ENGR 071
Digital Signal Processing

Class 02

01/23/2025

- Class Overview
  - Overview of Signals and Systems
    - Continuous Signals & Systems
    - Point out similarities for Discrete Time Signals

Assignment 1

Assignment 1: Review of Complex Variables

Due Sunday, Jan. 26

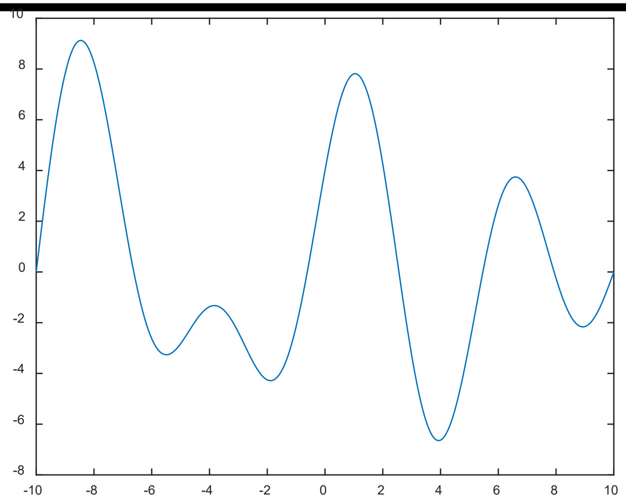
SIGNALS AND SYSTEMS

Signals

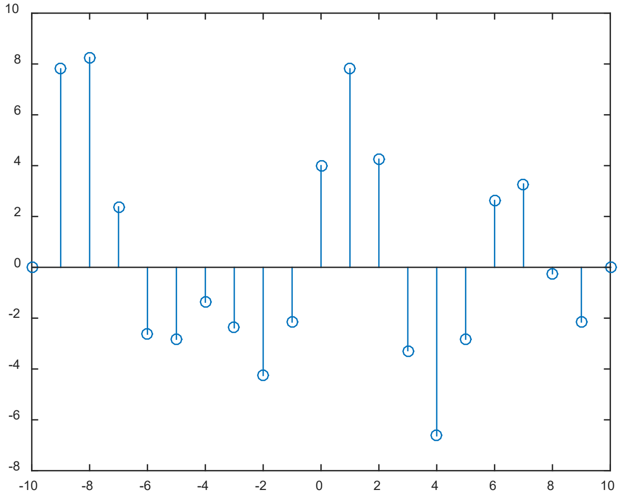
- Classification of signals
  - **Continuous** signals
    - Continuous values for amplitude and time
      - » $v(t)$, $t_{start} < t < t_{end}$, $v(t)$ is a real-valued function of a continuous variable, t.
  - **Discrete** signals
    - Continuous values for amplitude, discrete values for time
      - » Often sampled at fixed time interval, T_s
        - » $v(nT_s)$, $n_{\min} \le n \le n_{\max}$
        - » Discrete times associated with sample are nT_s , $n = \cdots, -2, -1, 0, 1, 2, \cdots$
        - » Do not need to explicitly denote sampling time, T_s
        - » Signal can be thought of as a sequence of number: $v(n)$ $\cdots, v(-2), v(-1), v(0), v(1), v(2), \cdots$
  - **Digital** signals
    - Quantized amplitudes, discrete value for time
      - » $v_q(nT_s)$, $v_q \in \{\text{fixed set of values}\}$, $n_{\min} \le n \le n_{\max}$

Signals

– Continuous: (matlab `plot`)



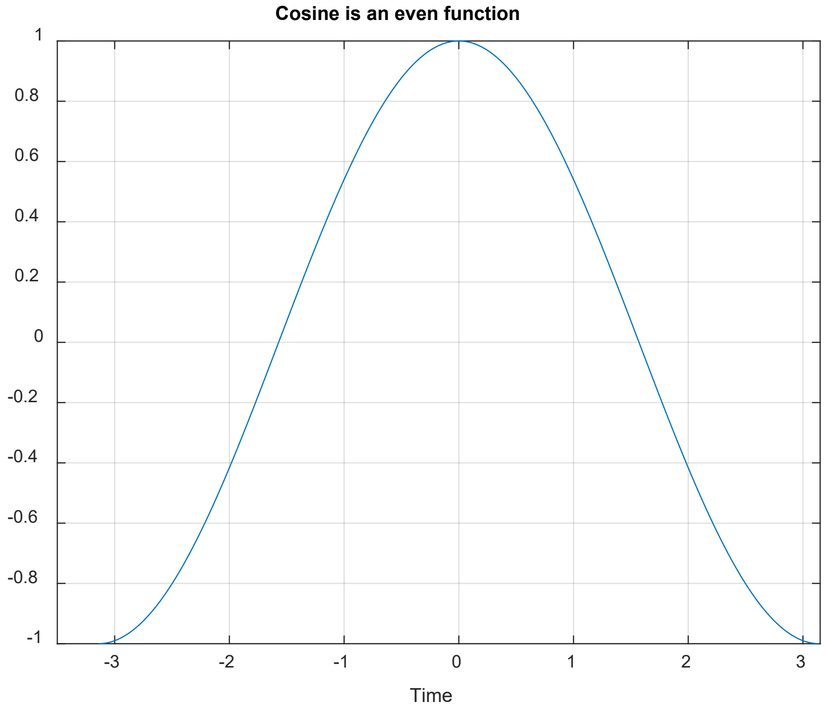
– Discrete (Matlab `stem`)

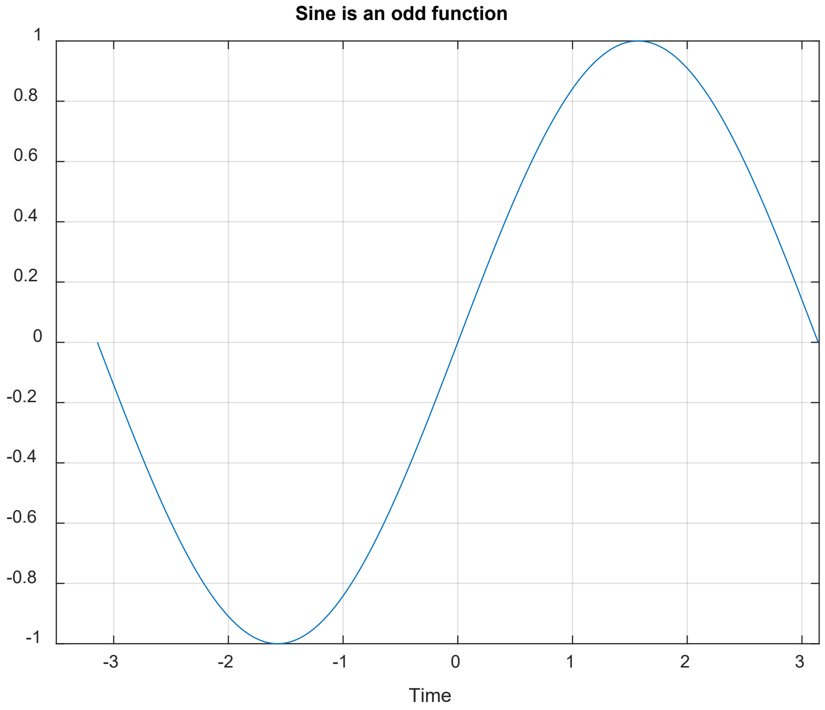


Signals

- Other attributes for signals:
  - Support: range of times for which signal is non-zero.
    - Finite support: $t_{\min} \le t \le t_{\max}$ $\left[n_{\min} \le n \le n_{\max}\right]$
    - Infinite duration: $-\infty \le t \le \infty$ $\left[-\infty \le n \le \infty\right]$
    - Semi-infinite: $0 \le t \le \infty$ $\left[0 \le n \le \infty\right]$
  - Deterministic or random
    - Deterministic would be something like voice
    - Random would be something like noise
  - Even or Odd
    - Even: $v(t) = v(-\mathrm{t})$ e.g. cosine $\left[v(n) = v(-n) \quad \text{e.g. cosine}\right]$
    - Odd: $v(t) = -v(-\mathrm{t})$ e.g. sine $\left[v(n) = -v(-n) \quad \text{e.g. sine}\right]$

Signals


Cosine is an even function
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-3
-2
-1
0
1
2
3
Time


Sine is an odd function
1
0.8
0.6
0.4
0.2
0
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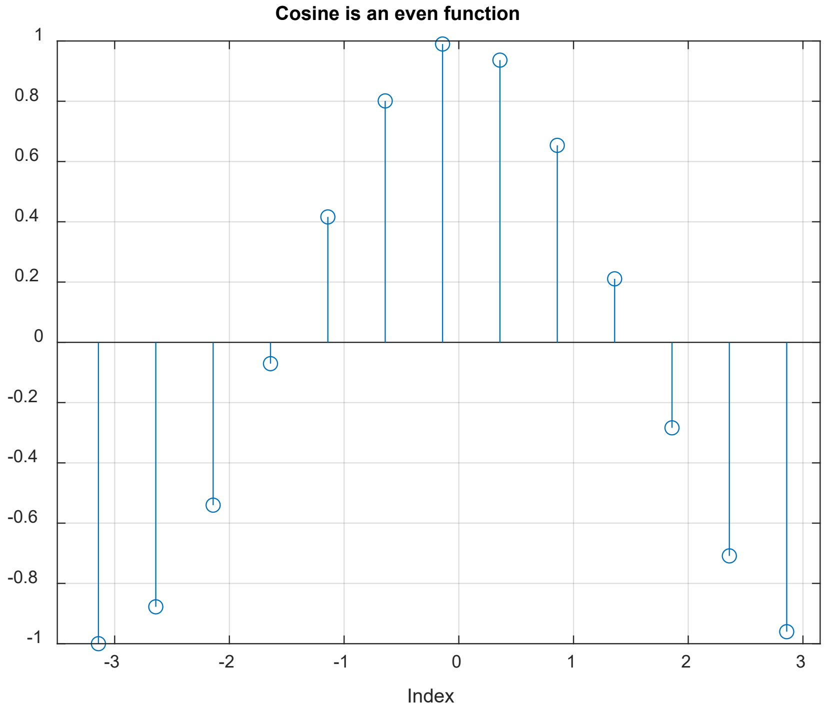
-0.8
-1
-3
-2
-1
0
1
2
3
Time

Signals



Cosine is an even function
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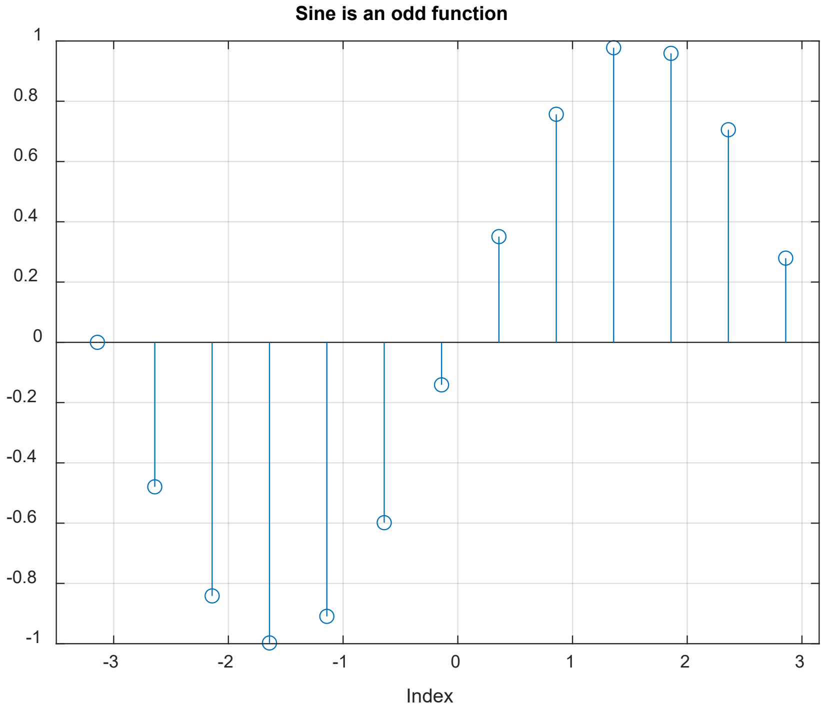
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-3
-2
-1
0
1
2
3
Index



Sine is an odd function
1
0.8
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0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-3
-2
-1
0
1
2
3
Index

