Sobolev maps between manifolds

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Let M, N be riemannians manifolds, $m = \dim M$, consider N embedded in $\mathbf{R}^\ell$ (by Nash-Moser theorem); for

$$u \,:\, M \to N \subset \mathbf{R}^\ell$$

we consider the functional of p-energy

$$E_p(u) := \int\limits_M |Du|^p \qquad p \geq 1.$$

In order to study such functional, let's introduce the Sobolev space

$$W^{1,p}(M,N) := \{u : M \to \mathbf{R}^\ell \mid u \in W^{1,p}(M, \mathbf{R}^\ell),\ u(x) \in N \text{ a.e.}\}\,.$$

We will consider the following problems, treated in [1], [2]:

1$^{\text{st}}$) is $C^\infty(M,N)$ dense in $W^{1,p}(M,N)$ w.r.t. the strong topology of $W^{1,p}$?

2$^{\text{nd}}$) if not, what is a "good" dense set?

3$^{\text{rd}}$) what are the properties of $\overline{C^\infty(M,N)}^{W^{1,p}(M,N)}$?

4$^{\text{th}}$) what about the weak topology?

5$^{\text{th}}$) assume $\partial M \neq \emptyset$ and let $g : \partial M \to N$ belong to the trace space

$$W^{1-1/p,p}(\partial M, N) = \{g \in W^{1-1/p,p}(\partial M, \mathbf{R}^\ell) \mid g(x) \in N \text{ a.e.}\}\,,$$

and try to minimize $\int_M |Du|^p$ on

$$W^{1,p}_g(M,N) = \{u \in W^{1,p}(M,N) \mid u = g \text{ on } \partial M\}\,;$$

when do we have $W^{1,p}_g(M,N) \neq \emptyset$ for any $g \in W^{1-1/p,p}(\partial M, N)$?

Remark 1 The last question is almost completely open, whether the 3^{rd} one is partially open.

Let's now consider the problem of density

1^{st} case: $p > \dim M = n$.

Theorem 1 *If $p > m$, then $C^\infty(M, N)$ is dense in $W^{1,p}(M, N)$ w.r.t. the strong topology.*

Proof. Assume that $M = B^m$. Let $\varphi : \mathbf{R}^m \to [0, 1]$ be a mollifier:

$$\varphi \in C_c^\infty(\mathbf{R}^m), \qquad \int_{\mathbf{R}^m} \varphi(x)\, dx = 1\,.$$

Let

$$\varphi_n(x) = \frac{1}{n^m}\varphi\left(\frac{x}{n}\right), \qquad x \in \mathbf{R}^m\,.$$

For $u \in W^{1,p}(M, N)$, consider the convolution $u_h := u * \varphi_h$; by standard theory we have $u_h \in C^\infty(M, \mathbf{R}^\ell)$, moreover $u_h \to u$ in $W^{1,p}(M, \mathbf{R}^\ell)$.

Thus, by Sobolev embedding theorem ($p > n$)

$$u_n \to u \qquad \text{in } C^0(M, \mathbf{R}^\ell),$$

and so

$$\forall\, \varepsilon_0 > 0\,, \quad \operatorname{dist}(u_h(x), N) \le |u_h(x) - u(x)| \le \varepsilon_0 \quad \text{if } h \le \overline{h},\ \forall\, x \in M\,.$$

Consider the tubular nbd. of N

$$\theta_\varepsilon = \{y \ : \ \operatorname{dist}(y, N) \le \varepsilon\}$$

and the nearest point projection $\pi : \theta_\varepsilon \to N$, which is well defined and smooth for sufficiently small ε if N is smooth.

Denote by $\widetilde{u}_h := \pi \circ u_h$; then we have

$$\widetilde{u}_h \in C^\infty(M, N) \quad \text{and} \quad \widetilde{u}_h \to u \text{ in } W^{1,p}(M, N).$$

For a general manifold M, use local charts and a partition of unity argument. ∎

2^{nd} case: $p = m$

Theorem 2 ([4]) *If $p = m$, then $C^\infty(M,N)$ is dense in $W^{1,p}(M,N)$.*

Proof. The proof is analogous to the previous one, except for the use of the Sobolev theorem; but in this case we use a Poincaré inequality to prove that $\operatorname{dist}(u_h(x), N) \overset{h \to 0}{\to} 0$ uniformly in x.

