Real-Time Embedded Digital Signal Processing
FINAL EXAM SOLUTION

Question 1
Consider a weighted moving average function: $y_n = [x_n + 0.4x_{n-1} + 0.2x_{n-2}]/1.6$

1. ***What is the transfer function H(z)=Y(z)/X(z)?***

Taking Z-transform of both sides of the moving average function gives:

$$Y(z) = [X(z) + 0.4z^{-1}X(z) + 0.2z^{-2}X(z)]/1.6$$

$$H(z) = Y(z)/X(z) = [1 + 0.4z^{-1} + 0.2z^{-2}]/1.6$$

$$H(z) = (z^2 + 0.4z + 0.2)/(1.6z^2)$$

2. ***Where is the first pole?***

There is a poles at z=0.

3. ***Where is the second pole?***

There is a second pole at z=0.

4. ***Where is the first zero?***

Zeros are given by: $z^2 + 0.4z + 0.2 = 0$

Roots of a quadratic of the form: $ax^2 + 2bx + c = 0$ are $z = \left(-b \pm \sqrt{b^2 - ac}\right)/a$

Thus, roots are at: $z = \left(-0.2 \pm \sqrt{0.2^2 - 0.2}\right) = -0.2 \pm \sqrt{-0.16} = -0.2 \pm j0.4$

First zero is at -0.2+0.4j (or -0.2-0.4j)

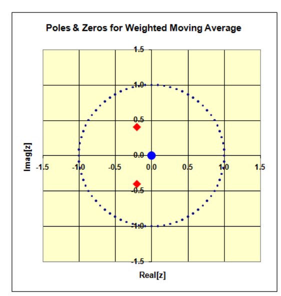
5. ***Where is the second zero?***

Second zero is at -0.2-0.4j (or -0.2+0.4j)

6. ***Stability***

Zeros (red diamond) and poles (blue dots) are plotted below.

As the poles are inside the unit circle the system is stable.



7. ***What is the real part of the frequency response H(ω) ?***

Frequency response is the transfer function evaluated on the unit circle:

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = (1 + 0.4e^{-j\omega} + 0.2e^{-2j\omega})/1.6$$

$$H(\omega) = [1 + 0.4(\cos\omega - j\sin\omega) + 0.2(\cos 2\omega - j\sin 2\omega)]/1.6$$

$$H(\omega) = 0.625 * [(1 + 0.4\cos\omega + 0.2\cos 2\omega) + j(-0.4\sin\omega - 0.2\sin 2\omega)]$$

Real part is:

$$\operatorname{Re} H(\omega) = 0.625 * (1 + 0.4\cos\omega + 0.2\cos 2\omega)$$

8. ***What is the imaginary part of the frequency response H(ω) ?***

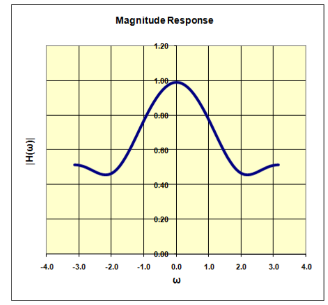
Imaginary part is:

$$\operatorname{Im} H(\omega) = 0.625 * (-0.4\sin\omega - 0.2\sin 2\omega)$$

9. ***What is the magnitude squared of the frequency response H(ω) ?***

Magnitude squared is:

$$|H(\omega)|^2 = [\operatorname{Re} H(\omega)]^2 + [\operatorname{Im} H(\omega)]^2$$



$$|H(\omega)|^2 = 0.625^2[(1+0.4\cos\omega+0.2\cos 2\omega)^2 + (-0.4\sin\omega - 0.2\sin 2\omega)^2]$$

$$|H(\omega)|^2 = \begin{bmatrix} 0.625^2[1+\underline{0.4^2}\cos^2\omega + \underline{\underline{0.2^2}}\cos^2 2\omega \\ +2*0.4\cos\omega + \underleftrightarrow{2*0.4\cos\omega*0.2\cos 2\omega} + 2*0.2\cos 2\omega \\ +\underline{0.4^2}\sin^2\omega + \underline{\underline{0.2^2}}\sin^2 2\omega + \underleftrightarrow{2*0.4\sin\omega*0.2\sin 2\omega}] \end{bmatrix}$$

$$|H(\omega)|^2 = 0.625^2[1+\underline{0.4^2}+\underline{\underline{0.2^2}}+0.8\cos\omega+0.4\cos 2\omega+\underleftrightarrow{0.16}\{\cos\omega\cos 2\omega+\sin\omega\sin 2\omega\}]$$

$$|H(\omega)|^2 = 0.625^2[1.2+0.8\cos\omega+0.4\cos 2\omega+0.16\{\cos\omega(1-2\sin^2\omega)+\sin\omega(2\sin\omega\cos\omega)\}]$$

