Question 1a)

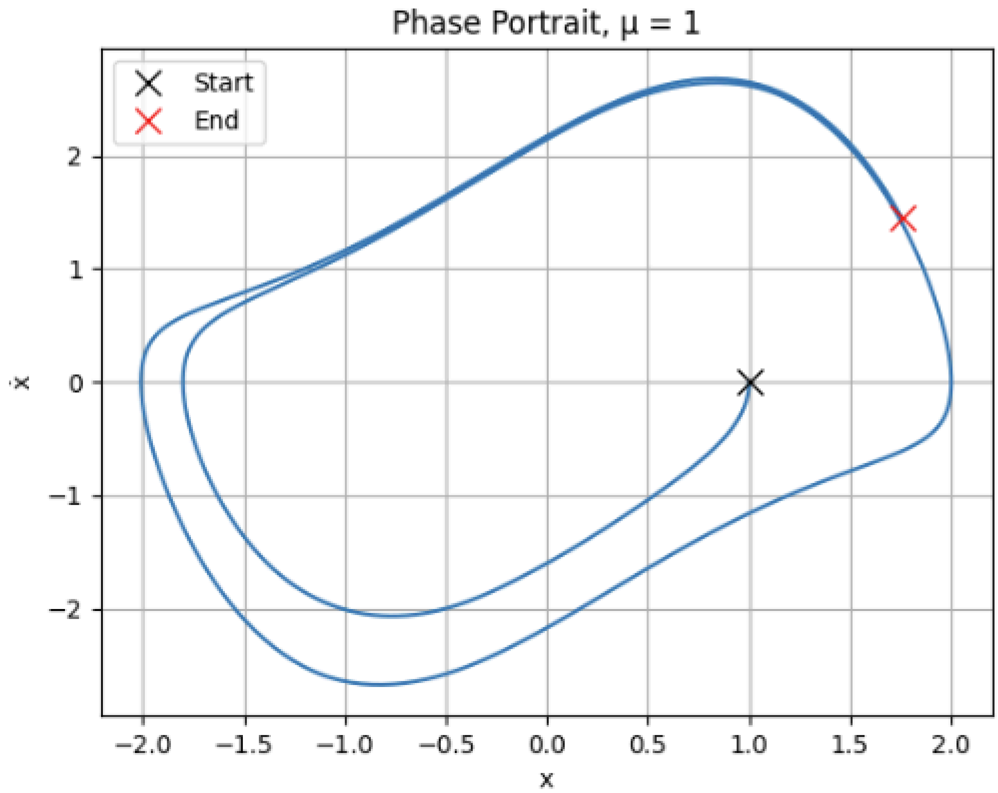


Figure 1. Phase Portrait Plot for **μ = 1**

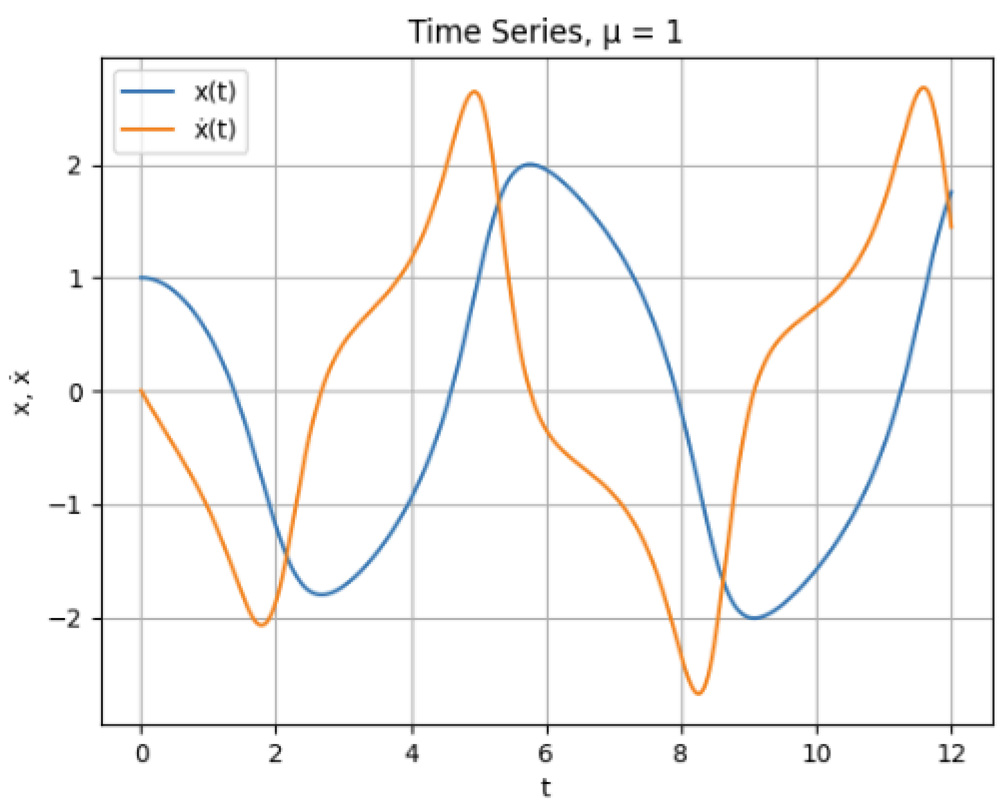


Figure 2. Time Series Plot for **μ =1**

Question 1b)