In fact,

$$(1) \qquad \frac{1}{h^m} \int_{B(x,h)} |u(y) - u_h(x)|^m \, dy \le c \int_{B(x,h)} |Du|^m \, dy \quad \forall\, x \in M\,,$$

then

$$\begin{aligned} \operatorname{dist}(u_h(x), N)^m &= \fint_{B(x,h)} \operatorname{dist}(u_h(x), N)^m \, dy \le \\ &\le \fint_{B(x,h)} |u_h(x) - u(y)|^m \, dy \xrightarrow{h \to 0} 0 \end{aligned}$$

because of (1) and the absolute continuity of the integral. ∎

3^{rd} case: $p < m = \dim M$.

Theorem 3 *If $p < m = \dim M$ then*

$$C^\infty(M,N) \text{ is dense in } W^{1,p}(M,N) \;\Leftrightarrow\; \pi_{[p]}(N) = 0\,,$$

where $[p]$ = greatest integer less or equal to p.

Remark 2 $\pi_k(N) \neq 0,\ \ k \in \mathbf{N} \;\Leftrightarrow\; \exists\, g : S^k \to N$ continuous (or smooth) such that g is *not* homotopic to a constant map.

Proof. $\Longrightarrow$
Suppose $\pi_{[p]}(N) \neq 0$, then $\exists\, g : S^{[p]} \to N$ g not homotopic to a constant map, and we can take $g \in C^1$.

Assume first that $m - 1 < p < m$ and $M = B^m$. The function $f(x) = g\left(\dfrac{x}{|x|}\right)$ belongs to $W^{1,p}(M,N)$, but f cannot be approximated by smooth maps . Moreover, $f \in C^\infty(M \setminus \{0\}, N)$ (one point singularity).

Assume by contradiction that there exists $f_n \in C^\infty(M,N)$, such that $f_n \to f$ in $W^{1,p}(M,N)$; by Fubini's theorem, we may choose some $r_0 \in [0,1]$, such that

$$f_{n|S^{m-1}_{r_0}} \longrightarrow f_{|S^{m-1}_{r_0}} \qquad \text{in } W^{1,p}\,,$$

where $S^{m-1}_{r_0} = \partial B^m_{r_0}(0)$.

Now, by Sobolev theorem we have

$$f_{n|S^{m-1}_{r_0}} \longrightarrow f_{|S^{m-1}_{r_0}} \qquad \text{in } C^0\,.$$

By definition $f_{|S^{m-1}_{r_0}}(x) = g\left(\frac{x}{|x|}\right)$, thus $f_{|S^{m-1}_{r_0}}$ is not homotopic to a constant; on the other hand, $f_{n|S^{m-1}_{r_0}}$ is homotopic to a constant: actually f_n is defined on the whole ball $B^m = M$, so we have the homotopy

$$\begin{aligned} F:\, S^{m-1}_{r_0} \times [0,1] &\longrightarrow N \\ F(x,r) &= f_n\left(\frac{rx}{r_0}\right); \end{aligned}$$

but homotopy-type is preserved under uniform convergence, and this leads to a contradiction.

In case $p = m-1$ we use the same argument, thanks to B. White's theorem, (see [5]), which says that homotopy classes are preserved by $W^{1,p}$ convergence, when $p =$ dimension of the domain (in our case, S^{m-1}).

Consider the case $p < m-1$. Assume for example, $m-2 \le p < m-1$. Suppose $M = B^{m-1} \times [0,1]$, and let $f : B^{m-1} \to N$ be constructed as in the previous case ($m-1 \le p < m$); define

$$\tilde{f}:\, B^{m-1} \times [0,1] \longrightarrow N \qquad \text{by} \qquad \tilde{f}(z,x) = f(z)\,.$$

Note that the singular set of $\tilde{f}$ is the 1-dimensional set $\{0\} \times [0,1]$, and $\tilde{f} \in W^{1,p}(B^{m-1} \times [0,1], N)$.

