Digital Signal Processing

Class 12
02/27/2025

ENGR 71

- Class Overview
  - Frequency Analysis of Discrete Signals
- Assignments
  - Reading:

    Chapter 4: Frequency Analysis of Signals

ENGR 71

- Homework 4
  - Problems: 3.2 (b & f), 3.4(d), 3.12,
    3.14(b), 3.16, 3.31
    C3.3 (use Matlab)
    C3.5 (use Matlab)

    Due Mar. 2

Frequency Decomposition of Signals

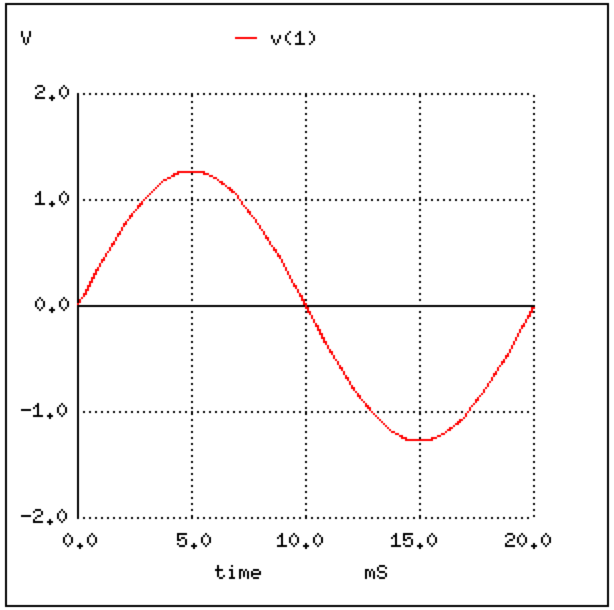
- Key concept behind frequency decomposition of signals:
  - Basis functions of sines and cosines (and complex exponential)
  - **Frequency components of signal are unchanged when passed through Linear Time Invariant systems**
    - Only amplitude and phase change

$$A\cdot\sin(\omega t+\varphi) \longrightarrow \boxed{H(s)} \longrightarrow M\cdot A\cdot\sin(\omega t+\varphi+\theta)$$

    - Any signal can be decomposed and reconstructed from its frequency components
  - Frequency and time domains are complementary representations of signals

Frequency Decomposition of Signals

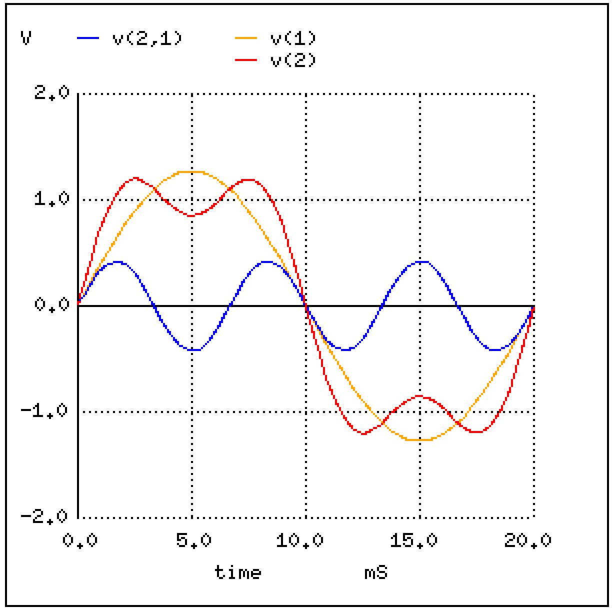
- Continuous periodic signals
  - Fourier series decomposes periodic signals into frequency components



1 50 Hz sine wave (1st harmonic)

Frequency Decomposition of Signals

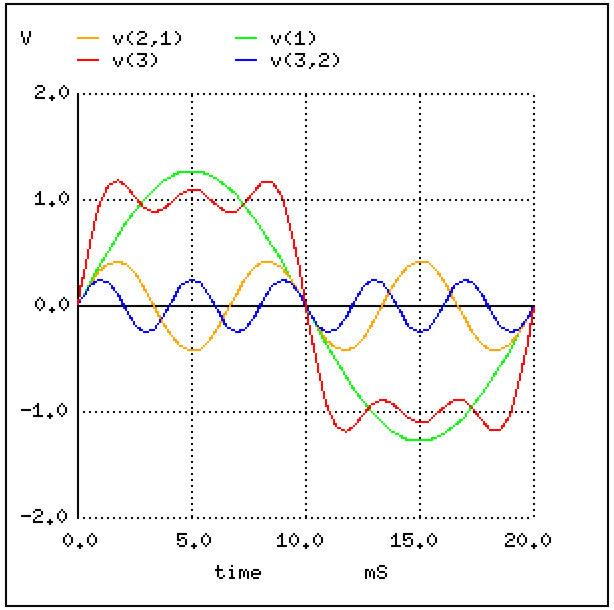
- Continuous periodic signals
  - Fourier series decomposes periodic signals into frequency components




```
  1     50 Hz sine wave (1st harmonic)
+ 1/3 150 Hz sine wave (3rd harmonic)
```

Frequency Decomposition of Signals

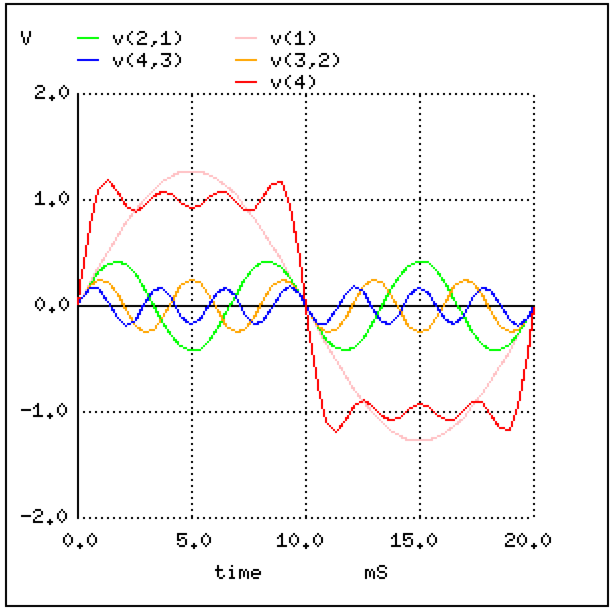
- Continuous periodic signals
  - Fourier series decomposes periodic signals into frequency components




```
  1     50 Hz sine wave (1st harmonic)
+ 1/3 150 Hz sine wave (3rd harmonic)
+ 1/5 250 Hz sine wave (5th harmonic)
```

Frequency Decomposition of Signals

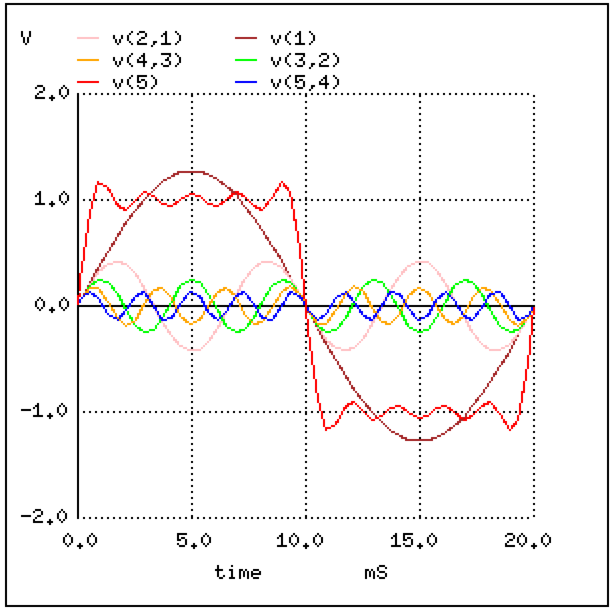
- Continuous periodic signals
  - Fourier series decomposes periodic signals into frequency components




```
  1     50 Hz sine wave (1st harmonic)
+ 1/3 150 Hz sine wave (3rd harmonic)
+ 1/5 250 Hz sine wave (5th harmonic)
+ 1/7 350 Hz sine wave (7th harmonic)
```

Frequency Decomposition of Signals

- Continuous periodic signals
  - Fourier series decomposes periodic signals into frequency components




```
  1     50 Hz sine wave (1st harmonic)
+ 1/3 150 Hz sine wave (3rd harmonic)
+ 1/5 250 Hz sine wave (5th harmonic)
+ 1/7 350 Hz sine wave (7th harmonic)
+ 1/9 450 Hz sine wave (9th harmonic)
```

Frequency Decomposition of Signals

- Continuous periodic signals $x(t) = x(t+T_0),\quad f_0 = \dfrac{1}{T_0},\quad \omega_0 = 2\pi f_0 = \dfrac{2\pi}{T_0}$
  - Fourier series: Trigonometric Form

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

(Synthesis Eq.)

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt$$

(Analysis Eqs.)

