Digital Signal Processing

Class 10
02/20/2025

ENGR 71

- Class Overview
  - Z-Transform
- Assignments
  - Reading:

    Chapter 3: The z-Transform and its Applications to the Analysis of LTI

ENGR 71

- Homework 4
  - Problems: 3.2 (b & f), 3.4(d), 3.12,
    3.14(b), 3.16, 3.31
    C3.3 (use Matlab)
    C3.5 (use Matlab)

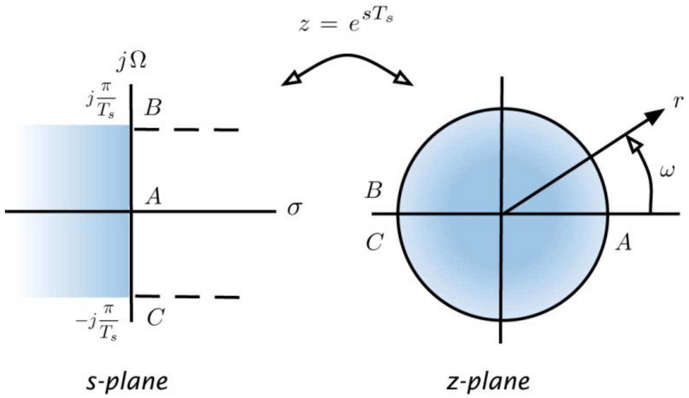
  Due Feb. 20

Class Information

- Z-Transform Topics
  - The z-Transform
  - Properties of the z-Transform
  - Rational z-Transforms
  - Inversion of the z-Transform
  - Analysis of Linear Time-Invariant Systems in the z-Domain
  - The One-sided z-Transform

Laplace and z-Transforms

- Laplace and Z-transforms:
  - Laplace: $X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$ $\qquad x(t) = \frac{1}{2\pi j} \lim_{T\to\infty} \int_{\gamma-jT}^{\gamma+jT} X(s) e^{st} ds$
  - z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ $\qquad x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$



  - Mainly concerned with causal signals and systems: $t \geq 0$ $\qquad n \geq 0$

$$x(t) = x(t)u(\mathrm{t})$$
$$x[n] = x[n]u[n]$$

    - Limits in sum and integral start at 0:

$$X(s) = \int_{0}^{+\infty} x(t) e^{-st} dt$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Z-Transform

- Definition of z-transform:
  - Bilateral

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

  - Unilateral (causal signals & systems)

$$X(z) = \sum_{n=0}^{\infty} \mathrm{x}(n)\, z^{-n}$$

  - Inverse:

$$x(n) = \frac{1}{2\pi j} \oint X(z)\, z^{n-1} dz$$

    - Rarely use this, although this is common integral in complex variables math courses.
    - We compute forward & inverse by use of transform pairs and properties.
    - Can also find inverse by long division.

Common z-transforms

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Z-Transform Properties

- Linearity

$$a_1x_1[n]+a_2x_2[n]\Leftrightarrow a_1X_1(z)+a_2X_2(z)$$

- Time shift

For $x[n]\Leftrightarrow X(z)$ $\qquad x[n-k]\Leftrightarrow z^{-k}X(z)$

- Scaling

For $x[n]\Leftrightarrow X(z)$ ROC $r_1<|z|<r_2$ $\qquad \alpha^n x[n]\Leftrightarrow X\left(\frac{z}{\alpha}\right)$ ROC $|\alpha|r_1<|z|<|\alpha|r_2$

- Time reversal

For $x[n]\Leftrightarrow X(z)$ ROC $r_1<|z|<r_2$ $\qquad x[-n]\Leftrightarrow X\left(\frac{1}{z}\right)$ ROC $\frac{1}{r_1}<|z|<\frac{1}{r_2}$

- Convolution

Z-Transform Properties

- Linearity

$$a_1 x_1(n) + a_2 x_2(n) \Leftrightarrow a_1 X_1(z) + a_2 X_2(z)$$

- Time shift

For $x(n) \Leftrightarrow X(z)$ $\qquad x(n-k) \Leftrightarrow z^{-k} X(z)$

- Scaling

For $x(n) \Leftrightarrow X(z)$ ROC $r_1 < |z| < r_2$ $\qquad \alpha^n x(n) \Leftrightarrow X\left(\frac{z}{\alpha}\right)$ ROC $|\alpha| r_1 < |z| < |\alpha| r_2$

- Time reversal

For $x(n) \Leftrightarrow X(z)$ ROC $r_1 < |z| < r_2$ $\qquad x(-n) \Leftrightarrow X\left(\frac{1}{z}\right)$ ROC $\frac{1}{r_1} < |z| < \frac{1}{r_2}$

Z-Transform Properties

- Convolution

$$\text{For } \; x(n) \Leftrightarrow X(z) \;\; \& \;\; y(n) \Leftrightarrow Y(z) \qquad x(n) * y(n) \Leftrightarrow X(z)Y(z)$$

- Correlation

$$\text{For } \; x(n) \Leftrightarrow X(z) \;\; \& \;\; y(n) \Leftrightarrow Y(z) \qquad r_{xy}(l) \Leftrightarrow X(z)Y(-z)$$

- Differentiation in z-domain

$$\mathrm{n}x(n) \Leftrightarrow -z\frac{dX(z)}{dz}$$

- Real & Imaginary parts

$$\mathrm{Re}\left[x(n)\right] \Leftrightarrow \tfrac{1}{2}\left[X(z) + X^*(z^*)\right] \qquad , \qquad \mathrm{Re}\left[x(n)\right] \Leftrightarrow \tfrac{1}{2}j\left[X(z) - X^*(z^*)\right]$$

Z-Transform Properties

- Initial value theorem (for causal signals)

$$x(0) = \lim_{z \to \infty} X(z)$$

- Final value theorem (if final value exists)

$$\lim_{n \to \infty} x(n) = \lim_{z \to 1} \left(z - 1 \right) X(z)$$

- Multiplication in time domain

$$x_1(n) x_2(n) \Leftrightarrow \frac{1}{2\pi j} \oint X_1\left(\upsilon\right) X_2\left(z/\upsilon\right) \upsilon^{-1} d\upsilon$$

- Parseval's relation

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) \Leftrightarrow \frac{1}{2\pi j} \oint X_1\left(\upsilon\right) X_2\left(1/\upsilon^*\right) \upsilon^{-1} d\upsilon$$

Table 3.2

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\text{Im}\{x(n)\}$	$\frac{1}{2}j[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\dfrac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}\,dv$	At least, $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

Z-Transform Rational Functions

- Rational function

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Can also be written with positive powers of z

$$H(z) = \left(\frac{b_0 z^{-M}}{a_0 z^{-N}}\right) \frac{z^M + (b_1/b_0) z^{M-1} + \cdots + (b_M/b_0)}{z^N + (a_1/a_0) z^{N-1} + \cdots + (a_N/a_0)}$$

- The numerator and denominator can be factor into product of linear terms:

$$H(z) = \left(\frac{b_0}{a_0}\right) z^{N-M} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_M)}$$

Z-Transform Rational Functions

- Zeros and Poles

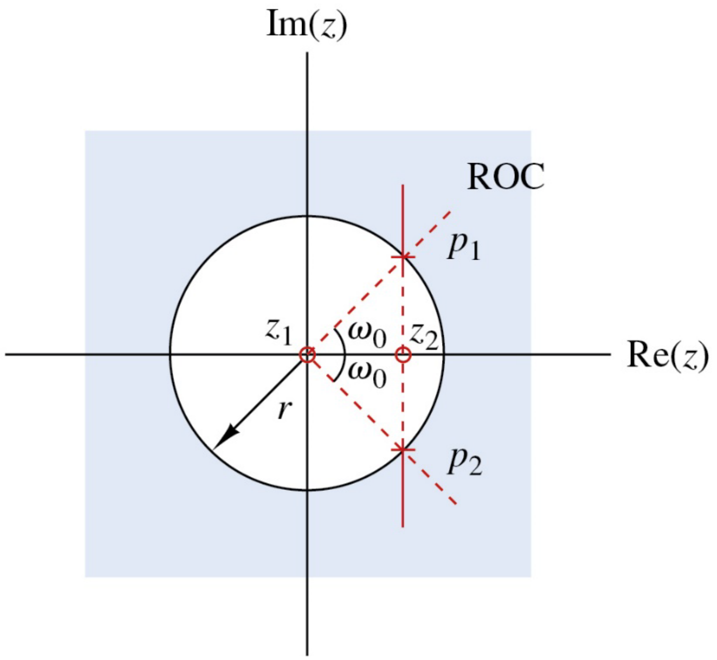
$$\frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_M)}$$

