SINGULARITIES OF MINIMIZING BIHARMONIC MAPS

KATARZYNA EWA MAZOWIECKA

Contents

The analysis of fourth order partial differential equations, even in the linear case, carries a lot of challenges. The system of biharmonic map equations, which have geometric and variational origins, provides an example of system of equations with *critical growth nonlinearities*. In such case the classical methods just fail to apply, making such problems the main issue of modern nonlinear analysis. The behavior of biharmonic maps, especially their *boundary regularity* and the onset of their *singularities*, are far from being fully understood - much less is known here than for geometric variational problems that lead to second order equations. We believe that the present project can contribute to this topic.

1. Objectives

Let $\Omega \subseteq \mathbb{R}^m$ be a smooth domain and $\mathcal{N}^n$ a smooth compact Riemannian manifold without boundary of dimension n. According to J. Nash's embedding theorem [24], we may assume that $\mathcal{N}$ is isometrically embedded in some Euclidean space $\mathbb{R}^N$ for N sufficiently large. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ we define the Sobolev spaces

$$W^{k,p}(\Omega, \mathcal{N}) := \left\{ u \in W^{k,p}(\Omega, \mathbb{R}^N) : u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega \right\},$$

equipped with the topology inherited from the topology of the linear Sobolev space $W^{k,p}(\Omega, \mathbb{R}^N)$.

For $k = 1$ we define the Dirichlet energy as

$$E(u) = \int_\Omega |\nabla u|^2 dx \quad \text{for } u \in W^{1,2}(\Omega, \mathcal{N}).$$

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We say that a map $u \in W^{1,2}(\Omega, \mathcal{N})$ is *harmonic* if it is a critical point of the Dirichlet integral with respect to compactly supported variations in the target manifold, i.e.

$$\frac{d}{dt}\Big|_{t=0} E(\Pi(u + t\Phi)) = 0$$

for all $\Phi \in C_0^\infty(\Omega, \mathbb{R}^N)$, where Π denotes the nearest point projection onto $\mathcal{N}$.

Unlike the unconstrained case the critical points of the Dirichlet integral may have singularities, are not necessarily minimizers of the energy, there is no uniqueness in general in the class of prescribed boundary condition and the corresponding system of Euler–Lagrange equations is a nonlinear system of second order equations. The topic of harmonic mappings has been extensively studied especially in the mid 80's of the last century. In the conformally invariant case $m = 2$ all harmonic maps are regular. For $m \geq 3$ the singularities may appear even in the case of minimizing maps. In the latter case the Hausdorff dimension of the singular set is at most $m - 3$ (see [27]), whereas on the other hand there are examples of everywhere discontinuous harmonic maps (the example of Rivière [25]).

Harmonic maps play a very important role in many branches of mathematics:

Differential geometry: For $m = 1$ harmonic maps are the geodesics of $\mathcal{N}$. Moreover, any submanifold of an affine Euclidean space has constant mean curvature if and only if its Gauss map is a harmonic map. A submanifold $\mathcal{M}$ of a manifold $\mathcal{N}$ is minimal if and only if the immersion of $\mathcal{M}$ in $\mathcal{N}$ is harmonic;

Partial differential equations: Harmonic maps are a model case of an elliptic system with a critically nonlinear right hand side. Such equations are very important in the theory of partial differential equations. It is not only interesting to study the regularity of solutions but also to understand the behavior of solutions near possible singularities;

Theoretical physics: Harmonic maps between surfaces and Lie groups have properties analogous to those (anti) self-dual Yang–Mills connections on 4-dimensional manifolds.

A generalization of harmonic maps, which we plan to investigate in this project is given by biharmonic maps. They are given as solutions to a nonlinear fourth order system of equations.

The Hessian energy (or the extrinsic biharmonic energy) is defined as

$$\mathbb{H}(u) := \int_\Omega |\Delta u|^2 dx \quad \text{for } u \in W^{\Omega,\mathcal{N}}, \tag{1}$$

where Δ is the standard Laplace operator on $\mathbb{R}^m$.

