HARMONIC MAPS INTO SPHERES — EQUATIONS

Notation: Let $A = (a_{ij})$, $B = (b_{ij})$ be $\ell \times n$ matrices, then

$$A : B = \sum_{i=1}^{n} \sum_{j=1}^{\ell} a_{ij} b_{ij}.$$

Lemma 0.1. *Let $u \in W^{1,2}(B^n, \mathbb{S}^{\ell-1})$, where B^n is an n-dimensional ball in $\mathbb{R}^n$ and $\mathbb{S}^{\ell-1}$ is the unit $(\ell-1)$-dimensional sphere in $\mathbb{R}^\ell$. Then $u = (u^1, u^2, \ldots, u^\ell)$ and the following are equivalent:*

(1) *u satisfies in the distibutional sense the equations*

$$-\Delta u = |\nabla u|^2 u, \tag{EL}$$

where

$$\nabla u = \begin{pmatrix} u^1_{x_1} & \ldots & u^1_{x_n} \\ & \ddots & \\ u^\ell_{x_1} & \ldots & u^\ell_{x_n} \end{pmatrix}_{\ell \times n}$$

is an $\ell \times n$ matrix, $|\nabla u|^2 = \nabla u : \nabla u = \sum_{i=1}^n \sum_{j=1}^\ell \left(u^j_{x_i}\right)^2$. (EL) is a short way to write down the system of ℓ equations

$$\begin{cases} -\Delta u^1 & = \sum_{i=1}^n \sum_{j=1}^\ell \left(u^j_{x_i}\right)^2 u^1 \\ & \vdots \\ -\Delta u^\ell & = \sum_{i=1}^n \sum_{j=1}^\ell \left(u^j_{x_i}\right)^2 u^\ell. \end{cases}$$

"Satisfied in the distributional sense" means that if $\Phi \in C_c^\infty(B^n, \mathbb{R}^\ell)$[1] is a vector-valued function, $\Phi = (\Phi^1, \ldots, \Phi^\ell)$ then one has

$$\int_{B^n} \nabla u : \nabla \Phi \, dx = \int_{B^n} |\nabla u|^2 u \cdot \Phi \, dx,$$

where $\nabla u : \nabla \Phi = \sum_{j=1}^\ell \sum_{i=1}^n u^j_{x_i} \Phi^i_{x_j}$ and $u \cdot \Phi = \sum_{j=1}^\ell u^j \Phi^j$.

Note that this implies that for every $j \in \{1, \ldots, \ell\}$ and every $\phi \in C_c^\infty(B^n, \mathbb{R})$ (note that here ϕ is a scalar function) we have

$$\int_{B^n} \nabla u^j \cdot \nabla \phi \, dx = \int_{B^n} |\nabla u|^2 u^j \phi \, dx.$$

On the right-hand side $\nabla u^j = (u^j_{x_1}, \ldots, u^j_{x_n})$, $\nabla \phi = (\phi_{x_1}, \ldots, \phi_{x_n})$ are vectors and $\nabla u^j \cdot \nabla \phi = \sum_{i=1}^n u^j_{x_i} \phi_{x_i}$.

[1] Actually it makes sense if we assume that $\Phi \in W_0^{1,2}(B^n, \mathbb{R}^\ell) \cap L^\infty$

(2) *u satisfies in the sense of distributions*

$$-\Delta u(x) \perp T_{u(x)}\mathbb{S}^{\ell-1}, \qquad i.e.,$$

$$\int_{B^n} \nabla u : \nabla \Phi \, dx = \int_{B^n} \lambda(x) u \cdot \Phi \, dx,$$

for all vector-valued functions $\Phi = (\Phi^1, \ldots, \Phi^\ell) \in C_c^\infty(B^n, \mathbb{R}^\ell)$ *and some scalar function* $\lambda\colon B^n \to \mathbb{R}$.

Here again, $\nabla u : \nabla \Phi = \sum_{i=1}^n \sum_{j=1}^\ell u^j_{x_i} \Phi^j_{x_i}$ *and* $u \cdot \Phi = \sum_{j=1}^\ell u^j \Phi^j$.

(3) *u satisfies in the sense of distributions*

$$\operatorname{div}(u^k \nabla u^j - u^j \nabla u^k) = 0, \qquad \text{for all } k,\, j \in \{1, \ldots, \ell\}.$$

That is, for any $\phi \in C_c^\infty(\mathbb{B}^n, \mathbb{R})$, $k\, j \in \{1, \ldots, \ell\}$ *we have*

$$\int_{B^n} (u^k \nabla u^j - u^j \nabla u^k) \nabla \phi \, dx = 0.$$

Here u^k, u^j *are scalars and* $\nabla u^k = (u^k_{x_1}, \ldots, u^k_{x_n}), \nabla u^j = (u^j_{x_1}, \ldots, u^j_{x_n}), \nabla \phi = (\phi_{x_1}, \ldots, \phi_{x_n})$ *are vectors. Thus,*

$$(u^k \nabla u^j - u^j \nabla u^k) \nabla \phi = \sum_{i=1}^n (u^k u^j_{x_i} - u^j u^k_{x_i}) \phi_{x_i}.$$