A plastic-damage model for concrete

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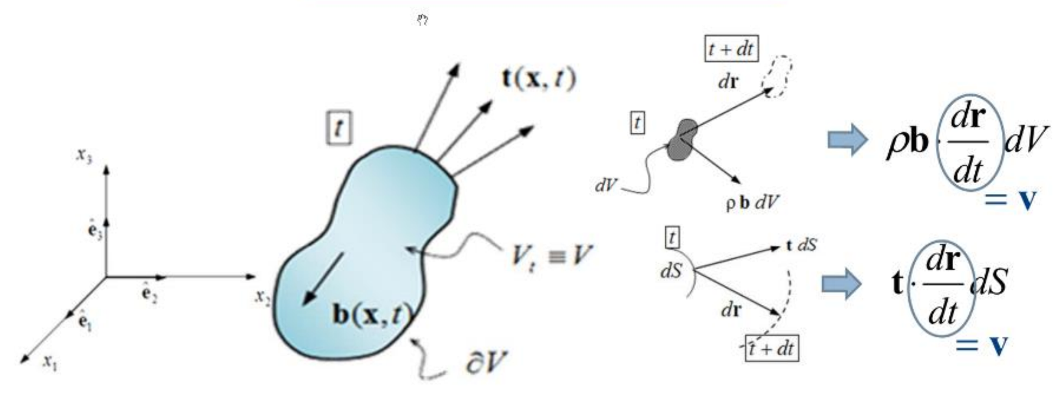
Online Course on Computational Solid Mechanics

Computational Solid Mechanics

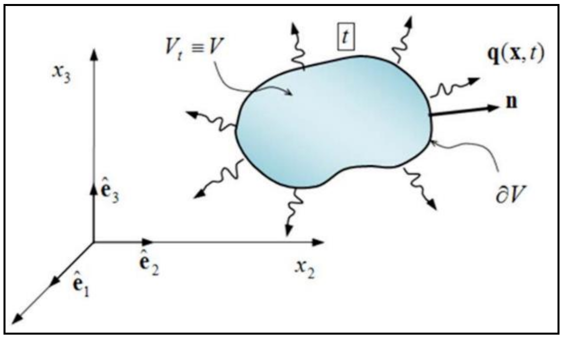
$$P_e(t)=\int_V \rho\mathbf{b}\cdot\mathbf{v}\,dV+\int_{\partial V}\mathbf{t}\cdot\mathbf{v}\,dS$$

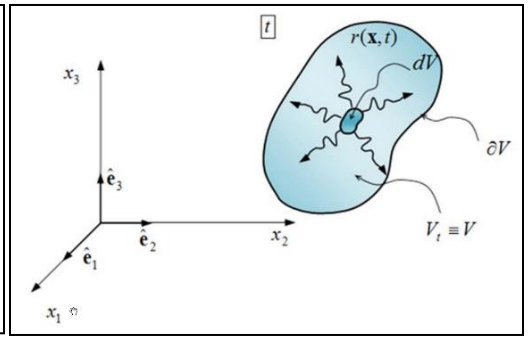


$$\boxed{P_e(t)=\int_V \rho\mathbf{b}\cdot\mathbf{v}\,dV+\int_{\partial V}\mathbf{t}\cdot\mathbf{v}\,dS=\frac{d}{dt}\underbrace{\int_{V_t\equiv V}\frac{1}{2}\rho v^2\,dV}_{\text{Kinetic energy }\mathcal{K}}+\underbrace{\int_V \boldsymbol{\sigma}:\mathbf{d}\,dV}_{\text{Stress power }\mathcal{P}_\sigma}}$$

external mechanical power entering the medium

$$\boxed{P_e(t)=\frac{d}{dt}\mathcal{K}(t)+\mathcal{P}_\sigma}$$



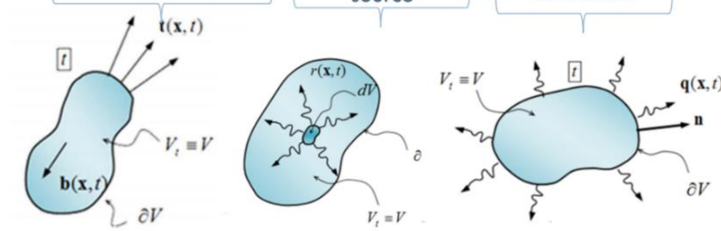


$$\frac{\text{non convective (conduction) incoming heat}}{\text{unit of time}} = -\int_{\partial V} \mathbf{q}\cdot\mathbf{n}\, dS$$

$$\frac{\text{heat generated by internal sources}}{\text{unit of time}} = \int_V \rho\, r\, dV$$

$$W_{input} = P_e + Q_e = \overbrace{\frac{d}{dt}\int_{V_t\equiv V}\frac{1}{2}\rho v^2 dV + \int_V \boldsymbol{\sigma}:\mathbf{d}\, dV}^{P_e(t)} + \overbrace{\int_V \rho\, r\, dV - \int_{\partial V}\mathbf{q}\cdot\mathbf{n}\, dS}^{Q_e(t)}$$

kinetic energy · stress power · Internal heat source · External heat conduction



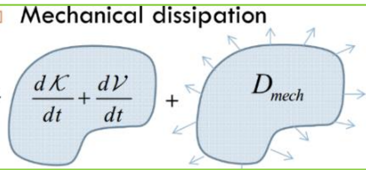
$$P_{stored} = \underbrace{\frac{d}{dt}\int_{V_t\equiv V}\frac{1}{2}\rho v^2 dV}_{\text{Kinetic energy } \mathcal{K}} + \underbrace{\frac{d}{dt}\int_V \rho\psi\, dV}_{\text{Stored mechanical energy } \mathcal{V}} = \frac{d\mathcal{K}}{dt} + \frac{d\mathcal{V}}{dt}$$

$$\rho_0\psi(\mathbf{x},t) \rightarrow \begin{cases}\text{Density of free energy}\\ \text{(Helmholtz energy)}\end{cases} \rightarrow \frac{\text{Stored mechancical energy}}{\text{unit of volume}}$$

$$D_{mech} = \underbrace{P_e}_{\frac{d\mathcal{K}}{dt}+\mathcal{P}_\sigma} - \underbrace{P_{stored}}_{\frac{d\mathcal{K}}{dt}+\frac{d\mathcal{V}}{dt}}$$

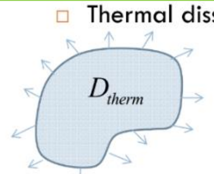
$$= \underbrace{\cancel{\frac{d\mathcal{K}}{dt}} + \int_V \boldsymbol{\sigma}:\mathbf{d}\, dV}_{P_e} - \Big(\cancel{\frac{d\mathcal{K}}{dt}} + \underbrace{\frac{d\mathcal{V}}{dt}}_{\int_V \rho\dot{\psi}\, dV}\Big) = \int_V \boldsymbol{\sigma}:\mathbf{d}\, dV - \int_V \rho\dot{\psi}\, dV$$

- Mechanical dissipation

$P_e \rightarrow$ 

$$\boxed{D_{mech} = -\int_V \rho\dot{\psi}\, dV + \int_V \boldsymbol{\sigma}:\mathbf{d}\, dV}$$

- Thermal dissipation :



$$\boxed{D_{therm} = -\int_V \rho s\dot{\theta}\, dV}$$

$$\underbrace{D = D_{mech} + D_{therm} = -\int_V [\rho(\dot{\psi} + s\dot{\theta}) + \boldsymbol{\sigma} : \mathbf{d}]\, dV \geq 0 \quad \forall \Delta V \subset V}_{\text{Global (integral) form of the second principle of thermodynamics}}$$

$$\underbrace{\mathcal{D}(\mathbf{x},t) = -\rho\left(\dot{\psi} + s\dot{\theta}\right) + \boldsymbol{\sigma} : \mathbf{d} \geq 0 \quad \forall \mathbf{x} \ \forall t}_{\text{Local (differential) form of the second principle of the thermodynamics}}$$

$$u(\mathbf{x},t) := \psi + s\theta \rightarrow \begin{cases} u(\mathbf{x},t) \longrightarrow \text{Total stored energy} \\ \psi(\mathbf{x},t) \longrightarrow \text{Mechanical stored stored energy} \\ s\theta(\mathbf{x},t) \longrightarrow \text{Thermal stored stored energy} \end{cases}$$

$$\mathcal{D} = -\rho\left(\dot{u} - \theta\dot{s}\right) + \boldsymbol{\sigma} : \mathbf{d} \geq 0$$

Clausius-Planck Inequality
in terms of the
specific internal energy

REMARK

For infinitessimal deformation, $\mathbf{d} = \dot{\boldsymbol{\varepsilon}}$,
the Clausius-Planck inequality
becomes: $-\rho(\dot{\psi} + s\,\dot{\theta}) + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \geq 0$

$$\boldsymbol{\sigma} = E\boldsymbol{\varepsilon}$$

$$\sigma_{ij} = \mathbb{C}_{ijkl}\varepsilon_{kl}$$

$$\mathbb{C}_{ijkl} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathrm{I}$$

$$[\mathbf{a} \otimes \mathbf{b}]_{i,j} = a_i b_j$$

$$[\mathbf{A} \otimes \mathbf{B}]_{ijkl} = A_{ij} B_{kl}$$

$$[\mathbf{1}]_{ij} = \delta_{ij}$$

$$[I]_{ijkl} = \frac{1}{2}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$$

$$\begin{cases} \boldsymbol{\sigma} = \lambda tr(\boldsymbol{\varepsilon}) + 2\mu\boldsymbol{\varepsilon} \\ \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \quad i, j \in \{1,2,3\} \end{cases}$$

Theorem

$$\mathcal{D}(x,y,\dot{x},\dot{y}) = f(x,y)\dot{x} + g(x,y)\dot{y} \geq 0 \quad \forall \dot{x}, \dot{y} \Rightarrow \begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$$

$$\mathcal{D} = f(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha) : \dot{\boldsymbol{\varepsilon}} + g(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha) \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}$$

Variables

$$\mathbb{V} := \{ \mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_{n_\mathrm{v}} \} \quad \mathrm{v}_i(\mathbf{x}, t) \quad i \in \{1, 2, \ldots, n_\mathrm{v}\}$$

$$\mathbb{V} := \{ \rho, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, u, \psi, s, \theta, \alpha \}$$

1) $$\mathbb{F} := \{ \lambda_1, \lambda_2, \ldots, \lambda_{n_F} \} \quad \lambda_i(\mathbf{x}, t) \quad i \in \{1, 2, \ldots, n_F\}$$

$$\dot{\lambda}_i(\mathbf{x}, t) = \frac{\partial \lambda_i(\mathbf{x}, t)}{\partial t} \to \text{any}$$

$$\mathbb{F} := \{ \rho, \boldsymbol{\varepsilon} \}$$

2) $$\mathbb{I} := \{ \alpha_1, \alpha_2, \ldots, \alpha_{n_I} \} \quad \alpha_i(\mathbf{x}, t) \quad i \in \{1, 2, \ldots, n_I\}$$

$$\dot{\alpha}_i = \frac{\partial \alpha_i(\mathbf{x}, t)}{\partial t} = \xi_i(\underbrace{\boldsymbol{\lambda}(\mathbf{x}, t), \boldsymbol{\alpha}(\mathbf{x}, t)}_{\substack{\text{instantaneous} \\ \text{values (at time } t)}}) \quad i \in \{1, 2, \ldots n_I\}$$

$$\mathbb{I} := \{ \alpha \} \qquad \dot{\alpha} = \gamma(\rho, \boldsymbol{\varepsilon}, \alpha)$$

3) $$\mathbb{D} := \{ d_1, d_2, \ldots, d_{n_D} \} \quad d_i(\mathbf{x}, t) \quad i \in \{1, 2, \ldots, n_D\}$$

$$d_i = \gamma_i(\boldsymbol{\lambda}, \boldsymbol{\alpha}) \to \dot{d}_i = \varphi_i(\boldsymbol{\lambda}, \boldsymbol{\alpha}, \dot{\boldsymbol{\lambda}}) \quad i \in \{1, 2, \ldots n_D\}$$

$$\mathbb{D} := \{ \cancel{\rho}, \boldsymbol{\sigma}, \cancel{\boldsymbol{\varepsilon}}, u, \psi, s, \theta, \cancel{\alpha} \}$$

$$\psi = \psi(\rho, \boldsymbol{\varepsilon}, \alpha)$$

$$\dot{\psi} = \gamma(\rho, \boldsymbol{\varepsilon}, \alpha, \dot{\rho}, \dot{\boldsymbol{\varepsilon}}, \dot{\alpha}(\rho, \boldsymbol{\varepsilon}, \alpha)) = \underbrace{\dot{\psi}(\rho, \boldsymbol{\varepsilon}, \alpha, \dot{\rho}, \dot{\boldsymbol{\varepsilon}})}_{\substack{\text{not depending on} \\ \text{the internal variable} \\ \text{evolution, } \dot{\alpha}}}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\rho, \boldsymbol{\varepsilon}, \alpha)$$

$$\dot{\boldsymbol{\sigma}} = \varphi(\rho, \boldsymbol{\varepsilon}, \alpha, \dot{\rho}, \dot{\boldsymbol{\varepsilon}}, \underbrace{\dot{\alpha}(\rho, \boldsymbol{\varepsilon}, \alpha)}_{\substack{\text{provided by the} \\ \text{evolution equation}}}) = \dot{\boldsymbol{\sigma}}(\rho, \boldsymbol{\varepsilon}, \alpha, \dot{\rho}, \dot{\boldsymbol{\varepsilon}})$$

4) Postulate a specific form of the free energy:

Elastic material

$\mathbb{F} := \{\boldsymbol{\varepsilon}\}$

$\mathbb{I} := \{\varnothing\}$

$\mathbb{D} := \{\boldsymbol{\sigma}, \psi\}$

$$\begin{cases} \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) \rightarrow \dot{\boldsymbol{\sigma}} = \dfrac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} \\ \rho_0 \psi(\boldsymbol{\varepsilon}) \rightarrow \rho_0 \dot{\psi} = \dfrac{\partial \rho_0 \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} \end{cases}$$

$$\rho_0 \psi(\boldsymbol{\varepsilon}) = \tfrac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}$$