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

Frequency Decomposition of Signals

- Continuous periodic signals $x(t) = x(t+T_0), \quad f_0 = \dfrac{1}{T_0}, \quad \omega_0 = 2\pi f_0 = \dfrac{2\pi}{T_0}$
  - Fourier series: Complex Exponential Form

$$x(t) = x(t+T_0) \qquad f_0 = \frac{1}{T_0} \qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t} \qquad \text{(Synthesis Eq.)}$$

$$X_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\, dt \qquad \text{(Analysis Eq.)}$$

Frequency Decomposition of Signals

- Dirichlet conditions guarantee Fourier series converges to function $x(t)$
  - Signal has finite number of discontinuities in a period
  - Signal contains finite number of maxima and minima in a period
  - Signal is absolutely integrable

    $$\int_{T_0} |x(t)| < \infty$$
  - Dirichlet conditions are sufficient, but not necessary.
    - There may be signals that do not satisfy these conditions, but still have convergent Fourier series
  - Square integrability is not a Dirichlet condition, but
    - If signal is square integrable (finite energy), energy will be zero (although signal may not be equal for all values of t)

$$\int_{T_0} |x(t)|^2 < \infty \Rightarrow \int_{T_0} |e(t)|^2 dt = 0 \quad \text{where } e(t) = x(t) - \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t}$$

Frequency Decomposition of Signals

- All signals that we deal with satisfy these conditions
- Out of curiosity, what would be examples of functions that do not satisfy Dirichlet conditions?

$$f(t) = \begin{cases} 1 & \text{for } t \in \text{ rational numbers} \\ 0 & \text{for } t \notin \text{ rational numbers} \end{cases}$$

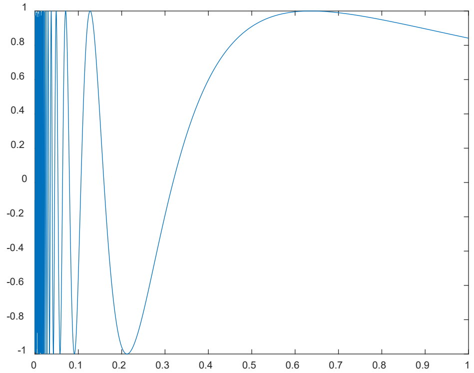
$$x(t) = f(t \mod T_0) \quad \text{(to make it periodic)}$$

This is discontinuous at every point

$$f(t) = \frac{1}{\sin(t)}, \ t \neq 0$$

$$x(t) = f(t \mod T_0) \quad \text{(to make it periodic)}$$

Infinite number of minima and maxima as $t \to 0$



Frequency Decomposition of Signals

- Continuous aperiodic signals
  - Fourier transform decomposes aperiodic signals into frequency components

$$X(\omega) = \mathscr{F}[x(t)] = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \qquad \text{(Analysis Equation)}$$

$$x(t) = \mathscr{F}^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{+j\omega t} d\omega \qquad \text{(Synthesis Equation)}$$

or, with $\omega = 2\pi f$

$$X(f) = \mathscr{F}[x(t)] = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt \qquad \text{(Analysis Equation)}$$

$$x(t) = \mathscr{F}^{-1}[X(f)] = \int_{-\infty}^{+\infty} X(f) e^{+j2\pi f t} df \qquad \text{(Synthesis Equation)}$$

Frequency Decomposition of Signals

– Dirichlet conditions also guarantee existence of Fourier transform for functions
  - Again, sufficient but not necessary

– Practically all physical signals have Fourier transforms

– Example of function not satisfying the Dirichlet conditions is sinc function

$$x(t) = \frac{\sin(\pi t)}{\pi t} \xleftrightarrow{\mathcal{F}} \mathrm{rect}(f) = \begin{cases} 1, & |f| \le \frac{1}{2} \\ 0, & |f| > \frac{1}{2} \end{cases}$$

Energy and Power Spectral Density

- Energy signals: (continuous signals)

$$E = \int_{-\infty}^{\infty} \left| x(t) \right|^2 dt < \infty$$

- Power signals: (continuous signals)
  - Power defined as:

$$P = \lim_{T \to \infty} \int_{-T}^{T} \left| x(t) \right|^2 dt$$

  - Periodic signals have infinite energy
    - Power can be calculated over one period as:

$$P = \frac{1}{T_p} \int_{T_P} \left| x(t) \right|^2 dt$$

Power and Energy Spectral Density

- Parseval's power and energy relations for continuous signals
  - For periodic signals

$$P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt = \sum_{k=-\infty}^{+\infty} |X_k|^2$$

  - Energy for aperiodic signals

$$\boxed{E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega}$$

  - Power spectral density: Power per frequency component $|X_k|^2$
  - Energy spectral density: Energy as function of frequency $|X(f)|^2$

Discrete signals

- Discrete-time sinusoidal signals

$$x(n) = A\cos\left(\omega n + \theta\right) \quad \text{or} \quad A\cos\left(2\pi f n + \theta\right)$$

- Differences between continuous and discrete signals
  - Discrete-time sinusoids are periodic only if frequency is a rational number

$$x(n+N) = x(n)$$

$$\cos\left(2\pi f_0 (n+N) + \theta\right) = \cos\left(2\pi f_0 n + 2\pi f_0 N) + \theta\right) \stackrel{?}{=} \cos\left(2\pi f_0 n + \theta\right)$$

  - This will only be true if there is some integer k, such that

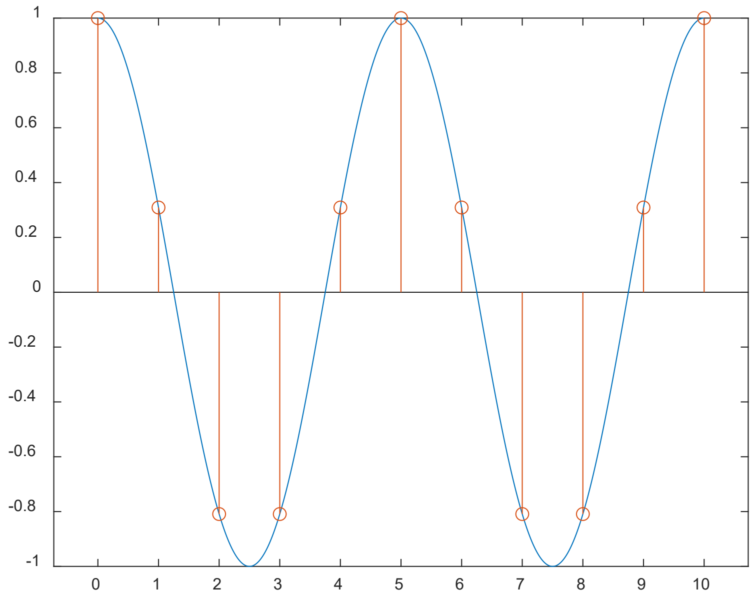
$$2\pi f_0 N = 2\pi k \quad \Rightarrow \quad f_0 = \frac{k}{N}$$

    - Cancel out common factors so k and N are relatively primed. N is the period

Discrete signals

- Example of periodic and non-periodic discrete sinusoids
  - Periodic: $x(n) = \cos\left(2\pi f_0\right)$

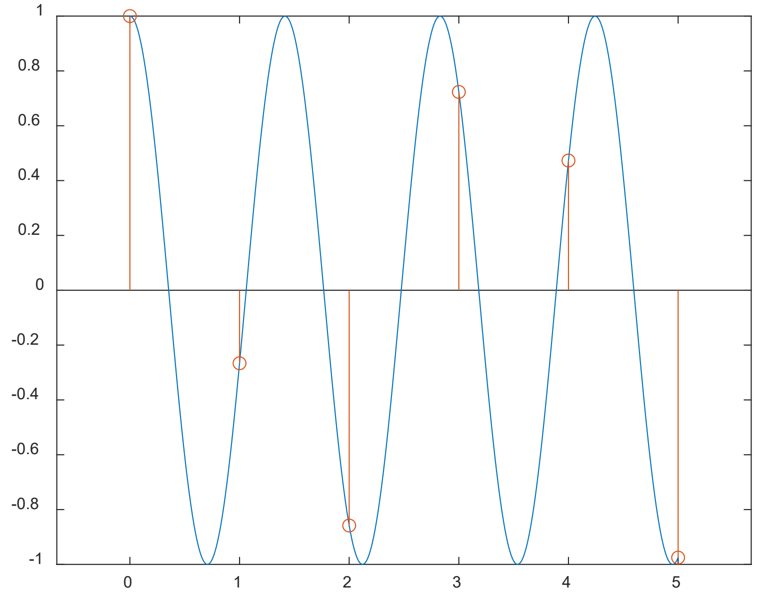
$$f_0 = 0.8 = \frac{8}{10} = \frac{4}{5} \quad , \text{ Period } N = 5$$



Discrete signals

- Example of periodic and non-periodic discrete sinusoids
  - Not periodic: $x(n) = \cos\left(2\pi \frac{1}{\sqrt{2}} n\right)$

$$f_0 = \frac{1}{\sqrt{2}} \approx 0.7071 \ , \ \text{Period } T = \sqrt{2}$$



Discrete signals

- Another difference between continuous and discrete sinusoids
  - Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical:

$$x(n) = \cos\left((\omega_0 + 2\pi k)n + \theta\right) = \cos\left(\omega_0 n + 2\pi kn + \theta\right) = \cos\left(\omega_0 n + \theta\right)$$

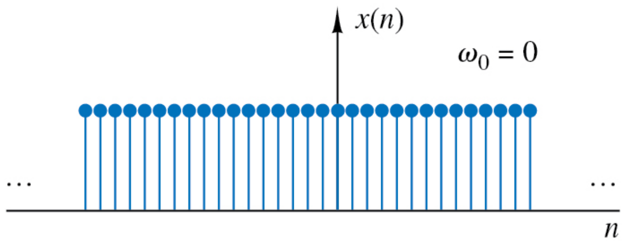
$$x_k(n) = A\cos\left(\omega_k n + \theta\right) \text{ where } \omega_k = \omega_0 + 2k\pi \quad -\pi \le \omega_0 \le \pi$$

These sequences are identical

  - This means that for discrete frequency, we only need to consider frequencies between $-\pi$ and π, or 0 and 2π

Discrete signals

- Representation of signals in by only need to consider frequencies between $-\pi$ and π, or 0 and 2π
  - Lowest frequency is 0, highest frequency is π (or $-\pi$)



