Fourth Edition

LINEAR ALGEBRA
AND ITS APPLICATIONS

Gilbert Strang

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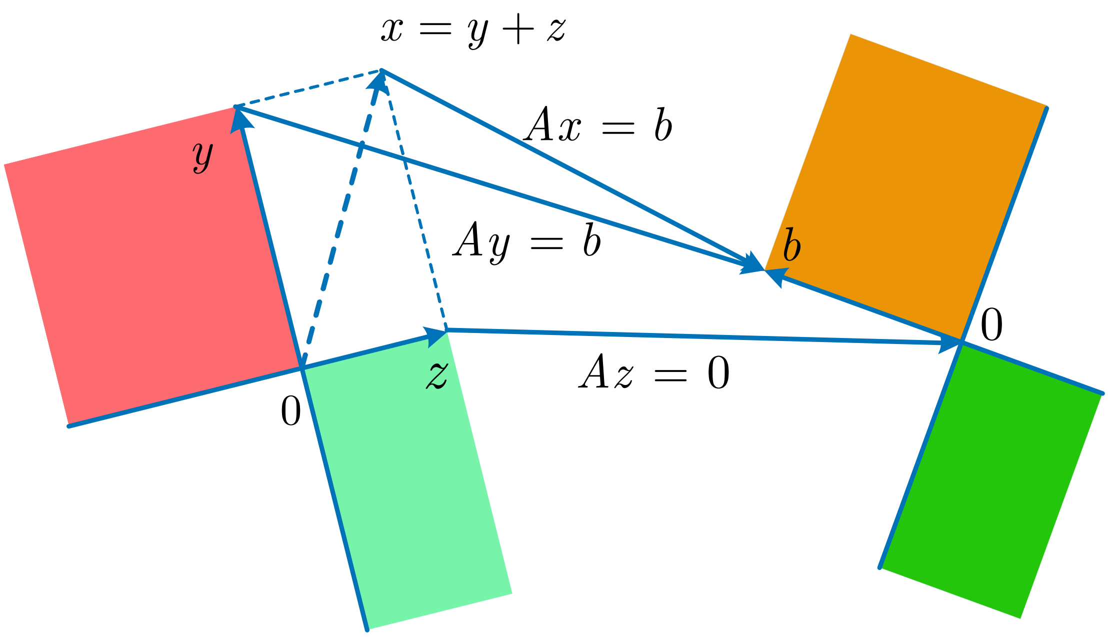

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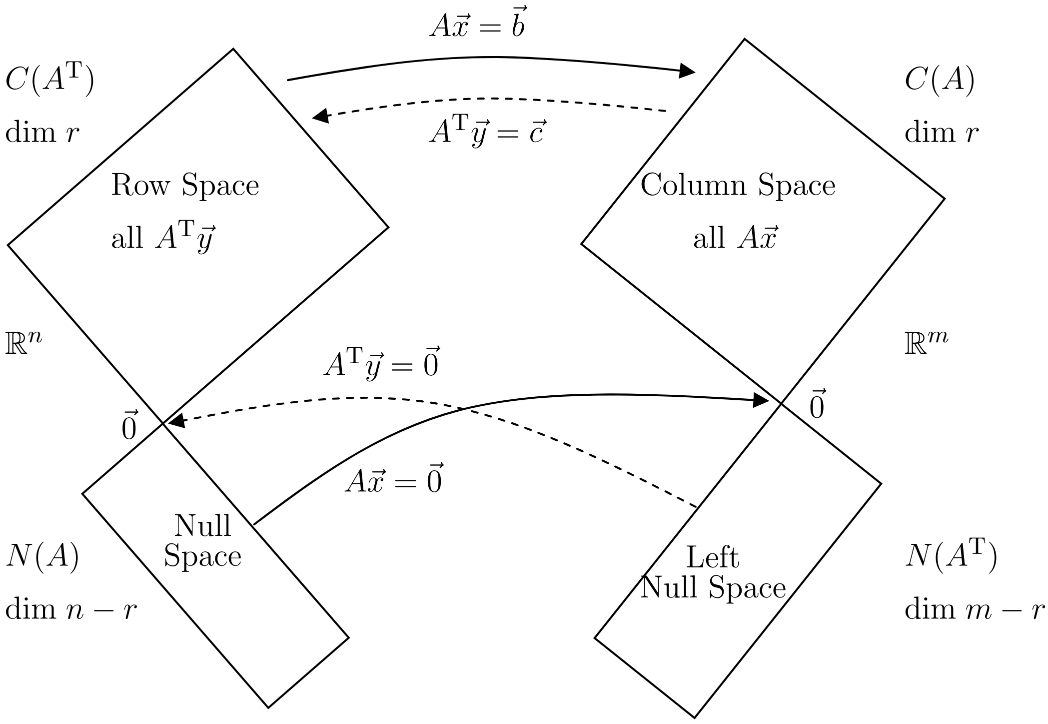
$A\vec{x} = \vec{b}$
$C(A^{\mathrm{T}})$
dim r
$A^{\mathrm{T}}\vec{y} = \vec{c}$
$C(A)$
dim r
Row Space
all $A^{\mathrm{T}}\vec{y}$
Column Space
all $A\vec{x}$
$\mathbb{R}^n$
$\mathbb{R}^m$
$A^{\mathrm{T}}\vec{y} = \vec{0}$
$\vec{0}$
$\vec{0}$
$A\vec{x} = \vec{0}$
Null Space
$N(A)$
dim $n - r$
Left Null Space
$N(A^{\mathrm{T}})$
dim $m - r$

