3.

a.

$$P(H_k \mid y_j) = \text{(Bayes's Theorem)}$$

$$\frac{P(y_j \mid H_k)\,P(H_k)}{\sum_{i=1}^{3} P(y_j \mid H_i)\,P(H_i)} =$$

$$\frac{P(H_k)}{\frac{1}{P(y_j \mid H_k)} \sum_{i=1}^{3} P(y_j \mid H_i)\,P(H_i)} =$$

$$\frac{P(H_k)}{P(H_1)\frac{P(y_j \mid H_1)}{P(y_j \mid H_k)} + P(H_2)\frac{P(y_j \mid H_2)}{P(y_j \mid H_k)} + P(H_3)\frac{P(y_j \mid H_3)}{P(y_j \mid H_k)}}$$

b.

$p(y_j|H_1) =$ (definition of marginal density)

$$\int_0^{\phi_T-\delta} p(y_j, \pi_j|H_1)\, d\pi_j =$$

$$\int_0^{\phi_T-\delta} p(y_j, \pi_j|H_1) \frac{p(\pi_j|H_1)}{p(\pi_j|H_1)}\, d\pi_j =$$

$$\int_0^{\phi_T-\delta} \frac{p(y_j, \pi_j|H_1)}{p(\pi_j|H_1)} p(\pi_j|H_1)\, d\pi_j = \text{(Bayes)}$$

$$\int_0^{\phi_T-\delta} p(y_j|\pi_j, H_1)\, p(\pi_j|H_1)\, d\pi_j =$$

(since $y_j|\pi_j \sim Bin(n_j, \pi_j)$ and $\pi_j|H_1 \sim Unif(0, \phi_T-\delta)$)

$$\int_0^{\phi_T-\delta} \binom{n_j}{y_j} \pi_j^{y_j} (1-\pi_j)^{n_j-y_j} \frac{1}{\phi_T-\delta}\, d\pi_j =$$

$$\frac{1}{(\phi_T-\delta)(n_j+1)} \int_0^{\phi_T-\delta} \binom{n_j}{y_j} \pi_j^{y_j} (1-\pi_j)^{n_j-y_j} (n_j+1)\, d\pi_j =$$

$$\frac{1}{(\phi_T-\delta)(n_j+1)} \int_0^{\phi_T-\delta} \frac{n_j!}{y_j!(n_j-y_j)!} \pi_j^{y_j} (1-\pi_j)^{n_j-y_j} (n_j+1)\, d\pi_j =$$

$$\frac{1}{(\phi_T-\delta)(n_j+1)} \int_0^{\phi_T-\delta} \frac{(n_j+1)!}{y_j!(n_j-y_j)!} \pi_j^{y_j} (1-\pi_j)^{n_j-y_j}\, d\pi_j =$$

$$\frac{1}{(\phi_T-\delta)(n_j+1)} \int_0^{\phi_T-\delta} \frac{\Gamma(n_j+2)}{\Gamma(y_j+1)\Gamma(n_j-y_j+1)} \pi_j^{y_j} (1-\pi_j)^{n_j-y_j} \, d\pi_j =$$

$$\frac{1}{(\phi_T-\delta)(n_j+1)} \int_0^{\phi_T-\delta} \frac{\pi_j^{y_j} (1-\pi_j)^{n_j-y_j}}{\frac{\Gamma(y_j+1)\Gamma(n_j-y_j+1)}{\Gamma(n_j+2)}} \, d\pi_j =$$

$$\frac{1}{(\phi_T-\delta)(n_j+1)} \int_0^{\phi_T-\delta} f_{Beta(y_j+1,\, n_j-y_j+1)}(\pi_j) \, d\pi_j =$$

$$\frac{1}{(\phi_T-\delta)(n_j+1)} F_{Beta(y_j+1,\, n_j-y_j+1)}(\phi_T-\delta)$$

c.

