Rubin, Jean E. (1967). *Set Theory for the Mathematician*. San Francisco: Holden-Day.
https://github.com/kmi-ne/Math-MyNotes

Chapter 1

Class algebra

1.1 Class

Definition 1.1 — x is a set/proper class

1. $$\mathsf{M}(x) :\leftrightarrow \exists u\ (x \in u)$$
2. $$\mathsf{Pr}(x) :\leftrightarrow \neg\, \mathsf{M}(x)$$

— label: `dfn_M`
`dfn_Pr`

Axiom 1.2 — Axiom of Extensionality

$$\forall u\ (u \in x \leftrightarrow u \in y) \to x = y$$

— label: `axm_ext`

Definition 1.3 — x is a subclass/proper subclass of y

1. $$x \subseteq y :\leftrightarrow \forall u\ (u \in x \to u \in y)$$
2. $$x \subset y :\leftrightarrow x \subseteq y \neq x$$

— label: `dfn_sbc`
`dfn_psbc`

Proposition 1.4

1. $$x \subseteq x$$
2. $$x \subseteq y \subseteq z \to x \subseteq z$$
3. $$x \subseteq y \subseteq x \to x = y$$

— label: `thm_sbc_tr`
`thm_sbc_atsy`

Proof:

1. By $\forall u\ (u \in x \to u \in x)$ and Definition 1.3.1.
2. Assume (A1) $x \subseteq y \subseteq z$.
   By Definition 1.3.1 and (A1), $\forall u\ (u \in x \to u \in y)$ and $\forall u\ (u \in y \to u \in z)$. Thus, $\forall u\ (u \in x \to u \in z)$.
   Thus, by Definition 1.3.1, $x \subseteq z$. Release (A1)
3. Assume (A1) $x \subseteq y \subseteq x$.
   By Definition 1.3.1 and (A1), $\forall u\ (u \in x \to u \in y)$ and $\forall u\ (u \in y \to u \in x)$. Thus, $\forall u\ (u \in x \leftrightarrow u \in y)$.
   Thus, by Axiom of Extensionality, $x = y$. Release (A1)

Proposition 1.5

1. $$\neg(x \subset x)$$
2. $$x \subset y \leftrightarrow \begin{cases} x \subseteq y \\ \exists u\ (u \in y \land u \notin x) \end{cases}$$
3. $$x \subset y \subseteq z \to x \subset z$$
4. $$x \subseteq y \subset z \to x \subset z$$
5. $$x \subset y \subset z \to x \subset z$$
6. $$x \subseteq y \leftrightarrow (x \subset y \lor x = y)$$

— label: thm_psbc_nin
thm_psbcsbc_tr

Proof:

1. By $x = x$ and Definition 1.3.2.
2. $(\leftarrow)$ Assume (A1) $x \subseteq y$ and (A2) $\exists u\ (u \in y \wedge u \notin x)$.
   By (A2), $x \neq y$. Thus, by (A1) and Definition 1.3.2, $x \subset y$. Release (A1, A2)
   $(\rightarrow)$ Assume (A1) $x \subset y$.
   By (A1) and Definition 1.3.2, $x \subseteq y$ and $x \neq y$.
   Thus, by Proposition 1.4.2, $\neg(y \subseteq x)$. Thus, by Definition 1.3.1, $\exists u\ (u \in y \wedge u \notin x)$. Release (A1)
3. Assume (A1) $x \subset y \subseteq z$.
   (1) By (A1) and Definition 1.3.2, $x \subseteq y \subseteq z$. Thus, by Proposition 1.4.2, $x \subseteq z$.
   (2) By (A1) and Proposition 1.5.2, $\exists u\ (u \in y \wedge u \notin x)$. Take such u.
   By (A1) and Definition 1.3.1, $u \in y \rightarrow u \in z$. Thus, $u \in z \wedge u \notin x$. Thus, $\exists u\ (u \in z \wedge u \notin x)$.
   Thus, by Proposition 1.5.2, $x \subset z$. Release (A1)
4. Assume (A1) $x \subseteq y \subset z$.
   (1) By (A1) and Definition 1.3.2, $x \subseteq y \subseteq z$. Thus, by Proposition 1.4.2, $x \subseteq z$.
   (2) By (A1) and Proposition 1.5.2, $\exists u\ (u \in z \wedge u \notin y)$. Take such u.
   By (A1) and Definition 1.3.1, $u \in x \rightarrow u \in y$. Thus, $u \in z \wedge u \notin x$. Thus, $\exists u\ (u \in z \wedge u \notin x)$.
   Thus, by Proposition 1.5.2, $x \subset z$. Release (A1)
5. By Definition 1.3.2, $x \subset y \subset z \rightarrow x \subseteq y \subset z$. Thus, by Proposition 1.5.3, $x \subset z$.

