# Cartesian currents.

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## 1 Introductory remarks

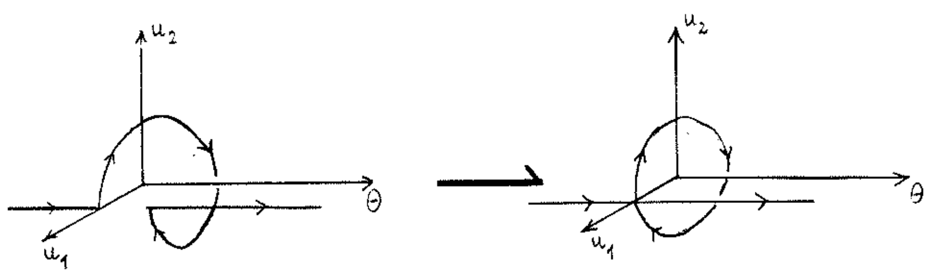
Simple examples show that information about functions (as the topological ones) are lost passing to the limit in the weak sense of functions, while this trouble doesn't occur if we consider weak convergence of graphs.

**Example 1** . Consider the sequence

$$u_k : [-\pi, \pi] \cong S^1 \to S^1 \subset \mathbf{R}^2$$

$$u_k(\theta) := \begin{cases} (1,0) & \text{if } \theta \leq 0 \text{ or } \theta \geq \frac{2\pi}{k} \\ (\cos k\theta, \sin k\theta) & \text{if } 0 < \theta < \frac{2\pi}{k} \end{cases}$$

Obviously the degree of each $u_k$ is one and $u_k \to u_\infty \equiv (1,0)$ which has degree 0; on the other hand the convergence of graphs preserves the degree, as the following picture shows:



## 2 Graph of a Lipschitz map

Let $n, N \geq 1$ and $\Omega$ be a bounded subset of $\mathbf{R}^n$. We recall that if

$$u : \Omega \subseteq \mathbf{R}^n \to \mathbf{R}^N$$

is a Lipschitz function and if $\mathcal{G}_{u,\Omega}$ denotes the graph of $u$, then the following area formula holds

$$\mathcal{H}^n(\mathcal{G}_{u,\Omega}) = \int_\Omega [\det(F^*F)]^{\frac{1}{2}} dx$$

where

$$F := \begin{pmatrix} Id \\ Du \end{pmatrix} = D(Id \oplus u).$$

Moreover, for a.e. $x \in \Omega$ there exists the tangent plane at $(x, u(x))$ to $\mathcal{G}_{u,\Omega}$ and it is given by

$$\mathrm{Tan}_{(x,u(x))}\mathcal{G}_{u,\Omega} = \{(z, y) | y = u(x) + Du(x) \cdot (z - x)\}.$$

A simple $n$-vector associated to the tangent plane, which is also the area element of $\mathcal{G}_{u,\Omega}$, is given by

$$M(Du) = \Lambda^n F(e_1 \wedge \ldots \wedge e_n) = (e_1 + v_1) \wedge \ldots \wedge (e_n + v_n)$$

where

$$v_j = \sum_{i=1}^N D_j u^i \varepsilon_i$$

and

$$\{e_i\}_{i=1}^n, \{\varepsilon_i\}_{i=1}^N$$

denote the standard bases of $\mathbf{R}^n$ and $\mathbf{R}^N$ respectively.

**Notation.** The standard notation for the multi-indeces in $I(n, n-k)$ and $I(n, k)$ will be $\alpha$ and $\beta$ respectively. $\bar{\alpha}$ will denote the multi-index complementar of $\alpha$ while $\sigma(\alpha, \bar{\alpha})$ will be the signature of the permutation $(\alpha, \bar{\alpha})$. We will write "$i \in \beta$ " to specify that the number $i$ is one of the $\beta$'s components. In such a case, "$\beta - i$ " will denote the $(|\beta| - 1)-$index obtained from $\beta$ by deleting $i$.
If $G = \{G_i^j\}_{i,j=1,\ldots,n}$ is an $N \times n$-matrix ($i$ is the row index, $j$ is the column one), then we set

$$G_0^0 := 1.$$

Given $\alpha \in I(n, n-k)$ and $\beta \in I(n,k)$, let

$$G_{\bar{\alpha}}^{\beta} := \{G_i^j\}_{j\in\bar{\alpha}, i\in\beta} \qquad \text{and} \qquad M_{\bar{\alpha}}^{\beta} G := \det G_{\bar{\alpha}}^{\beta}.$$

Moreover we will need the matrix of cofactors:

$$(\operatorname{adj} G_{\bar{\alpha}}^{\beta})_j^i := \sigma(i, \beta - i)\sigma(j, \bar{\alpha} - j) M_{\bar{\alpha}-j}^{\beta-i} G$$

where $j \in \bar{\alpha}$, $i \in \beta$.
We recall that

(1) $$\sum_{j\in\bar{\alpha}} G_j^h (\operatorname{adj} G_{\bar{\alpha}}^{\beta})_j^i = \delta_{ih} M_{\bar{\alpha}}^{\beta} G.$$

**Remark 1** . The following hold

(R1) $\det(F^*F)^{\frac{1}{2}} = |M(Du)|$

(R2) $M(Du) = \sum_{k=0}^{m} M_{(k)}(Du)$, where $m := \min(n, N)$ and

$$M_{(k)}(Du) := \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|=k}} \sigma(\alpha, \overline{\alpha}) M_{\bar{\alpha}}^{\beta}(Du) e_\alpha \wedge e_\beta$$

(R3) $\mathrm{Id} \oplus u$ maps null sets in $\Omega$ into null sets in $\mathcal{G}_{u,\Omega}$.

## 3 Graphs of non-smooth maps.

A class of maps with good tangential properties:

$$\{ \text{ a.e. approximately differentiable maps } \}.$$

Given an a.e. approximately differentiable map, one can define its graph. Let $u : \Omega \subset \mathbf{R}^n \to \mathbf{R}^N$ be such a map and let

$$\mathcal{A}_u = \{x \in \Omega \,|\, u \text{ is approximately differentiable at } x\}$$

$$\mathcal{G}_{u,\Omega} = (\mathrm{Id} \oplus u)\mathcal{A}_u$$

$$\mathrm{Tan}_{(x,u(x))}\mathcal{G}_{u,\Omega} = \{(z,y) \in \mathcal{A}_u \times \mathbf{R}^N | y = u(x) + \mathrm{ap}Du(x)\cdot(z-x)\}$$

We have the following theorem.

**Theorem 1** . *Let $u$ be a.e. approximately differentiable. Then*

*(i)* $\mathcal{H}^n(\mathcal{G}_{u,\Omega}) = \int_\Omega |M(\mathrm{ap}Du(x))|dx$;

*(ii)* $\mathcal{G}_{u,\Omega}$ *is countably rectifiable;*

*(iii)* $\mathrm{Tan}_{(x,y)}\mathcal{G}_{u,\Omega}$ *agrees* $\mathcal{H}^n$*-a.e. with the approximate tangent plane defined by blowing-up* $\mathcal{G}_{u,\Omega}$*;*

*(iv) If* $v = u$ *a.e., then* $\mathcal{H}^n(\mathcal{G}_{u,\Omega}\Delta\mathcal{G}_{v,\Omega}) = 0$.

The proof of Theorem 1 easily follows by the following result.

**Theorem 2 (Federer, 1945)** . *The following statements are equivalent:*

*(i)* $u$ *is a.e. approximately differentiable, i.e.* $|\Omega \setminus \mathcal{A}_u| = 0$;

*(ii)* $\exists\{\Omega_j\}_{j\in\mathbf{N}}$ *such that* $|\Omega \setminus \cup_j\Omega_j| = 0$ *and* $u_{|\Omega_j}$ *is Lipschitz.*

*(iii)* $\exists\{F_j\}_{j\in\mathbf{N}}$ *of closed sets and* $\{v_j\}_{j\in\mathbf{N}}$ *of* $C^1(\mathbf{R}^n, \mathbf{R}^N)$*-maps such that* $|\Omega \setminus \cup_j F_j| = 0$ *and* $u = v_j$ *on* $F_j$.

*Moreover* $\mathcal{A}_u \subset \cup_j\Omega_j$ *and*

$$apDu = D\tilde{u}_j$$

*a.e. in* $\mathcal{A}_u \cap \Omega_j$*, where* $\tilde{u}_j : \mathbf{R}^n \to \mathbf{R}^n$ *is any Lipschitz extension of* $u_{|\Omega_j}$.

## 4 Link with Sobolev maps

Given $u \in W^{1,1}(\Omega, \mathbf{R}^N), \Omega$ open subset of $\mathbf{R}^n$, let $\mathcal{L}_u$ and $\mathcal{L}_{Du}$ denote the sets of Lebesgue points of $u$ and $Du$ respectively. Setting

$$\mathcal{R}_u := \mathcal{L}_u \cap \mathcal{L}_{Du} = \{\text{regular points of } u\}$$

of course we have $|\Omega \setminus \mathcal{R}_u| = 0$.

**Theorem 3 (Calderon-Zygmund)** . *Let* $u \in W^{1,1}(\Omega, \mathbf{R}^N)$*. Then* $\mathcal{R}_u \subset \mathcal{A}_u$*,* $|\Omega \setminus \mathcal{R}_u| = 0$ *and (taking the Lebesgue value of $u$ as representant of $u$) apDu =Lebesgue value of $Du$ a.e. in* $\mathcal{R}_u$.

**Remark 2** . Let $u \in W^{1,1}(\Omega, \mathbf{R}^N)$. Then, by virtue of Theorem 3, it is reasonable to write simply $Du$ instead of ap $Du$. Moreover, in such a case one has

$$\mathcal{H}^n(\mathcal{G}_{u,\Omega} \Delta (\mathrm{Id} \oplus u)\mathcal{R}_u) = 0$$

whereby the graph of $u$ can be coherently defined without introducing the set of points at which $u$ is approximately differentiable, namely

$$\mathcal{G}_{u,\Omega} := (\mathrm{Id} \oplus u)\mathcal{R}_u$$

(taking the Lebesgue value of $u$ as a representant of $u$).

**Example 2** (of $u$ s.t. $\mathcal{G}_{u,\Omega}$ consists of an union of pieces of smooth graphs). Let $C$ and $v : (0,1) \to (0,1)$ be respectively the Cantor set and the Cantor-Vitali function. Then $v$ is differentiable at every $x \in (0,1) \setminus C$ and

$$\mathcal{G}_{v,(0,1)} = \text{graph of } v_{|[(0,1)\setminus C] \times (0,1)}$$

which consists of a measure one, countable union of horizontal segments. The remaining part of the continuous graph of $v$ (i.e. the so called Cantor part) is "vertical ".

