Digital Signal Processing

Class 15
03/18/2025

ENGR 71

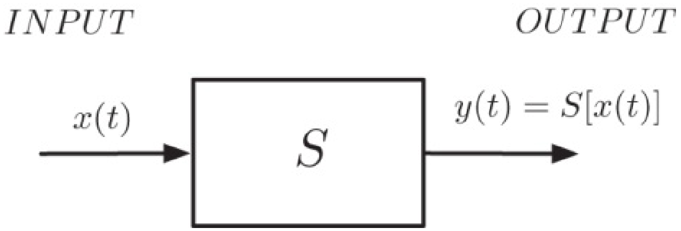
- Class Overview
  - Review for Exam 1
- Assignments
  - Exam 1: Due Sunday, March 23, 11:59 PM
  - Lab 2: Due Friday, March 28, 11:59 PM

ENGR 71

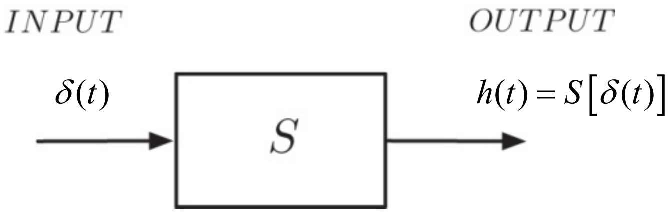
- Exam
  - Take home exam
  - Open book, open note
    - But, must be your own work
      - Don’t use AI

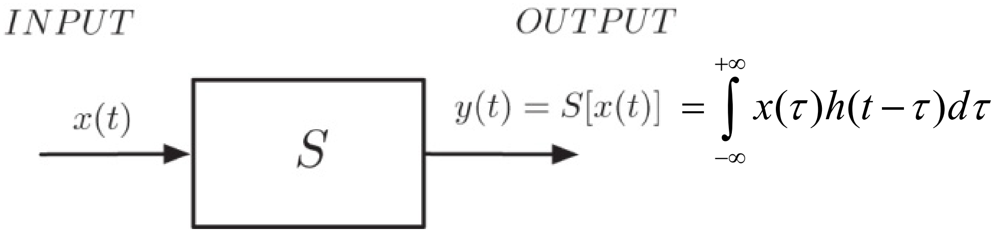
Review for Exam 1

- Continuous Linear Time-Invariant Systems
  - General system description



  - LTI systems characterized by impulse response





Review for Exam 1

- Continuous Linear Time-Invariant Systems
  - Most general form of LTI system is linear D.E. with constant coefficients

$$\frac{d^n y}{dt^n}+a_{n-1}\frac{d^{n-1}y}{dt^{n-1}}+a_{n-2}\frac{d^{n-2}y}{dt^{n-2}}+\cdots+a_1\frac{dy}{dt}+a_0 y$$
$$=b_m\frac{d^m x}{dt^m}+b_{m-1}\frac{d^{m-1}x}{dt^{m-1}}+b_{m-2}\frac{d^{m-2}x}{dt^{m-2}}+\cdots+b_1\frac{dx}{dt}+b_0 x \quad \text{(Initial conditions are all zero)}$$

OR

$$\boxed{\frac{d^n y}{dt^n}=-a_{n-1}\frac{d^{n-1}y}{dt^{n-1}}-a_{n-2}\frac{d^{n-2}y}{dt^{n-2}}-\cdots-a_1\frac{dy}{dt}-a_0 y+b_m\frac{d^m x}{dt^m}+b_{m-1}\frac{d^{m-1}x}{dt^{m-1}}+b_{m-2}\frac{d^{m-2}x}{dt^{m-2}}+\cdots+b_1\frac{dx}{dt}+b_0 x}$$

Review for Exam 1

- Continuous Linear Time-Invariant Systems
  - Laplace transform

$$F(s) = \mathscr{L}[f(t)] = \int_{-\infty}^{+\infty} f(t)e^{-st}\,dt \quad \text{where } s \text{ is a complex variable } (s = \sigma + j\omega)$$

  - Laplace transform turns linear differential equations into algebraic equations by differentiation property

$$\mathscr{L}\left[\frac{df(t)}{dt}u(t)\right] = sF(s) - f(0-)$$

$$\mathscr{L}\left[\frac{d^2 f(t)}{dt^2}u(t)\right] = s^2F(s) - sf(0-) - \left.\frac{df(t)}{dt}\right|_{t=0-}$$

$$\mathscr{L}\left[f^{(N)}(t)u(t)\right] = s^N F(s) - \sum_{k=0}^{N-1} f^{(k)}(0-)s^{N-1-k}$$

