Quiz W08

Bilinear Transform Filter Design

Use a **second-order** lowpass Butterworth filter and the bilinear transform to design a digital lowpass filter with bandwidth 1600 Hz. Assume sampling frequency is 7200 Hz.

- Derive H(z).
- Prepare an Excel spreadsheet & plot the impulse response.
- Draw a direct form realization for this filter.

Solution

BUTTERWORTH FILTER

A Butterworth filter (of order L and cutoff frequency ν_c) is defined by the equation:

$$H(s) = \frac{1}{\prod_{k=1}^{L} \frac{s - s_k}{\nu_c}}$$

$$where\ s_k = \nu_c e^{j(2k+L-1)\pi/2L} \quad k = 1,2,\ldots,L.$$

For a second-order Butterworth filter L=2. Thus, the poles are:

$$s_k = \nu_c e^{j(2k+1)\pi/4} \quad k = 1,2.$$

$$s_1 = \nu_c e^{j3\pi/4}, \quad s_2 = \nu_c e^{j5\pi/4}$$

$$s_1 = \nu_c e^{j3\pi/4} = \nu_c e^{j(4\pi-\pi)/4} = \nu_c e^{j\pi} e^{-j\pi/4} = \nu_c(-1)e^{-j\pi/4}$$

$$s_1 = -\nu_c(\cos(-\pi/4) + j\sin(-\pi/4)) = -\nu_c(1/\sqrt{2} - j/\sqrt{2}) = -\nu_c(1-j)/\sigma$$

$$where \quad \sigma = \sqrt{2}$$

Similarly,

$$s_2 = \nu_c e^{j5\pi/4} = \nu_c e^{j(4\pi+\pi)/4} = \nu_c e^{j\pi} e^{j\pi/4} = \nu_c(-1)e^{j\pi/4}$$

$$s_2 = -\nu_c(\cos(\pi/4) + j\sin(\pi/4)) = -\nu_c(1/\sqrt{2} + j/\sqrt{2}) = -\nu_c(1+j)/\sigma$$

$$s_1 = \beta(-1+j) \quad and \quad s_2 = \beta(-1-j) \quad where \quad \beta = \nu_c/\sigma = \nu_c/\sqrt{2}$$

And the transfer function is:

$$H(s)=\frac{1}{\prod_{k=1}^{2}\frac{s-s_k}{\nu_c}}=\frac{1}{\left(\frac{s-s_1}{\nu_c}\right)\left(\frac{s-s_2}{\nu_c}\right)}$$

$$H(s)=\frac{\nu_c^2}{(s-s_1)(s-s_2)}$$

FILTER PARAMETERS

Given the sampling frequency f_s=7200 Hz, the sampling interval is calculated as:

$$f_s=7200\ Hz \qquad T=1/f_s=0.139ms$$

Desired bandwidth = lowpass cutoff frequency = f_c = 1600 Hz. Converting this to normalized frequency $\boldsymbol{\omega_c}$:

$$\boldsymbol{\omega_c = 2\pi\,\frac{f_c}{f_s} = 2\pi\frac{1600\mathrm{Hz}}{7200\mathrm{Hz}} = 1.396\ \mathrm{radians}}$$

Since the bilinear transform is non-linear, it will warp all the frequencies, so 1.396 radians will end up somewhere other than 1.396 after the transforms. To offset this, we pre-warp the target cutoff frequency 1.396 so that after the transform it ends up at 1.396.

$$\nu_c=\frac{2}{T}\tan\left(\frac{\omega_c}{2}\right)=\frac{2}{T}\tan\left(\frac{1.396}{2}\right)=\frac{2\tan 0.698}{T}=\frac{1.678}{0.139ms}=12070Hz$$

BILINEAR TRANSFORM

Second order lowpass Butterworth filter with cutoff ν_c = 12070 Hz is:

$$H(s)=\frac{\nu_c{}^2}{(s-s_1)(s-s_2)} \quad \text{where}$$

$$s_1=\beta(-1+j) \quad and \quad s_2=\beta(-1-j) \quad where \quad \beta=\nu_c/\sigma=\nu_c/\sqrt{2}$$

