1. Complete proof of the convolution property for the laplace transform.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t)$$

$$Y(s) = \int_{-\infty}^{\infty} \Big\{ y(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right) e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\left(\int_{-\infty}^{\infty} h(t-\tau)\, e^{-st}\, dt \right)}\, d\tau$$

let $u = t-\tau \Rightarrow t = u+\tau$, $dt = du$, $\boxed{\text{When } t=\tau,\ u=0 \quad t\to\infty,\ u\to\infty}$

$$\int_{\tau}^{\infty} h(t-\tau)\, e^{-st}\, dt = \int_{0}^{\infty} h(u)\, e^{-s(u+\tau)}\, du = e^{-s\tau} \int_{0}^{\infty} h(u)\, e^{-su}\, du$$

We see that,

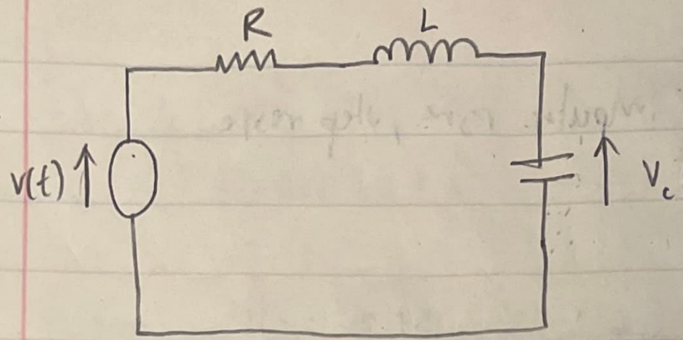
$$\int_{0}^{\infty} h(u)\, e^{-su}\, du = H(s)$$

The inner integral becomes $H(s) \cdot e^{-s\tau}$

$$= \int_{0}^{\infty} x(\tau) \left[e^{-s\tau} H(s) \right] d\tau = H(s) \overbrace{\int_{0}^{\infty} x(\tau)\, e^{-s\tau}\, d\tau}^{X(s)} = H(s)\, X(s)$$

ii. $Y(s) = H(s)\,X(s)$

2.



KVL: $V(t) = iR + L\dfrac{di}{dt} + V_c(t)$

$$V_c = \frac{1}{C}\int i\,dt$$

$$\frac{dv(t)}{dt} = \frac{di}{dt}R + L\frac{d^2 i}{dt^2} + \frac{i}{C}$$

b) $\omega_0 = 1/\sqrt{LC}$ $\quad \zeta = R/2\sqrt{C/L}$

$$\mathcal{L}\left[\frac{dv(t)}{dt}\right] = V(s) = sIR + s^2LI + \frac{I}{C} = I\left(sR + s^2L + \frac{1}{C}\right)$$

$$\frac{dV_c}{dt} = \frac{i}{C} \quad , \quad \mathcal{L}\left[\frac{dV_c}{dt}\right] = V_c(s) = \frac{I}{C}$$

$$\frac{V_c(s)}{V(s)} = \frac{I}{C}\,\frac{1}{I(sR+s^2L+1/C)} = \frac{1}{sRC+s^2LC+1} = \frac{1/LC}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Repl'ng w/ ω_0, ζ, we get $\dfrac{V_c(s)}{V(s)} = \dfrac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$

Find pole-zero map, impulse resre, step respe

c) $\omega_0 = 4$, $\zeta = 0.25$

$$\frac{16}{s^2 + 2s + 16}$$

$S + 2s = -16$

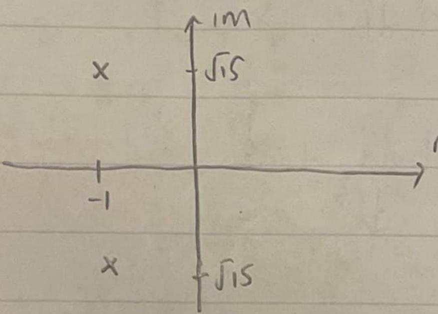
$S + 2s + (1)^2 = -16 + (1)^2$

$(s+1) = \sqrt{-15}$

$s = -1 \pm \sqrt{15}\, j$

No zeros

Poles: $-1 + \sqrt{15}\, j$, $-1 - \sqrt{15}\, j$



impulse response:

$$\frac{4^2}{s^2 + 2(0.25)4s + 4^2}$$

$$V_c(t) = \frac{4}{\sqrt{0.9375}} e^{-t} \sin(\sqrt{15}\, t) \qquad t \geq 0$$

step response

$$\frac{4^2}{s^2 + 2(0.25)4s + 4^2} \cdot \frac{1}{s} = \frac{A}{s} + \frac{B}{s^2 + 2s + 16}$$

$$16 = A(s^2 + 2s + 16) + Bs$$

$$\frac{4^2}{s^2 + 2(0.25)4s + 4^2} \cdot \frac{1}{s}$$

$$V_{c_{step}} = 1 - \frac{1}{\sqrt{0.9375}} e^{-t} \sin\left(\sqrt{15}\, t + \phi\right) \quad \text{where } \phi = \tan^{-1}\left(\frac{\sqrt{0.9375}}{0.25}\right)$$

