Rubin, Jean E. (1967). *Set Theory for the Mathematician*. San Francisco: Holden-Day.
https://github.com/kmi-ne/Math-MyNotes

Chapter 1

Class algebra

1.1 Class

Definition 1.1 — x is a set/proper class

1. $$\mathsf{M}(x) :\leftrightarrow \exists u\ (x \in u)$$
2. $$\mathsf{Pr}(x) :\leftrightarrow \neg\,\mathsf{M}(x)$$

— label: `dfn_M` `dfn_Pr`

Axiom 1.2 — Axiom of Extensionality

$$\forall u\ (u \in x \leftrightarrow u \in y) \to x = y$$

— label: `axm_ext`

Definition 1.3 — x is a subclass/proper subclass of y

1. $$x \subseteq y :\leftrightarrow \forall u\ (u \in x \to u \in y)$$
2. $$x \subset y :\leftrightarrow x \subseteq y \neq x$$

— label: `dfn_sbc` `dfn_psbc`

Proposition 1.4

1. $$x \subseteq x$$
2. $$x \subseteq y \subseteq z \to x \subseteq z$$
3. $$x \subseteq y \subseteq x \to x = y$$

— label: `thm_sbc_tr`

Proof:

1. By $\forall u\ (u \in x \to u \in x)$ and Definition 1.3.1.
2. Assume (A1) $x \subseteq y \subseteq z$.
   By Definition 1.3.1 and (A1), $\forall u\ (u \in x \to u \in y)$ and $\forall u\ (u \in y \to u \in z)$. Thus, $\forall u\ (u \in x \to u \in z)$.
   Thus, by Definition 1.3.1, $x \subseteq z$. Release (A1)
3. Assume (A1) $x \subseteq y \subseteq x$.
   By Definition 1.3.1 and (A1), $\forall u\ (u \in x \to u \in y)$ and $\forall u\ (u \in y \to u \in x)$. Thus, $\forall u\ (u \in x \leftrightarrow u \in y)$.
   Thus, by Axiom of Extensionality, $x = y$. Release (A1)

Proposition 1.5

1. $$\neg(x \subset x)$$
2. $$x \subset y \leftrightarrow \begin{cases} x \subseteq y \\ \exists u\ (u \in y \wedge u \notin x) \end{cases}$$
3. $$x \subset y \subseteq z \to x \subset z$$
4. $$x \subset y \subset z \to x \subset z$$
5. $$x \subset y \subset z \to x \subset z$$
6. $$x \subseteq y \leftrightarrow (x \subset y \vee x = y)$$

— label: `thm_psbc_nin`

thm_psbcsbc_tr

Proof:

1. By $x = x$ and Definition 1.3.2.
2. $(\leftarrow)$ Assume (A1) $x \subseteq y$ and (A2) $\exists u\ (u \in y \wedge u \notin x)$.
   By (A2), $x \neq y$. Thus, by (A1) and Definition 1.3.2, $x \subset y$. Release (A1, A2)
   $(\rightarrow)$ Assume (A1) $x \subset y$.
   By (A1) and Definition 1.3.2, $x \subseteq y$ and $x \neq y$.
   Thus, by Proposition 1.4.2, $\neg(y \subseteq x)$. Thus, by Definition 1.3.1, $\exists u\ (u \in y \wedge u \notin x)$. Release (A1)
3. Assume (A1) $x \subset y \subseteq z$.
   (1) By (A1) and Definition 1.3.2, $x \subseteq y \subseteq z$. Thus, by Proposition 1.4.2, $x \subseteq z$.
   (2) By (A1) and Proposition 1.5.2, $\exists u\ (u \in y \wedge u \notin x)$. Take such u.
   By (A1) and Definition 1.3.1, $u \in y \rightarrow u \in z$. Thus, $u \in z \wedge u \notin x$. Thus, $\exists u\ (u \in z \wedge u \notin x)$.
   Thus, by Proposition 1.5.2, $x \subset z$. Release (A1)
4. Assume (A1) $x \subseteq y \subset z$.
   (1) By (A1) and Definition 1.3.2, $x \subseteq y \subseteq z$. Thus, by Proposition 1.4.2, $x \subseteq z$.
   (2) By (A1) and Proposition 1.5.2, $\exists u\ (u \in z \wedge u \notin y)$. Take such u.
   By (A1) and Definition 1.3.1, $u \in x \rightarrow u \in y$. Thus, $u \in z \wedge u \notin x$. Thus, $\exists u\ (u \in z \wedge u \notin x)$.
   Thus, by Proposition 1.5.2, $x \subset z$. Release (A1)
5. By Definition 1.3.2, $x \subset y \subset z \rightarrow x \subseteq y \subset z$. Thus, by Proposition 1.5.3, $x \subset z$.

