Digital Signal Processing

Class 25
04/22/2025

ENGR 71

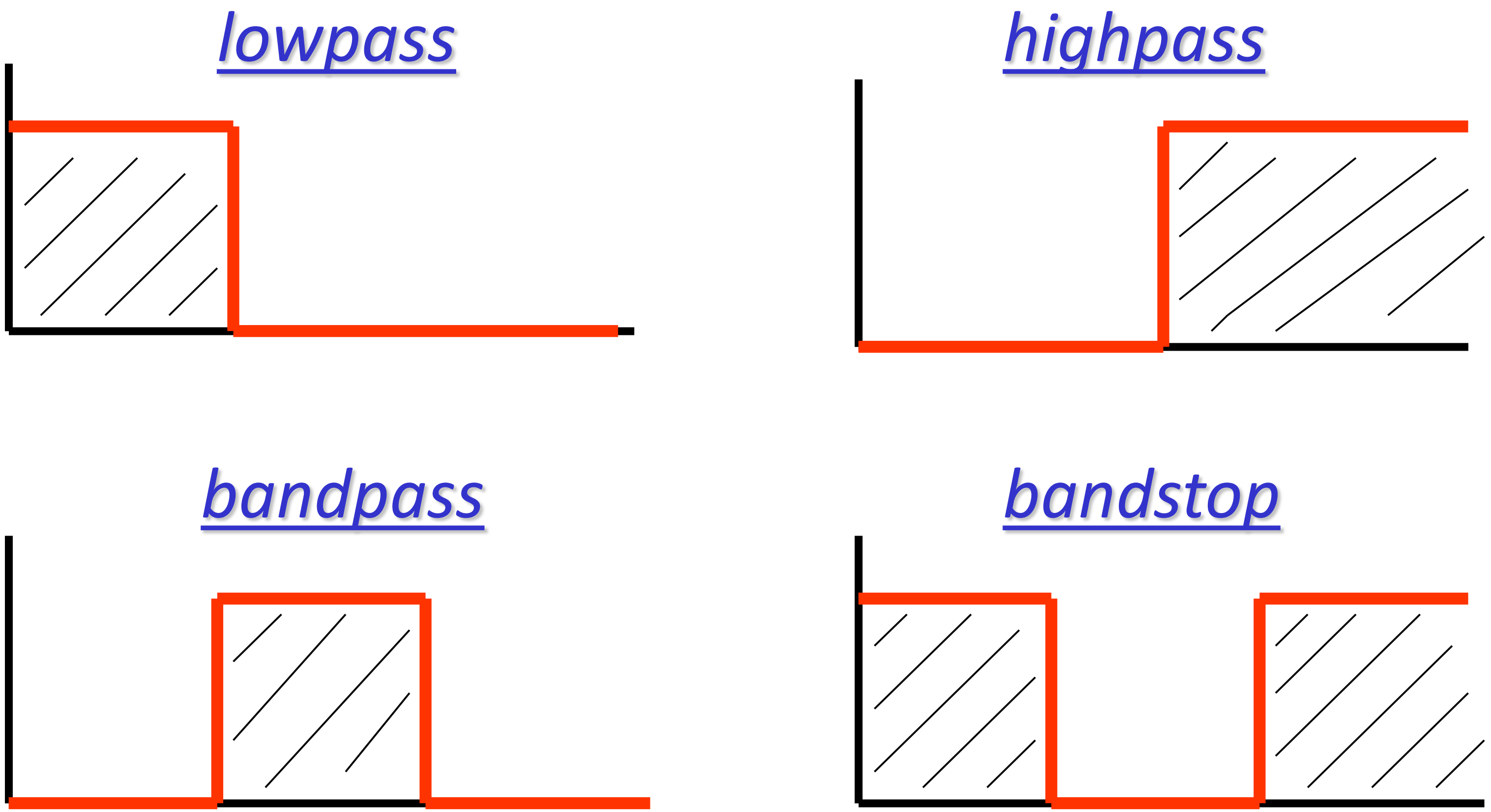
- Class Overview
  - Example for FIR frequency-sampling (
  - Digital Filter Design
    - IIR filters
- Assignments
  - Reading:
    Chapter 10: Design of Digital Filters
    https://www.mathworks.com/help/signal/ug/fir-filter-design.html
  - Problems: TBD
  - Lab 3: “Fun with Filters”
    - Due May 4 (Sunday)

Project

- Projects
  - You can work in groups if you wish
  - Presentation at time reserved for final exam
    - Friday, May 9, 7:00-10:00 PM
    - Science Center 264
  - Submit a brief description of your proposed project and who you will be working with to the Project Description Dropbox
  - Submit slides from presentation to Project Dropbox
  - Submit written report to Project Dropbox by end of semester (May 15)