•

**Definition 1** . *By $\mathcal{A}^1(\Omega, \mathbf{R}^N)$ we will denote the set of the functions $u \in L^1(\Omega, \mathbf{R}^N)$ which are a.e. ap-differentiable and such that $|M(Du)| \in L^1(\Omega)$.*

**Remark 3** . By the foregoing discussion it follows that, for $u \in \mathcal{A}^1(\Omega, \mathbf{R}^N)$, the following hold:

(R1) $\mathcal{G}_{u,\Omega}$ is well defined, it is rectifiable and $\mathcal{H}^n(\mathcal{G}_{u,\Omega}) < +\infty$;

(R2) the area formula holds; in fact

$$\int_\Omega \Phi(x,y) d\mathcal{H}^n \llcorner \mathcal{G}_{u,\Omega} = \int_\Omega \Phi(x, u(x)) |M(Du(x))| dx$$

holds for every Lipschitz function $f : \mathbf{R}^n \times \mathbf{R}^N \to \mathbf{R}$;

(R3) the unit $n$-vector $\xi(x, u(x)) \in \Lambda_n \mathbf{R}^{n+N}$ (for a.e. $x \in \Omega$), defined as

$$\xi(x, u(x)) := \frac{MDu(x)}{|MDu(x)|},$$

generates the tangent plane to $\mathcal{G}_{u,\Omega}$ at $(x, u(x))$. Moreover, the linear space $\mathrm{Tan}_{(x,u(x))}\mathcal{G}_{u,\Omega}$ agrees with the approximate tangent plane defined by blow-up.

## 5 The current integration over the graph

Let $u \in \mathcal{A}^1(\Omega, \mathbf{R}^N)$. We define the current $G_u \in \mathcal{D}_n(\mathbf{R}^n \times \mathbf{R}^N)$ as

$$G_u(\omega) = \int \langle \xi, \omega \rangle d\mathcal{H}^n \llcorner \mathcal{G}_{u,\Omega}$$

where

$$\xi = \frac{MDu(x)}{|MDu(x)|} \qquad \text{and} \qquad \mathcal{G}_{u,\Omega} = (\mathrm{Id} \oplus u)\mathcal{A}_u.$$

By the foregoing remark we see that $G_u$ is an integer rectifiable current in $\mathbf{R}^n \times \mathbf{R}^N$, with density 1. Namely $G_u = [\![\mathcal{G}_{u,\Omega}, \xi, 1]\!]$.
Also we have

(i) $\mathbf{M}(G_u) = \mathcal{H}^n(\mathcal{G}_{u,\Omega}) = \int_\Omega |MDu| dx$

(ii) $G_u(\omega) = \int_\Omega (\mathrm{Id} \oplus u)^\sharp \omega = \int_\Omega \langle e_1 \wedge \ldots \wedge e_n, (\mathrm{Id} \oplus u)^\sharp \omega \rangle dx$

By the area formula (see remark 3) we can rewrite $(ii)$ as follows

$$G_u(\omega) = \int_\Omega \langle MDu(x), \omega(x, u(x)) \rangle dx. \tag{2}$$

**Notation.** We introduce the "stratification" of an $n$–form $\omega$ in $\mathbf{R}^n \times \mathbf{R}^N$. Let

$$\omega_{\alpha\beta} := \omega \llcorner (dx^\alpha \wedge dy^\beta) \qquad \text{and} \qquad \omega^{(k)} := \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|=k}} \omega_{\alpha\beta} dx^\alpha \wedge dy^\beta.$$

Then

$$\omega = \sum_{k=0}^{m} \omega^{(k)}$$

where $m := \min(n, N)$ and $\omega^{(k)}$ will be called the "$k$–stratum " of $\omega$.

By (2) and recalling remark 1, we obtain

$$\begin{aligned} G_u(\omega^{(k)}) &= \int_\Omega \langle MDu(x), \omega^{(k)}(x, u(x)) \rangle dx \\ &= \sum_{\substack{|\alpha|+|\beta|=n \\ |\beta|=k}} \int_\Omega M_{\bar{\alpha}}^{\beta}(Du(x)) \omega_{\alpha\beta}(x, u(x)) dx. \\ &= \int_\Omega \langle M_{(k)} Du(x), \omega'^{(k)}(x, u(x)) \rangle dx \end{aligned}$$

where

$$\omega' = \sum_{|\alpha|+|\beta|=n} \sigma(\alpha,\overline{\alpha})\omega_{\alpha\beta} dx^{\alpha} \wedge dy^{\beta}.$$

In particular , for $\Phi \in C_c^{\infty}(\mathbf{R}^n \times \mathbf{R}^N)$, one has:

$$\begin{aligned} G_u^{(\alpha,\beta)}(\Phi) &:= G_u \llcorner dx^{\alpha} \wedge dy^{\beta}(\Phi) = G_u(\Phi dx^{\alpha} \wedge dy^{\beta}) \\ &= \sigma(\alpha,\overline{\alpha}) \int_{\Omega} \Phi(x,u(x)) M_{\overline{\alpha}}^{\beta}(Du(x))dx. \end{aligned}$$

For example:

$$G_u^{(\overline{0},0)}(\Phi) = \int_{\Omega} \Phi(x,u(x))dx.$$

The distributions $G_u^{(\alpha,\beta)}$ are called "components " of the graph-current $G_u$.

We will need the following lemma.

**Lemma 1** . *Let $u_k, u \in \mathcal{A}^1(\Omega, \mathbf{R}^N)$ be such that*

$$u_k \rightharpoonup u \;\; \textit{in } L^1 \qquad \textit{and} \qquad MDu_k \rightharpoonup MDu \;\; \textit{in } L^1.$$

*Then $G_{u_k} \rightharpoonup G_u$ in the sense of currents.*

The lemma's proof is a consequence of the Severini-Egoroff theorem.

## 6 Boundaries

For $T \in \mathcal{D}_n(\mathbf{R}^{n+N})$, the boundary current $\partial T \in \mathcal{D}_{n-1}(\mathbf{R}^{n+N})$ is defined as follows:

$$\partial T(\omega) := T(d\omega)$$

for all $\omega \in \mathcal{D}^{n-1}(\mathbf{R}^{n+N})$.

**Proposition 1** . *Let $u \in C^2(\overline{\Omega}, \mathbf{R}^N)$. Then $\partial G_u \llcorner (\Omega \times \mathbf{R}^N) = 0$.*

*Proof.* As $\Phi^{\sharp} d = d\Phi^{\sharp}$ whenever $\Phi \in C^2$, one has (let $\omega \in \mathcal{D}^n(\Omega \times \mathbf{R}^N)$ and recall $(ii)$ of the previous section):

$$\begin{aligned} \partial G_u(\omega) &= G_u(d\omega) = \int_{\Omega} \langle e_1 \wedge \ldots \wedge e_n, (Id \oplus u)^{\sharp} d\omega \rangle dx \\ &= \int_{\Omega} \langle e_1 \wedge \ldots \wedge e_n, d(Id \oplus u)^{\sharp} \omega \rangle dx = 0 \end{aligned}$$

by Green formulas. ∎

More generally the following holds.

**Proposition 2** . *Let* $u \in \mathcal{A}^1(\Omega, \mathbf{R}^N) \cap W^{1,m}(\Omega, \mathbf{R}^N)$, *where* $m := \min(n, N)$. *Then*

$$\partial G_u \llcorner (\Omega, \mathbf{R}^N) = 0.$$

*Proof.* (Idea: approximate $u$ with regular $u_k$ and use Proposition 1).

Let $\{u_k\}$ be a sequence of smooth maps such that $u_k \to u$ in $W^{1,m}$.

In particular

$$u_k \to u \ \text{ in } L^1 \qquad \text{and} \qquad MDu_k \to MDu \ \text{ in } L^1.$$

Therefore (by Lemma 1)

$$G_{u_k} \rightharpoonup G_u.$$

Since $\partial G_{u_k} \llcorner (\Omega \times \mathbf{R}^N) = 0$ (by Proposition 5), we get the thesis. ∎

Now, we give an example of function $u \in \mathcal{A}^1 \cap W^{1,p}$ such that $\partial G_u \llcorner (\Omega \times \mathbf{R}^N) \neq 0$. In particular, $u$ cannot be approximated by smooth functions.

**Example 3** . Consider $u : \Omega := B(0,1) \subset \mathbf{R}^n \to \mathbf{R}^n$ (so that $n = N$) defined as

$$u(x) = \frac{x}{|x|}.$$

Then

$$u \in W^{1,p}(B(0,1), \mathbf{R}^n)$$

for every $p < n$, and

$$u \in \mathcal{A}^1(B(0,1), \mathbf{R}^n)$$

in that $\det Du \equiv 0$ on $B(0,1) \setminus \{0\}$.

We shall show that

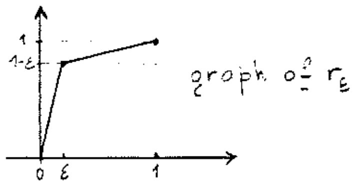
$$\partial G_u \llcorner (B(0,1) \times \mathbf{R}^n) = -\{0\} \times [\![S^{n-1}]\!]$$

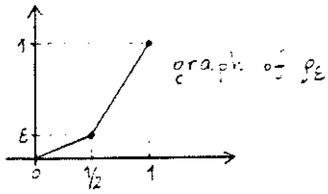
which is trivially not zero.

*Proof.* Consider the functions

$$r_\varepsilon, \rho_\varepsilon : [0,1] \to [0,1]$$

having the following graphs ($\varepsilon > 0$) :




Let $u_\varepsilon, \gamma_\varepsilon : B(0,1) \to B(0,1)$ defined by

$$\begin{cases} \gamma_\varepsilon(z) := \rho_\varepsilon(|z|)u(z) \\ u_\varepsilon(z) := r_\varepsilon(|z|)u(z) \end{cases}$$

and consider the map

$$\Gamma_\varepsilon : B(0,1) \to B(0,1) \times B(0,1)$$
$$z \mapsto \big(\gamma_\varepsilon(z), u_\varepsilon(\gamma_\varepsilon(z))\big)$$

As $\Gamma_\varepsilon = (\mathrm{Id} \oplus u_\varepsilon) \circ \gamma_\varepsilon$ it's easy to check that

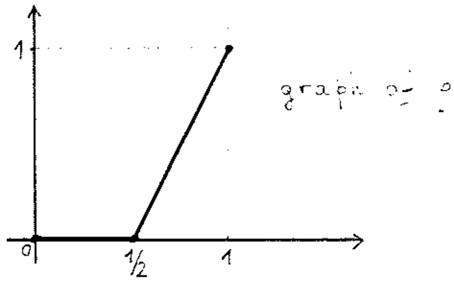
$$G_{u_\varepsilon} = \Gamma_{\varepsilon\sharp}[B(0,1)].$$

On the other hand $\{\Gamma_\varepsilon\}_\varepsilon$ are *equilipschitz* and converge to

$$\Gamma(z) := \begin{cases} (0, 2z) & if |z| \leq \frac{1}{2} \\ \big(\gamma(z), u(\gamma(z))\big) & if |z| > \frac{1}{2} \end{cases}$$

when $\varepsilon \to 0$, where $\gamma = \lim_{\varepsilon \to 0} \gamma_\varepsilon$, namely

$$\gamma(z) = \rho(|z|)\frac{z}{|z|}.$$




It follows that

$$\begin{aligned} G_{u_\varepsilon} \rightharpoonup \Gamma_\sharp[\![B(0,1)]\!] &= \Gamma_\sharp[\![B(0,1) \setminus B(0,\tfrac{1}{2})]\!] + \Gamma_\sharp[\![B(0,\tfrac{1}{2})]\!] \\ &= G_u + \{0\} \times [\![B^n]\!]. \end{aligned}$$

As $\partial G_{u_\varepsilon} \llcorner (B(0,1) \times \mathbf{R}^n) = 0$, we conclude that

$$\partial G_u \llcorner (B(0,1) \times \mathbf{R}^n) + \{0\} \times S^{n-1} = 0.$$

**Remark 4** **(a generalization of the previous example).**

(R1) With the same construction one can find that if $\phi : S^{n-1} \longrightarrow S^{n-1}$ is smooth and if $u : B(0,1) \longrightarrow S^{n-1}$ is defined by $u(x) = \phi(\frac{x}{|x|})$ then

$$\partial G_u \llcorner (B(0,1) \times \mathbf{R}^n) = -\mathrm{deg}\phi \, \{0\} \times [\![S^{n-1}]\!].$$

(R2) It's even possible to find $u \in \mathcal{A}^1$ such that

$$\mathbf{M}_{\Omega \times \mathbf{R}^n}(\partial G_u) = +\infty.$$

In the case $n = 1$, an example is given by the function of Cantor-Vitali.

