THE LAVRENTIEV GAP PHENOMENON
FOR HARMONIC MAPS INTO SPHERES
HOLDS ON A DENSE SET OF ZERO DEGREE BOUNDARY DATA

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ABSTRACT. We prove that for each positive integer N the set of smooth, zero degree maps $\psi\colon \mathbb{S}^2 \to \mathbb{S}^2$ which have the following three properties:

(i) there is a unique minimizing harmonic map $u\colon \mathbb{B}^3 \to \mathbb{S}^2$ which agrees with ψ on the boundary of the unit ball;

(ii) this map u has at least N singular points in $\mathbb{B}^3$;

(iii) the Lavrentiev gap phenomenon holds for ψ, i.e., the infimum of the Dirichlet energies $E(w)$ of all smooth extensions $w\colon \mathbb{B}^3 \to \mathbb{S}^2$ of ψ is strictly larger than the Dirichlet energy $\int_{\mathbb{B}^3} |\nabla u|^2$ of the (irregular) minimizer u,

is dense in the set $\mathcal{S}$ of all smooth zero degree maps $\phi\colon \mathbb{S}^2 \to \mathbb{S}^2$ endowed with the $W^{1,p}$–topology, where $1 \le p < 2$. This result is sharp: it fails in the $W^{1,2}$–topology of $\mathcal{S}$.

1. INTRODUCTION

In this note, we revisit a well-known topic, the study of singularities of maps $u\colon \mathbb{B}^3 \to \mathbb{S}^2$ which minimize the Dirichlet integral

$$(1.1) \qquad E(u) = \int_{\mathbb{B}^3} |\nabla u|^2 dx\,, \qquad u \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$$

under a prescribed boundary condition $u|_{\partial \mathbb{B}^3} = \varphi\colon \mathbb{S}^2 \to \mathbb{S}^2$. Here, $\mathbb{B}^3$ stands for the open unit ball in $\mathbb{R}^3$, $\mathbb{S}^2$ is the unit sphere, and

$$W^{1,2}(\mathbb{B}^3, \mathbb{S}^2) = \{v = (v_1, v_2, v_3) \in W^{1,2}(\mathbb{B}^3, \mathbb{R}^3)\colon |v(x)| = 1 \text{ for a.e. } x \in \mathbb{B}^3\}\,.$$

Moreover, for a map φ in the fractional Sobolev space $H^{1/2}(\mathbb{S}^2, \mathbb{S}^2)$ we write

$$W^{1,2}_\varphi(\mathbb{B}^3, \mathbb{S}^2) = \{v \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)\colon v|_{\partial \mathbb{B}^3} = \varphi \text{ in the trace sense}\}\,.$$

Minimizers of the Dirichlet integral (1.1) in $W^{1,2}_\varphi(\mathbb{B}^3, \mathbb{S}^2)$ satisfy the Euler–Lagrange system

$$(1.2) \qquad \begin{cases} -\Delta u & = \ |\nabla u|^2 u \qquad \text{in } \mathbb{B}^3, \\ u|_{\partial \mathbb{B}^3} & = \ \varphi\,. \end{cases}$$

The main motivation behind the present work was to reach a deeper understanding of the mechanisms governing the onset of singularities of solutions, and the cardinality and structure of the set of minimizing solutions for a fixed boundary condition. We also wanted

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to know whether the *Lavrentiev gap phenomenon*, cf. (1.3) below, occurs typically (in a precise topological meaning). Despite the work of numerous experts over the last three decades, this topic is still not fully understood. Our main result states, roughly speaking, that even in the case when there is no purely topological reason for the solution of (1.2) to be discontinuous, singularities of u *do* occur under arbitrarily small (in the $W^{1,p}$ sense, for $1 \leq p < 2$) perturbations of an *arbitrary* smooth boundary data φ.

Before giving formal statements of the results, let us sketch a broader perspective.

When $\deg \varphi \neq 0$, all solutions of (1.2) in $W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ obviously have singularities, as φ has no continuous extension $u \colon \mathbb{B}^3 \to \mathbb{S}^2$. By a celebrated classic theorem of Schoen and Uhlenbeck [10] the singular set of a *minimizing* solution of (1.2) consists of isolated points. By another theorem of Almgren and Lieb [2], if the boundary condition φ has square integrable derivatives on $\mathbb{S}^2$, then the number of these points does not exceed a constant multiple of the *boundary energy* $\int_{\mathbb{S}^2} |\nabla_T \varphi|^2 d\sigma$. (Non-minimizing solutions can behave in a wild way: Rivière [9] proves that for any non-constant boundary data φ there exists an everywhere discontinuous solution of the harmonic map system (1.2).)

However, even when $\varphi \colon \mathbb{S}^2 \to \mathbb{S}^2$ satisfies $\deg \varphi = 0$ — so that a priori there is no topological obstruction for a map $u \in W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ to be continuous — minimizers of E in $W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ might be singular because this is energetically preferable. Hardt and Lin [8] give an example of a smooth zero degree boundary data $\widetilde{\varphi} \colon \mathbb{S}^2 \to \mathbb{S}^2$ which is $H^{1/2}$-close to a constant map and has the following properties:

(a) Each minimizer v of E in $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$ has at least N singular points (the number N can be prescribed a priori);

(b) The Lavrentiev gap phenomenon holds for E in $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$. By this, we mean the following inequality:

$$(1.3) \qquad \mu(\widetilde{\varphi}) := \min_{W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)} E(u) < \overline{\mu}(\widetilde{\varphi}) := \inf_{W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2) \cap C^0(\overline{\mathbb{B}}^3)} E(u).$$

An immediate consequence of (1.3) is that $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2) \cap C^0(\overline{\mathbb{B}}^3)$ is not dense in $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$.

As Bethuel, Brezis and Coron have shown, cf. [3, Theorem 5], for boundary conditions φ of zero degree, the Lavrentiev gap phenomenon is equivalent to the fact that all minimizing harmonic maps in $W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ have singularities. Other examples of unexpected and counterintuitive behavior of singularities of minimizing harmonic maps have been given by Almgren and Lieb in [2]. In particular, a minimizer u of E in $W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ can have a large number of singular points even if $\det \nabla_T \varphi \equiv 0$ on $\mathbb{S}^2$ and φ maps the whole sphere $\mathbb{S}^2$ to a smooth curve γ. The abstract of [2] ends with the phrase: "in particular, singularities in u can be unstable under small perturbations of φ."[1]

Our main result ascertains that the message of the last sentence, *singularities can be unstable*, may be strengthened, i.e., replaced with a firm *singularities* are *unstable*, at least

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[1]We have just changed Almgren and Lieb's notation from φ, ψ to our u, φ respectively.

when one takes into account small perturbations of the boundary data in the topology of each of the space $W^{1,p}$, $1 \leq p < 2$. Here is the precise statement.

Theorem 1.1. *Assume that $\varphi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ is an arbitrary smooth map with $\deg \varphi = 0$ and $1 \leq p < 2$. Then, for each $\varepsilon > 0$ and each $N \in \mathbb{N}$ there exists a map $\widetilde{\varphi} \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ such that*

   (i) *$\deg \widetilde{\varphi} = 0$;*
   (ii) *$\|\varphi - \widetilde{\varphi}\|_{W^{1,p}} < \varepsilon$ and $\mathcal{H}^2(\{x \in \mathbb{S}^2 \colon \varphi(x) \neq \widetilde{\varphi}(x)\}) < \varepsilon$;*
   (iii) *the Dirichlet integral E has precisely one minimizer $\widetilde{u} \in W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$; moreover, $\widetilde{u}$ has at least N point singularities in $\mathbb{B}^3$.*

Combining the above result with Bethuel, Brezis and Coron, [3, Theorems 5–6], one immediately obtains the following.

