Rubin, Jean E. (1967). *Set Theory for the Mathematician*. San Francisco: Holden-Day.
https://github.com/kmi-ne/Math-MyNotes

Chapter 1

Class algebra

1.1 Class

Definition 1.1 — x is a set/proper class

1. $\mathsf{M}(x) :\leftrightarrow \exists y \; (x \in y)$
2. $\mathsf{Pr}(x) :\leftrightarrow \neg \mathsf{M}(x)$

— label: `Dfn_M` `Dfn_Pr`

Proposition 1.2

$x \in y \rightarrow \mathsf{M}(x)$ — label: `Thm_elem_is_set`

Proof:

1. **Show:** $\mathsf{M}(x)$ (2, Def. 1.1.1)
2. $\exists y \; (x \in y)$ (3)
3. **Assume:** $x \in y$

Axiom 1.3 — Axiom of Extensionality

$\forall u \; (u \in x \leftrightarrow u \in y) \rightarrow x = y$ — label: `Axm_ext`

Definition 1.4 — x is a subclass/proper subclass of y

1. $x \subseteq y :\leftrightarrow \forall u \; (u \in x \rightarrow u \in y)$
2. $x \subset y :\leftrightarrow x \subseteq y \land x \neq y$

— label: `Dfn_Sbc` `Dfn_Psbc`

Proposition 1.5

1. $x \subseteq x$
2. $x \subseteq y \subseteq z \rightarrow x \subseteq z$
3. $x \subseteq y \subseteq x \rightarrow x = y$

— label: `Thm_Sbc_is_reflRel` `Thm_Sbc_is_transRel` `Thm_Sbc_is_antisymRel`

Proof:

1. 1. **Show:** $x \subseteq x$ (2, Def. 1.1.1)
   2. $\forall u \; (u \in x \rightarrow u \in x)$
2. 1. **Show:** $x \subseteq z$ (2)
   2. $\forall u \; (u \in x \rightarrow u \in z)$ (3)
   3. $\begin{cases} \forall u \; (u \in x \rightarrow u \in y) \\ \forall u \; (u \in y \rightarrow u \in z) \end{cases}$ (4)
   4. **Assume:** $x \subseteq y \subseteq z$
3. 1. **Show:** $x = y$ (2, Axiom of Extensionality)

2. $\forall u\ (u \in x \leftrightarrow u \in y)$ (3)

3. $\begin{cases} \forall u\ (u \in x \to u \in y) \\ \forall u\ (u \in y \to u \in x) \end{cases}$ (4, Def. 1.4.1)

4. **Assume:** $x \subseteq y \subseteq x$

Proposition 1.6

1. $\neg(x \subset x)$
2. $x \subset y \leftrightarrow \begin{cases} x \subseteq y \\ \exists u\ (u \in y \land u \notin x) \end{cases}$
3. $x \subset y \subseteq z \to x \subset z$
4. $x \subseteq y \subset z \to x \subset z$
5. $x \subset y \subset z \to x \subset z$
6. $x \subseteq y \leftrightarrow x \subset y \lor x = y$

— label: Thm_Psbc_is_irreflRel
Thm_psbc_has_less_elem
Thm_psbc_of_sbc_is_psbc
Thm_sbc_of_psbc_is_psbc
Thm_psbc_of_psbc_is_psbc
Thm_sbc_eqv_psbc_or_Eq

Proof:

1\. 1. **Show:** $\neg(x \subset x)$ (2, Def. 1.4.2)
2. $\neg(x \subseteq x \land x \neq x)$ (3)
3. $x = x$

2\. 1. **Show:** $\begin{cases} x \subseteq y \\ \exists u\ (u \in y \land u \notin x) \end{cases} \to x \subset y$
2. **Show:** $x \subset y$ (5, 3)
3. $x \neq y$ (6, 4)
4. $x = y \to \neg\exists u\ (u \in y \land u \notin x)$
5. **Assume:** $x \subseteq y$
6. **Assume:** $\exists u\ (u \in y \land u \notin x)$
7. **Show:** $x \subset y \to \begin{cases} x \subseteq y \\ \exists u\ (u \in y \land u \notin x) \end{cases}$
8. **Show:** $\exists u\ (u \in y \land u \notin x)$ (9, Def. 1.4.1)
9. $\neg(y \subseteq x)$ (11, 10, Prop. 1.5.3)
10. $x \neq y$ (12, Def. 1.4.2)
11. $x \subseteq y$ (12, Def. 1.4.2)
12. **Assume:** $x \subset y$

3\. 1. **Show:** $x \subset z$ (5, 2, Prop. 1.6.2)
2. **Show:** $\exists u\ (u \in z \land u \notin x)$ (4, 3)
3. $u \in y \to u \in z$ (7, Def. 1.4.1)
4. $\exists u\ (u \in y \land u \notin x)$ (7, Prop. 1.6.2)
5. **Show:** $x \subseteq z$ (6, Prop. 1.5.2)
6. $x \subseteq y \subseteq z$ (7, Def. 1.4.2)
7. **Assume:** $x \subset y \subseteq z$

