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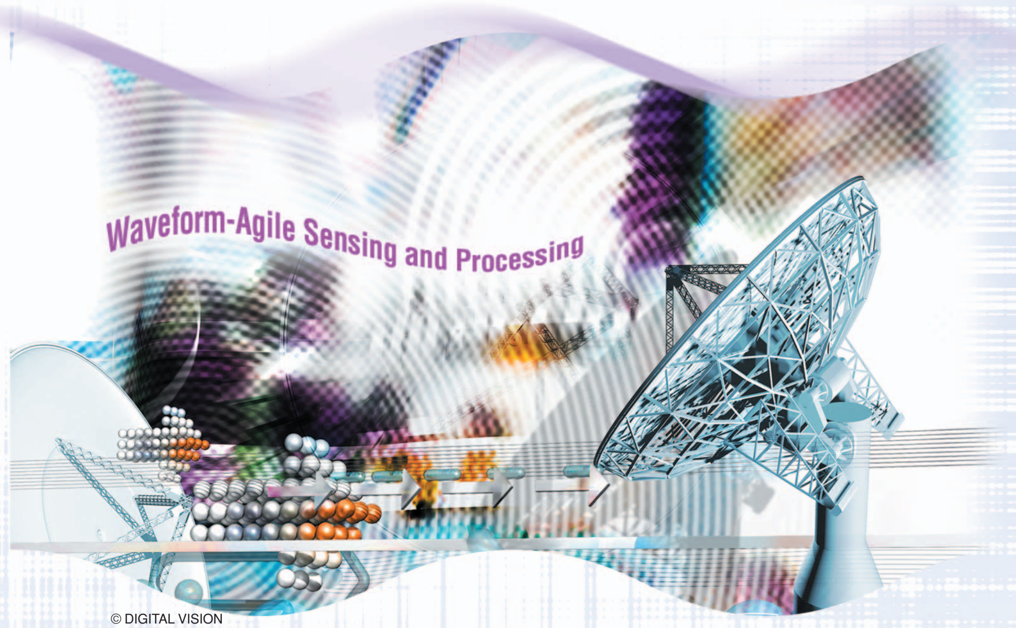

Phase-Coded Waveforms and Their Design

[The role of the ambiguity function]

The design of radar waveforms has received considerable attention since the 1950s. In 1953, P.M. Woodward [58], [59] defined the narrowband radar ambiguity function or, simply, ambiguity function. It is a device formulated to describe the effects of range and Doppler on matched filter receivers. Woodward acknowledged the influence that Shannon's communication theory from 1948 had on his ideas; and he explained the relevance of ambiguity in radar signal processing, perhaps best conceived in terms of a form of the uncertainty principle (see the sections "Motivation" and "Ambiguity Functions"). His book [58] ends with the astonishing, self-deprecating, and heartfelt, statement:

> The reader may feel some disappointment, not unshared by the writer, that the basic question of what to transmit remains substantially unanswered.

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However, in the 50 or so years since Woodward's book was published, radar signal processing has used the ambiguity function as an intricate and flexible tool in the design of waveforms to solve diverse problems in radar. In the process, substantial connections were established in mathematics, physics, and other areas of signal processing.

The rich history of the field and limitations of space preclude the presentation of a comprehensive summary on waveform design. Therefore, we focus on a specific piece of the waveform design problem as pertaining to the use of constant amplitude zero autocorrelation sequences in this paper. Constant amplitude (CA) sequences with zero autocorrelation (ZAC), which often serve as coefficients of the translates of the sampling function used to design a given waveform, arise naturally in a host of problems associated with radar, communications, coding theory, and various areas of signal processing. There are books and surveys, such as [40] and [31], in the area, and literally thousands of articles from the second half of the 20th century.

As such, we are introducing two new methods, discussed in sections "CAZAC Sequences" and "Aperiodic Simulations," respectively, which can be formulated in a relatively small space, but that fit in the broader context reflected by the diverse literature mentioned in the previous paragraph. Hopefully, these methods will serve as a catalyst for developing new waveform design methods to address the requirement of spectral efficiency imposed by a rapidly dwindling EM spectrum.

There have been quadratic phase CAZACs going back to Norbert Wiener [57], and possibly Gauss. We construct new quadratic sequences as well as nonchirp-like sequences, and provide a discrete periodic ambiguity function analysis of them. The section "Aperiodic Simulations" develops the analogue of the section "CAZAC Sequences," but for the aperiodic case. In the section "Motivation," we provide some encouragement for waveform design and an elementary background on radar. The section "Ambiguity Functions" records the definition and basic properties of the ambiguity function.

MOTIVATION

The area of waveform design enjoys continued attention due to advances in radar hardware, computational algorithms, and coding schemes. These advances have had a creative, synergistic impact on the design of radar waveforms, providing improved ambiguity function and target detection properties.

Two basic desiderata that must be satisfied in the design of radar waveforms are the following [17], [40], [58], and [59]:

- Short duration (in the time domain perforce large extent in the frequency domain) pulses are required for good range resolution.
- On the other hand, target detection calls for sufficient energy on target.

These two guiding principles impose conflicting requirements on the radar waveform design problem. The idea and associated methods of pulse compression are employed to satisfy these conflicting requirements. In turn, this calls for waveforms with large compression ratios (time-bandwidth product) [40]. Extension of this approach to multistatic radars is considered in [11].

Early work on the problem of radar waveform design was devoted to the use of a fixed transmit signal. This was largely due to a hardware limitation, which precluded waveform agile operation. Consequently, radar resolution in delay and Doppler was inhibited by the Heisenberg uncertainty principle. Thus, increased resolution in range comes about at the expense of resolution in Doppler and vice-versa [2], [3]. Information theoretic criteria for radar waveform design are outlined in [2] and [3]. The work of [3] sheds light on ameliorating the intrinsic limitations dictated by the Heisenberg uncertainty principle. Specifically, it draws an analogy of the problem of delay-Doppler imaging to an equivalent problem in high resolution optical image formation. More precisely, high-resolution optical images are formed by coherently combining several low-resolution images each having a specified point spread function. In this process, it is important to minimize the cross point spread functions between the combined images. In the radar context, the ambiguity function plays the role of the point spread function. The work of [3] exploits this concept for designing radar waveforms to produce high resolution delay-Doppler images. In that respect, it highlights the importance of designing libraries of waveforms with diverse characteristics in terms of their ambiguity functions. We address this issue in the section "CAZAC Sequences."