Corollary 1.2. *Assume that $\varphi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ and $\deg \varphi = 0$. Let $\widetilde{\varphi} \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ be given by Theorem 1.1. Then the Lavrentiev gap phenomenon (1.3) holds for $\widetilde{\varphi}$.*

It is a natural question whether the occurrence of such boundary data is a *typical property* in the class of all maps of degree zero, i.e., whether the set of mappings $\widetilde{\varphi} \colon \mathbb{S}^2 \to \mathbb{S}^2$ such that conditions (i) and (iii) of Theorem 1.1 hold *contains a countable intersection of open and dense sets of maps of zero degree* in $H^{1/2}$ (or in some other topology). In spite of some efforts, we have not been able to settle that question.

The main novelty of Theorem 1.1 and its proof is that (1) we show that the singularities are unstable in a generic sense, (2) in order to achieve that, we show how to combine an appropriately modified idea of Hardt and Lin, applied by them only to *constant* boundary conditions $\phi \colon \mathbb{S}^2 \to \{*\}$, with a revisited version of Almgren and Lieb's method of installing new singular points, see [2, Theorem 4.3]. A bridge between these two ingredients is provided by a brief topological argument which guarantees that for each boundary condition φ with $\deg \varphi = 0$ there exist two antipodal points $\pm q \in \mathbb{S}^2$ such that φ maps them to the same point of $\mathbb{S}^2$, $\varphi(q) = \varphi(-q)$. We select any pair of such points and, roughly speaking, show how to insert many tiny bubbles into φ close to those two antipodal points to obtain the new boundary condition $\tilde{\varphi}$. This way, φ is changed only in two little spherical caps centered at $\pm q \in \mathbb{S}^2$, so that the second statement in (ii) in Theorem 1.1 does hold.

To control the degree of $\tilde{\varphi}$ and to guarantee the uniqueness of minimizers of the Dirichlet integral in $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$, we employ the uniform boundary regularity of minimizing harmonic maps combined with the fact that harmonic maps are real analytic in the interior of the regular set.

Finally, the distance from φ to $\widetilde{\varphi}$ in $W^{1,p}$ is estimated by a technical, explicit analysis of the small bubbles. It is crucial here that $p < 2$: the computations in Lemma 3.7 break down for $p = 2$, and an application of Almgren and Lieb's [2, Theorem 2.12] shows that Theorem 1.1 indeed fails for $p = 2$, see Remark 3.10. On the other hand, Hardt and Lin's Stability Theorem [7] asserts that for a Lipschitz boundary mapping ψ with unique energy minimizer v, each minimizer u for a boundary mapping $\widetilde{\psi}$ sufficiently close to ψ in the

Lipschitz norm has the same number of singularities as v. In that sense, the $W^{1,p}$ topology for $1 \leq p < 2$ in Theorem 1.1 is optimal.

The notation throughout the paper is standard. $B(x_0, r) = \{x \in \mathbb{R}^3 : |x - x_0| < r\}$ is the standard Euclidean open ball. We write

$$\partial E(\varphi) = \int_{\mathbb{S}^2} |\nabla_T \varphi|^2 d\sigma$$

to denote the *boundary energy* of a map $\varphi \colon \mathbb{S}^2 \to \mathbb{S}^2$. For a map $u \colon \mathbb{B}^3 \to \mathbb{S}^2$ we set

$$J(x) \equiv J(u)(x) = \sqrt{\det \left(Du(x) Du(x)^T \right)}.$$

If the rank of $Du(x)$ is maximal, i.e., equal to 2, then $J(u)(x)$ measures how $Du(x)|_V$, where V is the orthogonal complement of $\ker Du(x)$, distorts the surface measure: for an arbitrary ball B centered at x, the Jacobian $J(u)(x)$ is equal to the ratio of $\mathcal{H}^2(Du(x)B)$ to $\mathcal{H}^2(B \cap V)$.

2. Installing new singularities

We start with a theorem of Almgren and Lieb, see [2, Theorem 4.3], which describes how to modify the boundary mapping so that its energy minimizer would have a singularity and the energy of the new minimizer would be almost the same as the energy of the initial one. This result will serve as a main tool in constructing $\widetilde{\varphi}$ in the proof of Theorem 1.1.

Before giving the statement, we introduce the notation which will be useful in several places below.

Definition 2.1. *For a fixed map $\psi \colon \mathbb{S}^2 \to \mathbb{S}^2$, which is smooth near a point $q \in \mathbb{S}^2$, and for a fixed number $\varrho > 0$, we let $[\psi]_{q,\varrho} \colon \mathbb{S}^2 \to \mathbb{S}^2$ denote any smooth boundary map which arises from ψ by a small deformation in a neighborhood of q so that the following four conditions are satisfied:*

   (a) *$[\psi]_{q,\varrho}(x) = \psi(x)$ whenever $|x - q| \geq \varrho$;*
   (b) *$[\psi]_{q,\varrho}(x) \equiv \psi(q)$ if $|x - q| = \varrho/2$;*
   (c) *The restriction of $[\psi]_{q,\varrho}$ to the annular region $\frac{\varrho}{2} < |x - q| < \varrho$ satisfies the Lipschitz condition with a Lipschitz constant L which depends only on ψ and <u>not on</u> ϱ;*
   (d) *$[\psi]_{q,\varrho}$ is a diffeomorphism of the spherical cap $\{|x - q| < \varrho/2\} \cap \mathbb{S}^2$ onto the punctured sphere $\mathbb{S}^2 \setminus \{\psi(q)\}$ such that the boundary Dirichlet integral energy of $[\psi]_{q,\varrho}$ on this cap equals $8\pi + o(1)$ as $\varrho \to 0$.*

It is well known that such maps exist, e.g. a modification of the mapping obtained in [1, Appendix A.2]. If we identify the spherical cap from (d) with a disc and assume that $\psi(q) = (0, 0, 1) = N$ we can map a concentric smaller disc to the whole sphere without two spherical caps centered at N and $-N$ consisting of points whose angular distance (in radians) from the point N are smaller or equal $\frac{\varrho}{2}$ and from the point $-N$ are greater or equal $\pi - \frac{\varrho}{2}$, respectively. To do this we use a properly rescaled and rotated inverse stereographic projection. It is a smooth conformal mapping and therefore its Dirichlet energy is equal

twice the Hausdorff measure of the image (and hence approaches $2 \cdot 4\pi$ as $\varrho \to 0$). The remaining annuli from the domain can be mapped into the punctured spherical cap left in the image without changing the Dirichlet integral too much. For details see the proof of Lemma 3.7.

We shall sometimes say that $[\psi]_{q,\varrho}$ arises from ψ by inserting a smooth bubble at q.

Theorem 2.2. *Suppose $u : \mathbb{B}^3 \to \mathbb{S}^2$ is a minimizer which is unique for its boundary mapping $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ and which has an interior singularity at $p \in \mathbb{B}^3$. Assume ψ has finite boundary Dirichlet integral energy and is smooth near $q \in \mathbb{S}^2$ and let $\psi_j : \mathbb{S}^2 \to \mathbb{S}^2$ be any sequence of continuous boundary mappings such that $\psi_j = [\psi]_{q,2/j}$ for all j sufficiently large.*

Finally, let u_j be any minimizer in $\mathbb{B}^3$ with boundary mapping ψ_j. Then, for all sufficiently large j, the mapping u_j will have at least two interior singular points q_j and p_j such that $q_j \to q$ and $p_j \to p$ as $j \to \infty$.

Since we had some trouble to follow the argument in [2] — in particular the lines 11–14 on page 521 — in full detail, we include here a more detailed variant of Almgren and Lieb's proof, explaining the parts which were unclear for us.

Proof. The proof consists of five steps.

