

Example 1. Find the derivative of the function

$$y = (x^2 - 2x + 3)^5.$$

Solution. Putting $y = u^5$, where $u = (x^2 - 2x + 3)$, by formula (1) we will have

$$y' = (u^5)'_u (x^2 - 2x + 3)'_x = 5u^4 (2x - 2) = 10(x - 1)(x^2 - 2x + 3)^4.$$

Example 2. Find the derivative of the function

$$y = \sin^3 4x.$$

Solution. Putting

$$y = u^3; \quad u = \sin v; \quad v = 4x,$$

we find

$$y' = 3u^2 \cdot \cos v \cdot 4 = 12 \sin^2 4x \cos 4x.$$

Find the derivatives of the following functions (the rule for differentiating a composite function is not used in problems 368-408).

A. Algebraic Functions

368. $y = x^5 - 4x^3 + 2x - 3.$ 375. $y = 3x^{\frac{2}{3}} - 2x^{\frac{5}{2}} + x^{-3}.$

369. $y = \frac{1}{4} - \frac{1}{3}x + x^2 - 0.5x^4.$ 376*. $y = x^2 \sqrt[3]{x^2}.$

370. $y = ax^2 + bx + c.$

377. $y = \frac{a}{\sqrt[3]{x^2}} - \frac{b}{x \sqrt[3]{x}}.$

371. $y = \frac{-5x^3}{a}.$

378. $y = \frac{a+bx}{c+dx}.$

372. $y = at^m + bt^{m+n}.$

379. $y = \frac{2x+3}{x^2-5x+5}.$

373. $y = \frac{ax^4+b}{\sqrt{a^2+b^2}}.$

380. $y = \frac{2}{2x-1} - \frac{1}{x}.$

374. $y = \frac{\pi}{x} + \ln 2.$

381. $y = \frac{1+\sqrt{z}}{1-\sqrt{z}}$

B. Inverse Circular and Trigonometric Functions

382. $y = 5 \sin x + 3 \cos x.$

386. $y = \arctan x + \text{arc cot } x.$

383. $y = \tan x - \cot x.$

387. $y = x \cot x.$

384. $y = \frac{\sin x + \cos x}{\sin x - \cos x}.$

388. $y = x \arcsin x.$

385. $y = 2t \sin t - (t^2 - 2) \cos t.$

389. $y = \frac{(1+x^2) \arctan x - x}{2}.$

C. Exponential and Logarithmic Functions

390. $y = x^7 \cdot e^x.$

396. $y = e^x \arcsin x.$

391. $y = (x-1)e^x.$

397. $y = \frac{x^2}{\ln x}.$

392. $y = \frac{e^x}{x^2}.$

398. $y = x^3 \ln x - \frac{x^3}{3}.$

393. $y = \frac{x^5}{e^x}.$

399. $y = \frac{1}{x} + 2 \ln x - \frac{\ln x}{x}.$

394. $f(x) = e^x \cos x.$

400. $y = \ln x \log x - \ln a \log_a x.$

395. $y = (x^2 - 2x + 2)e^x.$

D. Hyperbolic and Inverse Hyperbolic Functions

401. $y = x \sinh x.$

405. $y = \arctan x - \operatorname{arctanh} x.$

402. $y = \frac{x^2}{\cosh x}.$

406. $y = \arcsin x \operatorname{arsinh} x.$

403. $y = \tanh x - x.$

407. $y = \frac{\operatorname{arc cosh} x}{x}.$

404. $y = \frac{3 \coth x}{\ln x}$

408. $y = \frac{\operatorname{arc coth} x}{1-x^2}.$

E. Composite Functions

In problems 409 to 466, use the rule for differentiating a composite function with one intermediate argument.

Find the derivatives of the following functions:

409**. $y = (1 + 3x - 5x^2)^{30}.$

Solution. Denote $1 + 3x - 5x^2 = u$; then $y = u^{30}$. We have:

$y'_u = 30u^{29}; \quad u'_x = 3 - 10x;$

$u'_x = 30u^{29} \cdot (3 - 10x) = 30(1 + 3x - 5x^2)^{29} \cdot (3 - 10x).$

410. $y = \left(\frac{ax+b}{c}\right)^8.$

411. $f(y) = (2a + 3by)^8.$

412. $y = (3 + 2x^2)^4.$

413. $y = \frac{3}{56(2x-1)^7} - \frac{1}{24(2x-1)^6} - \frac{1}{40(2x-1)^5}.$

414. $y = \sqrt[4]{1-x^2}.$

415. $y = \sqrt[3]{a+bx^3}.$

416. $y = (a^{2/3} - x^{3/2})^{1/2}.$

417. $y = (3 - 2 \sin x)^5.$

Solution. $y' = 5(3 - 2 \sin x)^4 \cdot (3 - 2 \sin x)' = 5(3 - 2 \sin x)^4(-2 \cos x) = -10 \cos x(3 - 2 \sin x)^4.$

