Definition.
Determinism is true of the *world* if and only if, given a specified *way things are at a time t*, the way things go *thereafter* is *fixed* as a matter of *natural law*.
(Stanford Encyclopedia of Philosophy, Entry on Causal Determinism)

Laplace's Demon.
"We ought to regard the present state of the universe as the effect of its antecedent state and as the cause of the state that is to follow. An intelligence knowing all the forces acting in nature at a given instant, as well as the momentary positions of all things in the universe, would be able to comprehend in one single formula the motions of the largest bodies as well as the lightest atoms in the world, provided that its intellect were sufficiently powerful to subject all data to analysis; to it nothing would be uncertain, the future as well as the past would be present to its eyes. The perfection that the human mind has been able to give to astronomy affords but a feeble outline of such an intelligence. Discoveries in mechanics and geometry, coupled with those in universal gravitation, have brought the mind within reach of comprehending in the same analytical formula the past and the future state of the system of the world. All of the mind's efforts in the search for truth tend to approximate the intelligence we have just imagined, although it will forever remain infinitely remote from such an intelligence."
(1820)
(Essai Philosophique sur les Probabilités)

Principle of Sufficient Reason - Leibniz

"Free Will is an illusion" - Spinoza

Heisenberg Uncertainty Principle

Gödel's Incompleteness Theorem

$$\left\{ \begin{array}{l} \dot{x} = f(x,t) \\ x(0) = x_0 \end{array} \right\} *$$

$x \in \mathbb{R}^n$

Chaos is impossible if $n<3$

To "solve" this IVP means to find a function $x(t)$ that satisfies (*).

- analytical solution : use MATH
- numerical solution : use computer

A solution may exist for all $t \in \mathbb{R}$ or for a subset of $\mathbb{R}$

$$\left\{ \begin{array}{l} \dot{x} = f(x,t) \\ x(0) = x_0 \end{array} \right\} \quad (*)$$

$x \in \mathbb{R}^n$

Chaos is impossible if $n<3$

To "solve" this IVP means to find a function $x(t)$ that satisfies $(*)$.

- analytical solution : use MATH
- numerical solution : use computer

A solution may exist for all $t \in \mathbb{R}$ or for a subset of $\mathbb{R}$

Mon, Jan 27 Lecture 2

$\dot{x} = f(x,t)$: first-order

$\ddot{x} = f(x,\dot{x},t)$: 2^{nd}

$\dddot{x} = f(x,\dot{x},\ddot{x},t)$: 3^{rd}

e.g. $x(0) = ..$

$\dot{x}(0) = ..$

$\ddot{x}(0) = ..$

$\ddot{x} = f(x,\dot{x})$: autonomous

$\ddot{x} = f(x,\dot{x},t)$: nonautonomous

Equivalence of n^{th} order differential equations
and a system of n 1^{st} order " "

$$\frac{d^n x}{dt^n} = f\left(x, \dot{x}, \ddot{x}, x^{(3)}, x^{(4)}, \dots, x^{(n-1)}\right)$$

it is always possible to write an equivalent system of n 1^{st} order equations:

Define $y \in \mathbb{R}^n$

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= y_4 \\ &\vdots \\ \dot{y}_{n-1} &= y_n \\ \dot{y}_n &= f(y_1, y_2, y_3, \dots) \end{aligned}$$

Any dynamics problem can be written as

$$\dot{\vec{x}} = f(\vec{x}, t) \quad \text{where} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \dot{\vec{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$\dot{x} = f(x)$, $x \in \mathbb{R}^n$

$x(0) = x_0$

$n \times n$

is linear if it can be written as $\dot{x} = A x$

$n = 1$, linear $\longrightarrow$ increasing n

$\downarrow$

increasing nonlinearity

chaos lives here

$n \gg 1$, nonlinear

$n = 1$, nonlinear

$\dot{x} = \sin x$

$x(0) = x_0$

$\longrightarrow$ analytical : $x(t)$ function

numerical : $\{t_i, x_i\}$

geometric

① Analytical $\dfrac{dx}{dt} = \sin x$

$$\int \operatorname{cosec} x \, dx = \int dt$$

$$-\log\left|\operatorname{cosec} x + \cot x\right| + C = t$$

use $t = 0$, $x = x_0$ to find C

$$\log\left|\frac{\operatorname{cosec} x_0 + \cot x_0}{\operatorname{cosec} x + \cot x}\right| = t$$

2 Numerical

$$\frac{dx}{dt} = \sin x$$

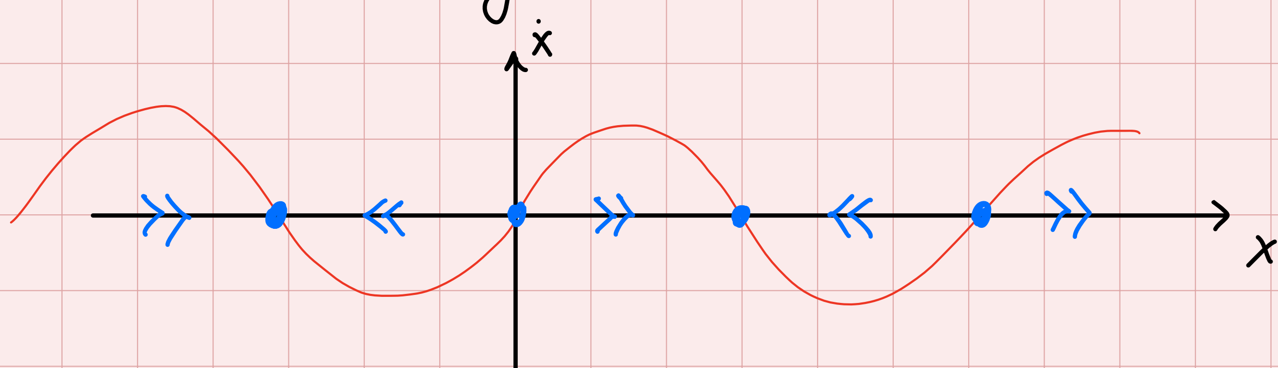
$$\frac{x_{n+1} - x_n}{\Delta t} = \sin x_n \Rightarrow x_{n+1} = \Delta t \sin(x_n) + x_n$$

$n = 0, 1, 2, 3, \ldots$

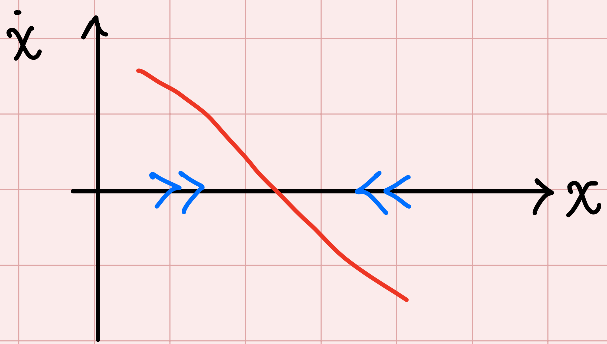
3 Geometric

- The state of system is a point on the x-axis
- $\dot{x} > 0$: moving to the right
  $\dot{x} < 0$: moving to the left.

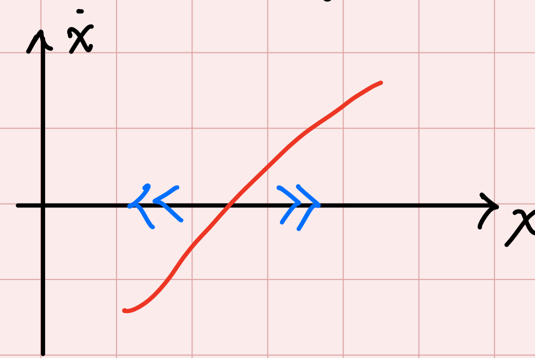
} call this "flow"



Two kinds of "fixed points" emerge:

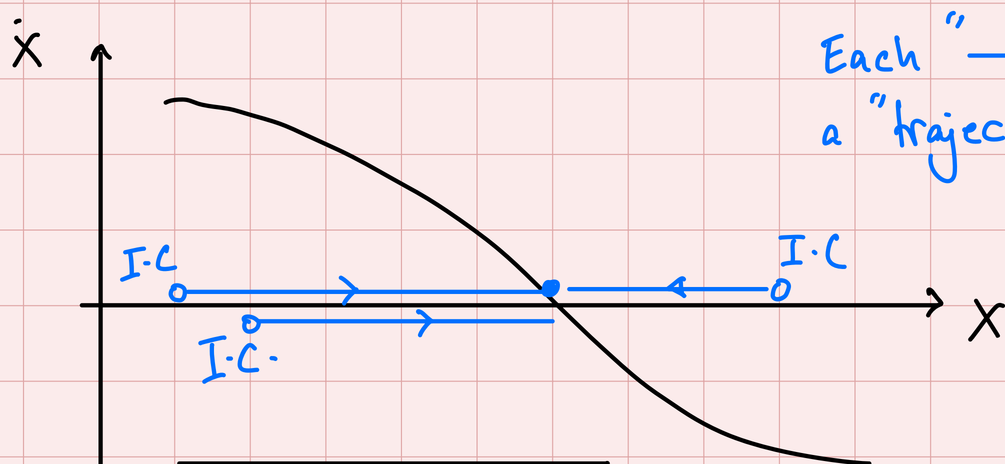


ATTRACTOR
SINK
(stable)



REPELLER
SOURCE
(unstable)

Phase Portrait (for 1-d systems)



Each "$\longrightarrow$" is a "trajectory".

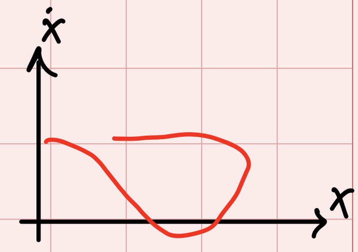
$\boxed{\text{System } \dot{x} = f(x)}$

- one phase portrait
- (a few ?) fixed points
- infinite trajectories

A diagram showing
- all qualitatively different trajectories
- all fixed points

Note : $f(x)$ must be a function

in addition, we will work with $[f(x)]$'s that are "nice"
= sufficiently smooth.

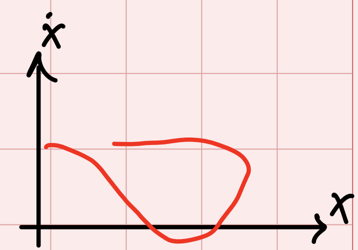


not allowed

Example $\dot{N} = rN\left[1 - \frac{N}{K}\right]$ Logistic Equation

A diagram showing
- all qualitatively different trajectories
- all fixed points

Note : $f(x)$ must be a function



not allowed

in addition, we will work with $[f(x)]$'s that are "nice"
= sufficiently smooth.

Wed, Jan 29 Lecture 3

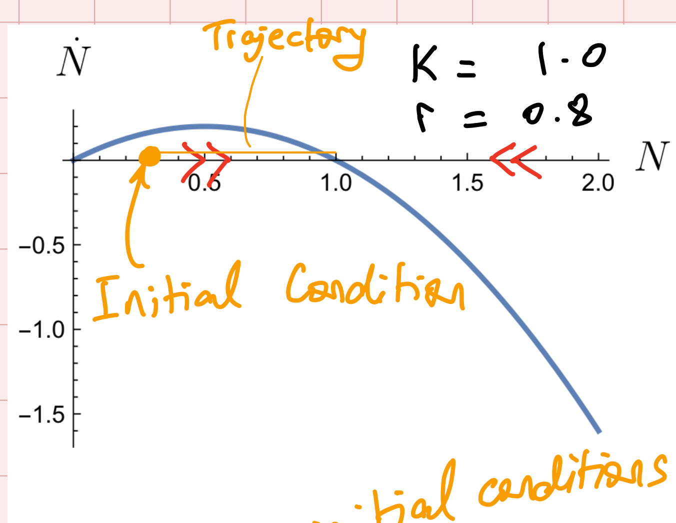
Example

$$\dot{N} = rN\left[1 - \frac{N}{K}\right]$$

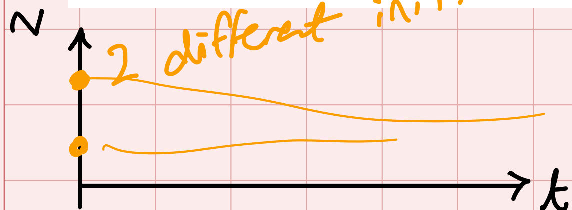
Logistic Equation

carrying capacity

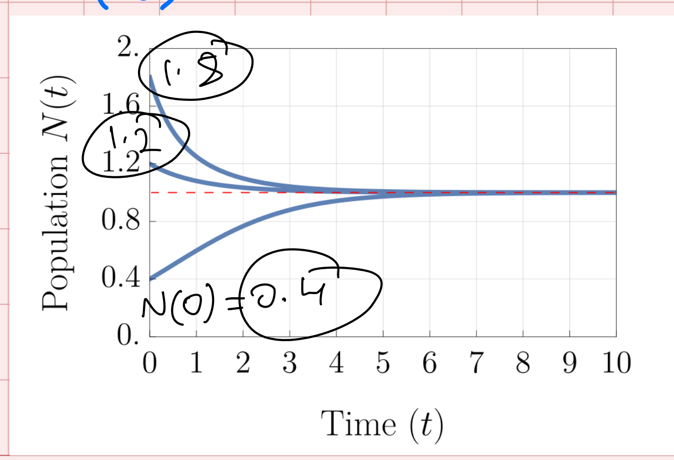
$[N]$ = people
$[r]$ = day^{-1}
$[K]$ = people
$[\dot{N}]$ = people/day



2 different initial conditions



N(t)

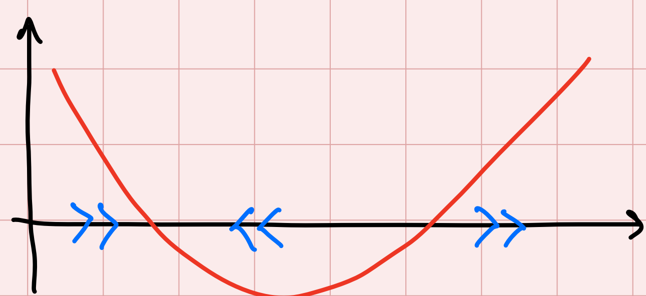


Calculating curves $N(t)$:

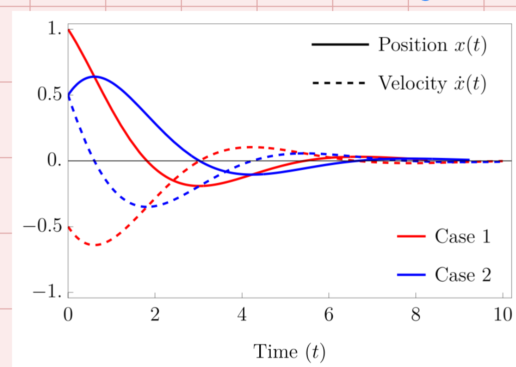
$$\frac{dx}{dt} = x(1-x) \qquad \text{with } x(0) = x_0 \text{ or, } 1/2$$

<u>Note</u> : It can be shown that oscillations and "overshoot", or other <u>nonmonotonic behaviour</u> is impossible in

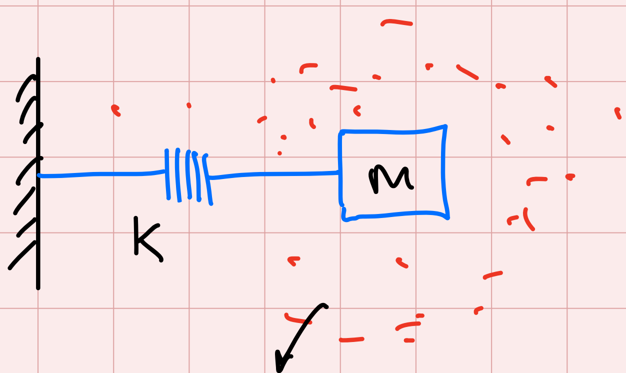
$$\dot{x} = f(x), \quad x \in \mathbb{R}^1$$



e.g.



in S.H.O.



