Rubin, Jean E. (1967). *Set Theory for the Mathematician*. San Francisco: Holden-Day.
https://github.com/kmi-ne/Math-MyNotes

Chapter 1

Class algebra

1.1 Class

Definition 1.1 — x is a set/proper class

1. $\mathsf{M}(x) :\leftrightarrow \exists y\ (x \in y)$
2. $\mathsf{Pr}(x) :\leftrightarrow \neg \mathsf{M}(x)$

— label: `Dfn_M`
`Dfn_Pr`

Proposition 1.2

$x \in y \to \mathsf{M}(x)$ — label: `Thm_elem_is_set`

Proof:

1. **Show:** $\mathsf{M}(x)$ ——— (2, Def. 1.1.1)
2. $\exists y\ (x \in y)$ ——— (3)
3. **Assume:** $x \in y$

Axiom 1.3 — Axiom of Extensionality

$\forall u\ (u \in x \leftrightarrow u \in y) \to x = y$ — label: `Axm_ext`

Definition 1.4 — x is a subclass/proper subclass of y

1. $x \subseteq y :\leftrightarrow \forall u\ (u \in x \to u \in y)$
2. $x \subset y :\leftrightarrow x \subseteq y \wedge x \neq y$

— label: `Dfn_Sbc`
`Dfn_Psbc`

Proposition 1.5

1. $x \subseteq x$
2. $x \subseteq y \subseteq z \to x \subseteq z$
3. $x \subseteq y \subseteq x \to x = y$

— label: `Thm_Sbc_is_reflRel`
`Thm_Sbc_is_transRel`
`Thm_Sbc_is_antisymRel`

Proof:

1. 1. **Show:** $x \subseteq x$ ——— (2, Def. 1.1.1)
   2. $\forall u\ (u \in x \to u \in x)$
2. 1. **Show:** $x \subseteq z$ ——— (2)
   2. $\forall u\ (u \in x \to u \in z)$ ——— (3)
   3. $\begin{cases} \forall u\ (u \in x \to u \in y) \\ \forall u\ (u \in y \to u \in z) \end{cases}$ ——— (4)
   4. **Assume:** $x \subseteq y \subseteq z$
3. 1. **Show:** $x = y$ ——— (2, Axiom of Extensionality)

2. $\forall u\ (u \in x \leftrightarrow u \in y)$ —— (3)

3. $\begin{cases} \forall u\ (u \in x \to u \in y) \\ \forall u\ (u \in y \to u \in x) \end{cases}$ —— (4, Def. 1.4.1)

4. **Assume:** $x \subseteq y \subseteq x$

Proposition 1.6

1. $\neg(x \subset x)$
2. $x \subset y \leftrightarrow \begin{cases} x \subseteq y \\ \exists u\ (u \in y \land u \notin x) \end{cases}$
3. $x \subset y \subseteq z \to x \subset z$
4. $x \subseteq y \subset z \to x \subset z$
5. $x \subset y \subset z \to x \subset z$
6. $x \subseteq y \leftrightarrow x \subset y \lor x = y$

— label: `Thm_Psbc_is_irreflRel`
`Thm_psbc_has_less_elem`
`Thm_psbc_of_sbc_is_psbc`
`Thm_sbc_of_psbc_is_psbc`
`Thm_psbc_of_psbc_is_psbc`
`Thm_sbc_eqv_psbc_or_Eq`

Proof:

1\. 1. **Show:** $\neg(x \subset x)$ —— (2, Def. 1.4.2)
2. $\neg(x \subseteq x \land x \neq x)$ —— (3)
3. $x = x$

2\. 1. **Show:** $\begin{cases} x \subseteq y \\ \exists u\ (u \in y \land u \notin x) \end{cases} \to x \subset y$
2. **Show:** $x \subset y$ —— (5, 3)
3. $x \neq y$ —— (6, 4)
4. $x = y \to \neg\exists u\ (u \in y \land u \notin x)$
5. **Assume:** $x \subseteq y$
6. **Assume:** $\exists u\ (u \in y \land u \notin x)$
7. **Show:** $x \subset y \to \begin{cases} x \subseteq y \\ \exists u\ (u \in y \land u \notin x) \end{cases}$
8. **Show:** $\exists u\ (u \in y \land u \notin x)$ —— (9, Def. 1.4.1)
9. $\neg(y \subseteq x)$ —— (11, 10, Prop. 1.5.3)
10. $x \neq y$ —— (12, Def. 1.4.2)
11. $x \subseteq y$ —— (12, Def. 1.4.2)
12. **Assume:** $x \subset y$

