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ON OPTIMIZATION OF INCOHERENT SCATTER MEASUREMENTS

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ABSTRACT

The design of incoherent scatter measurements, as well as other radar measurements is constrained by technical factors, the most important of which are radar power and duty cycle. If these are fixed, it is possible to improve detection accuracy by using different kinds of coding methods. We study a situation where radar backscatter autocorrelation functions (ACF) of a continuously distributed medium are to be measured with a specified radar range and autocorrelation lag resolution. As the measurement is disturbed by thermal noise, it is then shown that the accuracy of the resulting ACF estimates cannot be better than a certain limit, whatever kinds of coding schemes are used. The accuracies of alternating codes are compared to this limit for a range of pulse lengths used. We suppose that the noise power due to thermal fluctuations in the system and background noise is higher than the signal power itself. The proof of the limiting accuracy is based on linear statistical inversion theory.

INTRODUCTION

Incoherent scatter radar measurements of different parts of the ionosphere pose a large variety of radar measurement problems with varying resolutions and measurement range extents depending on the geophysical phenomena studied. Because the different objectives such as good range and time resolution and good parameter estimation accuracy are contradictory, there is no general method good enough for all possible situations. The incoherent scattering signal is very weak due to the general nature of the method itself and often the measurement can only be done by specially designed radar codes optimized for a specific type of experiment.

In this paper we discuss the design and optimization of incoherent scatter measurements in general. In particular, we elaborate on the situation encountered in the F-region measurements, where a solution close to the theoretical optimal limit is known for a rather large variation of possible resolution requirements.

We will not discuss the design of different measurement geometries, that is the directions of the antenna or antennas and possible sweeps of antennas to different directions. We suppose that the antenna is pointing in a given direction and discuss the design of the experiment so that it will give the backscatter ACFs or spectra with specified resolutions in range and ACF lag or spectral space.

FIRST PHASE: GEOPHYSICAL SPECIFICATION

To specify the resolutions of the radar experiment may require special knowledge of the measurement process itself and thus it may be good to divide the specification task into two parts. The duty of the geophysicist should be to consider the situation from the geophysical point of view without putting too much attention to the method of measurement. The geophysicist should be able to specify at least the following points:

1) *Range resolution*: The size of the smallest details of the ionosphere as a function of radar range the experimenter thinks are interesting.

2) *Range extent*: The interval between the closest and longest radar ranges that are interesting in the experiment.

3) *Interesting plasma parameters and their anticipated values*: The plasma parameters that the experimenter thinks have effect on the backscatter spectra. In addition to the parameters themselves it is most important to have some idea about their possible values to be able to choose proper spectral resolutions.

4) *Time resolution of the experiment*: The anticipated time scale in which changes in the ionosphere occur.

SECOND PHASE: SPECIFICATION OF THE RADAR RESOLUTIONS

When the information described in the previous section is available, it is the task of the radar specialist to specify more accurately the experimental resolutions required to achieve the goals set. While many of the resolutions can be directly inferred from the geophysical requirements, the job is not, however, totally straightforward. The resolutions to be specified at this stage and how to do the specification is desctribed in the following paragraphs.

1) *Range resolution*: Range resolution is essentially determined by point 1) in the previous chapter. There is, however, an additional factor that should be taken into account and that is the fact that if the measurement resolution volumes are large and there are gradients in the plasma parameters, it follows that the radar estimates obtained represent averages of spectra over a range of parameter values. As the plasma spectrum is not a linear function, these averages of spectra are not equal to a spectrum corresponding to the average of the parameters and there is an additional error contributing to the fit of plasma parameters to this averaged spectrum. The size of the error depends on the variation of the plasma parameters inside the measurement volume and this way on the range resolution itself. These errors are first studied in /10/, where they are called *(parameter) mixing errors.*

2) *Range extent*: This is just point 2) from the previous chapter with perhaps a more detailed consideration of how the codes used totally or only partially overlap the interesting range.