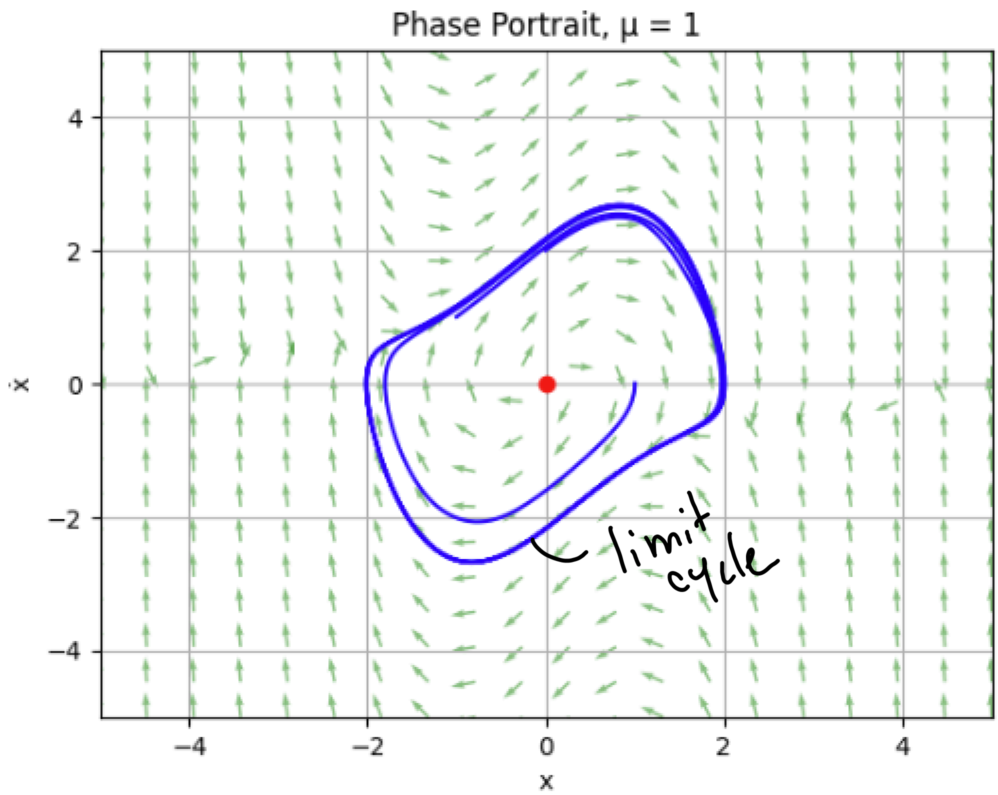


Figure 3. Phase Portrait Plot **μ=1**

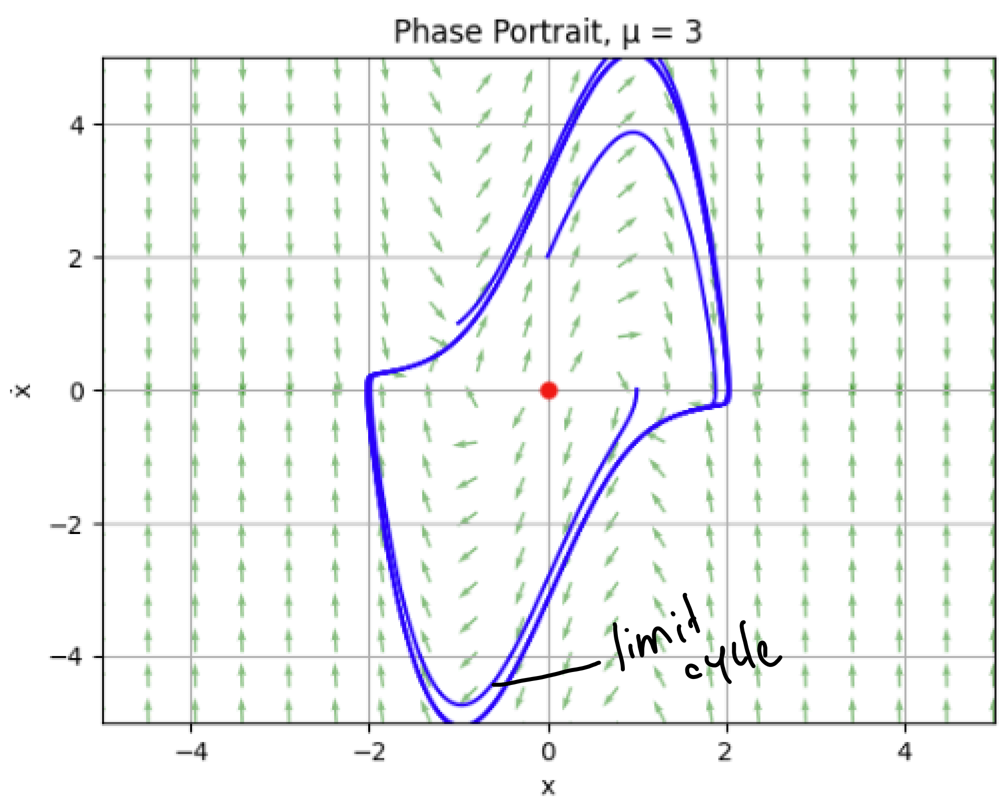


Figure 4. Phase Portrait Plot for **μ = 3**

Question 1c)