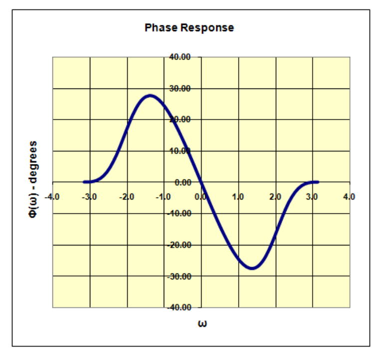
$$|H(\omega)|^2 = 0.625^2[1.2+0.8\cos\omega+0.4\cos 2\omega+0.16\cos\omega]$$

$$|H(\omega)|^2 = 0.625^2[1.2+0.96\cos\omega+0.4\cos 2\omega]$$

$$|H(\omega)|^2 = 0.391[1.2+0.96\cos\omega+0.4\cos 2\omega] \qquad \mathit{for}\ -\pi \le \omega \le \pi$$

10. ***What is the tangent of the phase of the frequency response H(ω) ?***

Tangent of phase is:



$$\tan\Phi(\omega) = \operatorname{Im} H(\omega)/\operatorname{Re} H(\omega)$$

$$\tan\Phi(\omega) = (-0.4\sin\omega - 0.2\sin 2\omega)/(1+0.4\cos\omega+0.2\cos 2\omega)$$

$$\tan\Phi(\omega) = -(2\sin\omega+\sin 2\omega)/(5+2\cos\omega+\cos 2\omega)$$

$$\tan\Phi(\omega) = -(2\sin\omega+2\sin\omega\cos\omega)/(5+2\cos\omega+2\cos^2\omega-1)$$

$$\tan\Phi(\omega) = -2\sin\omega(1+\cos\omega)/(2\cos^2\omega+2\cos\omega+4)$$

$$\tan\Phi(\omega) = \frac{-\sin\omega(1+\cos\omega)}{\cos^2\omega+\cos\omega+2} \qquad \mathit{for}\ -\pi \le \omega \le \pi$$

11. ***What is the group delay (defined as the negative derivative of the phase)?***

Time delay for the signal "envelope" (aka group delay) is defined as:

$$T_d(\omega) = -\frac{d\Phi(\omega)}{d\omega}$$

For our filter:

$$\Phi(\omega) = \tan^{-1}\Psi(\omega) \qquad \mathit{where} \quad \Psi(\omega) = \frac{g(\omega)}{h(\omega)} = \frac{-\sin\omega(1+\cos\omega)}{\cos^2\omega+\cos\omega+2} = \frac{-\sin\omega-\sin\omega\cos\omega}{\cos^2\omega+\cos\omega+2}$$

Since the derivative of $\tan^{-1}(x) = 1/(1+x^2)$, we have:

$$T_d(\omega) = -\frac{d\Phi(\omega)}{d\omega} = -\frac{1}{1+\Psi^2(\omega)}\frac{d\Psi(\omega)}{d\omega}$$

Since the derivative of $\Psi(\omega)$ $=g(\omega)/h(\omega) = [g'(\omega)h(\omega)-g(\omega)h'(\omega)]/h(\omega)^2$, we have:

$$T_d(\omega) = -\frac{1}{1+\Psi^2(\omega)}\frac{d\Psi(\omega)}{d\omega} = -\frac{1}{1+g(\omega)^2/h(\omega)^2}\frac{g'(\omega)h(\omega)-g(\omega)h'(\omega)}{h(\omega)^2}$$

$$T_d(\omega) = \frac{-g'(\omega)h(\omega) + g(\omega)h'(\omega)}{h(\omega)^2 + g(\omega)^2}$$