3) *Lag resolution of the measured ACFs*: This is equivalent to specifying the frequency extent of the spectra, but the problem has been most completely handled in /9/ in the ACF space. As was shown in /9/, the lag resolution is most heavily affected by the highest anticipated temperature of the lightest ion in the plasma. Also the number and combination of the parameters to be fitted have some effect on this requirement. Practical values of the necessary lag resolution can be read from the tables of /9/ in any situation. Practical examples are also given in terms of real measurements that may be used to scale the requirements to any situation using the known way the ACF width and thus also lag resolution change when temperatures are changed.

4) *Lag extent of the measured ACFs*: This is the longest lag that is necessary to measure in order to be able to make the fits reasonably stable. Again, it is equivalent to specify spectral resolution, but in /9/ this problem is solved in the ACF framework. The lag extent is determined by the lowest anticipated temperature of the heaviest ion in the plasma, and practical results can again be found in /9/ for any situation. In the same way as the lag resolution, the lag extent requirement also depends on the number of parameters fitted because the fit becomes more and more unstable as more parameters are fitted and the extent and resolution requirements become more strict. This problem is also solved in /9/.

5) *Necessary accuracy of the measured ACF estimates*: If the ACFs are measured to certain resolution and accuracy, the parameter estimates have widely different final estimation errors depending on the combination of the fitted parameters. If it is supposed that the lag extent and range are in the stable range as specified in /9/, it is possible to use a certain normalization of the ACF estimation errors and derive error estimates for all possible combinations of parameter fits. The ACF estimates actually produced should be accurate enough to provide a reasonable accuracy in the final fit of the desired parameter combination.

6) *Maximum integration time*: Maximum integration time is essentially the same as the point 4) in the previous chapter, but like in the case of 1), it may be necessary to use even smaller integration times in the experiment than those sufficient from the geophysical point of view because of the parameter mixing errors.

OBJECT OF OPTIMIZATION

The object of optimization of measurements is to find such a radar coding method that the above determined resolution and extent requirements become fulfilled and that the variance of the resulting ACF estimates after the specified integration time is as small as possible. If the resulting variances are small enough to make reasonably accurate parameter estimates, we can be satisfied with the design and use it. If they are too large, the experiment cannot be performed or at least some of the requirements have to be relaxed. If the resulting variances are smaller than would be necessary, it may be advisable to reconsider the experiment specifications and see, whether it would be useful to make some of the requirements even more stringent. In this case it might also be possible to redesign the experiment so that it could be useful for a more general range of situations than the original intention was.

In the design work above considered it is useful to know a theoretical limit to the accuracy of a measurement given prescribed resolutions and also a coding method by which one can come close to this limit. When deriving theoretical limits like this, the design of the measurement is constrained by technical factors such as radar power, antenna gains, transmitter duty cycle, technical details of the receiver, the signal processing system and software and eventually also such factors as manpower available to write new programs and to do the analysis.

It is of course impossible to take all these factors into account in a derivation of a limiting accuracy of the measurement,.but useful results can be found by considering only the most fundamental constraints such as radar peak power and duty cycle. Practical work /11/ has shown that it is possible to design experiments where the additional constraints do not come to the way and the accuracy obtained is very close to that which would be possible with only these constraints taken into account.

THE CONDITIONS FOR THE OPTIMAL SOLUTION AND THEORETICAL LIMIT

In the following derivations we make the following assumptions:

1) The signal-to-noise ratio (SNR) of the experiment is low, so that the main contribution to the errors in the ACF estimates is the background noise component, if ionospheric clutter may be ignored. This leads to an underestimation of the best possible error bars. If the clutter component in the ACF estimation errors is significant it is not possible to realize an experiment whose errors would be very close to the theoretical limits here described. The optimal way of designing the experiment is not known in this case.

2) The characteristic time scales of the situation are in the following order:

$$range\ resolution \leq lag\ resolution < lag\ extent < modulation\ time, \tag{1}$$

or, simultaneously defining the following symbols

$$\Delta r \leq \Delta \tau < \tau_{\mathrm{ext}} < t_{\mathrm{mod}} \tag{2}.$$

The modulation time t_{mod} is the time available for sending of codes before the system is switched to receiving. The last inequality excludes cases where correlation times are so long that pulse-to-pulse correlation schemes become useful. The middlemost inequality only says that we want more than one lag values to be measured. These assumptions are always true for measurements of the F-layer plasmas at least with the wavelengths used in the EISCAT radars.