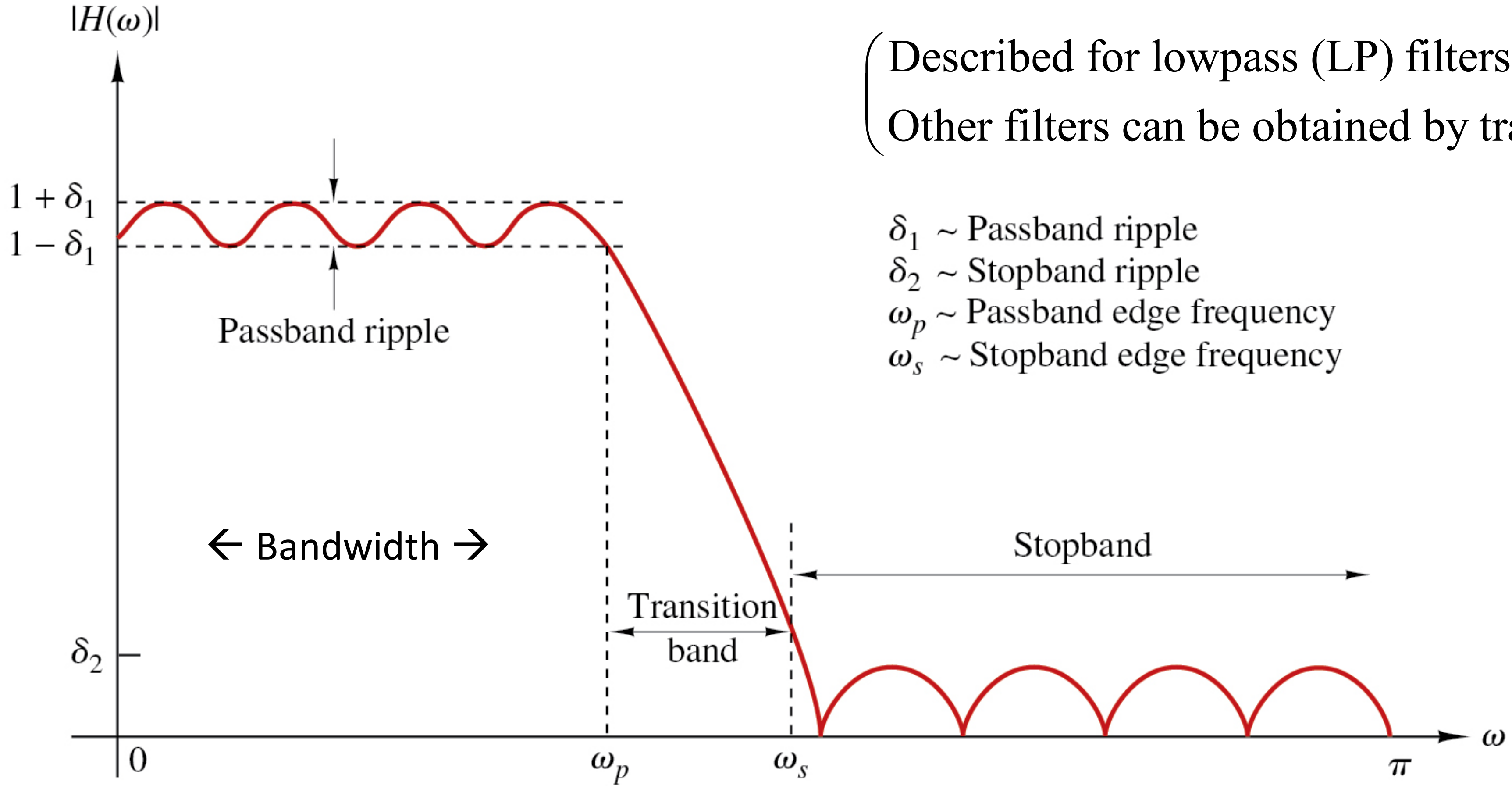
Filters

- Design of Digital Filters



Filter Design

- Specifications for physically realizable filters:



(Described for lowpass (LP) filters.
Other filters can be obtained by transforming LP filters)

δ_1 ~ Passband ripple
δ_2 ~ Stopband ripple
ω_p ~ Passband edge frequency
ω_s ~ Stopband edge frequency

Filters

- Design of Digital Filters
  - Finite Impulse Response (FIR)
    - Used when linear-phase required in passband
      - Constant time delay
      - No dispersion as a function of frequency
    - Three methods discussed
      - Windows, Frequency sampling, Iterative method for optimum equiripple filters
  - Infinite Impulse Response (IIR)
    - No requirement on linear-phase
    - Better characteristics for fewer parameters
    - Less memory
    - Lower computational complexity

IIR Filters

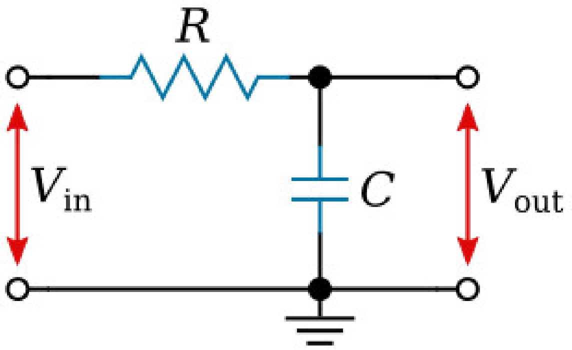
- Infinite Impulse Response Filters
  - Advantages
    - Usually require fewer coefficients to get similar response
    - Work faster
      - A consideration for hardware implementations
    - Require less memory
      - Again, probably on a consideration for hardware or firmware
  - Disadvantages
    - Nonlinear phase
      - Different frequency components have different delays
      - Causes distortion of signal's waveform shape

IIR Filters

- Methods for designing IIR filters
  - Start with analog filter and convert to a digital filter
    - Specified in terms of $H(s)$, transfer function in Laplace domain
    - In Laplace domain, derivatives become powers of s
  - Three methods
    - Approximation of derivatives in analog filter description
    - Impulse invariance
      - Involves sampling the continuous impulse response
    - Bilinear transformation

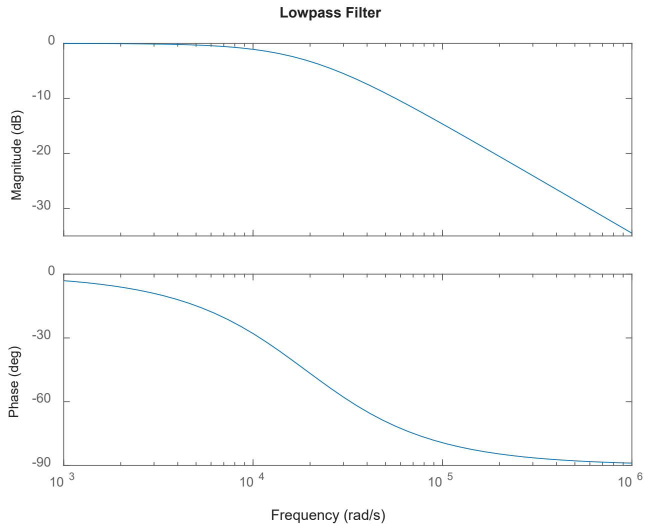
IIR Filters

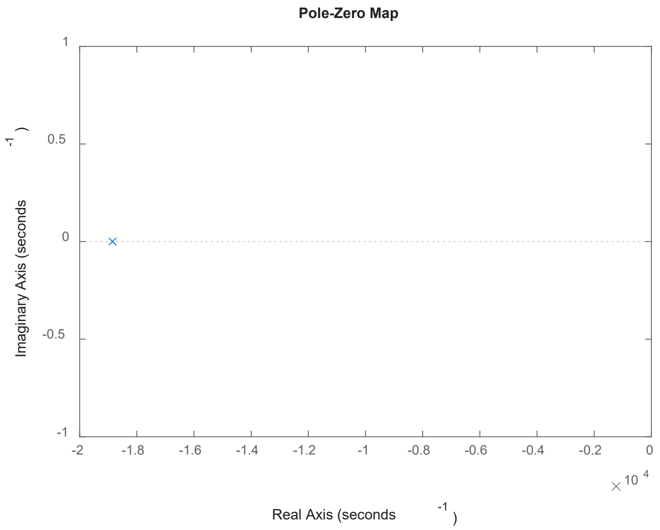
- Some simple analog filters
  - Lowpass



$$H(s) = \frac{1/RC}{s + (1/RC)} = \frac{\Omega_c}{s + \Omega_c}$$

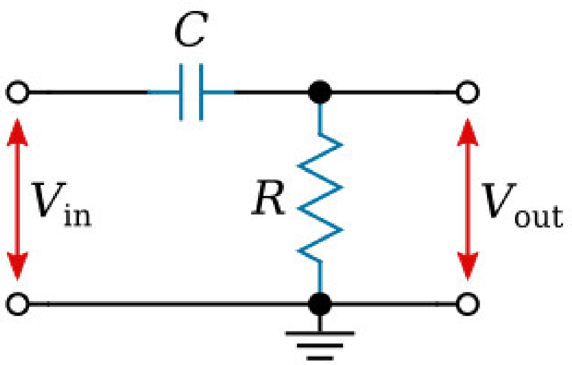
$$\Omega_c = 2\pi/RC \quad \text{(cutoff)}$$