418. $y = \tan x - \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x.$

419. $y = \sqrt{\cot x} - \sqrt{\cot a}.$

423. $y = \frac{1}{3 \cos^3 x} - \frac{1}{\cos x}.$

420. $y = 2x + 5 \cos^3 x.$

424. $y = \sqrt{\frac{3 \sin x - 2 \cos x}{5}}.$

421*. $x = \operatorname{cosec}^2 t + \sec^2 t.$

425. $y = \sqrt[3]{\sin^2 x} + \frac{1}{\cos^2 x}.$

422. $f(x) = -\frac{1}{6(1 - 3 \cos x)^2}.$

426. $y = \sqrt{1 + \operatorname{arc sin} x}.$

427. $y = \sqrt{\operatorname{arc tan} x} - (\operatorname{arc sin} x)^2.$

428. $y = \frac{1}{\operatorname{arc tan} x}.$

429. $y = \sqrt{x e^x + x}.$

430. $y = \sqrt[3]{2e^x - 2^x + 1} + \ln^3 x.$

431. $y = \sin 3x + \cos \frac{x}{5} + \tan \sqrt{x}.$

Solution. $y' = \cos 3x \cdot (3x)' - \sin \frac{x}{5} \left(\frac{x}{5}\right)' + \frac{1}{\cos^2 \sqrt{x}} (\sqrt{x})' = 3 \cos 3x - \frac{1}{5} \sin \frac{x}{5} + \frac{1}{2 \sqrt{x} \cos^2 \sqrt{x}}.$

432. $y = \sin(x^2 - 5x + 1) + \tan \frac{a}{x}.$

433. $f(x) = \cos(ax + \beta).$

434. $f(t) = \sin t \sin(t + \varphi).$

435. $y = \frac{1 + \cos 2x}{1 - \cos 2x}.$

436. $f(x) = a \cot \frac{x}{a}.$

437. $y = -\frac{1}{20} \cos(5x^2) - \frac{1}{4} \cos x^2.$

438. $y = \operatorname{arc sin} 2x.$

Solution. $y' = \frac{1}{\sqrt{1 - (2x)^2}} \cdot (2x)' = \frac{2}{\sqrt{1 - 4x^2}}.$

439. $y = \operatorname{arc sin} \frac{1}{x^2}.$

441. $y = \operatorname{arc tan} \frac{1}{x}.$

440. $f(x) = \operatorname{arc cos} \sqrt{x}.$

442. $y = \operatorname{arc cot} \frac{1+x}{1-x}.$

443. $y = 5e^{-x^2}.$

444. $y = \frac{1}{5^{x^2}}.$

445. $y = x^2 10^{x^2}.$

446. $f(t) = t \sin 2^t.$

452. $y = \ln(e^x + 5 \sin x - 4 \arcsin x).$

453. $y = \arctan(\ln x) + \ln(\arctan x).$

454. $y = \sqrt{\ln x + 1} + \ln(\sqrt{x} + 1).$

447. $y = \arccos e^x.$

448. $y = \ln(2x + 7).$

449. $y = \log \sin x.$

450. $y = \ln(1 - x^2).$

451. $y = \ln^2 x - \ln(\ln x).$

F. Miscellaneous Functions

455**. $y = \sin^2 5x \cos^2 \frac{x}{3}.$

456. $y = -\frac{11}{2(x-2)^2} - \frac{4}{x-2}.$

457. $y = -\frac{15}{4(x-3)^4} - \frac{10}{3(x-3)^3} - \frac{1}{2(x-3)^2},$

458. $y = \frac{x^8}{8(1-x^2)^4}.$

459. $y = \frac{\sqrt{2x^2 - 2x + 1}}{x}.$

460. $y = \frac{x}{a^2 \sqrt{a^2 + x^2}}.$

461. $y = \frac{x^3}{3 \sqrt[3]{(1+x^2)^4}}.$

462. $y = \frac{3}{2} \sqrt[3]{x^2} + \frac{18}{7} x \sqrt[6]{x} + \frac{9}{7} x \sqrt[3]{x^2} + \frac{6}{13} x^2 \sqrt[6]{x}.$