Axiom 1.6 — Axiom of Comprehension

(x is not free in NBG-formula ϕ)

$$\exists x\ \forall u\ (u \in x \leftrightarrow \phi \wedge \mathsf{M}(u))$$

— label: axm_comp

Theorem 1.7

(x is not free in NBG-formula ϕ)

$$\exists! x\ \forall u\ (u \in x \leftrightarrow \phi \wedge \mathsf{M}(u))$$

Proof:

Existence By Axiom of Comprehension.
Uniqueness Assume (A1) $\forall u\ (u \in x_1 \leftrightarrow \phi \wedge \mathsf{M}(u))$ and $\forall u\ (u \in x_2 \leftrightarrow \phi \wedge \mathsf{M}(u))$.
By (A1), $\forall u\ (u \in x_1 \leftrightarrow u \in x_2)$. Thus, by Axiom of Extensionality, $x_1 = x_2$. Release (A1)

本来はここに $\{u \mid \phi\}$ の定義などが入るが省略.

$$v \in \{u \mid \phi\} \leftrightarrow \phi[v/u] \wedge \mathsf{M}(v)$$

$$(\phi \rightarrow \psi) \rightarrow \{u \mid \phi\} \subseteq \{u \mid \psi\}$$

$$(\phi \leftrightarrow \psi) \rightarrow \{u \mid \phi\} = \{u \mid \psi\}$$

$$\{u \mid u \in \{v \mid \phi\}\} = \{u \mid \phi[u/v]\}$$

$$\{u \mid u \notin \{v \mid \phi\}\} = \{u \mid \neg\phi[u/v]\}$$

$$x = \{u \mid u \in x\}$$

Definition 1.8

1. $$\varnothing := \{u \mid u \neq u\}$$
2. $$\mathbf{U} := \{u \mid u = u\}$$
3. $$\mathbf{Ru} := \{u \mid u \notin u\}$$

— label: dfn_emp
dfn_univ
dfn_russ

$$u \in \varnothing \leftrightarrow u \neq u \wedge \mathsf{M}(u)$$
$$u \in \mathbf{U} \leftrightarrow u = u \wedge \mathsf{M}(u)$$
$$u \in \mathbf{Ru} \leftrightarrow u \notin u \wedge \mathsf{M}(u)$$

Proposition 1.9

1. $u \notin \varnothing$
2. $\mathsf{M}(u) \to u \in \mathbf{U}$
3. $\varnothing \subseteq x$
4. $x \subseteq \mathbf{U}$
5. $\mathsf{Pr}(\mathbf{Ru})$

— label: `thm_emp_nin` `thm_M_in_univ`

Proof:

1. By $u \in \varnothing \leftrightarrow u \neq u \wedge \mathsf{M}(u)$.
2. By $u \in \mathbf{U} \leftrightarrow u = u \wedge \mathsf{M}(u)$.
3. By Proposition 1.9.1, $\forall u\ (u \in \varnothing \to u \in x)$. Thus, by Definition 1.3.1, $\varnothing \subseteq x$.
4. By Definition 1.1.1, $u \in x \to \mathsf{M}(u)$. Thus, by Proposition 1.9.2, $u \in x \to u \in \mathbf{U}$. Thus, $\forall u\ (u \in x \to u \in \mathbf{U})$. Thus, by Definition 1.3.1, $x \subseteq \mathbf{U}$.
5. By $\mathbf{Ru} \in \mathbf{Ru} \leftrightarrow \mathbf{Ru} \notin \mathbf{Ru} \wedge \mathsf{M}(\mathbf{Ru})$, $\mathbf{Ru} \notin \mathbf{Ru} \leftrightarrow \neg(\mathbf{Ru} \notin \mathbf{Ru} \wedge \mathsf{M}(\mathbf{Ru}))$. Thus, $\neg\mathsf{M}(\mathbf{Ru})$. Thus, by Definition 1.1.2, $\mathsf{Pr}(\mathbf{Ru})$.