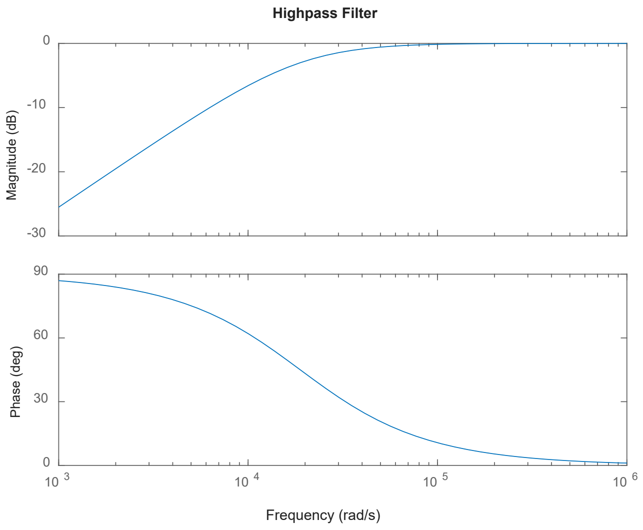
IIR Filters

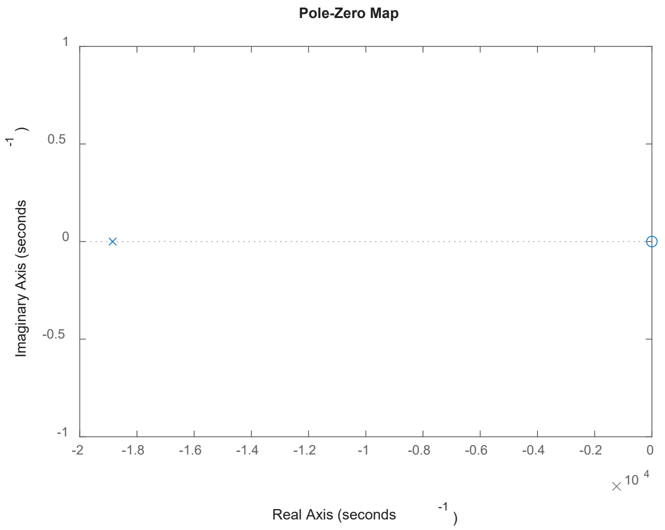
- Some simple analog filters
  - Highpass



$$H(s) = \frac{s}{s + (1/RC)} = \frac{s}{s + \omega_c}$$

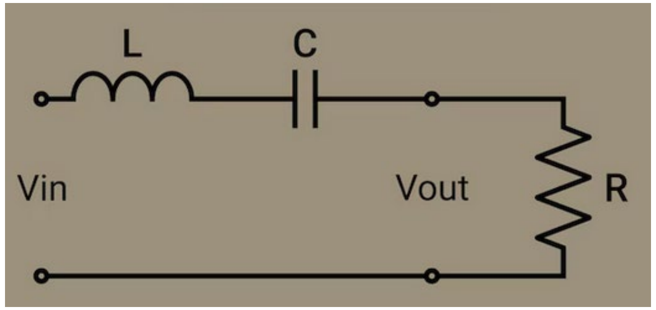
$$\omega_c = 1/RC \quad \text{(cutoff)}$$





IIR Filters

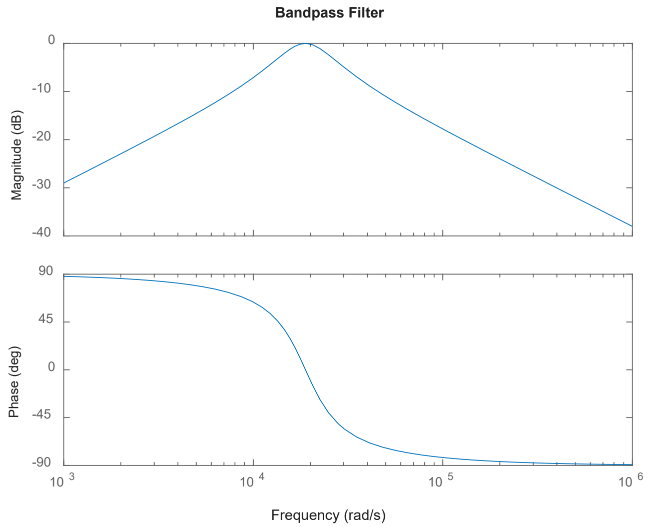
- Some simple analog filters
  - Bandpass:

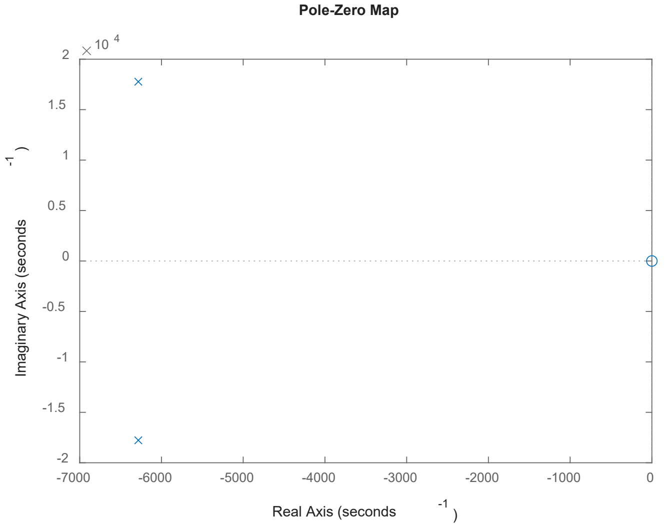


$$H(s) = \frac{(R/L)s}{s^2 + (R/L)s + (1/LC)} = \frac{\beta s}{s^2 + \beta s + \omega_0^2}$$

$$\omega_0 = \sqrt{1/LC} \quad \text{(center frequency)}$$

$$\beta = R/L \quad \text{(bandwidth)}$$





IIR Filters – Bilinear Transform

- Bilinear transformation
  - Useful transformation for analog →digital filter design because it can be used for all filter types (LP,HP,BP,BS)
  - Bilinear transformation:

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) = \frac{2}{T}\left(\frac{z - 1}{z + 1}\right) \quad ; \quad z = \frac{2}{T}\left(\frac{1 + sT/2}{1 - sT/2}\right)$$

  - In complex variable theory this is a linear fractional transformation that has the general form:

$$w = \frac{az + b}{cz + d} \quad ; \quad z = \frac{-dw + b}{cw - a}$$

IIR Filters – Bilinear Transform

- In complex variable theory this is a linear fractional transformation that has the general form:

$$w = \frac{az+b}{cz+d} \quad ; \quad z = \frac{-dw+b}{cw-a}$$

For the bilinear transformation shown:

$$w = sT/2, \quad a = 1, \quad b = -1, \quad c = 1, \quad d = 1$$

- This is a conformal mapping
  - Maps each point in the *w* domain to a unique point in the *z* domain (except at $w = a/c$)
  - Derivative in nonzero and analytic
  - Preserves local angle preservation

IIR Filters – Bilinear Transform

– Motivation for bilinear transformation for DSP

- Consider simple first-order system:

Differential equation: $y'(t) + ay(t) = bx(t) \Rightarrow \; y'(t) = -ay(t) + bx(t)$

