = \frac{Az + \frac{1}{4}A + Bz - \frac{3}{4}B}{(z-\frac{3}{4})(z+\frac{1}{4})}$$

$z = (A+B)z + \frac{1}{4}(A - 3B) \Rightarrow A + B = 1$

$\frac{1}{4}(A - 3B) = 0$

$B = 1 - A \quad ; \quad A - 3B = 0 \Rightarrow A - 3(1-A) = 0$

$4A - 3 = 0 \Rightarrow A = 3/4$

$B = 1 - 3/4 = 1/4$

$$\frac{H(z)}{z} = \frac{3/4 \cdot 1}{(z - 3/4)} + 1/4 \frac{1}{z + 1/4}$$

$$H(z) = 3/4 \frac{z}{(z - 3/4)} + 1/4 \frac{z}{z + 1/4}$$

From table: $Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n \mu[n]$

$$Z^{-1}\{H(z)\} = 3/4\,(3/4)^n \mu[n] + 1/4\,(-1/4)^n \mu[n]$$

$$\boxed{h[n] = \left[(3/4)^{n+1} + (-1)^n (1/4)^{n+1}\right] \mu[n]}$$

$h[0] = 1$

$h[1] = 9/16 - 1/16 = 1/2$

$h[2] = 7/16$

$h[3] = 5/16$

$\vdots$

4 Example of unstable system

$$H(z) = \frac{1+2z^{-1}+2z^{-2}}{1-3z^{-1}+2z^{-2}} = \frac{z^2+2z+2}{z^2-3z+2}$$

$$\frac{H(z)}{z} = \frac{z^2+2z+2}{(z)(z-1)(z-2)}$$

$$\frac{z^2+2z+2}{(z)(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

pole outside unit circle (unstable)

$$z^2+2z+2 = A(z-1)(z-2) + B(z)(z-2) + C(z)(z-1)$$

$$= Az^2 - 3Az + 2A + Bz^2 - 2Bz + Cz^2 - Cz$$

$$z^2+2z+2 = (A+B+C)z^2 + (-3A-2B-C)z + (2A)$$

$A+B+C = 1$; $-3A-2B-C = 2$; $2A = 2$
$\Rightarrow A = 1$

$1+B+C = 1 \Rightarrow B+C = 0 \Rightarrow B = -C$

$-3-2B-C = 2 \Rightarrow -3+2C-C = 2 \Rightarrow C = 5$
$\Rightarrow B = -5$

$$H(z)/z = 1/z - 5/(z-1) + 5/(z-2)$$

$$H(z) = 1 - \frac{5z}{z-1} + \frac{5z}{z-2}$$

$$h[n] = \delta[n] - 5\cdot 1^n \mu[n] + 5\cdot 2^n \mu[n]$$

$$\boxed{h[n] = \delta[n] - 5\mu[n] + 5\cdot 2^n \mu[n]}$$

5 More complex case with delays in input and output (IIR)

$$y[n] = y[n-1] - \tfrac{3}{16} y[n-2] + x[n] + \tfrac{1}{2} x[n-1] + \tfrac{3}{16} x[n-2]$$

Transfer function:

$$\mathcal{Z}\left\{ y[n] - y[n-1] + \tfrac{3}{16} y[n-2] = x[n] + \tfrac{1}{2} x[n-1] + \tfrac{3}{16} x[n-2] \right\}$$

$$Y(z) - z^{-1} Y(z) + \tfrac{3}{16} z^{-2} Y(z) = X(z) + \tfrac{1}{2} z^{-1} X(z) + \tfrac{3}{16} z^{-2} X(z)$$

$$\left(1 - z^{-1} + \tfrac{3}{16} z^{-2}\right) Y(z) = \left(1 + \tfrac{1}{2} z^{-1} + \tfrac{3}{16} z^{-2}\right) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{2} z^{-1} + \frac{3}{16} z^{-2}}{1 - z^{-1} + \frac{3}{16} z^{-2}}$$ Transfer function