STEP 1. We first show that $u_j \to u$ strongly in H^1. By [2, Theorem 1.1],

$$E(u_j) < C\sqrt{\partial E(\psi_j)} < C\sqrt{\partial E(\psi) + 8\pi + L},$$

so $\sup_j E(u_j) < \infty$ and $\sup_j \partial E(\psi_j) < \infty$. Therefore, by [2, Theorem 1.2 part (4)], after passing to subsequences u_j and ψ_j converge to some u_0 and ψ_0, respectively; the convergence (of each of these subsequences) is strong in H^1 and u_0 is a minimizer for its boundary mapping ψ_0. By the strong convergence we obtain the existence of another subsequence j_k such that $\psi_{j_k}(x) \to \psi_0(x)$ for a.e. $x \in \mathbb{S}^2$. However, by its very definition $\psi_j(x) \to \psi(x)$ for all $x \in \mathbb{S}^2 \setminus \{q\}$, so that $\psi_0 = \psi$ a.e. and by the uniqueness of u we obtain that $u_0 = u$.

STEP 2. Now the existence of interior singular points p_j of u_j for sufficiently large j, as well as the convergence $p_j \to p$, follows from [2, Theorem 1.8 part (2)]. (In short, if all u_j were regular in a small neighborhood of p, the scaled energy of u over a small ball $B(p, 2/j)$ would be small enough to guarantee the regularity of u at p.)

STEP 3. By the Boundary Regularity Theorem [11] and monotonicity formula (see e.g. [1, Corollary 1.7]), we may choose an $R > 0$ such that for each $r < R/2$ we have $\int_{B(q,2r)} |\nabla u(x)|^2 \, dx < 2\pi r$.

STEP 4. As $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ is continuous near q, for any $\varepsilon > 0$ we may find a $\delta > 0$ such that if $|x - q| < \delta$, $x \in \mathbb{S}^2$, then $|\psi(x) - \psi(q)| < \varepsilon$. Let us fix $\varepsilon > 0$ and assume that for a fixed small $r = \min(\delta, \frac{1}{2}R)$ independent of j there is no singularity for each u_j in the region $|x - q| < 2r$.

Combining the elementary inequality $2|J(x)| \leq |\nabla u(x)|^2$ and the co-area formula

$$\int_{\mathbb{B}^3} |J(u)(x)|\,dx = \int_{w \in \mathbb{S}^2} \mathcal{H}^1(u^{-1}\{w\})\,d\mathcal{H}^2(w)\,,$$

see [4, Chapter 3], one obtains

(2.1)
$$\int_{\mathbb{B}^3} |\nabla u(x)|^2\,dx \geq 2\int_{w \in \mathbb{S}^2} \mathcal{H}^1(u^{-1}\{w\})\,d\mathcal{H}^2(w).$$

For a fixed point $q \in \mathbb{S}^2$, to shorten the notation, we write $D(q,a) = B(q,a) \cap \mathbb{S}^2$ for the spherical cap formed by the intersection of the ball $B(q,a)$ and the unit sphere. $A(q;a,b) = \big(B(q,b) \setminus B(q,a)\big) \cap \mathbb{S}^2$ is the intersection of the annulus $B(q,b) \setminus B(q,a)$ with the unit sphere. We also write $\mathcal{U}_t = \partial\big(B(q,t) \cap \mathbb{B}^3\big)$ for the boundary of the intersection of the unit ball and the ball centered at q of radius t, and $\mathcal{U}_t^- = \partial B(q,t) \cap \mathbb{B}^3$ for the boundary of the ball centered at q of radius t intersected with the unit ball. Finally, $\mathcal{V}_\varepsilon = B(\psi(q),\varepsilon) \cap \mathbb{S}^2$ stands for the spherical cap established by the intersection of a ball centered at $\psi(q)$ of radius ε and a unit sphere.

We will use (2.1) to estimate the energy of u_j for sufficiently large j's in the region $r < |x-q| < 2r$. We consider $j > 2/r$, so that the strict inclusion $D(q,2/j) \subsetneq D_r := D(q,r)$ holds. By assumption (d), $\psi_j\big(D(q,1/j)\big) = \mathbb{S}^2 \setminus \{\psi(q)\}$ and ψ_j is injective in this small spherical cap, i.e., for any $y \in \mathbb{S}^2 \setminus \{\psi(q)\}$ the set $\psi|_{D(q,1/j)}^{-1}(y)$ consists of only one point. By (a) and (c), we also have $\psi_j\big(A(q;1/j,2/j)\big) \subsetneq \mathcal{V}_\varepsilon$ and $\psi_j\big(A(q;2/j,2r)\big) \subseteq \mathcal{V}_\varepsilon$.

Since, by the assumption above, u_j is continuous in the region $\{|x-q| < 2r\}$, we have $\deg\big(u_j|_{\mathcal{U}_t}\big) = 0$ for every $t < 2r$ because the set $\mathcal{U}_t$ is topologically a sphere. Now, choose a number $t \in (r,2r)$, fix a point $y \in \mathbb{S}^2 \setminus \{\psi\big(A(q;1/j,r)\big)\}$ and consider the set $(u_j|_{\mathcal{U}_t})^{-1}(y)$ of all its preimages. We know that there exists precisely one point $a \in D(q,1/j)$ such that $\psi_j(a) = u_j(a) = y$; since the degree is 0 we deduce that there must be another point $b \in \mathcal{U}_t^-$ such that $u_j(b) = y$ (with a reverse orientation than at a). This degree consideration shows that for each $t \in (r,2r)$ there exists a point $x_t \in \mathcal{U}_t^-$ such that $u_j(x_t) = y$. Since $\mathbb{S}^2 \setminus \{\psi\big(A(q;1/j,r)\big)\} \supset \mathbb{S}^2 \setminus \mathcal{V}_\varepsilon$, we have $\mathcal{H}^1(u_j^{-1}\{w\}) \geq r$ for all $w \in \mathbb{S}^2 \setminus \mathcal{V}_\varepsilon$.

A simple computation yields $\mathcal{H}^2(\mathbb{S}^2 \setminus \mathcal{V}_\varepsilon) = \pi\big(3 + \big(1 - \frac{\varepsilon}{2}\big)^2\big)$. Thus, for ε small, by formula (2.1) we obtain

$$\int_{\{r < |x-q| < 2r\}} |\nabla u_j(x)|^2 dx \geq 2\int_{\mathbb{S}^2} \mathcal{H}^1(u_j^{-1}\{w\})d\mathcal{H}^2(w)$$

$$\geq 2 \cdot r \cdot \pi \left(3 + \left(1 - \frac{\varepsilon}{2}\right)^2\right)$$

$$> 7\pi r.$$

Having in mind the inequality $\int_{B(q,2r)} |\nabla u|^2 dx < 2\pi r$ from Step 3, this is a contradiction to the strong convergence obtained in Step 1. Thus in the region $|x-q| < 2r$ for sufficiently large j's each u_j has a singularity q_j.
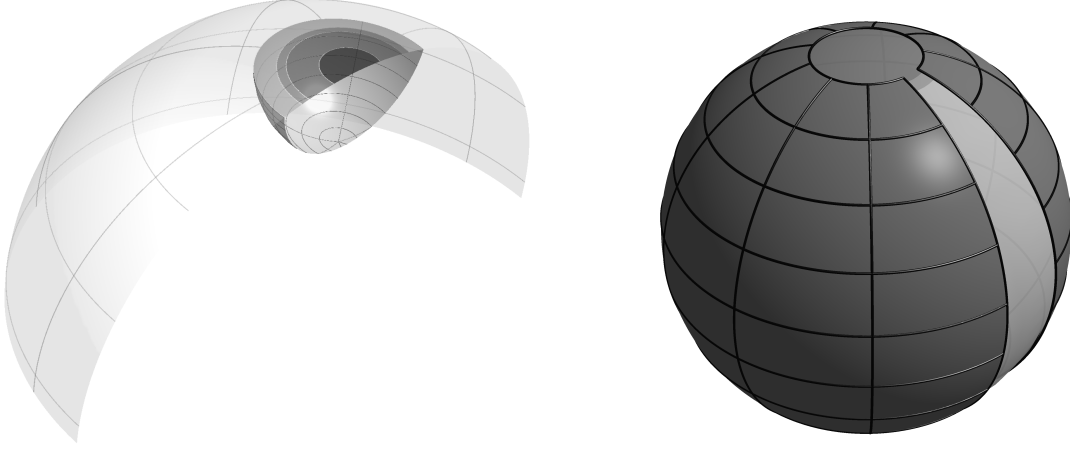
FIGURE 1. Left: the domain of u_j. On the top part of $\mathbb{S}^2$, four shaded areas are visible: the dark gray cap $D(q, 1/j)$, a lighter annulus $A(q; 1/j, 2/j)$, still lighter narrow annulus $A(q; 2/j, r)$, and the lightest $\mathcal{U}_t \setminus D_r$, with the rest of the boundary of $\mathbb{B}^3 \cap B(q, t)$ 'hanging below'. Right: the image of $u_j(\mathcal{U}_t)$, with corresponding shades of gray. The innermost dark cap $D(q, 1/j)$ is mapped to almost the whole sphere, like a blown-up piece of the bubble gum.