$z_1, z_2, \ldots$ are the zeros

$p_1, p_2, \ldots$ are the poles

Z-Transform Rational Functions

- Example:

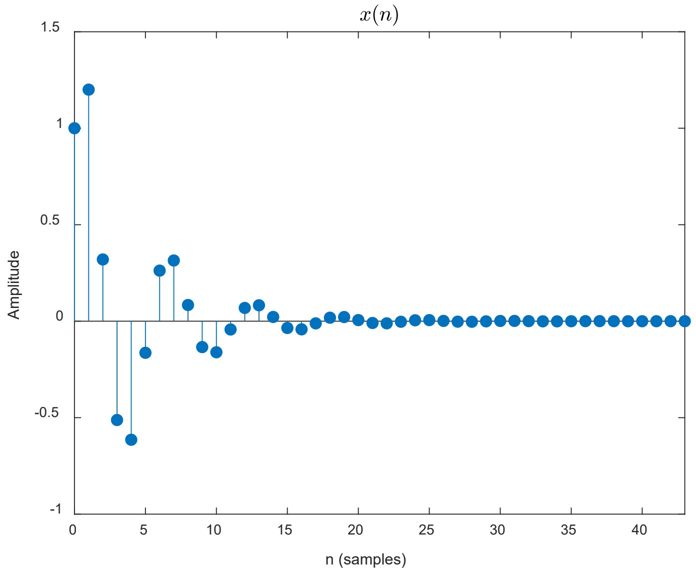


Zeros: $z_1 = 0, z_2 = r\cos\omega_0$

Poles: $p_1 = re^{j\omega_0}, p_2 = re^{-j\omega_0}$ (complex conjugate pair)

$$H(z) = G\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = \frac{z(z-r\cos\omega_0)}{(z-re^{j\omega_0})(z-re^{-j\omega_0})} \quad \text{ROC } |z|>r$$

$$h(n) = r^n\cos(n\omega_0)u(n)$$



Z-Transform Rational Functions

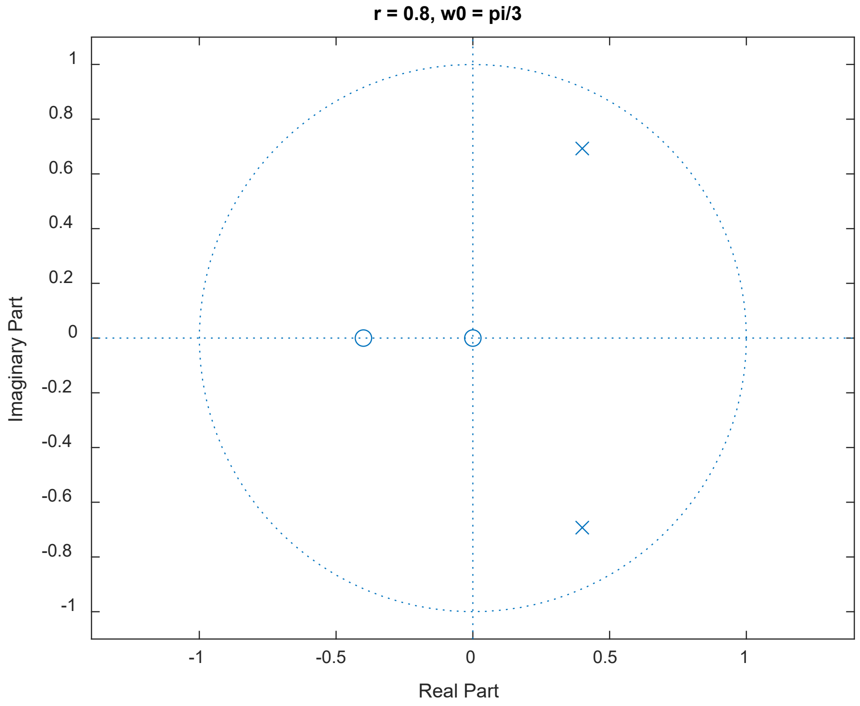
- Example:

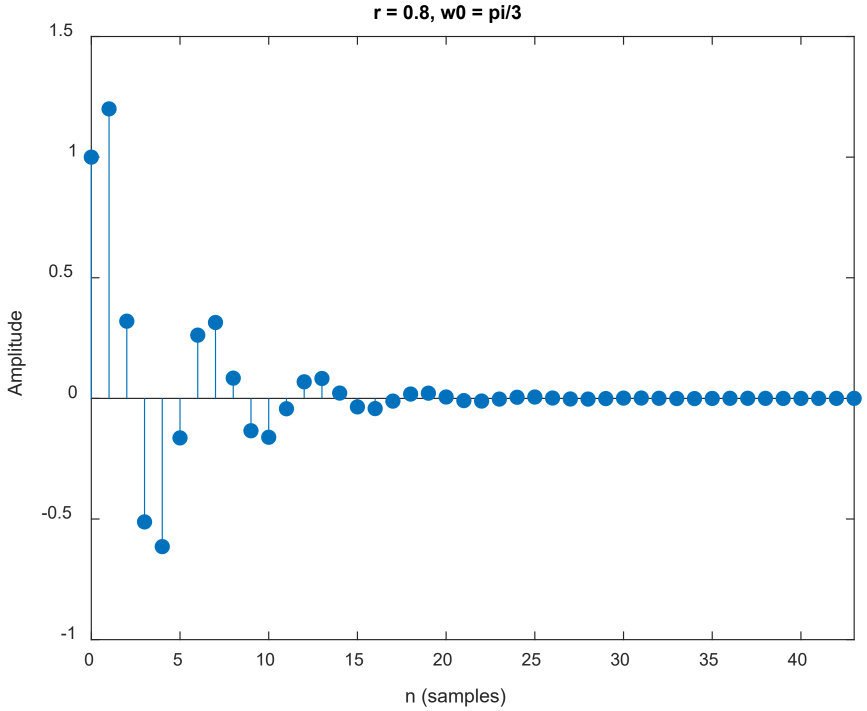
$$h(n) = r^n \cos(n\omega_0) u(n)$$

$$H(z) = \frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} \qquad \text{ROC} \quad |z| > r$$

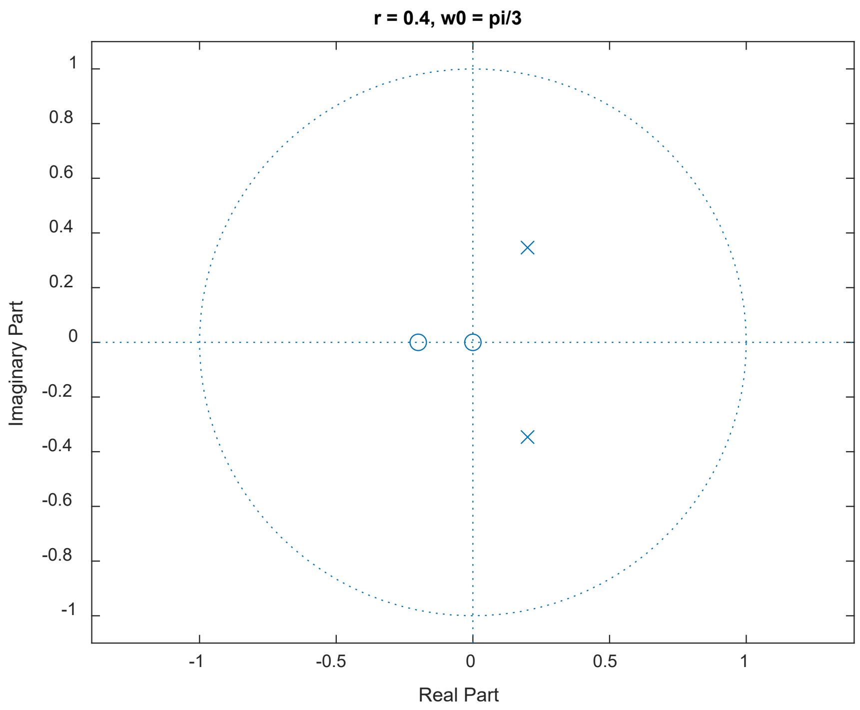
Try several values to see how rate of oscillation and decay vary with pole location

