Digital Signal Processing

Class 21
04/08/2025

ENGR 71

- Class Overview
  - Digital Filter Design
    - FIR filters
- Assignments
  - Reading:

    Chapter 10: Design of Digital Filters

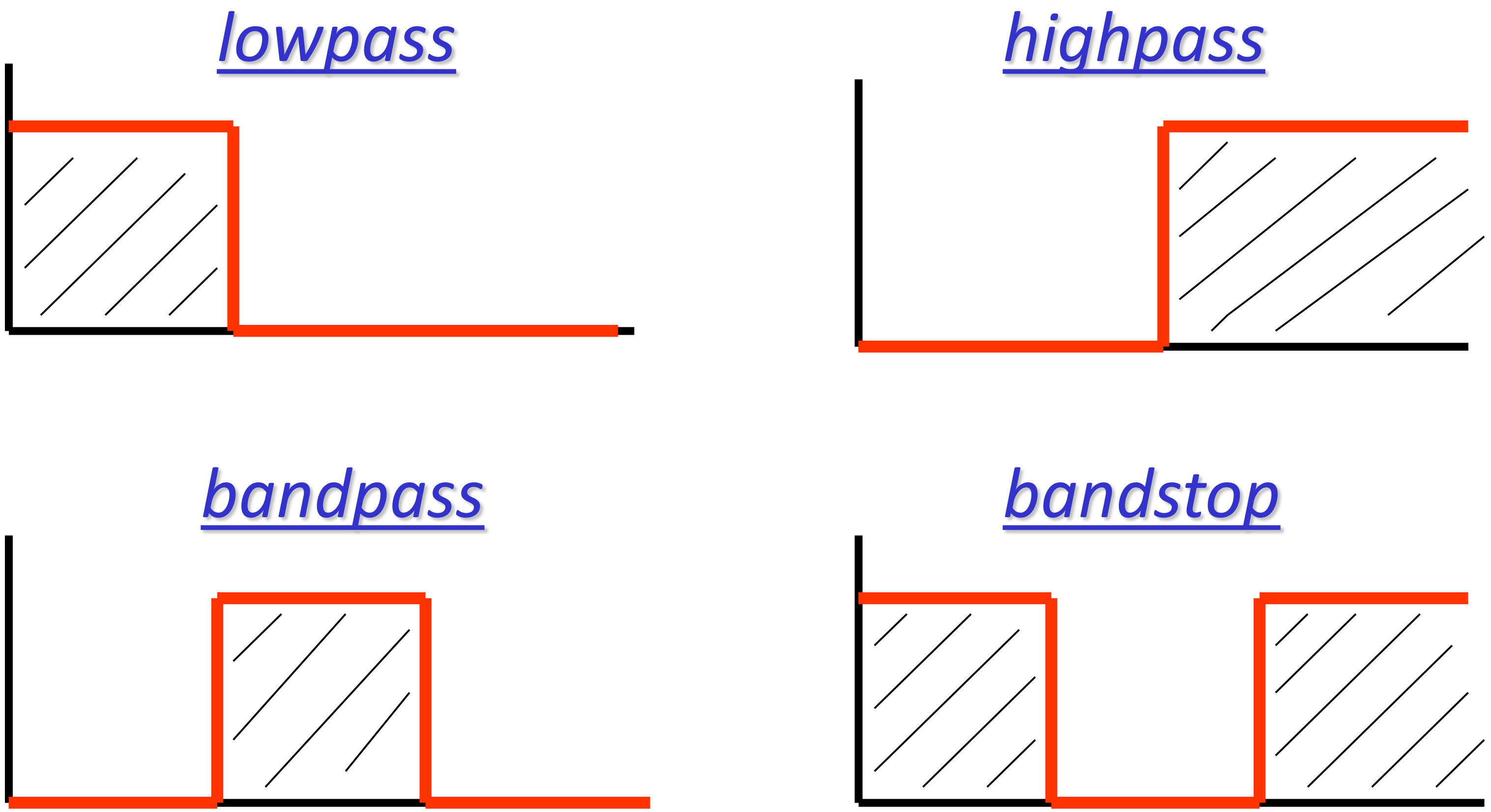
    https://www.mathworks.com/help/signal/ug/fir-filter-design.html
  - Problems: 10.2, 10.3,10.6
    - Due April 20 (Sunday)

Project

- Projects
  - You can work in groups if you wish
  - Presentation at time reserved for final exam
    - Friday, May 9, 7:00-10:00 PM
    - Science Center 264
  - Submit a brief description of your proposed project and who you will be working with to the Project Description Dropbox
  - Submit slides from presentation to Project Dropbox
  - Submit written report to Project Dropbox by end of semester (May 15)

Filters

- Design of Digital Filters



Filters

- Design of Digital Filters
  - Finite Impulse Response (FIR)
    - Used when linear-phase required in passband
      - Constant time delay
      - No dispersion as a function of frequency
  - Infinite Impulse Response (IIR)
    - No requirement on linear-phase
    - Better characteristics for fewer parameters
    - Less memory
    - Lower computational complexity

Causality and Its Implications

- Causality and Its Implications
  - Mathematical criterion for LTI causal system
    - Time domain (Discrete Systems):

      $h(n)$ has finite energy and $h(n) = 0$ for all $n < 0$.

    - Frequency domain: **Paley-Wiener Criterion**
      - In the frequency domain, the magnitude of the transfer function can be zero only at a discrete number of frequencies.
      - Mathematical description of Paley–Wiener criterion:
        For a realizable filter, necessary and sufficient condition for $|H(\omega)|$ is

$$\int_{-\pi}^{\pi} \left| \ln | H(\omega \right\|d\omega < \infty$$

Causality and Its Implications

– For causal systems the impulse response can be determined from just its even part
  (Or, its odd part plus the value at n=0)

$$h(n) = h_e(n) + h_o(n) \quad \text{where} \quad h_e(n) = \frac{1}{2}\big(h(n) + h(-n)\big) \;\; \text{and} \;\; h_o(n) = \frac{1}{2}\big(h(n) - h(-n)\big)$$

$$\text{Since } h(n) = 0 \;\; \text{for} \;\; n < 0$$

$$h(n) = 2h_e(n)u(n) - h_e(0)\delta(0)$$

and

$$h(n) = 2h_o(n)u(n) + h(0)\delta(0)$$

Causality and Its Implications

- Looking at this in the frequency domain:
  - Write the DTFT in terms of its real and imaginary components

$$H(\omega) = H_R(\omega) + jH_I(\omega)$$

  - Using the DTFT property

$$h_e(n) \leftrightarrow H_R(\omega) \text{ and } h_o(n) \leftrightarrow H_I(\omega)$$

    - Since $h(n)$ is completely determined by $h_e(n)$, $H(\omega)$ can be found from just $H_R(\omega)$. The same is true for $H_I(\omega)$.

  - The real and imaginary parts of the transfer function are interrelated for causal system

Causality and Its Implications

– The Discrete Hilbert transform:

Relating the imaginary part of the transfer function to the real part:

$$H_I(\omega) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} H_R(\lambda)\cot\left(\frac{\omega-\lambda}{2}\right)d\lambda$$

A similar expression for the real part in terms of the imaginary part:

$$H_R(\omega) = h(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} H_I(\lambda)\cot\left(\frac{\omega-\lambda}{2}\right)d\lambda$$

(There is a more complete derivation of this in the book by Oppenheim & Schafer)

– The Hilbert transform is useful in digital communications for things like Single-Sideband modulation

Causality and Its Implications

- Summary of the implications of causality
  - Frequency response cannot be zero except at a finite set of points
  - The magnitude of $|H(w)|$ cannot be constant in any finite range of frequencies
  - The transition from passband to stopband cannot be infinitely sharp
  - The imaginary and real parts of the transfer function are interdependent and related by the Hilbert transform
    - The magnitude and phase of the transfer function cannot be chosen arbitrarily

Filter Design

- Finite Impulse Response (FIR) Filters
  - In terms of impulse response

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \qquad \left(b_k\text{'s are } h(k)\text{'s} \right)$$

  - In terms of transfer function:

$$H(z) = \sum_{k=0}^{M-1} h(k)z^{-k}$$

  - Always stable since finite impulse.

Filter Design

- Advantage of FIR filters is the can be designed to have linear phase in the passband
  - Only introduces time delay in the filtered signal
    - No dispersion (time delay dependent of frequency)
- Condition to guarantee linear phase:
  - For length *M* FIR filter: $h(n) = \pm h(M-1-n)$
  - Four cases:
    - Symmetric: $h(n) = +h(M-1-n)$
      - *M* even or *M* odd
    - Antisymmetric: $h(n) = -h(M-1-n)$
      - *M* even or *M* odd

Filter Design

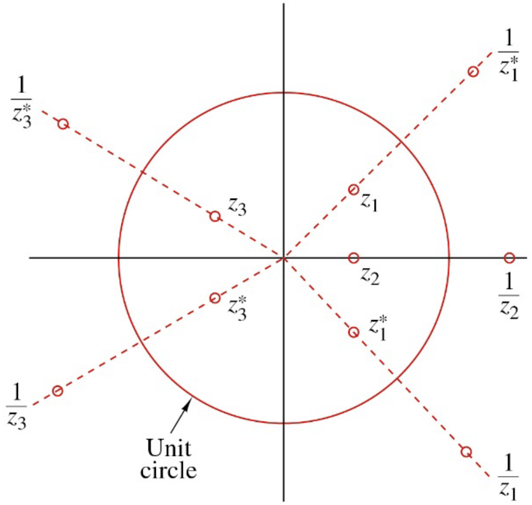
– For linear phase FIR filters

$$z^{-(M-1)}H(z^{-1}) = \pm H(z)$$

Roots of polynomial $H(z)$ are identical to roots of $H(z^{-1})$, so roots occur in reciprocal pairs.

If $h(n)$ is real, roots occur in complex conjugate pairs.