Signals

- Even or Odd
  - You can represent a signal by its even and odd parts

$$v(t) = v_{even}(t) + v_{odd}(t) \quad \text{where}$$

$$v_{even}(t) = \frac{1}{2}\big(v(t) + v(-t)\big)$$

$$v_{odd}(t) = \frac{1}{2}\big(v(t) - v(-t)\big)$$

  - Note that:

$$v_{even}(-t) = \frac{1}{2}\big(v(-t) + v(+t)\big) = \frac{1}{2}\big(v(t) + v(-t)\big) = v_{even}(t)$$

$$v_{odd}(-t) = \frac{1}{2}\big(v(-t) - v(+t)\big) = -\frac{1}{2}\big(v(t) - v(-t)\big) = -v_{odd}(t)$$

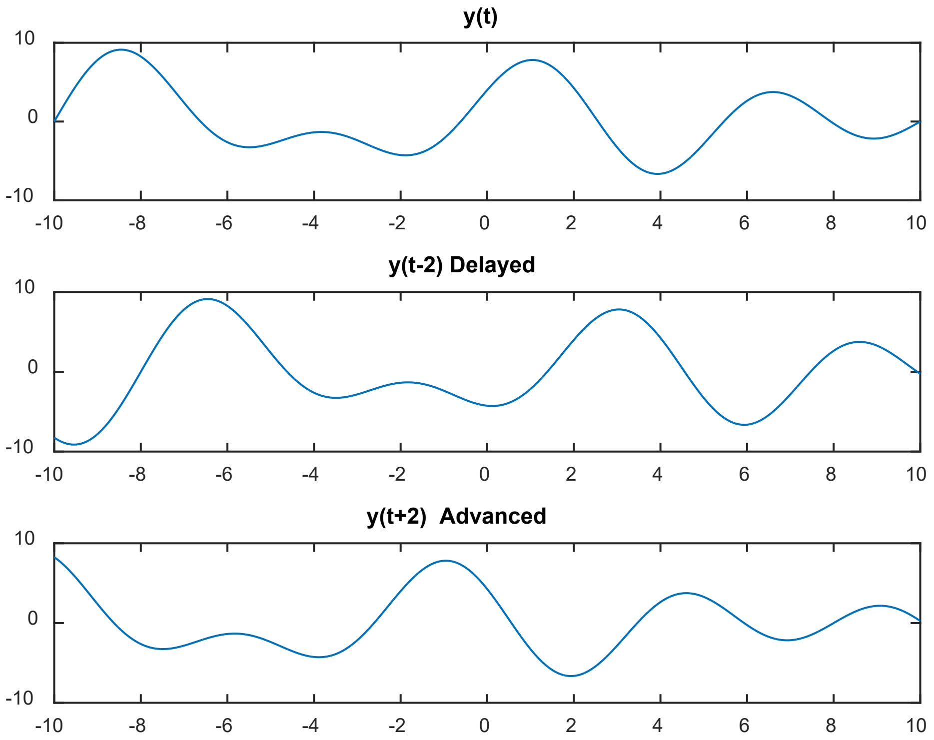
(It is understood that there are similar expressions for discrete signals, but I'll stop showing them.)

Signals

- Basic Signal Operations
  - Add signals: $w(t) = u(t) + v(t)$
  - Scale signal (multiply by a constant) $\alpha v(t)$
  - Time shift:
    - Delay by τ: $v(t - \tau)$
    - Advance by τ: $v(t + \tau)$
  - Time scaling: $v(\alpha t)$
    - If α = -1, then you reflect time axis (reverse time)
  - Windowing: multiply by some function *w(t)* that has finite support. $w(t)v(t)$
  - Integrate signal: $\int_0^t v(\tau)\,d\tau$

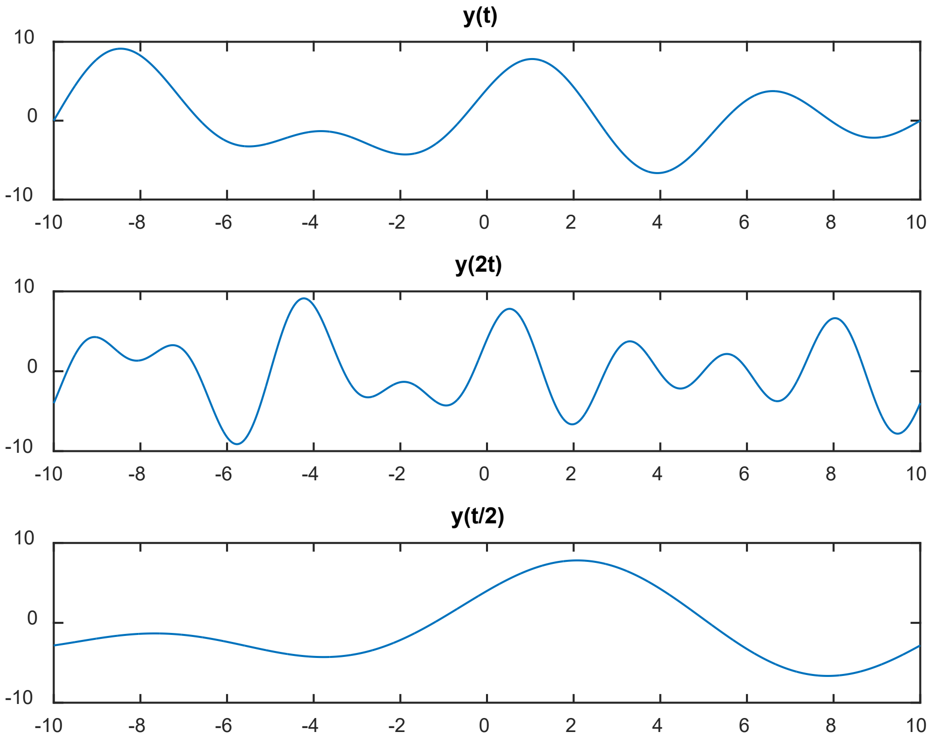
Signals

– Time shifting



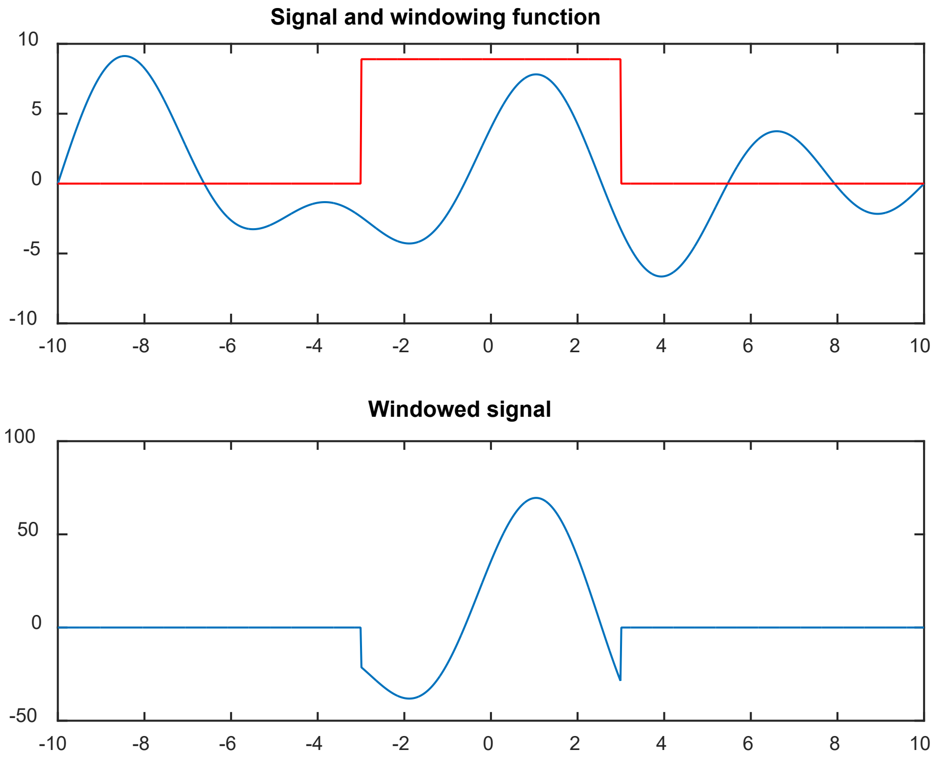
Signals

– Time scaling



Signals

– Windowing



Signals

- Other attributes for signals:
  - Periodic or aperiodic.
    - Periodic: Signal repeats after some time interval

$$v(t) = v(t + kT_P)\ k = 1,2,\ldots$$

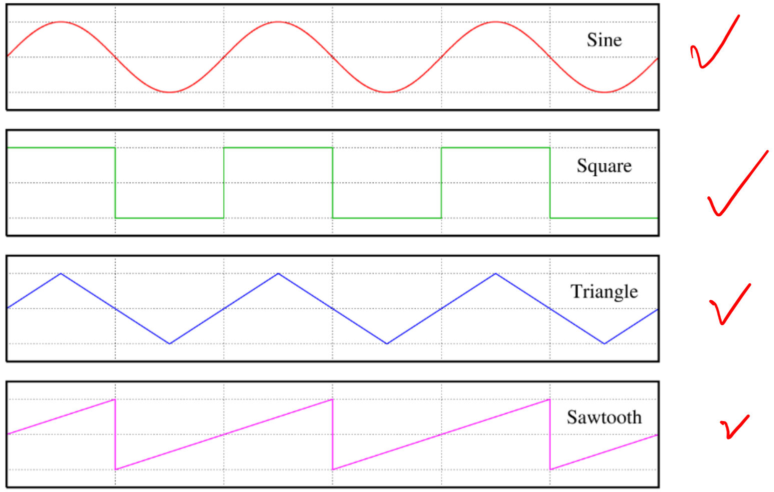
      - » Trigonometric or train of pulses would be examples
    - Aperiodic: Signal doesn't repeat
      - » A voice signal would be an example.

  - Casual: $v(t) = 0\ ,\ \ t < 0$

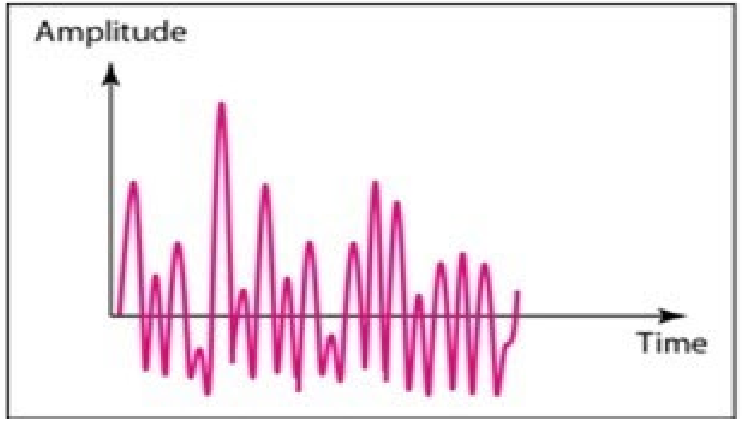
  (Idea of a casual signal is that it is a signal that can be the impulse response of a casual system.)

Signals

Periodic



Aperiodic



Signals

- Other attributes for signals:
  - Finite-energy called "Energy Signals"
    - Also called square integrable $$E = \int_{-\infty}^{+\infty} |v(t)|^2 dt < \infty$$
    - Examples:

      $v(t) = e^{-3t} \quad 0 < t < +\infty$ Finite energy

      $v(t) = e^{0.01t} \quad 0 < t < +\infty$ Not finite energy

    - Can also have absolutely integrable signals

$$\int_{-\infty}^{+\infty} |v(t)| dt < \infty$$

For discrete signals, integrals are sums

$v(t) = t^{-1} \quad 1 < t < +\infty$ Square integable? Absolutely integrable?

