Roaming Behavior of Unconstrained Particles

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***Abstract*—It has been shown recently that unconstrained particles that follow the position and velocity update rules of a standard global best particle swarm optimization algorithm leave the boundaries of the search space within the first few iterations of the search process. Provided that a better solution does not exist outside of the search boundaries, these roaming particles are eventually pulled back within the search boundaries. This article illustrates the consequence of roaming particles should better solutions exist outside of the search boundaries, namely that particles are pulled outside of the search boundaries and that such infeasible solutions are found. The article also evaluates the hypothesis that it is the roaming behavior of unconstrained particles that improves the ability of particle swarm algorithms to locate feasible solutions outside of the particle initialization space.**

I. Introduction

Recently, Engelbrecht [1] provided empirical evidence that particles that are not subject to any boundary constraints leave the boundaries of the search space within the first iterations of the optimization process. The study in [1] assumed that a better solution does not exist outside of the boundaries defined for the optimization problem. Under this assumption, it was shown that particles are eventually pulled back into the search boundaries to converge on a feasible solution within the search boundaries. This behavior of unconstrained particles was referred to as roaming behavior, and it was pointed out that, although particles are pulled back into feasible space, much computational effort is wasted by searching outside of the search boundaries. While not formally investigated, it was pointed out in [1] that this roaming behavior may result in infeasible solutions should a better solution exsit outside of the search boundaries and if no boundary constraint mechanism is employed to ensure feasible particle positions.

This article follows on the work presented in [1] to allude to the danger of infeasible solutions due to particle roaming behavior. Should a better solution exist outside of the search boundaries, unconstrained particles are pulled outside of the search boundaries, due to personal best and the global best positions being outside of the search boundaries. Empirical evidence of infeasible solutions due to particle roaming behavior is provided. Note that the main objective of this study is to emphasize the consequences of particle roaming behavior, so that particle swarm optimization (PSO) practitioners are aware of this. The aim is not to suggest remedies to prevent roaming behavior, as many boundary constraint mechanisms exist to prevent particles from leaving the search space [2], [3], [4].

As a second objective, this study shows that it is the roaming behavior of particles that allow particles to find feasible solutions outside of the particle initialization space, but still within the search boundaries.

The first behavior is illustrated for a standard global best (gbest) PSO [5], the guaranteed convergence PSO (GCPSO) [6], the constriction PSO (CPSO) [7], [8], and the barebones PSO (BBPSO) [9]. Due to space constraints, results for the second behavior are presented for only the gbest PSO and the BBPSO.

The remainder of this paper is organized as follows: Section II provides a compact summary of the algorithms considered in this study. An overview of the roaming behavior of particles is provided in Section III. Section IV provides empirical evidence that infeasible solutions are found by unconstrained particles should better solutions exist outside of the search boundaries. Section V provides empirical evidence that the roaming behavior has the advantage that solutions within bounds, but outside of the particle initialization space can be found.

II. Particle Swarm Optimization Algorithms

The standard PSO as introduced by [5] implemented one of two neighborhood topologies: either a star topology where a particle's neigborhood is the entire swarm, or a ring neighborhood topology where a particle's neighborhood is defined by its immediate neighbors. This study considers PSO algorithms that use the star topology. Under this assumption, for the gbest PSO, particle positions $\mathbf{x}_i$ are updated using

$$\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t+1) \tag{1}$$

and the velocities are updated using the inertia weight model [10] as follows

$$\begin{aligned} v_{ij}(t+1) \quad = \quad & wv_{ij}(t) + c_1 r_{1j}(t)[y_{ij}(t) - x_{ij}(t)] + \\ & c_2 r_{2j}(t)[\hat{y}_j(t) - x_{ij}(t)] \end{aligned} \tag{2}$$

where w is the inertia weight, $v_{ij}(t)$ is the velocity of particle i in dimension $j = 1, \ldots, n_x$, $x_{ij}(t)$ is the position of particle i in dimension j, $y_{ij}(t)$ is particle i's personal best position in dimension j, $\hat{y}_j(t)$ is the global best position in dimension j, c_1 and c_2 are positive acceleration constants used to scale the contribution of the cognitive and social components respectively, and $r_{1j}(t), r_{2j}(t) \sim U(0,1)$ are random values in the range $[0,1]$, sampled from a uniform distribution.

It is important to note that this study selects the global best position as the best personal best position, hence using a memory-based global best update. Particle positions are initialized within the given domain and velocities are initialized to zero [1]. No boundary constraint mechanism is employed for the first part of the study, i.e. to illustrate that roaming particles will find better positions outside of the bounds if they exist. For the second part of the study, i.e. to show that particles have the ability to find solutions outside of the initialization space (but within the boundaries), a constraint is imposed on personal best positions: A personal best position is only updated if the new position is better and within the search boundaries. Due to the memory-based global best update, this personal best constraint guarantees that the global best postion will be feasible.