Note $\pi_j = \rho_j^{\exp(\alpha)} = g^{-1}(\alpha) \Longleftrightarrow \log(\pi_j) = \exp(\alpha)\log(\rho_j) \Longleftrightarrow$

$\dfrac{\log(\pi_j)}{\log(\rho_j)} = \exp(\alpha) \Longleftrightarrow \alpha = \log\left(\dfrac{\log \pi_j}{\log \rho_j}\right) = g(\pi_j)$. By the

transformation theorem,

$f_\alpha(\alpha) = f_{\pi_j}(g^{-1}(\alpha)) \left| \dfrac{d}{d\alpha} g^{-1}(\alpha) \right| = - f_{\pi_j}(g^{-1}(\alpha))\, \rho_j^{\exp(\alpha)} \log(\rho_j)\exp(\alpha)$.

Under H_1 we have $f_{\pi_j}(\pi_j) = \dfrac{1}{\phi_T - \delta}$; $g(\pi_j)$ at the boundaries

$0, \phi_T - \delta$ of the support of π_j is

$\log\left(\dfrac{\log 0}{\log \rho_j}\right) = \log\left(\dfrac{-\infty}{\log \rho_j}\right) = \log(\infty) = \infty$ and $\log\left(\dfrac{\log(\phi_T - \delta)}{\log \rho_j}\right)$.

So $f(\alpha | H_1) = - \dfrac{\log \rho_j}{\phi_T - \delta} \rho_j^{\exp(\alpha)} \exp(\alpha)$, $\alpha > \log\left(\dfrac{\log(\phi_T - \delta)}{\log \rho_j}\right)$.

Under H_2 we have $f_{\pi_j}(\pi_j) = \dfrac{1}{\phi_T + \delta - (\phi_T - \delta)} = \dfrac{1}{2\delta}$; $g(\pi_j)$ at

the boundaries $\phi_T - \delta, \phi_T + \delta$ of the support of π_j is

$\log\left(\dfrac{\log(\phi_T - \delta)}{\log \rho_j}\right)$ and $\log\left(\dfrac{\log(\phi_T + \delta)}{\log \rho_j}\right)$. We have

$\phi_T - \delta < \phi_T + \delta \Rightarrow \log(\phi_T - \delta) < \log(\phi_T + \delta) \Rightarrow$

$\dfrac{\log(\phi_T - \delta)}{\log \rho_j} > \dfrac{\log(\phi_T + \delta)}{\log \rho_j} \Rightarrow \log\left(\dfrac{\log(\phi_T - \delta)}{\log \rho_j}\right) > \log\left(\dfrac{\log(\phi_T + \delta)}{\log \rho_j}\right)$, So $f(\alpha \mid H_2) = -\dfrac{\log \rho_j}{2\delta} \rho_j^{\exp(\alpha)} \exp(\alpha)$, $\log\left(\dfrac{\log(\phi_T+\delta)}{\log \rho_j}\right) < \alpha < \log\left(\dfrac{\log(\phi_T-\delta)}{\log \rho_j}\right)$.

Under H_3 we have $f_{\pi_j}(\pi_j) = \dfrac{1}{1-(\phi_T+\delta)} = \dfrac{1}{1-\phi_T-\delta}$; $g(\pi_j)$ at the boundaries $\phi_T+\delta$, 1 of the support of π_j is

$\log\left(\dfrac{\log(\phi_T+\delta)}{\log \rho_j}\right)$ and $\log\left(\dfrac{\log 1}{\log \rho_j}\right) = \log\left(\dfrac{0}{\log \rho_j}\right) = \log 0 = -\infty$,

so $f(\alpha \mid H_3) = -\dfrac{\log \rho_j}{1-\phi_T-\delta} \rho_j^{\exp(\alpha)} \exp(\alpha)$, $\alpha < \log\left(\dfrac{\log(\phi_T+\delta)}{\log \rho_j}\right)$.