We say that $u \in W^{2,2}(\Omega, \mathcal{N})$ is *(extrinsically) biharmonic* if it is a critical point of (1). A map $u \in W^{2,2}(\Omega, \mathcal{N})$ is called *(extrinsically) minimizing biharmonic* if among all maps $v \in W^{2,2}(\Omega, \mathcal{N})$ satisfying $u - v \in W_0^{2,2}$ we have

$$\mathbb{H}(u) \leq \mathbb{H}(v).$$

There is also an intermediate concept between minimizing biharmonic maps and biharmonic maps. These are *stationary biharmonic maps* defined as critical points of the energy $\mathbb{H}$ with respect to variations in the domain Ω.

Remark. We do not use the definition of *intrinsic biharmonic maps*, which are defined as critical points of the tension energy functional

$$\mathbb{H}_T := \int_\Omega |(\Delta u)^T|^2 dx,$$

where the tension field $(\Delta u)^T$ is the component of Δu tangent to $T_u\mathcal{N}$. Intrinsic biharmonic mappings may be seen as a direct generalization of harmonic maps, because harmonic mappings have vanishing tension field. Therefore, they are also intrinsic biharmonic. Moreover, for $\mathcal{N}$ with nonpositive sectional curvature every intrinsic biharmonic map is harmonic. For targets with positive sectional curvature there exist intrinsic biharmonic maps which are not harmonic. A survey on the intrinsic biharmonic maps can be found in S. Montaldo and C. Oniciuc article [23].

Unlike the Hessian energy, the tension field does not depend on the embedding $\mathcal{N} \hookrightarrow \mathbb{R}^N$. Therefore, from geometric point of view, the tension energy seems more natural. On the other hand, from the analytic point of view, the Hessian energy bounds the $W^{2,2}$ norm. This is no longer true for the tension energy.

In this proposal we do not consider intrinsic biharmonic maps, therefore, we omit the adjective "extrinsically".

In case $\mathcal{N} = \mathbb{R}^N$ the Euler–Lagrange system corresponding to biharmonic maps is simply

$$\Delta^2 u = 0$$

and the solution to this system is said to be a biharmonic *function*.

Let us introduce the notation before stating the system in general. Let $p \in \mathcal{N}$, $P(p) = \nabla\Pi(p)$ be the orthonormal projection onto the tangent space $T_p\mathcal{N}$. The orthonormal projection onto the normal space will be denoted by $P^\perp$. We notice that $P + P^\perp = id$. Moreover, let $A(\cdot)(\cdot,\cdot)$ be the second fundamental form of $\mathcal{N}$ in $\mathbb{R}^N$, given by

$$A(p)(X,Y) = P^\perp(\nabla_X(Y)) \text{ for } X, Y \in T_p\mathcal{N},$$

where X, Y have been extended to tangent vector fields of $\mathcal{N}$ in a neighborhood of p. The corresponding Euler–Lagrange equation takes the form (for derivation of this system see for example [34])

$$\Delta^2 u = \langle \Delta(P(u)), \Delta u\rangle - \Delta(A(u)(\nabla u, \nabla u)) + 2\langle \nabla(P(u)), \nabla\Delta u\rangle, \tag{2}$$

in the sense of distributions.

For $\mathcal{N} = \mathbb{S}^{N-1}$ the system (2) reduces to

$$\Delta^2 u = -(|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\nabla u \cdot \nabla\Delta u)u,$$

where $\cdot$ denotes the inner product on $\mathbb{R}^N$. The equation follows by differentiating four times $|u|^2 = 1$ and observing that $\Delta^2 u$ is parallel to u in $\mathcal{D}'$.

One can see that the system (2) is a nonlinear fourth order system of equations of critical growth, i.e. the right hand side of the equation is *a priori* only in L^1. The standard bootstrap methods do not yield any extra regularity of the solutions — regularity for such class of systems is a subtle issue.