## 7 Weak convergence of minors

We recall the following closure theorem.

**Theorem 4 (Reshetnyak, Ball)** . *Let $N \geq n$ and $\{u_k\} \subset W^{1,n}(\Omega, \mathbf{R}^N)$ such that*

$$u_k \longrightarrow u \quad \text{in } L^1$$

*and*

$$\mathrm{det} Du_k \quad \textit{are equi-integrable.}$$

*Then $u \in W^{1,n}(\Omega, \mathbf{R}^N)$ and, possibly by passing to subsequence,*

$$\mathrm{det} Du_k \rightharpoonup v.$$

*Moreover $v = \mathrm{det} Du$.*

We should like an analogous result for $\mathcal{A}^1$ functions. The following one is just what we need.

**Theorem 5** . *Let $u_k \in \mathcal{A}^1(\Omega, \mathbf{R}^n)$ and suppose that*

*(i) $u_k \longrightarrow u$ in $L^1$;*

*(ii) $M_{\bar{\alpha}}^{\beta} Du_k \rightharpoonup v_{\bar{\alpha}}^{\beta}$ in $L^1$;*

*(iii) $\mathbf{M}_{\Omega \times \mathbf{R}^n}(\partial G_{u_k}) \leq c$ for every $k$.*

*Then $u \in \mathcal{A}^1$ and*

$$v_{\bar{\alpha}}^{\beta} = M_{\bar{\alpha}}^{\beta}(Du).$$

*Proof.* By means of (i) and (ii) it isn't difficult to check that, for all $\Phi \in C_c^\infty(\Omega \times \mathbf{R}^n)$, one has

$$G_{u_k}^{(\alpha,\beta)}(\Phi) \to \int_\Omega \Phi(x, u(x)) v_{\bar{\alpha}}^{\beta}(x) dx$$

if $|\beta| > 0$, while

$$G_{u_k}^{(\bar{0},0)}(\Phi) \to \int_\Omega \Phi(x, u(x)) dx.$$

On the other hand, by (i) and (ii) again, we have

$$\sup_k \mathbf{M}_{\Omega\times\mathbf{R}^n}(G_{u_k}) < +\infty$$

so that, by (iii), the normal masses of $G_{u_k}$ are equibounded. By the compactness theorem of Federer and Fleming we find a rectifiable current $S$ such that $G_{u_k} \rightharpoonup S$ (possibly by passing to a subsequence).

It follows that

$$\left\{ \begin{array}{ll} S(\Phi dx^\alpha \wedge dy^\beta) = \int_\Omega \Phi(x, u(x)) v_{\bar{\alpha}}^{\beta}(x) dx & (\text{if } |\beta| > 0) \\ S(\Phi dx) = \int_\Omega \Phi(x, u(x)) dx & \end{array} \right. \tag{3}$$

By these relations we will be able to identify $S$ in terms of $u$. The idea is to rewrite them "on the graph of $u$".

With this purpose we consider the $n$-vector

$$V(x) := e_1 \wedge \ldots \wedge e_n + \sum_{\alpha,\beta} \sigma(\alpha, \bar{\alpha}) v_{\bar{\alpha}}^{\beta} e_\alpha \wedge e_\beta$$

Then, remarking that $u \in W^{1,1}$ (in that $u_k \xrightarrow{L^1} u$ and $D_j u_k^i \xrightharpoonup{L^1} v_j^i$) and by using the area formula in theorem 1, from (3) we obtain:

$$\begin{aligned} S(w) &= \int_\Omega \langle V(x), \omega(x, u(x)) \rangle dx \\ &= \int \langle \frac{V}{|MDu|}, \omega \rangle d\mathcal{H}^n \llcorner \mathcal{G}_{u,\Omega}. \end{aligned}$$

As $S$ is rectifiable, it must be

$$S = [\![\mathcal{G}_{u,\Omega}, \frac{V}{|MDu|\rho}, \rho]\!]$$

so that (by Theorem 1)

$$\frac{V}{|MDu|\rho} = \sigma \frac{M(Du)}{|MDu|} \tag{4}$$

where $\sigma : \mathcal{G}_{u,\Omega} \to \{\pm 1\}$.

As

$$V^{\bar{0}0} = M_{\bar{0}}^{0}(Du) = 1$$

it follows that $\rho \equiv \sigma$ i.e., trivially,

$$\rho \equiv \sigma \equiv 1.$$

By (4) again we conclude that $MDu = V$. ∎

**Remark 5** . Theorem 5 continues to hold also replacing the assumption of strong convergence (i) with the weaker one:

$$u_k \rightharpoonup u. \tag{5}$$

Indeed the following proposition holds.

**Proposition 3 (ws=ww)** . *If one has*

$$u_k \rightharpoonup u \quad (L^1)$$

$$Du_k \rightharpoonup w \quad (L^1)$$

*then (possibly passing to a subsequence)*

$$u_k \to u \quad (L^1).$$

*Proof.* As $\{u_k\}$ is bounded in $W^{1,1}$ then one can use Rellich theorem to deduce the thesis. ∎

**Definition 2** . $\mathrm{cart}^1(\Omega, \mathbf{R}^N) := \{u \in \mathcal{A}^1(\Omega, \mathbf{R}^N) | \partial G_u \llcorner (\Omega \times \mathbf{R}^N) = 0\}$

**Theorem 6 (closure of $\mathrm{cart}^1$)** . *The set* $\mathrm{cart}^1(\Omega, \mathbf{R}^N)$ *is closed w.r.t.* $L^1$ *weak convergence of* $u$ *and the minors.*

*Proof.* Let $\{u_k\} \subset \text{cart}^1(\Omega, \mathbf{R}^N)$ be such that

$$\begin{cases} u_k \xrightarrow{L^1} u \\ MDu_k \xrightarrow{L^1} v \end{cases}$$

then (theorem 5) $u \in \mathcal{A}^1$ and $v = MDu$. We conclude by Proposition 3 and Lemma 1. ■

**Example 4** . As an application one can prove the existence of minimizers for polyconvex functionals generated by positive, convex and l.s.c. functions

$$f : \Lambda_n \mathbf{R}^{n+N} \cap \{\xi^{\bar{0}0} = 1\} \to \bar{\mathbf{R}}_+$$

such that the following growth condition is satisfied:

$$f(\xi) \geq \Phi(|\xi|)$$

where $\Phi : \mathbf{R}^+ \to \mathbf{R}^+$ is an increasing and convex function such that

$$\lim_{t \to +\infty} \frac{\Phi(t)}{t} = +\infty.$$

The functional is defined by

$$\mathcal{F}(u) := \int_\Omega f(MDu)dx$$

and the following proposition holds. •

**Proposition 4** . *Let $u_0 \in \text{cart}^1(\Omega, \mathbf{R}^N)$. Then the problem of minimizing $\mathcal{F}$ among all $u \in \text{cart}^1(\Omega, \mathbf{R}^N)$ subject to the Dirichlet condition $\partial G_u = \partial G_{u_0}$ has a solution.*

As far as concerned the boundary condition, we remark that, in general, $u = v$ on $\partial\Omega$ doesn't imply $\partial G_u = \partial G_v$ $(u, v \in W^{1,1})$ whereas one can see that such an implication is true for $u, v \in W^{1,p}$ if $p \geq m = \min(n, N)$.
Here we give an example showing that $\partial G_u = \partial G_v$ can fail also if $\partial G_u \llcorner (\Omega \times \mathbf{R}^N) = \partial G_v \llcorner (\Omega \times \mathbf{R}^N)$

**Example 5** . Let

$$u(x_1, x_2) = (\frac{x_1 x_2}{|x|^2}, \frac{x_2^2}{|x|^2}) : B_+(0,1) \to B((0, \frac{1}{2}), \frac{1}{2})$$

where $B_+(0,1) := B(0,1) \cap \{x_2 > 0\} \subset \mathbf{R}^2$ $(n = N = 2)$.
We note that $u \in \mathcal{A}^1$ and $u \in W^{1,p}$ for every $p < 2$. As $u \in C^\infty(B_+(0,1))$, we have

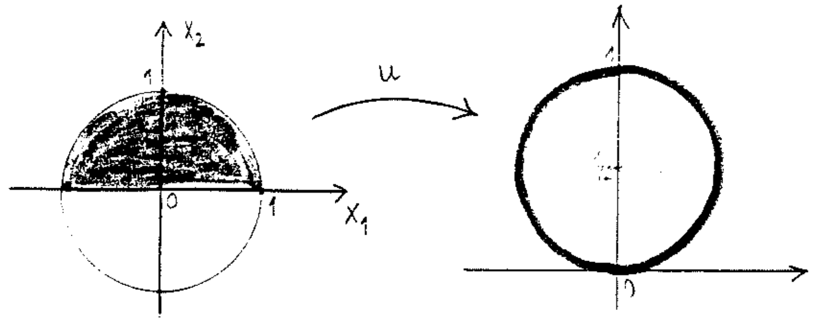
$$\partial G_u \llcorner (B_+(0,1) \times \mathbf{R}^2) = 0.$$

.Analogously, for $v \equiv 0$,

$$\partial G_v \llcorner (B_+(0,1) \times \mathbf{R}^2) = 0$$

holds.
But

$$\partial G_u = \partial G_v + \delta_0 \times \partial [\![ B((0, \frac{1}{2}), \frac{1}{2}) ]\!] \neq \partial G_v.$$

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## 8 Explicit formulas for the boundary

Let $u \in \mathcal{A}^1(\Omega, \mathbf{R}^N)$ and consider the "horizontal " form

$$\omega := \sum_{i=1}^{n} (-1)^{i+1} \Phi_i \, d\hat{x}_i$$

where $\Phi_i \in C_c^\infty(\mathbf{R}^n \times \mathbf{R}^N)$.

As

$$d\omega = \sum_{i=1}^{n} \frac{\partial \Phi_i}{\partial x_i} dx + \sum_{i=1}^{n} (-1)^{i+1} d_y \Phi_i \wedge d\hat{x}_i,$$

one has also

$$(\mathrm{Id} \oplus u)^\sharp \omega = \mathrm{div}[\vec{\Phi}\ (x, u(x))]$$

and then

$$\partial G_u(\omega) = \int_\Omega \mathrm{div}[\vec{\Phi}\ (x, u(x))] dx.$$

Consequently, the following proposition holds.

