ENGR 071
Digital Signal Processing

Class 03

01/28/2025

- Class Overview
  - Overview of Signals and Systems
    - Continuous Signals & Systems
    - Point out similarities for Discrete Time Signals

Assignment 2

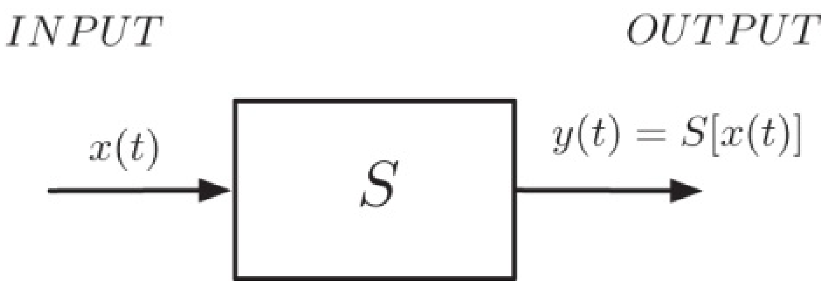
Reading: Chapters 1 and 2 in Proakis and Manolakis

Assignment 2: Due Sunday, Feb. 2

SIGNALS AND SYSTEMS

Systems

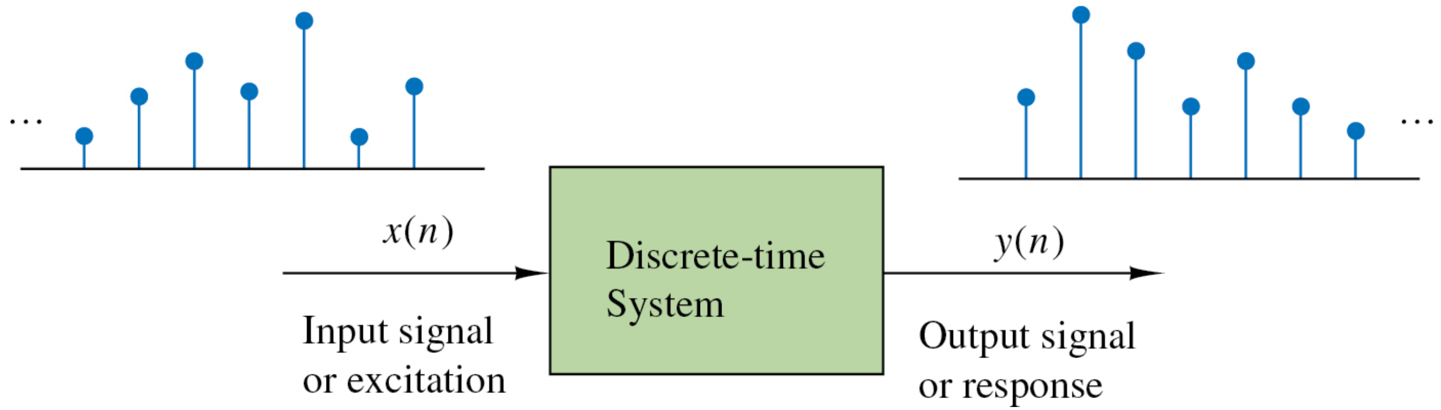
– System
  - Transforms input signal to output signal
  - Illustrated by “black box”



– The system can be thought of as a mathematical transformation mapping the input, *x(t)* to the output, *y(t)*.

Systems

– Discrete system have many of the same properties
  - Transforms input signal to output signal
  - Illustrated by “black box”



– The system can be thought of as a mathematical transformation mapping the input, $x(n)$ to the output, $y(n)$.

Classification of Systems

- **Causal Systems**
  - If output $y(t)$ at time t_0 only depends on input $x(t)$ for $t \le t_0$, system is causal.

    $$y(t_0) = F[x(t)] \text{ for } t \le t_0 \qquad t, t_0 \in \mathbb{R}$$

  - In other words, output only depends on past and current input.
  - For discrete system response only depends on past and current inputs

    $$y(n) = F[x(n), x(n-1), x(n-2), \ldots] \quad , \quad n \in \mathbb{Z}$$

Classification of Systems

- **Linear Systems**
  - If you scale the input to the system, the output scales by the same factor.
  - If you add to inputs and let the system operate on the inputs, the output is like you gave each input separately and sum the individual responses.
  - **For analog systems:**

$$S[a_1 x_1(t) + a_2 x_2(t)] = a_1 S[x_1(t)] + a_2 S[x_2(t)]$$

In general: $$S\left[\sum_m a_m x_m(t)\right] = \sum_m a_m S\left[x_m(t)\right]$$

  - If you superimpose two signals, output is superposition of two outputs.
    - Principle of superposition
  - **For discrete systems:**

$$S[a_1 x_1(n) + a_2 x_2(n)] = a_1 S[x_1(n)] + a_2 S[x_2(n)]$$

In general: $$S\left[\sum_m a_m x_m(n)\right] = \sum_m a_m S\left[x_m(n)\right]$$

Classification of Systems

- **Time Invariant Systems (Analog)**
  - Parameters of system do not change with time.
  - If you shift input time, output is shifted in same way
  - If system with input *x(t)* produces output *y(t)*, then input at *x(t-t₀)* produces output at *y(t-t₀)*
  - Examples
    - » Capacitor is time invariant since:

$$v(t) = \int_{-\infty}^{t} i(\tau)\,d\tau$$

If you consider input shifted by time t_0

$$v_{t_o}(t) = \int_{-\infty}^{t} i(\tau - t_0)\,d\tau = \int_{-\infty}^{t-t_0} i(\tau)\,d\tau = v(t-t_0)$$

  - Example that is not time invariant: $y(t) = x(t) + \sin \omega t$

$$y(t) = S[x(t)] = x(t) + \sin \omega t$$

$$S[x(t - t_0)] = x(t-t_0) + \sin \omega t$$

$$y(t-t_0) = x(t-t_0) + \sin \omega (t-t_0)$$

$$\therefore S[x(t - t_0)] \neq y(t-t_0)$$

Classification of Systems

- **Time (or Shift) Invariant Systems (Discrete)**
  - Parameters of system do not change with time.
  - If you shift input sample, output is shifted in same way
  - If system with input *x(n)* produces output *y(n)*,
    then input at *x(n-k)* produces output at *y(n-k)*

  - **Example of time invariant system:**

$$y(n) = x(n) - x(n-1)$$

If you consider input shifted by time k, output is: $x(n-k) - x(n-k-1)$.

Change argument of for output to $n - k$, substitute $n - k$ for n on both sides:

$y(n-k) = x\big((n-k)\big) - x\big((n-k)-1\big) = x(n-k) - x(n-k-1)$.

Time invariant since shifting input, produces output shifted by same amount.

Classification of Systems

- **Time (or Shift) Invariant Systems (Discrete)**
  - Parameters of system do not change with time.
  - If you shift input sample, output is shifted in same way
  - If system with input $x(n)$ produces output $y(n)$,
    then input at $x(n-k)$ produces output at $y(n-k)$

  - **Example of time-variant system:**

$$y(n) = x(-n)$$

Input shifted by time k, output is: $x(-n-k)$.

Change argument of for output to $n - k$, substitute $n - k$ for n on both sides:

$$y(n-k) = x\big(-(n-k)\big) = x(-n+k) \neq x(-n-k)$$

Not time invariant since shifting input, does not produce output shifted by same amount.

LTI Analog Systems

- **Linear Time Invariant Systems: Analog**
  - Important class of systems
  - Can be represented by ordinary linear differential equation with constant coefficients.
  - Not all Linear D.E.'s with constant coefficients correspond to LTI systems
    - » Must be causal and initially quiescent (initial conditions all zero)
  - The Zero-State response is what LTI systems produce in response to an input
  - Can a system be LTI if the initial conditions are not zero?
  - – No: If you double the input, the zero-state response will double, but the zero-input response will not change.

LTI Analog Systems

– Linear Time Invariant Systems: Analog

- General form of LTI system as differential equation:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_1\frac{dy}{dt} + a_0 y$$
$$= b_m\frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1} x}{dt^{m-1}} + b_{m-2}\frac{d^{m-2} x}{dt^{m-2}} + \cdots + b_1\frac{dx}{dt} + b_0 x \quad \text{(Initial conditions are all zero)}$$

OR

$$\boxed{\frac{d^n y}{dt^n} = -a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} - a_{n-2}\frac{d^{n-2} y}{dt^{n-2}} - \cdots - a_1\frac{dy}{dt} - a_0 y + b_m\frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1} x}{dt^{m-1}} + b_{m-2}\frac{d^{m-2} x}{dt^{m-2}} + \cdots + b_1\frac{dx}{dt} + b_0 x}$$

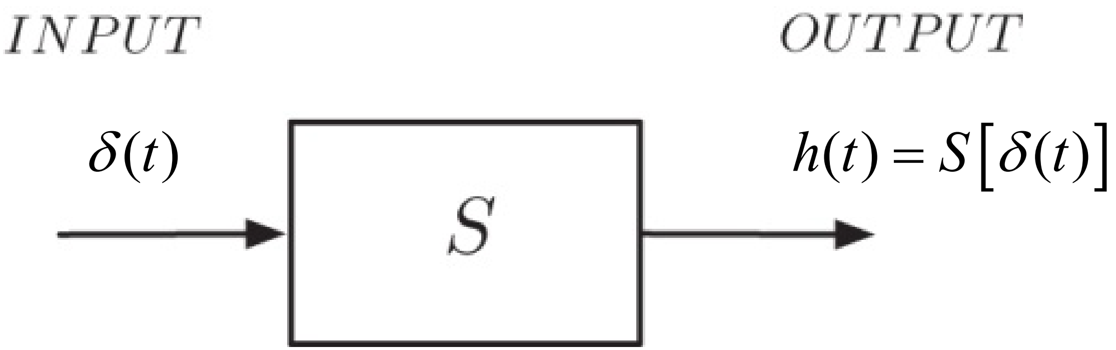
LTI Systems – Analog Case

- Impulse response
  - Impulse has zero width and infinite magnitude
  - Area "under curve" is 1

$$\delta(t) = 0, \quad t \neq 0$$
$$\delta(t) = \infty, \quad t = 0$$

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

$$f(0) = \int_{-\infty}^{+\infty} f(t)\delta(t)\,dt$$



LTI Systems – Analog Case

- Input-Output relationship for Linear Time Invariant System
  - Any arbitrary input signal, *x(t)*, can be written as:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

  - Think of *x(τ)* as weights (<u>not</u> functions of *t*)
  - The output of the system, *y(t),* is

$$y(t) = S\left[x(t)\right] = S\left[\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau\right] = \int_{-\infty}^{+\infty} x(\tau)S\left[\delta(t-\tau)\right]d\tau$$

Linearity

$$S[a_1x_1(t) + a_2x_2(t)] = a_1S[x_1(t)] + a_2S[x_2(t)]$$

  - Call the impulse response *h*(*t*): $h(t) = S\left[\delta(t)\right]$
  - Linearity and Time Invariance:

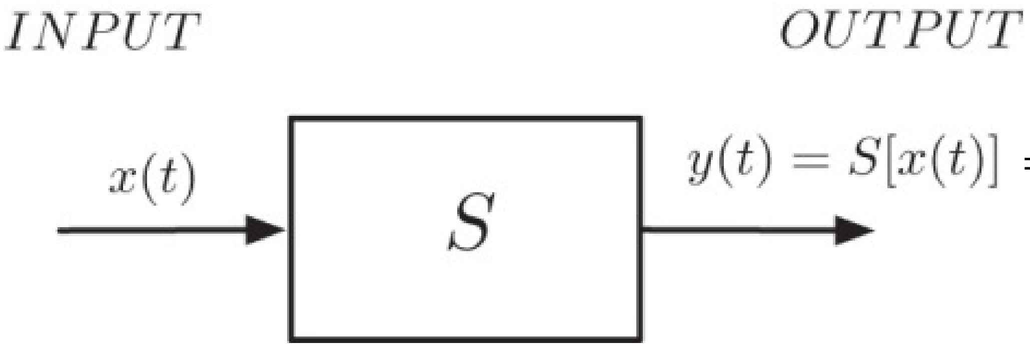
Time Invariance

$$h(t-\tau) = S\left[\delta(t-\tau)\right]$$

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau}$$

LTI Systems – Analog Case

- Output of system in terms of its input and impulse response:



$$= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

Convolution – Analog Case

- Convolution
  - The integral $\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

    is called the convolution of functions $x(t)$ and $h(t)$
    and denoted as: $x(t)*h(t)$

  - In general, for any functions, $f(t)$ and $g(t)$, the convolution is:

$$[f * g](t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau$$

LTI Systems – Analog Case

- Note that convolution is symmetric:
  - Use change of variables: $\tau \to t - \tau'$

$$[f * g](t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau = \int_{+\infty}^{-\infty} f(t-\tau')g(\tau')(-d\tau') = \int_{-\infty}^{+\infty} g(\tau')f(t-\tau')d\tau' = [g * f](t)$$

- Back to our system:
  - Output of the system can be written as: $y(t) = [x * h](t)$ or $y(t) = [h * x](t)$
  - **Impulse response is fundamental characterization of linear time-invariant systems**
  - **Convolving the input and impulse response is equivalent to finding the zero-state (zero initial conditions) solution when system is represented by linear D.E. with constant coefficients.**

Example of Convolution of Impulse Response (Analog)

Detailed example shown in class 2 for :

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \sin(t), \quad t \geq 0, \quad y(0) = 4/5, \quad y'(0) = 11/10$$

Solved using

1) Method of homogeneous and particular solution
2) Method of zero-input, zero-state :

The input to this system is $\sin(t),\ t \geq 0$

The impulse response for this system is $h(t) = \frac{1}{2}\left(e^{-t} - e^{-3t}\right), \quad t > 0$

Calculated the convolution integral:

$$\int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t \sin(\tau)\tfrac{1}{2}\left(e^{-(t-\tau)} - e^{-3(t-\tau)}\right)d\tau$$

Example of Convolution of Impulse Response (Analog)