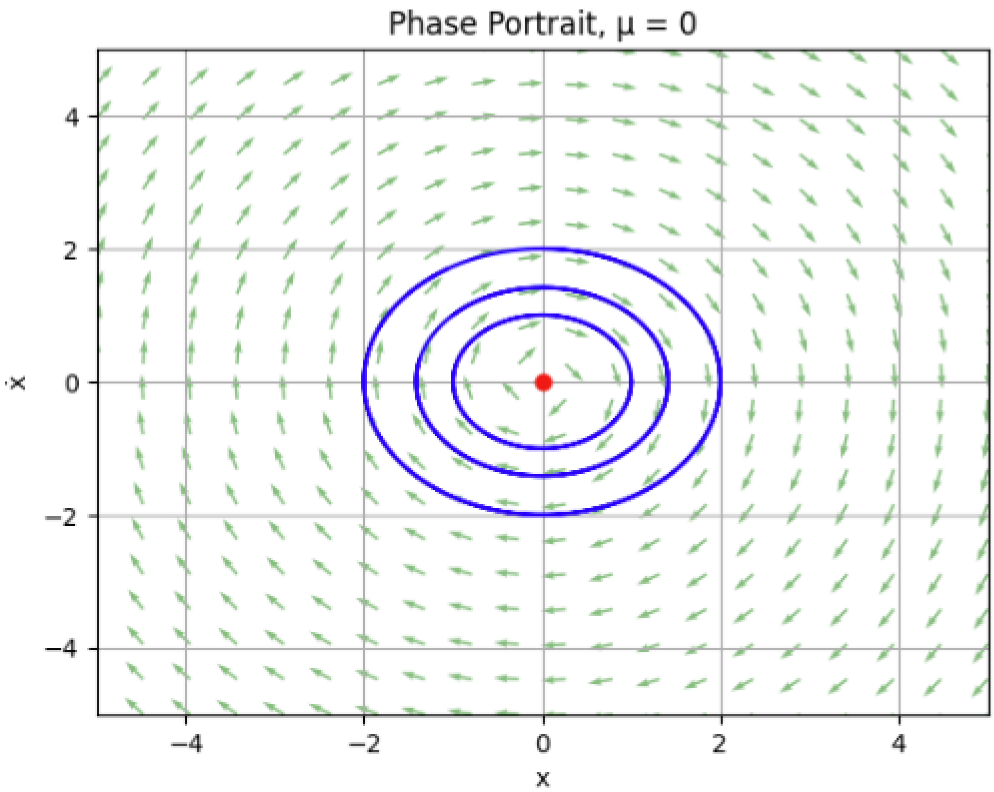
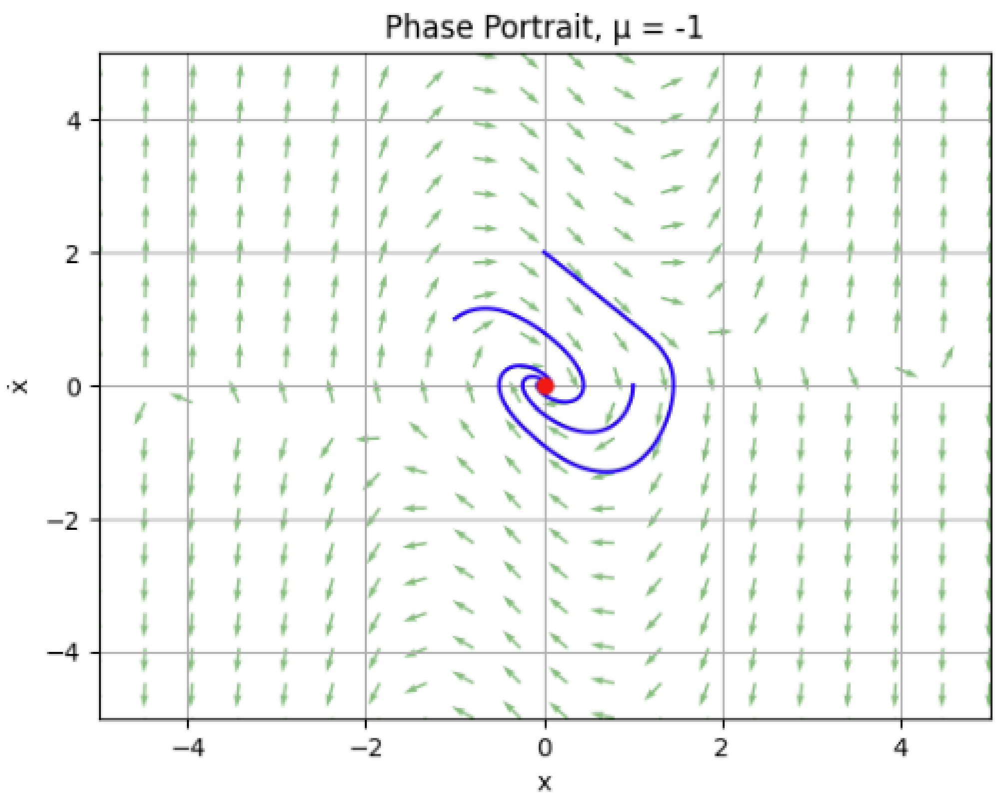


Figure 5. Phase Portrait Plot for **μ = 0**

For **μ = 0,** the van der Pol equation reduces to a simple harmonic oscillator. There is no limit cycle; trajectories are closed elliptical orbits around the origin.

Question 1d)



No limit cycle

Figure 6. Phase Portrait Plot for **μ = -1**

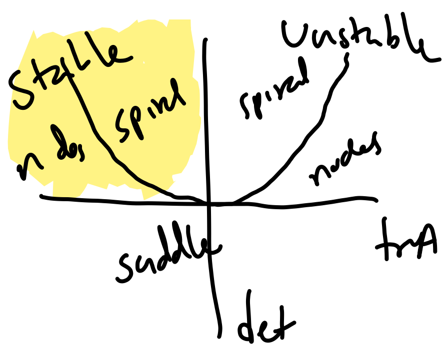
For **μ** < 0, trajectories spiral into the origin, indicating stability.

$\ddot{x} = -x - \mu(x^2-1)\dot{x}$ $\Rightarrow$ $\dot{x} = y$

let $y = \dot{x}$, $\dot{y} = \ddot{x}$ $\qquad$ $\dot{y} = -x - \mu(x^2-1)y$

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -1-2x\mu & -\mu(x^2-1) \end{bmatrix} ; \quad J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

$\left.\begin{array}{l} trA = \mu \Rightarrow -ve \\ det = 1 \end{array}\right\}$ indicates that it lies within the stable side of the trA-det plot



for all values of $\mu < 1$, we live in the shaded quadrant that is stable

2. $\dot{x} = \mu x + y + \sin x$

$\dot{y} = x - y$

a) $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \mu x + y + \sin x \\ x - y \end{bmatrix}$

$0 = \mu x + y + \sin x$

$y = -\mu x - \sin x$ $\qquad$ $x = -\mu x - \sin x$

$y = x$

b) $J = \begin{bmatrix} \mu + \cos x & 1 \\ 1 & -1 \end{bmatrix}$ $\qquad$ $J = \begin{bmatrix} df_1/dx & df_1/dy \\ df_2/dx & df_2/dy \end{bmatrix}$

$J(0,0) = \begin{bmatrix} \mu + 1 & 1 \\ 1 & -1 \end{bmatrix}$

c) $\mathrm{tr}A = \mu$ $\qquad$ $\det = -\mu - 1 - 1 = -\mu - 2$

$-\mu - 2 > 0 \rightarrow$ not a saddle

@ $\mu > -2$

When $\mu > -2$ we always get saddle fixed points

When $\mu < -2$ and $4(-\mu - 2) > \mu^2$ we get stable nodes for fixed points $(0,0)$