- Dissipation Isothermal case

$$\mathcal{D} = -\rho_0 (\dot{\psi} + \cancel{s\dot{\theta}}) + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \geq 0$$

$$\mathcal{D} = \underbrace{(\boldsymbol{\sigma} - \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}})}_{f(\boldsymbol{\varepsilon})} : \dot{\boldsymbol{\varepsilon}} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}} \Rightarrow f(\boldsymbol{\varepsilon}) = 0$$

Inelastic material

$$\boldsymbol{\varepsilon} = \underbrace{\boldsymbol{\varepsilon}^e}_{\substack{\text{Elastic}\\ \text{strain}}} + \underbrace{\boldsymbol{\varepsilon}^i}_{\substack{\text{Inelastic}\\ \text{strain}}}$$

$$\mathbb{F} := \{\boldsymbol{\varepsilon}\}$$

$$\mathbb{I} := \{\boldsymbol{\varepsilon}^i, \alpha\} \qquad \begin{cases} \dot{\alpha} = \dot{\alpha}(\boldsymbol{\varepsilon}, \cancel{\boldsymbol{\varepsilon}^i}, \alpha) \\ \dot{\boldsymbol{\varepsilon}}^i = \dot{\boldsymbol{\varepsilon}}^i(\boldsymbol{\varepsilon}, \cancel{\boldsymbol{\varepsilon}^i}, \alpha) \end{cases}$$

$$\mathbb{D} := \{\boldsymbol{\sigma}, \psi\} \qquad \begin{aligned} \dot{\boldsymbol{\sigma}} &= \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}^i} : \dot{\boldsymbol{\varepsilon}}^i + \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \alpha} \dot{\alpha} \\ \dot{\psi} &= \frac{\partial \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}^i} : \dot{\boldsymbol{\varepsilon}}^i + \frac{\partial \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \alpha} \dot{\alpha} \end{aligned}$$

$$\rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha) = \underbrace{\tfrac{1}{2} \overbrace{\boldsymbol{\varepsilon}^e}^{(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^i)} : \mathbb{C} : \overbrace{\boldsymbol{\varepsilon}^e}^{(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^i)}}_{\text{Elastic potential}} + \underbrace{\mathcal{H}(\alpha)}_{\substack{\text{Hardening}\\ \text{potential}}}$$

$$+ \cancel{s\dot{\theta}}) + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \geq 0$$

□ **Dissipation:** $\mathcal{D} = -\rho_0 \left(\dot{\psi} + \cancel{s\dot{\theta}} \right) + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \geq 0$

$$\mathcal{D} = \underbrace{(\boldsymbol{\sigma} - \frac{\partial(\rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha))}{\partial \boldsymbol{\varepsilon}}}_{f(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)} : \dot{\boldsymbol{\varepsilon}}) + \underbrace{\underbrace{\frac{\partial(\rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha))}{\partial \boldsymbol{\varepsilon}} : \overbrace{\dot{\boldsymbol{\varepsilon}}^i}^{\dot{\boldsymbol{\varepsilon}}^i(\boldsymbol{\varepsilon}, \alpha)}}_{g_1(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)} \underbrace{- \frac{\partial(\rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha))}{\partial \alpha} \overbrace{\dot{\alpha}}^{\dot{\alpha}(\boldsymbol{\varepsilon}, \alpha)}}_{g_2(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}}_{g(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}$$

$$\overbrace{\boldsymbol{\sigma} = \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : \underbrace{(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i)}_{\boldsymbol{\varepsilon}^e}}^{\text{Constitutive equation}}$$

$$\mathcal{D} = g(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha) = \underbrace{\frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}}}_{\frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\sigma}} : \dot{\boldsymbol{\varepsilon}}^i - \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \alpha} : \dot{\alpha} \geq 0$$

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^i - (\underbrace{\frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \alpha}}_{\mathcal{H}'(\alpha) := -q(\alpha)}) : \dot{\alpha} = \overbrace{\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^i + \underbrace{q(\alpha)}_{\substack{\text{Hardening}\\ \text{variable}}} \dot{\alpha}}^{\text{Dissipation } \mathcal{D}} \geq 0$$

Stress driven models

$$\boxed{\mathbb{F} := \{\boldsymbol{\sigma}\}}$$

$$\boxed{\mathbb{I} := \{q\} \quad \rightarrow \dot{q} = \zeta(\boldsymbol{\sigma}, q)}$$

$$\boxed{\mathbb{D} := \{\boldsymbol{\varepsilon}, G\}}$$

$$\overbrace{\rho_0 G(\boldsymbol{\sigma}) = \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \rho_0 \psi(\boldsymbol{\varepsilon}) \quad ; \quad \boldsymbol{\sigma} = \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}}^{\text{Legendre transform: } \boldsymbol{\varepsilon}, \psi(\boldsymbol{\varepsilon}) \rightarrow \boldsymbol{\sigma}, G(\boldsymbol{\sigma})}$$

$$\rho_0 \dot{G} = (\dot{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}) - \rho_0 \dot{\psi}$$

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \dot{\psi} \geq 0 \quad \Rightarrow \quad \boxed{\mathcal{D} = \rho_0 \dot{G} - \boldsymbol{\varepsilon} : \dot{\boldsymbol{\sigma}} \geq 0}$$

$$\begin{cases} \psi(x) \leftrightarrow G(y) \quad ; \quad G(y) = xy - \psi(x) \\ x \leftrightarrow y \; ; \quad y \equiv \dfrac{d\psi(x)}{dx} \; ; \; x \equiv \dfrac{dG(y)}{dy} \end{cases}$$

$$\rightarrow \frac{\partial \rho_0 G(\sigma)}{\partial \sigma} = \boldsymbol{\varepsilon}(\sigma)$$

Elastic material

$$\mathbb{F} := \{\sigma\} \qquad \mathbb{I} := \{\varnothing\} \qquad \mathbb{D} := \{\boldsymbol{\varepsilon}(\boldsymbol{\sigma}), G(\boldsymbol{\sigma})\}$$

$$\rho_0 G(\boldsymbol{\sigma}) = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) - \rho_0 \psi(\boldsymbol{\varepsilon}(\boldsymbol{\sigma})) \quad ; \quad \boldsymbol{\sigma} = \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}$$

$$\rightarrow \frac{\partial \rho_0 G(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \boldsymbol{\varepsilon}(\boldsymbol{\sigma})$$

$$\dot{\boldsymbol{\varepsilon}} = \frac{\partial \boldsymbol{\varepsilon}(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}$$

$$\dot{G} = \frac{\partial G(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}$$

$$\mathcal{D} = \rho_0 \dot{G} - \boldsymbol{\varepsilon} : \dot{\boldsymbol{\sigma}} \geq 0$$

$$\mathcal{D} = (\underbrace{\frac{\partial \rho_0 G(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}}_{=\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}) : \dot{\boldsymbol{\sigma}} = 0 \quad \forall \dot{\boldsymbol{\sigma}}$$

$$\overbrace{\boldsymbol{\varepsilon} = \frac{\partial \rho_0 G(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}}^{\text{Constitutive equation}}$$

$$\overbrace{\mathcal{D} = 0}^{\text{Dissipation}}$$

$$\overbrace{G(\boldsymbol{\sigma}) = \frac{1}{2} \boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma}}^{\text{Gibs potential}}$$

- Variable set definition
  - Free variables: $\mathbb{F} := \{\boldsymbol{\sigma}\}$
  - Internal variables: $\mathbb{I} := \{q\} \longrightarrow \dot{q} = \zeta(\boldsymbol{\sigma}, q)$
  - Dependent: $\mathbb{D} := \{\boldsymbol{\varepsilon}(\boldsymbol{\sigma}, q), G(\boldsymbol{\sigma}, q)\} \longrightarrow \dot{G} = \frac{\partial G(\boldsymbol{\sigma}, q)}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial G(\boldsymbol{\sigma}, q)}{\partial q} \dot{q}$
- Potential
  - Gibbs energy : $\rho_0 G = \frac{1}{2} \boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma} + \eta(q)$
- Dissipation

$$\mathcal{D} = \rho_0 \dot{G} - \boldsymbol{\varepsilon} : \dot{\boldsymbol{\sigma}} \geq 0 \longrightarrow \mathcal{D} = (\underbrace{\frac{\partial \rho_0 G(\boldsymbol{\sigma}, q)}{\partial \boldsymbol{\sigma}} - \boldsymbol{\varepsilon}}_{f(\boldsymbol{\sigma}, q)}) : \dot{\boldsymbol{\sigma}} + \underbrace{\frac{\partial G(\boldsymbol{\sigma}, q)}{\partial q} \overset{\zeta(\boldsymbol{\sigma}, q)}{\dot{q}}}_{g(\boldsymbol{\sigma}, q)} \geq 0 \quad \forall \dot{\boldsymbol{\sigma}}$$

$$f(\boldsymbol{\sigma}, q) = \frac{\partial \rho_0 G(\boldsymbol{\sigma}, q)}{\partial \boldsymbol{\sigma}} - \boldsymbol{\varepsilon} = 0$$

$$g(\boldsymbol{\sigma}, q) = \frac{\partial \rho_0 G(\boldsymbol{\sigma}, q)}{\partial q} \dot{q} \geq 0$$

$$\boldsymbol{\varepsilon} = \frac{\partial \rho_0 G(\boldsymbol{\sigma}, q)}{\partial \boldsymbol{\sigma}}$$

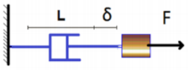
$$\mathcal{D} = \frac{\partial \rho_0 G(\boldsymbol{\sigma}, q)}{\partial q} \dot{q} \geq 0$$

$$\overbrace{\boldsymbol{\varepsilon} = \frac{\partial \rho_0 G(\boldsymbol{\sigma}, q)}{\partial \boldsymbol{\sigma}} = \mathbb{C}^{-1} : \boldsymbol{\sigma}}^{\text{Constitutive equation}}$$

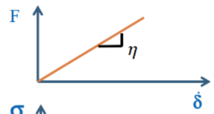
$$\overbrace{\mathcal{D} = \frac{\partial \rho_0 G(\boldsymbol{\sigma}, q)}{\partial q} \dot{q}}^{\text{Dissipation}} = \underbrace{\eta'(q)}_{\alpha(q)} \dot{q} \geq 0$$

$$\alpha(q) = \eta'(q) = \frac{d\eta(q)}{dq} \rightarrow \begin{cases} \text{conjugate} \\ \text{internal variable} \end{cases}$$

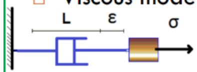
- Viscous element



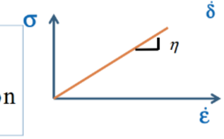
$$F = \eta \dot{\delta} \rightarrow \dot{\delta} = \frac{1}{\eta} F$$



- Viscous model



$$\begin{cases} \boldsymbol{\beta} = \boldsymbol{\varepsilon} \rightarrow \text{Internal variable} \\ \dot{\boldsymbol{\beta}} = \dfrac{1}{\eta} \boldsymbol{\sigma} \rightarrow \text{Evolution equation} \end{cases}$$



- Stress driven model: $\mathbb{F} := \{\boldsymbol{\sigma}\}$ $\quad \mathbb{I} := \{\boldsymbol{\beta}\}$ $\quad \mathbb{D} := \{\boldsymbol{\varepsilon}, G\}$

$$\begin{cases} \rho_0 G(\boldsymbol{\sigma}, \boldsymbol{\beta}) = \boldsymbol{\sigma} : \boldsymbol{\beta} \\ \boldsymbol{\varepsilon} = \dfrac{\partial \rho_0 G(\boldsymbol{\sigma}, \boldsymbol{\beta})}{\partial \boldsymbol{\sigma}} = \boldsymbol{\beta} \end{cases} \Rightarrow \mathcal{D} = \underbrace{\frac{\partial \rho_0 G(\boldsymbol{\sigma}, \boldsymbol{\beta})}{\partial \boldsymbol{\beta}}}_{=\boldsymbol{\sigma}} : \dot{\boldsymbol{\beta}} = \boldsymbol{\sigma} : \underbrace{\dot{\boldsymbol{\beta}}}_{\frac{1}{\eta}\boldsymbol{\sigma}} = \frac{1}{\eta} \underbrace{\boldsymbol{\sigma} : \boldsymbol{\sigma}}_{\geq 0} \geq 0 \Rightarrow \boxed{\eta \geq 0}$$

- Strain driven model $\mathbb{F} := \{\boldsymbol{\varepsilon}\}$

Free and internal variables must be different !!!!