$$H(z) = \frac{z^2 + \frac{1}{2} z + \frac{3}{16}}{z^2 - z + \frac{3}{16}}$$

$$H(z) = \frac{z^2 + \frac{1}{2} z + \frac{3}{16}}{(z - \frac{1}{4})(z - \frac{3}{4})}$$

$$\frac{H(z)}{z} = \frac{z^2 + \frac{1}{2} z + \frac{3}{16}}{z(z - \frac{1}{4})(z - \frac{3}{4})} = \frac{A}{z} + \frac{B}{z - \frac{1}{4}} + \frac{C}{z - \frac{3}{4}}$$

$$z^2 + \tfrac{1}{2} z + \tfrac{3}{16} = A(z - \tfrac{1}{4})(z - \tfrac{3}{4}) + Bz(z - \tfrac{3}{4}) + Cz(z - \tfrac{1}{4})$$

$$z^2 + \tfrac{1}{2} z + \tfrac{3}{16} = (A+B+C)z^2 - \left(A + \tfrac{3}{4}B + \tfrac{1}{4}C\right)z + \tfrac{3}{16}A$$

$$A = 1 \Rightarrow 1 = (1+B+C) \Rightarrow B = -C$$

$$\tfrac{1}{2} = -\left(1 - \tfrac{3}{4}C + \tfrac{1}{4}C\right) \Rightarrow \tfrac{3}{2} = \tfrac{1}{2}C \Rightarrow C = 3$$

$$\Rightarrow B = -3$$

$$\frac{H(z)}{z} = \frac{1}{z} - \frac{3}{z - 1/4} + \frac{3}{z - 3/4}$$

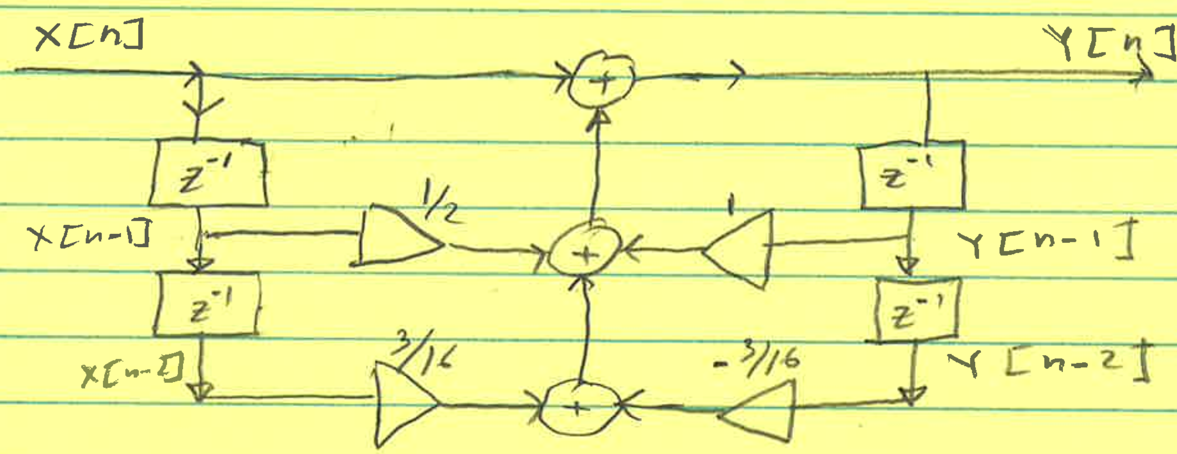
$$H(z) = 1 - \frac{3 \cdot z}{z - 1/4} + \frac{3 \cdot z}{z - 3/4}$$

Impulse response

$$\boxed{h[n] = \delta[n] - 3 \cdot \left(\tfrac{1}{4}\right)^n \mu[n] + 3\left(\tfrac{3}{4}\right)^n \mu[n]}$$

