Variational problems in homotopy classes

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First lecture

Let M and N be compact smooth manifolds, $\dim M = m$ and $\dim N = n$. We shall assume that M and N have a riemannian metric and that they are isometrically embedded in two euclidean spaces we denote by E_1 and E_2 respectively.

Given a function

$$F : M \times N \times \{ \text{ linear maps defined from } m\text{-planes in } E_1 \text{ to } E_2\} \longrightarrow \mathbf{R}$$

we consider the corresponding functional

$$F : Lip(M, N) \to \mathbf{R}$$

given by

$$F[u] = \int_M F(x, u(x), Du(x))\, dx.$$

We want to study maps u that are local minima for F, that is $F[u] \leq F[v]$ for all v sufficently "near" u in a sense to be made precise. In particular, we are interested in proving existence of local minimizers.

Let us see some classes of functionals of this kind:

1) PARAMETRIC FUNCTIONALS

We say that F is "parametric" if

$$F[u] = F[u \circ \varphi]$$

whenever φ is an orientation-preserving diffeomorphism, $\varphi : M \to M$.

Example 1 AREA.

$$\begin{aligned} F(x,y,L) &= A(x,y,L) = |Jac_m\, L| = \\ &= \text{area spanned?? by } L(e_1),\ldots,L(e_m), \end{aligned}$$

where $e_1,\ldots,e_m$ is an orthogonal basis of the m-plane where L is defined.
Thus we have $A[u] =$ area of $u(M)$, counting multiplicity.

2) ENERGY-TYPE FUNCTIONALS 2a) ENERGY OF u

$$E[u] = \int_M |Du|^2,$$

where $|L|$ is the euclidean norm of L. In local coordinates for M we have

$$|Du(x)|^2 = g^{i,j}(x)\frac{\partial u}{\partial x^i}(x)\frac{\partial u}{\partial x^j}(x)$$

and

$$E[u] = \int g^{i,j}(x)\frac{\partial u}{\partial x^i}(x)\frac{\partial u}{\partial x^j}(x)\sqrt{det(g_{i,j}(x))}\, dx^1 ... dx^m.$$

2b) p-ENERGY

$$E[u] = \int |Du|^p.$$

2c)

$$E[u] = \int (|Du(x)|^2 + g(u(x))\; d\; vol(x)$$

2d)

$$c_1|L|^p \le F(x,y,L) \le c_2|L|^p$$

or

$$c_1|L|^p \le F(x,y,L) \le c_2(1+|L|^p)\,.$$

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Now we investigate the existence of minima for energy or p-energy in different cases.

1. **Minimize among all maps**
   In this case the minimizers are just the constant maps.

2. **Minimize among lipschitz maps u homotopic to a given map $u_0 : M \to n$.**
   This idea may provide non-trivial minimizers.

Example 2 Let $\dim M = 1$ and $F =$ energy. In this case a minimizer u exists, and it is a constant-speed geodesic which realizes the minimum length in its homotopy class.

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Example 3 (*Eells-Sampson*). $F =$ energy.

If N has negative sectional curvatures, then there exists a unique smooth minimizer

Proposition 1 (Area-Energy inequality)

$$|Jac_m Du| \leq \frac{1}{m^{m/2}} |Du|^m .$$

The equality occours if and only if u is conformal.

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Example 4 $M = N = S^k = k$-sphere in $\mathbf{R}^{k+1}$, $F = p$-energy, $u_0 =$ identity map. We have

$$\begin{aligned} \mathrm{Area}\,(S^k) &= \int \det D \leq (x)\, dx \leq \int |Jac_k\, Du(x)|\, dx \quad (= \text{ iff } \det Du \geq 0) \\ &\leq \frac{1}{k^{k/2}} \int |Du|^k \quad (= \text{ iff } u \text{ is conformal}). \end{aligned}$$

By Hölder inequality we have that if $p > k$ then

$$\mathrm{Area}\,(S^k) \leq c\Big(\int |Du|^p\Big)^{k/p} \qquad (= \text{ iff } |Du| = \text{constant}).$$

If $u = u_0$, one has equality everywhere. Then, if $p > k$, the identity map is a p-energy minimizer.

The same is true for $u_0 =$ identity: $M^k \to M^k$.

Now we consider the case $p < k$. We note that there are many conformal maps from S^k to S^k. For example if S is the stereographic projection $S : S^k \to \mathbf{R}^k$, then the maps $S_l(z) = S^{-1}(lS(z))$ are conformal and homotopic to the identity map.