The guaranteed convergence PSO (GCPSO) developed by [6] forces the global best particle to search within a confined region for a better position, preventing early stagnation of particles. Let τ be the index of the global best particle, so that $\mathbf{y}_\tau = \hat{\mathbf{y}}$. GCPSO changes the position update of the global best particle to

$$x_{\tau j}(t+1) = \hat{y}_j(t) + wv_{\tau j}(t) + \rho(t)(1 - 2r_2(t)) \tag{3}$$

which is obtained using equation (1) if the velocity update of the global best particle changes to

$$v_{\tau j}(t+1) = -x_{\tau j}(t) + \hat{y}_j(t) + wv_{\tau j}(t) + \rho(t)(1 - 2r_{2j}(t)) \tag{4}$$

where $\rho(t)$ is a scaling factor that determines the size of the bounding box around the global best particle within which a better position is search for [6]. Note that only the global best particle is adjusted according to Equations (3) and (4); all other particles use Equations (1) and (2).

Clerc [7] and Clerc and Kennedy [8] developed the constriction PSO (CPSO), where the velocity update of Equation (2) changes to

$$v_{ij}(t+1) = \chi[v_{ij}(t) + \phi_1(y_{ij}(t) - x_{ij}(t)) + \phi_2(\hat{y}_j(t) - x_{ij}(t))] \tag{5}$$

with the constriction coefficient,

$$\chi = \frac{2\kappa}{|2 - \phi - \sqrt{\phi(\phi - 4)}|} \tag{6}$$

where $\phi = \phi_1 + \phi_2, \phi_1 = c_1 r_1$, and $\phi_2 = c_2 r_2$. Equation (6) is used under the constraints that $\phi \geq 4$ and $\kappa \in [0, 1]$.

In the barebones PSO (BBPSO), proposed by Kennedy [9], particles do not follow search trajectories, but are sampled from a Gaussian distribution centered around the average of the particle's personal best position and the global best position. If it is assumed that $c_1 = c_2$, the velocity update changes to

$$v_{ij}(t+1) \sim N\left(\frac{y_{ij}(t) + \hat{y}_{ij}(t)}{2}, \sigma\right),\ \sigma = |y_{ij}(t) - \hat{y}_{ij}(t)| \tag{7}$$

The position update changes to

$$x_{ij}(t+1) = v_{ij}(t+1) \tag{8}$$

III. Roaming Behavior

Helwig and Wanka [11] conducted a theoretical analysis of initial particle swarm behavior, providing formal proofs that most particles leave the boundaries of the search space within the first iterations. Empirical evidence of this roaming behavior of unconstrained particles was provided by Engelbrecht [1], and the following consequences of particle roaming behavior were identified:

- Should a better solution be found outside of the defined bounds of the optimization problem, and no boundary constraint mechanism employed, personal best positions are also pulled outside the bounds of the search space. The consequence of these infeasible solutions is that global best positions are also pulled outside of the search space, resulting in an infeasible optimum being found.
- If roaming particles do not find better solutions outside of the search space, they are eventually pulled back into feasible space. However, this is done over a large number of iterations, wasting effort searching infeasible space.
- Roaming particles have a negative influence on swarm diversity calculations, since diversity increases as particles move further outside the bounds of the search space. This is a problem for PSO algorithms that use measures of diversity to control the search trajectories of particles.

An additional consequence of the roaming behavior of particles, as identified in this study, is that it enables particles to locate solutions outside of the particle initialization space, but within boundary constraints provided that some form of boundary constraint handling mechanism is employed.

The focus of this study is to provide empirical evidence to illustrate the disadvantage of finding infeasible solutions if unconstrained particles are used and the advantage of finding feasible solutions outside of the initialization space if constrained particles are used.

IV. Infeasible Solutions outside of Boundaries

The purpose of this section is to provide empirical evidence that particles that are not subject to a boundary constraint will find a better, infeasible solution outside of the search space boundaries should such better solution exsit. This is as a direct result of the roaming behavior of particles. Section IV-A summarizes the experimental procedure, while Section IV-B provides and discusses the obtained results.

A. Experimental Procedure

The consequence of particle roaming behavior is illustrated for the gbest PSO, GCPSO, CPSO, and BBPSO. For this purpose, 10 benchmark functions have been used as summarized in Table I. Note that the domains of these functions were selected to ensure that better minima exist outside of the bounds. Each of the functions were evaluated in 30 dimensions. Shifting a function, f_l, was implemented using

$$f_l^S(\mathbf{x}) = f_l(\mathbf{z}) + \beta$$

where $\mathbf{z} = \mathbf{x} - \gamma$ and β is a constant. Rotation was done using Salomon's method [12], with $\mathbf{z} = \mathbf{x} - \gamma\mathbf{M}$, where $\mathbf{M}$ is the rotation matrix. A new rotation matrix was computed for each independent run of the algorithm. The rotated function is referred to as f_l^R, computed by multiplying the decision vector $\mathbf{x}$ with the transpose of the rotation matrix.