If $\tilde{f}_h$ is a sequence of smooth maps which approximates $\tilde{f}$ in $W^{1,p}$, then by Fubini's theorem, for some $r_0 \in [0,1]$, $\widetilde{f_{h|B^{m-1}\times\{r_0\}}} \longrightarrow \tilde{f}_{|B^{m-1}\times\{r_0\}}$ in $W^{1,p}$, then the contradiction follows as previously.

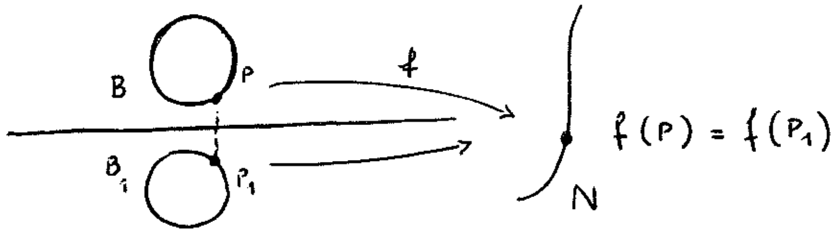
Similarly we treat the other cases, considering $M = B^{[p]+1} \times S^{m-[p]-1}$ anyway, notice that the singular set of the constructed function has dimension $m - [p] - 1$.

For a general manifold M, work locally as above, then extend the function f as follows:

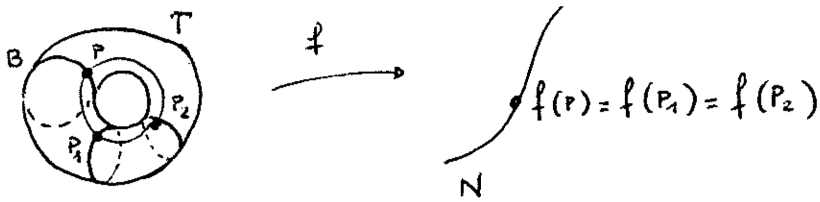
Case $m-1 \leq p < m$, $f : B^m \subset M \to N$: reflect f on a second ball $B_1^m \subset M$, then the extension of

$$f : B^m \cup B_1^m \to N$$

comes from standard theory.



Case $m-2 \leq p < m-1$, f can be extended in a natural way on a torus $T \subset M$ containing B^{m-1}, and the extension on the whole manifold comes again from standard arguments.



Similarly treat the other cases.

$\Longleftarrow$: the "if" part of the theorem will be proved in the case $m-1 < p < m$ and will be a consequence of the following two statements.

Proposition 1 *Consider the set*

$$R_0 = \{v \in W^{1,p}(M,N) \mid v \in C^0 \textit{ except at a finite number of points}\};$$

then R_0 is dense in $W^{1,p}(M,N)$ for $m-1 < p < m$

Remark 3 Here we *do not* assume $\pi_{m-1}(N) = 0$ ($[p] = m-1$).

Proposition 2 *Let $v \in R_0$, then v can be approximated by smooth maps $\Leftrightarrow$ $\pi_{m-1}(N) = 0$.*

Proof of Proposition 1. Assume $M = [0,1]^m$; in the general case we will consider a triangulation of the manifold made by such cubes (... a "cube-ulation" ...).

Thanks to an argument of B. White (see [5]), we may assume that any $u \in W^{1,p}(M,N)$ is such that $u_{|\partial C_i^{(m)}}$ is continuous, with $C_i^{(m)}$ a cube of the cube-ulation we will consider.

In fact, if U is a tubular nbd. of M, and $\pi : U \to M$ is the nearest point projection, then $\tilde{u} : U \to N$, $\tilde{u} := u \circ \pi$ is such that

$$\int_U |D\tilde{u}|^p \leq c \int_M |Du|^p .$$

If $X = \bigcup_i \partial C_i^{(m)}$ is the $(m-1)$-skeleton of a cube-ulation of M, and $h_v : X \to U$, $x \mapsto x + v$, for $|v| \leq r$ (r sufficiently small), we have

$$\int_{|v|\leq r} dv \int_X |D\tilde{u} \circ h_V|^p \, dx = \int_X dx \int_{|v|\leq r} |D\tilde{u} \circ h_V|^p \, dv \leq \int_X dx \int_U |D\tilde{u}(y)|^p \, dy \leq$$

$$\leq |X| \int_U |D\tilde{u}|^p \leq C \int_M |Du|^p < +\infty .$$

Hence, by Fubini's theorem we may choose v such that $\tilde{u} \circ h_V \in W^{1,p}(X,N)$; but $\dim X < p$, and by Sobolev theorem we conclude $\tilde{u} \circ h_V \in C^0(X,N)$, and finally $\tilde{u} \in C^0(X + v, N)$.