Noting derivative formulas:

d/dω [sinω] = cosω
d/dω [cosω] = –sinω
d/dω [sinω cosω] = cosω cosω + sinω(–sinω) = $\cos^2\omega - \sin^2\omega$ = $2\cos^2\omega$ - 1

$$T_d(\omega) = \frac{-(-\cos\omega - 2\cos^2\omega + 1)h(\omega) + g(\omega)(2\cos\omega * (-\sin\omega) - \sin\omega)}{h(\omega)^2 + g(\omega)^2}$$

$$T_d(\omega) = \frac{(\cos\omega + 2\cos^2\omega - 1)h(\omega) - \sin\omega(2\cos\omega + 1)g(\omega)}{h(\omega)^2 + g(\omega)^2}$$

$$T_d(\omega) = \frac{(2\cos^2\omega + \cos\omega - 1)(\cos^2\omega + \cos\omega + 2) - \sin\omega(2\cos\omega + 1)(-\sin\omega - \sin\omega\cos\omega)}{(\cos^2\omega + \cos\omega + 2)^2 + (-\sin\omega - \sin\omega\cos\omega)^2}$$

$$T_d(\omega) = \frac{(2\cos^2\omega + \cos\omega - 1)(\cos^2\omega + \cos\omega + 2) + \sin^2\omega(2\cos\omega + 1)(1 + \cos\omega)}{(\cos^2\omega + \cos\omega + 2)^2 + \sin^2\omega(1 + \cos\omega)^2}$$

Simplifying the numerator separately and defining β=cosω, and noting $\sin^2\omega = 1-\cos^2\omega = 1-\beta^2$:

$$NumT_d = (2\beta^2 + \beta - 1)(\beta^2 + \beta + 2) + (1 - \beta^2)(2\beta + 1)(\beta + 1)$$

Note that the roots of the first quadratic are real: (-1±√(1+8))/4=(-1±√9)/4=(-1±3)/4={0.5,-1}

$$NumT_d = (2\beta - 1)(\beta + 1)(\beta^2 + \beta + 2) + (1 - \beta^2)(2\beta + 1)(\beta + 1)$$

$$NumT_d = (\beta + 1)[(2\beta - 1)(\beta^2 + \beta + 2) + (1 - \beta^2)(2\beta + 1)]$$

$$NumT_d = (\beta + 1)[\underline{2\beta^3} + \underline{\underline{2\beta^2}} + 4\beta - \underline{\underline{\beta^2}} - \beta - 2 + 2\beta + 1 - \underline{2\beta^3} - \underline{\underline{\beta^2}}]$$

$$NumT_d = (\beta + 1)[+5\beta - 1] = 5\beta^2 + 4\beta - 1$$

$$NumT_d = 5\cos^2\omega + 4\cos\omega - 1$$

Simplifying the denominator separately:

$$DenT_d = (\beta^2 + \beta + 2)^2 + (1 - \beta^2)(1 + \beta)^2$$

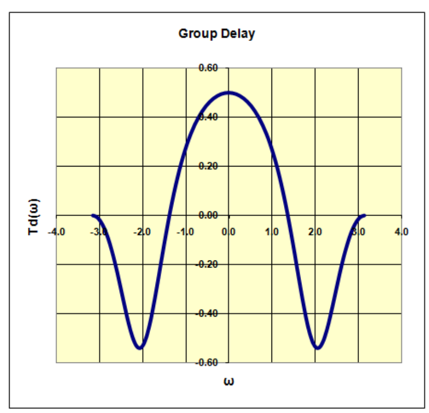
$$DenT_d = (\beta^4 + \beta^2 + 4 + 2\beta^3 + 4\beta + 4\beta^2) + (1 - \beta^2)(1 + 2\beta + \beta^2)$$

$$DenT_d = (\underline{\beta^4} + \underline{\underline{2\beta^3}} + 5\beta^2 + 4\beta + 4) + (1 + 2\beta + \beta^2 - \beta^2 - \underline{\underline{2\beta^3}} - \underline{\beta^4})$$

$$DenT_d = 5\beta^2 + 6\beta + 5 = 5\cos^2\omega + 6\cos\omega + 5$$

Thus, the group time delay is:

$$T_d(\omega) = \frac{5\cos^2\omega + 4\cos\omega - 1}{5\cos^2\omega + 6\cos\omega + 5}$$



Question 2

1. ***Compute the DFT of the sequence x(n) = {2, 3, -2, -4} using the matrix formulation***

Using the matrix formulation, the DFT **X** of a sequence **x** is defined as:

$$X = Wx$$

$$X = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$where: \quad W_N^n = e^{-j2\pi n/N} = \cos(2\pi n/N) - j\sin(2\pi n/N)$$