$\boxed{\text{If } z_1 \text{ is a root, so is } 1/z_1,\ z_1^*, \text{ and } 1/z_1^*}$



Filter Design

– Frequency response for Type I FIR filter

- Symmetric, *M* odd

$$h(n) = +h(M - 1 - n)$$

$$H(\omega) = \sum_{n=0}^{(M-1)/2} h(n) e^{-j\omega n}$$

$$H(\omega) = e^{-j\omega(M-1)/2} \sum_{n=0}^{(M-1)/2} a(n)\cos(\omega n)$$

where

$$a(0) = h\left(\frac{M-1}{2}\right)$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right), \quad n = 1, 2, \ldots, \frac{M-1}{2}$$

Type I is the most versatile form.
Can be used for all low-pass, high-pass, band-pass, and band-stop filters

Filter Design

– Frequency response for Type II FIR filter

- Symmetric, *M* even

$$h(n) = +h(M - 1 - n)$$

$$H(\omega) = \sum_{n=0}^{(M/2)-1} h(n) e^{-j\omega n}$$

$$H(\omega) = e^{-j\omega(M-1)/2} \sum_{n=1}^{M/2} b(n) \cos\left[\omega\left(n - \frac{1}{2}\right)\right]$$

where

$$b(n) = 2h\left(\frac{M}{2} - n\right), \quad n = 1, 2, \ldots, \frac{M}{2}$$

Type II is zero at $\omega=\pi$.
Cannot be used for high-pass filter

Filter Design

– Frequency response for Type III FIR filter

- Antisymmetric, *M* odd

$$h(n) = -h(M - 1 - n)$$

$$H(\omega) = \sum_{n=0}^{(M-1)/2} h(n) e^{-j\omega n}$$

$$H(\omega) = j e^{-j\omega(M-1)/2} \sum_{n=1}^{(M-1)/2} a(n) \sin(\omega n)$$

where

$$a(n) = 2h\left(\frac{M-1}{2} - n\right), \quad n = 1, 2, \ldots, \frac{M-1}{2}$$

Type III is zero at $\omega=0$ and $\omega=\pi$. Cannot be used for low-pass or high-pass filter

Filter Design

– Frequency response for Type IV FIR filter

- Antisymmetric, *M* even

$$h(n) = -h(M-1-n)$$

$$H(\omega) = \sum_{n=0}^{(M/2)-1} h(n) e^{-j\omega n}$$

$$H(\omega) = j e^{-j\omega(M-1)/2} \sum_{n=1}^{M/2} b(n) \sin\left[\omega\left(n - \frac{1}{2}\right)\right]$$

where

$$b(n) = 2h\left(\frac{M}{2} - n\right), \quad n = 1, 2, \ldots, \frac{M}{2}$$

Type IV is zero at ω=0.
Cannot be used for low-pass filter.

Filter Design

– Task for FIR filter design:

- Determine the M coefficients, b_k, for

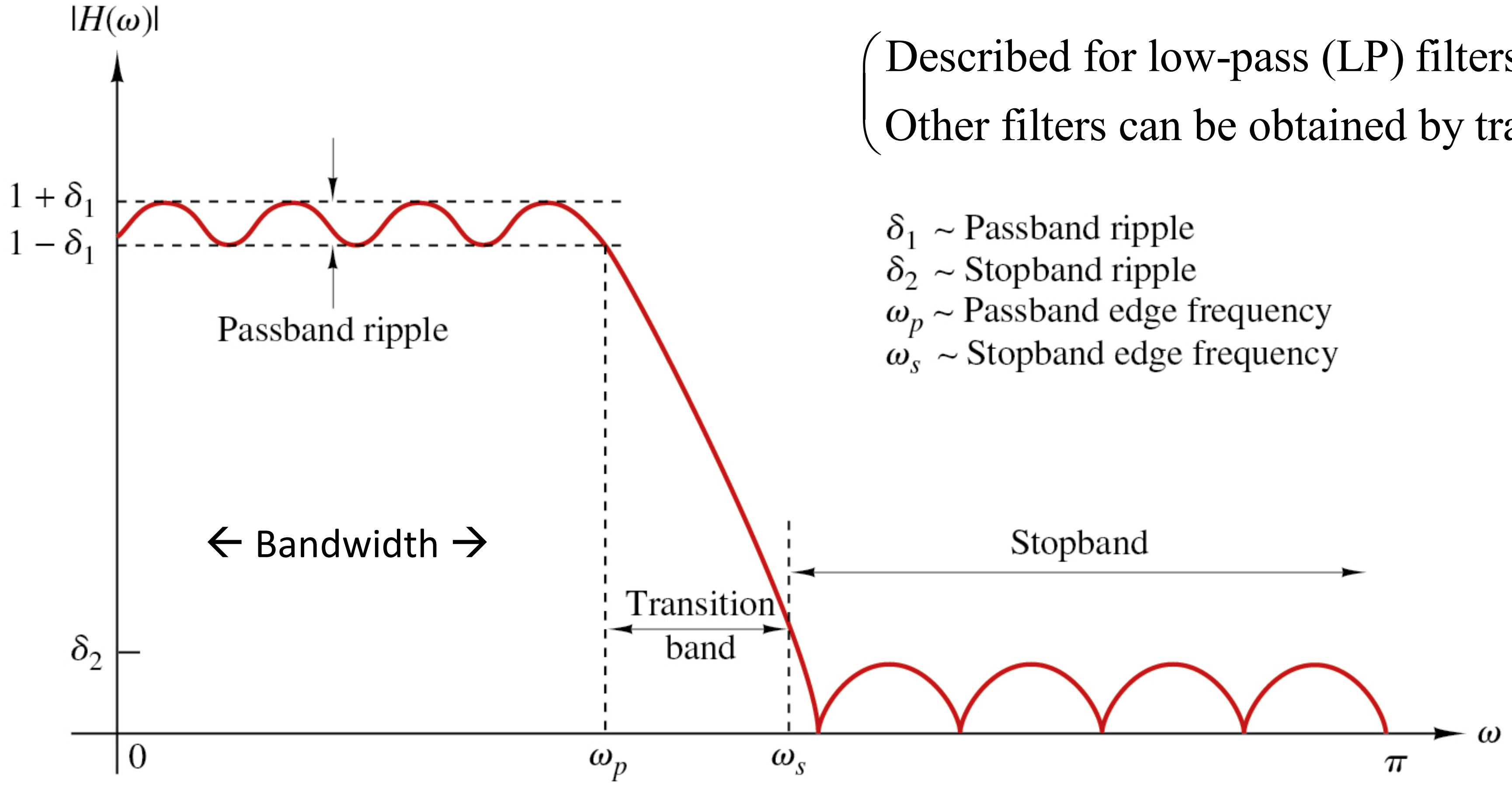
$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

that best match desired filter response, $H_d(\omega)$, in the frequency domain.

Filter Design

- Specifications for physically realizable filters:

Described for low-pass (LP) filters.
Other filters can be obtained by transforming LP filters



δ_1 ~ Passband ripple
δ_2 ~ Stopband ripple
ω_p ~ Passband edge frequency
ω_s ~ Stopband edge frequency

Linear-Phase FIR Filter Design

– Methods
  - Windowing impulse response
    - Specify desired response

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n) e^{-j\omega n}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

    - Cannot have an infinite impulse response, so window the $h_d(n)$
    - If truncate the series at some value of n, that is like a rectangular window
    - We will discuss different types of windowing functions

Linear-Phase FIR Filter Design

– Example: Low-pass filter

- Ideal linear-phase low-pass filter:
  - Specify desired response

$$H_d(\omega) = 1e^{-j\omega(M-1)/2} \quad \text{for } 0 \le |\omega| \le \omega_c$$

Notice that a time delay of (M-1)/2 samples is "built in" with the linear phase.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n} d\omega$$

  - For concreteness, consider $\omega_c = 3\pi/16$ (3/16 of Nyquist)
    Filter of length M=9 (odd)
    - For CD quality music, sampled at 44.1 kHz, This would correspond to ~ 8 kHz sampling which is what sampling rate is for digital phone.
    - This would correspond to a Nyquist frequency of ~4 kHz
      like listening to someone play a song over the phone

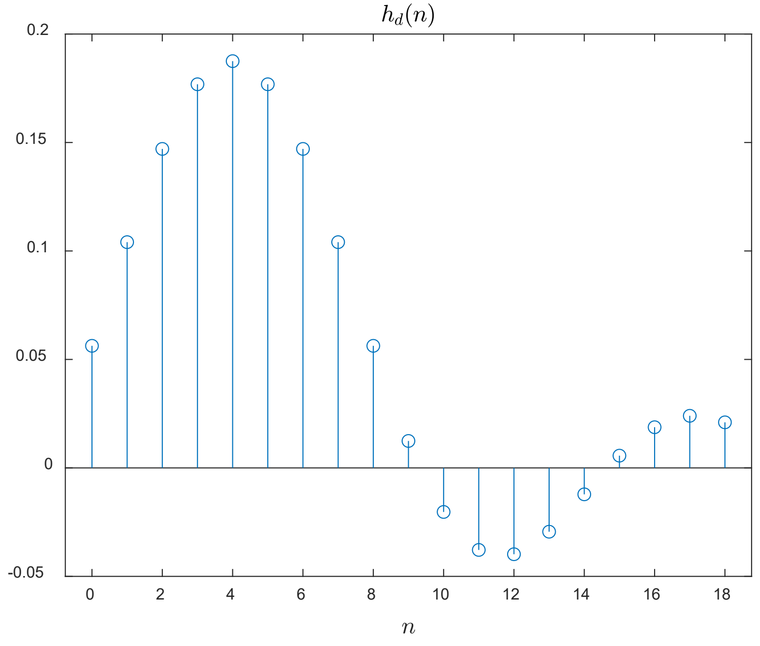
Linear-Phase FIR Filter Design

- Find time-domain impulse response

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H_d(\omega) e^{j\omega n} d\omega$$

$$\left(\text{On board}\right)$$

$$h_d(n) = \frac{\omega_c}{\pi}\text{sinc}\left[\omega_c\left(n-\frac{M-1}{2}\right)\right] = \frac{3}{16}\text{sinc}\left[\frac{3\pi}{16}(n-4)\right]$$



Caution! This is the unnormalized sinc function: sin(x)/x
You have to divide x by pi before calling Matlab's sinc function

