```
em;qmuo;fc;ft;fc0;          hp;qh0;pm0;   ah;bh;ch;dh;  as;ecc;gtol;itypey;  wf;wf1;ft1;gtol=1.0d-4;
irate_ekffect=0;     isotropic=2;  imaxsubinc=20;       iendflag=0    bs=1.d0;      df=0.85d0; istrrateflg=0

fb=1.50d0*fc**(-0.075d0)*fc
par=ft/fb*(fb**tw-fc**tw)/(fc**tw-ft**tw)

ecc=(on+par)/(tw-par)
pm0=thr*(fc**tw-ft**tw)/fc/ft*ecc/(ecc+on)

e0=ft/em;

tkp=statev(1);
de(1:6)=dstran;
old_e(1:6)=statev(21:26);

plen=CELENT;  e_rate=de;    isubinc_count=0;     isubinc_flag=0;      iconvrg=1


cc=zr;
p1=em/(on+qmuo)/(on-tw*qmuo);
cc(1:3,1:3)=p1*qmuo;
cc(1,1)=(on-qmuo)*p1;
cc(2,2)=(on-qmuo)*p1;
cc(3,3)=(on-qmuo)*p1;
Gm=Em/(on+qmuo)/tw;
cc(4,4)=Gm;
cc(5,5)=Gm;
cc(6,6)=Gm;
ddsdde=cc;
cin=zr;
cin(1:3,1:3)=-on*qmuo/Em;
cin(1,1)=on/Em;
cin(2,2)=on/Em;
cin(3,3)=on/Em;
cin(4,4)=on/Gm;
cin(5,5)=on/Gm;
cin(6,6)=on/Gm;
```

$$\boxed{e = \frac{1+\epsilon}{2-\epsilon}, \quad \text{where } \epsilon = \frac{\bar{f}_{\mathrm{t}}}{\bar{f}_{\mathrm{b}}} \frac{\bar{f}_{\mathrm{b}}^2 - \bar{f}_{\mathrm{c}}^2}{\bar{f}_{\mathrm{c}}^2 - \bar{f}_{\mathrm{t}}^2}}$$

$$\boxed{m_0 = \frac{3\left(f_{\mathrm{c}}^2 - f_{\mathrm{t}}^2\right)}{f_{\mathrm{c}} f_{\mathrm{t}}} \frac{e}{e+1}}$$

```
tot_e=old_e+de;
strain_rate=de;

e_p=statev(3:8);

tot_e1=tot_e;
ttot_e=tot_e;

do while (iconvrg.eq.1.or.isubinc_flag.eq.1)
       el_e=ttot_e-e_p;
       sg_tr=matmul(cc,el_e)
       call khaigh(sg_tr,sv_tr,ro_tr,theta_tr,dinv_dsig_pr);
       call kff(sv_tr,ro_tr,theta_tr,tkp,yield);
```

$\boxed{\bar{\boldsymbol{\sigma}}_{n+1} = \bar{\boldsymbol{\sigma}}^{trial}_{n+1} - \Delta\lambda \mathbf{C}_0\, \mathbf{m} \quad \text{with } \bar{\boldsymbol{\sigma}}^{trial}_{n+1} = \mathbf{C}_0 \left(\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}^p_n\right)}$

$\boxed{F\left(\bar{\boldsymbol{\sigma}}^{trial}_{n+1}, \boldsymbol{\kappa}_n\right) \leq \ 0, \text{ then this is an elastic state}}$

```
       apex_sg=zr
       if(yield>zr) then
           irtype=0
           call kcheckvertex(sv_tr,tkp,apex_sg,irtype)
           if (irtype.eq.1.or.irtype.eq.2) then   call kvertexreturn(sg_tr,apex_sg,tkp,irtype,iconvrg,sg_tr)
           if(irtype.eq.0) then call regular_return(sg_tr,tkp,iendflag,sg_tr,iconvrg,tkp,pj,rs,pnorm_rs,pp,pxx)
       else
              iconvrg=0;
              do jj=1,6     e_p(jj)=statev(jj+2) end do
              exit
       end if
```

```
     if (iconvrg.eq.1) then
              isubinc_counter=isubinc_counter+1;
              if (isubinc_counter>imaxsubinc) then
                    write(*,*) 'erorrrr'
c                   call xit
              else if (isubinc_counter > imaxsubinc-1.and.tkp < on) then
                    tkp=on
              end if
              isubinc_flag=1;dtot_e=half*dtot_e;
              ttot_e=conv_e+dtot_e;pj_prev=pj;
       else if (iconvrg.eq.0.and.isubinc_flag.eq.0) then
              el_e=matmul(cin,sg_tr)
              e_p=tot_e-el_e;
      else if (iconvrg.eq.0.and.isubinc_flag.eq.1) then
              el_e=matmul(cin,sg_tr)
              e_p=ttot_e-el_e;conv_e=ttot_e
              dtot_e=tot_e-ttot_e;ttot_e=tot_e;
      end if
end do
```

```
if(itypey.eq.3) then
       wt_o=0.d0;wc_o=0.d0;
       eps_t=0.d0;eps_c=0.d0;
       pkdt=0.d0;pkdt1=0.d0;pkdt2=0.d0;
       pkdc=0.d0;pkdc1=0.d0;pkdc2=0.d0;
       rate_fac=0.d0;alpha=0.d0
else
rate_fac=statev(17);eps_t=statev(19);eps_c=statev(20)
pkdt=statev(9);pkdt1=statev(10);pkdt2=statev(11)
pkdc=statev(12);pkdc1=statev(13);pkdc2=statev(14)
wt_o=statev(15);wc_o=statev(16);alpha=statev(18)

call kc_alpha(sg_tr,sig_ekff_t,sig_ekff_c,alpha);

do jj=1,6
       el_e_old(jj)=statev(20+jj)-statev(jj+2)
end do
sg_old=matmul(cc,el_e_old)
parrr=0.d0
do jj=3,8
       parrr=parrr+(e_p(jj-2)-statev(jj))**tw
end do
pnorm_inc_e_p=sqrt(parrr)
call kdamage(wc_o,wt_o,strain_rate,rate_fac,alpha,eps_t,eps_c,pkdt,pkdt1,pkdt2,pkdc,pkdc1,pkdc2,sg_tr,
tkp,pnorm_inc_e_p,plen,sg_old,statev(18),statev(2),
wc,wt,eps_t,eps_c,pkdt,pkdt1,pkdt2,pkdc,pkdc1,pkdc2,eps_new)!statev(18) old alpha
end if
if (isotropic.eq.0) then
       sig=(on-wt)*sig_ekff_t+(on-wc)*sig_ekff_c
else if (isotropic.eq.1) then
       sig=(on-wt)*sg_tr
else
       sig=(on-wt*(on-alpha))*(on-wc*alpha)*sg_tr
c      modified from original code
end if
statev(1)=tkp;statev(3:8)=e_p;statev(2)=eps_new
statev(9)=pkdt;statev(10)=pkdt1;statev(11)=pkdt2;
statev(12)=pkdc;statev(13)=pkdc1;statev(14)=pkdc2;
statev(15)=wt;statev(16)=wc;
statev(17)=rate_fac;statev(18)=alpha;
statev(19)=eps_t;statev(20)=eps_c;
statev(21:26)=tot_e;
```

INPUT DATA $[t_n, t_n + \Delta t = t_{n+1}] \rightarrow \boldsymbol{\varepsilon}_n, r_n, \boldsymbol{\varepsilon}_{t_n+1}$

$$\text{Step 1} \rightarrow \text{Compute} \begin{cases} \bar{\boldsymbol{\sigma}}_{n+1} = \mathbb{C} : \boldsymbol{\varepsilon}_{n+1} \rightarrow \begin{cases} \tau_{\boldsymbol{\varepsilon}_n} = \sqrt{\boldsymbol{\varepsilon}_n : \mathbb{C} : \boldsymbol{\varepsilon}_n} \\ \tau_{\boldsymbol{\varepsilon}_{n+1}} = \sqrt{\boldsymbol{\varepsilon}_{n+1} : \mathbb{C} : \boldsymbol{\varepsilon}_{n+1}} = \sqrt{\bar{\boldsymbol{\sigma}}_{n+1} : \boldsymbol{\varepsilon}_{n+1}} \end{cases} \\ \tau_{\boldsymbol{\varepsilon}_{n+\alpha}} = (1-\alpha)\tau_{\boldsymbol{\varepsilon}} + \alpha \tau_{\boldsymbol{\varepsilon}_{n+1}} \end{cases}$$

$$\text{Step 2} \rightarrow \text{If } \tau_{\boldsymbol{\varepsilon}+\alpha} \le r_n \rightarrow \left| \begin{array}{l} \text{Elastic} \\ \text{Unloading} \end{array} \right.$$

$$\rightarrow \left| \begin{array}{l} r_{n+1} = r_n \quad ; \quad d_{n+1} = d_n = 1 - \dfrac{q(r_{n+1})}{r_{n+1}} \quad ; \quad \boldsymbol{\sigma}_{n+1} = (1 - d_{n+1})\bar{\boldsymbol{\sigma}}_{n+1} \\ \mathbb{C}^{vd}_{\mathrm{alg},n+1} = (1 - d_{n+1})\mathbb{C} \end{array} \right.$$

$$\text{Step 3} \rightarrow \text{If } \tau_{\varepsilon_{n+\alpha}} > r_n \rightarrow \text{(Loading)}$$

$$\rightarrow \left| \begin{array}{l} r_{n+1} = \dfrac{[\eta - \Delta t(1-\alpha)]}{\eta + \alpha \Delta t} r_n + \dfrac{\Delta t}{\eta + \alpha \Delta t} \tau_{\varepsilon_{n+\alpha}} \quad ; \quad d_{n+1} = 1 - \dfrac{q(r_{n+1})}{r_{n+1}} \\ \boldsymbol{\sigma}_{n+1} = (1 - d_{n+1})\bar{\boldsymbol{\sigma}}_{n+1} \\ \mathbb{C}^{vd}_{\mathrm{alg},n+1} = (1 - d_{n+1})\mathbb{C} + \\ \qquad + \dfrac{\alpha \Delta t}{\eta + \alpha \Delta t} \dfrac{1}{\tau_{\varepsilon_{n+1}}} \dfrac{H_{n+1} r_{n+1} - q(r_{n+1})}{(r_{n+1})^2} (\bar{\boldsymbol{\sigma}}_{n+1} \otimes \bar{\boldsymbol{\sigma}}_{n+1}) \end{array} \right.$$

$$\dot{\lambda} = \frac{\frac{\partial \bar{\sigma}_1}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_\mathrm{e} : \dot{\boldsymbol{\varepsilon}}}{\frac{\partial \bar{\sigma}_1}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_\mathrm{e} : \frac{\partial \bar{\sigma}_1}{\partial \bar{\boldsymbol{\sigma}}} + H_\mathrm{p}}$$

$$\mathbf{D}_\mathrm{epd} = (1-\omega)\left(\mathbf{D}_\mathrm{e} - \frac{\mathbf{D}_\mathrm{e} : \frac{\partial \bar{\sigma}_1}{\partial \bar{\boldsymbol{\sigma}}} \otimes \frac{\partial \bar{\sigma}_1}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_\mathrm{e}}{\frac{\partial \bar{\sigma}_1}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_\mathrm{e}} : \frac{\partial \bar{\sigma}_1}{\partial \bar{\boldsymbol{\sigma}}} + H_\mathrm{p}\right) - g'_\mathrm{d}\left(\frac{\mathbf{D}_\mathrm{e} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p}) \otimes \frac{\partial \bar{\sigma}_1}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_\mathrm{e}}{\frac{\partial \bar{\sigma}_1}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_\mathrm{e} : \frac{\partial \bar{\sigma}_1}{\partial \bar{\boldsymbol{\sigma}}} + H_\mathrm{p}}\right)$$

$$\frac{\mathrm{d}\boldsymbol{\sigma}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}} = \sum_{A=1}^{3}\sum_{B=1}^{3}\frac{\mathrm{d}\hat{\sigma}_A}{\mathrm{d}\hat{\varepsilon}_B}\mathbf{m}_A^{\mathrm{T}}\mathbf{m}_B + \frac{(1-D_{n+1})}{2} \times \sum_{A=1}^{3}\sum_{B\neq A}\left[\left(\frac{\hat{\bar{\sigma}}_B-\hat{\bar{\sigma}}_A}{\hat{\varepsilon}_B-\hat{\varepsilon}_A}\right)\left(\mathbf{m}_{AB}^{\mathrm{T}}\mathbf{m}_{AB}+\mathbf{m}_{AB}^{\mathrm{T}}\mathbf{m}_{BA}\right)\right]$$

$$\mathbf{m}_A = \mathbf{v}_A^{\mathrm{T}}\mathbf{v}_A, \qquad \mathbf{m}_{AB} = \mathbf{v}_A^{\mathrm{T}}\mathbf{v}_B, \quad A \neq B.$$

$\left(\hat{\bar{\sigma}}_B - \hat{\bar{\sigma}}_A\right) / \left(\hat{\varepsilon}_B - \hat{\varepsilon}_A\right)$ by $\partial\left(\hat{\bar{\sigma}}_B - \hat{\bar{\sigma}}_A\right) / \partial\hat{\varepsilon}_B$

```
subroutine kregular_return(sg,pk,iendflag,sg2,iconvrg,pkp2,pj,resd,pnorm_res,pp,pxx)
resd(4),      pnorm_res(4), pincrmt(4), unkn1(4),
pkm,temp_sg,dth(3), sg_p_tr(3), sg_p(3),        sg(6), sg2(6), sg3(6), sig_ten(3,3),
dir(3,3),     dir_tem(3),
dgdinv(2),          ddkdldinv(2),       ddgddinv(2,2),       ddgdinvdk(2), dfdinv(2),    ddg_ddinv(2,2),
ddg_dinvdk(2),      ddk_dldinv(2),      ddgdinv(2,2),        dinv_dsig_pr(3,3),
pj(4,4),pj2(4,4),   dthi(3),      pp(3,3),pt(4,3),pr(3,3), pjr(3,3),pjr1(3),pxx(3,3)

iter=0;itot_iter=200;

pkm=em/thr/(on-tw*qmuo);gm=em/tw/(on+qmuo)

resd(1:4)=zr; pnorm_res(1:4)=zr;   pincrmt(1:4)=zr;
dl=zr;

call khaigh(sg,sv_tr1,ro_tr1,theta_tr1,dinv_dsig_pr)
call kvec_to_tens(sg,sig_ten)
call kjacobi_eigenvalue(3,sig_ten,dir,sg_p)

sg_p_tr=sg_p
pkp=pk;
tkp1=pk;
sv=sv_tr1;
ro=ro_tr1;
```

$$\boxed{\mathbf{x} = \left[\hat{\hat{\boldsymbol{\sigma}}}_{n+1} \quad \boldsymbol{\kappa}_{n+1} \quad \Delta\lambda\right]^{\mathrm{T}}}$$

$$\boxed{\mathbf{x}^{(0)} = \begin{bmatrix} \hat{\hat{\boldsymbol{\sigma}}}^{(0)}_{n+1} \\ \boldsymbol{\kappa}^{(0)}_{n+1} \\ \Delta\lambda^{(0)} \end{bmatrix} = \begin{bmatrix} \hat{\hat{\boldsymbol{\sigma}}}^{trial}_{n+1} \\ \boldsymbol{\kappa}_n \\ 0 \end{bmatrix}}$$

```
unkn1(1)=sv_tr1;
unkn1(2)=ro_tr1;
unkn1(3)=tkp1;
unkn1(4)=zr;

call kff(sv,ro,theta_tr1,tkp1,resd(4));
ppnorm_res=on;
ppnorm_res1=on;
iconvrg=0;
ggtol=gtol*1.d-2

do while (ppnorm_res>ggtol)
       iter=iter+1
       pnorm_res(1)=resd(1)/pkm;
       pnorm_res(2)=resd(2)/tw/gm
       pnorm_res(3)=resd(3);
       pnorm_res(4)=resd(4)
       ppnorm_res1=(pnorm_res(1)**tw+pnorm_res(2)**tw)**half
ppnorm_res=(pnorm_res(1)**tw+pnorm_res(2)**tw+pnorm_res(3)**tw+pnorm_res(4)**tw)**half
       if (iter.gt.1) then
              if (ppnorm_res1<gtol*gtol*10.d0) then  exit
       end if
       if(iter.eq.itot_iter) then
              if (ppnorm_res<gtol*1.d-2) then  exit
              iconvrg=1
              exit
```

```
       end if
       if (ppnorm_res>ggtol) then
                icomputeall=1
call kderv(sv,ro,theta_tr1,tkp1,icomputeall,dgdinv,dkdl,dfdinv,ddgddinv,dfdk,ddgdinvdk,ddkdldinv,ddk_dldk)
                pj(1,1)=on+pkm*dl*ddgddinv(1,1);
                pj(1,2)=pkm*dl*ddgddinv(1,2);
                pj(1,3)=pkm*dl*ddgdinvdk(1);
                pj(1,4)=pkm*dgdinv(1);
                pj(2,1)=tw*gm*dl*ddgddinv(2,1);
                pj(2,2)=on+tw*gm*dl*ddgddinv(2,2);
                pj(2,3)=tw*gm*dl*ddgdinvdk(2);
                pj(2,4)=tw*gm*dgdinv(2);pj(3,1)=dl*ddkdldinv(1);
                pj(3,2)=dl*ddkdldinv(2);pj(3,3)=dl*ddk_dldk-on;
                pj(3,4)=dkdl;pj(4,1)=dfdinv(1);
                pj(4,2)=dfdinv(2);pj(4,3)=dfdk;pj(4,4)=zr;
```