When $\mu < -2$ and $4(-\mu - 2) < \mu^2$ we get stable spirals for fixed pt $(0,0)$

d) $\ddot{x} + \mu \dot{x} + \sin x = 0$

$$x + \mu x + x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\dots = 0$$

$$x\left(\mu + 2 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots\right) = 0 \qquad ; \; x = 0$$

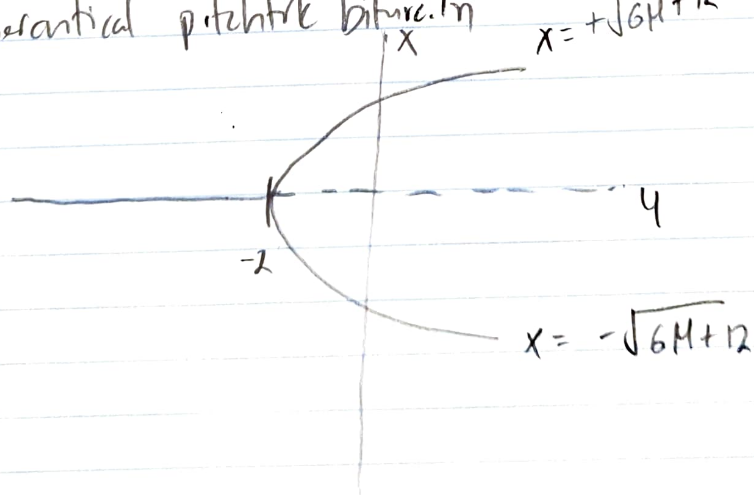
Since we are looking at values near 0, the higher order terms are approximately zero, we can retain the x^2 term

$$\mu + 2 - \frac{x^2}{6} = 0$$

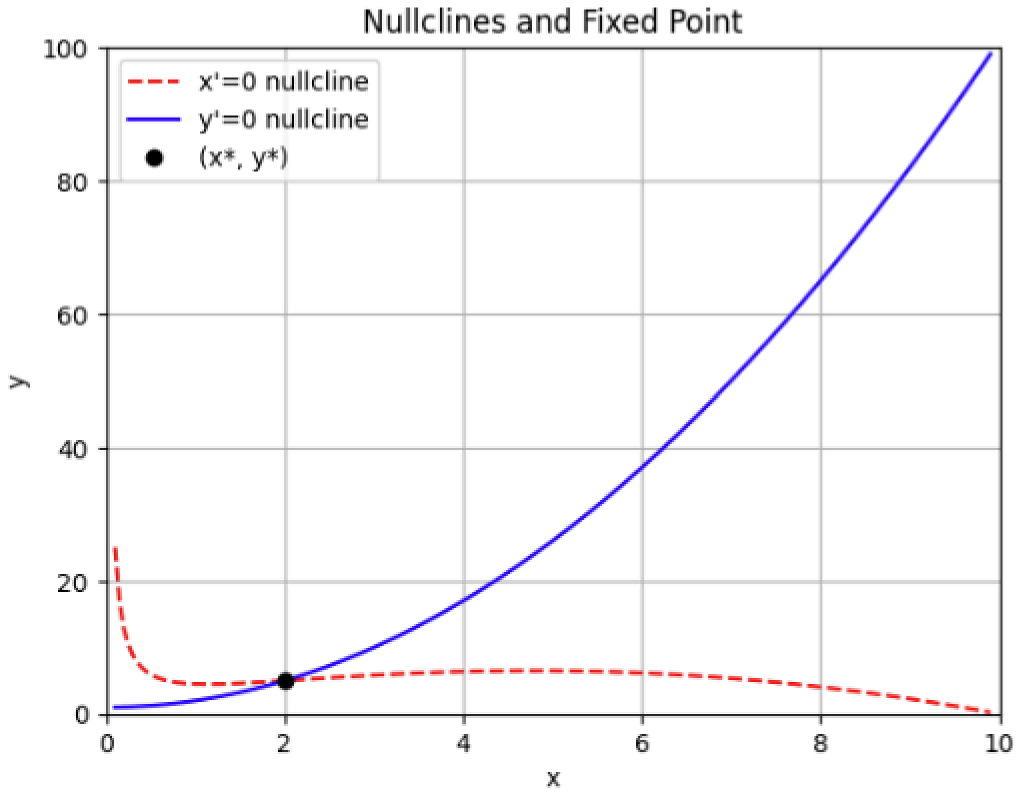
$$x^2 = 6\mu + 12$$

$$x = \pm\sqrt{6\mu + 12} \qquad , \; y = x$$

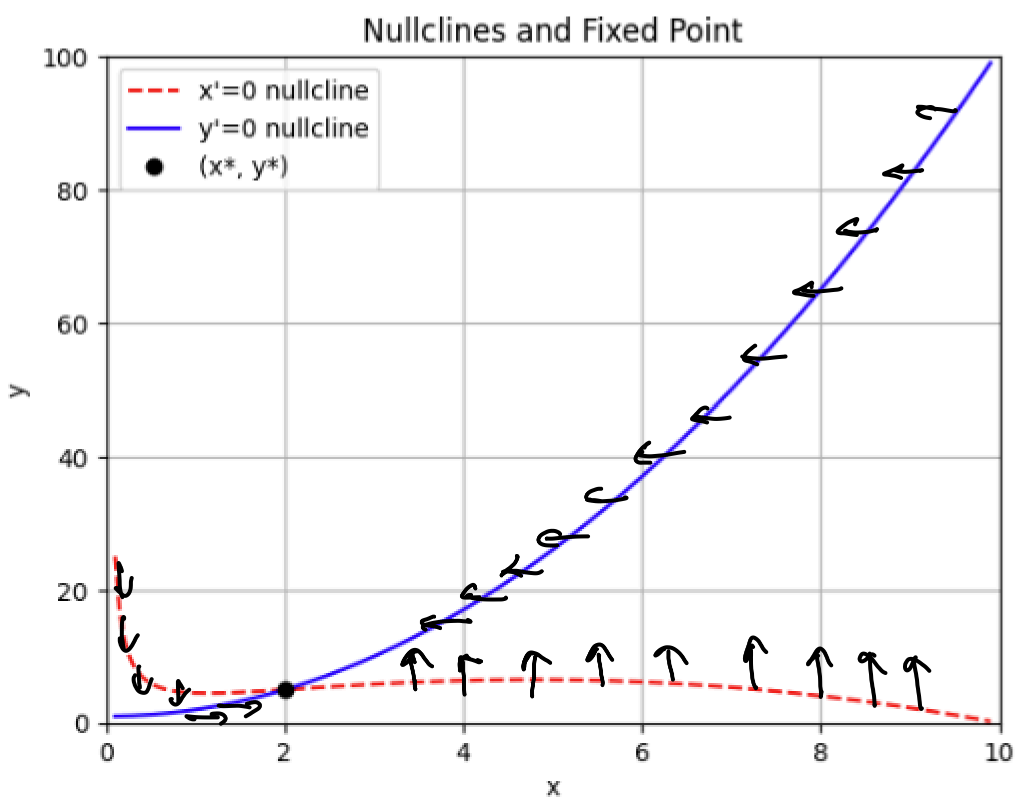
e) We get supercritical pitchfork bifurcation



