# Two-dimensional minimal surfaces and harmonic maps

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Geometric measure theory and
geometrical variational problems

Trento, Sept. 1–9, 1993

The emphasis here is on the dimension two of the domain $\Sigma$ in which we consider harmonic maps or parametrical minimal surfaces $u : \Sigma \to N$. We shall therefore describe here phenomena and techniques that are of special importance in the case of two independent variables, e.g.,

— conformal parametrizations

— the Courant-Lebesgue lemma

— bubbling of the spheres

As general references we recommend:

1. Dierkes & Hildebrandt & Küster & Wohlrab, Minimal surfaces I, II, Springer 1992.

2. Eells & Lemaire, Another report on harmonic maps, Bull. London Math. Soc. **20** (1988), 385–524.

3. Jost, Harmonic mappings between Riemannian manifolds, Proc. CMA, Vol. 4, ANU-Press Canberra 1984.

4. Jost, Two-dimensional geometric variational problems, Wiley- Interscience, Clichester, New York 1991.

## 1 Energy and area

By *energy* we here always mean the $p$-energy for $p = 2$, i.e. for $\Sigma$ open in $\mathbf{C} = \mathbf{R}^2$ and $u : \Sigma \to \mathbf{R}^n$

$$E_\Sigma(u) := \frac{1}{2}\int_\Sigma |Du|^2 \, d\mathcal{L}^2 = \frac{1}{2}\int_\Sigma (|u_x|^2 + |u_y|^2) \, dx \, dy \quad (\textit{Dirichlet's integral})$$

($z = x + iy$ Cartesian coordinates on $\Sigma$, $u_x := \dfrac{\partial u}{\partial x}$, etc.).

The *area* of the parametric surface $u$ is:

$$\begin{aligned} A_\Sigma(u) &:= \int_\Sigma Ju \, d\mathcal{L}^2 = \int_\Sigma |u_x \wedge u_y| \, dx \, dy = \\ &= \int_\Sigma \sqrt{|u_x|^2|u_y|^2 - u_x \cdot u_y} \, dx \, dy \,. \end{aligned}$$

Then we have

$$\begin{aligned} &A_\Sigma(u) \leq E_\Sigma(u) \ \text{ with equality if and only if } u \text{ is } \textit{conformal}, \text{ i.e.} \\ (1) \qquad &|u_x|^2 = |u_y^2| \quad \text{and} \quad u_x \cdot u_y = 0 \ \mathcal{L}^2\text{-almost everywhere on } \Sigma \,. \end{aligned}$$

This is a special feature in dimension 2 if we deal with the usual quadratic energy; the analog for $m \geq 3$ independent variables holds with the $n$-energy $\int_\Sigma |Du|^m \, d\mathcal{L}^m$ instead of $E_\Sigma(u)$ and a suitable constant in the inequality—cf. Brian White's lectures. Another special feature of dimension 2 is

$$\begin{aligned} &E \ \text{ is conformally invariant, i.e.} \\ &E_\Sigma(u) = E_{k\Sigma}(u \circ k^{-1}) \ \text{ for (anti)-conformal diffeomorphisms } k. \end{aligned}$$

This follows from the change of variable formula and the chain rule, the effect of the conformal transformation cancels pointwise in the integrand.

(For the $m$-energy on $m$-dimensional domains the same statement holds, but there are much fewer conformal diffeomorphisms in dimensions $m \geq 3$ than in dimension 2). The area is, of course, invariant with respect to all diffeomorphisms on the domain, conformal or not. Consequently,

> the energy $E_\Sigma(u)$ is well defined (by the above formula in local conformal coordinates $z = x + iy$) whenever $\Sigma$ is a Riemann surface [1] (with or without boundary)
>
> (It does depend on the conformal structure of $\Sigma$, however).

The area $A_\Sigma(u)$ is even defined whenever $\Sigma$ is an arbitrary manifold with no extra structure, and

(1) holds again.

Considering a *Riemannian target manifold* $N$ nothing changes if we assume $N \subset \mathbf{R}^n$ embedded isometrically. The energy of $u : \Sigma \to N$ is the same as the energy of the same map $u : \Sigma \to N \hookrightarrow \mathbf{R}^n$ viewed as a map into $\mathbf{R}^n$. Alternatively we can choose a Riemannian metric $g = (g_{ij})$ on $\mathbf{R}^n$ to represent a coordinate neighborhood in a Riemannian target manifold. Then for $u$ with image in that coordinate neighborhood, the above formulas hold for the energy and area with Euclidean norm $|\cdot|$ replaced by the Riemannian norm, i.e.

$$\begin{aligned} E_\Sigma(u) &= \frac{1}{2}\int_\Sigma [(g \circ u)(u_x, u_x) + (g \circ u)(u_y, u_y)]\, dx\, dy = \\ &= \frac{1}{2}\int_\Sigma [(g_{ij} \circ u)(u_x^i u_x^j + u_y^i u_j^j)\, dx\, dy \\ A_\Sigma(u) &= \int_\Sigma \left[\det \begin{pmatrix} (g \circ u)(u_x, u_x) & (g \circ u)(u_x, u_y) \\ (g \circ u)(u_x, u_y) & (g \circ u)(u_y, u_y) \end{pmatrix}\right]^{1/2} dx\, dy\,. \end{aligned}$$

In general one considers harmonic mappings between two Riemannian manifolds, so we should also consider putting a *Riemannian metric on the source manifold* $\Sigma$. In the definition of area and energy we then have to replace the partial derivatives of $u$ by the derivatives with respect to an orthonormal basis of tangent vectors at each point of $\Sigma$ and integration with respect to Lebesgue measure on $\Sigma$. However, by a well-known theorem of differential geometry one can choose local conformal coordinates on $\Sigma$ ("isothermal coordinates") and in such coordinates $z = x + iy$ the energy is given exactly by the same formulas as in the case $\Sigma \subset \mathbf{C} = \mathbf{R}^2$ above. It does not really depend on the Riemannian metric but only on the conformal class of the

[1] For simplicity we restrict to oriented parameter domains

metric. Therefore (letting aside questions of orientation) it is sufficient, in the 2-dimensional theory, to consider a Riemann surface without specified metric as source manifold.

There are more general functionals of interest. M. Grüter has classified all the *smooth conformally invariant first order variational integrals* with quadratic growth in the derivatives. They are of the form, for $u : \Sigma \subset \mathbf{C} \to \mathbf{R}^n$,

$$\frac{1}{2}\int_{\Sigma} ](g \circ u)(u_x, u_x) + (g \circ u)(u_y, u_y)]\, dx\, dy + \int_{\Sigma} (Q \circ u)(u_x \wedge u_y)\, dx\, dy$$

with a Riemannian metric $g$ in $\mathbf{R}^n$ and $Q : \mathbf{R}^n \to \Lambda_2\mathbf{R}^n$. Here the first integral is a conformally invariant energy and the second is invariant with respect to all orientation preserving diffeomorphisms of the domain.

## 2 First variation

If we want to minimize energy and understand what the minimizers are we have to compute the first variation and derive the Euler equations. This is very simple in the case of the Dirichlet integral. Given $v : \Sigma \to \mathbf{R}^n$ of the same class we compute:

$$\begin{aligned} \delta E_\Sigma(u, v) &:= \frac{d}{d\varepsilon}_{|\varepsilon=0} E_\Sigma(u + \varepsilon v) = \frac{d}{d\varepsilon}_{|\varepsilon=0} \frac{1}{2} \int_{\Sigma} |Du + \varepsilon Dv|^2\, dx\, dy \\ &= \int_{\Sigma} Du \cdot Dv\, dx\, dy = \int_{\Sigma} (u_x \cdot u_x + u_y \cdot u_y)\, dx\, dy\,. \end{aligned}$$

If $u$ is of class $C^2$ and $v$ has compact support we can integrate by parts to obtain

$$\delta E_\Sigma(u, v) = -\int_{\Sigma} (\Delta u) \cdot v\, dx\, dy\,.$$

Then $u$ is *stationary for the energy with respect to variations of the independent variables*, i.e. the *variational equation*

$$\delta E_\Sigma(u, v) = 0 \text{ holds for all } v \text{ with } \operatorname{spt} v \subset\subset \Sigma\,,$$

if and only if $u$ is a *harmonic vector function* on $\Sigma$. This is true also if we do not know a priori that $u$ is smooth, because the variational equation then says that $u$ is harmonic in the sense of distributions (weakly harmonic) and we have a regularity theorem for this (Weyl's lemma).

If we consider mappings $u : \Sigma \to N \subset \mathbf{R}^n$ into a Riemannian manifold and want to minimize among such mappings then we cannot, of course, use variations $u + \varepsilon v$ leading into the ambient space. In this situation it is natural to restrict to variational fields

> $v$ tangential to $N$ along $u$ (i.e. $v(z) \in T_{u(z)}N$ for $z \in \Sigma$ with $\operatorname{spt} v \subset\subset \Sigma$ (e.g. $v(z) =$ Proj. onto $T_{u(z)}N$ of $w(z)$ with $w \in C^1_{\text{cpt}}(\Sigma, \mathbf{R}^n)$ arbitrary)

and consider variations like

$$u_\varepsilon \;:=\; \pi_N \circ (u + \varepsilon v) \qquad (\pi_N \text{ nearest point retraction onto } N)\,,$$

or

$$u_\varepsilon(z) \;=\; \exp_{u(z)} \varepsilon v(z) \qquad (\exp \text{ the exponential mapping on } N)\,.$$

The calculation of the first variation of energy the leads to

$$\delta E_\Sigma(u, v) := \frac{d}{d\varepsilon}_{|\varepsilon=0} E_\Sigma(u_\varepsilon) = \int\limits_\Sigma Du \cdot Dv\, dx\, dy$$
$$\text{for } v \text{ tangential to } N \text{ along } u$$

and the corresponding Euler equation for $u$, in case it is stationary for the energy with respect to the variations considered, says that the Euler operator applied to $u$ must be orthogonal to the constraint (this is a general fact valid for variational problems with constraints), i.e.

$$\Delta u \perp N \qquad (\text{i.e. } \Delta u(z) \in T^\perp_{u(z)}N \text{ for } z \in \Sigma),$$

or equivalently, as one may calculate

$$\begin{aligned} \Delta u &= (II_N \circ u)(u_x, u_x) + (II_N \circ u)(u_y, u_y) \\ &= \operatorname{trace} u^* II_N \end{aligned}$$

where $II_N$ is the second fundamental form of $N$ (with values in the normal bundle of $N$).

**Definition 1** *Any classical solution $u$ to this equation (which is nonlinear with quadratic first order terms) is called a* harmonic mapping *from $\Sigma$ into $N$. If $u$ is only stationary with respect to the variations considered, i.e. the equation for harmonic mappings holds in a weak sense, then $u$ is called a* weakly harmonic mapping

The treatment of the first variation and the derivation of the Euler equation is completely analogous for mappings $u : M \to N \subset \mathbf{R}^n$ between general Riemannian manifolds, the only difference being that the Laplace operator $\Delta$ has to replaced by the Laplace-Beltrami operator $\Delta_\gamma$ associated with the Riemannian metric $\gamma$ on $M$. There is one big difference, however, between dimension 2 and higher dimensions of the domain here, because we have a recent *theorem of*

> **Helein:** *Weakly harmonic mapping from 2-dimensional domains are smooth, hence harmonic.*

(This is definitively not true in $m \geq 3$ independent variables). Thus we do not have to worry about (interior) regularity of maps which are minimizing or stationary for the energy in our 2-dimensional situation. We also note that the notion of a harmonic mapping $u : \Sigma \to N \subset \mathbf{R}^n$ makes sense for general Riemann surfaces $\Sigma$, although the Laplace operator is not well defined on $\Sigma$ (under a change of conformal coordinates the Laplacian in coordinates is multiplied by a positive function; thus the equation $\Delta u = 0$ has a meaning independent of coordinates).

Now let us look at mappings $u : \Sigma \to N \subset \mathbf{R}^n$ which are *conformal*, i.e. $(u_x)^2 = (u_y)^2$, $u_x \cdot u_y = 0$ on $\Sigma$. For variations $u_\varepsilon$ of $u = u_0$ we then have

$$A_\Sigma(u_\varepsilon) \leq E_\Sigma(u_\varepsilon) \qquad \text{with equality for } \varepsilon = 0.$$

It follows that

$$\delta A_\Sigma(u, v) = 0 \quad \text{if and only if} \quad \delta E_\Sigma(u, v) = 0,$$

provided $\delta A_\Sigma(u, v)$ exists (which is the case if $u$ is a regular parametric surface in the sense of differential geometry). Mapping $u$ which are minimal for the area integral, or only stationary in the sense that $\delta A_\Sigma(u, v) = 0$ for all $v$ with compact support holds, are called minimal surfaces. Therefore we define

**Definition 2** *$u : \Sigma \to N \subset \mathbf{R}^n$ is a* parametric minimal *surface if $u$ is a harmonic mapping and conformal.*

For conformal mappings $u : \Sigma \to \mathbf{R}^n$ the equation $\Delta u = 0$ has a differential geometric meaning (at least at points which are not "branch points", i.e. where $u$ has rank 2), namely we have

> for conformal $u : \Sigma \to \mathbf{R}^n$ (of max rank 2) $\Delta u$ is (twice) the mean curvature vector field of $u$.