463. $y = \frac{1}{8} \sqrt[3]{(1+x^3)^8} - \frac{1}{5} \sqrt[5]{(1+x^3)^5}.$

464. $y = \frac{4}{3} \sqrt[4]{\frac{x-1}{x+2}}.$

465. $y = x^4(a - 2x^3)^2.$

466. $y = \left(\frac{a+bx^n}{a-bx^n} \right)^m.$

467. $y = \frac{9}{5(x+2)^5} - \frac{3}{(x+2)^4} + \frac{2}{(x+2)^3} - \frac{1}{2(x+2)^2}.$

468. $y = (a+x) \sqrt{a-x}.$

469. $y = \sqrt{(x+a)(x+b)(x+c)}.$

470. $z = \sqrt[3]{y + \sqrt{y}}.$

471. $f(t) = (2t+1)(3t+2) \sqrt[3]{3t+2}.$

472. $x = \frac{1}{\sqrt{2ay - y^2}}.$

473. $y = \ln(\sqrt{1+e^x} - 1) - \ln(\sqrt{1+e^x} + 1).$

474. $y = \frac{1}{15} \cos^3 x (3 \cos^2 x - 5).$

475. $y = \frac{(\tan^2 x - 1)(\tan^4 x + 10 \tan^2 x + 1)}{3 \tan^3 x}.$

476. $y = \tan^5 5x.$

485. $y = \arcsin \frac{x^2 - 1}{x^2}.$

477. $y = \frac{1}{2} \sin(x^2).$

486. $y = \arcsin \frac{x}{\sqrt{1+x^2}}.$

478. $y = \sin^2(t^3).$

487. $y = \frac{\arccos x}{\sqrt{1-x^2}}.$

479. $y = 3 \sin x \cos^2 x + \sin^3 x.$

488. $y = \frac{1}{\sqrt{b}} \arcsin \left(x \sqrt{\frac{b}{a}} \right).$

480. $y = \frac{1}{3} \tan^3 x - \tan x + x.$

489. $y = \sqrt{a^2 - x^2} + a \arcsin \frac{x}{a}.$

481. $y = -\frac{\cos x}{3 \sin^2 x} + \frac{4}{3} \cot x.$

490. $y = x \sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a}.$

482. $y = \sqrt{a \sin^2 x + \beta \cos^2 x}.$

491. $y = \arcsin(1-x) + \sqrt{2x-x^2}.$

483. $y = \arcsin x^2 + \arccos x^2.$

484. $y = \frac{1}{2} (\arcsin x)^2 \arccos x.$

492. $y = \left(x - \frac{1}{2} \right) \arcsin \sqrt{x} + \frac{1}{2} \sqrt{x-x^2}.$

493. $y = \ln(\arcsin 5x).$

494. $y = \arcsin(\ln x).$

495. $y = \arctan \frac{x \sin \alpha}{1 - x \cos \alpha}.$

496. $y = \frac{2}{3} \arctan \frac{5 \tan \frac{x}{2} + 4}{3}.$

497. $y = 3b^2 \arctan \sqrt{\frac{x}{b-x}} - (3b + 2x) \sqrt{bx-x^2}.$

498. $y = -\sqrt{2} \arccot \frac{\tan x}{\sqrt{2}} - x.$

499. $y = \sqrt{e^{ax}}.$

500. $y = e^{\sin^2 x}.$

501. $F(x) = (2ma^{mx} + b)^p.$

502. $F(t) = e^{\alpha t} \cos \beta t.$

503. $y = \frac{(a \sin \beta x - \beta \cos \beta x) e^{\alpha x}}{\alpha^2 + \beta^2}.$