The work of [56] shows that the ambiguity function results from both a detection and an estimation problem for mono-static radars. Extension of this approach to bi-static radars is not straightforward due to the large number of free parameters arising from the geometry of the bi-static radar set up. The work of [53] formulates the ambiguity function for bi-static radars taking into account the geometry of a given setting. Furthermore, this work makes significant strides in terms of the discrete time ambiguity function with key considerations of sampling, reconstruction, and aliasing. The corresponding exercise for multistatic radar ambiguity function analysis is very much an open problem. Recent results point to the advantages of studying this problem in the context of frames and filter banks (see the sections "CAZAC Sequences" and "Epilogue").

Another important issue in this context is the design of discrete time radar waveforms with a given ambiguity function design specification. For a given waveform, the ambiguity function can be calculated. However, several waveforms can give rise to the same ambiguity function modulus. Consequently, the problem of radar waveform design for a given ambiguity function design specification becomes an ill-posed problem calling for suitable regularization techniques involving constraints on the time-bandwidth product. The design of constant modulus waveforms can shed valuable insight in tackling this important problem This is addressed in the next two sections.

NARROWBAND RADAR AMBIGUITY FUNCTION

Let $\mathbb{R}$, resp., $\mathbb{C}$, be the set of real, resp., complex, numbers. A function $u : \mathbb{R} \to \mathbb{C}$ is a finite energy signal if

$$\|u\|_2 = \left(\int_{\mathbb{R}} |u(s)|^2 ds \right)^{\frac{1}{2}} < \infty.$$

In this case, we write $u \in L^2(\mathbb{R})$. (For this article, it is not necessary to be concerned with the definition of the Lebesgue integral.) The Fourier transform, $\widehat{u} : \mathbb{R} \to \mathbb{C}$, of u can be well defined by the formal expression

$$\widehat{u}(\gamma) = \int_{\mathbb{R}} u(t) e^{-2\pi i t \gamma} dt, \;\; \gamma \in \mathbb{R},$$

where i denotes the imaginary unit in $\mathbb{C}$. For an explanation of this definition of the Fourier transform and the subtleties that can surround it see, for example, [4].

Let $\mathbb{R}^2$ be the direct product $\mathbb{R} \times \mathbb{R}$. The narrowband radar ambiguity function $A(u)$ of $u \in L^2(\mathbb{R})$ is defined by

$$\begin{aligned} A(u)(t, \gamma) &= \int_{\mathbb{R}} u(s+t)\overline{u(s)} e^{-2\pi i s \gamma} ds \\ &= e^{\pi i t \gamma} \int_{\mathbb{R}} u\left(s + \frac{t}{2}\right) \overline{u\left(s - \frac{t}{2}\right)} e^{-2\pi i s \gamma} ds, \end{aligned} \tag{1}$$

for $(t, \gamma) \in \mathbb{R}^2$. For simplicity, we refer to $A(u)$ as the ambiguity function of the signal u.

An elementary form of the uncertainty principle mentioned in the section "Motivation," apropos the ambiguity function, is the formula,

$$\iint_{\mathbb{R}^2} |A(u)(t, \gamma)|^2 dt d\gamma = \|u\|_2^4, \tag{2}$$

where $u \in L^2(\mathbb{R})$. Equation 2 asserts that $A(u)$ can not be concentrated arbitrarily close to the origin $(0, 0) \in \mathbb{R}^2$. A more refined form of (2), which is called the radar uncertainty principle, is the following assertion: If $\|u\|_2 = 1$, and if $X \subseteq \mathbb{R}^2$ and $\epsilon > 0$ have the property that

$$\iint_X |A(u)(t, \gamma)|^2 dt d\gamma \geq 1 - \epsilon,$$

then $|X| \geq 1 - \epsilon$, where $|X|$ is the area (Lebesgue measure) of X.

It is not difficult to prove that $A(u)$ is uniformly continuous on $\mathbb{R}^2$ *if* $u \in L^2(\mathbb{R})$.

Typically, in radar, the modulus $|A(u)(t, \gamma)|$ is the only quantity that is, or can be, measured. In 1968, Vakman [55] posed the narrowband radar ambiguity problem: For a given $u \in L^2(\mathbb{R})$, find all signals $v \in L^2(\mathbb{R})$ with the property that

$$|A(v)(t, \gamma)| = |A(u)(t, \gamma)| \tag{3}$$

for all $(t, \gamma) \in \mathbb{R}^2$. The signal $v(s) = c e^{i\omega s} u(s + x)$, for fixed $(x, \omega) \in \mathbb{R}^2$ and constant $c \in \mathbb{C}$ for which $|c| = 1$, is a solution of (3). However, the problem is not completely resolved, see [34] for recent results.

REMARK 1

There are applications in seismology and sonar, as well as other disciplines, where narrowband approximations, which give rise to (1), are not valid. This leads to the wideband ambiguity function $WA(u)$ of $u \in L^2(\mathbb{R})$, which is defined by

$$WA(u)(a, t) = \sqrt{a} \int_{\mathbb{R}} u(a(s - t))\overline{u(s)} ds, \tag{4}$$

for $a > 0$ and $t \in \mathbb{R}$, see [1] for its formulation. $WA(u)$ is a continuous wavelet transform of $u \in L^2(\mathbb{R})$. In particular, in the wideband case, $WA(u)$ is a time-scale operator, whereas the narrowband ambiguity function, $A(u)$, is a time-frequency operator. There is a corresponding wideband radar ambiguity problem that is at least as intractable as the narrowband case.

REMARK 2

The Wigner distribution $W(u)$ of $u \in L^2(\mathbb{R})$ was introduced by E. Wigner in 1932 in the context of quantum mechanics. It is defined by

$$W(u)(t, \gamma) = \int_{\mathbb{R}} u\left(t + \frac{s}{2}\right) \overline{u\left(t - \frac{s}{2}\right)} e^{-2\pi i s \gamma} ds,$$

for $(t, \gamma) \in \mathbb{R}^2$. It is a remarkable fact that, up to a rotation, $W(u)$ is the two-dimensional (2-D) Fourier transform of $A(u)$, e.g., see [28].

