On certain generalizations of the Gagliardo-Nirenberg inequality

Miniconference: Around Hardy and Poincare inequalities

Jan P. Peszek
joint works with Agnieszka Kałamajska

May 14, 2014

Introduction

The model inequality:

$$\int_{\mathbb{R}} |f^{'}(x)|^p dx \leq \left(\sqrt{p-1}\right)^p \int_{\mathbb{R}} \left(\sqrt{|f''(x)f(x)|}\right)^p dx.$$

Introduction

The model inequality:

$$\int_{\mathbb{R}} |f^{'}(x)|^p dx \leq \left(\sqrt{p-1}\right)^p \int_{\mathbb{R}} \left(\sqrt{|f''(x)f(x)|}\right)^p dx.$$

- $p \geq 2$,

Introduction

The model inequality:

$$\int_{\mathbb{R}} |f'(x)|^p dx \leq \left(\sqrt{p-1}\right)^p \int_{\mathbb{R}} \left(\sqrt{|f''(x)f(x)|}\right)^p dx.$$

- $p \geq 2$,
- $f \in C_0^\infty(\mathbb{R})$.

Introduction

The model inequality:

$$\int_{\mathbb{R}} |f^{'}(x)|^p dx \leq \left(\sqrt{p-1}\right)^p \int_{\mathbb{R}} \left(\sqrt{|f^{''}(x)f(x)|}\right)^p dx.$$

Applying Hölder's inequality to the right-hand side:

$$\left(\int_{\mathbb{R}} |f^{'}(x)|^p dx\right)^{\frac{2}{p}} \leq (p-1)\left(\int_{\mathbb{R}} |f(x)|^q dx\right)^{\frac{1}{q}} \cdot \left(\int_{\mathbb{R}} |f^{''}(x)|^r dx\right)^{\frac{1}{r}},$$

assuming that $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$.

The model inequality:

$$\int_{\mathbb{R}} |f'(x)|^p dx \le \left(\sqrt{p-1}\right)^p \int_{\mathbb{R}} \left(\sqrt{|f''(x)f(x)|}\right)^p dx.$$

Applying Hölder's inequality to the right-hand side:

$$\left(\int_{\mathbb{R}} |f'(x)|^p dx\right)^{\frac{2}{p}} \le (p-1) \left(\int_{\mathbb{R}} |f(x)|^q dx\right)^{\frac{1}{q}} \cdot \left(\int_{\mathbb{R}} |f''(x)|^r dx\right)^{\frac{1}{r}},$$

assuming that $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$.
This is the classic Gagliardo-Nirenberg inequality (1959).

Introduction

Maz'ya, *Sobolev Spaces*, 1985:

$$\int_{\{x:f(x)\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx,$$

Introduction

Maz'ya, *Sobolev Spaces*, 1985:

$$\int_{\{x:f(x)\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx,$$

- $p > 2$,

Maz'ya, *Sobolev Spaces*, 1985:

$$\int_{\{x:f(x)\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx,$$

- $p > 2$,
- $f \in C_0^\infty(\mathbb{R})$, $f \geq 0$

Introduction

Maz'ya, *Sobolev Spaces*, 1985

$$\int_{\{x:f(x)\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx$$

Maz'ya, *Sobolev Spaces*, 1985

$$\int_{\{x:f(x)\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx$$

$$\downarrow$$

Maz'ya, *Sobolev Spaces*, 1985

L_p capacitary estimates

Maz'ya, *Sobolev Spaces*, 1985

$$\int_{\{x:f(x)\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx$$

$\downarrow$

Maz'ya, *Sobolev Spaces*, 1985

L_p capacitary estimates

$\downarrow$

Maz'ya, *Sobolev Spaces*, 1985

L_p embeddings $\Leftrightarrow$ isoperimetric inequalities

Introduction

The model inequality

$\int_{\mathbb{R}} |f'|^p dx \leq$
$C \int_{\mathbb{R}} \left(\sqrt{|f'' f|}\right)^p dx$

↓

The Gagliardo-Nirenberg inequality, 1959

$\left(\int_{\mathbb{R}} |f'|^p dx\right)^{\frac{2}{p}} \leq$
$C \left(\int_{\mathbb{R}} |f|^q dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} |f''|^r dx\right)^{\frac{1}{r}}$

Maz'ya, *Sobolev Spaces*, 1985

$\int_{\{f\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq$
$C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx$