Preface

Revising this textbook has been a special challenge, for a very nice reason. So many people have read this book, and taught from it, and even loved it. The spirit of the book could never change. This text was written to help our teaching of linear algebra keep up with the enormous importance of this subject—which just continues to grow.

One step was certainly possible and desirable—*to add new problems*. Teaching for all these years required hundreds of new exam questions (especially with quizzes going onto the web). I think you will approve of the extended choice of problems. The questions are still a mixture of *explain and compute*—the two complementary approaches to learning this beautiful subject.

I personally believe that many more people need linear algebra than calculus. Isaac Newton might not agree! But he isn't teaching mathematics in the 21st century (and maybe he wasn't a great teacher, but we will give him the benefit of the doubt). Certainly the laws of physics are well expressed by differential equations. Newton needed calculus—quite right. But the scope of science and engineering and management (and life) is now so much wider, and linear algebra has moved into a central place.

May I say a little more, because many universities have not yet adjusted the balance toward linear algebra. Working with curved lines and curved surfaces, the first step is always to *linearize*. Replace the curve by its tangent line, fit the surface by a plane, and the problem becomes linear. The power of this subject comes when you have ten variables, or 1000 variables, instead of two.

You might think I am exaggerating to use the word "beautiful" for a basic course in mathematics. Not at all. This subject begins with two vectors v and w, pointing in different directions. The key step is to *take their linear combinations*. We multiply to get $3v$ and $4w$, and we add to get the particular combination $3v+4w$. That new vector is in the *same plane* as v and w. When we take all combinations, we are filling in the whole plane. If I draw v and w on this page, their combinations $cv+dw$ fill the page (and beyond), but they *don't go up* from the page.

In the language of linear equations, I can solve $cv+dw=b$ exactly when the vector b lies in the same plane as v and w.

Matrices

I will keep going a little more to convert combinations of three-dimensional vectors into linear algebra. If the vectors are $v = (1,2,3)$ and $w = (1,3,4)$, put them into the **columns of a matrix**:

$$\textbf{matrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}.$$

To find combinations of those columns, "**multiply**" the matrix by a vector (c,d):

$$\textbf{Linear combinations } cv + dw \qquad \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Those combinations fill a *vector space*. We call it the **column space** of the matrix. (For these two columns, that space is a plane.) To decide if $b = (2,5,7)$ is on that plane, we have three components to get right. So we have three equations to solve:

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \qquad \text{means} \qquad \begin{aligned} c + d &= 2 \\ 2c + 3d &= 5 \\ 3c + 4d &= 7 \end{aligned}.$$

I leave the solution to you. The vector $b = (2,5,7)$ does lie in the plane of v and w. If the 7 changes to any other number, then b won't lie in the plane—it will *not* be a combination of v and w, and the three equations will have no solution.