One of the main reasons for which the theory of higher order elliptic equations is by far less developed then the theory of analogous second order equations is the failure of the maximum principle (or truncation methods in general). For a discussion about recent developments on the theory of higher order elliptic equations see [9]. The aim of this proposal is to develop new tools suitable for boundary value problems of biharmonic maps. We hope that our research will result in better understanding of the behavior of solutions to biharmonic map systems near singular points.

1.1. **Partial regularity.** We note that in case $m = 4$ the Hessian energy is conformally invariant, and hence conformal maps of the Euclidean four space are biharmonic[1]. The investigation of regularity of biharmonic maps was initiated by S.-Y. A. Chang et al. in [7]. They have investigated mappings with values in the sphere $\mathbb{S}^{N-1}$. In case $m = 4$ they proved the regularity of *all biharmonic* maps, while for $m \geq 5$ they have proved that stationary biharmonic maps are C^∞ except a closed set Σ of Hausdorff dimension at most $m-4$. Their result was partially extended to general target manifold by C. Wang in [32, 33, 34]. Alternative proofs were given by P. Strzelecki [31] for $m = 4$, $\mathcal{N} = \mathbb{S}^{N-1}$, T. Lamm with T. Rivierè [19] for $m = 4$ and arbitrary $\mathcal{N}$, M. Struwe [30] for $m \geq 5$ and arbitrary target manifold $\mathcal{N}$.

Each of the proofs relies on another idea. In [7] S.-Y. A. Chang, L. Wang and P. Yang derive from the stationary assumption a monotonicity formula, although only for sufficiently regular maps. That formula was crucial in the proof of partial regularity for $m \geq 5$. A rigorous proof of the monotonicity formula was given by G. Angelsberg in [3].

In the case of minimizing biharmonic maps the partial regularity results may be strengthen. First it was observed by M.-C. Hong and C. Wang in [18] that for $\mathcal{N} = \mathbb{S}^{N-1}$ the singular set Σ has Hausdorff dimension at most $m - 5$. One can prove the optimality of this result considering a map $\frac{x}{|x|} : \mathbb{B}^5 \to \mathbb{S}^4$. Finally, C. Scheven in [26] reduced the dimension of singular set of minimizing mappings for an arbitrary target manifold $\mathcal{N}$. His result states that, as in the case $\mathcal{N} = \mathbb{S}^{N-1}$, the singular set Σ of minimizing biharmonic maps has $\dim_{\mathscr{H}} \Sigma \leq m - 5$.

In a recent paper C. Breiner and T. Lamm [5] prove that each minimizing biharmonic map is *locally* in $W^{4,p}$ for $1 \leq p \leq 5/4$.

1.2. **Open problem: boundary regularity.** In the harmonic and p-harmonic case[2] it was shown in [28] and [15], respectively, that the minimizing maps are regular in a full neighborhood of the boundary of Ω. It is natural to ask whether there is boundary regularity for biharmonic mappings. More precisely, let $\phi \in C^\infty(\Omega_\delta, \mathcal{N})$ be given for some $\delta > 0$, where $\Omega_\delta = \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) < \delta\}$. Consider a biharmonic map $u \in W^{2,2}(\Omega, \mathcal{N})$ satisfying

$$u = \phi \text{ on } \partial\Omega, \quad \nabla u = \nabla \phi \text{ on } \partial\Omega.$$

Problem 1. *Is it true that for every smooth boundary mapping ϕ the corresponding minimizing biharmonic mappings have no singularities in a full neighborhood of the boundary?*