Since for all conformal maps u we have

$$\mathrm{Area}\,(S^k) = \frac{1}{k^{k/2}} \int |Du|^k \geq c\Big(\int |Du|^p \Big)^{k/p},$$

with equality holding if and only if u is constant, the identity is a maximizer among all conformal maps homotopic to identity.

It is easy to check that

$$\lim_{l \to \infty} \int |DS_l|^p = 0\,.$$

Indeed $S_l(z) \to$ north pole, except when $z =$ south pole.

Thus we have the following difficulty:
$\inf\{F(u) \mid u$ homotopic to $u_0\}$ may well be 0 even if u_0 is not homotopic to a constant.

In the special case $p = k$ we have infinitely many minimizers which are all the conformal maps.

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Example 5 $F =$ energy, $M = T^2$, $N = S^2$.

We consider the function $u_0 : T^2 \to S^2$, which maps the torus minus a small disc D to the north pole of S^2, while $u_0(D)$ covers $S^2 \setminus \{\text{north pole}\}$ once. The degree of u_0 is one, hence u_0 is not homotopic to a constant map. We have

$$4\pi = \mathrm{Area}\,(S^2) \leq \frac{1}{2} \int_{T^2} |Du|^2\,.$$

We can show that 4π is actually the infimum, but it is not a minimum because there are no conformal maps between T^2 and S^2, hence the inequality above is strict.

It is not difficult to construct a minimizing sequence converging to a constant (which belongs to a different homotopy class).

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Example 6 $M = N = S^2$, $u_0 = \mathrm{id}$

$$F[u] = \frac{1}{2}\int |Du|^2 + \int u \cdot v$$

where v is the unit vector pointing towards the south pole. We have

$$\begin{aligned} F[u] &\geq \text{Area}\,(u) + \int u \cdot v \\ &\geq 4\pi + \int u \cdot v \geq 4\pi - 4\pi = 0\,. \end{aligned}$$

Clearly, the value zero is never attained in the homotopy class of u_0. On the other hand

$$\lim_{l \to \infty} F[S_l] = 0\,.$$

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For the abowe reasons, we wonder under which conditions one has

$$\inf \{ \int |Du|^p \mid u \text{ homotopic to } u_0 \} = 0,$$

with u_0 not homotopic to a constant map.

Second lecture

Let us begin with some remarks on homotopy.

1. If N is nice (that is, a lipschitz neighborhood retract) then any continuous map $f : A \to N$, with A a manifold or a polyhedral complex, is continuously homotopic to a lipschitz map $g : A \to N$.

2. If N and A are as above and two lipschitz maps $f_0, f_1 : A \to N$ are continuously homotopic, there exists also a lipschitz homotopy. (The idea is to regularize the homotopy by convolution, and then to project down on N by means of the lipschitz retraction).

We denote by $[A, N]$ the set of homotopy equivalence classes of maps from A to N. In the special case $A = S^k$, we write $\pi_k(N) = [S^k, N]$ (k-th homotopy group of N). π_k has a natural group structure, and it is abelian if $k \geq 2$.

Definition 1 *Let M, N be compact oriented manifolds without boundary of the same dimension. We want to define the* degree *of a continuous map $f : M \to N$. We take a smooth map $\tilde{f} : M \to N$ homotopic to f. By Sard's theorem we know that for almost every $p \in N$*

$$\tilde{f}^{-1}(p) = \{x_1, \ldots, x_k\}$$

is a finite set of regular points.

In this case we define

$$\mathrm{Deg}\,(f) = \sum_{q \in f^{-1}(p)} \mathrm{sign}\,(\det D\tilde{f}(q))\,.$$

This is a good definition because it does not depend on the regular value $p \in N$, and on the choice of the function $\tilde{f}$ homotopic to f. Indeed, the degree is a homotopy invariant.

If M or N is not orientable, then we may define

$$\mathrm{Deg}\,(f) = \#\,\{\tilde{f}^{-1}(p)\} \bmod 2\,,$$

where p is a regular value of $\tilde{f}$. This is again a well defined homotopic invariant.

Example 7 The map u_0 in example 5 has degree 1, hence it is not homotopic to a constant.