Now I can describe the first part of the book, about linear equations $Ax = b$. The matrix A has n columns and m rows. *Linear algebra moves steadily to n vectors in m-dimensional space*. We still want combinations of the columns (in the column space). We still get m equations to produce b (one for each row). Those equations may or may not have a solution. They always have a least-squares solution.

The interplay of columns and rows is the heart of linear algebra. It's not totally easy, but it's not too hard. Here are four of the central ideas:

1. The **column space** (all combinations of the columns).

2. The **row space** (all combinations of the rows).

3. The **rank** (the number of independent columns) (or rows).

4. **Elimination** (the good way to find the rank of a matrix).

I will stop here, so you can start the course.

Web Pages

It may be helpful to mention the web pages connected to this book. So many messages come back with suggestions and encouragement, and I hope you will make free use of everything. You can directly access *http://web.mit.edu/18.06*, which is continually updated for the course that is taught every semester. Linear algebra is also on MIT's OpenCourseWare site *http://ocw.mit.edu*, where 18.06 became exceptional by including videos of the lectures (which you definitely don't have to watch...). Here is a part of what is available on the web:

1. Lecture schedule and current homeworks and exams with solutions.
2. The goals of the course, and conceptual questions.
3. Interactive Java demos (audio is now included for eigenvalues).
4. Linear Algebra Teaching Codes and MATLAB problems.
5. Videos of the complete course (taught in a real classroom).

The course page has become a valuable link to the class, and a resource for the students. I am very optimistic about the potential for graphics with sound. The bandwidth for voiceover is low, and FlashPlayer is freely available. This offers a *quick review* (with active experiment), and the full lectures can be downloaded. I hope professors and students worldwide will find these web pages helpful. My goal is to make this book as useful as possible with all the course material I can provide.

Other Supporting Materials

Student Solutions Manual 0-495-01325-0 The Student Solutions Manual provides solutions to the odd-numbered problems in the text.

Instructor's Solutions Manual 0-030-10588-4 The Instructor's Solutions Manual has teaching notes for each chapter and solutions to all of the problems in the text.

Structure of the Course

The two fundamental problems are $Ax = b$ and $Ax = \lambda x$ for square matrices A. The first problem $Ax = b$ has a solution when A has *independent columns*. The second problem $Ax = \lambda x$ looks for *independent eigenvectors*. A crucial part of this course is to learn what "independence" means.

I believe that most of us learn first from examples. You can see that

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \qquad \textbf{does } \textbf{\textit{not}} \textbf{ have independent columns.}$$

Column 1 plus column 2 equals column 3. A wonderful theorem of linear algebra says that the three rows are not independent either. The third row must lie in the same plane as the first two rows. Some combination of rows 1 and 2 will produce row 3. You might find that combination quickly (I didn't). In the end I had to use elimination to discover that the right combination uses 2 times row 2, minus row 1.

Elimination is the simple and natural way to understand a matrix by producing a lot of zero entries. So the course starts there. But don't stay there too long! You have to get from combinations of the rows, to independence of the rows, to "dimension of the row space." That is a key goal, to see whole spaces of vectors: the ***row space*** and the ***column space*** and the ***nullspace***.

A further goal is to understand how the matrix *acts*. When A multiplies x it produces the new vector Ax. The whole space of vectors moves—it is "transformed" by A. Special transformations come from particular matrices, and those are the foundation stones of linear algebra: diagonal matrices, orthogonal matrices, triangular matrices, symmetric matrices.

The eigenvalues of those matrices are special too. I think 2 by 2 matrices provide terrific examples of the information that eigenvalues λ can give. Sections 5.1 and 5.2 are worth careful reading, to see how $Ax = \lambda x$ is useful. Here is a case in which small matrices allow tremendous insight.