STEP 5. Now it suffices to show that $q_j \to q$ as $j \to \infty$. Since $\varepsilon > 0$ was arbitrary, we may choose a sequence of $\varepsilon_j \searrow 0$ such that the corresponding $r_j \searrow 0$ and the regions $B(q, 2r_j)$ in which the singularity q_j appears will shrink to $\{q\}$. □

Remark 2.3. *The assertion of Theorem 2.2 holds true if we replace each ψ_j by a smooth approximation $\widetilde{\psi}_j$ such that the modification in the region $|x - q| < \frac{1}{j}$ from Definition 2.1 (d) remains a diffeomorphism of the smaller disc to the whole sphere without a small cap centered at $\psi(q)$, such that for sufficiently large j's this cap is contained in $\mathcal{V}_\varepsilon$ from Step 4. One may easily check that it does not affect the proof.*

3. Construction of $\widetilde{\varphi}$

The main idea is as follows: we will modify φ on two antipodal sets (in fact, on two little antipodal spherical caps in $\mathbb{S}^2$) of small measures. The modified $\widetilde{\varphi}$ will be arbitrarily close to φ in the space $W^{1,p}$, $1 \le p < 2$ although its oscillations on these discs will be large in C^0. In the first step of the construction, we shall perturb the original mapping slightly, to make it constant on those two discs. Next, roughly speaking, we repeat the construction of Hardt and Lin in [8] in those regions to obtain our $\widetilde{\varphi}$.

At the beginning of this section we recall without proofs a few known results which will be used in the proof of Theorem 1.1. In the second part we construct our boundary condition and we close the section with the proof of Theorem 1.1.

3.1. **Auxiliary propositions.** The following theorem is a restatement of boundary regularity criterion of Schoen and Uhlenbeck [11]. This form, convenient for our purposes, is taken from [2, Theorem 1.10 (2)].

Theorem 3.1. *There exists $\varepsilon > 0$ with the following properties. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is three times continuously differentiable with $f(0) = 0$, $|\nabla f(0)| = 0$, and each partial derivative of f up to order 3 does not exceed ε^2 in absolute value. Suppose also that $\varphi_0 : \mathbb{R}^2 \to \mathbb{S}^2$ is three times continuously differentiable and that each partial derivative of φ_0 up to order 3 does not exceed ε^2 in absolute value. Finally suppose that u^* is a minimizer in the region*

$$\left\{ (x, y, z) : x^2 + y^2 \le 1 \text{ and } f(x, y) \le z \le 1 \right\}$$

and the boundary mapping φ^* for u^* satisfies the condition that

$$\varphi^*(x, y, f(x, y)) = \varphi_0(x, y) \text{ whenever } x^2 + y^2 < 1.$$

Then there is a two times continuously differentiable mapping $u_0 : \mathbb{R}^3 \to \mathbb{S}^2$ such that each partial derivative of u_0 up to order 2 does not exceed ε in absolute value and u^* coincides with u_0 in the region

$$\left\{ (x, y, z) : x^2 + y^2 \le \frac{1}{2} \text{ and } f(x, y) \le z \le \varepsilon \right\}.$$

The next theorem was discovered by Almgren and Lieb; a precise statement can be found in [2, Theorem 4.1 (1)]. It asserts that the boundary mappings having unique minimizers are dense in $H^1(\partial \mathbb{B}^3)$. Theorem 3.2, and the trick used in its proof, will play an important role in our construction.

Theorem 3.2. *Suppose that q is a point in $\partial \mathbb{B}^3$, $\varepsilon > 0$, and that $\varphi : \partial \mathbb{B}^3 \to \mathbb{S}^2$ is a boundary mapping with $\partial E(\varphi) < \infty$. Then there is another mapping $\varphi^* : \partial \mathbb{B}^3 \to \mathbb{S}^2$ which coincides with φ except possibly on that part of $\partial \mathbb{B}^3$ within the ball $B(q, \varepsilon)$, which differs from φ in $H^1(\partial \mathbb{B}^3)$ norm by no more than ε, and for which there is exactly one minimizer $u^* : \mathbb{B}^3 \to \mathbb{S}^2$ having boundary mapping φ^*.*

The key observation in the proof of the above theorem is the following lemma which follows easily from the fact that harmonic maps into $\mathbb{S}^2$ are real analytic away from their singular points (see the proof of Theorem 4.1 in [2]).

Lemma 3.3. *Suppose Ω is a proper subdomain of a larger domain Ω^* and u is any minimizer (not necessary unique) in Ω^*. Then the restriction $u|_\Omega$ of u to Ω is the unique minimizer for its boundary mapping.*

3.2. **Construction of $\widetilde{\varphi}$.** We start with the observation that if $\deg \varphi = 0$ then there exist two antipodal points q, $-q \in \mathbb{S}^2$ such that $\varphi(q) = \varphi(-q)$. For the existence of such $\pm q \in \mathbb{S}^2$, see for instance Granas and Dugundji [6, Part II, p. 94, Thm. (6.1)]. For the convenience of the reader, we give here the gist of a quick argument: assume on the contrary that $\varphi(q) \ne \varphi(-q)$ for all $q \in \mathbb{S}^2$; one then easily constructs a homotopy from φ to another map φ_0 which preserves the antipodes, i.e., $\varphi_0(q) = -\varphi_0(-q)$ for each $q \in \mathbb{S}^2$. This is done as

follows: for a given $q \in \mathbb{S}^2$, if we already have $\varphi(q) = -\varphi(-q)$ for some $q \in \mathbb{S}^2$, then the homotopy changes nothing; if $\varphi(q) \neq -\varphi(-q)$, then the two *distinct* points $\varphi(\pm q) \in \mathbb{S}^2$ determine a unique arc γ of the great circle such that the length of γ is smaller than π, and we let $\varphi(\pm q)$ travel at equal, constant speeds towards two antipodal points $\pm \widetilde{q}$ on that great circle (note that γ is located symmetrically on one of the half-circles joining $\pm \widetilde{q}$). However, it is well known that each map which preserves the antipodes must be of odd degree, a contradiction.

In the remaining part of this section, we simply say that $\pm q \in \mathbb{S}^2$ are the antipodal points of φ. First, we perturb φ slightly by making it constant close to $\pm q$.