Thus, $\Delta u = 0$ just says that the surface $u$ has mean curvature zero which is the well-known condition from differential geometry for vanishing of the first variation of area. In fact, one calculates that the Euler operator of the area integral (in any number of independent variables) gives just the mean curvature vector of the surface (up to a dimension factor which is matter of convention). Similarly, for conformal $u : \Sigma \to N \subset \mathbf{R}^n$ the tangential component of $\Delta u$ relative to $N$ gives the mean curvature field of $u$ relative to $N$, and the equation for harmonic mappings in this case says that $u$ has mean curvature zero relative to $N$.

The Euler equation for general conformally invariant variational integrals in 2 independent variables can, for conformally parametrized surfaces, also be interpreted in terms of mean curvature. For example, the *energy functional for surfaces with prescribed mean curvature*

$$\frac{1}{2}\int_{\Sigma}((u_x)^2 + (u_y)^2)\,dx\,dy + \int_{\Sigma}(Q \circ u)\cdot(u_x \times u_y)\,dx\,dy$$

for $u : \Sigma \to \mathbf{R}^3$ has a Euler equation

$$\Delta u = 2(H \circ u)\, u_x \wedge u_y \qquad \text{with } H := \frac{1}{2}\operatorname{div} Q$$

which, for conformal $u$, expresses the fact that $u$ has *mean curvature prescribed* by the vector field $Q$. Similarly, the Euler equation for general conformally invariant integrals

$$\frac{1}{2}\int_{\Sigma}[(g \circ u)(u_x, u_x) + (g \circ u)(u_y, u_y) + 2(Q \circ u)\cdot u_x \wedge u_y]\,dx\,dy$$

can, for conformal $u$ of rank 2, be interpreted as equations determining the mean curvature field of $u : \Sigma \to \mathbf{R}^n$ in the Riemannian manifold $(\mathbf{R}^n, g)$ (see Jost's book)

## 3 Variations of the independent variables and conformality

Suppose $\Sigma$ is a domain in $\mathbf{C} = \mathbf{R}^2$, $h_\varepsilon(z) = z + \varepsilon\xi(z)$ for $z \in \Sigma$ and $|\varepsilon| < 1$ is a deformation of $\mathrm{id}_\Sigma$ with deformation vector field $\xi \in C^1_b(\Sigma, \mathbf{C})$, $u \in W^{1,2}(\Sigma, \mathbf{R}^n)$.

Then $(u \circ h_\varepsilon^{-1})_{|\varepsilon| \ll 1}$ is called an *inner variation* of $u$ (variation of the independent variables for $u$).

We want to compute the *first variation of the Dirichlet-Integral* of $u$ *for inner variations*:

$$\partial E_\Sigma(u;\xi) := \frac{d}{d\varepsilon}\Big|_0 \frac{1}{2} \int\limits_{h_\varepsilon \Sigma} |D(u \circ h_\varepsilon^{-1})|^2 \, dx\, dy \quad \text{(cannot be differentiated}$$

under the integral immediately, since $u$ is not smooth)

$$= \frac{d}{d\varepsilon}\Big|_0 \frac{1}{2} \int\limits_{h_\varepsilon \Sigma} |(Du(Dh_\varepsilon)^{-1}) \circ h_\varepsilon^{-1}??|^2 \, dx\, dy$$

$$= \frac{d}{d\varepsilon}\Big|_0 \frac{1}{2} \int\limits_{\Sigma} |Du(Dh_\varepsilon)^{-1}|^2 \, |\det Dh_\varepsilon| \, dx\, dy$$

$$= \frac{1}{2} \int\limits_{\Sigma} [-2Du \cdot (Du\, D\xi) + |Du|^2 \operatorname{div} \xi] \, dx\, dy$$

$$= \frac{1}{2} \int\limits_{\Sigma} \operatorname{trace} \left[ -2 \begin{pmatrix} |u_x|^2 & +2u_x \cdot u_y \\ 2u_x \cdot u_y & |u_y|^2 \end{pmatrix} D\xi + \begin{pmatrix} |Du|^2 & 0 \\ 0 & |Du|^2 \end{pmatrix} D\xi \right] dx\, dy$$

$$= \frac{1}{2} \int\limits_{\Sigma} \operatorname{trace} \left[ \begin{pmatrix} |u_y^2| - |u_x|^2 & -2u_x \cdot u_y \\ -2u_x \cdot u_y & |u_x|^2 - |u_y|^2 \end{pmatrix} D\xi \right] dx\, dy$$

$$= \frac{1}{2} \int\limits_{\Sigma} \left[ (|u_x|^2 - |u_y|^2)(\xi^1_x - \xi^2_y) + 2(u_x \cdot u_y)(\xi^1_y + \xi^2_y) \right] dx\, dy$$

$$= -\int\limits_{\Sigma} \operatorname{Re}(\Phi\, \overline{\partial} \xi) \, dx\, dy$$

where

$$\Phi := |u_x|^2 - |u_y|^2 - 2i u_x \cdot u_y$$

and

$$\overline{\partial} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

We then have the following well known

**Theorem 1** (i) *$u$ is* stationary with respect to compactly supported inner variations[2] *(i.e. $\partial E_\Sigma(u,\xi) = 0$ whenever* spt *$\xi \subset\subset \Sigma$) if and only if $\Phi := |u_x|^2 - |u_y|^2 - 2iu_x \cdot u_y$ is holomorphic on $\Sigma$.*

(ii) *Suppose $\Sigma$ is smooth at its boundary. Then $u$ is* stationary with respect to inner variations transforming $\Sigma$ into itself *(i.e. $\partial E_\Sigma(u,\xi) = 0$ whenever $\xi_{|\partial\Sigma}$ is tangential to $\partial\Sigma$) if and only if $\Phi$ is smooth up to the boundary and $\Phi_{|\partial\Sigma}(i\gamma)^2$ is real on $\partial\Sigma$ where $\nu : \partial\Sigma \to S^1 \subset \mathbf{C}$ is the exterior unit normal field of $\Sigma$.*

*Proof.* (i) If $\Phi$ is smooth, then integration by parts gives

$$\partial E_\Sigma(u,\xi) = -\int_\Sigma \mathrm{Re}\,(\Phi\overline{\partial}\xi)\,dx\,dy = \int_\Sigma \mathrm{Re}\,(\xi\overline{\partial}\Phi)\,dx\,dy$$

for all $\xi$ with spt $\xi \subset\subset \Sigma$, and it follows that $\overline{\partial}\Phi = 0$ on $\Sigma$ is equivalent with $\partial E_\Sigma(u,\xi) = 0$ for all such $\xi$.

In general, $\Phi$ is only in $L^1(\Sigma, \mathbf{C})$, and the variational equation

$$0 = -\int_\Sigma \mathrm{Re}\,(\Phi\,\overline{\partial}\xi)\,dx\,dy \quad \text{for all } \xi \in C^1(\Sigma,\mathbf{C}) \text{ with spt } \xi \subset\subset \Sigma$$

says that $\Phi$ is weakly holomorphic (in the distributional sense) and hence, by Weyl's Lemma, again holomorphic on $\Sigma$. (In fact, smoothing $\Phi$ by convolution with mollifying kernels $\varphi_\varepsilon$ one obtain the same variational equation for $\varphi_\varepsilon * \Phi$ instead of $\Phi$, hence $\varphi_\varepsilon * \Phi$ is holomorphic and converges to $\Phi$ in $L^1$-norm on compact subsets of $\Sigma$ as $\varepsilon \downarrow 0$. On the other hand, from Cauchy's integral formula one sees an $L^1$-convergent sequence of holomorphic functions converges locally uniformly together with its derivatives, therefore the limit $\Phi$ must coincide $\mathcal{L}^2$-almost everywhere with a holomorphic function).

(ii) By (i), $\Phi$ is holomorphic on $\Sigma$. If $\Phi$ is smooth up to the boundary of $\Sigma$ then we can apply the Gauß theorem to see

$$\partial E_\Sigma(u,\xi) = -\int_\Sigma \mathrm{Re}\,(\Phi\,\overline{\partial}\xi)\,dx\,dy = -\int_{\partial\Sigma} \mathrm{Re}\,(\Phi\xi\nu)\,d\mathcal{H}^1$$

[2]Harmonic mappings with this property have been called "stationary harmonic mapping" recently which is somewhat misleading.

for all $\xi \in C^1_b(\overline{\Sigma}, \mathbf{C})$ with compact support in $\overline{\Sigma}$ and $\xi$ tangential to $\partial\Sigma$ along $\partial\Sigma$, i.e. $\xi_{|\partial\Sigma} = i\varphi\nu$ with $\varphi : \partial\Sigma \to \mathbf{R}$ (smooth). It follows that $\partial E_\Sigma(u, \xi) = 0$ for all such $\xi$ if and only if $\mathrm{Re}\,(\Phi_{|\partial\Sigma} i\nu^2) \equiv 0$, i.e. iff $(\Phi_{|\partial\Sigma}(i\nu^2)$ is real (by approximation, then also $\partial E_\Sigma(u, \xi) = 0$ for all $\xi \in C^1_b(\overline{\Sigma}, \mathbf{C})$ with $\xi_{|\partial\Sigma}$ tangential along $\partial\Sigma$ but not necessarily $\mathrm{spt}\,\xi$ compact in $\overline{\Sigma}$).

To prove boundary regularity of $\Phi$ from the vanishing of the first variation $\partial E_\Sigma(u, \xi)$ for variational vector fields $\xi$ tangential along $\Sigma$ we localize by introducing conformal coordinates near a boundary point of $\Sigma$ such that $\partial\Sigma$ appears as a straight segment in these coordinates. (This uses the Riemann mapping theorem and corresponding boundary regularity results). It is then sufficient to consider the upper unit half disc $\mathcal{U}_+$ in $\mathbf{C}$ and an $L^1$-function $\Phi$ on $\mathcal{U}_+$ such that

$$\int_{\mathcal{U}_+} \mathrm{Re}\,(\Phi\, \overline{\partial}\xi)\, dx\, dy = 0$$

$$\text{for all } \xi \in C^1(\mathcal{U}, \mathbf{C}) \text{ with } \mathrm{spt}\,\xi \subset\subset \mathcal{U} \text{ and } \xi_{|]-1,1[} \text{ real.}$$

Since $\Phi$ is then real valued on $]-1,1[$ in a weak sense we extend $\Phi$ to the unit disc $\mathcal{U}$ as an $L^1$-function by

$$\Phi(z) := \Phi^*(z) := \overline{\Phi(\overline{z})} \quad \text{for } z \in \mathcal{U},\ \mathrm{Im}\, z < 0,$$

and prove that the extended function $\Phi$ is (weakly) holomorphic on $\mathcal{U}$. For this, we consider $\xi \in C^1(\mathcal{U}, \mathbf{C})$ arbitrary with compact support in $\mathcal{U}$, verify $\overline{\partial}(\xi^*) = (\overline{\partial}\xi)^*$ and compute

$$\begin{aligned}\int_{\mathcal{U}} \mathrm{Re}(\Phi\, \overline{\partial}\xi)\, dx\, dy &= \int_{\mathcal{U}_+} \mathrm{Re}(\Phi\, \overline{\partial}\xi)\, dx\, dy + \int_{\mathcal{U}_-} \mathrm{Re}(\Phi(\overline{z})(\overline{\partial}\xi)(\overline{z}))\, dx\, dy \\ &= \int_{\mathcal{U}_+} \mathrm{Re}\,(\Phi\, \overline{\partial}(\xi + \xi^*))\, dx\, dy = 0,\end{aligned}$$

because $\xi + \xi^*$ is real on $]-1,1[$. Thus, the extended function $\Phi$ coincides $\mathcal{L}^2$-almost everywhere with a holomorphic function $u$ in $\mathcal{U}$. (This argument is, of course, nothing else than the Schwarz reflection principle formulated for weakly holomorphic maps). ■

**Remarks.** (1) The hypotheses of (i) are satisfied, whenever $u$ minimizes $E_\Sigma$ subject to boundary conditions and topological constraints (preserved under homotopies of type $(u \circ h_\varepsilon)_{|\varepsilon| \ll 1}$. The hypotheses of (ii) are satisfied, whenever the boundary conditions give only geometric restrictions for the boundary curve $u_{|\partial\Sigma} : \partial\Sigma \to \mathbf{R}^n$ which do not depend on its parametrization. Typical examples are *Plateau type boundary conditions* ($u_{|\partial\Sigma}$ parametrizes a given Jordan curve or system of curves monotonically) or *free boundary conditions* ($u(\partial\Sigma)$ is required to lie in a given submanifold of $\mathbf{R}^n$). As the proof shows, local versions of (ii) hold at the boundary, e.g. one can treat *partially free boundary conditions* where one portion of $\partial\Sigma$ is mapped to a given Jordan arc and the remaining part of $\partial\Sigma$ into a submanifold carrying the endpoints of the arc. (Strictly speaking the Plateau boundary condition is also a free boundary condition with one degree of freedom, in contrast with a Dirichlet condition where the boundary mapping $u_{|\partial\Sigma}$ is prescribed pointwise on $\partial\Sigma$.)

(2) The theorem holds (with the same proof) for *Riemann surfaces* $\Sigma$ as domains. Then, in view of transformation properties,

$$\Phi := (|u_x|^2 - |u_y|^2 - 2iu_x \cdot u_y)\, dz^2 \qquad (z = x + iy)$$

defines (in terms of local complex coordinates $z = x + iy$ on $\Sigma$) a *holomorphic quadratic differential* on $\Sigma$ under the hypotheses (i), and this differential is real on the boundary $\partial\Sigma$ (i.e. $(|u_x|^2 - |u_y|^2 - 2iu_x \cdot u_y)_{|z_0} < dz, \tau >^2$ is real for coordinates around $z_0 \in \partial\Sigma$ and tangent vectors $\tau$ to $\partial\Sigma$ in $z_0$) under the assumptions (ii).