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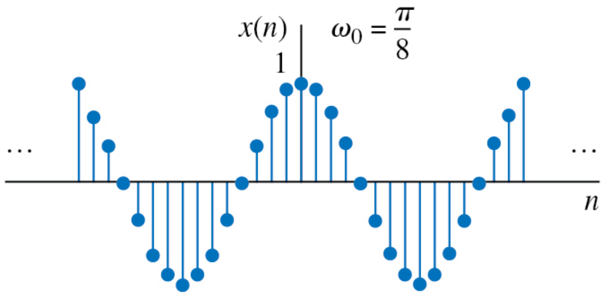
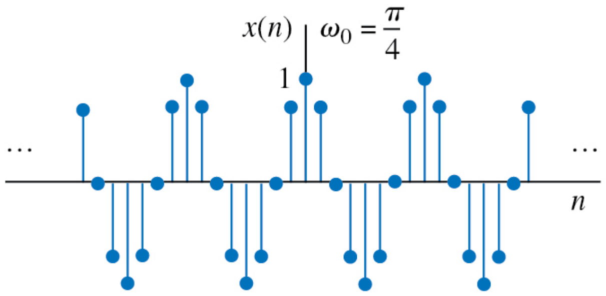
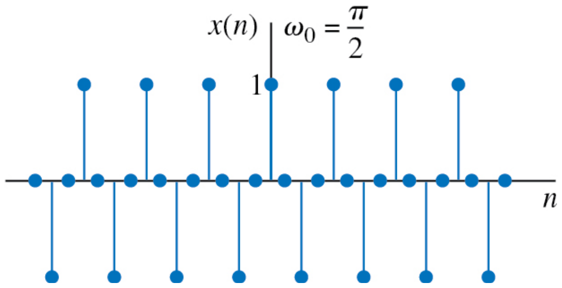
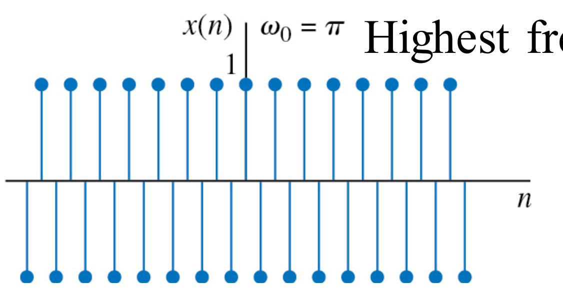








Highest frequency

Discrete signals

- Set of harmonically related discrete-time exponentials

$$s_k(n) = e^{j2\pi k f_0 n} \quad k = 0, \pm 1, \pm 2, \ldots \qquad \text{frequency } f_0, \text{ period } N = 1/f_0$$

  – Only N distinct elements in set.

    - If k exceeds N-1, they repeat:

$$s_N(n) = e^{j2\pi(k+N)n/N} = e^{j2\pi kn/N}\, e^{j2\pi n} = e^{j2\pi kn/N}$$

  – Just consider the set of N unique elements

$$s_k(n) = e^{j2\pi kn/N} \quad k = 0, 1, 2, \ldots, N-1$$

  – A linear combination of these elements is also periodic in N

$$x(n) = \sum_{k=0}^{N-1} c_k s_k(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

Frequency Analysis for Discrete Signals

- Frequency analysis for discrete signals:
  - Three transforms to consider:
    - Discrete Time Fourier Transform - DTFT
      - Fourier transform of sampled signal
    - Discrete Time Fourier Series - DTFS
      - Fourier series of sampled periodic signal
    - Discrete Fourier Transform – DFT
      - Create periodic extension of finite sequence
      - Then find the Fourier series.
      - This is the transform that is most often used
      - Fast algorithm to compute: Fast Fourier Transform (FFT)

Discrete-time Fourier transform

- Discrete Time Fourier Transform (DTFT)
  - Fourier transform of sampled signal

$$x_s(t) = \sum_n x(nT_s)\delta\left(t - nT_s\right)$$

$$\mathcal{F}\left\{x_s(t)\right\} = \sum_n x(nT_s)\mathcal{F}\left\{\delta\left(t - nT_s\right)\right\} = \sum_n x(nT_s)e^{-jn\Omega T_s}$$

$\left[\text{Using } \mathcal{F}\left\{\delta(t)\right\} = 1 \text{ and shift property } \mathcal{F}\left\{x(t-\tau)\right\} = X\left(\Omega\right)e^{-j\Omega\tau}\right]$

Ω is the analog frequency variable

Note that: $\mathcal{F}\left\{x_s(t)\right\}$ is periodic:

$$\sum_n x(nT_s)e^{-jn\Omega T_s} = \sum_n x(nT_s)e^{-jn\left(\Omega + \frac{2\pi k}{T_s}\right)T_s}$$

So, the spectrum of a sampled signal is periodic

Discrete-time Fourier transform

– Discrete Time Fourier Transform (DTFT)

Define $\omega = \Omega T_s$ as the frequency of the discrete signal (in radians)

and define $x[n] = x(nT_s)$ as samples of the sampled signal

- Fourier transform of sampled signal

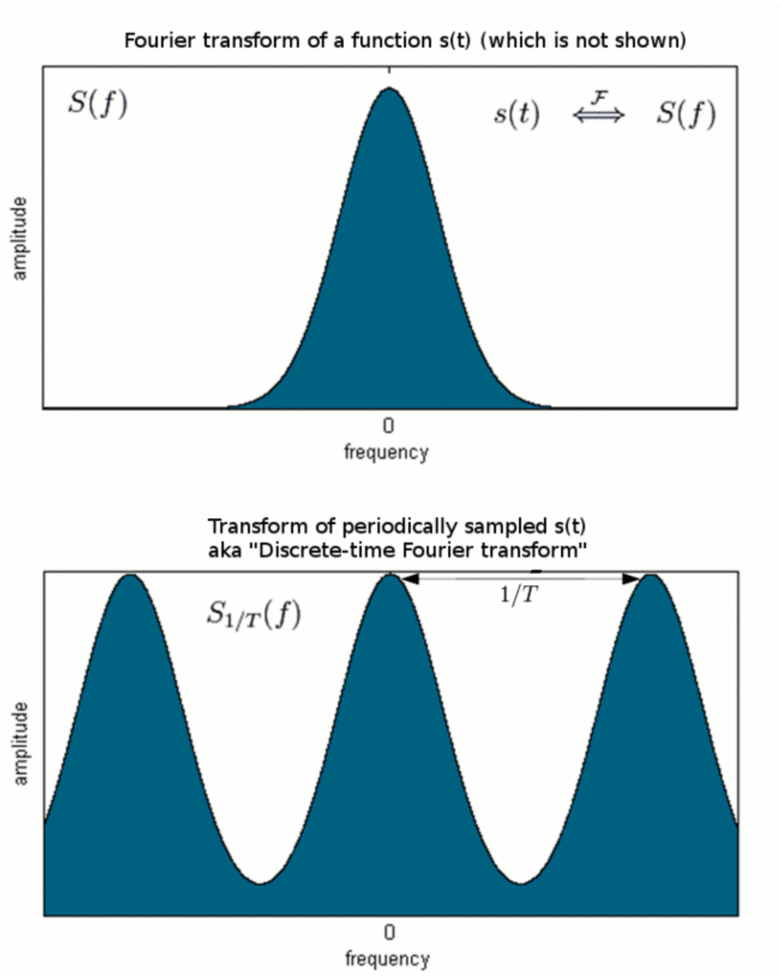
$$X(\mathrm{e}^{j\omega}) = \sum_n x[n] e^{-j\omega n} \qquad -\pi \le \omega < \pi$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Note that this is a continuous function in the variable ω
- Measures frequency content of discrete signal (Discrete frequency is in radians)
- DTFT is periodic in frequency ω

$$X(\mathrm{e}^{j(\omega+2\pi k)}) = \sum_n x[n] e^{-j(\omega+2\pi k)n} = \sum_n x[n] e^{-j\omega n} e^{-j2\pi k} = \sum_n x[n] e^{-j\omega n} = X(\mathrm{e}^{j\omega})$$

Discrete-time Fourier transform



Discrete-time Fourier transform

– DTFT exists if sequence is absolutely summable

$$\left|X(\mathrm{e}^{j\omega})\right| \le \sum_n |x[n]|\left|e^{-j\omega n}\right| = \sum_n |x[n]| < \infty$$

– Relationship of z-transform to DTFT:

$$X(z)\Big|_{z=e^{j\omega}} = \sum_n x[n]\, z^{-n}\Big|_{e^{j\omega}} \quad \Rightarrow \quad \sum_n x[n] e^{-j\omega n} = X(\mathrm{e}^{j\omega})$$

– i.e. Z-transform computed on unit circle.
(Region of Convergence (ROC) must include unit circle.)

