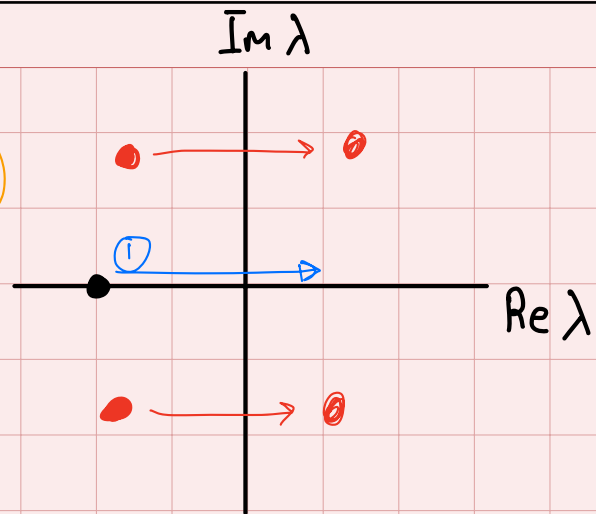


Wed, Mar 26 Lecture 16

Hopf Bifurcations (Limit Cycles)

- ① A stable node becomes unstable

Bifurcations seen so far involve $\lambda = 0$ at the critical point.



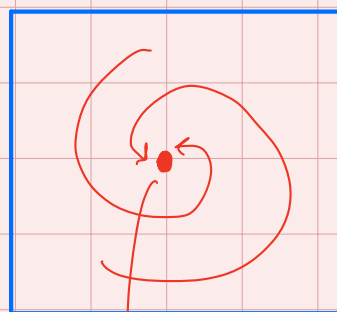
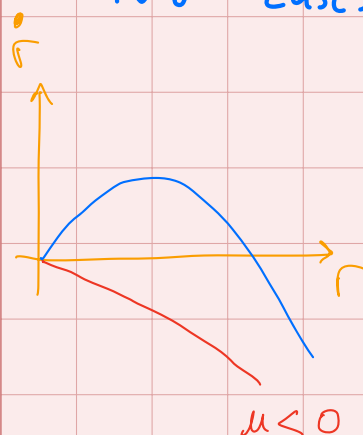
- ② Hopf Bifurcation

Bifurcations occur when $\text{Im}(\lambda) \neq 0$

$$\begin{aligned}\dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + br^2\end{aligned}$$

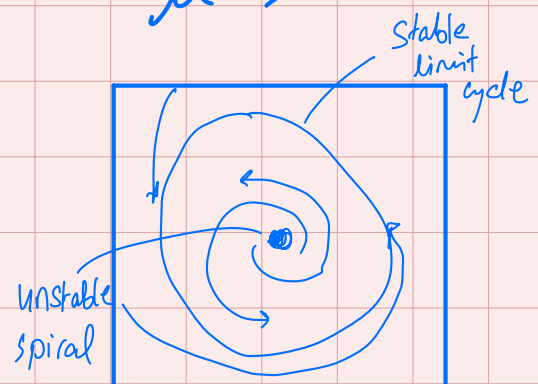
→ fixed points in r occur at $r=0, r = \pm\sqrt{\mu}$ (only if μ is positive)

Two cases: $\mu < 0$



stable spiral

$\mu > 0$



convert to x, y

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta \quad \text{plug in}$$

$$\begin{aligned} \dot{x} &= (\mu r - r^3) \cos \theta - r (\omega + b r^2) \sin \theta \\ &= \mu x - (x^2 + y^2) x - y (\omega + b(x^2 + y^2)) \\ &= \mu x - \omega y + \text{higher-order terms} \end{aligned}$$

$$\dot{y} = \omega x + \mu y + \dots \Rightarrow A = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}$$

Eigenvalues are $\boxed{\lambda = \mu \pm i\omega}$

As $\mu \uparrow$, eigenvalues cross the imaginary axis.

Types of Hopf Bifurcation:

supercritical
subcritical
saddle-node
 ∞ -period
homoclinic

<in-class exercise>