Details for the shifts and rotations are as follows:

- Shifted rotated Ackley: A linear rotation matrix was used with a condition number of 100, $\beta = -140$, and $\gamma = -32$.
- Shifted rotated Griewank: A linear rotation matrix was used with a condition number of 3, $\beta = -180$ and $\gamma = -60$.
- Shifted Norwegian: $\beta = 0$ and $\gamma = 10$.
- Shifted Rosenbrock: $\beta = 390$ and $\gamma = -60$.
- Shifted Schwefel 1.2: $\beta = 0$ and $\gamma = -50$.
- Shifted Spherical: $\beta = -450$ and $\gamma = -20$.

For all of the algorithms, swarm sizes of 30 particles were used, and each algorithm executed for 50 independent runs of 5000 iterations. The control parameters were set to the same values as follows: The inertia weight was set to 0.729844, while the values of the acceleration coefficients were set to 1.49618. This choice is based on [13], where it was shown that such parameter settings facilitate convergent behaviour. For the CPSO, $\kappa = 1.0$ and $c_1 = c_2 = 2.05$. These values are equivalent to the inertia weight and acceleration coefficient values as given above for the other PSO algorithms.

B. Results and Discussion

Table II summarizes the fitness of the best solution found at the last iteration and the swarm diversity as averages and deviations over the 50 independent runs for each function and algorithm. Figures 1 and 2 illustrate fitness, diversity, and percentage position violations for the Ackley and Eggholder functions respectively. These figures are representative of behaviors observed for the other functions. The first major observation from the table and these figures is that swarm diversity remained very high after 5000 iterations despite the fact that the average fitness of the global best particles converged. This observation exlcudes the Eggholder and Schwefel 2.26 functions for which an average fitness of -INF was obtained (this is due to the fact that particles continue to move outside of the defined boundaries to better positions, due to the shape of these functions), and an incomputable diversity. For these functions, particles continued to accelerate to positions of large negative fitness for each iteration. Figure 2(a) shows that swarm diversity for the Eggholder function exploded to extremely large values within the first few iterations.

These large swarm diversity values are indicative of large dispersion of the particles, and the fact that swarm diversity values are significantly larger than the extent of the domains points to the roaming behavior of particles being pulled further away from the search boundaries. The percentage of particles that violate boundaries, for at least one dimension, supports this observation. The consequence of particle roaming is clearly illustrated in the average percentage of personal best positions that violate boundary constraints and the percentage of simulations for which the global best position violates boundary constraints. In addition to the Ackley and Eggholder profiles, Figure 3 illustrates the global best position violations for the other functions. Note how the percentage of simulations for which the global best position violates boundary constraints reached 100% within the first few iterations. This also implies that all the personal best positions will violate the boundary constraints. These results provide clear support for the observation that particles find better solutions outside of the search boundaries if such better solutions exist.

It is interesting to note that only for the Salomon function did the diversity decrease over time, albeit still to a large value. For all the other functions, diversity continued to increase, indicating that particles were moving further away from one another. Note that diversity was caluclated as the average distance that particles are from the swarm center.

V. Feasible Solutions outside of Initialization Space

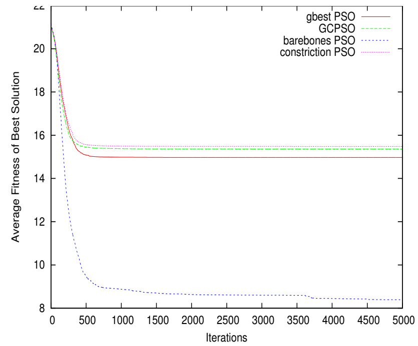
The aim of this section is to illustrate that the roaming behavior of boundary constrained particles allows particles to find feasible solutions outside of the particle initialization space. Section V-A provides the experimental procedure, and Section V-B presents and discusses the results.

A. Experimental Procedure

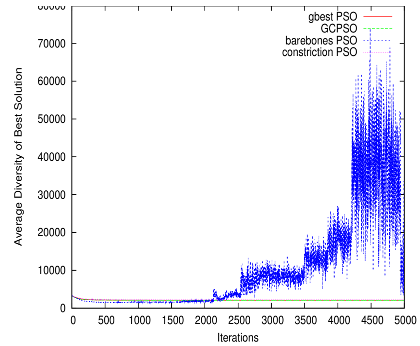
This section empirically analyzes the ability of PSO algorithms to locate optima outside of the range in which particle positions have been initialized. The hypothesis is that the roaming behavior of particles facilitate exploration outside of the initialization bounds to obtain the best solution, but still within the problem boundaries. For the purposes of this study, personal best positions were constrained to remain within the problem boundaries, which will ensure that global best positions are feasible due to the memory-based global best update strategy used. The algorithms used the same control parameter values as given above. Only the gbest PSO and BBPSO results are presented. The GCPSO and CPSO exhibited similar behavior. The functions used for this study include the Ackley and Salomon functions defined in Table I, but for domains of [-32.768,32.768] and [-100,100] respectively. In addition to these functions, the functions defined in Table III have been used. The rotations for the Griewank and Rosenbrock functions were done using a randomly generated orthonormal matrix with condition number of one.