3\. 1. **Show:** $x \subset z$ —— (5, 2, Prop. 1.6.2)
2. **Show:** $\exists u\ (u \in z \land u \notin x)$ —— (4, 3)
3. $u \in y \to u \in z$ —— (7, Def. 1.4.1)
4. $\exists u\ (u \in y \land u \notin x)$ —— (7, Prop. 1.6.2)
5. **Show:** $x \subseteq z$ —— (6, Prop. 1.5.2)
6. $x \subseteq y \subseteq z$ —— (7, Def. 1.4.2)
7. **Assume:** $x \subset y \subseteq z$

4\. 1. **Show:** $x \subset z$ —— (5, 2, Prop. 1.6.2)
2. **Show:** $\exists u\ (u \in z \land u \notin x)$ —— (4, 3)
3. $u \in x \to u \in y$ —— (7, Def. 1.4.1)
4. $\exists u\ (u \in z \land u \notin y)$ —— (7, Prop. 1.6.2)
5. **Show:** $x \subseteq z$ —— (6, Prop. 1.5.2)
6. $x \subseteq y \subseteq z$ —— (7, Def. 1.4.2)
7. **Assume:** $x \subseteq y \subset z$

5\. 1. **Show:** $x \subset z$ —— (2, Prop. 1.6.3)

2. ⇥ $x \subset y \subseteq z$ ——— (3, Def. 1.4.2)
3. **Assume:** $x \subset y \subset z$

6. 1. **Show:** $x \subseteq y \leftrightarrow x \subset y \lor x = y$ ——— (7, 2)
2. ⇥ **Show:** $\left[\begin{array}{l} x \subseteq y \land x \neq y \\ x \subseteq y \land x = y \end{array}\right. \leftrightarrow x \subset y \lor x = y$ ——— (4, 3)
3. ⇥ ⇥ $x \subseteq y \land x \neq y \leftrightarrow x \subset y$ ——— (Def. 1.4.2)
4. ⇥ ⇥ **Show:** $x \subseteq y \land x = y \leftrightarrow x = y$ ——— (6, 5)
5. ⇥ ⇥ ⇥ $x \subseteq y \land x = y \to x = y$
6. ⇥ ⇥ ⇥ $x = y \to x \subseteq y \land x = y$ ——— (Prop. 1.5.1)
7. ⇥ $x \subseteq y \leftrightarrow \left[\begin{array}{l} x \subseteq y \land x \neq y \\ x \subseteq y \land x = y \end{array}\right.$

Definition 1.7

$u \in \{v \mid \phi\} \leftrightarrow \phi[u/v] \land \mathsf{M}(u)$ — label: `Dfn_cls`

Proposition 1.8

1. $(\phi \to \psi) \to \{u \mid \phi\} \subseteq \{u \mid \psi\}$
2. $(\phi \leftrightarrow \psi) \to \{u \mid \phi\} = \{u \mid \psi\}$
3. $(\phi \to \mathsf{M}(u)) \to (u \in \{v \mid \phi\} \leftrightarrow \phi[u/v])$
4. $\{u \mid \phi \land \mathsf{M}(u)\} = \{u \mid \phi\}$
5. $\{u \mid u \in \{v \mid \phi\} \lor \psi\} = \{u \mid \phi[u/v] \lor \psi\}$
6. $\{u \mid u \in \{v \mid \phi\} \land \psi\} = \{u \mid \phi[u/v] \land \psi\}$
7. $\{u \mid u \in x\} = x$

— label: `Thm_imp_wff_yield_sbc`
`Thm_eqv_wff_yield_eq_cls`
`Thm_wff_imp_Mu_then_omit_Mu`
`Thm_cls_wff_and_Mu_eq_cls_wff`
`Thm_cls_in_cls_lor`
`Thm_cls_in_cls_land`
`Thm_cls_of_elem_of_x_eq_x`

Proof:

1. 1. **Show:** $\{u \mid \phi\} \subseteq \{u \mid \psi\}$ ——— (4, 3, 2, Def. 1.4.1)
2. ⇥ $\psi \land \mathsf{M}(u) \to u \in \{u \mid \psi\}$ ——— (Def. 1.7)
3. ⇥ $\phi \land \mathsf{M}(u) \to \psi \land \mathsf{M}(u)$ ——— (5)
4. ⇥ $u \in \{u \mid \phi\} \to \phi \land \mathsf{M}(u)$ ——— (Def. 1.7)
5. **Assume:** $\phi \to \psi$

2. 1. **Show:** $(\phi \leftrightarrow \psi) \to \{u \mid \phi\} = \{u \mid \psi\}$ ——— (3, 2, Prop. 1.5.3)
2. ⇥ $(\psi \to \phi) \to \{u \mid \psi\} \subseteq \{u \mid \phi\}$ ——— (Prop. 1.8.1)
3. ⇥ $(\phi \to \psi) \to \{u \mid \phi\} \subseteq \{u \mid \psi\}$ ——— (Prop. 1.8.1)

3. 1. **Show:** $u \in \{u \mid \phi\} \leftrightarrow \phi[u/v]$ ——— (3, 2)
2. ⇥ $\phi[u/v] \land \mathsf{M}(u) \leftrightarrow \phi[u/v]$ ——— (4)
3. ⇥ $u \in \{v \mid \phi\} \leftrightarrow \phi[u/v] \land \mathsf{M}(u)$ ——— (Def. 1.7)
4. **Assume:** $\phi \to \mathsf{M}(u)$

Definition 1.9

1. $\varnothing := \{u \mid u \neq u\}$
2. $\mathbf{U} := \{u \mid u = u\}$
3. $\mathbf{Ru} := \{u \mid u \notin u\}$

— label: `Dfn_emp`
`Dfn_univ`
`Dfn_russ`

Proposition 1.10

1. $u \notin \varnothing$
2. $\mathsf{M}(u) \to u \in \mathbf{U}$
3. $\varnothing \subseteq x$
4. $x \subseteq \mathbf{U}$

— label: `Thm_nothing_is_elem_of_emp`
`Thm_set_is_elem_of_univ`
`Thm_emp_is_sbc_of_everything`
`Thm_everything_is_sbc_of_univ`

Proof:

1. 1. **Show:** $u \notin \varnothing$ (3, 2)
   2. $u \in \varnothing \to u \neq u$ (Def. 1.9.1, Def. 1.7)
   3. $u = u$
2. 1. **Show:** $\mathsf{M}(u) \to u \in \mathbf{U}$ (3, 2)
   2. $u = u \wedge \mathsf{M}(u) \to u \in \mathbf{U}$ (Def. 1.9.2, Def. 1.7)
   3. $\mathsf{M}(u) \to u = u \wedge \mathsf{M}(u)$
3. 1. **Show:** $\varnothing \subseteq x$ (2, Def. 1.4.1)
   2. $u \in \varnothing \to u \in x$ (Prop. 1.10.1)
4. 1. **Show:** $x \subseteq \mathbf{U}$ (3, 2, Def. 1.4.1)
   2. $\mathsf{M}(x) \to x \in \mathbf{U}$ (Prop. 1.10.2)
   3. $u \in x \to \mathsf{M}(x)$ (Prop. 1.2)

Theorem 1.11

$\mathsf{Pr}(\mathbf{Ru})$ — label: `Thm_russ_is_pr`

Proof:

1. **Show:** $\mathsf{Pr}(\mathbf{Ru})$ (2, Def. 1.1.2)
2. **Show:** $\neg\mathsf{M}(\mathbf{Ru})$ (4, 3)
3. $\mathbf{Ru} \notin \mathbf{Ru} \wedge \mathsf{M}(\mathbf{Ru}) \to \mathbf{Ru} \in \mathbf{Ru}$ (Def. 1.9.3, Def. 1.7)
4. **Show:** $\mathbf{Ru} \notin \mathbf{Ru}$ (5)
5. $\mathbf{Ru} \in \mathbf{Ru} \to \mathbf{Ru} \notin \mathbf{Ru}$ (6)
6. $u \in \mathbf{Ru} \to u \notin u$ (Def. 1.9.3, Def. 1.7)

Definition 1.12

1. $x \cup y := \{u \mid u \in x \vee u \in y\}$
2. $x \cap y := \{u \mid u \in x \wedge u \in y\}$
3. $x \setminus y := \{u \mid u \in x \wedge u \notin y\}$
4. $x^{\complement} = \{u \mid u \notin x\}$