4\. 1. **Show:** $x \subset z$ (5, 2, Prop. 1.6.2)
2. **Show:** $\exists u\ (u \in z \land u \notin x)$ (4, 3)
3. $u \in x \to u \in y$ (7, Def. 1.4.1)
4. $\exists u\ (u \in z \land u \notin y)$ (7, Prop. 1.6.2)
5. **Show:** $x \subseteq z$ (6, Prop. 1.5.2)
6. $x \subseteq y \subseteq z$ (7, Def. 1.4.2)
7. **Assume:** $x \subseteq y \subset z$

5\. 1. **Show:** $x \subset z$ (2, Prop. 1.6.3)

2. ⇥ $x \subset y \subseteq z$ ———— (3, Def. 1.4.2)
3. **Assume:** $x \subset y \subset z$

6\. 1. **Show:** $x \subseteq y \leftrightarrow x \subset y \lor x = y$ ———— (7, 2)
2. ⇥ **Show:** $\left[\begin{array}{l} x \subseteq y \land x \neq y \\ x \subseteq y \land x = y \end{array}\right. \leftrightarrow x \subset y \lor x = y$ ———— (4, 3)
3. ⇥ ⇥ $x \subseteq y \land x \neq y \leftrightarrow x \subset y$ ———— (Def. 1.4.2)
4. ⇥ ⇥ **Show:** $x \subseteq y \land x = y \leftrightarrow x = y$ ———— (6, 5)
5. ⇥ ⇥ ⇥ $x \subseteq y \land x = y \to x = y$
6. ⇥ ⇥ ⇥ $x = y \to x \subseteq y \land x = y$ ———— (Prop. 1.5.1)
7. ⇥ $x \subseteq y \leftrightarrow \left[\begin{array}{l} x \subseteq y \land x \neq y \\ x \subseteq y \land x = y \end{array}\right.$

Definition 1.7

$u \in \{v \mid \phi\} \leftrightarrow \phi[u/v] \land \mathsf{M}(u)$ — label: `Dfn_cls`

Proposition 1.8

1.

Definition 1.9

1. $\varnothing := \{u \mid u \neq u\}$
2. $\mathbf{U} := \{u \mid u = u\}$
3. $\mathbf{Ru} := \{u \mid u \notin u\}$

— label: `Dfn_emp`
`Dfn_univ`
`Dfn_russ`

Proposition 1.10

1. $u \notin \varnothing$
2. $\mathsf{M}(u) \to u \in \mathbf{U}$
3. $\varnothing \subseteq x$
4. $x \subseteq \mathbf{U}$

— label: `Thm_nothing_is_elem_of_emp`
`Thm_set_is_elem_of_univ`
`Thm_emp_is_sbc_of_everything`
`Thm_everything_is_sbc_of_univ`

Proof:

1\. 1. **Show:** $u \notin \varnothing$ ———— (3, 2)
2. ⇥ $u \in \varnothing \to u \neq u$ ———— (Def. 1.9.1, Def. 1.7)
3. ⇥ $u = u$

2\. 1. **Show:** $\mathsf{M}(u) \to u \in \mathbf{U}$ ———— (3, 2)
2. ⇥ $u = u \land \mathsf{M}(u) \to u \in \mathbf{U}$ ———— (Def. 1.9.2, Def. 1.7)
3. ⇥ $\mathsf{M}(u) \to u = u \land \mathsf{M}(u)$

3\. 1. **Show:** $\varnothing \subseteq x$ ———— (2, Def. 1.4.1)
2. ⇥ $u \in \varnothing \to u \in x$ ———— (Prop. 1.10.1)

4\. 1. **Show:** $x \subseteq \mathbf{U}$ ———— (3, 2, Def. 1.4.1)
2. ⇥ $\mathsf{M}(x) \to x \in \mathbf{U}$ ———— (Prop. 1.10.2)
3. ⇥ $u \in x \to \mathsf{M}(x)$ ———— (Prop. 1.2)

Theorem 1.11

$\mathsf{Pr}(\mathbf{Ru})$

Proof:

1. **Show:** $\mathsf{Pr}(\mathbf{Ru})$ ———— (2, Def. 1.1.2)

2. ⇥ **Show:** $\neg\,\mathsf{M}(\mathbf{Ru})$ ——— (4, 3)
3. ⇥ ⇥ $\mathbf{Ru} \notin \mathbf{Ru} \wedge \mathsf{M}(\mathbf{Ru}) \rightarrow \mathbf{Ru} \in \mathbf{Ru}$ ——— (Def. 1.9.3, Def. 1.7)
4. ⇥ ⇥ **Show:** $\mathbf{Ru} \notin \mathbf{Ru}$ ——— (5)
5. ⇥ ⇥ ⇥ $\mathbf{Ru} \in \mathbf{Ru} \rightarrow \mathbf{Ru} \notin \mathbf{Ru}$ ——— (6)
6. ⇥ ⇥ ⇥ $u \in \mathbf{Ru} \rightarrow u \notin u$ ——— (Def. 1.9.3, Def. 1.7)

Definition 1.12

1. $x \cup y := \{u \mid u \in x \vee u \in y\}$
2. $x \cap y := \{u \mid u \in x \wedge u \in y\}$
3. $x \setminus y := \{u \mid u \in x \wedge u \notin y\}$
4. $x^{\complement} = \{u \mid u \notin x\}$

— label: `Dfn_cup`
`Dfn_cap`
`Dfn_cdif`
`Dfn_cmpl`