All of the functions except the Eggholder and Schwefel 2.26 functions have their global minimum at $\mathbf{x}^* = (0, 0, \ldots, 0)$, with $f(\mathbf{x}^*) = 0$. The global minimum of the Schwefel 2.26 function is at $\mathbf{x}^* = (420.9687, \ldots, 420.9687)$, with $f(\mathbf{x}^*) = -12569.5$. The global minimum of the Eggholder function is located near the maximum boundary of the search space.

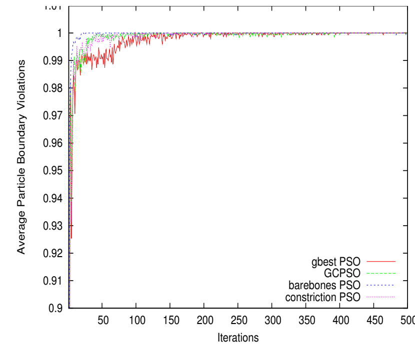
For all of the functions, except the Eggholder and Schwefel 2.26 functions, the intialization range for particle positions were taken as 5%, 10%, 15%, ..., 100% of the extent of the problem domain, starting from the maximum value. For both the Eggholder and Schwefel 2.26 functions, the global minimum is located close to the maximum boundary of the



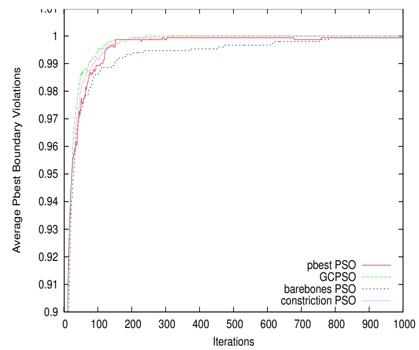
(a) Fitness



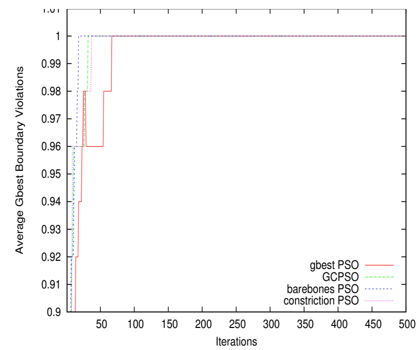
(b) Diversity



(c) Particle Boundary Violations

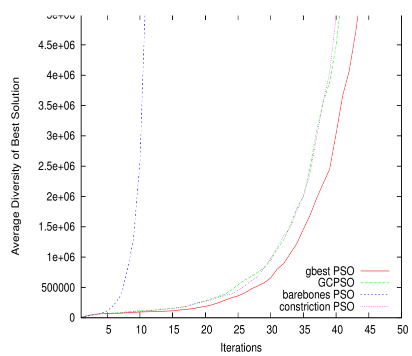


(d) Personal Best Boundary Violations

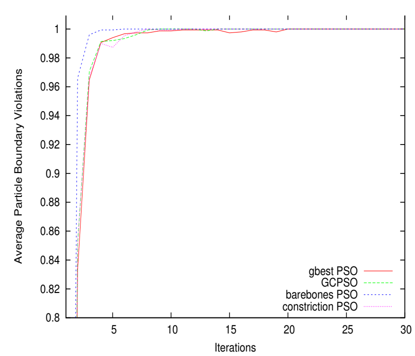


(e) Global Best Boundary Violations

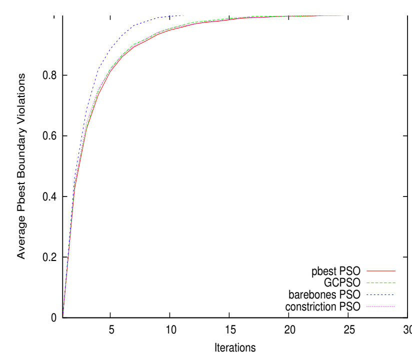
Fig. 1. Ackley Fitness, Diversity, and Boundary Violation Profiles for Infeasible Solutions Study