Discrete-time Fourier transform

- Eigenfunctions and the DTFT
  - Suppose input to system is $x[n] = e^{j\omega_o n}$
  - Output is

$$y[n] = \sum_k h[k]x[n-k] = \sum_k h[k]e^{j\omega_o(n-k)}$$
$$= e^{j\omega_o n}\sum_k h[k]e^{-j\omega_o k} = H\left(e^{j\omega_o}\right)e^{j\omega_o n}$$

  - Output is same as input multiplied by DTFT of the impulse response
  - That is to say, $x[n] = e^{j\omega_o n}$ are eigenvectors of systems with eigenvalues of $H\left(e^{j\omega_o}\right)$, the DTFT evaluated at ω_0

Discrete-time Fourier transform

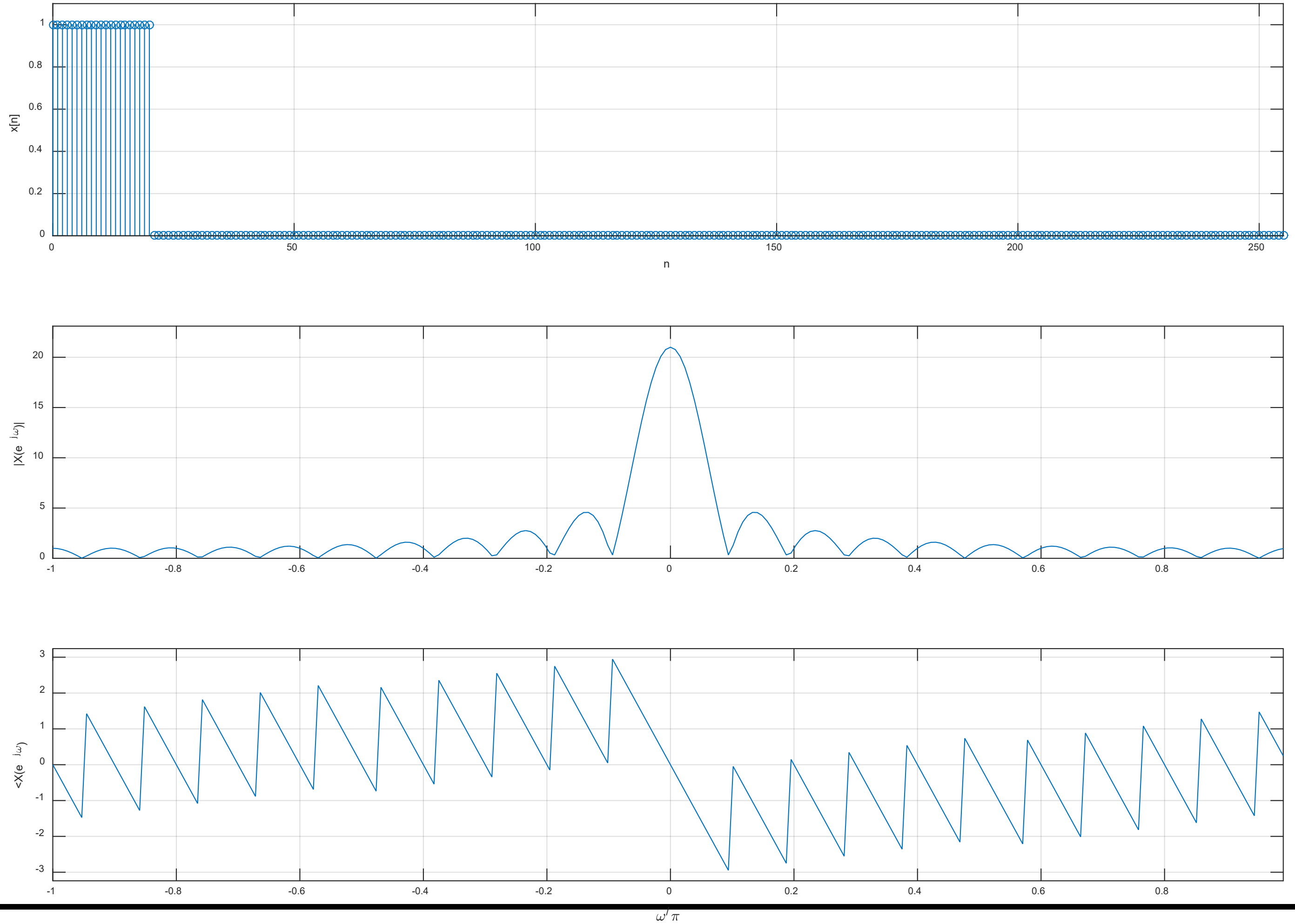
- Duality in Time and Frequency:
  - You can find DTFT of functions that are not absolutely summable using duality between domain
  - Consider: $e^{-j\omega_o n}$ which is not absolutely summable
  - What is inverse DTFT of $2\pi\delta(\omega+\omega_o)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega+\omega_o)\, e^{j\omega n}\, d\omega = e^{-j\omega_o n}$$

  - If the inverse DTFT of $2\pi\delta(\omega+\omega_o)$ is $e^{-j\omega_o n}$ then $2\pi\delta(\omega+\omega_o)$ must be the DTFT of $e^{-j\omega_o n}$
  - Even though $e^{-j\omega_o n}$ is not absolutely summable, its DTFT exists and is $2\pi\delta(\omega+\omega_o)$
  - This is what is meant by duality