Zero initial conditions $\Longrightarrow$

$$\mathscr{L}\left[\frac{df(t)}{dt}u(t)\right] = sF(s)$$

$$\mathscr{L}\left[\frac{d^2 f(t)}{dt^2}u(t)\right] = s^2F(s)$$

$$\mathscr{L}\left[f^{(N)}(t)u(t)\right] = s^N F(s)$$

Review for Exam 1

- Continuous Linear Time-Invariant Systems
  - Laplace transform turns linear differential equations into algebraic equations by differentiation property

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + a_{n-2}\frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_1\frac{dy}{dt} + a_0 y =$$

$$b_m\frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1} x}{dt^{m-1}} + b_{m-2}\frac{d^{m-2} x}{dt^{m-2}} + \cdots + b_1\frac{dx}{dt} + b_0 x$$

$$n > m$$

  - Taking Laplace transform of both sides:
    (with zero initial conditions, i.e. zero-state … aka quiescent)

$$\left(s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-1} + \cdots + a_1 s + a_0\right)Y(s) = \left(b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_1 s + b_0\right)X(s)$$

Review for Exam 1

- Continuous Linear Time-Invariant Systems
  - Impulse response:

$$\frac{d^n h(t)}{dt^n} + a_{n-1}\frac{d^{n-1}h(t)}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}h(t)}{dt^{n-2}} + \cdots + a_1\frac{dh(t)}{dt} + a_0 h(t) =$$

$$b_m\frac{d^m\delta(t)}{dt^m} + b_{m-1}\frac{d^{m-1}\delta(t)}{dt^{m-1}} + b_{m-2}\frac{d^{m-2}\delta(t)}{dt^{m-2}} + \cdots + b_1\frac{d\delta(t)}{dt} + b_0\delta(t)$$

For input, $x(t)$, output is convolution of impulse response with input:

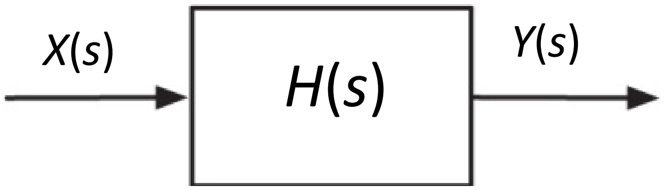
$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \equiv h(t) * x(t)$$

$Y(s) = H(s)X(s)$ convolution in time domain is multiplication in Laplace domain

Transfer function: $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-1} + \cdots + a_1 s + a_0}$

Review for Exam 1

- Continuous Linear Time-Invariant Systems
  - System in Laplace domain



  - To get impulse response, inverse Laplace transform:

$$h(t) = \mathcal{L}^{-1}\left[\frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-1} + \cdots + a_1 s + a_0}\right]$$

  - To find step response, inverse Laplace transform:

$$s(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\left(\frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-1} + \cdots + a_1 s + a_0}\right)\right] \quad \text{since } \mathcal{L}\left[u(t)\right] = \frac{1}{s}$$

Review for Exam 1

- Continuous Linear Time-Invariant Systems
  - Numerator and denominator as product of linear terms with zeros and poles (zeros at s =-z_k , poles at s = -p_k)

$$H(s)=\frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)}=\frac{\prod_{k=1}^{M}(s+z_k)}{\prod_{k=1}^{N}(s+p_k)_0} \qquad \text{text } N>M$$

  - To invert, do partial fraction expansion:

$$H(s)=\frac{A_1}{s+p_1}+\frac{A_2}{s+p_2}+\cdots+\frac{A_N}{s+p_N}$$

$$\mathscr{L}^{-1}\left[\frac{1}{s+p}\right]=e^{-pt}=e^{-(\sigma+j\omega)t}=e^{-\sigma t}e^{-jwt}$$

You can see why the poles must be in the left-hand side of the s-plane for the system to be stable.

Review for Exam 1

- Continuous Linear Time-Invariant Systems
  - Example from HW2:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$\left(s^2 + 4s + 5\right)Y(s) = \left(s + 2\right)X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^2 + 4s + 5}$$

zero at $s = -2$; poles at $s = -4 \pm \sqrt{16 - (4)(5)} \big/ 2 = -2 \pm j$

Stable since poles are in the left half of the s-plane

(see HW 2 problem 3 for Matlab example)