Diagram of system:

$$y[n] = y[n-1] - \tfrac{3}{16} y[n-2] + x[n] + \tfrac{1}{2} x[n-1] + \tfrac{3}{16} x[n-2]$$



4 - delays used

Minimum realization (Direct Form II)

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \frac{W(z)}{X(z)} = \frac{1 + \frac{1}{2} z^{-1} + \frac{3}{16} z^{-2}}{1 - z^{-1} + \frac{3}{16} z^{-2}}$$

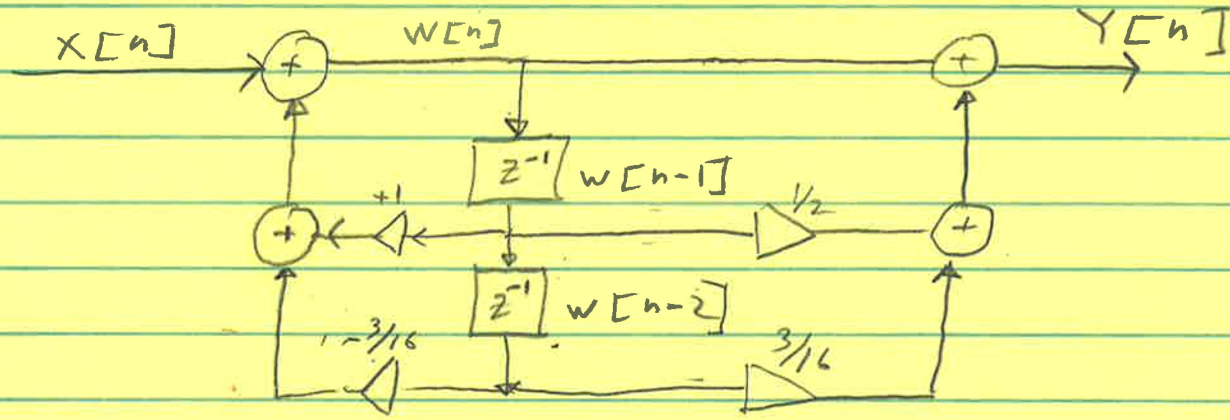
$$\frac{W(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{3}{16} z^{-2}} \Rightarrow W(z) - z^{-1} W(z) + \frac{3}{16} z^{-2} W(z) = X(z)$$

$$W(z) = X(z) + z^{-1} W(z) - \frac{3}{16} z^{-2} W(z)$$

$$\left(W[n] = X[n] + W[n-1] - \frac{3}{16} W[n-2] \right)$$

$$\frac{Y(z)}{W(z)} = 1 + \frac{1}{2} z^{-1} + \frac{3}{16} z^{-2} \Rightarrow Y(z) = W(z) + \frac{1}{2} z^{-1} W(z) + \frac{3}{16} z^{-2} W(z)$$

$$\left(Y[n] = W[n] + \frac{1}{2} W[n-1] + \frac{3}{16} W[n-2] \right)$$



$\boxed{\text{Uses only 2 delays}}$

6 Example with complex poles

$$H(z) = \frac{1}{z^2 - 1/2\, z + 1/2},$$

$$\frac{H(z)}{z} = \frac{1}{(z)(z^2 - 1/2\, z + 1/2)} = \frac{A}{z} + \frac{B}{(z-p_1)} + \frac{C}{(z-p_2)}$$

$$z^2 - 1/2\, z + 1/2 = 0, \quad z = \frac{1/2 \pm \sqrt{(-1/2)^2 - (4)(1/2)}}{2}$$