Definition 3.4. *For each $\varphi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ with $\deg(\varphi) = 0$, having two antipodal points $\pm q \in \mathbb{S}^2$, and for a fixed number $\delta > 0$ such that $\mathcal{H}^2\left(\varphi(B(q, 2\delta)) \cup \varphi(B(-q, 2\delta))\right) < 4\pi$, we let $\varphi_1 : \mathbb{S}^2 \to \mathbb{S}^2$ denote any intermediate smooth mapping such that*

(1) *$\varphi_1(x) \equiv \varphi(q)$ for $x \in \mathbb{S}^2 \cap \left(B(q, \delta) \cup B(-q, \delta)\right)$;*
(2) *$\varphi_1(x) = \varphi(x)$ on $\mathbb{S}^2 \setminus \left(B(q, 2\delta) \cup B(-q, 2\delta)\right)$;*
(3) *On each of the two annuli $B(\pm q, 2\delta) \setminus B(\pm q, \delta)$ the map φ_1 is given by a composition of φ with a smooth diffeomorphism from the annulus to a punctured disc.*

The parameter δ will be important in our further estimates. Therefore, we explain the choice of δ in the following lemma.

Lemma 3.5. *For each $\varepsilon > 0$ there is a $\delta > 0$ such that the map φ_1 specified in Definition 3.4 above has $\deg(\varphi_1) = 0$ and $\|\varphi - \varphi_1\|_{H^1(\partial \mathbb{B}^3)} < \frac{\varepsilon}{4}$.*

Proof. By Sard's theorem (and the assumption that $\varphi(B(q, 2\delta)) \cup \varphi(B(-q, 2\delta))$ is not of full measure) we may choose a regular value y of φ_1 such that $y \notin \varphi(B(\pm q, 2\delta))$; by definition, the preimages of y under φ_1 are the same as its preimages under φ, so that $\deg(\varphi_1) = \deg(\varphi) = 0$. Since $\varphi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$, we have $\max_{x \in \mathbb{S}^2} |\nabla_T \varphi(x)| < \infty$ and, as $\varphi \not\equiv \varphi_1$ only on the discs $B(\pm q, 2\delta)$ and $\nabla \varphi_1 \equiv 0$ on $B(\pm q, \delta)$,

$$
\begin{aligned}
\frac{1}{2} \|\varphi - \varphi_1\|_{H^1(\partial \mathbb{B}^3)}^2 \leq &\int_{\mathbb{S}^2 \cap B(q, 2\delta)} |\nabla_T \varphi(x)|^2 d\sigma + \int_{\mathbb{S}^2 \cap (B(q, 2\delta) \setminus B(q, \delta))} |\nabla_T \varphi_1(x)|^2 \, d\sigma \\
&+ \int_{\mathbb{S}^2 \cap B(-q, 2\delta)} |\nabla_T \varphi(x)|^2 d\sigma + \int_{\mathbb{S}^2 \cap (B(-q, 2\delta) \setminus B(-q, \delta))} |\nabla_T \varphi_1(x)|^2 \, d\sigma \\
&+ \frac{1}{2} \mathcal{H}^2\left(\mathbb{S}^2 \cap (B(q, 2\delta) \cup B(-q, 2\delta))\right) \cdot 2^2 \\
\leq &\, 2(4\pi\delta^2 + 3\pi\delta^2) \max_{x \in \mathbb{S}^2} |\nabla_T \varphi(x)|^2 + 16\pi\delta^2 \\
\leq &\, 16\pi\delta^2 \max_{x \in \mathbb{S}^2} (|\nabla_T \varphi(x)|^2 + 1).
\end{aligned}
$$

(3.1)

Thus choosing δ such that

$$
\delta < \frac{\varepsilon}{4\sqrt{16\pi \max_{x \in \mathbb{S}^2}(|\nabla_T \varphi(x)|^2 + 1)}}
$$

we obtain $\|\varphi - \varphi_1\|_{H^1(\partial\mathbb{B}^3)} < \frac{\varepsilon}{4}$.

$\square$

We now fix φ_1 as above, and, perturbing it, define a new intermediate map $\varphi_2\colon \mathbb{S}^2 \to \mathbb{S}^2$. Let $\alpha = 4\arcsin\frac{\delta}{2}$ denote the length of the arc $\gamma \cap B(q,\delta)$, where γ is any great circle through q. Without loss of generality suppose from now on that $q = (0,0,1) \in \mathbb{S}^2$. Roughly speaking, we are going to insert $2N$ appropriately small bubbles into φ_1, at points $\pm\xi_i$ close to $\pm q$, preserving the degree but forcing the minimizers to be singular at many points.

Recall also that $D(a, 2/j) \equiv B(a, 2/j) \cap \mathbb{S}^2$ denotes a spherical cap centered at a.

Definition 3.6. *Let $\xi_i = \left(0, \sin\left(\frac{i\alpha}{N+1}\right), \cos\left(\frac{i\alpha}{N+1}\right)\right) \in B(q,\delta)$ for $i = 1, \ldots, N$. For sufficiently large j's, with $2/j \ll \delta/2N$, we define $\varphi_2 : \mathbb{S}^2 \to \mathbb{S}^2$ as follows:*

(1) $\varphi_2(x) = [\varphi_1]_{\xi_i, 2/j}(x)$ for $x \in D(\xi_i, 2/j)$;
(2) $\varphi_2(x) = [\varphi_1]_{\xi_i, 2/j}(-x)$ for $x \in D(-\xi_i, 2/j)$;
(3) $\varphi_2 \equiv \varphi_1$ on $\mathbb{S}^2 \setminus \left(\bigcup_{i=1}^{N} D(\xi_i, 2/j) \cup D(-\xi_i, 2/j)\right)$,

where $[\psi]_{a,b}$ is the modification of ψ in the spherical cap $D(a,b)$, see Definition 2.1.

Note that φ_2 on each cap $D(\xi_i, 2/j)$ is either an orientation-preserving (degree 1) or an orientation-reversing (degree -1) map onto $\mathbb{S}^2$, while on $D(-\xi_i, 2/j)$ it is of opposite orientation (respectively degree -1 or degree 1) map onto $\mathbb{S}^2$. Since $\deg(\varphi_1) = 0$ we also have $\deg(\varphi_2) = 0$.

In the following lemma we will show that this procedure of inserting a single bubble to a map does not change the $W^{1,p}$ norm too much for $p < 2$.

Lemma 3.7. *Let $p < 2$, then for each $N \in \mathbb{N}$, $\varepsilon > 0$ there is a (sufficiently large) j such that $\left\|\nabla(\varphi_1 - [\varphi_1]_{\xi_i, 2/j})\right\|_{L^p} < \frac{\varepsilon}{8N}$ for each i.*

Proof. Fix $i \in \{1, \ldots, N\}$. Rotate $\mathbb{S}^2$ so that ξ_i is the south pole $S = (0,0,-1)$, and work in the spherical coordinates (ϕ, θ), where $\phi \in [0, \pi]$ stands for the polar angle ($\phi = \pi$ corresponds to S) and $\theta \in [0, 2\pi]$ – for the azimuthal angle.

Let $\Phi := [\varphi_1]_{S, 2/j}$ on the spherical cap $D(S, 2/j)$. Without loss of generality we assume that $\Phi(S) = \varphi_1(S) = (0,0,1)$. On the annulus $A(S; 1/j, 2/j)$ the map φ_1 is constant, and Φ is Lipschitz with a constant not depending on j. The main task is to estimate the p-energy of Φ on a smaller disk $D(S, 1/j)$, blown by Φ onto a punctured sphere. For the sake of explicit estimates, we shall extract the formula for Φ on $D(S, 1/j)$.