REMARK 3

If $u, v \in L^2(\mathbb{R})$, the narrowband cross-ambiguity function $A(u, v)$ of u and v is defined by

$$\begin{aligned} A(u, v)(t, \gamma) &= \int_{\mathbb{R}} u(s + t)\overline{v(s)} e^{-2\pi i s \gamma} ds \\ &= e^{2\pi i t \gamma} \int_{\mathbb{R}} u(s)\overline{v(s - t)} e^{-2\pi i s \gamma} ds. \end{aligned}$$

Thus, $A(u, v)$ is the short-time Fourier transform (STFT) of $u \in L^2(\mathbb{R})$ with window v. STFTs are a staple in spectral analysis where one analyzes spectrograms $|A(u, v)|$ recorded from various experiments, e.g., in speech analysis. They are also the basis for time-frequency or Gabor or Weyl-Heisenberg analysis.

It is not difficult to verify the time-frequency analysis equation,

$$A(u, v)(t, \gamma) = e^{2\pi i t \gamma} A(\hat{u}, \hat{v})(\gamma, -t),$$

for $u, v \in L^2(\mathbb{R})$ and $(t, \gamma) \in \mathbb{R}^2$.

DISCRETE AMBIGUITY FUNCTIONS

Let $\mathbb{Z}$ be the set of integers, let $N \geq 1$ be an integer, and denote the additive group of integers modulo N by $\mathbb{Z}_N$. For example, if $N = 6$, then $2 + 3 \equiv 5 \pmod 6$ and $4 + 5 \equiv 3 \pmod 6$.

Let $u = \{u[m] \ : \ m = 0, 1, \ldots, N-1\}$ be a sequence of complex numbers of length N. We shall denote by u_p the periodic extension of u to a bi-infinite sequence on all of $\mathbb{Z}$, i.e., for each $m \in \mathbb{Z}$, $u_p[m] = u[k]$, where $0 \leq k \leq N-1$ and $k \equiv m \,(\mathrm{mod} N)$. We shall denote by u_a the following aperiodic extension of u to $\mathbb{Z}$:

$$u_a[m] = \begin{cases} u[m], & m = 0, 1, \ldots, N-1, \\ 0, & \text{otherwise.} \end{cases}$$

- The discrete periodic ambiguity function $A_p(u) : \mathbb{Z}_N \times \mathbb{Z}_N \to \mathbb{C}$ of u is defined by

$$A_p(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u_p[m+k]\overline{u_p[k]}e^{-2\pi ikn/N}.$$

- The discrete aperiodic ambiguity function $A_a(u) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ of u is defined by

$$A_a(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} u_a[m+k]\overline{u_a[k]}e^{-2\pi ikn/N}.$$

Note that if $0 \leq m \leq N-1$ and $n \in \mathbb{Z}$, then

$$A_a(u)(m, n) = \frac{1}{N} \sum_{k=0}^{N-1-m} u[m+k]\overline{u[k]}e^{-2\pi ikn/N}.$$

EXAMPLE 1 (SHAPIRO-RUDIN POLYNOMIALS AND GOLAY PAIRS)

The Shapiro-Rudin polynomials, P_n, Q_n, $n = 0, 1, \ldots,$ are defined recursively as follows for $t \in \mathbb{R}/\mathbb{Z}$:

$$\begin{aligned} P_0(t) &= Q_0(t) = 1 \\ P_{n+1}(t) &= P_n(t) + e^{2\pi i 2^n t} Q_n(t), \\ Q_{n+1}(t) &= P_n(t) - e^{2\pi i 2^n t} Q_n(t). \end{aligned}$$

Two finite sequences, p and q, of length N, are a Golay complementary pair if

$$\begin{aligned} A_a(p)(0,0) + A_a(q)(0,0) &\neq 0, \quad \text{and} \\ A_a(p)(m,0) + A_a(q)(m,0) &= 0 \end{aligned}$$

for $1 \leq m \leq N-1$. It is interesting and not difficult to prove that, for each $n \geq 1$, the $N = 2^n$ coefficients for each of P_n and Q_n combine to form a Golay complementary pair.

Note that the Shapiro–Rudin polynomials obey a conjugate mirror filter pair condition, $|P_n(t)|^2 + |Q_n(t)|^2 = 2^{n+1}$, and thus, one can develop a theory of radar signal processing in a filter bank setting. In light of this, matched filter processing reflects the application of single channel inversion techniques.

CAZAC SEQUENCES

HISTORICAL BACKGROUND FROM ENGINEERING

A function $u : \mathbb{Z}_N \to \mathbb{C}$ is a CAZAC sequence if

$$\text{(CA)} \qquad |u[m]| = 1, \quad 0 \leq m \leq N-1,$$

and

$$\text{(ZAC)} \qquad A_p(u)(m, 0) = 0, \quad 1 \leq m \leq N-1.$$

There is extensive literature on CAZACs because of the importance of such sequences in communications, coding theory, cryptology, and radar, e.g., [15], [19], [20], [24], [30], [31], [35], [36], [39], [40], [42], [46], [48], and [54].

CAZAC sequences (and some of their close relatives) are also referred to as (among other names): polyphase sequences with good periodic or optimum correlation properties, e.g., [15], [19], [30], [38], [49], and [50]; constant amplitude optimal sequences, e.g. [44]; perfect autocorrelation or root-of-unity sequences, e.g., [21], [23], and [43]; generalized chirp-like polyphase sequences, e.g., [45]; bi-unimodular sequences, e.g., [9] and [10]; bent functions, e.g., [18] and [16]; and constant amplitude allpass sequences. As mentioned earlier, the literature in this area is extensive, and possibly overwhelming, see [31].

EXAMPLE 2 (QUADRATIC PHASE CAZAC SEQUENCES)

A quadratic phase CAZAC sequence $u : \mathbb{Z}_N \to \mathbb{C}$ is defined by

$$u[k] = e^{\pi i P(k)/N}, \quad 0 \leq k \leq N-1,$$

where $P(k)$ is a quadratic polynomial in k. The following are specific examples from the aforementioned literature:

- Chu sequences: $P(k) = k(k-1)$, N odd
- P4 sequences: $P(k) = k(k-N)$

The set of quadratic phase sequences is actually much richer than the classical nickname, chirps, implies. We shall quantify this assertion in the section "The Discrete Periodic Ambiguity Function of Wiener CAZAC Sequences," but set the stage with the following definition.