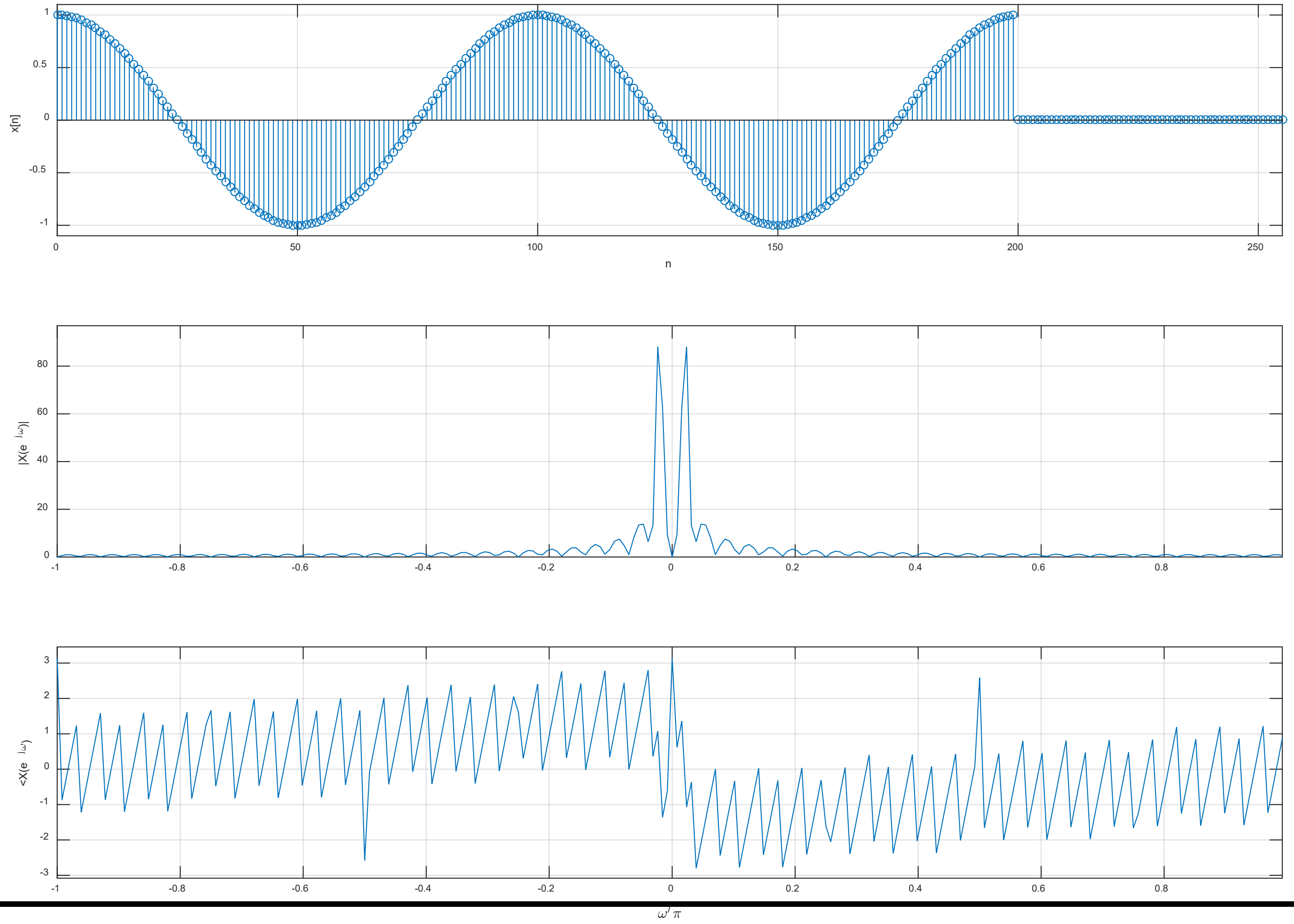
Discrete-time Fourier transform

- Example: Pulse



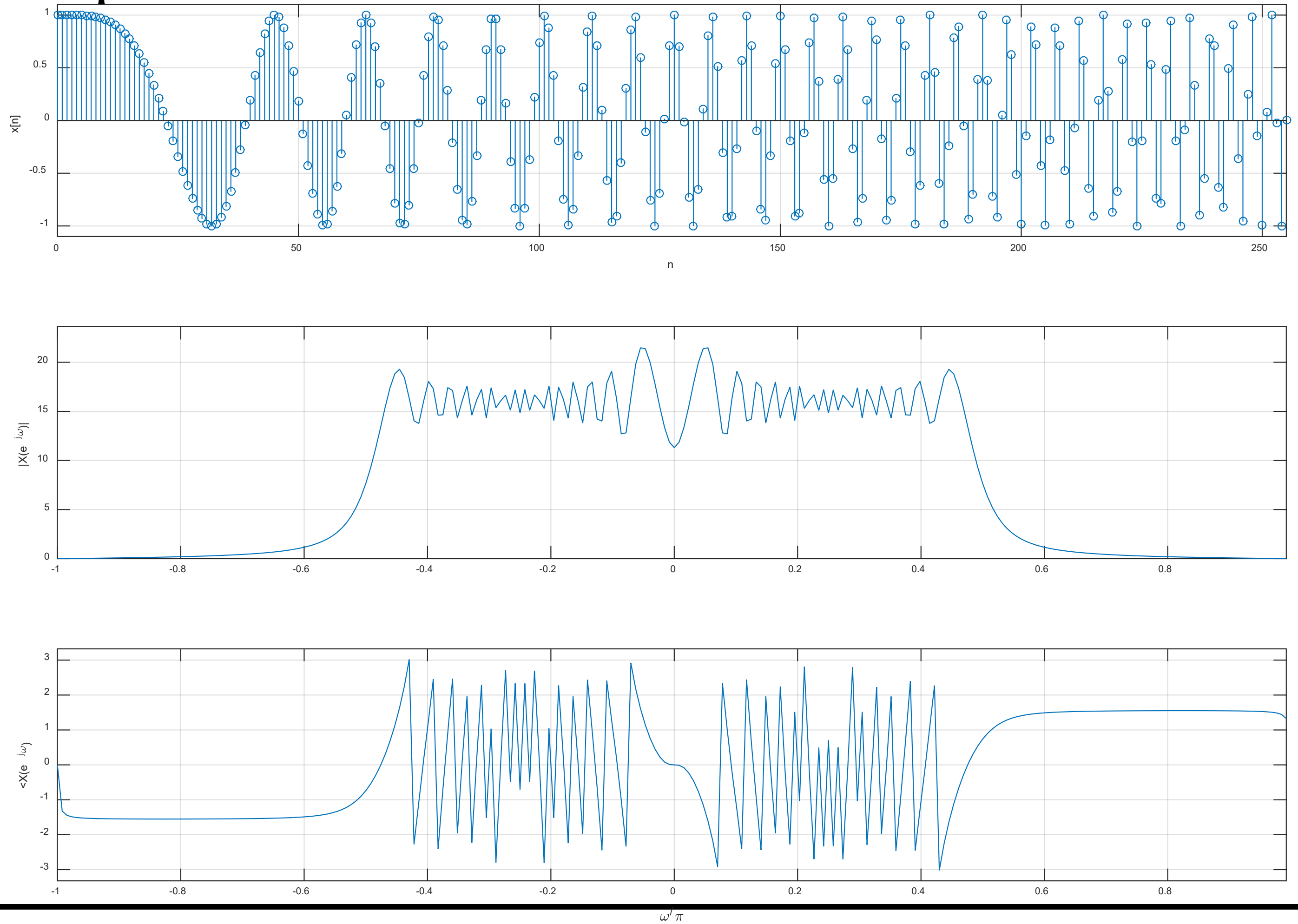
Discrete-time Fourier transform

- Example: Cosine



Discrete-time Fourier transform

- Example: Chirp



Discrete-time Fourier transform

- Frequency and time support:
  - For the DTFT of a signal, the frequency support is inversely proportional to time support (same relationship we saw for continuous signals)
  - Example: Consider DTFT of pulse function

$$p[n] = u[n] - u[n-N]$$

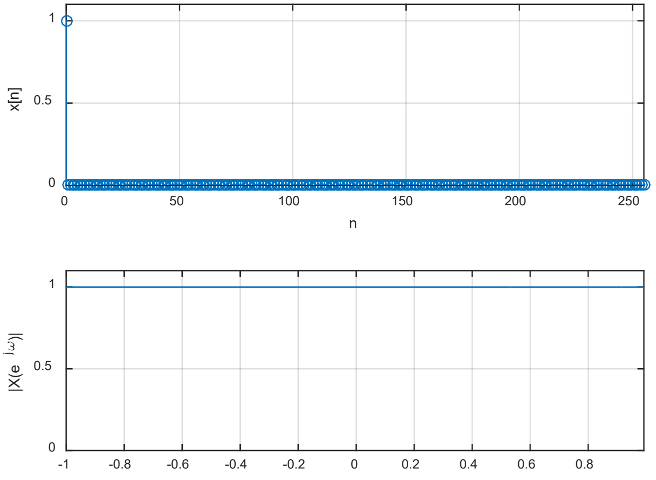
$$P(z) = \frac{1}{1-z^{-1}} - \frac{z^{-N}}{1-z^{-1}} = \frac{1-z^{-N}}{1-z^{-1}}$$

$$P\left(e^{j\omega}\right) = \frac{1-e^{-jN\omega}}{1-e^{-j\omega}} = \frac{e^{-jN\omega/2}\left(e^{+jN\omega/2} - e^{-jN\omega/2}\right)}{e^{-j\omega/2}\left(e^{+j\omega/2} - e^{-j\omega/2}\right)}$$

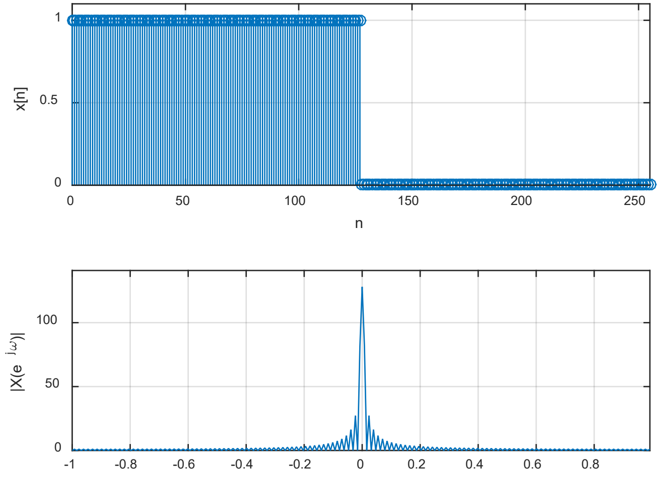
$$P\left(e^{j\omega}\right) = e^{-j(N-1)\omega/2} \frac{\left(\sin\left(N\omega/2\right)\right)}{\left(\sin\left(\omega/2\right)\right)}$$

Discrete-time Fourier transform

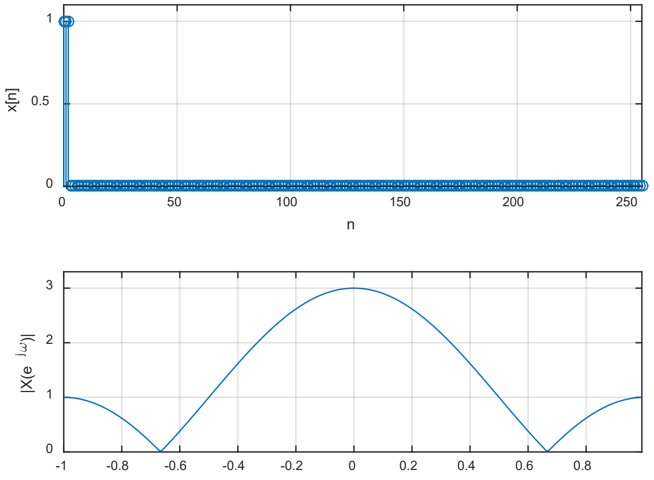
N=1



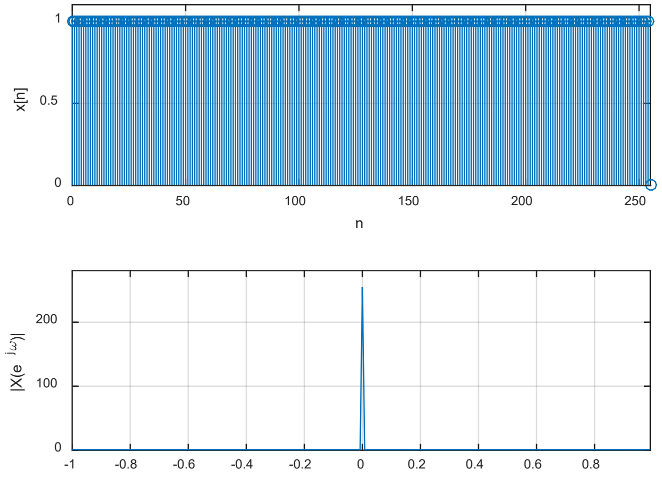
N=128



N=3



N=255



Discrete-time Fourier transform

- Since DTFT can be obtained from z-transform
  - Has same properties for time shifts, convolution, etc.
  - Expressed in terms of $e^{-j\omega}$ instead of z

Discrete-time Fourier transform

Discrete-time Fourier Transforms (DTFT)

	Discrete-time signal	DTFT $X(e^{j\omega})$, periodic of period 2π
(1)	$\delta[n]$	$1, -\pi \leq \omega < \pi$
(2)	A	$2\pi A\delta(\omega), -\pi \leq \omega < \pi$
(3)	$e^{j\omega_0 n}$	$2\pi\delta(w - \omega_0), -\pi \leq \omega < \pi$
(4)	$\alpha^n u[n], \|\alpha\| < 1$	$\frac{1}{1-\alpha\, e^{-j\omega}}, -\pi \leq \omega < \pi$
(5)	$n\,\alpha^n u[n], \|\alpha\| < 1$	$\frac{\alpha\, e^{-j\omega}}{(1-\alpha\, e^{-j\omega})^2}, -\pi \leq \omega < \pi$
(6)	$\cos(\omega_0 n)u[n]$	$\pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right], -\pi \leq \omega < \pi$
(7)	$\sin(\omega_0 n)u[n]$	$-j\pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right], -\pi \leq \omega < \pi$
(8)	$\alpha^{\|n\|}, \|\alpha\| < 1$	$\frac{1-\alpha^2}{1-2\alpha\cos(\omega)+\alpha^2}, -\pi \leq \omega < \pi$
(9)	$p[n] = u[n + N/2] - u[n - N/2]$	$\frac{\sin(\omega(N+1)/2)}{\sin(\omega/2)}, -\pi \leq \omega < \pi$
(10)	$\alpha^n \cos(\omega_0 n)u[n]$	$\frac{1-\alpha\cos(\omega_0)e^{-j\omega}}{1-2\alpha\cos(\omega_0)e^{-j\omega}+\alpha^2 e^{-2j\omega}}, -\pi \leq \omega < \pi$
(11)	$\alpha^n \sin(\omega_0 n)u[n]$	$\frac{\alpha\sin(\omega_0)e^{-j\omega}}{1-2\alpha\cos(\omega_0)e^{-j\omega}+\alpha^2 e^{-2j\omega}}, -\pi \leq \omega < \pi$