System transfer function: $H(s) = \dfrac{b}{s+a}$

Integrate the differential equation: $y(t) = \displaystyle\int_{t_0}^{t} y'(\tau)d\tau + y(t_0)$

Approximating the integral by the trapezoidal rule at $t = nT$:

$$\left(\begin{array}{l} Area = (b-a)\cdot\dfrac{1}{2}\left(f(a) + f(b)\right) \\ t_0 = nT - T; \quad b - a = T; \quad f(a) = y(nT); \quad f(b) = y(nT - T) \end{array} \right)$$

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT - T)\right] + y(nT - T)$$

IIR Filters – Bilinear Transform

Substituting $y'(t) = -ay(t) + bx(t)$ at $t = nT$ into $y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T)$

(labeling nT just by n)

$$y(nT) = \frac{T}{2}\left[\left(-ay(n) + bx(n)\right) + \left(-ay(n-1) + bx(n-1)\right)\right] + y(n-1)$$

Collect y on one side and x on the other:

$$y(n) + \frac{aT}{2}y(n) + \frac{aT}{2}y(n-1) - y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right]$$

$$\boxed{\left(1 + \frac{aT}{2}\right)y(n) - \left(1 - \frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right]}$$

IIR Filters – Bilinear Transform

Take the z-transform: $\mathcal{Z}\left\{\left(1+\frac{aT}{2}\right)y(n)-\left(1-\frac{aT}{2}\right)y(n-1)=\frac{bT}{2}\left[x(n)+x(n-1)\right]\right\}$

$$\left[\left(1+\frac{aT}{2}\right)-\left(1-\frac{aT}{2}\right)z^{-1}\right]Y(z)=\frac{bT}{2}\left[1+z^{-1}\right]X(z)$$

The system transfer function is:

$$H(z)=\frac{bT/2\left(1+z^{-1}\right)}{1+aT/2-\left(1-aT/2\right)z^{-1}}=\frac{b}{\dfrac{\left(1-z^{-1}\right)+aT/2\left(1+z^{-1}\right)}{T/2\left(1+z^{-1}\right)}}$$

The system transfer function is:

$$\boxed{H(z)=\frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}}$$

IIR Filters – Bilinear Transform

Compare: $H(z) = \dfrac{b}{\dfrac{2}{T}\dfrac{\left(1-z^{-1}\right)}{\left(1+z^{-1}\right)}+a}$ to continuous time transfer function: $H(s) = \dfrac{b}{s+a}$

The mapping from s to z plane is:

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

which is the bilinear transform

IIR Filters – Bilinear Transform

Compare: $H(z) = \dfrac{b}{\dfrac{2}{T}\dfrac{\left(1-z^{-1}\right)}{\left(1+z^{-1}\right)}+a}$ to continuous time transfer function: $H(s) = \dfrac{b}{s+a}$

The mapping from s to z plane is:

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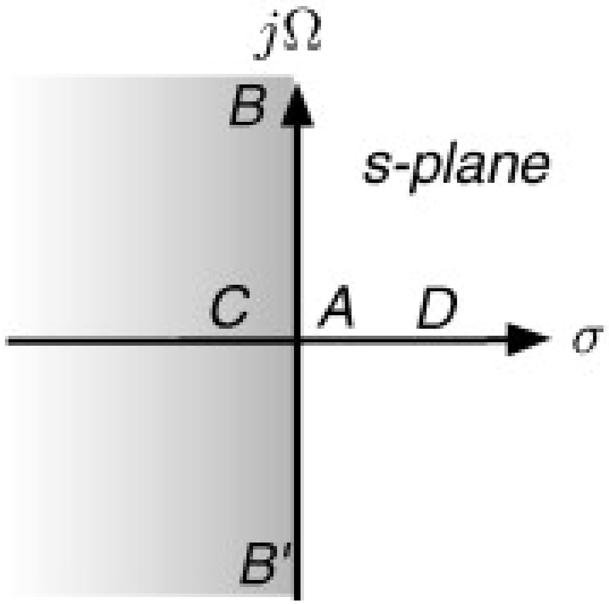
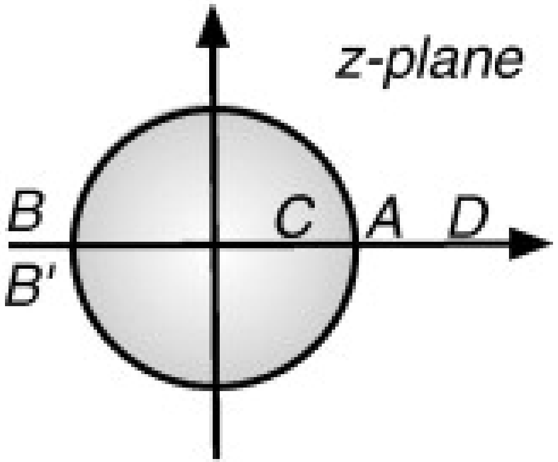
which is the bilinear transform

IIR Filters – Bilinear Transform

- Bilinear transform has some interesting properties





$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$s = \sigma + j\Omega; \quad z = re^{j\omega}$$

IIR Filters – Bilinear Transform

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \qquad s = \sigma + j\Omega; \quad z = re^{j\omega}$$

$$r < 1 \Leftrightarrow \sigma < 0 \quad \text{Stable system}$$

$$r > 1 \Leftrightarrow \sigma > 0 \quad \text{Unstable system}$$

$$r = 1 \Leftrightarrow \sigma = 0 \quad j\Omega \text{ axis in s-plane}$$

on the frequency axis (or unit circle)