By definition, $\tilde{u}|_{X+v} = u \circ \pi|_{X+v}$, π is smooth, so $u|_{\pi(X+v)}$ is continuous, $\pi(X+v)$ gives the $(m-1)$-skeleton of the desired cube-ulation.

As a consequence, we may suppose $u|_{\partial M} \in C^0$ where $M = [0,1]^m$.

Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbf{R}^m$; denote by $P_{j,a}$ the hyperplane orthogonal to the vector e_j passing through $A \equiv a e_j$.

For $a \in [0, 1/n]^m$, consider the set

$$W_{j,a_j} = \bigcup_{k=1}^{n} P_{j,a_j+(k-1)/m}; \;\; a = (a_j)_{j=1,\ldots,m}$$

then

$$\partial M \cup \bigcup_{j=1}^{m} W_{j,a_j} = \bigcup_{r=1} \partial C_r$$

with $\cup C_r$ a grid of mesh $1/n$.

Let $h = 1/n$. By Fubini's theorem, we have

$$E(u) = \int_M |Du|^p = \int_0^h da_j \int_{W_{j,a_j}} |Du|^p$$

$\forall\, j = 1, \ldots, m$; hence we may choose $a = (a_j)_{j=1\ldots,m}$ such that

$$\int_{W_{j,a_j}} |Du|^p \le \frac{1}{h} E(u) = nE(u);$$

this gives

$$\sum_r \int_{\partial C_r} |Du|^p \le \overline{C} n E(u) + \tilde{C} \int_{\partial M} |Du|^p \le CnE(u).$$

Denote by C a copy of $[0,1]^m$, then for every little cube C_r we define the scaled energy

$$\tilde{E}(u, C_r) = E(\tilde{u}_{n,r}, C) = \int_C |D\tilde{u}_{n,r}|^p,$$

where $\tilde{u}_{n,r} : C \to N$ is given by $\tilde{u}_{n,r}(x) = u(\frac{x}{n} + x_r)$ (where x_r is the barycenter of C_r), for the cubes which are not in contact with the boundary, and in a similar way for the cubes which are in contact with the boundary ∂M.

We also set $\tilde{E}(u, \partial C_r) = E(\tilde{u}_{n,r}, \partial C)$. We have the following scaling equalities:

$$\begin{aligned} \tilde{E}(u, C_r) &= n^{m-p} E(u, C_r) \\ \tilde{E}(u, \partial C_r) &= n^{m-p-1} E(u, \partial C_r). \end{aligned}$$

Applying Sobolev embedding theorem to the function $\tilde{u}_{n,r|\partial C}$ we get the following result

Lemma 1 *For any $x, y \in \partial C_r$*

$$\| u(x) - u(y) \|^p \le c\tilde{E}(u, \partial C_r).$$

Next we are going to divide the cubes in "good" ones and "bad" ones. Fix $0 < \nu < p$, $\varepsilon > 0$, then C_r is a good cube if

$$\tilde{E}(u, \partial C_r) \leq \varepsilon \tag{2}$$
$$\tilde{E}(u, C_r) \leq n^{-\nu} . \tag{3}$$

As for the bad cubes (the remaining ones), the following fact holds:

Lemma 2 *If $P_n = \cup$ bad cubes, then $|P_n| \to 0$ as $n \to +\infty$.*

In fact by a counting argument, if $P_{1,n}$ are the cubes violating (2), and $P_{2,n}$ the cubes violating (3), then

$$CnE(u) \geq \sum_{C_r \in P_{1,n}} E(u, \partial C_r) \geq N_1 \varepsilon h^{m-p-1} ,$$

where N_1 is the number of cubes in $P_{1,n}$, hence

$$|P_{1,n}| \leq N_1 h^m \leq \frac{CE(u)}{\varepsilon} h^p \longrightarrow 0 \text{ as } n \to +\infty \ (h = 1/n) .$$