Let us mention here two inconclusive results in this direction. Firstly, it was shown in [20] by T. Lamm and C. Wang that polyharmonic maps[3], in the conformal case $m = 2k$, enjoy the property of being continuous in a neighborhood of the boundary. Although, the proof is strongly dependent on the relation $m = 2k$ and one might not extend this method to the case $m > 2k$. The other results concerns partial boundary regularity for stationary maps. It was shown in [11] by H. Gong et al. that if we impose an additional condition on the boundary mapping then there exists a closed subset $\Sigma \subseteq \overline{\Omega}$, with $\mathcal{H}^{m-4}(\Sigma) = 0$ such that the stationary biharmonic map is smooth up to the boundary, except possibly the set Σ. The additional condition is the boundary monotonicity formula. Unlike the monotonicity formula,

[1] This fact does not hold true if one tries to generalize biharmonic maps to arbitrary smooth manifolds in the domain in the most obvious way: as critical points of $\int_{\mathcal{M}} |\Delta_{\mathcal{M}} u|^2 dvol_{\mathcal{M}}$, where the standard Laplace operator was replaced by the Laplace–Beltrami operator $\Delta_{\mathcal{M}}$. In order to have conformal invariance in dimension 4 one should consider the so called *Paneitz* functional (see e.g [9, Section 1.5] and references therein).

[2] *p-harmonic maps* are defined for $u \in W^{1,p}(\Omega, \mathcal{N})$ as critical points of the energy $\int_\Omega |\nabla u|^p dx$.

[3] *Polyharmonic maps* are defined for $u \in W^{k,2}(\Omega, \mathcal{N})$ as critical points of $\int_\Omega |\Delta^k u|^2 dx$.

the boundary monotonicity formula is an artificial assumption — it is unknown whether it can be deduced for all stationary maps.

1.3. **Compactness.** It was proven by P. Strzelecki in [31] in the case $\mathcal{N} = \mathbb{S}^{N-1}$ that a weak limit of a sequence of biharmonic maps is also biharmonic. In case of minimizing biharmonic maps and $\mathcal{N} = \mathbb{S}^{N-1}$ it was proven that if a sequence of minimizing biharmonic maps is weakly convergent in $W^{2,2}$ to a map $u \in W^{2,2}(\Omega, \mathbb{S}^{N-1})$, then the convergence is also strong in $W^{2,2}_{loc}(\Omega, \mathcal{N})$ and the limiting map is minimizing biharmonic.

In [26] for an arbitrary target domain $\mathcal{N}$ there is another compactness result. Let $M(\Omega)$ denote the closure in $W^{2,2}_{loc}$ of the set of minimizing biharmonic maps. If $\{u_j\} \in M(\Omega)$ is a sequence with $\sup_j \|u_j\|_{W^{2,2}} < \infty$, then there exist $u \in M(\Omega)$ such that $u_{j_i} \to u$ in $W^{2,2}_{loc}$, up to a subsequence $\{j_i\}$.

Up to our best knowledge the problem stated below is still open.

Problem 2. *Let $u_j \in W^{2,2}(\Omega, \mathcal{N})$ be a uniformly bounded sequence of minimizing biharmonic maps. Then there exist a subsequence $\{j_i\}$, such that $u_{j_i} \to u$ in $W^{2,2}_{loc}$ for a minimizing biharmonic $u \in W^{2,2}(\Omega, \mathcal{N})$.*

Let us briefly discuss the problems that appear in proving the compactness results for minimizing harmonic maps. Having a uniformly bounded sequence of minimizing harmonic maps $\{u_j\} \in W^{1,2}_{loc}(\Omega, \mathcal{N})$ we get by the Rellich–Kondrachov Lemma existence of a subsequence $\{j_i\}$ such that u_{j_i} converges strongly in L^2 to a map u on compact subsets and Du_{j_i} converges weakly in L^2 to Du. The fact that the limiting map u has a.e. value in $\mathcal{N}$ is a consequence of the pointwise a.e. convergence. The strong convergence of the derivatives may be provided by the regularity theory, but one does not simply prove that the limiting map is minimizing. Since the maps u_{j_i} and u slightly differs on the boundary one may not use directly the definition of minimizing map to compare their energies. A tool for comparing those energies was provided by S. Luckhaus and his lemma in [21].