In the spherical coordinates on $\mathbb{S}^2$ and the polar coordinates on the equatorial $\mathbb{R}^2$, the stereographic projection Φ_1 is given by $\Phi_1(\phi, \theta) = \left(\cot\left(\frac{\phi}{2}\right), \theta\right)$. We have $D(S, 1/j) = \{(\phi, \theta) : \gamma_j < \phi \leq \pi\}$ with the latitude angle $\gamma_j = 2\arccos\frac{1}{2j}$ on $\partial D(S, 1/j)$. Thus, in the polar coordinates (ρ, ϑ) in $\mathbb{R}^2$,

$$\Delta_j := \Phi_1(D(S, 1/j)) = \{(\rho, \vartheta) : 0 \leq \rho < d_j := (4j^2 - 1)^{-1/2}\}.$$

We now set $\Phi|_{D(S,1/j)} = \Phi_2 \circ \Phi_1|_{D(S,1/j)}$, where Φ_2 sends an annulus $\tilde{A}_j \Subset \Delta_j \setminus \{0\}$ onto the whole $\mathbb{S}^2$ without two small caps (by rescaling $\tilde{A}_j$ and then applying Φ_1^{-1}), and Φ_2 is Lipschitz with an absolute constant on $\Delta_j \setminus \tilde{A}_j$ (so that the resulting map Φ satisfies all the requirements of Definition 2.1).

Specifically, fix $0 < \beta = \beta_j = \frac{d_j}{2}$, and let $R = R_j = \cot \frac{\beta_j}{2}$ be the radius of the circle in $\mathbb{R}^2$ which is mapped to $\{\phi = \beta_j\}$ on $\mathbb{S}^2$ by the inverse stereographic projection Φ_1^{-1}. Set

$$\lambda_j = \frac{R_j}{\beta_j}, \qquad r_j = \beta_j \tan^2 \frac{\beta_j}{2}, \qquad \text{so that} \quad \lambda_j r_j = \tan \frac{\beta_j}{2} = \cot \frac{\pi - \beta_j}{2}.$$

The circle $\partial B^2(0, \lambda_j r_j) \subset \mathbb{R}^2$ is mapped by the inverse stereographic projection to the latitude circle $\pi - \beta_j$ near the south pole in $\mathbb{S}^2$. Hence, $\tilde{A}_j := \{(\rho, \vartheta) \colon r_j < \rho < \beta_j\} \subset \Delta_j$ satisfies $(\Phi_1^{-1} \circ \lambda_j \operatorname{Id})(\tilde{A}_j) = \mathbb{S}^2 \cap \{\beta_j < \phi < \pi - \beta_j\}$. We define the whole map $\Phi_2 \colon \Delta_j \to \mathbb{S}^2$ by setting

$$\Phi_2(\rho, \vartheta) = \begin{cases} \left(\pi - \rho \frac{\beta_j}{r_j}, \vartheta\right) & \text{for } 0 \leq \rho \leq r_j, \\ \left(2 \operatorname{arc cot}\left(\frac{R_j}{\beta_j} \rho\right), \vartheta\right) & \text{for } r_j < \rho < \beta_j \quad (\text{i.e., on } \tilde{A}_j), \\ (d_j - \rho, \vartheta) & \text{for } \beta_j \leq \rho < d_j. \end{cases}$$

Finally, $\Phi = \Phi_2 \circ \Phi_1$ on $D(S, 1/j)$. The key input to the p-energy of Φ comes from its behavior on the annular region $A_j^{(1)} := \Phi^{-1}\left(\mathbb{S}^2 \cap \{\beta_j < \phi < \pi - \beta_j\}\right)$. A computation shows that

$$A_j^{(1)} = A\left(S; \frac{2r_j}{\sqrt{r_j^2 + 1}}, \frac{2\beta_j}{\sqrt{\beta_j^2 + 1}}\right) \subset \mathbb{S}^2 \,;$$

on this set, Φ is given by

$$\Phi(\phi, \theta) = \left(2 \operatorname{arc cot}\left(\frac{R_j}{\beta_j} \cot \frac{\phi}{2}\right), \theta\right).$$

We estimate the p-energy of Φ on $A_j^{(1)}$ as follows, dropping the index j next to R, r, and β for the sake of brevity:

$$\int_{A_j^{(1)}} |\nabla_T \Phi|^p d\sigma$$

$$= \int_{2 \operatorname{arc cot} \beta}^{2 \operatorname{arc cot} r} \int_0^{2\pi} \left(\left(\frac{R\beta}{\beta^2 \sin^2(\phi/2) + R^2 \cos^2(\phi/2)}\right)^2 + \left(\frac{1}{\sin \phi}\right)^2\right)^{p/2} \sin \phi \, d\theta \, d\phi$$

$$\leq 2\pi \int_{2 \operatorname{arc cot} \beta}^{2 \operatorname{arc cot} r} \left(\left(\frac{R\beta}{\beta^2 + (R^2 - \beta^2) \cos^2(\phi/2)}\right)^p + \sin^{-p} \phi\right) \sin \phi \, d\phi \ =: \ 2\pi \, (I_1 + I_2).$$

Since $(ab)^{p/2} \leq (a+b)^p$ for positive a, b, we have

$$
\begin{aligned}
I_1 &\leq \frac{2 \cdot (R\beta)^p}{(\beta^2(R^2 - \beta^2))^{p/2}} \int_{2\arccot\beta}^{2\arccot r} \frac{\sin\left(\frac{\phi}{2}\right)}{\cos^{p-1}\left(\frac{\phi}{2}\right)} d\phi \\
&= \frac{4R^p}{(R^2 - \beta^2)^{p/2}} \int_{r/\sqrt{r^2+1}}^{\beta/\sqrt{\beta^2+1}} t^{1-p} \, dt \\
&= \frac{4}{2-p} \frac{R^p}{(R^2 - \beta^2)^{p/2}} \left(\left(\frac{\beta^2}{\beta^2+1}\right)^{2-p} - \left(\frac{r^2}{r^2+1}\right)^{2-p} \right).
\end{aligned}
$$

Therefore, $I_1 \to 0$ as $j \to \infty$, since $R = R_j \to \infty$ and $0 < r = r_j < \beta = \beta_j \to 0$ (note that we use $1 \leq p < 2$). Similarly, $I_2 \to 0$ as $j \to \infty$ by the absolute continuity of the integral, since $\int_0^\pi \sin^{1-p} \phi \, d\phi$ converges for $p < 2$, and both endpoints of I_2 go to π as $j \to \infty$.

Now to estimate the difference $\int_{\mathbb{S}^2} |\nabla_T(\varphi_1 - [\varphi_1]_{S,2/j})|^p d\sigma$ we note the following:

(a) φ_1 and $[\varphi_1]_{S,2/j}$ differ only on the spherical cap $D(S, \frac{2}{j})$ and φ_1 is constant on that cap;

(b) Φ is a composition of a conformal map (Φ_1) and a Lipschitz map (Φ_2) with constant β_j/r_j on the spherical cap $D_j := D(S, 2r_j/\sqrt{r_j^2 + 1})$;

(c) The p-energy of Φ on $A_j^{(1)}$ goes to zero as $j \to \infty$,

(d) Φ is a composition of a conformal map (Φ_1) and a Lipschitz map (Φ_2) with constant 1 on the annular region $A_j^{(2)} := A(S; \frac{2\beta_j}{\sqrt{\beta_j^2+1}}, \frac{1}{j})$;

(e) $\Phi = [\varphi_1]_{S,2/j}$ is Lipschitz on $A_j^{(3)} = A(S; \frac{1}{j}, \frac{2}{j})$ with constant L dependent only on φ_1 and not on j.