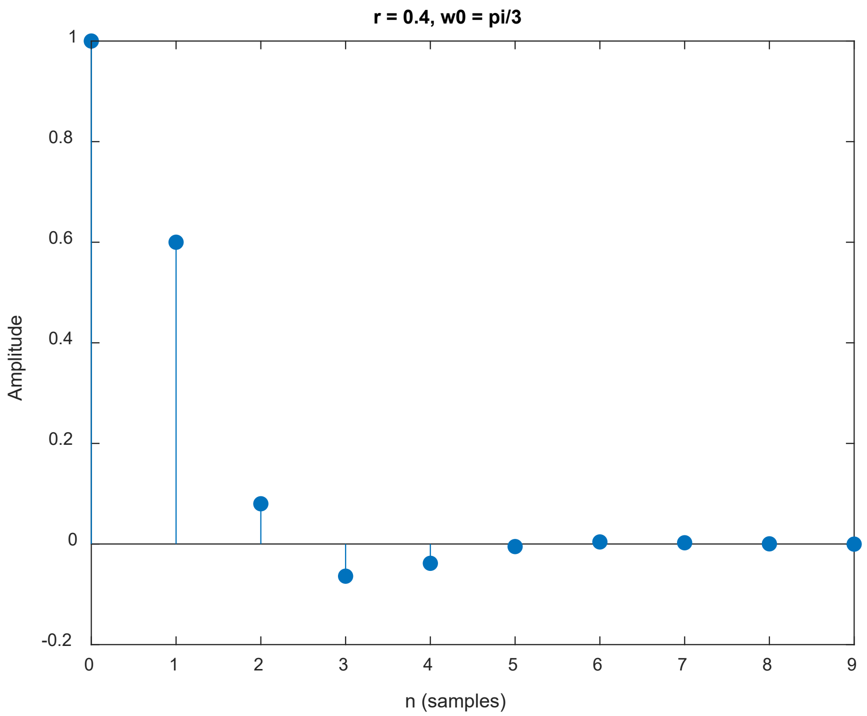
Z-Transform Rational Functions



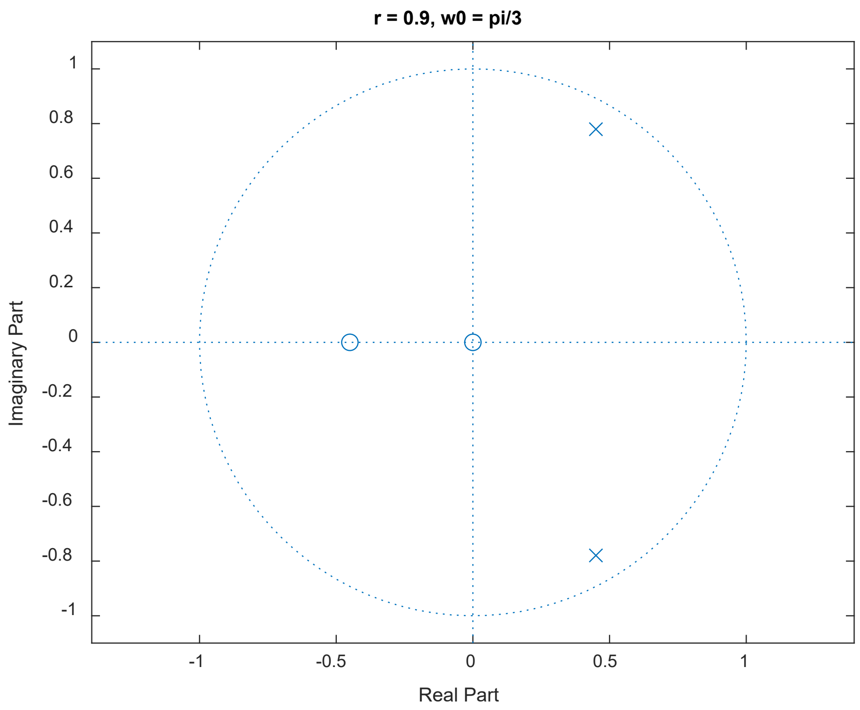


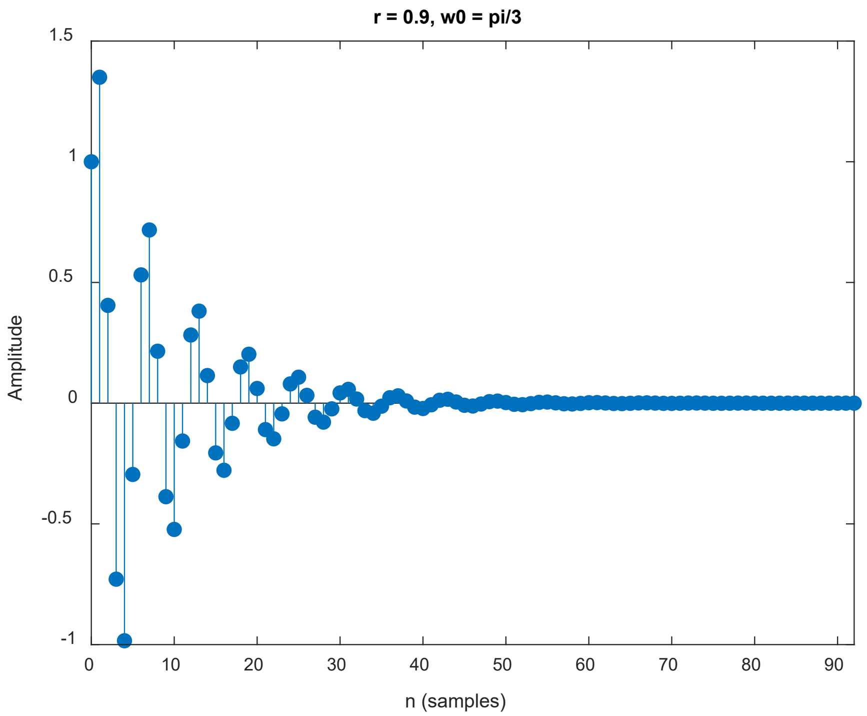
Z-Transform Rational Functions



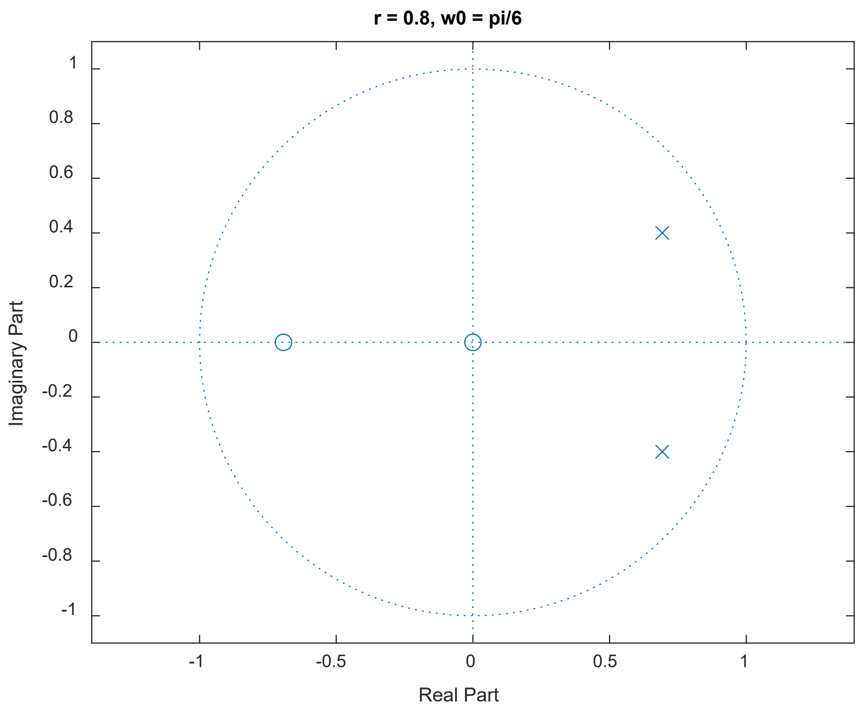


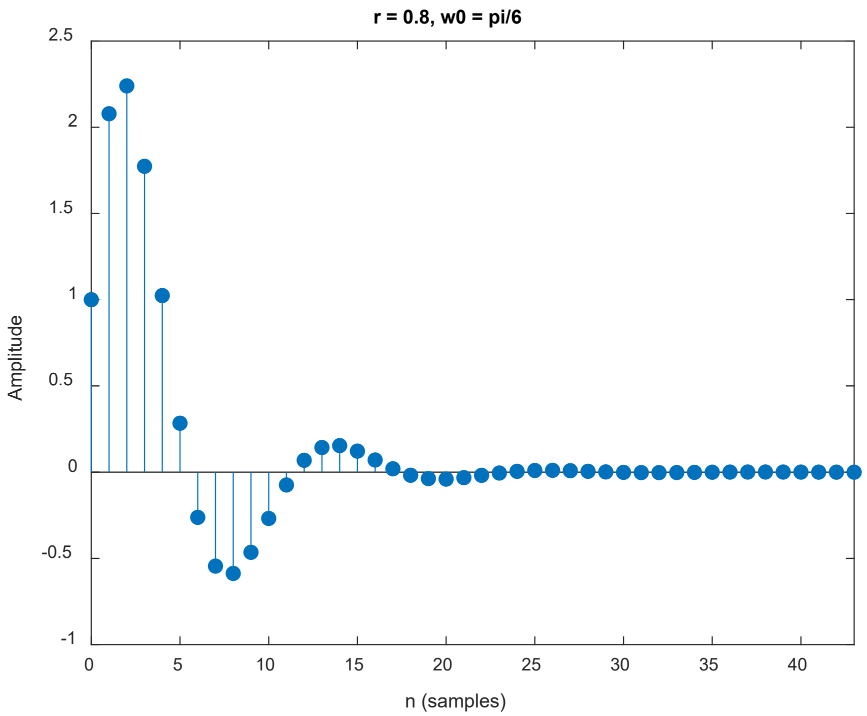
Z-Transform Rational Functions



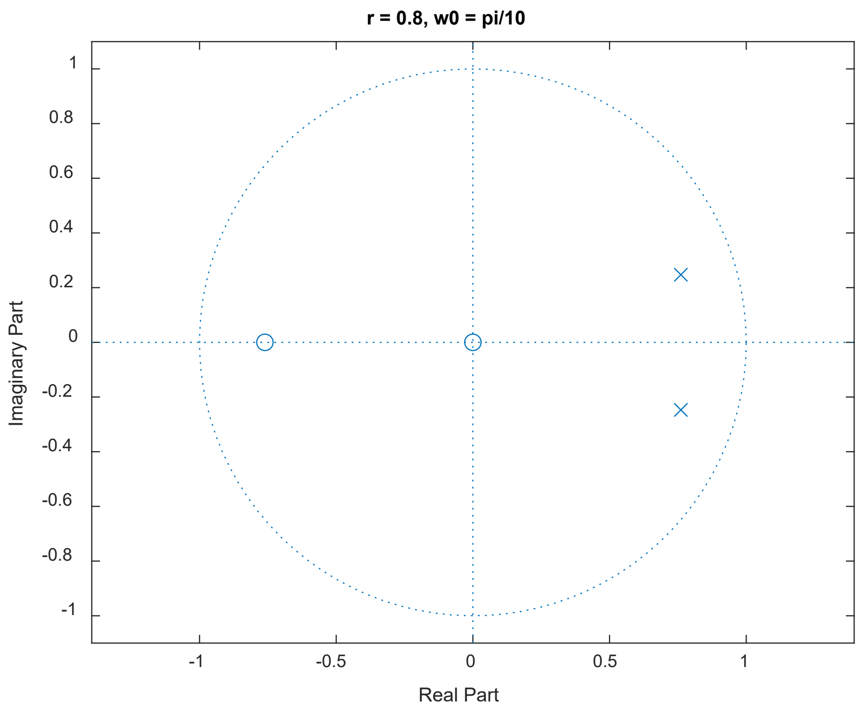


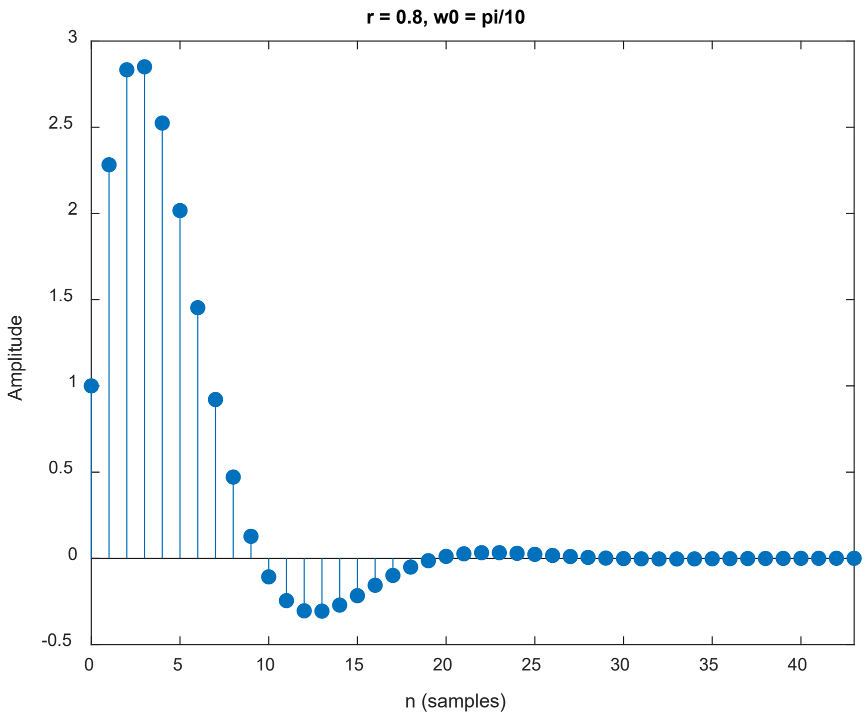
Z-Transform Rational Functions





Z-Transform Rational Functions





Z-Transform Inverse

- Three ways to find inverse
  - Contour integration in complex plane using Cauchy residue theorem (which we will not do)

$$h(n) = \frac{1}{2\ \pi\ j} \oint H(z)\ z^{n-1} dz$$

  - Power series expansion of function (which we will not do)

$$H(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n}$$

$$h(n) = c_n$$

Z-Transform Inverse

- Three ways to find inverse
  - Partial fraction expansion and then beat it into shape we recognize from table (which we will do)

$$H(z)=\frac{b_0+b_1z^{-1}+\cdots+b_M z^{-M}}{1+a_1z^{-1}+\cdots+a_N z^{-N}}$$

To be a proper rational expression,
order of numerator must be less than that
of the denominator

$$\frac{H(z)}{z}=\frac{b_0z^{N-1}+b_1z^{N-2}+\cdots+b_M z^{N-M-1}}{z^N+a_1z^{N-1}+\cdots+a_N}$$

Z-Transform Inverse

– First step – Make it a rational function (if it isn't already one)

- First, make it a rational function (if it isn't already one)

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- If $M > N$:

$$H(z) = c_0 + c_1 z^{-1} + \cdots c_{M-N} + \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

  – Where c's are found from long division of original polynomial
  – Also, assume you've factored a_0 out of the denominator:

Z-Transform Inverse

- Form to work with:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

It is easier to work with positive powers of z, so multiply through by z^N

$$H(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

Divide both sides by z

$$\frac{H(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

Why?