Moreover, if N_2 is the number of cubes in $P_{2,n}$,

$$E(u) \geq \sum_{C_r \in P_{2,n}} E(u, C_r) = h^{m-p} \sum_{C_r \in P_{2,n}} \tilde{E}(u, C_r) > N_2 \varepsilon h^{m-p+\nu} ,$$

hence

$$|P_{2,n}| \leq N_2 h^m \leq E(u) h^{p-\nu} \longrightarrow 0 \text{ per } n \to +\infty .$$

Let's construct the approximating sequence $\{u_n\} \subset R_0$ in the following way:

$$u_n = u \quad \text{on} \quad \partial C_r \quad \text{for any cube} \quad C_r .$$

If C_r is a good cube, and $\tilde{u}_n$ is a minimizer of

$$\int_{C_r} |Dv|^p , \quad v = u \text{ on } \partial C_r , v : C_r \to \mathbf{R}^\ell ,$$

$\tilde{u}_n \in C^0(C_r, \mathbf{R}^\ell)$. We have, for $x \in C_r$, $x_0 \in \partial C_r$

$$\begin{aligned} \operatorname{dist}(\tilde{u}_n(x), N) &\leq |\tilde{u}_n(x) - u(x_0)| \leq && \text{by maximum principle} \\ &\leq \sup_{z,y \in \partial C_r} |u(z) - u(y)| \leq && \text{by lemma 1} \\ &\leq C\tilde{E}(u, \partial C_r)^{1/r} \leq C\varepsilon^{1/p} && \text{by (2).} \end{aligned}$$

Hence $u_n = \pi \circ \tilde{u}_n : C_r \to N$ is well defined and continuous ($\pi : U \to N$ is the nearest point projection from a tubular nbd. U of N).

Let's prove that, up to a subsequence,

$$\int_{Q_n} |Du_n - Du|^p \longrightarrow 0 \qquad n \to +\infty$$

where $Q_n = \bigcup_{C_r \text{ good}} C_r$.

We have

$$\int_{Q_n} |D\tilde{u}_n|^p \leq \int_{Q_n} |Du|^p \tag{4}$$

because $\tilde{u}_n$ is minimizer; moreover

$$\| \tilde{u}_n - u \|_{L^p(Q_m;\mathbf{R}^\ell)} \longrightarrow 0 \qquad \text{as } n \to +\infty\,,$$

by Poincaré inequality, in fact

$$\frac{1}{h^{m-p}} \int_{C_r} |\tilde{u}_n - u|^p \leq C \int_{C_r} |D\tilde{u} - Du|^p \leq C \int_{C_r} |Du|^p\,,$$

hence

$$\sum_{\substack{\text{good}\\ \text{cubes}}} \int_{C_r} |\tilde{u}_n - u|^p \leq h^{m-p} E(u) \longrightarrow 0 \qquad \text{as } n \to +\infty \ (h = 1/n)\,;$$

then $\tilde{u}_n \rightharpoonup u$ weakly in $W^{1,p}(Q_m;\mathbf{R}^\ell)$ and by (4) $\tilde{u}_n \to u$ strongly in $W^{1,p}(Q_m;\mathbf{R}^\ell)$ $(p > 1)$, and finally $u_n \to u$ in $W^{1,p}(Q_m; N)$.

If C_r is a bad cube, define u_n on C_r so that

$$\int_{C_r} |Du_n|^p \leq C \int_{C_r} |Du|^p\,, \tag{5}$$

then by lemma 2 and Dominated Convergence Theorem,

$$\int_{P_n} |Du|^p \longrightarrow 0 \qquad \text{as } n \to +\infty\,,$$

hence $\| u_n - u \|_{W^{1,p}(P_n)} \to 0$.