(3) Similar statements and proofs apply to *energy- minimizing mappings into Riemannian manifolds* $N$; in the definition of $\Phi(w)$ above one simply has to replace the Euclidean norm "$|\cdot|$" and inner product "$\cdot$" by the Riemannian norm and inner product on $N$ (at the point $u(w) \in N$). If we assume $N \subset \mathbf{R}^n$ isometrically embedded into the Euclidean space, then it does not make a difference with regard to inner variations if we consider $u$ as mapping $\Sigma \to N$ or as mapping $\Sigma \to \mathbf{R}^n$. More generally, the theorem applies to *energy minimizing mappings with constraints* that restrict their image. For example, one can think of *obstacle problems, maps into manifolds with singularities*, etc.

(4) The theorem applies to *more general functionals* than energies. For example, we may add an arbitrary parametric integral (invariant with respect to arbitrary diffeomorphisms of the domain). Thus, it applies to all confor-

mally invariant variational integrals as described in Section 2, in particular, also to energy functionals for surfaces with prescribed mean curvature.

(5) If $u$ is a *smooth* solution of the Euler equation for the energy functional

$$\Delta u = 0 \qquad \text{on } \Sigma \subset \mathbf{C} = \mathbf{R}^2,$$

then $\Phi := |u_x|^2 - |u_y|^2 - 2iu_x \cdot u_y$ is holomorphic on $\Sigma$. This is immediate from ($\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$)

$$\Phi = 4\partial u \cdot \partial u, \qquad \overline{\partial}\Phi = 8\overline{\partial}\partial u \cdot \partial u = 2\Delta u \cdot \partial u$$

(where "$\cdot$" denotes the canonical $\mathbf{C}$-bilinear form on $\mathbf{C}^n$). Thus, *smooth* functions which are stationary with respect to variations of the dependent variables are also stationary with respect to variations of the independent variables. This is a general fact in the calculus of variations, known as "the first Euler equation implies the second Euler equation for smooth functions", and can be proved along the following lines:

$$\begin{aligned}\partial F(u;\xi) &:= \frac{d}{d\varepsilon}_{|0} F(u \circ (\mathrm{id} + \varepsilon\xi)^{-1}) = \frac{d}{d\varepsilon}_{|0} F(u \circ (\mathrm{id} - \varepsilon\xi + \varepsilon^2 \cdots)) \\ &= \frac{d}{d\varepsilon}_{|0} F(u - \varepsilon(Du)\xi + \varepsilon^2 \cdots) = \delta F(u; (Du)\xi) = 0.\end{aligned}$$

The analogous fact therefore also holds if $u$ is smooth harmonic mapping into a Riemannian manifold $N$ and $\Phi$ is defined correspondingly using the Riemannian metric of $N$. (One can, of course, also perform a similar, but more involved, calculation as above.)

Since, by the recent theorem of Helein every weakly harmonic map from a 2-dimensional Riemannian manifold into some Riemannian manifold is smooth, the holomorphy of the associated quadratic differential $\Phi$ therefore follows from the Euler equation. We emphasize, however, that the theorem can also be applied in situations where no Euler equation is available (e.g. obstacle problems, maps into manifolds with singularities) and if $u$ is stationary with respect to arbitrary inner variations then part (ii) of the theorem gives information about the boundary behaviour of $\Phi$ which cannot be seen from an Euler equation.

(6) The relation of the quadratic differential $\Phi = (|u_x|^2 - |u_y|^2 - 2iu_x \cdot u_y)\, dz^2$ associated with $u$ to the conformality of $u$ is, evidently,

$$\Phi \equiv 0 \quad \text{if and only if} \quad u \quad \text{is weakly conformal.}$$

Moreover, whenever $\Phi$ is holomorphic one can pass on simply connected subdomains $\mathcal{R}$ of the domain of $u$ to a mapping $\tilde{u} = (u, v)$ such that $v : \mathcal{R} \to \mathbf{C}$ is harmonic and $\tilde{u}$ is weakly conformal (and harmonic, in case $u$ was). Indeed, choosing $v$ with $v(z) = \overline{z} + w(z)$ where $\partial w = -\dfrac{1}{4}\Phi$ one has

$$\begin{aligned} |\tilde{u}_x|^2 - |\tilde{u}_y|^2 - 2i\tilde{u}_x \cdot \tilde{u}_y &= \Phi + 4\partial v \partial \overline{v} = \\ &= \Phi + 4\partial w = 0. \end{aligned}$$

This is an invention of Rick Schoen also used by M. Grüter.

**Examples.** (1) $\Sigma = \mathcal{U}$ *unit disc* in $\mathbf{C}$. Carries no holomorphic quadratic differential $\Phi \not\equiv 0$, which is real on $S^1 = \partial\mathcal{U}$, because $\Phi(z)(iz)^2$ real for $|z| = 1$ and holomorphic in $\overline{\mathcal{U}}$ implies $\Phi(z)z^2$ real and holomorphic on $\mathcal{U}$, hence constant, hence $\equiv 0$ (value at $z = 0$). Consequence: Maps from the unit disc which are stationary with respect to all inner variations are weakly conformal. This is a classical result used in the solution of Plateau's problem and it applies, in particular, to energy minimizing mappings from the unit disc subject to Plateau or free boundary conditions. Another application is the following *theorem of (Wente and) Lemaire*:

> If $u : \overline{\mathcal{U}} \to N$ (Riemannian manifold) is harmonic on $\mathcal{U}$ and constant on $\partial\mathcal{U}$ the $u \equiv$ const.

Indeed, from $u_{|\partial\mathcal{U}} \equiv$ const. one infers that $\Phi$ is real on $\partial\mathcal{U}$ (e.g. mapping $\mathcal{U}$ conformally onto the upper half plane), hence $\Phi \equiv 0$ on $\mathcal{U}$, i.e. $u$ is conformal. Then the total derivative of $u$ must vanish everywhere on $\partial\mathcal{U}$, one may reflect $u$ across $\partial\mathcal{U}$ to obtain a harmonic conformal map of $\mathbf{C}$ into $N$, this map turns out to coincide with a constant to infinite order on $\partial\mathcal{U}$ and hence, by a theorem of unique continuation for solutions to elliptic equations, $u$ must be constant. Later we shall use the case $N = S^2$ only and in that case it is, of course, much easier to see that a conformal map of $\mathcal{U}$ to $S^2$, constant on $\partial\mathcal{U}$, must be constant on $\mathcal{U}$ (using complex variables theory). Also, Wood has proved the same result for harmonic mappings $u : \overline{\mathcal{U}}^m \to N$ from the $m$-dimensional unit ball (with a quite simple calculation, see [EL, §12] for references).

(2) $\Sigma = S^2$ the *2-sphere.* Here every holomorphic quadratic differential must vanish, because identifying $S^2 = \mathbf{C} \cup \{\infty\}$ we can write it $\Phi(z)\,dz^2$ with a holomorphic function $\Phi$ such that $\Phi(\frac{1}{z})\frac{1}{z^2}$ is extendable holomorphically to $z = 0$. In particular, $\lim\limits_{|z| \to \infty} \Phi(z) z^2$ exists in $\mathbf{C}$ and $\lim\limits_{|z| \to \infty} \Phi(z) = 0$, so that $\Phi \equiv 0$ by Liouville's theorem. Consequence:

> Every harmonic map $u : S^2 \to N$ is conformal and therefore a parametric minimal surface (or constant).

(3) For general $\Sigma$ there exist nonvanishing holomorphic quadratic differentials which are real on the boundary.

For example we have $\Phi(z) = \dfrac{\alpha}{z^2}\,dz^2$ with $\alpha \in \mathbf{R}$ on an annulus in $\mathbf{C}$ centered at 0 and $\Phi(z) = b\,dz^2$ with $b \in \mathbf{C}$ on a torus $\mathbf{C}/\Gamma$. This is related (via Teichmüller theory) to the fact that there are different conformal structures on $\Sigma$. If then $u : \Sigma \to N$ is energy minimizing subject to topological constraints and Plateau or free boundary conditions we cannot expect that $u$ is conformal. It turns out that this will be true, however, if $u$ is minimizing also with respect to all possible conformal structures on $\Sigma$. (Note that changing the conformal structure will change the energy of $u$, in general.) For example, if we take an annulus $\Sigma$ and look for a minimal surface $u : \Sigma \to \mathbf{R}^3$ mapping $\partial\Sigma$ monotonically with degree 1 onto the union of two disjoint Jordan curves, then minimizing the energy subject to these conditions will produce a nonconformal harmonic map, in general. If we also minimize with respect to the conformal structure of $\Sigma$, that is we allow the inner radius of $\Sigma$ also to vary during the minimization process, then a minimal surface of annulus type will result, provided some reasonable geometric conditions on the Jordan curves are satisfied. Thus, to produce minimal surfaces of higher connectivity ot genus one must minimize the energy also with respect to the different choices of conformal structure on the parameter surfaces $\Sigma$. Therefore teichmüller theory—to describe the space of conformal structures on $\Sigma$—will naturally enter the picture.

## 4 The Courant-Lebesgue Lemma

It is a device that has already appeared in a much more general context in the lectures of B. White and F. Bethuel. It says roughly

> "Maps of finite energy have small oscillation on suitable small circles in their (two-dimensional) domains."

It is a very important technical tool in the theory of 2-dimensional minimal surfaces and harmonic mappings. Therefore, let me give a precise statement and proof:

**Lemma 1 (Courant-Lebesgue)** *Suppose $\Sigma$ open $\subset \mathbf{C} = \mathbf{R}^2$, $z_0 \in \mathbf{C}$, $0 < r < R < \infty$ and $u \in H^{1,2}(\Sigma, \mathbf{R}^n)$ with*

$$\frac{1}{2} \int\limits_{\Sigma \cap \mathcal{U}_R(z_0)} |Du|^2 \, dx \, dy \leq M < \infty.$$

*Then, for $\mathcal{L}^1$-almost all $\rho \in ]0, R[$ the mapping $\theta \mapsto u(z_0 + \rho e^{i\theta})$ (taken in the sense of traces) is of class $W^{1,2}$ and hence Hölder continuous with exponent $1/2$ on each interval in its domain $\Theta_\rho := \{\theta \in \mathbf{R} \ : \ z_0 + \rho e^{i\theta} \in \Sigma\}$. Moreover, there is a set of positive $\mathcal{L}^1$ measure in $]r, R[$ such that for $\rho$ from this set and $[\theta_1, \theta_2] \subset \Theta_\rho$ we have the estimate*

$$|u(z_0 + \rho e^{i\theta_1}) - u(z_0 + \rho e^{i\theta_2}| \leq \int\limits_{\theta_1}^{\theta_2} \left| \frac{\partial}{\partial\theta} u(z_0 + \rho e^{i\theta}) \right| d\theta$$

$$\leq \left[ \int\limits_{\theta_1}^{\theta_2} \left| \frac{\partial}{\partial\theta} u(z_0 + \rho e^{i\theta}) \right|^2 d\theta \right]^{1/2} |\theta_1 - \theta_2|^{1/2} \leq \sqrt{\frac{2M}{\ln(R/r)}} \, |\theta_1 - \theta_2|^{1/2}.$$

(One frequently takes $r = \delta_1 R = \sqrt{\delta}$ with $0 < \delta < 1$ in the applications.) Note that $\sqrt{2M/\ln(R/r)}$ becomes small as $r \downarrow$).

*Proof.* We assume $z_0 = 0$ for notational convenience. From

$$2M \geq \int\limits_{\Sigma \cap \mathcal{U}_R} |Du|^2 \, dx \, dy = \int\limits_0^R \int\limits_{[0,2\pi] \cap \Theta_\rho} \left[ \left| \frac{\partial}{\partial\rho} u(\rho e^{i\theta}) \right|^2 + \frac{1}{\rho^2} \left| \frac{\partial}{\partial\theta} u(\rho e^{i\theta}) \right|^2 \right] \rho \, d\rho \, d\theta$$

$$\geq \int_r^R \frac{1}{\rho} \underbrace{\int_{[0,2\pi]\cap\Theta_\rho} \left|\frac{\partial}{\partial\theta} u(\rho e^{i\theta})\right|^2 d\theta}_{=:\, I_\rho} \, d\rho$$

we deduce

$$I_\rho < \infty \qquad \text{for almost all } \rho \in ]0, R[$$

and

$$I_\rho \leq \frac{2M}{\ln(R/r)} \qquad \text{for a set of } \rho\text{'s with positive } \mathcal{L}^1\text{-measure in } ]r, R[\,,$$

because otherwise we would have $\int_r^R \frac{1}{\rho} I_\rho \, d\rho > 2M$. The claim then follows from the fundamental theorem of calculus and Hölder's inequality. ■

**Remark 1** We can choose $\rho$ simultaneously for infinitely many members of a sequence $(u_n)$ with uniformly bounded energy $E_\Sigma(u_n) \leq M$, if we replace $2M$ by $4M$ in the estimate. (The set $A_n$ of "good" $\rho$'s for $u_n$, such that the oscillation estimate is satisfied, then has $\mathcal{L}^1$ measure $\geq \frac{1}{2}(R-r)$ for each $u_n$, and therefore the set of $\rho$'s which are "good" for infinitely many $n$ has also measure $\geq \frac{1}{2}(R-r)$, as $\mathcal{L}^1(\bigcap_{m=1}^\infty \bigcup_{n=m+1}^\infty A_n) = \lim_{m\to\infty} \mathcal{L}^1(\bigcup_{n=m+1}^{\infty} A_n) \geq \frac{1}{2}(R-r)$.)