Complete response

$$y(t) = 2e^{-t} - e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

Homogeneous solution
(Natural response)

$$y_H(t) = 2e^{-t} - e^{-3t}$$

Zero-input
No input / Use initial conditions

$$y_{zi}(t) = \frac{7}{4}e^{-t} - \frac{19}{20}e^{-3t}$$

Particular solution
Forced response

$$y_P(t) = \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

Zero-state
Use input/zero initial conditions

$$y_{zs}(t) = \frac{1}{4}e^{-t} - \frac{1}{20}e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

Example of Convolution of Impulse Response (Analog)

For causal signals and systems: $x(t)=0$ and $h(t)=0$ for $t<0$

Note: $h(t-\tau)=0$ in the integral for $t-\tau<0$ or $\tau>t$

Similarly: $x(\tau)=0$ in the integral for $\tau<0$

So, the convolution integral is:

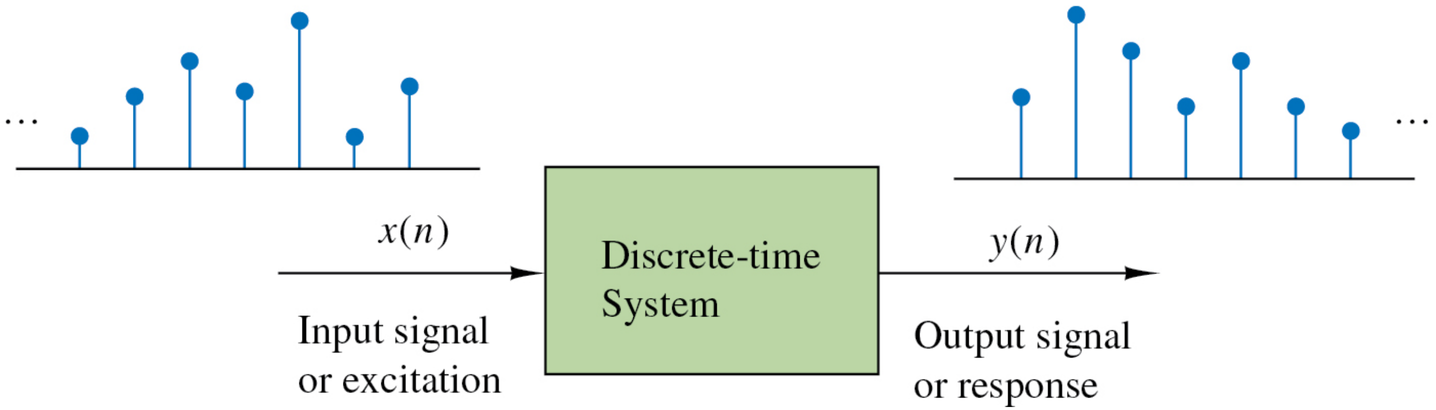
$$\int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t \sin(\tau)\tfrac{1}{2}\left(e^{-(t-\tau)} - e^{-3(t-\tau)}\right)d\tau$$

The result is: $$\mathrm{y}(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \frac{1}{4}e^{-t} - \frac{1}{20}e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

which is the zero-state solution.

Discrete Systems

– Discrete system is essentially the same
  - Transforms input signal to output signal
  - Illustrated by “black box”



– The system can be thought of as a mathematical transformation mapping the input, *x(n)* to the output, *y(n)*.

Discrete System

- Linearity for Discrete Signals
  - System: $y(n) = S[x(n)]$
  - Linearity:
    - System response to a sum of weighted input sequences is the same weighted sum outputs for the response to the individual inputs.
    - Principle of superpostion

$$S[a_1 x_1(n) + a_2 x_2(n)] = a_1 S[x_1(n)] + a_2 S[x_2(n)]$$

  - Time invariance:
    - If you delay the input by k samples, the output is the same as it was for the undelayed signal, except shifted by k samples

$$y(n-k) = S[x(n-k)]$$

LTI Systems – Discrete Case

– **Linear Time Invariant Systems: Discrete**

- Can be represented by constant-coefficient difference equations.
  - » Must be causal and initially quiescent
- General form of equation describing Discrete LTI system

$$y(n) + a_1 y(n-1) + a_2 y(n-2) + \cdots + a_N y(n-N)$$
$$= b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_M x(n-M) \quad \text{(Initial conditions are all zero)}$$

OR

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_M x(n-M)$$

OR

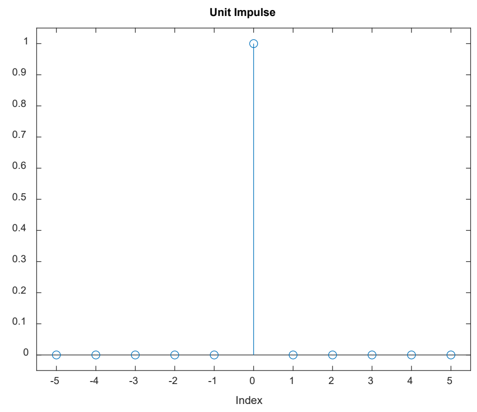
$$\boxed{y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)}$$

Discrete System

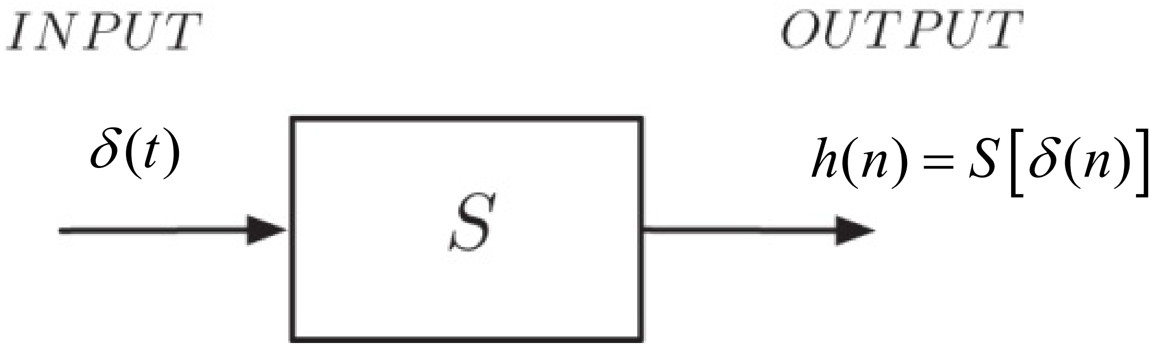
- Unit impulse

Impulse is much simpler in discrete case: $\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

More generally, $\delta(n-k) = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases}$



- Impulse response: $h(n) = S[\delta(n)]$

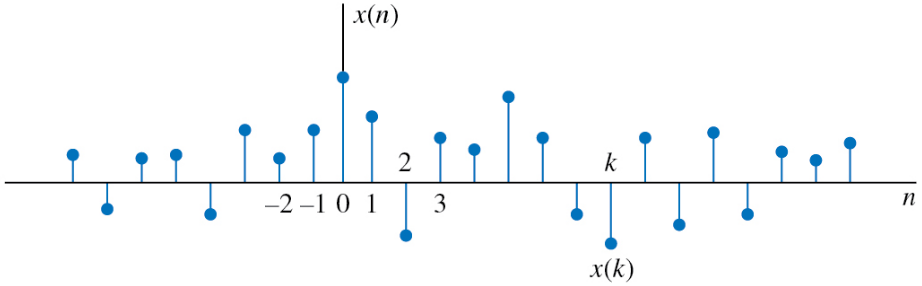


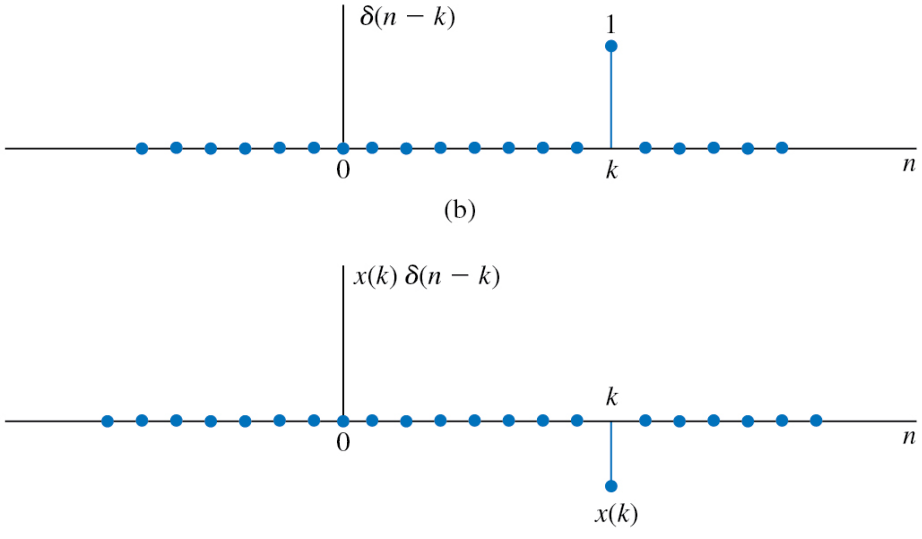
LTI Systems – Discrete Case

- Input-Output relationship for Linear Time (Shift) Invariant System
  - Any arbitrary input signal, *x(n)*, can be written as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

  - Think of $x(k)$ as the sample amplitude at shift k





(b)
(c)

LTI Systems – Discrete Case

- Input-Output relationship for Linear Time (Shift) Invariant System
  - Any arbitrary input signal, *x(n)*, can be written as:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

  - Think of $x(k)$ as the sample amplitude at shift k
  - The output of the system, *y(t),* is

$$y(n) = S\left[x(n)\right] = S\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k)S\left[\delta(n-k)\right]$$

  - Call the impulse response $h(n)$: $h(n) = S\left[\delta(n)\right]$
  - Linearity and Time Invariance:

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)}$$

Linearity

$S[a_1x_1(n) + a_2x_2(n)] = a_1S[x(n)] + a_2S[x_2(n)]$

Time Invariance

$h(n-k) = S\left[\delta(n-k)\right]$

Convolution – Discrete Case

- Convolution
  - The sum $\boxed{\sum_{k=-\infty}^{\infty} x(k)h(n-k)}$

    is the discrete convolution of functions $x(n)$ and $h(n)$
    and denoted as: $x(n)*h(n)$

  - In general, for any discrete sequences, $f(n)$ and $g(n)$, the convolution is:

    $$[f*g](n) = \sum_{k=-\infty}^{\infty} f(k)g(n-k)$$

LTI Systems – Discrete Case

– Note that convolution is symmetric:
  - Use change of variables: $k \rightarrow n-k'$

$$[f*g](n)=\sum_{k=-\infty}^{\infty} f(k)g(n-k)=\sum_{k=+\infty}^{-\infty} f(n-k')g(k')=\sum_{k=-\infty}^{+\infty} g(k')f(n-k')=[g*f](t)$$

- Back to our system:
  – Output of the system can be written as: $y(n)=[x*h](n)$ or $y(n)=[h*x](n)$
  – **Impulse response is fundamental characterization of discrete linear time-invariant systems**
  – **Convolving the input and impulse response is equivalent to finding the zero-state (zero initial conditions) solution when system is represented by constant coefficients difference equations.**

Review of Laplace Transform

- The Laplace Transform
  - Important method of analysis for signal & image processing and process control
  - Definition:

$$\boxed{F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{+\infty} f(t)e^{-st}\,dt \quad \text{where } s \text{ is a complex variable } (s = \sigma + j\omega)}$$

  - Things you can do with Laplace transform
    - Characterize system by a transfer function
    - Determine stability of system
    - Transform linear differential equations to algebraic equations
    - Launching point for frequency analysis

Laplace Transform

- The Laplace Transform
  - What does it mean?
  - Consider an input signal $x(t) = e^{st}$
    where s is a complex number: $s=\sigma+j\omega$
  - Consider the LTI system processing this input:

$$y(t) = S\left[x(t)\right] = S\left[e^{st}\right]$$

  - Using the impulse response of the system $h(t)$ and the convolution theorem:

$$y(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{+\infty} e^{s(t-\tau)}h(\tau)d\tau = e^{st}\int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)d\tau$$

$$y(t) = \left[\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau\right]e^{st} = H(s)x(t)$$

Laplace Transform

- The Laplace Transform
  - What does it mean? …
    - A way of characterizing LTI system in terms its eigenvalues & eigenfunctions

$$y(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{+\infty} e^{s(t-\tau)}h(\tau)d\tau = e^{st}\int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)d\tau$$

$$y(t) = \left[\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau\right]e^{st} = H(s)x(t)$$

  - The output is the input multiplied by the complex function *H(s)*

  - In mathematical terms:
    The function e^{st} is an eigenfunction of the LTI system
    H(s) is the eigenvalue for the LTI system

Laplace Transform

- Typically, Laplace transform is a rational polynomial

$$F(s) = \frac{N(s)}{D(s)} \quad \text{where } N(s) \text{ and } D(s) \text{ are polynomials in } s$$

$$\text{Example: } F(s) = \frac{2\left(s^2+1\right)}{s^2+2s+5} = \frac{2(s+j)(s-j)}{(s+1)^2+4} = \frac{2(s+j)(s-j)}{(s+1+2j)(s+1-2j)}$$

Written in this form to show poles and zeros of $F(s)$

Poles where denominator is zero, i.e., $D(s)=0$ ($F(s)$ becomes infinite)

Zeros where numerator is zero i.e., $\mathrm{N}(s)=0$ ($F(s)$ is zero)

For example:

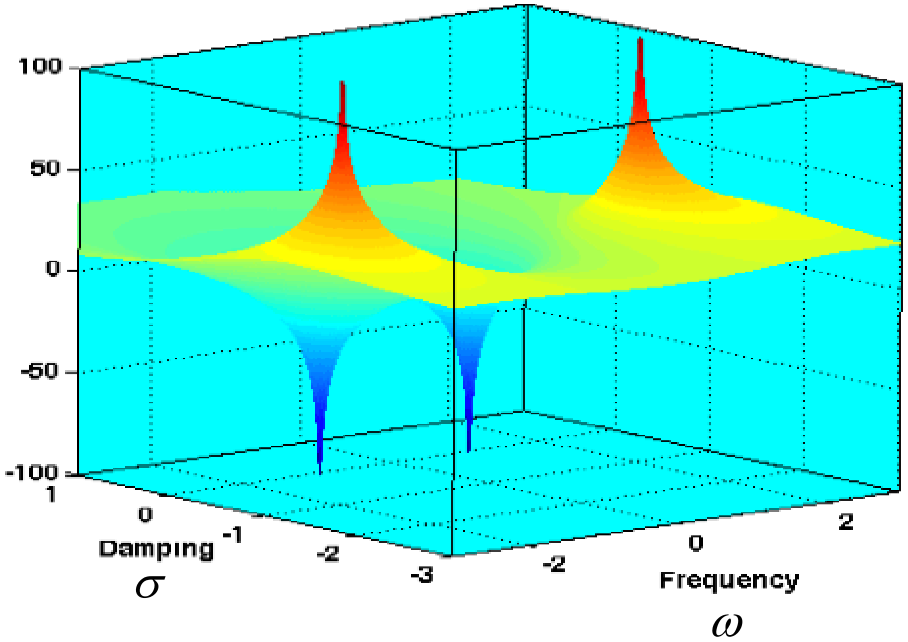
Poles at $s=-1+2j$ and $s=-1-2j$

Zeros at $s=j$ and $s=-j$

Laplace Transform

$$F(s)=\frac{2\left(s^2+1\right)}{s^2+2s+5}=\frac{2\left(s+j\right)\left(s-j\right)}{\left(s+1\right)^2+4}=\frac{2\left(s+j\right)\left(s-j\right)}{\left(s+1+2j\right)\left(s+1-2j\right)}$$

Plot of $\log F(s)$: zeros have $\log 0 \to -\infty$, poles have $\log \infty \to \infty$



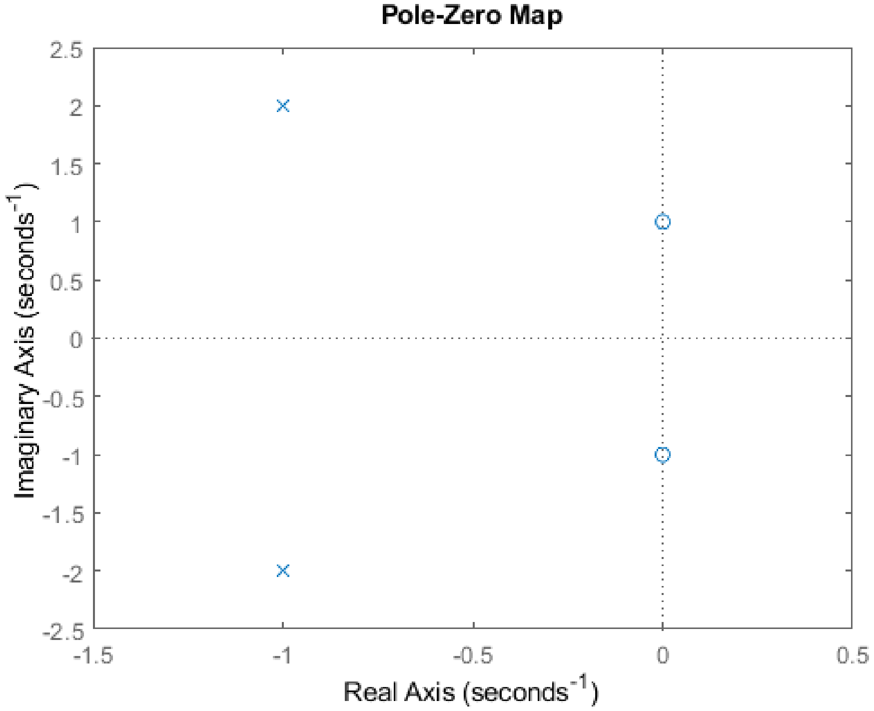
Laplace Transform

MATLAB has a nice function for plotting poles and zeros: `pzmap`

```
% Example of pzmap:
s = tf('s')
H1 = 2*(s^2+1)/(s^2+2*s+5)
figure(1)
pzmap(H1)
axis([-1.5,0.5,-2.5,2.5]);

% or
clear
H2 = tf([2,0,2],[1,2,5]);
figure(2)
pzmap(H2)
axis([-1.5,0.5,-2.5,2.5]);
```

s-plane



Laplace Transform

- Region of convergence $\left|\int_{-\infty}^{\infty} f(t)e^{-st}dt\right| = \left|\int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt\right| \le \int_{-\infty}^{\infty}\left|f(t)e^{-\sigma t}\right|dt < \infty$
  - You cannot have poles in the region of convergence
    - If you did, the integral would not converge absolutely
  - For a causal function

    $f(t) = 0$ for $t < 0$, ROC is part of s-plane to the right of the poles.

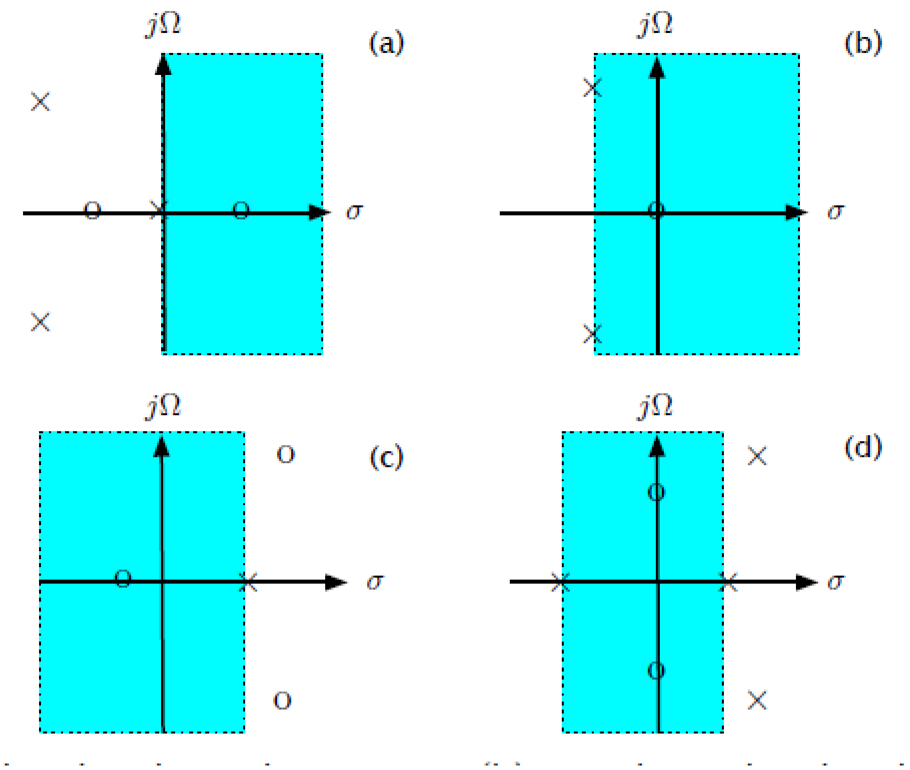
  - For anti-causal function

    $f(t) = 0$ for $t > 0$, ROC is part of s-plane to the left of the poles.

  - For non-causal:

  $f(t)$ is defined for $-\infty < t < \infty$ ROC is intersection of causal and anti-causal parts between the poles on the right and left

Laplace Transform



(a) Causal
(b) Causal with poles to left of imaginary axis
(c) Anti-causal
(d) Non-causal (ROC bounded by poles)

Laplace Transform

- The Laplace Transform (one-sided, unilateral)
  - Maps a real-valued function of time, t, into a function of a complex variable s.

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} f(t)e^{-st}\,dt \quad \text{where } s \text{ is a complex variable } (s = \sigma + j\omega)$$

  - Convergence:

$$\int_0^{\infty} f(t)e^{-st}\,dt = \int_0^{\infty} f(t)e^{-(\sigma+j\omega)t}\,dt = \int_0^{\infty} f(t)e^{-\sigma t}e^{j\omega t}\,dt$$

Converges if

$$\left|\int_0^{\infty} f(t)e^{-st}\,dt\right| = \left|\int_0^{\infty} f(t)e^{-(\sigma+j\omega)t}\,dt\right| \le \int_0^{\infty} \left|f(t)e^{-\sigma t}\right|dt < \infty$$

Laplace Transform

- The Inverse Laplace Transform
  - The formal mathematical definition is:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} \, ds$$

where σ is large enough that $F(s)$ is defined for $\mathrm{Re}(s) \geq \sigma$

  - This formula is rarely used to find inverse.
  - A more common way is to cast expression in the Laplace domain in a form that corresponds to entries in a table of Laplace transforms.
    - Often you have to reduce a complicated expression into a simpler one to do this.
    - Generally, involves operations like **partial fractions** and **completing the square**.

Inverse Laplace Transform

- Key problem in finding inverse Laplace transform for complicated expressions involving s
  - Need to get into simple form first
    - Usually involves partial fractions
    - Sometimes need to complete square
    - Sometimes need to be clever in rewriting terms
  - Often utilize the properties shown on following slides

LAPLACE TRANSFORM TABLE

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

SPECIFIC FUNCTIONS		GENERAL RULES	
$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\frac{1}{s}$	1	$\frac{e^{-as}}{s}$	$u(t-a)$
$\frac{1}{s^n}, \quad n \in \mathbb{Z}^+$	$\frac{t^{n-1}}{(n-1)!}$	$e^{-as}F(s)$	$f(t-a)u(t-a)$
$\frac{1}{s+a}$	e^{-at}	$F(s-a)$	$e^{at}f(t)$
$\frac{1}{(s+a)^n}, \quad n \in \mathbb{Z}^+$	$e^{-at}\frac{t^{n-1}}{(n-1)!}$	$sF(s)-f(0)$	$f'(t)$
$\frac{1}{s^2+\omega^2}$	$\frac{\sin(\omega t)}{\omega}$	$s^2F(s)-sf(0)-f'(0)$	$f''(t)$
$\frac{s}{s^2+\omega^2}$	$\cos(\omega t)$	$F'(s)$	$-tf(t)$
$\frac{1}{(s+a)^2+\omega^2}$	$\frac{e^{-at}\sin(\omega t)}{\omega}$	$F^{(n)}(s)$	$(-t)^n f(t)$
$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos(\omega t)$	$\frac{F(s)}{s}$	$\int_0^t f(u)\,du$
$\frac{1}{(s^2+\omega^2)^2}$	$\frac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^3}$	$F(s)G(s)$	$(f*g)(t)$
$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t\sin(\omega t)}{2\omega}$		

Common Laplace Transform Properties

Name	Illustration
Definition of Transform	$f(t) \xleftrightarrow{L} F(s)$ $F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$
Linearity	$Af_1(t) + Bf_2(t) \xleftrightarrow{L} AF_1(s) + BF_2(s)$
First Derivative	$\frac{df(t)}{dt} \xleftrightarrow{L} sF(s) - f(0^-)$
Second Derivative	$\frac{d^2f(t)}{dt^2} \xleftrightarrow{L} s^2F(s) - sf(0^-) - \dot{f}(0^-)$
n^{th} Derivative	$\frac{d^nf(t)}{dt^n} \xleftrightarrow{L} s^nF(s) - \sum_{i=1}^{n} s^{n-i}f^{(i-1)}(0^-)$
Integral	$\int_0^t f(\lambda)d\lambda \xleftrightarrow{L} \frac{1}{s}F(s)$
Time Multiplication	$tf(t) \xleftrightarrow{L} -\frac{dF(s)}{ds}$
Time Delay	$f(t-a)\gamma(t-a) \xleftrightarrow{L} e^{-as}F(s)$ $\gamma(t)$ is unit step
Complex Shift	$f(t)e^{-at} \xleftrightarrow{L} F(s+a)$
Scaling	$f\left(\frac{t}{a}\right) \xleftrightarrow{L} aF(as)$
Convolution Property	$f_1(t) * f_2(t) \xleftrightarrow{L} F_1(s)F_2(s)$
Initial Value	$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$
Final Value (if final value exists)	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

Laplace Transform Properties

- Straightforward to find some Laplace transforms from definitions
- Others can be found, starting from a simple function and using properties of transform
- Proof of Laplace transform properties is fairly straightforward starting from the definition.
  - We will not go through proofs of the properties
    - A good summary of the Laplace Transform and proofs of some properties can be found at: [The Laplace Transform](#) (Prof. Cheever's website)

Laplace Transform Properties

[Click here more details about properties of Laplace transforms]

Linearity

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace Transforms of $f_1(t)$ and $f_2(t)$

$$\boxed{L\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)}$$

Example:

$$\boxed{L\left[\cos(\omega t)u(t)\right] = L\left[\frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)u(t)\right] = \frac{s}{s^2 + \omega^2}}$$

Laplace Transform Properties

Time Shift

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$L\left[f(t-a)u(t-a)\right] = e^{-as}F(s)$$

Example:

$$L\left[\cos(\omega(t-a))u(t-a)\right] = e^{-as}\frac{s}{s^2+\omega^2}$$

Laplace Transform Properties

Frequency Shift

If $F(s)$ is the Laplace Transforms of $f(t)$, then

$$L\left[e^{-at} f(t) u(t)\right] = F(s+a)$$

Example:

$$L\left[e^{-at} \cos(\omega t) u(t)\right] = \frac{s+a}{(s+a)^2 + \omega^2}$$

Laplace Transform Properties

Scaling

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$\boxed{L[f(at)] = \frac{1}{a} F(\frac{s}{a})}$$

Example:

$$\boxed{L[\sin(2\omega t)u(t)] = \frac{2\omega}{s^2 + 4\omega^2}}$$

Laplace Transform Properties

Time Differentiation

If $F(s)$ is the Laplace Transforms of $f(t)$, then the Laplace Transform of its derivative is

$$L\left[\frac{df}{dt}u(t)\right] = sF(s) - f(0^-)$$

Example:

$$L[\sin(\omega t)u(t)] = \frac{\omega}{s^2 + \omega^2}$$

Laplace Transform Properties

Time Differentiation More Generally:

For a signal $f(t)$, with Laplace transform $F(s)$, the one-sided Laplace transform of its first- and second-order derivatives are

$$\mathcal{L}\left[\frac{df(t)}{dt}u(t)\right] = sF(s) - f(0-) \tag{3.14}$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}u(t)\right] = s^2F(s) - sf(0-) - \frac{df(t)}{dt}\Big|_{t=0-} \tag{3.15}$$

In general, if $f^{(N)}(t)$ denotes the Nth-order derivative of a function $f(t)$ that has a Laplace transform $F(s)$, we have that

$$\mathcal{L}[f^{(N)}(t)u(t)] = s^N F(s) - \sum_{k=0}^{N-1} f^{(k)}(0-)s^{N-1-k} \tag{3.16}$$

where $f^{(m)}(t) = d^m f(t)/dt^m$ is the mth-order derivative, $m > 0$, and $f^{(0)}(t) \triangleq f(t)$.