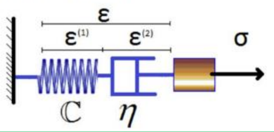
$$\overbrace{\mathbb{I} := \underbrace{\{\boldsymbol{\beta}\}}_{=\boldsymbol{\varepsilon}} = \mathbb{F}}$$

$\mathbb{D} := \{\boldsymbol{\sigma}, \psi\}$

$$\begin{cases} \rho_0 \psi = \rho_0 G - \boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \boldsymbol{\sigma} : \underbrace{\boldsymbol{\beta}}_{\boldsymbol{\varepsilon}} - \boldsymbol{\sigma} : \boldsymbol{\varepsilon} = 0 \rightarrow \text{Free enerfy does not exist} \\ \boldsymbol{\sigma} = \dfrac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\beta})}{\partial \boldsymbol{\varepsilon}} \end{cases}$$

A strain driven version of the model has no sense

- Viscous Model



Let's define

$$\begin{cases} \boldsymbol{\varepsilon} = \underbrace{\boldsymbol{\varepsilon}^e}_{\boldsymbol{\varepsilon}^{(1)}} + \underbrace{\boldsymbol{\varepsilon}^i}_{\boldsymbol{\varepsilon}^{(2)}} \\ \boldsymbol{\varepsilon}^i = \boldsymbol{\beta} \end{cases} \quad \text{being} \quad \begin{cases} \boldsymbol{\varepsilon}^e = \mathbb{C}^{-1} : \boldsymbol{\sigma} \\ \dot{\boldsymbol{\varepsilon}}^i = \dfrac{1}{\eta}\boldsymbol{\sigma} \end{cases}$$

$$\mathbb{F} := \{\boldsymbol{\varepsilon}\} \quad , \quad \mathbb{I} := \{\boldsymbol{\beta}\} \quad \text{and} \quad \mathbb{D} := \{\boldsymbol{\sigma}, \psi\} \qquad \dot{\boldsymbol{\beta}} = \frac{1}{\eta}\boldsymbol{\sigma}$$

$$\rho_0\psi(\boldsymbol{\varepsilon}) = \rho_0\psi^{(1)}(\boldsymbol{\varepsilon}) + \rho_0\psi^{(2)}(\boldsymbol{\varepsilon})$$

Elastic $\rho_0\psi^{(1)}(\boldsymbol{\varepsilon},\boldsymbol{\beta}) = \frac{1}{2}\boldsymbol{\varepsilon}^{(1)} : \mathbb{C} : \boldsymbol{\varepsilon}^{(1)} = \frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\beta}) : \mathbb{C} : (\boldsymbol{\varepsilon}-\boldsymbol{\beta})$

Viscous $\rho_0\psi^{(2)}(\boldsymbol{\varepsilon}) = 0$

$$\rho_0\psi(\boldsymbol{\varepsilon}) = \rho_0\psi^{(1)}(\boldsymbol{\varepsilon},\boldsymbol{\beta}) + \rho_0\psi^{(2)}(\boldsymbol{\varepsilon}) = \frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\beta}) : \mathbb{C} : (\boldsymbol{\varepsilon}-\boldsymbol{\beta})$$

$$\overbrace{\dot{\boldsymbol{\varepsilon}} = \mathbb{C}^{-1} : \dot{\boldsymbol{\sigma}} + \frac{1}{\eta}\boldsymbol{\sigma}}^{\text{Constitutive equation}}$$

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\beta}} = \eta\,\underbrace{\dot{\boldsymbol{\beta}} : \dot{\boldsymbol{\beta}}}_{\geq 0} \geq 0 \quad \Rightarrow \quad \boxed{\eta \geq 0}$$

$$\mathbb{F} := \{\boldsymbol{\sigma}\} \quad \mathbb{I} := \{\boldsymbol{\beta}\} \quad \mathbb{D} := \{\boldsymbol{\varepsilon}, G\} \qquad \dot{\boldsymbol{\beta}} = \frac{1}{\eta}\boldsymbol{\sigma}$$

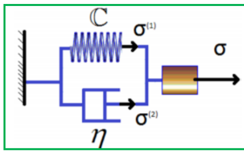
Elastic $\rho_0 G^{(1)}(\boldsymbol{\sigma},\boldsymbol{\beta}) = \frac{1}{2}\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma}$

Viscous $\rho_0 G^{(2)}(\boldsymbol{\sigma},\boldsymbol{\beta}) = \boldsymbol{\sigma} : \boldsymbol{\beta}$

$$\Rightarrow \boxed{\rho_0 G(\boldsymbol{\sigma},\boldsymbol{\beta}) = \frac{1}{2}\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma} + \boldsymbol{\sigma} : \boldsymbol{\beta}}$$

$$\overbrace{\dot{\boldsymbol{\varepsilon}} = \mathbb{C}^{-1} : \dot{\boldsymbol{\sigma}} + \frac{1}{\eta}\boldsymbol{\sigma}}^{\text{Constitutive equation}}$$

$$\boxed{\eta \geq 0}$$



$$\begin{cases} \boldsymbol{\sigma} = \underbrace{\boldsymbol{\sigma}^{(1)}}_{\boldsymbol{\sigma}^{e}} + \underbrace{\boldsymbol{\sigma}^{(2)}}_{\boldsymbol{\sigma}^{i}} \\ \boldsymbol{\varepsilon}^{(1)} = \boldsymbol{\varepsilon}^{(2)} = \boldsymbol{\beta} \end{cases} \quad \text{being} \quad \begin{cases} \boldsymbol{\sigma}^{(1)} = \boldsymbol{\sigma}^{e} = \mathbb{C} : \boldsymbol{\beta} \\ \boldsymbol{\sigma}^{(2)} = \eta\, \dot{\boldsymbol{\beta}} \end{cases}$$

- Stress driven model
  - Variable set definition

$$\mathbb{F} := \{\boldsymbol{\sigma}\} \quad \text{and} \quad \mathbb{I} := \{\boldsymbol{\beta}\} \qquad \dot{\boldsymbol{\beta}} = \frac{1}{\eta}\boldsymbol{\sigma}^{(2)} = \frac{1}{\eta}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(1)}) = \frac{1}{\eta}(\boldsymbol{\sigma} - \mathbb{C} : \boldsymbol{\beta})$$

- Potential $\rho_0 G(\boldsymbol{\sigma}, \boldsymbol{\beta}) = \rho_0 G^{(1)}(\boldsymbol{\sigma}, \boldsymbol{\beta}) + \rho_0 G^{(2)}(\boldsymbol{\sigma}, \boldsymbol{\beta})$

$$= \tfrac{1}{2}\boldsymbol{\sigma}^{(1)} : \mathbb{C}^{-1} : \boldsymbol{\sigma}^{(1)} = \tfrac{1}{2}(\underbrace{\boldsymbol{\beta} : \mathbb{C}}_{\boldsymbol{\sigma}^{(1)}}) : \mathbb{C}^{-1} : (\underbrace{\mathbb{C} : \boldsymbol{\beta}}_{\boldsymbol{\sigma}^{(1)}}) = \tfrac{1}{2}\boldsymbol{\beta} : \mathbb{C} : \boldsymbol{\beta}$$

Viscous $\rho_0 G^{(2)}(\boldsymbol{\sigma}, \boldsymbol{\beta}) = \boldsymbol{\sigma}^{(2)} : \boldsymbol{\beta} = (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(1)}) : \boldsymbol{\beta} = \boldsymbol{\sigma} : \boldsymbol{\beta} - \underbrace{\boldsymbol{\beta} : \mathbb{C}}_{\boldsymbol{\sigma}^{(1)}} : \boldsymbol{\beta}$

$$\boxed{\rho_0 G(\boldsymbol{\sigma}, \boldsymbol{\beta}) = \rho_0 G^{(1)} + \rho_0 G^{(2)} = \boldsymbol{\sigma} : \boldsymbol{\beta} - \frac{1}{2}\boldsymbol{\beta} : \mathbb{C} : \boldsymbol{\beta}}$$

Constitutive equation

$$\begin{cases} \dot{\boldsymbol{\beta}} = \dfrac{1}{\eta}\boldsymbol{\sigma}^{(2)} = \dfrac{1}{\eta}(\boldsymbol{\sigma} - \underbrace{\mathbb{C} : \boldsymbol{\beta}}_{\boldsymbol{\sigma}^{(1)}}) \rightarrow \boldsymbol{\beta}(\boldsymbol{\sigma}) \\ \boldsymbol{\varepsilon} = \boldsymbol{\beta}(\boldsymbol{\sigma}) \end{cases}$$

$$\boxed{\eta \geq 0}$$

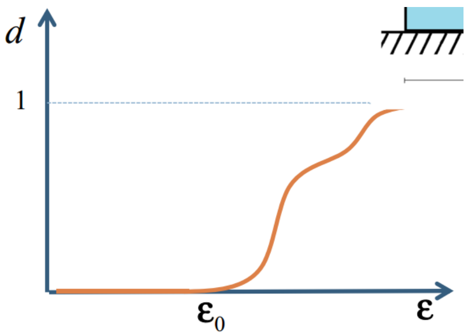
- Strain driven model
  - Variables set definition

$$\underbrace{\mathbb{F} := \{\boldsymbol{\varepsilon}\} \;,\; \mathbb{I} := \{\boldsymbol{\beta}\}} \qquad \mathbb{D} := \{\boldsymbol{\sigma}, \psi\} \qquad \dot{\boldsymbol{\beta}} = \frac{1}{\eta}\boldsymbol{\sigma}^{(2)}$$

NOT PHYSICAL

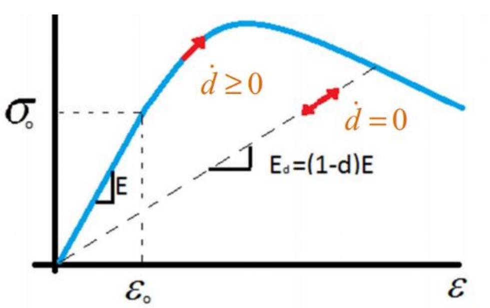
Free and internal variables must be different !!!!

Damage



- Damage or (degradation) is initiated when the strain (or stress) exceeds the initial damage threshold $(\varepsilon_0, \sigma_0)$

$$d = 0 \quad if \quad \boldsymbol{\varepsilon} \le \boldsymbol{\varepsilon}_0 \quad or \quad \boldsymbol{\sigma} \le \boldsymbol{\sigma}_0$$



$$\begin{cases} \boldsymbol{\sigma} = \underbrace{(1-d)\mathbb{C}}_{\mathbb{C}^d} : \boldsymbol{\varepsilon} = \mathbb{C}^d : \boldsymbol{\varepsilon} = (1-d)\bar{\boldsymbol{\sigma}} \\ \bar{\boldsymbol{\sigma}} = \mathbb{C} : \boldsymbol{\varepsilon} \rightarrow \text{effective stress} \end{cases}$$

$$\mathbb{C}^d = (1-d)\mathbb{C} = (1-d)[\lambda \mathbf{1} \otimes 1 + 2\mu \mathbb{I}]$$

$$\mathbb{F} := \{\boldsymbol{\varepsilon}\}$$

$$\mathbb{I} := \{r\} \qquad \dot{r} = \lambda(\boldsymbol{\varepsilon}, r)$$

$$\mathbb{D} := \{\boldsymbol{\sigma}(\varepsilon, r), \psi(\varepsilon, r), d(\varepsilon, r) \ldots\}$$

$$\psi(\boldsymbol{\varepsilon}, d(r)) = (1 - d(r))\psi_0(\boldsymbol{\varepsilon}) = (1 - d(r))\frac{1}{2}(\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon})$$

$$\dot{\psi} = \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial \mathrm{d}} \dot{d}$$

$$\mathcal{D} = (\boldsymbol{\sigma} - \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}}) : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \psi(\boldsymbol{\varepsilon}, \dot{d}(\mathrm{r}))}{\partial d} \dot{d}(r, \dot{r}) \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}$$

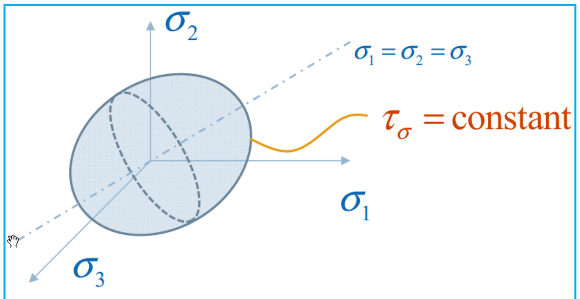
$$\psi(\boldsymbol{\varepsilon}, d(r)) = (1 - d(r))\psi_0(\boldsymbol{\varepsilon}) = (1 - d(r)) \underbrace{\frac{1}{2}(\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon})}_{\psi_0(\boldsymbol{\varepsilon}) \geq 0}$$

$$\boldsymbol{\sigma} = \frac{\partial \psi(\boldsymbol{\varepsilon}, d(\mathrm{r}))}{\partial \boldsymbol{\varepsilon}} = (1 - d(r)) \frac{\partial \psi_0(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}} = (1 - d(r)) \mathbb{C} : \boldsymbol{\varepsilon}$$

$$\mathcal{D} = -\frac{\partial \psi(\boldsymbol{\varepsilon}, d(\mathrm{r}))}{\partial \mathrm{d}} \dot{d} = \underbrace{\psi_0}_{\geq 0} \dot{d} \geq 0 \quad \Rightarrow \quad \boxed{\dot{d} \geq 0}$$

$$\tau_\sigma = \sqrt{\boldsymbol{\sigma} : \mathbb{M} : \boldsymbol{\sigma}} = \|\boldsymbol{\sigma}\|_{\mathbb{M}}$$

$$\mathbb{M} = \alpha \mathbf{1} \otimes \mathbf{1} + \beta \mathbb{I} \text{ with } \begin{cases} \alpha + \beta \geq 0 \\ \beta \geq 0 \end{cases}$$



Remark

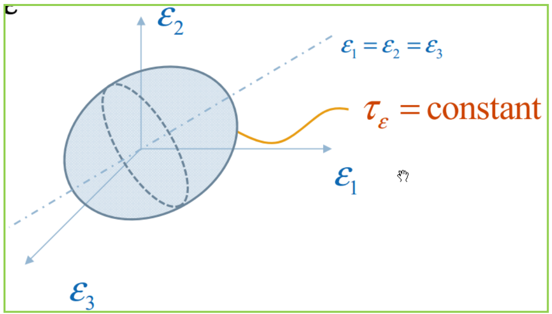
$\sigma_1 \geq \sigma_2 \geq \sigma_3$

High-Westergaard "stress space"

- STRAIN NORM

$$\tau_\varepsilon = \|\bar{\boldsymbol{\sigma}}\|_{\mathbb{M}} = \sqrt{\bar{\boldsymbol{\sigma}} : \mathbb{M} : \bar{\boldsymbol{\sigma}}} = (\bar{\boldsymbol{\sigma}} : \mathbb{M} : \underbrace{\bar{\boldsymbol{\sigma}}}_{\mathbb{C}:\boldsymbol{\varepsilon}})^{\frac{1}{2}} = (\boldsymbol{\varepsilon} : \underbrace{\mathbb{C} : \mathbb{M} : \mathbb{C}}_{\mathbb{C}^*} : \boldsymbol{\varepsilon})^{\frac{1}{2}} = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C}^* : \boldsymbol{\varepsilon}}$$

$$\tau_\varepsilon = \|\boldsymbol{\varepsilon}\|_{\mathbb{C}^*} = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C}^* : \boldsymbol{\varepsilon}}$$



$$\tau_\sigma = (1-d)\tau_\varepsilon$$

- **Damage function** (in stress space)

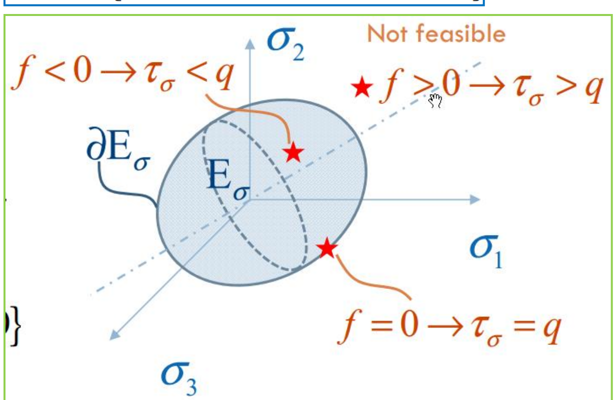
$$f(\boldsymbol{\sigma}, r) = \tau_\sigma - q(r)$$