$$\sigma = \frac{2}{T}\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}$$

$$\Omega = \frac{2}{T}\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}$$

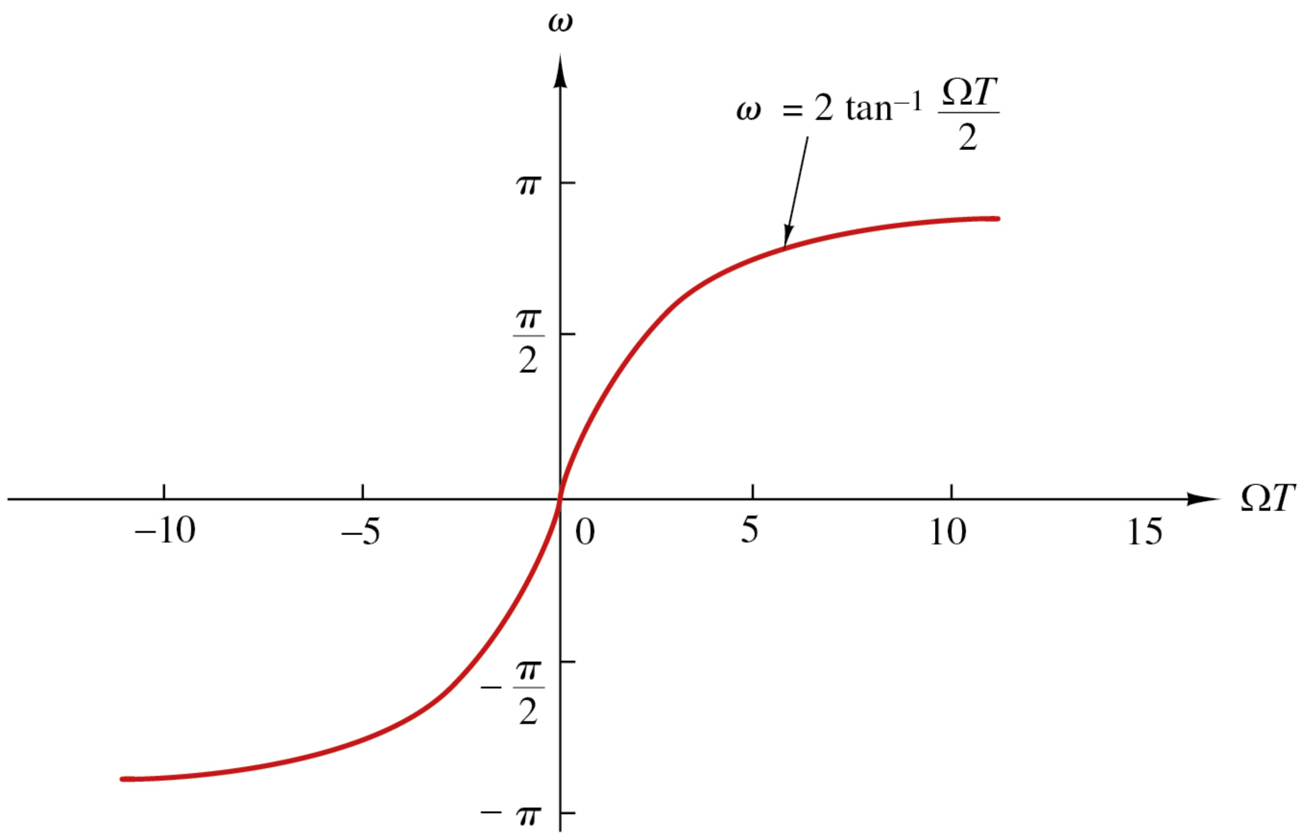
$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

very non-linear mapping of continuous to digital frequency

IIR Filters – Bilinear Transform

Frequency warping



IIR Filters – Bilinear Transform

- Method for designing digital filters using bilinear tranform
  - Establish requirements in the digital domain
    - e.g., cutoff frequency
  - Compute the equivalent analog frequency (prewar)

$$\Omega_c = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right)$$

  - Design an analog filter using these parameters

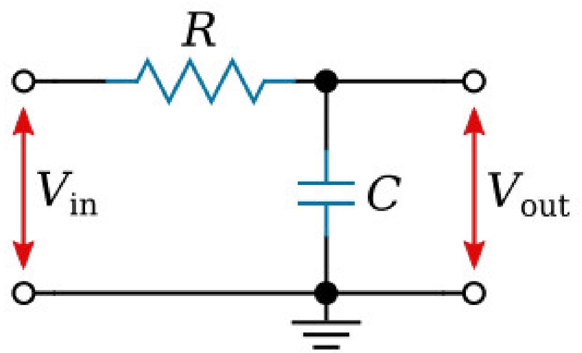
$$H(s)$$

  - Use bilinear transform to find digital filter

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

IIR Filters – Bilinear Transform

Example of 1’st order lowpass filter



$$H(s) = \frac{1/RC}{s + (1/RC)} = \frac{\Omega_c}{s + \Omega_c}$$

cut-off frequency: $f_c = 3000\text{Hz}$

$$\Omega_c = 2\pi f_c = 18850 \text{ rad/sec}$$

(On board)

Analog Filters

- Analog Filters
  - Four types of common analog filters
    - Butterworth
    - Chebyshev Type I
    - Chebyshev Type II
    - Elliptic

[Matlab filter functions]

Butterworth Filter

- Butterworth Filter
  - Magnitude is maximally flat at the origin and no ripples in either the passband or stopband
  - Magnitude changes monotonically with frequency
  - Compared to other types, has a slower roll-off
  - All pole filter
  - Frequency response of N'th order Butterworth filter

$$\left|H(\Omega)\right|^2 = \frac{1}{1+\left(\Omega/\Omega_c\right)^{2N}} = \frac{1}{1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}}$$

Ω_c is cut-off frequency

Ω_p is passband frequency

$1/\left(1+\varepsilon^2\right)$ is band-edge value frequency

Butterworth Filter

- Butterworth Filter

$$H(s)H(-s)\big|_{s=j\Omega} = \left|H(\Omega)\right|^2$$

$$s^2 = -\Omega^2$$

$$H(s)H(-s) = \frac{1}{1+\left(-s^2/\Omega_c^2\right)^N}$$

poles at

$$\left(-s^2/\Omega_c^2\right)^N = -1$$

$$-s^2/\Omega_c^2 = (-1)^{1/N} = e^{j(2k+1)\pi/N}, \quad k = 0,1,\ldots,N-1$$

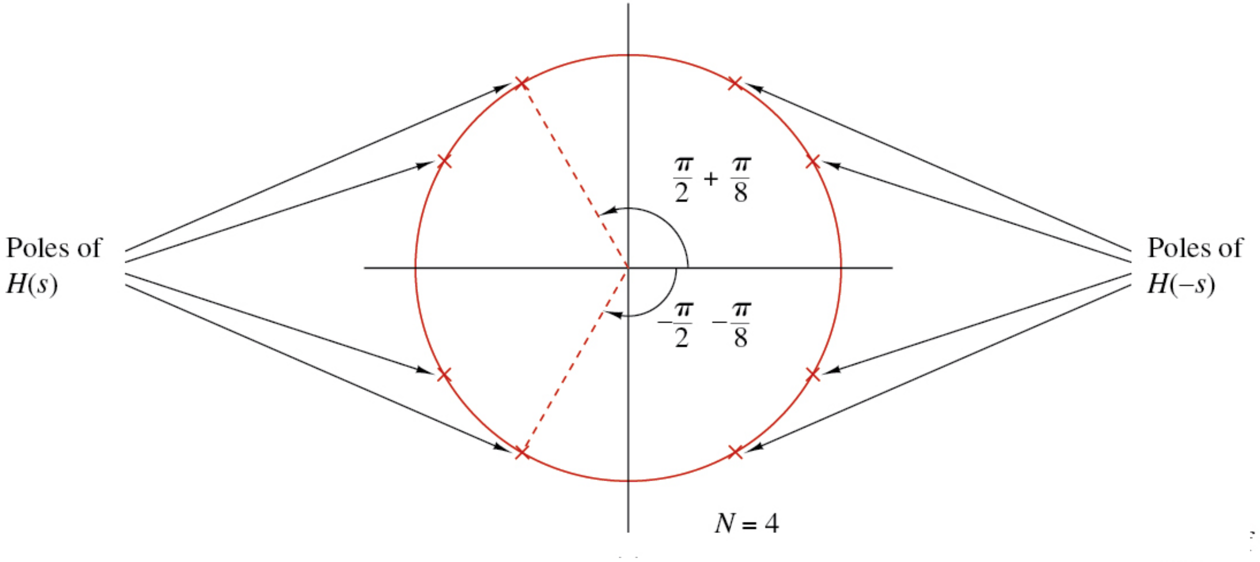
$$s^2 = -\Omega_c^2 e^{j(2k+1)\pi/N} = e^{-j\pi}\Omega_c^2 e^{j(2k+1)\pi/N}, \quad k = 0,1,\ldots,N-1$$