1) This is usually the case you have
2) Takes care of the case where $M=N$ (so avoids long division)
3) Makes it a rational function

Z-Transform Inverse

- Example where numerator and denominator are of same order:

$$H(z) = \frac{1 + 2z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

Easier to work with:

$$H(z) = \left(\frac{z^2}{z^2}\right)\frac{1 + 2z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = \frac{z^2 + 2z + 2}{z^2 - 3z + 2}$$

– Can make this rational by dividing by z on both sides

– Partial fraction expansion:

$$\frac{H(z)}{z} = \frac{z^2 + 2z + 2}{z\left(z^2 - 3z + 2\right)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

Z-Transform Inverse

- Example where numerator is higher order than denominator?
  - Convert improper rational function to rational one by long division
    - Example

$$H(z)=\frac{1+3z^{-1}+\frac{11}{6}z^{-2}+\frac{1}{3}z^{-3}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

  - Long division until order of numerator is less than denominator
    You want to end up with z^{-1} in numerator, so

$$\left(\frac{1}{6}z^{-2}+\frac{5}{6}z^{-1}+1\right)\overline{\Big)\frac{1}{3}z^{-3}+\frac{11}{6}z^{-2}+3z^{-1}+1}=1+2z^{-1}+\frac{\frac{1}{6}z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

Z-Transform Inverse

- Factor denominator into product of linear functions:

$$H(z)=\frac{b_0 z^{N-1}+b_1 z^{N-1-2}+\cdots+b_M z^{N-M-1}}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

  This is what you expand into partial fractions.

- Consider 3 cases
  1) Simplest case is distinct real poles
  2) Some complex poles
  3) Some poles are not distinct
  4) Repeated complex poles (reduces to combination of 2 & 3)

Z-Transform Inverse

Simple real poles: $z^2 - 5z + 6 = (z-2)(z-3)$

$$z^3 - 4z^2 + z + 6 = (z-3)(z-2)(z+1)$$

Repeated real poles: $z^3 - 3z^2 + 6z - 4 = (z-2)^2(z+1)$

$$z^4 - z^3 - 3z^2 + 5z - 2 = (z-1)^3(z+2)$$

Complex poles: $z^2 - 10z + 26 = \left(z - 5 - j\right)\left(z - 5 + j\right)$

$$z^3 - 8z^2 + 22z - 20 = (z-2)(z-3-j)(z-3+j)$$

Notice that to get a real polynomial, complex roots occur in complex conjugate pairs.

Can you determine how many real and complex roots you will have from just the coefficients of the polynomial?

Yes! **Descartes' rule of signs**

Z-Transform Inverse

To determine roots of polynomial numerically in Matlab:

$$z^2 - 5z + 6 = (z-2)(z-3)$$

```
>> roots([1 -5 6])
ans =
    3.0000
    2.0000
```

$$z^3 - 8z^2 + 22z - 20 = (z-2)(z-3-j)(z-3+j)$$

```
>> roots([1 -8 22 -20])
ans =
   3.0000 + 1.0000i
   3.0000 - 1.0000i
   2.0000 + 0.0000i
```

Can also do it symbolically

```
>>syms z
>> factor(z^3 - 8*z^2 + 22*z - 20,'FactorMode','full')
ans =
[z - 2, z - 3 + 1i, z - 3 - 1i]
```

Z-Transform Inverse

- Partial Fraction examples

$$H(z)=\frac{1+2z^{-1}+2z^{-2}}{1-3z^{-1}+2z^{-2}}=\frac{z^2+2z+2}{z^2-3z+2}$$

  - This is a case where the numerator and denominator have the same order
  - In this case, you can divide through by z to make it proper

$$\frac{H(z)}{z}=\frac{z^2+2z+2}{z\left(z^2-3z+2\right)}=\frac{A}{z}+\frac{B}{z-1}+\frac{C}{z-2}$$

  - Once in this form, it is easy to find inverse transform

Z-Transform Inverse

– Inverse z-transform of polynomial part of the expression can be done by inspection since powers of z

$$H(z) = 1 + 2z^{-1} + \frac{\frac{1}{6}z^{-1}}{z^{-2} + \frac{5}{6}z^{-1} + 1}$$

$$Z^{-1}\left[1 + 2z^{-1}\right] = \delta\left(n\right) + \delta(n-1)$$

Z-Transform Inverse

- Simplest case: Distinct real poles

$$\frac{H(z)}{z}=\frac{b_0 z^{N-1}+b_1 z^{N-1-2}+\cdots+b_M z^{N-M-1}}{(z-p_1)(z-p_2)\cdots(z-p_N)}=\frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\cdots+\frac{A_N}{z-p_N}$$

- The individual terms are easily inverted:

$$H(z)=\frac{A_1 z}{z-p_1}+\frac{A_2 z}{z-p_2}+\cdots+\frac{A_N z}{z-p_N}$$

$$h(n)=\left[A_1(p_1)^n+A_2(p_2)^n+\cdots+A_N(p_N)^n\right]u(n)$$

(assuming causal signal or system)

Z-Transform Inverse

- Simplest case: Distinct real poles

$$\frac{H(z)}{z}=\frac{b_0 z^{N-1}+b_1 z^{N-1-2}+\cdots+b_M z^{N-M-1}}{(z-p_1)(z-p_2)\cdots(z-p_N)}=\frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\cdots+\frac{A_N}{z-p_N}$$

- How do you find the *A's*?
  – Show through an example:

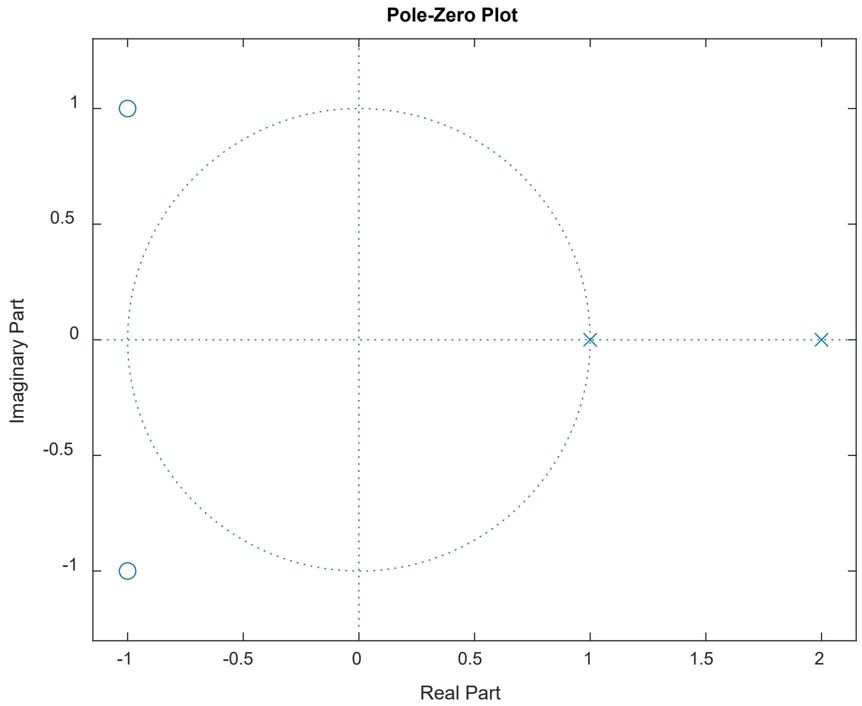
$$H(z)=\frac{1+2z^{-1}+2z^{-2}}{1-3z^{-1}+2z^{-2}}$$