504. $y = \frac{1}{10} e^{-x} (3 \sin 3x - \cos 3x).$ 507. $y = 3^{\cot \frac{1}{x}}.$
505. $y = x^n a^{-x^2}.$ 508. $y = \ln(ax^2 + bx + c).$
506. $y = \sqrt{\cos x} a^{\sqrt{\cos x}}.$ 509. $y = \ln(x + \sqrt{a^2 + x^2}).$
510. $y = x - 2\sqrt{x} + 2 \ln(1 + \sqrt{x}).$
511. $y = \ln(a + x + \sqrt{2ax + x^2}).$ 514*. $y = \ln \frac{(x-2)^5}{(x+1)^3}.$
512. $y = \frac{1}{\ln^2 x}.$ 515. $y = \ln \frac{(x-1)^3(x-2)}{x-3}.$
513. $y = \ln \cos \frac{x-1}{x}.$ 516. $y = -\frac{1}{2 \sin^2 x} + \ln \tan x.$
517. $y = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln(x + \sqrt{x^2 - a^2}).$
518. $y = \ln \ln(3 - 2x^3).$
519. $y = 5 \ln^3(ax + b).$
520. $y = \ln \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2} - x}.$
521. $y = \frac{m}{2} \ln(x^2 - a^2) + \frac{n}{2a} \ln \frac{x-a}{x+a}.$
522. $y = x \cdot \sin \left(\ln x - \frac{\pi}{4} \right).$
523. $y = \frac{1}{2} \ln \tan \frac{x}{2} - \frac{1}{2} \frac{\cos x}{\sin^2 x}.$
524. $f(x) = \sqrt{x^2 + 1} - \ln \frac{1 + \sqrt{x^2 + 1}}{x}.$
525. $y = \frac{1}{3} \ln \frac{x^2 - 2x + 1}{x^2 + x + 1}.$
526. $y = 2^{\arcsin x} + (1 - \arccos 3x)^2.$
527. $y = 3^{\frac{\sin ax}{\cos bx}} + \frac{1}{3} \frac{\sin^3 ax}{\cos^3 bx}.$
528. $y = \frac{1}{\sqrt[3]{3}} \ln \frac{\tan \frac{x}{2} + 2 - \sqrt[3]{3}}{\tan \frac{x}{2} + 2 + \sqrt[3]{3}}.$
529. $y = \arctan \ln x.$
530. $y = \ln \arcsin x + \frac{1}{2} \ln^2 x + \arcsin \ln x.$
531. $y = \arctan \ln \frac{1}{x}.$
532. $y = \frac{\sqrt{2}}{3} \arctan \frac{x}{\sqrt{2}} + \frac{1}{6} \ln \frac{x-1}{x+1}.$

533. $y = \ln \frac{1 + \sqrt{\sin x}}{1 - \sqrt{\sin x}} + 2 \arctan \sqrt{\sin x}.$

534. $y = \frac{3}{4} \ln \frac{x^2 + 1}{x^2 - 1} + \frac{1}{4} \ln \frac{x - 1}{x + 1} + \frac{1}{2} \arctan x.$

535. $f(x) = \frac{1}{2} \ln(1+x) - \frac{1}{6} \ln(x^2 - x + 1) + \frac{1}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}}.$

536. $f(x) = \frac{x \arcsin x}{\sqrt{1-x^2}} + \ln \sqrt{1-x^2}.$

537. $y = \sinh^2 2x.$

542. $y = \operatorname{arcosh} \ln x.$

538. $y = e^{ax} \cosh \beta x.$

543. $y = \operatorname{artanh}(\tan x).$

539. $y = \tanh^2 2x.$

544. $y = \operatorname{arccoth}(\sec x).$

540. $y = \ln \sinh 2x.$

545. $y = \operatorname{artanh} \frac{2x}{1+x^2}.$

541. $y = \operatorname{arsinh} \frac{x^2}{a^2}.$

546. $y = \frac{1}{2}(x^2 - 1) \operatorname{artanh} x + \frac{1}{2}x.$

547. $y = \left(\frac{1}{2}x^2 + \frac{1}{4}\right) \operatorname{arsinh} x - \frac{1}{4}x \sqrt{1+x^2}.$

548. Find y' , if:

a) $y = |x|;$

b) $y = x|x|.$

Construct the graphs of the functions y and y' .

549. Find y' if

$$y = \ln|x| \quad (x \neq 0).$$

550. Find $f'(x)$ if

$$f(x) = \begin{cases} 1-x & \text{for } x \leq 0, \\ e^{-x} & \text{for } x > 0. \end{cases}$$

551. Calculate $f'(0)$ if

$$f(x) = e^{-x} \cos 3x.$$

Solution. $f'(x) = e^{-x}(-3 \sin 3x) - e^{-x} \cos 3x;$

$$f'(0) = e^0(-3 \sin 0) - e^0 \cos 0 = -1.$$

552. $f(x) = \ln(1+x) + \operatorname{arsin} \frac{x}{2}.$ Find $f'(1).$

553. $y = \tan^2 \frac{\pi x}{6}.$ Find $\left(\frac{dy}{dx}\right)_{x=2}.$

554. Find $f'_+(0)$ and $f'_{-}(0)$ of the functions:

a) $f(x) = \sqrt{\sin(x^2)};$ d) $f(x) = x^2 \sin \frac{1}{x}, \quad x \neq 0; \quad f(0) = 0;$

b) $f(x) = \operatorname{arsin} \frac{a^2 - x^2}{a^2 + x^2};$ e) $f(x) = x \sin \frac{1}{x}, \quad x \neq 0; \quad f(0) = 0$

c) $f(x) = \frac{x}{1 + e^x}, \quad x \neq 0; \quad f(0) = 0;$