The classical application of the Courant-Lebesgue Lemma is the following.

## 5 Solution of Plateau's problem

The problem, in its simplest form, can be stated as follows:

Given a closed Jordan curve $\Gamma$ in $\mathbf{R}^n$, find a surface $u : \overline{\mathcal{U}} \to \Gamma$ of least area, parametrized on the unit disc, which spans $\Gamma$.

A direct approach to this problem based on the minimization of the area integral $\int_{\mathcal{U}} |u_x \wedge u_y| \, dx \, dy$ among surfaces spanning $u$ must fail for several reasons:

— there is a large ($\infty$-dimensional) symmetry group of self-diffeomorphisms of $u$.

Therefore, the set of minimizers is noncompact (if nonempty) and minimizing sequences do not converge, in general.

— the area functional is degenerate in the sense that $A_{\mathcal{R}}(u) = 0$ can occurs for $\mathcal{R}$ open $\neq \emptyset$ and $u$ nonconstant on $\mathcal{R}$.

For example, we may attach (infinitely many) "hairs" to a minimizer and parametrize this on the unit disc without increasing the area integral. Therefore the set of minimizers, if non empty, is enormous, and one can also produce minimizing sequences that way which converge to a prescribed point locally uniformly on $\mathcal{U}$ or which do not subconverge to a limit mapping in any reasonable sense at all.

— the area functional is not lower semicontinuous with respect to distributional (or, say, $\mathcal{L}^2$-almost everywhere) convergence of sequences $u_n \to u$ with uniformly bounded area (which would be a natural convergence for the problem).

All this is true also in dimension larger than 2, but for 2 independent variables there is another approach invented by Douglas and Radó (independently) in their solution of the Plateau problem, namely:

Minimize the energy $E_{\mathcal{U}}(u)$ instead of area $A_{\mathcal{U}}(u)$
among $u : \overline{\mathcal{U}} \to \mathbf{R}^n$, $\partial\mathcal{U} \to \Gamma$.

Because the energy functional does not have all the nasty properties of the area functional, this problem is much easier. But why do its solutions also solve the original Plateau problem? The reason is that the solutions will be conformal in view of the Plateau boundary condition which allows one to vary the independent variables freely also at the boundary of the unit disc (see Section 3). Thus

Minimizers $u_*$ will be conformal, hence satisfy
$A(u_*) = E(u_*) \leq E(u)$ for all comparison surfaces $u$,
but $E(u) \geq A(u)$

It still seems that one is stuck here, because the lost inequality goes in the wrong direction. However, for sufficiently regular immersions $u$ classical theorems (of Lichtenstein) assert that they may be reparametrized to become conformal, thus for a suitable diffeomorphism $h$ of $\overline{\mathcal{U}}$ one has

$$A(u_*) \leq E(\underbrace{u \circ h}_{\text{conformal}}) = A(u \circ h) = A(u),$$

and therefore $u_*$ is minimizing area when compared with regular immersions $u$, by approximation then also among all reasonable parametric surfaces $u$. Indeed, the following is true:

**Theorem 2 (Plateau's problem in $\mathbf{R}^n$, Douglas, Radó 1930/31)** *Suppose $\Gamma$ is an oriented rectifiable closed Jordan curve in $\mathbf{R}^n$. Then the class $\mathcal{S}(\Gamma) := \{u \in W^{1,2}(\mathcal{U}, \mathbf{R}^n) \;:\; u_{|\partial\mathcal{U}}$ maps $\partial\mathcal{U}$ monotonically with degree one onto $\Gamma\}$ in nonempty and contains energy minimizers. Each such minimizer $u$ satisfies*

(i) $\Delta u = 0$ *on* $\mathcal{U}$

(ii) $|u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y$ *on* $\mathcal{U}$

(iii) $u \in C^0(\overline{\mathcal{U}}, \mathbf{R}^n)$ *and* $u_{|\partial\mathcal{U}} : \partial\mathcal{U} \to \Gamma$ *is a homeomorphism.*

*Moreover, $u$ also minimizes the area integral in $\mathcal{S}(\Gamma)$.*

The so-called *Plateau-boundary condition*

$$u_{|\partial\mathcal{U}} : \partial\mathcal{U} \longrightarrow \Gamma \quad \text{monotonically with degree 1}$$

here means precisely, that the boundary trace of $u$ coincides with a continuous function $e^{i\theta} \mapsto \gamma(e^{i\varphi(\theta)})$ $\mathcal{H}^1$-almost everywhere on the unit circle $\partial\mathcal{U}$, where $\varphi : [0, \pi] \to [0, \pi]$ is nondecreasing and onto and $\gamma : \partial\mathcal{U} \to \Gamma$ is an orientation preserving homeomorphism. (Equivalent is that $u_{|\partial\mathcal{U}}$ is a uniform limit of orientation preserving homeomorphisms from $S^1$ onto $\Gamma$.)

*Proof.* (Courant) Since $\Gamma$ is rectifiable, one easily sees $\mathcal{S}(\Gamma) \neq \emptyset$ (by a cone construction, e.g.)[3]. Consider a minimizing sequence $(u_n)$ in $\mathcal{S}(\Gamma)$. Then $u_n$ subconverges weakly in $W^{1,2}$ to some $u_*$ with $E_{\mathcal{U}}(u_*) \leq \liminf_{n\to\infty} E_{\mathcal{U}}(u_n)$, and the problem is to show that $u_*$ belongs to $\mathcal{S}(\Gamma)$ again, i.e. $u_*$ satisfies the Plateau boundary condition. This will certainly be true if we can prove uniform convergence on the boundary circle $\partial\mathcal{U}$. (By Dirichlet's principle we also may assume that each $u_n$ is harmonic, because the energy of the

[3]For this and the theorem it is sufficient that $\Gamma$ admits a parametrization of class $W^{1/2,2}$.

harmonic function on $\mathcal{U}$ with the boundary values of $u_n$ does not exceed the energy of $u_n$. Then uniform convergence of the $u_n$ on the boundary implies uniform convergence on $\overline{\mathcal{U}}$ by the maximum principle.)

Consider now $z_0 \in \partial\mathcal{U}$ and apply the Courant-Lebesgue Lemma to find $0 < \rho \ll 1$ such that for the two points $z_1, z_2$ in $\partial\mathcal{U} \cap \partial\mathcal{U}_\rho(z_0)$ the corresponding points $u_n(z_1), u_n(z_2) \in \Gamma$ have uniformly small distance. These two points divide $\Gamma$ into two arcs, one of small diameter, the other one large. One wants to conclude, of course, that $u_n$ maps the whole arc $\partial\mathcal{U} \cap \overline{\mathcal{U}_\rho(z_0)}$ onto the smaller arc of $\Gamma$ thereby concluding uniform continuity of the $u_n$ and, by the Arzelá-Ascoli theorem, uniform subconvergence. This conclusion is not valid, however; there are surfaces in $\mathcal{S}(\Gamma)$ with energy bounded by a fixed constant which map arbitrarily small arcs of the boundary circle onto large arcs of $\Gamma$. This is due to the fact that we have still a noncompact symmetry group in our minimization problem, the three dimensional group of conformal self transformations of the unit disc. This group acts transitive on triples of distinct points on the unit circle, in particular, there are conformal self-mappings of the unit disc which map small boundary arcs onto very large ones, and reparametrizing a surface from $\mathcal{S}(\Gamma)$ by such mappings $h$ one obtains surfaces $u \circ h \in \mathcal{S}(\Gamma)$ of the same energy which map a very small arc in $\partial\mathcal{U}$ to a large arc in $\Gamma$.

To exclude this phenomenon one must *divide out the conformal group* by fixing the points $w_1, w_2, w_3$ on $\partial\mathcal{U}$ and $P_1, P_2, P_3$ on $\Gamma$ in the order of the orientation and requiring a *3-point-condition*

$$u(w_i) = P_i \quad \text{for} \quad i = 1, 2, 3.$$

Then the argument above can be carried out successfully, because the small arc $\partial\mathcal{U} \cap \mathcal{U}_\rho(z_0)$ from $z_1$ to $z_2$ and the circle can contain at most one of the $w_i$ and the larger of the two arcs on $\Gamma$ with endpoints $u_n(z_1), u_n(z_2)$ must contain at least two of the $P_i$ so that the arc on the circle cannot be mapped to this large arc on $\Gamma$ but only to the small one in view of the Plateau boundary condition.

To be precise we define the class of surfaces

$$\mathcal{S}^*(\Gamma) := \{u \in \mathcal{S}(\Gamma) \,:\, u(w_i) = P_i \text{ for } i = 1, 2, 3\}$$

and some $u \in \mathcal{S}^*(\Gamma)$ with $E_{\mathcal{U}}(u) \leq M < \infty$. Since $\Gamma$ is a Jordan curve, for $0 < \varepsilon < \min_{i \neq j} |P_i - P_j|$ we can determine $\delta > 0$ such that the smaller of the two arcs in which $\Gamma$ is divided by any two points with distance $\leq \delta$ has diameter at most $\varepsilon$. Choose $R < \frac{1}{2}\min_{i \neq j}|w_i - w_j|$ with $R < \sqrt{2}$ and $0 < r < R$ such that $\frac{2\pi M}{\ln(R/r)} \leq \delta^2$. For $z_0 \in \partial\mathcal{U}$ then apply the Courant-Lebesgue Lemma to find $\rho \in ]r, R[$ such that $u$ maps $\partial\mathcal{U} \cap \overline{\mathcal{U}_\rho(z_0)}$ does not contain (at least) two of the points $w_i$. Since $u_{|\partial\mathcal{U}}$ is weakly monotonic, it follows that the image of this arc in $\Gamma$ omits a subarc bounded by two of the $P_i$. By the choice of $\varepsilon$ this image therefore must be the smaller of the two arcs in which $\Gamma$ is divided by $P$ and $Q$ hence its diameter is $\leq \varepsilon$. In particular, we have determined $r$ depending on $\varepsilon, \Gamma$ and the energy bound $M$ only, such that $u \in \mathcal{S}^*(\Gamma)$ with $E_\mathcal{U}(u) \leq M$ maps each arc $\partial\mathcal{U} \cap \overline{\mathcal{U}_\rho(z_0)}$ in $\partial\mathcal{U}$ onto an arc of diameter $\leq \varepsilon$ in $\Gamma$. Thus energy bounded subsets of $\mathcal{S}^*(\Gamma)$ have equicontinuous boundary values, hence $\mathcal{S}^*(\Gamma)$ is closed with respect to weak $W^{1,2}$ convergence and energy minimizers $u_*$ exist in $\mathcal{S}^*(\Gamma)$. These $u_*$ are then, of course, also minimizers in $\mathcal{S}(\Gamma)$ because each $u \in \mathcal{S}^*(\Gamma)$ can be transformed to $u \circ h \in \mathcal{S}^*(\Gamma)$ by a conformal transformation of the unit disc into itself.

The remaining statements about the minimizers $u_*$ in $\mathcal{S}(\Gamma)$, have mostly been proved already in Sections 2 and 3. Since we can vary the independent variables by deformation fields with compact support in $\mathcal{U}$, the first variation of energy with respect to such variations must vanish, i.e. $u_*$ is harmonic. Since we can also vary the independent variables within $\mathcal{S}(\Gamma)$ by arbitrary diffeomorphisms of the unit disc, $u_*$ is also conformal. If $u_{*|\partial\mathcal{U}}$ were not a homeomorphism, then $u_*$ would be constant on an arc in the unit circle and could be extended, by reflection, to a conformal harmonic map across this arc. For the extension the angular derivative would vanish on the arc and, by conformality, the radial derivative also, thus $u_*$ would have to be constant which is clearly impossible for $u \in \mathcal{S}(\Gamma)$.

Finally, to prove that $u_*$ also minimizes the area, one uses the following *$\varepsilon$-conformality lemma of Morrey* (1948): For any $u \in W^{1,2}(\mathcal{U}, \mathbf{R}^n) \cap C^0(\overline{\mathcal{U}}, \mathbf{R}^n)$ and $\varepsilon > 0$ there exists a homeomorphism $h$ of $\overline{\mathcal{U}}$ such that $u \circ h \in W^{1,2}(\mathcal{U}, \mathbf{R}^n)$, $E_\mathcal{U}(u \circ h) \leq \varepsilon + A_\mathcal{U}(u)$. For our minimizers $u_*$ in $\mathcal{S}(\Gamma)$ this implies

$$A_\mathcal{U}(u_*) = E_\mathcal{U}(u_*) \leq E_\mathcal{U}(u \circ h) \leq \varepsilon + A_\mathcal{U}(u)$$

with $\varepsilon > 0$ arbitrarily small, hence $A_\mathcal{U}(u_*) \leq A_\mathcal{U}(u)$.

(It seems that the only source for Morrey's lemma is his original paper; the idea is to perturb $u$ to a smooth immersion $\tilde{u}$ and choose $h$ such that $\tilde{u} \circ h$ is conformal.) ∎

**Remarks.** (1) *Uniqueness of minimizers* (up to conformal reparametrization) is known only in very special cases, e.g. if $\Gamma \subset \mathbf{R}^3$ has a convex projection onto a plane (Radó). There are examples where non uniqueness holds, e.g. boundary curves with symmetries.

