Purely Functional Data Structures

CIS 352 — Spring 2020

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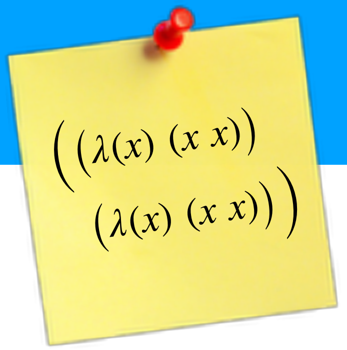
Logistics

- e2/e3 released over weekend
  - Both .25% bonus (not all exercises will be)
- a2 released today: due Monday after next
  - a3 will likely be released before a2 due
- Do e2/e3 before attempting a2
- **Coding exam 0 — Week after next**
  - More logistics soon
  - In-class programming exam (roughly half of class)
  - Email me soon if you need anything special for this

Example

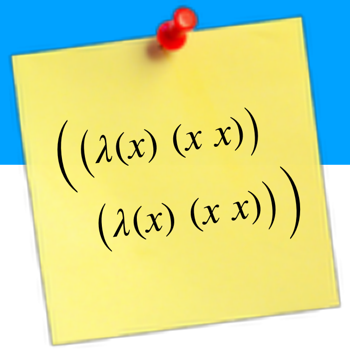


Warmup (observations on folds)

Assignment 1 defines a portion of PageRank as a sum…

$$\sum_{p_j \in M(p_i)} \frac{PR(p_j)}{L(p_j)}$$

Example



Warmup (observations on folds)

Consider a mathematical sum over a set, S

$$\sum_{e \in S} f(e)$$

The summation is readily translated to using foldl:

```
(define s (set 1 2 3))
(define (f x) (+ 1 x))
(foldl (λ (e acc) (+ (f e) acc)) 0 (set->list s))
```

Exercise

Write the following product using foldl and multiplication:

$$\prod_{e \in \{1,2,3\}} 2e$$

Exercise

Write the following product using foldl and multiplication:

$$\prod_{e \in \{1,2,3\}} 2e$$

```
(foldl (λ (e acc) (* 2 e acc))
       1
       (set->list (set 1 2 3)))
```

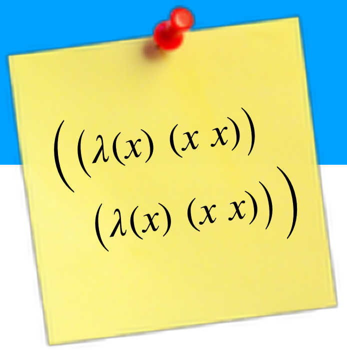
Data Structures

- A **data structure** is a representation of data
- **Constructors** build data
- **Destructors** (or matching) observes data
  - E.g., (empty?, cons?, car, cdr)
    - These four functions alone sufficient to define all functions that observe lists
- Defines various **operations** on the data
- **Abstract data type (ADT)** leaves form opaque, just operations
    - E.g., push, pop
    - Same ADT can have multiple concrete implementations

Purely Functional Data Structures

- A data structure is **purely functional** when all operations produce *new* data, rather than *changing* input data
- Otherwise the data structure is **imperative** or **stateful**
- Most of Racket's data structures are purely functional:
  - Cons cells, Lists, Immutable hashes, etc…
- Imperative variants have some potential advantages
  - Can be faster, allow more flexible access
- Reasoning about imperative data structures requires reasoning about the temporal patterns in its shape
  - This can be tricky!

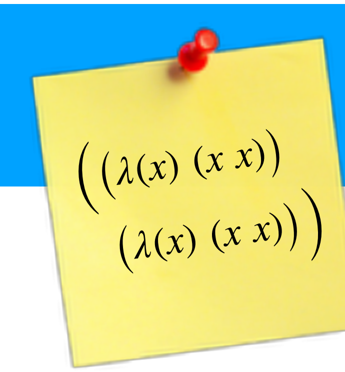
Example



A **queue** is a first-in, first-out data structure:

- **Enqueue** insert an element into queue
- **First** retrieves first element of the queue
- **Rest** retrieves the rest of the queue

Example



We can implement a queue as a **list**

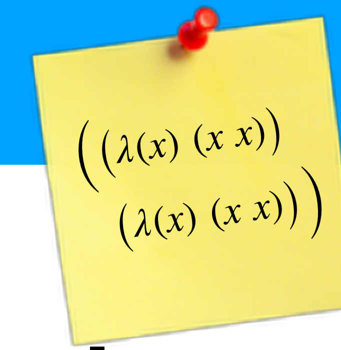
```
(define (empty-queue) '())

(define (queue-add queue elt)
  (append queue `(,elt)))

(define (queue-first queue) (first queue))

(define (queue-rest queue) (rest queue))
```