See example on Moodle

Square integrable, but not absolutely integrable

Signals

- Other attributes for signals:
  - Finite-power (Called "Power Signals")
    - Time-averaged energy

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{+T} |v(t)|^2 dt < \infty$$

Finite energy signals
have zero power

    - For a periodic signal, average energy over one period

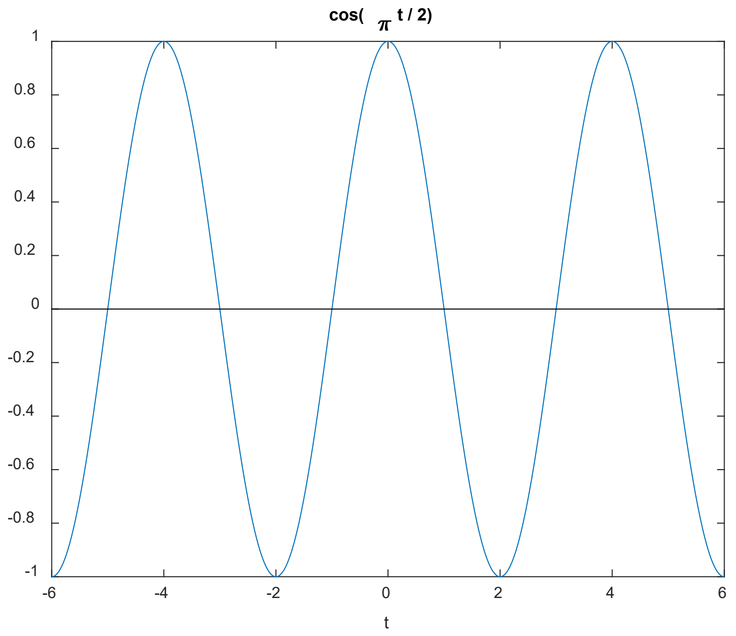
$$P = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |v(t)|^2 \, dt$$

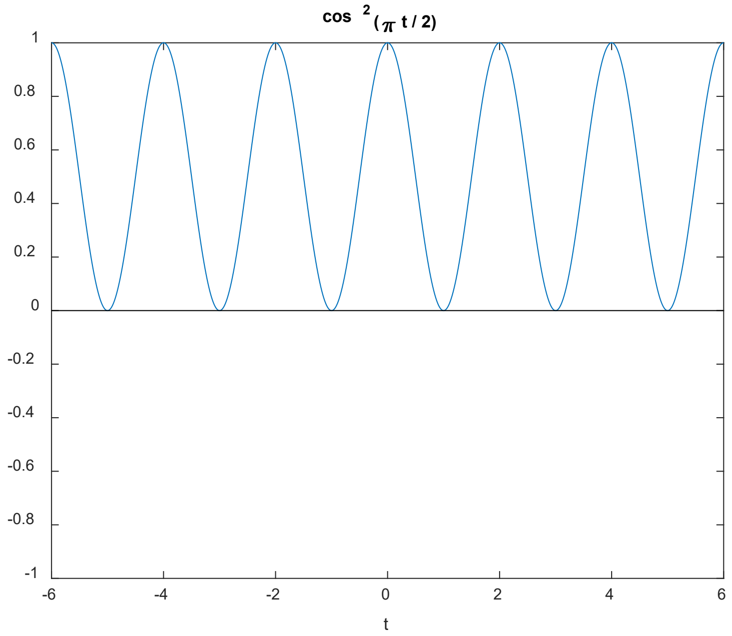
Periodic signals
Can have non-zero power

Example: $x(t) = \cos\left(\pi t / 2\right), -\infty < t < \infty \quad ; \quad E \to \infty \quad ; \quad P = 1/2$

See example on Moodle

Signals





$$E = \int_{-\infty}^{+\infty} \cos^2\left(\frac{\pi t}{2}\right) dt \to \infty$$

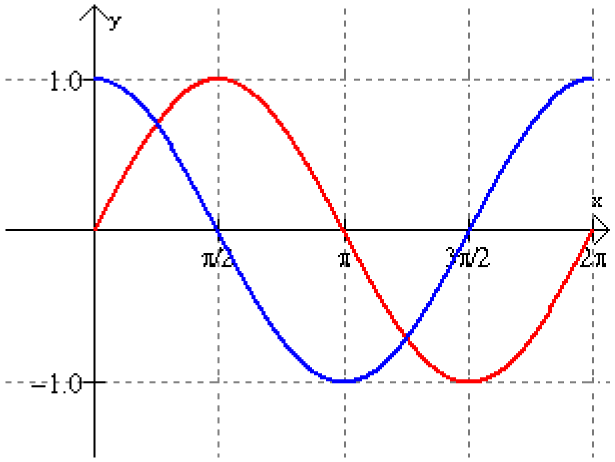
$$P = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \left|\cos\left(\frac{\pi t}{2}\right)\right|^2 dt = \frac{1}{4}\int_{-2}^{2} \cos^2\left(\frac{\pi t}{2}\right) dt = \frac{1}{2}$$

Signals

– "Special Signals

- Sinusoids

$$v(t) = A\cos(\omega t + \theta)$$



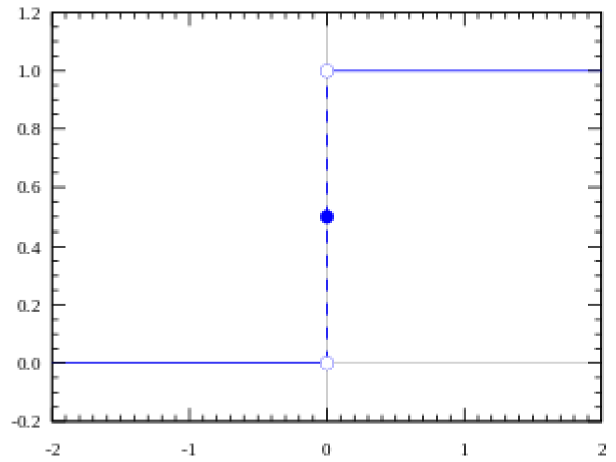
- Complex Exponential:

$$v(t) = Ce^{(r+j\omega)t} = |C|e^{j\theta}e^{rt}e^{j\omega t} = |C|e^{rt}\left[\cos(\omega t + \theta) + j\sin(\omega t + \theta)\right]$$

Signals

- "Special Signals
  - Heaviside Step function

$$H(t) = 0;\ \ 0 < t$$
$$H(t) = 1;\ \ t > 0$$



  - Also denoted as
    $u(t)$ or $\gamma(t)$ or $\theta(t)$

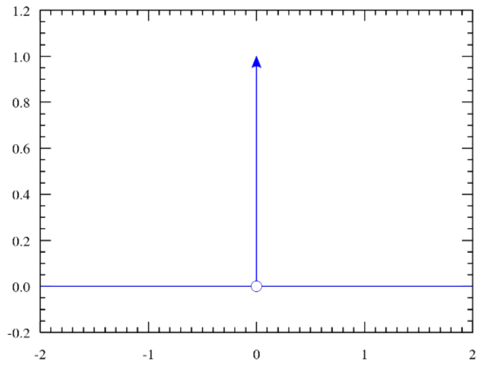
Signals

- "Special Signals"
  - Impulse function (Dirac δ-function)
    - Not really a function (a functional)

$$\delta(t) = 0, \quad t \neq 0$$

$$\delta(t) = \infty, \quad t = 0$$

Area under δ function is 1



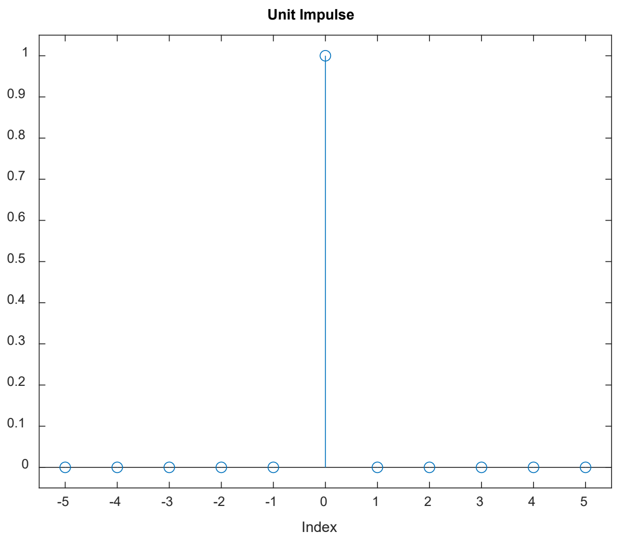
    - Actually defined by

$$f(0) = \int_{-\infty}^{+\infty} f(t)\delta(t)\,dt$$

Signals

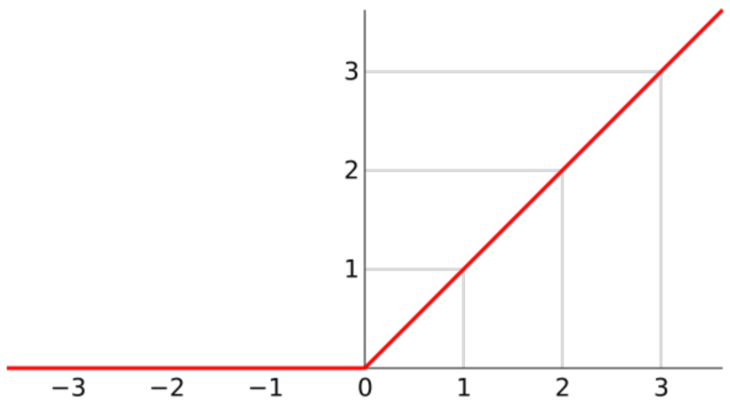
- This one is different (and simpler) for discrete signals
- Impulse function (or unit sample function)

$$\delta(n) = 0, \quad n \neq 0$$
$$\delta(n) = 1, \quad t = 0$$



Signals

– “Special Signals
  - Ramp



$$r(t) = 0, t < 0$$

$$r(t) = t, t \geq 0$$

or

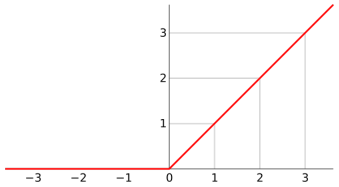
$$r(t) = tH(t)$$

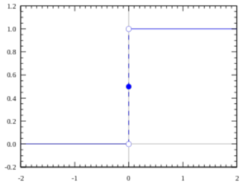
Signals

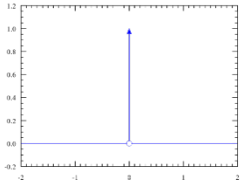
– "Special Signals
  - Relationships between step, impulse, & ramp

$$\frac{dr(t)}{dt} = H(t)$$

$$\frac{d^2 r(t)}{dt^2} = \frac{dH(t)}{dt} = \delta(t)$$



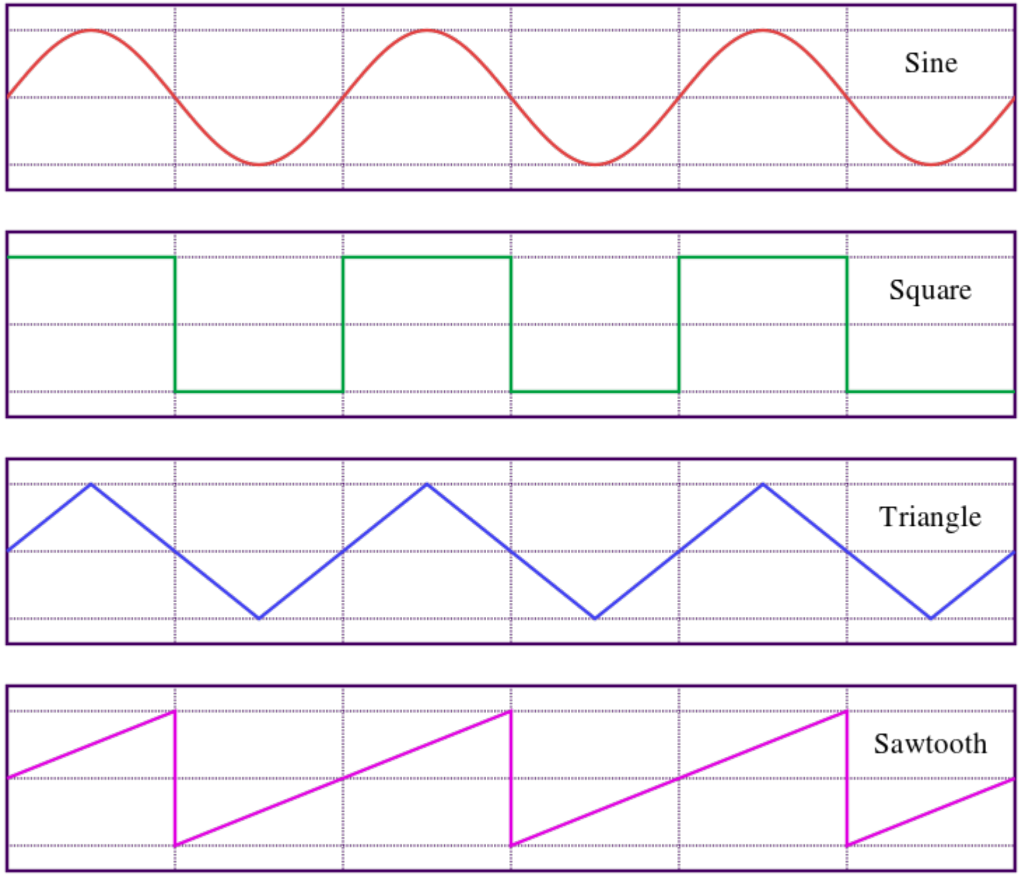




Derivatives do not exist for discrete signals,
but you can get similar relationships by subtracting
shifted ramps or step functions

Signals

– Other common (periodic) signals

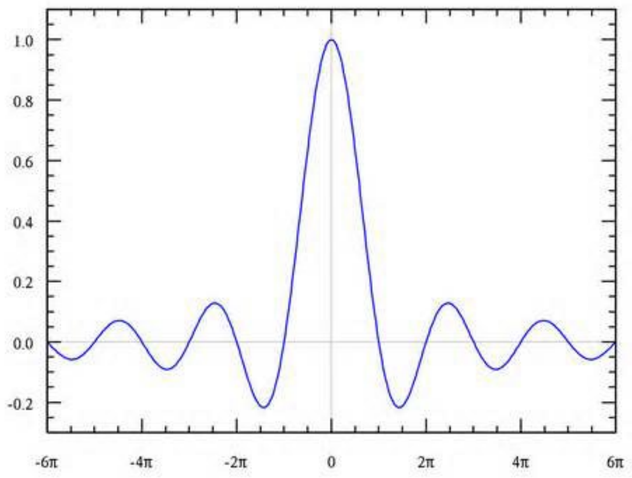


Signals

– “Special Signals

- Sinc function

$$S(t) = \frac{\sin(\pi t)}{\pi t}$$
$$S(0) = 1$$
$$S(k) \; = \; 0, k \text{ integer}$$



This is the normalized sinc function

$$\int_{-\infty}^{\infty} S^2(t)dt = 1$$

Matlab & NumPy use this form.