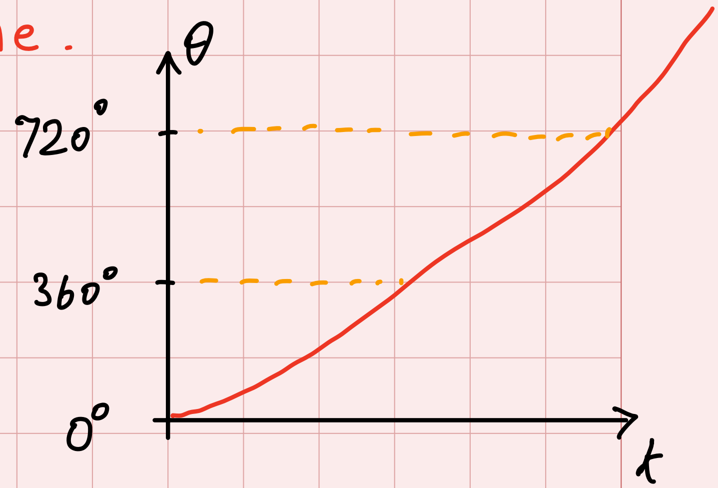
very viscous fluid, c

$$\underbrace{m\ddot{x}}_{\text{drop}} + c\dot{x} + kx = 0$$

But : $\dot{\theta} = f(\theta), \quad \theta \in [0, 2\pi)$ can have "oscillations"

we reinterpret $\theta = 370^\circ$ to mean
$\theta = 10^\circ$.

i.e. θ lives on the circle here, not on the real line.



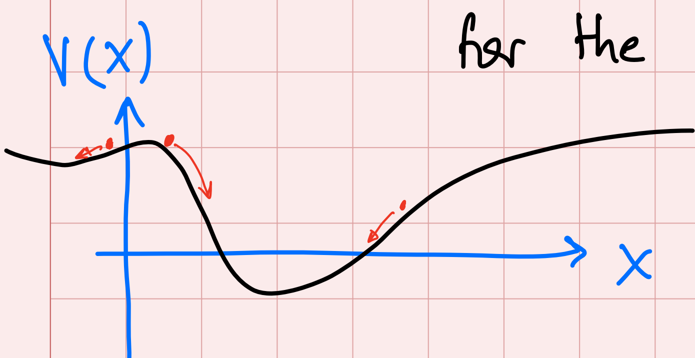
Potentials

if $\dot{x} = f(x)$ and $f(x)$ can be expressed as

$$f(x) = -\frac{d}{dx}V(x)$$

for some function $V(x)$

then $V(x)$ is called a "potential" for the dynamical system $\dot{x} = f(x)$

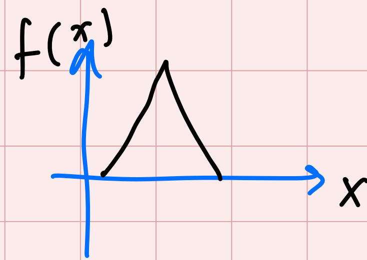


Flow occurs "downhill" in V.

Strogatz (2.7) shows that $\frac{dV}{dt} < 0$ along trajectories $x(t)$.

$$f(x) = 2x \longrightarrow V(x) = -x^2$$

$f(x) =$



Linear Stability Analysis of fixed points.

Suppose x^* is a value of x where $f(x^*)=0$

What happens to x if it is initialized close to x^*?

Let $\eta(t) \equiv x(t) - x^*$

$$\dot{\eta}(t) = \underbrace{\dot{x}(t)}_{f(x)} - 0$$

$$\dot{\eta} = \dot{x} = f(x)$$

$$\dot{\eta} = f(x) = f(x^* + \eta)$$

use Taylor series assuming η small.

$$f(x^* + \eta) = \cancelto{0}{f(x^*)} + \eta f'(x^*) + \underbrace{\frac{\eta^2}{2!} f''(x^*) + \cdots}_{O(\eta^2)}$$

$$\boxed{\dot{\eta} = \eta \underbrace{f'(x^*)}_{\text{const. number}} + O(\eta^2)}$$

Evolution equation for small perturbations η away from x^*.

Evolution equation for small perturbations η away from x^*.

Mon, Feb 3 Lecture 4

Bifurcations

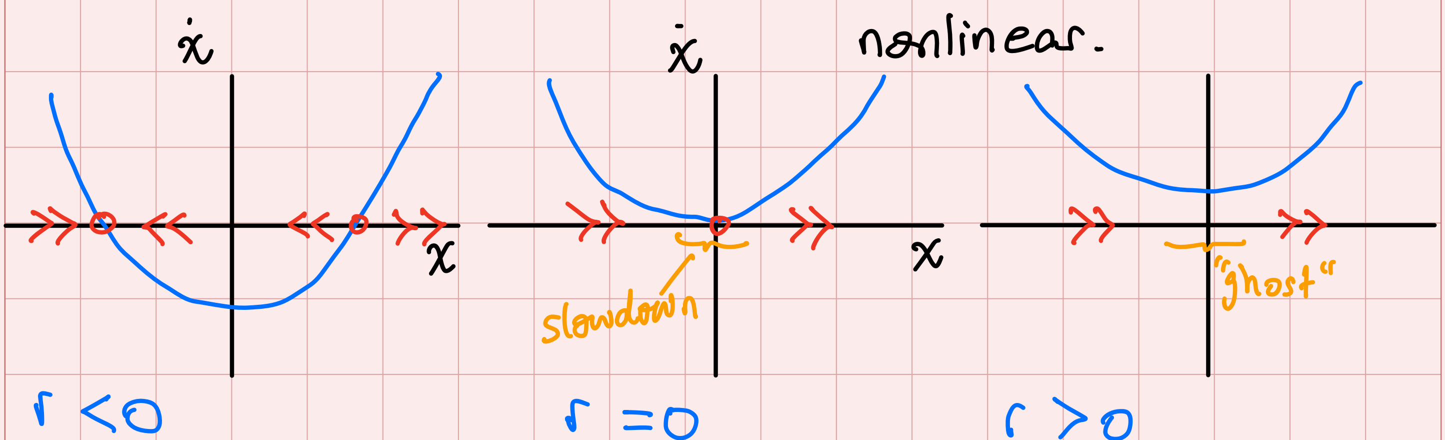
$$\dot{x} = f(\overbrace{x}^{\text{independent variable}}; \underbrace{r}_{\text{parameter.}})$$

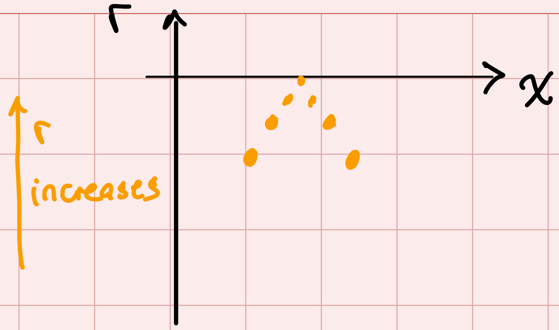
Let the system be parameterized by one or more parameters 'r'. How does the qualitative behaviour of the system change with r?

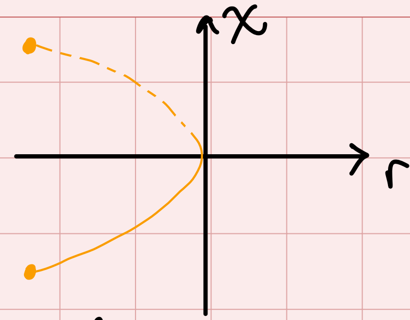
① Saddle - Node Bifurcation

$$\dot{x} = r + x^2$$

one-dimensional first-order nonlinear.



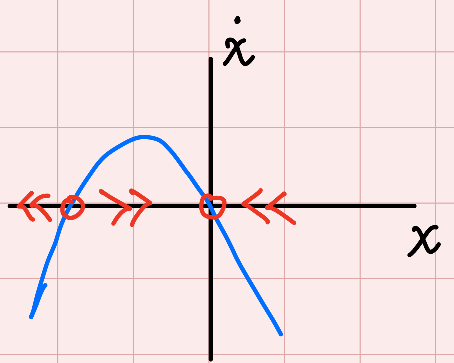




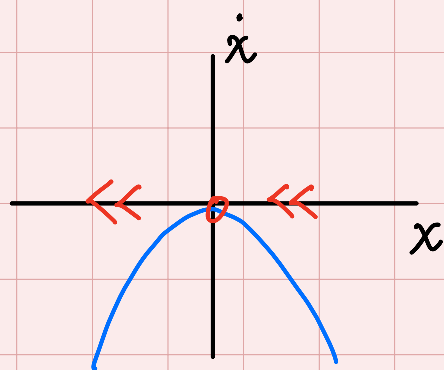
Bifurcation Diagram for Saddle-Node Bifurcation

② Transcritical Bifurcation

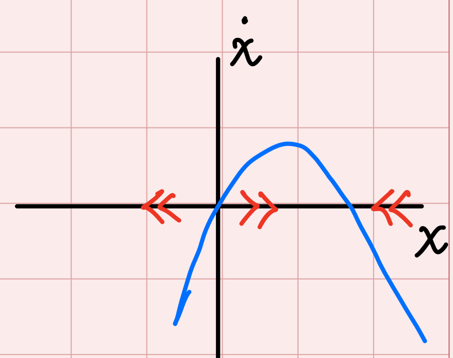
$$\dot{x} = rx - x^2$$



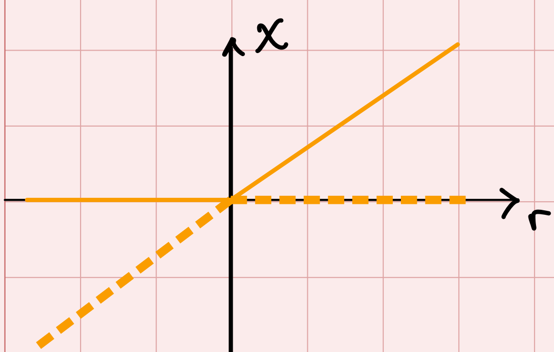
$r < 0$



$r = 0$



$r > 0$



"exchange of stability"

Bifurcation Diagram for Transcritical Bifurcation

How to calculate Bifurcation Curves

$$\dot{x} = -x + r \tanh x \qquad (A)$$

Find fixed points x^*, for which $f(x^*) = 0$

Here, f also has a parameter r.

Solve $\boxed{f(x^*; r) = 0}$ for many r's.

Gives $\{x^*, r\}$ pairs. Plot them.

with root-finding program or by hand.

e.g. for system (A)

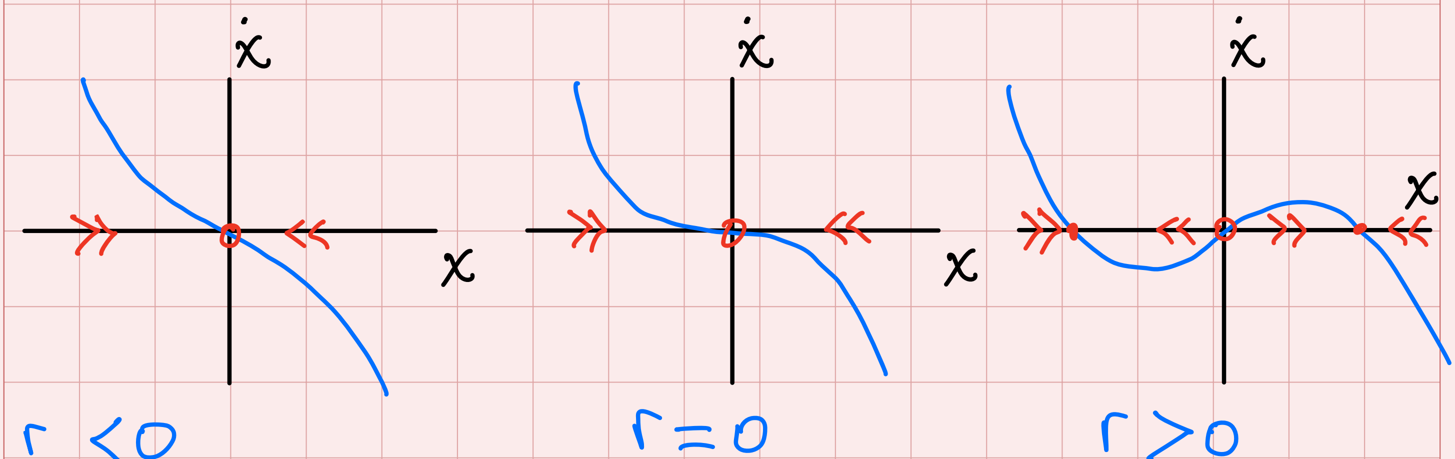
Solve $0 = -x^* + r \tanh x^*$ for x^*
after setting r to some value.

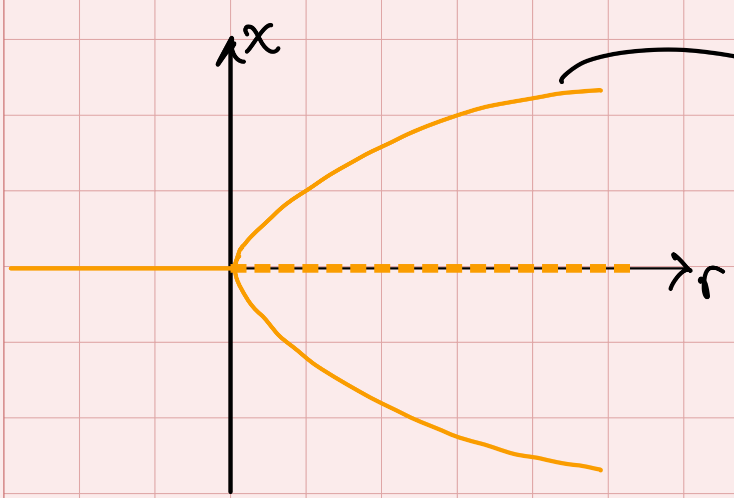
Set x^* to some value, find $r. = \dfrac{x^*}{\tanh x^*}$

③ Pitchfork Bifurcation

$$\dot{x} = rx - x^3 \quad \text{supercritical}$$
$$\dot{x} = rx + x^3 \quad \text{subcritical}$$

→ supercritical : new fixed pts. appear above critical r.