Consider in fact $u_n = u$ in ∂C_r and $u_n|_{\overset{\circ}{C_r}} = u_{\partial C_r}\left(\dfrac{x - x_r}{\| x - x_r \| h}\right)$, being x_r the center of C_r, where

$$\| x \| := \sup\{|x_i| \mid x = (x_1, \ldots, x_m)\}\,.$$

Remark 4 $u_n|_{C_r}$ has one point singularity in the center x_r of any bad cube C_r, therefore $u_n \in R_0$.

Since u_n is "radial", we have

$$\int_{C_r} |Du_n|^p \leq Ch \int_{\partial C_r} |Du|^p$$

and

$$\begin{aligned} \int_{P_n} |Du_n|^p &\leq \sum_{\substack{\text{bad}\\ \text{cubes}}} \int_{C_r} |Du_n|^p \leq \sum_{\substack{\text{bad}\\ \text{cubes}}} Ch \int_{\partial C_r} |Du|^p \leq \\ &\leq cE(u) \overset{n\to\infty}{\not\longrightarrow} 0\,; \end{aligned}$$

We are therefore led to divide C_r into smaller cubes $c_{r'}$ of mesh $h' \ll h$ and define u_n on $c_{r'}$ as

$$\begin{aligned} u_n &= u \quad \text{on } \cup\, \partial c_{r'}\,, \\ u_n|_{\overset{\circ}{c}_{r'}} &= u|_{\partial c_{r'}}\left(\frac{x - x_{r'}}{\| x - x_{r'} \| h'}\right) \quad (x_{r'} \text{ center of } c_{r'}). \end{aligned}$$

By the previous argument

$$\int_{C_r} |Du_n|^p \leq ch' \sum_{c_{r'} \subset C_r} \int_{\partial c_{r'}} |Du_n|^p\,; \tag{6}$$

from an argument as before (slicing method and Fubini's theorem) we then choose the refined grid of mesh h' so that

$$h' \sum \int_{\partial c_{r'} \setminus \partial C_r} |Du|^p \leq \int_{C_r} |Du|^p\,.$$

Then follows by (6)

$$\begin{aligned} \int_{C_r} |Du_n|^p &\leq h'C \sum \int_{\partial c_{r'} \setminus \partial C_r} |Du_n|^p + h'C \int_{\partial C_r} |Du_n|^p \leq \\ &\leq C \int_{C_r} |Du|^p + h'C \int_{\partial C_r} |Du|^p\,, \end{aligned}$$

and for h' sufficiently small we get (5), and the proof of Proposition 1 is complete.

■

Proof of Proposition 2. Let $m-1<p<m$, let $u \in R_0$ (constructed as in Proposition 1, with radial behaviour around each singularity), and $\pi_{m-1}(N)=0$.

Construct the approximate sequence as follows: if a is a singular point of u (assume $a=0$) and $u(x)=g\left(\frac{x}{\| x \|}\right)$ on $[-h,h]^m$, with $g : \partial([-1,1]^m) \to N$ and $g \in C^0$, by the hypothesis $\pi_{m-1}(N)=0$ we can extend g to a function $\tilde{g} \in C^0 \cap W^{1,p}([-1,1]^m;N)$.

Define $u_n \in C^0 \cap W^{1,p}(M,N)$ such that $u_n \to u$ in $W^{1,p}(M,N)$ as follows: for $h=\frac{1}{n} \leq h_0$

$$u_h(x)=\begin{cases} u(x) & \text{outside } [-h,h]^m \\ \tilde{g}\left(\frac{x}{h}\right) & \text{in } [-h,h]^m\,; \end{cases}$$

and repeat for each singular point; then

$$\begin{aligned} \int_M |Du_n - Du|^p &= (\#\text{ singular points of } u) \int_{[-h,h]^m} |Du_n - Du|^p \leq \\ &\leq Ch^{m-p} \longrightarrow 0 \qquad \text{as } n \to +\infty\,. \end{aligned}$$

■

Remark 5 In the proof of Proposition 2 it is sufficient to assume that the homotopy class of g is trivial; notice that we define the homotopy class of u at a singular point a (if isolated) as the homotopy class of $u|_{\partial B_r(a)}$, r small.