Also, an unnormalized form:

$$S(t) = \frac{\sin(t)}{t}$$

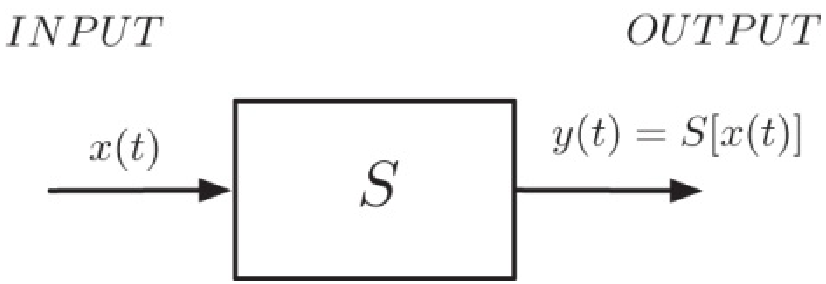
Mathematica and WolframAlpha use this form

Table 1.1 Basic Signals

Signal	Definition/Properties
Damped complex exponential	$\lvert A\rvert e^{t}[\cos(\Omega_0 t+\theta)+j\sin(\Omega_0 t+\theta)]\ -\infty<t<\infty$
Sinusoid	$A\cos(\Omega_0 t+\theta)=A\sin(\Omega_0 t+\theta+\pi/2)\quad -\infty<t<\infty$
Unit-impulse	$\delta(t)=0\ t\neq 0$, undefined at $t=0$, $\int_{-\infty}^{t}\delta(\tau)d\tau=1,\ t>0$, $\int_{-\infty}^{\infty}f(\tau)\delta(t-\tau)d\tau=f(t)$
Unit-step	$u(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$
Ramp	$r(t)=tu(t)=\begin{cases}t & t>0\\ 0 & t\leq 0\end{cases}$ $\delta(t)=du(t)/dt$ $u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau$ $r(t)=\int_{-\infty}^{t}u(\tau)d\tau$
Rectangular pulse	$p(t)=A[u(t)-u(t-1)]=\begin{cases}A & 0\leq t\leq 1\\ 0 & \text{otherwise}\end{cases}$
Triangular pulse	$\Lambda(t)=A[r(t)-2r(t-1)+r(t-2)]=\begin{cases}At & 0\leq t\leq 1\\ A(2-t) & 1<t\leq 2\\ 0 & \text{otherwise}\end{cases}$
Sampling signal	$\delta_{T_s}(t)=\sum_k\delta(t-kT_s)$
Sinc	$S(t)=\sin(\pi t)/(\pi t)$ $S(0)=1$ $S(k)=0, k$ integer $\neq 0$ $\int_{-\infty}^{\infty}S^2(t)dt=1$

Systems

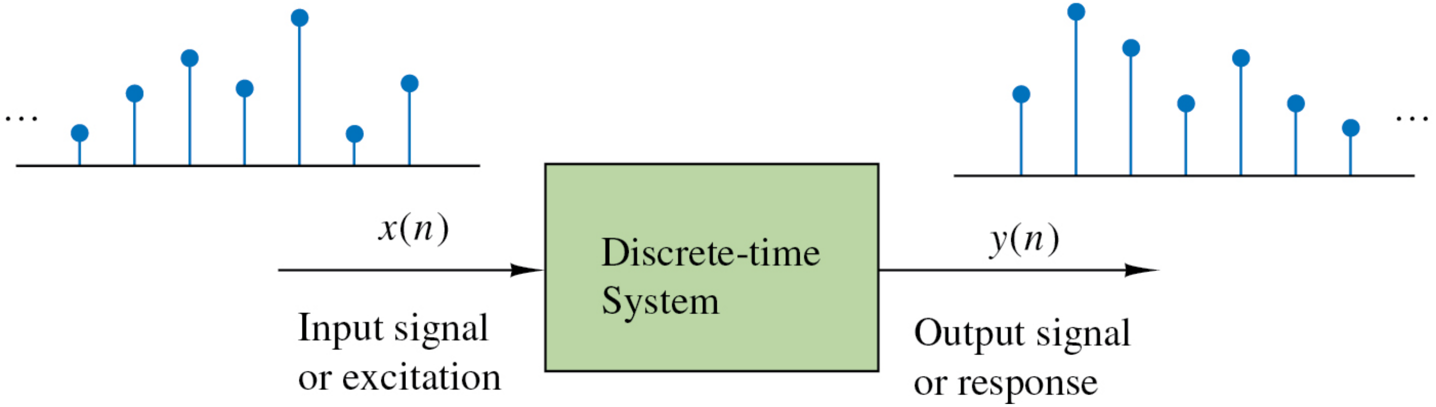
– System
  - Transforms input signal to output signal
  - Illustrated by "black box"



– The system can be thought of as a mathematical transformation mapping the input, *x(t)* to the output, *y(t)*.

Systems

- Discrete system is essential the same
  - Transforms input signal to output signal
  - Illustrated by "black box"



- The system can be thought of as a mathematical transformation mapping the input, *x(n)* to the output, *y(n)*.

Classification of Systems

- High level classification of systems
  - **Lumped or Distributed**
    - » Lumped means elements of systems are localized and you only need to consider the evolution of components in time.
      - » Example would be circuit with discrete elements (like R, L, C)
    - » Distributed means system is distributed over space, like transmission lines
    - » Lumped systems can be described with ordinary differential equations
    - » Distributed systems are described with partial differential equations.
  - **Passive or Active systems**
    - » Passive systems can not deliver energy outside the system
      - » Example R-L-C circuits
    - » Active systems can deliver energy outside the system
      - » Example: Op Amp circuits

Classification of Systems

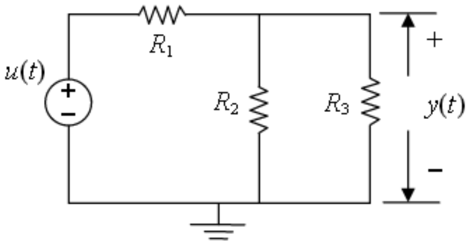
- More classifications of systems
  - **Continuous-time**
    - » Input and output signals are continuous time functions
  - **Discrete-time**
    - » Input and output signals consist of sampled times
  - **Digital**
    - » Inputs and outputs are discrete in time and amplitudes are quantized
  - **Hybrid**
    - » Input and output signals can be mixed
    - » Example Analog to Digital (A/D) converter

Classification of Systems

– **Static or Dynamic** (Also called **memoryless** or with **memory**)

– Static system only depends on the input at the present time

» Example: resistive circuit excited by input voltage



$$y(t) = \frac{R_2 R_3}{R_1\left(R_2 + R_3\right) + R_2 R_3} u(t)$$

– Dynamic system depends not only on the input at the current time, but also on the input at previous times.

» Example would be circuits with capacitors and inductors

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

» Another example would be a combination lock.
(i.e., needs to know two previous inputs plus present input to unlock.)

Classification of Systems

- **Causal Systems**
  - If output $y(t)$ at time t_0 only depends on input $x(t)$ for $t \le t_0$, system is causal.
  - In other words, output can only depends on past and current input.

Classification of Systems

- **Linear Systems**
  - If you scale the input to the system, the output scales by the same factor.
  - If you add to inputs and let the system operate on the inputs, the output is like you gave each input separately and sum the individual responses.
  - Mathematically:

$$S[\alpha x(t) + \beta y(t)] = \alpha S[x(t)] + \beta S[y(t)]$$

  - If you superimpose two signals, output is superposition of two outputs.
    - » Principle of superposition

Classification of Systems

- **Time Invariant Systems**
  - Parameters of system do not change with time.
  - If you shift input time, output is shifted in same way
  - If input *x(t)* produces output *y(t)*, then input at $x(t\text{-}t_0)$ produces output at $y(t\text{-}t_0)$
  - Examples
    - » Capacitor is time invariant since:

$$v(t) = \int_{-\infty}^{t} i(\tau)d\tau$$

If you consider input shifted by time t_0

$$v_{t_o}(t) = \int_{-\infty}^{t} i(\tau - t_0)d\tau = \int_{-\infty}^{t-t_0} i(\tau)d\tau = v(t - t_0)$$

  - Example that is not time invariant: $y(t) = x(t) + \sin\omega t$

$$y(t) = S[x(t)] = x(t) + \sin\omega t$$

$$S[x(t\text{-}t_0)] = x(t - t_0) + \sin\omega t$$

$$y(t - t_0) = x(t - t_0) + \sin\omega(t - t_0)$$

$$\therefore S[x(t\text{-}t_0)] \neq y(t - t_0)$$

LTI Systems

- **Linear Time Invariant Systems**
  - Important class of systems
  - Can be represented by ordinary linear differential equation with constant coefficients.
    - Not all Linear D.E.'s with constant coefficients correspond to LTI systems
      - Must be causal and initially quiescent
  - What is so special about LTI systems?
    - LTI systems can be completely characterized by impulse response

LTI Systems

Review of Linear Differential Equations

- Homogeneous, linear, constant coefficient:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_1\frac{dy}{dt} + a_0 y = 0$$

- Non-homogeneous, linear, constant coefficient:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_1\frac{dy}{dt} + a_0 y = x(t)$$

- Most general form:

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_1\frac{dy}{dt} + a_0 y$$

$$= b_m\frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1} x}{dt^{m-1}} + b_{m-2}\frac{d^{m-2} x}{dt^{m-2}} + \cdots + b_1\frac{dx}{dt} + b_0 x$$

LTI Systems

- Approaches to finding total solution:
  - Natural and forced response
    - Homogeneous and particular solutions
      - Need to solve for both homogeneous and particular solutions before using initial conditions.
  - Zero-input response and Zero-state response
    - Zero-input response:
      - Result due exclusively to the initial conditions since the input is zero
      - Set input side of equation to zero, but use initial conditions to find constants
    - Zero-state response:
      - Result due exclusively to the input since the initial conditions are zero
      - Find total solution, but with the initial conditions set to zero

LTI Systems

- Solving linear differential equations
  - Two ways to get total solution
    1. Homogeneous Solution + Particular Solution
       (Natural response + Forced response)

    2. Zero Input Solution + Zero State Solution
       (Response due just to initial conditions
       + Response due just to the input)

LTI Systems

- Zero Input / Zero State
  versus
  Homogeneous/Particular

  - Conceptually, the ZI/ZS approach has the advantage of a specific physical interpretation:
    - Zero input is the part of the response due to initial conditions alone (input to system is 0)
    - Zero state is the part of the response due to the system input alone (no initial conditions)
  - Advantage of Homogeneous/Particular approach is that it is simpler
    - You have to use this approach to find the zero state solution

(However, we will see that you can use convolution to find zero state solution directly)

LTI Systems

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = x(t)$$

Method 1:

Solve homogeneous equation:

$$\frac{d^n y_h}{dt^n} + a_{n-1}\frac{d^{n-1} y_h}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y_h}{dt^{n-2}} + \cdots + a_1 \frac{dy_h}{dt} + a_0 y_h = 0$$

(Solution has *n* unknown constants that are determined from initial conditions)

Solve non-homogeneous equation:

$$\frac{d^n y_p}{dt^n} + a_{n-1}\frac{d^{n-1} y_p}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y_p}{dt^{n-2}} + \cdots + a_1 \frac{dy_p}{dt} + a_0 y_p = x(t)$$

(Solution has no unknown constants)

LTI Systems

Method 1 (continued)