Now this analog filter may be transformed into the z domain using the bilinear transform:

$$s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = H(s)\big|_{s=2(z-1)/T(z+1)} = \frac{v_c^2}{\left[\frac{2(z-1)}{T(z+1)}-s_1\right]\left[\frac{2(z-1)}{T(z+1)}-s_2\right]}$$

$$H(z)=\frac{v_c^2 T^2 (z+1)^2}{\left[2(z-1)-s_1 T(z+1)\right]\left[2(z-1)-s_2 T(z+1)\right]}$$

$$H(z)=\frac{\alpha^2 (z+1)^2}{\left[2(z-1)-s_1 T(z+1)\right]\left[2(z-1)-s_2 T(z+1)\right]} \quad where \quad \alpha=v_c T$$

$$H(z)=\frac{\alpha^2 (z+1)^2}{\left[2z-2-\beta(-1+j)T(z+1)\right]\left[2z-2-\beta(-1-j)T(z+1)\right]}$$

$$H(z)=\frac{\alpha^2 (z^2+2z+1)}{\left[2z-2-\beta T(-z-1+jz+j)\right]\left[2z-2-\beta T(-z-1-jz-j)\right]}$$

$$H(z)=\frac{\alpha^2 (z^2+2z+1)}{\left[z(2+\beta T-j\beta T)+(-2+\beta T-j\beta T)\right]\left[z(2+\beta T+j\beta T)+(-2+\beta T+j\beta T)\right]}$$

$$H(z)=\frac{\alpha^2 (z^2+2z+1)}{\left[z(\beta T+2-j\beta T)+(\beta T-2-j\beta T)\right]\left[z(\beta T+2+j\beta T)+(\beta T-2+j\beta T)\right]}$$

$$H(z)=\frac{\alpha^2 (z^2+2z+1)}{\left[z(\kappa-j\beta T)+(\lambda-j\beta T)\right]\left[z(\kappa+j\beta T)+(\lambda+j\beta T)\right]}$$

$$where \quad \kappa=\beta T+2, \quad and \quad \lambda=\beta T-2$$

$$H(z)=\frac{\alpha^2 (z^2+2z+1)}{z^2(\kappa-j\beta T)(\kappa+j\beta T)+z\left[(\kappa-j\beta T)(\lambda+j\beta T)+(\kappa+j\beta T)(\lambda-j\beta T)\right]+(\lambda-j\beta T)(\lambda+j\beta T}$$

$$H(z)=\frac{\alpha^2 (z^2+2z+1)}{z^2(\kappa^2+\beta^2T^2)+z\left[(\kappa\lambda-j\lambda\beta T+j\kappa\beta T+\beta^2T^2)+(\kappa\lambda+j\lambda\beta T-j\kappa\beta T+\beta^2T^2)\right]+(\lambda^2+\beta^2T^2)}$$

$$H(z)=\frac{\alpha^2 (z^2+2z+1)}{z^2(2\beta^2T^2+4\beta T+4)+z\left[(2\kappa\lambda+2\beta^2T^2)\right]+(2\beta^2T^2-4\beta T+4)}$$

$$H(z)=\frac{\alpha^2 (z^2+2z+1)}{z^2(2\beta^2T^2+4\beta T+4)+2z\left[((\beta^2T^2-4)+\beta^2T^2)\right]+(2\beta^2T^2-4\beta T+4)}$$

$$H(z)=\frac{\alpha^2 (z^2+2z+1)}{z^2(2\beta^2T^2+4\beta T+4)+2z(2\beta^2T^2-4)+(2\beta^2T^2-4\beta T+4)}$$

