Opial-type inequalities

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Original Opial's result

Theorem (Opial, 1960)

Let x be a function in class $C^1((0,h))$ and $x(0) = x(h) = 0$, $x(t) > 0$ for $0 < t < h$. Then:

$$\int_0^h |x(t)x'(t)|dt \leq \frac{h}{4}\int_0^h x'(t)^2 dt.$$

The constant $h/4$ is sharp.

For those who like convex functions

Theorem (Godunova, Levin, 1967)

Let f be convex and increasing function on $[0, \infty)$ with $f(0) = 0$. Further, let x be an a.c. function with $x(\alpha) = 0$. Then the following inequality holds:

$$\int_{\alpha}^{\tau} f'(|x(t)|)|x'(t)|dt \leq f\left(\int_{\alpha}^{\tau} |x'(t)|dt\right).$$

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Theorem (Qi, 1985)

Let x be an absolutely continuous function on $[a, b]$ and $x(a) = 0$ or $x(b) = 0$. Let $P(u)$ and $Q(u)$ be non-decreasing functions on $[0, \infty]$, additionally, let Q be convex and $P(0) = Q(0) = 0$. If $Q^{-1}(v)$ denotes right-continuous inverse to Q, and $R(u) := \int_0^u P((b-a)Q^{-1}(s))ds$ then:

$$\int_a^b P(|x(t)|)Q(|x'(t)|)dt \leq$$

$$\leq (b-a)R\left(\frac{1}{b-a}\int_a^b Q(|x'(t)|)dt\right) \leq$$

$$\leq \int_a^b R(Q(|x'(t)|))dt.$$

For those who like weights

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$$\int_a^b q(t)|y(t)|^l|y'(t)|^m dt \leq K \int_a^b p(t)|y'(t)|^{l+m} dt$$

For those who like weights

Theorem (Beesack, Das, 1968)

Let $l, m > 0$ and $l + m > 1$, and let p, q be nonnegative, measurable weight functions on (a, b) such that: $\int_a^b (p(t))^{-\frac{1}{l+m-1}} dt < \infty$ and the constant

$$K := \left(\frac{m}{l+m}\right)^{\frac{m}{l+m}} \left[\int_a^b q(t)^{\frac{l+m}{l}} p(t)^{-\frac{m}{l}} \left(\int_a^t p(s)^{-\frac{1}{l+m-1}} ds\right)^{l+m-1} dt\right]^{\frac{l}{l+m}}$$

is finite. If y is an absolutely continuous function on $[a, b]$ and $y(a) = 0$, the following inequality holds:

$$\int_a^b q(t)|y(t)|^l |y'(t)|^m dt \leq K \int_a^b p(t)|y'(t)|^{l+m} dt$$

For those who like higher order derivatives

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Let $x \in C^n[0, a]$ be such that $x^{(i)}(0) = 0$ for $0 \leq i \leq n-1$. Then the following inequality holds:

$$\int_0^a |x(t)x^{(n)}(t)|dt \leq \frac{1}{2}a^n \int_0^a |x^{(n)}(t)|^2 dt.$$

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Let $l \geq 1$, $m > 0$, $r_k \geq 0$, $0 \leq k \leq n-1$, with $\sum_{k=0}^{n-1} r_k = 1$. Further, let $x \in C^{n-1}[0,a]$ be such that $x^{(i)} = 0$ for $0 \leq i \leq n-1$, $x^{(n-1)}(t)$ is a. c. and $\int_0^a |x^{(n)}(t)|^{l+m} dt \leq \infty$. Then,

$$\int_0^a \left(\prod_{k=0}^{n-1} |x^{(k)}(t)|^{r_k} \right)^m |x^{(n)}(t)|^l dt \leq \sum_{k=0}^{n-1} c^*_{n-k} r_k a^{(n-k)l} \int_0^a |x^{(n)}(t)|^{l+m} dt,$$

