Hinman, Peter G. (2005). *Fundamentals of Mathematical Logic*. A K Peters.
https://github.com/kmi-ne/Math-MyNotes

# Chapter 1

# Propositional Logic

connective, sentence symbol, $L$-symbol, $L$-expression:

**Definition 1.1** [a] [b]

1. $$\mathsf{Connec} := \{\neg, \vee, \wedge, \to, \leftrightarrow\}$$
2. $$\mathsf{SentSymb} := \{p_n \mid n \in \omega\}$$
3. $$\mathsf{Symb}_L := \mathsf{Connec} \cup \mathsf{SentSymb}$$
4. $$\mathsf{Expr}_L := {}^{<\omega}\mathsf{Sent}_L$$

— label: `dfn_Connec`
`dfn_SentSymb`
`dfn_Symb`
`dfn_Expr`

[a] $\neg, \vee, \wedge, \to, \leftrightarrow, p_n$ may be defined as $0, 1, 2, 3, 4, n+5$ (resp.).
[b] ${}^{<\omega}X := \{f \mid \exists N \in \omega\ (f\colon N \to X)\}$: the set of all finite sequences on $X$

concatenation:

**Definition 1.2**

$$f \frown g := \{\langle n + \mathrm{dom}(f), g(n)\rangle \mid n \in \mathrm{dom}(g)\}$$

— label: `dfn_conc`

**Convention 1.3** $[s_{\mathbf{0}}, \dots, s_{\mathbf{n}}]$ / $s_{\mathbf{0}} \dots s_{\mathbf{n}}$ — Syn. for $\{\langle 0, s_{\mathbf{0}}\rangle, \dots, \langle n, s_{\mathbf{n}}\rangle\}$

$L$-atomic sentence:

**Definition 1.4** [a]

$$\mathsf{AtSent}_L := {}^{1}\mathsf{SentSymb}$$

See: SentSymb — label: `dfn_AtSent`

[a] ${}^{n}X := \{f \mid f\colon n \to X\}$: the set of all $n$-term sequences on $X$

- $\mathsf{AtSent}_L = \{[s] \mid s \in \mathsf{SentSymb}\}$
- $\mathsf{AtSent}_L \subseteq \mathsf{Expr}_L$

definition by recursion:

**Theorem 1.5**

$$\begin{cases} z \in Z \\ G\colon Z \times \omega \to Z \end{cases} \to \exists! F\colon \omega \to Z \begin{cases} F(0) = z \\ \forall n \in \omega\ F(n^+) = G(F(n), n) \end{cases}$$

— label: thm_recdfn

$(X, A, \mathcal{H})$ is an induction system:

**Definition 1.6** [a]

$$\mathrm{Ind}(X, A, \mathcal{H}) :\leftrightarrow \begin{cases} A \subseteq X \neq \varnothing \\ \forall H \in \mathcal{H}\ \exists n \in \omega\ (H\colon X^n \to X) \end{cases}$$

— label: dfn_Ind

[a] By Theorem 1.5, $\exists! F\colon \omega \to \mathbf{U} \begin{cases} F(0) = 1 \\ \forall n \in \omega\ F(n^+) = \begin{cases} X & \text{if } n = 0 \\ F(n) \times X & \text{if } n > 0 \end{cases} \end{cases}$. Write $F(n)$ as $X^n$.

$!n \in \omega\ (\mathrm{dom}(H) = X^n)$. Define $k_{H,X}$ as $n$:

$$\exists n \in \omega\ (H\colon X^n \to X) \to \begin{cases} k_{H,X} \in \omega \\ \mathrm{dom}(H) = X^{k_{H,X}} \end{cases}$$

. Thus

$$\mathrm{Ind}(X, A, \mathcal{H}) \leftrightarrow \begin{cases} A \subseteq X \neq \varnothing \\ \forall H \in \mathcal{H}\ \begin{cases} k_{H,X} \in \omega \\ H\colon X^{k_{H,X}} \to X \end{cases} \end{cases}$$

. Write meta-$k_{H,X}$ as $\mathbf{k}_{H,X}$.