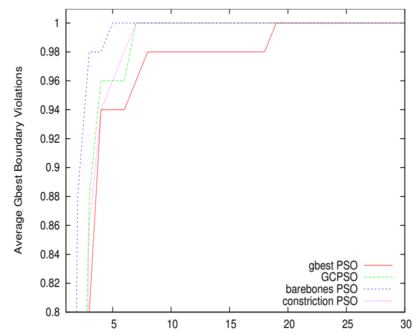
(a) Diversity



(b) Particle Boundary Violations

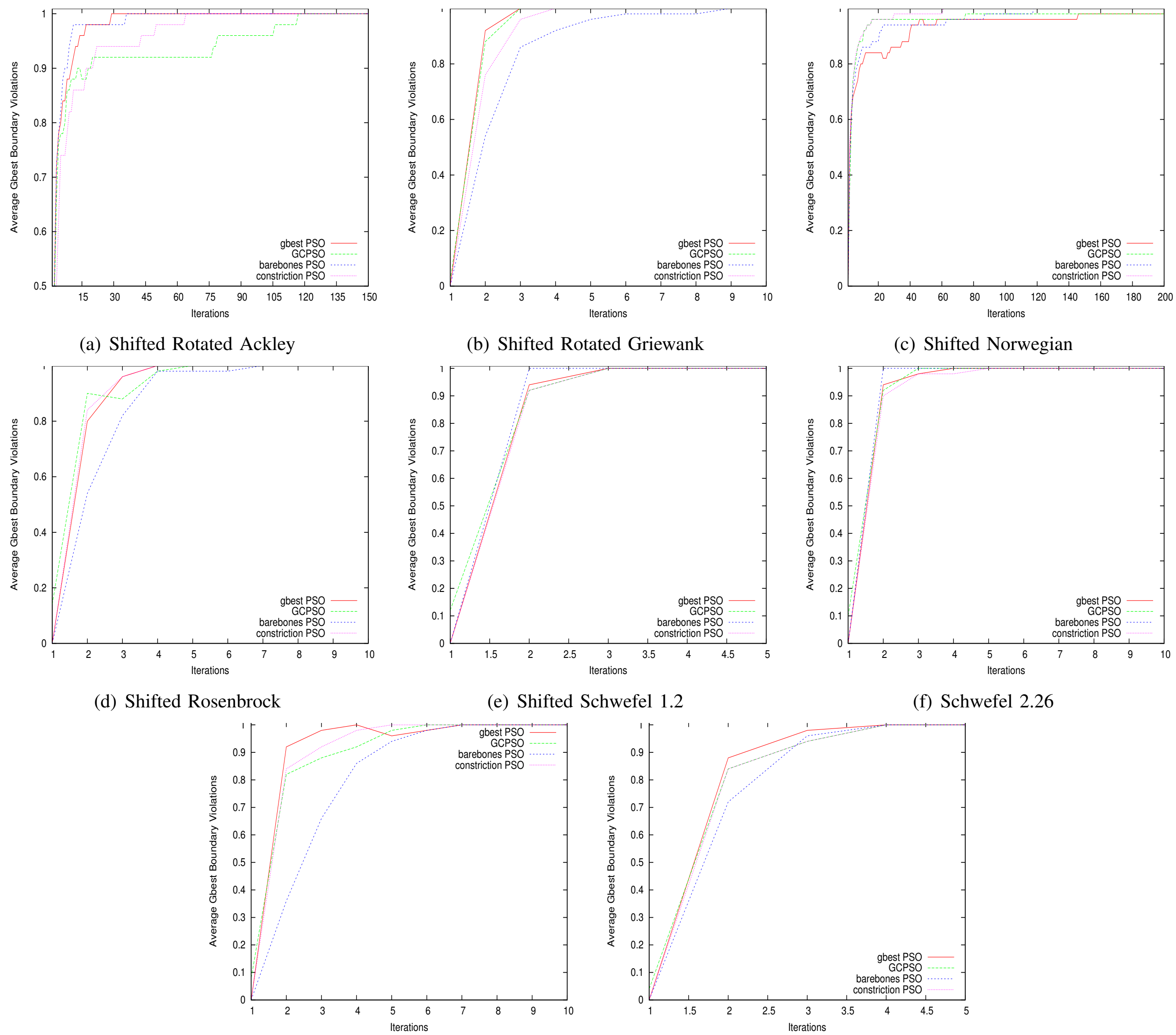


(c) Personal Best Boundary Violations



(d) Global Best Boundary Violations

Fig. 2. Eggholder Fitness, Diversity, and Boundary Violation Profiles for Infeasible Solutions Study



(a) Shifted Rotated Ackley (b) Shifted Rotated Griewank (c) Shifted Norwegian (d) Shifted Rosenbrock (e) Shifted Schwefel 1.2 (f) Schwefel 2.26 (g) Salomon (h) Shifted Spherical

Fig. 3. Global Best Boundary Violation Profiles for Infeasible Solutions Study

search space. That is, for the 5% initialization range, for the Griewank function, particles were uniformly initialized in the range $[540, 600]^{30}$, for the 10% initialization range particle were uniformly initialized in the range $[480, 600]^{30}$, and so on.

