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Quantization Noise

Digital Output Value (D) = Analog Input Value (A) + Quantization Error (Q)
or D = A + Q

Let R_A denote the range of the analog signal and σ_A its variance (square of std. dev.). Similarly, let R_Q denote the range of the quantization error and σ_Q its variance. As an example, assume that:

Analog Input is in the range [0,10] volts, R_A = 10 volts
ADC has 8 bits; B=8
Digital output levels are $2^B = 2^8 = 256$ levels.
Each bit spans $\Delta = R_A/2^B$ = 10V/256 = 39 mV

Assume that the levels are uniformly divided as follows:

```
0volt                                                                                   10volt
|---------|---------|-------------------------------|---------|---------|
0         1         2 …..                            254       255
```

Since each bit spans Δ = 39 mV, the conversion table looks like:

Analog Input	Digital Output
[0, 39) mV	0
[39, 78) mV	1
[78, 117) mV	2
[9922, 9961) mV	254
[9961,10000] mV	255

So the quantization error (Q) in this scenario is as little as 0 volts and as much as Δ volts. If we assume that probability of error is uniformly distributed over the [0, Δ] range, then the quantization error probability density function ρ(Q) will look like the following:

```
ρ(Q)          1/Δ      ______________
                      |              |
                      |              |
              0    ___|              |______________
                      0              Δ             →Q
```

Notice that the area under the curve is 1. The average Q is calculated as:

$$\bar{Q} = \int_{-\infty}^{+\infty} Q\rho(Q)dQ = \frac{1}{\Delta}\int_0^{\Delta} QdQ = \frac{1}{\Delta}\left[\frac{Q^2}{2}\right]_{Q=0}^{Q=\Delta} = \frac{1}{\Delta}\left[\frac{\Delta^2}{2} - 0\right] = \frac{\Delta}{2}$$

The variance of Q is calculated as:

$$\sigma_Q{}^2 = \int_{-\infty}^{+\infty} (Q-\bar{Q})^2\rho(Q)dQ = \frac{1}{\Delta}\int_0^{\Delta} (Q-\bar{Q})^2 dQ = \frac{1}{\Delta}\left[\frac{(Q-\bar{Q})^3}{3}\right]_{Q=0}^{Q=\Delta}$$

$$\sigma_Q{}^2 = \frac{1}{3\Delta}\left[(Q-\Delta/2)^3\right]_{Q=0}^{Q=\Delta} = \frac{1}{3\Delta}\left[(\Delta-\Delta/2)^3 - (-\Delta/2)^3\right]$$

$$\sigma_Q{}^2 = \frac{1}{3\Delta}\left[(\Delta/2)^3 + (\Delta/2)^3\right] = \frac{1}{3\Delta}\left[2(\Delta^3/8)\ \right] = \frac{\Delta^2}{12}$$

Since $\Delta = R_A/2^B$ this can be written as:

$$\sigma_Q{}^2 = \frac{\Delta^2}{12} = \frac{1}{12}\frac{R_A{}^2}{2^{\boldsymbol{B}}} = \frac{1}{3}\frac{R_A{}^2}{2^{\boldsymbol{2B}}}$$

Signal power can be represented by $\sigma_A{}^2$. Quantization noise power can be represented by $\sigma_Q{}^2$. The signal to quantization noise ratio can then be written as:

$$SQNR = \frac{\sigma_A{}^2}{\sigma_Q{}^2} = 3 * 2^{\boldsymbol{2B}}\ \frac{\sigma_A{}^2}{R_A{}^2}$$

In decibels, the SQNR is:

$$SQNR_{dB} = 10\ log_{10} SQNR = 10\ log_{10}\left(3 * 2^{\boldsymbol{2B}}\ \frac{\sigma_A{}^2}{R_A{}^2}\right)$$

$$SQNR_{dB} = 10\ log_{10} 3 + 10\ log_{10}(2^{\boldsymbol{2B}}\ ) + 10\ log_{10}\left(\frac{\sigma_A{}^2}{R_A{}^2}\right)$$

$$SQNR_{dB} = 4.77 + 20B\ log_{10}(2) + 10\ log_{10}\left(\frac{\sigma_A{}^2}{R_A{}^2}\right)$$

$$SQNR_{dB} = 4.77 + 6.02B + 10\ log_{10}\left(\frac{\sigma_A{}^2}{R_A{}^2}\right)$$

Assume the input signal is sinusoidal, thus $\frac{\sigma_A{}^2}{R_A{}^2} = 0.5$:

$$SQNR_{dB} = 4.77 + 6.02B + 10\ log_{10}(\ 0.5)$$

$$SQNR_{dB} = 4.77 + 6.02B + 10\ (-\ 0.301) = 6.02B + 1.76 \approx 6B$$

If B=8, then SQNR is 48 dB, but if B=16 then SQNR is 96 dB.