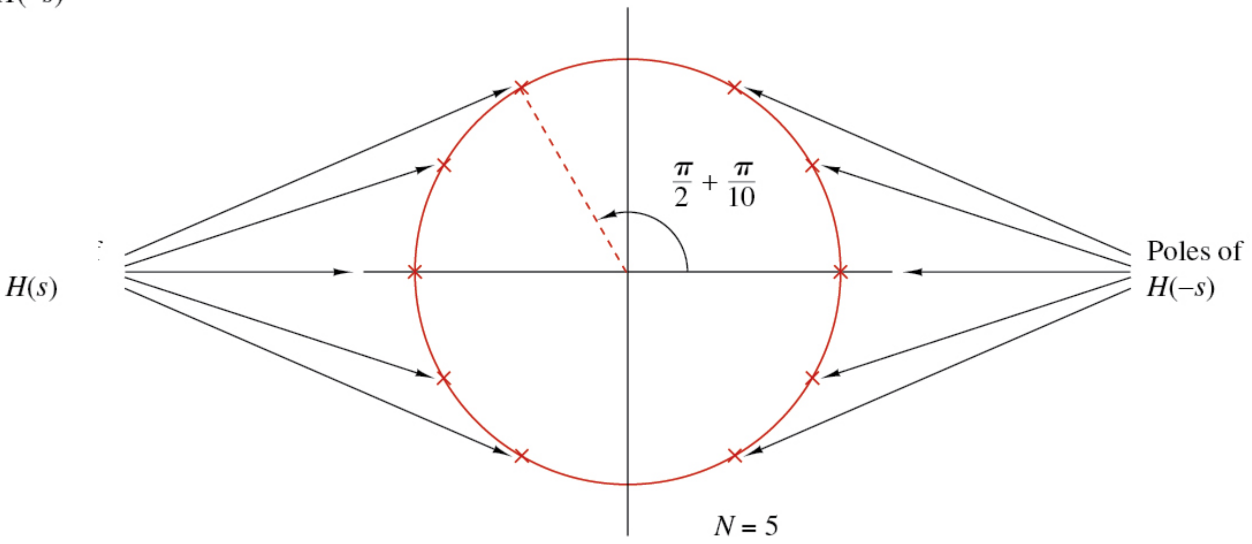
$$\boxed{s_k = e^{-j\pi/2}\Omega_c e^{j(2k+1)\pi/2N}, \quad k = 0,1,\ldots,N-1}$$

N equally spaced poles on circle of radius Ω_c

Butterworth Filter

- Butterworth Filter





(b)

Butterworth Filter

- How do you find what order filter you need?
  - Given passband specification:

$$\left|H\left(\Omega_p\right)\right|^2 \geq \frac{1}{1+\varepsilon^2} \quad \text{or passband ripple: } \mathrm{R}_p = -10\log_{10}\left(1+\varepsilon^2\right) \text{ dB}$$

  - Stopband specification:

$$\left|H\left(\Omega_s\right)\right|^2 \leq \frac{1}{1+A^2} \quad \text{or stopband attenuation is } R_s = -10\log_{10}\left(1+A^2\right)$$

Butterworth Filter

- Filter must satisfy:

$$\left|H\left(\Omega_p\right)\right|^2 = \frac{1}{1+\left(\Omega_p/\Omega_c\right)^{2N}} \geq \frac{1}{1+\varepsilon^2}$$

$$\left|H\left(\Omega_s\right)\right|^2 = \frac{1}{1+\left(\Omega_s/\Omega_c\right)^{2N}} \leq \frac{1}{1+A^2}$$

$$1+\left(\Omega_p/\Omega_c\right)^{2N} \leq 1+\varepsilon^2 \Rightarrow \left(\Omega_p/\Omega_c\right)^{2N} \leq \varepsilon^2$$

$$1+\left(\Omega_s/\Omega_c\right)^{2N} \geq 1+A^2 \Rightarrow \left(\Omega_s/\Omega_c\right)^{2N} \geq A^2$$

$$\frac{\left(\Omega_s/\Omega_c\right)^{2N}}{\left(\Omega_p/\Omega_c\right)^{2N}} \geq \frac{A^2}{\varepsilon^2} \Rightarrow \left(\frac{\Omega_s}{\Omega_p}\right)^{2N} \geq \frac{A^2}{\varepsilon^2}$$

$$2N\log\left(\frac{\Omega_s}{\Omega_p}\right) \geq \log\left(\frac{A^2}{\varepsilon^2}\right)$$

$$\boxed{N \geq \frac{1}{2}\frac{\log\left(\frac{A^2}{\varepsilon^2}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}}$$

Butterworth Filter

- Usually, you specify the stopband attenuation snd passband ripple in dB (where it is understood that they are negative)

$$\varepsilon^2 = 10^{\mathrm{R}_p/10} - 1$$

$$A^2 = 10^{\mathrm{R}_s/10} - 1$$

- You also specify the frequencies for the stopband and passband

$$\boxed{N \geq \frac{1}{2} \frac{\log\left(\frac{A^2}{\varepsilon^2}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}}$$

Butterworth Filter

- Issue of what are the parameters?

Ω_c : -3dB point (1/2 power)

Ω_p : passband frequency where parameter ε describes edgeband where

$$\left|H\left(\Omega_p\right)\right|^2 \geq \frac{1}{1+\varepsilon^2} \quad \text{or passband ripple: } R_p = \text{-}10\log_{10}\left(1+\varepsilon^2\right) \text{ dB}$$

Ω_s : stopband frequency where parameter A describes point where

$$\left|H\left(\Omega_s\right)\right|^2 \leq \frac{1}{1+A^2} \quad \text{or stopband attenuation is } R_s = \text{-}10\log_{10}\left(1+A^2\right)$$

- If you look at Matlab's filter designer or code generated by it

Butterworth Filter

```
function Hd = butterworth_example
%BUTTERWORTH_EXAMPLE Returns a discrete-time filter object.

% MATLAB Code
% Generated by MATLAB(R) 24.2 and Signal Processing Toolbox 24.2.
% Generated on: 21-Apr-2025 22:24:20

% Butterworth Lowpass filter designed using FDESIGN.LOWPASS.

% All frequency values are normalized to 1.

Fpass = 0.31830988618; % Passband Frequency
Fstop = 0.63661977237; % Stopband Frequency
Apass = 1; % Passband Ripple (dB)
Astop = 40; % Stopband Attenuation (dB)
match = 'stopband'; % Band to match exactly

% Construct an FDESIGN object and call its BUTTER method.
h = fdesign.lowpass(Fpass, Fstop, Apass, Astop);
Hd = design(h, 'butter', 'MatchExactly', match);
```

Notice it uses frequencies of stop and passband
And passband ripple and stopband attenuation.