(2) The minimizers are in fact *immersions* on $\mathcal{U}$. Points $z$ with $u_x(z) \wedge u_y(z) = 0$ are called *branch points* of a parametric minimal surface $u$; *true* if accompanied by lines of self intersection emanating from $u(z)$, *false* otherwise (line $z \mapsto u(z^2)$ at $z = 0$). It is known that branch points $z$ are isolated in $\mathcal{U}$, and one has an asymptotic development of $u$ around $z$ showing that the target plane to the surface can be extended continuously to $z$. As late as 1970 Osserman showed that branch points do not exist for energy minimizing parametric minimal surfaces in $\mathbf{R}^3$. His argument was not complete, however, with regard to false branch points; this was later rectified by Gulliver and by Alt. The idea in the case of a true branch point is to cut the surface along a branch line, reparametrize and obtain a new surface of the same energy with "edges", however, pushing in the "edges" would decrease the energy, a contradiction.

(3) The minimizers are *not embedded*, in general. If the Jordan curve $\Gamma$ is unknotted in $\mathbf{R}^3$, then there must be self intersections. However, these must also occur for certain knotted $\Gamma$'s.

It is known for *extreme curves* $\Gamma \subset \mathbf{R}^3$, i.e. Jordan curves on the boundary of a convex region in $\mathbf{R}^3$ (or if $\Gamma$ is a Jordan curve on the boundary of a $C^2$-domain in $\mathbf{R}^3$ with non-negative mean curvature with respect to the inward normal and $\Gamma$ is contractible in the closure of this domain) that the energy minimizers in $\mathcal{S}(\Gamma)$ are embedded discs (Meeks & and Yau; weaker related results has been obtained by Almgren & Simon and previously by Tomi & Tromba ).

(4) *Boundary regularity* for parametric minimal surfaces $u$ was proved by Hildebrandt (and improved by Nitsche): If $\Gamma$ is of class $C^{m,\alpha}$ with $m \geq 1$ and $0 < \alpha < 1$ then $u$ is of class $C^{m,\alpha}$ on $\overline{\mathcal{U}}$. However, *branch points on the boundary* can be only excluded in special cases, e.g. if $\Gamma \subset \mathbf{R}^3$ is real analytic (Gulliver & Lesley) or $\Gamma \subset \mathbf{R}^3$ has total curvature $\leq 4\pi$ (Nitsche).

(5) The *Plateau problem for surfaces of prescribed mean curvature $H$ in $\mathbf{R}^3$* where the Euler equation $\Delta u = 0$ is replaced by $\Delta u = 2H u_x \times u_y$ and the energy $E_{\mathcal{U}}(u)$ by $\frac{1}{2}\int_{\mathcal{U}}(|u_x|^2 + |u_y|^2)\,dx\,dy + \int_{\mathcal{U}}(Q \circ u)\cdot(u_x \times u_y)\,dx\,dy$ with $\operatorname{div} Q = 2H$ has also been solved by an analogous, but more involved reasoning under various natural conditions on the prescribed mean curvature $H$ and boundary curve $\Gamma$, e.g. $\Gamma$ contained in the unit ball and $|H| \le 1$ on $\mathbf{R}^3$.

## 6 Morrey's theorem on harmonic mappings in Riemannian manifolds

In 1948 Morrey published a paper which was the first at all treating minimal surfaces in Riemannian manifolds. In trying to solve Plateau's problem in a Riemannian manifold he found that it was not possible to prove uniform convergence of minimizing sequences for the energy as was the case in the Plateau problem in a Euclidean space. Thus, Morrey had to consider Sobolev spaces of mappings into a Riemannian manifold and prove the continuity (and higher regularity) of minimizers for the energy among such mappings. A large part of his work is therefore devoted to the definition and study of Sobolev spaces of mappings from 2-dimensional domains in a Riemannian target manifold $N$. Interestingly enough, Morrey did not assume that this target manifold was embedded in a Euclidean space (the Nash isometric embedding theorem was not yet proved at that time), but had to introduce his Sobolev-type spaces of mapping into $N$ in an intrinsic (complicated) way. Nowadays it has become a custom to circumvent all these difficulties by assuming $N \subset \mathbf{R}^n$ embedded isometrically and defining

$$W^{1,2}(\Sigma, N) := \{u \in W^{1,2}(\Sigma, \mathbf{R}^n) \,:\, u(z) \in N \text{ for almost all } z \in \Sigma\}.$$

Here $\Sigma$ is a compact Riemann surface with or without boundary, so that $W^{1,2}(\Sigma, \mathbf{R}^n)$ is a well defined space. (The $W^{1,2}$-norm depends on the choice of a Riemannian metric $\Sigma$, but it is well defined up to equivalence of norms.)

Morrey's theorem then is the following

**Theorem 3 (Morrey)** *Suppose $\partial\Sigma \neq \emptyset$, and $N \subset \mathbf{R}^n$ is compact or closed and homogeneously regular (in the sense of Morrey, see below). Let*

*$u_0 : \partial\Sigma \to N$ be the trace of some $W^{1,2}$-map from $\partial\Sigma$ to $N$. Then there exists an energy minimizing mapping $u$ in the class of all $W^{1,2}$-mappings from $\Sigma$ to $N$ with boundary trace $u_0$. Each such energy minimizer $u$ is smooth and a harmonic mapping on $\Sigma$. Moreover, if $u_0$ is continuous, then $u$ is continuous on $\Sigma \cup \partial\Sigma$.*

The condition of *homogeneous regularity* introduced by Morrey says that each point $P \in N$ is the center of a coordinate system which maps a coordinate ball around $P$ biLipschitz onto the unit ball in $\mathbf{R}^{\dim N}$, where the biLipschitz constant is bounded independently of the point $P \in N$. This is clearly satisfied for compact $N$, hence it is to be interpreted as a condition at infinity. If $N \subset \mathbf{R}^n$ has bounded second fundamental form, for example, the condition may be verified.

*Proof.* There is no problem with the minimization process here since energy bounded sequences in $\{u \in W^{1,2}(\Sigma, N) : u_{|\partial\Sigma} = u_0\}$ are bounded in $W^{1,2}$ (in view of $\partial\Sigma \neq \emptyset$, $\Sigma$ connected, Poincaré's inequality), hence weakly and, by Rellich's theorem, almost everywhere on $\Sigma$ subconvergent. The lower semicontinuity of energy then provides a minimizer $u$, and the problem is to prove the regularity of $u$. Since $u$ is stationary with respect to variations of the dependent variables, as explained in Section 2, $u$ must be weakly harmonic. The mentioned theorem of Helein then implies that $u$ is regular on $\Sigma$. However, this theorem requires (I think) compactness of $N$, and does not give continuity at the boundary, so we follow Morrey's original proof.

Morrey used, of course, one of his favorite ideas: *harmonic replacement.* It is no restriction to assume that $\Sigma$ is a domain in $\mathbf{C}$ (introducing coordinates on $\Sigma$). We then look at a ball $\mathcal{U}_\rho(a) \subset\subset \Sigma$ and want to produce a comparison surface for $u$ replacing it on $\mathcal{U}_\rho(a)$ by a harmonic function with respect to coordinates on $N$.

If $\displaystyle\int_0^{2\pi} \left|\frac{\partial}{\partial\theta} u(a + \rho e^{i\theta})\right|^2 d\theta$ is small then $u(\partial\mathcal{U}_\rho(a))$ has small diameter in $N$, hence is contained in a coordinate ball mapped onto the unit ball in $\mathbf{R}^{\dim N}$ by $X$ with $\operatorname{Lip} X + \operatorname{Lip} X^{-1} \leq \operatorname{const}(N)$. Letting $h : \mathcal{U}_\rho(a) \to \mathbf{R}^{\dim N}$ be the harmonic function with boundary values $X \circ u$ on $\partial\mathcal{U}_\rho(a)$ we can define $\tilde{u} := \begin{cases} u & \text{on } \Sigma \setminus \mathcal{U}_\rho(a) \\ X^{-1} \circ h & \text{on } \mathcal{U}_\rho(a) \end{cases}$

This is an admissible comparison surface, hence $E_\Sigma(u) \leq E_\Sigma(\tilde{u})$. Using

Fourier series one easily sees $E_{\mathcal{U}_\rho(a)}(h) \leq \frac{1}{2}\int\limits_0^{2\pi} \left|\frac{\partial}{\partial\theta} X \circ u(a+\rho e^{i\theta})\right|^2 d\theta$, hence we have

$$\varphi(\rho) := E_{\mathcal{U}_\rho(a)}(u) \leq \operatorname{const}(N) \underbrace{\frac{1}{2}\int\limits_0^{2\pi} \left|\frac{\partial}{\partial\theta} u(a+\rho e^{i\theta})\right|^2 d\theta}_{\leq \rho\varphi'(\rho)}.$$

This implies

$$\varphi(\rho) \leq \underbrace{\operatorname{const}(N)(1+E_\Sigma(u))}_{=: \frac{1}{2\alpha},\ 0<\alpha<1} \rho\varphi'(\rho)$$

for (almost) all $\rho$ with $\mathcal{U}_\rho \subset \Sigma$, and integrating this differential inequality one obtains a *Morrey growth condition* for the Dirichlet integrals:

$$\frac{1}{2}\int\limits_{\mathcal{U}_\rho(a)} (|u_x|^2 + |u_y|^2)\, dx\, dy \leq E_\Sigma(u)\left(\frac{\rho}{r}\right)^{2\alpha}$$

for $0 < \rho < r$, $\mathcal{U}_r(a) \subset \Sigma$.

It is a well known fact, also due to Morrey, that such a condition implies *Hölder continuity*

$$|u(z) - u(\tilde{z})| \leq \operatorname{const}(\alpha)\sqrt{E_\Sigma(u)}\,\frac{|z-\tilde{z}|^\alpha}{r^\alpha}$$
$$\text{for } r > 0,\ z, \tilde{z} \in \Sigma \text{ with } \operatorname{dist}([z,\tilde{z}], \partial\Sigma) \geq r$$

and

continuity of $u$ on $\Sigma \cup \partial\Sigma$, if $u_{|\partial\Sigma} = u_0$ is continuous.

Once $u$ is proved to be continuous one may localize in the target introducing coordinates. Higher regularity is then a matter of regularity for systems of (nonlinear) elliptic equations, and we do not go into this subject here. (Assuming something more on the boundary values than continuity, e.g. Lipschitz condition, one can reduce the boundary regularity to interior regularity by a biLipschitz transformation.) ■

**Remarks.** (1) With the same hypotheses on $N$ one can now solve the *Plateau problem for Jordan curves* $\Gamma$ *in the Riemannian manifold* $N$, if $\Gamma$ bounds a parametrized surface of finite energy in $N$. One define $\mathcal{S}(\Gamma)$, $\mathcal{S}^*(\Gamma)$ as subclasses of $W^{1,2}(\mathcal{U}, N)$ exactly as in Section 5 and proves that minimizing sequences in $\mathcal{S}^*(\Gamma)$ are equicontinuous on the boundary using the Courant-Lebesgue Lemma also exactly as in Section 5. Minimizers in $\mathcal{S}^*(\Gamma)$ and $\mathcal{S}(\Gamma)$ therefore exist and are, in addition to being harmonic, also invariant with respect to arbitrary variations of the independent variables, hence conformal according to Section 3.

Thus, minimizers are parametric minimal surfaces in $\mathcal{S}(\Gamma)$.

(2) That some global condition on the target $N$ is necessary for the theorem to be true was already demonstrated with an example by Morrey:

$$N := \{\xi \in \mathbf{R}^3 \ : \ \xi_3 = -\ln\sqrt{\xi_1^2 + \xi_2^2}\}$$
$$u_0(x, y) := (x, y, 0) \quad \textit{resp.} \quad \Gamma := S^1 \times \{0\} \subset N$$

$N \cap \mathbf{R}^2 \times [0, \infty[$ has finite area and can be parametrized conformally and rotationally symmetric (by explicit formulas) on $\mathcal{U} \setminus \{0\}$. Hence $u_0$ admits an extension into $N$ of finite energy which is a parametric surface satisfying the Plateau boundary condition for $\Gamma$, but obviously $u_0$ cannot be extended continuously to $N$ and $\Gamma$ does not bound a continuous parametric surface of type of the disc in $N$.

## 7 Harmonic mappings with topological constraints

The energy minimization method of Morrey gives nonconstant harmonic mappings into a Riemannian manifold only because one has prescribed boundary values (nonconstant). If we consider a compact Riemann surface $\Sigma$ without boundary as parameter domain then every energy minimizing $u : \Sigma \to N \subset \mathbf{R}^n$ is constant, of course, which is not very interesting. Thus, one wants to impose some additional topological restrictions on the mappings which prevent them from degenerating to constants in the limit of the minimizing procedure.

Now, in B. White's lectures we have already seen that $W^{1,2}$ mappings between manifolds have a well-defined 1-homotopy type, hence it makes sense for a $W^{1,2}$-mapping $u : \Sigma \to N$ to require

$$u_\#[\alpha_i] = [\beta_i] \quad \text{for given loops } \alpha_i \text{ in } \Sigma \text{ and } \beta_i \text{ in } N.$$

This was actually first used in a paper by Schoen & Yau which started the subsequent intensive study of topological invariants for Sobolev mappings between manifolds about which we have heard a lot in the lectures of B. White and F. Bethuel. For our purpose here let me recall, how the action of a $W^{1,2}$-mapping on loops is defined. (This discussion is actually not restricted to 2-dimensional parameter domains, but we shall stick to that case here, for simplicity.)