**Theorem 1.7**

$$\mathrm{Ind}(X, A, \mathcal{H}) \to$$

$$\exists! F\colon \omega \to \wp(X) \begin{cases} F(0) = A \\ \forall n \in \omega\ F(n^+) = F(n) \cup \left\{ H(x_{\mathbf{1}}, \dots, x_{\mathbf{k}_{H,X}}) \;\middle|\; \begin{cases} H \in \mathcal{H} \\ x_{\mathbf{1}}, \dots, x_{\mathbf{k}_{H,X}} \in F(n) \end{cases} \right\} \end{cases}$$

See: $\mathrm{Ind}(X, A, \mathcal{H})$ — label: thm_recmap

**Definition 1.8** Define: $(X, A, \mathcal{H})$ as $F$ in Theorem 1.7:

$$\mathrm{Ind}(X, A, \mathcal{H}) \to$$

$$\begin{cases} (X, A, \mathcal{H})\colon \omega \to \wp(X) \\ (X, A, \mathcal{H})(0) = A \\ \forall n \in \omega\ (X, A, \mathcal{H})(n^+) = (X, A, \mathcal{H})(n) \cup \left\{ H(x_{\mathbf{1}}, \dots, x_{\mathbf{k}_{H,X}}) \;\middle|\; \begin{cases} H \in \mathcal{H} \\ x_{\mathbf{1}}, \dots, x_{\mathbf{k}_{H,X}} \in (X, A, \mathcal{H})(n) \end{cases} \right\} \end{cases}$$

See: $\mathrm{Ind}(X, A, \mathcal{H})$ — label: dfn_recmap

**Convention 1.9**

1. $(X, A, \mathcal{H})_n$ / $X_n$ — Syn. for $(X, A, \mathcal{H})(n)$
2. $\mathcal{X}$ — Syn. for $(X, A, \mathcal{H})$

See: $\mathrm{Ind}(X, A, \mathcal{H})$ / $(X, A, \mathcal{H})$

$\mathcal{X}_n \equiv (X, A, \mathcal{H})_n$

inductive closure of $A$ under $\mathcal{H}$:

**Definition 1.10**

$$\overline{(X, A, \mathcal{H})} := \bigcup_{n \in \omega} (X, A, \mathcal{H})_n$$

See: $(X, A, \mathcal{H})$ — label: `dfn_clos`

$\overline{(X, A, \mathcal{H})} = \{x \in X \mid \exists n \in \omega \ (x \in (X, A, \mathcal{H})_n)\}$

**Theorem 1.11** $\mathrm{Ind}(X, A, \mathcal{H}) \to$

1. $$\forall n \in \omega \ (A \subseteq (X, A, \mathcal{H})_n \subseteq X)$$
2. $$A \subseteq \overline{(X, A, \mathcal{H})} \subseteq X$$

See: $\mathrm{Ind}(X, A, \mathcal{H})$ / $(X, A, \mathcal{H})$ / $\overline{(X, A, \mathcal{H})}$ — label: `thm_recmap_minmax` `thm_clos_minmax`

**Definition 1.12**

1. $$H_\neg \colon \mathsf{Expr}_L \ni \phi \mapsto [\neg] ^\frown \phi$$
2. $\bullet \equiv \vee, \wedge, \to, \leftrightarrow$
$$H_\bullet \colon \mathsf{Expr}_L \times \mathsf{Expr}_L \ni \langle \phi, \psi \rangle \mapsto [\bullet] ^\frown \phi ^\frown \psi$$
3. $$\mathcal{H}_{\mathsf{Sent}} := \{H_\neg, H_\vee, H_\wedge, H_\to, H_\leftrightarrow\}$$

See: $\mathsf{Expr}_L$ / $f ^\frown g$ — label: `dfn_H_lnot` `dfn_H_connec` `dfn_HSent`

- $H_\neg(\phi),\ H_\bullet(\phi) \in \mathsf{Expr}_L$
- $k_{H_\neg, \mathsf{Expr}_L} = 1,\ k_{H_\bullet, \mathsf{Expr}_L} = 2$