↓

Maz'ya, *Sobolev Spaces*, 1985

L_p capacitary estimates

↓

Maz'ya, *Sobolev Spaces*, 1985

L_p embeddings $\Leftrightarrow$ isoperimetric inequalities

Introduction

The model inequality

$$\int_{\mathbb{R}} |f'|^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f'' f|}\right)^p dx$$

The Gagliardo-Nirenberg inequality, 1959

$$\left(\int_{\mathbb{R}} |f'|^p dx\right)^{\frac{2}{p}} \leq C \left(\int_{\mathbb{R}} |f|^q dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} |f''|^r dx\right)^{\frac{1}{r}}$$

(KP)

Linking inequality

Maz'ya, *Sobolev Spaces*, 1985

$$\int_{\{f\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx$$

Maz'ya, *Sobolev Spaces*, 1985

L_p capacitary estimates

Maz'ya, *Sobolev Spaces*, 1985

L_p embeddings $\Leftrightarrow$ isoperimetric inequalities

Introduction

The model inequality

$\int_{\mathbb{R}} |f'|^p dx \leq$
$C \int_{\mathbb{R}} \left(\sqrt{|f'' f|}\right)^p dx$

The Gagliardo-Nirenberg inequality, 1959

$\left(\int_{\mathbb{R}} |f'|^p dx\right)^{\frac{2}{p}} \leq$
$C \left(\int_{\mathbb{R}} |f|^q dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} |f''|^r dx\right)^{\frac{1}{r}}$

(KP)

Linking inequality

(KP)

Orlicz generalization

Maz'ya, *Sobolev Spaces*, 1985

$\int_{\{f\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq$
$C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx$

Maz'ya, *Sobolev Spaces*, 1985

L_p capacitary estimates

Maz'ya, *Sobolev Spaces*, 1985

L_p embeddings $\Leftrightarrow$ isoperimetric inequalities

Introduction

The model inequality

$$\int_{\mathbb{R}} |f'|^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f'' f|}\right)^p dx$$

The Gagliardo-Nirenberg inequality, 1959

$$\left(\int_{\mathbb{R}} |f'|^p dx\right)^{\frac{2}{p}} \leq C \left(\int_{\mathbb{R}} |f|^q dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} |f''|^r dx\right)^{\frac{1}{r}}$$

(KP)

Linking inequality

(KP)

Orlicz generalization

(KP)

Orlicz capacitary estimates

Maz'ya, *Sobolev Spaces*, 1985

$$\int_{\{f\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx$$

Maz'ya, *Sobolev Spaces*, 1985

L_p capacitary estimates

Maz'ya, *Sobolev Spaces*, 1985

L_p embeddings $\Leftrightarrow$ isoperimetric inequalities

Introduction

The model inequality

$$\int_{\mathbb{R}} |f'|^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f'' f|}\right)^p dx$$

The Gagliardo-Nirenberg inequality, 1959

$$\left(\int_{\mathbb{R}} |f'|^p dx\right)^{\frac{2}{p}} \leq C \left(\int_{\mathbb{R}} |f|^q dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} |f''|^r dx\right)^{\frac{1}{r}}$$

(KP)

Orlicz embeddings $\Leftrightarrow$ isoperimetric inequalities

(KP)

Linking inequality

(KP)

Orlicz generalization

(KP)

Orlicz capacitary estimates

Maz'ya, *Sobolev Spaces*, 1985

$$\int_{\{f\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \leq C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx$$

Maz'ya, *Sobolev Spaces*, 1985

L_p capacitary estimates

Maz'ya, *Sobolev Spaces*, 1985

L_p embeddings $\Leftrightarrow$ isoperimetric inequalities

Introduction

The model inequality

$$\int_{\mathbb{R}} |f'|^p dx \le C \int_{\mathbb{R}} \left(\sqrt{|f'' f|}\right)^p dx$$

The Gagliardo-Nirenberg inequality, 1959

$$\left(\int_{\mathbb{R}} |f'|^p dx\right)^{\frac{2}{p}} \le C \left(\int_{\mathbb{R}} |f|^q dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} |f''|^r dx\right)^{\frac{1}{r}}$$

(KP)

Orlicz embeddings $\Leftrightarrow$ isoperimetric inequalities

(KP) – Bonus

Regularity theory

(KP)

Linking inequality

(KP)

Orlicz generalization

(KP)