The first inequality is not necessarily true for all F-region measurement situations and it is not really needed in the proof of the theoretical accuracy limit either. It is posed here because the optimal solution candidate now known, that is the alternating codes of /3/ or /11/, does not produce short lags well if the first inequality is not true. It is not known to the author whether some other coding scheme comes as close to the theoretical accuracy as the alternating codes come in the situation above. In the extreme case, where the range resolution is very large (longer than the lag extent), ordinary long pulse codes are known to be optimal, but there may be some slack between the theoretical limits here derived and practical codes known to the author in intermediate cases, however not more than an additional factor of two or three in integration times or variances.

It would be possible to generalize the theoretical limits to the pulse-to-pulse correlation situation, but this is not considered here.

DESCRIPTION OF THE RADAR BACKSCATTER SIGNAL

The incoherent scatter signal $e(t)$ is a Gaussian random process whose autocorrelation function is given by

$$\langle e(t)\overline{e(t')}\rangle = P_{\mathrm{res}} \frac{1}{\Delta r} \int_{-\infty}^{\infty} env(t-r)\overline{env(t'-r)}\, \frac{r_0^2}{r^2} \sigma_{\mathrm{eff}}(t-t';r)dr + \kappa T \delta(t-t'), \tag{3}$$

where $\sigma_{\mathrm{eff}}(t-t';r)$ gives the plasma backscatter cross section density as a function of range r and correlation lag $t-t'$ and P_{res} is the signal power scattered back from range interval $[r_0, r_0+\Delta r]$ supposing that the transmission would be an unmodulated sine wave. It is given by

$$P_{\mathrm{res}} = A_{beam} \frac{c}{2} \Delta r \frac{N_0}{2} P_0, \tag{4}$$

where P_0 is the single electron backscatter power at range r_0 and in the middle of the beam, A_{beam} is the radar beam cross section at range r_0, and T is the receiver system noise temperature. The constants are given by

$$P_0(r) = 4\pi\rho_0^2 \sin^2\chi \frac{P_t G}{4\pi r_0^2} \frac{G}{4\pi r_0^2} \frac{\lambda^2}{4\pi} \tag{5}$$

and

$$A_{beam} = C_{beam} \frac{4\pi r_0^2}{G} . \tag{6}$$

Here P_t is the transmitter power, $4\pi\rho_0^2 = 10^{-28}m^2$ is the classical electron cross section, χ is the polarization angle, G is the antenna gain factor and C_{beam} is a numerical factor depending on the exact form of the antenna beam. It is equal to 1/2 for Gaussian beam forms, 0.460 for a beam form of a uniformly illuminated circular aperture and close to these values for any reasonable beam form (see /2/ for more details). Radar wave length is denoted by λ and env is the transmission envelope.

It is useful to note that the constants were chosen so that for $T_e/T_i < 4$

$$\sigma_{\text{eff}}(0,r) = \frac{2N_e(r)/N_0}{(1+(kD)^2)(1+(kD)^2+T_e/T_i)}. \tag{7}$$

Thus, in the case of Debye lengths much smaller than the scattering wavelengths, $kD \approx 0$, and equal ion and electron temperatures, $T_e = T_i$, we have $\sigma(0,r) = N_e(r)/N_0$. This normalization corresponds to the conventions used in /9/ so that any resulting error estimates for our σ can be directly used to make estimates of parameter estimation errors as shown in /9/. The reason for the introduction of reference scales like reference density N_0 and reference range scale Δr is to make the resulting formulae nondimensional.