All this yields

$$
\begin{aligned}
\int_{\mathbb{S}^2} |\nabla_T(\varphi_1 - [\varphi_1]_{S,2/j})|^p d\sigma &\leq \left(\int_{D_j} + \int_{A_j^{(1)}} + \int_{A_j^{(2)}} + \int_{A_j^{(3)}} \right) |\nabla_T \Phi|^p d\sigma \\
&\leq \left(\frac{\beta_j}{r_j}\right)^p \left(2\mathcal{H}^2(\Delta_j)\right)^{p/2} \cdot \left(\mathcal{H}^2(D_j)\right)^{(2-p)/2} + \int_{A_j^{(1)}} |\nabla_T \Phi|^p d\sigma \\
&\quad + \left(2\mathcal{H}^2(\Phi_1(A_j^{(2)}))\right)^{p/2} \cdot \mathcal{H}^2(A_j^{(2)})^{(2-p)/2} + L^p \mathcal{H}^2(A_j^{(3)})
\end{aligned}
$$

(we have applied Hölder's inequality and conformality of Φ_1 to estimate the integrals over D_j and $A_j^{(2)}$). One easily sees that all the terms above converge to 0 as $j \to \infty$. Thus, we can choose j such that

$$
(3.2) \qquad \int_{\mathbb{S}^2} |\nabla_T(\varphi_1 - [\varphi_1]_{S,2/j})|^p d\sigma < \left(\frac{\varepsilon}{8N}\right)^p,
$$

thereby concluding the proof. $\square$

Remark 3.8. *The map $[\varphi_1]_{\xi_i,2/j}$ given in the proof of the above lemma by an exact formula is not smooth. In order to obtain a smooth map we choose an approximation $\widetilde{\Phi}$ of Φ, so that $\widetilde{\Phi} \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$, the degree of Φ remains the same and the inequality (3.2) holds true.*

Lemma 3.9. *Fix $\delta_1 > 0$ sufficiently small. One may modify $\varphi_2 : \mathbb{S}^2 \to \mathbb{S}^2$ in a spherical cap of radius δ_1, located away from all $D(\xi_i, 2/j)$, obtaining a new map $\varphi_3 \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ such that $\|\varphi_2 - \varphi_3\|_{H^1(\partial\mathbb{B}^3)} < 10\delta_1$ and $\deg(\varphi_3) = 0$, for which there is exactly one minimizer $\widetilde{u} : \mathbb{B}^3 \to \mathbb{S}^2$ with $\widetilde{u}\mid_{\partial\mathbb{B}^3} = \varphi_3$.*

We essentially follow Almgren and Lieb's proof of Theorem 3.2; the only important difference is that we have to make sure that our φ_3 is of degree 0. For the sake of completeness we state the argument in full.

Proof. We extend the ball $\mathbb{B}^3$ slightly to obtain a new smooth domain $\Omega \supsetneq \mathbb{B}^3$, so that Ω coincides with $\mathbb{B}^3$ except in a ball $B(q^*, \delta_1)$, where $\delta_1 < \frac{1}{4}\mathrm{dist}(q, q^*)$ and, to fix the ideas, we choose

$$q^* = \left(0, -\sin\left(\frac{\alpha}{2N}\right), \cos\left(\frac{-\alpha}{2N}\right)\right)$$

away from all the ξ_i and from the caps where the bubbles are inserted into φ_1. Roughly speaking, the new Ω is the union of $\mathbb{B}^3$ and of a tiny and very flat bump of width $2\delta_1$ and height δ_1^5, which is added close to q^*. It is convenient to imagine $\partial\Omega$ as the graph of a smooth nonnegative function $\theta : \mathbb{S}^2 \to [0, \infty)$ such that θ vanishes on $\mathbb{S}^2 \setminus B(q^*, \delta_1)$ and close to q^*, after we flatten the sphere locally,

$$\theta(\cdot) = \delta_1^5 \eta\left(\frac{\cdot}{\delta_1}\right) : \mathbb{R}^2 \to [0, \infty),$$

where η is a smooth nonnegative cutoff function supported in the unit disc with $\eta(0) > 0$. Formally, we let $T : \mathbb{S}^2 \setminus \{-q^*\} \to \mathbb{R}^2$ be a stereographic projection such that $T(q^*) = 0$ and set

$$\Omega = \mathbb{B}^3 \cup \left\{ y : T(\Pi(y)) \in B(0, \delta_1) \subseteq \mathbb{R}^2 \text{ and } \mathrm{dist}(y, \mathbb{S}^2) < \delta_1^5 \eta\left(\frac{T(\Pi(y))}{\delta_1}\right) \right\},$$

where Π stands for the nearest point projection from $\partial\Omega$ to $\partial\mathbb{B}^3$. Multiplying η by a positive constant, we may obviously assume that each partial derivative up to order 3 of $\delta_1^5 \eta\left(\frac{\cdot}{\delta_1}\right)$ does not exceed δ_1^2 in absolute value.

Next we define a new mapping on the boundary of Ω, $\varphi^* : \partial\Omega \to \mathbb{S}^2$, by setting $\varphi^*(x) = \varphi_2(\Pi(x))$. By this definition we have $\varphi^* \equiv \varphi(q)$ on $B(q^*, 4\delta_1) \cap \partial\Omega$. In particular, each partial derivative of φ^* is equal to 0 on that set and therefore does not exceed δ_1^2 in absolute value.

Let $u^* : \Omega \to \mathbb{S}^2$ be any minimizer for φ^*. Then, $u^*|_{\mathbb{B}^3} : \mathbb{B}^3 \to \mathbb{S}^2$ is the unique minimizer for its boundary mapping $\varphi_3 := u^*|_{\partial\mathbb{B}^3}$ by Lemma 3.3. Note that by Theorem 3.1 u^* is of class C^2 up to the boundary of $B(q^*, 2\delta_1) \cap \Omega$. This regularity assertion can easily be improved. To this end, we fix any smooth bounded domain $V \subset B(q^*, 2\delta_1) \cap \Omega$ with, say,
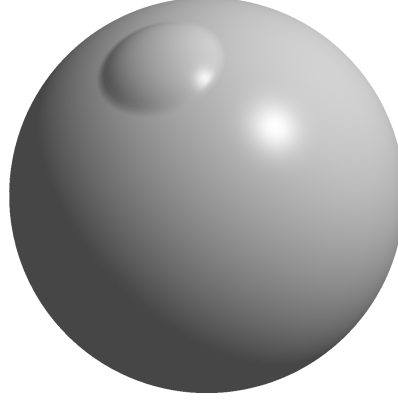
FIGURE 2. Ω is the union of a ball and a small flat bump.

$V \supset \Omega \cap B(q^*, \frac{3}{2}\delta_1)$, and with a C^∞ boundary $\partial V \supset B(q^*, \frac{3}{2}\delta_1) \cap \partial\Omega$. An easy inductive argument using linear Schauder theory, see [5, Thm. 6.19], applied to $u^*|_V$ and the elliptic system $-\Delta u = |\nabla u|^2 u \equiv f$ on V, shows that in fact u^* is of class $C^\infty(\overline{V})$. Therefore φ_3 is of class $C^\infty(\mathbb{S}^2, \mathbb{S}^2)$.

Next we show that $\deg(\varphi_3) = 0$. By the Uniform Boundary Regularity Theorem 3.1, the energy minimizer u^* is two times continuously differentiable at least on $B(q^*, \delta_1)$ and each of its partial derivatives does not exceed δ_1, so that $|u^*(x) - u^*(y)| \le \sqrt{3}\,\delta_1|x-y|$ by the mean value theorem. Thus, if $x \in \partial\Omega \cap B(q^*, \delta_1)$ and $y \in \mathbb{S}^2 \cap B(q^*, \delta_1)$, then

$$|\varphi(q) - \varphi_3(y)| = |u^*(x) - u^*(y)| \le \sqrt{3}\,\delta_1|x-y| < 4\delta_1^2.$$

To compute the degree of φ_3, choose any regular value of φ_3 away from $\mathbb{S}^2 \cap B(\varphi(q), 4\delta_1^2)$. Its preimages under φ_3 will be the same as those under φ_2. Thus, the degree of φ_3 must be the same as that of φ_2, i.e., equal to zero.