1.2 Class algebra

Definition 1.10

1. $x \cup y := \{u \mid u \in x \vee u \in y\}$
2. $x \cap y := \{u \mid u \in x \wedge u \in y\}$
3. $x \setminus y := \{u \mid u \in x \wedge u \notin y\}$
4. $x^{\complement} := \{u \mid u \notin x\}$

— label: `dfn_cup` `dfn_cap` `dfn_cdif` `dfn_cmpl`

Proposition 1.11

1. $x \cup y = y \cup x$
2. $x \cap y = y \cap x$
3. $(x \cup y) \cup z = x \cup (y \cup z)$
4. $(x \cap y) \cap z = x \cap (y \cap z)$
5. $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$
6. $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$
7. $x \cup x = x$
8. $x \cap x = x$
9. $x \subseteq x \cup y$
10. $x \cap y \subseteq x$
11. $x \subseteq y \leftrightarrow x \cup y = y$
12. $x \subseteq y \leftrightarrow x \cap y = x$

— label: `thm_cup_sbc` `thm_cap_sbc`

Proof:

1. $$\begin{aligned} x \cup y &= \{u \mid u \in x \vee u \in y\} \\ &= \{u \mid u \in y \vee u \in x\} \\ &= y \cup x \end{aligned}$$

2.
$$\begin{aligned} x \cap y &= \{u \mid u \in x \wedge u \in y\} \\ &= \{u \mid u \in y \wedge u \in x\} \\ &= y \cap x \end{aligned}$$

3.
$$\begin{aligned} (x \cup y) \cup z &= \{u \mid u \in x \cup y \vee u \in z\} \\ &= \{u \mid u \in x \vee u \in y \vee u \in z\} \\ &= \{u \mid u \in x \vee u \in y \cup z\} \end{aligned}$$

4.
$$\begin{aligned} (x \cap y) \cap z &= \{u \mid u \in x \cap y \wedge u \in z\} \\ &= \{u \mid u \in x \wedge u \in y \wedge u \in z\} \\ &= \{u \mid u \in x \wedge u \in y \cap z\} \end{aligned}$$

5.
$$\begin{aligned} x \cap (y \cup z) &= \{u \mid u \in x \wedge u \in y \cup z\} \\ &= \{u \mid u \in x \wedge (u \in y \vee u \in z)\} \\ &= \{u \mid (u \in x \wedge u \in y) \vee (u \in x \wedge u \in z)\} \\ &= \{u \mid u \in x \cap y \vee u \in x \cap z\} \\ &= (x \cap y) \cup (x \cap z) \end{aligned}$$

6.
$$\begin{aligned} x \cup (y \cap z) &= \{u \mid u \in x \vee u \in y \cap z\} \\ &= \{u \mid u \in x \vee (u \in y \wedge u \in z)\} \\ &= \{u \mid (u \in x \vee u \in y) \wedge (u \in x \vee u \in z)\} \\ &= \{u \mid u \in x \cup y \wedge u \in x \cup z\} \\ &= (x \cup y) \cap (x \cup z) \end{aligned}$$

7.
$$\begin{aligned} x \cup x &= \{u \mid u \in x \vee u \in x\} \\ &= \{u \mid u \in x\} \\ &= x \end{aligned}$$

8.
$$\begin{aligned} x \cap x &= \{u \mid u \in x \wedge u \in x\} \\ &= \{u \mid u \in x\} \\ &= x \end{aligned}$$

9.
$$\begin{aligned} x &= \{u \mid u \in x\} \\ &\subseteq \{u \mid u \in x \vee u \in y\} \\ &= x \cup y \end{aligned}$$

10.
$$\begin{aligned} x \cap y &= \{u \mid u \in x \wedge u \in y\} \\ &\subseteq \{u \mid u \in x\} \\ &= x \end{aligned}$$

11. ($\leftarrow$) Assume (A1) $x \cup y = y$.

$$\begin{aligned} x &\subseteq x \cup y && \text{by Proposition 1.11.9} \\ &= y && \text{by (A1)} \end{aligned}$$

Release (A1)

($\rightarrow$) Assume (A1) $x \subseteq y$.