Orlicz capacitary estimates

Maz'ya, *Sobolev Spaces*, 1985

$$\int_{\{f\neq 0\}} \left(\frac{|f'|}{\sqrt{f}}\right)^p dx \le C \int_{\mathbb{R}} \left(\sqrt{|f''|}\right)^p dx$$

Maz'ya, *Sobolev Spaces*, 1985

L_p capacitary estimates

Maz'ya, *Sobolev Spaces*, 1985

L_p embeddings $\Leftrightarrow$ isoperimetric inequalities

Linking inequality

$$\int_a^b |f'|^p h(f)dx \leq \left(\sqrt{p-1}\right)^p \int_a^b \left(\sqrt{|f'' \mathcal{T}_h(f)|}\right)^p h(f)dx,$$

where:

Linking inequality

$$\int_a^b |f'|^p h(f)dx \le \left(\sqrt{p-1}\right)^p \int_a^b \left(\sqrt{|f'' \mathcal{T}_h(f)|}\right)^p h(f)dx,$$

where:

- $-\infty \le a < b \le +\infty$, $p \ge 2$,

Linking inequality

$$\int_a^b |f'|^p h(f)dx \le \left(\sqrt{p-1}\right)^p \int_a^b \left(\sqrt{|f'' \mathcal{T}_h(f)|}\right)^p h(f)dx,$$

where:

- $-\infty \le a < b \le +\infty$, $p \ge 2$,
- $f \in \mathcal{R}$, $C_0^\infty(a,b) \subseteq \mathcal{R} \subseteq W^{2,1}_{loc}(a,b)$,

Linking inequality

$$\int_a^b |f'|^p h(f)dx \le \left(\sqrt{p-1}\right)^p \int_a^b \left(\sqrt{|f''\mathcal{T}_h(f)|}\right)^p h(f)dx,$$

where:

- $-\infty \le a < b \le +\infty$, $p \ge 2$,
- $f \in \mathcal{R}$, $C_0^\infty(a,b) \subseteq \mathcal{R} \subseteq W^{2,1}_{loc}(a,b)$,
- h satisfies certain technical assumptions,

Linking inequality

$$\int_a^b |f'|^p h(f)dx \leq \left(\sqrt{p-1}\right)^p \int_a^b \left(\sqrt{|f'' \mathcal{T}_h(f)|}\right)^p h(f)dx,$$

where:

- $-\infty \leq a < b \leq +\infty$, $p \geq 2$,
- $f \in \mathcal{R}$, $C_0^\infty(a,b) \subseteq \mathcal{R} \subseteq W^{2,1}_{loc}(a,b)$,
- h satisfies certain technical assumptions,
- $\mathcal{T}_h = \frac{H}{h}$.

Orlicz generalisation

Main result:

$$\int_{\mathbb{R}} M(|f^{'}|h(f))dx \le C \int_{\mathbb{R}} M\left(\sqrt{|f'' \mathcal{T}_h(f)|} \cdot h(f)\right) dx,$$

where:

Orlicz generalisation

Main result:

$$\int_{\mathbb{R}} M(|f^{'}|h(f))dx \leq C \int_{\mathbb{R}} M\left(\sqrt{|f^{''}\mathcal{T}_h(f)|} \cdot h(f)\right) dx,$$

where:

- $f \in W^{2,1}_{loc}(\mathbb{R})$, $f^{'}$ has a compact support,

Orlicz generalisation

Main result:

$$\int_{\mathbb{R}} M(|f^{'}|h(f))dx \leq C \int_{\mathbb{R}} M\left(\sqrt{|f'' \mathcal{T}_h(f)|} \cdot h(f)\right) dx,$$

where:

- $f \in W^{2,1}_{loc}(\mathbb{R})$, $f^{'}$ has a compact support,
- M is an *admissible* N-function,

Orlicz generalisation

Main result:

$$\int_{\mathbb{R}} M(|f^{'}|h(f))dx \leq C \int_{\mathbb{R}} M\left(\sqrt{|f^{''}\mathcal{T}_h(f)|} \cdot h(f)\right) dx,$$

where:

- $f \in W^{2,1}_{loc}(\mathbb{R})$, $f^{'}$ has a compact support,
- M is an *admissible* N-function,
- $\mathcal{T}_h(\lambda) = \frac{H(\lambda)}{h(\lambda)}$,

Orlicz generalisation

Main result:

$$\int_{\mathbb{R}} M(|f^{'}|h(f))dx \leq C \int_{\mathbb{R}} M\left(\sqrt{|f^{''}\mathcal{T}_h(f)|} \cdot h(f)\right) dx,$$

where:

- $f \in W^{2,1}_{loc}(\mathbb{R})$, $f^{'}$ has a compact support,
- M is an *admissible* N-function,
- $\mathcal{T}_h(\lambda) = \frac{H(\lambda)}{h(\lambda)}$,
- C is a constant that depends on M and h.