Question 3a)



Question 3b)



3. Poincare - Bendixson Theorem

$$\dot{x} = a - x - \frac{4xy}{1+x^2} \qquad a = 10,\ b = 2$$

$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$$

$$0 = 10 - x - \frac{4xy}{1+x^2}$$

$$x - 10 = -\frac{4xy}{1+x^2}$$

$$(x-10)(1+x^2) = -4xy$$

$$y = \frac{(x-10)(1+x^2)}{-4x} = \frac{(1+x^2)(10-x)}{4x} \qquad x \neq 0$$

$$0 = 2x\left(1 - \frac{y}{1+x^2}\right)$$

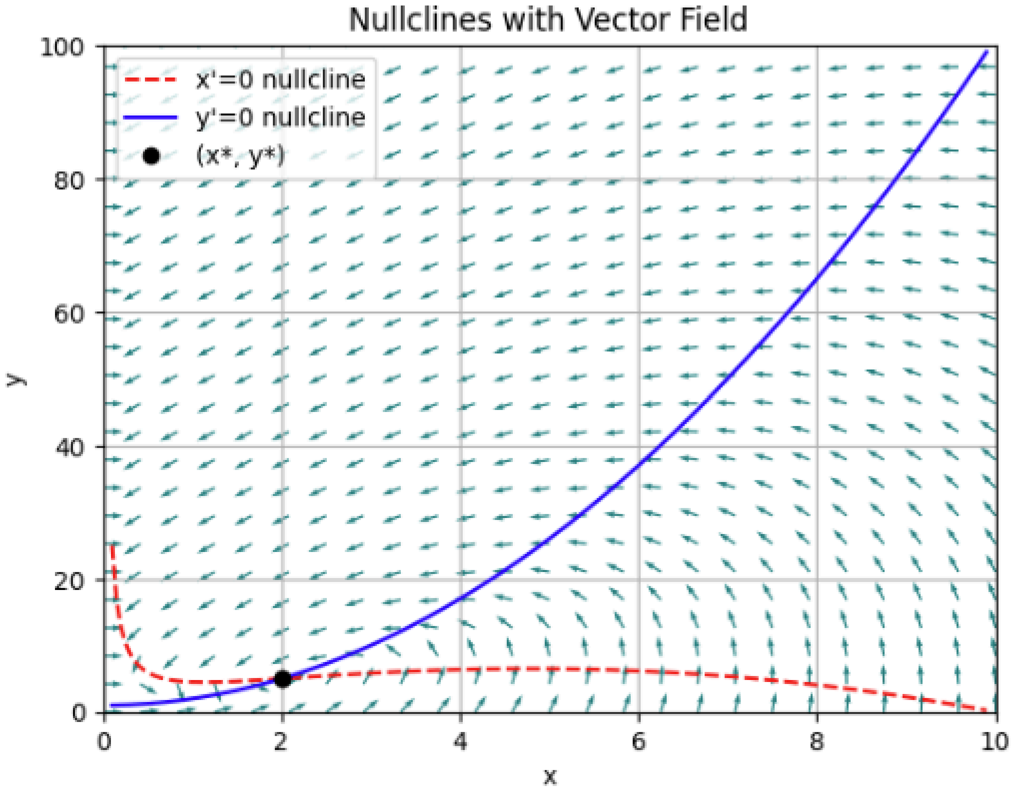
$2x = 0$ $\qquad$ $1 - \frac{y}{1+x^2} = 0$

$x = 0$ $\qquad$ $1 + x^2 = y$

(1) $\quad y = \frac{(1+x^2)(10-x)}{4x}, \quad y = 1+x^2$

Fixed point @ $(2, 5)$

Question 3c)



Nullclines with Vector Field
x'=0 nullcline
y'=0 nullcline
(x*, y*)
y
x
100
80
60
40
20
0
0
2
4
6
8
10

d) Using linearization, determine the fixed point that occurs at the point you determined in (a).

$f_1 = 10 - x - \frac{4x}{1+x^2} y$ $\qquad$ $f_2 = 2x - \frac{2x}{1+x^2} y$

$\frac{df_1}{dy} = \frac{-4x}{1+x^2}$ $\qquad$ $\frac{df_2}{dy} = \frac{-2x}{1+x^2}$ $\qquad$ $\frac{-8}{1+4} = -8/5$

$-4/5$

$\frac{df_1}{dx} = -1 + \frac{4(x^2-1)}{(x^2+1)^2} y$ $\qquad$ $\frac{df_2}{dx} = 2 + \frac{2(x^2-1)}{(x^2+1)^2} y$

$$\begin{bmatrix} df_1/dx & df_1/dy \\ df_2/dx & df_2/dy \end{bmatrix} \quad @ \ (2,5)$$

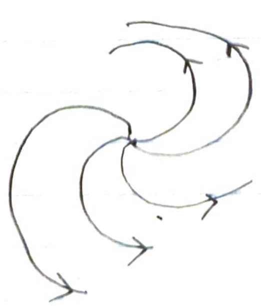
$$\begin{bmatrix} 1.4 & -8/5 \\ 3.2 & -4/5 \end{bmatrix} = \begin{bmatrix} 1.4 & -1.6 \\ 3.2 & -0.8 \end{bmatrix}$$

$\text{tr}A = 1.4 - 0.8 = 0.6$

$\det = (1.4)(-0.8) + (3.2 \times 1.6) = 4$

$\text{tr}A > 0$, $\det > 0$ $\qquad$ $4\det > \text{tr}A^2$

unstable spiral



e) The rectangle $0 < x < 10$, $0 < y < 101$ has inward pointing vectors on all boundaries. @ $x = 0$, $\dot{x} > 0$, @ $x = 10$: $\dot{x} < 10$, @ $y = 0$, $\dot{y} > 0$ and @ $y = 101$, $\dot{y} < 0$ indicating that the trajectories can't leave.