Laplace Transform Properties

Time Integration

If $F(s)$ is the Laplace Transforms of $f(t)$, then the Laplace Transform of its integral is

$$\boxed{L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)}$$

Example:

$$\boxed{L\left[t^n\right] = \frac{n!}{s^{n+1}}}$$

Find this recursively, starting from $L[1] \;= \frac{1}{s}$

$$t = \int_0^t 1 d\tau \Rightarrow L\left[t\right] = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

$$\frac{t^2}{2} = \int_0^t \tau d\tau \Rightarrow L\left[t^2\right] = \frac{1}{s}\cdot\frac{2}{s^2} = \frac{2}{s^3}$$

Laplace Transform Properties

Frequency Differentiation

If *F(s)* is the Laplace Transforms of *f (t),* then the derivative with respect to s, is

$$L[tf(t)] = -\frac{dF(s)}{ds}$$

Example:

$$L[te^{-at}u(t)] = \frac{1}{(s+a)^2}$$

Laplace Transform Properties

Initial and Final Values

The initial-value and final-value properties allow us to find the initial value *f(0)* and *f(∞)* of *f(t)* directly from its Laplace transform *F(s)*.

$$\boxed{f(0) = \lim_{s \to \infty} sF(s)}$$

Initial-value theorem

$$\boxed{f(\infty) = \lim_{s \to 0} sF(s)}$$

Final-value theorem

Laplace Transform Properties

The Convolution Integral

Defined as $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ or $y(t) = x(t) * h(t)$

Given two functions, $f_1(t)$ and $f_2(t)$ with Laplace Transforms $F_1(s)$ and $F_2(s)$, respectively

$y(t) = 4e^{-t}$ and $h(t) = 5e^{-2t}$

$$\boxed{F_1(s)F_2(s) = L\left[f_1(t) * f_2(t)\right]}$$

Example: $h(t) = 5e^{-2t}u(t)$; $x(t) = 4e^{-t}u(t)$

$$\boxed{h(t) * x(t) = L^{-1}\left[H(s)X(s)\right] = L^{-1}\left[\left(\frac{5}{s+2}\right)\left(\frac{4}{s+1}\right)\right] = 20(e^{-t} - e^{-2t}),\quad \mathrm{t} \geq 0}$$

Laplace Transform Properties

Proof of Convolution Property

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad \text{or } y(t) = x(t) * h(t)$$

$$Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} h(t-\tau)e^{-st}dt\right)d\tau$$

$$\vdots$$

$$\boxed{= H(s)X(s)}$$

(Complete this proof for problem 1 of HW2)

Analysis of LTI systems in the Laplace domain

- Most general form of LTI System

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_1\frac{dy}{dt} + a_0 y =$$

$$b_m\frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1} x}{dt^{m-1}} + b_{m-2}\frac{d^{m-2} x}{dt^{m-2}} + \cdots + b_1\frac{dx}{dt} + b_0 x$$

$$n > m$$

- Taking the Laplace transform of both sides:
  (with zero initial conditions, i.e., zero-state … aka quiescent)

$$\left(s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-1} + \cdots + a_1 s + a_0\right)Y(s) =$$

$$\left(b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_1 s + b_0\right)X(s)$$

Analysis of LTI systems in the Laplace domain

With initial conditions, more complicated because:

$$f(t) \quad \Leftrightarrow \quad F(s)$$

$$\mathcal{L}\left[\frac{df(t)}{dt}u(t)\right] = sF(s) - f(0-)$$

$$\mathcal{L}\left[\frac{d^2 f(t)}{dt^2}u(t)\right] = s^2F(s) - sf(0-) - \frac{df(t)}{dt}\Big|_{t=0-}$$

If $f^{(N)}(t)$ denotes Nth-order derivative of a function $f(t)$

$$\mathcal{L}[f^{(N)}(t)u(t)] = s^N F(s) - \sum_{k=0}^{N-1} f^{(k)}(0-)s^{N-1-k}$$

Analysis of LTI systems in the Laplace domain

Analysis of LTI systems – Differential Equation representation

Complete response $y(t)$ of system represented by an N^{th}-order linear differential equation

$$y^{(N)}(t) + \sum_{k=0}^{N-1} a_k y^{(k)}(t) = \sum_{\ell=0}^{M} b_\ell x^{(\ell)}(t) \qquad N > M$$

$$x(t) \;\; \textit{input}, \;\; y(t) \;\; \textit{output} \;\; \textit{and initial conditions} \{y^{(k)}(t), \;\; 0 \le k \le N-1\}$$

is obtained by inverting the Laplace transform

$$\begin{aligned} Y(s) &= \frac{B(s)}{A(s)}X(s) + \frac{1}{A(s)}I(s) \qquad Y(s) = \mathcal{L}[y(t)], X(s) = \mathcal{L}[x(t)] \\ A(s) &= \sum_{k=0}^{N} a_k s^k \qquad a_N = 1 \\ B(s) &= \sum_{\ell=0}^{M} b_\ell s^\ell \\ I(s) &= \sum_{k=1}^{N} a_k \left(\sum_{m=0}^{k-1} s^{k-m-1} y^{(m)}(0) \right) \end{aligned}$$

Analysis of LTI systems in the Laplace domain

$$\textit{Letting} \quad H(s) = \frac{B(s)}{A(s)} \quad \textit{and} \quad H_1(s) = \frac{1}{A(s)}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)]$$
$$Y(s) = H(s)X(s) + H_1(s)I(s)$$

which gives

$$y(t) = y_{zs}(t) + y_{zi}(t)$$
$$\textit{zero-state response} \quad y_{zs}(t) = \mathcal{L}^{-1}[H(s)X(s)]$$
$$\textit{zero-input response} \quad y_{zi}(t) = \mathcal{L}^{-1}[H_1(s)I(s)]$$

In terms of convolution integrals

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau + \int_0^t i(\tau)h_1(t-\tau)d\tau$$
$$h(t) = \mathcal{L}^{-1}[H(s)], \ \textit{and} \ h_1(t) = \mathcal{L}^{-1}[H_1(s)]$$
$$i(t) = \mathcal{L}^{-1}[I(s)] = \sum_{k=1}^{N} a_k \left(\sum_{m=0}^{k-1} y^{(m)}(0)\delta^{(k-m-1)}(t) \right)$$

Analysis of LTI systems in the Laplace domain

- Transfer Function

$$H(s)=\frac{Y(s)}{X(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+b_{m-2}s^{m-2}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+a_{n-2}s^{n-1}+\cdots+a_1 s+a_0}$$

- Poles and zeros completely describe transfer function
- If any poles are in the right half of the s-plane, the system is unstable:
  - Why?

$$L\left[e^{-at}\right]=\frac{1}{s+a}$$

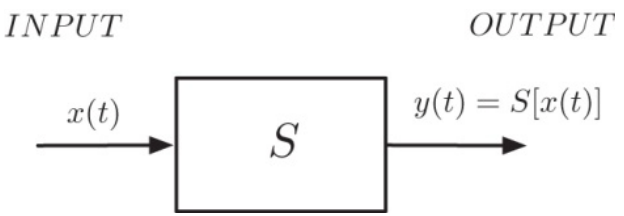
Examples: (for causal signals, $t \geq 0$)

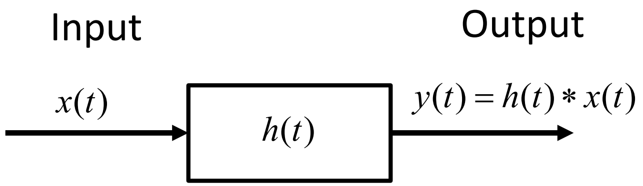
Pole at $s=-2$: $\quad F(s)=\frac{1}{s+2}, \quad f(t)=e^{-2t}\rightarrow 0$ as $t\rightarrow\infty$

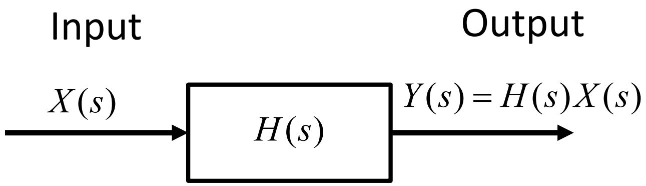
Pole at $s=+2$: $\quad F(s)=\frac{1}{s-2}, \quad f(t)=e^{+2t}\rightarrow\infty$ as $t\rightarrow\infty$

Analysis of LTI systems in the Laplace domain

- System:







– Consider a system implementing the differential equation:

$$y''(t) + 7y'(t) + 12y(t) = x'(t) + 2x(t)$$

$$L\left[y''(t) + 7y'(t) + 12y(t) = x'(t) + 2x(t)\right] \Rightarrow s^2Y(s) + 7sY(s) + 12Y(s) = sX(s) + 2X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^2 + 7s + 12} = \frac{s+2}{(s+3)(s+4)}$$

$$\boxed{h(t) = L^{-1}[H(s)] = ?} \qquad \boxed{h(t) = 2e^{-4t} - e^{-3t}}$$

Example: Real Poles

- Example of pole-zero map and relationship to response: (real poles)

$$H(s) = \frac{s+2}{s^2+7s+12} = \frac{s+2}{(s+3)(s+4)}$$

Step response:

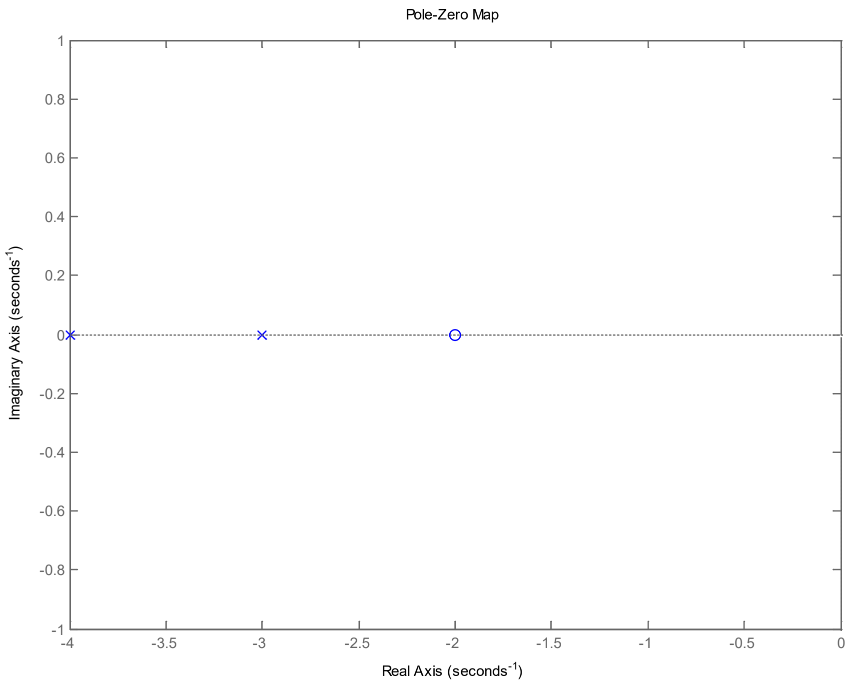
$$X(s) = \frac{1}{s} \quad \text{input is a step function, } u(t)$$

$$\mathrm{Y}(s) = H(s)X(s) = \frac{s+2}{\left(s^2+7s+12\right)}\frac{1}{s}$$

$$y(t) = L^{-1}\left[\frac{s+2}{\left(s^2+7s+12\right)}\frac{1}{s}\right] = \frac{1}{6} + \frac{1}{3}e^{-3t} - \frac{1}{2}e^{-4t}$$

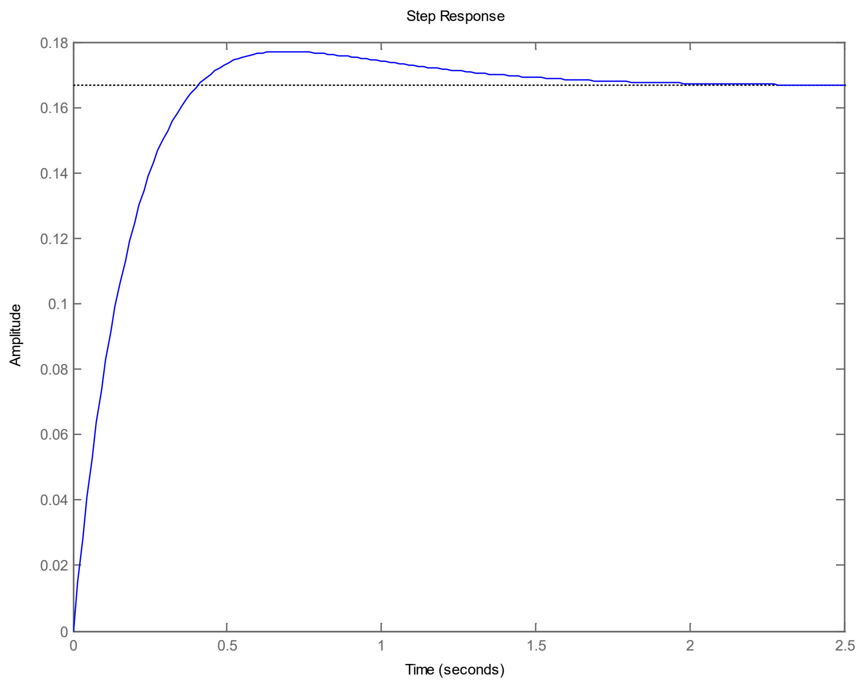
Example: Real Poles

- Example of pole-zero map and relationship to response: using Matlab function “pzmap”



Example: Real Poles

- Step response: (using Matlab function “step”)



(See example: Example_step_real_poles.m)