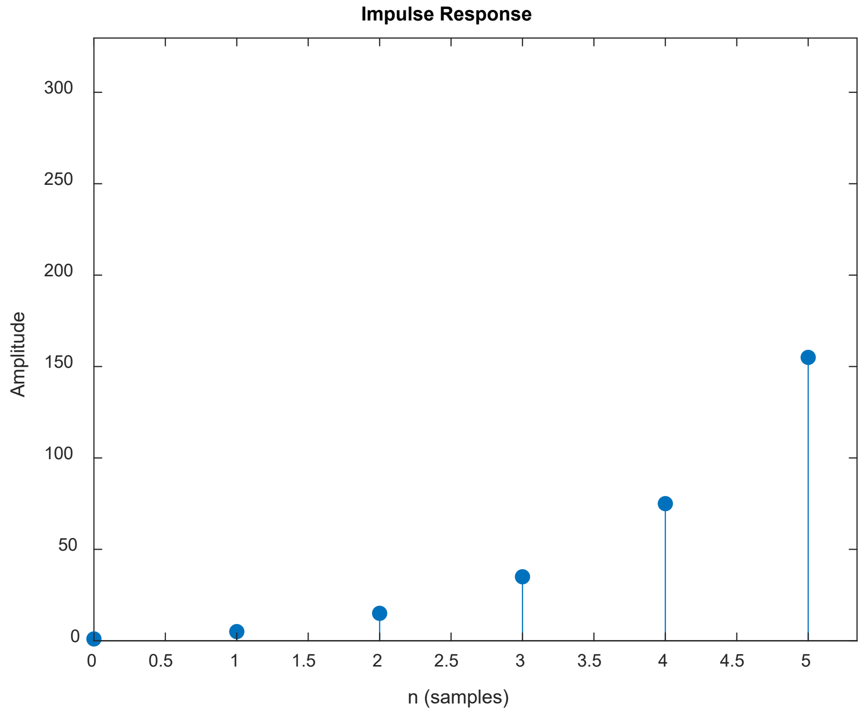
$$h(n)=\delta(n)+5\left[(2)^n-1\right]u(n)$$

Z-Transform Inverse

$$H(z)=\frac{1+2z^{-1}+2z^{-2}}{1-3z^{-1}+2z^{-2}}$$

$$h(n)=\delta(n)+5\left[(2)^{n}-1\right]u(n)$$





Z-Transform Inverse

- Simplest case: Distinct real poles

$$\frac{H(z)}{z}=\frac{b_0z^{N-1}+b_1z^{N-1-2}+\cdots+b_Mz^{N-M-1}}{(z-p_1)(z-p_2)\cdots(z-p_N)}=\frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\cdots+\frac{A_N}{z-p_N}$$

- Another example

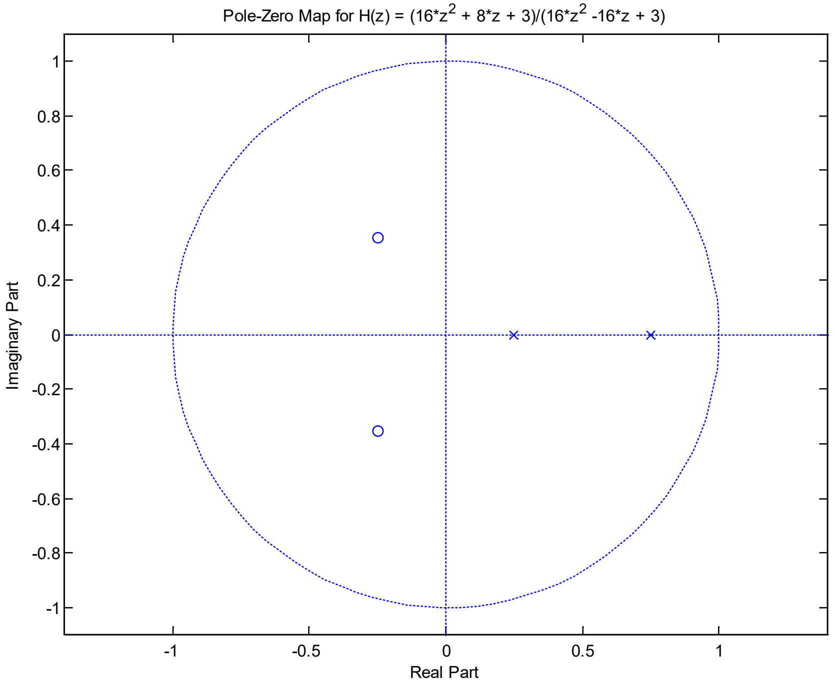
$$H(z)=\frac{1-(1/2)z^{-1}+(3/16)z^{-2}}{1-z^{-1}+(3/16)z^{-2}}=1-\frac{3z}{z-1/4}+\frac{3z}{z-3/4}$$

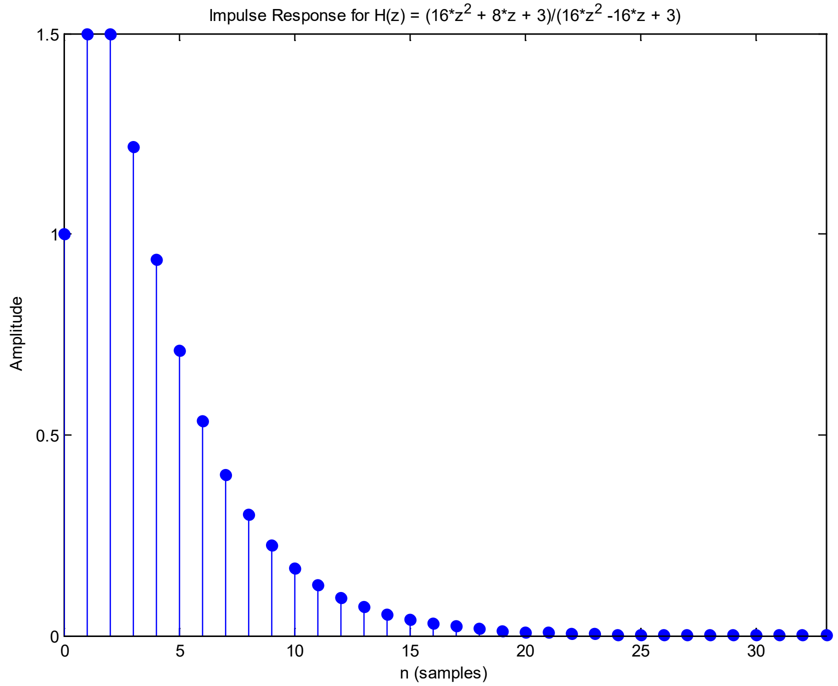
$$h(n)=\delta[n]-3\cdot\left(\tfrac{1}{4}\right)^n u(n)+3\cdot\left(\tfrac{3}{4}\right)^n u(n)$$

Z-Transform Inverse

$$H(z)=\frac{1-(1/2)z^{-1}+(3/16)z^{-2}}{1-z^{-1}+(3/16)z^{-2}}$$

$$h(n)=\delta[n]-3\cdot\left(\tfrac{1}{4}\right)^{n}u(n)+3\cdot\left(\tfrac{3}{4}\right)^{n}u(n)$$





Z-Transform Inverse

- Case where order of numerator is larger than denominator

$$H(z)=\frac{1+3z^{-1}+\frac{11}{6}z^{-2}+\frac{1}{3}z^{-3}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}=1+2z^{-1}+\frac{\frac{1}{6}z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

$$Z^{-1}\left[1+2z^{-1}\right] \;=\delta\left(n\right)+2\delta(n-1)$$

Then deal with rational polynomial part: $X_2(z)=\dfrac{\frac{1}{6}z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$

$$Z^{-1}\left[H(z)\right] \;=h(n)=\delta\left(n\right)+2\delta(n-1)+\left[\left(-\frac{1}{3}\right)^{n}-\left(-\frac{1}{2}\right)^{n}\right]u(n)$$

Z-Transform Inverse

$$H(z)=\frac{1+3z^{-1}+\frac{11}{6}z^{-2}+\frac{1}{3}z^{-3}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