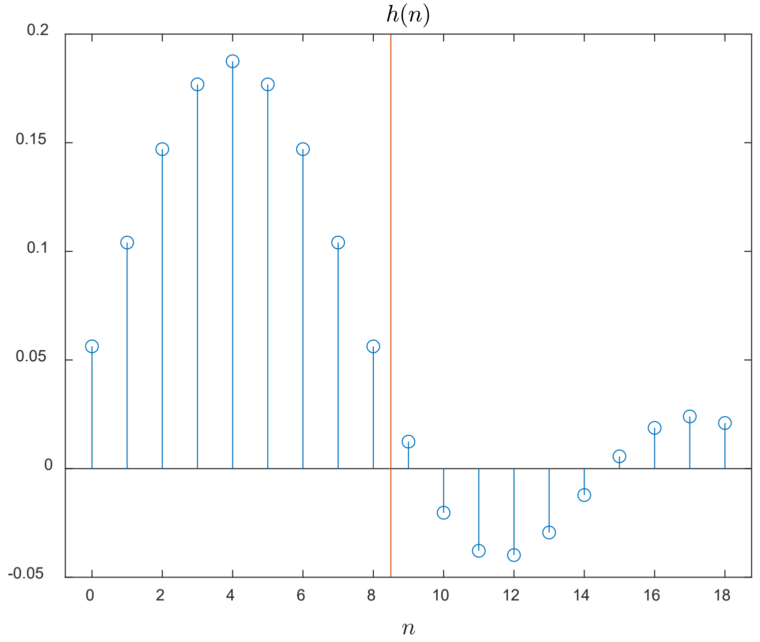
Linear-Phase FIR Filter Design

- Rectangular window:

$$w(n)=\begin{cases}1, & n=0,1,\ldots,M-1\\ 0, & \text{otherwise}\end{cases} \Rightarrow \quad w(n)=\begin{cases}1, & n=0,1,\ldots,8\\ 0, & \text{otherwise}\end{cases}$$

$$h_d(n)=\begin{cases}\operatorname{sinc}\left[\omega_c\left(n-\dfrac{M-1}{2}\right)\right], & n=0,1,\ldots,M-1\\ 0, & \text{otherwise}\end{cases}$$

$$h(n)=\begin{cases}\dfrac{3}{16}\operatorname{sinc}\left[\dfrac{3\pi}{16}(n-4)\right], & n=0,1,\ldots,8\\ 0, & \text{otherwise}\end{cases}$$

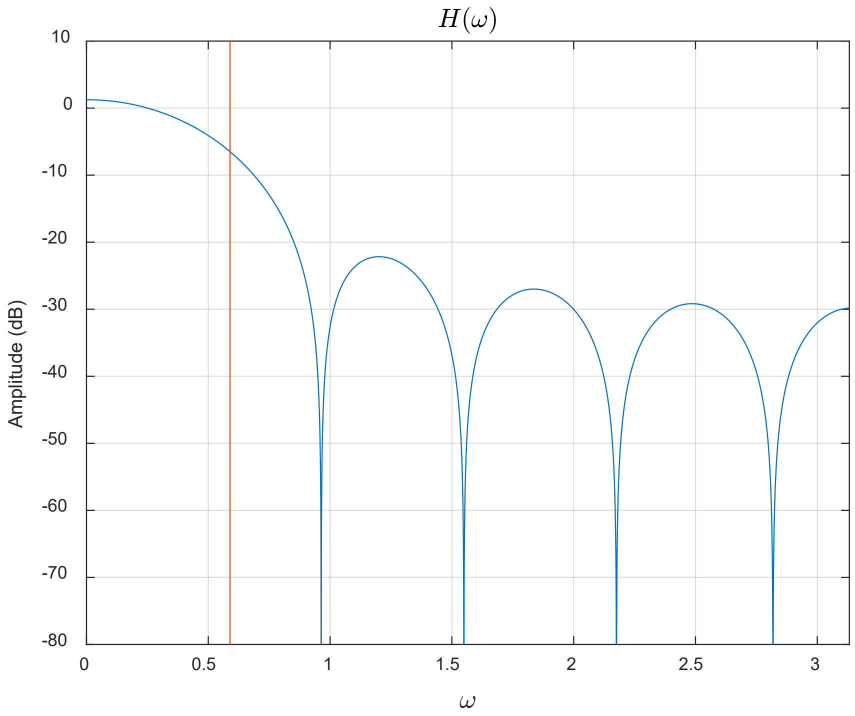


Notice $h(M-1-n)=h(n)$

$\{0.0563 \quad 0.1041 \quad 0.1470 \quad 0.1768 \quad 0.1875 \quad 0.1768 \quad 0.1470 \quad 0.1041 \quad 0.0563\}$

Linear-Phase FIR Filter Design

- It would be nice to know what the frequency domain transfer function looks like after windowing
  - Can do it numerically in Matlab



Linear-Phase FIR Filter Design

- In general, you can see the effect of the window by convolving the window function in the frequency domain with the ideal filter transfer function.

$$H(\omega) = H_d(\omega) \otimes W\left(\omega\right)$$

For a window of lenght M

$$W\left(\omega\right) = \sum_{n=0}^{M-1} w(n)e^{-j\omega n}$$

For a rectangular window of length M $w(n)$ is 1 in the sum

$$W\left(\omega\right) = \sum_{n=0}^{M-1} e^{-j\omega n}$$

Linear-Phase FIR Filter Design

Use the expression for the finite geometric series:

$$S=\sum_{n=0}^{M-1} r^n=\frac{\left(1-r^M\right)}{1-r}$$

$$W(\omega)=\sum_{n=0}^{M-1} e^{-j\omega n}=\frac{\left(1-e^{-j\omega M}\right)}{1-e^{-j\omega}}$$

(on board)

$$W(\omega)=e^{-j\omega(M-1)/2}\frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

Magnitude: $\left|W(\omega)\right|=\frac{\sin(\omega M/2)}{\sin(\omega/2)}$ for $-\pi \le \omega \le \pi$

Phase: $\Theta(\omega)=\begin{cases}-\omega(M-1)/2 & \text{for } \sin(\omega M/2)\ge 0\\ -\omega(M-1)/2+\pi & \text{for } \sin(\omega M/2)<0\end{cases}$

Linear-Phase FIR Filter Design

The frequency domain transfer function is:

$$H(\omega)=\int_{-\pi}^{\pi} H_d(\nu)W(\omega-\nu)d\nu=\int_{-\pi}^{\pi} W(\omega)H_d(\omega-\nu)d\nu$$

For rectangular window:

$$H(\omega)=\int_{-\pi}^{\pi} W(\omega)H_d(\omega-\nu)d\nu=\int_{-\omega_c}^{\omega_c} e^{-j\nu(M-1)/2}\frac{\sin(\nu M/2)}{\sin(\nu/2)}1e^{-j(\omega-\nu)(M-1)/2}d\nu$$

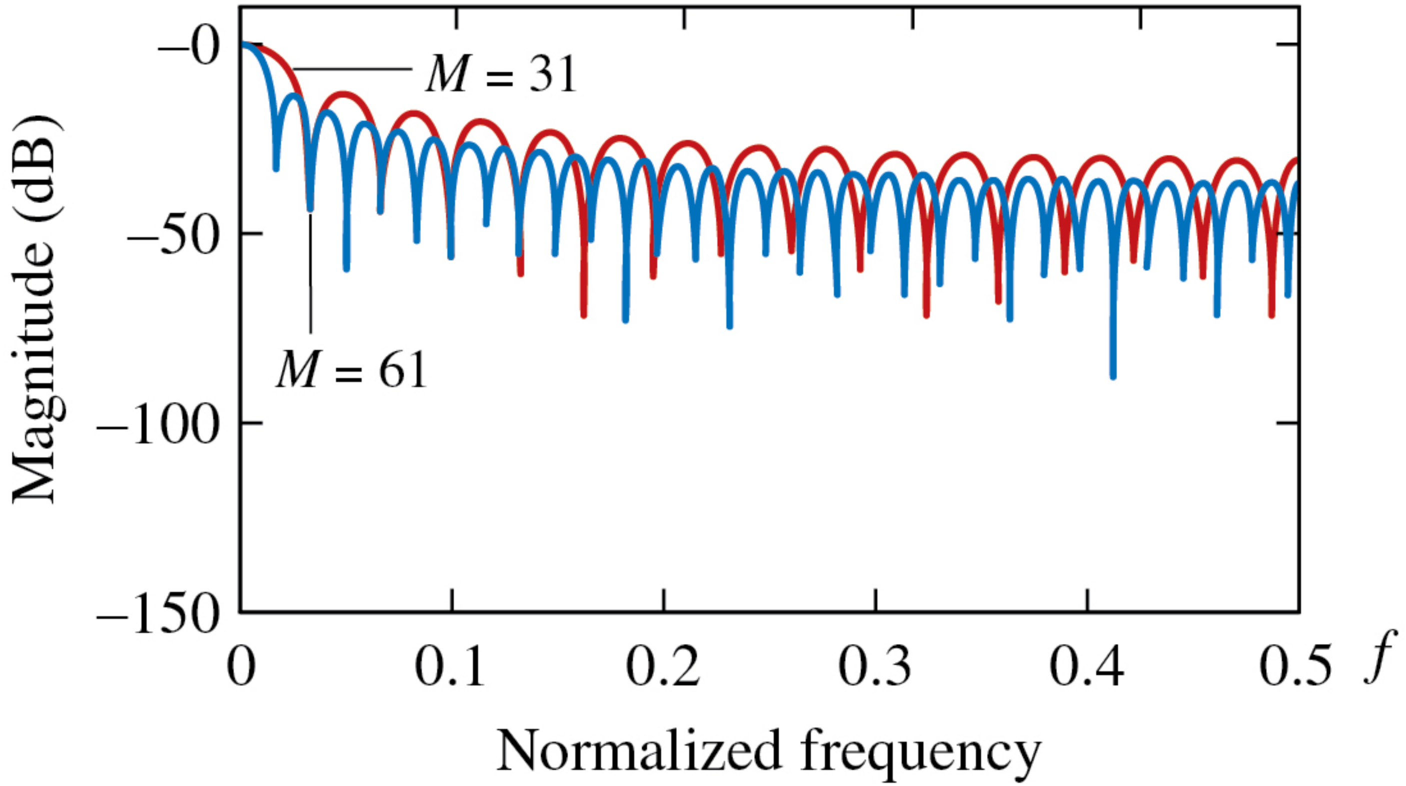
$$H(\omega)=\int_{-\omega_c}^{\omega_c} e^{-j\nu(M-1)/2}e^{+j\nu(M-1)/2}e^{-j\omega(M-1)/2}\frac{\sin(\nu M/2)}{\sin(\nu/2)}d\nu$$

$$H(\omega)=e^{-j\omega(M-1)/2}\int_{-\omega_c}^{\omega_c}\frac{\sin(\nu M/2)}{\sin(\nu/2)}d\nu$$