Example: Real Poles

```
% Example of Step Response with real poles
% y'' + 7y' + 12y = x' + 2x
clear
close all
H = tf([1 2],[1 7 12])
figure(1)
pzmap(H)
axis([-5,1,-2,2])
figure(2)
step(H)
figure(3)
impulse(H)
```

```
syms s t
Hs = (s + 2)/(s*(s^2 + 7*s + 12))
ht = ilaplace(Hs)
figure(4)
fplot(t,ht)
axis([0,2.5,0,0.18])
title('Step response from inverse Laplace')

Himp = (s +2)/(s^2 + 7*s +12);
htimp = ilaplace(Himp)
figure(5)
fplot(t,htimp)
axis([0,2.5,-0.2,1.0])
hold on
plot([0,2.5],[0,0],'--')
title('Impulse response from inverse Laplace')
```

Example Complex Poles

- Complex poles:

$$H(s) = \frac{s+2}{s^2+2s+3} = \frac{s+2}{(s+1+j\sqrt{2})(s+1-j\sqrt{2})}$$

Step response:

$$X(s) = \frac{1}{s}$$

$$\mathrm{Y}(s) = H(s)X(s) = \frac{s+2}{\left(s^2+2s+3\right)}\frac{1}{s}$$

$$y(t) = L^{-1}\left[\frac{s+2}{\left(s^2+2s+3\right)}\frac{1}{s}\right] = \frac{2}{3}\left[1-e^{-t}\left(\cos\left(\sqrt{2}t\right)-\frac{\sqrt{2}}{4}\sin\left(\sqrt{2}t\right)\right)\right]$$

Example_complex_poles.m

Example Complex Poles

- Complex poles:

$$H(s)=\frac{s+2}{s^2+2s+3}=\frac{s+2}{(s+1+j\sqrt{2})(s+1-j\sqrt{2})}$$

Impulse Response:

$$X(s)=1$$

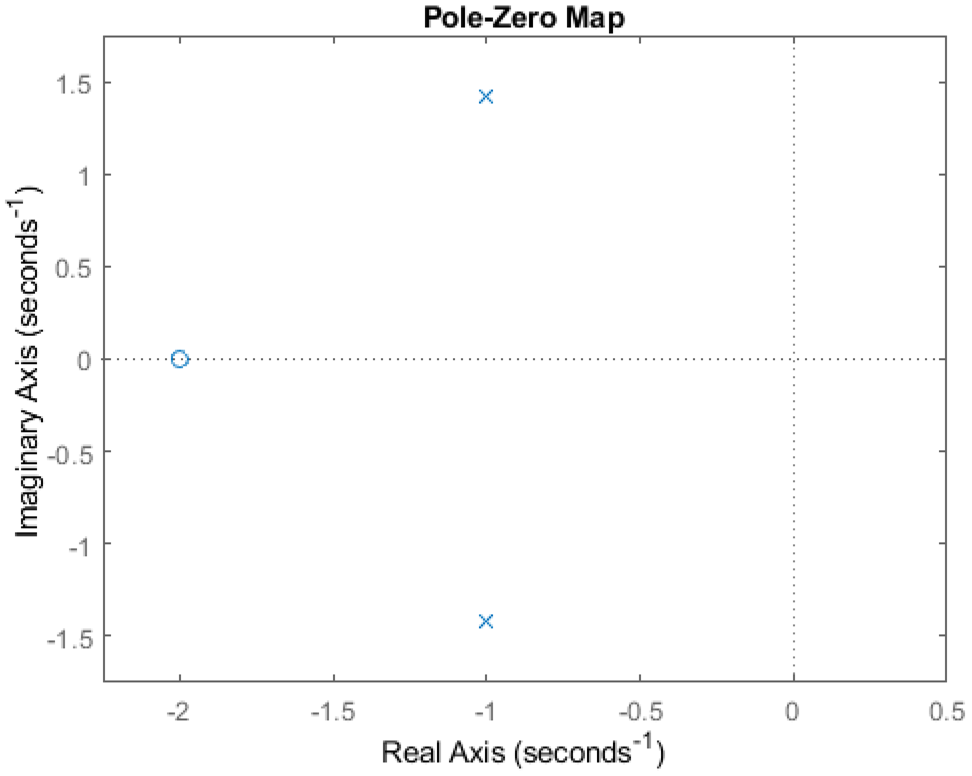
$$\mathrm{Y}(s)=H(s)X(s)=\frac{s+2}{\left(s^2+2s+3\right)}$$

$$y(t)=L^{-1}\left[\frac{s+2}{\left(s^2+2s+3\right)}\right]=L^{-1}\left[\frac{s+1}{\left(s^2+2s+1\right)+\left(\sqrt{2}\right)^2}+\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\left(s^2+2s+1\right)+\left(\sqrt{2}\right)^2}\right]$$

$$y(t)=e^{-t}\left(\cos\left(\sqrt{2}t\right)-\frac{1}{\sqrt{2}}\sin\left(\sqrt{2}t\right)\right)$$

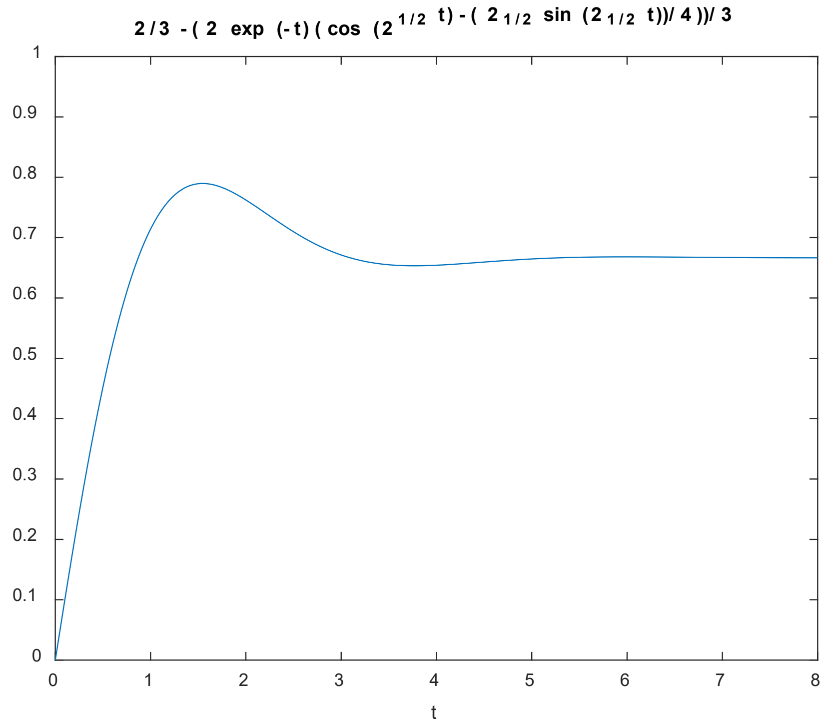
Example: Complex Poles

- Example of pole-zero map and relationship to response: (complex poles) using Matlab function “pzmap”



Example: Complex Poles

- Step response using Matlab function "step"



Example: Complex Poles

```
% Example of Step Response with complex poles
% y'' + 2y' + 3y = x' + 2x
clear
close all
H = tf([1 2],[1 2 3])
figure(1)
pzmap(H)
axis([-5,1,-2,2])
figure(2)
step(H)
figure(3)
impulse(H)
```

```
syms s t
Hs = (s + 2)/(s*(s^2 + 2*s + 3))
ht = ilaplace(Hs)
figure(4)
fplot(t,ht,[0,6])
axis([0,6,0,0.8])
title('Step response from inverse Laplace')


Himp = (s +2)/(s^2 + 2*s +3);
htimp = ilaplace(Himp)
figure(5)
fplot(t,htimp,[0,7])
axis([0,7,-0.2,1.0])
hold on
plot([0,7],[0,0],'--')
title('Impulse response from inverse Laplace')
```

Example: Improper transfer function

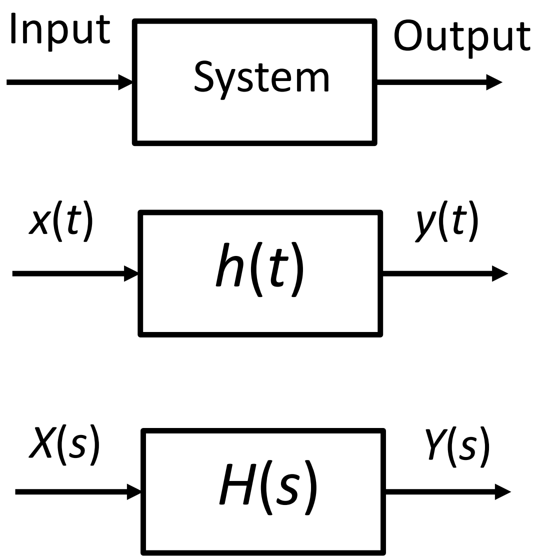
Differential Equation

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=2\frac{d^2x}{dt^2}+2x$$

Laplace Transform

$$\left(s^2+2s+5\right)Y(s)=\left(2s^2+0s+2\right)X(s)$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{2\left(s^2+1\right)}{s^2+2s+5}=\frac{2s^2+0s+2}{s^2+2s+5}$$



This is not a "proper" systems since order of output is not less than that of the input. (Both sides are second order)

Output will have an impulse.

Example: Improper transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2\left(s^2+1\right)}{s^2+2s+5} = \frac{2s^2+0s+2}{s^2+2s+5}$$

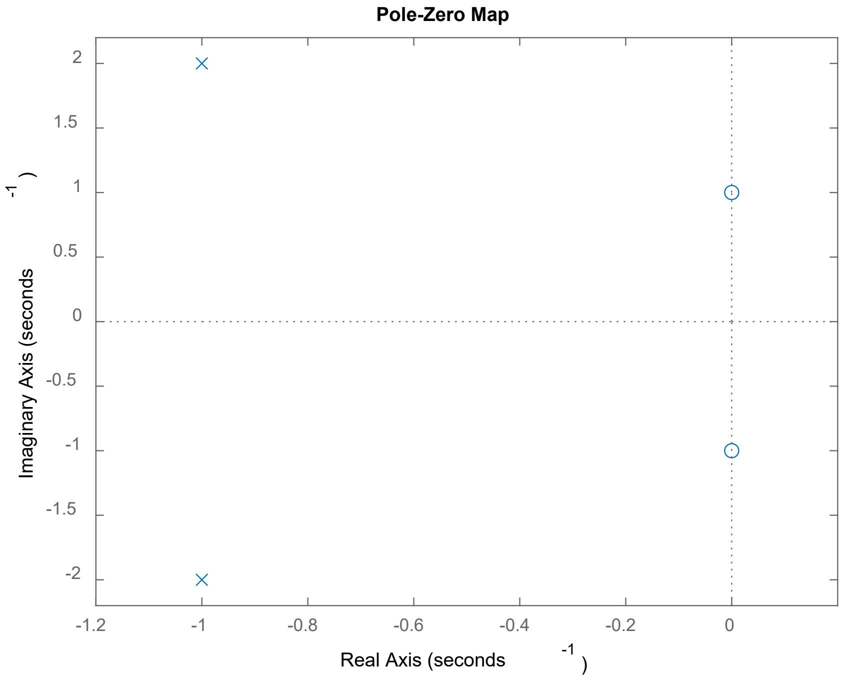
How do you deal with a system that is not proper?

Synthetic Division: $\left(s^2+2s+5\right)\overline{\big)2s^2+0s+2}$

$$\frac{2s^2+0s+2}{s^2+2s+5} = 2 - \frac{4\left(s+2\right)}{s^2+2s+5}$$

Where are the poles and zeros?

$$2 - \frac{4\left(s+2\right)}{s^2+2s+5} = 2 - \frac{4\left(s+2\right)}{\left(s+1+2j\right)\left(s+1-2j\right)}$$



Example: Improper transfer function

Inverse Laplace transform:

$$L^{-1}\left[2-\frac{4(s+2)}{s^2+2s+5}\right]=L^{-1}[2]-L^{-1}\left[\frac{4(s+2)}{s^2+2s+5}\right]$$

$$=2L^{-1}[1]-4L^{-1}\left[\frac{s+2}{s^2+2s+5}\right]$$

The inverse Laplace transform of the first term is easy: $L^{-1}[1]=\delta(t)$

For the second term, complete the square

$$\frac{s+2}{s^2+2s+5}=\frac{(s+1)}{(s^2+2s+1)+4}+\frac{1}{(s^2+2s+1)+4}$$

$$=\frac{(s+1)}{(s+1)^2+2^2}+\frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

Example: Improper transfer function

Use the frequency shift property: $L\left[e^{-at}f(t)\right]=F(s+a)$

Laplace transform of $\cos(\omega t)$ and $\sin(\omega t)$:

$$L[\cos\omega t]=\frac{s}{s^2+\omega^2} \quad ; \quad L[\sin\omega t]=\frac{\omega}{s^2+\omega^2}$$

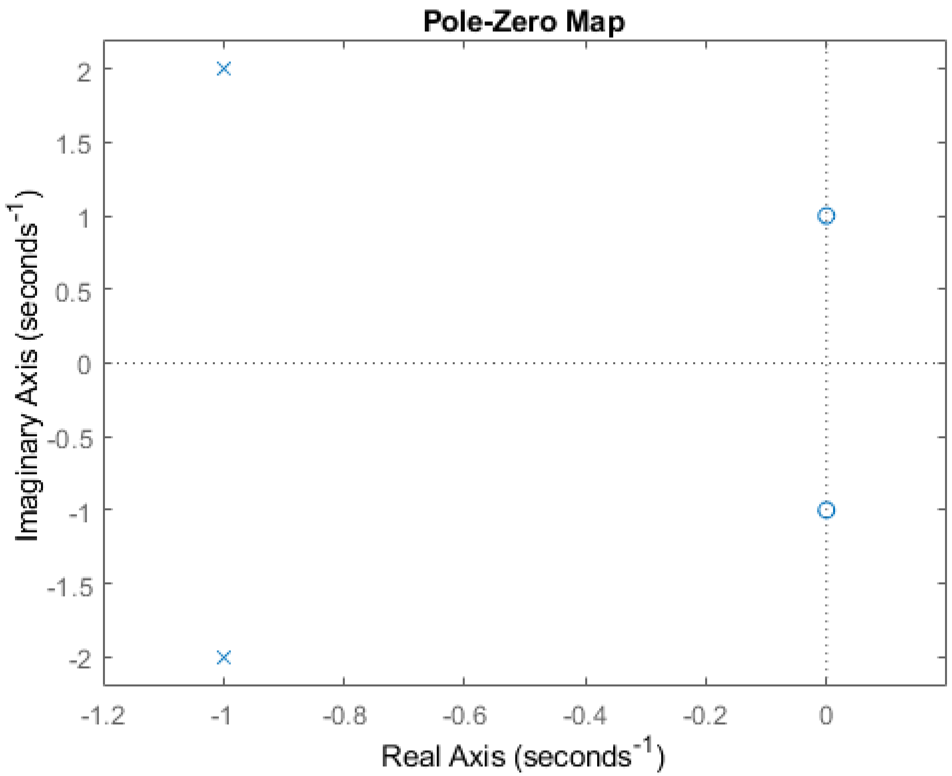
$$L^{-1}\left[\frac{(s+1)}{(s+1)^2+2^2}+\frac{1}{2}\frac{2}{(s+1)^2+2^2}\right]=e^{-1t}\cos(2t)+\frac{1}{2}e^{-1t}\,\mathrm{sins}(2t)$$