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Theorem 1 *If* $\dim M = k$, *two maps from M into S^k are homotopic iff their degrees coincide (see [2])*

Definition 2 *Let $f, g : M \to N$, we say that f is k-homotopic to g, $f \sim_k g$, if and only if $f_{|X}$ is homotopic to $g_{|X}$, whenever X is the k-skeleton of a triangulation of M.*

Proposition 2 *If Σ is a polyhedral complex of dimension $\leq k$ and $\varphi : \Sigma \to M$, $f, g : M \to N$ are lipschitz functions with $f \sim_k g$, then $f \circ \varphi$ is homotopic to $g \circ \varphi$.*

Proof. By a deformation theorem, φ is homotopic to a map $\tilde{\varphi}$ such that $\tilde{\varphi}(\Sigma) = X$, where X is a k-skeleton of M. ∎

Corollary 1 *If $f, g : M \to N$ and $f \sim_k g$, then they induce the same homomorphisms $\pi_i(M) \to \pi_i(N)$ for $i \leq k$.*

The same result holds for homology groups.

Theorem 2 (Homotopy extension theorem) *If A is a nice subset of M and $f_0 : M \to N$, then any homotopy of $f_{0|A}$ can be extended to a homotopy of f_0.*

Theorem 3 *Let $f_0 : M \to N$ be lipschitz (with M and N compact manifolds), then*

$$\inf \left\{ \int_M |Df|^p \quad \middle| \quad f : M \to N \text{ lipschitz and } f \text{ homotopic to } f_0 \right\} = 0$$

if and only if f_0 is $[p]$-homotopic to a constant map, where $[p]$ is the greatest integer less or equal to p.

Example 8 The identity map from $S^3 \times S^3$ into $S^3 \times S^3$ is 2- homotopic to a constant, while the identity map of $S^2 \times S^4$ is not.

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Example 9 If $k > p$ the identity map of S^k is $[p]$-homotopic to a constant, hence the infimum of p-energy in its homotopy class is 0.

This is not longer true if we consider the identity map of $\mathbf{P}^k(\mathbf{R})$. Indeed $\pi_1(\mathbf{P}^k(\mathbf{R})) = \mathbf{Z}_2$, hence the identity cannot be $[p]$-homotopic to a constant.

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Proof of theorem 3. Let us assume that $f_0 \sim_{[p]}$ constant. We will prove that the infimum of p-energy is 0.

We shall give in some details the proof, in case $m-1 \leq p < m = \dim M$.

It is more convenient to work with a "cube-ulation" instead of a triangulation of M. By changing the metric we may think of each one of these cubes as $[0,1]^m$: the change of metric does not affect the infimum being 0 or not.

We are given a homotopy of $f_{0|X^{m-1}}$ to a constant (X^{m-1} is a $(m-1)$-skeleton of M).

By homotopy extension theorem we can show that f_0 is homotopic to a function $f : M \to N$ which is constant on X^{m-1}.

Let $\varphi_\varepsilon : [0,1]^m \to [0,1]^m$ (we may suppose without loss of generality that $M = [0,1]^m$) be the lipschitz maps which carries a small cube of edge ε at the center of $[0,1]^m$, into the whole cube $[0,1]^m$, while it projects what is left onto the boundary.

φ_ε :

This dilation is easily seen to be homotopic to the identity.

Let us define $g_\varepsilon = f \circ \varphi_\varepsilon$, which is homotopic to f_0. We have

$$\begin{aligned}\int |Dg_\varepsilon|^p &= \int |Dg_\varepsilon|^p + \int_{\varepsilon\,\text{cube}} |Dg_\varepsilon|^p \\ &= 0 + \varepsilon^{-(p-m)} \int_{[0,1]^m} |Df|^p \longrightarrow 0 \text{ as } \varepsilon \to 0\,.\end{aligned}$$

The case $p < m-1$ can be treated by the same method, with a slightly different construction.

For the proof of the converse, we need Poincaré inequality for polyhedral complex:

Let X be a regular polyhedral complex of dimension p, and $f : X \to \mathbf{R}^h$ lipschitz, then

$$\text{(1)} \qquad \exists\, v \in \mathbf{R}^h : \int_X |f(x) - v|^p \le c(x,p) \int_X |Df|^p$$

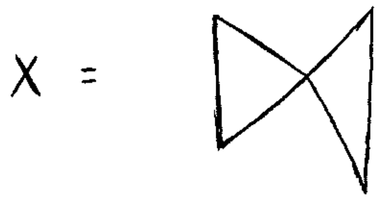
$$\text{(2)} \qquad \exists\, v(r,y) : \quad r^{-p} \int_{B(y,r)} |f(x) - v|^p \le c(x) \int_{B(y,3r)} |Df(x)|^p\,.$$

We recall that X is said to be regular if

a) $X \setminus X^{p-2}$ is connected;

b) if $U \subset X$ is connected and open, then $U \setminus X^{p-2}$ is still connected.