Linear-Phase FIR Filter Design

Linear-Phase FIR Filter Design

Here is the Magnitude of the window function for a couple values of M

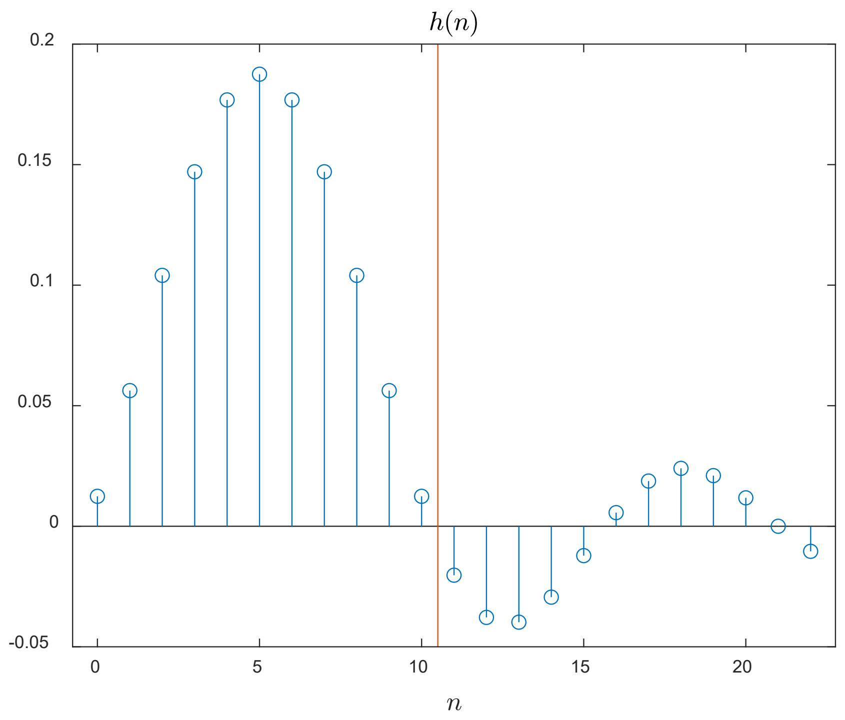


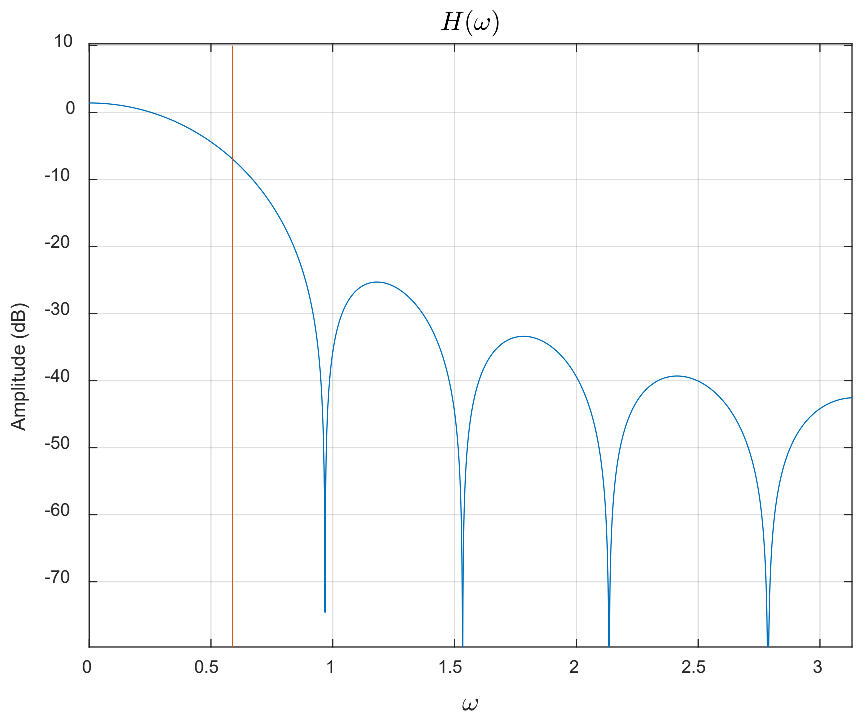
Linear-Phase FIR Filter Design

- Increasing *M* makes sidelobes narrower, but height doesn't decrease much.
- How does value of M affect our example: $\omega_c = 3\pi/16$