The difficulty of proving an analogous fact for biharmonic maps is, again, the proof of the minimizing property of the limiting map. Unfortunately we may not use directly S. Luckhaus's lemma to maps from $W^{2,2}$, as in this case we cannot extend maps from $W^{2,2}$ by Lipschitz maps.

We note here that the compactness problems are very challenging even in the second order setting. For example, it is not known, for an arbitrary target manifold, whether the limit of a sequence of harmonic maps convergent weakly in $W^{1,2}$ is again harmonic.

1.4. **Examples, special cases.** There are very few examples illustrating the peculiar behavior of biharmonic maps. Let us mention some of them.

In [18] M.-C. Hong and C. Wang proved that the mapping $\frac{x}{|x|} : \mathbb{B}^5 \to \mathbb{S}^4$ is a unique *minimizing* map for its boundary condition. Moreover, they proved that there exist infinitely many solutions to system of biharmonic map equations for the boundary condition $\frac{x}{|x|}$. It seems that a positive answer to the Problem 1 would allow us to establish a similar result for general boundary data.

G. Angelsberg proved in [4] in case $\Omega = \mathbb{B}^4$, $\mathcal{N} = \mathbb{S}^4$ that for a class of boundary data there exist at least two distinct minimizers of the Hessian energy.

Problem 3. *For which boundary data there is uniqueness in the class of minimizing biharmonic mappings? Is the set of boundary data for which we have this uniqueness dense in some space?*

Note that, by Hopf theorem, the mapping $\phi : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ admits a continuous extension if and only if $\deg(\phi) = 0$. However, even without the topological obstruction the singularities sometimes occurs, simply because it is energy preferable. This fact was first observed in the harmonic setting by R. Hardt and F.H. Lin [14] and generalized to biharmonic maps by M.-C. Hong and C. Wang [18]. Moreover, there exist boundary mappings for which the *Lavrentiev gap phenomenon holds*:

$$\min_{u \in W^{2,2}(\Omega, \mathbb{S}^4)} \mathbb{H}(u) < \inf_{u \in C^0(\Omega, \mathbb{S}^4)} \mathbb{H}(u),$$

were the minimum and infimum was taken over mappings having the same boundary values.

C. Breiner and T. Lamm give in [5] a uniform bound on the number of singular points on compact subsets of the domain in case $\Omega = \mathbb{B}^5$ and $\mathcal{N} = \mathbb{S}^4$. Since the singularities appear in the case $m = 5$ it is interesting not only to study the behavior of the map near its singular points but also to find a bound on their number.

Problem 4. *How many singular points can a minimizing biharmonic map $u : \mathbb{B}^5 \to \mathbb{S}^4$ have? Can we estimate the number of singular points in terms of the boundary condition, e.g. $\|\phi\|_{W^{2,\infty}}$?*

2. Significance

The theory of biharmonic functions is an old and rich subject: they have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity.

The study of biharmonic mappings in this project is purely cognitive. The original inspiration to began the work on biharmonic mappings where the problems in conformal geometry. Harmonic mappings are conformally invariant only in dimension two. It is natural to study the properties of conformally invariant energy integrands. The functional associated with the p-Laplacian: $\int_\Omega |\nabla u|^p$ is invariant for $p = m$, although the regularity for arbitrary target manifold is not known. The Hessian energy appears as a higher order functional conformally invariant in dimension 4. However, stripped of its geometric significance it is also an intriguing PDE problem. The system of biharmonic map equations is an example of a nonlinear fourth order system of equations with critical exponent (the right hand side of the system (2) is only in L^1). Such problems are far from being fully understood and the study of them fit in with the mainstream of modern theory of partial differential equations.