Let us define lag profiles by

$$M(t,\tau) = e(t+\tau)\overline{e(t)}/P_{\text{res}} \tag{8}$$

so that, ignoring the contribution from the background noise part,

$$\langle M(t,\tau)\rangle = \int_{-\infty}^{\infty} W_{t,\tau}(r)\sigma(\tau;r)dr/\Delta r = \int_{-\infty}^{\infty} W_{0,\tau}(r-t)\sigma(\tau;r)dr/\Delta r, \tag{9}$$

where

$$\sigma(\tau,r) = \frac{r_0^2}{r^2}\sigma_{\text{eff}}(\tau,r) \tag{10}$$

and

$$W_{t,\tau}(r) = env(t-r)\overline{env(t'-r)}. \tag{11}$$

Using the fourth moments theorem for complex random processes, we can find the variances of experimental estimates of $M(t,\tau)$. First, if we define the estimation fluctuations of any random variables f by

$$\Delta f = f - \langle f\rangle, \tag{12}$$

it follows that

$$\begin{aligned} \left\langle \Delta(e(t)\overline{e(\tau)})\, \Delta(e(t')\overline{e(\tau')})\right\rangle &= \langle(e(t)\overline{e(\tau')})\rangle\, \langle(e(t')\overline{e(\tau)})\rangle \quad \text{and} \\ \left\langle \Delta(e(t)\overline{e(\tau)})\, \overline{\Delta(e(t')\overline{e(\tau')})}\right\rangle &= \langle(e(t)\overline{e(t')})\rangle\, \langle(e(\tau')\overline{e(\tau)})\rangle \end{aligned} \tag{13}$$

and after some algebra we get

$$\left\langle \mathrm{Re}\Delta(e(t)\overline{e(\tau)})\, \mathrm{Re}\Delta(e(t')\overline{e(\tau')})\right\rangle = \frac{1}{2}\mathrm{Re}\left(\langle(e(t)\overline{e(\tau')})\rangle\langle(e(t')\overline{e(\tau)})\rangle + \langle(e(t)\overline{e(t')})\rangle\langle(e(\tau')\overline{e(\tau)})\rangle\right). \tag{14}$$

with similar results for cross correlations between real and imaginary parts and also for two imaginary parts of the crossed products. Taking only second moments of the background noise part into account, it follows that for $\tau > 0$ and $\tau' > 0$,

$$\langle \Delta\mathrm{Re}M(t,\tau)\, \Delta\mathrm{Re}M(t',\tau')\rangle = \frac{1}{2}\left(\frac{\kappa T}{P_{\text{res}}}\right)^2 \delta(t-t')\delta(\tau-\tau') \tag{15}$$

with identical results when one or both of the real parts are replaced with imaginary parts. There is no need to study negative values of τ because $M(t,-\tau) = \overline{M(t-\tau,\tau)}$ exactly. The delta functions still

have the dimension of $1/seconds$ and we will get a more invariant form of the formulas if we instead of the time variables t and τ consider the dimensionless variables defined by $\tilde{t} = t/\Delta r$ and $\tilde{\tau} = \tau/\Delta r$. All the functions considered should also be defined in terms of the new scaled time variables, for example $\tilde{\sigma}(\tilde{\tau}, \tilde{r}) = \sigma(\tau\Delta r, r\Delta r)$, $\tilde{W}_{\tilde{t},\tilde{\tau}}(\tilde{r}) = W_{\tau\Delta r, r\Delta r}(r\Delta r)$ etc. If we, however, forget all the tildes and suppose that all functions are already given in terms of the scaled time variables, we get the final relations giving the expectations and variances of the signal lag profiles:

$$\langle M(t,\tau)\rangle = \int_{-\infty}^{\infty} W_{t,\tau}(r)\sigma(\tau;r)dr = \int_{-\infty}^{\infty} W_{0,\tau}(r-t)\sigma(\tau;r)dr, \tag{16}$$

and

$$\langle \Delta \mathrm{Re}M(t,\tau)\, \Delta \mathrm{Re}M(t',\tau')\rangle = \frac{1}{2}\left(\frac{\kappa T}{P_{\mathrm{res}}}\right)^2 \Delta r^{-2}\delta(t-t')\delta(\tau-\tau') = A^2\delta(t-t')\delta(\tau-\tau'). \tag{17}$$

INFORMATION CONTENT OF A SET OF RADAR MEASUREMENTS

To simplify the derivations we suppose in the following that the function $\sigma(\tau,r)$ is real as well as the ambiguity functions $W_{t,\tau}(r)$ so that we do not have to keep the real parts in the previous equation. Because the above equation is true with one or both real parts replaced by imaginary parts, it is clear that it would be possible to make the following derivations also in the general case and by a transformation to a basis in a complex plane, locally defined by the directions of the modulus and phase of the lagged profiles, it can be seen that the results here derived for purely real data will also be valid for any estimation situations concerning the modulus part of the data only. In /2/ it was shown that the plasma velocity part in the estimation problem is independent of the determination of the other parameters so that the considerations here work at least in estimation problems studied in /9/.