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where $c_i^* := \xi m^{m\xi}[\frac{i(1-\xi)}{(i-\xi)}^{l(1-\xi)}](i!)^{-l}$, $\xi = (l+m)^{-1}$

For those who like vector-valued functions

Theorem (Pachpatte, 1986)

Let $p(t)$ be positive and continuous on $[\alpha, \tau]$ with $\int_{\alpha}^{\tau} \frac{1}{p(s)} ds < \infty$ and let $q(t)$ be positive, bounded and non-increasing on $[\alpha, \tau]$. Further, let $\mathbf{x}(t) = (x_1(t), x_2(t))$ where x_i are absolutely continuous on $[\alpha, \tau]$ and $\mathbf{x}(\alpha) = \mathbf{0}$. Then the following inequality holds:

$$\int_{\alpha}^{\tau} q(t)(|x_1(t)x_2'(t)| + |x_1'(t)x_2(t)|)dt \leq$$

$$\leq \frac{1}{2} \int_{\alpha}^{\tau} \frac{dt}{p(t)} \int_{\alpha}^{\tau} p(t)q(t)||\mathbf{x}'(t)||_2^2 dt.$$

For those who like PDE's

Theorem (Pachpatte, 1989)

Let $x_i(t,s) \in C^{(1,1)}(R)$, $1 \le i \le m$ be such that $x_i(a,s) = x_i(b,s) = 0$, $x_i(t,c) = x_i(t,d) = 0$. Further, let $f_i(r)$, $1 \le i \le m$ be continuously differentiable on $[0,\infty)$ with $f_i(0) = 0$, $f_i'(r) \le 0$ and non-decreasing on $[0,\infty)$. Then the following inequality holds:

$$\int\int_R \sum_{i=1}^{m} \left(\prod_{j=1, j\neq i}^{m} f_j(|x_j(t,s)|) \right) f_i'(|x_i(t,s)|) |D_1 D_2 x_i(t,s)| dtds \le$$

$$\le \sum_{k=1}^{4} \prod_{i=1}^{m} f_i \left(\int\int_{R_k} |D_1 D_2 x_i(t,s)| dtds \right)$$

Where: $R = [a,b] \times [c,d]$, $R_1 := [a,T] \times [c,S]$, $R_2 := [T,b] \times [S,d]$, $R_3 := [a,T] \times [S,d]$, $R_4 := [T,b] \times [S,d]$

For those who like polar coordinates

Theorem (Nečaew, 1973)

Let $u(x)$ be differentiable function defined on a convex bounded domain $V \in \mathbb{R}^n$ in which exist a point x_0 such that $u(x_0) = 0$. Further, let l and m be positive numbers with $l + m \leq 1$ and let $p(x), q(x)$ be non-negative measurable functions on V such that:

$$\int_0^{r(\phi)} \left[p(\rho, \phi)\rho^{n-1}\right]^{-1/(l+m-1)} d\rho < \infty,$$

where $r(\phi) := max\{||(\rho, \phi) - x_0||, (\rho, \phi) \in V\}$. Then the following inequality holds:

$$\int_V q(x)|u(x)|^l ||\nabla u(x)||^m dx \leq \left(\frac{m}{l+m}\right)^{\frac{m}{l+m}} \int_V Kp(x)||\nabla u(x)||^{l+m} dx,$$

Where

$$K = K(\phi, l, m) = \left(\int_0^{r(\phi)} \rho^{n-1} (q(\rho, \phi))^{(l+m)/l} p(\rho, \phi)^{-m/l} \times \right.$$
$$\left. \times \left[\int_0^{\rho} (p(\sigma, \phi) \sigma^{n-1})^{-1/(l+m-1)} d\sigma \right]^{l+m-1} d\rho \right)^{l/(l+m)} .$$

The bible

R. P. Agarwal, P. Y. Pang *Opial Inequalities with Applications in Differential and Difference Equations*, 1995