Admissible N-function

We assume that M is a differentiable N-function.

Admissible N-function

We assume that M is a differentiable N-function.

(M1)

There exist constants $D_M \geq d_M > 1$, such that

$$d_M \frac{M(\lambda)}{\lambda} \leq M^{'}(\lambda) \leq D_M \frac{M(\lambda)}{\lambda} \text{ for } \lambda > 0,$$

Examples

Example

$M(\lambda) = \lambda^p$, $p > 1$

Examples

Example

$M(\lambda) = \lambda^p$, $p > 1$

- λ^p – convex and $\frac{\lambda^p}{\lambda} = \lambda^{p-1}$ – N-function,

Examples

Example

$M(\lambda) = \lambda^p$, $p > 1$

- λ^p – convex and $\frac{\lambda^p}{\lambda} = \lambda^{p-1}$ – N-function,
- $D_M = d_M = p$ and $p\lambda^{p-1} = (\lambda^p)'$ – condition **(M1)**.

Example

$p > q > 1$

$$M(\lambda) := \begin{cases} \lambda^p & \text{for} \quad 0 \leq \lambda < (\frac{q}{p})^{\frac{1}{p-q}}, \\ \lambda^q + \left(\frac{q}{p}\right)^{\frac{p}{p-q}} - \left(\frac{q}{p}\right)^{\frac{q}{p-q}} & \text{for} \quad (\frac{q}{p})^{\frac{1}{p-q}} \leq \lambda \end{cases}$$

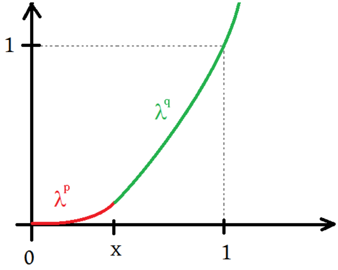
is an N function satisfying **(M1)** – $D_M = p$, $d_M = q$.

Example

$p > q > 1$

$$M(\lambda) := \begin{cases} \lambda^p & \text{for} \quad 0 \le \lambda < \left(\frac{q}{p}\right)^{\frac{1}{p-q}}, \\ \lambda^q + \left(\frac{q}{p}\right)^{\frac{p}{p-q}} - \left(\frac{q}{p}\right)^{\frac{q}{p-q}} & \text{for} \quad \left(\frac{q}{p}\right)^{\frac{1}{p-q}} \le \lambda \end{cases}$$

is an N function satisfying **(M1)** – $D_M = p$, $d_M = q$.



Example

$p > q > 1$, $M(\lambda) = \lambda^p + \lambda^q$ is an N-function satisfying **(M1)** – $D_M = p$, $d_M = q$.

Applications to the capacitary estimates and the isoperimetric inequalities

Capacity

- E – compact subset of $F \subset \Omega$,

Applications to the capacitary estimates and the isoperimetric inequalities

Capacity

- E – compact subset of $F \subset \Omega$,
- M satisfies **(M1)**,

Applications to the capacitary estimates and the isoperimetric inequalities

Capacity

- E – compact subset of $F \subset \Omega$,
- M satisfies **(M1)**,
- $\mathcal{G} := \{u \in C_0^\infty(F), u \geq 0 \text{ on } F,\ u \equiv 1 \text{ in a neighborhood of } E\}$.

Applications to the capacitary estimates and the isoperimetric inequalities

Capacity

- E – compact subset of $F \subset \Omega$,
- M satisfies **(M1)**,
- $\mathcal{G} := \{u \in C_0^\infty(F), u \geq 0 \text{ on } F,\ u \equiv 1 \text{ in a neighborhood of } E\}$.

We define the capacity of E with respect to F:

$$\mathrm{cap}_M^+(E,F) \quad := \quad \inf\left\{\int_\Omega M(|\nabla^{(2)}u|)dx : u \in \mathcal{G}\right\}.$$

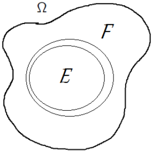
Applications to the capacitary estimates and the isoperimetric inequalities

Capacity

- E – compact subset of $F \subset \Omega$,
- M satisfies **(M1)**,
- $\mathcal{G} := \{u \in C_0^\infty(F), u \geq 0 \text{ on } F,\ u \equiv 1 \text{ in a neighborhood of } E\}$.