Discrete-time Fourier transform

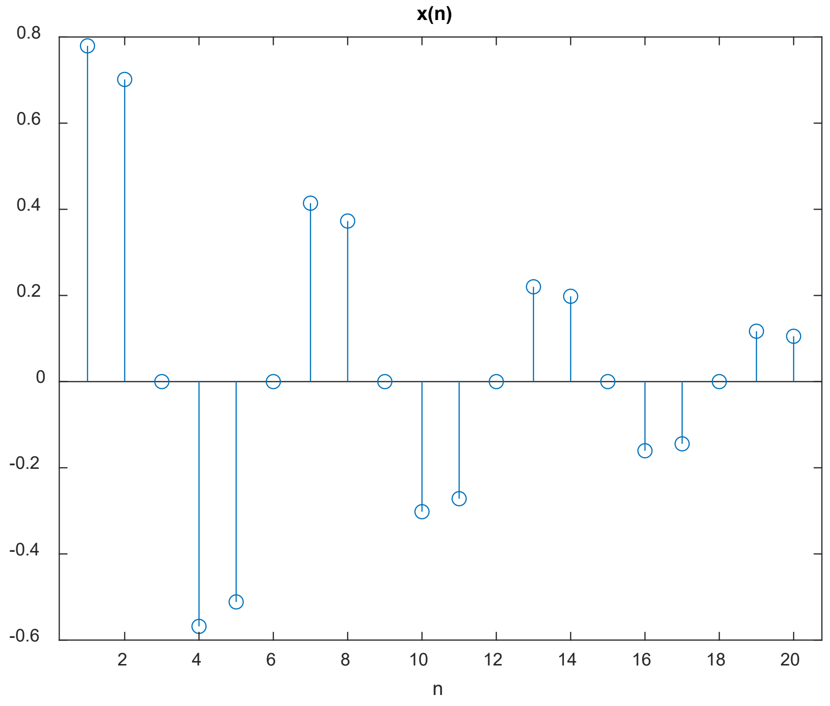
Properties of the DTFT

Z-transform:	$x[n], X(z), \lvert z\rvert = 1 \in ROC$	$X(e^{j\omega}) = X(z)\lvert_{z=e^{j\omega}}$
Periodicity:	$x[n]$	$X(e^{j\omega}) = X(e^{j(\omega+2\pi k)}),\ k\ integer$
Linearity:	$\alpha x[n] + \beta y[n]$	$\alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
Time-shifting:	$x[n-N]$	$e^{-j\omega N}X(e^{j\omega})$
Frequency-shift:	$x[n]e^{j\omega_o n}$	$X(e^{j(\omega-\omega_0)})$
Convolution:	$(x * y)[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication:	$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Symmetry:	$x[n]$ real-valued	$\lvert X(e^{j\omega})\rvert$ even function of ω
		$\angle X(e^{j\omega})$ odd function of ω
Parseval's relation:	$\sum_{n=\infty}^{\infty}\lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\omega}\rvert^2 d\omega$	

Discrete-time Fourier transform

- Example: Find the DTFT of

$$x(n) = \alpha^n \sin(\omega_0) u(n) \quad \text{for} \quad |\alpha| < 1$$

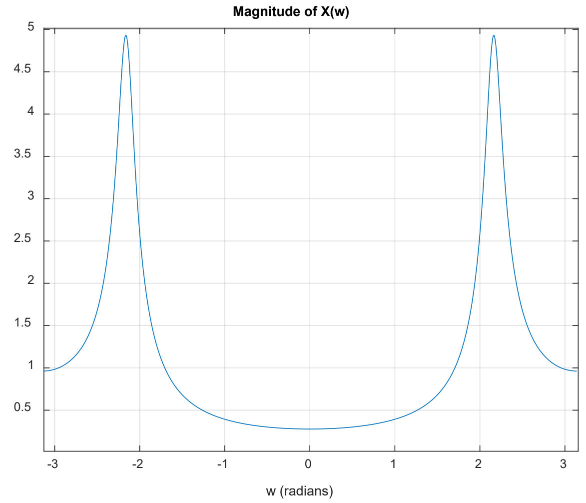


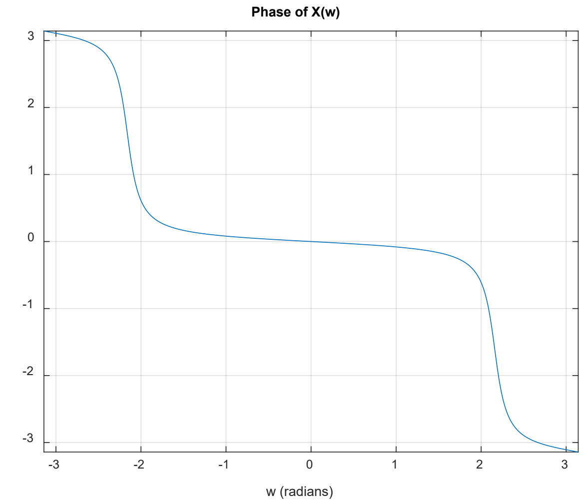
Discrete-time Fourier transform

- Example: Find the DTFT of

$$x(n) = \alpha^n \sin(\omega_0) u(n) \quad \text{for} \quad |\alpha| < 1$$

$$X(\omega) = \frac{\alpha \sin \omega_0 e^{-j\omega}}{1 + 2\cos \omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega}}$$





Discrete-time Fourier transform

- Example: Find the inverse DTFT of

$$X(\omega) = \begin{cases} 0, & \text{for } \quad 0 < |\omega| \leq \omega_0 \\ 1, & \text{for } \omega_0 < |\omega| \leq \pi \end{cases}$$

Discrete-time Fourier series

- Consider the frequency representation of a periodic sequence where N is the period. $x[n+kN]=x[n]$
  - A periodic sequence can be represented in terms of a sum over basis functions:

$$\phi[k,n]=e^{j2\pi k n/N} \quad \left(\text{Different notation, but same as } s_k(n) \text{ in Proakis and Manolakis}\right)$$

  - These basis functions are periodic in k and n with period N
    - Easy to show. Substitute $k=k+rN$; substitute $n=n+rN$ where r is an integer
  - Basis functions are orthogonal over period N

$$\sum_{n=0}^{N-1}\phi[k,n]\times\phi^*[l,n]=\sum_{n=0}^{N-1}e^{j\frac{2\pi}{N}kn}\times e^{-j\frac{2\pi}{N}ln}=\begin{cases}N & k=l\\ 0 & k\neq l\end{cases}$$

Discrete-time Fourier series

– You can show orthogonality using our old friend, the geometric series, but not consider the finite geometric series:

$$\boxed{1+r+r^2+r^3+\cdots+r^{N-1}=\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r} \text{ for } r \neq 1}$$

$$\sum_{n=0}^{N-1} \phi[k,n]\times\phi^*[l,n]=\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn}\times e^{-j\frac{2\pi}{N}ln}=\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)n}$$

$$=\sum_{n=0}^{N-1}\left(e^{j\frac{2\pi(k-l)}{N}}\right)^n=\frac{1-e^{j\frac{2\pi(k-l)N}{N}}}{1-e^{j\frac{2\pi(k-l)}{N}}}=\frac{1-e^{j2\pi(k-l)}}{1-e^{j\frac{2\pi(k-l)}{N}}}=0 \text{ if } k\neq l$$

$$\boxed{\begin{array}{l}\text{If } k=l,\\ \displaystyle\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)n}=\sum_{n=0}^{N-1}\left(e^{j\frac{2\pi}{N}n}\right)^0=\sum_{n=0}^{N-1}1=N\end{array}}$$

Discrete-time Fourier series

– The orthogonality of $\phi[k,n] = e^{j2\pi kn/N}$ can be used to represent a periodic sequence (of period N) as:

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \quad \text{where} \quad X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

The nomenclature in the book is different than that shown here. $c_k \equiv X[k]$

which is the Fourier Series of $x[n]$.