DEFINITION 1

Given $N > 1$, and let $M = N$ if N is odd, $M = 2N$ if N is even. Assume that ω is a primitive M-th root of unity, i.e., $\omega = e^{2\pi ik/M}$, where k and M are relatively prime, denoted by $(k, M) = 1$. The function $u : \mathbb{Z}_N \to \mathbb{C}$ defined by

$$u[m] = \omega^{m^2}, \quad 0 \leq m \leq N-1,$$

is a Wiener sequence, see [7].

It is not difficult to prove that Wiener sequences are CAZAC sequences.

The discrete Fourier transform (DFT) of a function $u : \mathbb{Z}_N \to \mathbb{C}$ is defined by

$$\widehat{u}[n] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} u[m]e^{-2\pi imn/N}.$$

The following facts are elementary to verify:

- u CA $\Longleftrightarrow$ DFT of u is ZAC.
- u CAZAC $\Longleftrightarrow$ DFT of u is CAZAC.
- If u is CA, then $\widehat{u}$ can have zeros.

MATHEMATICAL PROBLEMS

We have defined CAZAC sequences in terms of functions $u : \mathbb{Z}_N \to \mathbb{C}$. Of course, such a function can be identified with a point $(u[0], \dots, u[N-1]) \in \mathbb{C}^N$. Thus, the set of all CAZAC sequences defines a subset of $\mathbb{C}^N$. The periodicity inherent in $\mathbb{Z}_N$ and the fact that $\mathbb{C}^N$ consists of N-tuples leads us to consider circulant matrices, i.e., $N \times N$ matrices A_u whose first row is $u[0], \dots, u[N-1]$, whose second row is $u[N-1], u[0], \dots, u[N-2]$, etc., where each $u[m] \in \mathbb{C}$. A circulant matrix H_u is called a Hadamard matrix if $H_u H_u^\star = N\,Id$, where $H_u^\star$ is the Hermitian conjugate of H_u and Id is the $N \times N$ identity matrix.

We begin this section with the following well-known fact.

PROPOSITION 1

A function $u : \mathbb{Z}_N \to \mathbb{C}$ is a CAZAC sequence if and only if A_u is a Hadamard matrix.

We shall say that two CAZAC sequences $u, v : \mathbb{Z}_N \to \mathbb{C}$ are equivalent if $v = cu$ for some $c \in \mathbb{C}$ for which $|c| = 1$. There are other equivalence relations one can impose on the category of CAZAC sequences from which one can pose problems related to natural nonabelian groups that arise, see [7]. For example,

- shifts: for every $n = 0, \dots, N-1$ and $m \in \mathbb{Z}$,

$$u[n] = v[m+n];$$

- cyclic permutations: for k with $(k, N) = 1$,

$$u[n] = v[kn];$$

- multiplication by powers of N-th roots of unity: for ω with $\omega^N = 1$,

$$u[n] = v[n]\omega^n.$$

In all of these cases, u is a CAZAC sequence if v is also CAZAC.

There is the following compelling problem: For a given N, compute or estimate the number of nonequivalent CAZAC sequences. The problem has been investigated by Gabidulin [21], [22]. Björck and Saffari [10] proved that *if* $N = MK^2$, then there are infinitely many nonequivalent CAZAC sequences, e.g., $N = 4$, 8, 9, or 12. On the other hand, Haagerup [29] has given a complete mathematical proof that if N is prime, then there are only finitely many nonequivalent CAZAC sequences.

REMARK 4

Haagerup's theorem, as well as Gabidulin's work, uses a refinement of Chebotorev's theorem (1924) that if p is prime, then all square submatrices of the $p \times p$ DFT matrix are nonsingular. Such refinements, with intermediate contributions by Dieudonné, as well as Donoho and Stark, and a recent formulation by Tao, are critical in compressive sampling and especially the theory of Candès, Romberg, and Tao, see [51].

EXAMPLE 3

- Let N be odd. Binary CAZAC waveforms $u : \mathbb{Z}_N \to \{\pm 1\}$ can not exist.
- Let N be arbitrary. It is well known that if $u : \mathbb{Z}_N \to \{\pm 1\}$, then $A(u)(m,0) \equiv N \pmod 4$. Thus, if $A(u)(m,0)$ has zeros, and, in particular, if u is a CAZAC sequence, then four divides N.
- Let N be arbitrary. The only known binary CAZAC sequence $u : \mathbb{Z}_N \to \{\pm 1\}$, up to any translation and multiplication by -1, is $\{1, 1, 1, -1\}$. However, there do exist periodic complex binary sequences, not ± 1, which are CAZAC sequences, e.g., Björck [9] and Golomb (1992) [23], cf., [29] (see the section "Björck Sequences"). In fact, Saffari [47] was able to determine all such complex binary sequences.

THE DISCRETE PERIODIC AMBIGUITY FUNCTION OF WIENER CAZAC SEQUENCES

We begin this section with the following fact:

THEOREM 1

Let $j \in \mathbb{Z}$. Define $u_j : \mathbb{Z}_N \to \mathbb{C}$ by $u_j(k) = e^{2\pi i jk^2/M}$, where $M = 2N$ if N is even and $M = N$ if N is odd. If N is even, then

$$A(u_j)(m,n) = \begin{cases} e^{2\pi i jm^2/(2N)}, & jm + n \equiv 0 \pmod{N}, \\ 0, & \text{otherwise}. \end{cases}$$

If N is odd, then

$$A(u_j)(m,n) = \begin{cases} e^{2\pi i jm^2/N}, & 2jm + n \equiv 0 \pmod{N}, \\ 0, & \text{otherwise}. \end{cases}$$

The discrete periodic ambiguity function, $A(u)$, of a Wiener CAZAC sequence $u : \mathbb{Z}_N \to \mathbb{C}$ has a simple behavior, since for any fixed value of n, $A(u)(m,n)$ is zero for all except one value of m. That is, for each fixed n, the graph of $A(u)(\cdot, n)$ as a function of m consists of a single peak. In fact, we have the following consequence of Theorem 1.