We define the capacity of E with respect to F:

$$\mathrm{cap}_M^+(E,F) \quad := \quad \inf\left\{\int_\Omega M(|\nabla^{(2)}u|)dx : u \in \mathcal{G}\right\}.$$



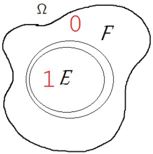
Applications to the capacitary estimates and the isoperimetric inequalities

Capacity

- E – compact subset of $F \subset \Omega$,
- M satisfies **(M1)**,
- $\mathcal{G} := \{u \in C_0^\infty(F), u \geq 0 \text{ on } F,\ u \equiv 1 \text{ in a neighborhood of } E\}$.

We define the capacity of E with respect to F:

$$\mathrm{cap}_M^+(E, F) \quad := \quad \inf \left\{ \int_\Omega M(|\nabla^{(2)} u|)dx : u \in \mathcal{G} \right\}.$$



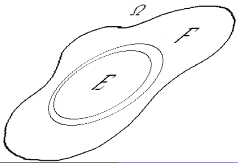
Applications to the capacitary estimates and the isoperimetric inequalities

Capacity

- E – compact subset of $F \subset \Omega$,
- M satisfies **(M1)**,
- $\mathcal{G} := \{u \in C_0^\infty(F), u \geq 0 \text{ on } F,\ u \equiv 1 \text{ in a neighborhood of } E\}$.

We define the capacity of E with respect to F:

$$\mathrm{cap}_M^+(E, F) \quad := \quad \inf \left\{ \int_\Omega M(|\nabla^{(2)} u|)dx : u \in \mathcal{G} \right\}.$$



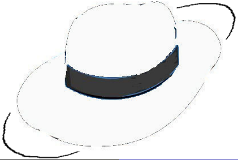
Applications to the capacitary estimates and the isoperimetric inequalities

Capacity

- E – compact subset of $F \subset \Omega$,
- M satisfies **(M1)**,
- $\mathcal{G} := \{u \in C_0^\infty(F), u \geq 0 \text{ on } F,\ u \equiv 1 \text{ in a neighborhood of } E\}$.

We define the capacity of E with respect to F:

$$\mathrm{cap}_M^+(E,F) \quad := \quad \inf \left\{ \int_\Omega M(|\nabla^{(2)} u|)dx : u \in \mathcal{G} \right\}.$$



Applications to the capacitary estimates and the isoperimetric inequalities

Capacitary estimate:

$$\int_0^\infty \mathrm{cap}_M^+(\mathcal{N}_t, \Omega) dM(t) \leq C \int_\Omega M\left(|\nabla^{(2)} u|\right) dx,$$

where

Capacitary estimate:

$$\int_0^\infty \mathrm{cap}_M^+(\mathcal{N}_t,\Omega)dM(t) \leq C\int_\Omega M\left(|\nabla^{(2)}u|\right)dx,$$

where

- M satisfies **(M1)**,

Applications to the capacitary estimates and the isoperimetric inequalities

Capacitary estimate:

$$\int_0^\infty \mathrm{cap}_M^+(\mathcal{N}_t, \Omega) dM(t) \leq C \int_\Omega M\left(|\nabla^{(2)} u|\right) dx,$$

where

- M satisfies **(M1)**,
- $u \in C_0^\infty(\Omega)$ – nonnegative,

Applications to the capacitary estimates and the isoperimetric inequalities

Capacitary estimate:

$$\int_0^\infty \mathrm{cap}_M^+(\mathcal{N}_t, \Omega) dM(t) \leq C \int_\Omega M\left(|\nabla^{(2)} u|\right) dx,$$

where

- M satisfies **(M1)**,
- $u \in C_0^\infty(\Omega)$ – nonnegative,
- $\mathcal{N}_t := \{x \in \Omega : |u(x)| \geq t\}$,