$$z = \frac{1/2 \pm \sqrt{-7/4}}{2} = \frac{1 \pm i\sqrt{7}}{4}$$

$$p_1 = \frac{1 + i\sqrt{7}}{4}, \quad p_2 = \frac{1 - i\sqrt{7}}{4}$$

$$|p_1|^2 = \frac{1+7}{16} = 1/2, \quad |p_2|^2 = 1/2$$

$$p_1 = \frac{1}{\sqrt{2}} e^{j\theta}, \quad \theta = \tan^{-1}(\sqrt{7})$$

$$p_2 = \frac{1}{\sqrt{2}} e^{-j\theta}$$

$$A(z-p_1)(z-p_2) + Bz(z-p_2) + Cz(z-p_1) = 1$$

$$(A+B+C)z^2 - \left[(p_1+p_2)A + p_2 B + p_1 C\right] z + p_1 p_2 A = 1$$

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$$A = \frac{1}{p_1 p_2} = \frac{1}{1/2} = 2$$

$$(p_1 + p_2) A + p_2 B + p_1 C = 0$$

$$\tfrac{1}{2} A + p_2 B + p_1 C = 0 \Rightarrow p_2 B + p_1 C = -1$$

$$A + B + C = 0 \Rightarrow B + C = -2$$

$$B = \frac{\begin{vmatrix} -1 & p_1 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} p_2 & p_1 \\ 1 & 1 \end{vmatrix}} = \frac{-1 + 2p_1}{p_2 - p_1} = \frac{-1 + \frac{2}{4}(1 + i\sqrt{7})}{-\frac{1}{2} i \sqrt{7}}$$

$$B = \frac{-1/2}{-1/2\, i\sqrt{7}} + \frac{1/2\, i\sqrt{7}}{-1/2\, i\sqrt{7}} = -1 - i/\sqrt{7}$$

$$C = -2 - (-1 - i/\sqrt{7}) = -1 + i/\sqrt{7}$$

$$B = -\sqrt{\frac{8}{7}}\, e^{j\phi}, \quad \phi = \tan^{-1}(1/\sqrt{7})$$

$$C = -\sqrt{\frac{8}{7}}\, e^{-j\phi}$$

$$\frac{H(z)}{z} = \frac{2}{z} - \frac{2\sqrt{2/7}\, e^{j\phi}}{z - 1/\sqrt{2}\, e^{j\theta}} - \frac{2\sqrt{2/7}\, e^{-j\phi}}{z - 1/\sqrt{2}\, e^{-j\theta}}$$

$$h[n] = 2\delta[n] - 2\sqrt{2/7}\, e^{j\phi}\left(1/\sqrt{2}\, e^{j\theta}\right)^n - 2\sqrt{2/7}\, e^{-j\phi}\left(\tfrac{1}{\sqrt{2}}\, e^{-j\theta}\right)^n$$

$$h[n] = 2\,\delta[n] - \frac{2\sqrt{2}}{\sqrt{7}}\left[\left(\frac{1}{\sqrt{2}}\right)^n e^{j(n\theta+\phi)} + \left(\frac{1}{\sqrt{2}}\right)^n e^{-j(n\theta+\phi)}\right]$$

$$= 2\,\delta[n] - \frac{4\sqrt{2}}{\sqrt{7}}\left(\frac{1}{\sqrt{2}}\right)^n \cos(n\theta+\phi)\,\mu[n]$$

$$\boxed{\therefore\ h[n] = 2\left(\delta[n] - \frac{1}{\sqrt{7}}\left(\frac{1}{\sqrt{2}}\right)^{n-3}\cos(n\theta+\phi)\,\mu[n]\right) \quad \text{where } \theta = \tan^{-1}(\sqrt{7}),\ \phi = \tan^{-1}\left(1/\sqrt{7}\right)}$$

Note:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^2 - 1/2\,z + 1/2} = \frac{z^{-2}}{1 - \frac{1}{2}z^{-1} + 1/2\,z^{-2}}$$

The system is:

$$y[n] - 1/2\,y[n-1] + \frac{1}{2}y[n-2] = x[n-2]$$

or

$$\boxed{y[n] = \frac{1}{2}\left(y[n-1] - y[n-2]\right) + x[n-2]}$$