$$\boxed{\begin{bmatrix} 1+K\Delta\lambda\dfrac{\partial m_{\mathrm{V}}}{\partial\bar{\sigma}_{\mathrm{V}}} & K\Delta\lambda\dfrac{\partial m_{\mathrm{V}}}{\partial\bar{\rho}} & K\Delta\lambda\dfrac{\partial m_{\mathrm{V}}}{\partial\kappa_{\mathrm{p}}} & Km_{\mathrm{V}} \\ 2G\Delta\lambda\dfrac{\partial m_{\mathrm{D}}}{\partial\bar{\sigma}_{\mathrm{V}}} & 1+2G\Delta\lambda\dfrac{\partial m_{\mathrm{D}}}{\partial\bar{\rho}} & 2G\Delta\lambda\dfrac{\partial m_{\mathrm{D}}}{\partial\kappa_{\mathrm{p}}} & 2Gm_{\mathrm{D}} \\ -\Delta\lambda\dfrac{\partial k_{\mathrm{p}}}{\partial\bar{\sigma}_{\mathrm{V}}} & -\Delta\lambda\dfrac{\partial k_{\mathrm{p}}}{\partial\bar{\rho}} & 1 & -k_{\mathrm{p}} \\ \dfrac{\partial f_{\mathrm{p}}}{\partial\bar{\sigma}_{\mathrm{V}}} & \dfrac{\partial f_{\mathrm{p}}}{\partial\bar{\rho}} & \dfrac{\partial f_{\mathrm{p}}}{\partial\kappa_{\mathrm{p}}} & 0 \end{bmatrix}}$$

$$\boxed{\begin{aligned} \mathbf{R}_{\hat{\bar{\sigma}}} &= \hat{\bar{\boldsymbol{\sigma}}}_{n+1} + \Delta\lambda\hat{\mathbf{C}}_0\mathbf{m} - \hat{\bar{\boldsymbol{\sigma}}}_{n+1}^{trial} \\ \mathbf{R}_{\kappa} &= \boldsymbol{\kappa}_{n+1} - \Delta\lambda\,\mathbf{h} - \boldsymbol{\kappa}_n \\ \mathbf{R}_{\Delta\lambda} &= F\left(\hat{\bar{\boldsymbol{\sigma}}}_{n+1}, \boldsymbol{\kappa}_{n+1}\right) \end{aligned}}$$

```
                call Kdet44(pj,parry)
                if (abs(parry)<1.0d-10) then  iconvrg=1  exit   end if

                call kmatinv4(pj,pj2)

                pincrmt=-matmul(pj2,resd)

                unkn1=unkn1+pincrmt

                if (unkn1(2)<zr) unkn1(2)=zr
                if (unkn1(3)<pkp) unkn1(3)=pkp
                if (unkn1(4)<zr) unkn1(4)=zr
                sv=unkn1(1);ro=unkn1(2);
                tkp1=unkn1(3);dl=unkn1(4);

call kderv(sv,ro,theta_tr1,tkp1,0,dgdinv,dkdl,dfdinv,ddg_ddinv,dfdk,ddg_dinvdk,ddk_dldinv,ddk_dldk)
```

$$\boxed{\begin{aligned} \mathbf{R}_{\hat{\bar{\sigma}}} &= \hat{\bar{\boldsymbol{\sigma}}}_{n+1} + \Delta\lambda\hat{\mathbf{C}}_0\mathbf{m} - \hat{\bar{\boldsymbol{\sigma}}}_{n+1}^{trial} \\ \mathbf{R}_{\kappa} &= \boldsymbol{\kappa}_{n+1} - \Delta\lambda\,\mathbf{h} - \boldsymbol{\kappa}_n \\ \mathbf{R}_{\Delta\lambda} &= F\left(\hat{\bar{\boldsymbol{\sigma}}}_{n+1}, \boldsymbol{\kappa}_{n+1}\right) \end{aligned}}$$

$$\boxed{\begin{aligned} \bar{\sigma}_{\mathrm{V}} &= \bar{\sigma}_{\mathrm{V}}^{\mathrm{tr}} - K\Delta\lambda m_{\mathrm{V}}(\bar{\boldsymbol{\sigma}}, \kappa_{\mathrm{p}}) \\ \bar{\rho} &= \bar{\rho}^{\mathrm{tr}} - 2G\Delta\lambda m_{\mathrm{D}}(\bar{\boldsymbol{\sigma}}, \kappa_{\mathrm{p}}) \end{aligned}}$$

```
                resd(1)=sv-sv_tr1+pkm*dl*dgdinv(1)
                resd(2)=ro-ro_tr1+tw*gm*dl*dgdinv(2)
                resd(3)=pkp-tkp1+dl*dkdl
                call kff(sv,ro,theta_tr1,tkp1,resd(4))
          end if
```

```
        end do
```

```
if(iconvrg.eq.0) then
     sg_p=zr;
     sg_p(1)=sv+sqrt(tw/thr)*ro*dcos(theta_tr1)
     sg_p(2)=sv+sqrt(tw/thr)*ro*dcos(theta_tr1-tw/thr*pi)
     sg_p(3)=sv+sqrt(tw/thr)*ro*dcos(theta_tr1+tw/thr*pi);

     sig_ten=zr;
     sig_ten(1,1)=sg_p(1);
     sig_ten(2,2)=sg_p(2);
     sig_ten(3,3)=sg_p(3);

call ktens_to_vec(matmul(dir,matmul(sig_ten,transpose(dir))),sg)

pk=tkp1
pj2(1:4,3)=pj2(1:4,3)*-1.d0;
pj(1,1)=pkm;
pj(1,2)=pkm;
pj(1,3)=pkm

s1=sg_p_tr(1)*tw/thr-sg_p_tr(2)/thr-sg_p_tr(3)/thr
s2=sg_p_tr(2)*tw/thr-sg_p_tr(1)/thr-sg_p_tr(3)/thr
s3=sg_p_tr(3)*tw/thr-sg_p_tr(2)/thr-sg_p_tr(1)/thr;

par=tw*gm/ro_tr1;
pj(2,1)=par*s1;
pj(2,2)=par*s1;%%%%
pj(2,3)=par*s3;
pj(3,1)=zr;pj(3,2)=zr;pj(3,3)=zr

call kff(sv,ro,theta_tr1,pk,yield)
theta_tr3=theta_tr1-10.0d-3

call kff(sv,ro,theta_tr3,pk,yield1)
fb_th=(yield-yield1)/10.0d-3

sg3=zr;sg3(1)=sg_p(1);sg3(2)=sg_p(2);sg3(3)=sg_p(3);
call khaigh(sg3,sv_tr1,ro_tr2,theta_tr3,dinv_dsig_pr)
sg3(1)=sg_p(1)+tw/thr*10.0d-3;sg3(2)=sg_p(2)-10.0d-3/thr;
sg3(3)=sg_p(3)-10.0d-3/thr;

call khaigh(sg3,sv_tr1,ro_tr2,theta_tr2,dinv_dsig_pr)
par1=(theta_tr2-theta_tr3)/10.0d-3
sg3(2)=sg_p(2)+tw/thr*10.0d-3;
sg3(1)=sg_p(1)-10.0d-3/thr;
sg3(3)=sg_p(3)-10.0d-3/thr;

call khaigh(sg3,sv_tr1,ro_tr2,theta_tr2,dinv_dsig_pr)
par2=(theta_tr2-theta_tr3)/10.0d-3
sg3(3)=sg_p(3)+tw/thr*10.0d-3;
sg3(2)=sg_p(2)-10.0d-3/thr;
sg3(1)=sg_p(1)-10.0d-3/thr;

call khaigh(sg3,sv_tr1,ro_tr2,theta_tr2,dinv_dsig_pr)
par3=(theta_tr2-theta_tr3)/10.0d-3

dth(1)=(tw*par1-par2-par3)/thr;
dth(2)=(tw*par2-par1-par3)/thr;
dth(3)=(tw*par3-par2-par1)/thr;
```

```
pj(4,1:3)=dth(1:3)*-fb_th;
pjr(1:2,1:3)=matmul(pj2(1:2,1:4),pj(1:4,1:3))
pjr(3,1:3)=dth



     pj(1,1)=on;
     pj(2,1)=on;
     pj(3,1)=on;
     pj(1,2)=on*sqrt(tw/thr)*dcos(theta_tr1);
     pj(2,2)=on*sqrt(tw/thr)*dcos(theta_tr1-tw/thr*pi);
     pj(3,2)=on*sqrt(tw/thr)*dcos(theta_tr1+tw/thr*pi);
     pj(1,3)=-on*sqrt(tw/thr)*ro*dsin(theta_tr1);
     pj(2,3)=-on*sqrt(tw/thr)*ro*dsin(theta_tr1-tw/thr*pi);
     pj(3,3)=-on*sqrt(tw/thr)*ro*dsin(theta_tr1+tw/thr*pi);

     pjr=matmul(pj(1:3,1:3),pjr);

     pt(1,1)=pkm;pt(1,2)=pkm;pt(1,3)=pkm

     par=tw*gm/ro_tr1;

     s1=sg_p_tr(1)*tw/thr-sg_p_tr(2)/thr-sg_p_tr(3)/thr
     s2=sg_p_tr(2)*tw/thr-sg_p_tr(1)/thr-sg_p_tr(3)/thr
     s3=sg_p_tr(3)*tw/thr-sg_p_tr(2)/thr-sg_p_tr(1)/thr;

     par=tw*gm/ro_tr1;

     pt(2,1)=par*s1;pt(2,2)=par*s2;pt(2,3)=par*s3
     pt(3,1)=zr;pt(3,2)=zr;pt(3,3)=zr;

     par=tw*gm/sqrt(tw/thr)/ro_tr1/dsin(theta_tr1);

     par1=par*(s1**tw/ro_tr1**tw-on);
     par2=par*(s1*s2/ro_tr1**tw);
     par3=par*(s1*s3/ro_tr1**tw);

     if (theta_tr1**tw<gtol) then
        par=tw*gm/sqrt(tw/thr)/ro_tr1/dsin(theta_tr1-tw/thr*pi);
        par1=par*(s1*s2/ro_tr1**tw);
        par2=par*(s2**tw/ro_tr1**tw-on);
        par3=par*(s2*s3/ro_tr1**tw);
     end if

     pt(4,1)=(tw*par1-par2-par3)/thr;
     pt(4,2)=(tw*par2-par1-par3)/thr;
     pt(4,3)=(tw*par3-par2-par1)/thr;
     pp(1,1)=on;pp(2,1)=on;pp(3,1)=on;
     pp(1,2)=on*sqrt(tw/thr)*dcos(theta_tr1);
     pp(2,2)=on*sqrt(tw/thr)*dcos(theta_tr1-tw/thr*pi);
     pp(3,2)=on*sqrt(tw/thr)*dcos(theta_tr1+tw/thr*pi);

     par=(sg_p(1)-sv)
     pp(1,3)=-par*dsin(theta_tr1)/dcos(theta_tr1);
     pp(2,3)=-par*dsin(theta_tr1-tw/thr*pi)/dcos(theta_tr1);
     pp(3,3)=-par*dsin(theta_tr1+tw/thr*pi)/dcos(theta_tr1);

    pr(1:2,1:2)=pj2(1:2,1:2)
```

```
        pr(1:2,3)=pj2(1:2,4)*-fb_th*0
        pr(3,1)=zr;
        pr(3,2)=zr;
        pr(3,3)=on;

        pt(3,1:3)=pt(4,1:3);
        pp=matmul(pp,matmul(pr,pt(1:3,1:3)))

        pjr1(1:2)=pj2(3,1:2);pjr1(3)=pj2(3,4)*-fb_th*0;
        pjr1(1:3)=matmul(pjr1,pt(1:3,1:3))
        do i=1,3
            do j=1,3
                pxx(i,j)=sg_p(i)*pjr1(j)
            end do
        end do
    end if
    return
    end
```

$$\boxed{\delta\bar{\boldsymbol{\sigma}} = \sum_{I=1}^{3} \delta\bar{\sigma}_I \mathbf{n}_I \otimes \mathbf{n}_I + \sum_{I=1}^{3} \bar{\sigma}_I (\delta\mathbf{n}_I \otimes \mathbf{n}_I + \mathbf{n}_I \otimes \delta\mathbf{n}_I)}$$

$$\boxed{\delta\mathbf{n}_1 = \frac{\delta\varepsilon_{12}}{\varepsilon_{\mathrm{e}1}^{\mathrm{tr}} - \varepsilon_{\mathrm{e}2}^{\mathrm{tr}}}\mathbf{n}_2 + \frac{\delta\varepsilon_{13}}{\varepsilon_{\mathrm{e}1}^{\mathrm{tr}} - \varepsilon_{\mathrm{e}3}^{\mathrm{tr}}}\mathbf{n}_3}$$

$$\boxed{\begin{aligned} \delta\bar{\sigma}_{11} &= \mathbf{n}_1 \cdot \bar{\boldsymbol{\sigma}} \cdot \mathbf{n}_1 = \delta\bar{\sigma}_1 \\ \delta\bar{\sigma}_{12} &= \mathbf{n}_1 \cdot \bar{\boldsymbol{\sigma}} \cdot \mathbf{n}_2 = \frac{\bar{\sigma}_1 - \bar{\sigma}_2}{\varepsilon_{\mathrm{e}1}^{\mathrm{tr}} - \varepsilon_{\mathrm{e}2}^{\mathrm{tr}}}\delta\varepsilon_{12} \end{aligned}}$$

$$\boxed{2G_{12} = \frac{\bar{\sigma}_1 - \bar{\sigma}_2}{\varepsilon_{\mathrm{e}1}^{\mathrm{tr}} - \varepsilon_{\mathrm{e}2}^{\mathrm{tr}}}}$$

$$\boxed{\bar{\sigma}_1 - \bar{\sigma}_2 = \sqrt{\frac{2}{3}}\bar{\rho}[\cos\theta^{\mathrm{tr}} - \cos(\theta^{\mathrm{tr}} - 2\pi/3)]}$$

$$\boxed{\varepsilon_{\mathrm{e}1}^{\mathrm{tr}} - \varepsilon_{\mathrm{e}2}^{\mathrm{tr}} = \sqrt{\frac{2}{3}}\rho_{\mathrm{e}}^{\mathrm{tr}}[\cos\theta^{\mathrm{tr}} - \cos(\theta^{\mathrm{tr}} - 2\pi/3)]}$$

$$\boxed{\delta\bar{\sigma}_1 = \delta\bar{\sigma}_{\mathrm{V}} + \sqrt{\frac{2}{3}}\cos\theta^{\mathrm{tr}}\delta\bar{\rho} - \sqrt{\frac{2}{3}}\bar{\rho}\sin\theta^{\mathrm{tr}}\delta\theta^{\mathrm{tr}}}$$

$$\boxed{\delta\theta^{\mathrm{tr}} = \frac{\sqrt{3}}{2J_{\mathrm{e}2}^{3}\sin 3\theta^{\mathrm{tr}}}\left(\frac{3}{2}\sqrt{J_{\mathrm{e}2}}J_{\mathrm{e}3}\delta J_{\mathrm{e}2} - J_{\mathrm{e}2}^{3/2}\delta J_{\mathrm{e}3}\right) = \frac{\sqrt{3}}{2J_{\mathrm{e}2}^{5/2}\sin 3\theta^{\mathrm{tr}}}\left(\frac{3}{2}J_{\mathrm{e}3}\mathbf{e}_{\mathrm{e}}^{\mathrm{tr}} - J_{\mathrm{e}2}\mathbf{e}_{\mathrm{e}}^{\mathrm{tr}} \cdot \mathbf{e}_{\mathrm{e}}^{\mathrm{tr}}\right) : \delta\mathbf{e}}$$

