Chevalley, Claude. (1956). *Fundamental Concepts of Algebra*. New York: Academic Press.
https://github.com/kmi-ne/Math-MyNotes

Chapter 1

Monoids

1.1 Definition of a monoid

Convention 1.1 Syn. for $\top(a,b)$ — $a \top b$

Definition 1.2 **($\top$ is associative)**

$$\mathsf{Assoc}(\top; A) :\leftrightarrow \begin{cases} \top\colon A \times A \to A \\ \forall a,b,c \in A\ ((a \top b) \top c = a \top (b \top c)) \end{cases}$$

— label: `dfn_Assoc`

Example: [a]

1. $\mathsf{Assoc}(\top_{+}; \mathbb{Z})$
2. $\mathsf{Assoc}(\top_{\cdot}; \mathbb{Z})$
3. $\mathsf{Assoc}\left(\top_{\circ}; {}^{S}S\right)$
4. $\neg\,\mathsf{Assoc}(\top_{-}; \mathbb{Z})$

[a] $\top_{\mathbf{f}}$: the function corresponding to function symbol **f**

Definition 1.3 **(e is a neutral element)**

$$\mathsf{Neut}(e; \top, A) :\leftrightarrow \begin{cases} \top\colon A \times A \to A \\ e \in A \\ \forall a \in A\ (a \top e = e \top a = a) \end{cases}$$

— label: `dfn_Neut`

Example:

1. $\mathsf{Neut}(0; \top_{+}, \mathbb{Z})$
2. $\mathsf{Neut}(1; \top_{\cdot}, \mathbb{Z})$
3. $\mathsf{Neut}\left(\mathrm{id}_S; \top_{\circ}, {}^{S}S\right)$
4. $\forall x \in \mathbb{Z}\ \neg\,\mathsf{Neut}(x; \top_{-}, \mathbb{Z})$

Theorem 1.4 (Uniqueness of neutral element)

$$!e\ \ \mathsf{Neut}(e; \top, A)$$

See: $\mathsf{Neut}(e; \top, A)$ — label: `thm_neut_unq`

Proof: Assume $\mathsf{Neut}(e_{\mathbf{1}}; \top, A)$ and $\mathsf{Neut}(e_{\mathbf{2}}; \top, A)$. By Definition 1.3,

$$\begin{gathered} e_{\mathbf{1}}, e_{\mathbf{2}} \in A \\ \forall a \in A\ (e_{\mathbf{1}} \top a = a) \\ \forall a \in A\ (a \top e_{\mathbf{2}} = a) \end{gathered}$$

Thus $e_{\mathbf{1}} = e_{\mathbf{1}} \top e_{\mathbf{2}} = e_{\mathbf{2}}$.

Definition 1.5 (neutral element) Define $e_{\top,A}$ as e in Theorem 1.4

$$\exists e\ \ \mathsf{Neut}(e; \top, A) \to \mathsf{Neut}\big(e_{\top,A}; \top, A\big)$$

See: $\mathsf{Neut}(e; \top, A)$ — label: `dfn_neut`

$$\exists e\ \ \mathsf{Neut}(e; \top, A) \leftrightarrow \mathsf{Neut}\big(e_{\top,A}; \top, A\big)$$

Definition 1.6

$$e_\top := e_{\top,\mathsf{dom}(\mathsf{dom}(\top))}$$

See: $e_{\top,A}$ — label: `dfn_neut2`

$$\top\colon A \times A \to A \to e_\top = e_{\top,A}$$

Definition 1.7 (A is a monoid)

$$\mathsf{Monoid}(A; \top) :\leftrightarrow \begin{cases} \mathsf{Assoc}(\top; A) \\ \exists e\ \ \mathsf{Neut}(e; \top, A) \end{cases}$$

See: $\mathsf{Assoc}(\top; A)$ / $\mathsf{Neut}(e; \top, A)$ — label: `dfn_Monoid`

$$\mathsf{Monoid}(A; \top) \leftrightarrow \begin{cases} \top\colon A \times A \to A \\ \forall a, b, c \in A\ ((a \top b) \top c = a \top (b \top c)) \\ \exists e \in A\ \forall a \in A\ (a \top e = e \top a = a) \end{cases}$$

Definition 1.8 (Composite of finite sequences)

$$\mathop{\top}_{\mathbf{i}=\mathbf{m}}^{\mathbf{n}^+} \tau :\equiv \begin{cases} e_\top & \text{if } \mathbf{m} > \mathbf{n}^+ \\ \displaystyle\mathop{\top}_{\mathbf{i}=\mathbf{m}}^{\mathbf{n}} \tau \top \tau_{\mathbf{i}\to\mathbf{n}^+} & \text{if } \mathbf{m} < \mathbf{n}^+ \end{cases}$$

See: $e_\top$ — label: `dfn_compSeq`

Theorem 1.9 (General associativity theorem) Assume $\mathbf{1} = \mathbf{k_1} \leq ... \leq \mathbf{k_h} < \mathbf{k_{h+1}} = \mathbf{n} + \mathbf{1}$.

$$\begin{cases} a_{\mathbf{1}}, \ldots, a_{\mathbf{n}} \in A \\ \top\colon A \times A \to A \end{cases} \to \mathop{\top}_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} a_{\mathbf{i}} = \mathop{\top}_{\mathbf{i}=\mathbf{1}}^{\mathbf{h}} \mathop{\top}_{\mathbf{j}=\mathbf{k_i}}^{\mathbf{k_{i+1}}-\mathbf{1}} a_{\mathbf{j}}$$

See: $\displaystyle\mathop{\top}_{\mathbf{i}=\mathbf{m}}^{\mathbf{n}} \tau$ — label: `thm_GenAssoc`

Proof: (Induction on n)

(1) Assume $\mathbf{n} = \mathbf{0}$. $\mathbf{h} = \mathbf{0}$ by Assump. (continue)

Definition 1.10 (**A is a [commutative/Abelian] monoid**)

$$\mathsf{CommMonoid}(A;\top) :\leftrightarrow \begin{cases} \mathsf{Monoid}(A;\top) \\ \forall a,b \in A\ (a \top b = b \top a) \end{cases}$$

See: $\mathsf{Monoid}(A;\top)$ — label: `dfn_CommMonoid`

$$\mathsf{CommMonoid}(A;\top) \leftrightarrow \begin{cases} \top : A \times A \to A \\ \forall a,b,c \in A\ ((a \top b) \top c = a \top (b \top c)) \\ \exists e \in A\ \forall a \in A\ (a \top e = e \top a = a) \\ \forall a,b \in A\ (a \top b = b \top a) \end{cases}$$

Theorem 1.11 General commutativity theorem

1.2 Submonoids. Generators

Definition 1.12 (**B is stable**)

$$\mathsf{Stable}(B;A) :\leftrightarrow \begin{cases} B \subseteq A \\ \forall a,b \in B\ (a \top b \in B) \end{cases}$$

— label: `dfn_Stable`

Definition 1.13 (**induced law of composition**)

$$\top_B := \top \restriction (B \times B)$$

— label: `dfn_indComp`

Proposition 1.14

1. $$\mathsf{Assoc}(\top) \to \mathsf{Assoc}(\top_B)$$
2.