— label: `Dfn_cup`
`Dfn_cap`
`Dfn_cdif`
`Dfn_cmpl`

Proposition 1.13

1. $x \cup y = y \cup x$
2. $x \cap y = y \cap x$
3. $(x \cup y) \cup z = x \cup (y \cup z)$
4. $(x \cap y) \cap z = x \cap (y \cap z)$
5. $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$
6. $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$
7. $x \cup x = x$
8. $x \cap x = x$

— label: `Thm_cup_is_comm`
`Thm_cap_is_comm`
`Thm_cup_is_assoc`
`Thm_cap_is_assoc`
`Thm_cup_cap_disr`
`Thm_cap_cup_disr`
`Thm_cup_is_idemp`
`Thm_cap_is_idemp`

Proof:

1\. 1. **Show:** $x \cup y = y \cup x$ (4, 3, 2)
   2. $\{u \mid u \in y \lor u \in x\} = y \cup x$ (Def. 1.12.1)
   3. $\{u \mid u \in x \lor u \in y\} = \{u \mid u \in y \lor u \in x\}$ (Prop. 1.8.2)
   4. $x \cup y = \{u \mid u \in x \lor u \in y\}$ (Def. 1.12.1)

2\. 1. **Show:** $x \cap y = y \cap x$ (4, 3, 2)
   2. $\{u \mid u \in y \land u \in x\} = y \cap x$ (Def. 1.12.2)
   3. $\{u \mid u \in x \land u \in y\} = \{u \mid u \in y \land u \in x\}$ (Prop. 1.8.2)
   4. $x \cap y = \{u \mid u \in x \land u \in y\}$ (Def. 1.12.2)

3\. 1. **Show:** $(x \cup y) \cup z = x \cup (y \cup z)$
   2. $\{u \mid u \in x \lor u \in y \cup z\} = x \cup (y \cup z)$ (Def. 1.12.1)
   3. $\{u \mid u \in x \lor u \in y \lor u \in z\} = \{u \mid u \in x \lor u \in y \cup z\}$ (Def. 1.12.1, Prop. 1.8.5)
   4. $\{u \mid u \in x \cup y \lor u \in z\} = \{u \mid u \in x \lor u \in y \lor u \in z\}$ (Def. 1.12.1, Prop. 1.8.5)
   5. $(x \cup y) \cup z = \{u \mid u \in x \cup y \lor u \in z\}$ (Def. 1.12.1)

4\. 1. **Show:** $(x \cap y) \cap z = x \cap (y \cap z)$ (5, 4, 3, 2)
   2. $\{u \mid u \in x \land u \in y \cap z\} = x \cap (y \cap z)$ (Def. 1.12.2)
   3. $\{u \mid u \in x \land u \in y \land u \in z\} = \{u \mid u \in x \land u \in y \cap z\}$ (Def. 1.12.2, Prop. 1.8.6)
   4. $\{u \mid u \in x \cap y \land u \in z\} = \{u \mid u \in x \land u \in y \land u \in z\}$ (Def. 1.12.2, Prop. 1.8.6)
   5. $(x \cap y) \cap z = \{u \mid u \in x \cap y \land u \in z\}$ (Def. 1.12.2)

5\. 1. **Show:** $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$ (6, 5, 4, 3, 2)
   2. $\{u \mid u \in x \cap y \lor u \in x \cap z\} = (x \cap y) \cup (x \cap z)$ (Def. 1.12.1)
   3. $\left\{ u \;\middle|\; \left[\begin{array}{l} u \in x \land u \in y \\ u \in x \land u \in z \end{array}\right. \right\} = \{u \mid u \in x \cap y \lor u \in x \cap z\}$ (Prop. 1.8.5, Def. 1.12.2)
   4. $\{u \mid u \in x \land (u \in y \lor u \in z)\} = \left\{ u \;\middle|\; \left[\begin{array}{l} u \in x \land u \in y \\ u \in x \land u \in z \end{array}\right. \right\}$ (Prop. 1.8.2)
   5. $\{u \mid u \in x \land u \in y \cup z\} = \{u \mid u \in x \land (u \in y \lor u \in z)\}$ (Def. 1.12.1, Prop. 1.8.6)
   6. $x \cap (y \cup z) = \{u \mid u \in x \land u \in y \cup z\}$ (Def. 1.12.2)