Since N=4,

$$W_4^n = e^{-j2\pi n/4} = \cos(\pi n/2) - j\sin(\pi n/2)$$

Thus,

$$X = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0-j & -1-0 & 0+j \\ 1 & -1-0 & 1-0 & -1-0 \\ 1 & 0+j & -1-0 & 0-j \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -2 \\ -4 \end{bmatrix}$$

$$X = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 2+3-2-4 \\ 2-3j+2-4j \\ 2-3-2+4 \\ 2+3j+2+4j \end{bmatrix} = \begin{bmatrix} -1 \\ 4-7j \\ +1 \\ 4+7j \end{bmatrix}$$

So for x(n) = {2, 3, -2, -4} , the DFT is X(n) = {-1, 4-7j, +1, 4+7j}

2. ***How many multiplies were needed (not counting multiplies needed to generate the W matrix & assuming x is real)?***

No. of Multiplies = (no. of equations) * (no. of terms/equation) * (no. of multiplies/term)
= (4) * (4) * (1 complex number multiplied by a 1 real number)
= (4) * (4) * (2 real multiplies)
= 32 real multiplies

Note that if x was not assumed to be real, then the number of multiples per term would be 4, as multiplying two complex numbers requires four real multiplies: (a+jb)*(c+jd)=(ac-bd)+j(bc+ad).

3. ***How many adds were needed?***

No. of Adds = (no. of equations) * (no. of complex adds/equation) *(no. of real adds/complex add)
= (4) * (3) * (2)
= 24 real adds

4. ***<u>The inverse DFT may be computed as: x = W* X / N where N = 4. What is the first row of the matrix W* ?</u>***

$$W^* = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}^*$$

$$W^* = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ W_4^0 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ W_4^0 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix}$$

$$W^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Note that:

$$x = \frac{1}{N} W^* X$$

$$x = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}^* \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

$$x = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} -1 \\ 4-7j \\ 1 \\ 4+7j \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1+4-7j+1+4+7j \\ -1+4j+7-1-4j+7 \\ -1-4+7j+1-4-7j \\ -1-4j-7-1+4j-7 \end{bmatrix}$$

$$x = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 8 \\ 12 \\ -8 \\ -16 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \\ -4 \end{bmatrix} \text{ giving back the original sequence!}$$

The first row of W* is [1 1 1 1].

5. ***What is the second row of the matrix W* ?***
The second row of W* is [1 +j -1 -j].

6. ***What is the third row of the matrix W* ?***
The second row of W* is [1 -1 1 -1].

7. ***What is the fourth row of the matrix W* ?***
The second row of W* is [1 -j -1 j].

8. ***Given x(n) = [2, 3, -2, -4], is split into even and odd numbered sequences xe(n) and xo(n), what is the DFT of xe?***
Given x(n) = {2, 3, -2, -4}, split sequence into even and odd numbered sequences:
xe(n) = {2, -2}
xo(n) = {3, -4}

DFT of xe:

$$XE = \begin{bmatrix} XE(0) \\ XE(1) \end{bmatrix} = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \begin{bmatrix} xe(0) \\ xe(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

9. ***Given x(n) = [2, 3, -2, -4], is split into even and odd numbered sequences xe(n) and xo(n), what is the DFT of xo?***

DFT of xo:

$$XO = \begin{bmatrix} XO(0) \\ XO(1) \end{bmatrix} = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \begin{bmatrix} xo(0) \\ xo(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 3-4 \\ 3+4 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

10. ***Using the DFT of xe and xo, compute the FFT by decimation in time of x(n) = [2, 3, -2, -4]***
FFT by decimation in time (where N=2M):

$$X(k) = \left\{ \begin{matrix} XE(k) + W_N^k XO(k) & k = 0,1,\ldots, M-1 \\ XE(k-M) - W_N^{k-M} XO(k-M) & k = M, M+1, \ldots, 2M-1 \end{matrix} \right\}$$

In our case M=2, N=4:

$$X(k) = \left\{ \begin{matrix} XE(k) + W_4^k XO(k) & k = 0,1 \\ XE(k-2) - W_4^{k-2} XO(k-2) & k = 2,3 \end{matrix} \right\}$$

Thus:

$$X(0) = XE(0) + W_4^0 XO(0) = 0 + 1(-1) = -1$$
$$X(1) = XE(1) + W_4^1 XO(1) = 4 + (-j)(7) = 4 - j7$$
$$X(2) = XE(0) - W_4^0 XO(0) = 0 - 1(-1) = 1$$
$$X(3) = XE(1) - W_4^1 XO(1) = 4 - (-j)(7) = 4 + j7$$

Thus, the FFT is [-1, 4-7j, +1, 4+7j]