On negative results concerning Hardy means

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Mean

a *mean* is simply a function $\mathfrak{A}\colon \bigcup_{n=1}^{\infty} I^n \to \mathbb{R}_+$, where I is an interval.

Hardy Mean [Definition introduced by Pales and Persson]

Let $I \subset \mathbb{R}_+$ be an interval, $\inf I = 0$. A mean $\mathfrak{A}$ defined on I is *Hardy* if there exists a constant C such that for any $a \in l^1(I)$

$$\sum_{n=1}^{\infty} \mathfrak{A}(a_1, \dots, a_n) < C \sum_{n=1}^{\infty} a_n.$$

Power Means

- Hardy 1920 – p-th Power Mean ($\mathcal{P}_p$) is Hardy if and only if $p < 1$ (with a constant $(p(1-p))^{-1/p}$ for $p \in (0,1)$).
- Landau 1921 – optimal constant for $p \in (0,1)$ (equal $(1-p)^{-1/p}$).
- Carleman 1923 – optimal constant for $p = 0$ (equals e).
- Knopp 1928 – optimal constant for $p < 0$ (equal $(1-p)^{-1/p}$).

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Paper by Páles and Persson (2004)

- some nessesary and some sufficient condition for a deviation mean to be Hardy [omitted in this talk].
- some nessesary and some sufficient condition for Gini means to be Hardy [postponed until applications].

Theorem (P. 2013)

Let $\mathfrak{A}$ be a mean defined on an interval I, (a_n) be a sequence of positive numbers in I satisfying $\sum_{n=1}^{\infty} a_n = +\infty$.
If $\lim_{n\to\infty} a_n^{-1}\mathfrak{A}(a_1,\ldots,a_n) = \infty$ then $\mathfrak{A}$ is not Hardy.

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In all examples $a_n = \frac{1}{n}$ and we will estimate $n \cdot \mathfrak{A}(1, \frac{1}{2}, \dots, \frac{1}{n})$ from below.

Suppose conversely that $\mathfrak{A}$ is a Hardy mean with a constant $C > 0$. By

$$\sum_{n=1}^{\infty} a_n = +\infty \text{ and } \lim_{n\to\infty} a_n^{-1}\mathfrak{A}(a_1,\ldots,a_n) = \infty$$

there exist n_0 and $n_1 > n_0$ such that

$$a_n^{-1}\mathfrak{A}(a_1,\ldots a_n) > 2\,C \text{ for any } n > n_0,$$

$$\sum_{n=n_0+1}^{n_1-1} a_n > \sum_{n=1}^{n_0} a_n.$$

Let $b_n = \begin{cases} a_n & \text{, for } n \le n_1, \\ a_{n_1} 2^{-n} & \text{, for } n > n_1 \end{cases}$. The sequence $(b_n) \in l^1(I)$ will give a contradiction.

$$\mathfrak{G}_{p,q}(a_1,\dots,a_n) := \begin{cases} \left(\dfrac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^q}\right)^{1/(p-q)} & \text{if } p \neq q\,, \\ \exp\left(\dfrac{\sum_{i=1}^n a_i^p \ln a_i}{\sum_{i=1}^n a_i^p}\right) & \text{if } p = q\,. \end{cases}$$

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Proposition, Pales & Persson 2004

Let $p, q \in \mathbb{R}$. If $\mathfrak{G}_{p,q}$ is a Hardy mean, then

$$\min(p,q) \le 0 \text{ and } \max(p,q) \le 1.$$

Conversely, if

$$\min(p,q) \le 0 \text{ and } \max(p,q) < 1$$

then $\mathfrak{G}_{p,q}$ is a Hardy mean.

Remaining case[Páles & Persson Conjecture]

If $\min(p, q) \leq 0$ and $\max(p, q) = 1$ then $\mathfrak{G}_{p,q}$ is not Hardy.

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Proof.

In this case, using the equality $\mathfrak{G}_{p,q} = \mathfrak{G}_{q,p}$ one may suppose that $p = 1$ and $q \leq 0$. Moreover, it might be proved that

$$n\mathfrak{G}_{1,q}(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{n}) \geq (\ln n)^{1/(1-q)} \text{ for any } q \leq 0.$$

Whence $\mathfrak{G}_{1,q}$ is Hardy for no $q \leq 0$. $\square$

Gaussian Product

Let $\lambda = (\lambda_0, \dots, \lambda_p) \in \mathbb{R}^{p+1}$ and v be an all-positive-components vector. One defines a sequence

$$v^{(0)} = v,$$
$$v^{(i+1)} = \left(\mathcal{P}_{\lambda_0}(v^{(i)}), \mathcal{P}_{\lambda_1}(v^{(i)}), \dots, \mathcal{P}_{\lambda_p}(v^{(i)})\right).$$

Then it is known that the limit $\lim_{i\to\infty} \mathcal{P}_{\lambda_k}(v^{(i)})$ exists and does not depend on k. This common limit is denoted by $\mathcal{P}_{\lambda_0} \otimes \cdots \otimes \mathcal{P}_{\lambda_p}(v)$.

Theorem

$\mathcal{P}_{\lambda_0} \otimes \cdots \otimes \mathcal{P}_{\lambda_p}$ is Hardy **iff** $\max(\lambda_0, \ldots, \lambda_p) < 1$.