Combine two solutions:

$$y(t) = y_h(t) + y_p(t)$$

Apply initial conditions to *total* solution to solve for *n* constants

$$y(0), y'(0), y''(0), \cdots y^{(n-2)}(0), y^{(n-1)}(0)$$

LTI Systems

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_1\frac{dy}{dt} + a_0 y = x(t)$$

Method 2:

Solve zero-input equation:

$$\frac{d^n y_{zi}}{dt^n} + a_{n-1}\frac{d^{n-1} y_{zi}}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y_{zi}}{dt^{n-2}} + \cdots + a_1\frac{dy_{zi}}{dt} + a_0 y_{zi} = 0$$

Apply initial conditions to <u>*zero-input*</u> solution to solve for *n* constants:

$$y(0), y'(0), y''(0), \cdots y^{(n-2)}(0), y^{(n-1)}(0)$$

LTI Systems

Method 2 (continued)

Solve zero-state equation:

$$\frac{d^n y_{zs}}{dt^n} + a_{n-1}\frac{d^{n-1} y_{zs}}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y_{zs}}{dt^{n-2}} + \cdots + a_1 \frac{dy_{zs}}{dt} + a_0 y_{zs} = x(t)$$

Apply initial conditions that are all zero:

$$y_{zs}(0) = 0, y'_{zs}(0) = 0, y''_{zs}(0) = 0, \cdots y_{zs}^{(n-2)}(0) = 0, y_{zs}^{(n-1)}(0) = 0$$

to solve for n constants

Note that this means you must solve as in method 1 for
homogeneous + particular solutions but with the initial conditions set to zero

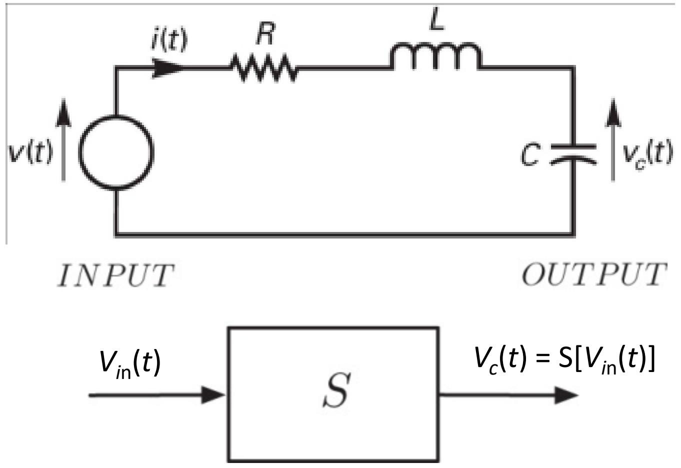
Combine zero-input and zero-state solutions for final result:

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

LTI Systems

Example: Solve the following O.D.E.

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\sin\left(t\right),\quad t\geq 0,\quad y(0)={}^{4}\!/_{5},\quad y'(0)={}^{11}\!/_{10}$$



$$\frac{d^2v_c(t)}{dt^2}+\frac{R}{L}\frac{dv_c(t)}{dt}+\frac{1}{LC}v_c(t)=\frac{1}{LC}v_{in}\left(t\right)$$

<u>Method 1</u>

Solve homogeneous equation:

$$y_H''(t)+4y_H'(t)+3y_H(t)=0$$

$$\lambda^2+4\lambda+3=0$$

$$\left(\lambda+1\right)\left(\lambda+3\right)=0\quad\Rightarrow\quad\lambda=-1,\quad\lambda=-3$$

$$y_H(t)=C_1e^{-t}+C_2e^{-3t}$$

LTI Systems

Method 1 (continued)

Solve particular equation:

$$y_P''(t)+4y_P'(t)+3y_P(t)=\sin(t)$$

Method of undetermined coefficients

$$y_P(t)=A\sin(t)+B\cos(t)$$

$$y_P'(t)=A\cos(t)-B\sin(t)$$

$$y_P''(t)=-A\sin(t)-B\cos(t)$$

Plug into D.E. :

$$\left(-A\sin(t)-B\cos(t)\right)+4\left(A\cos(t)-B\sin(t)\right)+3\left(A\sin(t)+B\cos(t)\right)=\sin(t)$$

Solve for A and B :

$$(2A-4B)\sin(t)+(4A+2B)\cos(t)=1\sin(t)+0\cos(t)$$

$$2A-4B=1 \quad \& \quad 4A+2B=0 \quad \Rightarrow \quad A={}^{1}\!/_{10} \quad ; \quad B=-{}^{1}\!/_{5}$$

$$y_P(t)=\frac{1}{10}\sin(t)-\frac{1}{5}\cos(t)$$

LTI Systems

Method 1 (continued)

Total Solution: $y(t) = y_H(t) + y_P(t) = C_1 e^{-t} + C_2 e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$

Use initial conditions to solve for C_1 and C_2:

$$y(t) = C_1 e^{-t} + C_2 e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

$$y'(t) = -C_1 e^{-t} - 3C_2 e^{-3t} + \frac{1}{10}\cos(t) + \frac{1}{5}\sin(t)$$

$$y(0) = C_1 + C_2 - \frac{1}{5} = \frac{4}{5} \qquad \Rightarrow \quad C_1 + C_2 = 1$$

$$y'(0) = -C_1 - 3C_2 + \frac{1}{10} = \frac{11}{10} \quad \Rightarrow -C_1 - 3C_2 = 1$$

$$C_1 = 2, \quad C_2 = -1$$

$$\boxed{\text{Method 1 solution: } \; y(t) = 2e^{-t} - e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)}$$

LTI Systems

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\sin\left(t\right), \quad t\geq 0, \quad y(0)=4/5, \quad y'(0)=11/10$$

Method 2

Solve the zero-input equation

$$\frac{d^2y_{zi}}{dt^2}+4\frac{dy_{zi}}{dt}+3y_{zi}=0$$

$$y_{zi}\left(t\right)=C_1e^{-t}+C_2e^{-3t}$$

(Since we've already solved the homogeneous Equation, we can write down the answer)

Apply initial conditions to ***zero-input*** solution to solve for constants:

$$y_{zi}(0)=C_1+C_2 \quad =4/5$$

$$y'_{zi}(0)=-C_1-3C_2=11/10$$

$$C_1=\frac{7}{4} \quad , \quad C_2=-\frac{19}{20}$$

$$y_{zi}(t)=\frac{7}{4}e^{-t}-\frac{19}{20}e^{-3t}$$

LTI Systems

Method 2 (continued)

Solve the zero-state equation (with input signal and with all initial conditions set to zero

$$\frac{d^2 y_{zs}}{dt^2} + 4\frac{dy_{zs}}{dt} + 3y_{zs} = \sin(t) \quad , \quad y_{zs}(0) = 0, \quad y'_{zs}(0) = 0$$

Must go through whole procedure of Method 1 (homogeneous + particular) !

Homogeneous equation:

$$y''_{Hzs}(t) + 4y'_{Hzs}(t) + 3y_{Hzs}(t) = 0$$

$$y_{Hzs}(t) = D_1 e^{-t} + D_2 e^{-3t}$$

Particular solution:

$$y_{Pzs}(t) = \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

LTI Systems

Method 2 (continued)

Total zero-state solution:

$$y_{zs}(t) = y_{Hzs}(t) + y_{Pzs}(t) = D_1 e^{-t} + D_2 e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

Apply all zero initial conditions:

$$y_{zs}(t) = D_1 e^{-t} + D_2 e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

$$y'_{zs}(t) = -D_1 e^{-t} - 3D_2 e^{-3t} + \frac{1}{10}\cos(t) + \frac{1}{5}\sin(t)$$

$$y_{zs}(0) = D_1 + D_2 - \frac{1}{5} = 0 \qquad \Rightarrow \quad D_1 + D_2 = \frac{1}{5}$$

$$y'_{zs}(0) = -D_1 - 3D_2 + \frac{1}{10} = 0 \quad \Rightarrow -D_1 - 3D_2 = -\frac{1}{10}$$

$$D_1 = \frac{1}{4}, \quad D_2 = -\frac{1}{20}$$

$$y_{zs}(t) = \frac{1}{4}e^{-t} - \frac{1}{20}e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

LTI Systems

Method 2 (continued)

Total solution:

$$y(t) = y_{zi}(t) + y_{zs}(t) = \frac{7}{4}e^{-t} - \frac{19}{20}e^{-3t} + \frac{1}{4}e^{-t} - \frac{1}{20}e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

$$\boxed{\text{Method 2 solution:} \quad y(t) = 2e^{-t} - e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)}$$

Which (luckily) is the same as the result from Method 1

LTI Systems

Complete response

$$y(t) = 2e^{-t} - e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

Homogeneous solution
(Natural response)

$$y_H(t) = 2e^{-t} - e^{-3t}$$

Zero-input
No input / Use initial conditions

$$y_{zi}(t) = \frac{7}{4}e^{-t} - \frac{19}{20}e^{-3t}$$

Particular solution
Forced response

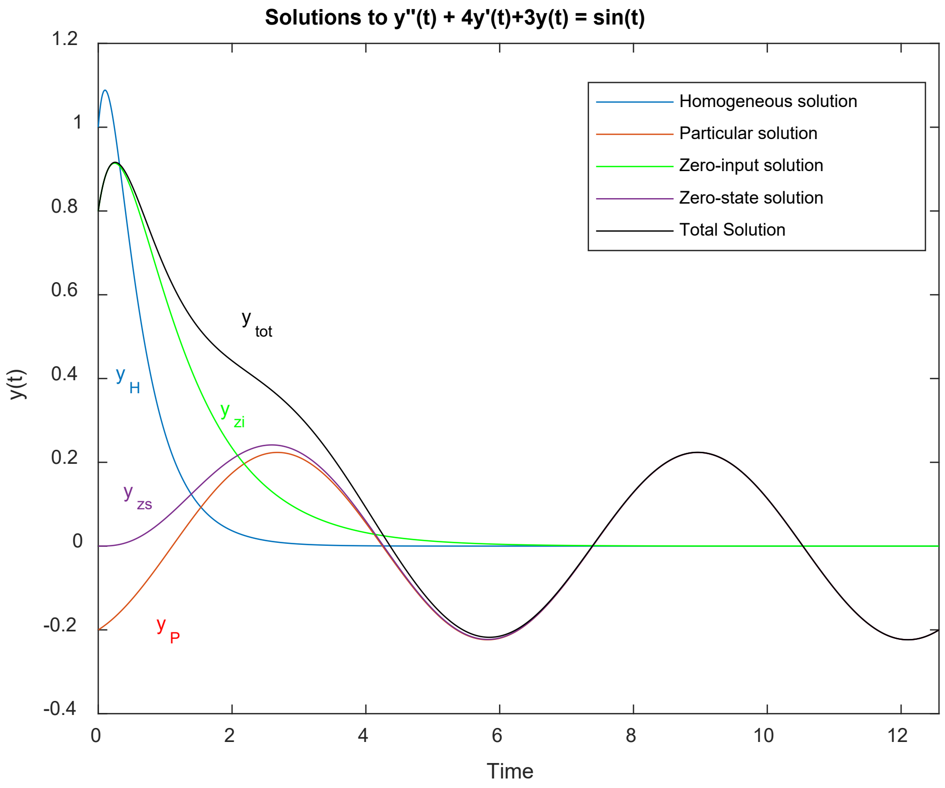
$$y_P(t) = \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

Zero-state
Use input/zero initial conditions

$$y_{zs}(t) = \frac{1}{4}e^{-t} - \frac{1}{20}e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

LTI Systems

$$y(t) = 2e^{-t} - e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$



$$y_H(t) = 2e^{-t} - e^{-3t}$$

$$y_P(t) = \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

$$y_{zi}(t) = \frac{7}{4}e^{-t} - \frac{19}{20}e^{-3t}$$

$$y_{zs}(t) = \frac{1}{4}e^{-t} - \frac{1}{20}e^{-3t} + \frac{1}{10}\sin(t) - \frac{1}{5}\cos(t)$$

LTI Systems

- Why is this important?
  - The Zero-State response is what LTI systems produce in response to an input
  - Can a system be LTI if the initial conditions are not zero?
    - No: If you double the input, the zero-state response will double, but the zero-input response will not change.
  - The zero-state response can be obtained by convolving the impulse response with the input.
    - You do not need to solve the differential equation

LTI Systems

- Convolution
  - Use LTI to find system response to sum of time-delayed inputs
  - System response

$$y(t) = S\left[x(t)\right]$$

  - Response to weighted sum of delayed inputs using LTI

$$S\left[\sum_k A_k x(t-\tau_k)\right] = \sum_k A_k S\left[x(t-\tau_k)\right] = \sum_k A_k y(t-\tau_k)$$

    - Response is weighted sum of time-delayed outputs
  - Consider sum going to integral:

$$S\left[\int g(\tau)x(t-\tau)d\tau\right] = \int g(\tau)S\left[x(t-\tau)\right]d\tau = \int g(\tau)y(t-\tau)d\tau$$