Linear-Phase FIR Filter Design

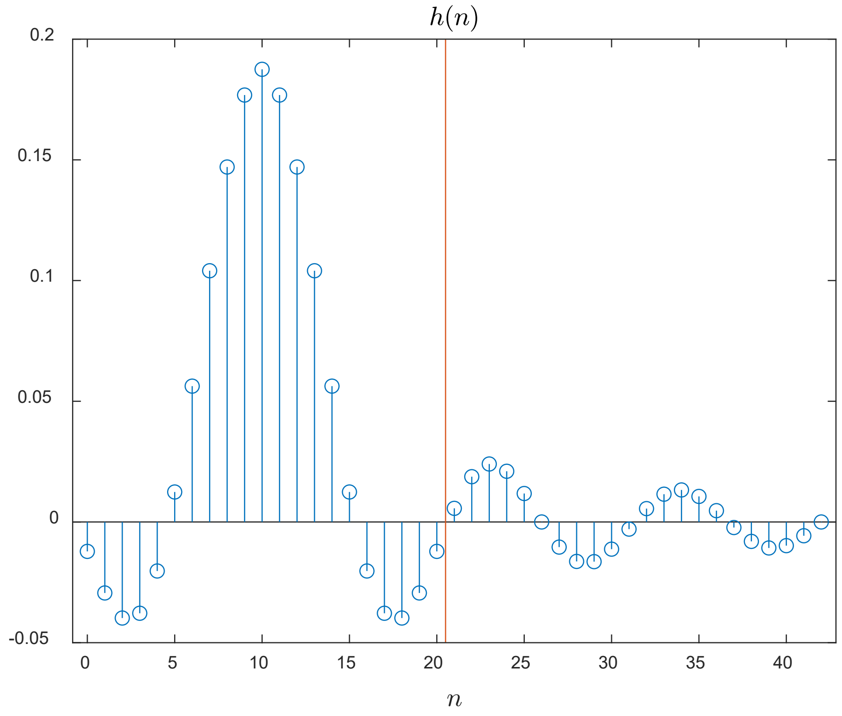
$M = 11$

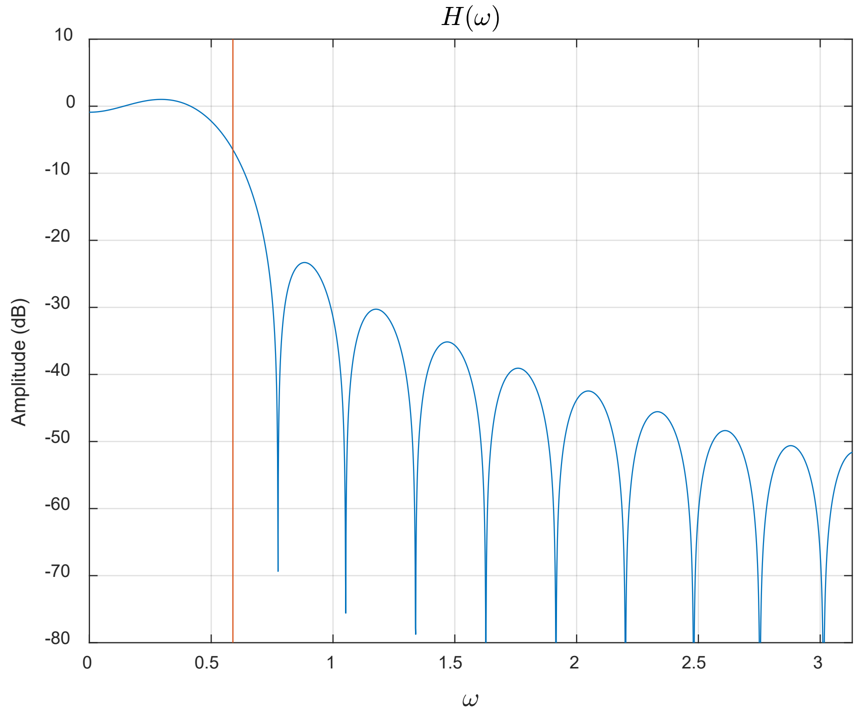




Linear-Phase FIR Filter Design

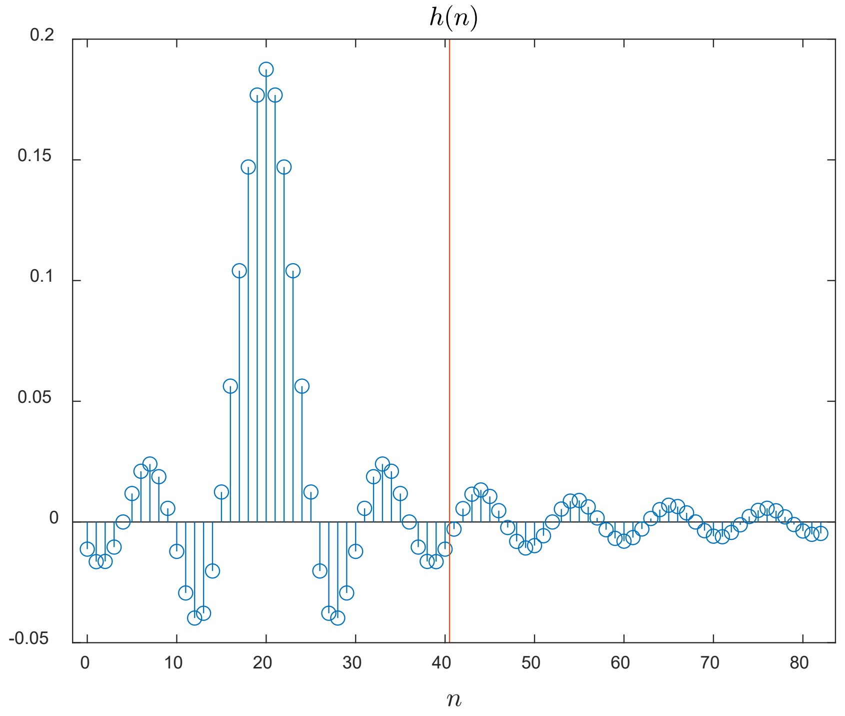
$M = 21$

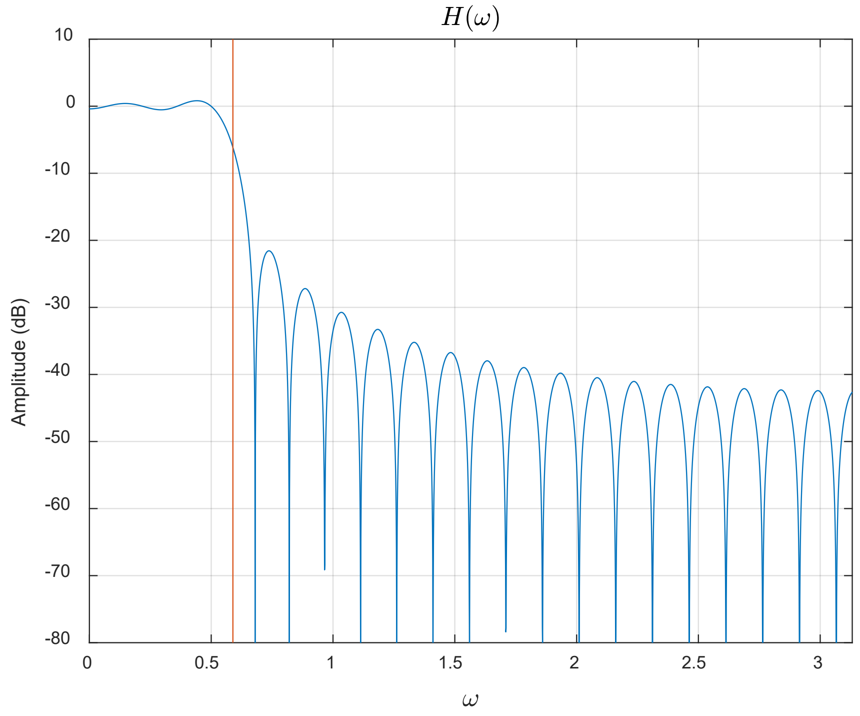




Linear-Phase FIR Filter Design

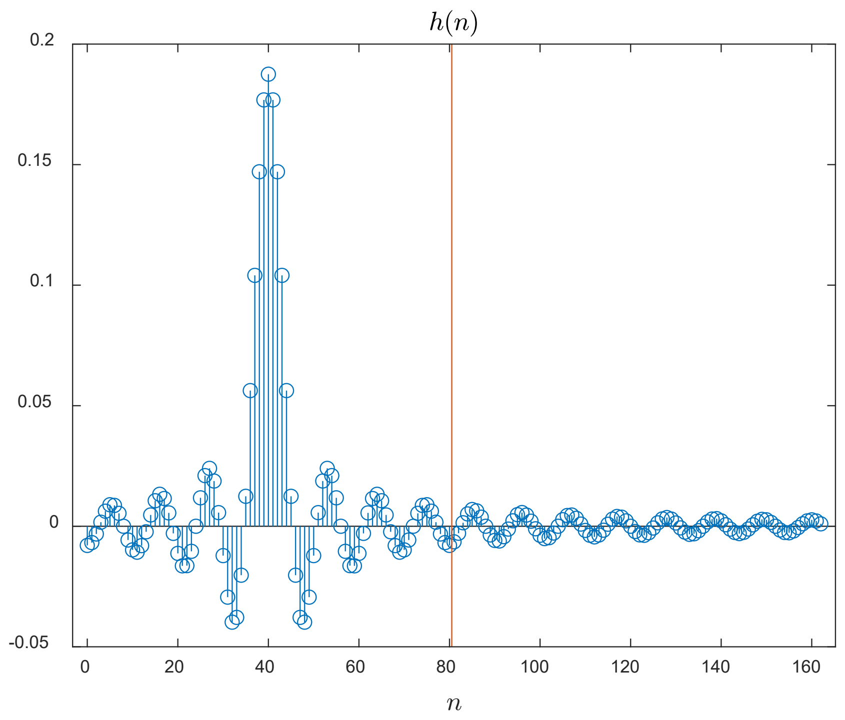
M = 41

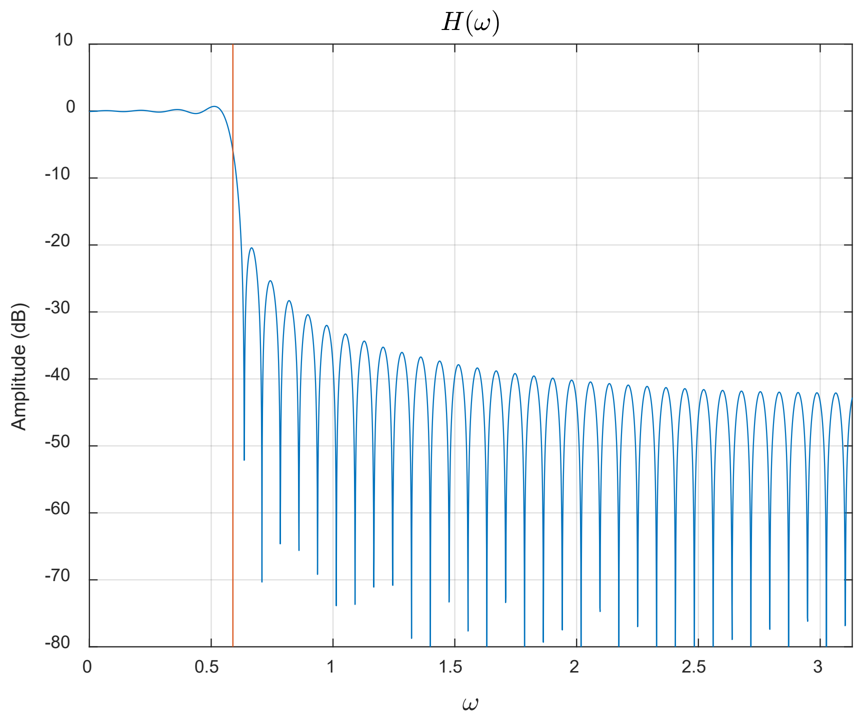




Linear-Phase FIR Filter Design

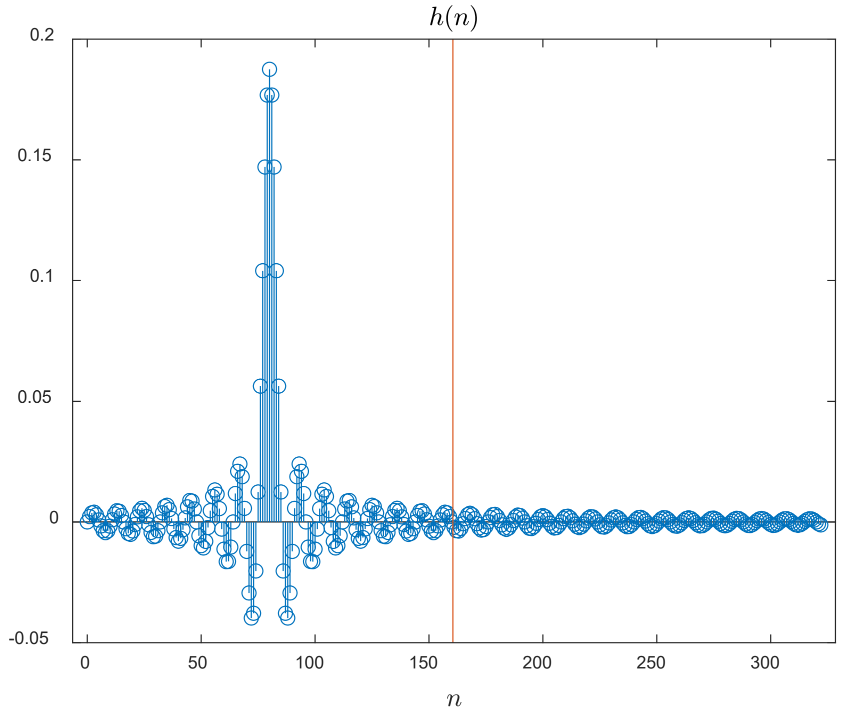
M = 81

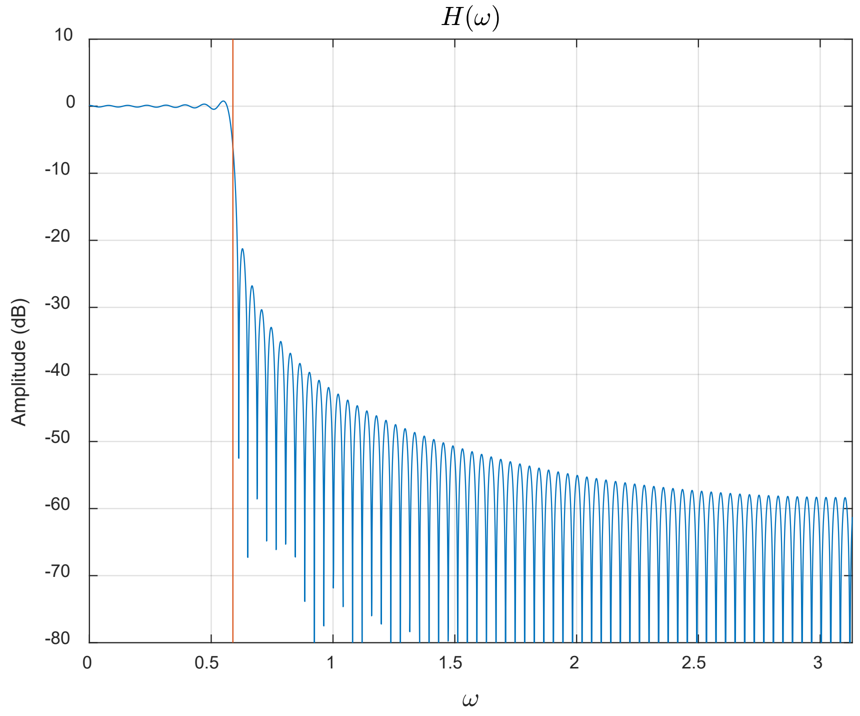




Linear-Phase FIR Filter Design

$M = 161$





Linear-Phase FIR Filter Design

- Better windows are ones that do not have abrupt discontinuities in the time domain
  - Result in lower sidelobes, less ringing in passband, and steeper fall off