Example



Unfortunately this is **slow**, as **append** is O(n). Thus `(queue-add)` is O(n)

```
(define (empty-queue) '())

(define (queue-add queue elt)
  (append queue `(,elt)))

(define (queue-first queue) (first queue))

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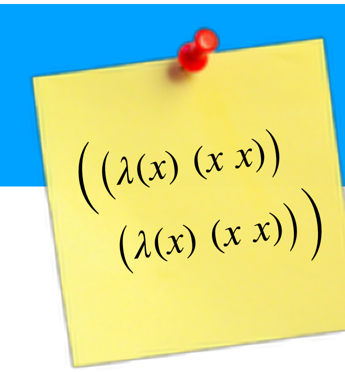
Example

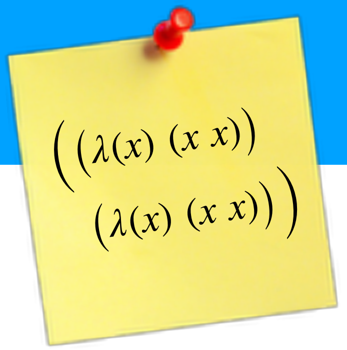


Let’s build some code to test our queue

```
;; build a queue of size i
(define (build-random-queue i)
  (define (loop num-left acc)
    (match num-left
      [0 acc]
      [_ (loop (- num-left 1) (queue-add acc (random 0 200)))]))
  (loop i (empty-queue)))

;; get nth element from the head of the queue
(define (get-nth queue n)
  (match n
    [0 (queue-first queue)]
    [_ (get-nth (queue-rest queue) (- n 1))]))
```

Example



And now build a queue of size 20,000, then retrieve its last element

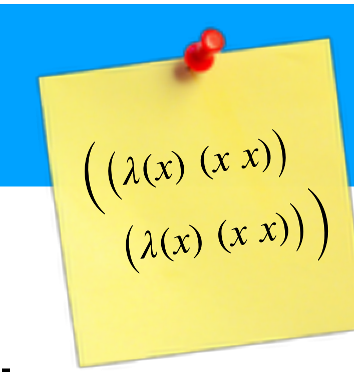
```
;; build a queue of size n, then destruct it
(define (n-firsts-and-rests n)
  (get-nth (build-random-queue n) (- n 1)))

(time
 (n-firsts-and-rests 20000))

;; cpu time: 4885 real time: 4825 gc time: 2824
```

4.8 seconds!

Example



Observation: to build queue O(n) calls to `(queue-add …)`, we do $O(n^2)$ work

```
;; build a queue of size n, then destruct it
(define (n-firsts-and-rests n)
  (get-nth (build-random-queue n) (- n 1)))

(time
 (n-firsts-and-rests 20000))

;; cpu time: 4885 real time: 4825 gc time: 2824
```

4.8 seconds!

Okasaki's Lazy Queues

- Our queue is **purely functional**, but it is slow
    - `(make-queue …)` is O(n), which is unacceptable
    - Imperative implementations perform O(1) insert
- Chris Oksaki presents **lazy queues**
    - Insert, first, and rest all have O(1) **amortized** time.
        - O(n) calls to insert (first, and reset) perform O(n) work
        - But an individual call may take up to O(n) time
    - Achieves this by using **two** lists rather than one
        - One you cons on to (the head) to insert
        - One you pull leaves from (call cdr on) to dequeue

Queue is a pair of a front (in order) and back (in reverse order)

Empty queue is just pair of empty lists

```
(define (empty-lazy-queue) (cons '() '()))

(define (lqueue-add queue elt)
  (match queue
    [(cons '() '()) (cons `(,elt) '())]
    [(cons front end)
     (cons front (cons elt end))]))
```

To add to queue: build new queue that conses new element to reversed end, O(1)

Tricky! Need to be careful when front is empty. In that case, first **is** end. We always want to be able to access first via **car**

```
(define (empty-lazy-queue) (cons '() '()))

(define (lqueue-add queue elt)
  (match queue
    [(cons '() '()) (cons `(,elt) '())]
    [(cons front end)
     (cons front (cons elt end))]))
```

Front is kept in order, and using cons ensures we get O(1) time for first

```
(define (lqueue-first queue)
  (match queue
    [(cons front end) (car front)]))
```

Rest must consider three cases:

- No more list left (heap underflow)
- Front empty, but back nonempty
  - Reverse back, make it front
- Front nonempty, pair its rest with back

```
(define (lqueue-rest queue)
  (match queue
    [(cons '() '()) (error 'underflow)]
    [(cons '() back)
     (queue-rest (cons '() (reverse back)))]
    [(cons front back)
     (cons (cdr front) back)]))
```

- Consider a queue that looks like…
  - `(cons `(0 … 10000) `(0 … 10000))`
- **Rest** will take O(1) time for the first 10,001 calls
- Then, 10,002nd call will reverse `` `(0 … 10000) `` and make it `` `(10000 … 0) ``, taking time proportional to 10k
- Then, 10,003rd call and onward take O(1) time: as they are back in first case

```
(define (lqueue-rest queue)
  (match queue
    [(cons '() '()) (error 'underflow)]
    [(cons '() back)
     (queue-rest (cons '() (reverse back)))]
    [(cons front back)
     (cons (cdr front) back)]))
```

Amortized Runtime

- **Amortization**: pay fee “up front” so next calls cheaper
- We say a function has **amortized** O(1) complexity if:
  - O(n) calls takes O(n) time
- O(f(n)) amortized if O(n) calls take O(f(n)*n) time
- Several methods for reasoning about amortized data
  - Won’t discuss specifics in this class
  - Basis for several popular functional data structures
- Imperative languages can often achieve O(1) complexity easier, as they can use pointers
  - But good functional data structures are usually fine