By (A1), $u \in x \rightarrow u \in y$. Thus,

$$\begin{aligned} x \cup y &= \{u \mid u \in x \vee u \in y\} \\ &= \{u \mid u \in y\} \\ &= y \end{aligned}$$

Release ((A1))

12. ($\leftarrow$) Assume (A1) $x \cap y = x$.

$$\begin{aligned} x &= x \cap y && \text{by (A1)} \\ &\subseteq y && \text{by Proposition 1.11.10} \end{aligned}$$

Release (A1)

(→) Assume (A1) $x \subseteq y$.
By (A1), $u \in x \to u \in y$. Thus,

$$\begin{aligned} x \cap y &= \{u \mid u \in x \land u \in y\} \\ &= \{u \mid u \in x\} \\ &= x \end{aligned}$$

Release (A1)

Proposition 1.12

1. $$x \cup \varnothing = x$$
2. $$x \cap \varnothing = \varnothing$$
3. $$x \cup \mathbf{U} = \mathbf{U}$$
4. $$x \cap \mathbf{U} = x$$

Proof:

1. $$\begin{aligned} x \cup \varnothing &= \{u \mid u \in x \lor u \in \varnothing\} \\ &= \{u \mid u \in x \lor u \neq u\} \\ &= \{u \mid u \in x\} \\ &= x \end{aligned}$$
2. $$\begin{aligned} x \cap \varnothing &= \{u \mid u \in x \land u \in \varnothing\} \\ &= \{u \mid u \in x \land u \neq u\} \\ &= \{u \mid u \neq u\} \\ &= \varnothing \end{aligned}$$
3. $$\begin{aligned} x \cup \mathbf{U} &= \{u \mid u \in x \lor u \in \mathbf{U}\} \\ &= \{u \mid u \in x \lor u = u\} \\ &= \{u \mid u = u\} \\ &= \mathbf{U} \end{aligned}$$
4. $$\begin{aligned} x \cap \mathbf{U} &= \{u \mid u \in x \land u \in \mathbf{U}\} \\ &= \{u \mid u \in x \land u = u\} \\ &= \{u \mid u \in x\} \\ &= x \end{aligned}$$

Proposition 1.13

1. $$(x^{\complement})^{\complement} = x$$
2. $$x \cup x^{\complement} = \mathbf{U}$$
3. $$x \cap x^{\complement} = \varnothing$$
4. $$\mathbf{U} \setminus x = x^{\complement}$$
5. $$x \setminus y = x \cap y^{\complement}$$
6. $$x \subseteq y \leftrightarrow y^{\complement} \subseteq x^{\complement}$$
7. $$x \subset y \leftrightarrow y^{\complement} \subset x^{\complement}$$

Proof:

1. $$\begin{aligned} (x^{\complement})^{\complement} &= \{u \mid u \notin x^{\complement}\} \\ &= \{u \mid \neg(u \notin x)\} \\ &= \{u \mid u \in x\} \\ &= x \end{aligned}$$

2. $$\begin{aligned} x \cup x^{\complement} &= \{u \mid u \in x \vee u \in x^{\complement}\} \\ &= \{u \mid u \in x \vee u \notin x\} \\ &= \{u \mid u = u\} \\ &= \mathbf{U} \end{aligned}$$

3. $$\begin{aligned} x \cap x^{\complement} &= \{u \mid u \in x \wedge u \in x^{\complement}\} \\ &= \{u \mid u \in x \wedge u \notin x\} \\ &= \{u \mid u \neq u\} \\ &= \varnothing \end{aligned}$$

4. $$\begin{aligned} \mathbf{U} \setminus x &= \{u \mid u \in \mathbf{U} \wedge u \notin x\} \\ &= \{u \mid x = x \wedge u \notin x\} \\ &= \{u \mid u \notin x\} \\ &= x^{\complement} \end{aligned}$$

5. $$\begin{aligned} x \setminus y &= \{u \mid u \in x \wedge u \notin y\} \\ &= \{u \mid u \in x \wedge u \in y^{\complement}\} \\ &= x \cap y^{\complement} \end{aligned}$$

Proposition 1.14

1. $$(x \cup y)^{\complement} = x^{\complement} \cap y^{\complement}$$
2. $$(x \cap y)^{\complement} = x^{\complement} \cup y^{\complement}$$