If $\alpha : S^1 \to \Sigma$ is a smooth immersed loop in $\Sigma$ then we thicken it considering an immersion $(\omega, s) \mapsto \alpha_s(\omega) \in \Sigma$ of $S^1 \times ]-1,1[$ into $\Sigma$. We can then speak of the "parallel curves" $\alpha_s$ of $\alpha$. As in the Courant-Lebesgue Lemma one sees that for $u \in W^{1,2}(\Sigma, N)$,

$$u \circ \alpha_s : S^1 \longrightarrow N \text{ is continuous for almost all } s.$$

Moreover

$$u \circ \alpha_s \text{ is homotopic in } N \text{ to } u \circ \alpha_{s'} \text{ for any two such } s, s'.$$

This can be seen from Morrey's theorem, for example, because $u \circ \alpha_s$ and $u \circ \alpha_{s'}$ are the boundary values of $S^1 \times [s, s'] \ni (\omega, \sigma) \mapsto u(\alpha_\sigma(\omega)) \in N$ which is a $W^{1,2}$-mapping, hence there exists also an energy minimizing mapping with these boundary values which is continuous and gives the required homotopy between $\alpha_s$ and $\alpha_{s'}$. With a similar argument one sees that for loops $\tilde{\alpha}$ in $\Sigma$ homotopic to $\alpha$ also $\tilde{\alpha}_{s'}$ is homotopic to $\alpha_s$ for almost all $s, s'$. In particular, if $\alpha$ is homotopically trivial, the so are $u \circ \alpha_s$ for almost all $s$. (because there is a Lipschitz map $\varphi : \overline{\mathcal{U}} \to \Sigma$ with $\varphi_{|S^1} = \alpha_s$ and then $u \circ \varphi : \overline{\mathcal{U}} \to N$ has finite energy, so that it bounds a continuous surface of disc type, by Morrey's theorem).

We are assuming here, of course, that $N$ satisfies the hypotheses required in Morrey's theorem, e.g. $N$ compact. The discussion so far shows that

> For $u \in W^{1,2}(\Sigma, N)$ and homotopy classes $[\alpha] \in [S^1, \Sigma]$ in $N$ we have a well defined image $u_\#[\alpha] \in [S^1, N]$ by taking the homotopy class of $u \circ \alpha_s$ for $\mathcal{L}^1$-almost all $s$.

It is also not difficult to see (and was elaborated in B. White's lectures), that this action of $u$ on homotopy classes of loops in $\Sigma$ is preserved in the limit under weak convergence $u_n \to u$ in $W^{1,2}(\Sigma, N) \subset W^{1,2}(\Sigma, \mathbf{R}^n)$.

Indeed, by Fubini's and Sobolev's theorem $u_n$ converges to $u$ uniformly on almost all of the parallel curves $\alpha_s$, hence $u_n \circ \alpha_s$ and $u \circ \alpha_s$ are in the same homotopy class for sufficiently large $n$ and therefore

$$\begin{gathered} u_{n\#}[\alpha] = u_{\#}[\alpha] \text{ for } n \gg 1, \\ \text{if } u_n \to u \text{ weakly in } W^{1,2}(\Sigma, N). \end{gathered}$$

It is now clear that one can minimize the energy subject to topological conditions prescribing the action on homotopy classes of loops. For example, one has the following

**Theorem 4 (Schoen &Yau)** (one should also mention here Lemaire, Sacks & Uhlenbeck as predecessors) *Suppose $\Sigma$ is a compact Riemann surface and either $\partial\Sigma = \emptyset$, $N$ compact, or $\partial\Sigma \neq \emptyset$, $N$ homogeneously regular (in Morrey's sense) and $u_0 : \partial\Sigma \to N$ continuous. Let $\alpha_i$ be finitely many loops in $\Sigma$ and $\beta_i$ corresponding loops in $N$. Then there exist energy minimizing maps in $\{u \in W^{1,2}(\Sigma, N) : u_{\#}[\alpha_i] = [\beta_i]$ for all $i$, $u_{|\partial\Sigma} = u_0$ in case $\partial\Sigma \neq \emptyset\}$, provided this class of mappings is non empty.*
*Each such minimizing mapping is energy minimizing without the topological constraints on the interior of each closed topological disc in $\Sigma$ and hence harmonic on $\Sigma$ and continuous on $\Sigma \cup \partial\Sigma$.*

*Proof.* The existence of minimizers should be clear after the discussion above which shows that the class of admissible surfaces is weakly closed in $W^{1,2}(\Sigma, N)$. By the Poincaré inequality minimizing sequences are also weakly compact. (For this one has to assume that $N$ is compact in case no boundary conditions are imposed on $u$, otherwise minimizing sequences could escape to infinity.) Finally, given a topological disc in $\Sigma$, we can perturb the loops $\alpha_i$ to homotopic loops not meeting the interior of the disc; therefore variations of a $W^{1,2}(\Sigma, N)$-map inside the disc do not affect its action on the homotopy classes of the loops, hence all minimizers are also minimizing with respect to arbitrary variations on the topological disc. The assertions then follow from Morrey's theorem in Section 5. ∎

**Remarks.** (1) The theorem holds analogously for Riemannian manifolds of dimension $m > 2$ as domains.

(2) One can also treat nonorientable domains. In this case there first seems to be some problem with the action on loops of the kind of the "soul" in a

Möbius strip, because there are no "parallel loops" here. But this problem can be circumvented by a somewhat more involved construction (or by B. White's trick of considering "parallel loops" in an ambient Euclidean space for the domain manifold).

(3) In case $\partial\Sigma \neq \emptyset$ one can also consider paths $\alpha_i$ connecting boundary points of $\Sigma$ and corresponding paths $\beta_i$ in $N$ connecting the $u_0$-image of these points and then prescribe the action of the admissible mappings on homotopy classes of such paths (with fixed end points) by $u_{\#}[\alpha_i] = [\beta_i]$.

(4) The boundary conditions can also be varied. For example one may choose free boundary conditions on part of $\partial\Sigma$ requiring that the admissible $u$ maps this part to a given closed submanifold of $N$ (including $N$ itself, so that no boundary conditions at all are imposed on this part of the boundary). If the boundary conditions are preserved under reparametrization of $u$ with diffeomorphisms of the domain $\Sigma$, the the minimizers will also be stationary w.r.t. variations of the independent variables (but not necessarily conformal; cf Section 3).

(5) The point of the topological constraints is, of course, that nontrivial, i.e. nonconstant, harmonic mappings are produced from the minimization process, if one of the loops $\beta_i$ is homotopically nontrivial. The theorem in the case $\partial\Sigma \neq \emptyset$ is of value only if $N$ has nontrivial fundamental group. (If $\partial\Sigma \neq \emptyset$, then boundary conditions not satisfied by constants exclude also trivial minimizers, of course.) **2**

There are situations where the homotopy class of a mapping is completely determined by its action on the fundamental group of the domain. This is so, for example, if we have a 2-dimensional domain $\Sigma$ and a target $N$ with second homotopy group $\pi_2(N) = 0$. Therefore, choosing the $\alpha_i$ as generators of the fundamental group of $\Sigma$ and the $\beta_i := u_0 \circ \alpha_i$ (with $u_0$ below), we obtain the

**Corollary 1 (Sacks-Uhlenbeck for $\partial\Sigma = \emptyset$, Lemaire)** *Suppose $\Sigma$ is a compact Riemann surface, $N$ is homogeneously regular, $\pi_2(N) = 0$, $N$ compact in case $\partial\Sigma = \emptyset$. Let $u_0 : \Sigma \cup \partial\Sigma \to N$ be continuous with finite energy. Then there exists a harmonic map $u \in C^0(\Sigma \cup \partial\Sigma, N)$ such that $u$ minimizes energy in the class of continuous $W^{1,2}$-maps from $\Sigma$ to $N$ which are homotopic to $u_0$ through mappings with the same boundary values as $u_0$.*

**Remark.** If $N$ has a contractible universal cover then the homotopy class of a continuous $u : M \to N$, where $M$ is a compact Riemannian manifold, is also determined by the action of $u$ on the fundamental group of $M$. By the Hadamard-Cartan theorem this condition on $N$ is satisfied, in particular, if $N$ has nonpositive sectional curvature. (Being a closed isometrically embedded submanifold of $\mathbf{R}^n$, $N$ is complete.) Therefore we obtain as another corollary (from the preceding theorem, valid for parameter manifolds of any dimension) the classical

**Theorem 5 (Eells & Sampson)** *Any homotopy class of mappings from a compact Riemannian manifold $M$ into a compact Riemannian manifold $N$ of nonpositive curvature contains an energy minimizing harmonic mapping.*

(**Hamilton's theorem** for $\partial M \neq \emptyset$ and Dirichlet or Neumann boundary conditions.)

Actually, the direct method gives in dimension $\geq 3$ only energy minimizing weakly harmonic mappings. But there is also a regularity theory which proves that these are smooth in the case of a nonpositively curved target. Eells & Sampson used the heat flow associated with the (gradient of the) energy functional in their proof and so did Hamilton in his version of the Eells-Sampson theorem for source manifolds with boundary. It seems that *Hamilton's theorem* can also be proved in a much simpler way along the lines indicated above.

## 8 Local replacement by harmonic mappings

Here we describe another method used by J. Jost to prove the theorem of the previous section and many more results on harmonic maps. It is similar to the so called Perron or balayage process to solve the Dirichlet problem for harmonic functions and also the curve shortening process in Riemannian geometry to produce geodesics of minimal length in homotopy classes.

*Harmonic replacement*: Modify a minimizing sequence $u_n : \Sigma \to N$ by replacing $u_{n|\mathcal{U}_\rho(z)}$ for balls $\mathcal{U}_\rho(z)$ in $\Sigma \cup \partial\Sigma$ through the energy minimizing harmonic mapping $\mathcal{U}_\rho(z) \to N$ of Morrey with same boundary values as $u$ repeatedly.

In this way get (hopefully) an improved minimizing sequence, e.g. an equicontinuous one, so that homotopy is preserved in the limit.

To make this work one needs the Courant-Lebesgue Lemma and some estimates for the energy minimizing mappings of Morrey contained in

**Lemma 2 (Jost)** *Let $N \subset \mathbf{R}^n$ be a compact manifold (for simplicity). Then there exists $r(N) > 0$ such that if $\Omega$ is a bounded Lipschitz domain in $\mathbf{C}$, $u : \Omega \to N$ is energy minimizing among $W^{1,2}$ mappings with prescribed continuous boundary values $\varphi$, and $\varphi(\partial\mathcal{R})$ is contained in a closed ball $B_r(P)$ in $N$ with radius $0 < r < r(N)$, then $u(\Omega) \subset B_r(P)$ and the modulus of continuity of $u$ on $\overline{\Omega}$ can be estimated in terms of $r(N)$, $E_\Omega(u)$ and the modulus of continuity of $\varphi$ and on compact $K \subset\subset \Omega$ in terms of $r(N)$, $E_\Omega(u)$ and $K$. (By a* uniqueness theorem of Jäger & Kaul *the harmonic maps $u$ are uniquely determined by the boundary values and the condition that the image is in a not too large ball in $N$. But this we shall not need.)*

*Proof.* Retract $N$ Lipschitz onto $B_r(P)$ as follows: segments of geodesic rays emanating from $P$ in $N$ contained in $B_r(P) \backslash \mathring{R}_r(P)$ are mapped affinely into the corresponding segments in $B_r(P)$ and $N \setminus \mathring{B}_{2r}(P)$ is mapped into $\{P\}$. This is well-defined and distance-decreasing outside $B_r(P)$ (with respect to the Riemannian metric on $N$), provided $r \leq r(N)$ is small enough. The proof requires some Riemannian geometry, one can determine $r(N)$ in terms of the injectivity radius and an upper bound for the sectional curvature of $N$ (see the book by Jost). If $u$ is minimizing with $u(\partial\Omega) \subset B_r(P)$ then $\pi \circ u$ is an admissible comparison mapping for $u$. Since $\pi$ is distance decreasing outside of $B_r(P)$ we must have $Du(z) = 0$ for almost all $z$ with $u(z) \in N \setminus B_r(P)$. It follows that $u - \pi \circ u$ has zero derivative almost everywhere on $\Omega$, hence $u = \pi \circ u$ (by Poincaré's inequality and thus $u(\overline{\Omega}) \subset B_r(P)$.