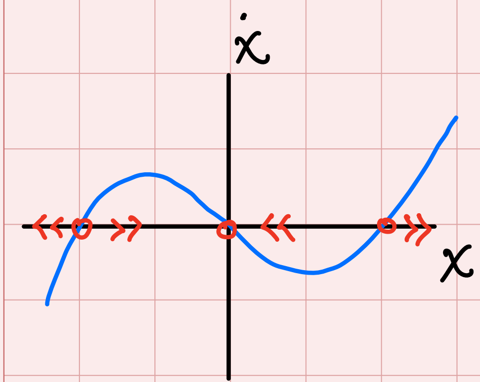


$x^*(r) = \cdots ? \sqrt{r} \cdots$

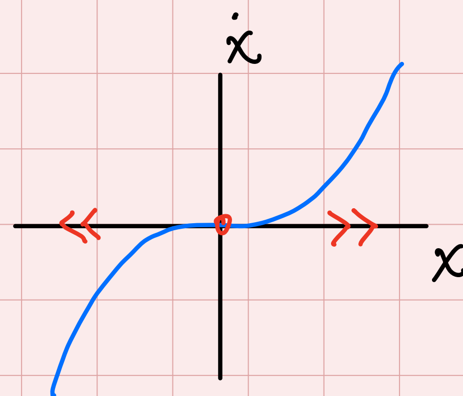
Wed, Feb 5　Lecture 5

→ subcritical pitchfork

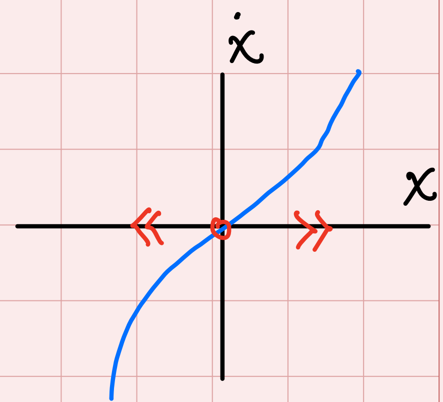
$$\dot{x} = rx + x^3$$



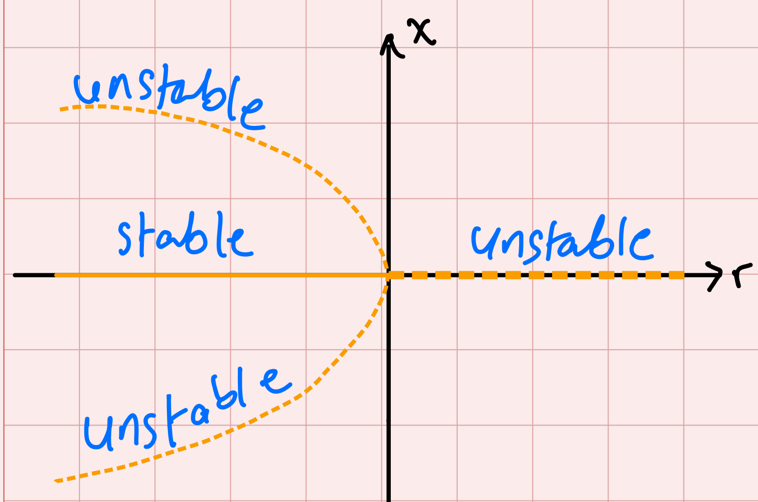
$r<0$



$r=0$

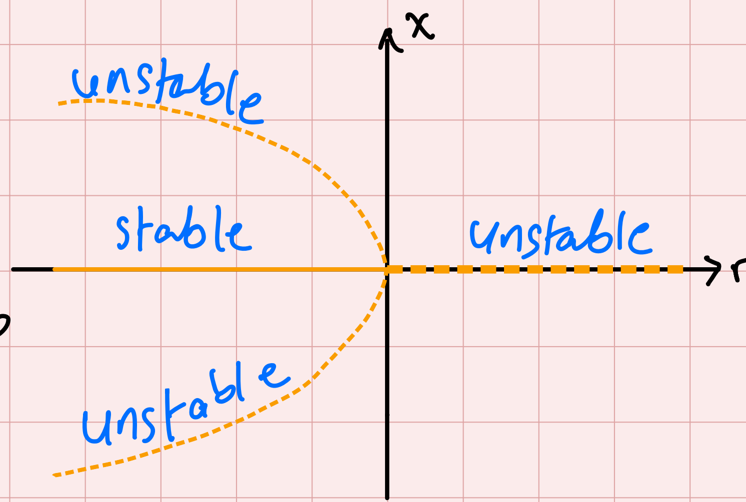


$r>0$



	$\dot{x} = f(x;r)$	$r<0$	$r>0$
Saddle-node	$\dot{x} = r + x^2$	1 stable 1 unstable	None None
Transcritical	$\dot{x} = rx - x^2$	1 stable 1 unstable	→ 1 unstable → 1 stable
Pitchfork:			
Supercrit.	$\dot{x} = rx - x^3$	None 1 stable None	→ 1 stable → 1 unstable → 1 stable
subcrit.	$\dot{x} = rx + x^3$	1 unstable 1 stable 1 unstable	→ None → 1 unstable → None

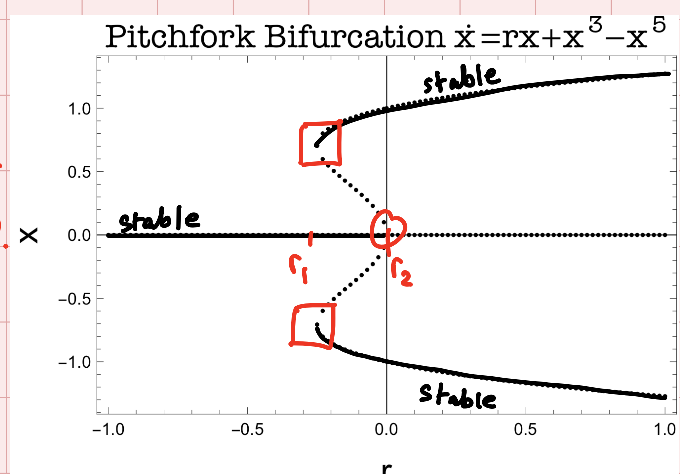
In practice, systems with a subcritical pitchfork bifurcation don't actually go off to $\pm\infty$ instead, higher-order terms play a stabilizing role.



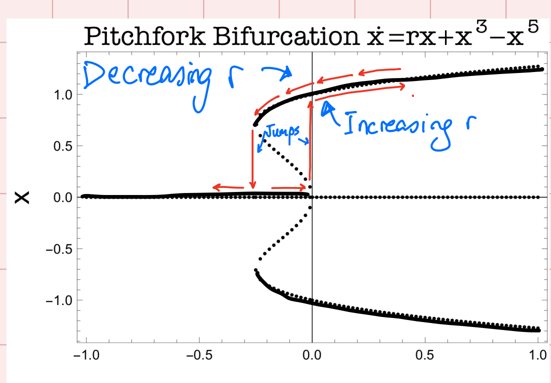
subcritical pitchfork

$$\underbrace{\dot{x} = rx + x^3}\; - x^5$$

for small x, this governs the dynamics.
for large x, the x^5 term plays a role also.



→ Saddle-node bifurcation at □.
→ Subcritical pitchfork bif. at ○.



Hysteresis

Mon, Feb 10   Lecture 6

Show that, with appropriate non-dimensionalization,

$\dot{u} = a u + b u^3 - c u^5$ is equivalent to

$\dot{x} = r x + x^3 - x^5$.   where $\dot{x} = \dfrac{dx}{d\tau}$

$U = \sqrt{b/c}$

$T = c/b^2$

$r = ac/b^2$

$x = u/U$

$\tau = t/T$

Dynamics with $\underline{n = 2}$

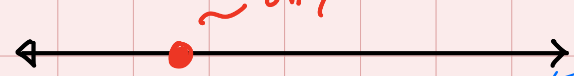
$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}) \qquad \underset{\sim}{x} \in \mathbb{R}^2$

<u>Conventions</u>   the state can be written as

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ or $\begin{bmatrix} x \\ y \end{bmatrix}$ or $\underset{\sim}{x}$

<u>Note</u>: f is a vector-valued function of a vector argument.

1-dimensional state

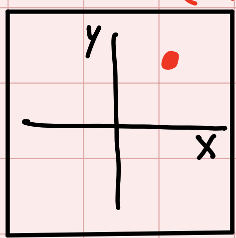


only one variable determines state

change: $\dot{x}$

2-dimensional state

(x,y) — determines state



change: $\dot{x}, \dot{y}$

The solutions of $\left\{ \dot{\underset{\sim}{x}} = f(\underset{\sim}{x}),\ \underset{\sim}{x} \in \mathbb{R}^2,\ \underset{\sim}{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right\}$

can be visualized as trajectories on the <u>phase plane</u>

$$\ddot{x} = -\omega^2 x \qquad \begin{matrix} \dot{x} \\ \ddot{x} \end{matrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

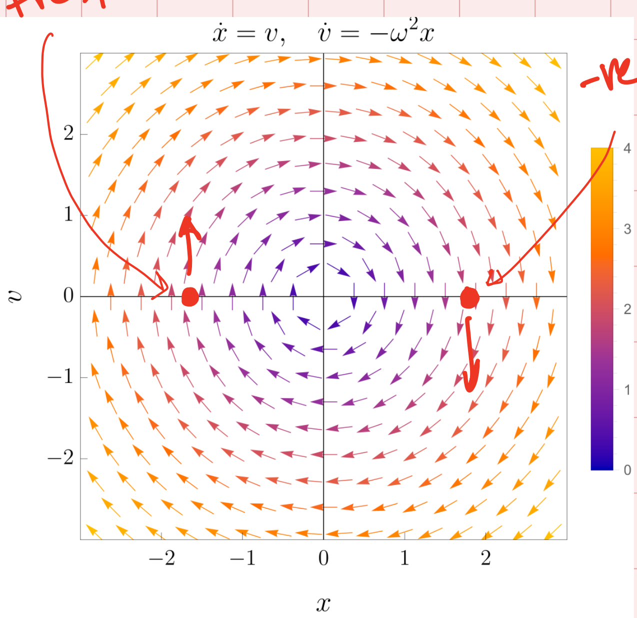
state: x, v

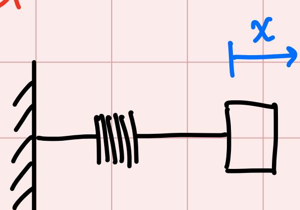
position, velocity

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ -\omega^2 x \end{bmatrix}$$

+ve speed

-ve speed





→ $\dot{\underset{\sim}{x}}$ is a vector with 2 components, defined for any point (x,y) on the plane.

i.e. $\dot{\underset{\sim}{x}}$ is a vector field

→ Trajectories are $\{x_1(t), x_2(t)\}$ functions of time parameterized by the initial condition.

or, numerically, ordered pairs parameterized by t.

→ Trajectories are everywhere tangent to vector field.

---

if $f(\underset{\sim}{x})$ is linear, without loss of generality we can express $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ as:

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note

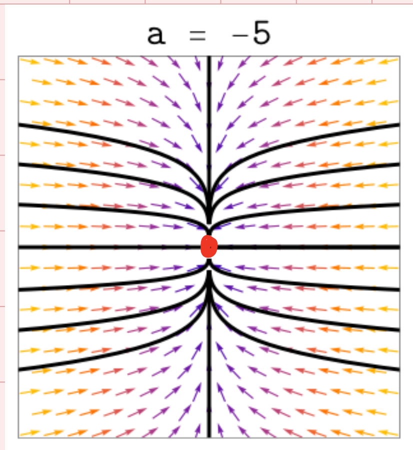
$\underset{\sim}{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is always a fixed pt

study a particular linear system:

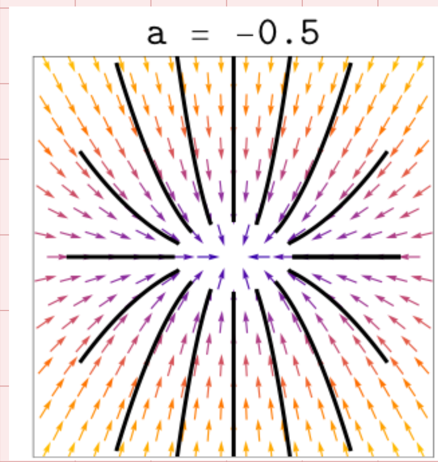
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} \dot{x} &= ax \\ \dot{y} &= -y \end{aligned} \longrightarrow \begin{aligned} x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{-t} \end{aligned}$$

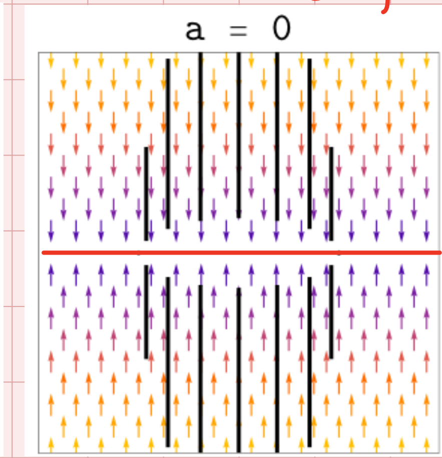
stable node

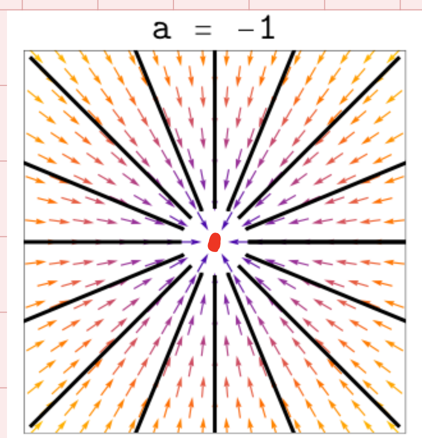


stable node ?

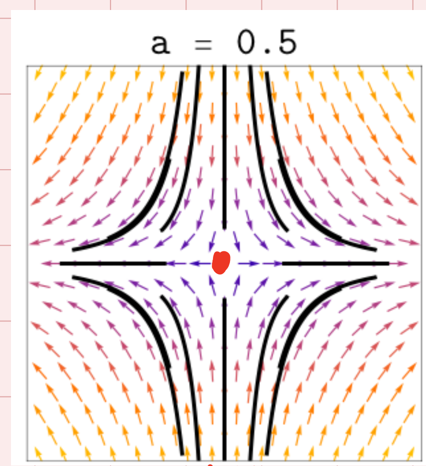


line of fixed pts



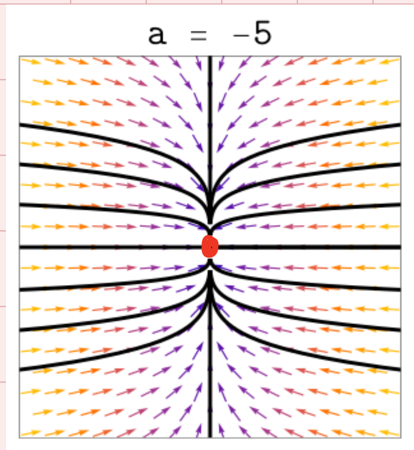


symmetric node
(star)

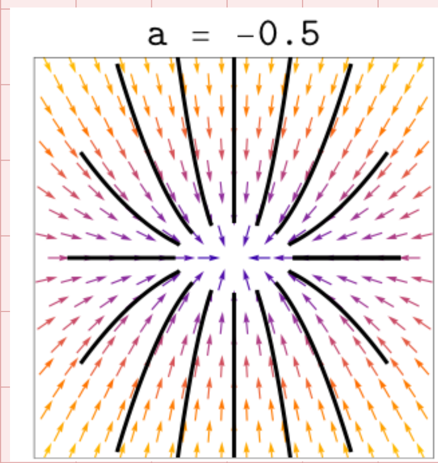


saddle point

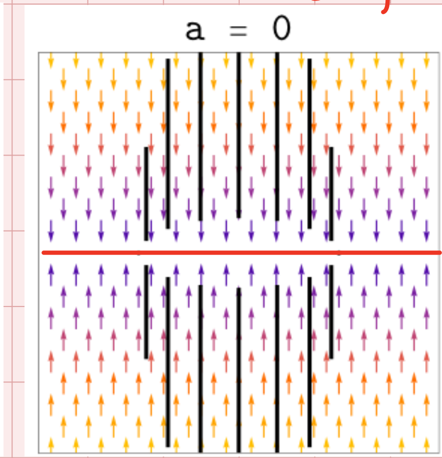
stable node

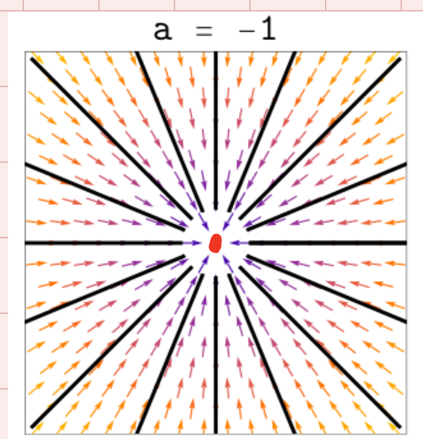


stable node

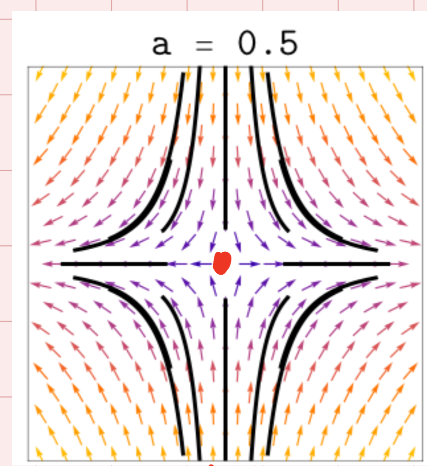


line of fixed pts





symmetric node (star)



saddle point

Wed, Feb 12, Lecture 7

General $n=2$ Linear Systems $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \overbrace{\begin{bmatrix}a & b\\ c & d\end{bmatrix}}^{A}\begin{bmatrix}x\\ y\end{bmatrix}$

Recall that, for $A = \begin{bmatrix}a & 0\\ 0 & -1\end{bmatrix}$, the x and y axes were $\underbrace{\text{special directions}}_{\text{related to eigenvectors of } A.}$ with $\underbrace{\text{straight-line}}$ trajectories

exponential in time

Are there such trajectories for general A ?
What directions do those trajectories travel in ?
i.e. is there a vector $\underset{\sim}{v}$ and λ such that

$$\underset{\sim}{x}(t) = e^{\lambda t}\underset{\sim}{v} \quad ?$$

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$$

$$\lambda \cancel{e^{\lambda t}}\underset{\sim}{v} = A\cancel{e^{\lambda t}}\underset{\sim}{v}$$

$$\lambda \underset{\sim}{v} = A\underset{\sim}{v}$$

: λ eigenvalues of A
$\underset{\sim}{v}$ eigenvectors of A.