**Proposition 5** . $u \in W^{1,1}(\Omega, \mathbf{R}^N)$ *if and only if* $\partial G_{u_{(0)}} \llcorner (\Omega \times \mathbf{R}^N) = 0$.

**Corollary 1** . $\mathrm{cart}^1(\Omega, \mathbf{R}^N) \subset W^{1,1}(\Omega, \mathbf{R}^N)$.

Now consider the problem of representing also the other boundary components

$$\partial G_{u_{(k)}} \qquad (k \geq 1).$$

Let $\alpha \in I(n, n-k)$, $\beta \in I(n, k)$ and $i \in \beta$. Then, by recalling the formula (1) of section 2, we obtain:

$$\partial G_u(\Phi\, dx^\alpha \wedge dy^{\beta - i}) =$$

$$= (-1)^{|\alpha|} \sigma(\alpha, \bar{\alpha})\, \sigma(i, \beta - i) \int_\Omega \sum_{j \in \bar{\alpha}} D_j [\vec{\Phi}\ (x, u(x))] (\mathrm{adj}(Du)^{\beta}_{\bar{\alpha}})^i_j dx.$$

By choosing $\Phi(x, y) = \phi(x) \chi_R(|y|)$, where

$$\chi_R(t) := \begin{cases} 1 & \text{if } t \in [0, R] \\ 0 & \text{if } t \in [2R, +\infty) \\ 2 - \frac{t}{R} & \text{if } t \in (R, 2R), \end{cases}$$

and by letting $R \to +\infty$, we obtain the following proposition.

**Proposition 6 (Piola's identity)** . *If* $u \in \mathrm{cart}^1(\Omega, \mathbf{R}^N)$ *then*

$$\sum_{j=1}^{n} \int_{\Omega} D_j\phi(x)(\mathrm{adj}(Du)_{\bar{\alpha}}^{\beta})_j^i dx = 0$$

*for* $i = 1, \ldots, n$ *and for every* $\phi \in C_c^\infty(\Omega)$.

**Remark 6** . By an approximation argument, the Piola's identity can be proved also for $u \in W^{1,n-1}(\Omega, \mathbf{R}^N)$. We note that in this case the foregoing proof can't work in that $\partial G_u \llcorner (\Omega, \mathbf{R}^N)$ can be different from 0 (e.g. $u(x) = \frac{x}{|x|} \in W^{1,n-1}$ but $\partial G_u \llcorner (B(0,1) \llcorner \mathbf{R}^n) \neq 0$).

## 9 Subclasses of $W^{1,n-1}(\Omega, \mathbf{R}^n)$

Throughout this section we will assume $N = n$. Consider the set

$$\mathcal{A}_{p,q}(\Omega, \mathbf{R}^n) := \{u \in W^{1,p}(\Omega, \mathbf{R}^n) | \mathrm{adj}\, Du \in L^q\}$$

which has been introduced by Šverak. By a standard approximation argument one can show that

$$\mathcal{A}_{p,q} \subset \mathrm{cart}^1$$

whenever

$$p > n-1 \qquad \text{and} \qquad q > \frac{p}{p-1}.$$

Moreover the following result holds.

**Theorem 7 (Müller, Tang, Ye)** .

$$\mathcal{A}_{n-1,\frac{n}{n-1}}(\Omega, \mathbf{R}^n) \subset \mathrm{cart}^1(\Omega, \mathbf{R}^n)$$

*Proof.* (sketch). Let $u \in \mathcal{A}_{n-1,\frac{n}{n-1}}$.

<u>First.</u> As $(\mathrm{det}Du)Id = Du(\mathrm{adj}Du)^T$,
it follows that

$$(\mathrm{det}Du)^n = \mathrm{det}Du \cdot \mathrm{det}(\mathrm{adj}Du).$$

Hence

$$|\mathrm{det}Du|^{n-1} \leq \|\mathrm{adj}Du\|^n$$

and then

$$\|\det Du\|_{L^1} \leq \|\mathrm{adj} Du\|_{L^q}^q$$

where $q := \frac{n}{n-1}$, so that $u \in \mathcal{A}^1(\Omega, \mathbf{R}^n)$.

Second. We have to prove that $\partial G_u \llcorner (\Omega \times \mathbf{R}^n) = 0$.

As $u \in W^{1,n-1}$, one can check that

$$\partial G_{u(0)} = \partial G_{u(1)} = \ldots = \partial G_{u(n-2)} = 0$$

in $\Omega \times \mathbf{R}^n$. Then, by the representation formula of the boundary given in Section 8 one deduces that

$$\partial G_u \llcorner (\Omega \times \mathbf{R}^n) = 0$$

is equivalent to

$$(6) \qquad \int_\Omega \{\sum_{j=1}^n \Phi(x, u(x))(\mathrm{adj} Du(x))_j^i + D_{y_i}\Phi(x, u(x))\det Du(x)\}dx = 0$$

for every $\Phi \in C_c^1(\Omega \times \mathbf{R}^n)$ $(i = 1, \ldots, n)$.

To prove that (6) holds, it's enough to consider $\Phi(x, y)$ of the form $\phi(x)g(y)$ with $\phi \in C_c^1(\Omega)$ and $g \in C_c^1(\mathbf{R}^n)$. So we have to prove that

$$(7) \qquad \int_\Omega \{g(u)\, D_j\phi(\mathrm{adj} Du)_j^i + \phi D_{y_i} g(u)\det Du\}dx = 0$$

for all $g \in C_c^1(\mathbf{R}^n)$, $\phi \in C_c^1(\Omega)$ $(i = 1, \ldots, n)$.

*Proof of (7).* Let $i$ be fixed, $\vec{\sigma}$ be such that

$$\sigma_j = (\mathrm{adj}\, Du)_j^i \qquad \text{and} \qquad v(x) = g(u(x)).$$

Then (7) is equivalent (in the sense of distributions) to

$$\mathrm{Div}(v\, \vec{\sigma}) = D_{y_i} g(u)\det Du$$

i.e., by applying (1) to $G = Du$,

$$(8) \qquad \mathrm{Div}(v\, \vec{\sigma}) = \sum_h \left(D_{y_h} g(u) \sum_j D_j u^h (\mathrm{adj} Du)_j^i\right) = \vec{\sigma} \cdot Dv.$$

(Warning: in the sense of distributions !)

To prove (8) one verifies (by regularization) that $\mathrm{Div}(v\, \vec{\sigma}) \in L^1$ and recalls that $\mathrm{Div}\, \vec{\sigma} = 0$ (by remark 6) after which the conclusion comes from the theorem below of formal integration by parts. ∎

**Theorem 8 (S.Müller)** . *Let* $v \in W^{1,p}(\Omega, \mathbf{R}^n)$ *and* $\vec{\sigma} \in L^q(\Omega, \mathbf{R}^n)$, *where*

$$\frac{1}{p} + \frac{1}{q} - \frac{1}{n} \leq 1.$$

*Assume that*

$$\mathrm{Div}(v\, \vec{\sigma}) \in L^1 \qquad \textit{and} \qquad \mathrm{Div}\, \vec{\sigma} = 0.$$

*Then*

$$\mathrm{Div}(v\, \vec{\sigma}) = Dv \cdot \vec{\sigma}\ .$$

Traces of maps in $\mathcal{A}_{n-1,\frac{n}{n-1}}$ are well behaved too. Indeed we have the following theorem.

**Theorem 9** . *Let* $u, v \in \mathcal{A}_{n-1,\frac{n}{n-1}}(\Omega, \mathbf{R}^n)$ *be such that*

$$u = v \qquad \textit{on } \partial\Omega.$$

*Then*

$$\partial G_u = \partial G_v.$$

**Definition 3** . *We shall write:*
(D1) *"$u_k \to u$ (strongly) in cart$^1$ " if $u_k \xrightarrow{L^1} u$ and $MDu_k \xrightarrow{L^1} MDu$;*
(D2) *"$u_k \rightharpoonup u$ (weakly) in cart$^1$" if $u_k \xrightarrow{L^1} u$ and $MDu_k \overset{L^1}{\rightharpoonup} MDu$.*

**Remark 7** . By proposition 3 (ws=ww) the definition of weak convergence

$$u_k \overset{\mathrm{cart}^1}{\rightharpoonup} u$$

is equivalent to

$$u_k \xrightarrow{L^1} u \qquad \text{and} \qquad MDu_k \overset{L^1}{\rightharpoonup} MDu.$$

**Definition 4** .

$$\mathrm{cart}^1_{\mathrm{strong}}(\Omega, \mathbf{R}^N) := \overline{\mathrm{Lip}(\Omega, \mathbf{R}^N)}^{\mathrm{cart}^1_{\mathrm{strong}}};$$

$$\mathrm{cart}^1_{\mathrm{weak}}(\Omega, \mathbf{R}^N) := \overline{\mathrm{Lip}(\Omega, \mathbf{R}^N)}^{\mathrm{cart}^1_{\mathrm{weak}}}.$$

The following inclusions hold

$$\text{Lip} \subset \text{cart}^1_{\text{strong}} \subset \text{cart}^1_{\text{weak}} \subset \text{cart}^1. \tag{9}$$

**Theorem 10 (Maly)** . *One has*

$$\text{cart}^1_{\text{strong}} = \text{cart}^1_{\text{weak}}.$$

Now we shall show, by an example, that the last inclusion in (9) is proper.

**Example 6** .
First step. Let $B \subset \mathbf{R}^2$ be a ball and consider a smooth map

$$\phi : S^1 \to \mathbf{R}^2 \setminus B.$$

We have the following proposition.

**Proposition 7** . *If there exists* $\{u_k\} \subset \text{Lip}(B(0,1), \mathbf{R}^2)$ *such that*

$$u_k \overset{\text{cart}^1}{\to} u := \phi\Big(\frac{x}{|x|}\Big)$$

*then* $\phi(S^1)$ *is homotopic (in* $\mathbf{R}^2 \setminus B$*) to a constant map.*

*Proof.* One has to verify that $\phi(S^1)$ doesn't "contain" $B$.

(Idea: if $\phi(S^1)$ should contain $B$ then, being also $u(\partial B_r) = \phi(S^1)$, it ought to be $u_k(B_r) \supset B$ (for $k$ great enough). Hence $|u_k(B_r)| > |B|$ which is an absurd if $r$ is small enough).

The complete proof is based on two facts that follow from the convergence assumptions:

(i) for all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that: $E \subset B(0,1)$ and $|E| < \delta_\varepsilon$ imply $\int_E |\text{det} Du_k| dx < \varepsilon$ for all $k$;
(ii) $u_{k|\partial B_r} \to u_{|\partial B_r}$ uniformly on $\partial B_r$ for a.e. $r \in [0,1]$.

Suppose that $\phi(S^1)$ is not homotopic (in $\mathbf{R}^2 \setminus \bar{B}$) to a constant map. Then we will find a contradiction.