(⇐) is straightforward. To prove the (⇒) one may assume $\lambda_0 = 1$, $\lambda_1 = \lambda_2 = \ldots = \lambda_p = -\lambda$ for certain $\lambda > 0$. Then it might be proved [main part of proof] that there exists $C, D > 0$ s.t.

$$n\mathcal{P}_1 \otimes \underbrace{\mathcal{P}_{-\lambda} \otimes \cdots \otimes \mathcal{P}_{-\lambda}}_{p}(1, \tfrac{1}{2}, \ldots, \tfrac{1}{n}) > C(\ln n)^D \text{ for any } n \geq 1.$$

Fix $\theta > 1$ and let

$$F\colon (a,b) \mapsto \mathcal{P}_1 \otimes \underbrace{\mathcal{P}_{-\lambda} \otimes \cdots \otimes \mathcal{P}_{-\lambda}}_{p}(a, \underbrace{b, \dots, b}_{p}),$$

$$G\colon (a,b) \mapsto \left(a^{\log_{p+1}\left(\frac{p+1}{\theta^{-\lambda}+p}\right)} b^{\lambda} \right)^{1/\left(\lambda+\log_{p+1}\left(\frac{p+1}{\theta^{-\lambda}+p}\right)\right)},$$

$$\tau\colon (a,b) \mapsto \left(\tfrac{1}{p+1}a, \left(\frac{p+1}{\theta^{-\lambda}+p}\right)^{1/\lambda} b \right).$$

Then

- $G(a,b) \in (\min(a,b), \max(a,b))$,
- $F(a,b) \in (\min(a,b), \max(a,b))$,
- $G \circ \tau(a,b) = G(a,b)$,
- F, G and τ are homogeneous.

Moreover, for $a > \theta b$,

$$
\begin{aligned}
F(a,b) &= \mathfrak{A}\Bigg(\frac{a+pb}{p+1}, \underbrace{\left(\frac{p+1}{a^{-\lambda}+pb^{-\lambda}}\right)^{1/\lambda}, \dots, \left(\frac{p+1}{a^{-\lambda}+pb^{-\lambda}}\right)^{1/\lambda}}_{p}\Bigg) \\
&\geq \mathfrak{A}\Bigg(\tfrac{1}{p+1}a, \underbrace{\left(\frac{p+1}{(\theta b)^{-\lambda}+pb^{-\lambda}}\right)^{1/\lambda}, \dots, \left(\frac{p+1}{(\theta b)^{-\lambda}+pb^{-\lambda}}\right)^{1/\lambda}}_{p}\Bigg) \\
&= \mathfrak{A}\Bigg(\tfrac{1}{p+1}a, \underbrace{\left(\frac{p+1}{\theta^{-\lambda}+p}\right)^{1/\lambda} b, \dots, \left(\frac{p+1}{\theta^{-\lambda}+p}\right)^{1/\lambda} b}_{p}\Bigg) \\
&= F\Bigg(\tfrac{1}{p+1}a, \left(\frac{p+1}{\theta^{-\lambda}+p}\right)^{1/\lambda} b\Bigg) = F\circ\tau(a,b)
\end{aligned}
$$

Next, we will prove that

$$F(a,b) > \tfrac{1}{\theta(p+1)} G(a,b) \text{ for any } a > b:$$

The case when $\frac{a}{b} < \theta(p+1)$ is simply implied by first and second property.
Otherwise, let $a_0 = a$, $b_0 = b$, $(a_{i+1}, b_{i+1}) = \tau(a_i, b_i)$. By the definition of τ, $a_n \to 0$ and $b_n \to +\infty$. Denote by N the smallest natural number such that $a_N \le \theta b_N$. Obviously $a_{N-1} > \theta b_{N-1}$, thus

$$\begin{aligned} a_N = \tfrac{1}{p+1} a_{N-1} > \tfrac{\theta}{p+1} b_{N-1} &= \tfrac{\theta}{p+1} \left(\frac{\theta^{-\lambda} + p}{p+1} \right)^{1/\lambda} b_N \\ &> \tfrac{\theta}{p+1} \left(\frac{\theta^{-\lambda} + \theta^{-\lambda} p}{p+1} \right)^{1/\lambda} b_N = \tfrac{1}{p+1} b_N. \end{aligned}$$

Hence $\frac{1}{p+1} b_N < a_N \leq \theta b_N$ so

$$\begin{aligned} F(a,b) &= F(a_0,b_0) \geq F \circ \tau^N(a_0,b_0) = F(a_N,b_N) \geq \min(a_N,b_N) \\ &> \tfrac{1}{\theta} a_N \geq \tfrac{1}{\theta(p+1)} \max(a_N,b_N) \geq \tfrac{1}{\theta(p+1)} G(a_N,b_N) \\ &= \tfrac{1}{\theta(p+1)} G \circ \tau^N(a_0,b_0) = \tfrac{1}{\theta(p+1)} G(a_0,b_0) = \tfrac{1}{\theta(p+1)} G(a,b). \end{aligned}$$

So

$$n\mathcal{P}_1 \otimes \underbrace{\mathcal{P}_{-\lambda} \otimes \cdots \otimes \mathcal{P}_{-\lambda}}_{p}(1, \tfrac{1}{2}, \dots, \tfrac{1}{n}) > nF(\tfrac{\ln n}{n}, \tfrac{1}{n}) > \tfrac{1}{\theta(p+1)} G(\ln n, 1).$$

- W. Gustin, *Gaussian Means*, Amer. Math. Monthly **54** (1947), 332–335.
- A. Kufner, L. Maligranda, L.-E. Persson, *The Hardy Inequality: About its History and Some Related Results*, Vydavatelský Servis, Pilsen 2007.
- J. A. Oguntuase, L-E. Persson, *Hardy type inequalities via convexity – the journey so far*, Aust. J. Math. Anal. Appl. **7** (2011), 1–19.
- Zs. Páles, L.-E. Persson, *Hardy-type inequalities for means*, Bull. Austral. Math. Soc. **70** (2004), 521–528.
- P. Pasteczka *On negative results concerning Hardy means*, arXiv:1311.2155 [math.CA].