Capacitary estimate:

$$\int_0^\infty \mathrm{cap}_M^+(\mathcal{N}_t, \Omega)\, dM(t) \leq C \int_\Omega M\left(|\nabla^{(2)} u| \right) dx,$$

where

- M satisfies **(M1)**,
- $u \in C_0^\infty(\Omega)$ – nonnegative,
- $\mathcal{N}_t := \{x \in \Omega : |u(x)| \geq t\}$,
- $L := \max\left\{ \limsup_{\lambda \to 0} \frac{M(\lambda)}{\lambda^{D_M}}, \limsup_{\lambda \to \infty} \frac{M(\lambda)}{\lambda^{d_M}} \right\} < \infty.$ **(L)**

Applications to the capacitary estimates and the isoperimetric inequalities

(L)

$$L := \max\left\{\limsup_{\lambda\to 0}\frac{M(\lambda)}{\lambda^{D_M}}, \limsup_{\lambda\to\infty}\frac{M(\lambda)}{\lambda^{d_M}}\right\} < \infty$$

Applications to the capacitary estimates and the isoperimetric inequalities

(L)

$$L := \max\left\{\limsup_{\lambda\to 0} \frac{M(\lambda)}{\lambda^{D_M}}, \limsup_{\lambda\to\infty} \frac{M(\lambda)}{\lambda^{d_M}}\right\} < \infty$$

Example

$M(\lambda) = \lambda^p$, $p > 1$ – N-function satisfying **(M1)**

Applications to the capacitary estimates and the isoperimetric inequalities

(L)

$$L := \max\left\{\limsup_{\lambda\to 0}\frac{M(\lambda)}{\lambda^{D_M}}, \limsup_{\lambda\to\infty}\frac{M(\lambda)}{\lambda^{d_M}}\right\} < \infty$$

Example

$M(\lambda) = \lambda^p$, $p > 1$ – N-function satisfying **(M1)** and **(L)** with $L = 1$.

Applications to the capacitary estimates and the isoperimetric inequalities

(L)

$$L := \max\left\{\limsup_{\lambda\to 0}\frac{M(\lambda)}{\lambda^{D_M}}, \limsup_{\lambda\to\infty}\frac{M(\lambda)}{\lambda^{d_M}}\right\} < \infty$$

Example

$p > q > 1$,

$$M(\lambda) := \begin{cases} \lambda^p & \text{for} \quad 0 \le \lambda < \left(\frac{q}{p}\right)^{\frac{1}{p-q}}, \\ \lambda^q + \left(\frac{q}{p}\right)^{\frac{p}{p-q}} - \left(\frac{q}{p}\right)^{\frac{q}{p-q}} & \text{for} \quad \left(\frac{q}{p}\right)^{\frac{1}{p-q}} \le \lambda \end{cases}$$

– N-function satisfying **(M1)**

Applications to the capacitary estimates and the isoperimetric inequalities

(L)

$$L := \max\left\{\limsup_{\lambda\to 0}\frac{M(\lambda)}{\lambda^{D_M}}, \limsup_{\lambda\to\infty}\frac{M(\lambda)}{\lambda^{d_M}}\right\} < \infty$$

Example

$p > q > 1$,

$$M(\lambda) := \begin{cases} \lambda^p & \text{for} \quad 0 \le \lambda < (\frac{q}{p})^{\frac{1}{p-q}}, \\ \lambda^q + \left(\frac{q}{p}\right)^{\frac{p}{p-q}} - \left(\frac{q}{p}\right)^{\frac{q}{p-q}} & \text{for} \quad (\frac{q}{p})^{\frac{1}{p-q}} \le \lambda \end{cases}$$

– N-function satisfying **(M1)** and **(L)**.

Applications to the capacitary estimates and the isoperimetric inequalities

(L)

$$L := \max\left\{\limsup_{\lambda\to 0}\frac{M(\lambda)}{\lambda^{D_M}}, \limsup_{\lambda\to\infty}\frac{M(\lambda)}{\lambda^{d_M}}\right\} < \infty$$

Example

$M(\lambda) = \lambda^p + \lambda^q$, $p > q > 1$ – N-function satisfying **(M1)**

Applications to the capacitary estimates and the isoperimetric inequalities

(L)

$$L := \max\left\{\limsup_{\lambda\to 0}\frac{M(\lambda)}{\lambda^{D_M}}, \limsup_{\lambda\to\infty}\frac{M(\lambda)}{\lambda^{d_M}}\right\} < \infty$$

Example

$M(\lambda) = \lambda^p + \lambda^q$, $p > q > 1$ – N-function satisfying **(M1)** but not **(L)**.