$t \geq 0$

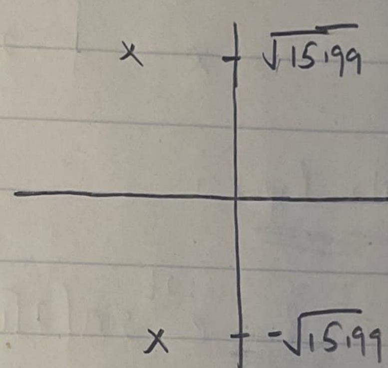
$\phi = 1.318$ rad

d) $\omega_0 = 4$ $\quad \xi = 0.025$

No zeros

Poles $= -\xi\omega_0 \pm j\,\omega_0\sqrt{1-\xi^2}$

$= -0.1 \pm j\sqrt{15.99}$



Impulse response $\quad \frac{4^2}{s^2 + 2(0.025)4s + 4^2}$

$$V_c = \frac{4}{\sqrt{0.999375}} e^{-0.1t} \sin\left(\sqrt{15}\, t\right) \quad t \geq 0$$

Step response $\quad \frac{1}{s} \frac{4^2}{s^2 + 2(0.025)4s + 4^2}$

$$\phi = \tan^{-1}\left(\frac{\sqrt{0.999375}}{0.025}\right)$$

$$V_{c,step} = 1 - \frac{1}{\sqrt{0.999375}} e^{-0.1t} \sin\left(\sqrt{15.99}\, t + \phi\right) \quad \phi = 1.54579 \text{ rad}$$

$t \geq 0$

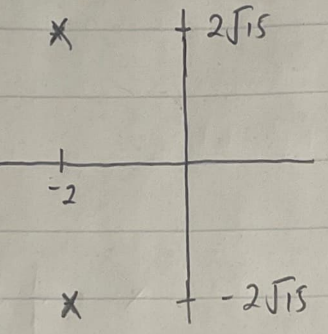
e) $\omega_0 = 8$, $\zeta = 0.25$

$$\frac{8^2}{s^2 + 2(0.25)8s + 8^2}$$

No zeros

Poles $= -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$

$= -2 \pm j\sqrt{60} = -2 \pm 2\sqrt{15}\,j$



Impulse Response :

$$V_c = \frac{8}{\sqrt{0.9375}} e^{-2t} \sin(2\sqrt{15}\,t) \qquad t \geq 0$$

Step Response :

$$\frac{8^2}{s^2 + 2(0.25)8s + 8^2} \cdot \frac{1}{s}$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{0.9375}}{0.25}\right)$$

$$V_{c_{step}} = 1 - \frac{1}{\sqrt{0.9375}} \sin(2\sqrt{15}\,t + \phi)$$

$\phi = 1.3181$ rad.

$(-i)^2$

$(-1)^2(-i)^2$

3. $$X(s) = \frac{s+2}{(s+2)^2+1}$$

a)

$$((s+2)^2+1)\,X(s) = (s+2)\,Y(s)$$

$$(s^2+4s+4+1)\,X(s) = (s+2)\,Y(s)$$

$$(s^2+4s+5)\,X(s) = (s+2)\,Y(s)$$

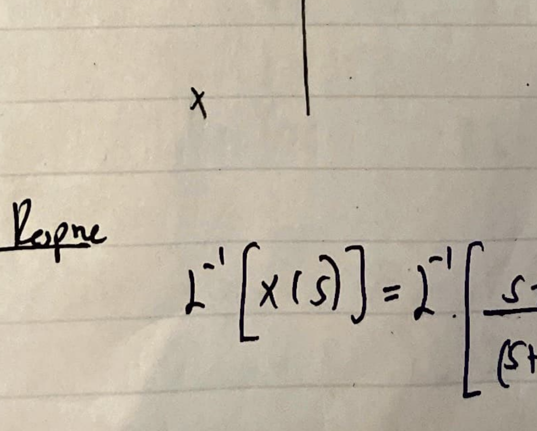
$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = \frac{dy}{dt} + 2y$$

b) Pole-zero map

poles = $-2+i$, $-2-i$

zeros = -2

$$X(s) = \frac{s+2}{(s+2)^2+1}$$



$a = 2$

$\omega_d = 1$

Impulse Response

$$\mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] = e^{-2t}\cos t\,u(t)$$

$$= e^{-2t}\cos t \quad t \geq 0$$

Step Respore

$$X(s)\,\frac{1}{s} = \frac{s+2}{(s+2)^2+1}\,\frac{1}{s} = \frac{s+2}{s\left((s+2)^2+1\right)}$$

$$\frac{s+2}{(s+2)^2+1}\,\frac{1}{s} = \frac{A}{s} + \frac{Bs+C}{(s+2)^2+1}$$

$$\frac{s+2}{\cancel{(s+2)^2+1}} = A\left((s+2)^2+1\right) + Bs^2 + Cs$$

$s=0$

$$2 = 5A \qquad A = 2/5$$

$s=1$

$$3 = 4 + B + C$$

$$B + C = -1$$

$s=2$

$$4 = \frac{34}{5} + 4B + 2C$$

$$\frac{20}{5} - \frac{34}{5} = 4B + 2C \qquad 4B + 2C = -\frac{14}{5}$$

Usiy a System of Equatus Solvr

$$B = -2/5 \quad , \quad C = -3/5$$

$$X(s) = \frac{2}{5}\frac{1}{s} + \frac{-2/5\, s + -3/5}{(s+2)^2 + 1}$$

$$\mathcal{L}^{-1}[X(s)] = \frac{2}{5}u(t) + e^{-2t}\left[\frac{-2}{5}\cos t + \frac{1}{5}\sin t\right]$$

Step Response $= \frac{2}{5} + e^{-2t}\left[-2/5 \cos t + \frac{1}{5}\sin t\right] \quad t \geq 0$