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$(a-\lambda)(d-\lambda) - cb = 0$$

$$\lambda^2 - \lambda a - \lambda d + ad - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{\mathrm{tr}(A),\ \tau}\lambda + \underbrace{ad-bc}_{\det(A),\ \Delta} = 0$$

tr: trace
det: determinant.

$$\boxed{\lambda^2 - \tau\lambda + \Delta = 0}$$

Equation for eigenvalues of A.

$$\boxed{\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$$

<u>Note</u> : once you know λ, it is straightforward to calculate $\underset{\sim}{v}$ by solving $\lambda\underset{\sim}{v} = A\underset{\sim}{v}$ for the two components of $\underset{\sim}{v}$.

As long as $\lambda_1 \neq \lambda_2$, any state of the system $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can be written as a linear combination of the eigenvectors $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$.

$$\underset{\sim}{x} = a_1 \underset{\sim}{v_1} + a_2 \underset{\sim}{v_2} \quad \text{for scalars } a_1 \text{ and } a_2.$$

$$\underset{\sim}{x}(t) = (a_1(t))\underset{\sim}{v_1} + (a_2(t))\underset{\sim}{v_2}$$

As $\underset{\sim}{x}$ varies over time, these scalars vary exponentially in time.

and it's possible to write a general solution $\underset{\sim}{x}(t)$ for the differential equation $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$.

$$\underset{\sim}{x}(t) = c_1 e^{\lambda_1 t} \underset{\sim}{v_1} + c_2 e^{\lambda_2 t} \underset{\sim}{v_2}$$

→ λ's, $\underset{\sim}{v}$'s are eigenvalues and eigenvectors.

→ c's are const. coefficients that depend on initial condition $\underset{\sim}{x}(0)$.

No such general solution exists for $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ if f is not linear in $\underset{\sim}{x}$.

<u>Exercise</u> <u>Solve</u>

$$\dot{x} = x + y \qquad x_0 = 2$$
$$\dot{y} = 4x - 2y \qquad y_0 = -3$$

$\dot{x} = Ax$ with $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$. $\tau = -1$, $\Delta = -6$

First, find eigenvalues.

$$\lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda^2 + 3\lambda - 2\lambda - 6 = 0$$

$$(\lambda + 3)(\lambda - 2) = 0$$

$$\Rightarrow \underline{\lambda = 2, -3}$$

Then find eigenvectors.

$$\underset{\sim}{A}\,\underset{\sim}{u} = \underbrace{\lambda}_{=2}\,\underset{\sim}{u}$$

$$\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ 2u_2 \end{bmatrix} \Rightarrow \begin{aligned} u_1 + u_2 &= 2u_1 \\ 4u_1 - 2u_2 &= 2u_2 \end{aligned} \Rightarrow \begin{aligned} u_1 &= 1 \\ u_2 &= 1 \end{aligned}$$

$$\Rightarrow \underline{\underset{\sim}{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

one eigenvector, associated with $\lambda = 2$

similarly .... $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is the 2^{nd} eigenvector associated with $\lambda = -3$

$$\underset{\sim}{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

at $t = 0$, $\underset{\sim}{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

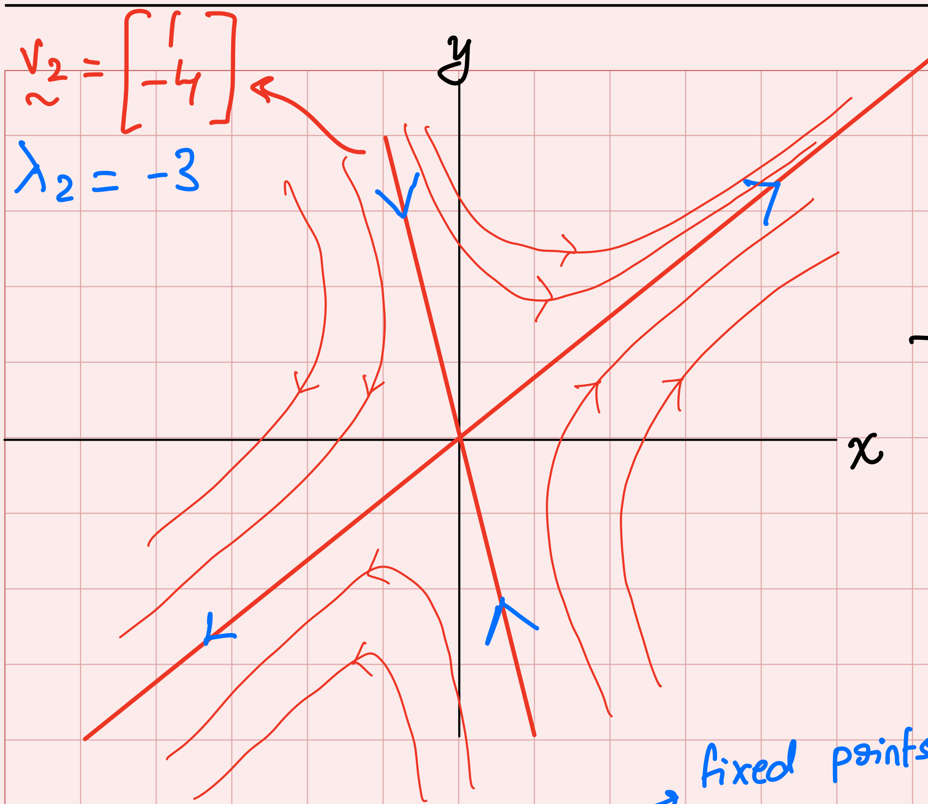
at $t = 0$

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^0 \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$c_1 + c_2 = 2$$
$$c_1 - 4c_2 = -3$$

$$\Rightarrow c_1 = 1$$
$$c_2 = 1$$

$$\underset{\sim}{x}(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$



$\underset{\sim}{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 = +2$

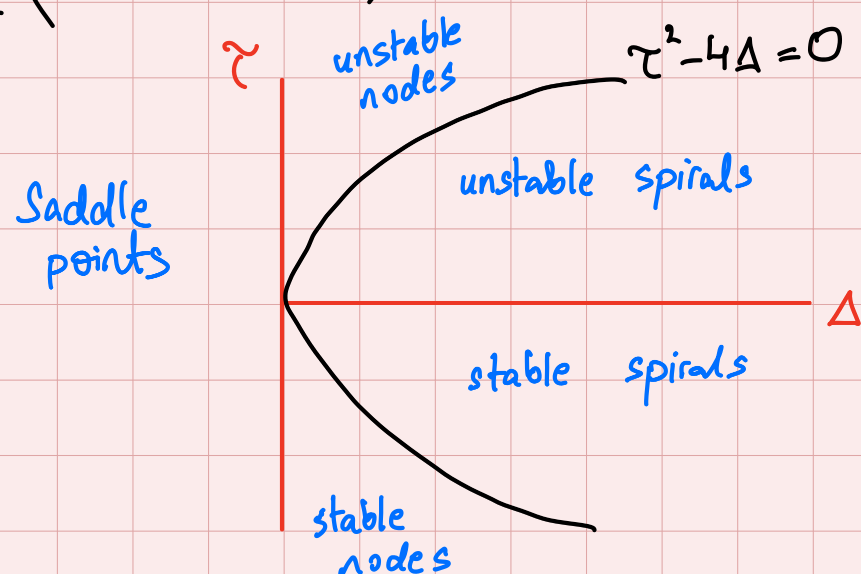
$\rightarrow \tau = -1$

$\Delta = -6$

fixed points that occur in

A classification of any $n=2$ linear system

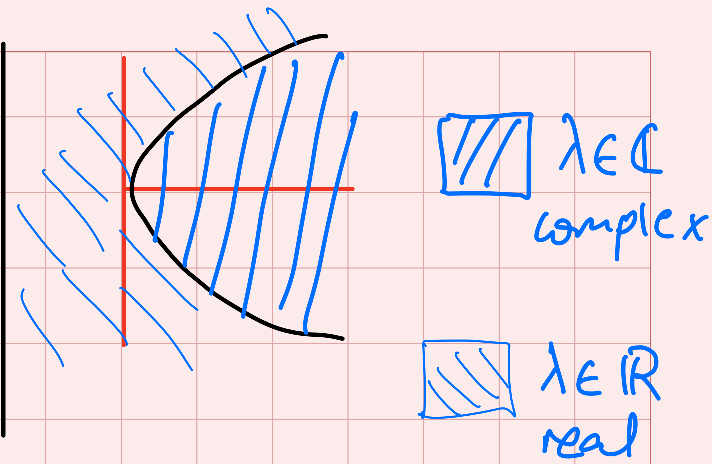
$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$



$0 < Re(\lambda_1) < Re(\lambda_2)$

$Re(\lambda_1) < 0 < Re(\lambda_2)$

$Re(\lambda_1) < Re(\lambda_2) < 0$



Mon, Feb 17 Lecture 8

Romeo & Juliet

$R(t)$: $\dot{R} = \_\_\_ R + \_\_\_ J \; + \underbrace{?}_{\text{constants}}$

$J(t)$: $\dot{J} = \_\_\_ R + \_\_\_ J \; + ?$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{constants}} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} \; \\ \; \end{bmatrix}}_{\text{constants.}}$$

+ve : love
−ve : hate

R : Romeo's love / hate for Juliet
J : Juliet's love / hate for Romeo

rate of change of R depends on value of R
" " J

<u>NOT</u> on rate of change of R, J.

"Romantic styles"

$\dot{x} = +x + y$
$\dot{x} = +x - y$
$\dot{x} = -x + y$
$\dot{x} = -x - y$

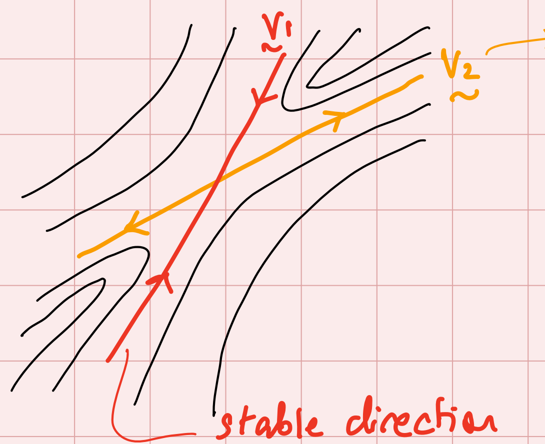
x : one lover
y : other one

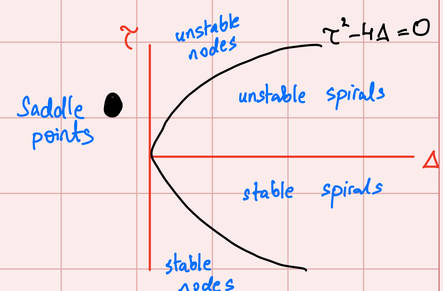
Wed, Feb 19　Lecture 9

Fixed Point Types and eigenvalues/eigenvectors

1 Saddle Point

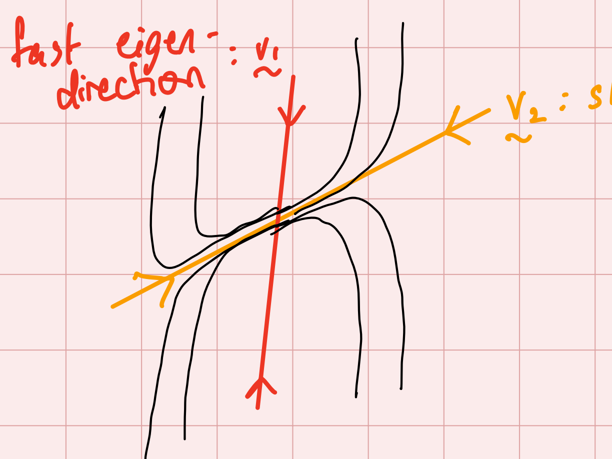
$$\underbrace{\lambda_1}_{\underset{\sim}{v_1}} < 0 < \underbrace{\lambda_2}_{\underset{\sim}{v_2}}$$

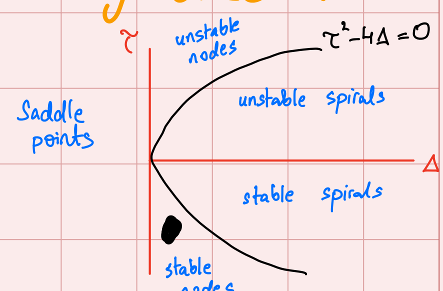




2 Nodes

$$\underbrace{\lambda_1}_{\underset{\sim}{v_1}} < \underbrace{\lambda_2}_{\underset{\sim}{v_2}} < 0$$

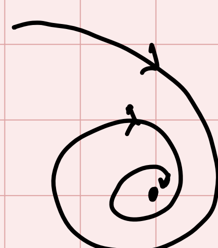


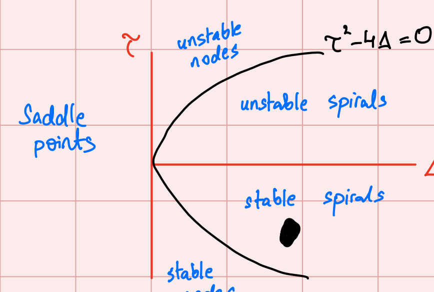


3 Spirals

Re(λ) : exponential decay

Im(λ) : oscillation



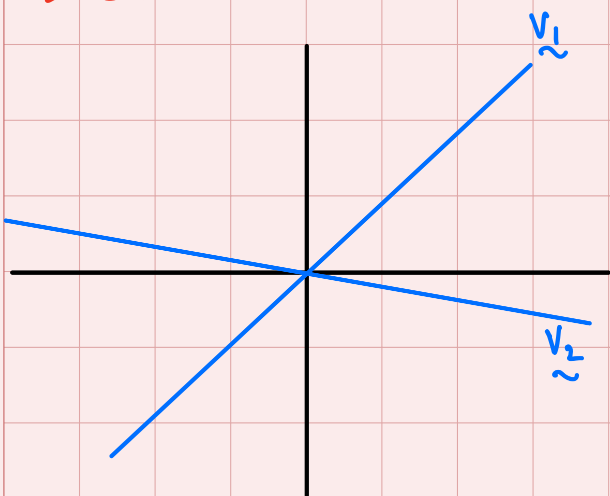


The phase plane (nonlinear $n=2$)

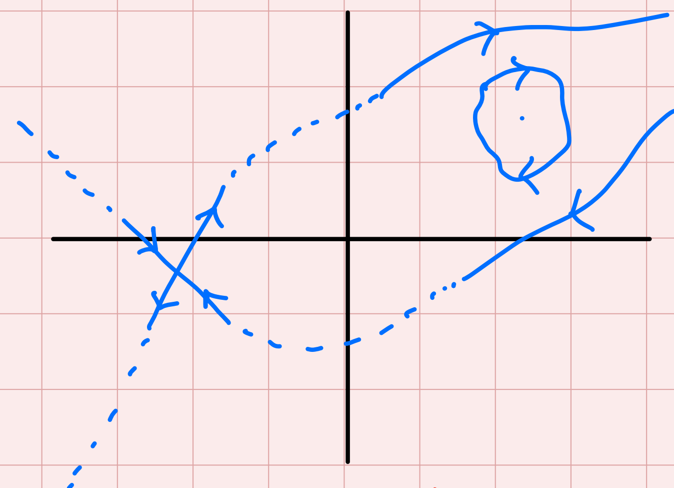
$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$$

$$x_1 = f_1(x_1, x_2)$$
$$x_2 = f_2(x_1, x_2)$$

Linear



Nonlinear



n eigenvectors paint a global picture of the phase plane.

n eigenvectors do not capture global picture.