B. Results and Discussion

Figure 4 illustrates the average best fitness obtained over the 50 independent runs for the gbest PSO and BBPSO. Also indicated in the figures is the deviation in best fitness over the 50 runs.

For the rotated Griewank, rotated Rosenbrock, Salomon, and Schwefel 1.2 functions, both algorithms showed no significant difference in average best fitness over the different initialization ranges. For all of the initialization ranges, the average best fitness values were very similar, with small deviations except for the 5% initialization range. For initialization ranges less than 50%, which exclude the global minimum, the algorithms were able to locate better solutions outside of the initialization range.

For the Ackley function, initialization ranges 5% to 35% showed no significant difference in average best fitness, at a high average value of just below 20. From initialization range of 40% and higher, solutions with an average fitness of around 2.5 were obtained. The Eggholder function showed a similar trend, but with performance improving only for larger initialization ranges. Note that the global minimum is located close to the maximum boundary of the search space, and that particles for small initialization ranges are initialized very far from the global minimum. Despite this, particles were able to locate positions of better quality outside of the initialization range. The Schwefel 2.26 function also has its global minimum located close to the maximum boundary. For this function all initialization ranges resulted in good solutions, with no significant difference in performance. Note that initialization in the entire domain resulted in the worst performance, clearly indicating success in locating better solutions outside of initialization ranges.

The Schaffer 6 function exhibited an interesting behavior, with performance deteriorating with increase in the initialization range. The best performance was obtained for a small initialization range of 5% with initial particle positions far from the global minimum.

VI. Conclusions

The main objective of this paper was to point out a disadvantage of the roaming behavior of particles in standard particle swarm optimizations. If no boundary constraint mechanism is employed, and if a solution of better quality exists outside of the boundaries, then roaming particles pull their personal best positions, and consequently the global best position, outside of the boundaries into infeasible space, never to return to feasible space. The important outcome of this observation is that a boundary constraint mechanism has to be used to ensure that a solution within the boundaries defined for the optimization problem is found.

In addition to this observation, empirical results were presented to show an advantage of the roaming behavior of particles, in that better positions outside of the initialization range can be obtained. Should a boundary constraint mechanism be used, then such solutions will be feasible. This finding shows that PSO can obtain good solutions even if the optimum does not lie within the initialization range.

The ability of particles to locate solutions outside of the initialization range is important for optimization problems that do not have boundary constraints, such as neural network training. The roaming behavior of particles may also play a positive role in constrained optimization problems where pockets of feasible space exist within infeasible space, as roaming may aid in particles crossing infeasible space into feasible space. These advantages to roaming behavior will be investigated in future studies. Additionally, it will be interesting to analyze roaming behavior for evolutionary algorithms such as differential evolution and evolutionary programming.

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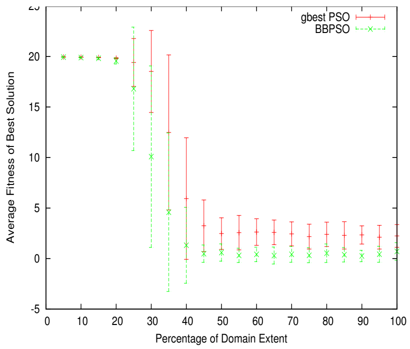
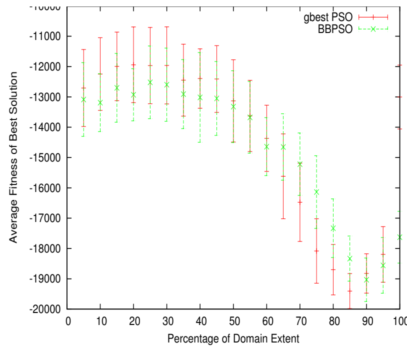
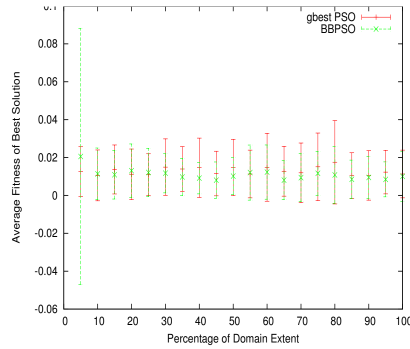


