Inverse z-transform for multiple-order pole

Consider the difference equation:

$$y(n) - y(n-1) - y(n-2) + y(n-3) = x(n)$$

The z-transform is:

$$Y(z) - z^{-1}Y(z) - z^{-2}Y(z) + z^{-3}Y(z) = X(z)$$
$$\left(1 - z^{-1} - z^{-2} + z^{-3}\right)Y(z) = X(z)$$

The transfer function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - z^{-1} - z^{-2} + z^{-3}} = \frac{z^2}{z^3 - z^2 - z + 1}$$

The denominator can be factored as:

$$H(z) = \frac{z^3}{z^3 - z^2 - z + 1} = \frac{z^3}{(z-1)\left(z^2-1\right)} = \frac{z^3}{(z+1)(z-1)^2}$$

so, there is a 2nd-order pole at $z = 1$.

The partial fraction expansion is:

$$\frac{H(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$\frac{H(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A(z-1)^2 + B(z+1)(z-1) + C(z+1)}{(z+1)(z-1)^2}$$

To solve for *A*, *B*, and *C*:

$$z^2 = A(z-1)^2 + B(z+1)(z-1) + C(z+1)$$

Set $z = -1$: $\quad (-1)^2 = A(-1-1)^2 + B(-1+1)(-1-1) + C(-1+1) = 4A \Rightarrow A = \frac{1}{4}$

Set $z = 1$: $\quad (1)^2 = A(1-1)^2 + B(1+1)(1-1) + C(1+1) = 2C \Rightarrow C = \frac{1}{2}$

Differentiate the equation:

$$2z = A2(z-1) + B\left[1(z-1) + (z+1)1\right] + C = 1 = 2A(z-1) + 2zB + C$$

Set $z = 1$: $\quad 2 = 2A(1-1) + 2B + \frac{1}{2} \Rightarrow 2B = 2 - \frac{1}{2} \Rightarrow B = \frac{3}{4}$

The partial fraction expansion is:

$$\frac{H(z)}{z} = \frac{1}{4}\frac{1}{z+1} + \frac{3}{4}\frac{1}{z-1} + \frac{1}{2}\frac{1}{(z-1)^2}$$

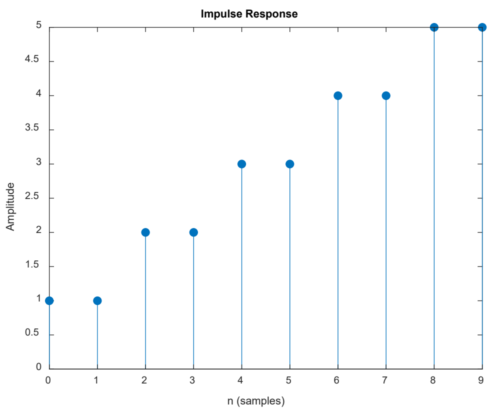
$$H(z) = \frac{1}{4}\frac{z}{z+1} + \frac{3}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}$$

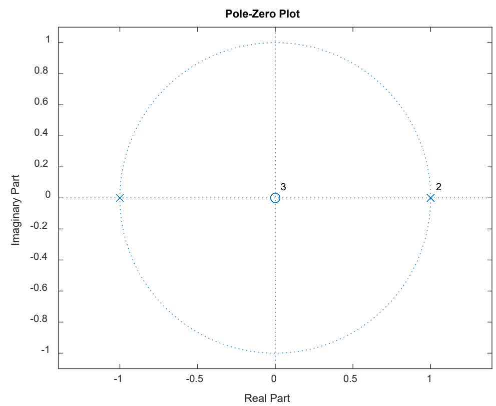
Using the entries in the z-transform table:

$$\mathcal{Z}^{-1}\left[\frac{z}{z-a}\right] = a^n u(n) \qquad \text{and} \qquad \mathcal{Z}^{-1}\left[\frac{az}{(z-a)^2}\right] = na^n u(n)$$

$$h(n) = \left[\frac{1}{4}(-1)^n + \frac{3}{4}(1)^n + \frac{1}{2}n(1)^n\right]u(n)$$

$$\boxed{h(n) = \frac{1}{4}\left[(-1)^n + 2n + 3\right]u(n)}$$





Note that the impulse response for a single pole at z=1 does not blow up.