It is interesting to note that for each τ the measured lag profile $M(t,\tau)$ is a convolution of the unknown function $\sigma(\tau;r)$ and the ambiguity function $W_{0,\tau}(r)$. This means that if we take a Fourier transform with respect to the range variable, we will get a particularly simple result. Defining the Fourier transform of any function of range $f(r)$ as $\hat{f}(k) = \int e^{ikr} f(r)dr$, we get

$$\langle \hat{M}(k,\tau)\rangle = \hat{W}_{0,\tau}(-k)\hat{\sigma}(\tau,k) \tag{18}$$

and

$$\begin{aligned} &\langle \Delta\hat{M}(k,\tau)\overline{\Delta\hat{M}(k',\tau)}\rangle = 2\pi A^2\delta(k-k')\delta(\tau-\tau') \text{ and} \\ &\langle \Delta\hat{M}(k,\tau)\Delta\hat{M}(k',\tau)\rangle = 0 \end{aligned} \tag{19}$$

if $k \geq 0$ and $k' \geq 0$ (for negative k, $\hat{M}(-k,\tau) = \overline{\hat{M}(k,\tau)}$).

Let us suppose that N different measurements have been performed, each with possibly different modulation envelopes $env^i(t)$. Then the Fourier transforms of our data satisfy

$$\langle \hat{M}^i(k,\tau)\rangle = \hat{W}^i_{0,\tau}(-k)\hat{\sigma}(\tau,k) \tag{20}$$

and

$$\begin{aligned} &\langle \Delta\hat{M}^i(k,\tau)\overline{\Delta\hat{M}^j(k',\tau)}\rangle = 2\pi A^2\delta_{ij}\delta(k-k')\delta(\tau-\tau') \text{ and} \\ &\langle \Delta\hat{M}^i(k,\tau)\Delta\hat{M}^j(k',\tau)\rangle = 0 \end{aligned} \tag{21}$$

All different Fourier components $\hat{\sigma}(\tau,k)$ are independent of each other and we may solve the inversion problem for these components independently from each other. From linear inversion theory (see /5/, /7/, /8/ or /12/) (or by standard statistical estimation theories) it is easy to see that the center point of the *a posteriori* distribution (or the properly weighted estimator) for the unknown coefficients $\hat{\sigma}(\tau,k)$ is given by the sum

$$\sigma_{\mathrm{est}}(\tau,k) = \left(\sum_{i=1}^{N}|\hat{W}^i_{0,\tau}(k)|^2\right)^{-1}\sum_{i=1}^{N}\hat{W}^i_{0,\tau}(k)^{-1}|\hat{W}^i_{0,\tau}(k)|^2\hat{M}(k,\tau) \tag{22}$$

and the variance of this estimator is given by

$$\begin{aligned} &\langle \Delta\sigma_{\mathrm{est}}(\tau,k)\overline{\Delta\sigma_{\mathrm{est}}(\tau',k')}\rangle = \left(\sum_{i=1}^{N}|\hat{W}^i_{0,\tau}(k)|^2\right)^{-1} 2\pi A^2\delta(k-k')\delta(\tau-\tau') \text{ and} \\ &\langle \Delta\sigma_{\mathrm{est}}(\tau,k)\Delta\sigma_{\mathrm{est}}(\tau',k')\rangle = 0 \end{aligned} \tag{23}$$

The inverse of the variance coefficients

$$Q(k,\tau) = (2\pi A^2)^{-1} \sum_{i=1}^{N} |\hat{W}^i_{0,\tau}(k)|^2 \tag{24}$$

may be called the Fisher information for the Fourier coefficients. Let us now suppose that the modulation interval is $[0, t_{\mathrm{mod}}]$ so that $env^i(t) = 0$, if $t \notin [0, t_{\mathrm{mod}}]$ and $|env^i(t)| \leq 1$ in this interval. It is then easy to use Parseval's equation and see that the possible Fisher information functions must all satisfy

$$\int_{-\infty}^{\infty} Q(k,\tau)dk \leq A^{-2}N(t_{\mathrm{mod}} - \tau), \tag{25}$$

with equality true, if $|env(t)| = 1$ for all t in the modulation interval.

