First and second order nonlinear Poincaré type inequalities

Around Hardy and Poincaré inequalities

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First order nonlinear Poincaré inequality

$$\int_{\{f>0\}} |f(x)|^p |f(x)|^{\theta p} dx \leq C \int_{\{f>0\}} |f'(x)|^p |f(x)|^{\theta p} dx$$

Second order nonlinear Poincaré inequalities

$$\int_{\{f>0\}} |f(x)|^p |f(x)|^{\theta p} dx \leq C \int_{\{f>0\}} |f''(x)|^p |f(x)|^{\theta p} dx,$$

$$\int_{\{f>0\}} |f'(x)|^p |f(x)|^{\theta p} dx \leq C \int_{\{f>0\}} |f''(x)|^p |f(x)|^{\theta p} dx.$$

Poincaré inequality for functions vanishing at one of the endpoints

If

- $p > 1$
- $f \in W^{1,p}((a,b))$
- $f(a) = 0$ or $f(b) = 0$

then

$$\int_a^b |f(x)|^p dx \leq C_p(a,b) \int_a^b |f'(x)|^p dx$$

holds with best constant $C_p(a,b) = \frac{\left(p(b-a)\sin\left(\frac{\pi}{p}\right)\right)^p}{(p-1)\pi^p}$.

If $f \in W_0^{1,p}((a,b))$ then the best constant is

$$C_p(a,b) = \frac{\left(p(b-a)\sin\left(\frac{\pi}{p}\right)\right)^p}{2^p(p-1)\pi^p}.$$

Nonlinear Beppo–Levi set

Let $\theta \in \mathbb{R}$, $p \geq 1$

$$L^{1,p,\theta}((a,b)) := \Big\{ f \in W^{1,1}_{loc}((a,b)) : \int_a^b |f'(x)|^p |f(x)|^{p\theta} dx < \infty \Big\}.$$

Let $p \geq 1$, $\theta \in \mathbb{R}$, $f \in C((a,b))$, $f \geq 0$ and let either

- $\theta > -1$;
- $f \in L^{1,p,\theta}(I_f)$ where $I_f = \{x \in (a,b) : f(x) > 0\}$;
- $f \in C([a,b])$ and $f(a) = 0$ or $f(b) = 0$.

or

- $\theta < -1$;
- $f > 0$;
- $f \in L^{1,p,\theta}((a,b))$;
- There exist $\lim_{x \to a} f(x)$, $\lim_{x \to b} f(x) \in (0, \infty]$ and one of the limits is infinite.

Then the **first order nonlinear Poincaré type inequality** holds with (optimal) constant $C = C_p(a,b)|1+\theta|$.

Proof

We show that $f^{1+\theta} \in W^{1,p}_{loc}((a,b))$ and $(f^{1+\theta})' = (1+\theta) f^{\theta} f' \chi_{f>0}$ and apply the classical Poincaré inequality.

Remark

The set $X = L^{1,p,\theta}(I_f) \cap C_{\geq}([a,b])$ is the optimal set for which the property

$$X \ni f \mapsto f^{1+\theta} \in W^{1,p}((a,b)) \cap C_{\geq}([a,b]),$$

holds.

Multiplicative inequality

$$\int_a^b |f'(x)|^p (f(x))^{\theta p} dx \leq$$

$$\left(\frac{p-1}{|1+\theta p|}\right)^{\frac{p}{2}} \int_a^b \left(\sqrt{|f(x)f''(x)|}\right)^p (f(x))^{\theta p} dx.$$

- $p \geq 2$;
- $\theta \neq -\frac{1}{p}$;
- $f \geq 0$;
- $f \in \mathcal{R}$ and $C_0^\infty((a,b)) \subseteq \mathcal{R} \subseteq W^{2,1}_{loc}((a,b))$.

Let $p \geq 2$, $\theta \notin \{-1, -\frac{1}{p}\}$, $f \in C((a,b))$, $f \geq 0$. Moreover let either

- $\theta > -1$;
- $f \in W^{2,1}_{loc}(I_f)$, where $I_f = \{f > 0\}$ and $f \in \mathcal{R}$;
- f continuous at a $z \in \{a, b\}$ and $f(z) = 0$;
- if $\theta < -\frac{1}{p}$ then $f > 0$

or

- $\theta < -1$, $f > 0$;
- $f \in W^{2,1}_{loc}((a,b))$ and $f \in \mathcal{R}$;
- $\lim_{x \to z} f(x) = \infty$ for at least one of the endpoints $z \in \{a, b\}$.

Then the **second order nonlinear Poincaré type inequality** holds

The inequality

$$\int_{(a,b)\cap\{x:f(x)>0\}} (f(x))^p (f(x))^{\theta p} dx \leq \widetilde{A} \int_{(a,b)\cap\{x:f(x)>0\}} |f''(x)|^p (f(x))^{\theta p} dx$$

holds with constant $\widetilde{A} = \left(C_p^2(a,b)(1+\theta)^2 \cdot \frac{p-1}{|1+\theta p|} \right)^p$
and

$$\int_{(a,b)\cap\{x:f(x)>0\}} |f'(x)|^p (f(x))^{\theta p} dx \leq \widetilde{B} \int_{(a,b)\cap\{x:f(x)>0\}} |f''(x)|^p (f(x))^{\theta p} dx$$

holds with constant $\widetilde{B} = \left(C_p(a,b)|1+\theta| \cdot \frac{p-1}{|1+\theta p|} \right)^p$.

Applications: Emden–Fowler equation with irregular data

Consider

$$f''(x) + g(x)f^{-\theta}(x) = 0,$$

$g \in L^p$.

- **Nonlinear Poincaré type inequalities** $\Rightarrow$ *a priori* estimates.

Thank you for your attention!