Review for Exam 1

- Discrete systems: Sampling
  - In frequency domain, spectrum repeats

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\Omega_s)$$

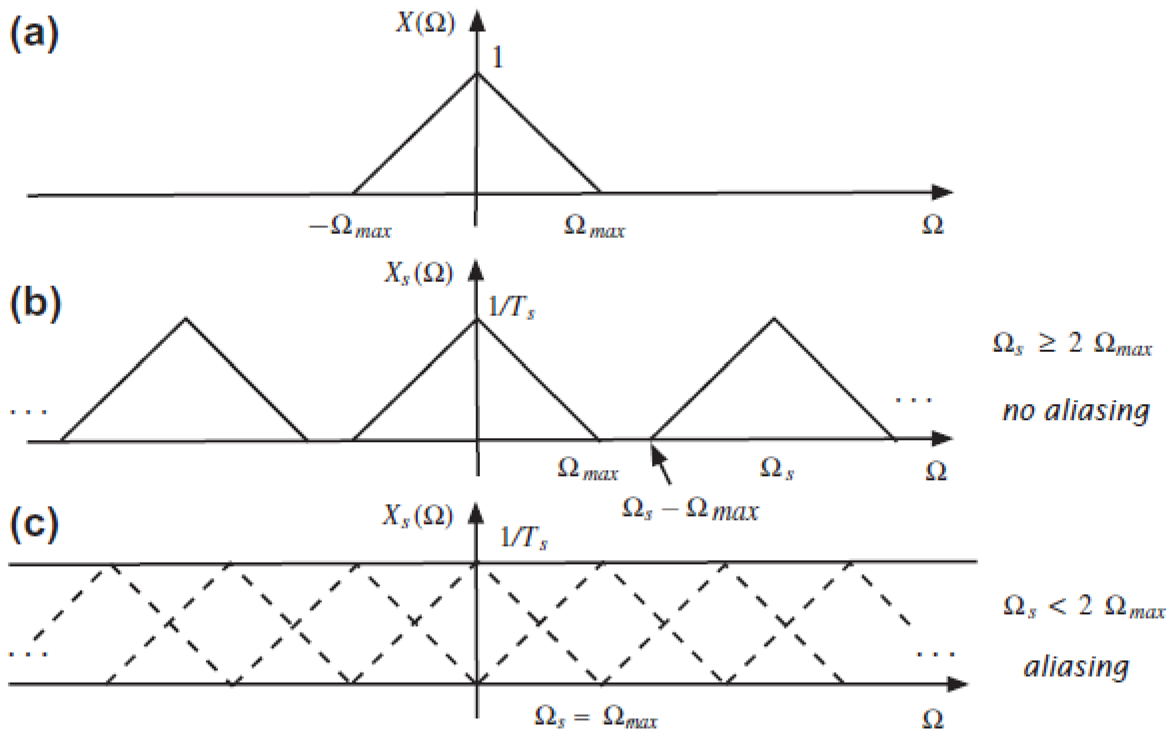
where T_s is sampling interval, and Ω_s is sampling rate (in radians)

$$\Omega_s = 2\pi f_s = \frac{2\pi}{T_s}$$

  - Nyquist sampling theorem states that analog signal can be recovered from sampled signal if you sample at twice the highest frequency in the analog signal.

Review for Exam 1

- Discrete systems: Sampling



spectra do not overlap if

$$\Omega_s - \Omega_{\max} \geq \Omega_{\max} \quad \Rightarrow \quad \Omega_s \geq 2\Omega_{\max}$$

$$f_s \geq 2 f_{\max}$$

Nyquist sampling rate is minimum rate at which signal can be sampled to accurately reconstruct it from its samples

Review for Exam 1

- Fourier Series:
  - Periodic signals can be expressed as a Fourier series:

$$x(t) = x(t+T_0) \qquad f_0 = \frac{1}{T_0} \qquad \omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} \qquad \text{(Synthesis Eq.)}$$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\, dt \qquad \text{(Analysis Eq.)}$$

  - Power of a signal:

$$\text{Time domain: } P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2\, dt \qquad \text{Fourier domain: } P_x = \sum_{k=-\infty}^{+\infty} |c_k|^2$$

Review for Exam 1

- A useful reminder about geometric series:

$$\sum_{k=0}^{N} r^k = \frac{1-r^{N+1}}{1-r} \qquad \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

$$\sum_{k=1}^{N} r^k = \frac{r\left(1-r^N\right)}{1-r} \qquad \sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$$

Think about $P_x = \sum_{k=-\infty}^{+\infty} \left|c_k\right|^2$ in the context of a geometric series if c_k can be cast in the form of r