Applications to the capacitary estimates and the isoperimetric inequalities

Applications to the capacitary estimates and the isoperimetric inequalities

- $\Omega \subset \mathbb{R}^n$ – open, μ – a Borel measure,

Applications to the capacitary estimates and the isoperimetric inequalities

- $\Omega \subset \mathbb{R}^n$ – open, μ – a Borel measure,
- M – N- function satisfying **(M1)** and **(L)**,

Applications to the capacitary estimates and the isoperimetric inequalities

- $\Omega \subset \mathbb{R}^n$ – open, μ – a Borel measure,
- M – N- function satisfying **(M1)** and **(L)**,
- N – some other N-function

Applications to the capacitary estimates and the isoperimetric inequalities

- $\Omega \subset \mathbb{R}^n$ – open, μ – a Borel measure,
- M – N- function satisfying **(M1)** and **(L)**,
- N – some other N-function

Then the following are equivalent:

Applications to the capacitary estimates and the isoperimetric inequalities

- $\Omega \subset \mathbb{R}^n$ – open, μ – a Borel measure,
- M – N- function satisfying **(M1)** and **(L)**,
- N – some other N-function

Then the following are equivalent:

Embedding

$\| M(|u|)\|_{L_N(\Omega,\mu)} \leq A \int_\Omega M(|\nabla^{(2)} u|)dx$ for all nonnegative $u \in C_0^\infty(\Omega)$; with the best constant A.

Applications to the capacitary estimates and the isoperimetric inequalities

- $\Omega \subset \mathbb{R}^n$ – open, μ – a Borel measure,
- M – N- function satisfying **(M1)** and **(L)**,
- N – some other N-function

Then the following are equivalent:

Embedding

$\| M(|u|)\|_{L_N(\Omega,\mu)} \leq A \int_\Omega M(|\nabla^{(2)} u|)dx$ for all nonnegative $u \in C_0^\infty(\Omega)$; with the best constant A.

Isoperimetric inequality

$\mu(E) N^{*-1}(\frac{1}{\mu(E)}) \leq B \mathrm{cap}_M^+(E, \Omega)$
for all $E \subset \Omega$– compact, such that $\mathrm{cap}_M^+(E,\Omega) > 0$; with the best constant B.

Applications to the capacitary estimates and the isoperimetric inequalities

- $\Omega \subset \mathbb{R}^n$ – open, μ – a Borel measure,
- M – N- function satisfying **(M1)** and **(L)**,
- N – some other N-function

Then the following are equivalent:

Embedding

$\| M(|u|)\|_{L_N(\Omega,\mu)} \leq A \int_\Omega M(|\nabla^{(2)} u|)dx$ for all nonnegative $u \in C_0^\infty(\Omega)$; with the best constant A.

Isoperimetric inequality

$\mu(E) N^{*-1}(\frac{1}{\mu(E)}) \leq B\mathrm{cap}_M^+(E,\Omega)$
for all $E \subset \Omega$– compact, such that $\mathrm{cap}_M^+(E,\Omega) > 0$; with the best constant B.

Moreover we have $M(1)B \leq A \leq BC$, where C is the constant from our capacitary estimate.

Applications to the regularity of the nonlinear eigenvalue problems

We consider the following problem:

$$\begin{cases} f''(x) = g(x)\tau(f(x)) \text{ a.e. in } (a,b), \\ f \in \mathcal{R} \end{cases}$$

where $-\infty \le a < b \le +\infty$ and:

Applications to the regularity of the nonlinear eigenvalue problems

We consider the following problem:

$$\begin{cases} f''(x) &= g(x)\tau(f(x)) \text{ a.e. in } (a,b), \\ f \in \mathcal{R} & \end{cases}$$

where $-\infty \le a < b \le +\infty$ and:

- $\tau : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ is an interval,

Applications to the regularity of the nonlinear eigenvalue problems

We consider the following problem:

$$\begin{cases} f''(x) & = & g(x)\tau(f(x)) \text{ a.e. in } (a, b), \\ f \in \mathcal{R} \end{cases}$$

where $-\infty \leq a < b \leq +\infty$ and:

- $\tau : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ is an interval,
- $g \in L^q(a, b)$, $q \in [1, \infty]$,

Applications to the regularity of the nonlinear eigenvalue problems

We consider the following problem:

$$\begin{cases} f^{''}(x) &= g(x)\tau(f(x)) \text{ a.e. in } (a,b), \\ f \in \mathcal{R} & \end{cases}$$

where $-\infty \leq a < b \leq +\infty$ and:

- $\tau : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ is an interval,
- $g \in L^q(a,b)$, $q \in [1,\infty]$,
- $f \in W^{2,1}_{loc}((a,b))$, $f(x) \in A$,

Applications to the regularity of the nonlinear eigenvalue problems

We consider the following problem:

$$\begin{cases} f''(x) = g(x)\tau(f(x)) \text{ a.e. in } (a,b), \\ f \in \mathcal{R} \end{cases}$$

where $-\infty \leq a < b \leq +\infty$ and:

- $\tau : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ is an interval,
- $g \in L^q(a,b)$, $q \in [1,\infty]$,
- $f \in W^{2,1}_{loc}((a,b))$, $f(x) \in A$,
- $\mathcal{R}$ defines the boundary conditions (admissible in our inequality).

Applications to the regularity of the nonlinear eigenvalue problems

We consider the following problem:

$$\begin{cases} f''(x) = g(x)\tau(f(x)) \text{ a.e. in } (a,b), \\ f \in \mathcal{R} \end{cases}$$

Applications to the regularity of the nonlinear eigenvalue problems

We consider the following problem:

$$\begin{cases} f^{''}(x) & = & g(x)\tau(f(x)) \text{ a.e. in } (a,b), \\ f \in \mathcal{R} & & \end{cases}$$

We find a function $h = h_\tau$, such that

$$|g(x)|^q = \left| \frac{f^{''}(x)}{\tau(f(x))} \right|^q = |\mathcal{T}_h(f(x)) f^{''}(x)|^{\frac{2q}{2}} h(f(x)),$$

Applications to the regularity of the nonlinear eigenvalue problems

We consider the following problem:

$$\begin{cases} f''(x) = g(x)\tau(f(x)) \text{ a.e. in } (a,b), \\ f \in \mathcal{R} \end{cases}$$

We find a function $h = h_\tau$, such that

$$|g(x)|^q = \left| \frac{f''(x)}{\tau(f(x))} \right|^q = |\mathcal{T}_h(f(x)) f''(x)|^{\frac{2q}{2}} h(f(x)),$$

then (assuming $p = 2q$) we apply our "Linking inequality":

$$\int_{(a,b)} |f'|^p h(f) dx \leq \left(\sqrt{p-1}\right)^p \int_{(a,b)} \left(\sqrt{|f'' \mathcal{T}_h(f)|}\right)^p h(f) dx = C\|g\|_q^q$$

Applications to the regularity of the nonlinear eigenvalue problems

This way obtain:

Applications to the regularity of the nonlinear eigenvalue problems

This way obtain:

- $\int_{(a,b)} |f^{'}|^{p} h(f) dx \leq C\|g\|_q^q,$

Applications to the regularity of the nonlinear eigenvalue problems

This way obtain:

- $\int_{(a,b)} |f^{'}|^p h(f) dx \leq C\|g\|_q^q$,
- $G(f)$ is γ-Hölder continuous, where $\gamma = 1 - \frac{1}{2q}$ and $G = G_\tau$ is a transform of τ, such that $|(G(f))^{'}|^{2q} = |f^{'}|^{2q} \cdot h(f)$,

Applications to the regularity of the nonlinear eigenvalue problems

This way obtain:

- $\int_{(a,b)} |f^{'}|^p h(f)dx \leq C\|g\|_q^q$,
- $G(f)$ is γ-Hölder continuous, where $\gamma = 1 - \frac{1}{2q}$ and $G = G_\tau$ is a transform of τ, such that $|(G(f))^{'}|^{2q} = |f^{'}|^{2q} \cdot h(f)$,
- we obtain a number of corollaries concerning regularity of solutions and their asymptotic behaviour.

Thank you for your attention!

References:

1. A. Kałamajska, J. Peszek, *On some nonlinear extensions of the Gagliargo-Nirenberg inequality with applications to nonlinear eigenvalue problems*, Asymptotic Anal. 77, No. 3-4, 169-196 (2012)
2. V. G. Maz'ya, *Sobolev Spaces*, Springer-Verlag, 1985.
3. A. Kałamajska, J. Peszek, *On certain generalizations of the Gagliardo-Nirenberg inequality and their applications to capacitary estimates and isoperimetric inequalities*, Journal of Fixed Point Theory and Applications, March 2013, Volume 13, Issue 1, pp 271-290,.