To estimate the modulus of continuity of $u$ we observe for given $r > 0$ that each $z \in \Omega \cup \partial\Omega$ is contained in a disc $\mathcal{U}_\rho(z)$ of radius $\geq \delta > 0$, where $\delta$ depends on $r$, $E_\Omega(u)$ and continuity modulus of $\varphi = u|\partial\Omega$ on the compact $K \subset\subset \Omega$ only such that $u_{|\partial(\Omega\cap\mathcal{U}_\rho(z))}$ is a closed curve of diameter $\leq r$. This follows immediately from the Courant-Lebesgue Lemma (Section 4). Then $u(\partial(\Omega \cap \mathcal{U}_\rho(z)))$ is contained in some ball $B_r(P) \subset N$ and, by the reasoning above, $u(\overline{\mathcal{R} \cap \mathcal{U}_\rho(z)}) \subset B_r(P)$ follows if $r \leq r(N)$. Thus, for $0 < r < r(N)$

we have determined $\delta > 0$ with $\operatorname{diam}[u(\overline{\Omega \cap \mathcal{U}_\delta(z)})] \leq 2r$ for all $z \in \mathcal{R} \cup \partial\Omega$ proving the estimate on the modulus of continuity of $u$. ■

Now consider an energy minimizing sequence $u_n : \Sigma \to N \subset \mathbf{R}^n$ subject to boundary conditions $u_{n|\partial\Sigma} = \varphi$ and topological constraints. Cover the Riemann surface $\Sigma \cup \partial\Sigma$ by coordinate discs $\mathcal{U}_\delta(z_i) \subset \Sigma \cup \partial\Sigma$, $i = 1, \ldots, m$, such that one can find $\delta < \rho_i < \sqrt{\delta} \ll 1$ with $u_n(\partial\mathcal{U}_{\rho_i}(z)) \subset$ some ball of radius $r$ in $N$, $r \leq r(N)$. This is possible by the Courant-Lebesgue Lemma. (One can choose the same $\rho_i$ simultaneously for all $n$. All the $\mathcal{U}_{\rho_i}(z_i)$ are coordinate discs if $0 < \delta \ll 1$.)

*1st replacement step*: Replace the $u_n$ on $\mathcal{U}_{\rho_1}(z_1)$ by the energy minimizing harmonic mappings $\mathcal{U}_{\rho_1}(z_1) \to N$ with the same boundary values as $u_n$ to obtain $u_n^1$ equicontinuous on $\overline{\mathcal{U}_\delta(z_1)} \subset\subset \mathcal{U}_{\rho_1}(z_1)$ and with $E(u_n^1) \leq E(u_n)$.

*2nd step*: Replace the $u_n^1$ on $\mathcal{U}_{\rho_2}(z_2)$ by the energy minimizing harmonic mappings $\mathcal{U}_{\rho_2}(z_2) \to N$ with the same boundary values as $u_n^1$ to obtain $u_n^2$ equicontinuous on $\mathcal{U}_\delta(z_1) \cup \mathcal{U}_\delta(z_2)$ with $E(u_n^2) \leq E(u_n^1) \leq E(u_n)$. (Here one uses the fact that the boundary values for this second replacement step are equicontinuous on $\partial\mathcal{U}_{\rho_2}(z_2) \cap \mathcal{U}_\rho(z_1)$, $\delta < \rho < \rho_1$.)

⋮

*After the mth step*: One has $u_n^m$ equicontinuous on $\Sigma$ with $E_\Sigma(u_n^m) \leq E_\Sigma(u_n)$ for all $n$.

Passing to a subsequence we may assume $u_n^m \to u$ weakly in $W^{1,2}$ and uniformly on $\Sigma$, $E_\Sigma(u) \leq \liminf\limits_{n\to\infty} E_\Sigma(u_n)$, $u_{|\partial\Sigma} = \varphi$.

What happened to the topological constraints during the replacement steps? Nothing, if they were of the type $u_{n\#}[\alpha_j] = [\beta_j]$, i.e. condition on the action on homotopy classes of loops or paths connecting boundary components of $\Sigma$. (This is clear, if $\delta$ is chosen small enough; because at each replacement step one can deform the loops such that they don't pass through the coordinate discs on which the replacement take place. Also nothing happened, if conditions on the homotopy classes $[u_n]$ was prescribed and at each replacement step (for large $n$) $u^k_{n|\overline{\mathcal{U}_{\rho_k}(z_k)}}$ was replaced by a *homotopic* harmonic mapping on $\overline{\mathcal{U}_{\rho_k}(z_k)}$ to obtain $u_{n^{k+1}}$. This is certainly the case, if $\pi_2(N) = 0$. Thus, in these cases the $u_n^m$, and by uniform convergence also $u$, satisfy the same topological conditions as the $u_n$, hence $u$ is energy minimizing with respect to the topological constraints and, in particular, locally energy minimizing and therefore harmonic. This is Jost's proof of the theorems of Schoen & Yau, Sacks & Uhlenbeck.

Now let us analyse what happens if this replacement method does not preserve the homotopy class of the mappings for all sufficiently large $n$. Then we find minimizing sequences $(v_n)$ in the homotopy class and discs $\mathcal{U}_\rho(z_n)$ with $\delta < \rho < \sqrt{\delta}$ in $\Sigma \cup \partial\Sigma$ for infinitely many $n$ such that $v_{n|\overline{\mathcal{U}_\rho(z_n)}}$ is not homotopic to the minimizing harmonic mapping on $\overline{\mathcal{U}_\rho(z_n)}$ with the boundary values of $v_n$ ($(v_n)$ is one of the sequences $(u_n^k)$ above). If such a situation holds for arbitrary small $\delta > 0$, then we can even find a minimizing sequence $w_n$ in the homotopy class and discs $\mathcal{U}_\rho(z_n)$ in $\Sigma \cup \partial\Sigma$ with $\rho_n \downarrow 0$ such that $w_{n|\overline{\mathcal{U}_{\rho n(z_n)}}}$ is not homotopic to the minimizing harmonic mapping on $\overline{\mathcal{U}_{\rho_n}(z_n)}$ with the boundary values of $w_n$. If not, then the preceding arguments show that we can minimize energy in the given homotopy class. Thus we have the following alternative:

*Either* the energy attains a minimum in the given homotopy class of mappings $\Sigma \to N$ (with prescribed boundary values if $\partial\Sigma \neq \emptyset$) considered,

*or* there exists a minimizing sequence $(w_n)$ in this class and $\rho_n \downarrow 0$, $z_n \in \Sigma \cup \partial\Sigma$ such that $w_n$ is not homotopic on $\overline{\mathcal{U}}_n := \overline{\mathcal{U}_{\rho_n}(z_n)}$ to the energy minimizing harmonic mapping $h_n$ from $\mathcal{U}_n$ to $N$ with boundary values of $u$. Thus $h_n$ and $w_{n|\mathcal{U}_n}$ define a nontrivial element in $\pi_2(N)$ (a "bubble").

We may also assume, in the case of bubbling, that

$$I_n := \int\limits_{z_n+\rho_n e^{i\theta}\in\Sigma} \left|\frac{\partial}{\partial\theta} w_n(z_n + \rho_n e^{i\theta})\right|^2 d\theta \longrightarrow 0 \quad \text{as } n \to \infty$$

and hence

$$\begin{aligned} E_{\mathcal{U}_n}(h_n) &\longrightarrow 0 \quad \text{as } n \to \infty \\ \operatorname{diam} h_n(\overline{\mathcal{U}}_n) &\longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned}$$

This follows from the fact that the $\rho_n$ were chosen as in the Courant-Lebesgue Lemma and the construction of a mapping with boundary values of $w_n$ on $\mathcal{U}_n$ into $N$ and with energy $\leq$ const $I_n$. (One can produce such mappings if $\mathcal{U}_n$ is a Euclidean disc simply by the harmonic extension of $w_{n|\mathcal{U}_n}$ with respect to the Euclidean metric on a suitable coordinate patch in $N$; the estimate of the energy of this mapping by $I_n$ can be seen, for example, from a Fourier series development. If $\mathcal{U}_n$ is a half disc in $\Sigma$ centered at the boundary point $z_n \in \partial\Sigma$, then one may use a Lipschitz transformation to a disc and argue similarly, provided the prescribed boundary values are of class $W^{1,2}$ on $\partial\Sigma$.) The conclusion for the modified mappings

$$\tilde{w}_n := \begin{cases} h_n & \text{on } \mathcal{U}_n \\ w_n & \text{on } (\Sigma \cup \partial\Sigma) \setminus \mathcal{U}_n \end{cases}$$

is then

$$\begin{aligned} \liminf_{n\to\infty} E_\Sigma(\tilde{w}_n) &= \lim_{n\to\infty} E_\Sigma(w_n) - \limsup_{n\to\infty} [E_{\mathcal{U}_n}(w_n) + E_{\mathcal{U}_n}(h_n)] = \\ &= \lim_{n\to\infty} E_\Sigma(w_n) - \limsup_{n\to\infty} E_{\mathcal{U}_n}(w_n) \end{aligned}$$

Now consider the special case of a target $N \approx S^2$ (diffeomorphic but possibly with different metric). Since $w_{n|\mathcal{U}_n}$ and $h_n$ together define a homotopically nontrivial map into $S^2$ their images must cover $N$ and thus, by the inequality between area and energy,

$$\begin{aligned} e &:= \inf_{\substack{u\in W^{1,2}(\Sigma,N) \\ u_{|\partial\Sigma}=\varphi}} E_\Sigma(u) \leq \liminf_{n\to\infty} E_\Sigma(\tilde{w}_n) \leq \\ &\leq \underbrace{lim_{n\to\infty} E_\Sigma(w_n)}_{=:\, e'} - \text{Area}\,(N). \end{aligned}$$

This is an impossibility, if the infimum $e'$ of energy in the prescribed homotopy class satisfies $e' < e + \text{Area}(N)$. In such case bubbling cannot occur and therefore the infimum $e'$ in the prescribed homotopy class is attained.

Now, if $\Sigma \neq \emptyset$ and $\varphi_{|\partial\Sigma} \neq$ const, then we can consider the harmonic map $u : \Sigma \to N \approx S^2$, $u_{|\partial\Sigma} = \varphi$, $E_\Sigma(u) = e$ of Morrey and produce a map $v \in W^{1,2}(\Sigma, N)$ not homotopic to $u$ with $E_\Sigma(v) < e + \text{Area}\,(N)$ by parametrizing suitable a large part of the sphere $N$ conformally (this uses the uniformization theorem if $N$ is not Euclidean sphere $S^2$) on a small disc $D$ centered at a point in $\Sigma$ where $u$ has derivative $\neq 0$ ($u_{|\partial\Sigma} = \varphi$ is not constant, hence such a point exists), setting $v = u$ outside a somewhat large disc $\widetilde{D}$ and interpolating suitably in between. This is a somewhat bubble matter.

The result is the following

**Theorem 6 (Jost, Brezis & Coron)** *If $\Sigma$ is a compact Riemann surface with $\partial\Sigma \neq \emptyset$, $N$ diffeomorphic to the 2-sphere and $\varphi : \partial\Sigma \to N$ not constant, then there exist at least two harmonic mappings from $\Sigma$ into $N$ with boundary values $\varphi$, both minimizing in their respective different homotopy classes.*

**Remarks.** (1) For small oscillation of $\varphi$ one imagines one of these harmonic mappings as being "small", the other one "large".
(2) For constant $\varphi$ the statement is definitively false. We have already seen that harmonic mappings from the unit disc with constant boundary values must be constant (because they are automatically conformal). **2**

Much more precise information has been obtained recently by Soyeur, Kuwert and others about harmonic mappings in two-spheres with prescribed boundary values.

## 9 More consequences from the analysis of bubbling

If "bubbling" occur in a minimizing sequence in a homotopy class of mappings $\Sigma \to N$ (with prescribed boundary values if $\partial\Sigma \neq \emptyset$), then we have seen in Section 8 that a homotopically nontrivial sphere splits off.

If bubbling occurs in an $E$ minimizing sequence $(u_n)$ on $\Sigma$, then there exist $\mathcal{U}_\rho(z_n) \subset \Sigma$, $\rho_n \downarrow 0$, such that $u_{n|\mathcal{U}_\rho(z_n)}$ is approximately (up to a piece of surface with small energy and diameter) a homotopically nontrivial sphere in $N$.

The idea is then to

rescale $\tilde{u}_n(z) := u_n(z_n + l_n z)$ with $l_n \to 0$, to obtain a harmonic map $\tilde{u} : \mathbf{C} \to N$ in the limit with finite energy and not constant (bubble!).

By a theorem of Sacks & Uhlenbeck, the isolated singularity at $\infty$ of $\tilde{u} : S^2 = \mathbf{C} \cup \{\infty\} \to N$ is removable, thus one obtains a harmonic map from $S^2$ into $N$ which then (according to Section 3) is also conformal, i.e. a parametric minimal surface (a branched immersion, in fact), and not homotopically constant.

Thus one has not succeeded in finding a minimizing in the original homotopy class, but found a minimal sphere in $N$, instead, which is also nice.

Making these ideas precise is not easy but was done to give, e.g. (for $N$ compact manifold $\subset \mathbf{R}^n$, as usual):

**Theorem 7 (Sacks & Uhlenbeck, also Schoen &Yau for (ii))**

(i) *Suppose $\pi_2(N) \neq 0$, then there exists a parametric minimal surface $u : S^2 \to N$ which is not homotopic to a constant.*

(ii) *In fact, $\pi_2(N)$ is generated as a $\mathbf{Z}[\pi_1(N)]$-module by homotopy classes containing energy minimizing parametric minimal spheres.*

(iii) *If $\pi_k(N) \neq 0$ for same $k \geq 3$, then the conclusion of (i) is also valid.*

The proof of (i) is as outlined above, for (ii) one has to prove the fact that the change in the homotopy class of mappings $S^2 \to N$ which can occur in the limit of the minimization process is exactly equal to the sum of the homotopy classes of the spheres that split off. (iii) requires the extension of all the reasoning to a situation where one does not look for a minimum but for a critical minimax value in a homotopy class; such a minimax procedure can be done if $\pi_k(N) \neq 0$ for some $k \geq 3$ (looking at suitable $(k-2)$-parameter families of parametric spheres in $N$).