    - The last integral is the convolution of g and y, also written:

$$\int g(\tau)y(t-\tau)d\tau = \left[g*y\right](t)$$

LTI Systems

- Impulse response
  - Any arbitrary input signal, *x(t)*, can be written as:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

  - Think of $x(\tau)$ as weights (<u>not</u> functions of *t*)
  - The output of the system, *y(t),* is

$$y(t) = S\left[x(t)\right] = S\left[\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau\right] = \int_{-\infty}^{+\infty} x(\tau)S\left[\delta(t-\tau)\right]d\tau$$

  - Define the impulse response as: $h(t) = S\left[\delta(t)\right]$
  - For LTI system:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \left[x * h\right](t)$$

LTI Systems

- Note that convolution is symmetric:
  - Use change of variables: $\tau \to t - \tau'$

$$[x * h](t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty} h(\tau)x(\mathrm{t}-\tau)d\tau = [h * x](t)$$

- Output of the system can be written as

$$y(t) = [x * h](t) \quad \text{or} \quad y(t) = [h * x](t)$$

- Impulse response is fundamental characterization of linear time-invariant systems
- **Equivalent to zero-state (zero initial conditions) response when system is represented by linear D.E. with constant coefficients.**

Example of Convolution of Impulse Response

Example: $\dfrac{d^2y(t)}{dt^2} + 4\dfrac{dy(t)}{dt} + 3y(t) = \sin(t), \quad t \geq 0$

The impulse response of the (causal) system represented by:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t)$$

is $\quad h(t) = \tfrac{1}{2}\left(e^{-t} - e^{-3t}\right), \quad t > 0$

(We will see in a bit how we get this using the Laplace transform)

Convolution of input with impulse response: $\displaystyle\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$

For causal signals and systems: $x(t) = 0$ and $h(t) = 0$ for $t < 0$

Note: $h(t-\tau) = 0$ in the integral for $t - \tau < 0$ or $\tau > t$

Similarly: $x(\tau) = 0$ in the integral for $\tau < 0$

Example of Convolution of Impulse Response

For causal signals and systems: $x(t) = 0$ and $h(t) = 0$ for $t < 0$

Note: $h(t-\tau) = 0$ in the integral for $t - \tau < 0$ or $\tau > t$

Similarly: $x(\tau) = 0$ in the integral for $\tau < 0$

So, the convolution integral is:

$$\int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t \sin(\tau)\tfrac{1}{2}\left(e^{-(t-\tau)} - e^{-3(t-\tau)}\right)d\tau$$

$$= \tfrac{1}{2}e^{-t}\int_0^t \sin(\tau)e^{\tau}d\tau - \tfrac{1}{2}e^{-3t}\int_0^t \sin(\tau)e^{3\tau}d\tau$$

Example of Convolution of Impulse Response

$$\int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t \sin(\tau)\tfrac{1}{2}\left(e^{-(t-\tau)} - e^{-3(t-\tau)}\right)d\tau$$

$$= \tfrac{1}{2}e^{-t}\int_0^t \sin(\tau)e^{\tau}d\tau - \tfrac{1}{2}e^{-3t}\int_0^t \sin(\tau)e^{3\tau}d\tau$$

A couple of ways to do these integrals:

(1) Integration by parts (twice) for each integral

(2) Use Euler's formula for sine: $\sin(\tau) = \tfrac{1}{2i}\left(e^{i\tau} - e^{-i\tau}\right)$

so that you only have integrals of exponential functions (which are easier)

Example of Convolution of Impulse Response

$$\int_0^t \sin(\tau)e^{\tau}d\tau = \int_0^t \frac{e^{i\tau}-e^{-i\tau}}{2i}e^{\tau}d\tau = \frac{1}{2i}\left(\int_0^t e^{(1+i)\tau}d\tau - \int_0^t e^{(1-i)\tau}d\tau\right)$$

$$= \frac{e^{\tau}}{2i}\left(\frac{e^{i\tau}}{1+i} - \frac{e^{-i\tau}}{1-i}\right)\Bigg|_0^t = \frac{e^{\tau}}{2i}\frac{(1-i)e^{i\tau}-(1+i)e^{-i\tau}}{(1+i)(1-i)}\Bigg|_0^t = \frac{e^{\tau}}{2i}\left(\frac{(e^{i\tau}-e^{-i\tau})-i(e^{i\tau}+e^{-i\tau})}{2}\right)\Bigg|_0^t$$

$$= \frac{e^{\tau}}{2}\left(\sin(\tau)-\cos(\tau)\right)\Big|_0^t = \frac{1}{2}\left(e^t\sin(t)-e^t\cos(t)-e^0\sin(0)+e^0\cos(0)\right) = \frac{1}{2}\left(1+e^t\sin(t)-e^t\cos(t)\right)$$

Looking promising since you can see first term gives you:

$$\frac{1}{2}e^{-t}\int_0^t \sin(\tau)e^{\tau}d\tau = \frac{1}{4}e^{-t}+\frac{1}{4}\sin(t)-\frac{1}{4}\cos(t)$$

The other term can be done in a similar fashion:

$$-\frac{1}{2}e^{-3t}\int_0^t \sin(\tau)e^{3\tau}d\tau = -\frac{1}{20}e^{-3t}-\frac{3}{20}\sin(t)+\frac{1}{20}\cos(t)$$

Example of Convolution of Impulse Response

Combining the terms gives a final result of:

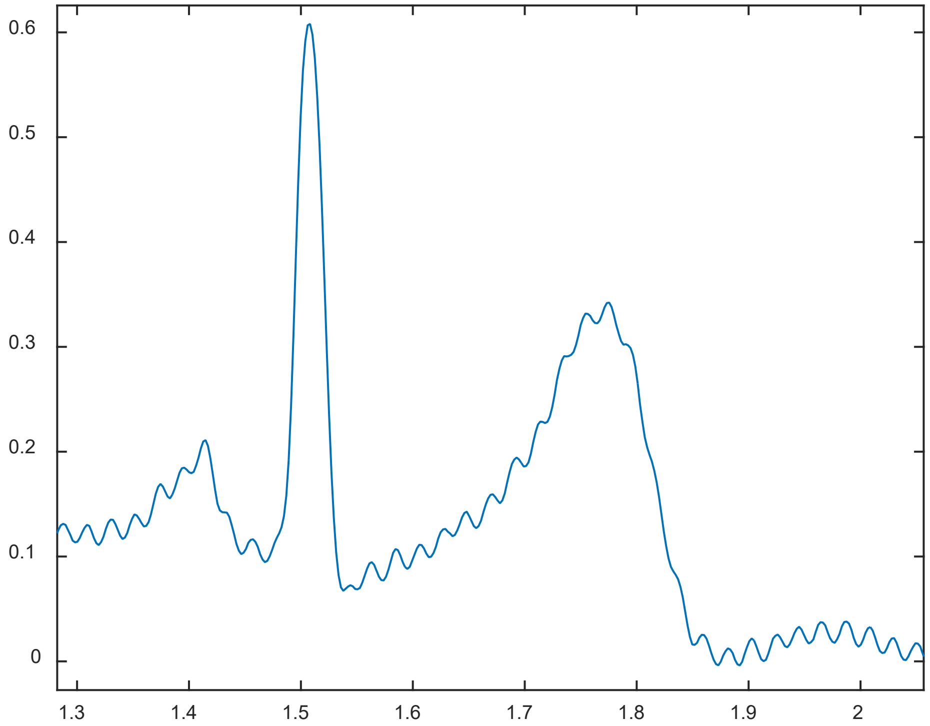
$$\frac{1}{4}e^{-t}-\frac{1}{20}e^{-3t}+\frac{1}{10}\sin(t)-\frac{1}{5}\cos(t)$$

Recall that the zero-state solution was:

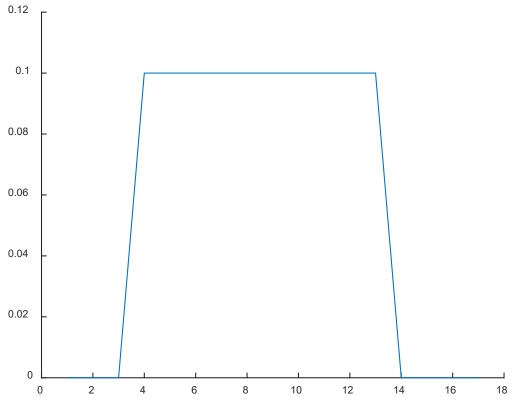
$$\mathrm{y}_{zs}(t)=\frac{1}{4}e^{-t}-\frac{1}{20}e^{-3t}+\frac{1}{10}\sin(t)-\frac{1}{5}\cos(t)$$

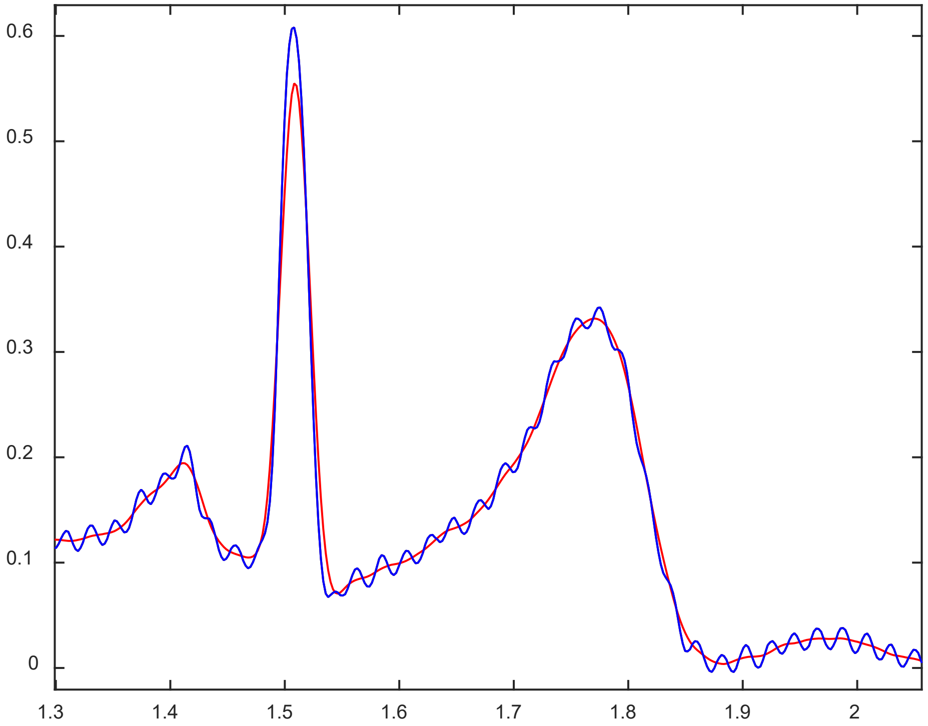
So, this example shows that the convolution of the impulse response with the input signal gives the zero-state response of the system.