This is to avoid the following situation:



In this case (1) is false if f takes two different constant values on each of the triangles!

Proof of (1). List the $[p]$-simplices of X ($X = D_1 \cup D_2 \cup \ldots \cup D_k$), in such a way that D_i, D_{i+1} have a common $(p-1)$-face. Of course this listing does not need to be injective. Take $\widetilde{D}_i$ an isometric copy of D_i, $\widetilde{X}_i = \cup \widetilde{D}_i$ with $\widetilde{D}_i$ glued to $\widetilde{D}_{i+1}$ only on one of the common faces.

Let $\pi : \widetilde{X} \to X$ be a local isometry (not necessarily 1-1).

On $\widetilde{X}$ we have the usual Poincaré inequality, since it is bilipschitz equivalent to a ball, hence

$$\begin{aligned} \int_X |f(x) - v|^p &\leq \int_{\widetilde{X}} |f \circ \pi - v|^p \\ &\leq \int_{\widetilde{X}} |D(f \circ \pi)|^p \leq n \int_X |Df|^p \end{aligned}$$

where n is the maximum number of preimages of π.

Third lecture

Now we conclude the proof of the theorem: we suppose

$$\inf\left\{ \int |Df|^p \mid f \sim f_0 \right\} = 0$$

and we prove that $f_0 \sim_{[p]}$ constant. (In the sequel we shall denote by C any constant independent of f and ε). Let us denote by X a $[p]$-skeleton of M and by

$$\pi : \mathcal{U} \to M \qquad \Pi : \mathcal{V} \to N$$

two lipschitz retractions of open neighbourhoods of M and N respectively.

For every $\varepsilon > 0$ we find $f \sim f_0$ such that

$$\int_M |Df|^p < \varepsilon .$$

Let $\tilde{f} = f \circ \pi : \mathcal{U} \to M$. We have

$$\int_{\mathcal{U}} |D\tilde{f}|^p \leq C \int_M |Df|^p \leq C\varepsilon .$$

Let $h_v : X \to \mathcal{U}$, $h_v(x) = x + v$. Then

$$\begin{aligned} \int_{|v|\leq r} \int_X |D(\tilde{f} \circ h_v)|^p &= \int_X \int_{|v|\leq r} |D(\tilde{f} \circ h_v)|^p \\ &\leq |X| \int_{\mathcal{U}} |D\tilde{f}|^p \leq C\varepsilon , \end{aligned}$$

hence we may find v with $|v| \leq r$ such that

$$\int_X |D(\tilde{f} \circ h_v)|^p \leq \frac{C\varepsilon}{\mathrm{vol}\,(B_r)} \leq C\varepsilon .$$

We define $g = \tilde{f} \circ h_v : X \to N$. Since $\tilde{f} \circ h_0 = \tilde{f}_{|X} = f_{|X}$ we have that $f_{|X} \sim g$ (via the homotopy $\tilde{f} \circ h_{tv}$). Moreover

$$\int_X |Dg|^p < C\varepsilon$$

by definition of g.

For simplicity we will assume that p be an integer ($p = [p] = \dim X$).

We define now a function

$$H : X \times [0, \text{intrinsic diam}\, X] \longrightarrow E_2$$

(recall that $N \subset \mathcal{V} \subset E_2$) in the following way: $H(x,t)$ is the point $w \in E_2$ that minimizes

$$\int_{\mathcal{B}(x,t)} |g(y) - w)|^p \, dy ,$$

where $\mathcal{B}(x,t)$ is the intrinsic ball in X, if $t > 0$. If $t = 0$ we define $H(x,0) = g(x)$.

H is a continuous homotopy, but it takes values in E_2 and not in N. We have, recalling Poincaré inequality

$$\begin{aligned} d(H(x,t),N)^p &\leq \fint_{\mathcal{B}(x,t)} |H(x,t) - g(y)|^p \, dy \\ &\leq Ct^{-p} \fint_{\mathcal{B}(x,t)} |H(x,t) - g(y)|^p \, dy \\ &\leq C \fint_{\mathcal{B}(x,3t)} |Dg(y)|^p \, dy \leq C \fint_X |Dg|^p < \widehat{C}\varepsilon \, . \end{aligned}$$

By choosing $\varepsilon < \left(\frac{r}{\widehat{c}}\right)^{1/p}$ we have

$$d(H(x,t),N) \leq r \, ,$$

and $H(x,t) \in \mathcal{U}$. Hence we may define

$$\widetilde{H} = \pi \circ H \, .$$

Since $H(x, \operatorname{diam} X)$ does not depend on x because $\mathcal{B}(x, \operatorname{diam} X) = X$, $\widetilde{H}$ is a homotopy from g to a constant. ∎

Sobolev spaces

Let X be a polyhedral complex (regular), e.g. a manifold, or its k- skeleton.