We may draw the following conclusions: The Fisher information function $Q(k,\tau)$ gives the inverse of the variance of the independent components of the Fourier transforms that can be inferred from the measurement. By choosing a proper set of modulation envelopes $env^i(t)$, the experiment designer may choose how well different wavelengths of the lag profiles are resolved. Due to a limited radar power and modulation time, there is however the restriction concerning the total area under the Fisher information curve as a function of k for each fixed τ. If we study two different measurement sets, with Fisher information functions $Q(k,\tau)$ and $Q'(k,\tau)$, and if $Q'(k,\tau) > Q(k,\tau)$ for some k, there must thus be another k' such that $Q'(k',\tau) < Q(k',\tau)$.

The experiment designer has rather much freedom in deciding the behaviour of $Q(\tau,k)$ through the choise of the set of transmitter modulation envelopes $env^i(t)$ used, but due to the above limitation he has to tackle with the fact that if he wants to have more information from some wavelengths of the lag profiles, he has to be satisfied with less information from other wavelengths. The distribution of the information mass as specified by Q determines the resolution of the experiment; for high resolution experiments $Q(k,\tau)$ should be a wider function of k, while for coarser resolutions it is best chosen to be tall and narrow.

It is also seen that any experiment, where $|env(t)| = 1$ in the whole modulation interval is optimal in the sense that for any other experiment, the Fisher information functions must be equal or there has to be such values of k that the Fisher information function of the second experiment is less than the information function of the first experiment. Which one of all these possible optimal choises to choose depends on which Fourier coefficients of the lag profiles are most important.

ACCURACY LIMIT FOR PRESCRIBED RESOLUTIONS

It is possible to fix a range resolution and use the results of Fourier components to derive limits of accuracy for the inferred lag profiles using methods in the Backus-Gilbert style. There is, however a more straightforward solution and it is based on modelling the lag profiles as step functions or piecewise linear functions with a certain resolution. For this, let

$$\psi(r) = 1, \text{ if } 0 < r < 1 \text{ and } \psi(r) = 0 \text{ otherwise} \tag{26}$$

for the step function case or

$$\psi(r) = 1 - |r| \text{ if } -1 < r < 1 \text{ and } \psi(r) = 0 \text{ otherwise} \tag{27}$$

for the piecewise linear case.

We can use the base splines $\psi_i(r) = \psi(r - i)$ to model our unknown function

$$\sigma(\tau, r) = \sum_{i=0}^{\infty} \sigma_i(\tau)\psi(r - i), \tag{28}$$

and the coefficients $\sigma_i(\tau)$ become our new unknowns. The problem of estimating these coefficients σ_i from the measured lag profiles $M(t,\tau)$ is a linear estimation problem, whose Fisher information matrix (=inverse of the *a posteriori* covariance matrix) is determined by the coefficients of $\sigma_i\sigma_j$ in the quadratic form

$$\begin{aligned} &\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dr W_{0,\tau}(r - t)\sigma(\tau, r) \int_{-\infty}^{\infty} dr' W_{0,\tau}(r' - t')\sigma(\tau', r')A^{-2}\delta(t - t')\delta(\tau - \tau') \\ &= \sum_{i,j} \sigma_i(\tau)\sigma_j(\tau') \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dr W_{0,\tau}(r - t)\psi_i(r) \int_{-\infty}^{\infty} dr' W_{0,\tau'}(r' - t)\psi_j(r')\, A^{-2}\delta(\tau - \tau'). \end{aligned} \tag{29}$$