Let

$$d := \text{dist}(\phi(S^1), B)$$

and consider

$$U := \Big\{ x \in \mathbf{R}^2 | \mathrm{dist}(x, \phi(S^1)) < \frac{d}{2} \Big\}.$$

By the assumption (ii) it follows that, for a.e. $r \in [0,1]$, there exists $\bar{k}(r)$ such that

$$\sup_{x \in \partial B_r} |u_{\bar{k}(r)}(x) - u(x)| < \frac{d}{2}.$$

Now, the map $H(t,\theta) := u(\theta)+t(v(\theta)-u(\theta))$ ( where $v(\theta) := u_{\bar{k}(r)}(r\cos\theta, r\sin\theta)$ and $(t,\theta) \in [0,1] \times S^1$) is a continous map with values in $U$, so that it is a homotopy map between $u$ and $v$. It follows that also $u_{\bar{k}(r)}(\partial B_r)$ isn't homotopic to a constant map in $\mathbf{R}^2 \setminus \bar{B}$ (for a.e. $r \in [0,1]$). then it must be $u_{\bar{k}(r)}(B_r) \supset B$.

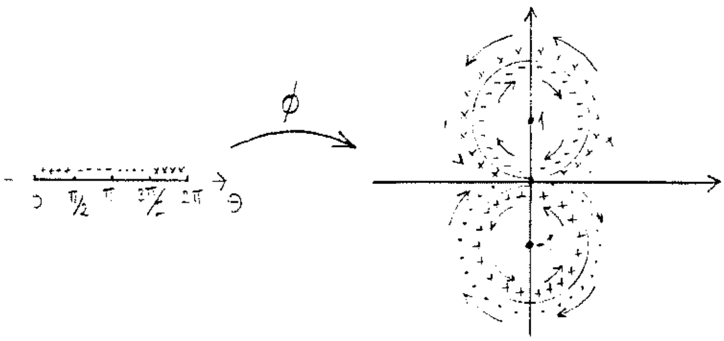
Hence one obtains

$$|u_{\bar{k}(r)}(B_r)| \geq |B|$$

for a.e. $r \in [0,1]$, which contradicts the assumption (i).

■

Second step. Take now $\phi$ defined according to the following picture



i.e.

$$\phi(\theta) := \begin{cases} \text{point on } \partial B((0,-1),1) - \text{anticlockwise} & \text{if } \theta \in [0, \frac{\pi}{2}) \\ \text{point on } \partial B((0,1),1) - \text{clockwise} & \text{if } \theta \in [\frac{\pi}{2}, \pi) \\ \text{point on } \partial B((0,-1),1) - \text{clockwise} & \text{if } \theta \in [\pi, \frac{3\pi}{2}) \\ \text{point on } \partial B((0,1),1) - \text{anticlockwise} & \text{if } \theta \in [\frac{3\pi}{2}, 2\pi). \end{cases}$$

Then

$$u = \phi(\frac{x}{|x|}) \in \mathcal{A}^\infty \text{ and } \partial G_u \llcorner (B(0,1) \times \mathbf{R}^2) = 0$$

in that $\deg \phi = 0$ and also by recalling remark 4. So we have $u \in \mathrm{cart}^1$. As $\phi(S^1)$ isn't homotopic to a constant, Proposition 7 implies that $u$ can't be approximated in $\mathrm{cart}^1$ by $u_k \in \mathrm{Lip}$.

•

## 10 Variational characterization of the strong closure $\mathrm{cart}^1_{\mathrm{strong}}$

Let $u \in \mathrm{cart}^1(\Omega, \mathbf{R}^N)$, then we recall that one has

$$M(G_u) = \int_\Omega |MDu| dx = \int_\Omega \sqrt{1 + \sum (M_\alpha^\beta Du)^2}\, dx$$

**Definition 5** . $\mathcal{A}(u) := \inf \left\{ \liminf \mathbf{M}(G_{u_k}) \,|\, u_k \in \mathrm{Lip}\,(\Omega, \mathbf{R}^N), u_k \xrightarrow{L^1} u \right\}$.

**Theorem 11 (Acerbi-Dal Maso)** . *Let $u \in W^{1,1}(\Omega, \mathbf{R}^N)$, then*

$$u \in \mathrm{cart}^1_{\mathrm{strong}}(\Omega, \mathbf{R}^N) \qquad \textit{if and onli if} \qquad M(G_u) = \mathcal{A}(u) < +\infty.$$

*Proof.* First of all, let us prove that if $M(G_u) = \mathcal{A}(u)$ then

$$u \in \mathrm{cart}^1_{\mathrm{strong}}(\Omega \times \mathbf{R}^N).$$

By the definition of $\mathcal{A}(u)$ there is a sequence $\{u_k\} \subset \mathrm{Lip}\,(\Omega)$ such that

$$u_k \xrightarrow{L^1} u\,, \qquad M(G_{u_k}) \to \mathcal{A}(u)$$

and

$$\sup_k \mathbf{M}(G_{u_k}) \leq C.$$

Hence

$$G_{u_k} \xrightarrow{\mathrm{cart}} T \in \mathrm{cart}(\Omega \times \mathbf{R}^N)$$

and by theorem 14 below we have

$$T = G_u + S.$$

Therefore we have

$$M(G_u) \leq M(T) \leq \liminf M(G_{u_k}) = \mathcal{A}(u) = M(G_u)$$

which implies $S = 0$ and thus

$$G_{u_k} \rightharpoonup G_u \qquad \text{and} \qquad M(G_{u_k}) \to M(G_u).$$

But then we have

$$\int \phi(x, u_k(x)) M_{\bar\alpha}^{\beta} Du_k dx \to \int \phi(x, u(x)) M_{\bar\alpha}^{\beta} Du\, dx \tag{10}$$

for all $\phi \in C_0^\infty(\Omega \times \mathbf{R}^N)$.

Since

$$M(G_{u_k}) \to M(G_u)$$

one has that (10) holds even if $\phi \in C_0^\infty(\Omega)$. Hence the measures $M_{\bar\alpha}^{\beta} Du_k dx$ converge weakly to $M_{\bar\alpha}^{\beta} Du$ and as the function

$$(M_{\bar\alpha}^{\beta}(Du))_{\alpha,\beta} \to |M(Du)| := \sqrt{1 + \sum_{\alpha,\beta} |M_{\bar\alpha}^{\beta} Du|^2}$$

is strictly convex we can recall a result of Reshetnyak to deduce that $MDu_k \xrightarrow{L^1} MDu$. The proof of the converse implication is obvious by definition. ■

Some open questions:

(Q1) to give a geometric characterization of $\mathrm{cart}^1_{\mathrm{strong}}$;

(Q2) to find an explicit formula for $\mathcal{A}(u)$;

(Q3) if we define, for any $T \in \mathrm{cart}(\Omega \times \mathbf{R}^N)$,

$$\mathcal{A}(T) = \inf\{\liminf_k \mathbf{M}(G_{u_k}) | u_k \in \mathrm{Lip} \text{ and } G_{u_k} \rightharpoonup T\},$$

which is the relation between $\mathcal{A}(T)$ and $\mathbf{M}(T)$ (remark: the inequality $\mathbf{M}(T) \leq \mathcal{A}(T)$ is obvious) ?

## 11 Space of cartesian currents

If we consider a sequence $\{u_k\} \subset \mathrm{cart}^1$ such that

$$\sup_k \mathbf{M}(G_{u_k}) \leq C$$

then we have, possibly by passing to a subsequence,

$$G_{u_k} \rightharpoonup T,$$

where $T \in D_n(\Omega \times \mathbf{R}^N)$. The weak limit $T$ satisfies the following properties:

(P1) $\mathbf{M}(T) < +\infty$;

(P2) $\partial T \llcorner (\Omega \times \mathbf{R}^N) = 0$ in that, for all $k$, one has $\partial G_{u_k} \llcorner (\Omega \times \mathbf{R}^N) = 0$;

(P3) $T$ is a rectifiable current, by Federer-Fleming theorem;

(P4) $T^{\bar{0}0} \geq 0$ in that the following formula

$$G_{u_k}^{\bar{0}0}(\phi) = \int_\Omega \phi(x, u_k(x)) \geq 0$$

holds for all $k$ and for all $\phi \in C_0^\infty(\Omega)$ such that $\phi \geq 0$.

If we add the condition $\sup_k \int |u_k| \leq C$ and we denote by $y$ the variable in $\mathbf{R}^N$, then we have

(P5) $\|T\|_1 = \sup\{T(\varphi(x,y)|y|dx) \mid \varphi \leq C_c^(\mathbf{R}^n), \; \|\varphi\| \leq 1\}$
(remark: if $T = G_u$ then $\|T\|_1 = \|u\|_{L^1}$).
and also

(P6) $\pi_\sharp T = [\![\Omega]\!]$, as follows from the lemma below.

**Lemma 2** . *If $T_k, T \in \mathcal{D}_n(\Omega \times \mathbf{R}^N)$ with $\|T_k\|_1 \leq C$ and $T_k \to T$, then*

$$\pi_\sharp T_k = \pi_\sharp T.$$

**Definition 6** . *cart($\Omega \times \mathbf{R}^N$) will denote the set of $T \in \mathcal{D}_n(\Omega \times \mathbf{R}^N)$ such that the following hypotheses are stisfied*
(H1) $\mathbf{M}(T) + \|T\|_1 < +\infty$
(H2) $\partial T \llcorner (\Omega \times \mathbf{R}^N) = 0$
(H3) *$T$ is rectifiable with integer multiplicity*
(H4) *$T^{\bar{0}0} \geq 0$ and $\pi_\sharp T = [\![\Omega]\!]$.*

**Definition 7** . *We say that a sequence $\{T_k\}$ converges weakly in* cart($\Omega \times \mathbf{R}^N$) *if $T_k \rightharpoonup T$ as current and $\mathbf{M}(T_k) + \|T_k\|_1 \leq C$. In such a case we write $T_k \overset{\text{cart}}{\rightharpoonup} T$.*

From the definitions above we get easily the following theorem of closure.

**Theorem 12** . $\mathrm{cart}(\Omega \times \mathbf{R}^N)$ *is closed under weak-cart convergence.*

**Theorem 13** . *The set* $\{T \in \mathcal{D}^n(\Omega \times \mathbf{R}^N) \mid \|T\|_1 + \mathbf{M}(T) < +\infty\}$ *is the dual of a separable Banach space. In particular* $\mathrm{cart}(\Omega \times \mathbf{R}^N)$ *is a closed subspace of the dual of a separable Banach space.*

**Remark 8** . As a consequence, the bounded sets

$$\{T \in \mathrm{cart}(\Omega \times \mathbf{R}^N) \mid \|T\|_1 + \mathbf{M}(T) \leq C\}$$

are metrizable.

## 12 Some examples

(E1) If $u \in \mathrm{cart}^1(\Omega \times \mathbf{R}^N)$, then $G_u \in \mathrm{cart}(\Omega \times \mathbf{R}^N)$.
(E2) Let $u \in \mathcal{A}^1$ such that $\partial G_u \neq 0$. If there is an $n$-rectifiable current $S$ such that $S^{\bar{0}0} = 0$, $\mathbf{M}(S) < +\infty$ and $\partial S \llcorner (\Omega \times \mathbf{R}^N) = -\partial G_u \llcorner (\Omega \times \mathbf{R}^N)$, then the current $T := G_u + S \in \mathrm{cart}(\Omega \times \mathbf{R}^N)$.