If $f \in \operatorname{Lip}(X; \mathbf{R}^\ell)$, we define its $W^{1,p}$-norm as follows

$$\| f \|_{1,p} = \| f \|_p + \| Df \|_p \, .$$

The Sobolev space $W^{1,p}(X, \mathbf{R}^\ell)$ is defined as the completion of $\operatorname{Lip}(X; \mathbf{R}^\ell)$ with respect to this norm.

If we fix now a target manifold N isometrically embedded into $\mathbf{R}^\ell$, we have several ways of defining Sobolev spaces of maps from X into N.

Definition 3

$$(3) \qquad W^{1,p}(X;N) := \{\, f \in W^{1,p}(X;\mathbf{R}^\ell) \;\mid\; f(x) \in N \textit{ for a.a. } x \in X\}$$

$$(4) \quad W^{1,p}_{\mathrm{strong}}(X;N) := \{\, \textit{closure of } \mathrm{Lip}\,(X;N) \textit{ in the } W^{1,p}\textit{-norm}\}$$

$$(5) \quad W^{1,p}_{\mathrm{weak}}(X;N) := \{\, \textit{closure of } \mathrm{Lip}\,(X;N) \textit{ with respect to the weak convergence of } W^{1,p}(X;\mathbf{R}^\ell)\}$$

Clearly, $W^{1,p}_{\mathrm{strong}}(X;N) \subset W^{1,p}_{\mathrm{weak}}(X;N) \subset W^{1,p}(X;N)$.

We shall see examples of triples X, N, p for which each one of the inclusions is strict.

Theorem 4 (Compactness theorem) *Let $\{f_i\} \subset W^{1,p}(X,\mathbf{R}^\ell)$ be a sequence such that*

$$\| f_i \|_{1,p} \leq K < +\infty\,.$$

Then there exists a subsequence f_i' and $g \in W^{1,p}(X,\mathbf{R}^\ell)$ such that

$$\begin{array}{ccll} f_i' & \longrightarrow & g & \textit{in } L^p(X,\mathbf{R}^\ell) \textit{ strong and} \\ \int_X |Df_i'| & \longrightarrow & \int_X |Dg|^p\,. & \end{array}$$

This theorem can easily be applied to both $W^{1,p}(X,N)$ and $W^{1,p}_{\mathrm{weak}}(X,N)$, while no such theorem holds for $W^{1,p}_{\mathrm{strong}}(X,N)$: for this reason the last space is less used in the calculus of variations to prove existence of minimizers.

We need to define homotopy classes for maps in Sobolev spaces.

Theorem 5 *Let M be a manifold.*

1. *Maps in $W^{1,p}_{\mathrm{strong}}(M,N)$ have a well-defined $[p]$-homotopy type, which is preserved under strong convergence in $(1,p)$-norm.*
2. *Maps in $W^{1,p}_{\mathrm{weak}}(M,N)$ have a well-defined d-homotopy type, where d is the greatest integer strictly less than p. These classes are invariant under weak convergence.*
3. *Maps in $W^{1,p}(M,N)$ have a well-defined $[p-1]$-homotopy type, which is invariant under weak convergence.*

Remark 1 Before beginning the proof of the theorem, let us say some words of comment. Assuming the theorem to be true, one sees easily how to define the homotopy class of f in the cases 1 and 2: in both cases, f has an approximating sequence $\{f_i\} \subset \mathrm{Lip}\,(M,N)$, which converges to f in the strong or in the weak topology of $W^{1,p}$ respectively. For each one of the approximant f_i the d-homotopy type is classically defined, so that we are forced to define

$$[f]_d = [f_i]_d \qquad \text{for } i \text{ large enough.}$$

Of course, we have to show that this is a good definition, i.e., that for i large enough $[f]_d$ does not change, provided $f \in W^{1,p}_{\mathrm{strong}}(M,N)$ and $d \le [p]$, or $f \in W^{1,p}_{\mathrm{weak}}(M,N)$ and $d < [p]$.

In case 3 it is not so obvious what to do: we shall discuss this case later.