Substituting

$$\beta=v_c/\sigma \quad and \quad \beta T=v_c T/\sigma=\alpha/\sigma \quad where \quad \sigma=\sqrt{2}$$

$$H(z)=\frac{\alpha^2(z^2+2z+1)}{z^2(2\alpha^2/\sigma^2+4\alpha/\sigma+4)+2z(2\alpha^2/\sigma^2-4)+(2\alpha^2/\sigma^2-4\alpha/\sigma+4)}$$

$$Let \quad \delta=\alpha/\sigma=\alpha/\sqrt{2}=\nu_c T/\sqrt{2}$$

$$H(z)=\frac{2\delta^2(z^2+2z+1)}{z^2(2\delta^2+4\delta+4)+4(\delta^2-2)z+(2\delta^2-4\delta+4)}$$

$$where \quad \delta=\nu_c T/\sqrt{2}=12070Hz\times 0.139ms/1.414=1.187$$

$$H(z)=\frac{2.816(z^2+2z+1)}{z^2(2.816+4.746+4)+4(1.409-2)z+(2.816-4.746+4)}$$

$$H(z)=\frac{2.816(z^2+2z+1)}{11.562z^2-2.364z+2.07}$$

$$H(z)=0.2436\frac{z^2+2z+1}{z^2-0.205z+0.179}$$

$$H(z)=0.244\frac{1+2z^{-1}+z^{-2}}{1-0.205z^{-1}+0.179z^{-2}}$$

$$H(z)=\frac{0.244+0.488z^{-1}+0.244z^{-2}}{1-0.205z^{-1}+0.179z^{-2}}$$

The IIR filter coefficients are:

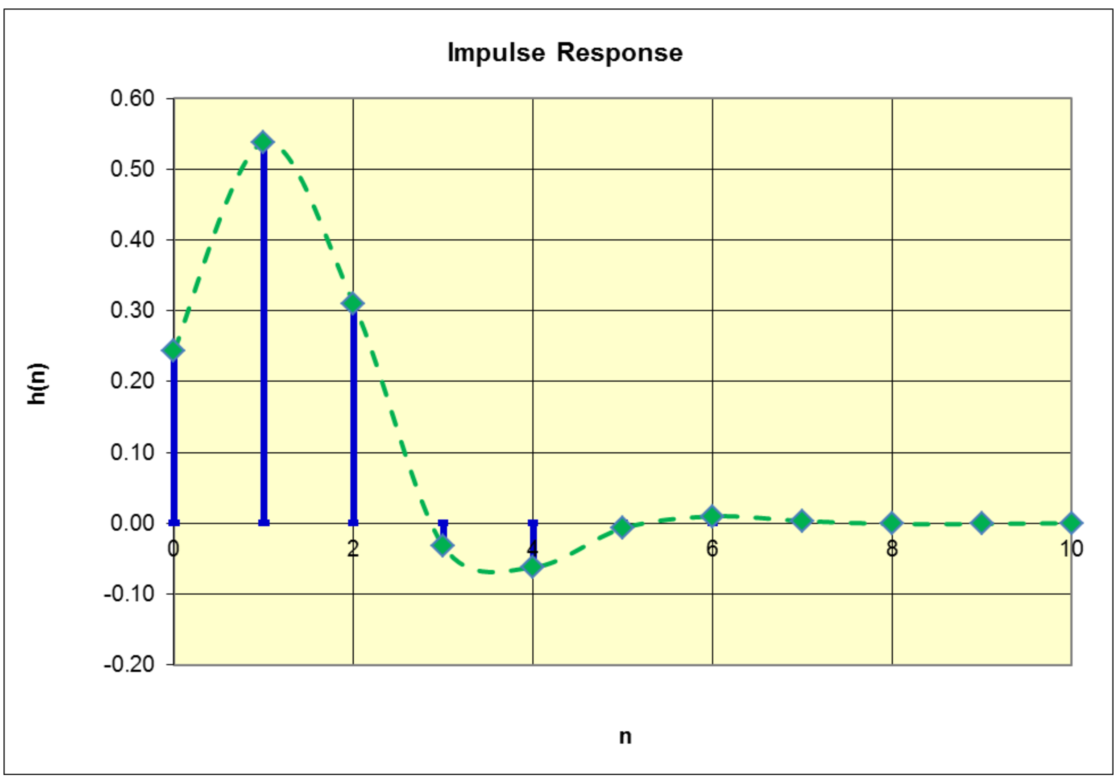
$$b_0=0.244, \quad b_1=0.488, \quad b_2=0.244,$$
$$a_1=-0.205, \quad a_2=0.179.$$