Thus, the Fisher information matrix between the unknowns $\sigma_i(\tau)$ and $\sigma_j(\tau')$ is given by

$$Q_{ij}(\tau,\tau') = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dr W_{0,\tau}(r-t)\psi_i(r) \int_{-\infty}^{\infty} dr' W_{0,\tau'}(r'-t)\psi_j(r')\, A^{-2}\delta(\tau-\tau'). \tag{30}$$

In the following we are interested in the diagonal elements Q_{ii}. Now, however, because $|W_{0,\tau}| \leq 1$, it is easy to see that the value of both two innermost integrals is at most 1. Moreover, since the length of the support of $W_{0,\tau}(r)$ is at most $t_{\mathrm{mod}} - \tau$, it follows that we have the relation

$$Q_{ij}(\tau,\tau') \leq A^{-2}(t_{\mathrm{mod}} - \tau)\delta(\tau-\tau')N, \tag{31}$$

where N is the number of repetitions of the experiment, possibly with a different modulation for each repetition. Because $\Sigma_{ii} = (Q^{-1})_{ii} \geq (Q_{ii})^{-1}$, it follows that for any estimator $\sigma_i^{\mathrm{est}}(\tau)$ of the coefficients $\sigma_i(\tau)$, the estimation variances must satisfy

$$\langle \Delta\sigma_i^{\mathrm{est}}(\tau)\Delta\sigma_j^{\mathrm{est}}(\tau')\rangle \geq \frac{A^2}{t_{\mathrm{mod}} - \tau}\delta(\tau-\tau')/N. \tag{32}$$

Let us suppose that we aim at a determination of $\sigma_i(\tau)$ with a lag resolution $\Delta\tau$, so that the final aim might be to find estimates for the variables

$$\sigma_{ij} = \int_{j\Delta\tau}^{(j+1)\Delta\tau} \sigma_i(\tau)d\tau/\Delta\tau. \tag{33}$$

It follows that any estimates for these variables have variances governed by

$$\langle(\Delta\sigma_{ij}^{\mathrm{est}})^2\rangle \geq \frac{A^2}{t_{\mathrm{mod}} - \tau}\Delta\tau^{-1}/N = \frac{1}{2}\left(\frac{\kappa T}{P_{\mathrm{res}}}\right)^2 \Delta r^{-2}(t_{\mathrm{mod}} - \tau)^{-1}\Delta\tau^{-1}/N, \tag{34}$$

and in addition it is known that the different lag values are independent. This is our final accuracy limit formula, but we have to remember that the variables were scaled so that Δr was used as a unit. Real physical value of the modulation length is $t_{\mathrm{mod}}\Delta r$ and lag resolution $\Delta\tau\Delta r$ so that in terms of unscaled variables, the result reads

$$\langle(\Delta\sigma_{ij}^{\mathrm{est}})^2\rangle \geq \frac{1}{2}\left(\frac{\kappa T}{P_{\mathrm{res}}}\right)^2 (t_{\mathrm{mod}} - \tau)^{-1}\Delta\tau^{-1}/N. \tag{35}$$

Here the ratio $P_{\mathrm{res}}/(\kappa T\Delta r^{-2})$ can be understood as some kind of invariant signal-to-noise ratio of the experiment. The noise bandwidth is then determined by the size of range resolution cell and the signal power is defined as the power that is scattered back from a single resolution cell. One has to be aware of the fact that this SNR needs not have anything to do with other ways of defining SNRs in practical experiments usually based on total backscatter power and receiver filters used.

We have not explicitly considered the possibility to divide the modulation interval between different frequency channels to measure the data. This is a method much used in EISCAT, where such pulse codes that contain gaps are interspersed with other codes sent with differing carrier frequencies. It is, however, quite a straightforward task to check that in such a case the total information matrix or function will satisfy the same limitations as in the considerations above. In fact because cross correlations between different frequencies are zero, it follows that the total information produced by such experiments will necessarily be lower than that produced by experiments, where all the power is sent at a single frequency, as is the case with alternating codes. In practice, some of the best frequency commutated multipulse codes may produce variances 5 to 10 times larger than the corresponding alternating codes would.