Remark 2 (on case 1) $W^{1,p}_{\mathrm{strong}}(M,N)$ is a metric space, which is union of its connected components. Actually, two maps f,g belong to the same connected component if and only if they are $[p]$-homotopic.

It can be proved also that

$$\inf\left\{ \int_M |Df|^p \;\mid\; f \in \mathrm{Lip}\,(M,N),\; f \sim f_0 \in \mathrm{Lip}\,(M,N)\right\}$$

depends only on $[f_0]_{[p]}$.

Before beginning the proof of Theorem 5, we state a couple of useful results.

Theorem 6 (Sobolev-Morrey) *Let X be a regular d-dimensional polyhedral complex, $d < p$ and $0 < \gamma < 1 - d/p$. Then for all $\varepsilon > 0$ there is a constant C_ε (depending only on ε, X and p) such that*

$$\| f \|_{C^{0,\gamma}} \le \varepsilon \| Df \|_p + C_\varepsilon \| f \|_p$$

whenever $f \in W^{1,p}(X, \mathbf{R}^\ell)$.

(This is a slightly unusual version of Sobolev's embedding theorem, easily proved arguing by contradiction)

Lemma 1 $\forall\, K > 0\ \exists\, \varepsilon > 0$ *such that, if* $f_1, f_2 \in \mathrm{Lip}\,(M, N)$ *are maps such that*

$$\| f_1 \|_{1,p} < K\,, \qquad \| f_2 \|_{1,p} < K$$
$$\| f_1 - f_2 \|_p < \varepsilon\,.$$

Then f_1 *is* d*-homotopic to* f_2*, provided* $d < p$.

Proof. As usual, we suppose $M \subset \mathcal{U} \subset \mathbf{R}^n$, $\mathcal{U}$ a neighborhood of M, $\pi : \mathcal{U} \to M$ a lipschitz projection. We denote by X the d-skeleton of M.

By setting $F_i = f_i \circ \pi$, f_i are extended to maps $F_i : \mathcal{U} \to N$. Moreover, as π is lipschitz,

$$\| F_i \|_{1,p} < CK\,, \qquad \| F_1 - F_2 \|_p < C\varepsilon\,.$$

With a Fubini argument we see easily that there is a set of positive measure of vectors $v \in \mathbf{R}^n$ such that

$$\| F_i \circ h_v \|_{1,p} < CK \qquad \text{and} \qquad \| F_1 \circ h_v - F_2 \circ h_v \|_p < C\varepsilon\,,$$

where $h_v : X \to \mathcal{U}$ is defined as $h_v(x) = x + v$. (Notice that the norms in the last inequality are over X).

If we define $g_i := F_i \circ h_v$, with v "good" in the sense above, then

$$g_{i|X} \sim f_{i|X} \qquad \text{and}$$
$$\| g_i \|_{W^{1,p}(X,\mathbf{R}^\ell)} \leq CK\,, \qquad \| g_1 - g_2 \|_{L^p(X)} \leq C\varepsilon\,.$$

By Sobolev-Morrey theorem:

$$\begin{aligned} \| g_1 - g_2 \|_{C^0} &\leq \eta \| D(g_1 - g_2) \|_p + C(\eta) \| g_1 - g_2 \|_p \leq \\ &\leq \eta 2CK + C(\eta) C\varepsilon\,. \end{aligned}$$

We choose η in such a way that $\eta 2CK < r/2$ and ε such that $C(\eta)C\varepsilon < r/2$. In this case

$$\| g_1 - g_2 \|_{C^0(X)} < r$$

and the two maps are therefore homotopic. ∎

The "Fubini argument" used above says what can be sintetically expressed in the following lemma:

Lemma 2 *Let X be a regular polyhedral complex, $\mathcal{U} \subset \mathbf{R}^n$ an open set, $\varphi \in \mathrm{Lip}\,(X,\mathcal{U})$ such that $\varphi(X) \subset\subset \mathcal{U}$ and $f \in W^{1,p}(\mathcal{U},\mathbf{R}^\ell)$. If $\varphi_v : X \to \mathcal{U}$ is defined as $\varphi_v(x) = \varphi(x) + v$, then*

1. *For almost every small $v \in \mathbf{R}^n$, $f \circ \varphi_v \in W^{1,p}(X,\mathbf{R}^\ell)$ and*

$$D(f \circ v)(x) = Df(\varphi_v(x))\, D\varphi_v(x) \qquad \text{for a.e. } x \in X.$$

2. *If $f_i \to f$ in $W^{1,p}$ "rapidly" enough, then for almost every small v, $f_i \circ \varphi_v \to f \circ \varphi_v$ in $W^{1,p}(X,\mathbf{R}^\ell)$.*

The proof is a simple application of Fubini's theorem.