The filter output is:

$$y(n)=\sum_{l=0}^{L-1}b_l x(n-l)-\sum_{m=1}^{M}a_m y(n-m)$$
$$y(n)=b_0x(n)+b_1x(n-1)+b_2x(n-2)-a_1y(n-1)-a_2y(n-2)$$
$$y(n)=0.244x(n)+0.488x(n-1)+0.244x(n-2)+0.205y(n-1)-0.179y(n-2)$$

EXCEL SPREADSHEET

	Coefficients=	**-0.2050**	**0.1790**		**0.2440**	**0.4880**	**0.2440**
n	**y(n)**	**y(n-1)**	**y(n-2)**	**....**	**x(n)**	**x(n-1)**	**x(n-2)**
	-	-		-	-		
	-	-		-	-		
	-	-		-	-		
0	**0.2440**	0.0000	0.0000		**1.0000**	0.0000	0.0000
1	**0.5380**	0.2440	0.0000		**0.0000**	1.0000	0.0000
2	**0.3106**	0.5380	0.2440		**0.0000**	0.0000	1.0000
3	**-0.0326**	0.3106	0.5380		**0.0000**	0.0000	0.0000
4	**-0.0623**	-0.0326	0.3106		**0.0000**	0.0000	0.0000
5	**-0.0069**	-0.0623	-0.0326		**0.0000**	0.0000	0.0000
6	**0.0097**	-0.0069	-0.0623		**0.0000**	0.0000	0.0000
7	**0.0032**	0.0097	-0.0069		**0.0000**	0.0000	0.0000
8	**-0.0011**	0.0032	0.0097		**0.0000**	0.0000	0.0000
9	**-0.0008**	-0.0011	0.0032		**0.0000**	0.0000	0.0000
10	**0.0000**	-0.0008	-0.0011		**0.0000**	0.0000	0.0000
11	**0.0001**	0.0000	-0.0008		**0.0000**	0.0000	0.0000
12	**0.0000**	0.0001	0.0000		**0.0000**	0.0000	0.0000
13	**0.0000**	0.0000	0.0001		**0.0000**	0.0000	0.0000
14	**0.0000**	0.0000	0.0000		**0.0000**	0.0000	0.0000
15	**0.0000**	0.0000	0.0000		**0.0000**	0.0000	0.0000
16	**0.0000**	0.0000	0.0000		**0.0000**	0.0000	0.0000
17	**0.0000**	0.0000	0.0000		**0.0000**	0.0000	0.0000
18	**0.0000**	0.0000	0.0000		**0.0000**	0.0000	0.0000
19	**0.0000**	0.0000	0.0000		**0.0000**	0.0000	0.0000
20	**0.0000**	0.0000	0.0000		**0.0000**	0.0000	0.0000



DIRECT FORM REALIZATION

Second order IIR Filter:

$$y(n) = \sum_{l=0}^{2} b_l x(n-l) - \sum_{m=1}^{2} a_m y(n-m)$$

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) - a_1 y(n-1) - a_2 y(n-2)$$

$$b_0 = 0.244$$

$$b_1 = 0.488$$

$$b_2 = 0.244$$

$$a_1 = -0.205$$

$$a_2 = 0.179$$

$$y(n) = 0.244x(n) + 0.488x(n-1) + 0.244x(n-2) + 0.205y(n-1) - 0.179y(n-2)$$