6\. 1. **Show:** $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$ (6, 5, 4, 3, 2)
   2. $\{u \mid u \in x \cup y \land u \in x \cup z\} = (x \cup y) \cap (x \cup z)$ (Def. 1.12.2)
   3. $\left\{ u \;\middle|\; \left\{\begin{array}{l} u \in x \lor u \in y \\ u \in x \lor u \in z \end{array}\right. \right\} = \{u \mid u \in x \cup y \land u \in x \cup z\}$ (Prop. 1.8.5, Def. 1.12.1)
   4. $\{u \mid u \in x \lor (u \in y \land u \in z)\} = \left\{ u \;\middle|\; \left\{\begin{array}{l} u \in x \lor u \in y \\ u \in x \lor u \in z \end{array}\right. \right\}$ (Prop. 1.8.2)
   5. $\{u \mid u \in x \lor u \in y \cap z\} = \{u \mid u \in x \lor (u \in y \land u \in z)\}$ (Def. 1.12.2, Prop. 1.8.5)
   6. $x \cup (y \cap z) = \{u \mid u \in x \lor u \in y \cap z\}$ (Def. 1.12.1)

7\. 1. **Show:** $x \cup x = x$ (4, 3, 2)
   2. $\{u \mid u \in x\} = x$ (Prop. 1.8.7)
   3. $\{u \mid u \in x \lor u \in x\} = \{u \mid u \in x\}$ (Prop. 1.8.2)
   4. $x \cup x = \{u \mid u \in x \lor u \in x\}$ (Def. 1.12.1)

8\. 1. **Show:** $x \cap x = x$ (4, 3, 2)
   2. $\{u \mid u \in x\} = x$ (Prop. 1.8.7)
   3. $\{u \mid u \in x \land u \in x\} = \{u \mid u \in x\}$ (Prop. 1.8.2)
   4. $x \cap x = \{u \mid u \in x \land u \in x\}$ (Def. 1.12.2)

Proposition 1.14

1. $x \subseteq x \cup y$
2. $x \cap y \subseteq x$

Proposition 1.15

1. $x \cup \varnothing = x$
2. $x \cap \varnothing = x$
3. $x \cup \mathbf{U} = \mathbf{U}$
4. $x \cap \mathbf{U} = x$

Proposition 1.16

1. $x \subseteq y \leftrightarrow (x \cup y = y)$
2. $x \subseteq y \leftrightarrow (x \cap y = x)$

Chapter 2

Functions, Relations

Definition 2.1

1. $\{x, y\} := \{u \mid u = x \lor u = y\}$
2. $\{x\} := \{u \mid u = x\}$

— label: `Dfn_pair` `Dfn_unit`

Proposition 2.2

$\{x, x\} = \{x\}$

Proposition 2.3

1. $\begin{cases} \mathsf{M}(x_1) \land \mathsf{M}(x_2) \land \mathsf{M}(y_1) \land \mathsf{M}(y_2) \\ \{x_1, y_2\} = \{y_1, y_2\} \end{cases} \to \left[\begin{array}{l} x_1 = y_1 \land x_2 = y_2 \\ x_1 = y_2 \land x_2 = y_1 \end{array}\right.$
2. $\begin{cases} \mathsf{M}(x) \land \mathsf{M}(y) \\ \{x\} = \{y\} \end{cases} \to x = y$

Axiom 2.4 — Axiom of Pairing

$\mathsf{M}(x) \land \mathsf{M}(y) \to \mathsf{M}(\{x, y\})$

Proposition 2.5

$\mathsf{M}(x) \to \mathsf{M}(\{x\})$

Definition 2.6

$\langle x, y \rangle := \{\{x\}, \{x, y\}\}$

Theorem 2.7

$$\begin{cases} \mathsf{M}(x_1) \land \mathsf{M}(x_2) \land \mathsf{M}(y_1) \land \mathsf{M}(y_2) \\ \langle x_1, x_2 \rangle = \langle y_1, y_2 \rangle \end{cases} \to \begin{cases} x_1 = y_1 \\ x_2 = y_2 \end{cases}$$

Proposition 2.8

$\mathsf{M}(x) \land \mathsf{M}(y) \to \mathsf{M}(\langle x, y \rangle)$