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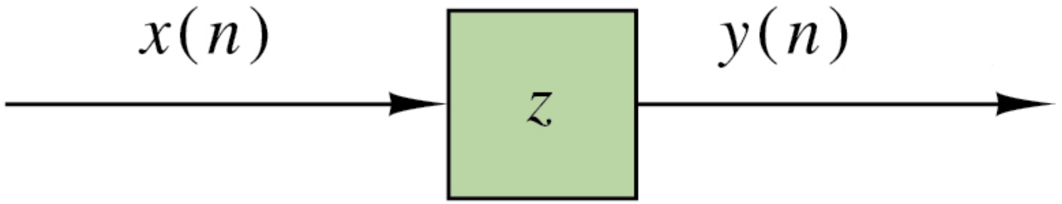
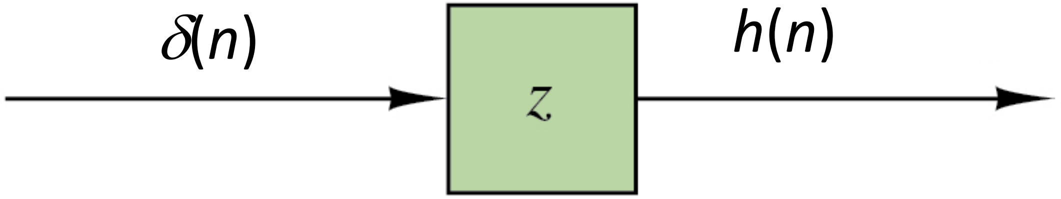
- Discrete Signals and Systems



  - Linear time invariant systems are characterized by their impulse response



Output for input $x(n)$ is found from discrete convolution:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Review for Exam 1

- Discrete Linear Time-Invariant Systems
  - Most general form of LTI system is linear difference equation with constant coefficients

$$y(n)+a_1 y(n-1)+a_2 y(n-2)+\ldots+a_N y(n-k)$$
$$=\ b_0 x(n)+b_1 x(n-1)+b_2 x(n-2)+\ldots+b_M x(n-k)$$

or

$$y(n)=-a_1 y(n-1)-a_2 y(n-2)-\ldots a_N y(n-k)$$
$$+b_0 x(n)+b_1 x(n-1)+b_2 x(n-2)+\ldots+b_M x(n-k)$$

N'th order difference equation

N'th order system

Review for Exam 1

- Discrete Linear Time-Invariant Systems
  - Most general form of LTI system is linear difference equation with constant coefficients

$$y(n)+\sum_{k=1}^{N}a_k y(n-k)=\sum_{k=0}^{M}b_k x(n-k)$$

$$y(n)=-\sum_{k=1}^{N}a_k y(n-k)+\sum_{k=0}^{M}b_k x(n-k)$$

N'th order difference equation

N'th order system

Review for Exam 1

- Discrete Linear Time-Invariant Systems
  - Finite Impulse Response System:
    Output stops at some point given an input
  - Non-recursive:

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

Current output only depends on present and past inputs.

Finite Impulse Response $\left(\text{FIR}\right)$ system

Review for Exam 1

- Discrete Linear Time-Invariant Systems
  - Infinite Impulse Response System: Output goes on forever
  - Generally, recursive:

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

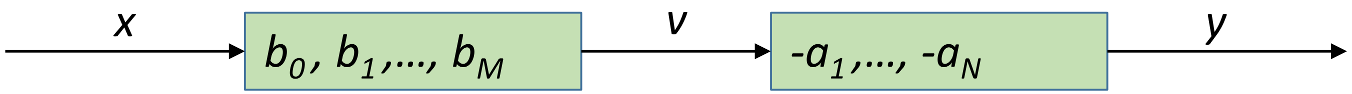
Current output only depends on previous outputs and present and past inputs.

Infinite Impulse Response $\left(\mathrm{IIR}\right)$ system

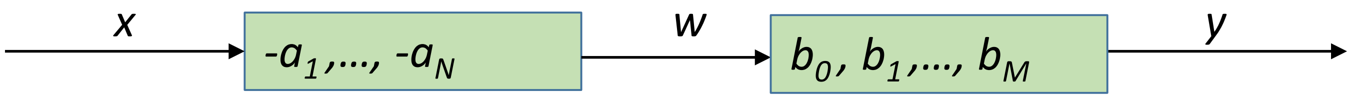
(Could be infinite if non-causal and depends on an infinite number of past inputs.)

