Characterizing Information Value of Energy via an Optimal Control Problem

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1 Introduction

This document defines an optimal control problem (OCP) for generating ergodic trajectories in a solar-powered robot. The robot's state includes its position (x, y) and battery state of charge (SOC). The control input is the robot's velocity $u = (v_x, v_y)$. The battery discharge rate is proportional to $\|u\|$, and the solar charging model follows a diurnal cycle. The goal is to compute an information-optimal trajectory over a finite time horizon T, balancing exploration and energy consumption.

We reference Kaleb's Eclares work [1] for the ergodicity formulation.

2 Preliminaries

2.1 Variable Names

Define the following states:

- $x(t)$: position of the robot at time t
- $u(t)$: control input (velocity) at time t
- $b(t)$: state of charge at time t
- $r(t)$: solar irradiance at time t

Define T_h to be the long-horizon exploration time window, such that

$$T \leq t \leq T + T_h \tag{1}$$

Further define:

- $\phi(x)$ as the spatial distribution of information
- $p(x, t)$ as the time-averaged trajectory distribution over T_h

2.2 Ergodicity

The ergodicity metric $\mathcal{E}(p, \phi)$ quantifies the mismatch between the trajectory distribution $p(x)$ and the information density $\phi(x)$. The ergodic control problem aims to minimize this metric.

We can measure ergodicity via the spectral norm difference:

$$\epsilon = \sum_k \lambda_k \left| c_k - \hat{c}_k \right|^2 \tag{2}$$

where c_k are the Fourier coefficients of $\phi(x)$ (the target information distribution), $\hat{c}_k$ are the Fourier coefficients of $p(x)$ (the actual information distribution), and λ_k are weights (Fourier coefficients) for the spatial frequencies.

> This ensures that the robot's motion explores the environment in a manner consistent with the desired information distribution.

To incorporate energy constraints, we introduce a time-dependent weight:

$$\mathcal{E}_w(p, \phi) = \int_0^T \omega(t) \sum_k \lambda_k \left| c_k - \hat{c}_k \right|^2 dt \tag{3}$$

where $\omega(t) = \frac{b(t)}{b_{\max}}$, meaning information gain is prioritized when energy is available.

2.3 Energy Dynamics

The energy dynamics of the system are given by:

$$\dot{b} = -\alpha \|u\|^3 + r(t) \tag{4}$$

where $\alpha > 0$ is a scaling factor for velocity-based energy consumption, and $r(t)$ is the solar charging rate, typically sinusoidal or piecewise defined.

The solar irradiance profile, denoted as $r(t)$, varies throughout the day following a diurnal pattern. A simple model that captures this variation is:

$$r(t) = r_{\max} \cdot \max\left(0, \sin\left(\frac{\pi(t - t_{\text{sunrise}})}{t_{\text{sunset}} - t_{\text{sunrise}}}\right)\right) \tag{5}$$

where:

- $r_{\max}$ is the peak irradiance at solar noon (typically around 1000 W/m^2).
- t_{sunrise} and t_{sunset} define the daylight hours.
- The sine function ensures that irradiance rises in the morning, peaks at noon, and falls in the evening.

This model assumes a clear-sky condition, without atmospheric effects or seasonal variations.

3 Optimal Control Problem Formulation

Minimize the ergodic metric:

$$\min_u \int_0^T \sum_k \lambda_k \left|c_k - \hat{c}_k\right|^2 dt \tag{6}$$

subject to the dynamics:

$$\dot{x} = u \tag{7}$$

$$\dot{b} = -\alpha \|u\|^3 + r(t) \tag{8}$$

and the constraints:

$$\|u(t)\| \leq u_{\max} \tag{9}$$

$$b(t) \geq 0 \tag{10}$$

$$b(T) = b_T \tag{11}$$

4 Indirect Methods Solution

We proceed using indirect methods. We first compute the following Hamiltonian:

$$H = \sum_k \lambda_k \left|c_k - \hat{c}_k\right|^2 + \lambda_x^T u + \lambda_b \left(-\alpha \|u\|^3 + r(t)\right) \tag{12}$$

where λ_x and λ_b are the costate variables.

This yields the following costate dynamics:

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} \tag{13}$$

$$\dot{\lambda}_b = -\frac{\partial H}{\partial b} \tag{14}$$

The optimal control u^* must satisfy:

$$\frac{\partial H}{\partial u} = 0 \tag{15}$$

and must satisfy the following boundary conditions:

$$x(0) = x_0 \tag{16}$$

$$b(0) = b_0 \tag{17}$$

$$b(T) = b_T \tag{18}$$

$$\lambda_x(T) = 0 \tag{19}$$

$$\lambda_b(T) = 0 \tag{20}$$

Thus, we have derived a two-point boundary value problem. We can use numerical methods such as the shooting method or collocation method to solve the resulting two-point boundary value problem (BVP).

5 Information Value of Energy

To compute the information value of energy, define:

$$\mathcal{I}(T_h, b_T) = \frac{1}{T_h - T} \int_{T}^{T+T_h} \sum_k \lambda_k \left| c_k - \hat{c}_k \right|^2 dt \tag{21}$$

The marginal information value of energy is:

$$\frac{d}{d(b_T)} \mathcal{I}(T_h, b_T) \tag{22}$$

which quantifies the additional information gain per unit SOC.

5.1 Brute Force Approach

This essentially sets up an independent OCP for computing the information value of energy. We can solve the same ergodic optimal control formulation from before, but varying the boundary conditions:

- No terminal condition on b. $b(T) = b_T$ is the only constraint.
- Sweep through increasing values of T_h
- Sweep through various values of b_T

This will give us a massive surface of information value of energy, from which we can quantify the marginal information value of energy.

If we allow for charging dynamics, I anticipate that there may be some periodic nature that arises for increasing values of T_h.

References

[1] K. B. Naveed, D. Agrawal, C. Vermillion, and D. Panagou, "Eclares: Energy-Aware Clarity-Driven Ergodic Search," in *2024 IEEE International Conference on Robotics and Automation (ICRA)*, May 2024, pp. 14 326–14 332.