For instance let us suppose

$$u = \frac{x}{|x|}$$

so that $\partial G_u = -\delta_0 \times S^{n-1}$. In this case we can choose

$$S := \delta_0 \times B(0,1) \qquad \text{or also} \qquad S := L \times S^{n-1}$$

where $L$ is a 1-dimensional rectifiable current such that $\partial L = \delta_0 - \delta_{x_0}$ with $x_0 \in \partial B(0,1)$. Then

$$G_u + S \in \mathrm{cart}^1(B(0,1) \times \mathbf{R}^n).$$

**Definition 8** . *Let* $T = [\![\mathcal{M}, \vec{T}, \theta]\!] \in \mathrm{cart}(\Omega \times \mathbf{R}^N)$. *Then we put*

$$\mathcal{M}_+ := \{z \in \mathcal{M} \mid \vec{T}^{\bar{0}0}(z) > 0\}$$

*and*

$$u^j(T)(\varphi(x)) := T(\varphi(x) y^j dx) \quad j = 1, \ldots, N.$$

**Theorem 14** . *Let* $T = [\![\mathcal{M}, \vec{T}, \theta]\!] \in \mathrm{cart}(\Omega \times \mathbf{R}^N)$. *Then, for* $j = 1, \ldots, N$, *the measures* $u^j(T)$ *are absolutely continuous with respect to the Lebesgue measure in* $\Omega$. *If* $u_T^j$ *denotes the density function, then one has*

$$u_T := (u_T^1, \ldots, u_T^N) \in \mathcal{A}^1(\Omega \times \mathbf{R}^N) \qquad \textit{and} \qquad T \llcorner \mathcal{M}_+ = G_{u_T}.$$

*Moreover*

$$u_T \in BV(\Omega \times \mathbf{R}^N) \qquad \textit{and} \qquad \|u_T\|_{BV} \leq \mathbf{M}(T) + \|T\|_1.$$

*Finally, if* $T_k \rightharpoonup T$ *in cart then* $u_{T_k} \rightharpoonup u_T$ *in* $BV$.

## 13 Geometry of maps $B^3 \to S^2$

Let us suppose $u : B^3 \to S^2$ smooth with

$$\frac{1}{2}\int |Du|^2 < +\infty.$$

Let $n := u(x)$ and $G := Du(x)$. Then, the 3-vector which orients the graph is given by

$$M(n, G) := (e_1 + Ge_1) \wedge (e_2 + Ge_2) \wedge (e_3 + Ge_3),$$

and the component $M_{(2)}(n, G)$ is

$$M_{(2)}(n, G) = e_1 \wedge Ge_2 \wedge Ge_3 - e_2 \wedge Ge_1 \wedge Ge_3 + e_3 \wedge Ge_1 \wedge Ge_2 = D(n, G) \wedge \varepsilon_2(n)$$

where

$$\varepsilon_2 := \text{ 2-unit vector orienting the sphere}$$

$$D(n, G) := (n \cdot b \times c)e_1 - (n \cdot a \times c)e_2 + (n \cdot a \times b)e_3$$

and $a, b, c$ are respectively the first, second and third column of $G$.
We note that

$$|D(n, G)| = |M_{(2)}(n, G)|$$

and the rank of $G$ is 2 if and only if $|D(n, G)| \neq 0$, moreover $D(n, G)$ generates $\ker G$ for $G$ such that $G^T \cdot n = 0$.

Let $u : B^3 \to S^2$ be smooth and consider a 3-form $\omega \in \mathcal{D}^3(B^3 \times \mathbf{R}^3)$. We will denote by $\omega^T$ its tangential part with respect to the graph of $u$.
Recall that $\omega^T$ can be written in the form

$$\phi_0 \, dx_1 \wedge dx_2 \wedge dx_3 + \sum_{i,j=1}^{3} \phi_{ij} \, d\hat{x}_i \wedge dy_j + \sum_{i=1}^{3} \phi_i \, dx_i \wedge \hat{\pi}^\sharp \omega_{S^2}$$

where:

$$\omega_{S^2} := \text{ volume form of } S^2$$

$$\hat{\pi} : \mathbf{R}^3_x \times \mathbf{R}^3_y \to \mathbf{R}^3_y \text{ is the usual projection}$$

and

$$\begin{cases} \phi_0 = \omega_{\bar{0}0} \\ \phi_{ij} = \omega_{\bar{i}j} - \omega_{\bar{i}h} n_h n_j \text{ ( hence } \sum_{j=1}^3 \phi_{ij}(x,y) n_j(y) = 0) \\ \phi_i = \sum_{|\beta|=2} \omega_{i\beta} (-1)^{\bar{\beta}-1} n_\beta. \end{cases}$$

One can verify that

$$G_u(\omega) = G_u(\omega^T).$$

Now we can calculate the components of $G_u$ when $u \in W^{1,2}$:

$$\begin{aligned}
(G_u)_{(0)}(\omega) = (G_u)(\omega^{(0)}) &= \int_{B^3} \phi(x, u(x)) dx \\
(G_u)_{(1)}(\omega) = (G_u)(\omega^{(1)}) &= \sum_{i,j=1}^{3} \int_{B^3} (-1)^{i-1} D_i u^j \phi_{ij}(x, u(x)) dx \\
(G_u)_{(2)}(\omega) = (G_u)(\omega^{(2)}) &= \sum_{i=1}^{3} \int_{B^3} \phi_i(x, u(x)) dx_i \wedge u^\sharp \omega_{S^2} \\
&= \sum_{i,j=1}^{3} \int_{B^3} \phi_i(x, u(x)) (-1)^{i+j} n_j M_{\bar{j}\bar{i}}(Du) dx \\
&= \sum_{i=1}^{3} \int_{B^3} \phi_i(x, u(x)) D_i(x) dx
\end{aligned}$$

where

$$D(x) := D_u(x) := D(u(x), Du(x)) = M Du(x)_{(2)} \llcorner \varepsilon_2(u(x))$$

and, eventually,

$$(G_u)_{(3)}(\omega) = (G_u)(\omega^{(3)}) = 0.$$

**Definition 9** . *Given $u \in W^{1,2}(B^3, S^2)$, we define $\mathbf{D}_u \in \mathcal{D}_1(B^3)$ as follows:*

$$\mathbf{D}_u(\alpha) = \frac{1}{4\pi} \int_{B^3} \langle \alpha, D_u \rangle dx$$

*for all $\alpha \in \mathcal{D}^1(B^3)$.*

**Remark 9** . By the expression of $(G_u)_{(2)}$, we obtain

$$\pi_\sharp(G_u \llcorner \hat{\pi}^\sharp \omega_{S^2}) = 4\pi \mathbf{D}_u. \tag{11}$$

If $u$ is smooth, as $\partial G_u \llcorner (B^3 \times S^2) = 0$, the equation (11) implies

$$\partial \mathbf{D}_u = 0$$

i.e.

$$0 = \mathbf{D}_u(df) = \frac{1}{4\pi} \int_{B^3} \langle df, D_u \rangle du = -\frac{1}{4\pi} \int_{B^3} f \mathrm{div} D_u dx$$

for all $f \in C_c^1(B^3)$.
It follows that

$$\mathrm{div} D_u \equiv 0 \qquad \text{in } B^3.$$

**Proposition 8** . *If $u \in W^{1,2}(B^3, S^2)$ then one has*

$$\partial G_u = \partial \mathbf{D}_u \times [\![S^2]\!].$$

*Proof.* First, by a standard approximation argument, one can prove that

$$(\partial G_u)_{(0)} = (\partial G_u)_{(1)} = 0.$$

Then, the thesis will follow once proved that

$$G_u(d\omega) = \partial \mathbf{D}_u \times [\![S^2]\!](\omega) \tag{12}$$

whenever $\omega \in \mathcal{D}^2(\mathbf{R}^3 \times \mathbf{R}^3)$ is totally vertical, i.e. such that $\omega = \omega^{(2)}$. Furthermore, as $G_u(\omega) = G_u(\omega^T)$, it is enough to consider forms of the form

$$\omega(x, y) = a(x)\omega_{S^2} + d_y \eta(x, y)$$

where $a \in C_c^\infty(\mathbf{B}^3)$ and $\eta \in \mathcal{D}^1(B^3 \times S^2)$.

Thus we have

$$\begin{aligned}\partial G_u(\omega) = G_u(d\omega) &= G_u(d(a(x) \wedge \omega_{S^2}) + G_u(d(d_y\,\eta)) \\ &= G_u(da \wedge \omega_{S^2} + a \wedge d(\omega_{S^2})) + G_u(d(d_x\,\eta)).\end{aligned}$$

Now

$$d\omega_{S^2} = 0$$

in that $\omega_{S^2}$ is closed, and

$$G_u(d(d_x\,\eta)) = 0$$

since $(\partial G_u)_{(1)} = 0$. Hence we have

$$\partial G_u(\omega) = 4\pi \mathbf{D}_u(da) = 4\pi \partial \mathbf{D}_u(a).$$

To get the thesis of the theorem we have to compute the right member in (12):

$$\begin{aligned}\partial \mathbf{D}_u \times [\![S^2]\!](\omega) &= \partial \mathbf{D}_u \times [\![S^2]\!](\omega^{(2)}) \\ &= \partial \mathbf{D}_u \times [\![S^2]\!](a\omega_{S^2} + d_y\,\eta) \\ &= 4\pi\partial \mathbf{D}_u(a) + (\partial \mathbf{D}_u \times [\![S^2]\!])(d_y \eta).\end{aligned}$$

The conclusion follows in that $(\partial \mathbf{D}_u \times [\![S^2]\!])(d_y\,\eta) = 0$ ($[\![S^2]\!]$ has no boundary!). ■

If $u : B^3 \to S^2$ is smooth except at a point $x_0 \in B^3$, then we can define the degree of $u$ in $x_0$ as the integer number $\deg(u, x_0)$ such that

$$u_\sharp[\![\partial B_r]\!] = \deg(u, x_0)[\![S^2]\!]$$

where $B_r := B_r(x_0) \subset B^3$; in other words, $\deg(u, x_0)$ is the degree of the function $u_r := u_{|\partial B_r} : \partial B_r \to S^2$ which doesn't depend on the choice of the radius $r$.
We have the following proposition.

**Proposition 9** . *Let $u \in W^{1,2}(B^3, S^2)$. We suppose $u$ is smooth with a finite number of singularities $x_1, \ldots, x_n$. Then the following are equivalent:*

*(i)* $\deg(u, x_i) = k_i \in \mathbf{Z}$;

*(ii)* $\partial G_u \llcorner (B^3 \times S^2) = -k_i \delta_{x_i} \times [\![S^2]\!]$;

*(iii)* $\partial \mathbf{D}_u = -k_i \delta_{x_i}$;

*(iv)* $\mathrm{div} D_u = k_i \delta_{x_i}$.

*The number $k_i$ will be said "the topological charge" of $\mathbf{D}_u$ at $x_i$.*

*Proof.* We prove only the equivalence (i)$\Leftrightarrow$(ii), the others being easy.