Proof of Theorem 5, Statements 1 and 2. Let $f \in W^{1,p}_{\text{weak}}(M,N)$. Lemma 1 shows that every sequence $\{f_i\} \subset \mathrm{Lip}\,(M,N)$ approximating f in the weak $W^{1,p}$ topology has constant d-homotopy type for i large, provided $d < p$.

As a consequence, d-homotopy types are well-defined in $W^{1,p}_{\text{weak}}(M,N)$ for every integer $d < p$.

By the way, this shows also that $[p]$-homotopy types are well defined in $W^{1,p}_{\text{strong}}(M,N)$ when p is *not* integer.

In the case of p integer, a slightly different argument is required, using mollification. ∎

We have the following theorem:

Theorem 7 *Let M be a manifold $\subset \mathcal{U}$, $\pi : \mathcal{U} \to M$ the nearest point projection, X a d-skeleton of M, $f \in W^{1,p}_{\text{weak}}(M,N)$, with $p > d$, $\varphi : X \to M$ the inclusion map. Then*

1. *$f \circ \pi \circ \varphi_v \in W^{1,p}(X,N) \hookrightarrow C^0(X,N)$ and $f \circ \pi \circ \varphi_v \sim f \circ \pi \circ \varphi_{\tilde v}$ for almost every small $v, \tilde v$ ($\varphi_v(x) = \varphi(x) + v$);*
2. *for almost every v small, the continuous map $f \circ \pi \circ \varphi_v : X \to N$ extends to a continuous map $M \to N$.*

Remark 3 Since the homotopy class does not depend on v, we define the d- homotopy class of a function $f \in W^{1,p}_{\text{weak}}(M,N)$ as follows

$$[f]_d = [f \circ \pi \circ \varphi_v]\,,.$$

for v good.

Remark 4 If f happens to be continuous on a neighborhood of X then

$$[f]_d = [f_{|X}].$$

Remark 5 By a theorem of F. Bethuel

$$W^{1,p}_{\text{strong}}(M,N) = W^{1,p}_{\text{weak}} \qquad \text{if } \ p \notin \mathbf{N}$$

Example 10 Let $f : \mathbf{RP}^n \to \mathbf{RP}^{n-1}$ be the following map: we may think of $\mathbf{RP}^n$ as a disc $\mathbf{B}^n$ modulo the antipodal identification on its boundary (which becomes a $\mathbf{RP}^{n-1}$ embedded in $\mathbf{RP}^n$). The function f carries the boundary of B^n into itself, while it projects radially points of B^n into the boundary.

We remark that $f \in W^{1,p}(\mathbf{RP}^n, \mathbf{RP}^{n-1})$, if $1 < p < n$. We claim that $f \notin W^{1,p}_{\text{weak}}(\mathbf{RP}^n, \mathbf{RP}^{n-1})$.

Indeed, if not, f would have a well define d-homotopy type, $d = [p]$,

$$[f]_d = [f_{|\mathbf{RP}^d}]$$

(note that $\mathbf{RP}^d \subset \mathbf{RP}^n = \partial B^n / \sim$ is a d-skeleton of $\mathbf{RP}^n$).

Moreover, as $f_{|\mathbf{RP}^d}$ is continuous, it would have a continuous extension defined on all $\mathbf{RP}^n$, but it is impossible for obvious topological reason.

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Mollification

Theorem 8 *Let X be a p-dimensional regular polihedral complex, $g : X \to N \subseteq \mathbf{R}^\ell$ belonging to $W^{1,p}$.*
We define $g_s(x)$, for $s > 0$, as the vector $v \in \mathbf{R}^\ell$ which minimizes

$$\fint_{B(x,s)} |g(y) - v|^p .$$

Then

1. *$g_s(x)$ is continuous in both x and s;*
2. $\operatorname{dist}(g_s(x), N) \to 0$ *as $s \to 0$;*

3. if g is continuous on $U \subset X$, then

$$\lim_{s\to 0} \| g_s - g \|_{L^\infty(K)} = 0$$

for all compact $K \subset U$.

The proof of 3 is based on Poincaré inequality.

Now we are able to prove Theorem 5, statement 3. Indeed, the following result holds.