Elastic domain (in stress space)

$$\mathrm{E}_\sigma := \{\boldsymbol{\sigma} \in \mathbb{S} \mid f(\boldsymbol{\sigma}, r) = \tau_\sigma - q(r) < 0\}$$

Damage surface (in stress space)

$$\partial \mathrm{E}_\sigma := \{\boldsymbol{\sigma} \in \mathbb{S} \mid f(\boldsymbol{\sigma}, r) = \tau_\sigma - q(r) = 0\}$$



Damage function (in strain space)

$$g(\varepsilon, r) = \tau_\varepsilon - r$$

Elastic domain (in strain space)

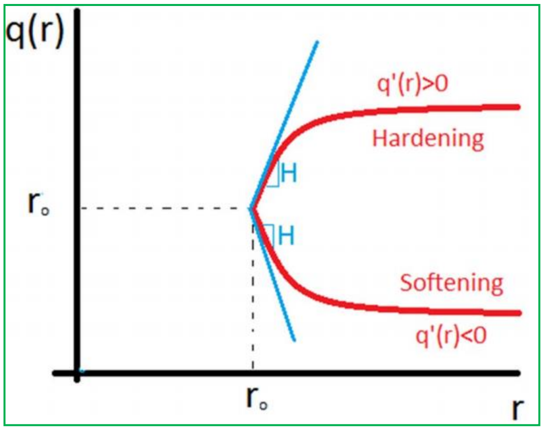
$$\mathrm{E}_\varepsilon := \{\varepsilon \in \mathbb{S} \mid g(\varepsilon, r) \equiv \tau_\varepsilon - r < 0\}$$

Damage surface (in strain space)

$$\partial \mathrm{E}_\varepsilon := \{\varepsilon \in \mathbb{S} \mid g(\varepsilon, r) \equiv \tau_\varepsilon - r = 0\}$$

THEOREM : The elastic domains ins stress and strain spaces are equivalent, i.e.:

$$\text{If } \boldsymbol{\sigma} \in \mathrm{E}_\sigma \Leftrightarrow \varepsilon \in \mathrm{E}_\varepsilon$$



$$\dot{r} = \lambda(\varepsilon, r) \geq 0, \quad r\big|_{t=0} = r_0 > 0, \quad r \in [r_0, \infty]$$

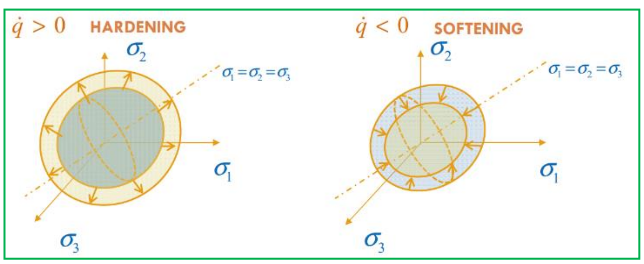
Damage variable

$$d := 1 - \frac{q(r)}{r} \qquad 0 \leq d \leq 1$$

$$q(r_0) = r_0 \quad \rightarrow d(r_0) = 0$$

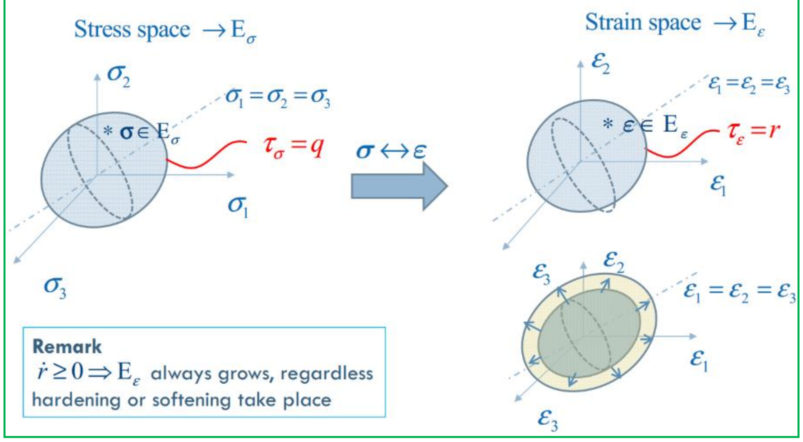
- Hardening parameter

$$H := \frac{dq(r)}{dr} = q'(r) \qquad \begin{array}{l} H > 0 \\ \text{Hardening} \\ \\ H < 0 \\ \text{Softening} \end{array}$$



$$H \leq \frac{q(r)}{r}$$

- Elastic domain (in stress and strain spaces)



Remark
$\dot{r} \geq 0 \Rightarrow \mathrm{E}_\varepsilon$ always grows, regardless hardening or softening take place

Constitutive equation-additional elements: Karush/Kuhn/Tucker conditions

MATHEMATICAL EXPRESSIONS OF THE K-K-T CONDITIONS

$$\lambda \geq 0 \;;\; g \leq 0 \;;\; \lambda g = 0 \rightarrow \text{Loading/unloading conditions}$$

$$\text{for } g = 0 \quad \lambda \dot{g} = 0 \rightarrow \text{Persistency/consistency conditions}$$

$$\psi(\boldsymbol{\varepsilon}, d(r)) = (1 - d(r))\psi_0(\boldsymbol{\varepsilon}) = (1 - d(r))\frac{1}{2}(\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon})$$

$$d := 1 - \frac{q(r)}{r} \quad 0 \leq d \leq 1 \qquad q(r_0) = r_0 \quad \text{for } d = 0$$

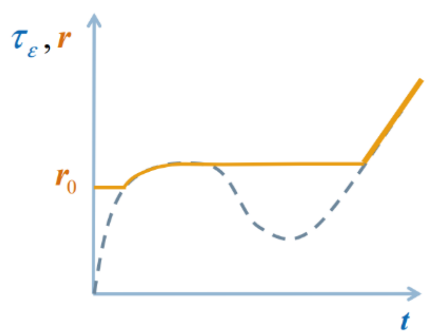
$$\boldsymbol{\sigma} = \frac{\partial \psi(\boldsymbol{\varepsilon}, \mathrm{d(r)})}{\partial \boldsymbol{\varepsilon}} = (1 - d(r))\frac{\partial \psi_0(\boldsymbol{\varepsilon}, \mathrm{d(r)})}{\partial \boldsymbol{\varepsilon}} = (1 - d(r))\mathbb{C} : \boldsymbol{\varepsilon}$$

$$r(t) = \underbrace{\max(r_0, \tau_\varepsilon(s)) \quad s \in [0, t]}_{\text{historical máximum of } [r_0, \tau_\varepsilon(s)]}$$

$$\boldsymbol{\varepsilon}_s \rightarrow \tau_\varepsilon(s) \rightarrow r_t \rightarrow q_t = q(r_t) \rightarrow$$

$$d_t = 1 - \frac{q_t}{r_t} \rightarrow \boldsymbol{\sigma}_t = (1 - d_t)\mathbb{C} : \boldsymbol{\varepsilon}_t$$

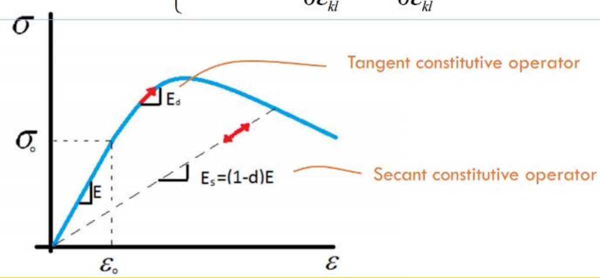
The integration is exact (closed form).
No dependence on Δt



$$\text{SUMMARY}: \left| \begin{array}{l} \text{1) the initial value of } r \text{ is } r_o > 0 \\ \text{2) the initial value of } \tau_\varepsilon \text{ is } \tau_\varepsilon(0) = 0 \\ \text{3) } r \text{ never decreases } (\dot{r} \geq 0) \\ \text{4) } \tau_\varepsilon \text{ is always smaller or equal than } r \quad (g \equiv \tau_\varepsilon - r \leq 0) \\ \text{5) When } \tau_\varepsilon \text{ equals } r \text{ then} \begin{cases} \text{if } \tau_\varepsilon \text{ grows} \rightarrow \dot{\tau}_\varepsilon = \dot{r} \geq 0 \\ \text{if } \tau_\varepsilon \text{ decreases} \rightarrow \dot{\tau}_\varepsilon < 0\,;\ \dot{r} = 0 \end{cases} \end{array} \right.$$

$$\boldsymbol{\varepsilon}(\mathbf{x},t) \rightarrow \boldsymbol{\sigma}(\mathbf{x},t) = \boldsymbol{\Sigma}(\boldsymbol{\varepsilon}(\mathbf{x},t))$$

$$\text{Tangent constitutive operator} \rightarrow \begin{cases} \mathbb{C}^d_{\text{tang}}(\boldsymbol{\varepsilon}) = \dfrac{\partial \boldsymbol{\Sigma}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \equiv \dfrac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \\ [\mathbb{C}^d_{\text{tang}}]_{ijkl} = \dfrac{\partial \boldsymbol{\Sigma}_{ij}(\boldsymbol{\varepsilon})}{\partial \varepsilon_{kl}} \equiv \dfrac{\partial \sigma_{ij}(\boldsymbol{\varepsilon})}{\partial \varepsilon_{kl}} \quad i,j,k,l \in \{1,2,3\} \end{cases}$$



$$\dot{\boldsymbol{\sigma}} = (1-d)\mathbb{C} : \dot{\boldsymbol{\varepsilon}} - \dot{d}\mathbb{C} : \boldsymbol{\varepsilon} = \mathbb{C}^d_{\text{tang}} : \dot{\boldsymbol{\varepsilon}}$$

$$\dot{r} = 0 \rightarrow \dot{d} = 0 \rightarrow \dot{\sigma} = \underbrace{(1-d)\mathbb{C}}_{\mathbb{C}^d_{\text{tang}}} : \dot{\boldsymbol{\varepsilon}}$$

$$\mathbb{C}^d_{\text{tang}} = (1-d)\mathbb{C} = \mathbb{C}^d_{\text{sec}} \quad \text{(secant constitutive operator)}$$

$$\dot{d}(t) = \frac{q - Hr}{r^3}\dot{r} \geq 0$$

$$\dot{r} = \frac{1}{r}\bar{\boldsymbol{\sigma}} : \mathbb{A} : \dot{\boldsymbol{\varepsilon}}$$

$$\dot{d} = \frac{q - Hr}{r^3}\bar{\boldsymbol{\sigma}} : \mathbb{A} : \dot{\boldsymbol{\varepsilon}}$$

$$\dot{\boldsymbol{\sigma}} = (1-d)\mathbb{C} : \dot{\boldsymbol{\varepsilon}} - \dot{d}\,\underbrace{\mathbb{C} : \boldsymbol{\varepsilon}}_{\bar{\boldsymbol{\sigma}}} = (1-d)\mathbb{C} : \dot{\boldsymbol{\varepsilon}} - \underbrace{\dot{d}\,\bar{\boldsymbol{\sigma}}}_{\bar{\boldsymbol{\sigma}} \otimes \dot{d}}$$

$$\mathbb{C}^d_{\text{tang}}(\boldsymbol{\varepsilon}) = (1-d)\mathbb{C} - \frac{q - Hr}{r^3}(\bar{\boldsymbol{\sigma}} \otimes [\bar{\boldsymbol{\sigma}} : \mathbb{A}])$$

$$\begin{array}{l} \text{Symmetric model} \\ \text{(tension/compression)} \end{array} \rightarrow \begin{cases} \mathbb{M} = \mathbb{C}^{-1} \\ \mathbb{A} = \mathbb{M} : \mathbb{C} \end{cases} \rightarrow \begin{cases} \mathbb{A} = \mathbb{C}^{-1} : \mathbb{C} = \mathbb{I} \\ \bar{\boldsymbol{\sigma}} : \mathbb{A} = \bar{\boldsymbol{\sigma}} : \mathbb{I} = \bar{\boldsymbol{\sigma}} \end{cases}$$

$$\mathbb{C}^d_{\text{tang}}(\boldsymbol{\varepsilon}) = \begin{cases} (1-d)\mathbb{C} \rightarrow \text{(elastic/unloading)} \\ (1-d)\mathbb{C} - \dfrac{q - Hr}{r^3}(\bar{\boldsymbol{\sigma}} \otimes \bar{\boldsymbol{\sigma}}) \rightarrow \text{(loading)} \end{cases}$$

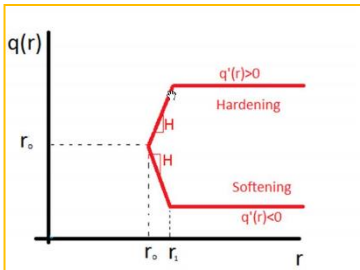
$$\text{Step 1} \rightarrow \text{Compute}\begin{cases}\bar{\boldsymbol{\sigma}}_{t+\Delta t} = \mathbb{C} : \boldsymbol{\varepsilon}_{t+\Delta t} \\ \tau_{\varepsilon_{t+\Delta t}} = \sqrt{\boldsymbol{\varepsilon}_{t+\Delta t} : \mathbb{C} : \boldsymbol{\varepsilon}_{t+\Delta t}} = \sqrt{\boldsymbol{\varepsilon}_{t+\Delta t} : \bar{\boldsymbol{\sigma}}_{t+\Delta t}}\end{cases}$$