Now we give an alternative proof of Proposition 1, which makes use of regularity theory: for $u \in W^{1,p}(M,N)$ define the functional

$$F_h(v)=\int |Dv|^p + \frac{|u-v|}{h}\,, \qquad h \text{ small parameter,}$$

and let v_h be the minimizer of F_h on $W^{1,p}(M,N)$. Then by regularity theory, $v_h \in R_0$ (see Schoen-Uhlenbeck ($p=2$) [4], Hardt-Lin ($p>2$) [3], Luckhaus, Fuchs-Hutchinson, Dusaar, ...).

Let's prove that $v_h \to u$ in $W^{1,p}(M,N)$. We have

$$F_h(v_h) \leq F_h(u) = \int |Du|^p\,;$$

in particular,

$$\int |Dv_n|^p \leq \int |Du|^p \tag{7}$$

and

$$\int |u - v_n|^p \leq h \int |Du|^p \longrightarrow 0 \qquad \text{as } h \to 0$$

then, passing to a subsequence,

$$v_h \rightharpoonup u \quad \text{weakly in } W^{1,p}$$

and by (7)

$$v_h \to u \quad \text{strongly in } W^{1,p} \quad (p > 1).$$

∎

Remark 6 The proof of Proposition 1 in the case $p \leq m-1$ considers the d-skeleton of a grid ($d = [p]$ if $p \notin \mathbf{Z}$, $d = p-1$ if $p \in \mathbf{Z}$) and the dipole method (see Brézis, Coron, Lieb : Harmonic maps with defects, Comm. Math. Phys. , **107** 649-705, 1986) to remove singularities.

Applications

Define

$$\begin{aligned} W^{1,p}_{\text{strong}}(M,N) &= \overline{C^\infty(M,N)}^{W^{1,p}\text{-strong}}, \\ W^{1,p}_{\text{weak}}(M,N) &= \overline{C^\infty(M,N)}^{W^{1,p}\text{-weak}}, \end{aligned}$$

then we have

$$W^{1,p}_{\text{strong}} \subset W^{1,p}_{\text{weak}} \subset W^{1,p}.$$

If $\pi_{[p]}(N) \neq 0$ the second inclusion is strict.

We now prove that if $p \notin \mathbf{Z}$, $W^{1,p}_{\text{strong}}$ is stable under sequentially weak convergence, and therefore $W^{1,p}_{\text{strong}} = W^{1,p}_{\text{weak}}$, (even in the case $\pi_{[p]}(N) \neq 0$).

Proof. Let $u_h \in C^\infty(M,N)$, $u_h \rightharpoonup u$ in $W^{1,p}$ weak, $m-1 < p < m$.

We only have to check that u can be strongly approximated by smooth maps. Making use of the decomposition into cubes as in Proposition 1, we can approximate u strongly in $W^{1,p}$ by $v_h \in R_0$; since u is a weak limit of smooth maps, the homotopy class of v_h at any singular point is trivial (p is not integer); then by the remark of page 11, Proposition 2 holds for v_h, then by a diagonal argument we're done. ∎

As another application of the main theorem, we have:

Theorem 4 *If $\pi_{[p]}(N) \neq 0$, $p \notin \mathbf{N}$, then there are infinitely many p-harmonic maps in $W^{1,p}(M,N)$.*

Proof. Consider $\{a_1, \dots, a_k\} \subset M$, and the set of the maps in R_0 having $\{a_1, \dots, a_k\}$ as singular set, with prescribed homotopy-type around the singularities. Take the strong closure in $W^{1,p}$, and then minimize the p-Energy. For each choice of the singular set and of the homotopic behaviour around the singularities you will get a different minimizer (see remark on page 11) ■

The extension problem for Sobolev mappings [2]

Let M, N be riemannian manifolds, $n = \dim M$, $N \subset \mathbf{R}^\ell$ isometrically embedded by the Nash theorem.

Assume $\partial M \neq \emptyset$, and consider the map

$$g : \partial M \to N .$$

Our aim is to extend g to a map $u : M \to N$, with $u \in W^{1,p}(M,N)$ for some fixed p.

Then a natural assumption for g is

$$g \in W^{1-1/p,p}(\partial M, N) := \{h \in W^{1-1/p,p}(\partial M, \mathbf{R}^\ell),\ h(x) \in N \text{ a.e.}\},$$

where $W^{1-1/p,p}(M, \mathbf{R}^\ell)$ is the space of traces of functions in $W^{1,p}(M, \mathbf{R}^\ell)$.