Features of phase plane:

→ Fixed points equilibrium solutions $f(\underset{\sim}{x}^*) = \underset{\sim}{0}$

→ Closed orbits periodic solutions $\underset{\sim}{x}(t+T) = \underset{\sim}{x}(t)$

→ Behavior of solutions <u>near</u> fixed pts & closed orbits

solutions: aka trajectories in this context.

near fixed pts → Linearize!

Consider $\dot{x} = x + e^{-y}$

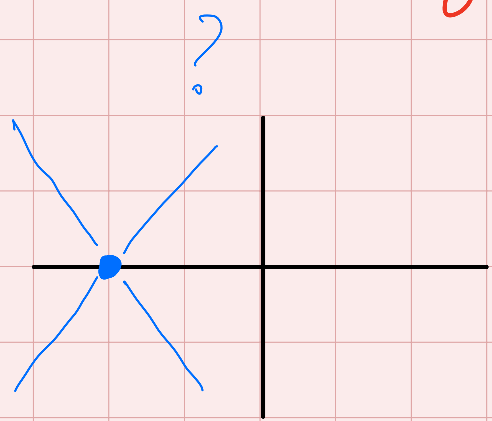
$\dot{y} = -y$

$e^{-y} = 1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \cdots$

$e^{-y} \approx 1 - y$

- Find fixed pt: $0 = x + e^{-y}$, $0 = -y$ $\Rightarrow$ $y = 0,\ x = -1$

$(x^*, y^*) = (-1, 0)$



$\dot{x} \approx x + 1 - y$

$\dot{y} = -y$

Define $x + 1 \longrightarrow x'$ NOT Derivative

$\dot{x}' = x' - y$

$\dot{y} = -y$

$$\begin{bmatrix} \dot{x}' \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\tau = 0,\quad \Delta = -1 \Rightarrow$ saddle

$\dot{x} = f_1(x, y)$

$\dot{y} = f_2(x, y)$

and $f_1(x^*, y^*) = f_2(x^*, y^*) = 0$

Let $u = x - x^*$

$v = y - y^*$

$\dot{u} = \dot{x}$

$= f_1(x^* + u, y^* + v)$

$$= \underbrace{f_1(x^*, y^*)}_{\substack{0 \text{ by def.} \\ \text{of fixed pt.}}} + u \left.\frac{\partial f_1}{\partial x}\right|_{\substack{x=x^* \\ y=y^*}} + v \left.\frac{\partial f_1}{\partial y}\right|_{\substack{x=x^* \\ y=y^*}} + \underbrace{\text{higher order terms.}}_{\text{ignore}}$$

$$\Rightarrow \boxed{\dot{u} = u \left.\frac{\partial f_1}{\partial x}\right|_{\substack{x^* \\ y^*}} + v \left.\frac{\partial f_1}{\partial y}\right|_{\substack{x^* \\ y^*}}}$$

by a similar argument,

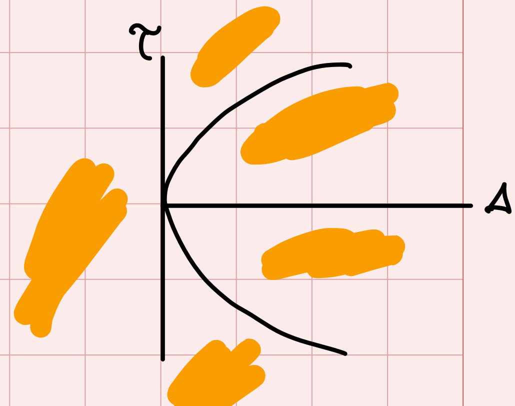
$$\boxed{\dot{v} = u \left.\frac{\partial f_2}{\partial x}\right|_{\substack{x^* \\ y^*}} + v \left.\frac{\partial f_2}{\partial y}\right|_{\substack{x^* \\ y^*}}}$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \underbrace{\overbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}}^{A_{ij} = \frac{\partial f_i}{\partial x_j}}\Bigg|_{\substack{x=x^* \\ y=y^*}}}_{\text{Jacobian matrix for the system } \dot{\underset{\sim}{x}} = f(\underset{\sim}{x}) \text{ evaluated at } \underset{\sim}{x}^*.} \cdot \begin{bmatrix} u \\ v \end{bmatrix}$$

A

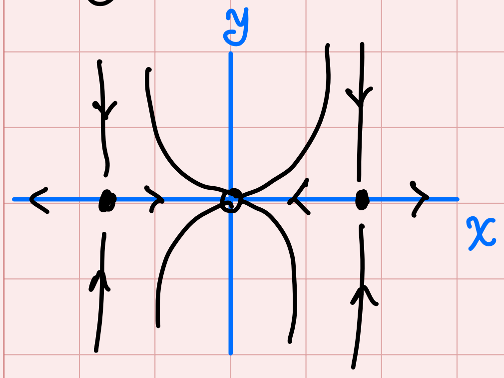
For saddles, spirals & nodes the system $\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}$ is a good representation of the nonlinear system $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ near (x^*, y^*)

evaluated at (x^*, y^*).



For other types of fixed points, the system $\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}$ gives a questionable representation of the nonlinear system $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$ near (x^*, y^*)

$$\dot{x} = -x + x^3$$
$$\dot{y} = -2y$$



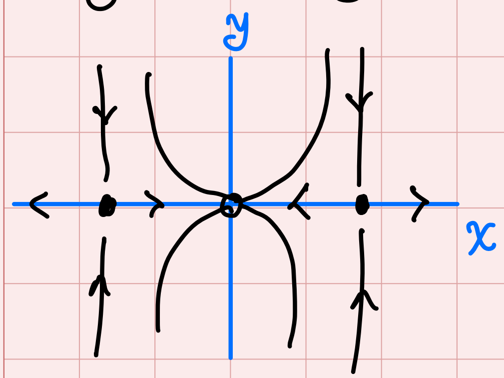
1) Find fixed pts.
2) Characterize each.

What kind of fixed pt. is it?

calculate Jacobian matrix, evaluate at each fixed pt.

where is this matrix on τ-Δ plane?

$$\dot{x} = -x + x^3$$
$$\dot{y} = -2y$$



1) Find fixed pts.

2) Characterize each.

What kind of fixed pt. is it?

calculate Jacobian matrix, evaluate at each fixed pt.

where is this matrix on τ-Δ plane?

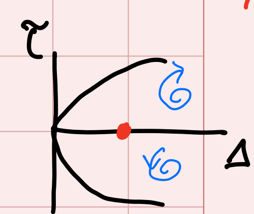
Mon, Feb 24　Lecture 10

Ex.
$$\dot{x} = -y + ax(x^2+y^2)$$
$$\dot{y} = x + ay(x^2+y^2)$$

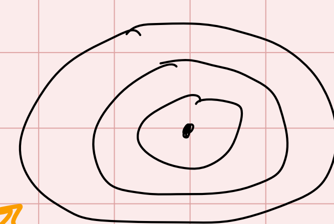
$\rightarrow$ (0,0) is a fixed pt. Classify its stability

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

evaluate at (0,0)



Linear theory predicts



$$\dot{x} = -y + ax(x^2+y^2)$$
$$\dot{y} = x + ay(x^2+y^2)$$

Near the origin, looks like a center.

But — notice the axes

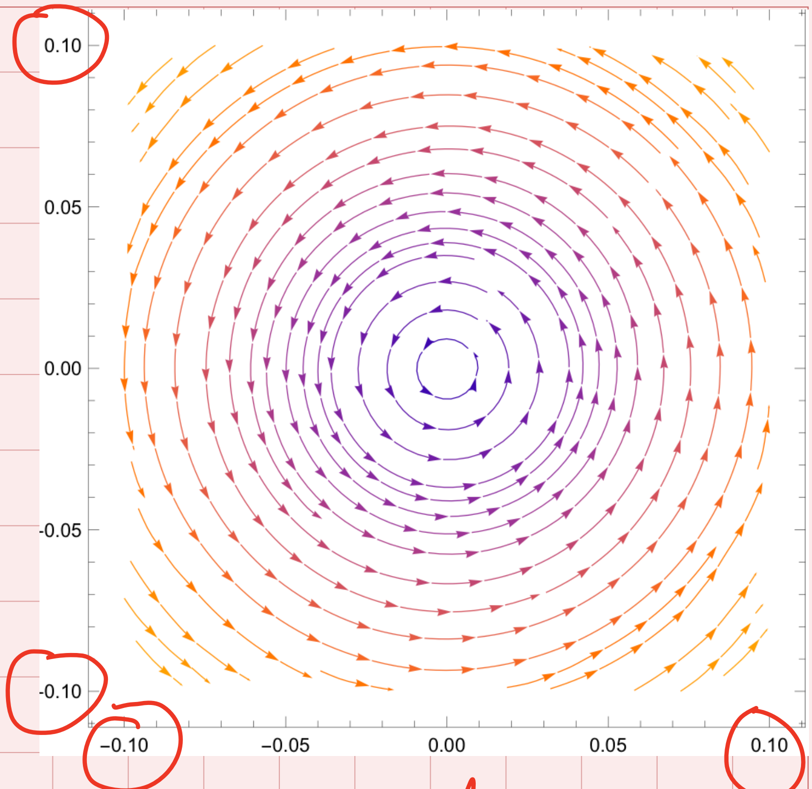
... zoom out:

and it looks like an unstable spiral.

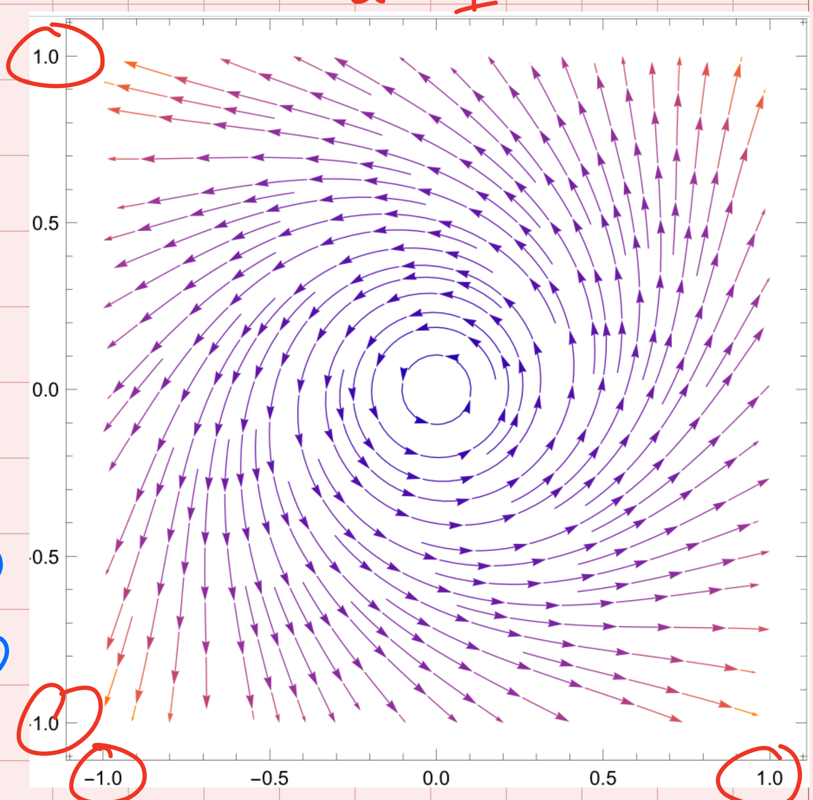
Strogatz 6.3.2 proves that this system is like an unstable spiral if $a > 0$, stable spiral if $a < 0$

(using r, θ coordinates)

$a = 1$



$a = 1$



Hyperbolic Fixed Pts.

Fixed pts that remain unchanged, qualitatively, by small nonlinear terms, relative to their linearized phase portraits.

"Local phase portrait near a hyperbolic fixed pt. is topologically equivalent to the phase portrait of its linearized version"

a homeomorphism exists between the two.

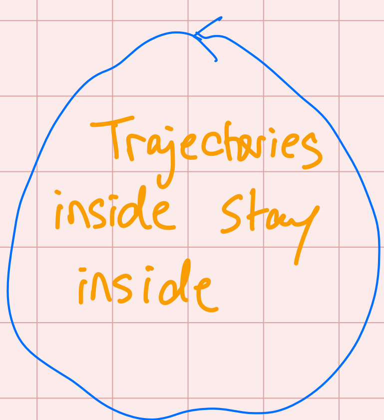
For hyperbolic fixed pts, all eigenvalues have non-zero real part.

↳ if one or more eigenvalue has zero real part, the fixed pt. is nonhyperbolic.

$$\dot{\underset{\sim}{x}} = f(\underset{\sim}{x}) \quad , \quad \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \qquad \underset{\sim}{x} \in \mathbb{R}^n$$

if f is continuous and all partial derivatives of f are continuous on a subset $D \subset \mathbb{R}^n$, then for $\underset{\sim}{x}_0$ in D, the I.V.P above has a <u>unique</u> solution $\underset{\sim}{x}(t)$ at least for some time.

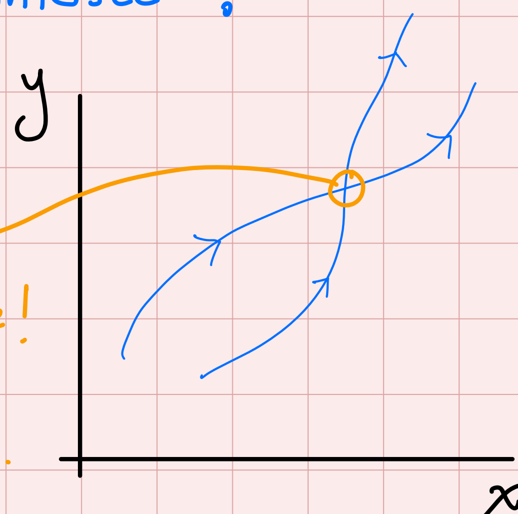
$\Longrightarrow$ Trajectories cannot intersect!



closed orbit.