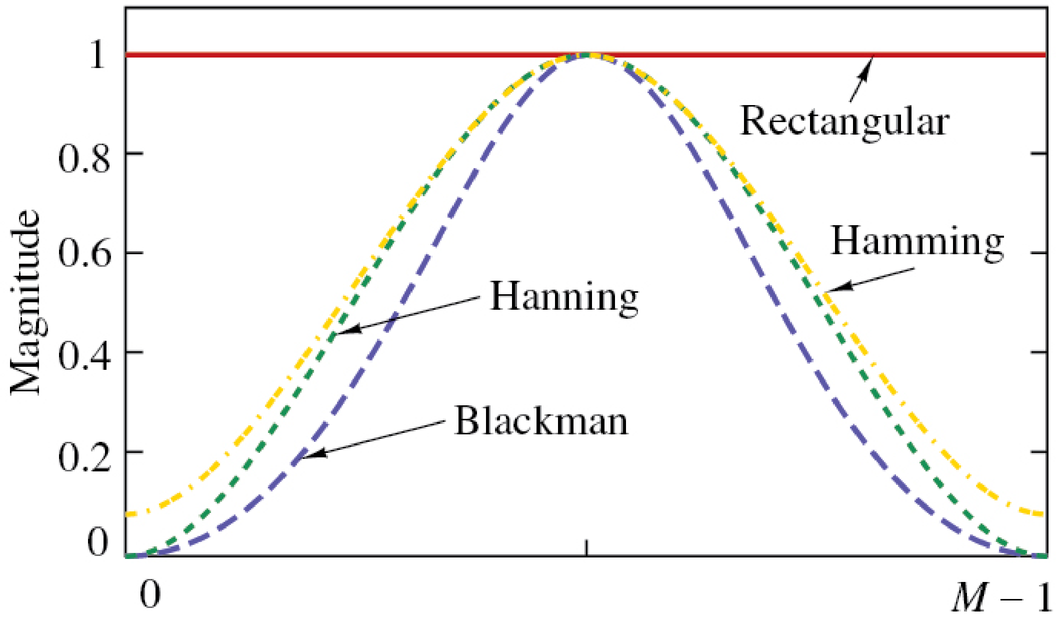
Linear-Phase FIR Filter Design

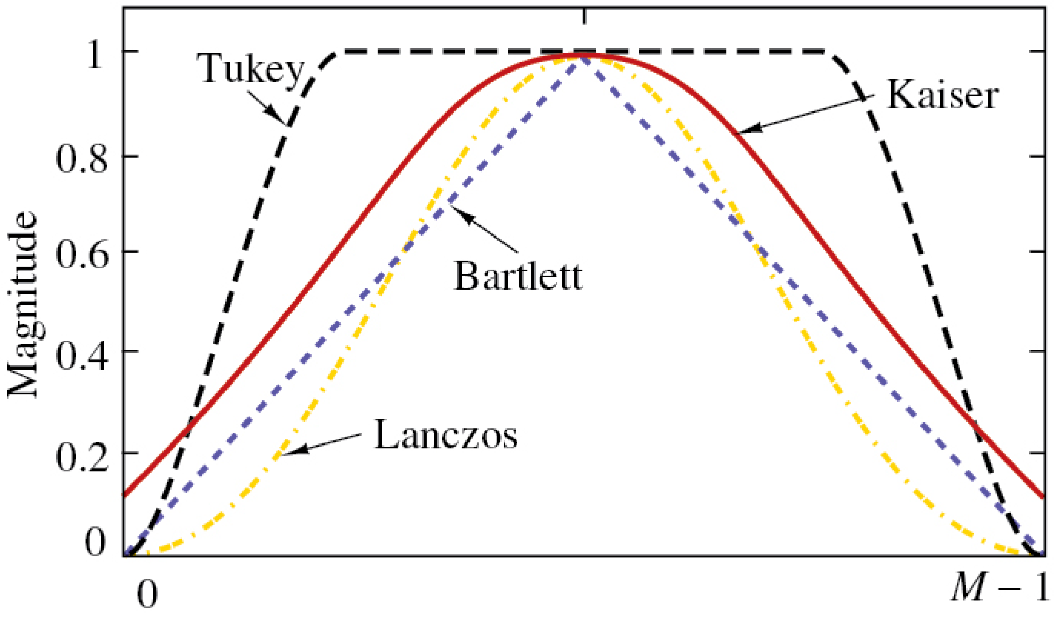
- Table of windows from the book

Name of window	Time-domain sequence, $h(n), 0 \le n \le M-1$
Bartlett (triangular)	$1 - \dfrac{2\left\lvert n - \frac{M-1}{2} \right\rvert}{M-1}$
Blackman	$0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}$
Hamming	$0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$
Kaiser	$\dfrac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]}$
Lanczos	$\left\{\dfrac{\sin\left[2\pi\left(n - \frac{M-1}{2}\right) \Big/ (M-1)\right]}{2\pi\left(n - \frac{M-1}{2}\right) \Big/ \left(\frac{M-1}{2}\right)}\right\}^L, \quad L > 0$
	$1, \left\lvert n - \frac{M-1}{2} \right\rvert \le \alpha\frac{M-1}{2}, \qquad 0 < \alpha < 1$
Tukey	$\dfrac{1}{2}\left[1 + \cos\left(\dfrac{n - (1+a)(M-1)/2}{(1-\alpha)(M-1)/2}\pi\right)\right]$
	$\alpha(M-1)/2 \le \left\lvert n - \frac{M-1}{2} \right\rvert \le \frac{M-1}{2}$

Linear-Phase FIR Filter Design

- Example of windows in book:





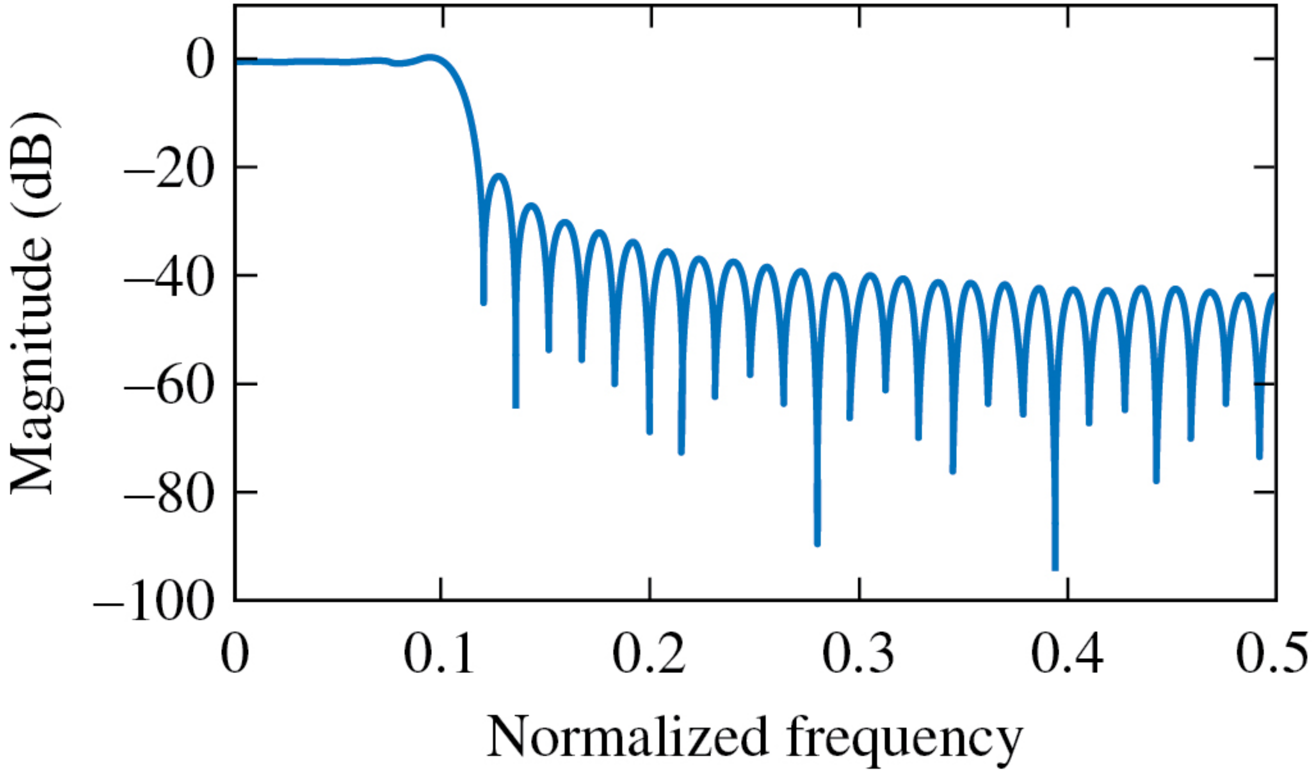
Linear-Phase FIR Filter Design

- Comparison of FIR filters with different windows

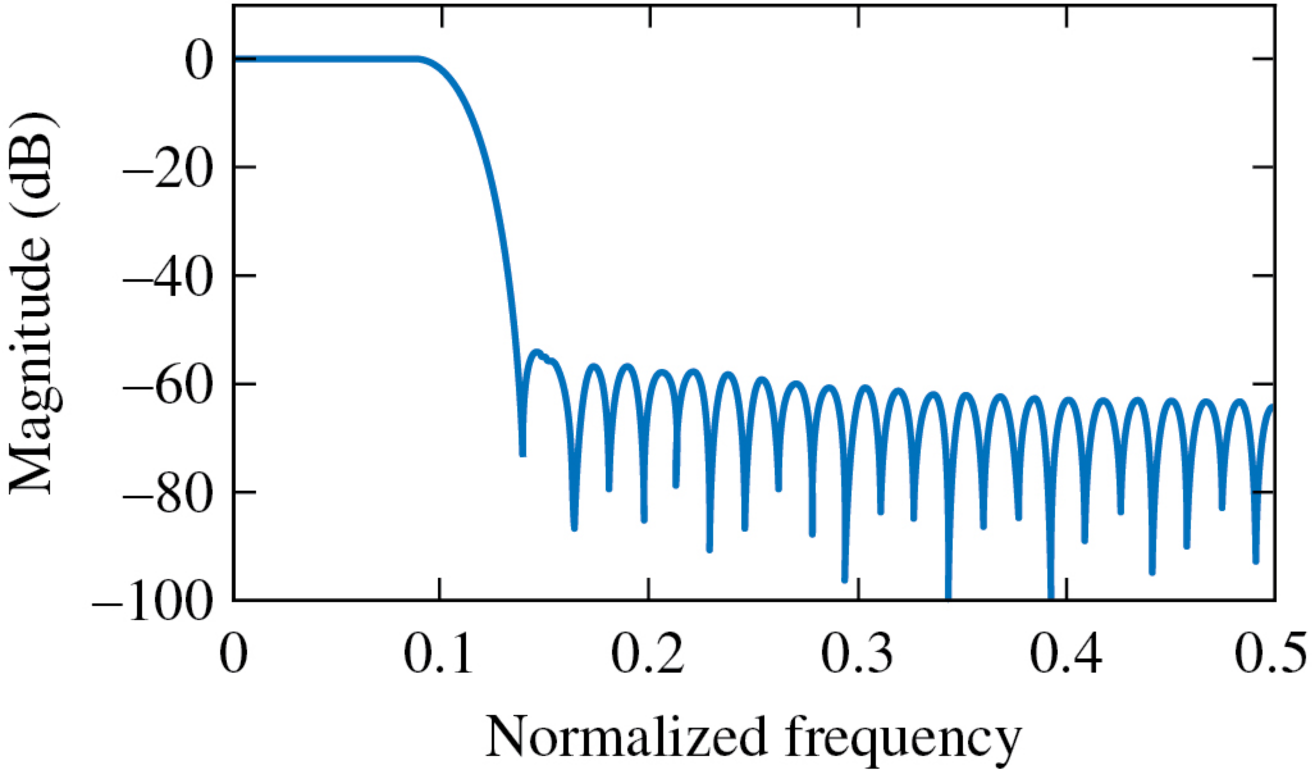
Type of window	Approximate transition width of main lobe	Peak sidelobe (dB)
Rectangular	$4\pi/M$	−13
Bartlett	$8\pi/M$	−25
Hanning	$8\pi/M$	−31
Hamming	$8\pi/M$	−41
Blackman	$12\pi/M$	−57

Wider main lobe – more smoothing and wider transition region
Larger sidelobes – more ripple
Making *M* larger makes transition narrower at expense of complexity and time delay introduced

Rectangular (M=61)

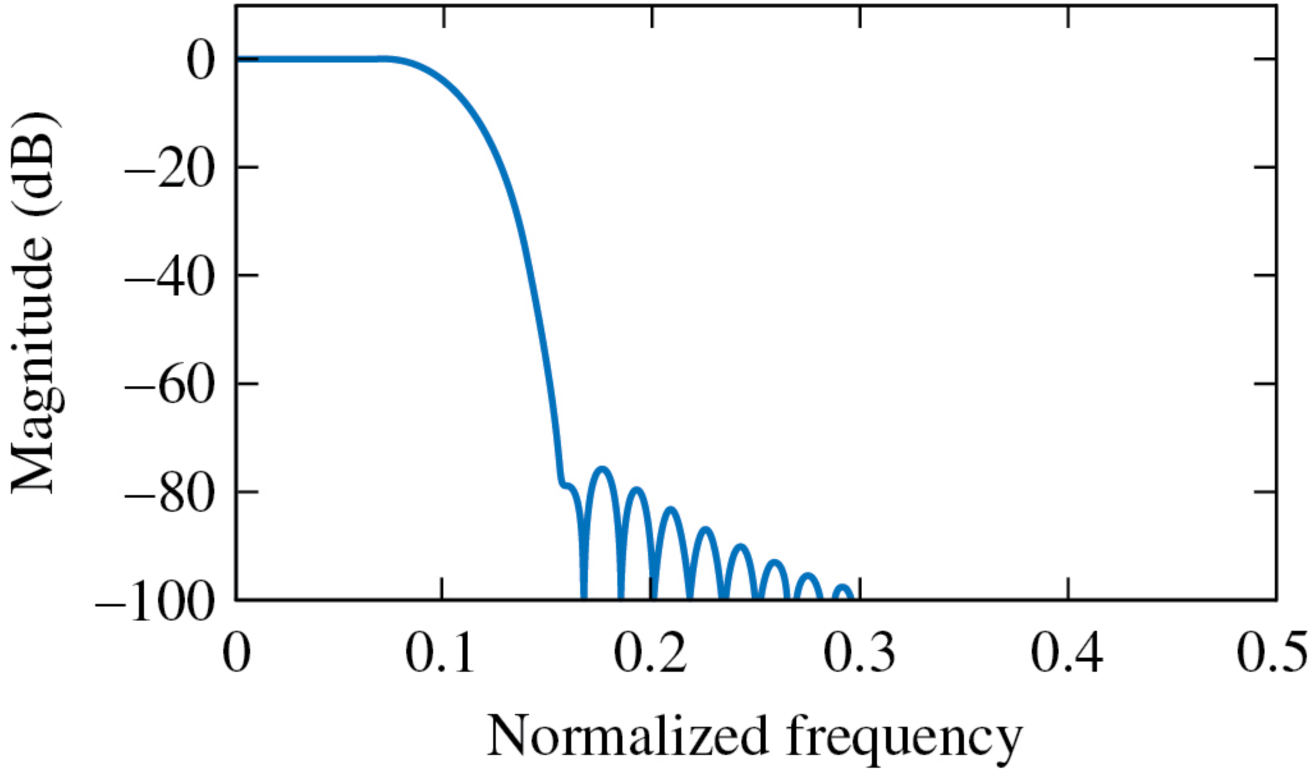


Hamming (M=61)

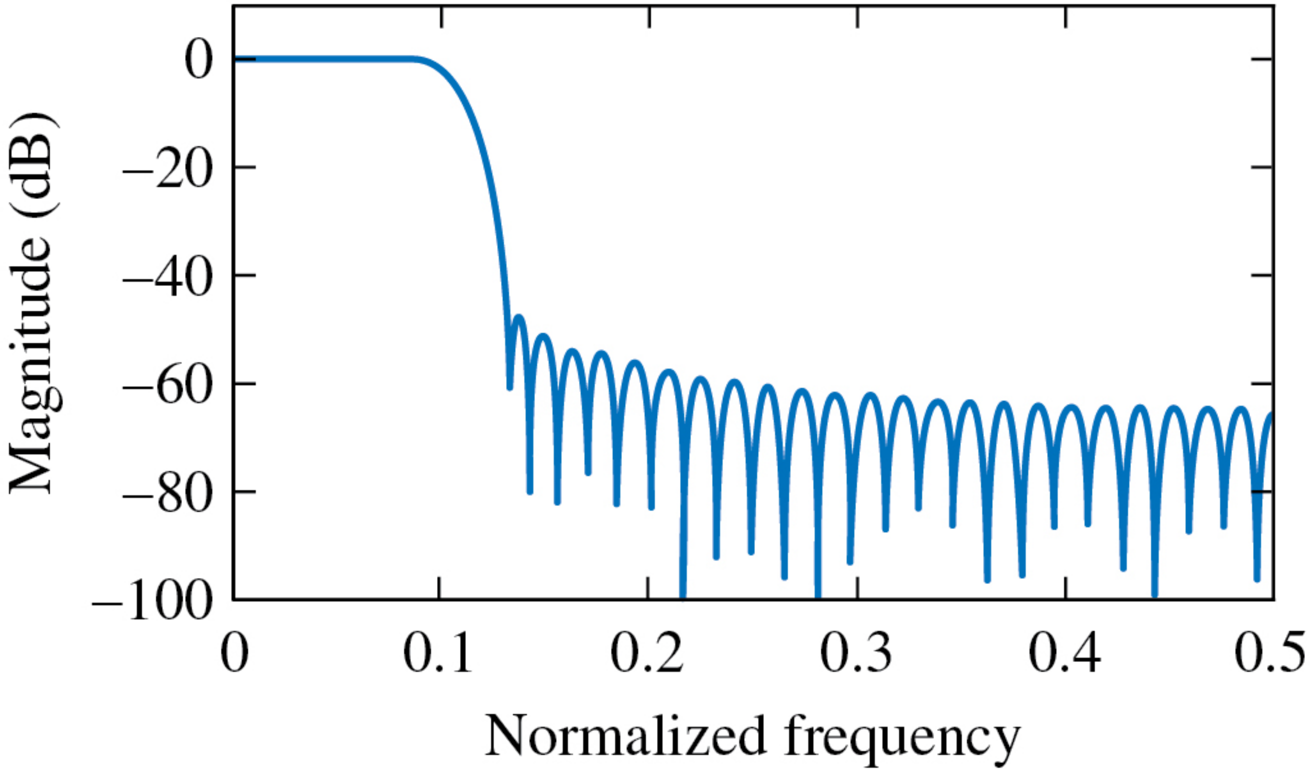


Magnitude (dB)
0
–20
–40
–60
–80
–100
0
0.1
0.2
0.3
0.4
0.5
Normalized frequency

Blackman (M=61)