The bounded region contains one fixed point that is unstable, with trajectories that do not leave indicating that there must be at least one limit cycle in the region.

The region D is where $0 < x < 10$ & $0 < y < 101$

Consider the dynamical system

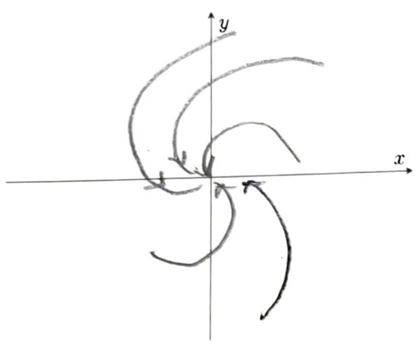
$$\dot{r} = \mu r - r^3 \tag{1a}$$

$$\dot{\theta} = \omega + br^2 \tag{1b}$$

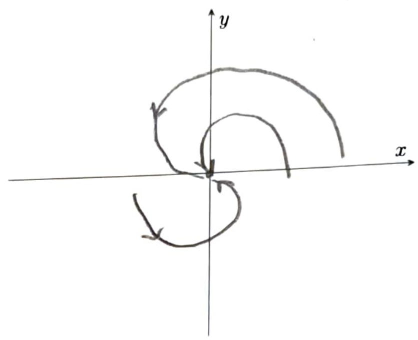
An interactive view of the phase portrait for this system is shown at https://tinyurl.com/E91supercriticalhopf This interactive version performs the necessary coordinate transformation from r, θ to x, y coordinates.

- Identify a bifurcation, and sketch the phase portrait before, during, and after the bifurcation.

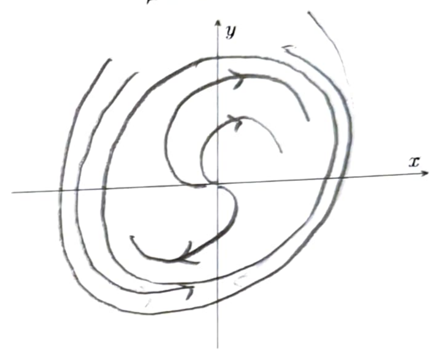
μ below critical



μ critical

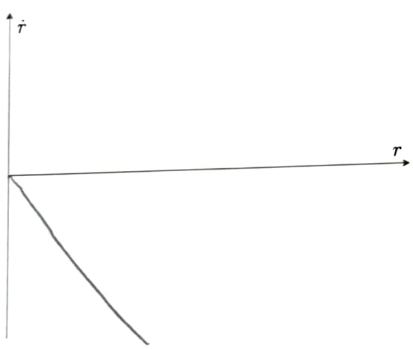


μ above critical

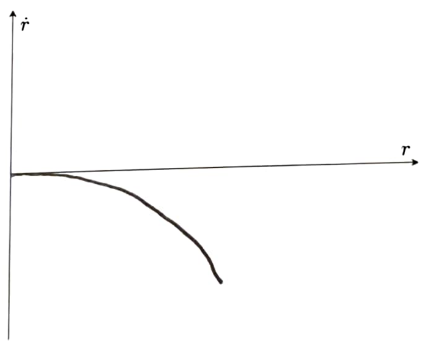


- Also plot $\dot{r}$ as a function of r for each of the three cases above.

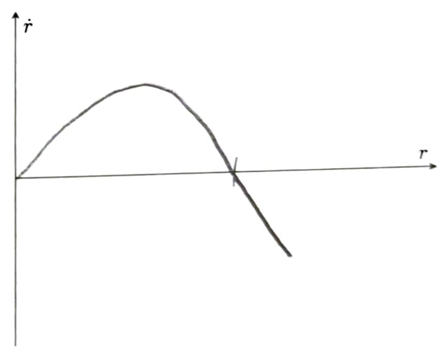
μ below critical



μ critical



μ above critical



- What is the radius of the limit cycle that is formed after the bifurcation?

$r = \sqrt{\mu}$

- Is the limit cycle stable or unstable? Re-write (1) so that the Hopf bifurcation leads to a limit cycle with the opposite stability. The limit cycle is stable

Consider the dynamical system

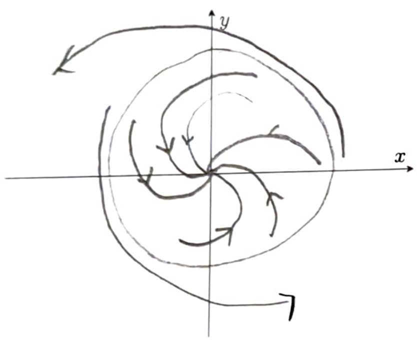
$$\dot{r} = \mu r + r^3 \tag{2a}$$

$$\dot{\theta} = \omega + br^2 \tag{2b}$$

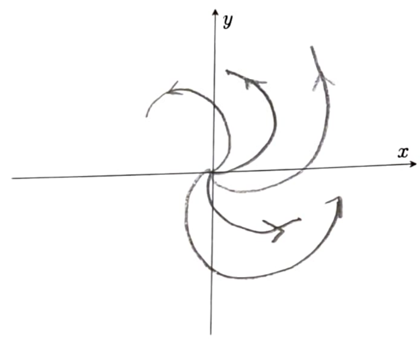
An interactive view of the phase portrait for this system is shown at https://tinyurl.com/E91subcriticalhopf. This interactive version performs the necessary coordinate transformation from r, θ to x, y coordinates.

- Identify a bifurcation, and sketch the phase portrait before, during, and after the bifurcation.