$$\boxed{\begin{bmatrix} 1 + K\Delta\lambda\dfrac{\partial m_{\mathrm{V}}}{\partial\bar{\sigma}_{\mathrm{V}}} & K\Delta\lambda\dfrac{\partial m_{\mathrm{V}}}{\partial\bar{\rho}} & K\Delta\lambda\dfrac{\partial m_{\mathrm{V}}}{\partial\kappa_{\mathrm{p}}} & Km_{\mathrm{V}} \\ 2G\Delta\lambda\dfrac{\partial m_{\mathrm{D}}}{\partial\bar{\sigma}_{\mathrm{V}}} & 1 + 2G\Delta\lambda\dfrac{\partial m_{\mathrm{D}}}{\partial\bar{\rho}} & 2G\Delta\lambda\dfrac{\partial m_{\mathrm{D}}}{\partial\kappa_{\mathrm{p}}} & 2Gm_{\mathrm{D}} \\ -\Delta\lambda\dfrac{\partial k_{\mathrm{p}}}{\partial\bar{\sigma}_{\mathrm{V}}} & -\Delta\lambda\dfrac{\partial k_{\mathrm{p}}}{\partial\bar{\rho}} & 1 & -k_{\mathrm{p}} \\ \dfrac{\partial f_{\mathrm{p}}}{\partial\bar{\sigma}_{\mathrm{V}}} & \dfrac{\partial f_{\mathrm{p}}}{\partial\bar{\rho}} & \dfrac{\partial f_{\mathrm{p}}}{\partial\kappa_{\mathrm{p}}} & 0 \end{bmatrix} \begin{Bmatrix} \delta\bar{\sigma}_{\mathrm{V}} \\ \delta\bar{\rho} \\ \delta\kappa_{\mathrm{p}} \\ \delta\lambda \end{Bmatrix} = \begin{Bmatrix} \delta\bar{\sigma}_{\mathrm{V}}^{\mathrm{tr}} \\ \delta\bar{\rho}^{\mathrm{tr}} \\ 0 \\ -\dfrac{\partial f_{\mathrm{p}}}{\partial\theta}\delta\theta^{\mathrm{tr}} \end{Bmatrix}}$$

$$\delta\bar{\sigma}_{\mathrm{V}}^{\mathrm{tr}} = K\boldsymbol{\delta} : \delta\boldsymbol{\varepsilon}$$
$$\delta\bar{\rho}^{\mathrm{tr}} = \frac{2G}{\bar{\rho}^{\mathrm{tr}}}\mathbf{s}^{\mathrm{tr}} : \delta\mathbf{e}$$

$$\delta\boldsymbol{\sigma} = (1-\omega)\delta\bar{\boldsymbol{\sigma}} - \bar{\boldsymbol{\sigma}}\,\delta\omega$$

$$\delta\omega = g_{\mathrm{d}}'\,\delta\tilde{\varepsilon} = \frac{g_{\mathrm{d}}'}{x_{\mathrm{s}}}\left[\Delta\lambda\left(\frac{\partial m_{\mathrm{V}}}{\partial\bar{\sigma}_{\mathrm{V}}} - \frac{m_{\mathrm{V}}x_{\mathrm{s}}'}{x_{\mathrm{s}}}\right)\delta\bar{\sigma}_{\mathrm{V}} + \Delta\lambda\frac{\partial m_{\mathrm{V}}}{\partial\bar{\rho}}\,\delta\bar{\rho} + \Delta\lambda\frac{\partial m_{\mathrm{V}}}{\partial\kappa_{\mathrm{p}}}\,\delta\kappa_{\mathrm{p}} + m_{\mathrm{V}}\,\delta\lambda\right]$$

$g_{\mathrm{d}}' = \mathrm{d}g_{\mathrm{d}}/\mathrm{d}\kappa_{\mathrm{d}}$ and $x_{\mathrm{s}}' = \mathrm{d}x_{\mathrm{s}}/\mathrm{d}\bar{\sigma}_{\mathrm{V}}$

$$g_{\mathrm{p}}(\bar{\sigma}_{\mathrm{V}}, \bar{\rho}; \kappa_{\mathrm{p}}) - g_{\mathrm{p}}(0, 0; \kappa_{\mathrm{p}}) \geqslant 0$$

```
subroutine khaigh(sg,sv,ro,thi,dinv_dsig_pr)


sv=(sg(1)+sg(2)+sg(3))/thr;
s=sg;
s(1:3)=sg(1:3)-sv
pj2=half*(s(1)**tw+s(2)**tw+s(3)**tw)+s(4)**tw+s(5)**tw+s(6)**tw


if (pj2<=gtol) THEN
       thi=zr;ro=zr
else
       ro=sqrt(tw*pj2);
pj3=s(1)**3+s(2)**3+s(3)**3+3*s(1)*(s(4)**2+s(6)**2)+sx*s(4)*s(5)*s(6)+3*s(2)*(s(4)**2+s(5)**tw)+3*s(3)*(s(5)**
2+s(6)**2)
       pj3=pj3/thr;
       thi=thr*sqrt(thr)/tw*pj3/(pj2**(1.5d0));
end if

if (thi>on) then
       thi=on
else if (thi<-on) then
       thi=-on
end if
thi= on/thr*dacos(thi)
```

$$\bar{\sigma}_{\mathrm{V}} = \frac{I_1}{3}$$

$$\bar{\rho} = \sqrt{2J_2}$$

$$J_2 = \tfrac{1}{2}\bar{\mathbf{s}} : \bar{\mathbf{s}} = \tfrac{1}{2}\bar{\mathbf{s}}^2 : \boldsymbol{\delta} = \tfrac{1}{2}\bar{s}_{i}$$

$$J_3 = \tfrac{1}{3}\bar{\mathbf{s}}^3 : \boldsymbol{\delta} = \tfrac{1}{3}\bar{s}_{ij}\bar{s}_{jk}\bar{s}_{ki}$$

$$\bar{\mathbf{s}} = \bar{\boldsymbol{\sigma}} - \boldsymbol{\delta} I_1/3.$$

$$\bar{\theta} = \frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\,\frac{J_3}{J_2^{3/2}}\right)$$

$$I_1 = \bar{\boldsymbol{\sigma}} : \boldsymbol{\delta} = \bar{\sigma}_{ij}\delta_{ij}$$

```
subroutine kequ_e(sv,ro,th,equ_e)
par1=(on-ecc**tw);par2=(tw*ecc-on)
rcos=(fr*par1*dcos(th)**tw+par2**tw)
rcos=rcos/(tw*par1*dcos(th)+par2*sqrt(fr*par1*dcos(th)**tw+5.d0*ecc**tw-fr*ecc))
par_p=-pm0*(ro*rcos/sqrt(sx)/fc+sv/fc);
par_q=-1.5d0*ro**tw/fc**tw
equ_e=e0*(-half*par_p+sqrt(par_p**tw/fr-par_q))
if (equ_e.le.zr) equ_e=zr
```

$$\boxed{\begin{aligned} f_{\mathrm{p}}(\bar{\sigma}_{\mathrm{V}}, \bar{\rho}, \bar{\theta}; \kappa_{\mathrm{p}}) = {} & \left\{ [1 - q_{\mathrm{h1}}(\kappa_{\mathrm{p}})] \left(\frac{\bar{\rho}}{\sqrt{6} f_{\mathrm{c}}} + \frac{\bar{\sigma}_{\mathrm{V}}}{f_{\mathrm{c}}} \right)^2 + \sqrt{\frac{3}{2}} \frac{\bar{\rho}}{f_{\mathrm{c}}} \right\}^2 \\ & + m_0 q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}}) q_{\mathrm{h2}}(\kappa_{\mathrm{p}}) \left[\frac{\bar{\rho}}{\sqrt{6} f_{\mathrm{c}}} r(\cos \bar{\theta}) + \frac{\bar{\sigma}_{\mathrm{V}}}{f_{\mathrm{c}}} \right] \\ & - q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}}) q_{\mathrm{h2}}^2(\kappa_{\mathrm{p}}) \end{aligned}}$$

$$\boxed{\begin{aligned} \tilde{\varepsilon} = {} & \frac{\varepsilon_0 m_0}{2} \left(\frac{\bar{\rho}}{\sqrt{6} f_{\mathrm{c}}} r(\cos\theta) + \frac{\bar{\sigma}_{\mathrm{V}}}{f_{\mathrm{c}}} \right) \\ & + \sqrt{\frac{\varepsilon_0^2 m_0^2}{4} \left(\frac{\bar{\rho}}{\sqrt{6} f_{\mathrm{c}}} r(\cos\theta) + \frac{\bar{\sigma}_{\mathrm{V}}}{f_{\mathrm{c}}} \right)^2 + \frac{3 \varepsilon_0^2 \bar{\rho}^2}{2 f_{\mathrm{c}}^2}} \end{aligned}}$$

$$\boxed{\tilde{\varepsilon} = \varepsilon_0 \frac{\bar{\sigma}_{\mathrm{t}}}{f_{\mathrm{t}}} = \bar{\sigma}_{\mathrm{t}} / E}$$

$$\boxed{\tilde{\varepsilon} = \frac{\bar{\sigma}_{\mathrm{c}} \varepsilon_0}{f_{\mathrm{c}}} = \frac{\bar{\sigma}_{\mathrm{c}} f_{\mathrm{t}}}{E f_{\mathrm{c}}}}$$

$$\textbf{\textit{Ductility}} \quad \chi_h(\bar{\sigma}_V) = \begin{cases} A_h - (A_h - B_h)\exp(-R_h(\bar{\sigma}_V)/C_h), & if\ R_h(\bar{\sigma}_V) \geq 0 \\ E_h \exp(R_h(\bar{\sigma}_V)/F_h) + D_h, & if\ R_h(\bar{\sigma}_V) < 0 \end{cases}, R_h(\bar{\sigma}_V) = -\frac{\bar{\sigma}_V}{f_c} - \frac{1}{3}, E_h = B_h - D_h, F_h = \frac{(B_h - D_h)C_h}{A_h - B_h}$$

$$x_{\mathrm{h}}(\bar{\sigma}_{\mathrm{V}}) = \begin{cases} A_{\mathrm{h}} - (A_{\mathrm{h}} - B_{\mathrm{h}})\exp(-R_{\mathrm{h}}(\bar{\sigma}_{\mathrm{V}})/C_{\mathrm{h}}) & \text{if } R_{\mathrm{h}}(\bar{\sigma}_{\mathrm{V}}) \geqslant 0 \\ E_{\mathrm{h}}\exp(R_{\mathrm{h}}(\bar{\sigma}_{\mathrm{V}})/F_{\mathrm{h}}) + D_{\mathrm{h}} & \text{if } R_{\mathrm{h}}(\bar{\sigma}_{\mathrm{V}}) < 0 \end{cases}$$

$$R_{\mathrm{h}}(\bar{\sigma}_{\mathrm{V}}) = -\frac{\bar{\sigma}_{\mathrm{V}}}{\bar{f}_{\mathrm{c}}} - \frac{1}{3}$$

$$E_{\mathrm{h}} = B_{\mathrm{h}} - D_{\mathrm{h}}$$

$$F_{\mathrm{h}} = \frac{(B_{\mathrm{h}} - D_{\mathrm{h}})C_{\mathrm{h}}}{A_{\mathrm{h}} - B_{\mathrm{h}}}$$

```
subroutine kductility(sv,th,icomput_deriv,duct_m,dduct_dinv)
```

$$\dot{\kappa}_{\mathrm{p}} = \frac{\|\dot{\varepsilon}_{\mathrm{p}}\|}{x_{\mathrm{h}}(\bar{\sigma}_{\mathrm{V}})}\left(2\cos\bar{\theta}\right)^2 = \frac{\dot{\lambda}\|\mathbf{m}\|}{x_{\mathrm{h}}(\bar{\sigma}_{\mathrm{V}})}\left(2\cos\bar{\theta}\right)^2$$

```
par1=(tw*dcos(th))**tw;x=-on*(sv+fc/thr)/fc;
if (x<zr) then
       eh=bh-dh;fh=eh*ch/(ah-bh)
       duct_m=(eh*exp(x/fh)+dh)/par1
       if (icomput_deriv.eq.1) then      dduct_dinv(1)=eh/fh*exp(x/fh)/par1*(-on)/fc
       else                              dduct_dinv(1)=zr      end if

else
       duct_m=(ah-(ah-bh)*exp(-x/ch))/par1
       if (icomput_deriv.eq.1) then  dduct_dinv(1)=(bh-ah)/ch*exp(-x/ch)/par1/fc
       else                          dduct_dinv(1)=zr         end if
end if
```

```
subroutine kff(sv,ro,th,pkp,f)
```

$$f_{\mathrm{p}}(\bar{\sigma}_{\mathrm{V}},\bar{\rho},\bar{\theta};\kappa_{\mathrm{p}}) = \left\{[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]\left(\frac{\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}}+\frac{\bar{\sigma}_{\mathrm{V}}}{f_{\mathrm{c}}}\right)^2+\sqrt{\frac{3}{2}}\frac{\bar{\rho}}{f_{\mathrm{c}}}\right\}^2 + m_0 q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})q_{\mathrm{h2}}(\kappa_{\mathrm{p}})\left[\frac{\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}}r(\cos\bar{\theta})+\frac{\bar{\sigma}_{\mathrm{V}}}{f_{\mathrm{c}}}\right] - q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})q_{\mathrm{h2}}^2(\kappa_{\mathrm{p}})$$

$$e=\frac{1+\epsilon}{2-\epsilon},\quad \text{where } \epsilon=\frac{\bar{f}_{\mathrm{t}}}{\bar{f}_{\mathrm{b}}}\frac{\bar{f}_{\mathrm{b}}^2-\bar{f}_{\mathrm{c}}^2}{\bar{f}_{\mathrm{c}}^2-\bar{f}_{\mathrm{t}}^2}$$

```
par1=(on-ecc**tw);par2=(tw*ecc-on)
```

$$r(\cos\bar{\theta})=\frac{4(1-e^2)\cos^2\bar{\theta}+(2e-1)^2}{2(1-e^2)\cos\bar{\theta}+(2e-1)\sqrt{4(1-e^2)\cos^2\bar{\theta}+5e^2-4e}}$$

```
rcos=(fr*par1*dcos(th)**tw+par2**tw)
rcos=rcos/(tw*par1*dcos(th)+par2*sqrt(fr*par1*dcos(th)**tw+5.d0*ecc**tw-fr*ecc))

call kcqh1(pkp,int(0),qh1,dqh1_dk)
call kcqh2(pkp,int(0),qh2,dqh2_dk)

par1=ro/(fc*sqrt(sx))+sv/fc;
par1=(on-qh1)*par1**tw+sqrt(1.5d0)*ro/fc
f=par1**tw+qh1**tw*qh2*pm0*(sv/fc+ro*rcos/(sqrt(tw*thr)*fc))-qh1**tw*qh2**tw
```

```
subroutine kcqh2(par,icomput_deriv,qh1,dqh1_dk)
```

$$\boxed{q_{h2}(\kappa_p) = \begin{cases} 1 & \text{if } \kappa_p < 1 \\ 1 + H_p(\kappa_p - 1) & \text{if } \kappa_p \geqslant 1 \end{cases}}$$

```
if (par<on) then  qh1=on
else              qh1=on+(par-on)*hp      end if


dqh1_dk=zr
if (icomput_deriv.eq.1) then
     if (par<=on) then     dqh1_dk=zr
     else                dqh1_dk=hp  end if
end if
```

```
subroutine kcqh1(par,icomput_deriv,qh2,dqh2_dk)
```

$$\boxed{q_{h1}(\kappa_p) = \begin{cases} q_{h0} + (1 - q_{h0})\left(\kappa_p^3 - 3\kappa_p^2 + 3\kappa_p\right) - H_p\left(\kappa_p^3 - 3\kappa_p^2 + 2\kappa_p\right) & \text{if } \kappa_p < 1 \\ 1 & \text{if } \kappa_p \geqslant 1 \end{cases}}$$

```
if (par<=zr) then  qh2=qh0
else if (par>zr.and. par<on)
              then  qh2=(on-qh0-hp)*par**thr-thr*(on-qh0-hp)*par**tw+(thr*(on-qh0)-tw*hp)*par+qh0
else    qh2=on       end if


dqh2_dk=zr
if (icomput_deriv.eq.1) then
     if (par<=on) then  dqh2_dk=thr*(on-qh0-hp)*par**tw-tw*thr*(on-qh0-hp)*par+(thr*(on-qh0)-tw*hp)
     else               dqh2_dk=zr        end if
end if
```