Overall, the beauty of linear algebra is seen in so many different ways:

1. **Visualization.** Combinations of vectors. Spaces of vectors. Rotation and reflection and projection of vectors. Perpendicular vectors. Four fundamental subspaces.

2. **Abstraction.** Independence of vectors. Basis and dimension of a vector space. Linear transformations. Singular value decomposition and the best basis.

3. **Computation.** Elimination to produce zero entries. Gram-Schmidt to produce orthogonal vectors. Eigenvalues to solve differential and difference equations.

4. **Applications.** Least-squares solution when $Ax = b$ has too many equations. Difference equations approximating differential equations. Markov probability matrices (the basis for Google!). Orthogonal eigenvectors as principal axes (and more...).

To go further with those applications, may I mention the books published by Wellesley-Cambridge Press. They are all linear algebra in disguise, applied to signal processing and partial differential equations and scientific computing (and even GPS). If you look at *http://www.wellesleycambridge.com*, you will see part of the reason that linear algebra is so widely used.

After this preface, the book will speak for itself. You will see the spirit right away. The emphasis is on understanding—*I try to explain rather than to deduce*. This is a book about real mathematics, not endless drill. In class, I am constantly working with examples to teach what students need.

Acknowledgments

I enjoyed writing this book, and I certainly hope you enjoy reading it. A big part of the pleasure comes from working with friends. I had wonderful help from Brett Coonley and Cordula Robinson and Erin Maneri. They created the LaTeX files and drew all the figures. Without Brett's steady support I would never have completed this new edition.

Earlier help with the Teaching Codes came from Steven Lee and Cleve Moler. Those follow the steps described in the book; MATLAB and Maple and Mathematica are faster for large matrices. All can be used (*optionally*) in this course. I could have added "Factorization" to that list above, as a fifth avenue to the understanding of matrices:

[L, U, P] = lu(A)	for linear equations
[Q, R] = qr(A)	to make the columns orthogonal
[S, E] = eig(A)	to find eigenvectors and eigenvalues.

In giving thanks, I never forget the first dedication of this textbook, years ago. That was a special chance to thank my parents for so many unselfish gifts. Their example is an inspiration for my life.

And I thank the reader too, hoping you like this book.

Gilbert Strang

Chapter 1

Matrices and Gaussian Elimination

1.1 Introduction

This book begins with the central problem of linear algebra: *solving linear equations*. The most important ease, and the simplest, is when the number of unknowns equals the number of equations. We have n **equations in** n **unknowns**, starting with $n = 2$:

$$\begin{array}{rrcrcr} \textbf{Two equations} & 1x & + & 2y & = & 3 \\ \textbf{Two unknowns} & 4x & + & 5y & = & 6. \end{array} \tag{1}$$

The unknowns are x and y. I want to describe two ways, *elimination* and *determinants*, to solve these equations. Certainly x and y are determined by the numbers 1, 2, 3, 4, 5, 6. The question is how to use those six numbers to solve the system.

1. **Elimination** Subtract 4 times the first equation from the second equation. This eliminates x from the second equation. and it leaves one equation for y:

   $$(\textbf{equation 2}) - 4(\textbf{equation 1}) \qquad -3y = -6. \tag{2}$$

   Immediately we know $y = 2$. Then x comes from the first equation $1x + 2y = 3$:

   $$\textbf{Back-substitution} \qquad 1x + 2(2) = 3 \quad \text{gives} \quad x = -1. \tag{3}$$

   Proceeding carefully, we cheek that x and y also solve the second equation. This should work and it does: 4 times $(x = -1)$ plus 5 times $(y = 2)$ equals 6.

2. **Determinants** The solution $y = 2$ depends completely on those six numbers in the equations. There most be a formula for y (and also x) It is a "ratio of determinants" and I hope you will allow me to write it down directly:

   $$y = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{1 \cdot 6 - 3 \cdot 4}{1 \cdot 5 - 2 \cdot 4} = \frac{-6}{-3} = 2. \tag{4}$$