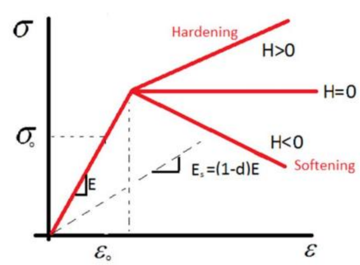
$$\text{Step 2} \rightarrow \text{If } \tau_{\varepsilon_{t+\Delta t}} \le r_t \rightarrow \left| \begin{array}{l}\text{Elastic} \\ \text{Unloading} \\ \text{Neutral loading}\end{array}\right.$$

$$\rightarrow \left| \begin{array}{l} r_{t+\Delta t} = r_t \\ d_{t+\Delta t} = d_t = 1 - \dfrac{q(r_{t+\Delta t})}{r_{t+\Delta t}} \\ \boldsymbol{\sigma}_{t+\Delta t} = (1 - d_{t+\Delta t})\bar{\boldsymbol{\sigma}}_{t+\Delta t} \\ (\mathbb{C}^{d}_{\text{tang}})_{t+\Delta t} = (1 - d_{t+\Delta t})\mathbb{C} \end{array}\right.$$

$$\text{Step 3} \rightarrow \text{If } \tau_{\varepsilon_{t+\Delta t}} > r_t \rightarrow \text{(Loading)}$$

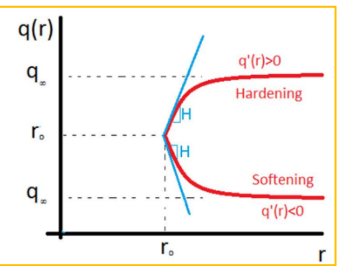
$$\rightarrow \left| \begin{array}{l} r_{t+\Delta t} = \tau_{\varepsilon_{t+\Delta t}} \\ d_{t+\Delta t} = 1 - \dfrac{q(r_{t+\Delta t})}{r_{t+\Delta t}} \\ \boldsymbol{\sigma}_{t+\Delta t} = (1 - d_{t+\Delta t})\bar{\boldsymbol{\sigma}}_{t+\Delta t} \\ (\mathbb{C}^{d}_{\text{tang}})_{t+\Delta t} = (1 - d_{t+\Delta t})\mathbb{C} - \dfrac{q(r_{t+\Delta t}) - H_{t+\Delta t} r_{t+\Delta t}}{(r_{t+\Delta t})^3}(\bar{\boldsymbol{\sigma}}_{t+\Delta t} \otimes \bar{\boldsymbol{\sigma}}_{t+\Delta t}) \end{array}\right.$$





$$q(r) = q_\infty - (q_\infty - r_0)e^{A(1-\frac{r}{r_0})} \quad (A > 0)$$

$$H(r) = \frac{dq(r)}{dr} = A\frac{(q_\infty - r_0)}{r_0}e^{A(1-\frac{r}{r_0})}$$

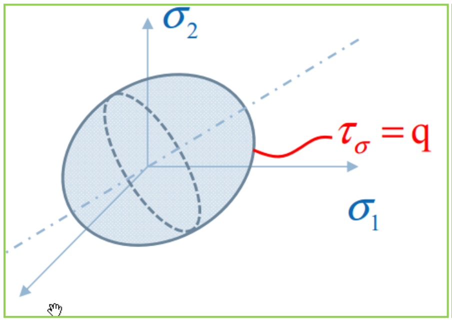


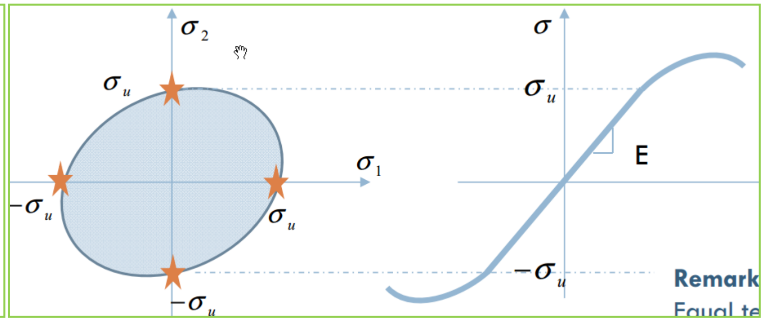
1. **Symmetric (tension/compression) model**

$$\mathbb{M} = \mathbb{C}^{-1}$$

$$\tau_\sigma = \sqrt{\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma}} = (1-d)\tau_\varepsilon = (1-d)\sqrt{\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}}$$

$$\tau_\varepsilon = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}}$$





$$[\sigma] \rightarrow \text{diagonalization} \rightarrow [\sigma]_{diag} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$[\sigma]^+_{diag} = \begin{bmatrix} \langle\sigma_1\rangle & 0 & 0 \\ 0 & \langle\sigma_2\rangle & 0 \\ 0 & 0 & \langle\sigma_3\rangle \end{bmatrix} \rightarrow \begin{matrix}\text{return to original} \\ \text{system of} \\ \text{coordinates}\end{matrix} \rightarrow \underbrace{[\sigma]^+}_{\text{Positive counterpart of } [\sigma]}$$

$$\boldsymbol{\sigma} = \sum_{i=1}^{i=3} \underbrace{\sigma_i}_{\text{eigenvalue "}i\text{"}} \overbrace{\hat{\mathbf{p}}_i}^{\text{eigenvector "}i\text{"}} \otimes \hat{\mathbf{p}}_i$$

$$\boldsymbol{\sigma}^+ = \sum_{i=1}^{i=3} \langle\sigma_i\rangle \hat{\mathbf{p}}_i \otimes \hat{\mathbf{p}}_i$$

$$\boldsymbol{\sigma} = \underbrace{(1-d)}_{\geq 0}\bar{\boldsymbol{\sigma}} \Rightarrow \boldsymbol{\sigma}^+ = (1-d)\bar{\boldsymbol{\sigma}}^+$$

$$\left| \begin{array}{l} \boldsymbol{\sigma} \textit{ and } \boldsymbol{\sigma}^+ \textit{ have the same eigenvectors} \\ \boldsymbol{\sigma}^+ \textit{ shares the positive eigenvalues of } \boldsymbol{\sigma} \\ \boldsymbol{\sigma}^+ \textit{ has those negative eigenvalues of } \boldsymbol{\sigma} \textit{ null} \end{array} \right.$$

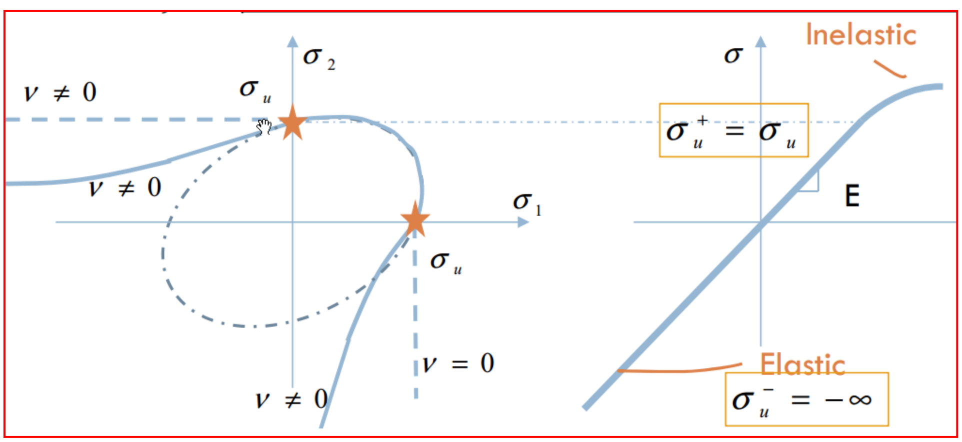
- Strain norm redefinition

$$\tau_\sigma^+ \equiv \sqrt{\boldsymbol{\sigma}^+ : \mathbb{C}^{-1} : \boldsymbol{\sigma}} = \sqrt{(1-d)^2 \bar{\boldsymbol{\sigma}}^+ : \mathbb{C}^{-1} : \bar{\boldsymbol{\sigma}}} = (1-d)\overbrace{\sqrt{\bar{\boldsymbol{\sigma}}^+ : \mathbb{C}^{-1} : \bar{\boldsymbol{\sigma}}}}^{\equiv \tau_\varepsilon^+} = (1-d)\tau_\varepsilon^+$$

$$\tau_\varepsilon^+ = (\bar{\boldsymbol{\sigma}}^+ : \underbrace{\mathbb{C}^{-1} : \bar{\boldsymbol{\sigma}}}_{\boldsymbol{\varepsilon}})^{\frac{1}{2}} = \sqrt{\bar{\boldsymbol{\sigma}}^+ : \boldsymbol{\varepsilon}}$$

$$\textit{Pure tensile state} \rightarrow \boldsymbol{\sigma} = \boldsymbol{\sigma}^+$$
$$\rightarrow \tau_\sigma^+ \equiv \sqrt{\boldsymbol{\sigma}^+ : \mathbb{C}^{-1} : \boldsymbol{\sigma}} = \tau_\sigma$$

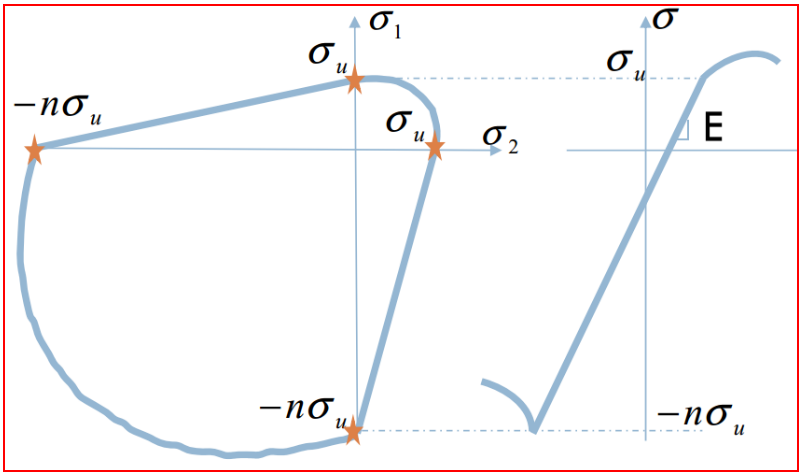
$$\textit{Pure compressive state}$$
$$\sigma^+ = 0 \rightarrow \tau_\sigma^+ \equiv \sqrt{\boldsymbol{\sigma}^+ : \mathbb{C}^{-1} : \boldsymbol{\sigma}} = 0 \rightarrow f = \overbrace{\tau_\sigma^+}^{=0} - \sigma_u < 0$$
$$\rightarrow \text{The state is always elastic}$$



$$\tau_\sigma = \left[\theta + \frac{1-\theta}{n}\right]\sqrt{\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma}}$$

$$\theta = \frac{\sum_1^3 \langle \sigma_i \rangle}{\sum_1^3 |\sigma_i|} = \frac{\sum_1^3 \langle \bar{\sigma}_i \rangle}{\sum_1^3 |\bar{\sigma}_i|}$$

$$\begin{cases} \sigma_1, \sigma_2, \sigma_3 > 0 \rightarrow \theta = 1 \rightarrow \tau_\sigma = \sqrt{\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma}} \\ \sigma_1, \sigma_2, \sigma_3 < 0 \rightarrow \theta = 0 \rightarrow \tau_\sigma = \dfrac{1}{n}\sqrt{\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma}} \end{cases}$$



Karush-Kuhn –Tucker and persistency conditions

$$\dot{r} = \lambda \geq 0, \quad g \leq 0 \Rightarrow \lambda g \leq 0$$

$$\text{if} \quad g = 0 \Rightarrow \lambda \dot{g} \leq 0$$

Visco regularization (Perzyna's Regularization)

Obtained by replacing $\lambda \rightarrow \frac{1}{\eta}\langle g \rangle$ in the inviscid model

Evolution equation

$$\dot{r} = \lambda(\boldsymbol{\varepsilon}, r) = \frac{1}{\eta}\langle g \rangle$$

Constitutive equation

where $\eta \geq 0$ is the viscosity

The only change is the evolution law

$\langle \ \rangle$ is the McAuley bracket

Damage function

$$g(\boldsymbol{\varepsilon}, r) \equiv \tau_{\varepsilon} - r$$

Karush – Kuhn –Tucker and persistency condition

Not necessary

- Remark 1

$$\frac{1}{\eta} \geq 0; \quad \langle g \rangle \geq 0 \Rightarrow \boxed{\lambda = \frac{1}{\eta}\langle g \rangle \geq 0}$$

$$\lambda g = \frac{1}{\eta}\underbrace{\langle g \rangle g}_{\langle g \rangle^2} = \frac{1}{\eta}\langle \underbrace{g}_{\lambda\eta} \rangle^2 = \frac{1}{\eta}(\lambda\eta)^2 = \eta \underbrace{\lambda}_{=\dot{r}}{}^2 = \eta \dot{r}^2 \qquad \text{As } \eta \rightarrow 0 \quad \boxed{\lambda g = 0}$$

- Remark 2

$$\langle g \rangle = \lambda \eta \geq 0$$

$$\text{As} \quad \eta \rightarrow 0 \ \langle g \rangle = 0 \Rightarrow \boxed{g \leq 0}$$

- Remark 3

$$\lambda g = 0 \qquad \dot{\lambda} g + \lambda \dot{g} = 0$$

$$\text{If} \quad g = 0 \Rightarrow \boxed{\lambda \dot{g} = 0}$$

$$\eta \rightarrow 0 \begin{cases} \lambda \geq 0 \quad g \leq 0 \quad \lambda g = 0 \rightarrow \text{Loading/unloading conditions} \\ \text{If } g = 0 \qquad \lambda \dot{g} = 0 \rightarrow \text{Persistency conditions} \end{cases}$$

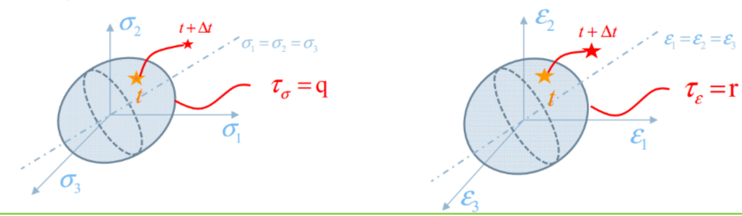
As $\eta \rightarrow 0$ (inviscid case) the rate independent continuum damage model is recovered

- Remark 3

For $\eta \neq 0$ and $\lambda \neq 0$ then $\langle g \rangle = \lambda\eta \neq 0$ $\Rightarrow$ $\begin{cases} g \equiv \tau_\varepsilon - r > 0 \rightarrow \tau_\varepsilon > r \\ f \equiv \tau_\sigma - q > 0 \rightarrow \tau_\sigma > q \end{cases}$

$$\begin{cases} f \leq 0 \Leftrightarrow g \leq 0 \rightarrow \dot{r} = \dfrac{1}{\eta}\langle g \rangle = 0 \rightarrow \textit{No evolution (unloading / neutral loading)} \\ f > 0 \Leftrightarrow g > 0 \rightarrow \dot{r} = \dfrac{1}{\eta}\langle g \rangle > 0 \rightarrow \textit{Evolution (loading)} \end{cases}$$