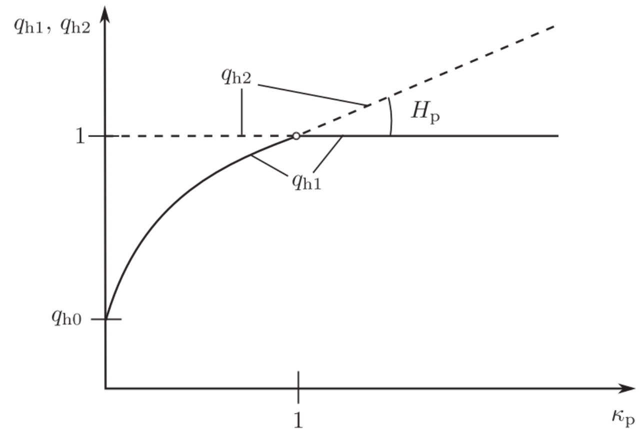


Fig. 3. The two hardening laws q_{h1} (solid line) and q_{h2} (dashed line).

```fortran
subroutine kc_alpha(s,st,sc,alpha)            !input s
call kvec_to_tens(s,sig_ten)
call kjacobi_eigenvalue(3,sig_ten,dir,sig_pr)
pnorm_s=(sig_pr(1)**tw+sig_pr(2)**tw+sig_pr(3)**tw)
alpha_t=zr;s_pt=zr;s_pc=zr;
if (pnorm_s>zr) then
       do i=1,3
              if (sig_pr(i)>=zr) then
                     s_pt(i)=sig_pr(i);s_pc(i)=zr
              else
                     s_pc(i)=sig_pr(i);s_pt(i)=zr
              end if
              alpha_t=alpha_t+s_pt(i)*(s_pt(i)+s_pc(i))
       end do
       alpha_t=alpha_t/pnorm_s
end if
alpha=on-alpha_t

call kvec_to_tens(s_pt,sig_ten)
call ktens_to_vec(matmul(dir,matmul(sig_ten,transpose(dir))),st)
call kvec_to_tens(s_pc,sig_ten)
call ktens_to_vec(matmul(dir,matmul(sig_ten,transpose(dir))),sc)
```

$$[\sigma] \rightarrow \text{diagonalization} \rightarrow [\sigma]_{diag} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$[\sigma]^{+}_{diag} = \begin{bmatrix} \langle\sigma_1\rangle & 0 & 0 \\ 0 & \langle\sigma_2\rangle & 0 \\ 0 & 0 & \langle\sigma_3\rangle \end{bmatrix} \rightarrow \begin{matrix}\text{return to original}\\ \text{system of}\\ \text{coordinates}\end{matrix} \rightarrow \underbrace{[\sigma]^{+}}_{\substack{\text{Positive}\\ \text{counterpar}\\ \text{of } [\sigma]}}$$

$$\boldsymbol{\sigma} = \sum_{i=1}^{i=3} \underbrace{\sigma_i}_{\text{eigenvalue "}i\text{"}} \overbrace{\hat{\mathbf{p}}_i}^{\text{eigenvector "}i\text{"}} \otimes \hat{\mathbf{p}}_i$$

$$\boldsymbol{\sigma}^{+} = \sum_{i=1}^{i=3} \langle\sigma_i\rangle \hat{\mathbf{p}}_i \otimes \hat{\mathbf{p}}_i$$

$$\boldsymbol{\sigma} = \underbrace{(1-d)}_{\geq 0}\bar{\boldsymbol{\sigma}} \Rightarrow \boldsymbol{\sigma}^{+} = (1-d)\bar{\boldsymbol{\sigma}}^{+}$$

$$\boldsymbol{\sigma} = (1-\omega_{\mathrm{t}})\bar{\boldsymbol{\sigma}}_{\mathrm{t}} + (1-\omega_{\mathrm{c}})\bar{\boldsymbol{\sigma}}_{\mathrm{c}}$$

```
subroutine kderv(sv,ro,th,tkp,icomputeall,dg_dinv,dkdl,dfdinv,ddg_ddinv,dfdk,ddg_dinvdk,ddk_dldinv,ddk_dldk)

call kcqh1(tkp,int(1),qh1,dqh1_dk);call kcqh2(tkp,int(1),qh2,dqh2_dk)
```

$$\boxed{A_{\mathrm{g}} = \frac{3f_{\mathrm{t}}q_{\mathrm{h2}}}{f_{\mathrm{c}}} + \frac{m_0}{2}}$$

```
ag=ft*qh2*thr/fc+pm0/tw;
bg_top=qh2/thr*(on+ft/fc)
bg_bottom=log(ag)-log(tw*df-on)-log(thr*qh2+pm0/tw)+log(df+on)
bg=bg_top/bg_bottom
```

$$B_{\mathrm{g}} = \frac{B_{\mathrm{g1}}}{B_{\mathrm{g2}}} = \frac{\left(\frac{q_{\mathrm{h2}}}{3}\right)\left(1+\frac{f_{\mathrm{t}}}{f_{\mathrm{c}}}\right)}{\ln A_{\mathrm{g}} - \ln(2D_{\mathrm{f}}-1) - \ln\left(3q_{h2}+\frac{m_0}{2}\right) + \ln(D_f+1)}$$

$$\boxed{\frac{\partial m_g}{\partial \bar{\sigma}_{\mathrm{v}}}} = \frac{A_g(\kappa_{\mathrm{p}})}{f_c}\exp\frac{\bar{\sigma}_{\mathrm{v}} - q_{\mathrm{h1}}(\kappa_{\mathrm{p}})f_t/3}{B_g(\kappa_{\mathrm{p}})f_c}, \qquad \boxed{\frac{\partial m_g}{\partial \kappa_{\mathrm{p}}}} = -\frac{A_g(\kappa_{\mathrm{p}})f_t}{3f_c}\frac{dq_{\mathrm{h1}}}{d\kappa_{\mathrm{p}}}\exp\frac{\bar{\sigma}_{\mathrm{v}} - q_{\mathrm{h1}}(\kappa_{\mathrm{p}})f_t/3}{B_g(\kappa_{\mathrm{p}})f_c}$$

$$\boxed{\frac{dB_{\mathrm{g1}}}{d\kappa_{\mathrm{p}}}} = \frac{(1+f_{\mathrm{t}}/f_{\mathrm{c}})}{3}\frac{dq_{\mathrm{h2}}}{d\kappa_{\mathrm{p}}}, \qquad \boxed{\frac{dB_{\mathrm{g2}}}{d\kappa_{\mathrm{p}}}} = \frac{1}{A_{\mathrm{g}}}\frac{dA_{\mathrm{g}}}{d\kappa_{\mathrm{p}}} - \frac{1}{\left(q_{h2}+\frac{m_0}{6}\right)}\frac{dq_{\mathrm{h2}}}{d\kappa_{\mathrm{p}}}, \qquad \boxed{\frac{dB_{\mathrm{g}}}{d\kappa_{\mathrm{p}}}} = \left(B_{\mathrm{g2}}\right)^{-1}\frac{dB_{\mathrm{g1}}}{d\kappa_{\mathrm{p}}} - B_{\mathrm{g1}}\left(B_{\mathrm{g2}}\right)^{-2}\frac{dB_{\mathrm{g2}}}{d\kappa_{\mathrm{p}}}$$

```
r=(sv-ft/thr*qh2)/fc/bg;
```

$$m_g(\bar{\sigma}_{\mathrm{v}},\kappa_{\mathrm{p}}) = A_g(\kappa_{\mathrm{p}})B_g(\kappa_{\mathrm{p}})\exp\frac{\bar{\sigma}_{\mathrm{v}} - q_{\mathrm{h1}}(\kappa_{\mathrm{p}})f_t/3}{B_g(\kappa_{\mathrm{p}})f_c}$$

$$\boxed{\frac{\partial m_g}{\partial \bar{\sigma}_{\mathrm{v}}}} = \frac{A_g(\kappa_{\mathrm{p}})}{f_c}\exp\frac{\bar{\sigma}_{\mathrm{v}} - q_{\mathrm{h1}}(\kappa_{\mathrm{p}})f_t/3}{B_g(\kappa_{\mathrm{p}})f_c}$$

```
pmQ=ag*exp(r);
```

$$B_1 = \frac{\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}} + \frac{\bar{\sigma}_{\mathrm{v}}}{f_{\mathrm{c}}}, \qquad B_y = \frac{\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}}r(\cos\bar{\theta}) + \frac{\bar{\sigma}_{\mathrm{v}}}{f_c},$$

$$A_1 = [1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})](B_x)^2 + \sqrt{\frac{3}{2}}\frac{\bar{\rho}}{f_{\mathrm{c}}},$$

$$g_{\mathrm{p}}(\bar{\sigma}_{\mathrm{V}},\bar{\rho};\kappa_{\mathrm{p}}) = (A_1)^2 + q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})\left[\frac{m_0\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}} + \frac{m_g(\bar{\sigma}_{\mathrm{v}},\kappa_{\mathrm{p}})}{f_{\mathrm{c}}}\right]$$

$$\frac{\partial g_p}{\partial \bar{\rho}} = 2A_1\frac{\partial A_1}{\partial \bar{\rho}} + \frac{m_0 q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})}{\sqrt{6}f_c}, \qquad \frac{\partial g_p}{\partial \bar{\sigma}_{\mathrm{V}}} = 2A_1\frac{\partial A_1}{\partial \bar{\sigma}_{\mathrm{V}}} + \frac{q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})}{f_c}\left(\frac{\partial m_g}{\partial \bar{\sigma}_{\mathrm{V}}}\right),$$

$$\frac{\partial A_1}{\partial \bar{\rho}} = \frac{2[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]B_1}{\sqrt{6}f_c} + \frac{\sqrt{1.5}}{f_{\mathrm{c}}}, \qquad \frac{\partial A_1}{\partial \bar{\sigma}_{\mathrm{V}}} = \frac{2[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]B_1}{f_c},$$

$$\frac{\partial^2 A_x}{\partial \bar{\rho}^2} = \frac{[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]}{3f_c^2}, \qquad \frac{\partial^2 A_x}{\partial \bar{\sigma}_{\mathrm{V}}{}^2} = \frac{2[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]}{f_c^2}, \qquad \frac{\partial^2 A_x}{\partial \bar{\rho}\partial \bar{\sigma}_{\mathrm{V}}} = \frac{2[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]}{\sqrt{6}f_c^2}$$

$$g_{\mathrm{p}}(\bar{\sigma}_{\mathrm{V}},\bar{\rho};\kappa_{\mathrm{p}}) = \left\{[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]\left(\frac{\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}} + \frac{\bar{\sigma}_{\mathrm{V}}}{f_{\mathrm{c}}}\right)^2 + \sqrt{\frac{3}{2}}\frac{\bar{\rho}}{f_{\mathrm{c}}}\right\}^2 + q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})\left(\frac{m_0\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}} + \frac{m_{\mathrm{g}}(\bar{\sigma}_{\mathrm{V}},\kappa_{\mathrm{p}})}{f_{\mathrm{c}}}\right)$$

```
Bl=sv/fc+ro/(fc*sqrt(sx));
Al=(1.0-qh1)*Bl**tw+sqrt(1.5)*ro/fc;
dg_dinv(1)=4.0*(1.0-qh1)/fc*Al*Bl+qh1**tw*pmQ/fc;
dg_dinv(2)=Al/(sqrt(sx)*fc)*(4.0*(1.0-qh1)*Bl+sx)+pm0*qh1**tw/(sqrt(sx)*fc);
```

$$\|\boldsymbol{m}\| = \left\|\frac{\partial g_p}{\partial \bar{\boldsymbol{\sigma}}}\right\|$$

$$\boldsymbol{m} = \frac{\partial g}{\partial \bar{\boldsymbol{\sigma}}} = \frac{\partial g}{\partial \bar{\sigma}_V}\frac{\partial \bar{\sigma}_V}{\partial \bar{\boldsymbol{\sigma}}} + \frac{\partial g}{\partial \bar{\rho}}\frac{\partial \bar{\rho}}{\partial \bar{\boldsymbol{\sigma}}}, \qquad \frac{\partial \bar{\sigma}_V}{\partial \bar{\boldsymbol{\sigma}}} = \frac{\boldsymbol{\delta}}{3}, \qquad \frac{\partial \bar{\rho}}{\partial \bar{\boldsymbol{\sigma}}} = \frac{\bar{\boldsymbol{s}}}{\bar{\rho}},$$

$$\bar{\rho} = \sqrt{2J_2} = \sqrt{\bar{\boldsymbol{s}}:\bar{\boldsymbol{s}}}, \qquad \bar{\sigma}_V = \frac{I_1}{3}, \qquad J_2 = \frac{1}{2}\bar{\boldsymbol{s}}:\bar{\boldsymbol{s}},$$

$$\boldsymbol{\delta}:\boldsymbol{\delta} = \mathbf{3}, \boldsymbol{\delta}:\bar{\boldsymbol{s}} = 0, \bar{\boldsymbol{s}}:\bar{\boldsymbol{s}} = \bar{\rho}$$

$$\|\boldsymbol{m}\|^2 = \left(\frac{\partial g}{\partial \bar{\sigma}_V}\frac{\boldsymbol{\delta}}{3} + \frac{\partial g}{\partial \bar{\rho}}\frac{\bar{\boldsymbol{s}}}{\bar{\rho}}\right):(\ldots) = \left(\left[\frac{\partial g}{\partial \bar{\sigma}_V}\right]^2\frac{1}{3} + \left[\frac{\partial g}{\partial \bar{\rho}}\right]^2\right)$$

```
equivaplentdg_dsg=sqrt(on/thr*dg_dinv(1)**tw+dg_dinv(2)**tw)
```

$$\dot{\kappa}_p = \frac{\|\dot{\boldsymbol{\varepsilon}}_p\|}{x_h(\bar{\sigma}_V)}\cos^2\bar{\theta} = \frac{\dot{\lambda}\|\mathbf{m}\|}{x_h(\bar{\sigma}_V)}\cos^2\bar{\theta} \qquad \dot{\kappa}_p = \dot{\lambda}k_p(\bar{\boldsymbol{\sigma}}, \kappa_p) \qquad k_p(\bar{\boldsymbol{\sigma}}, \kappa_p) = \frac{\|\mathbf{m}(\bar{\boldsymbol{\sigma}}, \kappa_p)\|}{x_h(\bar{\boldsymbol{\sigma}}:\boldsymbol{\delta}/3)}\cos^2\bar{\theta}$$

```
call kductility(sv,th,1,duct_m,dduct_dinv)
dkdl=equivaplentdg_dsg/duct_m;
```

$$\mathbf{m} = \frac{\partial g}{\partial \bar{\boldsymbol{\sigma}}} = \frac{\partial g}{\partial \bar{\sigma}_V}\frac{\partial \bar{\sigma}_V}{\partial \bar{\boldsymbol{\sigma}}} + \frac{\partial g}{\partial \bar{\rho}}\frac{\partial \bar{\rho}}{\partial \bar{\boldsymbol{\sigma}}} \qquad \mathbf{m} = \frac{\partial g}{\partial \bar{\boldsymbol{\sigma}}} = \frac{\partial m_g}{\partial \bar{\sigma}_V}\frac{\boldsymbol{\delta}}{3f_c} + \left(\frac{3}{f_c} + \frac{m_0}{\sqrt{6}\bar{\rho}}\right)\frac{\bar{\mathbf{s}}}{f_c}$$

$$\begin{bmatrix} 1 + K\,\Delta\lambda\dfrac{\partial m_V}{\partial \bar{\sigma}_V} & K\,\Delta\lambda\dfrac{\partial m_V}{\partial \bar{\rho}} & K\,\Delta\lambda\dfrac{\partial m_V}{\partial \kappa_p} & K m_V \\ 2G\,\Delta\lambda\dfrac{\partial m_D}{\partial \bar{\sigma}_V} & 1 + 2G\,\Delta\lambda\dfrac{\partial m_D}{\partial \bar{\rho}} & 2G\,\Delta\lambda\dfrac{\partial m_D}{\partial \kappa_p} & 2G m_D \\ -\Delta\lambda\dfrac{\partial k_p}{\partial \bar{\sigma}_V} & -\Delta\lambda\dfrac{\partial k_p}{\partial \bar{\rho}} & 1 & -k_p \\ \dfrac{\partial f_p}{\partial \bar{\sigma}_V} & \dfrac{\partial f_p}{\partial \bar{\rho}} & \dfrac{\partial f_p}{\partial \kappa_p} & 0 \end{bmatrix}$$

```
if (icomputeall.eq.1) then
```

$$r(\cos\bar{\theta}) = \frac{4(1-e^2)(\cos\bar{\theta})^2 + (2e-1)^2}{2(1-e^2)(\cos\bar{\theta}) + (2e-1)\sqrt{4(1-e^2)(\cos\bar{\theta})^2 + 5e^2 - 4e}}$$

```
        par1=(on-ecc**tw);par2=(tw*ecc-on)
        rcos=(fr*par1*dcos(th)**tw+par2**tw)
        rcos=rcos/(tw*par1*dcos(th)+par2*sqrt(fr*par1*dcos(th)**tw+5.d0*ecc**tw-fr*ecc))
```