3. Work plan

Familiarization with the topic of harmonic maps is crucial to work with biharmonic maps. Known regularity results for harmonic maps have biharmonic analogues and many of the proofs in the biharmonic setting are inspired by the proofs for harmonic maps. The necessary knowledge has been gained by the principal investigator of this proposal during the work as a member of the research team on another project: *Nonlinear elliptic systems: regularity, singularities and related topics.*

First of all we plan to work on the uniform boundary regularity. We plan to follow the proofs in harmonic [28] and p-harmonic settings [15]. We need to ensure that our mappings have higher regularity, therefore, we plan to replace reflections by higher order reflections (as in the general extension theorem for Sobolev spaces $W^{m,p}$ with $m \geq 2$, see [10, Lemma 6.37]) of minimizing biharmonic mappings defined on $(\mathbb{B}^m)^+ = \mathbb{B}^m \cap \{x_m \geq 0\}$. Preliminary results obtained by the principal investigator of this project as well as an in-depth analysis of the papers [28, 15] allow us to hope that we can obtain the boundary regularity at least for $\mathcal{N} = \mathbb{S}^{N-1}$. We plan to focus on the case of a sphere and an arbitrary target manifold separately. The work plan in general consists of the following steps:

Step 1: Obtaining an ε-regularity result for mappings defined on $(\mathbb{B}^m)^+$ (by considering the higher order reflections);

Step 2: The study of properties of boundary tangent maps. For $a \in (\mathbb{B}^m)^+ \cap \{x_m = 0\}$, $\lambda > 0$ we say that v is a *boundary tangent map* at the point a if there is a sequence $\lambda_j \searrow 0$ with $u(a + \lambda_j x) \to v$ in $W^{2,2}(\mathbb{R}^m, \mathcal{N})$ as $i \to \infty$. We plan to show that if u is minimizing then also is v;

Step 3: We want to prove that the only minimizing biharmonic maps defined on $(\mathbb{B}^m)^+$, that are homogeneous of degree 0, are constant maps.

The second step seems to be the most challenging part of the plan. In the case $\mathcal{N} = \mathbb{S}^{N-1}$ the compactness theory allows us to complete this step, while for an arbitrary target manifold one way to tackle this step would be proving Problem 2. Perhaps the special form of the rescaled maps $u(a + \lambda_j)$ will allow us to deduce step two without proving in general that the limit of a sequence of minimizing biharmonic maps is a minimizing biharmonic map.

Next, we plan to focus on the model case $\mathcal{M} = \mathbb{B}^5$, $\mathcal{N} = \mathbb{S}^4$ (in which C. Scheven has proved that there are finitely many singular points [26]). We divide our investigation into several stages:

- Prescribing the *topological degree of singularities*. In the harmonic case $u : \mathbb{B}^3 \to \mathbb{S}^2$ it was proven that small spheres around any singular point map to $\mathbb{S}^2$ with topological degree plus or minus one and that near a singularity (say at the origin) the minimizing map looks like $\pm R(x/|x|)$, where $R \in SO(3)$, see [6]. The proof relied on the classification of all harmonic maps from $\mathbb{S}^2$ to $\mathbb{S}^2$. A direct generalization of those results is impossible as the full characterization of biharmonic maps from $\mathbb{S}^4$ to $\mathbb{S}^4$ seems far more complicated. Fortunately, for our reasons it is enough to prove that the singularities can not have zero topological degree.
- A quest for conditions for the uniqueness of biharmonic mappings. Our hypothesis is that the set of boundary mappings for which there is exactly one minimizing mapping is dense in $H^{3/2}(\mathbb{S}^4)$;
- Convergence of singular points. We plan to investigate whether for a sequence of minimizing biharmonic maps u_j strongly convergent in $W^{2,2}$ to a minimizing u, there holds: (a) If y_j is a singular point of u_j such that $y_j \to y$, then y is a singular point of u; (b) If y is a singular point of u then, for all sufficiently large j, u_j has a singular point at some y_j with $y_j \to y$;
- Finding a *method of installing new singular points* into a mapping (we hope to find a method that will allow us, after perturbing the boundary condition on a small piece of the sphere $\mathbb{S}^4$, conclude that the minimizer for the newly created boundary mapping has exactly one more singular point, see [2] for the harmonic setting). The example of M.-C. Hong and C. Wang in [18], where they construct a boundary mapping for which the Lavrentiev gap phenomenon holds, has a similar construction to that of