Axiom 1.6 — Axiom of Comprehension

(x is not free in NBG-formula ϕ)

$$\exists x\ \forall u\ (u \in x \leftrightarrow \phi \wedge \mathsf{M}(u))$$

— label: axm_comp

Theorem 1.7

(x is not free in NBG-formula ϕ)

$$\exists! x\ \forall u\ (u \in x \leftrightarrow \phi \wedge \mathsf{M}(u))$$

Proof:

Existence By Axiom of Comprehension.
Uniqueness Assume (A1) $\forall u\ (u \in x_1 \leftrightarrow \phi \wedge \mathsf{M}(u))$ and $\forall u\ (u \in x_2 \leftrightarrow \phi \wedge \mathsf{M}(u))$.
By (A1), $\forall u\ (u \in x_1 \leftrightarrow u \in x_2)$. Thus, by Axiom of Extensionality, $x_1 = x_2$. Release (A1)

本来はここに $\{u \mid \phi\}$ の定義と定理が入るが省略.

Definition 1.8

1. $$\varnothing := \{u \mid u \neq u\}$$
2. $$\mathbf{U} := \{u \mid u = u\}$$
3. $$\mathbf{Ru} := \{u \mid u \notin u\}$$

— label: dfn_emp
dfn_univ
dfn_russ

$$u \in \varnothing \leftrightarrow u \neq u \wedge \mathsf{M}(u)$$
$$u \in \mathbf{U} \leftrightarrow u = u \wedge \mathsf{M}(u)$$
$$u \in \mathbf{Ru} \leftrightarrow u \notin u \wedge \mathsf{M}(u)$$

Proposition 1.9

1. $$u \notin \varnothing$$
2. $$\mathsf{M}(u) \rightarrow u \in \mathbf{U}$$
3. $$\varnothing \subseteq x$$
4. $$x \subseteq \mathbf{U}$$
5. $$\mathsf{Pr}(\mathbf{Ru})$$

— label: thm_emp_nin

thm_M_in_univ

Proof:

1. By $u \in \varnothing \leftrightarrow u \neq u \wedge \mathsf{M}(u)$.
2. By $u \in \mathbf{U} \leftrightarrow u = u \wedge \mathsf{M}(u)$.
3. By Proposition 1.9.1, $\forall u\ (u \in \varnothing \rightarrow u \in x)$. Thus, by Definition 1.3.1, $\varnothing \subseteq x$.
4. By Definition 1.1.1, $u \in x \rightarrow \mathsf{M}(u)$. Thus, by Proposition 1.9.2, $u \in x \rightarrow u \in \mathbf{U}$. Thus, $\forall u\ (u \in x \rightarrow u \in \mathbf{U})$. Thus, by Definition 1.3.1, $x \subseteq \mathbf{U}$.
5. By $\mathbf{Ru} \in \mathbf{Ru} \leftrightarrow \mathbf{Ru} \notin \mathbf{Ru} \wedge \mathsf{M}(\mathbf{Ru})$, $\mathbf{Ru} \notin \mathbf{Ru} \leftrightarrow \neg(\mathbf{Ru} \notin \mathbf{Ru} \wedge \mathsf{M}(\mathbf{Ru}))$. Thus, $\neg\,\mathsf{M}(\mathbf{Ru})$. Thus, by Definition 1.1.2, $\mathsf{Pr}(\mathbf{Ru})$.