$$f_{\mathrm{p}}(\bar{\sigma}_{\mathrm{V}},\bar{\rho},\bar{\theta};\kappa_{\mathrm{p}}) = \left\{[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]\left(\frac{\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}}+\frac{\bar{\sigma}_{\mathrm{v}}}{f_{\mathrm{c}}}\right)^2+\sqrt{\frac{3}{2}}\frac{\bar{\rho}}{f_{\mathrm{c}}}\right\}^2 + m_0 q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})q_{\mathrm{h2}}(\kappa_{\mathrm{p}})\left[\frac{\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}}r(\cos\bar{\theta})+\frac{\bar{\sigma}_{\mathrm{v}}}{f_{\mathrm{c}}}\right] - q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})q_{\mathrm{h2}}^2(\kappa_{\mathrm{p}})$$

$$f_{\mathrm{p}}(\bar{\sigma}_{\mathrm{V}},\bar{\rho},\bar{\theta};\kappa_{\mathrm{p}}) = (A_1)^2 + m_0 q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})q_{\mathrm{h2}}(\kappa_{\mathrm{p}})B_1 - q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})q_{\mathrm{h2}}^2(\kappa_{\mathrm{p}})$$

$$\frac{\partial f_{\mathrm{p}}}{\partial\bar{\sigma}_{\mathrm{V}}} = 2A_x\frac{\partial A_x}{\partial\bar{\sigma}_{\mathrm{V}}}+\frac{m_0 q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})q_{\mathrm{h2}}(\kappa_{\mathrm{p}})}{f_c}, \qquad \frac{\partial f_{\mathrm{p}}}{\partial\bar{\rho}} = 2A_x\frac{\partial A_x}{\partial\bar{\rho}}+\frac{m_0 q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})q_{\mathrm{h2}}(\kappa_{\mathrm{p}})}{\sqrt{6}f_c}r(\cos\,\bar{\theta})$$

```
dfdinv(1)=4.0*(1.0-qh1)/fc*Al*Bl+qh2*qh1**tw*pm0/fc
dfdinv(2)=Al/(sqrt(sx)*fc)*( 4.0*(1.0-qh1)*Bl+sx)+rcos*pm0*qh2*qh1**tw/(sqrt(sx)*fc);
```

$$\frac{\partial A_x}{\partial\bar{\rho}} = \frac{2[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]B_x}{\sqrt{6}f_c}+\frac{\sqrt{1.5}}{f_{\mathrm{c}}}, \frac{\partial A_x}{\partial\bar{\sigma}_{\mathrm{V}}} = \frac{2[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]B_x}{f_c},$$

$$\frac{\partial^2 A_x}{\partial\bar{\rho}^2} = \frac{[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]}{3f_c^2}, \qquad \frac{\partial^2 A_x}{\partial\bar{\sigma}_{\mathrm{V}}{}^2} = \frac{2[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]}{f_c^2}, \qquad \frac{\partial^2 A_x}{\partial\bar{\rho}\partial\bar{\sigma}_{\mathrm{V}}} = \frac{2[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]}{\sqrt{6}f_c^2}, \frac{\partial A_x}{\partial\kappa_{\mathrm{p}}} = -\frac{\partial q_{\mathrm{h1}}}{\partial\kappa_{\mathrm{p}}}(B_x)^2,$$

$$\frac{\partial^2 A_x}{\partial\kappa_{\mathrm{p}}^2} = -\frac{\partial^2 q_{\mathrm{h1}}}{\partial\kappa_{\mathrm{p}}^2}(B_x)^2, \qquad \frac{\partial^2 A_x}{\partial\bar{\rho}\partial\kappa_{\mathrm{p}}} = -2\frac{\partial q_{\mathrm{h1}}}{\partial\kappa_{\mathrm{p}}}\frac{B_x}{\sqrt{6}f_{\mathrm{c}}}, \qquad \frac{\partial^2 A_x}{\partial\bar{\sigma}_{\mathrm{V}}\partial\kappa_{\mathrm{p}}} = -2\frac{\partial q_{\mathrm{h1}}}{\partial\kappa_{\mathrm{p}}}\frac{B_x}{f_{\mathrm{c}}}$$

$$\frac{\partial f_{\mathrm{p}}}{\partial\kappa_{\mathrm{p}}} = -2A_x\left[\frac{\partial q_{\mathrm{h1}}}{\partial\kappa_{\mathrm{p}}}(B_x)^2\right] + m_0\left(2\frac{\partial q_{\mathrm{h1}}}{\partial\kappa_{\mathrm{p}}}q_{\mathrm{h1}}+q_{\mathrm{h1}}^2\frac{\partial q_{\mathrm{h2}}}{\partial\kappa_{\mathrm{p}}}\right)B_y - 2\left(q_{\mathrm{h1}}^2 q_{\mathrm{h2}}\frac{\partial q_{\mathrm{h2}}}{\partial\kappa_{\mathrm{p}}}+q_{\mathrm{h1}}q_{\mathrm{h2}}^2\frac{\partial q_{\mathrm{h1}}}{\partial\kappa_{\mathrm{p}}}\right)$$

```
!ddgddinv
dmQ_dsv=ag/(bg*fc)*exp(r);
dAl_dsv=tw*(on-qh1)*Bl/fc
```

$$m_g(\bar{\sigma}_{\rm v},\kappa_{\rm p}) = A_g(\kappa_{\rm p})B_g(\kappa_{\rm p})\exp\frac{\bar{\sigma}_{\rm v}-q_{\rm h1}(\kappa_{\rm p})f_t/3}{B_g(\kappa_{\rm p})f_c}, \quad \boxed{\frac{\partial m_g}{\partial\bar{\sigma}_{\rm v}}} = \frac{A_g(\kappa_{\rm p})}{f_c}\exp\frac{\bar{\sigma}_{\rm v}-q_{\rm h1}(\kappa_{\rm p})f_t/3}{B_g(\kappa_{\rm p})f_c}, \quad \boxed{\frac{\partial m_g}{\partial\kappa_{\rm p}}} = -\frac{A_g(\kappa_{\rm p})f_t}{3f_c}\frac{dq_{\rm h1}}{d\kappa_{\rm p}}\exp\frac{\bar{\sigma}_{\rm v}-q_{\rm h1}(\kappa_{\rm p})f_t/3}{B_g(\kappa_{\rm p})f_c}$$

```
dBl_dsv=on/fc;

dAl_dro=tw*(on-qh1)*Bl/(fc*sqrt(sx))+sqrt(1.5d0)/fc
dBl_dro=on/(fc*sqrt(sx))
```

$$g_{\rm p}(\bar{\sigma}_{\rm V},\bar{\rho};\kappa_{\rm p}) = (A_1)^2 + q_{\rm h1}^2(\kappa_{\rm p})\left[\frac{m_0\bar{\rho}}{\sqrt{6}f_{\rm c}} + \frac{m_g(\bar{\sigma}_{\rm v},\kappa_{\rm p})}{f_c}\right]$$

$$\frac{\partial g_p}{\partial\bar{\rho}} = 2A_1\frac{\partial A_1}{\partial\bar{\rho}} + \frac{m_0 q_{\rm h1}^2(\kappa_{\rm p})}{\sqrt{6}f_c}, \qquad \frac{\partial g_p}{\partial\bar{\sigma}_V} = 2A_1\frac{\partial A_1}{\partial\bar{\sigma}_{\rm V}} + \frac{q_{\rm h1}^2(\kappa_{\rm p})}{f_c}\left(\frac{\partial m_g}{\partial\bar{\sigma}_{\rm V}}\right),$$

$$\frac{\partial^2 g_p}{\partial\bar{\rho}^2} = 2A_1\frac{\partial^2 A_1}{\partial\bar{\rho}^2} + 2\left(\frac{\partial A_1}{\partial\bar{\rho}}\right)^2,$$

$$\frac{\partial^2 g_p}{\partial\bar{\sigma}_{\rm V}^2} = 2A_1\frac{\partial^2 A_1}{\partial\bar{\sigma}_{\rm V}^2} + 2\left(\frac{\partial A_1}{\partial\bar{\sigma}_{\rm V}}\right)^2 + \frac{q_{\rm h1}^2(\kappa_{\rm p})}{f_c}\left(\frac{\partial^2 m_g}{\partial\bar{\sigma}_{\rm V}^2}\right),$$

$$\frac{\partial^2 g_p}{\partial\bar{\rho}\partial\bar{\sigma}_{\rm V}} = 2A_1\frac{\partial^2 A_1}{\partial\bar{\rho}\partial\bar{\sigma}_{\rm V}} + 2\frac{\partial A_1}{\partial\bar{\rho}}\frac{\partial A_1}{\partial\bar{\sigma}_{\rm V}}$$

$$\frac{\partial A_1}{\partial\bar{\rho}} = \frac{2[1-q_{\rm h1}(\kappa_{\rm p})]B_1}{\sqrt{6}f_c} + \frac{\sqrt{1.5}}{f_{\rm c}}, \qquad \frac{\partial A_1}{\partial\bar{\sigma}_{\rm V}} = \frac{2[1-q_{\rm h1}(\kappa_{\rm p})]B_1}{f_c},$$

$$\frac{\partial^2 A_1}{\partial\bar{\rho}^2} = \frac{[1-q_{\rm h1}(\kappa_{\rm p})]}{3f_c^2}, \frac{\partial^2 A_1}{\partial\bar{\sigma}_{\rm V}^2} = \frac{2[1-q_{\rm h1}(\kappa_{\rm p})]}{f_c^2}, \frac{\partial^2 A_1}{\partial\bar{\rho}\partial\bar{\sigma}_{\rm V}} = \frac{2[1-q_{\rm h1}(\kappa_{\rm p})]}{\sqrt{6}f_c^2}$$

```
ddg_ddsv=fr*(on-qh1)/fc*(dAl_dsv*Bl+Al*dBl_dsv)+qh1**tw*dmQ_dsv/fc
ddg_ddro=dAl_dro/(sqrt(sx)*fc)*(fr*(on-qh1)*Bl+sx)+Al*dBl_dro*fr*(on-qh1)/(sqrt(sx)*fc)
ddg_dsvdro=fr*(on-qh1)/fc*(dAl_dro*Bl+Al*dBl_dro)
ddg_drodsv=dAl_dsv/(sqrt(sx)*fc)*(fr*(on-qh1)*Bl+sx)+Al/(sqrt(sx)*fc)*(fr*(on-qh1)*dBl_dsv)

ddg_ddinv(1,1)=ddg_ddsv;  ddg_ddinv(1,2)=ddg_drodsv
ddg_ddinv(2,1)=ddg_drodsv;ddg_ddinv(2,2)=ddg_ddro
```

```
!dfdk
```

$$\boxed{f_{\mathrm{p}}\left(\bar{\sigma}_{\mathrm{V}}, \bar{\rho}, \bar{\theta}; \kappa_{\mathrm{p}}\right) = (A_1)^2 + m_0 q_{\mathrm{h}1}^2(\kappa_{\mathrm{p}}) q_{\mathrm{h}2}(\kappa_{\mathrm{p}}) B_1 - q_{\mathrm{h}1}^2(\kappa_{\mathrm{p}}) q_{\mathrm{h}2}^2(\kappa_{\mathrm{p}})}$$

$$\boxed{\begin{aligned} &A_1 = \left[1 - q_{\mathrm{h}1}\left(\kappa_{\mathrm{p}}\right)\right](B_x)^2 + \sqrt{\frac{3}{2}}\frac{\bar{\rho}}{f_{\mathrm{c}}}, \\ &\frac{\partial f_{\mathrm{p}}}{\partial \kappa_{\mathrm{p}}} = -2A_x\left[\frac{\partial q_{\mathrm{h}1}}{\partial \kappa_{\mathrm{p}}}(B_x)^2\right] + m_0\left(2\frac{\partial q_{\mathrm{h}1}}{\partial \kappa_{\mathrm{p}}} q_{\mathrm{h}1} + q_{\mathrm{h}1}^2\frac{\partial q_{\mathrm{h}2}}{\partial \kappa_{\mathrm{p}}}\right)B_y - 2\left(q_{\mathrm{h}1}^2 q_{\mathrm{h}2}\frac{\partial q_{\mathrm{h}2}}{\partial \kappa_{\mathrm{p}}} + q_{\mathrm{h}1} q_{\mathrm{h}2}^2\frac{\partial q_{\mathrm{h}1}}{\partial \kappa_{\mathrm{p}}}\right)\end{aligned}}$$

```
dfdqh1=-tw*Al*(Bl**tw)+tw*qh1*qh2*pm0*(sv/fc+ro*rcos/(sqrt(sx)*fc))-tw*qh1*(qh2**tw)
dfdqh2=(qh1**tw)*pm0*(sv/fc+ro*rcos/(sqrt(sx)*fc))-tw*qh2*(qh1**tw)
dfdk=dqh1_dk*dfdqh1+dqh2_dk*dfdqh2
if(dfdk>zr)    dfdk=zr
```

```
!ddgdinvdk
```

$$g_{\mathrm{p}}(\bar{\sigma}_{\mathrm{V}},\bar{\rho};\kappa_{\mathrm{p}})=\left\{[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]\left(\frac{\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}}+\frac{\bar{\sigma}_{\mathrm{v}}}{f_{\mathrm{c}}}\right)^2+\sqrt{\frac{3}{2}}\frac{\bar{\rho}}{f_{\mathrm{c}}}\right\}^2+q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})\left[\frac{m_0\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}}+\frac{m_g(\bar{\sigma}_{\mathrm{v}},\kappa_{\mathrm{p}})}{f_{\mathrm{c}}}\right]=(A_x)^2+q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})\left[\frac{m_0\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}}+\frac{m_g(\bar{\sigma}_{\mathrm{v}},\kappa_{\mathrm{p}})}{f_{\mathrm{c}}}\right]$$

$$m_g(\bar{\sigma}_{\mathrm{v}},\kappa_{\mathrm{p}})=A_g(\kappa_{\mathrm{p}})B_g(\kappa_{\mathrm{p}})\exp\frac{\bar{\sigma}_{\mathrm{v}}-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})f_t/3}{B_g(\kappa_{\mathrm{p}})f_c},\qquad A_{\mathrm{g}}=\frac{3f_{\mathrm{t}}q_{\mathrm{h2}}}{f_{\mathrm{c}}}+\frac{m_0}{2}\to\boxed{\frac{dA_{\mathrm{g}}}{d\kappa_{\mathrm{p}}}=\frac{3f_{\mathrm{t}}}{f_{\mathrm{c}}}\frac{dq_{\mathrm{h2}}}{d\kappa_{\mathrm{p}}}}$$

```
dag_dk=dqh2_dk*thr*ft/fc;
```

$$B_{\mathrm{g}}=\frac{B_{\mathrm{g1}}}{B_{\mathrm{g2}}}=\frac{\left(\frac{q_{\mathrm{h2}}}{3}\right)\left(1+\frac{f_{\mathrm{t}}}{f_{\mathrm{c}}}\right)}{\ln A_{\mathrm{g}}-\ln(2D_{\mathrm{f}}-1)-\ln\left(3q_{h2}+\frac{m_0}{2}\right)+\ln(D_f+1)},$$

$$\boxed{\frac{dB_{\mathrm{g1}}}{d\kappa_{\mathrm{p}}}}=\frac{(1+f_{\mathrm{t}}/f_{\mathrm{c}})}{3}\frac{dq_{\mathrm{h2}}}{d\kappa_{\mathrm{p}}},\qquad\boxed{\frac{dB_{\mathrm{g2}}}{d\kappa_{\mathrm{p}}}}=\frac{1}{A_{\mathrm{g}}}\frac{dA_{\mathrm{g}}}{d\kappa_{\mathrm{p}}}-\frac{1}{\left(q_{h2}+\frac{m_0}{6}\right)}\frac{dq_{\mathrm{h2}}}{d\kappa_{\mathrm{p}}},\qquad\boxed{\frac{dB_{\mathrm{g}}}{d\kappa_{\mathrm{p}}}}=(B_{\mathrm{g2}})^{-1}\frac{dB_{\mathrm{g1}}}{d\kappa_{\mathrm{p}}}-B_{\mathrm{g1}}(B_{\mathrm{g2}})^{-2}\frac{dB_{\mathrm{g2}}}{d\kappa_{\mathrm{p}}}$$

```
dbg_top_dk=dqh2_dk/thr
dbg_bottom_dk=-thr*dqh2_dk/(thr*qh2+pm0/tw)
dbg_dk=(dbg_top_dk*bg_bottom-bg_top*dbg_bottom_dk)/(bg_bottom**tw)
```

$$\boxed{\frac{\partial m_g}{\partial\bar{\sigma}_{\mathrm{v}}}}=\frac{A_g(\kappa_{\mathrm{p}})}{f_c}\exp\frac{\bar{\sigma}_{\mathrm{v}}-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})f_t/3}{B_g(\kappa_{\mathrm{p}})f_c},\qquad\boxed{\frac{\partial m_g}{\partial\kappa_{\mathrm{p}}}}=-\frac{A_g(\kappa_{\mathrm{p}})f_t}{3f_c}\frac{dq_{\mathrm{h1}}}{d\kappa_{\mathrm{p}}}\exp\frac{\bar{\sigma}_{\mathrm{v}}-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})f_t/3}{B_g(\kappa_{\mathrm{p}})f_c}$$