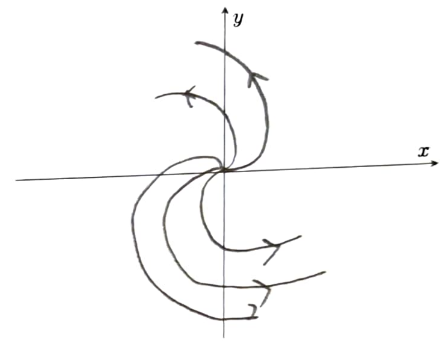
μ below critical



μ critical

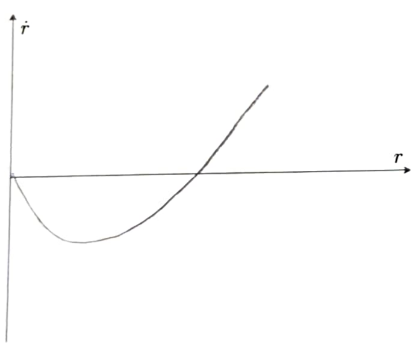


μ above critical

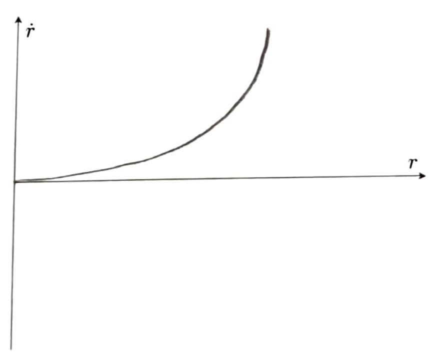


- Also plot $\dot{r}$ as a function of r for each of the three cases above.

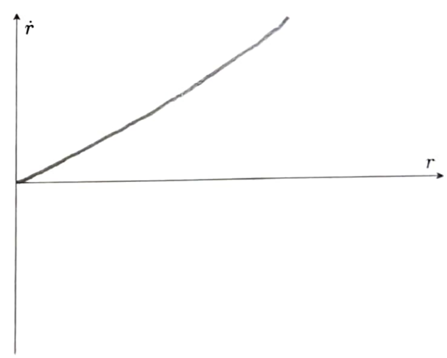
μ below critical



μ critical



μ above critical



- What is the radius of the limit cycle that is formed after the bifurcation?

$r = \sqrt{-\mu} = \sqrt{|\mu|}$

- Is the limit cycle stable or unstable?

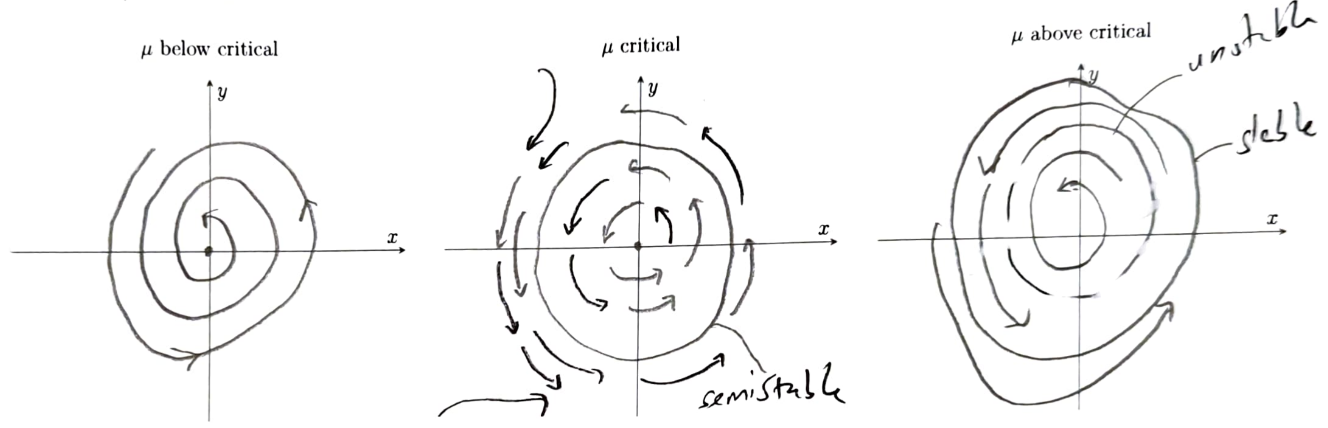
Limit cycle is unstable

Consider the dynamical system

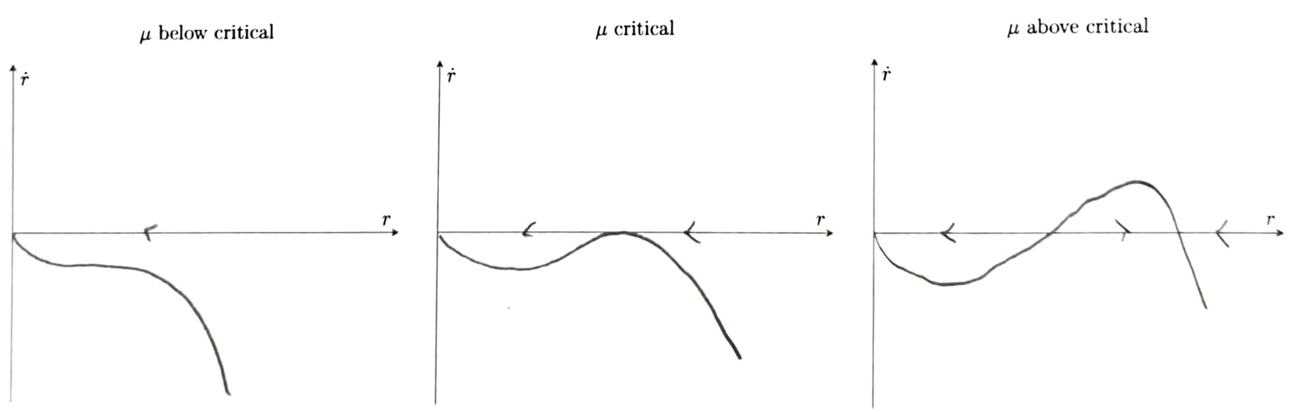
$$\dot{r} = \mu r + r^3 - r^5 \tag{3a}$$
$$\dot{\theta} = \omega + br^2 \tag{3b}$$

An interactive view of the phase portrait for this system is shown at https://tinyurl.com/E91subcriticalhigherorderhopf. This interactive version performs the necessary coordinate transformation from r, θ to x, y coordinates.