Review for Exam 1

- Discrete Linear Time-Invariant System Diagrams
  - Direct Forms 1 and 2:



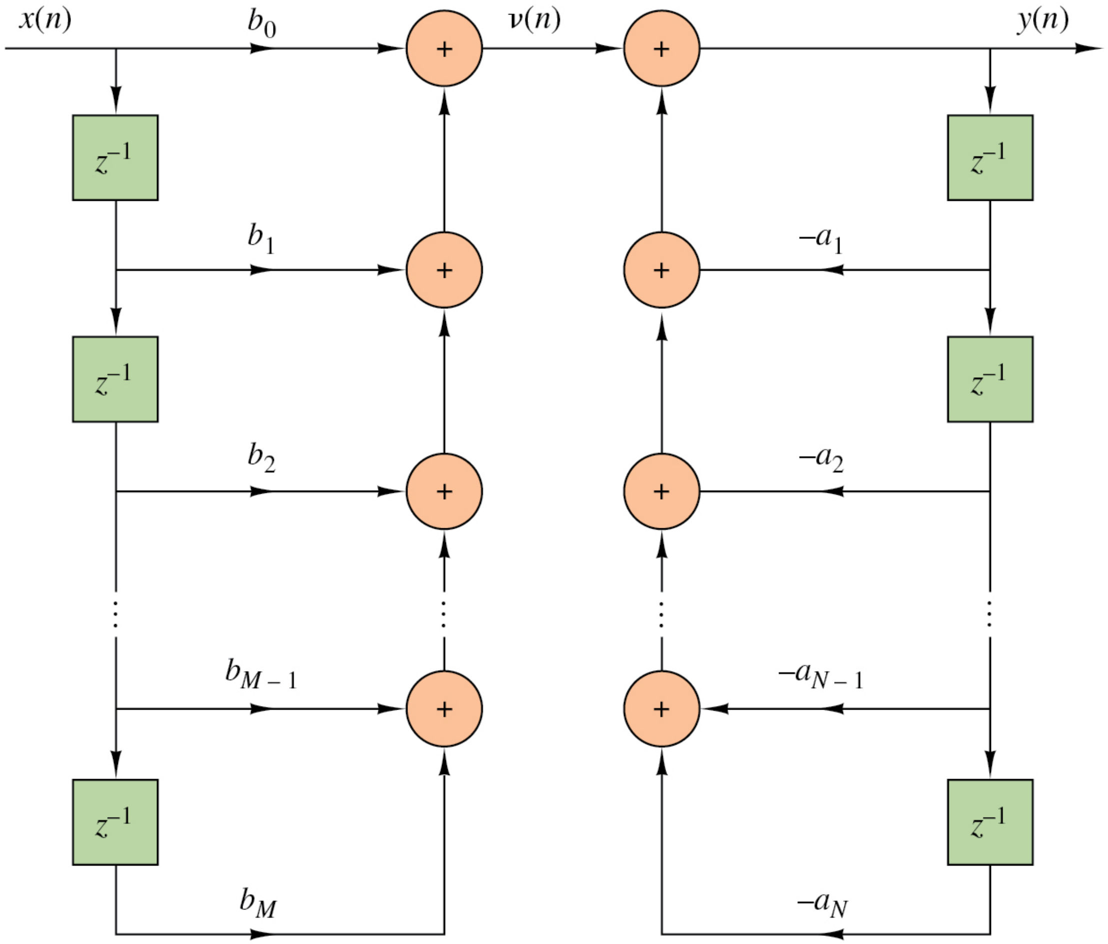
$$v(n) = +b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \ldots + b_M x(n-k)$$
$$y(n) = \left[-a_1 y(n-1) - a_2 y(n-2) - \ldots a_N y(n-k)\right] + \left[v(n)\right]$$



$$w(n) = \left[-a_1 w(n-1) - a_2 w(n-2) - \ldots a_N w(n-k)\right] + x(n)$$
$$y(n) = \left[b_0 w(n) + b_1 w(n-1) + \ldots b_M y(n-M)\right]$$