$$\boldsymbol{m}=\frac{\partial g}{\partial\bar{\boldsymbol{\sigma}}}=\frac{\partial g}{\partial\bar{\sigma}_{\mathrm{V}}}\frac{\partial\bar{\sigma}_{\mathrm{V}}}{\partial\bar{\boldsymbol{\sigma}}}+\frac{\partial g}{\partial\bar{\rho}}\frac{\partial\bar{\rho}}{\partial\bar{\boldsymbol{\sigma}}}$$

$$\frac{\partial\bar{\sigma}_{\mathrm{V}}}{\partial\bar{\boldsymbol{\sigma}}}=\frac{\boldsymbol{\delta}}{3},\qquad\frac{\partial\bar{\rho}}{\partial\bar{\boldsymbol{\sigma}}}=\frac{\bar{\boldsymbol{s}}}{\bar{\rho}},\qquad\bar{\rho}=\sqrt{2J_2}=\sqrt{\bar{\boldsymbol{s}}:\bar{\boldsymbol{s}}},\qquad\bar{\theta}=\frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right),\qquad\bar{\sigma}_{\mathrm{V}}=\frac{I_1}{3},\qquad J_2=\frac{1}{2}\bar{\boldsymbol{s}}:\bar{\boldsymbol{s}},\qquad J_3=\frac{1}{3}\bar{s}_{ij}\bar{s}_{jk}\bar{s}_{ki}$$

$$\boxed{\frac{\partial m_g}{\partial\bar{\sigma}_{\mathrm{v}}}}=\frac{A_g(\kappa_{\mathrm{p}})}{f_c}\exp\frac{\bar{\sigma}_{\mathrm{v}}-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})f_t/3}{B_g(\kappa_{\mathrm{p}})f_c}=\frac{A_g(\kappa_{\mathrm{p}})}{f_c}\exp\left(\frac{R_{top}}{R_{bottom}}\right)\to\frac{\partial R_{top}}{\partial\kappa_{\mathrm{p}}}=-\frac{dq_{\mathrm{h1}}(\kappa_{\mathrm{p}})}{d\kappa_{\mathrm{p}}}\frac{f_t}{3},\frac{\partial R_{bottom}}{\partial\kappa_{\mathrm{p}}}=f_c\frac{dB_{\mathrm{g}}}{d\kappa_{\mathrm{p}}}$$

```
R_top=(sv-ft/thr*qh2);R_bottom=fc*bg

dR_top_dk=-ft/thr*dqh2_dk;dR_bottom_dk=fc*dbg_dk
dR_dk=(dR_top_dk*R_bottom-R_top*dR_bottom_dk)/(R_bottom**tw)
```

$$\frac{\partial^2 m_g(\bar{\sigma}_{\mathrm{v}},\kappa_{\mathrm{p}})}{\partial\bar{\sigma}_{\mathrm{v}}\partial\kappa_{\mathrm{p}}}=\frac{dA_{\mathrm{g}}}{d\kappa_{\mathrm{p}}f_c}\exp\boldsymbol{r}+\frac{A_g(\kappa_{\mathrm{p}})}{f_c}\exp\boldsymbol{r}*\frac{d\boldsymbol{r}}{d\kappa_{\mathrm{p}}}$$

```
dmQ_dk=dag_dk*exp(r)+ag*dR_dk*exp(r);
```

$A_1 = [1-q_{h1}(\kappa_p)](B_1)^2 + \sqrt{\frac{3}{2}}\frac{\bar{\rho}}{f_c}, B_1 = \frac{\bar{\rho}}{\sqrt{6}f_c} + \frac{\bar{\sigma}_V}{f_c}, \boxed{\frac{\partial A_1}{\partial \bar{\rho}} = \frac{2[1-q_{h1}(\kappa_p)]B_1}{\sqrt{6}f_c} + \frac{\sqrt{1.5}}{f_c}}$

$\boxed{\frac{\partial A_1}{\partial \kappa_p} = -(B_1)^2 \frac{dq_{h1}}{d\kappa_p}} \rightarrow \boxed{\frac{\partial^2 A_1}{\partial \bar{\rho} \partial \kappa_p} = -2B_1 \frac{dq_{h1}}{d\kappa_p}\left(\frac{1}{\sqrt{6}f_c}\right)}, \boxed{\frac{\partial^2 A_1}{\partial \bar{\sigma}_V \partial \kappa_p} = -2B_1 \frac{dq_{h1}}{d\kappa_p}\left(\frac{1}{f_c}\right)}$

$$\boxed{g_p(\bar{\sigma}_V, \bar{\rho}; \kappa_p) = (A_x)^2 + q_{h1}^2(\kappa_p)\left[\frac{m_0\bar{\rho}}{\sqrt{6}f_c} + \frac{m_g(\bar{\sigma}_V, \kappa_p)}{f_c}\right]}$$

$$\frac{\partial g_p}{\partial \kappa_p} = 2A_x \frac{\partial A_x}{\partial \kappa_p} + 2\frac{\partial q_{h1}}{\partial \kappa_p} q_{h1}(\kappa_p)\left[\frac{m_0\bar{\rho}}{\sqrt{6}f_c} + \frac{m_g(\bar{\sigma}_V, \kappa_p)}{f_c}\right] + q_{h1}^2(\kappa_p)\left[\frac{1}{f_c}\frac{\partial m_g(\bar{\sigma}_V, \kappa_p)}{\partial \kappa_p}\right]$$

$$\frac{\partial g_p}{\partial \bar{\sigma}_V \partial \kappa_p} = 2A_1 \frac{\partial^2 A_x}{\partial \bar{\sigma}_V \partial \kappa_p} + 2\frac{\partial A_x}{\partial \kappa_p}\frac{\partial A_x}{\partial \bar{\sigma}_V} + 2\frac{\partial q_{h1}}{\partial \kappa_p} q_{h1}(\kappa_p)\frac{1}{f_c}\left(\frac{\partial m_g}{\partial \bar{\sigma}_V}\right) + q_{h1}^2(\kappa_p)\left[\frac{1}{f_c}\frac{\partial^2 m_g(\bar{\sigma}_V, \kappa_p)}{\partial \kappa_p \partial \bar{\sigma}_V}\right]$$

```
ddg_dinvdk(1)=(-4*Al*Bl/fc+4*(1-qh1)/fc*-(Bl**tw)*Bl)*dqh1_dk+ tw*dqh1_dk*qh1*pmQ/fc+qh1*dmQ_dk/fc
```

$$\boxed{\frac{\partial g_p}{\partial \bar{\rho} \partial \kappa_p} = 2A_1 \frac{\partial^2 A_1}{\partial \bar{\rho} \partial \kappa_p} + 2\frac{\partial A_1}{\partial \kappa_p}\frac{\partial A_1}{\partial \bar{\rho}} + 2\frac{\partial q_{h1}}{\partial \kappa_p} q_{h1}(\kappa_p)\frac{m_0}{\sqrt{6}f_c} = \left(-4A_1B_1 - (B_1)^2(4[1-q_{h1}(\kappa_p)]B_1 + 6) + m_0\right)\left(\frac{1}{\sqrt{6}f_c}\right) * 2\frac{\partial q_{h1}}{\partial \kappa_p} q_{h1}(\kappa_p),}$$

```
ddg_dinvdk(2)=( -4.0*Al*Bl -(Bl**tw)*(4.0*(on-qh1)*Bl+sx) +pm0)*tw*qh1*dqh1_dk/(sqrt(sx)*fc);
```

$$\boxed{\frac{\partial^2 A_x}{\partial \bar{\rho}^2} = \frac{[1-q_{h1}(\kappa_p)]}{3f_c^2}, \quad \frac{\partial^2 A_x}{\partial \bar{\sigma}_V{}^2} = \frac{2[1-q_{h1}(\kappa_p)]}{f_c^2}, \quad \frac{\partial^2 A_x}{\partial \bar{\rho} \partial \bar{\sigma}_V} = \frac{2[1-q_{h1}(\kappa_p)]}{\sqrt{6}f_c^2}}$$

$$\boxed{\dot{\kappa}_p = \frac{\|\dot{\varepsilon}_p\|}{x_h(\bar{\sigma}_V)}(2\cos\bar{\theta})^2 = \frac{\dot{\lambda}\|\mathbf{m}\|}{x_h(\bar{\sigma}_V)}(2\cos\bar{\theta})^2} \qquad \boxed{\dot{\kappa}_p = \dot{\lambda}k_p(\bar{\boldsymbol{\sigma}}, \kappa_p)}$$

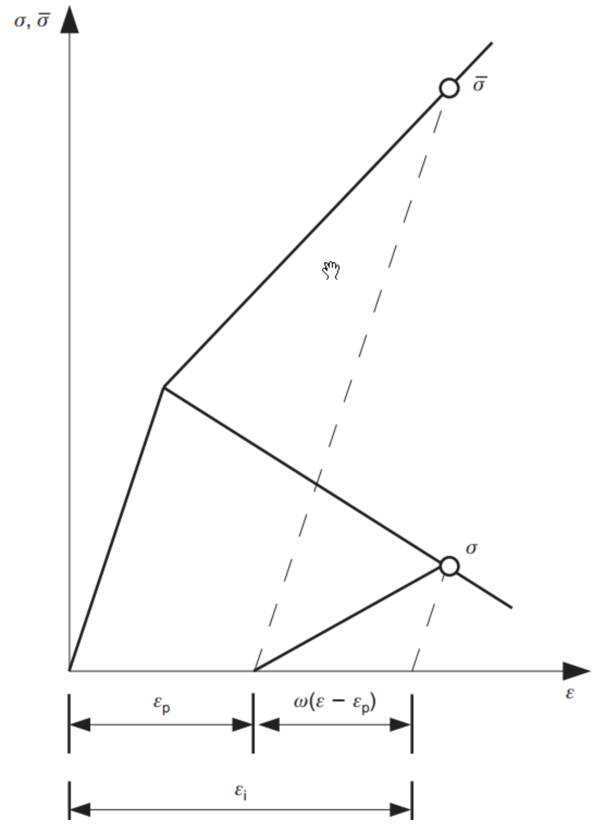
$$\boxed{\|\boldsymbol{m}\|^2 = \left(\frac{\partial g}{\partial \bar{\sigma}_{\mathrm{V}}}\frac{\boldsymbol{\delta}}{3} + \frac{\partial g}{\partial \bar{\rho}}\frac{\bar{\boldsymbol{s}}}{\bar{\rho}}\right) : (\ldots) = \left(\left[\frac{\partial g}{\partial \bar{\sigma}_{\mathrm{V}}}\right]^2 \frac{1}{3} + \left[\frac{\partial g}{\partial \bar{\rho}}\right]^2\right)}$$

```
!ddk_dldk
par1=dg_dinv(1)/equivaplentdg_dsg*ddg_dinvdk(1)
par1=tw/thr*par1

dEquivaplentdg_dstress_dk=(par1+tw*dg_dinv(2)/equivaplentdg_dsg*ddg_dinvdk(2))/tw
ddk_dldk= dEquivaplentdg_dstress_dk/duct_m
```

```
!ddKappadDeltaLambdadInv
dEquivdg_dstress_dinv(1)=tw/thr*dg_dinv(1)*ddg_ddinv(1,1)+tw*dg_dinv(2)*ddg_ddinv(2,1)

dEquivdg_dstress_dinv(1)=dEquivdg_dstress_dinv(1)/(tw*equivaplentdg_dsg)

dEquivdg_dstress_dinv(2)=tw/thr*dg_dinv(1)*ddg_ddinv(1,2)+tw*dg_dinv(2)*ddg_ddinv(2,2)
dEquivdg_dstress_dinv(2)=dEquivdg_dstress_dinv(2)/(tw*equivaplentdg_dsg)

ddk_dldinv(1)=(dEquivdg_dstress_dinv(1)*duct_m-equivaplentdg_dsg*dduct_dinv(1))/(duct_m**2)
ddk_dldinv(2)=(dEquivdg_dstress_dinv(2)*duct_m-equivaplentdg_dsg*dduct_dinv(2))/(duct_m**2);

else
      dfdinv=zr;ddg_ddinv=zr;dfdk=zr;ddg_dinvdk=zr;ddk_dldinv=zr;ddk_dldk=zr
end if
return
end
```

Damaget and Damgec

4	$\dot{\kappa} = \|\dot{\boldsymbol{\varepsilon}}_{\mathrm{p}}\| = \dot{\lambda}\left\|\frac{\partial g}{\partial \bar{\boldsymbol{\sigma}}}\right\|$
5	$\sigma_{\mathrm{y}} = f_{\mathrm{t}}(1 + H_{\mathrm{p}}\kappa)$
6	$\boldsymbol{\sigma} = \boldsymbol{D}_{\mathrm{e}} : (\boldsymbol{\varepsilon} - (\boldsymbol{\varepsilon}_{\mathrm{p}} + \omega(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{\mathrm{p}})))$
7	$\boldsymbol{\varepsilon}_{\mathrm{i}} = \boldsymbol{\varepsilon}_{\mathrm{p}} + \omega(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{\mathrm{p}})$
8	$\kappa_{\mathrm{d1}} = \|\boldsymbol{\varepsilon}_{\mathrm{p}}\|$
9	$\kappa_{\mathrm{d2}} = \max_{\tau<t}\|\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{\mathrm{p}}\|$
10	$f = \sigma - \sigma_{\mathrm{y}}$
11	$\sigma = f_{\mathrm{t}}\left(1 - \frac{\varepsilon_{\mathrm{i}}}{\varepsilon_{\mathrm{f}}}\right)$
12	$\varepsilon_{\mathrm{i}} = \kappa_{\mathrm{d1}} + \omega\kappa_{\mathrm{d2}}$
13	$\sigma = (1 - \omega)E\kappa_{\mathrm{d2}}$
14	$\omega = \frac{f_{\mathrm{t}}\kappa_{\mathrm{d1}} + \varepsilon_{\mathrm{f}}E\kappa_{\mathrm{d2}} - \varepsilon_{\mathrm{f}}f_{\mathrm{t}}}{\kappa_{\mathrm{d2}}(\varepsilon_{\mathrm{f}}E - f_{\mathrm{t}})}$
15	$\kappa_{\mathrm{d2}} = \sigma_{\mathrm{y}}/E = \frac{f_{\mathrm{t}}}{E}(1 + H_{\mathrm{p}}\kappa_{\mathrm{d1}})$
16	$\omega = \frac{f_{\mathrm{t}}E\kappa_{\mathrm{d1}}(1 + \varepsilon_{\mathrm{f}}H_{\mathrm{p}})}{(1 + H_{\mathrm{p}}\kappa_{\mathrm{d1}})(\varepsilon_{\mathrm{f}}Ef_{\mathrm{t}} - f_{\mathrm{t}}^2)}$



$$\sigma = (1-\omega)\bar{\sigma} = (1-\omega)E(\varepsilon - \varepsilon_{\mathrm{p}})$$

$$\sigma = E\left\{\varepsilon - \left[\varepsilon_{\mathrm{p}} + \omega(\varepsilon - \varepsilon_{\mathrm{p}})\right]\right\} = E(\varepsilon - \varepsilon_{\mathrm{i}})$$

$$\sigma = \begin{cases} f_{\mathrm{t}} - \frac{f_{\mathrm{t}}-\sigma_1}{\varepsilon_{\mathrm{f1}}}\varepsilon_{\mathrm{i}} & \text{if } 0 < \varepsilon_{\mathrm{i}} \leqslant \varepsilon_{\mathrm{f1}} \\ \sigma_1 - \frac{\sigma_1}{\varepsilon_{\mathrm{f}}-\varepsilon_{\mathrm{f1}}}(\varepsilon_{\mathrm{i}} - \varepsilon_{\mathrm{f1}}) & \text{if } \varepsilon_{\mathrm{f1}} < \varepsilon_{\mathrm{i}} \leqslant \varepsilon_{\mathrm{f}} \\ 0 & \text{if } \varepsilon_{\mathrm{f}} \leqslant \varepsilon_{\mathrm{i}} \end{cases}$$

$$\varepsilon_{\mathrm{i}} = \kappa_{\mathrm{dt1}} + \omega_{\mathrm{t}}\kappa_{\mathrm{dt2}}$$

$$\sigma = (1-\omega_{\mathrm{t}})E\kappa_{\mathrm{dt}}$$

$$\omega_{\mathrm{t}} = \begin{cases} \dfrac{(E\kappa_{\mathrm{dt}} - f_{\mathrm{t}})\varepsilon_{\mathrm{f1}} - (\sigma_1 - f_{\mathrm{t}})\kappa_{\mathrm{dt1}}}{E\kappa_{\mathrm{dt}}\varepsilon_{\mathrm{f1}} + (\sigma_1 - f_{\mathrm{t}})\kappa_{\mathrm{dt2}}} & \text{if } 0 < \varepsilon_{\mathrm{i}} \leqslant \varepsilon_{\mathrm{f1}} \\ \dfrac{E\kappa_{\mathrm{dt}}(\varepsilon_{\mathrm{f}} - \varepsilon_{\mathrm{f1}}) + \sigma_1(\kappa_{\mathrm{dt1}} - \varepsilon_{\mathrm{f}})}{E\kappa_{\mathrm{dt}}(\varepsilon_{\mathrm{f}} - \varepsilon_{\mathrm{f1}}) - \sigma_1\kappa_{\mathrm{dt2}}} & \text{if } \varepsilon_{\mathrm{f1}} < \varepsilon_{\mathrm{i}} \leqslant \varepsilon_{\mathrm{f}} \\ 0 & \text{if } \varepsilon_{\mathrm{f}} < \varepsilon_{\mathrm{i}} \end{cases}$$

$$\omega_{\mathrm{t}} = \begin{cases} \dfrac{(E\kappa_{\mathrm{dt}} - f_{\mathrm{t}})w_{\mathrm{f1}} - (\sigma_1 - f_{\mathrm{t}})\kappa_{\mathrm{dt1}}h}{E\kappa_{\mathrm{dt}}w_{\mathrm{f1}} + (\sigma_1 - f_{\mathrm{t}})\kappa_{\mathrm{dt2}}h} & \text{if } 0 < h\varepsilon_{\mathrm{i}} \leqslant w_{\mathrm{f1}}h \\ \dfrac{E\kappa_{\mathrm{dt}}(w_{\mathrm{f}} - w_{\mathrm{f1}}) + \sigma_1(\kappa_{\mathrm{dt1}}h - w_{\mathrm{f}})}{E\kappa_{\mathrm{dt}}(w_{\mathrm{f}} - w_{\mathrm{f1}}) - \sigma_1\kappa_{\mathrm{dt2}}h} & \text{if } w_{\mathrm{f1}} < h\varepsilon_{\mathrm{i}} \leqslant w_{\mathrm{f}} \\ 0 & \text{if } w_{\mathrm{f}} < h\varepsilon_{\mathrm{i}} \end{cases}$$

$$\sigma = f_{\mathrm{t}}\exp\left(-\frac{\varepsilon_{\mathrm{i}}}{\varepsilon_{\mathrm{fc}}}\right) \quad \text{if } 0 < \varepsilon_{\mathrm{i}}$$