COROLLARY 1

Let u be a Wiener CAZAC sequence.
If N is even, then

$$A(u)(m,n) = \begin{cases} \omega^{m^2}, & m \equiv -n \pmod{N}, \\ 0, & \text{otherwise}. \end{cases}$$

If N is odd, then

$$A(u)(m,n) = \begin{cases} \omega^{m^2}, & m \equiv -n(N+1)/2 \pmod{N}, \\ 0, & \text{otherwise}. \end{cases}$$

EXAMPLE 4:

- Let N be odd and let $\omega = e^{2\pi i/N}$. Then, $u(k) = \omega^{k^2}$, $0 \le k \le N-1$, is a CAZAC sequence. By Corollary 1, $|A(u)(m,n)| = |\omega^{m^2}| = 1$ if $2m+n = \ell_{m,n}N$ for some $\ell_{m,n} \in \mathbb{Z}$ and $|A(u)(m,n)| = 0$ otherwise, i.e., $A(u)(m,n) = 0$ on $\mathbb{Z}_N \times \mathbb{Z}_N$ unless $2m+n \equiv 0\ (\mathrm{mod} N)$. In the case $2m+n = \ell_{m,n}N$ for some $\ell_{m,n} \in \mathbb{Z}$, we have the following phenomenon. If $0 \le m \le (N-1/2)$ and $2m+n = \ell_{m,n}N$ for some $\ell_{m,n} \in \mathbb{Z}$, then n is odd; and if $(N+1/2) \le m \le N-1$ and $2m+n = \ell_{m,n}N$ for some $\ell_{m,n} \in \mathbb{Z}$, then n is even. Thus, the values (m,n) in the domain of the discrete periodic ambiguity function $A(u)$, for which $A(u)(m,n) = 0$, appear as two parallel discrete lines. The line whose domain is $0 \le m \le (N-1/2)$ has odd function values n; and the line whose domain is $(N+1/2) \le m \le N-1$ has even function values n.
- The behavior observed in part *a* has extensions for primitive and nonprimitive roots of unity.

Let $u : \mathbb{Z}_N \to \mathbb{C}$ be a Wiener sequence. Thus, $u[k] = \omega^{k^2}$, $0 \le k \le N-1$, and $\omega = e^{2\pi ij/M}$, $(j, M) = 1$, where M is defined in terms of N in Definition 1. By Corollary 1, for each fixed $n \in \mathbb{Z}_N$, the function $A(u)(\cdot, n)$ of m vanishes everywhere except for a unique value $m_n \in \mathbb{Z}_N$ for which $|A(u)(m_n, n)| = 1$. See Figure 1, where $N = 101$ and $j = 49, 4$, respectively.

- The hypotheses of Theorem 1 do not assume that $e^{2\pi ij/M}$ is a primitive Mth root of unity. In fact, in the case that $e^{2\pi ij/M}$ is not primitive, then, for certain values of n, $A(u)(\cdot, n)$ will be identically 0 and, for certain values of n, $|A(u)(\cdot, n)| = 1$ will have several solutions. For example, if $N = 100$ and $j = 2$, then, for each odd n, $A(u)(\cdot, n) = 0$ as a function of m. If $N = 100$ and $j = 3$, then $(100, 3) = 1$ so that $e^{2\pi i3/100}$ is a primitive 100th root of unity; and, in this case, for each $n \in \mathbb{Z}_N$ there is a unique $m_n \in \mathbb{Z}_N$ such that $|A(u)(m_n, n)| = 1$ and $A(u)(m, n) = 0$ for each $m \ne m_n$.

BJÖRCK SEQUENCES

We shall now define a class of CAZAC sequences that are not equivalent to Wiener CAZAC sequences. This result came as a surprise in several fields, and proves false the conjectures of Popa and Enflo, see [29]. This subsection presents a brief summary of Björck's results.

Let $N = p$ be prime. The Legendre symbol $(\frac{k}{p})$ is defined by

$$\left(\frac{k}{p}\right) = \begin{cases} 0, & \text{if } k \equiv 0\ (\mathrm{mod} p), \\ 1, & \text{if } k \equiv n^2\ (\mathrm{mod} p) \quad \text{for some } n \in \mathbb{Z}, n \not\equiv 0\ (\mathrm{mod} p) \\ -1, & \text{if } k \not\equiv n^2\ (\mathrm{mod} p) \quad \text{for all } n \in \mathbb{Z}. \end{cases}$$

EXAMPLE 5

Let $p = 7$, noting that $7 \equiv -1\ (\mathrm{mod}\ 4)$. Then $(\frac{k}{7}) = -1$ for $k = 3, 5, 6$. Björck made the remarkable discovery that $u : \mathbb{Z}_7 \to \mathbb{C}$, defined by

$$u[0] = u[1] = u[2] = u[4] = 1,$$
$$u[3] = u[5] = u[6] = e^{i\theta},$$

where $\theta = \arccos(-3/4)$, is a CAZAC sequence!

DEFINITION 2

Given $N = p$, a prime. The function $u : \mathbb{Z}_p \to \mathbb{C}$ defined by