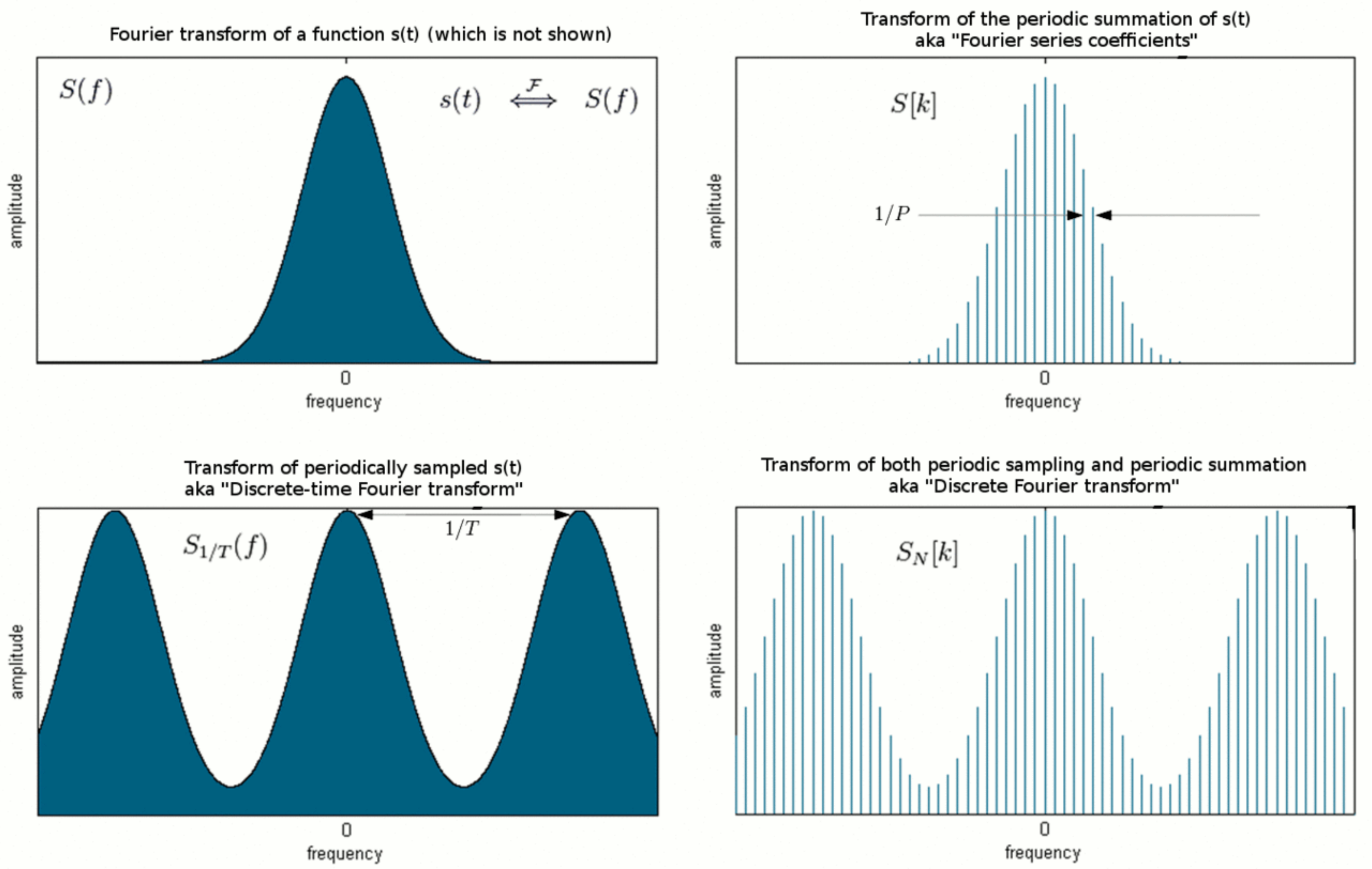
The fundamental frequency is $\omega_o = 2\pi/N$

Notice that the frequency components for $X[k]$ are discrete

Both signal and Fourier series are discrete sequences.

(In contrast to Discrete Time Fourier Transform)

Discrete-time Fourier series



Discrete-time Fourier series

– Power spectrum

$$P_x = \frac{1}{N}\sum_{k=0}^{N-1}\left|x[n]\right|^2$$

Also

$$P_x = \frac{1}{N}\sum_{k=0}^{N-1} x[n]x^*[n] = \sum_{k=0}^{N-1} X^*[k]\left(\frac{1}{N}\sum_{k}^{N} x[n]e^{-j\frac{2\pi}{N}kn}\right)$$

$$P_x = \sum_{k=0}^{N-1} X^*[k]X[k]$$

$$P_x = \sum_{k=0}^{N-1}\left|X[k]\right|^2$$

$$\boxed{P_x = \frac{1}{N}\sum_{k=0}^{N-1}\left|x[n]\right|^2 = \sum_{k=0}^{N-1}\left|X[k]\right|^2}$$

Discrete-time Fourier series

– Energy spectrum

$$E_x = \sum_{k=0}^{N-1} \left| x[n] \right|^2$$

Also

$$E_x = \sum_{k=0}^{N-1} x[n] x^*[n] = \sum_{k=0}^{N-1} X^*[k] \left(\frac{N}{N} \sum_{k}^{N} x[n] e^{-j\frac{2\pi}{N}kn} \right)$$

$$E_x = N \sum_{k=0}^{N-1} X^*[k] X[k]$$

$$E_x = N \sum_{k=0}^{N-1} \left| X[k] \right|^2$$

$$\boxed{E_x = \sum_{k=0}^{N-1} \left| x[n] \right|^2 = \mathrm{N} \sum_{k=0}^{N-1} \left| X[k] \right|^2}$$

Discrete-time Fourier series

– Symmetry for real signals

$$X^*[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{+j\frac{2\pi}{N}kn} = X[-k]$$

$$\left|X[-k]\right| = \left|X[-k]\right|$$

$$-\measuredangle X[-k] = \measuredangle X[k]$$

$$\left|X[0]\right| = \left|X[N]\right|$$

$$\left|X[1]\right| = \left|X[N-1]\right|$$

$$\left|X[N/2]\right| = \left|X[N/2]\right| \qquad N \text{ even}$$

$$\left|X[(N-1)/2]\right| = \left|X[(N+1)/2]\right| \quad N \text{ odd}$$

$X[k]$ is also periodic

$$X[k+N] = X[k] \Rightarrow X[N-k] = X[-k]$$

$$\left|X[k]\right| = \left|X[N-k]\right|$$

$$\measuredangle X[k] = -\measuredangle X[N-k]$$

$$\measuredangle X[0] = -\measuredangle X[N]$$

$$\measuredangle X[1] = -\measuredangle X[N-1]$$

$$\measuredangle X[N/2] = 0 \qquad N \text{ even}$$

$$\measuredangle X[(N-1)/2] = -\measuredangle X[(N+1)/2] \quad N \text{ odd}$$

Discrete-time Fourier series

- Obtaining Fourier series coefficients for discrete sequences from the z-transform is similar to what you do for continuous signals from the Laplace transform.

For $x_1[n] = x[n]\left(u[n] - u[n-N]\right)$

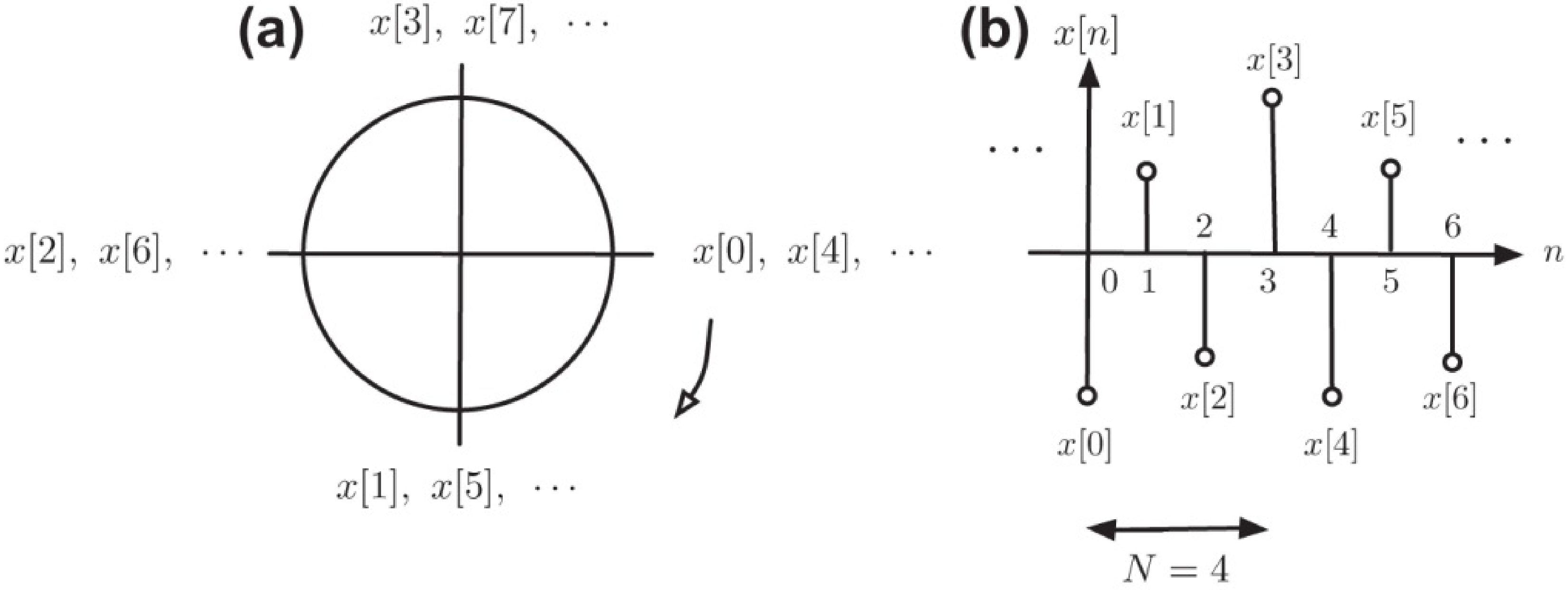
(i.e., one period of the periodic sequence $x[n]$)

$$Z\left\{x_1[n]\right\} = \sum_{n=0}^{N-1} x[n] z^{-n}$$

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} Z\left\{x_1[n]\right\}\Big|_{z=e^{j\frac{2\pi}{N}k}}$$

Discrete-time Fourier series

– For periodic sequences, it is convenient to think of the sequence values as being on circle



Discrete-time Fourier series

- Periodic convolution
  - For periodic sequence, convolution is a bit different
    - The product of two periodic sequences is also periodic
  - Periodic convolution:

$$v[n] = \sum_{m=0}^{N-1} x[m]\, y[n-m] \quad \Leftrightarrow \quad V[k] = N X[k] Y[k]$$

$$w[n] = x[n] y[n] \quad \Leftrightarrow \quad W[k] = \sum_{m=0}^{N-1} X[m] Y[n-m]$$