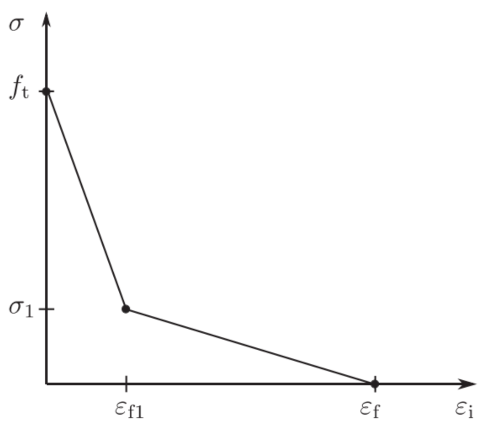


Fig. 5. Bilinear softening.

```
subroutine kdamaget(pkdt,pkdt1,pkdt2,plen,wt_old,wt)!pkdt,pkdt1,pkdt2,plen,wt_old output wt

inewton_iter=100;ytol=gtol*10.d0;

if (pkdt>e0*(on-ytol)) then
       if (itypey.eq.0) then
              wt=(Em*pkdt*wf-ft*wf+ft*pkdt1*plen)/(Em*pkdt*wf-ft*plen*pkdt2)
```

$$\omega_{\mathrm{t}} = \begin{cases} \dfrac{(E\kappa_{\mathrm{dt}} - f_{\mathrm{t}})w_{\mathrm{f1}} - (\sigma_1 - f_{\mathrm{t}})\kappa_{\mathrm{dt1}}h}{E\kappa_{\mathrm{dt}}w_{\mathrm{f1}} + (\sigma_1 - f_{\mathrm{t}})\kappa_{\mathrm{dt2}}h} & \text{if } 0 < h\varepsilon_{\mathrm{i}} \leqslant w_{\mathrm{f1}}h \\ \dfrac{E\kappa_{\mathrm{dt}}(w_{\mathrm{f}} - w_{\mathrm{f1}}) + \sigma_1(\kappa_{\mathrm{dt1}}h - w_{\mathrm{f}})}{E\kappa_{\mathrm{dt}}(w_{\mathrm{f}} - w_{\mathrm{f1}}) - \sigma_1\kappa_{\mathrm{dt2}}h} & \text{if } w_{\mathrm{f1}} < h\varepsilon_{\mathrm{i}} \leqslant w_{\mathrm{f}} \\ 0 & \text{if } w_{\mathrm{f}} < h\varepsilon_{\mathrm{i}} \end{cases}$$

```
       else if (itypey.eq.1) then
              wt=(Em*pkdt*wf1-ft*wf1-(ft1-ft)*pkdt1*plen)/(Em*pkdt*wf1+(ft1-ft)*plen*pkdt2)
              pari=plen*pkdt1+plen*wt*pkdt2
              if (pari>wf1.and.pari<wf) then
                     wt=(Em*pkdt*(wf-wf1)-ft1*(wf-wf1)+ft1*pkdt1*plen-ft1*wf1)
                     wt=wt/(Em*pkdt*(wf-wf1)-ft1*plen*pkdt2)
                     pari=plen*pkdt1+plen*wt*pkdt2
              else if (pari>wf) then
                     wt=on
              end if




       else if (itypey.eq.2) then
              !Exponential: Newton-Raphson
```

$$\sigma = f_{\mathrm{t}} \exp\left(-\frac{\varepsilon_{\mathrm{i}}}{\varepsilon_{\mathrm{fc}}}\right) \quad \text{if } 0 < \varepsilon_{\mathrm{i}}$$

```
              wt=on;
              residual=zr;
              residualDerivative=zr;
              iter=0;
              pari=on;
              do while (pari.eq.on)
                     iter=iter+1;
                     residual=(on-wt)*Em*pkdt-ft*EXP(-plen*(wt*pkdt2+pkdt1)/wf)
                     residualDerivative=-Em*pkdt+ft*plen*pkdt2*EXP(-plen*(wt*pkdt2+pkdt1)/wf)/wf
                     wt=wt-residual/residualDerivative
                      if(abs(residual/ft)<1.0d-8) pari=zr
              end do
       else
              wt=zr
       end if

       if(wt>on)                    wt=on
       if(wt<zr.or.wt<wt_old)       wt=wt_old

else
wt=zr
end if
return
```

$f_{\mathrm{dt}} = \tilde{\varepsilon}_{\mathrm{t}}(\bar{\boldsymbol{\sigma}}) - \kappa_{\mathrm{dt}}$ $\boxed{\omega_{\mathrm{t}} = g_{\mathrm{dt}}(\kappa_{\mathrm{dt}}, \kappa_{\mathrm{dt1}}, \kappa_{\mathrm{dt2}})}$

$f_{\mathrm{dc}} = \alpha_{\mathrm{c}}\tilde{\varepsilon}_{\mathrm{c}}(\bar{\boldsymbol{\sigma}}) - \kappa_{\mathrm{dc}}$ $\boxed{\omega_{\mathrm{c}} = g_{\mathrm{dc}}(\kappa_{\mathrm{dc}}, \kappa_{\mathrm{dc1}}, \kappa_{\mathrm{dc2}})}$

$\sigma = f_{\mathrm{t}} \exp\left(-\frac{\varepsilon_{\mathrm{i}}}{\varepsilon_{\mathrm{fc}}}\right)$ if $0 < \varepsilon_{\mathrm{i}}$

```
subroutine kdamagec(pkdc,pkdc1,pkdc2,wc_old,wc)       !pkdc,pkdc1,pkdc2,wc_old
if (isotropic.eq.1) then
       wc=zr;pkdc1=zr;pkdc2=zr;pkdc=zr
else
       inewton_iter=200;tol=gtol;ytol=gtol*10.d0;nite=0;
       residual=zr;dResdw=zr;
       if (pkdc>e0*(on-ytol)) then
              do while(nite<inewton_iter)
                     nite=nite+1;residual=(on-wc)*em*pkdc-ft*exp(-(pkdc1+wc*pkdc2)/efc)
                     dResdw=-em*pkdc+ft*pkdc2/efc*exp(-(pkdc1+wc*pkdc2)/efc);
                     wc=wc-residual/dResdw
                      errorOld = residual/ft
                      if(wc<zr) then    wc=zr    exit    end if
                               if(nite.eq.inewton_iter) then
                                     if(residual<zr) then wc=wc_old     exit
                                     else !disp('error')
                                     end if
                     end if
                    if(abs(residual/ft)<tol) then exit
              end do
       else
              wc=zr
       end if
end if
if(wc>on) wc=on
if(wc<zr.or.wc<wc_old) wc=wc_old
return
```

```
      subroutine kdamage(wc_old,wt_old,strain_rate,rate_fc,alpha,eps_t,eps_c,
pkdt_old,pkdt1,pkdt2,pkdc_old,pkdc1,pkdc2,sg_ekff,tkp,pnorm_inc_e_p,plen,sg_old,alpha_old,eps_old,
wc,wt,eps_t1,eps_c1, pkdt_new,     pkdt1t,pkdt2t,pkdc_new,pkdc1t,pkdc2t,eps_new)
```

$$\tau_\varepsilon = \|\boldsymbol{\varepsilon}\|_{\mathbb{C}^*} = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C}^* : \boldsymbol{\varepsilon}}$$

Damage function (in strain space)

$$g(\varepsilon, r) = \tau_\varepsilon - r$$

Elastic domain (in strain space)

$$\mathrm{E}_\varepsilon := \{\varepsilon \in \mathbb{S} \mid g(\varepsilon, r) \equiv \tau_\varepsilon - r < 0\}$$

Damage surface (in strain space)

$$\partial\mathrm{E}_\varepsilon := \{\varepsilon \in \mathbb{S} \mid g(\varepsilon, r) \equiv \tau_\varepsilon - r = 0\}$$

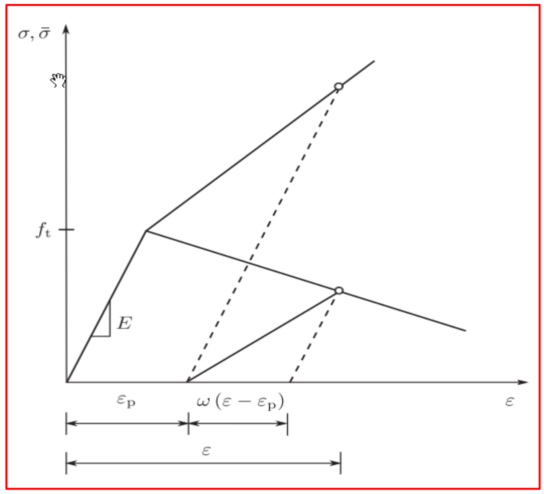
$$f_{\mathrm{dt}} = \tilde{\varepsilon}_{\mathrm{t}}(\bar{\boldsymbol{\sigma}}) - \kappa_{\mathrm{dt}}$$

$$\dot{\tilde{\varepsilon}}_{\mathrm{t}} = \dot{\tilde{\varepsilon}}$$

$$\kappa_{\mathrm{dt}} = \max_{\tau \le t} \tilde{\varepsilon}_{\mathrm{t}},$$

$$\dot{\kappa}_{\mathrm{dt1}} = \begin{cases} \frac{1}{x_{\mathrm{s}}} \|\dot{\boldsymbol{\varepsilon}}_{\mathrm{p}}\| & \text{if } \dot{\kappa}_{\mathrm{dt}} > 0 \text{ and } \kappa_{\mathrm{dt}} > \varepsilon_0 \\ 0 & \text{if } \dot{\kappa}_{\mathrm{dt}} = 0 \text{ or } \kappa_{\mathrm{dt}} < \varepsilon_0 \end{cases}$$

$$\dot{\kappa}_{\mathrm{dt2}} = \frac{\dot{\kappa}_{\mathrm{dt}}}{x_{\mathrm{s}}(\bar{\sigma}_{\mathrm{v}}, \bar{\rho})}$$



```
tol=gtol*10
if (rf.eq.zr) then       istep1flag=1;  rf=on
else                     istep1flag=0             end if

call kcheckunload(sg_ekff,sg_old,eps_old,gtol,pmin_equ_e,t_equ_e,iunload_flag)

eps_new=t_equ_e;
if(iunload_flag.eq.0) then  pmin_equ_e=eps_old   end if

dt=t_equ_e-eps_old;
dc=(pmin_equ_e-eps_old)*alpha_old+(t_equ_e-pmin_equ_e)*alpha

if(istrrateflg.eq.1.and.wc_old.eq.zr.and.wt_old.eq.zr)then

       call krate_fac(strain_rate,alpha,tol, t_rf)
       eps_t1=eps_t+dt/ t_rf;
       eps_c1=eps_c+dc/ t_rf
       if((eps_c1>e0.or.eps_t1>e0).and.istep1flag.ne.1) then
              eps_t1=eps_t+dt/rf;
              eps_c1=eps_c+dc/rf
       else
              rf=t_rf
       end if
else
       eps_t1=eps_t+dt/rf;
       eps_c1=eps_c+dc/rf
end if

call khaigh(sg_ekff,sv_el,ro_el,theta_el,dinv_dsig_pr)

if(sv_el<zr) then    rs1=-6.0**0.5 *sv_el/max(ro_el,1.0d-16)   else   rs1=zr        end if xs=on+(as-on)*rs1;

d_pkdt=(eps_t1-pkdt_old);   d_pkdc=(eps_c1-pkdc_old);                wt=zr;wc=zr;
```

$$\boxed{f_{\mathrm{dt}} = \tilde{\varepsilon}_{\mathrm{t}}(\bar{\boldsymbol{\sigma}}) - \kappa_{\mathrm{dt}}}$$

$$\boxed{\dot{\tilde{\varepsilon}}_{\mathrm{t}} = \dot{\tilde{\varepsilon}}} \quad \boxed{\dot{\kappa}_{\mathrm{dt1}} = \begin{cases} \frac{1}{x_{\mathrm{s}}} \|\dot{\boldsymbol{\varepsilon}}_{\mathrm{p}}\| & \text{if } \dot{\kappa}_{\mathrm{dt}} > 0 \text{ and } \kappa_{\mathrm{dt}} > \varepsilon_0 \\ 0 & \text{if } \dot{\kappa}_{\mathrm{dt}} = 0 \text{ or } \kappa_{\mathrm{dt}} < \varepsilon_0 \end{cases}}$$

$$\boxed{\dot{\kappa}_{\mathrm{dt2}} = \frac{\dot{\kappa}_{\mathrm{dt}}}{x_{\mathrm{s}}}} \quad \boxed{\kappa_{\mathrm{dt}} = \max_{\tau \le t} \tilde{\varepsilon}_{\mathrm{t}}, \quad \dot{\kappa}_{\mathrm{dt1}} = \frac{\|\dot{\epsilon}_{\mathrm{p}}\|}{x_{\mathrm{s}}(\overline{\sigma}_{\mathrm{v}}, \overline{\rho})}, \quad \dot{\kappa}_{\mathrm{dt2}} = \frac{\dot{\kappa}_{\mathrm{dt}}}{x_{\mathrm{s}}(\overline{\sigma}_{\mathrm{v}}, \overline{\rho})}}$$

$$\boxed{x_{\mathrm{s}} = 1 + (A_{\mathrm{s}} - 1) R_{\mathrm{s}}^{\xi}} \quad \boxed{R_{\mathrm{s}} = \begin{cases} -\frac{\sqrt{6}\bar{\sigma}_{\mathrm{V}}}{\bar{\rho}} & \text{if } \bar{\sigma}_{\mathrm{V}} \leqslant 0 \\ 0 & \text{if } \bar{\sigma}_{\mathrm{V}} > 0 \end{cases}}$$

$$\boxed{f_{\mathrm{dt}} = \tilde{\varepsilon}_{\mathrm{t}}(\bar{\boldsymbol{\sigma}}) - \kappa_{\mathrm{dt}}}$$

```
if(d_pkdt>=-e0*tol) then
       if(eps_t1<e0*(on-tol)) then                                  fac=zr
       else if (eps_t1>e0*(on-tol).and.pkdt_old<e0*(on-tol)) then   fac=(on-(e0-pkdt_old)/(eps_t1-pkdt_old))
       else                                                         fac=on        end if

       pkdt1t=pkdt1+pnorm_inc_e_p*fac/xs/rf;
       pkdt2t=pkdt2+d_pkdt/xs

       pkdt_new=eps_t1
       call kdamaget(pkdt_new,pkdt1t,pkdt2t,plen,wt_old,wt)         end if
```