- Identify a bifurcation near $\mu = -0.25$, and sketch the phase portrait before, during, and after the bifurcation. Note that there is more than one bifurcation in this system; we are only concerned with the one that occurs near $\mu = -0.25$.



- Also plot $\dot{r}$ as a function of r for each of the three cases above.



- What is the radius of the limit cycles that are formed after the bifurcation?

$r(\mu + r^2 - r^4) = 0$ $\quad r^4 - r^2 = \mu$ → solve for r gives radius $\left(r = \pm\sqrt{\frac{1 \pm \sqrt{1+4a}}{2}} \right)$

- Are the limit cycles stable or unstable?

The limit cycles for $\mu > \mu_c$ split into an unstable and stable one

Consider the dynamical system

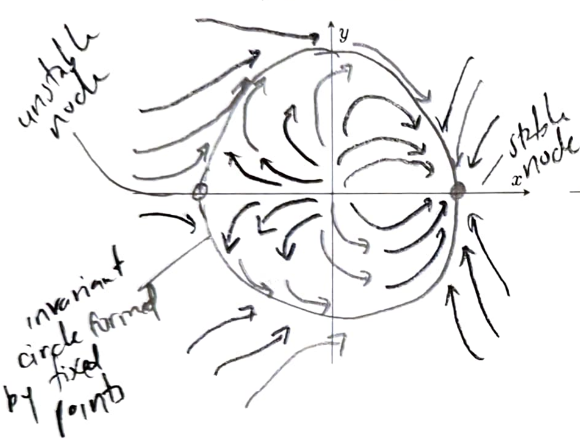
$$\dot{r} = r(1 - r^2) \tag{4a}$$

$$\dot{\theta} = \mu - \sin\theta \tag{4b}$$

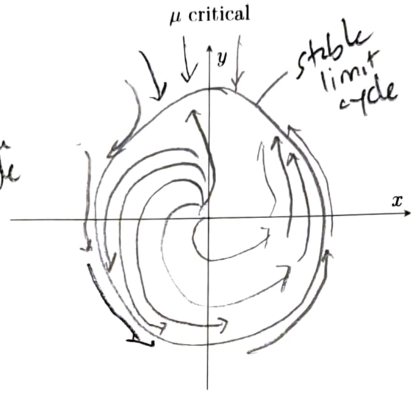
An interactive view of the phase portrait for this system is shown at `https://tinyurl.com/E91infiniteperiodhopf` This interactive version performs the necessary coordinate transformation from r, θ to x, y coordinates.

- ✍ Identify a bifurcation and sketch the phase portrait before, during, and after the bifurcation. It is recommneded that you use values of μ that are relatively close to the critical value.

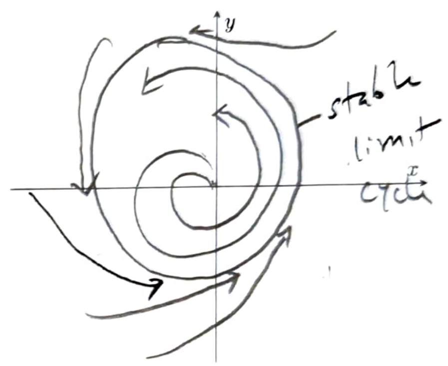
μ below critical



μ critical



μ above critical



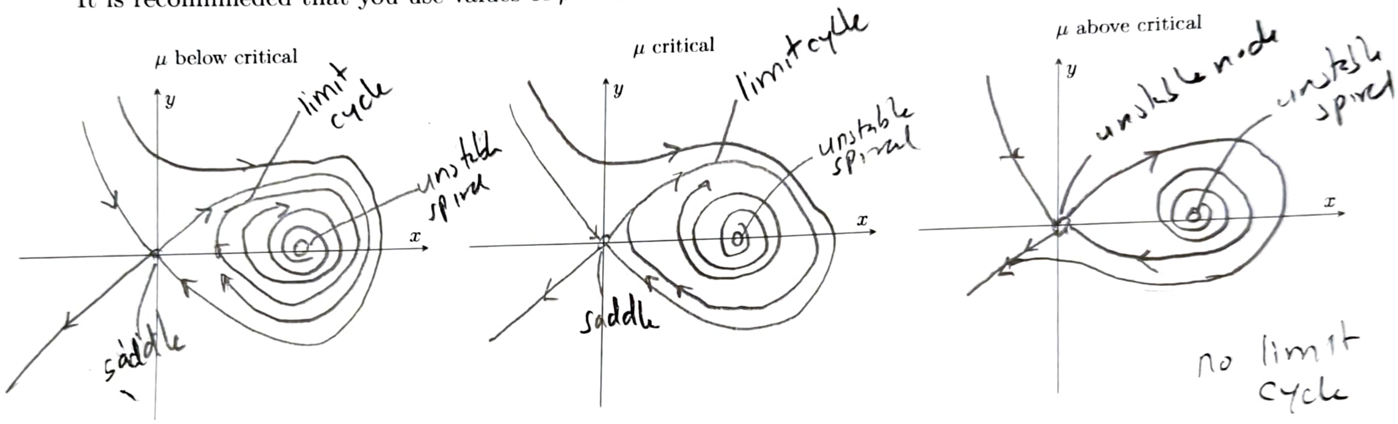
- ✍ Label the fixed points and limit cycles, and visually determine the stability of each.

Consider the dynamical system

$$\dot{x} = y \tag{5a}$$
$$\dot{y} = \mu y + x - x^2 + xy \tag{5b}$$

An interactive view of the phase portrait for this system is shown at https://tinyurl.com/E91homoclinicbifurcation. You are also encouraged to plot this system on pplane.

- Identify a bifurcation near $\mu \approx -0.8645$, and sketch the phase portrait before, during, and after the bifurcation. It is recommneded that you use values of μ that are relatively close to the critical value.



- Label the fixed points and limit cycles, and classify them visually.