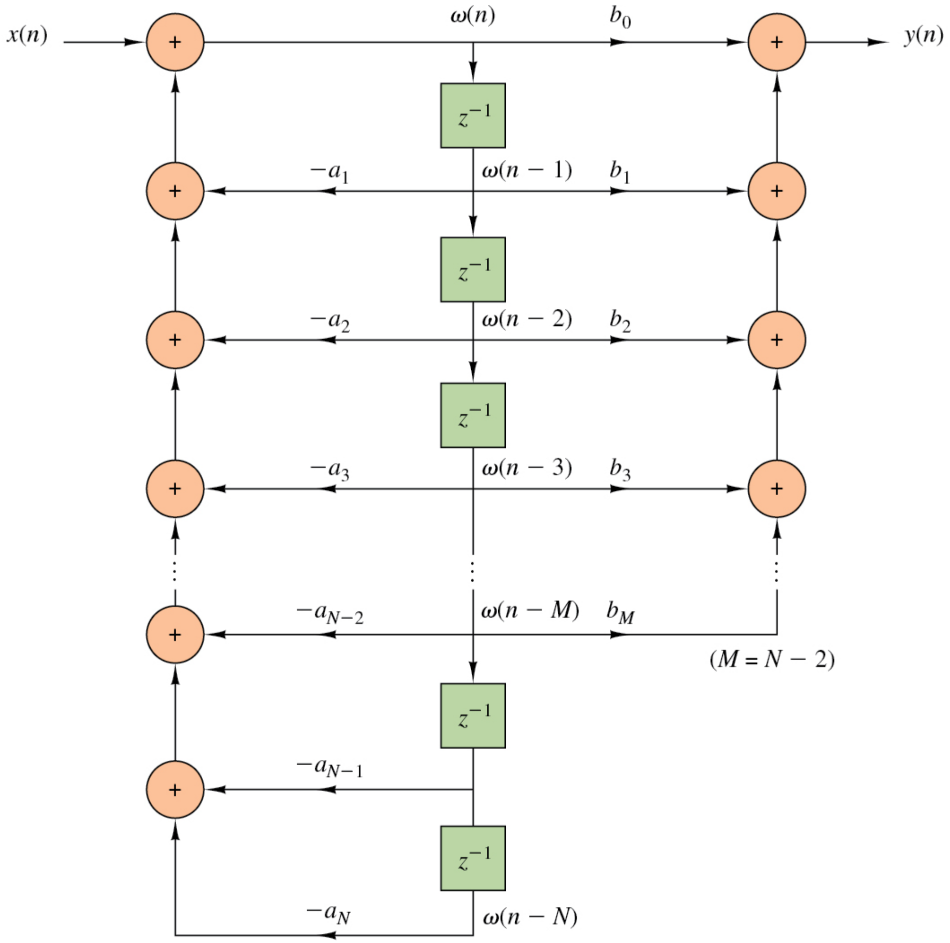
Review for Exam 1

- Discrete Linear Time-Invariant System Diagrams
  - Direct Form I



Review for Exam 1

- Discrete Linear Time-Invariant System Diagrams
  - Direct Form II



Review for Exam 1

- z-transform:
  - Definition of z-transform:
  - Bilateral

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

When you find z-transform, should also state the region of convergence (ROC)

  - Unilateral (causal signals & systems)

$$X(z) = \sum_{n=0}^{\infty} \mathrm{x}(n)\, z^{-n}$$

  - Inverse:

$$x(n) = \frac{1}{2\,\pi\, j} \oint X(z)\, z^{-n+1} dz$$

    - Rarely use this, although this is common integral in complex variables math courses.
    - We compute forward & inverse by use of transform pairs and properties.
    - Can also find inverse by long division.

Review for Exam 1

- z-transform:

$$\mathcal{Z}\left[x(n)\right] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\mathcal{Z}\left[x(n-k)\right] = \sum_{n=-\infty}^{\infty} x[n-k] z^{-n} = \sum_{n=-\infty}^{\infty} x[n-k] z^{-(n-k)} z^{-k} = z^{-k} \sum_{n=-\infty}^{\infty} x[n-k] z^{-(n-k)} = z^{-k} \sum_{m=-\infty}^{m} x[m] z^{-m}$$

$$\mathcal{Z}\left[x(n-k)\right] = z^{-k} Z\left[x(n)\right]$$

  - Shift property:

$$\boxed{\begin{array}{l} \text{For } x[n] \Leftrightarrow X(z) \\ x[n-k] \Leftrightarrow z^{-k} X(z) \end{array}}$$

Review for Exam 1

- z-transform transforms difference equations into algebraic equations:

Linear time invariant system described by difference equation:

$$y(n) + a_1 y(n-1) + a_2 y(n-2) + \cdots + a_N y(n-N)$$
$$= b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + bx_M (n-M)$$

z-transform:

$$\mathcal{Z}\left[y(n) + a_1 y(n-1) + a_2 y(n-2) + \cdots + a_N y(n-N)\right]$$
$$= \mathcal{Z}\left[b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_M x(n-M)\right]$$

$$\left(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}\right) Y(z) = \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}\right) X(z)$$

Review for Exam 1

- Transfer function:

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = b_0 \frac{1 + \left(b_1 / b_0\right) z^{-1} + \cdots + \left(b_M / b_0\right) z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

  – Factored into a product of zeros and poles:

$$H(z) = b_0 \frac{\left(1 - z_1 z^{-1}\right)\left(1 - z_2 z^{-1}\right)\cdots\left(1 - z_M z^{-1}\right)}{\left(1 - p_1 z^{-1}\right)\left(1 - p_2 z^{-1}\right)\cdots\left(1 - p_M z^{-1}\right)} = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - p_k z^{-1}\right)}$$