Solution is non-unique!

NOT ALLOWED.



Lotka-Volterra Population Dynamics

- Two species competing for a resource (limited)
- Each species has a growth rate, carrying capacity

→ $\dot{N} = rN(1 - N/K)$ logistic eqn.

- Two logistic eqns + competition.

x : rabbits

y : sheep

rabbits grow faster than sheep

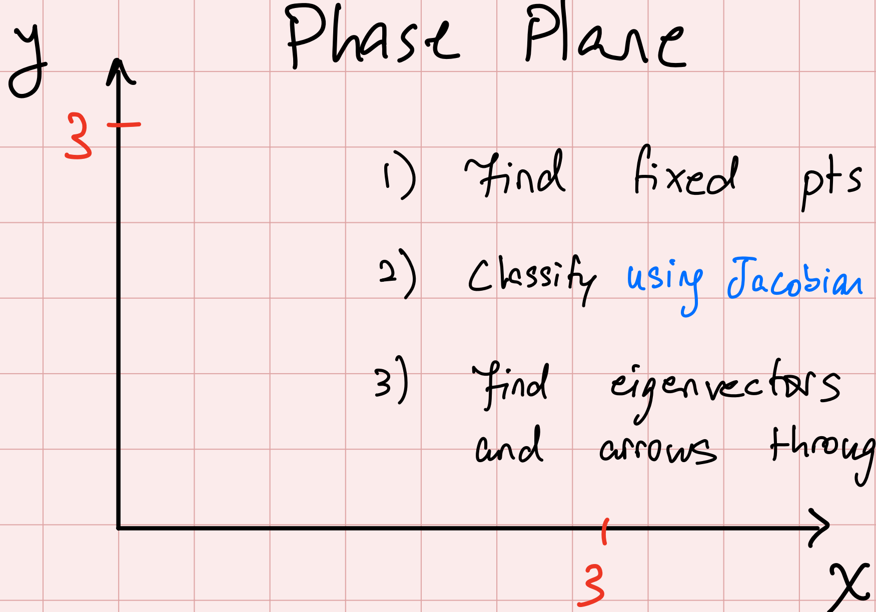
$$\dot{x} = x\left(3 - x/2 - 2y\right)$$

$$\dot{y} = y\left(2 - x - y\right)$$

Rabbits have higher carrying capacity

Sheep stronger than rabbits.

Phase Plane



1) Find fixed pts $(0,2)$ $(0,0)$ $(6,0)$ $(2/3, 4/3)$
2) Classify using Jacobian
3) Find eigenvectors to put lines and arrows through the fixed pts.

Wed, Feb 26 Lecture 11

Building up nonlinear phase plane using linearizations at each fixed pt.

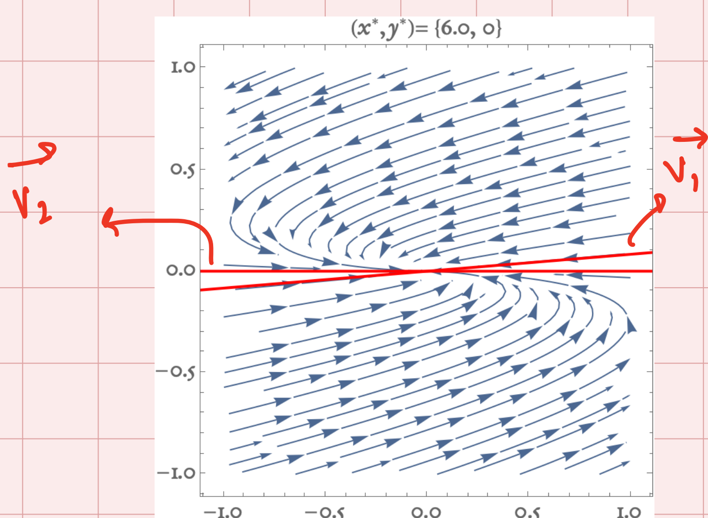
$(0,2) \longrightarrow$ $A = [\quad]$ $\lambda_{1,2} = \ldots$ $\vec{v}_{1,2} = [\,]$

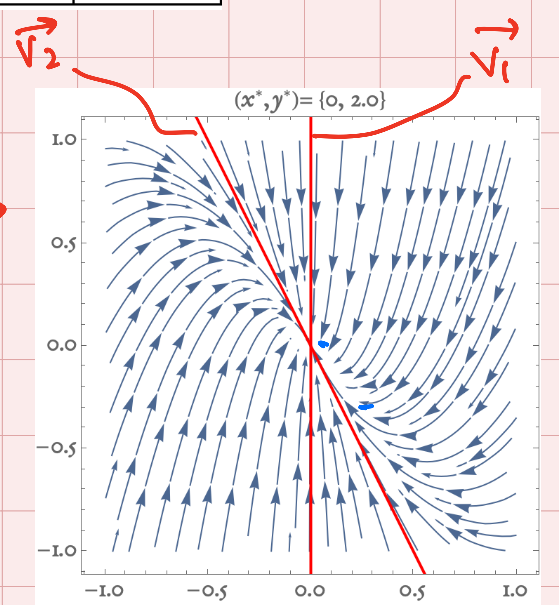
$(6,0) \longrightarrow$

$(0,0) \longrightarrow$

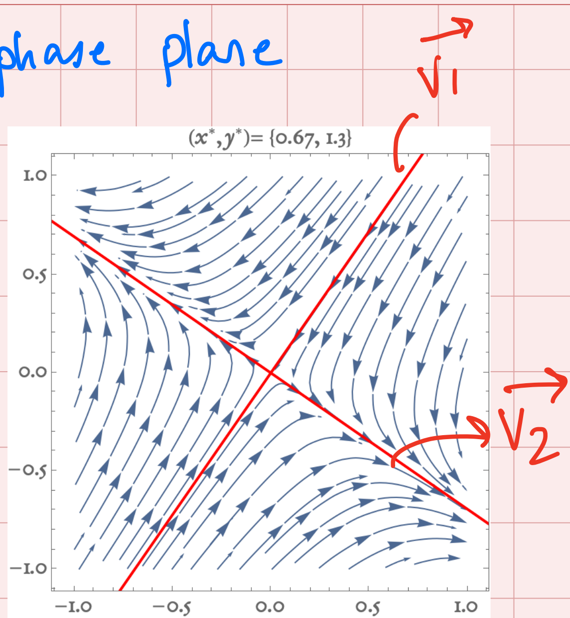
$(2/3, 4/3) \longrightarrow$ $\lambda = \{-2.26, +0.591\}$

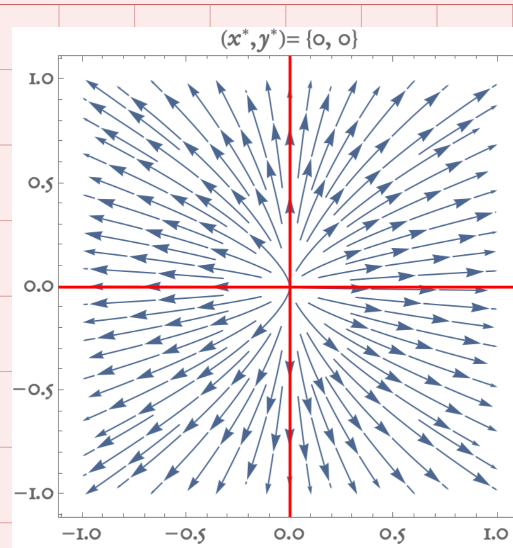
Fixed Point	Jacobian	τ	Δ	λ_1	$\vec{v}_1$	λ_2	$\vec{v}_2$	
$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$	-3	2	-2.00	$\begin{pmatrix} 0 \\ 1.00 \end{pmatrix}$	-1.00	$\begin{pmatrix} -1.00 \\ 2.00 \end{pmatrix}$	stable node
$\begin{pmatrix} \frac{2}{3} \\ \frac{4}{3} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{3} & -\frac{4}{3} \\ -\frac{4}{3} & -\frac{4}{3} \end{pmatrix}$	$-\frac{5}{3}$	$-\frac{4}{3}$	-2.26	$\begin{pmatrix} 0.693 \\ 1.00 \end{pmatrix}$	0.591	$\begin{pmatrix} -1.44 \\ 1.00 \end{pmatrix}$	saddle
$\begin{pmatrix} 6 \\ 0 \end{pmatrix}$	$\begin{pmatrix} -3 & -12 \\ 0 & -4 \end{pmatrix}$	-7	12	-4.00	$\begin{pmatrix} 12.0 \\ 1.00 \end{pmatrix}$	-3.00	$\begin{pmatrix} 1.00 \\ 0 \end{pmatrix}$	stable node
$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$	5	6	3.00	$\begin{pmatrix} 1.00 \\ 0 \end{pmatrix}$	2.00	$\begin{pmatrix} 0 \\ 1.00 \end{pmatrix}$	unstable node

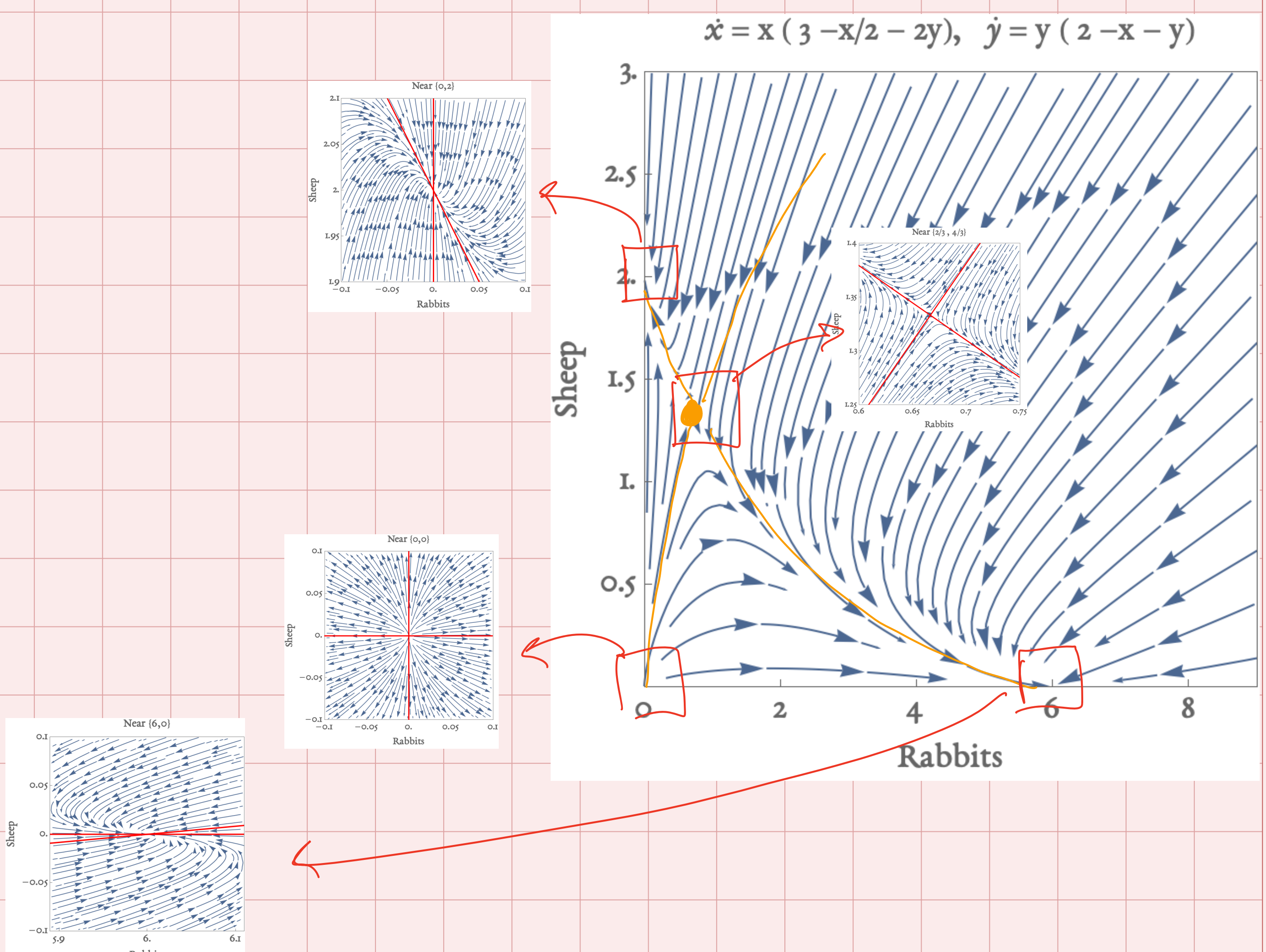




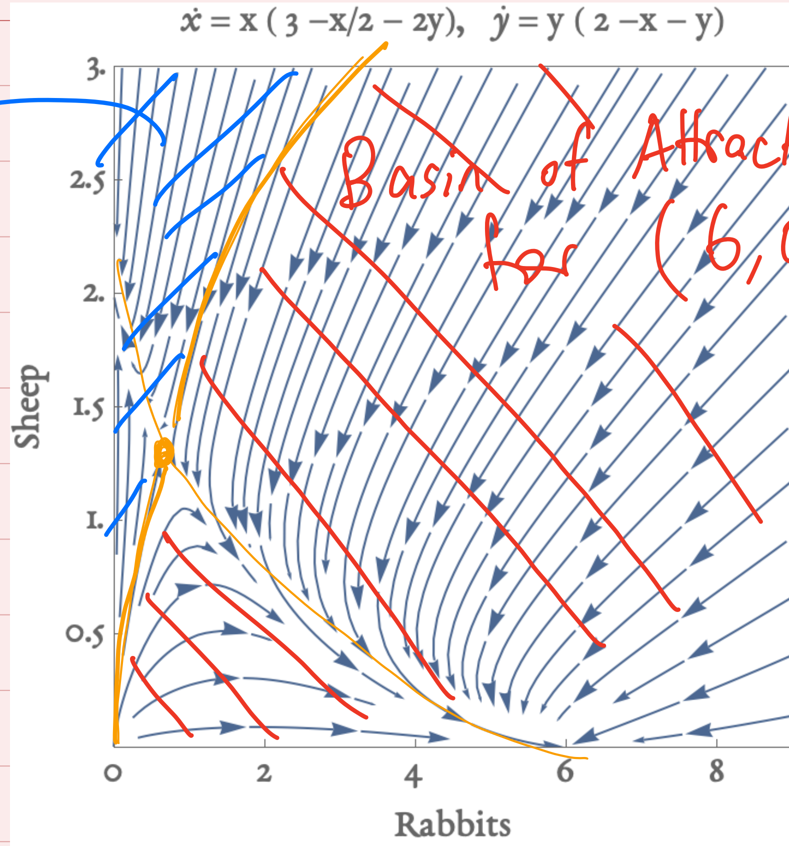
Linear phase plane







Basin of Attraction for (0,2)



Conservative Systems

Given $\dot{\underset{\sim}{x}} = f(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^n$, a conserved quantity is a real-valued continuous scalar function $E(\underline{x}) \neq$ const. that is constant on trajectories, i.e. $\frac{dE}{dt} = 0$

If a system has a conserved quantity, it is called a conservative system.

$$\underbrace{\ddot{x}}_{\text{"accel."}} = \underbrace{x^3 - x}_{\text{"force" } F(x)}$$

Find a conserved quantity for this system.

$$F(x) = -\frac{dV}{dx}$$

potential $V(x)$

$$x^3 - x = -\frac{dV}{dx}$$

$$\ddot{x} = F(x) = -dV/dx$$

$$\int (x^3 - x)\,dx = \int -\,dV$$

$$\ddot{x} + \frac{dV}{dx} = 0$$

$$\frac{x^4}{4} - \frac{x^2}{2} = -V + c$$

$$\dot{x}\ddot{x} + \dot{x}\frac{dV}{dx} = 0$$

$$\Rightarrow \boxed{V(x) = \frac{x^2}{2} - \frac{x^4}{4} + c}$$

$$\frac{d}{dt}\Big(\underbrace{\frac{1}{2}\dot{x}^2 + V(x)}_{E(x,\dot{x})}\Big) = 0$$

we have found that

$\frac{1}{2}\dot{x}^2 + V(x)$ is a conserved quantity.

