# Equivalency of Poincaré inequality and functional on the weighted Sobolev spaces

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### Definition

*Let $K$ be a compact set in $\Omega \Subset \mathbb{R}^n$ and let $\Phi(x,\xi) \geq 0$ be a continuous function in $\Omega \times \mathbb{R}^n$ and positive homogeneous of the first degree with respect to $\xi$. Denote $\mathcal{W}(K,\Omega) := \{u \in \mathcal{D}(\Omega) : \ u \geq 1 \text{ on } K\}$. The number*

$$(p,\Phi)\text{-}\mathrm{cap}(K,\Omega) := \inf\left\{ \int_\Omega [\Phi(x,\nabla u)]^p \, dx : \ u \in \mathcal{W}(K,\Omega) \right\}, \tag{1}$$

*where $p \geq 1$, is called the $(p,\Phi)$-capacity of $K$ relative to $\Omega$ and is denoted by $(p,\Phi)\text{-}\mathrm{cap}(K,\Omega)$.*

**Theorem**

1. *If there exists a constant $\beta$ such that for any compact set $F \subset \Omega$*

$$\mu(F)^{\alpha p} \leq \beta \cdot (p, \Phi)\text{-cap}(F, \Omega), \qquad (2)$$

*where $p \geq 1,\ \alpha > 0,\ \alpha p \leq 1$, then for all $u \in \mathcal{D}(\Omega)$*

$$\|u\|^p_{L^q(\Omega,\mu)} \leq C \int_\Omega [\Phi(x, \nabla u)]^p \ dx, \qquad (3)$$

*where $q = \alpha^{-1}$ and $C \leq p^p(p-1)^{1-p}\beta$.*

2. *If (3) holds for any $u \in \mathcal{D}(\Omega)$ and if the constant $C$ does not depend on $u$, then (2) is valid for all compact set $F \subset \Omega$ with $\alpha = q^{-1}$ and $\beta \leq C$.*

## Corollary

*Assume* $\Phi(x,\lambda)=\rho(x)^{1/p}|\lambda|,\ p=q$ *and for p-admissible weight we have* $d\mu(x)=\rho(x)dx$ *(see Chapter 1.1 in [?]), we have*

i) *if there exists a constant* $\beta$ *such that for any compact set* $F\subset\Omega$

$$\mu(F):=\int_F \rho(x)\ dx \leq \beta\cdot(p,\Phi)\text{-}\mathrm{cap}(F,\Omega), \tag{4}$$

*then for all* $u\in\mathcal{D}(\Omega)$

$$\|u\|_{L^p_\rho(\Omega)} \leq C\|\nabla u\|_{L^p_\rho(\Omega)}, \tag{5}$$

*where* $C\leq p(p-1)^{(1-p)/p}$.

ii) *If* (4) *holds for* $u\in\mathcal{D}(\Omega)$ *and if the constant* $C$ *does not depend on* $u$, *then* (5) *is valid for all compact set* $F\subset\Omega$ *and* $\beta\leq C$.

## Definition

*Let $\Omega$ be any open subset of $\mathbb{R}^n$ and $\Psi$ be the locally integrable function defined on $\Omega$ such that for every nonnegative compactly supported $w \in W^{1,p}(\Omega)$,*

$$\int_\Omega \Psi w \, dx > -\infty. \tag{6}$$

*Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $u \neq 0$ a.e. We say that*

$$-\Delta_p u \geq \Psi, \tag{7}$$

*if for every non-negative compactly supported $w \in W^{1,p}(\Omega)$ we have*

$$\langle -\Delta_p u, w \rangle := \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla w \rangle \, dx \geq \int_\Omega \Psi w \, dx. \tag{8}$$

## Definition

**$(\Psi, p)$-condition:** *Suppose $u$ and $\Psi$ are as in Definition 1.4 and moreover there exists*

$$(\Psi, p)\sigma_0 := \inf \{\sigma \in \mathbb{R} : \Psi \cdot u + \sigma |\nabla u|^p \geq 0 \text{ a.e. in } \Omega \cap \{u > 0\}\} \in \tag{9}$$

*where we set* $\inf \emptyset = +\infty$.

## Theorem (I. SKRZYPCZAK)

*Suppose $1 < p < \infty$ and there exists a non-negative solution $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$, to PDI in the sense of Definition 1.4*

$$\begin{cases} \Delta_p v(x) \geq \Psi \\ v(x) > 0 \qquad \text{in } \Omega, \end{cases} \tag{10}$$

*where $\Psi$ is locally integrable and satisfies $(\Psi, p)$ with $\sigma_0 \in \mathbb{R}$ given by (9). Assume further that $\beta$ and $\sigma$ are arbitrary numbers such that $\beta > 0$ and $\beta > \sigma \geq \sigma_0$. Then for every Lipschitz function $\xi$ with compact support in $\Omega$ we have*

$$\int_\Omega |\xi|^p \, \mu_1(dx) \leq \int_\Omega |\nabla \xi|^p \, \mu_2 dx. \tag{11}$$

*where*

**Theorem (I. SKRZYPCZAK)**

$$\mu_1(dx) := \left(\frac{\beta-\sigma}{p-1}\right)^{p-1} [\Psi \cdot v + \sigma|\nabla v|^p] \cdot v^{-\beta-1}\chi_{\{v>0\}}\ dx$$

$$\mu_2(dx) := v^{p-\beta-1}\chi_{\{|\nabla v|\neq 0\}}\ dx =$$

Applying this theorem for $0 \leq \Psi = \varphi(x)|v(x)|^{p-2}v(x),\ \sigma = 0$, where $\varphi$ is a non-negative function. Assume

$$\mu_1(dx) = \rho_1(x)dx \ \text{ and } \mu_2(dx) = \rho_2(x)dx. \tag{12}$$

Then we have

$$\begin{aligned} \rho_1(x) &= \left(\frac{\beta}{p-1}\right)^{p-1} [\Psi \cdot v] \cdot v^{-\beta-1}\chi_{\{v>0\}}, \\ &\geq \left(\frac{\beta}{p-1}\right)^{p-1} \left[\varphi|v|^{p-2}v \cdot v\right] \cdot v^{-\beta-1}\chi_{\{v>0\}}, \\ &= \left(\frac{\beta}{p-1}\right)^{p-1} \varphi|v|^{p-\beta-1}\chi_{\{|\nabla v|\neq 0\}} = \left(\frac{\beta}{p-1}\right)^{p-1} \phi\rho_2(x) \end{aligned}$$

Then we have for $\phi \geq C$, some constant and from (11),

$$\int_\Omega |\xi|^p \ \rho_1(x) \ (dx) \leq C \left(\frac{\beta}{p-1}\right)^{p-1} \int_\Omega |\nabla \xi|^p \ \rho_1(x) \ dx. \quad (13)$$

and from (ii) of Corollary 1.3, then we have for (1.6), there exists a constant $\beta^{'}$ such that for any compact set $F \subset \Omega$

$$\mu(F) := \int_F \rho_1(x) dx \leq \beta^{'} \cdot (p, \Phi)\text{-}\mathrm{cap}(\Omega, F). \quad (14)$$

# Thank you