Putting this all together the impulse response is:

$$L^{-1}\left[\frac{2(s^2+1)}{s^2+2s+5}\right]=L^{-1}\left[2-\frac{4(s+2)}{s^2+2s+5}\right]= \quad \boxed{2\delta(t)+e^{-t}\left(4\cos(2t)+2\sin(2t)\right)}$$

Example: Improper transfer function

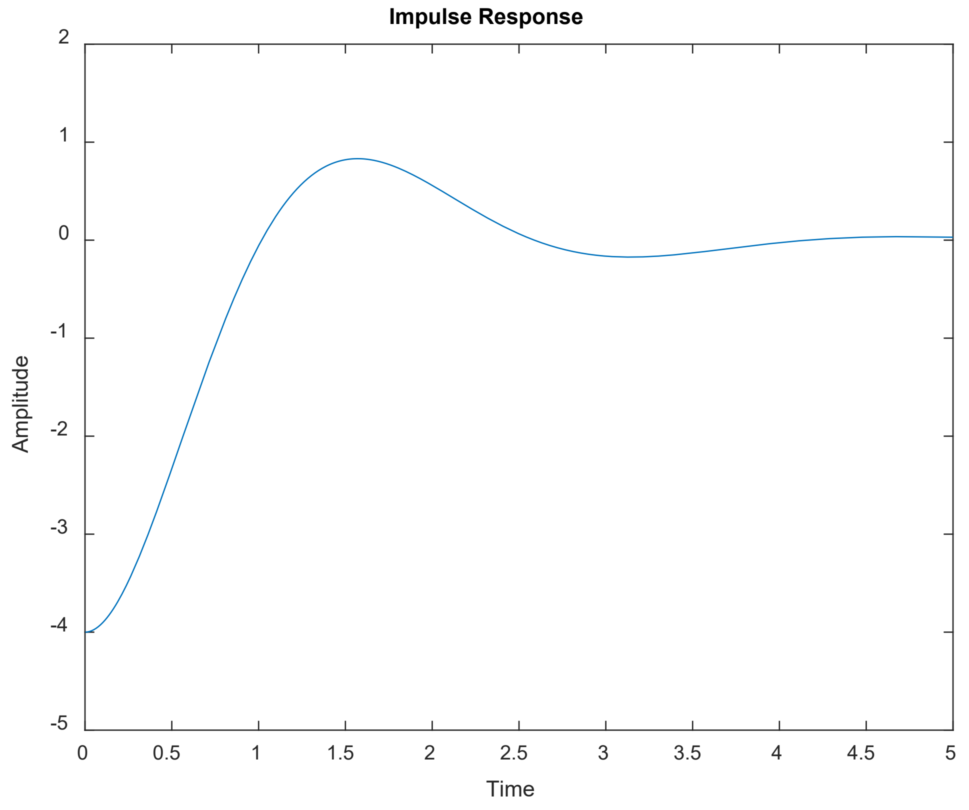
```
% D.E. y'' + 2*y + 5*y = 2*x'' + 2*x
% Laplace transform: (s^2 + 2*s + 5)*Y(s) =(2s^2 + 2)*X(s)
clear
close all
% Create transfer function for system
H = tf([2 0 2],[1 2 5])
% Plot the pole-zero map
pzmap(H)
axis([-1.2,0.2,-2.2,2.2])
syms s t
Hs = 2*(s^2 + 1)/(s^2 + 2*s +5)
ht = ilaplace(Hs)
partfrac(Hs,s,'FactorMode','complex')
figure(2)
fplot(t,ht)
axis([0,5,-5,2])
```



Example_1a.m

Example: Improper transfer function

```
% D.E. y'' + 2*y + 5*y = 2*x'' + 2*x
% Laplace transform: (s^2 + 2*s + 5)*Y(s) =(2s^2 + 2)*X(s)
clear
close all
% Create transfer function for system
H = tf([2 0 2],[1 2 5])
% Plot the pole-zero map
pzmap(H)
axis([-1.2,0.2,-2.2,2.2])
syms s t
Hs = 2*(s^2 + 1)/(s^2 + 2*s +5)
ht = ilaplace(Hs)
partfrac(Hs,s,'FactorMode','complex')
figure(2)
fplot(t,ht)
axis([0,5,-5,2])
```



Example: Improper transfer function

Suppose you wanted the step response?

$$Y_{step}(s) = H(s)U(s)$$

$$U(s) = L\left[u(t)\right] = \frac{1}{s}$$

$$Y_{step}(s) = \frac{2\left(s^2+1\right)}{s\left(s^2+2s+5\right)} = 2\left[\frac{A}{s} + \frac{Bs+C}{s^2+2s+5}\right]$$

Find A, B, and C using partial fractions.

$$A = \frac{1}{5} \quad ; \quad B = \frac{4}{5} \quad ; \quad C = -\frac{2}{5}$$

Complete the square

A lot of work

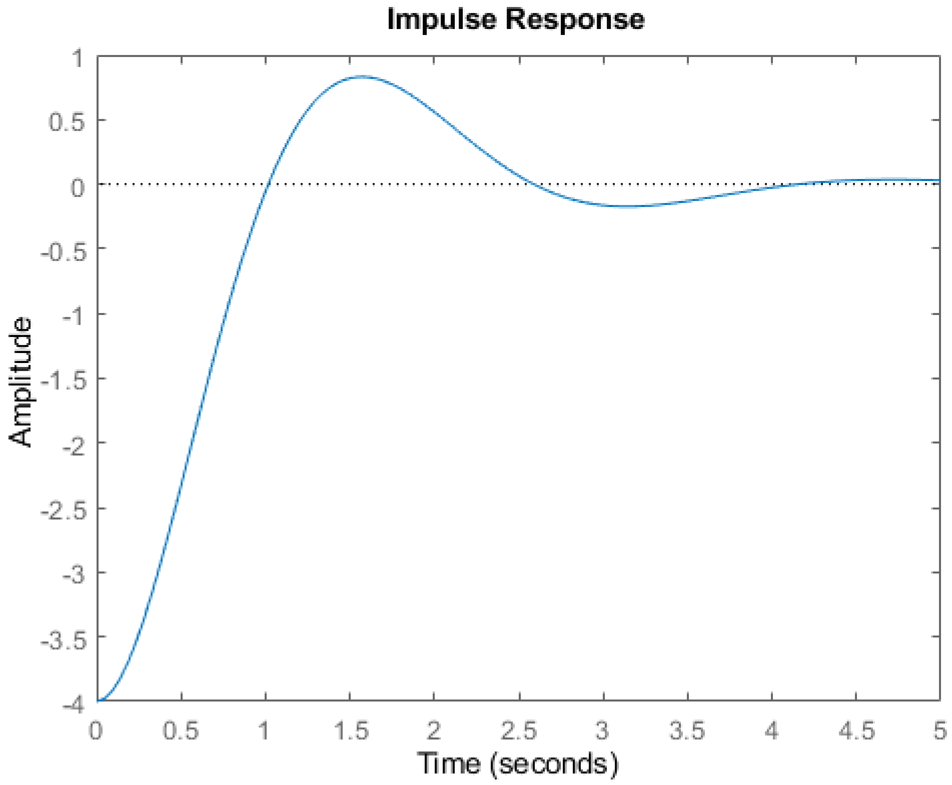
Or, use Matlab

$$y(t) = \frac{1}{5}\left[8e^{-t}\left(\cos 2t - \frac{3}{4}\sin 2t\right) + 2\right]$$

Example_1b.m

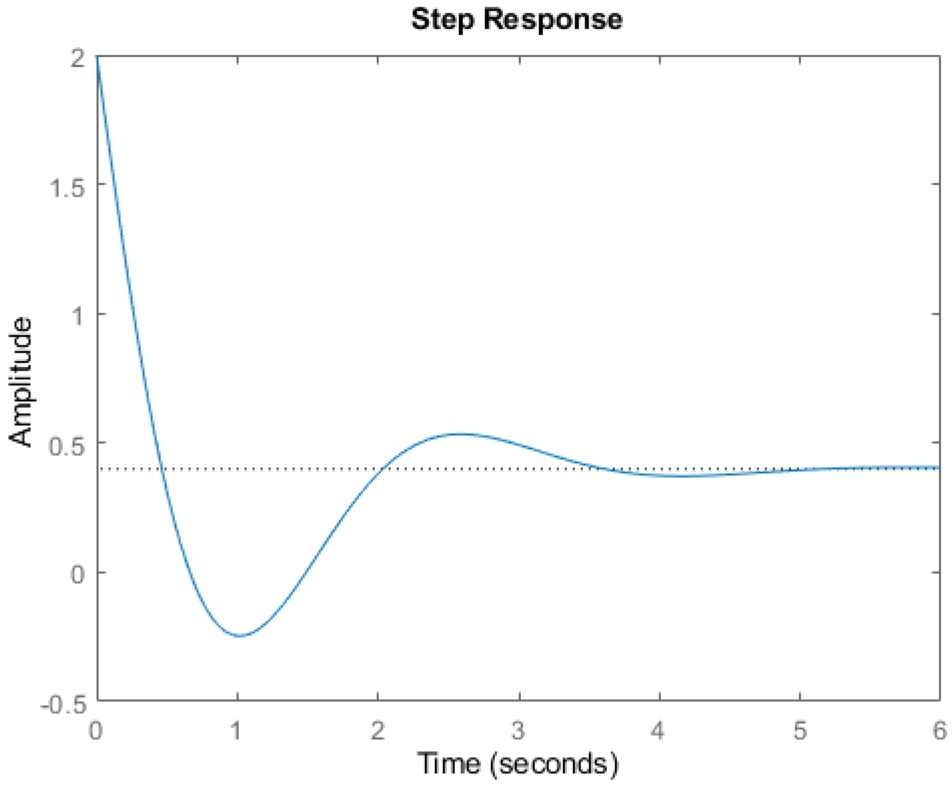
Example (Using Matlab)

```
H = tf([2 0 2],[1 2 5])
figure(1)
impulse(H)
figure(2)
step(H)
syms s t
H_impulse = 2*(s^2 + 1)/(s^2 + 2*s + 5)
ht_impulse = ilaplace(H_impulse)
H_step = 2*(s^2 + 1)/(s*(s^2 + 2*s + 5))
ht_step = ilaplace(H_step)
figure(3)
fplot(t,ht_impulse,[0,5])
axis([0,5,-4,1])
hold on
plot([0,5],[0,0],':')
title('Impulse Response (from inverse Laplace)')
figure(4)
fplot(t,ht_step,[0,6])
axis([0,6,-0.5,2])
hold on
plot([0,6],[2/5,2/5],':')
title('Step Response (from inverse Laplace)')
```



Example: Improper transfer function (Using Matlab)

```
H = tf([2 0 2],[1 2 5])
figure(1)
impulse(H)
figure(2)
step(H)
syms s t
H_impulse = 2*(s^2 + 1)/(s^2 + 2*s + 5)
ht_impulse = ilaplace(H_impulse)
H_step = 2*(s^2 + 1)/(s*(s^2 + 2*s + 5))
ht_step = ilaplace(H_step)
figure(3)
fplot(t,ht_impulse,[0,5])
axis([0,5,-4,1])
hold on
plot([0,5],[0,0],':')
title('Impulse Response (from inverse Laplace)')
figure(4)
fplot(t,ht_step,[0,6])
axis([0,6,-0.5,2])
hold on
plot([0,6],[2/5,2/5],':')
title('Step Response (from inverse Laplace)')
```



Wrapping discussion on solving Diff. Eqns.

- Looking at solution of differential equations representing system in Laplace domain:
  - Using differentiation property

For a signal $f(t)$, with Laplace transform $F(s)$, the one-sided Laplace transform of its first- and second-order derivatives are

$$\mathcal{L}\left[\frac{df(t)}{dt}u(t)\right] = sF(s) - f(0-) \tag{3.14}$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}u(t)\right] = s^2F(s) - sf(0-) - \frac{df(t)}{dt}\Big|_{t=0-} \tag{3.15}$$

In general, if $f^{(N)}(t)$ denotes the Nth-order derivative of a function $f(t)$ that has a Laplace transform $F(s)$, we have that

$$\mathcal{L}[f^{(N)}(t)u(t)] = s^N F(s) - \sum_{k=0}^{N-1} f^{(k)}(0-)s^{N-1-k} \tag{3.16}$$

where $f^{(m)}(t) = d^m f(t)/dt^m$ is the mth-order derivative, $m > 0$, and $f^{(0)}(t) \triangleq f(t)$.

Wrapping discussion on solving Diff. Eqns.

The **complete response** $y(t)$ of a system represented by an Nth-order linear ordinary differential equation with constant coefficients,

$$y^{(N)}(t) + \sum_{k=0}^{N-1} a_k y^{(k)}(t) = \sum_{\ell=0}^{M} b_\ell x^{(\ell)}(t) \qquad N > M \tag{3.38}$$

where $x(t)$ is the input and $y(t)$ the output of the system, and the initial conditions are

$$\{y^{(k)}(t), \quad 0 \leq k \leq N-1\} \tag{3.39}$$

is obtained by inverting the Laplace transform

$$Y(s) = \frac{B(s)}{A(s)}X(s) + \frac{1}{A(s)}I(s) \tag{3.40}$$

where $Y(s) = \mathcal{L}[y(t)], X(s) = \mathcal{L}[x(t)]$ and

$$A(s) = \sum_{k=0}^{N} a_k s^k, \qquad a_N = 1$$

$$B(s) = \sum_{\ell=0}^{M} b_\ell s^\ell$$

$$I(s) = \sum_{k=1}^{N} a_k \left(\sum_{m=0}^{k-1} s^{k-m-1} y^{(m)}(0) \right), \qquad a_N = 1$$

i.e., $I(s)$ depends on the initial conditions.