Finally, since by Theorem 3.1 each partial derivative of $u^*|_{\partial\mathbb{B}^3} = \varphi_3$ does not exceed δ_1 on $B(q^*, \delta_1)$, and on the set $\{\varphi_2 \ne \varphi_3\}$ the mapping φ_2 is constant, we have the estimate

$$\|\varphi_2 - \varphi_3\|_{H^1(\partial\mathbb{B}^3)}^2 = \int_{\{\varphi_2 \ne \varphi_3\}} \left(|\nabla_T\varphi_2 - \nabla_T\varphi_3|^2 + |\varphi_2 - \varphi_3|^2 \right) d\sigma$$

$$\le 2 \int_{\{\varphi_2 \ne \varphi_3\}} |\nabla_T\varphi_3|^2 \, d\sigma + 2^2 \mathcal{H}^2(\{\varphi_2 \ne \varphi_3\})$$

$$< 10\pi\delta_1^2 \qquad \text{for } \delta_1 < 1.$$

Therefore, for sufficiently small δ_1 we conclude that $\|\varphi_2 - \varphi_3\|_{H^1(\partial\mathbb{B}^3)} < 10\delta_1$. $\square$

Proof of Theorem 1.1. It remains to check that the mapping φ_3 given in Lemma 3.9 has the properties (i)–(iii).

(i) and (iii): By Lemma 3.9, a minimizer u_3 for the boundary condition φ_3 is unique and of degree 0. The proof that u_3 has at least $2N$ singularities is essentially the same as in Theorem 2.2, therefore we skip it.

(ii): Fix $\varepsilon > 0$. We now attune δ, δ_1 and j to obtain $\|\varphi - \varphi_3\|_{W^{1,p}} < \varepsilon$. We first choose $\delta > 0$ as in the proof of Lemma 3.5, then j as in the proof of Lemma 3.7. Next, we fix $\delta_1 < \frac{1}{10}$. Finally, we recall that φ differs from φ_3 only on the two spherical caps $\mathbb{S}^2 \cap B(\pm q, 2\delta)$ whose $\mathcal{H}^2$ measure is $8\pi\delta^2$, shrinking δ if necessary we obtain $\mathcal{H}^2(\{x \in \mathbb{S}^2 : \varphi(x) \neq \varphi_3(x)\}) < \frac{\varepsilon}{4}$ and hence

$$\|\varphi - \varphi_3\|_{W^{1,p}} < \|\varphi - \varphi_3\|_{L^p} + \|\nabla(\varphi - \varphi_1)\|_{L^2} \cdot \left(\mathcal{H}^2(\{\varphi \neq \varphi_1\})\right)^{\frac{2-p}{2p}}$$
$$+ \|\nabla(\varphi_1 - \varphi_2)\|_{L^p} + \|\nabla(\varphi_2 - \varphi_3)\|_{L^2} \cdot \left(\mathcal{H}^2(\{\varphi_2 \neq \varphi_3\})\right)^{\frac{2-p}{2p}}$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \cdot \left(\frac{\varepsilon}{4}\right)^{\frac{2-p}{2p}} + 2N \cdot \left\|\nabla(\varphi_1 - [\varphi_1]_{\xi_i, 2/j})\right\|_{L^p} + 10\delta_1 \cdot \frac{\varepsilon}{4} < \varepsilon.$$

$\square$

Remark 3.10. *Theorem 1.1 does not hold if we replace the norm $W^{1,p}(\mathbb{S}^2, \mathbb{S}^2)$ for $1 \leq p < 2$ by $W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$.*

Proof. Let $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ be a constant map. For this boundary condition of degree 0 there exists exactly one minimizer $u : \mathbb{B}^3 \to \mathbb{S}^2$, $u \equiv const$ for which the Lavrentiev gap phenomenon does not hold. If we modify the boundary map into $\widetilde{\psi}$, without changing its degree, so that the gap phenomenon would hold we would have to install at least 2 singular points into each minimizer. By [2, Theorem 2.12] the number of singularities is bounded by a universal constant C_{AL} times the boundary energy, which in our case would give $\int_{\mathbb{S}^2} |\nabla_T \widetilde{\psi}|^2 d\sigma \geq \frac{2}{C_{AL}}$. Therefore the modified boundary mapping $\widetilde{\psi}$ with singularities for each corresponding minimizer cannot be arbitrary close to the constant map ψ in the $W^{1,2}$ norm. $\square$

4. A REMARK ON THE NONUNIQUENESS IN THE CLASS OF MINIMIZING MAPPINGS

In the following we explain how the boundary mapping constructed in Theorem 1.1 leads to a nonuniqueness example, similar (in the construction) to that of [8, Section 5].

Remark 4.1. *Fix any $M \in \mathbb{N}$. There exist a mapping $\varphi_\tau \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$, $\deg(\varphi_\tau) = 0$, which serves as a boundary data for at least two energy minimizing maps from $\mathbb{B}^3$ to $\mathbb{S}^2$ having different number of singularities (one of them at most M; the other one at least $M + 2$).*

Indeed, let $\psi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ be any mapping having exactly $M \in \mathbb{N}$ singular points such that $\deg(\psi) = 0$ and for which there exists unique energy minimizer $w \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$. We construct $\widetilde{\psi} \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ as in Theorem 1.1 for which $\deg(\widetilde{\psi}) = 0$ and there exists precisely one energy minimizing mapping $\widetilde{w} \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ with at least $M + 2$ singularities.

Since the mappings ψ and $\widetilde{\psi}$ are homotopic, there exist a smooth family of smooth mappings $\{\varphi_t\}_{t\in[0,1]}$ such that $\varphi_0 = \psi$ and $\varphi_1 = \widetilde{\psi}$.

From the Stability Theorem obtained in [7] we deduce that for t sufficiently close to 0 each energy minimizer with boundary data φ_t has exactly M singular points. Let

$$\tau = \sup\left\{t \in [0,1] : \text{ each energy minimizer with boundary data } \varphi_t\right.$$
$$\left.\text{has at most } M \text{ singular points in } \mathbb{B}^3\right\}.$$

Then, $0 < \tau < 1$. We may choose a sequence $s_i \nearrow \tau$ and a sequence of energy minimizing maps $u_i \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ having at most M singular points such that $u_i|_{\mathbb{S}^2} = \varphi_{s_i}$. Similarly we choose $t_i \searrow \tau$ with a sequence of minimizing mappings $v_i \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ having at least $M + 2$ singularities, $v_i|_{\mathbb{S}^2} = \varphi_{t_i}$. (Since we consider boundary maps of degree zero, and it is known that the degree of a minimizing harmonic map on a small sphere around a singular point is ± 1, the number of singular points must jump at least by 2.) Passing to subsequences without changing notation, we obtain $u_i \to u$ and $v_i \to v$, the convergence is strong in $W^{1,2}$ and $u|_{\mathbb{S}^2} = \varphi_\tau = v|_{\mathbb{S}^2}$.

The mapping u has at most M singularities. (It is plausible that one might prove that the number of singularities equals M, by choosing the homotopy appropriately.) Indeed, assume u has at least $M+2$ singular points. Then, by [2, Theorem 1.8 (2)], in an arbitrarily small ball around each singularity of u there would be a singularity of u_i for i sufficiently large, a contradiction.

On the other hand, v has at least $M + 2$ point singularities. Recall that each v_i has at least $M + 2$ singularities and again by [2, Theorem 1.8] we know that singular points converge to singular points. To see that v has at least $M + 2$ singularities we must exclude the possibility that some singularities of the v_i's come together and cancel. By [2, Theorem 2.1] there exists a universal constant C such that if d denotes the distance from a singularity a to the boundary of the ball then there is no other singularity within distance Cd from a. Thus, the singularities of v_i cannot merge in the interior of $\mathbb{B}^3$. Moreover, by Theorem 3.1, there is a neighborhood of the boundary which contains no singularities of v and of the v_i's sufficiently close to v (as the φ_{t_i}'s and φ_τ are close to each other in C^∞). This precludes the case of singularities merging in the limit at the boundary.

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