HOW TO CHOOSE PULSE LENGTHS IN ALTERNATING CODES

The theoretical accuracy limit derived above is based on rather heavy assumptions about relationships between diagonal elements of the Fisher information matrix and the corresponding *a posteriori* covariance matrix. A priori it is not certain whether it is possible at all to design a code that comes close to this limit obtained. To check the situation we have evaluated the whole information matrix and the *a posteriori* covariance matrix in the situation described in the previous chapter. We have supposed that a very long

alternating code or a corresponding random code is used and have calculated the ratios of the final estimation variances and the theoretical limit variances as a function of pulse length used. As in the previous chapter, the resolution of the experiment (that is the node interval of the step function or linear interpolation expansion) was fixed to $\Delta r = 1$ and the pulse length used in the alternating codes was allowed to vary from 0.1 to 10.

Alternating codes have the same information matrices as corresponding random codes like those introduced in /6/ so that these results are directly applicable. In fact, mathematically speaking the set of alternating codes is just a random ensemble of codes, whose distribution is such that suitable sign products vanish as an ensemble average even though this ensemble is finite.

From Figure 1. we see that the best results are found when the pulse length is 1 to 2 times as long as the resolution aimed at. Then the resulting variances will be approximately 2 or 3 times larger than those given by the limit formula depending on which way of modelling is used. If the pulse length is much too big, say a factor a larger than the resolution desired, it can be seen from the curves drawn that the resulting variances will also be a factor a larger than those given by the limit formula. If the pulse length chosen is much shorter than the resolution, say $pulselength = \Delta r/b$, it can be seen from the figure (and it can also be proven) that the resulting variances will be a factor $3b/2$ larger than those ones given by the limit formula.

Summarizing, we see that the curves calculated can be used to estimate variances in situations where the experiment resolution is not matched to the resolution finally required.

Choosing $\Delta\tau$ equal to the first zero crossing τ_0 of the ACF measured, we see that the variance estimate obtained corresponds directly to the noise level as defined in /9/. This means that if we calculate the ratio

$$\lambda = \frac{1}{2}\left(\frac{\kappa T}{P_{\text{res}}}\right) t_{\text{mod}}{}^{-1/2}\tau_0^{-1/2}/N/0.01, \tag{36}$$

we can directly multiply the estimation error bars calculated in the tables in /9/ by λ to obtain parameter estimation accuracies for any resolutions. If we include the additional factor from Figure 1. depending on the ratio of the pulse length used and the desired resolution, we should have very accurate estimates of the error bars in any practical situations where alternating codes are used.

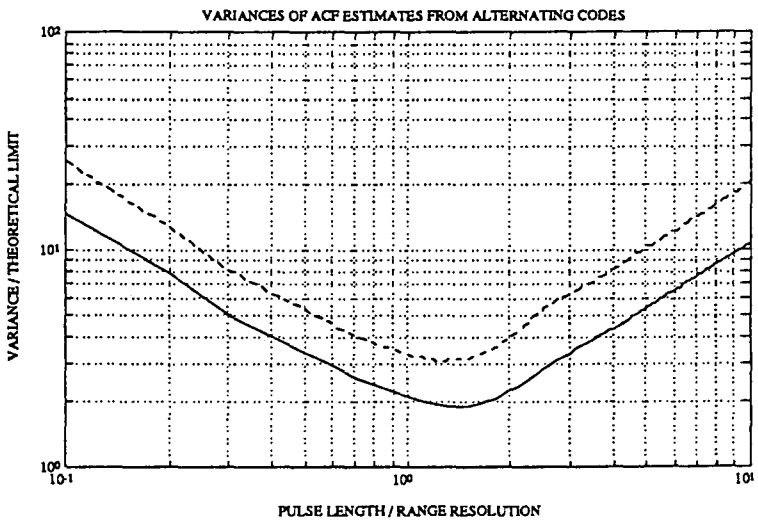

Fig 1. The ratio of the estimation variances of autocorrelation lags and their theoretical limits for experiments of the alternating code type. The horizontal axis is the ratio of the pulse length used and the resolution of the analysis. The solid curve has been calculated for the case where a step-function model is used for the unknowns σ_{ij}, while the dashed curve has been claculated for the case of a linear interpolation between node points.

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