$$u[m] = e^{i\theta_p[m]}, \quad 0 \le m \le p-1,$$

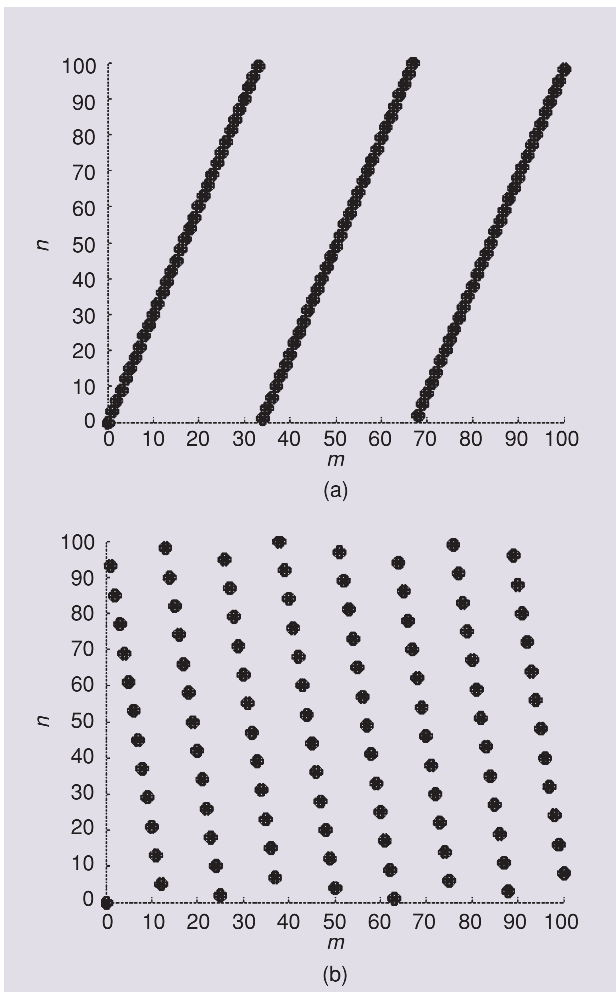
is a Björck sequence if,

- for $p \equiv 1\ (\mathrm{mod} 4)$, we have

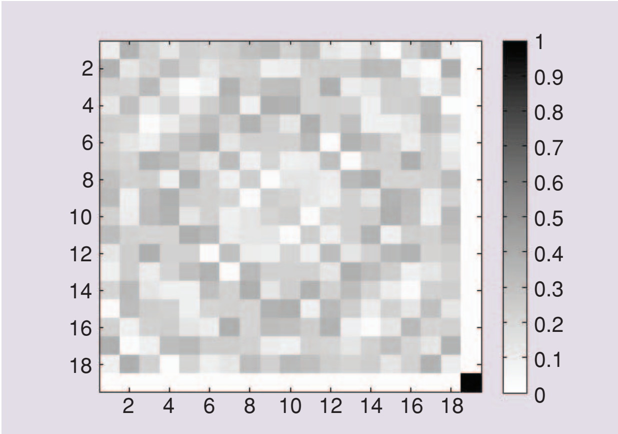
$$\theta_p[m] = \left(\frac{m}{p}\right) \arccos\left(\frac{1}{1+\sqrt{p}}\right),$$

- for $p \equiv -1\ (\mathrm{mod} 4)$, we have

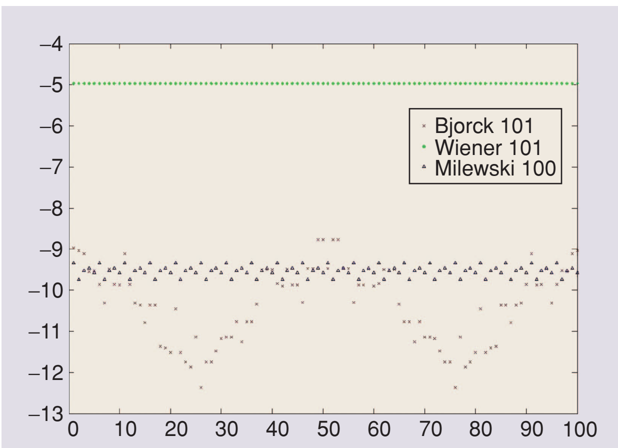
$$\theta_p[m] = \begin{cases} \arccos\left(\frac{1-p}{1+p}\right), & \text{if } \left(\frac{m}{p}\right) = -1, \\ 0, & \text{otherwise.} \end{cases}$$



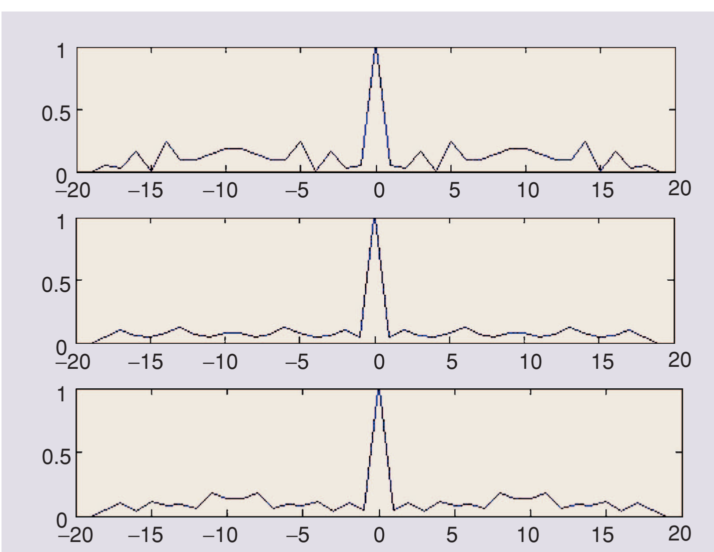
[FIG1] Discrete periodic ambiguity plot for Wiener sequences of length $N = 101$ and (a) $j = 49$ and (b) $j = 4$.



[FIG2] Discrete periodic ambiguity plot (in dB) for a Björck sequence of length $N = 19$.



[FIG3] Effect of cyclic shifts τ_m on the discrete aperiodic autocorrelation $A_a(u)(\cdot, 0)$ of different equivalence classes of CAZAC sequences; PSL(u) (in dB) versus shift length m. Red x: Björck ($N = 101$), Blue triangles: Milewski ($N = 100$), Green squares: Wiener ($N = 101$).



[FIG4] Discrete aperiodic autocorrelation for Björck sequences of length $N = 19$. Three shifts τ_m, with $m = 0, 4, 9$, respectively.

Björck (1985) went on to prove the following theorem, see [9], [10], [47].

THEOREM 2

Björck sequences are CAZAC sequences.

REMARK 5

Two-valued CAZACs have been classified by Saffari [47] in terms of Hadamard-Paley and Hadamard-Menon difference sets. The result of Saffari states that two-valued CAZACs exist for lengths $N \geq 3$ if and only if a) $N \equiv 3 \pmod 4$ and there exists a Hadamard-Paley difference set of length N, or b) $N \equiv 0 \pmod 4$ and there exists a Hadamard-Menon difference set of length N. In either case, explicit formulas are provided for the construction of the CAZAC sequence. A similar classification exists for two-level autocorrelation Legendre sequences [24], with several results relating to the existence of Hadamard-Paley difference sets. Existence of Hadamard-Menon sets is rather more difficult to prove, as it relates to the Hadamard circulant conjecture. It follows that two-valued CAZACs cannot exist for lengths $N \equiv 1 \pmod 4$. However, note that in this case Björck CAZAC sequences are almost two-valued.