① $m\ddot{x} + c\dot{x} + kx = 0$ ② $\ddot{\theta} = \sin\theta$

find $V(x)$, not $V(x,\dot{x})$

None exists

$$m\ddot{x} + kx = 0$$

$$E = \frac{1}{2}\dot{\theta}^2 + \cos\theta$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Mon, Mar 3 Lecture 12

When are systems conservative?

↳ Physically, they correspond to frictionless mechanical systems + others

↳ Mathematically, when you can find $E(\underline{x})$.

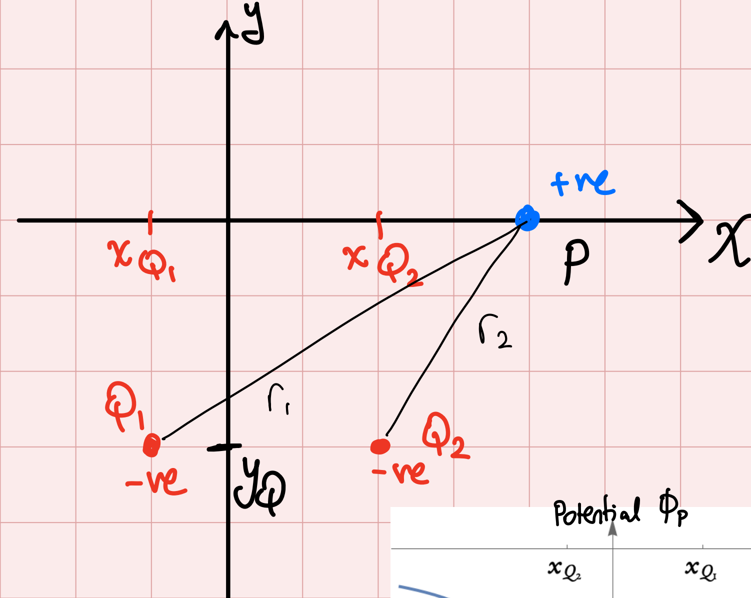
Ex

A positive charge P is confined to move on x-axis.

$$r_1 = \sqrt{(x - x_{Q_1})^2 + y_Q^2}$$

Location of P

$$r_2 = \sqrt{(x - x_{Q_2})^2 + y_Q^2}$$

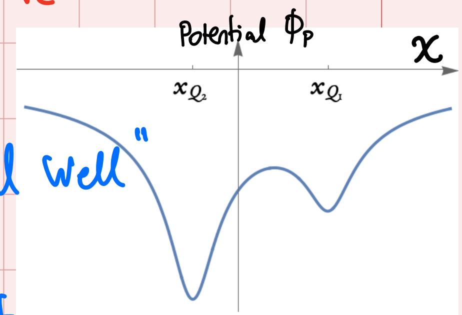


Electric Potential

$$\phi_P = \frac{Q_1}{r_1} + \frac{Q_2}{r_2}$$

Force $= -\nabla \phi$

"Potential Well"



$$= \frac{Q_1}{r_1^2}\hat{e}_{r_1} + \frac{Q_2}{r_2^2}\hat{e}_{r_2}$$ → horz component

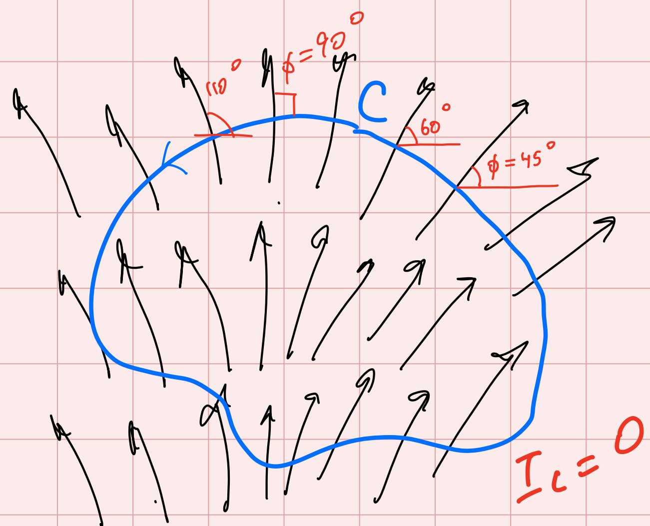
Horz force: $$\frac{Q_1}{(x-x_{Q_1})^2 + y_Q^2}\,\frac{x - x_{Q_1}}{\sqrt{(x-x_{Q_1})^2+y_Q^2}} + \frac{Q_2}{(x-x_{Q_2})^2 + y_Q^2}\,\frac{x - x_{Q_2}}{\sqrt{(\dots)}}$$

" $\ddot{x}$ = force "

Index of a closed curve in a vector field.

A Measure of the "winding" of the vector field.

Let $\underbrace{\dot{\underline{x}} = f(\underline{x})}_{\text{a smooth vector field}}$, $\underline{x} \in \mathbb{R}^2$ and C a closed curve
(does not self-intersect)
(does not pass thru fixed pts of the vector field)



Let ϕ be the angle between the vector field $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}$ and the horizontal for any point on C.

$\Rightarrow \phi = \tan^{-1}(\dot{y}/\dot{x})$

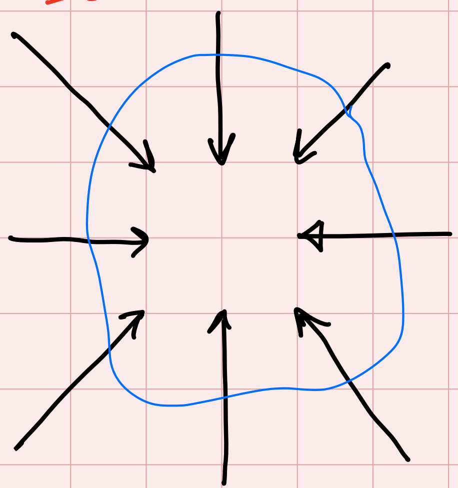
$[\phi]_C$: net change in ϕ over one c.c.w. loop around C.

Index of C is:

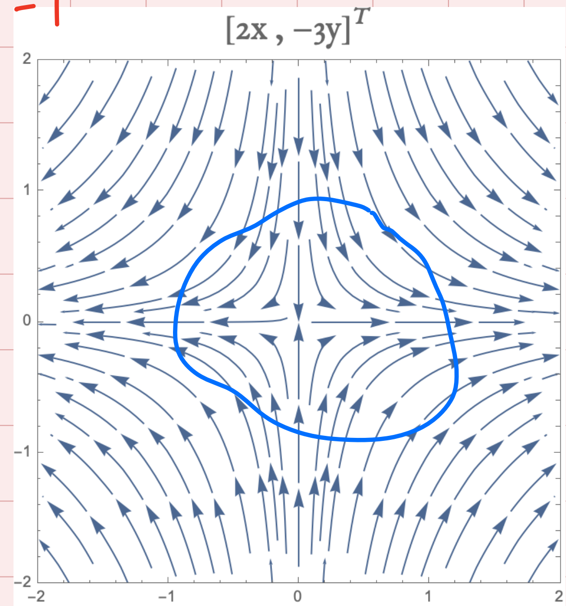
$$I_C = \frac{1}{2\pi}[\phi]_C$$

net number of counter-clockwise revolutions made by the vector field $f(\underline{x})$ as $\underline{x}$ moves counterclockwise around C.

$I_c = +1$



$I_c = -1$



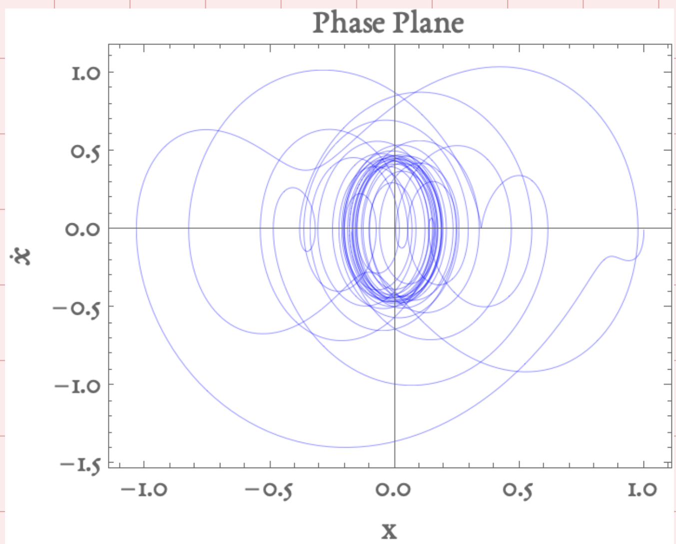
spirals, centers, stars, degenerate nodes : $I_c = +1$
Saddles $I_c = -1$

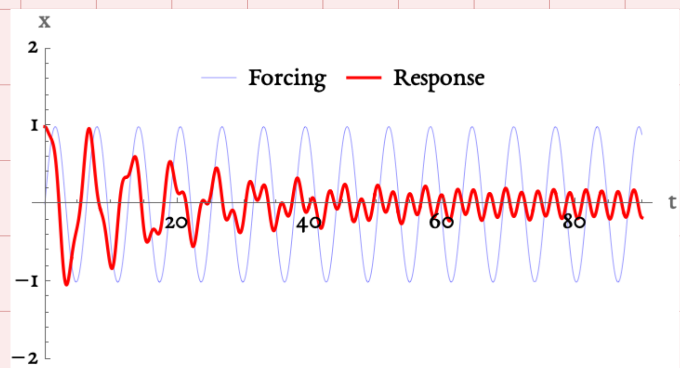
- $I_c = 0$ if no fixed pts inside C.
- If C is a trajectory, $I_c = +1$
- If C can be continuously deformed into C' without passing thru a fixed pt, $I_c = I_{c'}$.
- If all arrows in the vector field reverse direction $\vec{f} \to -\vec{f}$ then I does not change.
- The index of a fixed point = I_c for any C that encloses only that fixed pt.
- Any closed orbit in phase plane must enclose fixed pts whose indices sum up to $+1$.
- If C surrounds multiple fixed pts, $I_c = \sum$ index of each fixed pt.

Wed, Mar 5 Lecture 13

Limit Cycles

$$m\ddot{x} + c\dot{x} + kx = \sin(\omega t)$$





This is not an $n=2$ dynamical system $\{\dot{\underline{x}} = f(\underline{x})\}$

An n^{th} order $\underbrace{\text{time-dependent}}$ $_{= \text{nonautonomous}}$ eqn. is a special case of an $(n+1)^{\text{th}}$ order autonomous dynamical system

$$\ddot{x} + \dot{x} + x = \sin \omega t$$

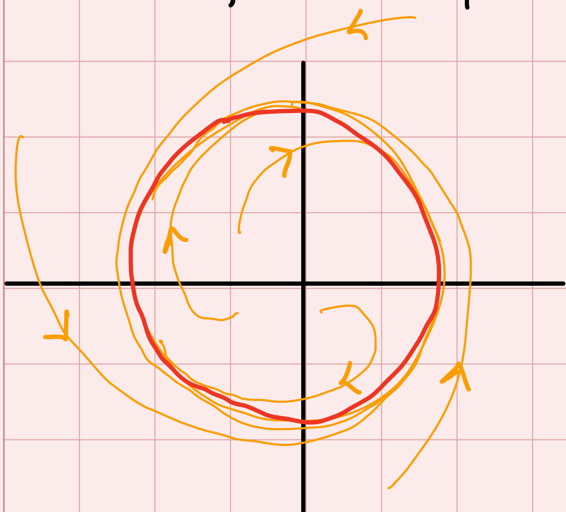
$$x_1 = x \qquad x_2 = \dot{x} \qquad x_3 = \omega t$$

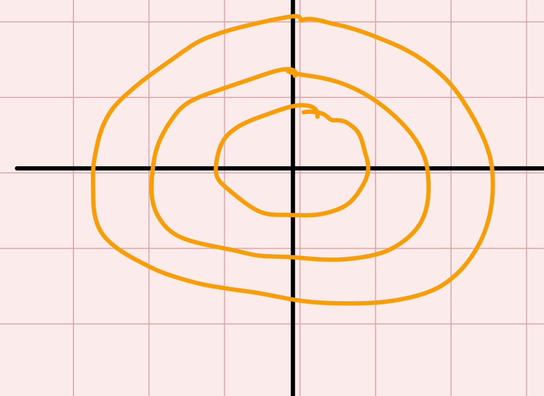
$$\dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - x_2 + \sin x_3 \\ \omega \end{bmatrix}$$

closed orbit

Limit Cycle is an isolated closed trajectory in phase space

$m\ddot{x} + \cancel{c}\dot{x} + kx = 0$, with $c \to 0$





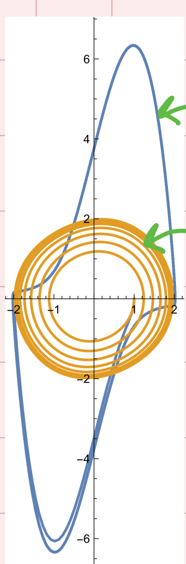
closed orbit that is a limit cycle.

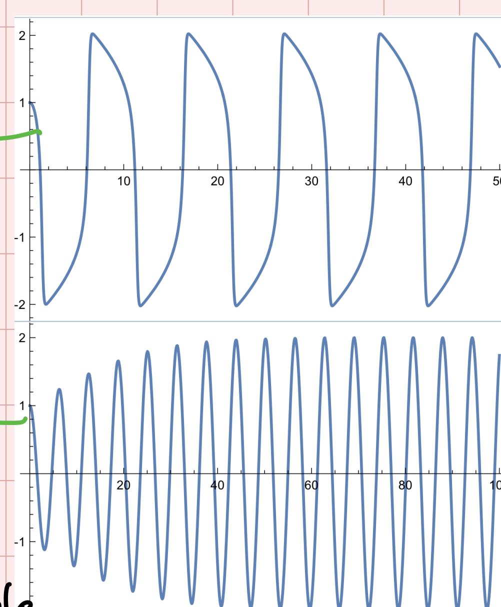
closed orbits but NOT limit cycles

<u>van der Pol oscillator</u>　　$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$

2 diff limit cycles

A distinctive feature of nonlinear systems





All trajectories approach the limit cycle (b/c it's <u>stable</u>). Opposite for unstable ones.

Higher-order pitchfork bifurcations In practice, systems with a subcritical pitchfork bifurcation don't actually go off to $\pm\infty$; instead, higher-order terms play a stabilizing role. Consider the system

$$\dot{x} = rx + x^3 - x^5, \tag{1}$$

where r is a single parameter that accounts for all the qualitatively different dynamics. Phase portraits are shown below for two values of r.