Example of convolution

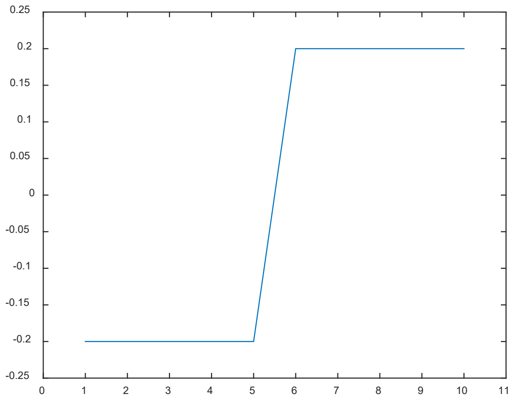


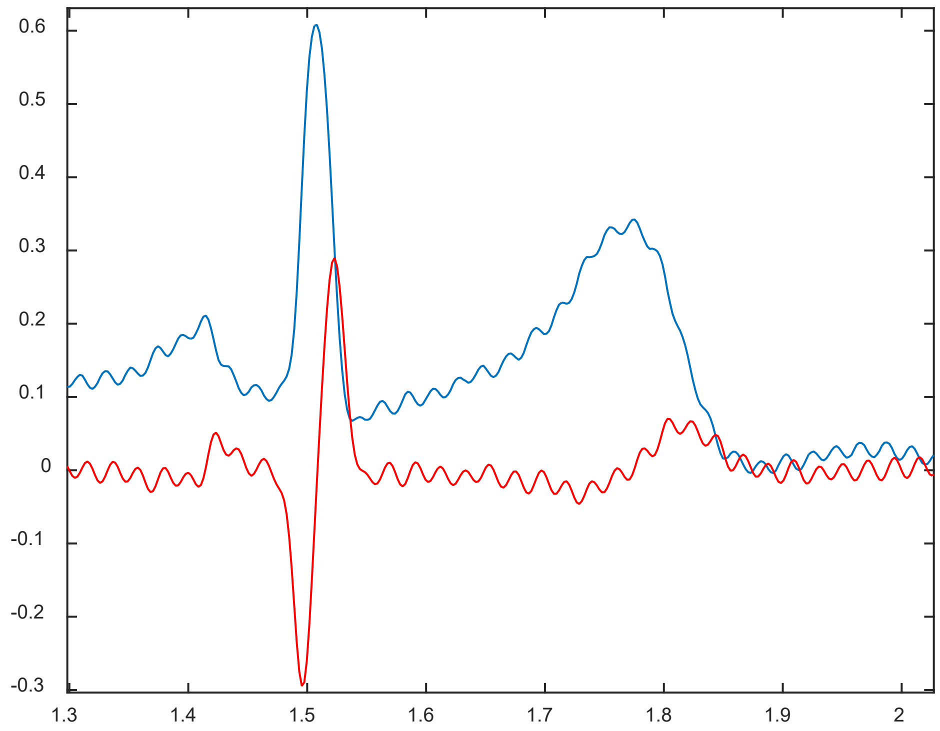
Example of convolution





Example of convolution





LTI Systems

- Step response:
  - Step response is: $s(t) = S[u(t)]$
  - If *h(t)* is impulse response $h(t) = S[\delta(t)]$
  - Since $\delta(t) = \dfrac{du(t)}{dt}$

$$h(t) = S[\delta(t)] = S\left[\frac{du(t)}{dt}\right] = \frac{dS[u(t)]}{dt}$$

$$h(t) = \frac{d\,s(t)}{dt}$$

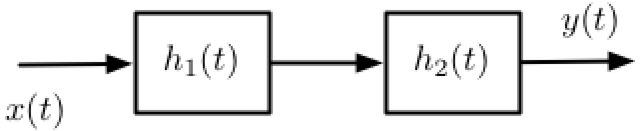
  - Similarly, for a ramp response of $\rho(t)$

$$h(t) = \frac{d^2\rho(t)}{dt^2}$$

LTI Systems

– Interconnection of systems

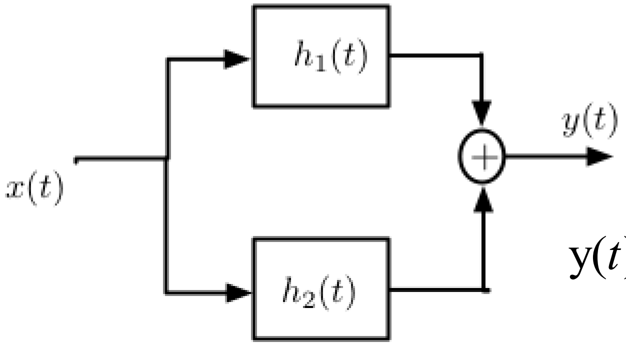
Cascade



(a) $h(t) = [h_1 * h_2](t) = [h_2 * h_1](t)$

$$y(t) = \big[x * [h_1 * h_2]\big](t) = \big[x * [h_2 * h_1]\big](t)$$

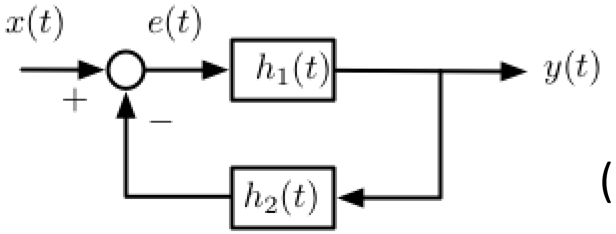
Parallel



(b) $h(t) = h_1(t) + h_2(t)$

$$y(t) = [x * h_1](t) + [x * h_2](t) = [x * (h_1 + h_2)](t)$$

Feedback



(c) $h(t) = [h_1 - h * h_1 * h_2](t)$

(Implicit expression for impulse response)

Causal Systems

– Causal Systems:

Continuous-time system S is **causal** *if*

- *for $x(t) = 0$ and no initial conditions, output $y(t) = 0$,*
- *$y(t)$ does not depend on future inputs.*

A LTI system represented by impulse response $h(t)$ is **causal** *if*

$$h(t) = 0 \qquad for\ t < 0$$

The output of a causal LTI system with a causal input $x(t)$, i.e., $x(t) = 0$ for $t < 0$, is

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

BIBO Systems

– Bounded Systems

Bounded-input Bounded-output (BIBO) Stability

BIBO stability: for a bounded (that is what is meant by well-behaved) input $x(t)$ the output of a BIBO stable system $y(t)$ is also bounded. This means that if there is a finite bound $M < \infty$ such that $|x(t)| < M$ (i.e., $x(t)$ in an envelope $[-M, M]$) the output is also bounded.
A LTI system with an absolutely integrable impulse response, i.e.,

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

is BIBO stable.

Review of Laplace Transform

- The Laplace Transform
  - Important method of analysis for signal & image processing and process control
  - Definition:

$$\boxed{F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{+\infty} f(t) e^{-st} dt \quad \text{where } s \text{ is a complex variable } (s = \sigma + j\omega)}$$

  - Things you can do with Laplace transform
    - Characterize system by a transfer function
    - Determine stability of system
    - Transform linear differential equations to algebraic equations
    - Launching point for frequency analysis

Laplace Transform

- The Laplace Transform
  - What does it mean?
  - Consider an input signal $x(t) = e^{st}$
    where s is a complex number: $s=\sigma+j\omega$
  - Consider the LTI system processing this input:

$$y(t) = S\left[x(t)\right] = S\left[e^{st}\right]$$

  - Using the impulse response of the system $h(t)$ and the convolution theorem:

$$y(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{+\infty} e^{s(t-\tau)}h(\tau)d\tau = e^{st}\int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)d\tau$$

$$y(t) = \left[\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau\right]e^{st} = H(s)x(t)$$

Laplace Transform

- The Laplace Transform
  - What does it mean? …
    - A way of characterizing LTI system in terms its eigenvalues & eigenfunctions

$$y(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{+\infty} e^{s(t-\tau)}h(\tau)d\tau = e^{st}\int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)d\tau$$

$$y(t) = \left[\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau\right]e^{st} = H(s)x(t)$$

  - The output is the input multiplied by the complex function *H(s)*

  - In mathematical terms:
    The function e^{st} is an eigenfunction of the LTI system
    H(s) is the eigenvalue for the LTI system

Laplace Transform

- Typically Laplace transform is a rational polynomial

$$F(s)=\frac{N(s)}{D(s)} \quad \text{where } N(s) \text{ and } D(s) \text{ are polynomials in } s$$

$$\text{Example: } F(s)=\frac{2\left(s^2+1\right)}{s^2+2s+5}=\frac{2(s+j)(s-j)}{(s+1)^2+4}=\frac{2(s+j)(s-j)}{(s+1+2j)(s+1-2j)}$$

Written in this form to show poles and zeros of $F(s)$

Poles where denominator is zero, i.e., $D(s)=0$ ($F(s)$ becomes infinite)

Zeros where numerator is zero i.e., $\mathrm{N}(s)=0$ ($F(s)$ is zero)

For example:

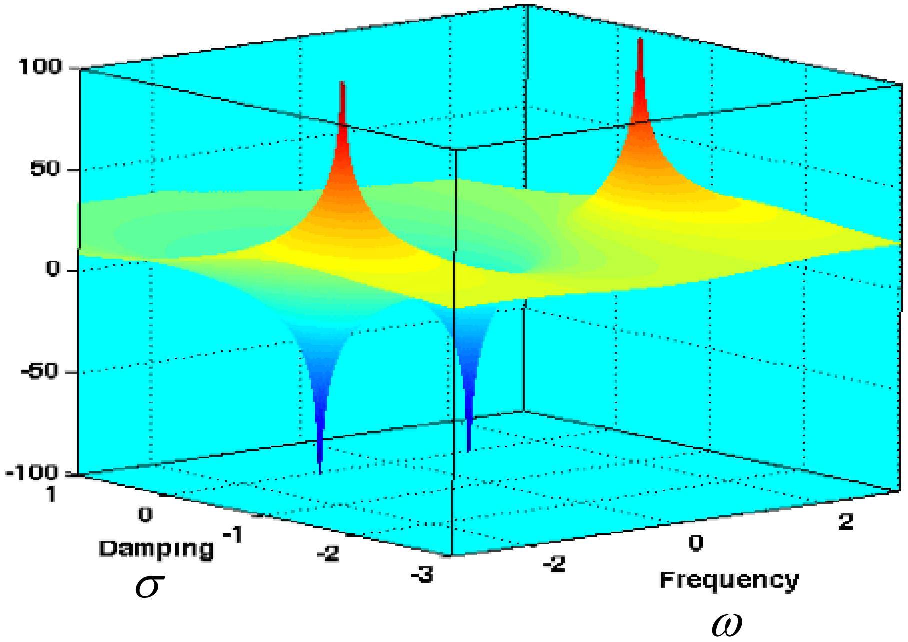
Poles at $s=-1+2j$ and $s=-1-2j$

Zeros at $s=j$ and $s=-j$

Laplace Transform

$$F(s) = \frac{2\left(s^2+1\right)}{s^2+2s+5} = \frac{2(s+j)(s-j)}{(s+1)^2+4} = \frac{2(s+j)(s-j)}{(s+1+2j)(s+1-2j)}$$

Plot of $\log F(s)$: zeros have $\log 0 \to -\infty$, poles have $\log \infty \to \infty$



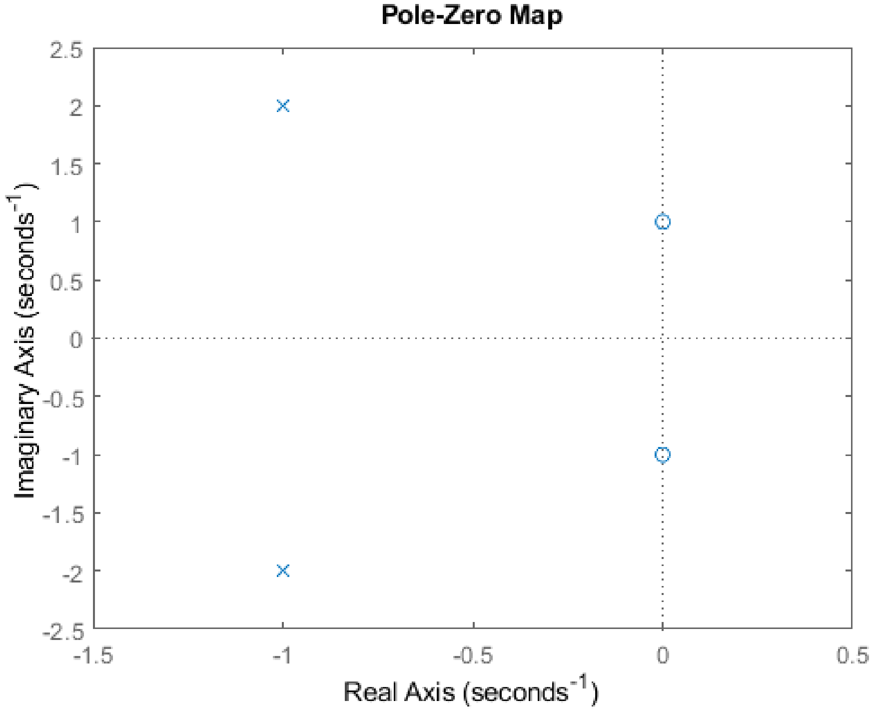
Laplace Transform

MATLAB has a nice function for plotting poles and zeros: `pzmap`

```
% Example of pzmap:
s = tf('s')
H1 = 2*(s^2+1)/(s^2+2*s+5)
figure(1)
pzmap(H1)
axis([-1.5,0.5,-2.5,2.5]);

% or
clear
H2 = tf([2,0,2],[1,2,5]);
figure(2)
pzmap(H2)
axis([-1.5,0.5,-2.5,2.5]);
```

s-plane



Laplace Transform

- Region of convergence $\left|\int_{-\infty}^{\infty} f(t)e^{-st}dt\right| = \left|\int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt\right| \le \int_{-\infty}^{\infty} \left|f(t)e^{-\sigma t}\right| dt < \infty$
  - You cannot have poles in the region of convergence
    - If you did, the integral would not converge absolutely
  - For a causal function

    $f(t) = 0$ for $t < 0$, ROC is part of s-plane to the right of the poles.