Wrapping discussion on solving Diff. Eqns.

Letting

$$H(s) = \frac{B(s)}{A(s)} \quad \text{and} \quad H_1(s) = \frac{1}{A(s)}$$

the **complete response** $y(t) = \mathcal{L}^{-1}[Y(s)]$ of the system is obtained by the inverse Laplace transform of

$$Y(s) = H(s)X(s) + H_1(s)I(s) \tag{3.42}$$

which gives

$$y(t) = y_{zs}(t) + y_{zi}(t) \tag{3.43}$$

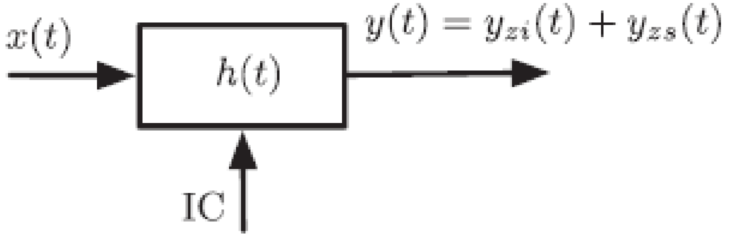
where

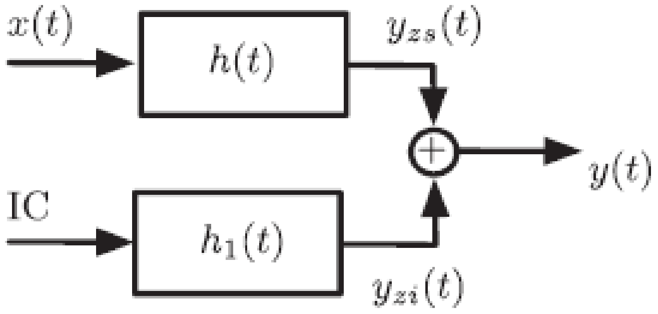
$y_{zs}(t) = \mathcal{L}^{-1}[H(s)X(s)]$ is the system's zero-state response

$y_{zi}(t) = \mathcal{L}^{-1}[H_1(s)I(s)]$ is the system's zero-input response

Wrapping discussion on solving Diff. Eqns.

$$H(s) = \frac{Y(s)}{X(s)}$$





Example: Improper transfer function

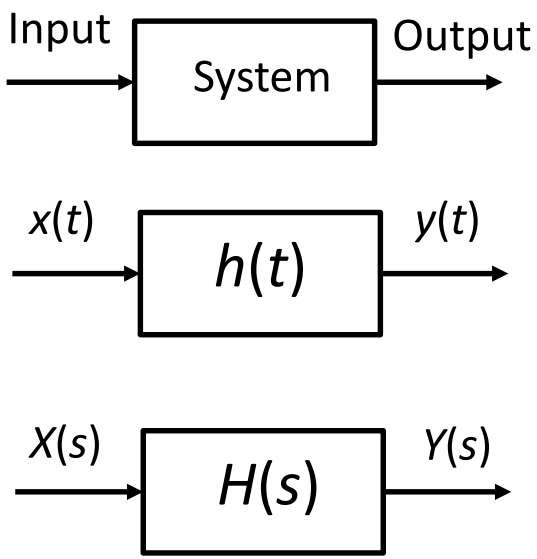
Differential Equation

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=2\frac{d^2x}{dt^2}+2x$$

Laplace Transform

$$\left(s^2+2s+5\right)Y(s)=\left(2s^2+0s+2\right)X(s)$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{2\left(s^2+1\right)}{s^2+2s+5}=\frac{2s^2+0s+2}{s^2+2s+5}$$



This is not a "proper" systems since order of output is not less than that of the input. (Both sides are second order)

Output will have an impulse.

Example: Improper transfer function

$$H(s)=\frac{Y(s)}{X(s)}=\frac{2\left(s^2+1\right)}{s^2+2s+5}=\frac{2s^2+0s+2}{s^2+2s+5}$$

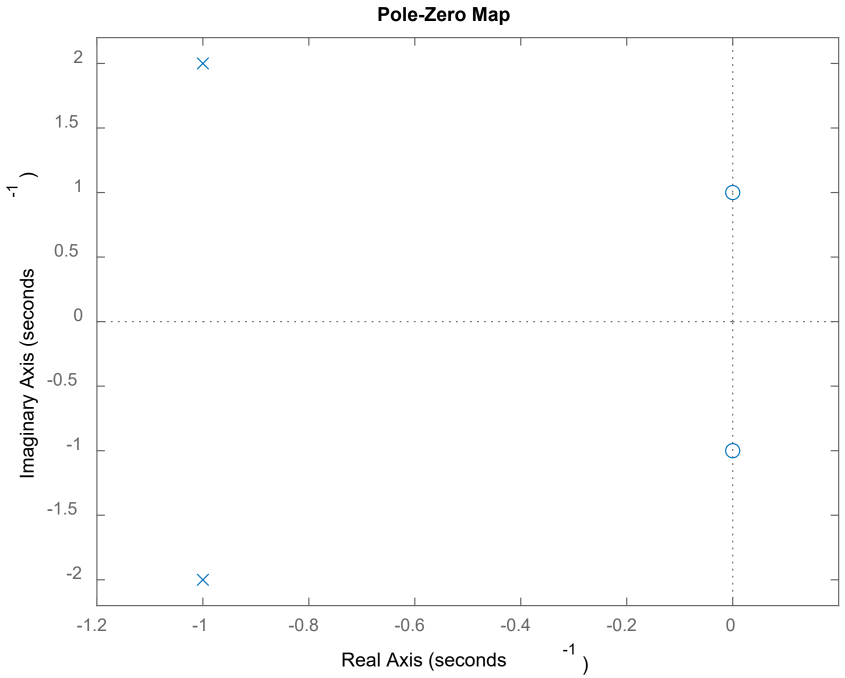
How do you deal with a system that is not proper?

Synthetic Division: $\left(s^2+2s+5\right)\overline{\big)2s^2+0s+2}$

$$\frac{2s^2+0s+2}{s^2+2s+5}=2-\frac{4\left(s+2\right)}{s^2+2s+5}$$

Where are the poles and zeros?

$$2-\frac{4\left(s+2\right)}{s^2+2s+5}=2-\frac{4\left(s+2\right)}{\left(s+1+2j\right)\left(s+1-2j\right)}$$



Example: Improper transfer function

Inverse Laplace transform:

$$L^{-1}\left[2-\frac{4(s+2)}{s^2+2s+5}\right]=L^{-1}[2]-L^{-1}\left[\frac{4(s+2)}{s^2+2s+5}\right]$$

$$=2L^{-1}[1]-4L^{-1}\left[\frac{s+2}{s^2+2s+5}\right]$$

The inverse Laplace transform of the first term is easy: $L^{-1}[1]=\delta(t)$

For the second term, complete the square

$$\frac{s+2}{s^2+2s+5}=\frac{(s+1)}{(s^2+2s+1)+4}+\frac{1}{(s^2+2s+1)+4}$$

$$=\frac{(s+1)}{(s+1)^2+2^2}+\frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

Wrapping discussion on solving Diff. Eqns.

- Steady-state response

$$y_{SS}(t) = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)$$

  - If all of the poles are in open left-hand s-plane, then steady state is zero
    - All damped exponential terms
    - On real axis, just exponential
    - Off real axis, damped exponential sinusoidal terms
  - This is the transient response of system
- Steady-state response is due to poles on the imaginary axis.
- The further the poles are in the negative real direction, the faster the decay
- The further poles are in the imaginary direction, the faster the oscillation

Frequency response

- What if we only look at the imaginary axis?
- Frequency response of system (filter)
  - Examine case where $s = j\omega$

$$H(j\omega) = \left|H(j\omega)\right| e^{j\angle H(j\omega)}$$

  - Consider the impulse response:

$$h(t) = tu(t) - 2u(t-1) - (t-2)u(t-2)$$

(a) Draw a sketch of $h(t)$

(b) Determine the transfer function $H(s)$.

Using $H(s)$, determine the magnitude of the frequency response, $|H(j\omega)|$.

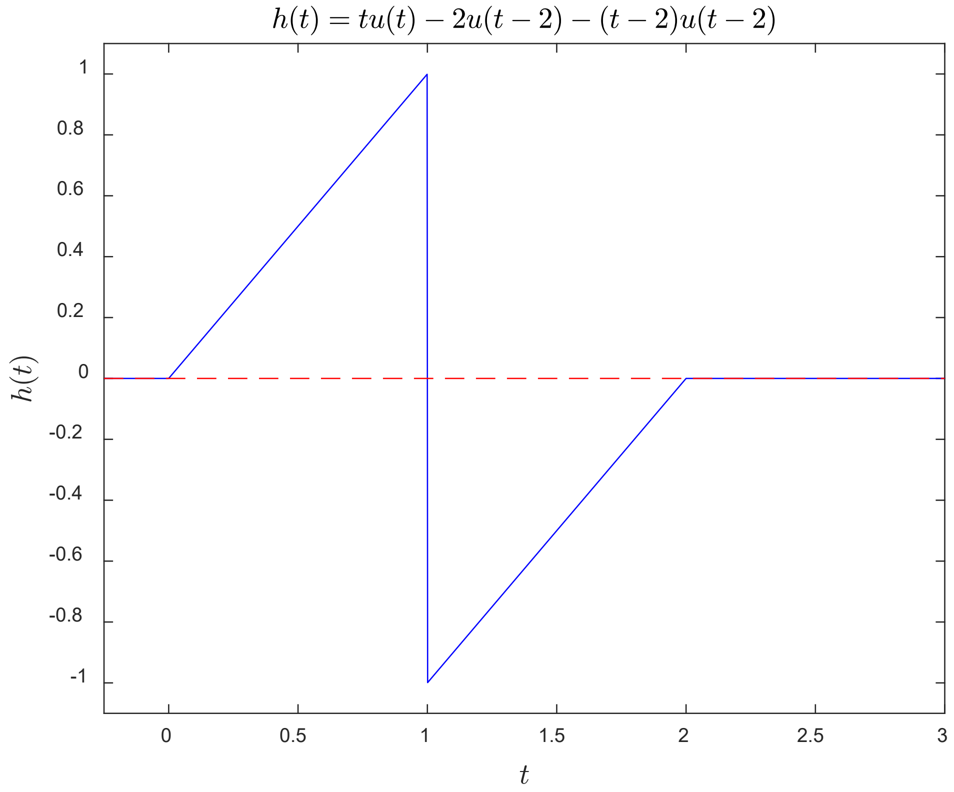
Does this act as a low-pass , high-pass, or band-pass filter?

(c) What are the poles and zeros of $H(s)$?

Example for frequency response

(1) For impulse response: $h(t) = tu(t) - 2u(t-1) - (t-2)u(t-2)$

(a) Draw a sketch of h(t)



Example for frequency response

(1) For impulse response: $h(t) = tu(t) - 2u(t-1) - (t-2)u(t-2)$

(b) Determine the transfer function $H(s)$.

Using $H(s)$, determine the magnitude of the frequency response, $|H(j\omega)|$. Does this act as a low-pass , high-pass, or band-pass filter?

$$L[tu(t)] = -\frac{dU(s)}{ds} = -\frac{d(1/s)}{ds} = \frac{1}{s^2} \quad \text{(using property 7: multiplication by } t\text{)}$$

$$L[2u(t-1)] = 2e^{-s}U(s) = \frac{2e^{-s}}{s} \quad \text{(using property 2: time shift)}$$

$$L[(t-2)u(t-2)] = e^{-2s}L[tu(t)] = \frac{e^{-2s}}{s^2} \quad \text{(using properties 2 and 7)}$$

$$\boxed{H(s) = \frac{1}{s^2} - \frac{2e^{-s}}{s} - \frac{e^{-2s}}{s^2} = \frac{1 - 2se^{-s} - e^{-2s}}{s^2}}$$

Poles and zeros at $s = 0$

Example for frequency response

$$H(s) = \frac{1 - 2se^{-s} - e^{-2s}}{s^2} = \frac{e^{-s}}{s^2}\left(e^{s} - e^{-s} - 2s\right)$$

$$H\left(j\omega\right) = \frac{e^{-j\omega}}{(j\omega)^2}\left(e^{j\omega} - e^{-j\omega} - 2j\omega\right) = -\frac{2je^{-j\omega}}{\omega^2}\left(\frac{e^{j\omega} - e^{-j\omega}}{2j} - \omega\right)$$

$$H\left(j\omega\right) = \frac{2je^{-j\omega}}{\omega^2}\left(\omega - \sin\omega\right)$$

$$\left|H\left(j\omega\right)\right|^2 = H(j\omega)H^*(j\omega) = \frac{\left(\omega - \sin\omega\right)^2}{\omega^4}\left(2je^{-j\omega}\right)\left(2(-j)e^{j\omega}\right) = \frac{4\left(\omega - \sin\omega\right)^2}{\omega^4}$$

$$\boxed{\left|H\left(j\omega\right)\right| = \frac{2\left(\omega - \sin\omega\right)}{\omega^2}}$$

Example for frequency response

Or, you could do it this way

Example_sawtooth.m

```
%*************************************************************************
%   SCRIPT:         laplace_example.m
%   DESCRIPTION:    Demonstrate ploting and Laplace
%   COURSE:         ENGR 51 - Biomedical Signals
%   AUTHOR:         Allan Moser
%   DATE CREATED:   28-Feb-2021
%   LAST CHANGED:   28-Feb-2021
%*************************************************************************
clear    % Clear all variables
syms t
h = t*heaviside(t) - heaviside(t - 2)*(t - 2) - 2*heaviside(t - 1);
tn = [-0.25:0.001:3];
hn = subs(h,tn);
figure(1)
hold off
plot(tn,hn,'b')
title('$$h(t)=tu(t)-2u(t-2)-(t-2)u(t-2)$$','interpreter','latex')
xlabel('$$t$$','interpreter','latex')
ylabel('$$h(t)$$','interpreter','latex')
axis([-0.25,3,-1.1,1.1])
hold on
plot([-0.25,3],[0,0],'r--')

% Find the Laplace transform symbolically
hs = laplace(h)
```

Example for frequency response

Does this act as a low-pass , high-pass, or band-pass filter?