If $M > N$, do the usual division to get a sum of terms and a proper rational function

$$H(z) = \sum_{k=0}^{M-N} c_k z^{-k} + H_{pr}\left(z\right)$$

Review for Exam 1

- Impulse response from transfer function:

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{\left(1 - p_1 z^{-1}\right)\left(1 - p_2 z^{-1}\right)\cdots\left(1 - p_M z^{-1}\right)}$$

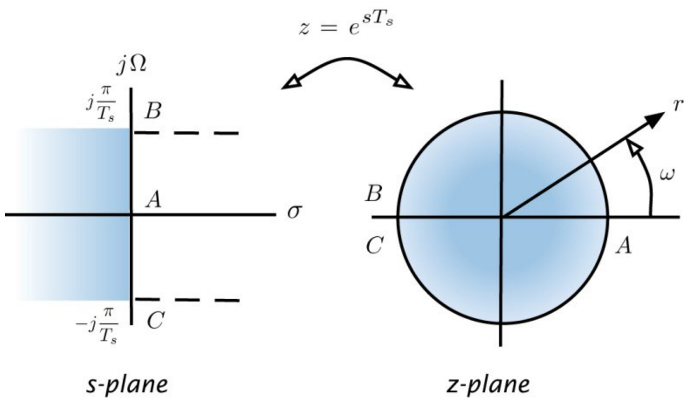
$$H(z) = A_1 \frac{1}{1 - p_1 z^{-1}} + A_2 \frac{1}{1 - p_2 z^{-1}} + \cdots + A_N \frac{1}{1 - p_N z^{-1}}$$

$$\mathcal{Z}^{-1}\left[\frac{1}{1 - a z^{-1}}\right] = (a)^n u(n)$$

Review for Exam 1

- System is stable if all poles are inside the unit circle.
  - That is $|p_k|<1$

$$\mathcal{Z}^{-1}\left[\frac{1}{1-az^{-1}}\right]=(a)^n u(n)\to 0 \text{ as } n\to\infty$$



Stable for continuous system if poles are in left half-plane
Stable for discrete systems if poles are inside the unit circle

Review for Exam 1

- Often z^{-N} is factored out:

$$H(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

- Actually, what you want to do is make last term constant, in either numerator of denominator.
  Multiple through by z^N or z^M whichever is larger

Review for Exam 1

- A couple of points about z-transform
  - Always make the leading coefficient of denominator 1
  - Regardless of whether you have H(z) in terms of positive or negative powers of z, the order of coefficients is written as:

$$b = [b_0, b_1, \cdots, b_M] \qquad\qquad a = [1, a_1, \cdots, a_N]$$

  - Important difference from continuous systems:
    Matlab assumes that order of numerator and denominator is the same! (add leading zeros if necessary).

Review for Exam 1

Example: How many zeros does this have?

How many poles does it have?

$$H(z)=\frac{z^{-2}}{\left(1-0.5z^{-1}\right)^{4}}$$

$$H(z)=\frac{z^{4}}{z^{4}}\frac{z^{-2}}{\left(1-0.5z^{-1}\right)^{4}}=\frac{z^{2}}{\left(z-0.5\right)^{4}}=\frac{z^{2}}{z^{4}-2z^{3}+\frac{3}{2}z^{2}-\frac{1}{2}z+\frac{1}{16}}$$

Might be tempted to write it as:
b=[1,0,0]
A=[1,-2,3/2,-1/2,1/16]

$$\frac{z^{2}+0z+0}{z^{4}-2z^{3}+\frac{3}{2}z^{2}-\frac{1}{2}z+\frac{1}{16}}$$

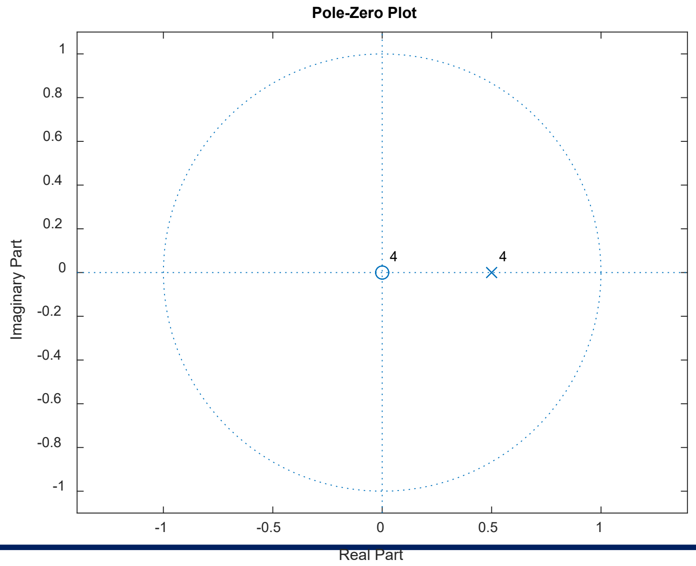
Review for Exam 1

Example: How many zeros does this have?