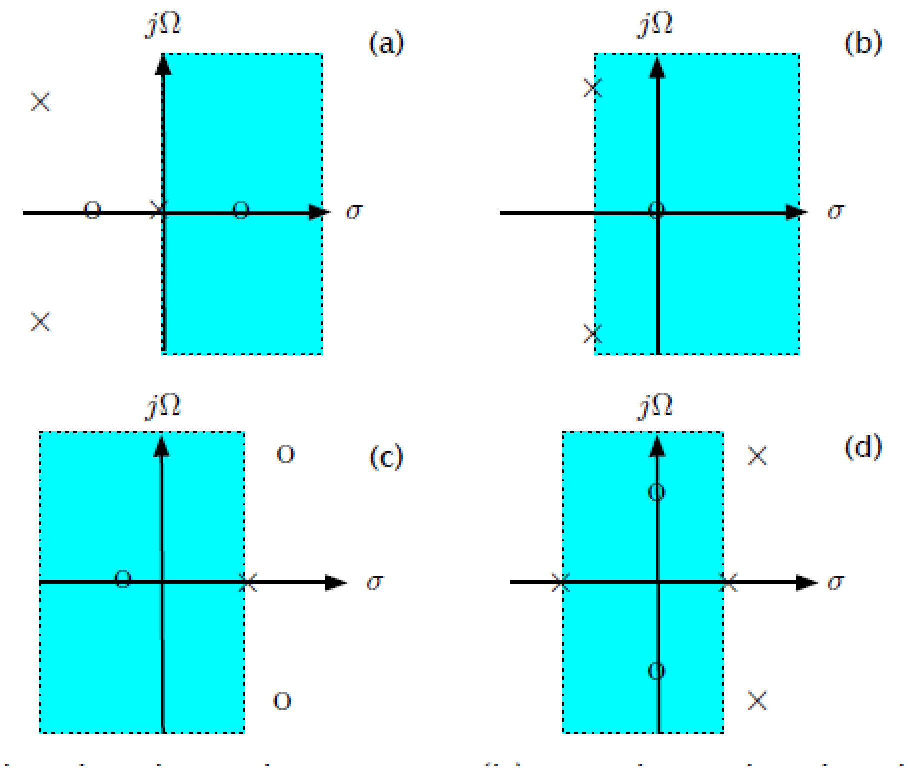
  - For anti-causal function

    $f(t) = 0$ for $t > 0$, ROC is part of s-plane to the left of the poles.

  - For non-causal:

    $f(t)$ is defined for $-\infty < t < \infty$ ROC is intersection of causal and anti-causal parts between the poles on the right and left

Laplace Transform



(a) Causal
(b) Causal with poles to left of imaginary axis
(c) Anti-causal
(d) Non-causal (ROC bounded by poles)

Laplace Transform

- The Laplace Transform (one-sided, unilateral)
  - Maps a real-valued function of time, *t*, into a function of a complex variable *s*.

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} f(t)e^{-st}\,dt \quad \text{where } s \text{ is a complex variable } (s = \sigma + j\omega)$$

  - Convergence:

$$\int_0^{\infty} f(t)e^{-st}\,dt = \int_0^{\infty} f(t)e^{-(\sigma+j\omega)t}\,dt = \int_0^{\infty} f(t)e^{-\sigma t}e^{j\omega t}\,dt$$

Converges if

$$\left|\int_0^{\infty} f(t)e^{-st}\,dt\right| = \left|\int_0^{\infty} f(t)e^{-(\sigma+j\omega)t}\,dt\right| \le \int_0^{\infty} \left|f(t)e^{-\sigma t}\right|dt < \infty$$

Laplace Transform

- The Inverse Laplace Transform
  - The formal mathematical definition is:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} \, ds$$

where σ is large enough that $F(s)$ is defined for $\mathrm{Re}(s) \geq \sigma$

  - This formula is rarely used to find inverse.
  - A more common way is to cast expression in the Laplace domain in a form that corresponds to entries in a table of Laplace transforms.
    - Often you have to reduce a complicated expression into a simpler one to do this.
    - Generally involves operations like **partial fractions** and **completing the square**.

Inverse Laplace Transform

- Key problem in finding inverse Laplace transform for complicated expressions involving s
  - Need to get into simple form first
    - Usually involves partial fractions
    - Sometimes need to complete square
    - Sometimes need to be clever in rewriting terms
  - Often utilize the properties shown on following slides

LAPLACE TRANSFORM TABLE

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

SPECIFIC FUNCTIONS		GENERAL RULES	
$F(s)$	$f(t)$	$F(s)$	$f(t)$
$\frac{1}{s}$	1	$\frac{e^{-as}}{s}$	$u(t-a)$
$\frac{1}{s^n}, \quad n \in \mathbb{Z}^+$	$\frac{t^{n-1}}{(n-1)!}$	$e^{-as}F(s)$	$f(t-a)u(t-a)$
$\frac{1}{s+a}$	e^{-at}	$F(s-a)$	$e^{at}f(t)$
$\frac{1}{(s+a)^n}, \quad n \in \mathbb{Z}^+$	$e^{-at}\frac{t^{n-1}}{(n-1)!}$	$sF(s)-f(0)$	$f'(t)$
$\frac{1}{s^2+\omega^2}$	$\frac{\sin(\omega t)}{\omega}$	$s^2F(s)-sf(0)-f'(0)$	$f''(t)$
$\frac{s}{s^2+\omega^2}$	$\cos(\omega t)$	$F'(s)$	$-tf(t)$
$\frac{1}{(s+a)^2+\omega^2}$	$\frac{e^{-at}\sin(\omega t)}{\omega}$	$F^{(n)}(s)$	$(-t)^n f(t)$
$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos(\omega t)$	$\frac{F(s)}{s}$	$\int_0^t f(u)\,du$
$\frac{1}{(s^2+\omega^2)^2}$	$\frac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^3}$	$F(s)G(s)$	$(f*g)(t)$
$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t\sin(\omega t)}{2\omega}$		

Common Laplace Transform Properties

Name	Illustration
Definition of Transform	$\mathrm{f(t)} \xleftrightarrow{\mathrm{L}} \mathrm{F(s)}$ $\mathrm{F(s)} = \int_{0^-}^{\infty} \mathrm{f(t)e^{-st}dt}$
Linearity	$Af_1(t) + Bf_2(t) \xleftrightarrow{L} AF_1(s) + BF_2(s)$
First Derivative	$\frac{df(t)}{dt} \xleftrightarrow{L} sF(s) - f(0^-)$
Second Derivative	$\frac{d^2 f(t)}{dt^2} \xleftrightarrow{L} s^2F(s) - sf(0^-) - \dot{f}(0^-)$
n^{th} Derivative	$\frac{d^n f(t)}{dt^n} \xleftrightarrow{L} s^nF(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0^-)$
Integral	$\int_0^t f(\lambda)d\lambda \xleftrightarrow{L} \frac{1}{s}F(s)$
Time Multiplication	$tf(t) \xleftrightarrow{L} -\frac{dF(s)}{ds}$
Time Delay	$\mathrm{f(t-a)\gamma(t-a)} \xleftrightarrow{\mathrm{L}} \mathrm{e^{-as}F(s)}$ γ(t) is unit step
Complex Shift	$f(t)e^{-at} \xleftrightarrow{L} F(s+a)$
Scaling	$f\left(\frac{t}{a}\right) \xleftrightarrow{L} aF(as)$
Convolution Property	$f_1(t) * f_2(t) \xleftrightarrow{L} F_1(s)F_2(s)$
Initial Value	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$
Final Value (if final value exists)	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Laplace Transform Properties

- Straightforward to find some Laplace transforms from definitions
- Others can be found, starting from a simple function and using properties of transform
- Proof of Laplace transform properties is fairly straightforward starting from the definition.
  - We will not go through proofs of the properties
    - A good summary of the Laplace Transform and proofs of some properties can be found at:
      The Laplace Transform (Prof. Cheever's website)

Laplace Transform Properties

Click here more details about properties of Laplace transforms

Linearity

If $F_1(s)$ and $F_2(s)$ are, respectively, the Laplace Transforms of $f_1(t)$ and $f_2(t)$

$$\boxed{L\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)}$$

Example:

$$\boxed{L\left[\cos(\omega t)u(t)\right] = L\left[\frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)u(t)\right] = \frac{s}{s^2 + \omega^2}}$$

Laplace Transform Properties

Time Shift

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$\boxed{L\left[f(t-a)u(t-a)\right] = e^{-as}F(s)}$$

Example:

$$\boxed{L\left[\cos(\omega(t-a))u(t-a)\right] = e^{-as}\frac{s}{s^2+\omega^2}}$$

Laplace Transform Properties

Frequency Shift

If $F(s)$ is the Laplace Transforms of $f(t)$, then

$$\boxed{L\left[e^{-at} f(t)u(t)\right] = F(s+a)}$$

Example:

$$\boxed{L\left[e^{-at}\cos(\omega t)u(t)\right] = \frac{s+a}{(s+a)^2+\omega^2}}$$

Laplace Transform Properties

Scaling

If *F (s)* is the Laplace Transforms of *f (t)*, then

$$\boxed{L\left[f(at)\right] = \frac{1}{a} F\left(\frac{s}{a}\right)}$$

Example:

$$\boxed{L\left[\sin(2\omega t)u(t)\right] = \frac{2\omega}{s^2 + 4\omega^2}}$$

Laplace Transform Properties

Time Differentiation

If *F (s)* is the Laplace Transforms of *f (t),* then the Laplace Transform of its derivative is

$$\boxed{L\left[\frac{df}{dt}u(t)\right] = sF(s) - f(0^-)}$$

Example:

$$\boxed{L[\sin(\omega t)u(t)] = \frac{\omega}{s^2 + \omega^2}}$$

Laplace Transform Properties

Time Differentiation More Generally:

For a signal $f(t)$, with Laplace transform $F(s)$, the one-sided Laplace transform of its first- and second-order derivatives are

$$\mathcal{L}\left[\frac{df(t)}{dt}u(t)\right] = sF(s) - f(0-) \tag{3.14}$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}u(t)\right] = s^2F(s) - sf(0-) - \frac{df(t)}{dt}\Big|_{t=0-} \tag{3.15}$$

In general, if $f^{(N)}(t)$ denotes the Nth-order derivative of a function $f(t)$ that has a Laplace transform $F(s)$, we have that

$$\mathcal{L}[f^{(N)}(t)u(t)] = s^N F(s) - \sum_{k=0}^{N-1} f^{(k)}(0-)s^{N-1-k} \tag{3.16}$$

where $f^{(m)}(t) = d^m f(t)/dt^m$ is the mth-order derivative, $m > 0$, and $f^{(0)}(t) \triangleq f(t)$.

Laplace Transform Properties

Time Integration

If *F (s)* is the Laplace Transforms of *f (t)*, then the Laplace Transform of its integral is

$$\boxed{L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)}$$

Example:

$$\boxed{L\left[t^n\right] = \frac{n!}{s^{n+1}}}$$

Find this recursively, starting from $L[1] = \frac{1}{s}$

$$t = \int_0^t 1 d\tau \Rightarrow L\left[t\right] = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

$$\frac{t^2}{2} = \int_0^t \tau d\tau \Rightarrow L\left[t^2\right] = \frac{1}{s}\cdot\frac{2}{s^2} = \frac{2}{s^3}$$

Laplace Transform Properties

Frequency Differentiation

If *F(s)* is the Laplace Transforms of *f (t),* then the derivative with respect to s, is

$$L\left[tf(t)\right] = -\frac{dF(s)}{ds}$$

Example:

$$L\left[te^{-at}u(t)\right] = \frac{1}{(s+a)^2}$$

Laplace Transform Properties

Initial and Final Values

The initial-value and final-value properties allow us to find the initial value *f(0)* and *f(∞)* of *f(t)* directly from its Laplace transform *F(s)*.

$$\boxed{f(0) = \lim_{s \to \infty} sF(s)}$$

Initial-value theorem

$$\boxed{f(\infty) = \lim_{s \to 0} sF(s)}$$

Final-value theorem

Laplace Transform Properties

The Convolution Integral

Defined as $y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$ or $y(t) = x(t) * h(t)$

Given two functions, $f_1(t)$ and $f_2(t)$ with Laplace Transforms $F_1(s)$ and $F_2(s)$, respectively

$y(t) = 4e^{-t}$ and $h(t) = 5e^{-2t}$

$$\boxed{F_1(s)F_2(s) = L\left[f_1(t) * f_2(t)\right]}$$

Example: $h(t) = 5e^{-2t}u(t)$; $x(t) = 4e^{-t}u(t)$

$$\boxed{h(t) * x(t) = L^{-1}\left[H(s)X(s)\right] = L^{-1}\left[\left(\frac{5}{s+2}\right)\left(\frac{4}{s+1}\right)\right] = 20(e^{-t} - e^{-2t}), \quad t \ge 0}$$