(i)$\Rightarrow$(ii): suppose $x_0$ is a singular point for $u$ and $B_r$ a small sphere of center $x_0$. We have

$$\hat{\pi}_\sharp G_{u_r} = u_\sharp [\![\partial B_r]\!] = \deg(u, x_0)[\![S^2]\!].$$

Since $u$ is singular in $x_0$, the support of $\partial G_u \llcorner (B_r \times S^2)$ has to be a subset of $\{x_0\} \times S^2$; then, by constancy theorem, it follows

$$\partial G_u \llcorner (B_r \times S^2) = c\, \delta_{x_0} \times [\![S^2]\!].$$

Moreover we have

$$\partial(G_u \llcorner (B_r \times S^2)) = \partial G_u \llcorner (B_r \times S^2) + G_y \llcorner (\partial B_r \times S^2) = \partial G_u \llcorner (B_r \times S^2) + G_{u_r}$$

whence, projecting by $\hat{\pi}$ both sides of the equality, we get

$$0 = \deg(u, x_0)[\![S^2]\!] + c[\![S^2]\!]. \tag{13}$$

Hence we have $c = k_0$ and adding on all the singularities points we get (2).

(ii)$\Rightarrow$(i): we have only to use (13) and the hypothesis

$$\partial G_u \llcorner (B_r \times S^2) = k_0 \delta_{x_0} \times [\![S^2]\!].$$

∎

**Remark 10** . If $u = \varphi$ on $\partial B^3$ and $\varphi \in C^1(B^3, S^2)$ then $\sum_i k_i = 0$

## 14 The space $\mathrm{cart}^{2,1}(B^3, S^2)$

Let $u_k : B^3 \to S^2$ be a sequence of Lipschitz functions such that

$$\int |Du_k|^2 dx \leq C.$$

Then the number sequence $\{\mathbf{M}(G_{u_k}\}$ is equibounded in that

$$|M_{(2)} Du_k| \leq \frac{1}{2}|Du_k|^2 \qquad \text{and} \qquad \det Du_k = 0.$$

Thus the following holds (possibly for a subsequence):

$$G_{u_k} \rightharpoonup T \in \mathrm{cart}(B^3 \times \mathbf{R}^3).$$

If we suppose that $u_k \rightharpoonup u$ in $W^{1,2}$ (which implies $u_k \xrightarrow{L^2} u$) then ,by theorem 14, we obtain $T = G_u + S$.
Furthermore, the couple of convergences

$$u_k \xrightarrow{L^2} u \qquad \text{and} \qquad Du_k \stackrel{L^2}{\rightharpoonup} Du$$

implies that

$$(G_{u_k})_{(0)}(\phi(x,y)\, dx) = \int \phi(x, u_k(x))dx \longrightarrow \int \phi(x, u(x)) = G_{u_{(0)}}(\phi)$$

and analogously

$$G_{u_{k_{(1)}}}(\phi\, d\hat{x}^i \wedge dy^j) \to G_{u_{(1)}}(\phi\, d\hat{x}^i \wedge dy^j).$$

These preliminaries provide the motivation for the following definition.

**Definition 10** . *By* $\mathrm{cart}^{2,1}(B^3, S^2)$ *we denote the family of the currents* $T$ *such that the following hypotheses are satisfied:*

*(H1)* $T \in \mathrm{cart}(B^3 \times \mathbf{R}^3)$*;*

*(H2)* $u_T \in W^{1,2}(B^3, \mathbf{R}^3)$*;*

*(H3)* $T_{(0)} = G_{u_{T(0)}}$ *and* $T_{(1)} = G_{u_{T(1)}}$*;*

*(H4) the support of* $T$ *is contained in* $B^3 \times S^2$*.*

**Example 7** . $T = G_{x/|x|} + \delta_0 \times B^3 \in \text{cart}^{2,1}(B^3, S^2)$.

•

**Definition 11** . *Let $T_k \in \text{cart}^{2,1}(B^3, S^2)$. We say that $T_k$ converges weakly in* $\text{cart}^{2,1}$ *to $T$ if*

$$T_k \rightharpoonup T \qquad \textit{and} \qquad \sup_k \{\mathbf{M}(T_k) + \frac{1}{2}\int |Du_{T_k}|^2\} \leq \text{const.}$$

By taking into account Theorem 14, it is easy to check that the following result holds.

**Theorem 15** . $\text{cart}^{2,1}(B^3, S^2)$ *is weakly closed with respect the weak convergence in* $\text{cart}^{2,1}$.

The structure of the currents belonging to $\text{cart}^{2,1}$ is described in the following theorem.

**Theorem 16 (structure theorem)** . *If $T \in \text{cart}^{2,1}(B^3, S^2)$ then there exists a one dimensional rectifiable current $L_T \in \mathcal{D}_1(B^3)$ such that*

$$T = G_{u_T} + L_T \times [\![S^2]\!].$$

*Proof.* By theorem 14, we obtain that

$$T = G_{u_T} + S$$

where, just by definition, $S_{(0)} = S_{(1)} = S_{(3)} = 0$.

As $\partial G_{u_T} = \partial \mathbf{D}_{u_T} \times [\![S^2]\!]$ holds by proposition 8, one has

$$\partial S = -\partial \mathbf{D}_{u_T} \times [\![S^2]\!].$$

Let us define

$$L_T(\alpha) := \frac{1}{4}\pi_\sharp(S(\alpha \wedge \omega_{S^2}))$$

for all $\alpha \in \mathcal{D}^1(B^3)$ so that

$$\partial L_T = -\partial \mathbf{D}_{u_T}.$$

In order to check that $L_T \times [\![S^2]\!] + S = 0$ we can consider only forms of the type

$$\omega := \alpha \wedge \omega_{S^2} + d_y\eta$$

with $\eta \in \mathcal{D}^{(2,1)}(B^3, S^2)$.

Then we have

$$\begin{aligned}(S - L_T \times [\![S^2]\!])(\omega) &= (S - L_T \times [\![S^2]\!])(d_y\eta) = (S - L_T \times [\![S^2]\!])(d\eta) \\ &= \partial S(\eta) - (\partial L_T \times [\![S^2]\!])(\eta) = 0.\end{aligned}$$

The rectifiability of $L_T$ follows, by slicing, from the rectifiability of $S$. ■

## 15 Approximation by smooth data

**Theorem 17 (Approximation theorem)** . *Let $T \in \mathrm{cart}^{2,1}(B^3 \times S^2)$. Then there exists a sequence of smooth maps $u_k : B^3 \to S^2$ such that*

$$\begin{cases} G_{u_k} \rightharpoonup T & \text{in } \mathrm{cart}^{2,1}(B^3 \times S^2) \\ \frac{1}{2}\int_{B^3} |Du_k|^2 dx \to \frac{1}{2}\int_{B^3} |Du_T|^2 dx + 4\pi \mathbf{M}(L_T) \end{cases}$$

*where $T = G_{u_T} + L_T \times [\![S^2]\!]$ is the decomposition provided from structure theorem.*

We shall give a sketch of the proof in the semplified "case of smooth Dirichlet data", even if the general statement can be proved following the same path. We need some notation. Let $\mathcal{B}^3 \supset \bar{B}^3$ be an open ball concentric to $B^3$, let $\varphi : \mathcal{B}^3 \to S^2_y$ be smooth (so that $\deg \varphi_{|S^2_x} = 0$) and consider the set

$$\mathrm{cart}^{2,1}_\varphi(\mathcal{B}^3 \times S^2) := \{T \in \mathrm{cart}^{2,1}(\mathcal{B}^3 \times S^2) | (T - G_\varphi) \llcorner (\mathcal{B}^3 \backslash \bar{B}^3) = 0\}$$

We shall prove that Theorem holds for $T \in \mathrm{cart}^{2,1}_\varphi(\mathcal{B}^3 \times S^2)$; in this case, the sequence $u_k$ can be chosen such that $u_k = \varphi$ in $\mathcal{B}^3 \backslash \bar{B}^3$.
We need the following.

**Theorem 18 (Bethuel)** . *Let*

$$\begin{aligned} W^{1,2}(\mathcal{B}^3, S^2) &:= W^{1,2}(\mathcal{B}^3, S^2) \cap \{u | u = \varphi \text{ in } \mathcal{B}^3 \backslash \bar{B}^3\} \\ R^\infty_{2,\varphi}(\mathcal{B}^3, S^2) &:= \{u \in W^{1,2}_\varphi(\mathcal{B}^3, S^2) | u \text{ is smooth except at a finite number} \\ &\qquad \text{of points}\}. \end{aligned}$$

*Then*

$$\overline{R^{\infty}_{2,\varphi}(\mathcal{B}^3, S^2)}^{W^{1,2}} = W^{1,2}_{\varphi}(\mathcal{B}^3, S^2)$$

**Sketch of the proof of the approximation theorem.** We begin by introducing some useful notation and by recalling a theorem due to Federer. For $u \in W^{1,2}_{\varphi}(\mathcal{B}^3, S^2)$, we define

$$\mathbf{P}_u := \pi_\sharp \left( \partial G_u \llcorner \hat{\pi}^\sharp (\frac{\omega_{S^2}}{4\pi}) \right) = \partial \mathbf{D}_u.$$

If $\mathbf{P}$ is a zero dimensional current in $\mathcal{B}^3$ with support in $\bar{B}^3$, we set

$$F_{\bar{B}^3}(\mathbf{P}) := \sup \{ \mathbf{P}(\xi) \mid \xi \in \mathcal{D}(\mathcal{B}^3),\ \|d\xi\| \leq 1 \text{ in } \bar{B}^3 \},$$

$$m_r(\mathbf{P}) := \inf \{ \mathbf{M}(\mathbf{D}) \mid \mathbf{D} \in \mathcal{D}_1(\mathcal{B}^3),\ \operatorname{spt} \mathbf{D} \subset \bar{B}^3,\ \partial \mathbf{D} = \mathbf{P} \}$$

and

$$m_i(\mathbf{P}) := \inf \{ \mathbf{M}(L) \mid L \in \mathcal{D}_1(\mathcal{B}^3) \text{ is rectifiable, } \operatorname{spt} L \subset \bar{B}^3,\ \partial L = \mathbf{P} \}.$$

**Theorem 19 (Federer)** . *If $L$ is integral, then $m_r(\partial L) = m_i(\partial L)$.*

Moreover it can be proved that $F_{\bar{B}^3}(\mathbf{P}) = m_r(\mathbf{P})$.

Let us start with the sketch. By Bethuel theorem one can find $u_k \in R^{\infty}_{2,\varphi}$ such that

$$u_k \overset{W^{1,2}}{\to} u_T \qquad \text{and} \qquad G_{u_k} \rightharpoonup G_{u_T}.$$

In particular, it follows that

$$\mathbf{M}(\mathbf{D}_{u_k} - \mathbf{D}_{u_T}) = \int_{\mathcal{B}^3} |Du_k - Du_T| \, dx \to 0$$

as $k \to +\infty$.