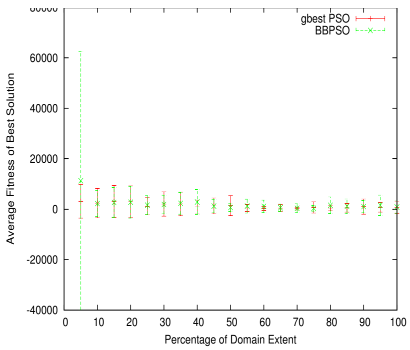
(a) Ackley



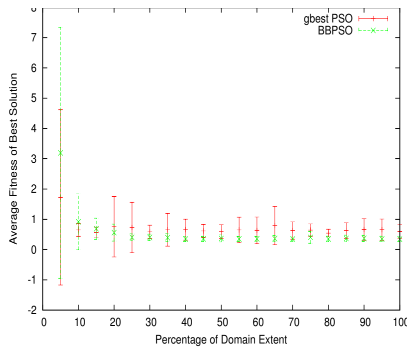
(b) Eggholder



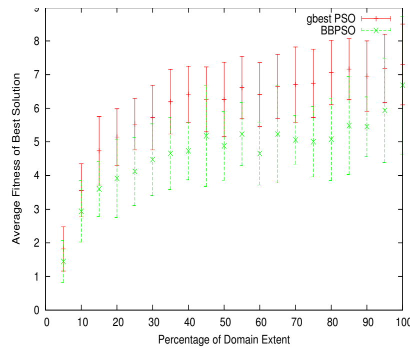
(c) Rotated Griewank



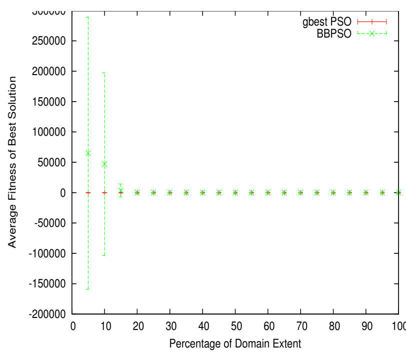
(d) Rotated Rosenbrock



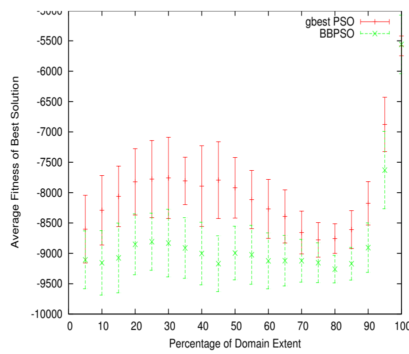
(e) Salomon



(f) Schaffer 6



(g) Schwefel 1.2



(h) Schwefel 2.26

Fig. 4. Global Best Boundary Violation Profiles for Infeasible Solutions Study

TABLE I
BENCHMARK FUNCTIONS USED FOR INFEASIBLE SOLUTIONS STUDY

Function	**Domain**	**Function Definition**
Ackley	[10,32.768]	$f(\mathbf{x}) = -20e^{-0.2\sqrt{\frac{1}{n}\sum_{j=1}^{n_x} x_j^2}} - e^{\frac{1}{n}\sum_{j=1}^{n_x}\cos(2\pi x_j)} + 20 + e$
Shifted rotated Ackley	[-32.768,0]	$f(\mathbf{x}) = -20e^{-0.2\sqrt{\frac{1}{n}\sum_{j=1}^{n_x} z_j^2}} - e^{\frac{1}{n}\sum_{j=1}^{n_x}\cos(2\pi z_j)} + 20 + e$
Eggholder	[-512,512]	$f(\mathbf{x}) = \sum_{j=1}^{n_x-1}\left(-(x_{j+1}+47)\sin(\sqrt{\lvert x_{j+1}+x_j/2+47\rvert}) + \sin(\sqrt{\lvert x_j-(x_{j+1}+47)\rvert})(-x_j)\right)$
Shifted rotated Griewank	[0,600]	$f(\mathbf{x}) = 1 + \frac{1}{4000}\sum_{j=1}^{n_x} z_j^2 - \prod_{j=1}^{n_x}\cos\left(\frac{z_j}{\sqrt{j}}\right)$
Shifted Norwegian	[-1.1,1.1]	$f(\mathbf{x}) = \prod_{j=1}^{n_x}\left(\cos(\pi z_j^3)\left(\frac{99+z_j}{100}\right)\right)$
Shifted Rosenbrock	[-30,30]	$f(\mathbf{x}) = \sum_{j=1}^{n_x-1}\left(100(z_{j+1}-z_j^2)^2 + (z_j-1)^2\right)$
Shifted Schwefel 1.2	[0,100]	$f(\mathbf{x}) = \sum_{j=1}^{n_x}\left(\sum_{k=1}^{j} z_k\right)^2$
Schwefel 2.26	[-50,50]	$f(\mathbf{x}) = -\sum_{j=1}^{n_x}\left(x_j\sin\left(\sqrt{\lvert x_j\rvert}\right)\right)$
Shifted Spherical	[0,100]	$f(\mathbf{x}) = \sum_{j=1}^{n_x} z_i^2$
Salomon	[-100,5]	$f_{14}(\mathbf{x}) = -\cos(2\pi\sum_{j=1}^{n_x} x_j^2) + 0.1\sqrt{\sum_{j=1}^{n_x} x_j^2} + 1$