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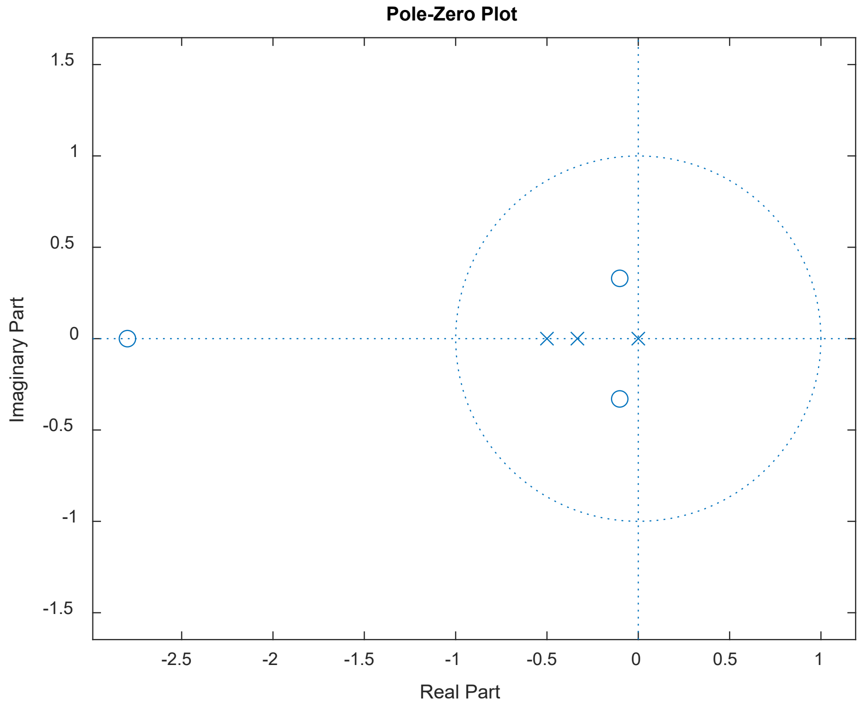

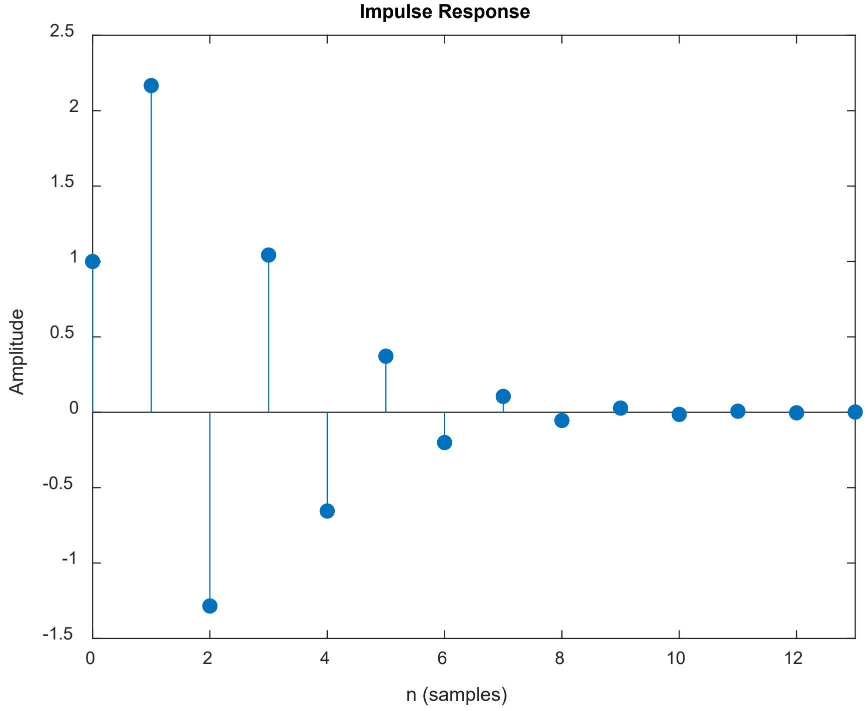

Z-Transform Inverse

- Complex (distinct) poles
  - Example: (Approach the same way as for real poles)

$$H(z) = \frac{z^{-2}}{1 - \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$$h(n) = 2\left[\delta(n) + \frac{2}{\sqrt{7}}\left(\frac{1}{\sqrt{2}}\right)^{(n-1)} \sin\left((n-1)\theta\right)u(n)\right]$$

Z-Transform Inverse

$$H(z)=\frac{z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

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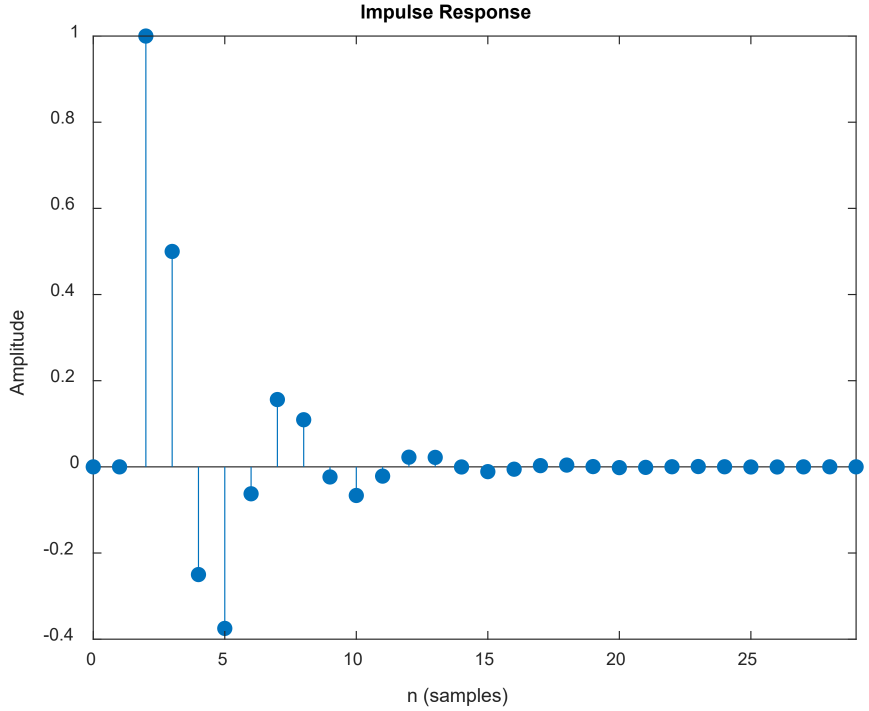

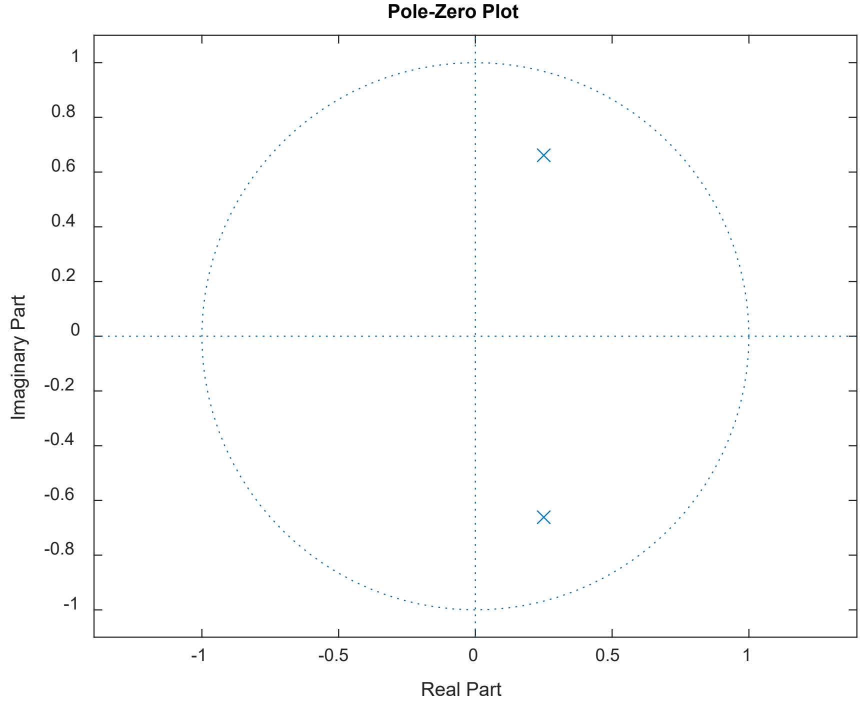

Z-Transform Inverse

- Multiple-order poles
  - Example:

$$H(z) = \frac{1}{1 - z^{-1} - z^{-2} + z^{-3}}$$

$$h(n) = \frac{1}{4}\left[(-1)^n + 2n + 3\right]u(n)$$

Z-Transform Inverse

$$H(z) = \frac{1}{1 - z^{-1} - z^{-2} + z^{-3}}$$

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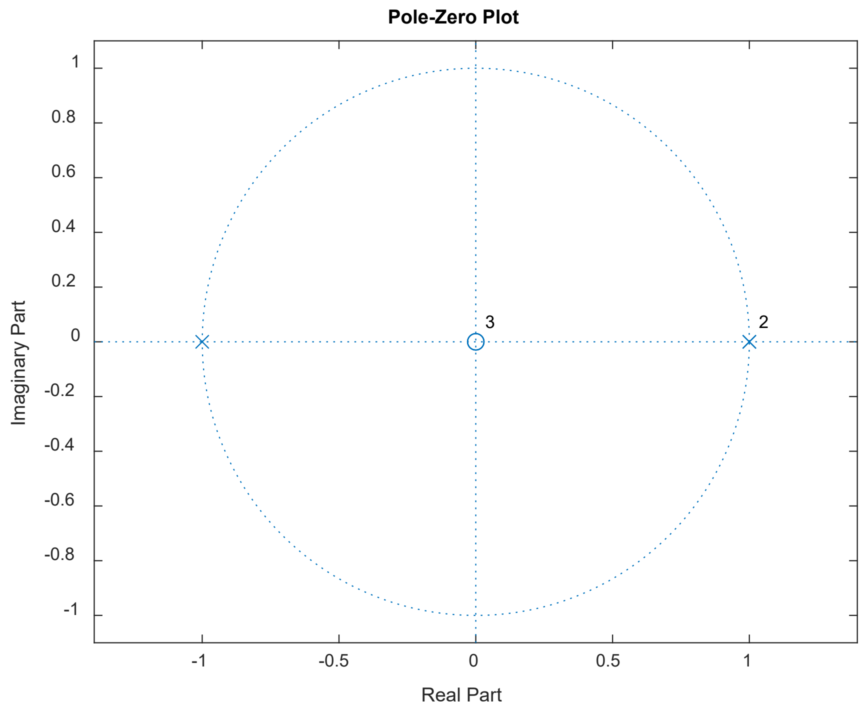

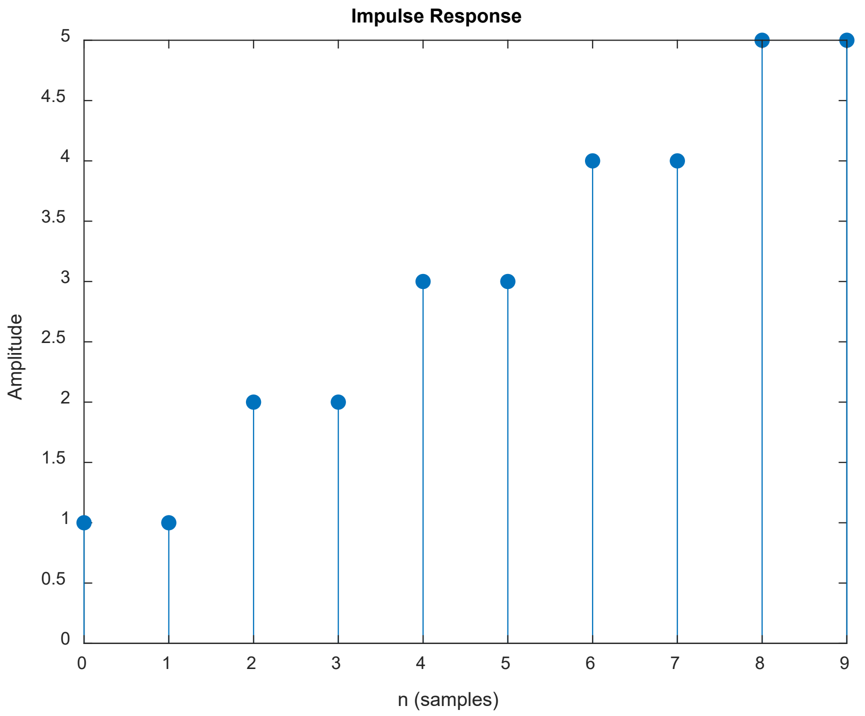

Decomposition of Rational z-transform

- It is useful to decompose rational z-transforms into product of first-order and second-order terms:

$$H(z)=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}=b_0\frac{1+\left(b_1/b_0\right)z^{-1}+\cdots+\left(b_M/b_0\right)z^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}$$

$$H(z)=b_0\frac{\left(1-z_1z^{-1}\right)\left(1-z_2z^{-1}\right)\cdots\left(1-z_Mz^{-1}\right)}{\left(1-p_1z^{-1}\right)\left(1-p_2z^{-1}\right)\cdots\left(1-p_Mz^{-1}\right)}=b_0\frac{\prod_{k=1}^{M}\left(1-z_kz^{-1}\right)}{\prod_{k=1}^{N}\left(1-p_kz^{-1}\right)}$$

If $M > N$, do the usual division to get a sum of terms and a proper rational function

$$H(z)=\sum_{k=0}^{M-N}c_kz^{-k}+H_{pr}\left(z\right)$$

Decomposition of Rational z-transform

If $M > N$, do the usual division to get a sum of terms and a proper rational function

$$H(z) = \sum_{k=0}^{M-N} c_k z^{-k} + H_{pr}(z)$$

Do partial fraction expansion of proper part (assume no multiple poles)

$$H_{pr}(z) = A_1 \frac{1}{1 - p_1 z^{-1}} + A_2 \frac{1}{1 - p_2 z^{-1}} + \cdots + A_N \frac{1}{1 - p_N z^{-1}}$$

Break this out into real poles and complex conjugate pairs of poles

For complex conjugate pairs:

$$\frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}} = \frac{A\left(1 - p^* z^{-1}\right) + A^*\left(1 - pz^{-1}\right)}{\left(1 - pz^{-1}\right)\left(1 - p^* z^{-1}\right)}$$

Decomposition of Rational z-transform

Break this out into real poles and complex conjugate pairs of poles

For complex conjugate pairs:

$$\frac{A}{1-pz^{-1}}+\frac{A^*}{1-p^*z^{-1}}=\frac{A\left(1-p^*z^{-1}\right)+A^*\left(1-pz^{-1}\right)}{\left(1-pz^{-1}\right)\left(1-p^*z^{-1}\right)}$$

$$\frac{A\left(1-p^*z^{-1}\right)+A^*\left(1-pz^{-1}\right)}{\left(1-pz^{-1}\right)\left(1-p^*z^{-1}\right)}=\frac{A-Ap^*z^{-1}+A^*-A^*pz^{-1}}{1-pz^{-1}-p^*z^{-1}+pp^*z^{-2}}$$

$$=\frac{b_0+b_1z^{-1}}{1+a_1z^{-1}+a_2z^{-2}}$$

$$b_0=2\,\mathrm{Re}(A) \quad , \quad a_1=-2\,\mathrm{Re}(p)$$

$$b_1=2\,\mathrm{Re}(A^*) \quad , \quad a_2=\left|p\right|^2$$

Decomposition of Rational z-transform

Now, write entire thing out in terms of real and complex poles (and delays if M>N)

$$H(z) = \sum_{k=0}^{M-N} c_k z^{-k} + \sum_{k=1}^{K_1} \frac{b_k}{1 + a_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{0k} + b_{1k} z^{-1}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}}$$