APERIODIC SIMULATIONS

CAZAC sequences are important in waveform design because of their defining properties: CA ensures optimal transmission efficiency while ZAC provides tight time localization at zero Doppler. In the section "CAZAC Sequences," we commented on their characterization in terms of the discrete periodic ambiguity function A_p. By definition, all CAZAC sequences have the same periodic autocorrelation $A_p(u)(\cdot, 0)$, but we have shown that they exhibit significant diversity of behavior once the full time-frequency landscape is considered as in Figure 2. Periodic autocorrelation is also referred to as cyclic autocorrelation, which permits computation by loading finite sequences into a circular register and rotating the register.

In the aperiodic case, as expected, ZAC is unattainable, and sidelobes appear. In order to quantify this behavior, we utilize two measures of sidelobe magnitude, that capture various aspects of the aperiodic autocorrelation $A_a(u)(\cdot, 0)$

DEFINITION 3

The peak sidelobe level (PSL) of a sequence u is defined by

$$\mathrm{PSL}(u) = \frac{1}{|A_a(u)(0,0)|} \sup_{1 \leq m \leq N-1} |A_a(u)(m,0)|.$$

The integrated sidelobe level (ISL) of a sequence u is defined by

$$\mathrm{ISL}(u) = \frac{1}{|A_a(u)(0,0)|^2} \sum_{m=1}^{N-1} |A_a(u)(m,0)|^2.$$

Both the PSL and the ISL differ within the equivalence classes of CAZAC sequences, and across classes.

This is analogous to results obtained for Legendre sequences [12], [13]. Figure 3 illustrates this effect on PSL with respect to shifts τ_m, where $(\tau_m u)[n] = u[m+n]$. Note that Wiener CAZACs are not affected at all, whereas Björck sequences exhibit the most variability. Similar results arise when one considers other measures, such as ISL.

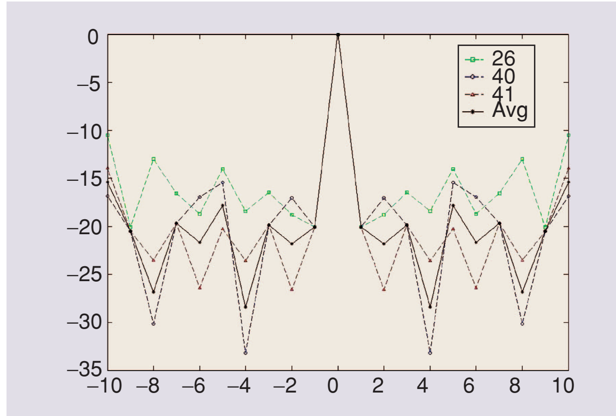
The variability in sidelobe behavior for Björck CAZACs is not limited to their energy levels. As Figure 4 shows, shifting also affects sidelobe location.

EXAMPLE 6

By averaging the autocorrelation $A_a(\tau_m u)(\cdot, 0)$ of several different shifts τ_m of the same sequence u, we can obtain an overall lowering of the energy levels off of the main lobe. In this example, we present the noncoherent average over two selected shifts, i.e., $|A_a(\tau_{40} u)(\cdot, 0)| + |A_a(\tau_{41} u)(\cdot, 0)|$, (see Figure 5). Note that we can improve over the shift that achieves the lowest PSL globally (shift by 28).

EXAMPLE 7

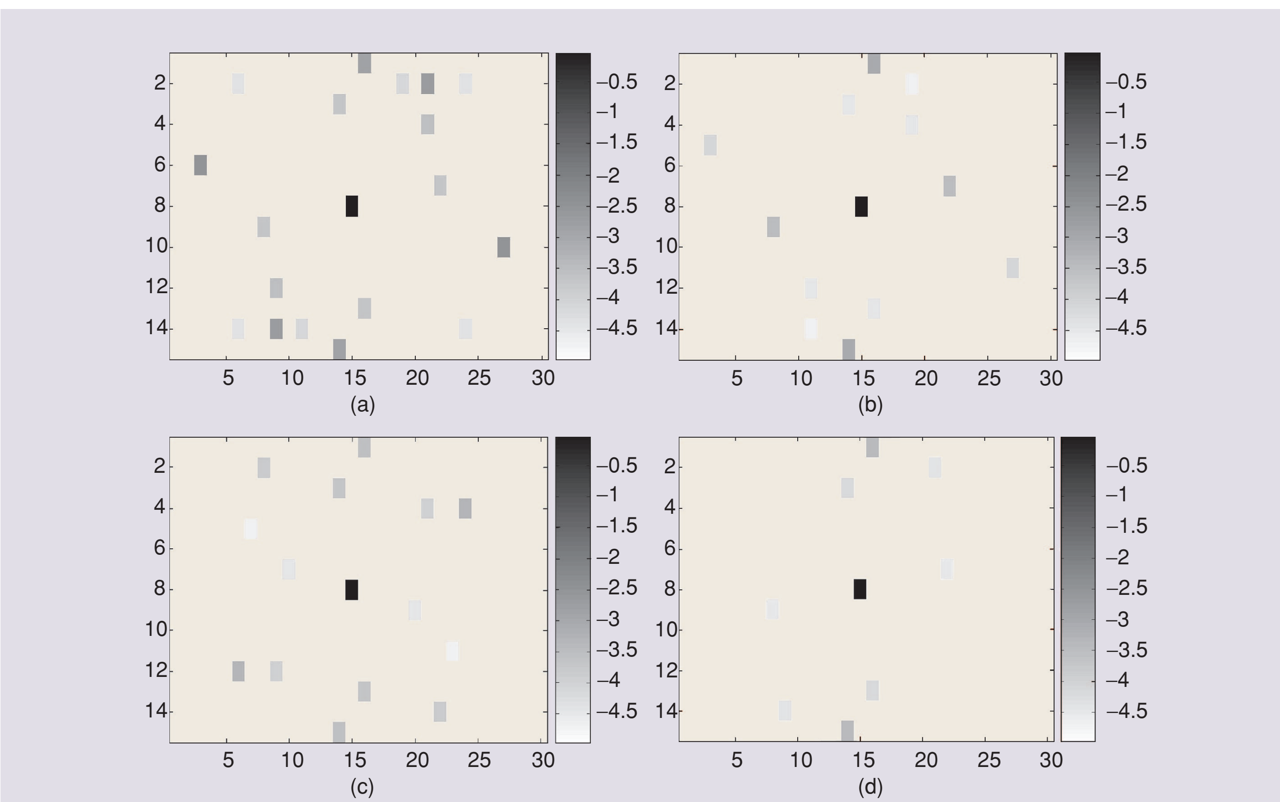
The same averaging technique can be used with the discrete aperiodic ambiguity functions of shifted sequences $A_a(\tau_m u)$. Figure 6(a–c) shows plots of the discrete aperiodic ambiguity functions for shifts τ_m of the Björck CAZAC of length 29 ($m = 0, 7, 12$), with a threshold of -10 dB. The area plotted is centered at the origin. We observe that the peak locations vary, and, by averaging all three, we obtain Figure 6(d).