Hence, possibly by passing to a subsequence, we can assume that

$$\mathbf{M}(\mathbf{D}_{u_{k+1}} - \mathbf{D}_{u_k}) \leq 2^{-k}.$$

Also, we can find a one dimensional rectifiable current $\tilde{L}_k$ carried by pieces of geodesics in $\bar{B}^3$ in such a way that

$$\mathbf{P}_{u_k} - \mathbf{P}_{u_{k+1}} = \partial \tilde{L}_k$$

and $\mathbf{M}(\tilde{L}_k)$ is equal to the sum of the lengths of those pieces of geodesics. By virtue of the recalled Federer's theorem, one obtains

$$\mathbf{M}(\tilde{L}_k) = m_i(\mathbf{P}_{u_k} - \mathbf{P}_{u_{k+1}}) = F_{\bar{B}^3}(\mathbf{P}_{u_k} - \mathbf{P}_{u_{k+1}}) \leq \mathbf{M}(\mathbf{D}_{u_{k+1}} - \mathbf{D}_{u_k}) \leq 2^{-k}.$$

Hence, by setting

$$L_{u_k,u_T} := -\sum_{j=k+1}^{+\infty} \tilde{L}_j \qquad \text{and} \qquad L_k := L_T - L_{u_k,u_T},$$

we obtain that

$$\mathbf{M}(L_{u_k,u_T}) = 0(k^{-1}), \qquad \partial L_{u_k,u_T} = \mathbf{P}_{u_T} - \mathbf{P}_{u_k} \tag{14}$$

and then also

$$\partial L_k = \mathbf{P}_{u_k}. \tag{15}$$

So, we have found a sequence $\{L_k\}$ of 1-dimensional currents suitably approximating the current $L_T$ and having the singular set of $u_k$ as support of the boundary.
Hence one has

$$\frac{1}{2}\int_{B^3} |Du_T|^2 dx + 4\pi\mathbf{M}(L_T) = \frac{1}{2}\int_{B^3} |Du_k|^2 dx + 4\pi\mathbf{M}(L_k) + 0(k^{-1}). \tag{16}$$

Now, since spt $\partial L_k$ contains only a finite number of points, and by means of Federer's approximation theorem, one can find (for all $k$) a polyhedral chain $P_k$ close to $L_k$ and a perturbation of identity $\phi_k : \mathcal{B}^3 \to \mathcal{B}^3$ such that

$$\phi_{k|\mathcal{B}^3\setminus\bar{B}^3} = id$$

and

$$\begin{cases} v_k \to u_T \quad \text{in } W^{1,2} \\ T_k := G_{v_k} + P_k \times [\![S^2]\!] \rightharpoonup T \\ \frac{1}{2}\int_{B^3} |Dv_k|^2 + 4\pi\mathbf{M}(P_k) \to \frac{1}{2}\int_{B^3} |Du_T|^2 + 4\pi\mathbf{M}(L_T) \end{cases}$$

where $v_k = u_k \cdot \phi_k \in R^{\infty}_{2,\varphi}(B^3, S^2)$.
Also, we can suppose that $P_k$ is simple, i.e. composed by a finite number of not intersecting, multiplicity one segments.

To conclude the proof, now we need to introduce some facts from the theory of the so called "dipole problem".

Let $l$ be a positive number and consider the couple of points in $\mathbf{R}^3$:

$$a_- := (0,0,0), \quad a_+ := (0,0,l)$$

Moreover, for $\sigma \in (0,1)$, let $\varphi_\sigma(x_3) := \min\{x_3, l - x_3, \sigma\}$ and define

$$\Omega_\sigma := \left\{x \in \mathbf{R}^3 | \sqrt{x_1^2 + x_2^2} < \varphi_\sigma(x_3),\ x_3 \in (0,l)\right\}$$

$$\tilde{\Omega}_\sigma := \left\{x \in \mathbf{R}^3 | \sqrt{x_1^2 + x_2^2} < 2\varphi_\sigma(x_3),\ x_3 \in (0,l)\right\} = \{(2x_1, 2x_2, x_3) | x \in \Omega_\sigma\}.$$

Then the following theorem holds:

**Theorem 20** . *Let $U$ be a neighborhood of the segment joining $a_-$ to $a_+$ and let $u : U \to S^2$ be a smooth map. Then for every positive and sufficiently small $\varepsilon$ and $\sigma$ there exists a smooth map $\bar{u} : U \backslash \{a_-, a_+\} \to S^2$ such that*

$$\bar{u} = u \ \ in\ U \backslash \tilde{\Omega}_\sigma, \quad \deg(\bar{u}, a_\pm) = \pm 1$$

*and*

$$\frac{1}{2} \int_{\tilde{\Omega}_\sigma} |D\bar{u}|^2 dx \leq 4\pi l + \varepsilon.$$

Now, we can use Theorem 20 to modify $v_j$ in a suitable neighbourhood of any interior segment $S$ of $P_j$ (i.e. such that $\operatorname{spt} \partial P_j \cap \operatorname{spt} S = \emptyset$) whereby the mass of the segment $S$ "changes into" the Dirichet energy of the modified map near to $\operatorname{spt} S$. Similarly we can proceed when $S$ is a boundary segment, by changing $u$ near the boundary.

All this can be made in such a way that, eventually, one obtains a map $w_j$ which is smooth except on a finite number of points where it has degree zero and such that $G_{w_j} \rightharpoonup T$ and

$$\frac{1}{2} \int_{B^3} |Dv_j|^2 + 4\pi \mathbf{M}(P_j) = \frac{1}{2} \int_{B^3} |Dw_j|^2 + 0(j^{-1}).$$

Hence

$$\frac{1}{2} \int_{B^3} |Dw_j|^2 \to \frac{1}{2} \int_{B^3} |Du_T|^2 + 4\pi \mathbf{M}(L_T). \tag{17}$$

Now we are able to eliminate the singularities of $w_j$ as the degree of $w_j$ at the singular points is zero.

## 16 The polyconvex lower semicontinuous extension of the Dirichlet integral

Let $e_1, e_2, e_3$ be the canonical basis of $\mathbf{R}^3 \supset B^3$ and let $u \in \mathbf{C}^1(B^3, S^2)$. Then a simple 3-vector tangent to the graph of $u$ is given by

$$\begin{aligned} M(Du(x)) &= \Lambda^3(id \oplus Du(x))(e_1 \wedge e_2 \wedge e_3) \\ &= (e_1 + D_1 u(x)) \wedge (e_2 + D_2 u(x)) \wedge (e_3 + D_3 u(x)) \\ &\in \Lambda^3(\mathbf{R} \times T_{u(x)} S^2). \end{aligned}$$

Then, by setting

$$\eta_u := \frac{M(Du)}{|M(Du)|}$$

one has

$$\int_{B^3} |Du|^2 dx = \int_{G_u} \frac{|M_{(1)}(Du)|^2}{|M(Du)|^2} d\mathcal{H}^3 = \int_{G_u} |\eta_{u(1)}|^2 d\mathcal{H}^3$$

i.e. the integrand of the Dirichlet integral, written on the graph, is

$$f(x, y; p) = f(p) = \frac{1}{2}|p_{(1)}|^2.$$

Once fixed $(x, y) \in B^3 \times S^2$, we can think $f$ defined on the set of simple vectors

$$\begin{aligned} \Sigma_1(x, y) &:= \left\{\xi \in \Lambda^3(\mathbf{R}^3 \times T_y S^2) \,|\, \xi \text{ is simple and } \xi_0 = e_1 \wedge e_2 \wedge e_3\right\} \\ &= \left\{M(G) \,|\, G : \mathbf{R}^3 \to T_y S^2 \text{ is linear}\right\}. \end{aligned}$$

Let us define, for all $\xi \in \Lambda_1 := \{\xi \in \Lambda^3(\mathbf{R}^3 \times \mathbf{R}^3) \,|\, \langle \xi_{(0)}, e_1 \wedge e_2 \wedge e_3 \rangle = 1\}$,

$$\tilde{f}(x, y; \xi) : \sup\{\phi(\xi) \,|\, \phi : \Lambda_1 \to \mathbf{R} \text{ is affine and } \phi(\eta) \leq f(\eta)\ \forall \eta \in \Sigma_1(x, y)\}$$

and then, for $\xi \in \Lambda_+ := \{\xi \in \Lambda^3(\mathbf{R}^3 \times \mathbf{R}^3) \,|\, \langle \xi_{(0)}, e_1 \wedge e_2 \wedge e_3 \rangle > 0\}$, we set

$$F(x, y; \xi) := |\xi_{(0)}| \tilde{f}\left(\frac{\xi}{|\xi_{(0)}|}\right).$$

$F$ is the largest convex and l.s.c. minorant of $f$ on $(x, y) \times \Sigma_1(x, y)$ for all fixed $(x, y) \in \mathbf{R}^3 \times S^2$ and it can be explicitly computed. In fact, one can prove that

$$F(x, y; \xi) = +\infty \qquad \text{if } |\xi| \neq 1$$

while, if $|\xi| = 1$,

$$F(x, y; \xi) = \begin{cases} \frac{1}{2}|G_\xi|^2 + \|\xi_{(2)} - |\xi_{(0)}| M_{(2)}(G_\xi)\| & \text{if } \xi_{(0)} \neq 0 \text{ and } \frac{\xi}{|\xi_{(0)}|} \in \Sigma_1(x, y) \\ 1 & \text{if } \xi_{(0)} = \xi_{(1)} = 0 \text{ and } \xi \text{ is simple} \\ +\infty & \text{otherwise} \end{cases}$$

where $G_\xi = M^{-1}\left(\frac{\xi}{|\xi_{(0)}|}\right)$.

Now, given any rectifiable three dimensional current $T = [\![R, \eta, \Theta]\!]$ in $B^3 \times \mathbf{R}^3_y$, we can write explicitly the polyconvex extension of the Dirichlet integral

$$\mathcal{F}(T) := \int F(x, y; \eta)\Theta \, d\mathcal{H}^3 \llcorner R$$

and conclude that $\mathcal{F}(T) < +\infty$ if and only if $T \in \operatorname{cart}^{2,1}(B^3, S^2)$, in which case

$$\mathcal{F}(T) = \frac{1}{2}\int_{B^3} |Du_T|^2 dx + 4\pi \mathbf{M}(L_T)$$

by the structure theorem.

As a consequence of the previous considerations, taking into account the approximation theorem we conclude that the relaxed Dirichlet functional $\bar{\mathcal{D}}$, with respect to the weak convergence in the class of rectifiable currents, agrees with $\mathcal{F}$ at every $T \in \operatorname{cart}^{2,1}(B^3, S^2)$.

In fact, consider $\mathcal{D} : \mathcal{R}_3(B^3 \times \mathbf{R}^3_y) \to \bar{\mathbf{R}}$ defined by:

$$\mathcal{D}(T) = \begin{cases} \frac{1}{2}\int_{B^3} |Du|^2 & \text{if } T = [\![\text{graph of } u]\!],\ u \in \mathbf{C}^1(B^3, S^2) \\ +\infty & \text{otherwise} \end{cases}$$

and recall that

$$\bar{\mathcal{D}}(T) = \inf\{\liminf_k \mathcal{D}(T_k) \,|\, T_k \in \mathcal{R}_3(B^3 \times \mathbf{R}^3),\ T_k \rightharpoonup T.\}$$

Then, for $T \in \text{cart}^{2,1}(B^3, S^2)$ (let $u_k$ be the sequence existing by the approximation theorem), one has

$$\begin{aligned}\mathcal{F}(T) &\leq \bar{\mathcal{D}}(T) \leq \liminf_k \mathcal{D}(G_{u_k}) = \liminf_k \frac{1}{2}\int_{B^3} |D_{u_k}|^2 dx = \\ &= \frac{1}{2}\int_{B^3} |D_{u_T}|^2 dx + \mathbf{M}(L_T) = \mathcal{F}(T)\end{aligned}$$

i.e. just

$$\mathcal{F}(T) = \bar{\mathcal{D}}(T).$$