The most general theorems about bubbling for some special minimizing sequences have been proved by Jost so far; he can also treat the case of prescribed Dirichlet-type or free (including Plateau-type) boundary values and prove in many cases that the sum of the energies of the bubbles that have split off is exactly equal to the jump in energy which occurs on the minimizing or minimaxing sequence in the limit. This method is harmonic replacement. Sacks & Uhlenbeck used the method of regular perturbations of the energy functional $E$ to

$$E_\alpha(u) := \int_\Sigma (1 + |Du|^2)^\alpha \, dx\, dy \qquad (\alpha > 0)$$

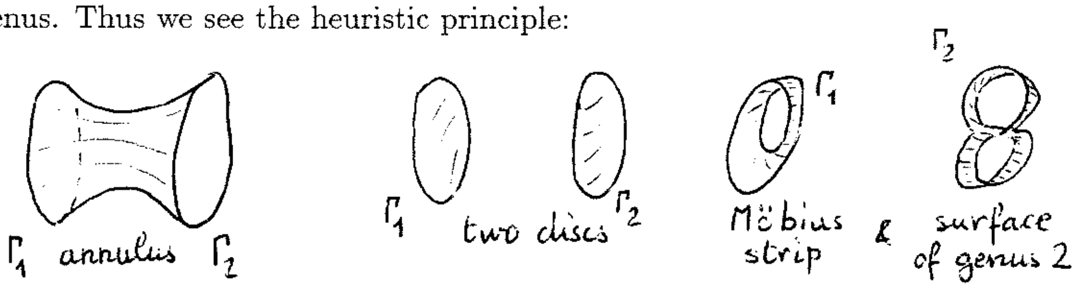
and then analyse the convergence and bubbling of sequences of $u_\alpha$ of critical points for $E_\alpha$ as $\alpha \downarrow 1$ to harmonic maps and parametric minimal spheres. Struwe has analysed the bubbling process in the heat flow associated with the energy; this may be viewed as another method of regularization.

## 10 The general Plateau-Douglas Problem

is to find a parametric minimal surface $u : \Sigma \to N$ (compact Riemannian manifold) on a domain $\Sigma$ of given topological type (compact Riemann surface of genus $p$ with the boundary components) such the boundary components $\gamma_1, \dots, \gamma_k$ $(k \geq 0)$ of $\Sigma$ are mapped onto given disjoint oriented Jordan curves $\Gamma_1, \dots, \Gamma_k$ in $N$, i.e. $u_{|\gamma_i} : \gamma_i \to \Gamma_i$ monotonically with degree 1 for $i = 1, \dots, k$. Moreover, if $\pi_1(N) \neq 0$ one may want to prescribe also the action of $u$ on the fundamental group of $\Sigma$ by some continuous $u_0 : \Sigma \to N$ satisfying these boundary conditions (i.e. $u_\# = u_{0\#} : \pi_1(\Sigma) \to \pi_1(N)$. $(\Sigma, p, k, N\Gamma_1, \dots \Gamma_k$ and $u_0$ are the *data of the problem.* They determine the *admissible surfaces.*)

This was treated in $\mathbf{R}^n$ by Douglas, Courant and Shiffman in the 40's. But the paper were criticized some years ago by Tromba for incomplete arguments at several points. Tromba gave a new treatment, but with somewhat weaker results. The best result which we have now is due to Jost. The genus 0 case (without topological constraints) was treated by Morrey (1942) in a Riemannian manifold.

To see what answer one has to exspect, we imagine a soap film experiment which was described already. If we try to form a soap film bounded by two Jordan curves in $\mathbf{R}^3$, then surfaces of different topological types will form depending on the (linked or not) geometry (distance, shape, ...) of the Jordan curves. If the two curves are far apart, no film of the type of the annulus will form (and this can be proved rigorously for parametric minimal surfaces of type of the annulus); we will obtain two disc type (or other type) disjoint soap films. If the two Jordan curves are close, then a doubly connected soap film may be seen, although, in the mathematical analysis, the disc-type surfaces bounding each Jordan curve individually are still present. In fact, even for a single Jordan curve it is not true that we always find a soap film of disc-type in the experiments. Indeed, if the curve is knotted, this is (by definition of knottedness) impossible. But also for complicated unknotted Jordan curves we may easily obtain films of large genus. Thus we see the heuristic principle:




A minimal surface picks its topological type (domain of parametrization) according to the topological and geometric data.

We cannot expect, therefore, to be able to solve the problem with a parameter domain $\Sigma$ of given topology unless we make suitable restrictions on the data.

A second fact that one may have in mind when dealing with the problem is that a soap film not only picks its topology but as submanifold of $\mathbf{R}^3$, say, also an induced Riemannian metric on it and therefore a conformal structure.

A minimal surface also picks its conformal structure.

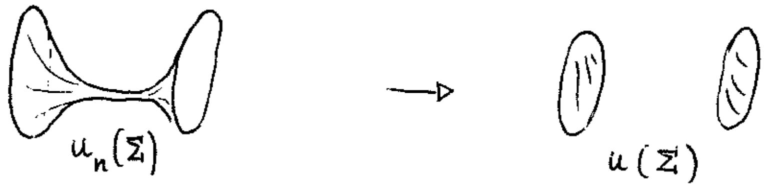
Therefore, if we want to describe for example a doubly connected soap film bounded by two given Jordan curves in $\mathbf{R}^3$ we cannot represent it as a parametric minimal surface on a *given* annulus in $\mathbf{C}$, but only on an annulus with the same conformal structure as the soap film (i.e. the quotient of the radii of the annulus is determined by the soap film). Thus, we cannot work with a fixed Riemann surface $\Sigma$ here, even after we have fixed the topological type, but we need to vary the conformal structures of $\Sigma$. In other words: If we minimize energy in a class of mappings defined by the data of the Plateau-Douglas problem above on a fixed Riemann surface, then the minimizer (if it exists) will have an associated holomorphic quadratic differential which is real on the boundary but not identically zero, in general, i.e. not conformal. Only if we also minimize energy with respect to all conformal structures on $\Sigma$, the minimizers will actually be conformal and hence a parametric minimal surface. (One way to see this, is to represent the conformal structures on $\Sigma$ by Riemannian metrics $\gamma$—each conformal structure corresponding to an infinite dimensional equivalence class of Riemannian metrics—and to write the energy as a function of the mapping $u$ and the metric $\gamma$, i.e. in local coordinates

$$E(u,\gamma) = \int \frac{\gamma_{22} u_x \cdot u_x - 2\gamma_{12} u_x \cdot u_y + \gamma_{11} u_y \cdot u_y}{\gamma_{11}\gamma_{22} - \gamma_{12}^2} \sqrt{\det \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix}}\, dx\, dy$$

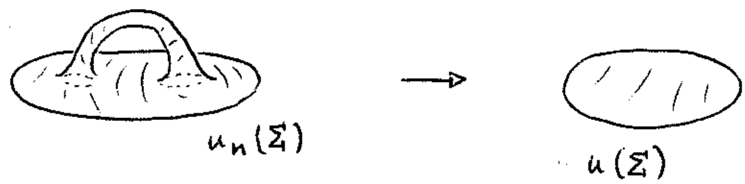
in local differentiable coordinates $x, y$ on $\Sigma$.

One can then compute the first variation of $E$ with respect to variations of $u$ and with respect to variations of $\gamma$, and it turns out that the latter first variation vanishes at $\gamma$ iff $u$ is conformal with respect to $\gamma$.)

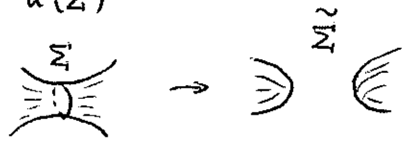
To set an idea about the nature of the conditions on the data one has to assume to make the solution of the Plateau-Douglas problem possible we consider what can happen to a minimizing sequence if we start with a domain of the wrong topology. In the example above we might obtain a minimizing sequence of doubly connected surfaces bounded by two Jordan curves which decomposes into a disconnected surface in the limit, because it is "cheaper" with regard to energy (or area) to choose that topological type. Another thing that could happen is that a handle might get pinched




and disappear in the limit. This will be the case, for example, if $\Sigma$ is the disc with one handle (Torus with one hole) and $\Gamma$ a planar circle. In these

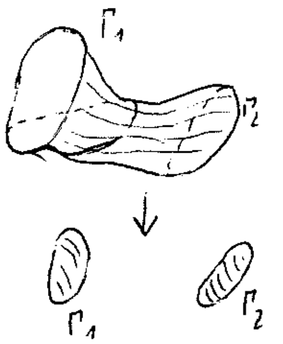



examples the topological type changes by a



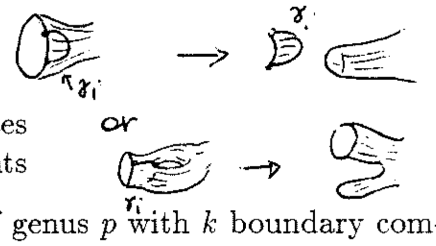

*Reduction of type I of* $\Sigma$: reduces genus or increases number of components.

But there is also another degeneration phenomenon that can occur minimizing sequences in the limit. Namely, the pinching could occur at a boundary curve $\Gamma_i$ like this:



(a loop from an arc on $\Gamma_i$ and a path on the surface might shrink to a point in the limit).

This is reflected in considering



*Reductions of the type II of* $\Sigma$: reduces genus or increases number of components

Now fix an oriented model surface $\Sigma$ of genus $p$ with $k$ boundary components $\Gamma_1, \ldots, \gamma_k$ $(k \geq 0)$, $k$ disjoint oriented (sufficiently smooth) Jordan curves $\Gamma_1, \ldots, \Gamma_k$ in $N$ and $u_0 : \Sigma \to n$ continuous on $\Sigma \cup \partial\Sigma$ mapping $\gamma_i$ monotonically with degree 1 onto $\Gamma_i$ for $i = 1, \ldots, k$, i.e. satisfying the so called *Plateau-Douglas boundary condition.* Set

$d(\Gamma_1, \ldots, \Gamma_k; p, u_0) :=$ inf of energies of all mappings $u : \Sigma \to N$ with respect to all conformal structures (or metrics) on $\Sigma$ such that $u$ satisfies the Plateau-Douglas boundary condition and the action $u_\#$ on homotopy classes of loops in $\Sigma$ equals $u_{0\#}$

| energy infimum on model surface |
|---|

$(= a(\Gamma_1, \ldots, \Gamma_k; p, u_0) :=$ same inf with Area replacing Energy)

Furthermore set

$d^*(\Gamma_1, \ldots, \Gamma_k; p, u_0) :=$ inf of energies of all mappings $u : \tilde{\Sigma} \to N$ with respect to all surfaces $\tilde{\Sigma}$ resulting from $\Sigma$ by reductions of type I or II, and all conformal structures on $\tilde{\Sigma}$, such that $u$ satisfies the Plateau-Douglas boundary condition on $\tilde{\Sigma}$ and the action $u_\#$ on homotopy classes of loops in $\tilde{\Sigma}$ compatible with $u_{0\#}$ and the reduction used.

| energy infimum on parametric surfaces of "simpler" topological type |
|---|

$(= a(\Gamma_1, \ldots, \Gamma_k; p, u_0) :=$ same thing for area).

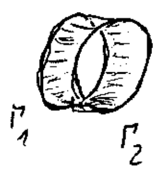
Note that either $\tilde{\Sigma}$ is connected with genus$(\tilde{\Sigma})$ <genus$(\Sigma)$ or $\tilde{\Sigma}$ has two components with sum of genus $\leq$genus$(\Sigma)$. In this sense the $\tilde{\Sigma}$ are of "simpler topological type". Set $d^* := \infty$ $(a^* := \infty)$ if no reductions of type I or II are possible.

**Theorem 8 (Douglas, Courant, Shiffman, Tomi & Tromba, Jost)** *If*

$$d(\Gamma_1, \ldots, \Gamma_k; p; u_0) < d^*(\Gamma_1, \ldots, \Gamma_k; p; u_0)$$

*(the* **Douglas-condition**; *equivalent is* $a(\dots) \le a^*(\dots)$*), then there exists a conformal structure on* $\Sigma$ *and a harmonic mapping* $u : \Sigma \to N$ *conformal with respect to this structure which solves the Plateau-Douglas problem and is energy and area minimizing in the class of surfaces admitted in the problem.*
*Moreover, if* $\pi_2(N) = 0$*, then* $u$ *is homotopic to* $u_0$ *(by a homotopy respecting the Douglas boundary condition).*

The classical Douglas theorem is the case $N = \mathbf{R}^n$; $u_0$ is not of relevance then, of course. The Douglas-condition can be verified in many situations, as Brian White has mentioned.


**Corollary 2** *For a given Jordan curve* $\Gamma$ *in* $N$ *one can find for every* $p \ge 0$ *a parametric minimal surface spanning* $\Gamma$ *of minimal area among surfaces of genus* $\le p$. *(True for* $k \ge 2$ *only, if disconnected surfaces are admitted).*

*Proof.* Minimize for $p = 0$, if Douglas condition holds for $p = 1$, take that minimizer, otherwise the one for $p = 0$. Proceed in this way. ∎

**Corollary 3 (Schoen & Yau)** *If* $\partial\Sigma = \emptyset$ *and* $u_{0\#}$ *is injective on the fundamental group of* $\Sigma$*, then there exist an area minimizing parametric minimal surface* $u : \Sigma \to N$ *among the parametric surfaces with same actions as* $u_{0\#}$ *on* $\pi_1(\Sigma)$.