EXTENSION PROPERTY

Theorem 0.1 (Extension Property)**.** *Let Ω be a bounded domain and let $v \in W^{1,2}(\Omega, \mathbb{R}^3)$ with $v(x) \in \mathbb{S}^2$ for a.e. $x \in \partial\Omega$. Then there exists a map $u \in W^{1,2}(\Omega, \mathbb{S}^2)$,*

$$u\Big|_{\partial\Omega} = v\Big|_{\partial\Omega}$$

with the estimate

$$\|\nabla u\|_{L^2(\Omega)} \leq C\,\|\nabla v\|_{L^2(\Omega)}$$

for a uniform constant C.

Proof. Let $a \in \mathbb{R}^3,\ |a| < 1$, consider the maps

$$u_a(x) = \frac{v(x) - a}{|v(x) - a|}, \quad x \in \Omega.$$

Then,

$$\nabla u_a(x) = \frac{\nabla v(x)}{|v(x) - a|} - \frac{(v(x) - a) \otimes (v(x) - a)}{|v(x) - a|^3} \nabla v(x), \quad \text{for } x \in \Omega.$$

Thus,

$$|\nabla u_a(x)| \leq \sqrt{2}\,\frac{|\nabla v(x)|}{|v(x) - a|}.$$

Hence,

$$\int_{B_{\frac{1}{2}}} |\nabla u_a|^2 \, da \leq 2 \int_{B_{\frac{1}{2}}} \frac{|\nabla v|^2}{|v - a|^2} \, da \leq 8\pi |\nabla v|^2,$$

because for any $p \in \mathbb{R}^3$ we have, for a $q \in B_{\frac{1}{2}}$, that

$$\int_{B_{\frac{1}{2}}} \frac{1}{|p - a|^2} \, da \leq \int_{B_{\frac{1}{2}}} \frac{1}{|q - a|^2} \, da \leq \int_{B_{\frac{1}{2}}(q)} \frac{1}{|y|^2} \, dy \leq \int_{B_1} \frac{1}{|y|^2} \, dy = 4\pi.$$

Integrating the above inequality over Ω we get by Fubini's theorem

$$\int_{B_{\frac{1}{2}}} \int_\Omega |\nabla u_a|^2 \, dx \, da = \int_\Omega \int_{B_{\frac{1}{2}}} |\nabla u_a|^2 \, da \, dx \leq 8\pi \int_\Omega |\nabla v|^2 \, dx.$$

Thus, again by Fubini's theorem, we infer the existence of $a_0 \in B_{\frac{1}{2}}$ for which

$$\int_\Omega |\nabla u_{a_0}|^2 \, dx \leq 8\pi \left| B_{\frac{1}{2}} \right|^{-1} \int_\Omega |\nabla v|^2 \, dx = 48 \int_\Omega |\nabla v|^2 dx. \tag{0.1}$$

For all $a \in B_{\frac{1}{2}}$ let

$$\Pi_a(\xi) = \frac{\xi - a}{|\xi - a|} \quad \text{for } \xi \in \mathbb{S}^2.$$

This is a C^1 homeomorphism of $\mathbb{S}^2$ onto itself. Indeed, the inverse map is given by

$$\Pi_a^{-1}(\xi) = a + [(a\cdot\xi)^2 + (1-|a|^2)]^{1/2}\xi. \tag{0.2}$$

Thus, after simple computations

$$|\nabla \Pi_a^{-1}(\xi)| \leq 2, \quad \text{for all } |a| \leq \frac{1}{2}.$$

We now observe that since $|v| \equiv 1$ almost everywhere on $\partial\Omega$ we have

$$u_a = \Pi_a \circ v \quad \text{on } \partial\Omega. \tag{0.3}$$

Finally, we define $u := \Pi_{a_0}^{-1} \circ u_{a_0}$. By (0.3) we have $u\big|_{\partial\Omega} = v\big|_{\partial\Omega}$ and combining (0.2) with (0.1) we conclude

$$\int_\Omega |\nabla u|^2\,dx \leq \left(\mathrm{Lip}\,(\Pi_{a_0}^{-1})\right)^2 \int_\Omega |\nabla u_{a_0}|^2\,dx \leq 192 \int_\Omega |\nabla v|^2\,dx.$$

□