This example uses normalized frequency,
Fpass = 1 rad/sec ➔ 1/pi
Fstop = 2 rad/sec ➔2/pi

Chebyshev Filters

- Chebyshev Filters
  - Two types
    - Type I: all pole filter that has equiripple in passband, monotonic in stopband
    - Type II: poles & zeros. Monotonic in passband, equiripple in stopband

Chebysev Type I:

$$\left|H(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\Omega/\Omega_p\right)}$$

Chebysev Type II:

$$\left|H(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2\left[\dfrac{T_N^2\left(\Omega_s/\Omega_p\right)}{T_N^2\left(\Omega_s/\Omega\right)}\right]}$$

Ω_p is passband frequency

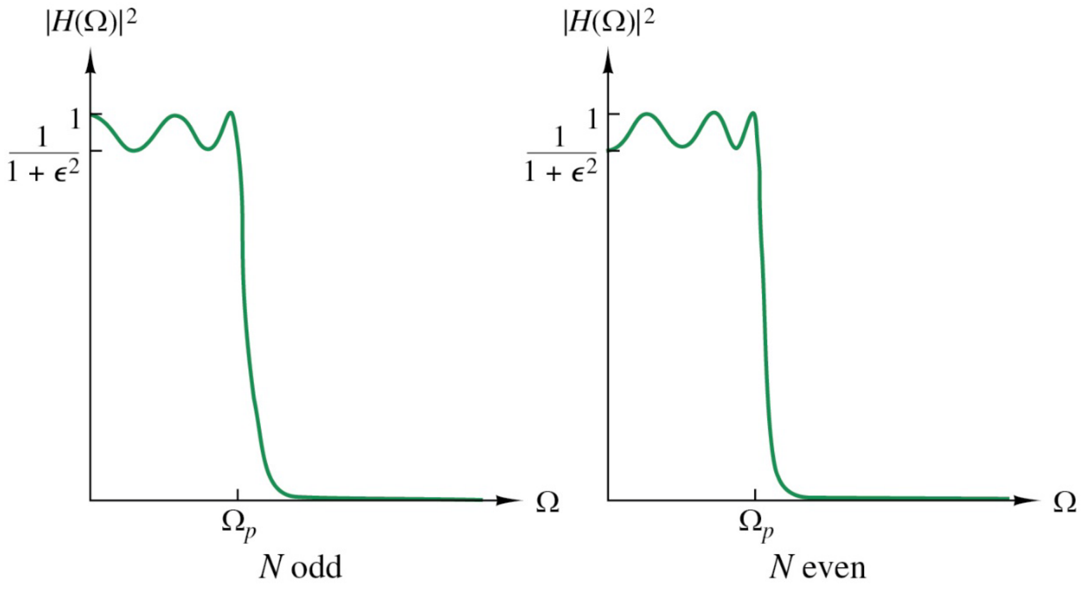
Ω_s is stopband frequency

ε is the ripple factor

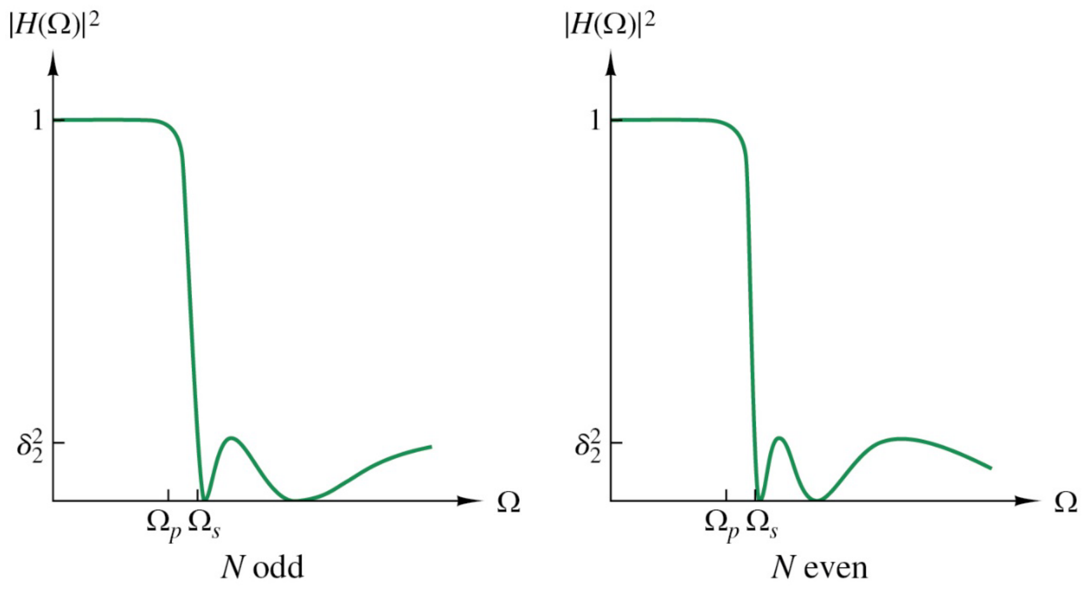
T_N is a Chebyshev polynomial

Chebyshev Filters

- Chebyshev Filters



Type 1



Type 2

Elliptic Filters

- Elliptic Filters
  - Equiripple in pass and stop bands
  - Smallest order filter for given set of specifications
  - Smallest transition band
  - Phase is more nonlinear in passband than Butterworth and Chebyshev filters

$$\left|H(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 U_N\left(\Omega/\Omega_p\right)}$$