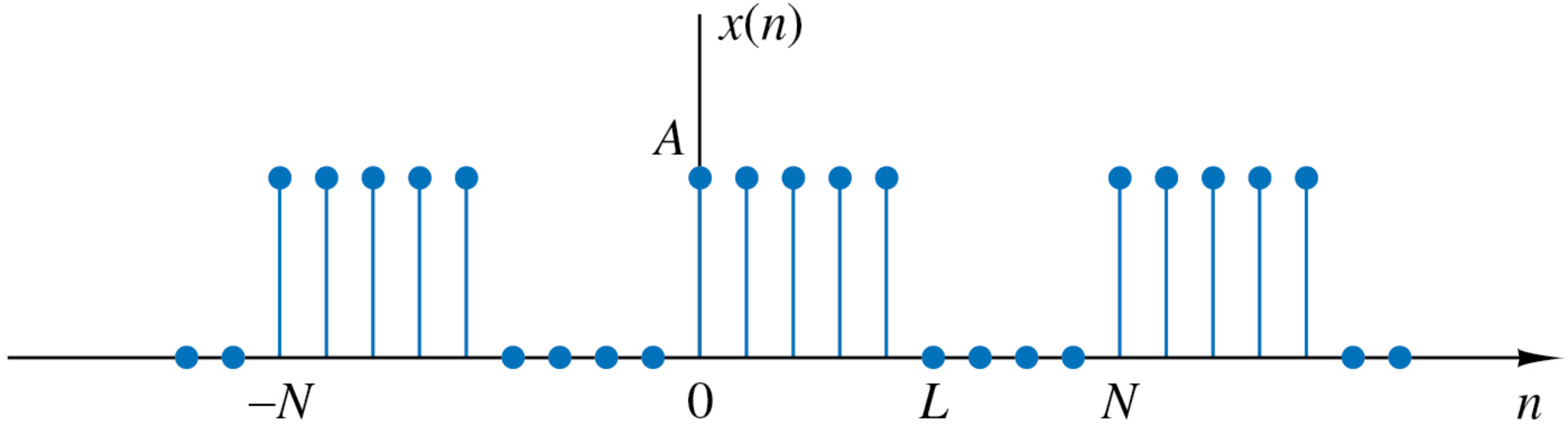
All are periodic with period N

Discrete-time Fourier series

Fourier Series of Discrete-time Periodic signals		
	$x[n]$ periodic signal of period N	$X[k]$ periodic FS coefficients of period N
Z-transform	$x_1[n] = x[n](u[n] - u[n-N])$	$X[k] = \frac{1}{N}\mathcal{Z}(x_1[n])\big\vert_{z=e^{j2\pi k/N}}$
DTFT	$x[n] = \sum_k X[k]e^{j2\pi nk/N}$	$X(e^{j\omega}) = \sum_k 2\pi X[k]\delta(\omega - 2\pi k/N)$
LTI response	input $x[n] = \sum_k X[k]e^{j2\pi nk/N}$	output: $y[n] = \sum_k X[k]H(e^{jk\omega_0})e^{j2\pi nk/N}$ $H(e^{j\omega})$ (frequency response of system)
Time-shift (circular shift)	$x[n-M]$	$X[k]e^{-j2\pi kM/N}$
Modulation	$x[n]e^{j2\pi Mn/N}$	$X[k-M]$
Multiplication	$x[n]y[n]$	$\sum_{m=0}^{N-1} X[m]Y[k-m]$ periodic convolution
Periodic convolution	$\sum_{m=0}^{N-1} x[m]y[n-m]$	$NX[k]Y[n]$

Discrete-time Fourier transform

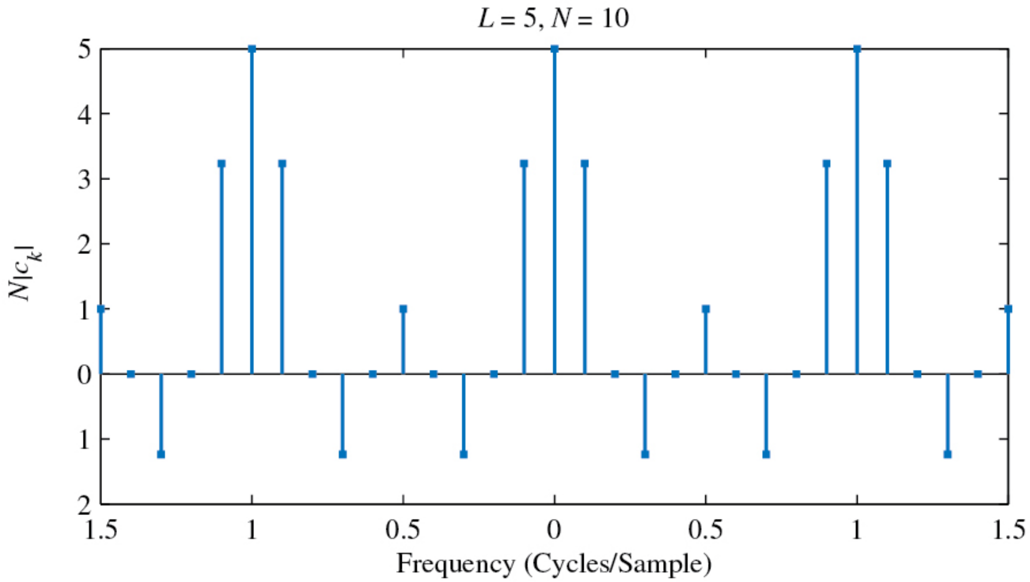
- Example: Find the inverse DTFS



$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \quad ; \quad X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

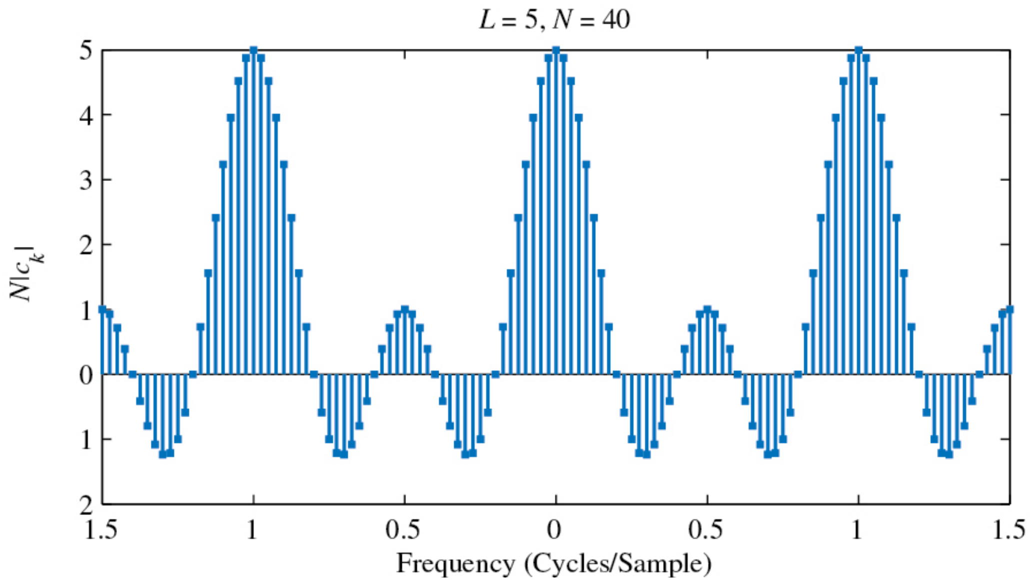
Discrete-time Fourier transform

- Example: Find the inverse L = 5, N=10, A=1 (power spectrum)



Discrete-time Fourier transform

- Example: Find the inverse   L = 5, N=10, A=1   (power spectrum)

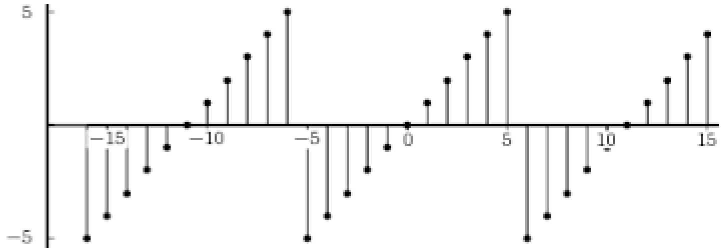


Discrete Fourier Transform (DFT)

- The step from the Discrete Fourier Series to the Discrete Fourier Transform is a short one.
  - Consider a periodic sequence $x[n]$ (period N)
    - It has a Fourier series
  - Consider a finite length sequence $\mathrm{x}[n], \quad 0 \le n \le N-1$
  - One can think of making a periodic extension of this sequence and then take it's Fourier series.
    - This is essentially the Discrete Fourier transform
    - Except … traditionally, the 1/N goes with the sum over the DFT coefficients.

Discrete Fourier Transform

- Discrete Fourier Transform (DFT)
  - Signals may not be periodic, but are generally finite in length
    - In practice, all signals are finite.
    - If you are working with really long signals, you can always break it up into shorter length sections.
  - Although signal is not periodic, you can create a periodic extension of the signal by repeating the signal before and after real signal.
    - Create a periodic signal
  - You can then find the Discrete Time Fourier Series of the periodic extension of the signal



Discrete Fourier Transform

- Discrete Fourier Transform (DFT)
  - The DFT is usually written a little differently than the DTFS
  - For a finite length signal of length L, one often pads it out to a larger number of samples, N, that is L or greater:
  - The factor of 1/N is usually put with the "inverse" transform

$$
\boxed{
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi nk}{N}} \qquad 0 \le k \le N-1 \\
x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{\frac{j2\pi nk}{N}} \qquad 0 \le n \le N-1
\end{aligned}
}
$$

We will discuss the Discrete Fourier Transform in more detail later