How many poles does it have?

Might be tempted to write it as:
b=[1,0,0]
A=[1,-2,3/2,-1/2,1/16]

$$\frac{z^2+0z+0}{z^4-2z^3+\frac{3}{2}z^2-\frac{1}{2}z+\frac{1}{16}}$$



Oops!
4 zeros?

Review for Exam 1

Example: How many zeros does this have?

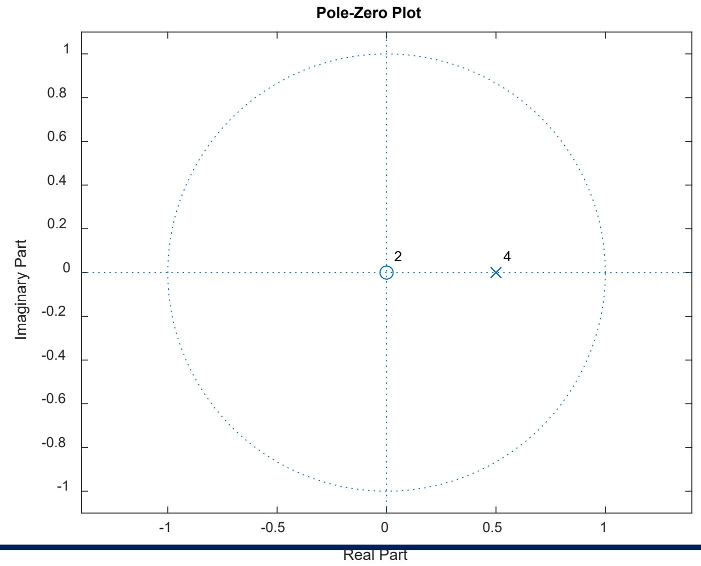
How many poles does it have?

write it as:

b=[0,0,1,0,0]

A=[1,-2,3/2,-1/2,1/16]

$$\frac{0 + 0z + 1z^2 + 0z^1 + 0}{z^4 - 2z^3 + \frac{3}{2}z^2 - \frac{1}{2}z + \frac{1}{16}}$$



2 zeros!

Review for Exam 1

- More Matlab stuff
  - Discrete systems:

    zplane(b,a) - pole-zero map for discrete system
    impz(b,a) – impulse response
    stepz(b,a) – step response
    freqz(b,a) - magnitude and phase of frequency response

  - Continuous systems:

    sys = tf(b,a) (creates transfer function model)
    pzmap(sys) - pole-zero map for continuos system
    impulse(sys) – impulse response
    step(sys) – step response
    bode(sys) - magnitude and phase of frequency response

Review for Exam 1

- Continuing with suggestions:
  - If you have the transfer function, find the impulse response using partial fractions
  - If numerator and denominator have the same order, which is often the case:

    Do partial fractions for $\frac{H(z)}{z}$

    then multiply through by z to look up inverse Z transform from table.

Review for Exam 1

- Continuing with suggestions:
  - What about finding the step response?
    The z-transform of u(n) is $1/(1-z^{-1})$
    So, could find inverse z-transform of:

$$H(z)\frac{1}{\left(1-z^{-1}\right)}=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}\frac{1}{\left(1-z^{-1}\right)}$$

or if working with positive powers of z:

$$H(z)\frac{z}{(z-1)}=\frac{b_0z^{N}+b_1z^{N-1}+\cdots+b_Mz^{N-M}}{z^{N}+a_1z^{N-1}+\cdots+a_N}\frac{z}{(z-1)}$$

Easier way if you already have the impulse response: $s(n)=\sum_{k=-\infty}^{n}h(k)$
and use sum of finite geometric series.

Review for Exam 1

- Discrete time Fourier transform:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

  - z-transform on unit circle
  - If you find $H(\omega)$ to use Matlab freqz you need the b and a coefficients of the corresponding z-transform.

- One more reminder: Euler's formula:

$$\cos(\omega_0 n) = \frac{e^{j\omega_o n} + e^{-j\omega_o n}}{2}$$

$$\sin(\omega_0 n) = \frac{e^{j\omega_o n} - e^{-j\omega_o n}}{2j}$$

Review for Exam 1

- Review HW4