The stress/strain state can lay outside the elastic domain



$$\psi(\boldsymbol{\varepsilon}, r) = (1-d)\psi_0(\boldsymbol{\varepsilon}) = (1-d)\frac{1}{2}(\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon})$$

$$\boxed{\dot{d} \geq 0}$$

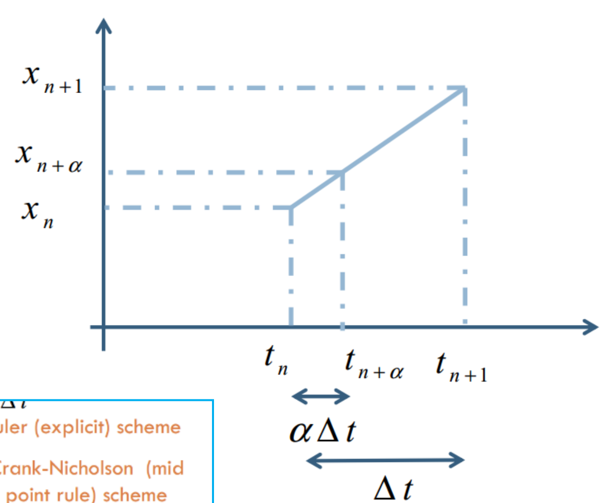
$$\mathbb{C}^{vd}_{\text{tang}} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} = (1-d)\mathbb{C}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial t} = \frac{1}{\eta} d'(r)\langle g(\boldsymbol{\varepsilon}, r)\rangle \mathbb{C} : \boldsymbol{\varepsilon}$$

$$\dot{r} = \frac{1}{\eta}\langle \tau_\varepsilon(\boldsymbol{\varepsilon}(t)) - r(t) \rangle \rightarrow \begin{cases} \text{Data :} \quad \underset{s \in [0,t]}{\boldsymbol{\varepsilon}(s)} \\ \text{Result:} \quad r(t) \end{cases}$$

$$x_n := x(t_n)$$
$$x_{n+1} := x(t_{n+1})$$
$$x_{n+\alpha} := x(t_n + \alpha\Delta t)$$

$$x_{n+\alpha} \simeq (1-\alpha)x_n + \alpha x_{n+1}$$



- Integration

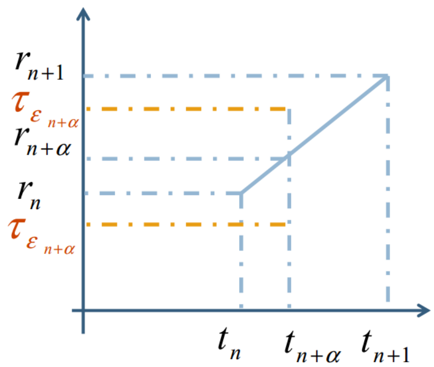
$$\dot{x}_{n+1} = \frac{x_{n+1} - x_n}{\Delta t} = f(x_{n+\alpha}, t_{n+\alpha}) \begin{cases} \alpha = 0 & x_{n+\alpha} = x_n & \text{Forward Euler (explicit) scheme} \\ \alpha = 0.5 & x_{n+\alpha} = \dfrac{x_{n+1} + x_n}{2} & \text{Crank-Nicholson (mid point rule) scheme} \\ \alpha = 1 & x_{n+\alpha} = x_{n+1} & \text{Backward Euler (implicit) scheme} \end{cases}$$

$$\boxed{\dot{r} = \frac{1}{\eta}\left\langle \tau_{\varepsilon}(\boldsymbol{\varepsilon}(t)) - r(t) \right\rangle}$$

- **Data**
$$\begin{cases} r\,|_{t=t_0} = r_0 \\ r_n,\ \boldsymbol{\varepsilon}_n,\ \boldsymbol{\varepsilon}_{n+1} \Rightarrow \tau_{\varepsilon_n}(\boldsymbol{\varepsilon}_n), \tau_{\varepsilon_{n+1}}(\boldsymbol{\varepsilon}_{n+1}) \end{cases}$$

- **Unknowns**
$$r_{n+1} \Rightarrow q_{n+1} \Rightarrow d_{n+1} \Rightarrow \sigma_{n+1}$$

- **Integration**
$$\begin{cases} \dot{r}_{n+1} = \dfrac{r_{n+1} - r_n}{\Delta t} = \dfrac{1}{\eta}\langle \tau_{\varepsilon_{n+\alpha}} - r_{n+\alpha} \rangle \\ \tau_{\varepsilon_{n+\alpha}} = (1-\alpha)\tau_{\varepsilon_n} + \alpha \tau_{\varepsilon_{n+1}} \to Data \\ r_{n+\alpha} = (1-\alpha) r_n + \alpha r_{n+1} \end{cases}$$



THEOREM: $\boxed{\tau_{\varepsilon_{n+a}} \leq r_{n+\alpha} \Leftrightarrow \tau_{\varepsilon_{n+a}} \leq r_n} \to (equivalence)$

COROLLARY: $\boxed{\tau_{\varepsilon_{n+a}} > r_{n+\alpha} \Leftrightarrow \tau_{\varepsilon_{n+a}} > r_n} \to (equivalence)$

1) Unloading / Neutral loading

$$\Leftrightarrow \quad \boxed{\tau_{\varepsilon_{n+a}} \leq r_n}$$

$$\boxed{r_{n+1} = r_n}$$

2) Loading

$$\boxed{\tau_{\varepsilon_{n+a}} > r_n}$$

$$\boxed{r_{n+1} = \frac{[\eta - \Delta t(1-\alpha)]r_n + \Delta t \tau_{\varepsilon_{n+\alpha}}}{\eta + \alpha \Delta t}}$$

Inviscid case and implicit integration

$$\boxed{r_{n+1} = \tau_{\varepsilon_{n+1}}}$$

Numerical integration inherits the properties of the model and recovers the solution of the rate independent problem for the **inviscid case and implicit integration**

Numerical integration: stability analysis

Cranck-Nicholson (mid-point rule)

$$\boxed{\frac{1}{2} \leq \alpha \leq 1}$$

Backward-Euler

$$\begin{cases} f(x(t), \dot{x}(t)) = 0 \\ x|_{t=0} = x_0 \end{cases} \rightarrow x_{n+1} = a x_n + b \qquad \boxed{|a| \leq 1} \rightarrow \begin{cases} \text{errors do not propagate} \\ \text{(stable integration)} \end{cases}$$

Numerical integration: accuracy analysis

- Accuracy

$$\dot r(t) = \frac{1}{\eta}\left(\tau_\varepsilon(t) - r(t)\right)$$

First order accuracy/consistency:
Numerical solution fulfills this equation as $\Delta t \to 0$

$$\ddot r(t) = \frac{1}{\eta}\left(\dot\tau_\varepsilon(t) - \dot r(t)\right)$$

Second order accuracy:
Numerical solution fulfills this equation as $\Delta t \to 0$

- Alpha method

$$r_{n+\alpha} = (1-\alpha) r_n + \alpha r_{n+1} \;\Rightarrow\; r_{n+\alpha}(t_{n+1}) = (1-\alpha) r_n + \alpha r_{n+1}(t_{n+1})$$

$$\dot r_{n+\alpha} = \frac{\partial r_{n+\alpha}}{\partial t_{n+1}} = \alpha \frac{\partial r_{n+1}}{\partial t_{n+1}} = \alpha \dot r_{n+1} \quad ; \quad \ddot r_{n+\alpha} = \frac{\partial \dot r_{n+\alpha}}{\partial t_{n+1}} = \alpha \frac{\partial \dot r_{n+1}}{\partial t_{n+1}} = \alpha \ddot r_{n+1}$$

$$\tau_{n+\alpha} = (1-\alpha)\tau_n + \alpha \tau_{n+1} \;\Rightarrow$$

$$\dot\tau_{n+\alpha} = \frac{\partial \tau_{n+\alpha}}{\partial t_{n+1}} = \alpha \dot\tau_{n+1} \quad ; \quad \ddot\tau_{n+\alpha} = \frac{\partial \dot\tau_{n+\alpha}}{\partial t_{n+1}} = \alpha \frac{\partial \dot\tau_{n+1}}{\partial t_{n+1}} = \alpha \ddot\tau_{n+1}$$

$$\boxed{\dot r(t) = \frac{1}{\eta}\left(\tau_\varepsilon(t) - r(t)\right) \quad ; \quad \ddot r(t) = \frac{1}{\eta}\left(\dot\tau_\varepsilon(t) - \dot r(t)\right)}$$

Stable: $\alpha = [\frac{1}{2}, 1]$ First order accurate: $\alpha = [0,1]$ Second order accurate: $\alpha = \frac{1}{2}$

$$\begin{cases} \mathbb{C}^{vd}_{\text{tang}} = \dfrac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = (1-d)\mathbb{C} \rightarrow \text{ Analytic tangent operator} \\ \mathbb{C}^{vd}_{\text{alg},n+1} := \dfrac{\partial \boldsymbol{\sigma}_{n+1}(\boldsymbol{\varepsilon}_{n+1})}{\partial \boldsymbol{\varepsilon}_{n+1}} \rightarrow \text{Algorithmic tangent operator} \end{cases} \qquad \begin{cases} \mathbb{C}^{vd}_{\text{alg},n+1} \neq \mathbb{C}^{vd}_{\text{tang}}(\boldsymbol{\varepsilon}_{n+1}) \\ \lim\limits_{\Delta t \to 0} \mathbb{C}^{vd}_{\text{alg},n+1} = \mathbb{C}^{vd}_{\text{tang}}(\boldsymbol{\varepsilon}_{n+1}) \end{cases}$$

$$\begin{cases} \boldsymbol{\sigma}_{n+1} = [1-d_{n+1}]\mathbb{C} : \boldsymbol{\varepsilon}_{n+1} \\ d_{n+1}(r_{n+1}) = 1 - \dfrac{q(r_{n+1})}{r_{n+1}} \\ r_{n+1} = \begin{cases} r_n \text{ (elastic/unloading)} \\ \dfrac{[\eta - \Delta t(1-\alpha)]}{\eta + \alpha \Delta t} r_n + \dfrac{\Delta t}{\eta + \alpha \Delta t} \tau_{\varepsilon_{n+\alpha}}(\boldsymbol{\varepsilon}_{n+1}) \text{ (loading)} \end{cases} \end{cases} \rightarrow \begin{cases} \boldsymbol{\sigma}_{n+1} = \boldsymbol{\Sigma}_{n+1}(\boldsymbol{\varepsilon}_{n+1}) \\ \delta\boldsymbol{\sigma}_{n+1} = \mathbb{C}^{vd}_{\text{alg},n+1}(\boldsymbol{\varepsilon}_{n+1}) : \delta\boldsymbol{\varepsilon}_{n+1} \end{cases}$$

$$\delta\boldsymbol{\sigma}_{n+1} = (1-d_{n+1})\mathbb{C} : \delta\boldsymbol{\varepsilon}_{n+1} - d'(r_{n+1})\delta r_{n+1} \otimes \underbrace{\mathbb{C} : \boldsymbol{\varepsilon}_{n+1}}_{\bar{\sigma}_{n+1}} \; ; \; \delta r_{n+1} = \begin{cases} 0 \text{ (elastic/unloading)} \\ \dfrac{\alpha \Delta t}{\eta + \alpha \Delta t} \delta \tau_{\varepsilon_{n+1}}) \text{ (loading)} \end{cases}$$

$$\tau_{\varepsilon_{n+1}} = \sqrt{\boldsymbol{\varepsilon}_{n+1} : \mathbb{C} : \boldsymbol{\varepsilon}_{n+1}} \Rightarrow \delta\tau_{\varepsilon_{n+1}} = \frac{1}{\tau_{\varepsilon_{n+1}}} \underbrace{\boldsymbol{\varepsilon}_{n+1} : \mathbb{C}}_{\bar{\boldsymbol{\sigma}}_{n+1}} : \delta\boldsymbol{\varepsilon}_{n+1} = \frac{1}{\tau_{\varepsilon_{n+1}}} \bar{\boldsymbol{\sigma}}_{n+1} : \delta\boldsymbol{\varepsilon}_{n+1}$$

$$\delta\boldsymbol{\sigma}_{n+1} = \begin{cases} (1-d_n)\mathbb{C} : \delta\boldsymbol{\varepsilon}_{n+1} \text{ (elastic/unloading)} \\ [(1-d_{n+1})\mathbb{C} - \dfrac{\alpha\Delta t}{\eta+\alpha\Delta t} \dfrac{1}{\tau_{\varepsilon_{n+1}}} d'_{n+1}\, \boldsymbol{\sigma}_{n+1} \otimes \boldsymbol{\sigma}_{n+1}] : \delta\boldsymbol{\varepsilon}_{n+1} \text{ (loading)} \end{cases} = \mathbb{C}^{vd}_{\text{alg},n+1} : \delta\boldsymbol{\varepsilon}_{n+1}$$

$$\mathbb{C}^{vd}_{\text{alg},n+1} = \begin{cases} (1-d_n)\mathbb{C} \text{ (elastic/unloading)} \\ \underbrace{(1-d_{n+1})\mathbb{C}}_{\mathbb{C}^{vd}_{\text{tang,n+1}}} - \underbrace{\dfrac{\alpha\Delta t}{\eta+\alpha\Delta t} \dfrac{1}{\tau_{\varepsilon_{n+1}}} d'_{n+1}\, \bar{\boldsymbol{\sigma}}_{n+1} \otimes \bar{\boldsymbol{\sigma}}_{n+1}}_{\text{Additional term} \to \mathcal{O}(\Delta t)} \text{ (loading)} \quad \left(d'_{n+1} = -\dfrac{H_{n+1}r_{n+1} - q(r_{n+1})}{(r_{n+1})^2}\right) \end{cases}$$

$$\mathbb{C}^{vd}_{\text{alg},n+1} = \begin{cases} (1-d_n)\mathbb{C} \text{ (elastic/unloading)} \\ \underbrace{(1-d_{n+1})\mathbb{C}}_{\mathbb{C}^{vd}_{\text{tang,n+1}}} - \underbrace{\dfrac{\alpha\Delta t}{\eta+\alpha\Delta t} \dfrac{1}{\tau_{\varepsilon_{n+1}}} d'_{n+1}\, \bar{\boldsymbol{\sigma}}_{n+1} \otimes \bar{\boldsymbol{\sigma}}_{n+1}}_{\text{Additional term} \to \mathcal{O}(\Delta t)} \text{ (loading)} \end{cases}$$