Theorem 9 *Let $f \in W^{1,p}(M,N)$, $d = [p-1]$, $M \subset \mathcal{U}$, $\pi : \mathcal{U} \to M$ the usual retraction, $X = M_d$ = d-skeleton of M and $M_{d+1} = (d+1)$-skeleton of M, $h_v : X \to \mathcal{U}$, $h_v(x) = x + v$.*

Then for almost every v_1 and v_2 small,

$$f \circ \pi \circ h_{v_1} \sim f \circ \pi \circ h_{v_2}$$

(as continuous maps $X \to M$) and they extend to continuous maps of $M_{d+1} \to M$.

Proof. We define $H_v : X \times [-1,2] \to \mathcal{U}$

$$H_v(x,t) := \begin{cases} h_v(x) = x + v & \text{if } -1 \le t \le 0 \\ h_{v+tw}(x) = x + v + tw & \text{if } 0 \le t \le 1 \\ h_{v+w}(x) = x + v + w & \text{if } 1 \le t \le 2\,. \end{cases}$$

Notice that, as X is a polyhedral complex, $X \times [-1,2]$ has the same property. If p is *not* an integer, then $p > [p] = \dim(X \times [-1,2])$.

Hence $f \circ \pi \circ H_v : X \times [-1,2] \to N$ is continuous for almost all small w, v, and the function $f \circ \pi \circ H_v$ is a homotopy between $f \circ \pi \circ H_v$ and $f \circ \pi \circ H_{v+w}$.

On the other hand, if p is an integer, $p = \dim(X \times [-1,2])$. Then for almost every small v, $f \circ \pi \circ H_v \in W^{1,p}(X \times [-1,2], N)$.

This function is not necessarily continuous, hence we mollify it obtaining a continuous map

$$\begin{aligned} (f \circ \pi \circ H_v)_s &: X \times [-1,2] \longrightarrow V \supset N \\ \pi_N (f \circ \pi \circ H_v)_s &: X \times [-1,2] \longrightarrow N \end{aligned}$$

(here we denote by π_N the nearest point projection from $\mathcal{U}$ to N: property (2) of mollification guarantees that we can reproject on N if s is small enough).

The map thus obtained is a homotopy from $(f\circ\pi\circ h_v)_s$ to $(f\circ\pi\circ h_{v+w})_s$. By property 3 of mollification, these two maps are close in L^∞ to $(f\circ\pi\circ h_v)$ and $(f\circ\pi\circ h_{v+w})$: indeed the two latter maps are continuous for almost all choice of v and w small. Hence, for s small enough we have

$$\begin{aligned}(f\circ\pi\circ h_v)_s &\sim (f\circ\pi\circ h_v)\\ (f\circ\pi\circ h_{v+w})_s &\sim (f\circ\pi\circ h_{v+w}),\end{aligned}$$

which concludes the proof. ■

We define therefore

$$[f]_{[p-1]} = [f\circ\pi\circ h_{v|X}].$$

As before, we can show that homotopy classes are preserved under weak convergence.

Moreover, the $[p-1]$-homotopy class of $f\in W^{1,p}(M,N)$ does not depend on the choice of the $[p-1]$-skeleton of M.

Conjecture: *Let $d=[p-1]$, $f\in W^{1,p}(M,N)$. Then $f\in W^{1,p}_{\text{weak}}(M,N)$ if and only if $[f]_d$ consists of continuous maps $X\to N$ which extend to continuous maps $M\to N$.*

Remark 6 In general, $W^{1,p}_{\text{strong}}(M,N)\subsetneq W^{1,p}_{\text{weak}}(M,N)$: there is a counterexample by Bethuel with $p=2$, $M=B^3$ and $N=S^2$.

Remarks on area-type functionals

If we search for least area surfaces on $\mathbf{R}^n$, we cannot expect stability of topological type in minimizing sequences.

On the other hand, it is always possible to find minimal $2d$-surfaces of given boundary with genus$\leq g$.

If the dimension is ≥ 3, the situation is much worse:

Theorem 10 *Let $d \geq 3$, M be a compact manifold of dimension d with boundary. Let $f : \partial M \hookrightarrow \mathbf{R}^\ell$ be any smooth embedding, S any "reasonable" surface in $\mathbf{R}^\ell$ such that $\partial S = f(\partial M)$. Then f can be extended to a lipschitz map $f : M \to S \cup \{\textit{set of dimension } d-1\}$ in such a way that*

$$\mathrm{Area}\,(f) = \mathrm{Area}\,(S)\,.$$

This result holds without any constraint on the topology of M and S!