Consider $\lim\limits_{\omega\to 0}\left|H\left(j\omega\right)\right| = \dfrac{2\left(\omega - \sin\omega\right)}{\omega^2} \to \dfrac{0}{0}$

Use L'Hospital's rule: $\lim\limits_{\omega\to 0}\dfrac{d\left(2\left(\omega-\sin\omega\right)\right)}{d\left(\omega^2\right)} = \dfrac{2(1-\cos\omega)}{2\omega} \to \dfrac{0}{0}$

Use L'Hospital's rule again: $\lim\limits_{\omega\to 0}\dfrac{d\left(1-\cos\omega\right)}{d\left(\omega\right)} = \dfrac{\sin\omega}{1} \to 0$

High pass filter since DC ($\omega = 0$) is filtered out.

Consider $\lim\limits_{\omega\to \infty}\left|H\left(j\omega\right)\right| = \dfrac{2\left(\omega - \sin\omega\right)}{\omega^2} \to \dfrac{2\omega}{\omega^2} = \dfrac{2}{\omega} \to 0$

Filters high frequencies out too.

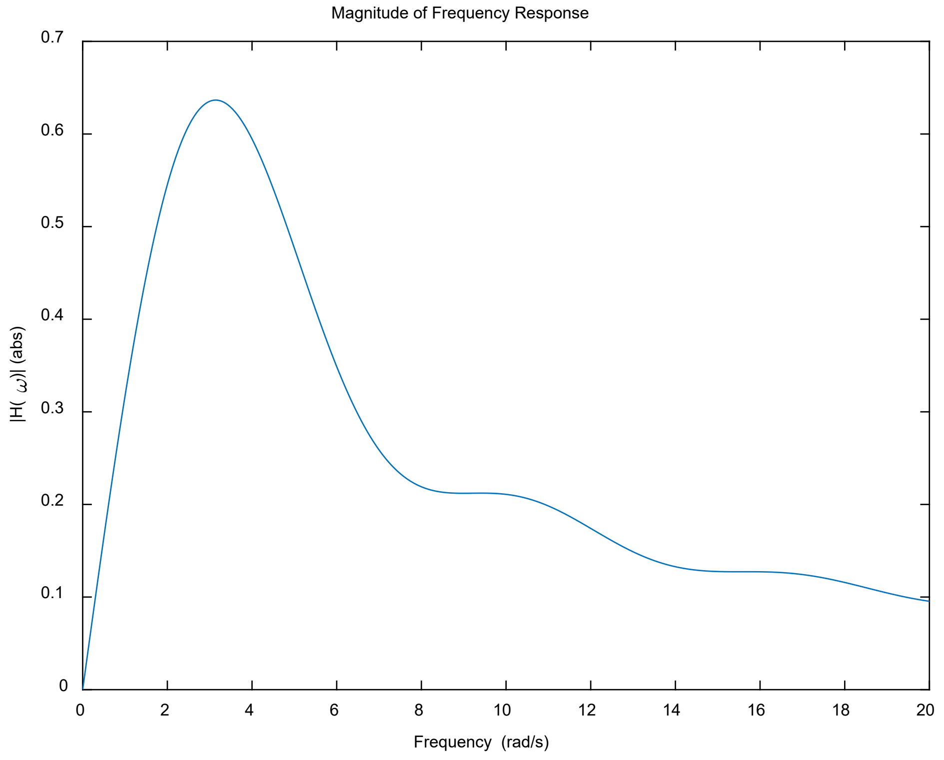
Class 03

```
s = tf('s')
H = 1/s^2 -2*exp(-s)/s - exp(-2*s)/s^2;
opts = bodeoptions;
opts.MagUnits = 'abs'
opts.MagScale = 'linear'
opts.FreqScale = 'linear'

bodemag(H,[0:0.001:20],opts)
figure(3)
bode(H,[0:0.001:20],opts)
```

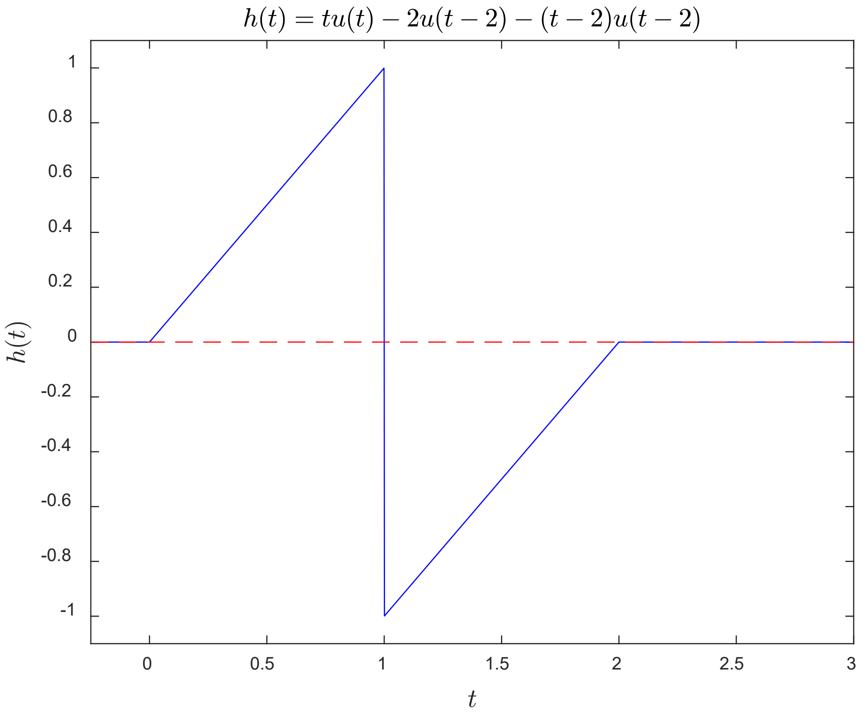
Example for frequency response

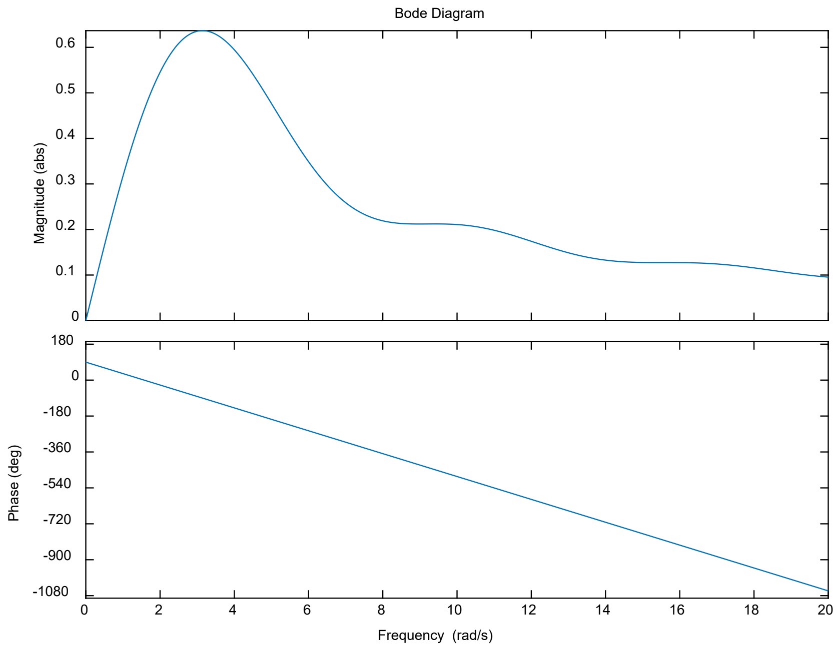
A funny filter. Filters out very low frequencies, enhances frequencies around $f = 0.1 \rightarrow 1.0$ Hz, then drops off.



Frequency response

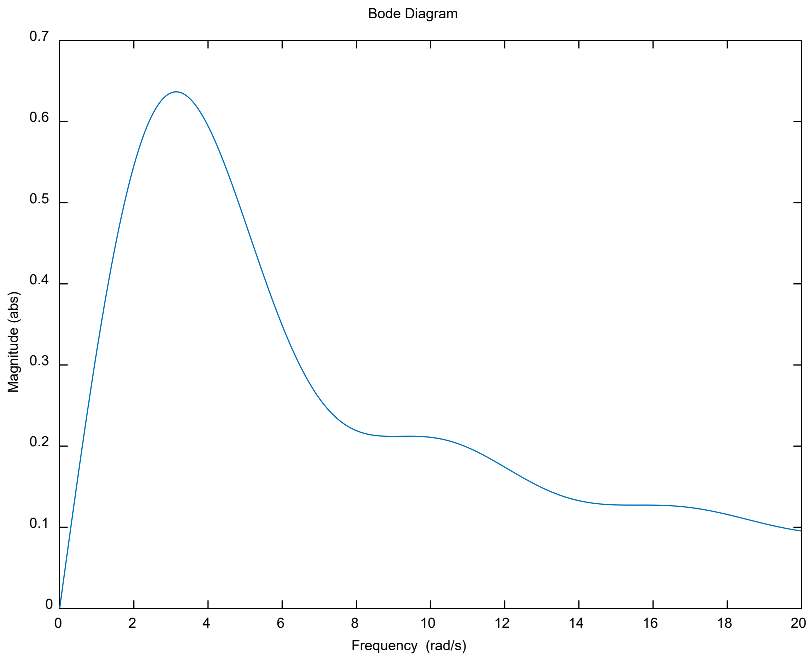
- Example:

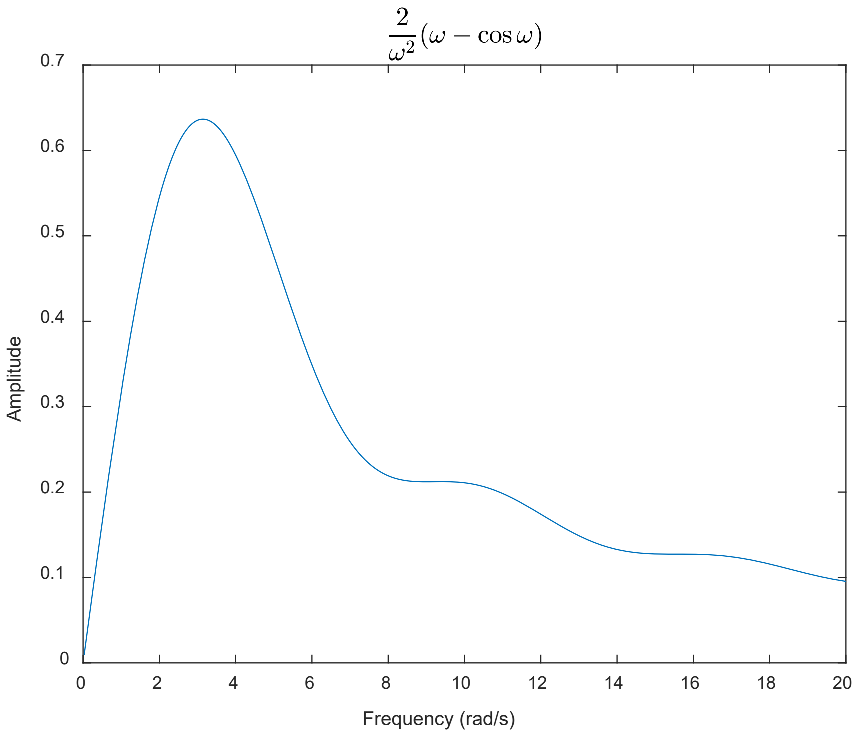




Frequency response

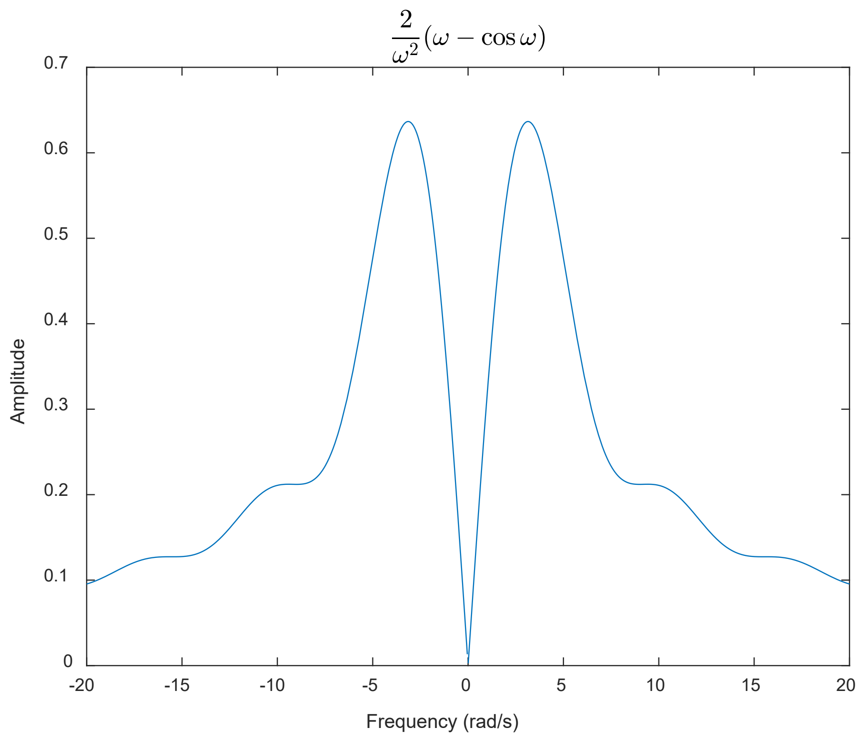
- Example:





Frequency response

- Example:



Frequency response

- Frequency response of system (filter)
  - Consider $s = j\omega$

$$H(\mathrm{j}\omega) = \left|H(\mathrm{j}\omega)\right| e^{j\angle H(j\omega)}$$

- Example:

$$w(t) = \cos(2\pi t)\left[u(t+1) - u(t-1)\right]$$

  - Find Laplace transform
  - Find frequency response