TABLE II
FITNESS AND DIVERSITY RESULTS FOR INFEASIBLE SOLUTIONS STUDY

		Fitness		**Diversity**	
Function	**Algorithm**	**Average**	**Deviation**	**Average**	**Deviation**
Ackley	gbest PSO	1.50E+001	6.93E+000	2.09E+003	1.30E+003
	GCPSO	1.54E+001	6.68E+000	2.10E+003	1.23E+003
	CPSO	1.55E+001	7.00E+000	2.21E+003	1.24E+003
	BBPSO	8.39E+000	9.19E+000	6.50E+003	3.82E+004
Shifted Rotated Ackley	gbest PSO	-1.19E+002	6.71E-002	5.08E+003	2.92E+003
	GCPSO	-1.19E+002	5.86E-002	4.50E+003	2.11E+003
	CPSO	-1.19E+002	7.63E-002	4.89E+003	3.37E+003
	BBPSO	-1.19E+002	4.80E-002	2.79E+004	4.84E+004
Eggholder	gbest PSO	-INF	-	-	-
	GCPSO	-INF	-	-	-
	CPSO	-INF	-	-	-
	BBPSO	-INF	-	-	-
Shifted Rotated Griewank	gbest PSO	-1.80E+002	1.53E-002	9.52E+003	2.70E+001
	GCPSO	-1.80E+002	1.37E-002	9.53E+003	3.06E+001
	CPSO	-1.80E+002	1.32E-002	9.53E+003	2.55E+001
	BBPSO	-1.80E+002	1.02E-002	9.53E+003	2.35E+001
Shifted Norwegian	gbest PSO	-4.43E+033	3.14E+034	1.40E+039	9.92E+039
	GCPSO	-2.27E+010	1.61E+011	1.55E+016	1.09E+017
	CPSO	-4.72E+017	3.34E+018	6.55E+022	4.63E+023
	BBPSO	-INF	-	1.05E+024	5.67E+024
Shifted Rosenbrock	gbest PSO	4.44E+002	1.16E+002	9.52E+003	4.31E+002
	GCPSO	4.83E+002	1.77E+002	9.47E+003	3.91E+002
	CPSO	4.71E+002	2.85E+002	9.60E+003	1.23E+003
	BBPSO	6.47E+002	5.17E+002	1.25E+004	9.16E+003
Salomon	gbest PSO	6.32E-001	3.80E-001	1.82E+002	1.10E+002
	GCPSO	6.56E-001	2.73E-001	1.90E+002	7.92E+001
	CPSO	5.80E-001	2.57E-001	1.67E+002	7.45E+001
	BBPSO	3.64E-001	1.38E-001	1.00E+002	4.08E+001
Shifted Schwefel 1.2	gbest PSO	5.78E-008	1.03E-007	7.94E+003	1.75E-001
	GCPSO	9.83E-008	3.58E-007	7.94E+003	2.39E-001
	CPSO	7.82E-008	1.79E-007	7.94E+003	4.80E-001
	BBPSO	1.39E-001	1.57E-001	7.99E+003	1.07E+002
Schwefel 2.6	gbest PSO	-INF	-	-	-
	GCPSO	-INF	-	-	-
	CPSO	-INF	-	-	-
	BBPSO	-INF	-	-	-
Shifted Spherical	gbest PSO	-4.50E+002	1.37E-013	3.18E+003	1.71E-006
	GCPSO	-4.50E+002	7.31E-014	3.18E+003	1.83E-006
	CPSO	-4.50E+002	4.80E-014	3.18E+003	1.78E-006
	BBPSO	-4.50E+002	2.69E-014	3.18E+003	1.44E-006

TABLE III
BENCHMARK FUNCTIONS USED FOR FEASIBLE SOLUTIONS STUDY

Function	**Domain**	**Function Definition**
Rotated Griewank	[-600,600]	$f(\mathbf{x}) = 1 + \frac{1}{4000}\sum_{j=1}^{n_x} z_j^2 - \prod_{j=1}^{n_x}\cos\left(\frac{z_j}{\sqrt{j}}\right)$
Rotated Rosenbrock	[-100,100]	$f(\mathbf{x}) = \sum_{j=1}^{n_x-1}\left(100(z_{j+1} - z_j^2)^2 + (z_j - 1)^2\right)$
Schaffer 6	[-100,100]	$f(\mathbf{x}) = \sum_{j=1}^{n_x-1}\left(0.5 + \frac{sin^2(x_j^2+x_{j+1}^2)-0.5}{(1+0.001(x_j^2+x_{j+1}^2))^2}\right)$
Schwefel 1.2	[-100,100]	$f(\mathbf{x}) = \sum_{j=1}^{n_x}\left(\sum_{k=1}^{j} x_k\right)^2$