Kaiser (α=4, M=61)



Linear-Phase FIR Filter Design

- Matlab has even more (see filterDesigner)
  - All of these windows are symmetric (or antisymmetric) so they have linear phase
- There is an optimal window that is maximally concentrated in both the time and frequency domain:
  - Discrete Prolate Spheroidal Sequences (DPSS)
  - The Kaiser window is a good approximation to this

Linear-Phase FIR Filter Design

- Comparison of Kaiser to DPSS

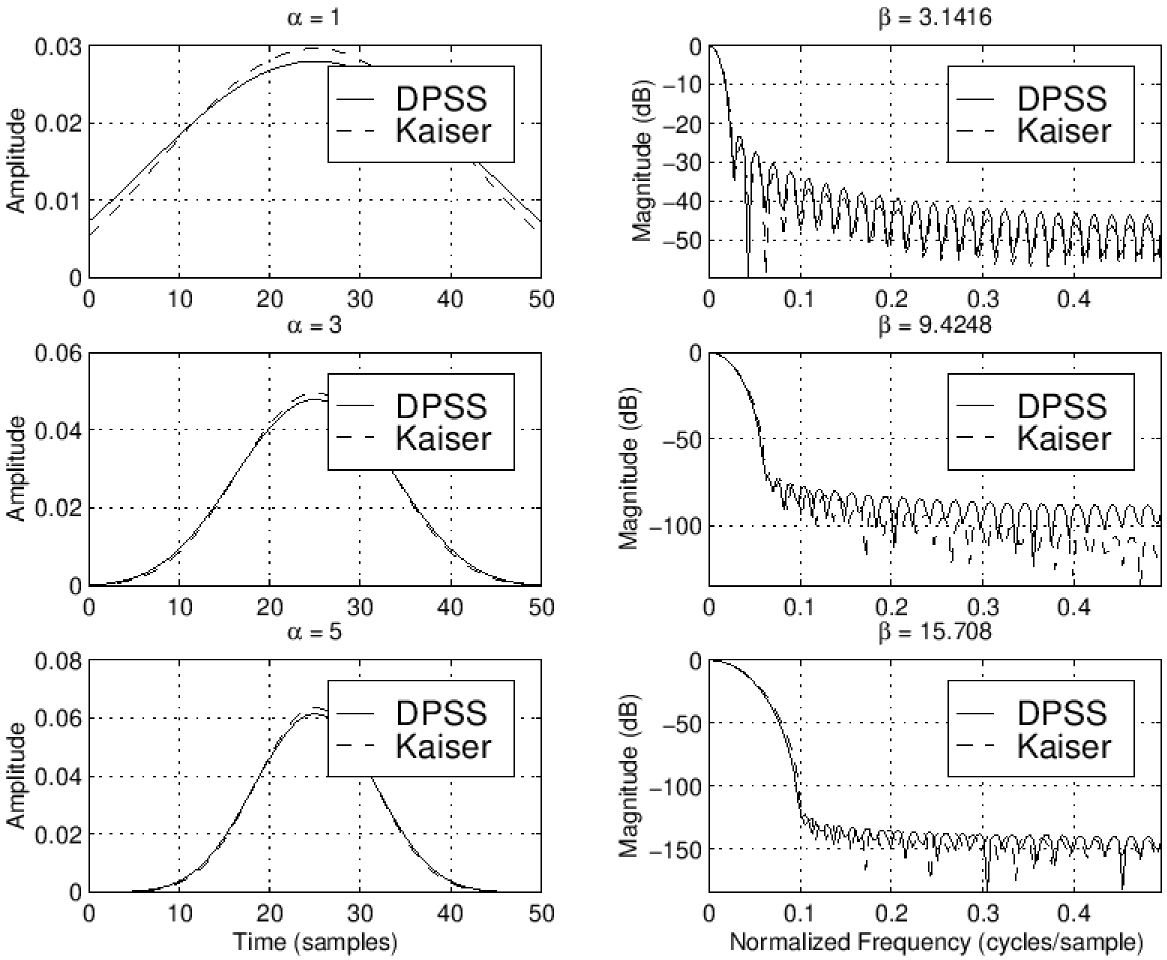


Figure: Comparison of length 51 DPSS and Kaiser windows for $\alpha = 1, 3, 5$.

Linear-Phase FIR Filter Design

- Comparison of Kaiser to DPSS

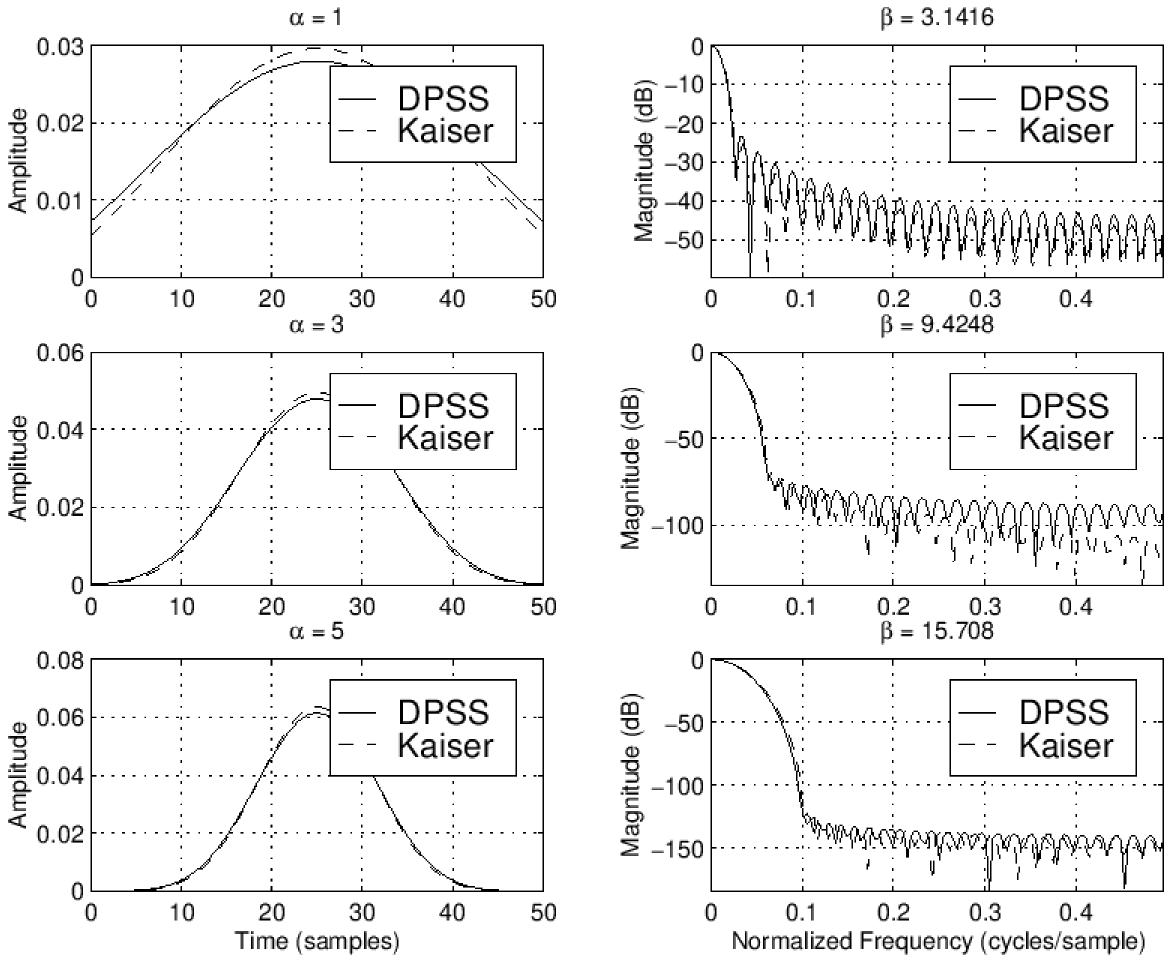


Figure: Comparison of length 51 DPSS and Kaiser windows for $\alpha = 1, 3, 5$.

Linear-Phase FIR Filter Design

- Ideal linear-phase band-stop filter:
  - Specify desired response

$$H_d(\omega) = 1e^{-j\omega(M-1)/2} \quad \text{for } \omega_c \le |\omega| \le \pi$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n}\, d\omega$$

- Ideal linear-phase band-pass filter:
  - Specify desired response

$$H_d(\omega) = 1e^{-j\omega(M-1)/2} \quad \text{for } \omega_{c1} \le |\omega| \le \omega_{c2}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n}\, d\omega$$

Linear-Phase FIR Filter Design

- Example:

10.1 Design an FIR linear-phase, digital filter approximating the ideal frequency response

$$H_d(\omega) = \begin{cases} 1, & \text{for } |\omega| \le \dfrac{\pi}{6} \\ 0, & \text{for } \dfrac{\pi}{6} < |\omega| \le \pi \end{cases}$$

a. Determine the coefficients of a 25-tap filter based on the window method with a rectangular window.

b. Determine and plot the magnitude and phase response of the filter.

c. Repeat parts (a) and (b) using the Hamming window.

d. Repeat parts (a) and (b) using a Bartlett window.

Bartlett (triangular)	$1 - \dfrac{2\left\|n - \dfrac{M-1}{2}\right\|}{M-1}$
Blackman	$0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}$
Hamming	$0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$

Linear-Phase FIR Filter Design

- Methods
  - Frequency sampling methods
    - Specify desired response at set of equally spaced frequencies
    - Solve for $h(n)$ from the specified response
  - Optimal Equiripple linear-phase FIR filters
    - Enable more precise control of pass and stop-band critical frequencies