- Remark 1

  If: $\alpha = 0$ $\quad \boxed{\mathbb{C}^{vd}_{\text{alg},n+1} = \mathbb{C}^{vd}_{\text{tang},n+1}}$

  If: $\Delta t = 0$ $\quad \boxed{\mathbb{C}^{vd}_{\text{alg},n+1} = \mathbb{C}^{d}_{n+1}}$

  Algorithmic and analytical tangent operators match

- Remark 2

  If: $\eta \to 0$

$$\boxed{\mathbb{C}^{vd}_{\text{alg},n+1} = (1-d_{n+1})\mathbb{C} - \frac{1}{\tau_{\varepsilon_{n+1}}} d'_{n+1}\, \bar{\boldsymbol{\sigma}}_{n+1} \otimes \bar{\boldsymbol{\sigma}}_{n+1}}$$

Algorithmic viscous tangent operator match (in the inviscid limit) the one for the rate independent damage model

INPUT DATA $[t_n, t_n + \Delta t = t_{n+1}] \rightarrow \boldsymbol{\varepsilon}_n, r_n, \boldsymbol{\varepsilon}_{t_n+1}$

$$\text{Step 1} \rightarrow \text{Compute} \begin{cases} \bar{\boldsymbol{\sigma}}_{n+1} = \mathbb{C} : \boldsymbol{\varepsilon}_{n+1} \rightarrow \begin{cases} \tau_{\boldsymbol{\varepsilon}_n} = \sqrt{\boldsymbol{\varepsilon}_n : \mathbb{C} : \boldsymbol{\varepsilon}_n} \\ \tau_{\boldsymbol{\varepsilon}_{n+1}} = \sqrt{\boldsymbol{\varepsilon}_{n+1} : \mathbb{C} : \boldsymbol{\varepsilon}_{n+1}} = \sqrt{\bar{\boldsymbol{\sigma}}_{n+1} : \boldsymbol{\varepsilon}_{n+1}} \end{cases} \\ \tau_{\boldsymbol{\varepsilon}_{n+\alpha}} = (1-\alpha)\tau_{\boldsymbol{\varepsilon}} + \alpha \tau_{\boldsymbol{\varepsilon}_{n+1}} \end{cases}$$

$\text{Step 2} \rightarrow \text{If } \tau_{\boldsymbol{\varepsilon}+\alpha} \leq r_n \rightarrow$ | Elastic Unloading

$$\rightarrow \left| \begin{array}{l} r_{n+1} = r_n \ ; \quad d_{n+1} = d_n = 1 - \dfrac{q(r_{n+1})}{r_{n+1}} \ ; \quad \boldsymbol{\sigma}_{n+1} = (1 - d_{n+1})\bar{\boldsymbol{\sigma}}_{n+1} \\ \mathbb{C}^{vd}_{\mathrm{alg},n+1} = (1 - d_{n+1})\mathbb{C} \end{array} \right.$$

$\text{Step 3} \rightarrow \text{If } \tau_{\varepsilon_{n+\alpha}} > r_n \rightarrow$ (Loading)

$$\rightarrow \left| \begin{array}{l} r_{n+1} = \dfrac{[\eta - \Delta t(1-\alpha)]}{\eta + \alpha \Delta t} r_n + \dfrac{\Delta t}{\eta + \alpha \Delta t} \tau_{\varepsilon_{n+\alpha}} \ ; \ d_{n+1} = 1 - \dfrac{q(r_{n+1})}{r_{n+1}} \\ \boldsymbol{\sigma}_{n+1} = (1 - d_{n+1})\bar{\boldsymbol{\sigma}}_{n+1} \\ \mathbb{C}^{vd}_{\mathrm{alg},n+1} = (1 - d_{n+1})\mathbb{C} + \\ \qquad + \dfrac{\alpha \Delta t}{\eta + \alpha \Delta t} \dfrac{1}{\tau_{\varepsilon_{n+1}}} \dfrac{H_{n+1} r_{n+1} - q(r_{n+1})}{(r_{n+1})^2} (\bar{\boldsymbol{\sigma}}_{n+1} \otimes \bar{\boldsymbol{\sigma}}_{n+1}) \end{array} \right.$$

OUTPUT DATA $[t_n, t_n + \Delta t = t_{n+1}] \rightarrow r_{n+1}, \boldsymbol{\sigma}_{n+1}, \mathbb{C}^{vd}_{\mathrm{alg},n+1}$

For $\eta = 0$ and $\alpha = 1$ the inviscid model is recovered !!!!

$$\boxed{\dot{\varepsilon}^{pl}_v = \frac{1}{\mu}(\varepsilon^{pl} - \varepsilon^{pl}_v)}$$

- Taking into account the symmetry of the stress and strain tensors, theey can be written in vector form:

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz}\\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z\end{bmatrix}\overset{not.}{=}\begin{bmatrix}\varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz}\\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz}\\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z\end{bmatrix} \Rightarrow \boxed{\{\boldsymbol{\varepsilon}\}\overset{def}{=}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}\in\mathbb{R}^6}$$

VOIGT'S NOTATION

$$\boldsymbol{\sigma}\equiv\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{xy} & \sigma_y & \tau_{yz}\\ \tau_{xz} & \tau_{yz} & \sigma_z\end{bmatrix} \Rightarrow \boxed{\{\boldsymbol{\sigma}\}\overset{def}{=}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{Bmatrix}\in\mathbb{R}^6}$$

REMARK

The double contraction $(\boldsymbol{\sigma}:\boldsymbol{\varepsilon})$ is equivalent to the scalar (dot) product $(\{\boldsymbol{\sigma}\}\cdot\{\boldsymbol{\varepsilon}\})$:

$$\underbrace{\boldsymbol{\sigma}:\boldsymbol{\varepsilon}}_{\text{2nd order tensors}}=\overbrace{\{\boldsymbol{\sigma}\}\cdot\{\boldsymbol{\varepsilon}\}}^{\text{vectors}} \Leftrightarrow \sigma_{ij}\varepsilon_{ij}=\sigma_i\varepsilon_i$$

- Hooke's law in terms of the stress and strain vectors:

$$\begin{cases}\boldsymbol{\sigma}=\mathbb{C}:\boldsymbol{\varepsilon}\\ \mathbb{C}=\lambda\mathbf{1}\otimes\mathbf{1}+2\mu\mathbb{I}\end{cases} \Rightarrow \boxed{\begin{cases}\{\boldsymbol{\sigma}\}=\boldsymbol{D}\cdot\{\boldsymbol{\varepsilon}\}\\ \sigma_i=D_{ij}\varepsilon_j \quad i\in\{1,....6\}\end{cases}}$$

Where $\boldsymbol{D}$ is the matrix of elastic constants:

$$\boldsymbol{D}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0\\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0\\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)}\end{bmatrix}$$

$$\begin{cases}\boldsymbol{\varepsilon}=\mathbb{C}^{-1}:\boldsymbol{\sigma}\\ \mathbb{C}=-\frac{\nu}{E}\mathbf{1}\otimes\mathbf{1}+\frac{1+\nu}{2E}\mathbb{I}\end{cases} \Rightarrow \boxed{\begin{cases}\{\boldsymbol{\varepsilon}\}=\boldsymbol{D}^{-1}\cdot\{\boldsymbol{\sigma}\}\\ \varepsilon_j=(\boldsymbol{D}^{-1})_{ij}\sigma_i \quad i\in\{1,...,6\}\end{cases}}$$

Where $\boldsymbol{D}^{-1}$ is the elastic compliance matrix:

$$\boldsymbol{D}^{-1}=\begin{bmatrix}\frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0\\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0\\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G}\end{bmatrix} \quad ; \quad G=\frac{E}{2(1+\nu)}$$

- The expressions used in the algorithms of the model read:

$$\tau_\varepsilon=\sqrt{\boldsymbol{\varepsilon}:\mathbb{C}:\boldsymbol{\varepsilon}}=\sqrt{\bar{\boldsymbol{\sigma}}:\boldsymbol{\varepsilon}}=\sqrt{\{\bar{\boldsymbol{\sigma}}\}\cdot\{\boldsymbol{\varepsilon}\}}=\sqrt{\{\boldsymbol{\varepsilon}\}\cdot\{\boldsymbol{D}\}\cdot\{\boldsymbol{\varepsilon}\}}=\sqrt{\{\bar{\boldsymbol{\sigma}}\}\cdot\{\boldsymbol{D}^{-1}\}\cdot\{\bar{\boldsymbol{\sigma}}\}}$$

$$\{\boldsymbol{\sigma}\}_{n+1}=(1-d(r_{n+1}))\{\bar{\boldsymbol{\sigma}}\}$$

$$\{\bar{\boldsymbol{\sigma}}\}_{n+1}=\boldsymbol{D}\cdot\{\boldsymbol{\varepsilon}\}_{n+1}$$

$$\tau_{\varepsilon_{n+1}}=\sqrt{\{\boldsymbol{\varepsilon}\}_{n+1}\cdot\boldsymbol{D}\cdot\{\boldsymbol{\varepsilon}\}_{n+1}}=\sqrt{\{\bar{\boldsymbol{\sigma}}\}_{n+1}\cdot\boldsymbol{D}^{-1}\cdot\{\bar{\boldsymbol{\sigma}}\}_{n+1}}=\sqrt{\{\bar{\boldsymbol{\sigma}}\}_{n+1}\cdot\{\boldsymbol{\varepsilon}\}_{n+1}}$$

$$\underbrace{\boldsymbol{D}^{vd}_{\text{alg},n+1}}_{6\times6}=\underbrace{\frac{\partial\{\boldsymbol{\sigma}\}_{n+1}}{\partial\{\boldsymbol{\varepsilon}\}_{n+1}}}_{6\times6}=(1-d_{n+1})\underbrace{\boldsymbol{D}}_{6\times6}-$$

$$-\frac{\alpha\Delta t}{\eta+\alpha\Delta t}\frac{1}{\tau_{\varepsilon_{n+1}}}\frac{H_{n+1}r_{n+1}-q(r_{n+1})}{(r_{n+1})^2}(\underbrace{\{\bar{\boldsymbol{\sigma}}\}_{n+1}}_{6\times1}\otimes\underbrace{\{\bar{\boldsymbol{\sigma}}\}_{n+1}}_{1\times6})$$

Plasticity model based on effective stress

$$
\begin{aligned}
&f_{\mathrm{p}}(\bar{\boldsymbol{\sigma}}, \kappa_{\mathrm{p}}) = \tilde{\sigma}(\bar{\boldsymbol{\sigma}}) - \sigma_{\mathrm{Y}}(\kappa_{\mathrm{p}}) \\
&\dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} = \dot{\lambda} \frac{\partial g_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} (\bar{\boldsymbol{\sigma}}, \kappa_{\mathrm{p}}) \\
&\dot{\kappa}_{\mathrm{p}} = \dot{\lambda} k_{\mathrm{p}}(\bar{\boldsymbol{\sigma}}, \kappa_{\mathrm{p}}) \\
&f_{\mathrm{p}} \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda} f_{\mathrm{p}} = 0
\end{aligned}
$$

$$
\begin{aligned}
&f_{\mathrm{d}}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{\mathrm{p}}, \kappa_{\mathrm{d}}) = \tilde{\varepsilon}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_{\mathrm{p}}) - \kappa_{\mathrm{d}} \\
&\omega = g_{\mathrm{d}}(\kappa_{\mathrm{d}}) \\
&f_{\mathrm{d}} \leqslant 0, \quad \dot{\kappa}_{\mathrm{d}} \geqslant 0, \quad \dot{\kappa}_{\mathrm{d}} f_{\mathrm{d}} = 0
\end{aligned}
$$

$$\dot{f}_{\mathrm{p}} \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda} \dot{f}_{\mathrm{p}} = 0$$

$$\dot{f}_{\mathrm{p}} = \frac{\partial f_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} : \dot{\bar{\boldsymbol{\sigma}}} + \frac{\partial f_{\mathrm{p}}}{\partial \kappa_{\mathrm{p}}} \dot{\kappa}_{\mathrm{p}} = \frac{\partial f_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_{\mathrm{e}} : \dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \left(\frac{\partial f_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_{\mathrm{e}} : \frac{\partial g_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} + H_{\mathrm{p}} k_{\mathrm{p}} \right)$$

$$H_{\mathrm{p}} = -\frac{\partial f_{\mathrm{p}}}{\partial \kappa_{\mathrm{p}}} = \frac{\mathrm{d}\sigma_{\mathrm{Y}}}{\mathrm{d}\kappa_{\mathrm{p}}}$$

$$\frac{\partial f_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_{\mathrm{e}} : \frac{\partial g_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} + H_{\mathrm{p}} k_{\mathrm{p}} > 0$$

$$\dot{\mathbf{f}} \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}^{\mathrm{T}}\dot{\mathbf{f}} = 0$$

$$\dot{\mathbf{f}} = \begin{Bmatrix} \dot{f}_{\mathrm{p}} \\ \dot{f}_{\mathrm{d}} \end{Bmatrix} \dot{\boldsymbol{\lambda}} = \begin{Bmatrix} \dot{\lambda} \\ \dot{\kappa}_{\mathrm{d}} \end{Bmatrix}$$

$$\dot{\mathbf{f}} = \mathbf{b} - \mathbf{A}\dot{\boldsymbol{\lambda}}$$

$$\dot{f}_{\mathrm{d}} = \frac{\partial f_{\mathrm{d}}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial f_{\mathrm{d}}}{\partial \boldsymbol{\varepsilon}_{\mathrm{p}}} : \dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} + \frac{\partial f_{\mathrm{d}}}{\partial \kappa_{\mathrm{d}}} \dot{\kappa}_{\mathrm{d}} = \frac{\partial f_{\mathrm{d}}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \dot{\lambda} \frac{\partial f_{\mathrm{d}}}{\partial \boldsymbol{\varepsilon}_{\mathrm{p}}} : \frac{\partial g_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} + \frac{\partial f_{\mathrm{d}}}{\partial \kappa_{\mathrm{d}}} \dot{\kappa}_{\mathrm{d}}$$