R. Hardt and F.H. Lin in [14]. It gives a glimmer of hope that an in-depth analysis of this example will lead us to a such general method of installing singularities. In the harmonic case, based on the method of installing singular points for harmonic maps it was proven by the principal investigator and her supervisor that the Lavrentiev gap phenomenon holds on a dense set of boundary data, see [22]. We hope to extend this result into biharmonic case;

- We are planing to investigate whether there is a "boiling water" type example for biharmonic maps, similar to the one for harmonic discovered by F. Almgren and E. Lieb in [2, Section 4] (the "boiling water" example states that there exist a boundary map from $\mathbb{S}^2$ to $\mathbb{S}^2$ with a unique minimizer having many singular points stacked up almost vertically near $\partial\mathbb{B}^3$). After preliminary analysis the problem seems technical albeit tractable since most of the tools used in the proofs have analogues for biharmonic maps;
- A research on the distances between singular points. We want to determine whether there is a universal constant C such that if y is a singularity of a minimizing mapping $u : \mathbb{B}^5 \to \mathbb{S}^4$ then there is no other singularity within distance CD of y, where by D we denote the minimal distance of y to the boundary $\mathbb{S}^4$. (If the above mentioned example of "boiling water" type occurs we may not expect that the distance of singularities is independent of the distance to the boundary);
- Finding a counterpart of the Stability Theorem of R. Hardt and F.H. Lin [16] (the result states that small perturbations in the Lipschitz norm of the boundary data does not affect the number of singular points of the corresponding minimizer). We would like to examine how does the number of singularities change after small, in various topologies, perturbations;
- Finally we would like to make a significant progress in specific constructions of nonunique solutions and examples of biharmonic maps with "large" sets of singularities.

4. Methodology

We plan to use and develop techniques of functional analysis, differential geometry and typical techniques of nonlinear partial differential equations (Sobolev–Gagliardo–Nirenberg inequalities, Morrey–Campanato iteration, weak convergence methods). In particular we plan to use the methods described in the book of L. Simon [29]. The various methods of proving the partial regularity provides several useful tools, such as:

- **Sobolev mappings between manifolds.** One of the main tool in partial differential equations is the *approximation of Sobolev functions* by smooth ones. In general, in the case of mappings between manifolds, the approximation is not always possible. A full characterization of Sobolev mappings that may be approximated by smooth mappings can be found in [13]. Moreover we will often use the basics of homotopy theory (in particular the notion of the degree of a map). We will also apply tools of Sobolev mappings from the paper [12];
- **Geometric measure theory.** The concept of *currents* (a general k-dimensional current is a continuous linear functional on an appropriate space of smooth differential k forms in $\mathbb{R}^n$) is closely related to harmonic mappings and often allows to simplify the notation and the proofs (for a use of currents in the theory of harmonic mappings and a link to minimal surfaces see [1]). Another useful tool is the co-area formula (see [22] for an exemplary application);

- **Harmonic analysis.** The celebrated paper of R. Coifman et al. [8] initiated numerous new applications of the *duality of Hardy and BMO spaces*, i.a. to problems of geometric and variational origins (see e.g. [17, Chapters 3 and 4]). A common feature of all those works is that in the critical case $kp = n$, where n is the dimension of the domain, the Sobolev imbedding theorem gives us only $W^{k,p} \subset \mathrm{BMO}$. Nevertheless in many instances the functions BMO norm is small whereas its L^∞ norm is only finite. For a use in the biharmonic setting see [31].
- Methods taken from the field theory: namely the notion of **Coulomb gauge** turns out to be relevant (see [19, 30]).

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