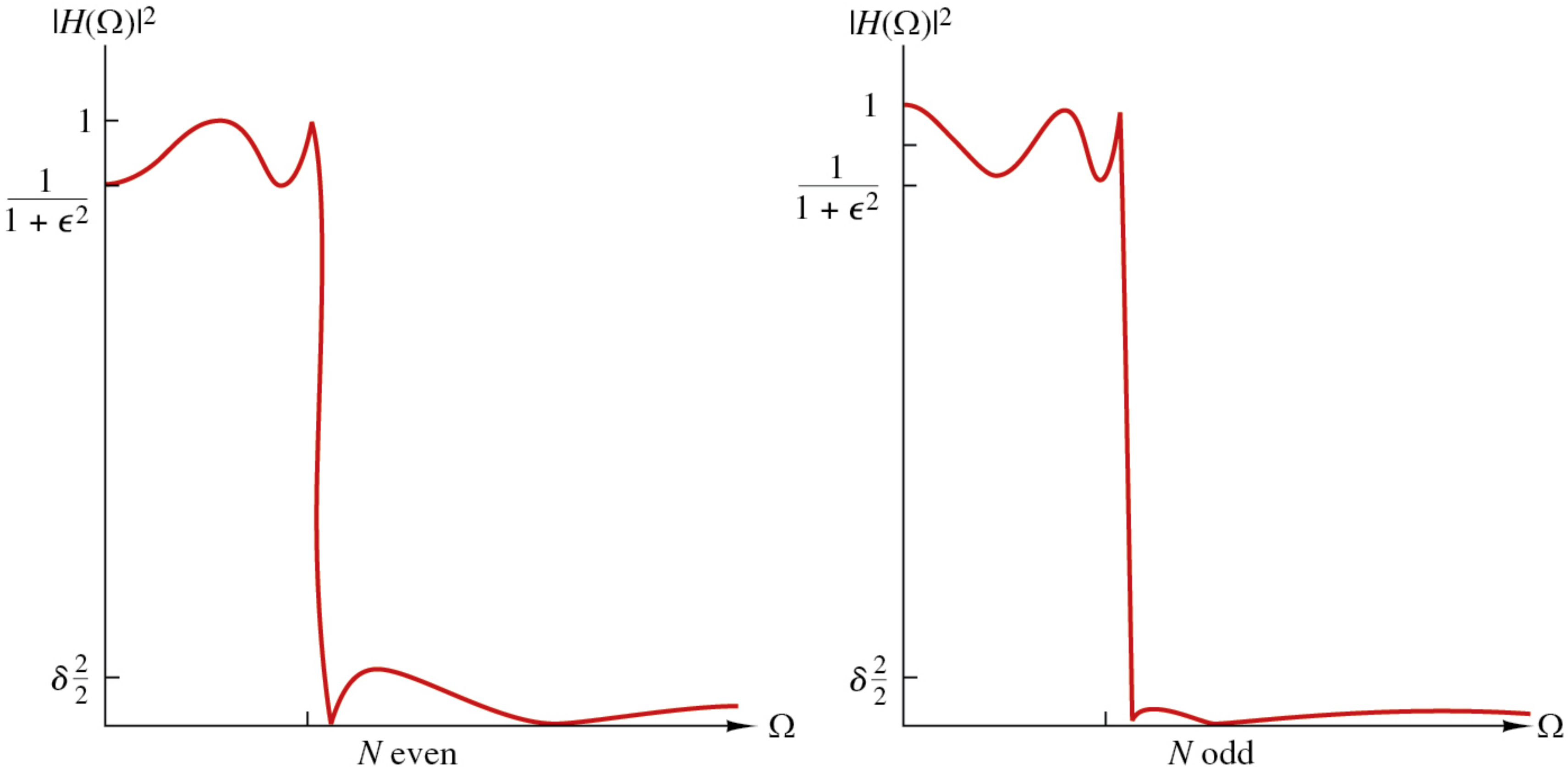
Ω_p is passband frequency

ε is the ripple factor

U_N is a Jacobian elliptic function of order N

Elliptic Filters

- Elliptic Filters



Bessel Filters

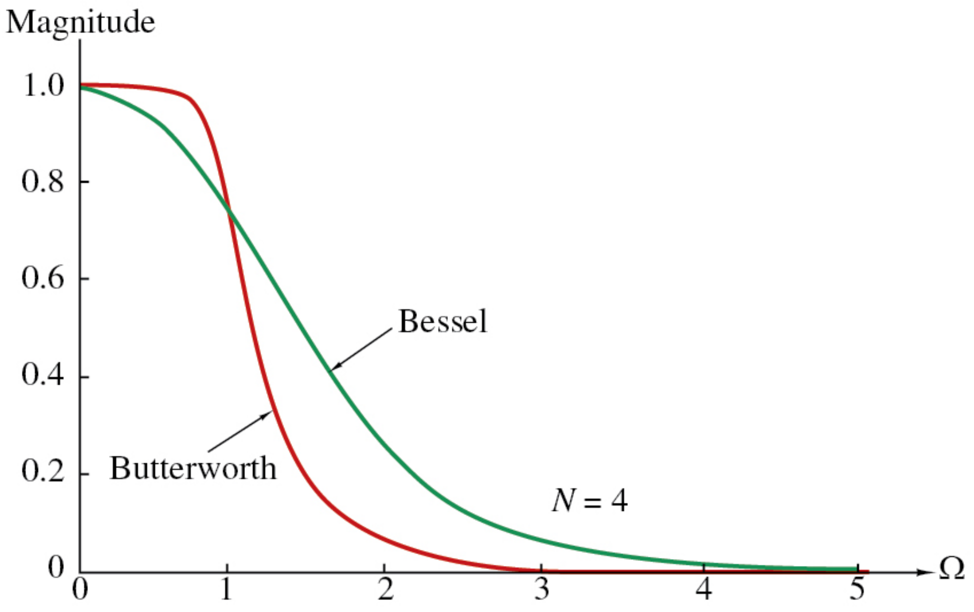
- Bessel Filters
  - All pole filters
  - Linear phase over passband
    - But when you transform it to digital filter, you lose that feature

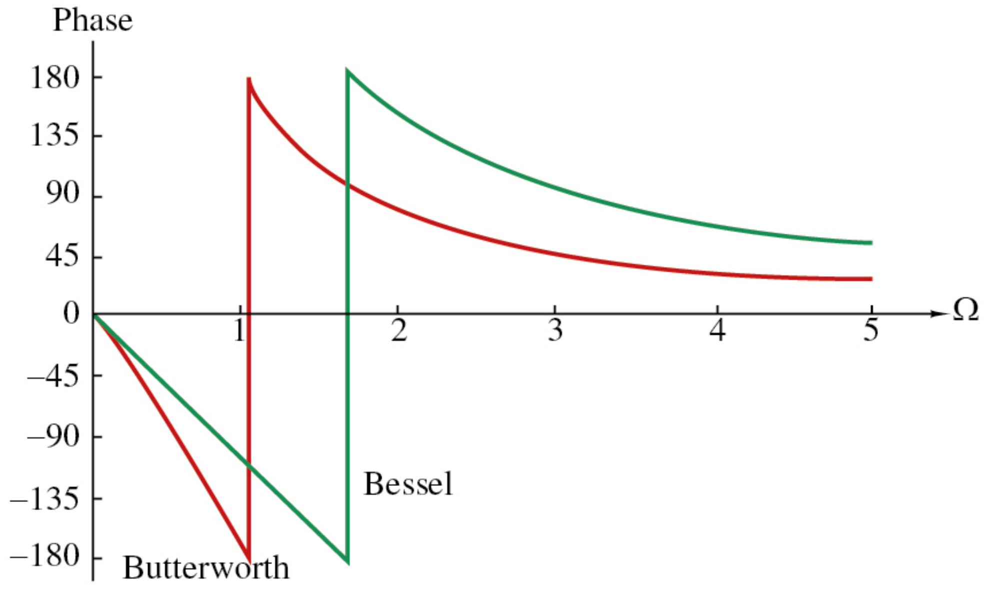
$$H(s) = \frac{1}{B_N(s)}$$

$B_N(s)$ is N'th order Bessel function

Bessel Filters

- Bessel Filters





Summary of Analog Filters

Analog Filter	Passband	Stopband	Transition Band	Specification
Butterworth	Monotonic	Monotonic	Broad	Pass/Stop band
Chebyshev-I	Equiripple	Monotonic	Narrow	Passband
Chebyshev-II	Monotonic	Equiripple	Narrow	Stopband
Elliptic	Equiripple	Equiripple	Very Narrow	Passband