$$\mathbf{A} = \begin{pmatrix} \dfrac{\partial f_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_{\mathrm{e}} : \dfrac{\partial g_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} + H_{\mathrm{p}} k_{\mathrm{p}} & 0 \\ -\dfrac{\partial f_{\mathrm{d}}}{\partial \boldsymbol{\varepsilon}_{\mathrm{p}}} : \dfrac{\partial g_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} & -\dfrac{\partial f_{\mathrm{d}}}{\partial \kappa_{\mathrm{d}}} \end{pmatrix}$$

$$\mathbf{b} = \begin{Bmatrix} \dfrac{\partial f_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_{\mathrm{e}} : \dot{\boldsymbol{\varepsilon}} \\ \dfrac{\partial f_{\mathrm{d}}}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} \end{Bmatrix}$$

$$\mathbf{A}\dot{\boldsymbol{\lambda}} - \mathbf{b} \geqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}^{\mathrm{T}}(\mathbf{A}\dot{\boldsymbol{\lambda}} - \mathbf{b}) = 0$$

$$A_{11} > 0, \quad A_{22} > 0, \quad A_{11}A_{22} - A_{21}A_{12} > 0$$

$$\frac{\partial f_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} : \mathbf{D}_{\mathrm{e}} : \frac{\partial g_{\mathrm{p}}}{\partial \bar{\boldsymbol{\sigma}}} + H_{\mathrm{p}} k_{\mathrm{p}} > 0$$

$$-\frac{\partial f_{\mathrm{d}}}{\partial \kappa_{\mathrm{d}}} > 0$$

$$
\begin{aligned}
& f_{\mathrm{p}}(\boldsymbol{\sigma}, \kappa_{\mathrm{p}}) = \tilde{\sigma}(\boldsymbol{\sigma}) - \sigma_{\mathrm{Y}}(\kappa_{\mathrm{p}}) \\
& \dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} = \dot{\lambda} \frac{\partial g_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} \\
& \dot{\kappa}_{\mathrm{p}} = \dot{\lambda} k_{\mathrm{p}}(\boldsymbol{\sigma}, \kappa_{\mathrm{p}})
\end{aligned}
$$

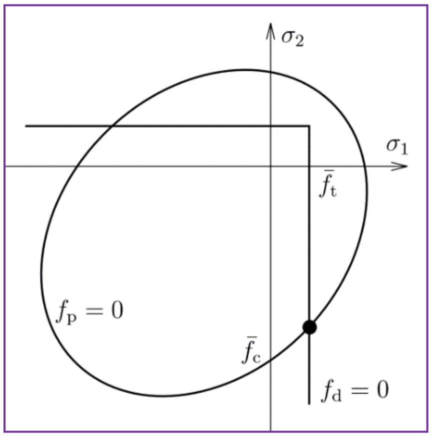
$$
\dot{f}_{\mathrm{p}} = \frac{\partial f_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f_{\mathrm{p}}}{\partial \kappa_{\mathrm{p}}} \dot{\kappa}_{\mathrm{p}} = (1-\omega) \frac{\partial f_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} : \mathbf{D}_{\mathrm{e}} : \dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \left[(1-\omega) \frac{\partial f_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} : \mathbf{D}_{\mathrm{e}} : \frac{\partial g_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} + H_{\mathrm{p}} k_{\mathrm{p}} \right] - \dot{\kappa}_{\mathrm{d}} \frac{\mathrm{d} g_{\mathrm{d}}}{\mathrm{d} \kappa_{\mathrm{d}}} \frac{\partial f_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}}
$$

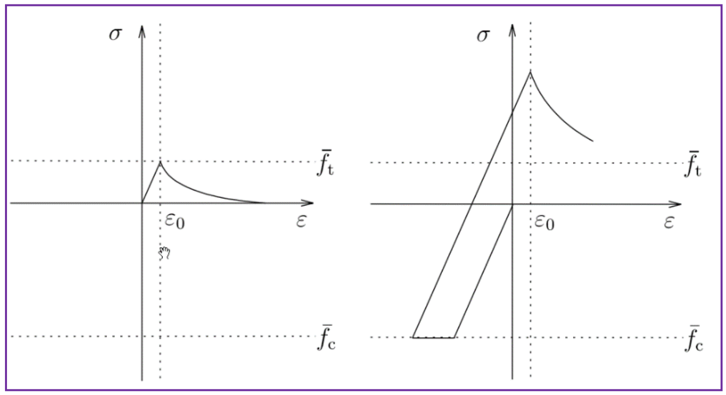
$$
\mathbf{A} = \begin{pmatrix} (1-\omega) \frac{\partial f_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} : \mathbf{D}_{\mathrm{e}} : \frac{\partial g_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} + H_{\mathrm{p}} k_{\mathrm{p}} & \frac{\mathrm{d} g_{\mathrm{d}}}{\mathrm{d} \kappa_{\mathrm{d}}} \frac{\partial f_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}} \\ -\frac{\partial f_{\mathrm{d}}}{\partial \boldsymbol{\varepsilon}_{\mathrm{p}}} : \frac{\partial g_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} & 1 \end{pmatrix}
$$

$$
\begin{aligned}
& (1-\omega) \frac{\partial f_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} : \mathbf{D}_{\mathrm{e}} : \frac{\partial g_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} + H_{\mathrm{p}} k_{\mathrm{p}} > 0 \\
& 1 > 0 \\
& (1-\omega) \frac{\partial f_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} : \mathbf{D}_{\mathrm{e}} : \frac{\partial g_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} + H_{\mathrm{p}} k_{\mathrm{p}} > -\frac{\partial f_{\mathrm{d}}}{\partial \boldsymbol{\varepsilon}_{\mathrm{p}}} : \frac{\partial g_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} \left(\frac{\mathrm{d} g_{\mathrm{d}}}{\mathrm{d} \kappa_{\mathrm{d}}} \frac{\partial f_{\mathrm{p}}}{\partial \boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}} \right)
\end{aligned}
$$

$$
H_{\mathrm{p}} k_{\mathrm{p}} > 0
$$

$$f_\mathrm{p}(\boldsymbol{\sigma}, \kappa_\mathrm{p}) = c_\phi I_1(\boldsymbol{\sigma}) + \sqrt{J_2(\boldsymbol{\sigma})} - \sigma_\mathrm{Y}(\kappa_\mathrm{p})$$
$$g_\mathrm{p}(\boldsymbol{\sigma}) = c_\psi I_1(\boldsymbol{\sigma}) + \sqrt{J_2(\boldsymbol{\sigma})}$$





$$\tilde{\varepsilon}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_\mathrm{p}) = \frac{1}{E} \max_{I=1,2,3} \langle \bar{\sigma}_I(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_\mathrm{p}) \rangle$$

$$g_\mathrm{d}(\kappa_\mathrm{d}) = \begin{cases} 0 & \text{if } \kappa_\mathrm{d} \leqslant \varepsilon_0 \\ 1 - \dfrac{\varepsilon_0}{\kappa_\mathrm{d}} \exp\left(-\dfrac{\kappa_\mathrm{d} - \varepsilon_0}{\varepsilon_\mathrm{f} - \varepsilon_0}\right) & \text{if } \kappa_\mathrm{d} \geqslant \varepsilon_0 \end{cases}$$

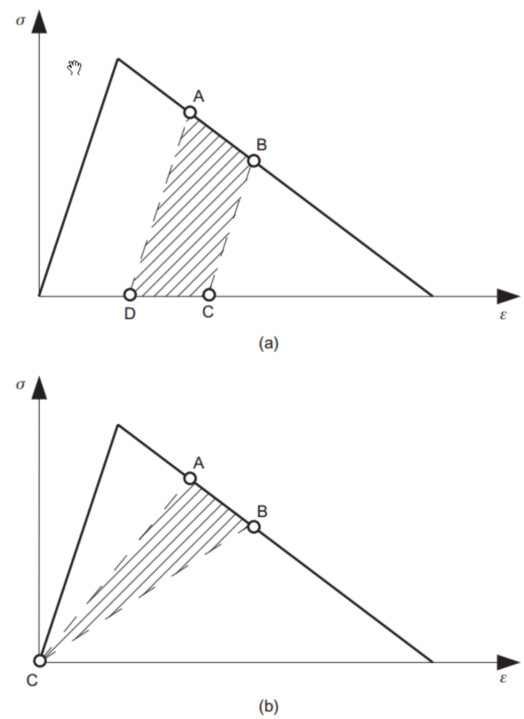
$$\dot{f}_\mathrm{d} = \frac{\partial f_\mathrm{d}}{\partial \boldsymbol{\varepsilon}_\mathrm{e}} : \dot{\boldsymbol{\varepsilon}}_\mathrm{e} + \frac{\partial f_\mathrm{d}}{\partial \kappa_\mathrm{d}} \dot{\kappa}_\mathrm{d} = \frac{\partial \tilde{\varepsilon}}{\partial \boldsymbol{\varepsilon}_\mathrm{e}} : \left(\dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \frac{\partial g_\mathrm{p}}{\partial \boldsymbol{\sigma}}\right) - \dot{\kappa}_\mathrm{d}$$

$$(1-\omega) \frac{\partial f_\mathrm{p}}{\partial \boldsymbol{\sigma}} : \mathbf{D}_\mathrm{e} : \frac{\partial g_\mathrm{p}}{\partial \boldsymbol{\sigma}} + H_\mathrm{p} k_\mathrm{p} > \frac{\partial \tilde{\varepsilon}}{\partial \boldsymbol{\varepsilon}_\mathrm{e}} : \frac{\partial g_\mathrm{p}}{\partial \boldsymbol{\sigma}} \left(\frac{\mathrm{d} g_\mathrm{d}}{\mathrm{d} \kappa_\mathrm{d}} \frac{\partial f_\mathrm{p}}{\partial \boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}}\right)$$

$$(1-\omega) E \left(\frac{3 c_\phi c_\psi}{1-2\nu} + \frac{1}{2(1+\nu)}\right) + H_\mathrm{p} k_\mathrm{p} > \left(\frac{1}{\kappa_\mathrm{d}} + \frac{1}{\varepsilon_\mathrm{f} - \varepsilon_0}\right) \left(\frac{c_\psi}{(1-2\nu)} + \frac{s_1}{2(1+\nu)\sqrt{J_2}}\right) \sigma_\mathrm{Y}(\kappa_\mathrm{p})$$

$$H_\mathrm{p} k_\mathrm{p} > \left(\frac{1.8257}{1 - \varepsilon_0/\varepsilon_\mathrm{f}} - 0.4342\right) E > 1.3915 E$$

25	$$\rho\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_\mathrm{p}, \omega) = \frac{1}{2}(1-\omega) \times (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p}) : \boldsymbol{D}_\mathrm{e} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p})$$
26	$$\dot{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho\dot{\psi} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \left(\frac{\partial \rho\psi}{\partial \boldsymbol{\varepsilon}}\dot{\boldsymbol{\varepsilon}} + \frac{\partial \rho\psi}{\partial \boldsymbol{\varepsilon}_\mathrm{p}}\dot{\boldsymbol{\varepsilon}}_\mathrm{p} + \frac{\partial \rho\psi}{\partial \omega}\dot{\omega}\right)$$
27	$$\frac{\partial \rho\psi}{\partial \boldsymbol{\varepsilon}}\dot{\boldsymbol{\varepsilon}} = (1-\omega)(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p}) : \boldsymbol{D}_\mathrm{e} : \dot{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$$
28	$$\dot{D} = \dot{D}^\mathrm{p} + \dot{D}^\mathrm{e} = \frac{\partial \rho\psi}{\partial \boldsymbol{\varepsilon}_\mathrm{p}}\dot{\boldsymbol{\varepsilon}}_\mathrm{p} + \frac{\partial \rho\psi}{\partial \omega}\dot{\omega}$$
29	$$\dot{D}^\mathrm{p} = \frac{\partial \rho\psi}{\partial \boldsymbol{\varepsilon}_\mathrm{p}}\dot{\boldsymbol{\varepsilon}}_\mathrm{p} = (1-\omega)(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p}) : \boldsymbol{D}_\mathrm{e} : \dot{\boldsymbol{\varepsilon}}_\mathrm{p}$$
30	$$\dot{D}^\mathrm{d} = \frac{\partial \rho\psi}{\partial \omega}\dot{\omega} = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p}) : \boldsymbol{D}_\mathrm{e} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p})\dot{\omega}$$



$$\rho\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_\mathrm{p}, \omega) = \tfrac{1}{2}(1-\omega)(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p}) : \mathbf{D}_\mathrm{e} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p})$$

$$\mathscr{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho\dot{\psi} = \left(\boldsymbol{\sigma} - \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}\right) : \dot{\boldsymbol{\varepsilon}} - \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}_\mathrm{p}} : \dot{\boldsymbol{\varepsilon}}_\mathrm{p} - \rho\frac{\partial\psi}{\partial\omega}\dot{\omega}$$

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}} = (1-\omega)\mathbf{D}_\mathrm{e} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p})$$

$$\mathscr{D} = -\rho\frac{\partial\psi}{\partial\boldsymbol{\varepsilon}_\mathrm{p}} : \dot{\boldsymbol{\varepsilon}}_\mathrm{p} - \rho\frac{\partial\psi}{\partial\omega}\dot{\omega} = (1-\omega)(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p}) : \mathbf{D}_\mathrm{e} : \dot{\boldsymbol{\varepsilon}}_\mathrm{p} + \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p}) : \mathbf{D}_\mathrm{e} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p})\dot{\omega} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_\mathrm{p} + Y\dot{\omega} \geqslant 0$$

$$Y = -\rho\frac{\partial\psi}{\partial\omega} = \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p}) : \mathbf{D}_\mathrm{e} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p})$$

$$\mathscr{D}_\mathrm{p} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_\mathrm{p} = \dot{\lambda}\boldsymbol{\sigma} : \frac{\partial g_\mathrm{p}}{\partial\bar{\boldsymbol{\sigma}}} = \dot{\lambda}(1-\omega)\bar{\boldsymbol{\sigma}} : \frac{\partial g_\mathrm{p}}{\partial\bar{\boldsymbol{\sigma}}}$$

$$\bar{\boldsymbol{\sigma}} = \mathbf{D}_\mathrm{e} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\mathrm{p})$$