For $g \in W^{1-1/p,p}(\partial M; \mathbf{R}^\ell)$ we have

Theorem 5 (Trace theorem) *Given $g \in W^{1-1/p,p}(\partial M; \mathbf{R}^\ell)$ there exists $u \in W^{1,p}(M; \mathbf{R}^\ell)$ such that $u = g$ on ∂M, and*

$$\| u \|_{W^{1,p}(M;\mathbf{R}^\ell)} \leq C \| g \|_{W^{1-1/p,p}(\partial M;\mathbf{R}^\ell)},$$

where

$$\begin{aligned} \| g \|^p_{W^{1-1/p,p}(\partial M;\mathbf{R}^\ell)} &:= \| g \|_{L^p(\partial M,\mathbf{R}^\ell)} \\ &+ \int_{\partial M}\int_{\partial M} \frac{|g(x) - g(y)|^p}{|x-y|^{p+m-2}}\, d\mathcal{H}^{m-1}(x)\, d\mathcal{H}^{n-1}(y) . \end{aligned}$$

Let's denote by

$$T^p(\partial M, N) := \{\, g \in W^{1-1/p,p}(\partial M; N) \mid g = u|_{\partial M}, \\ \text{for some } u \in W^{1,p}(M, N)\} .$$

The question is: when do we have

$$T^p(\partial M, N) = W^{1-1/p,p}(\partial M, N) \qquad ?$$

Let's take a look on the cases in which this question has been answered.

First consider the following topological property $\mathcal{P}(M, N)$:

Definition 1 *We say $\mathcal{P}(M, N)$ holds if for any continuous map $g : \partial M \to N$ there exists a continuous map $u : M \to N$ such that $u|_{\partial M} = g$.*

Remark 7 In general it is not easy to determine if $\mathcal{P}(M, N)$ holds, for given manifolds M, N; if $M \equiv B^m$, then we have that $\mathcal{P}(M, N)$ holds $\Leftrightarrow$ $\pi_{m-1}(N) = 0$.

This topological property enter the question we deal in view of the following statement:

Theorem 6 *Assume $p \geq m = \dim M$, then $T^p(\partial M, N) = W^{1-1/p,p}(\partial M, N)$ $\Leftrightarrow$ $\mathcal{P}(M, N)$ holds.*

Remark 8 In case $p > m$ the proof is based on Sobolev theorem, and the extension of $g \in W^{1-1/p,p}(\partial M, N)$ is constructed via its p-harmonic extension $U \in W^{1,p}(M, \mathbf{R}^\ell)$ (U minimizes $\int_M |Dv|^p$ among all $v \in W^{1,p}(M, \mathbf{R}^\ell)$, $v|_{\partial M} = g$);

in case $p = m$ an argument due to Schoen-Uhlenbeck is adapted to the situation.

Remark 9 The problem is much more complex for $p < m$. The following problem due to Hardt and Lin gives a partial answer.

Theorem 7 *Assume $1 \leq p < m$. If $\pi_0(N) = \pi_1(N) = \cdots = \pi_{[p]-1}(N) = 0$ then $T^p(\partial M, N) = W^{1-1/p,p}(\partial M, N)$.*

On the other hand, the following can be easily established

Theorem 8 *If* $\pi_{[p-1]}(N) \neq 0$ *then* $T^p(\partial M, N) \neq W^{1-1/p,p}(\partial M, N)$.

Remark 10 In other words, the condition $\pi_{[p]-1}(N) = 0$ is necessary in order to have

$$T^p(\partial M, N) = W^{1-1/p,p}(\partial M, N)\,. \tag{8}$$

Therefore we are led to the question: is this condition sufficient in order to have (8)? The answer is no, if we make no restriction on the topology of M, and the condition stated in last theorem 7 is necessary, in the sense that if there exists some $j \leq [p] - 1$, $j \in \mathbf{N}$, such that $\pi_j(N) \neq 0$, then there exists a manifold M of dimension m such that

$$T^p(\partial M, N) \neq W^{1-1/p,p}(\partial M, N)\,.$$

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