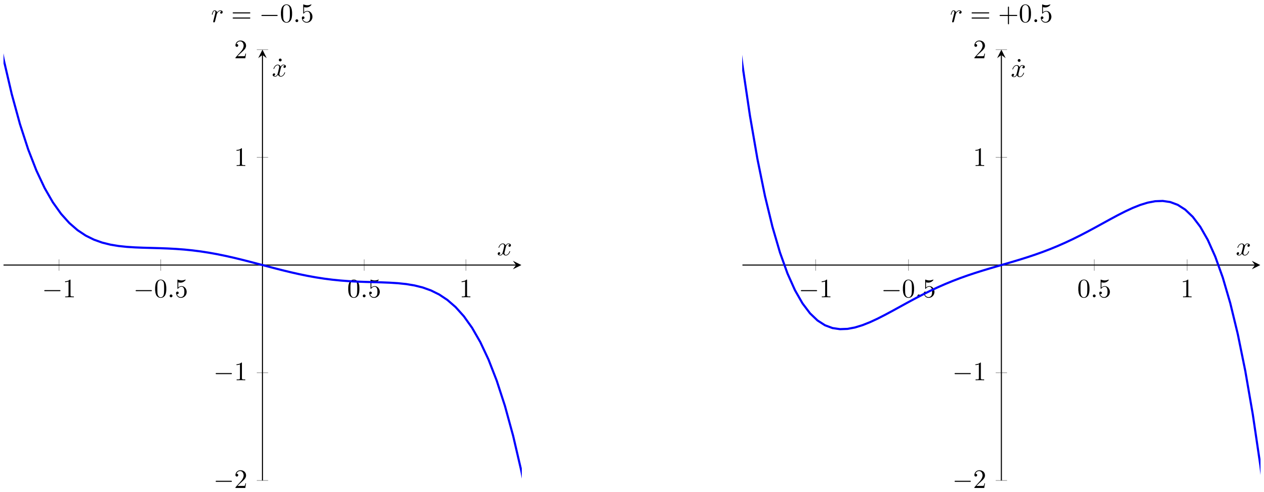


Figure 1: Phase portraits for $\dot{x} = rx + x^3 - x^5$.

✍ Draw arrows and fixed points to complete the phase portraits above.

✍ Draw some possible trajectories below. Indicate fixed points with horizontal lines.

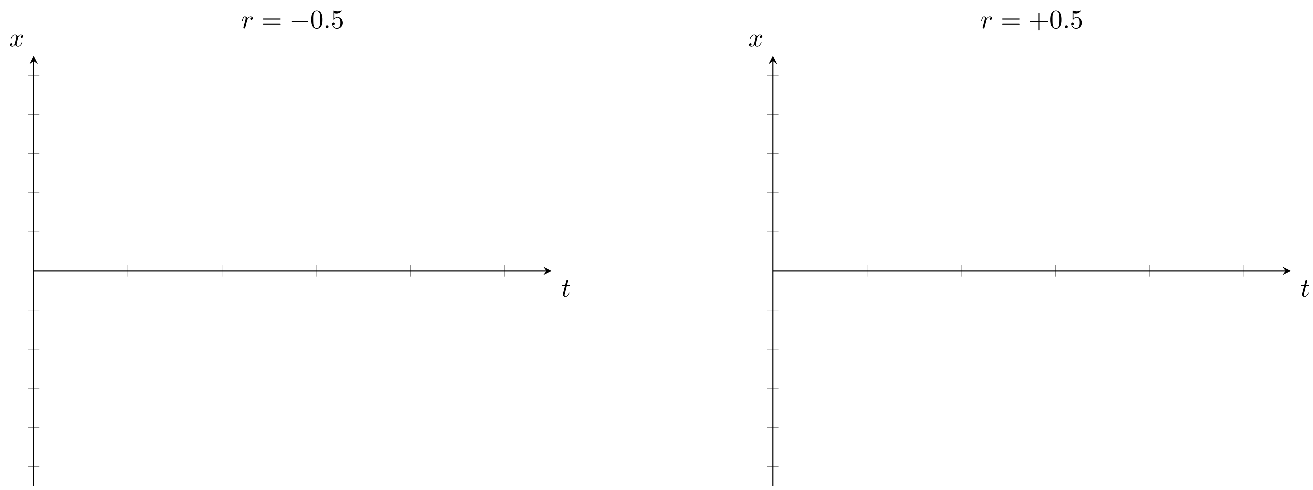
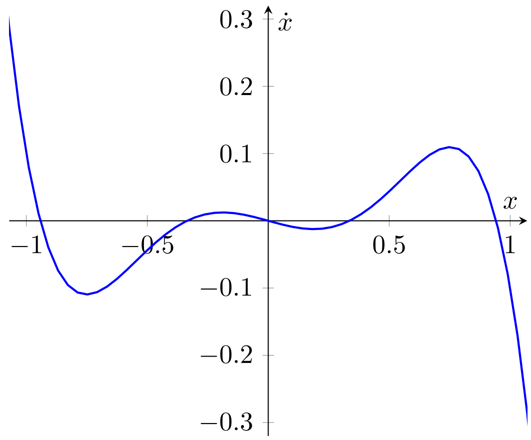


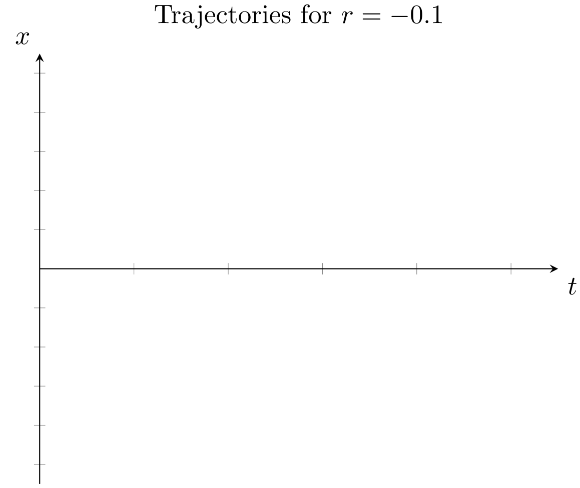
Figure 2: Trajectories for $\dot{x} = rx + x^3 - x^5$.

Next, we will consider the case when $r = -0.1$.

Phase portrait for $r = -0.1$



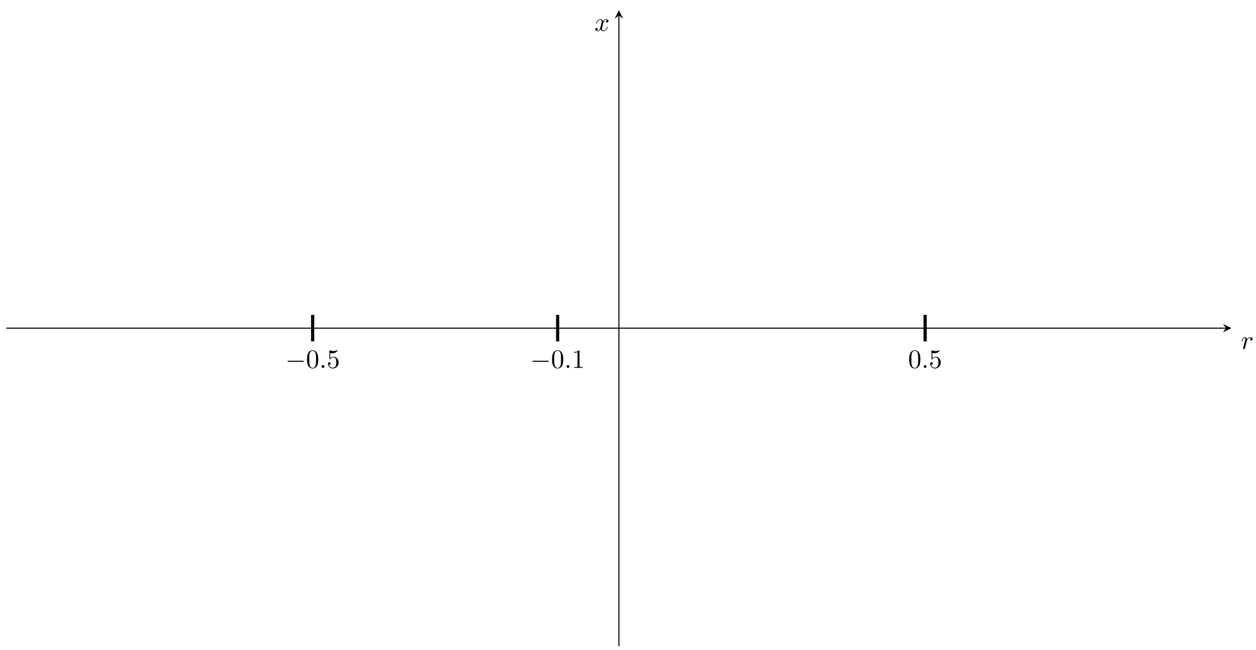
Trajectories for $r = -0.1$



✍ Complete the phase portrait and sketch some possible trajectories.

✍ Use what you have learned to fill in some points on the bifurcation curve shown below.

Bifurcation diagram for $\dot{x} = rx + x^3 - x^5$



✍ Use the Mathematica 'Manipulate' panel at `https://tinyurl.com/higherorderbifurcation1` to determine some other points on the bifurcation diagram.

✍ Do this for at least one value of $r < -0.5$, one value in the range $-0.5 < r < -0.1$, one in the range $-0.1 < r < 0.5$, and one value of $r > 0.5$.

✍ Note the stability and instability of each point you draw.

✍ Attempt to connect the points using a curve or curves.

Non-dimensionalization Consider the two equations

$$\dot{u} = au + bu^3 - cu^5, \tag{2a}$$
$$\dot{x} = rx + x^3 - x^5. \tag{2b}$$

These two equations are similar in their polynomial form, except that (2a) has three parameters whereas (2b) has only one. It is possible to show that (2b) is a 're-scaled' or nondimensionalized form of (2a). To see how this works, let us re-write the above equations explicitly,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = au + bu^3 - cu^5, \tag{3a}$$
$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = rx + x^3 - x^5, \tag{3b}$$

where we have chosen a different time variable, τ, for x than for u. This is because both the dependent variable and the independent variable will be re-scaled.
Now, let

$$x \equiv \frac{u}{U}, \text{ and} \tag{4a}$$
$$\tau \equiv \frac{t}{T} \tag{4b}$$

for some constants U and T. You can think of U and T as some characteristic values of u and t respectively; it doesn't matter what they are, as long as they have the same units as the quantities that they are dividing. These two quantities, U and T, are *constants* and do not change with time; $\mathrm{d}U$ and $\mathrm{d}T$ will be zero.

✍ Substitute (4) into (3a) to arrive at an equation for $\frac{\mathrm{d}x}{\mathrm{d}\tau}$ in terms of a, b, c, T and U.

✍ Equate the coefficients of your equation from the last part to the coefficients of (3b), and use the resulting equations to solve for r, U and T in terms of a, b and c.

✍ Assuming that u has units of length ('L') and t has units of time ('T'), show that the units of a, b and c are T^{-1}, $L^{-2}T^{-1}$ and $L^{-4}T^{-1}$ respectively. Then, check your expressions for T and U from above to ensure that T and U have the correct units.

Consider the differential equation

$$m\ddot{x} + c\dot{x} + kx = \sin(\omega t). \tag{1}$$

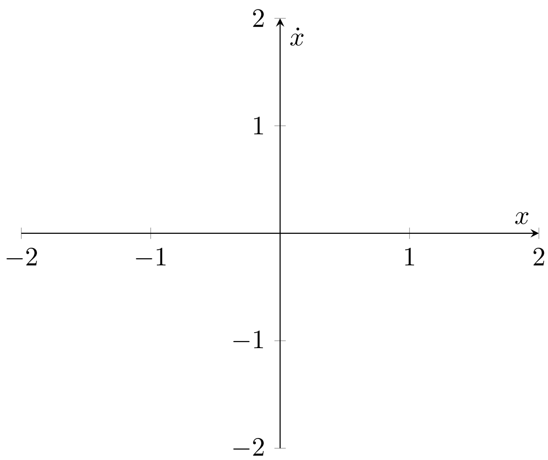
✍ What is the **order** of this differential equation?

✍ Is this an autonomous differential equation or a non-autonomous differential equation?

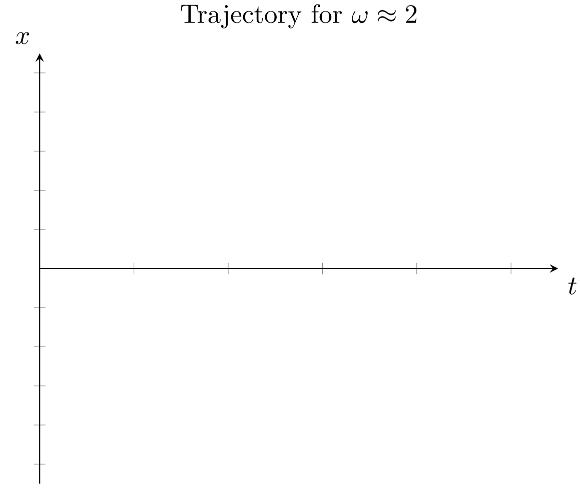
✍ The term on the left represents a spring-mass-dashpot system as usual. What is the physical meaning of the term on the right side?

✍ Visit `https://tinyurl.com/E91limitcycle2` and observe the dynamics at $\omega \approx 2$ and $\omega \approx 1/4$. Alternatively, you can visit `https://emadmasroor.github.io/classes/E91_S25/Resources/ForcedHarmonicOscillator.nb` to download the Mathematica notebook directly. Sketch x against time and $\dot{x}$ against x for long times below. Let the initial condition for your plots be $x(0) = 1, \dot{x}(0) = 0$.

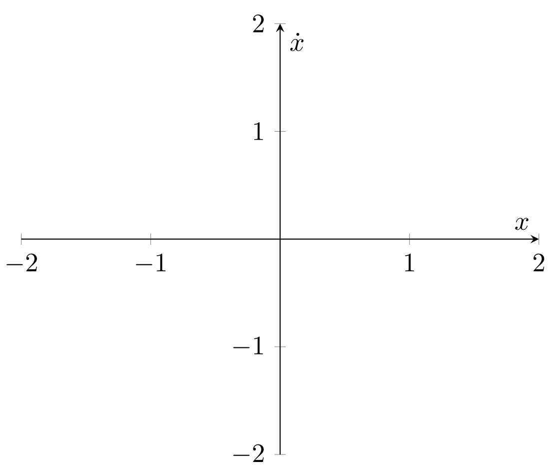
Phase portrait for $\omega \approx 2$



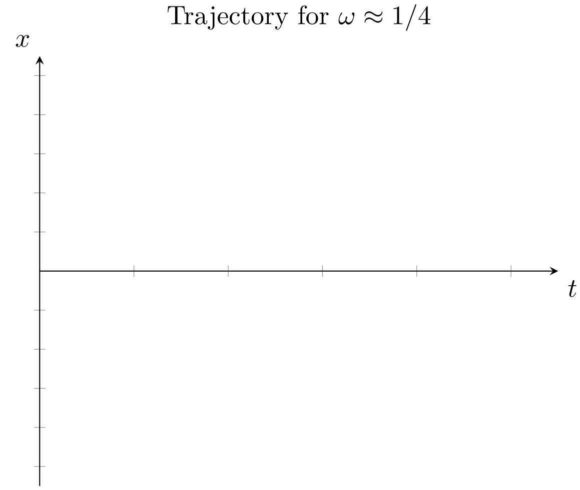
Trajectory for $\omega \approx 2$



Phase portrait for $\omega \approx 1/4$



Trajectory for $\omega \approx 1/4$



Consider the differential equation

$$\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0. \tag{2}$$

✍ What is the **order** of this differential equation?

✍ Is this an autonomous differential equation or a non-autonomous differential equation?

✍ Interpret the terms in this equation using the usual language of oscillators. What do they each mean?

✍ For two values of $\mu = 0.1, 4$, and using the initial condition $x(0) = 1, \dot{x}(0) = 0$, numerically integrate these equations using a computer program of your choice, and sketch the resulting trajectories $x(t)$. Use the accompanying graph paper to sketch what your computer program tells you.
For $\mu = 0.1$, plot $t = 0$ to $t = 100$. For $\mu = 4$, plot $t = 0$ to $t = 50$.