$$\alpha_c = \sum_{i=1}^{3} \frac{\bar{\sigma}_{\text{pc i}}(\bar{\sigma}_{\text{pt i}} + \bar{\sigma}_{\text{pc i}})}{\|\bar{\boldsymbol{\sigma}}_{\text{p}}\|^2}$$

$$\dot{\kappa}_{\text{dc1}} = \begin{cases} \frac{\alpha_c \beta_c}{x_s} \|\dot{\boldsymbol{\varepsilon}}_{\text{p}}\| & \text{if } \dot{\kappa}_{\text{dt}} > 0 \wedge \kappa_{\text{dt}} > \varepsilon_0 \\ 0 & \text{if } \dot{\kappa}_{\text{dt}} = 0 \vee \kappa_{\text{dt}} < \varepsilon_0 \end{cases}$$

and

$$\dot{\kappa}_{\text{dc2}} = \frac{\dot{\kappa}_{\text{dc}}}{x_s}$$

In (48), the factor β_c is

$$\beta_c = \frac{f_t q_{h2} \sqrt{2/3}}{\bar{\rho}\sqrt{1 + 2D_f^2}}$$

$$\dot{\tilde{\varepsilon}}_c = \alpha_c \dot{\tilde{\varepsilon}}_t$$

$$\kappa_{\text{dc}} = \max_{\tau \le t} \tilde{\varepsilon}_c, \quad \dot{\kappa}_{\text{dc1}} = \frac{\alpha_c \beta_c \|\dot{\epsilon}_p\|}{x_s(\overline{\sigma}_v, \overline{\rho})}, \quad \dot{\kappa}_{\text{dc2}} = \frac{\dot{\kappa}_{\text{dc}}}{x_s(\overline{\sigma}_v, \overline{\rho})}$$

```
if(d_pkdc>=-e0*tol) then
       if (eps_c1<e0*(on-tol)) then                              fac=zr
       elseif (eps_c1>e0*(on-tol).and.pkdc_old<e0*(on-tol)) then     fac=(on-(e0-pkdc_old)/(eps_c1-pkdc_old))
       else                                                      fac=on     end if
       call kcqh2(tkp,int(0),qh2,dqh2_dk)
       fac2=ft*qh2*sqrt(tw/thr)/max(ro_el,1.0d-16)/sqrt(on+tw*(df**tw));
       pkdc1t=pkdc1+pnorm_inc_e_p*fac*fac2*alpha/xs/rf;
       pkdc2t=pkdc2+d_pkdc/xs;
       pkdc_new=eps_c1
       call kdamagec(pkdc_new,pkdc1t,pkdc2t,wc_old,wc)
end if
```

```
subroutine kcheckunload(sg,sg_old,eps_old,gtol,pmin_equ_e,equ_e_new,iunload_flag)

call khaigh(sg,sv,ro,th,dinv_dsig_pr);                call kequ_e(sv,ro,th,equ_e_new)

dsg=sg-sg_old;
sg_plus=sg_old+0.01d0*dsg;
sg_minus=sg_old+0.99d0*dsg

call khaigh(sg_plus, sv,ro,th,dinv_dsig_pr);          call kequ_e(sv,ro,th,equ_e_plus)
call khaigh(sg_minus,sv,ro,th,dinv_dsig_pr);          call kequ_e(sv,ro,th,equ_e_minus)


pmin_equ_e=eps_old;

p=equ_e_plus;
pm=equ_e_minus;

o=eps_old;
pn=equ_e_new;


iunload_flag=0;grgtol=gtol*1.d-3

if ((p<o.and.pm<pn).and.(abs(p-o)>grgtol.and.abs(pm-pn)>grgtol)) then

       iunload_flag=1
       do i=1,100
              sg1=sg_old+dsg*i/100.0d0
              call khaigh(sg1,sv,ro,th,dinv_dsig_pr)    call kequ_e(sv,ro,th,equ_e1)
              if (equ_e1<=pmin_equ_e) then      pmin_equ_e=equ_e1      else        EXIT     end if
       end do
end if
return
```

The response of concrete is strongly rate dependent.
If the loading rate is increased, the apparent tensile and compressive strength increase.

This increase is more pronounced in tension than in compression.
The greater the rate factor α_{r}, the greater is the delay of the onset of damage and, therewith, the strength.

For tension, Malvar and Ross.

For compression, CEB-FIP Model Code 1990.

$$\boxed{\alpha_{\mathrm{r}} = (1-\alpha_{\mathrm{c}})\,\alpha_{\mathrm{rt}} + \alpha_{\mathrm{c}}\alpha_{\mathrm{rc}}}$$

Accordingly, the factor α_{r} is defined as

tension

$$\alpha_{\mathrm{rt}} = \begin{cases} 1 & \text{for } \dot{\varepsilon}_{\max} \leq 30 \times 10^{-6}\ \mathrm{s}^{-1} \\ \left(\dfrac{\dot{\varepsilon}_{\max}}{\dot{\varepsilon}_{\mathrm{t0}}}\right)^{\delta_{\mathrm{s}}} & \text{for } 30 \times 10^{-6}\ \mathrm{s}^{-1} \leq \dot{\varepsilon}_{\max} \leq 1\ \mathrm{s}^{-1} \\ \beta_{\mathrm{s}}\left(\dfrac{\dot{\varepsilon}_{\max}}{\dot{\varepsilon}_{\mathrm{t0}}}\right)^{1/3} & \text{for } 1\ \mathrm{s}^{-1} \leq \dot{\varepsilon}_{\max} \end{cases}$$

$$\boxed{\delta_{\mathrm{s}} = \frac{1}{1+8f_{\mathrm{c}}/f_{\mathrm{c0}}}} \qquad \boxed{\log \beta_{\mathrm{s}} = 6\delta_{\mathrm{s}} - 2}$$

compression

$$\alpha_{\mathrm{rc}} = \begin{cases} 1 & \text{for } |\dot{\varepsilon}_{\min}| \leq 30 \times 10^{-6}\ \mathrm{s}^{-1} \\ \left(\dfrac{\|\dot{\varepsilon}_{\min}\|}{\dot{\varepsilon}_{\mathrm{c0}}}\right)^{1.026\alpha_{\mathrm{s}}} & \text{for } 30 \times 10^{-6}\ \mathrm{s}^{-1} \leq |\dot{\varepsilon}_{\min}| \leq 30\ \mathrm{s}^{-1} \\ \gamma_{\mathrm{s}}\left(\dfrac{\|\dot{\varepsilon}_{\min}\|}{\dot{\varepsilon}_{\mathrm{c0}}}\right)^{1/3} & \text{for } 30\ \mathrm{s}^{-1} \leq |\dot{\varepsilon}_{\min}| \end{cases}$$

$$\boxed{\alpha_{\mathrm{s}} = \frac{1}{5+9f_{\mathrm{c}}/f_{\mathrm{c0}}}} \qquad \boxed{\log \gamma_{\mathrm{s}} = 6.156\alpha_{\mathrm{s}} - 2}$$

```
subroutine krate_fac(strain_rate,alpha,tol,rate_fac)    !strain_rate,alpha,tol
            alphas=on/(5.d0+9.d0*fc/fc0);
            deltas=on/(on+8.d0*fc/fc0)
            gammas=exp((6.156d0*alphas-tw)*log(10.d0))!check log
            betas=exp((sx*deltas-tw)*log(10.d0))
            rate_t0=1.0d-6;
            rate_c0=-30.0d-6;
            rate_t=on;
            rate_c=on
            tmp=strain_rate/tw;
            tmp(1)=tmp(1)*tw;
            tmp(2)=tmp(2)*tw;
            tmp(3)=tmp(3)*tw
            call kvec_to_tens(tmp,sig_ten)
            call kjacobi_eigenvalue(3,sig_ten,dir,strain_pr)
            pmax=-1.0d-20;
            pmin=1.0d20
            do i=1,3
                  if (pmax<strain_pr(i))      pmax=strain_pr(i)
                  if (pmin>strain_pr(i))      pmin=strain_pr(i)
            end do
            if ((on-alpha)>tol) then      rate =pmax
            else                          rate =pmin      end if
            ratio_t=rate/rate_t0;ratio_c=rate/ratio_c0;
```

tension

$$\alpha_{\mathrm{rt}} = \begin{cases} 1 & \text{for } \dot{\varepsilon}_{\max} \leq 30 \times 10^{-6}\ \mathrm{s}^{-1} \\ \left(\dfrac{\dot{\varepsilon}_{\max}}{\dot{\varepsilon}_{\mathrm{t0}}}\right)^{\delta_{\mathrm{s}}} & \text{for } 30 \times 10^{-6}\ \mathrm{s}^{-1} \leq \dot{\varepsilon}_{\max} \leq 1\ \mathrm{s}^{-1} \\ \beta_{\mathrm{s}}\left(\dfrac{\dot{\varepsilon}_{\max}}{\dot{\varepsilon}_{\mathrm{t0}}}\right)^{1/3} & \text{for } 1\ \mathrm{s}^{-1} \leq \dot{\varepsilon}_{\max} \end{cases}$$

```
            if (rate<30.0d-6) then                        rate_t=on
```

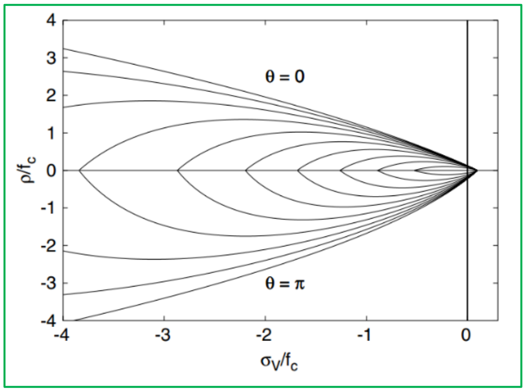
```
          else if (rate>30.0d-6.and.rate<on) then          rate_t=ratio_t**deltas
          else                                              rate_t=betas*ratio_t**(0.3333d0)    end if
compression
```

$$\boxed{\alpha_{rc} = \begin{cases} 1 & \text{for } |\dot{\varepsilon}_{min}| \leq 30 \times 10^{-6}\ \mathrm{s}^{-1} \\ \left(\dfrac{\|\dot{\varepsilon}_{min}\|}{\dot{\varepsilon}_{c0}} \right)^{1.026\alpha_s} & \text{for } 30 \times 10^{-6}\ \mathrm{s}^{-1} \leq |\dot{\varepsilon}_{min}| \leq 30\ \mathrm{s}^{-1} \\ \gamma_s \left(\dfrac{\|\dot{\varepsilon}_{min}\|}{\dot{\varepsilon}_{c0}} \right)^{1/3} & \text{for } 30\ \mathrm{s}^{-1} \leq |\dot{\varepsilon}_{min}| \end{cases}}$$

```
          if (rate>-30.0d-6) then                           rate_c=on
          else if (rate>-30.0d0 .and. rate<-30.0d-6) then   rate_c=ratio_c**(1.026d0*alphas)
          else                                              rate_c=gammas*ratio_c**(0.3333d0)    end if
          rate_fac=(on-alpha)*rate_t+alpha*rate_c
         return
```

```
subroutine kcheckvertex(sv_tr,tkp,apex_sg,irtype)       !sv_tr,tkp
if(sv_tr>zr) then
       irtype=1
       if(tkp<on) then      apex_sg=zr
       else                 call kcqh2(tkp,int(0),qh2,dqh2_dk)     apex_sg=qh2*fc/pm0       end if
else if (sv_tr <zr .and. tkp<on) then
       irtype=2;apex_sg=zr
else
       irtype=0;apex_sg=zr
end if
```

$$f_{\mathrm{p}}(\bar{\sigma}_{\mathrm{V}},\bar{\rho},\bar{\theta};\kappa_{\mathrm{p}}) = \left\{[1-q_{\mathrm{h1}}(\kappa_{\mathrm{p}})]\left(\frac{\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}}+\frac{\bar{\sigma}_{\mathrm{V}}}{f_{\mathrm{c}}}\right)^2+\sqrt{\frac{3}{2}}\frac{\bar{\rho}}{f_{\mathrm{c}}}\right\}^2 + m_0 q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})q_{\mathrm{h2}}(\kappa_{\mathrm{p}})\left[\frac{\bar{\rho}}{\sqrt{6}f_{\mathrm{c}}}r(\cos\bar{\theta})+\frac{\bar{\sigma}_{\mathrm{V}}}{f_{\mathrm{c}}}\right] - q_{\mathrm{h1}}^2(\kappa_{\mathrm{p}})q_{\mathrm{h2}}^2(\kappa_{\mathrm{p}})$$



```
subroutine kvertexreturn(sg,apex_sg,tkp,irtype,iconvrg,sg_ekff)     !sg,apex_sg,tkp,irtype,iconvrg

ytol=gtol*1.d-2;

pkp0=tkp;

maxiter=250

call khaigh(sg,sv,ro,theta,dinv_dsig_pr)
call kpp(pkp0,sv,ro,sv ,tkpi);           call kff(sv ,zr,zr,tkpi,y)

sv2 = apex_sg;
call kpp(pkp0,sv,ro,sv2,tkpi);           call kff(sv2,zr,zr,tkpi,y_mid)

pari=zr
if(y*y_mid>=zr) then
       iconvrg=1;irtype=0;
else
  if(y<zr) then
       dsv=sv2-sv;svAnswer=sv2
  else
       dsv=sv-sv2;svAnswer=sv2
  end if

  do j=1,maxiter
   `dsv=half*dsv
    sv_mid=svAnswer+dsv
    call kpp(pkp0,sv,ro,sv_mid,tkpi)                         call kff(sv_mid,zr,zr,tkpi,y_mid)





    if(y_mid<=zr) then      svAnswer=sv_mid       end if
       if (abs(y_mid)<ytol.and.y_mid<=zr) then
          call kratiopotential(svAnswer,tkpi,r1)
          r_trial=ro/(sv-svAnswer)

          if((r1>= r_trial.and.irtype.eq.1).or.(r1<= r_trial.and.irtype.eq.2))then exit
          else       iconvrg=1;  irtype=0;  pari=on        exit         end if

       end if
   end do

   if (pari.eq.zr) then
       sg_ekff(1:3)=svAnswer;      sg_ekff(4:6)=zr;     tkp=tkpi             iconvrg=zr
   end if
end if
```

```
subroutine kratiopotential(sv,pkp,ratio)  !input sv,pkp
ro=zr

call kcqh1(pkp,int(0),qh1,dqh1_dk)
call kcqh2(pkp,int(0),qh2,dqh2_dk)

ag=thr*ft*qh2/fc+pm0/tw;
bg_top=qh2/thr*(on+ft/fc)

!log vs exp
bg_bottom=log(ag)-log(tw*df-on)-log(thr*qh2+pm0/tw)+log(df+on)
bg=bg_top/bg_bottom
r=(sv-qh2*ft/thr)/fc/bg

!log vs exp
pmg=ag*bg*fc*exp(r);
dmg=ag*exp(r);
```

$$\boxed{\frac{\partial m_g}{\partial \bar{\sigma}_{\mathrm{v}}}} = \frac{A_g(\kappa_{\mathrm{p}})}{f_c} \exp \frac{\bar{\sigma}_{\mathrm{v}} - q_{\mathrm{h}1}(\kappa_{\mathrm{p}}) f_t/3}{B_g(\kappa_{\mathrm{p}}) f_c}$$

```
par1=ro/(fc*sqrt(sx))+sv/fc
par2=(on-qh1)*par1**tw+sqrt(1.5d0)*ro/fc
```

$$g_{\mathrm{p}}(\bar{\sigma}_{\mathrm{V}}, \bar{\rho}; \kappa_{\mathrm{p}}) = \left\{ [1 - q_{\mathrm{h}1}(\kappa_{\mathrm{p}})] \left(\frac{\bar{\rho}}{\sqrt{6} f_{\mathrm{c}}} + \frac{\bar{\sigma}_{\mathrm{V}}}{f_{\mathrm{c}}} \right)^2 + \sqrt{\frac{3}{2}} \frac{\bar{\rho}}{f_{\mathrm{c}}} \right\}^2 + q_{\mathrm{h}1}^2(\kappa_{\mathrm{p}}) \left(\frac{m_0 \bar{\rho}}{\sqrt{6} f_{\mathrm{c}}} + \frac{m_{\mathrm{g}}(\bar{\sigma}_{\mathrm{V}}, \kappa_{\mathrm{p}})}{f_{\mathrm{c}}} \right)$$

```
dgdinv(1)=fr*(on-qh1)/fc*par2*b1+qh1**tw*pmg/fc
dgdinv(2)=par2/(sqrt(sx)*fc)*(fr*(on-qh1)*par1+6)+pm0*qh1**tw/(sqrt(sx)*fc)

ratio=dgdinv(2)/dgdinv(1)*thr*(on-tw*qmuo)/(on+qmuo)
```