[FIG5] Effect of cyclic shifts on the aperiodic autocorrelation function of Björck CAZAC sequences; plot of absolute value (dB) in the area of the main lobe for two different shifts (by 40 and 41), and their average. Also plotted for reference is the shift that achieves the lowest PSL globally.

The previous examples illustrate the following facts:

- Waveforms associated to Chu-Zadoff and P4 CAZACs are known for their low sidelobes at zero Doppler shift, but their ambiguity functions exhibit strong coupling in the time-frequency plane.
- Waveforms associated to Björck CAZACs can more effectively decouple the effect of time and frequency shifts. However, at zero Doppler shift, their sidelobe behavior is less desirable than quadratic phase CAZACs.



[FIG6] Plot of the discrete aperiodic ambiguity function of shifts of the Björck CAZAC of length 29, thresholded at -10 dB; darker color denotes higher value. (a) Zero shift, (b) shift by seven (c) shift by 12, and (d) their average.

These differences led to our concatenation idea. A concatenation of partial CAZACs u and v is $w = Mix(r\%, u, v)$ defined as

$$w[m] = u[m], \text{ if } m = 0, \ldots, M$$

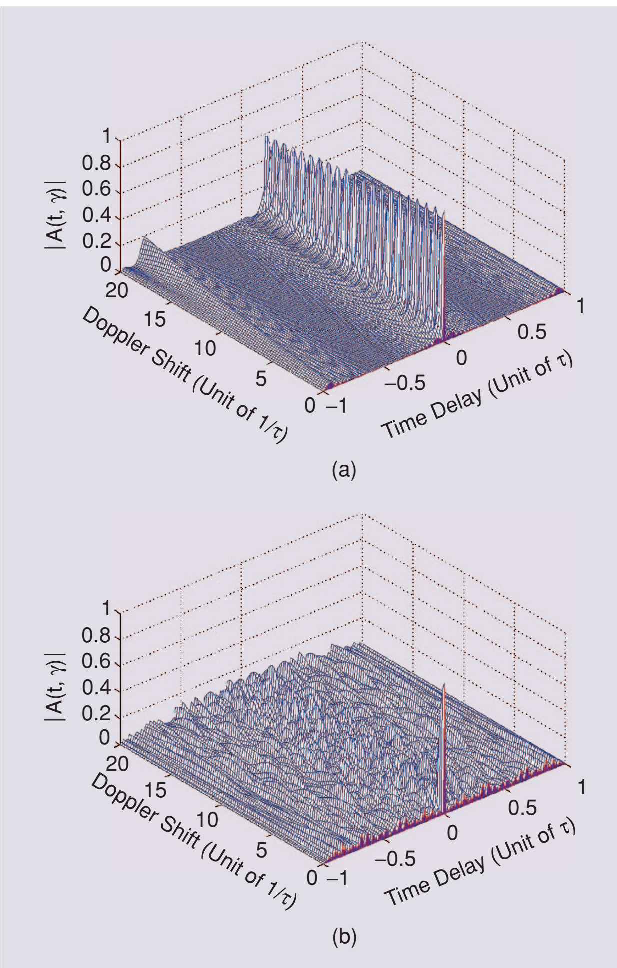
and

$$w[m] = v[m], \text{ if } m = M+1, \ldots, N-1,$$

where M is the nearest integer to $r \times N/100$. It can be shown that the ambiguity function of a concatenation of partial CAZACs exhibits improved PSL and ISL.

EPILOGUE

The purpose of this article was to observe the complexity of CAZAC sequences, resulting from diverse periodic and aperiodic ambiguity behavior (see Figure 7). Although we restricted ourselves to two special cases, viz., Wiener and Björck sequences, we ultimately want to fathom the depth of all nonequivalent CAZAC sequences. The supporting mathematical theory must be developed significantly to achieve this goal, and the engineering criteria, particularly in communications, coding, and radar, must interleave with the mathematics to ensure relevance and focus.



[FIG7] Ambiguity plot of a waveform that is phase coded with (a) Chu-Zadoff 101 and (b) Björck 101.

Even with these mathematical and engineering challenges to understanding and quantifying the ambiguity function diversity rooted in the mysteries surrounding nonequivalent CAZAC sequences $u : \mathbb{Z}_N \to \mathbb{C}$, there is also the next level of generalization that must be addressed. In fact, one can envision a theory of ambiguity vector fields to deal with multidimensional and vector-valued problems, e.g., in light of vector sensor and MIMO settings and capabilities. Some contemporary works [7], [8], and [41] illustrate the role of the theory of frames for ambiguity or Wigner distribution analysis, fitting in with its use in a host of recent applications, e.g., [5] and [37].

With regard to frames, especially finite unit norm tight frames, they have arisen in dealing with the robust transmission of data over erasure channels such as the internet [14], [27], [33], and in both multiple antenna code design for wireless communications [32] as well as multiple description coding [25], [26], [52]. There are also recent applications of FUN-TFs in quantum detection, Σ-Δ quantization, and Grassmanian min-max waveforms, e.g., [6]. Frames give redundant signal representation to compensate for machine imperfections, to ensure numerical stability, and to minimize the effects of noise.

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