Semantics Intro

CIS700 — Fall 2022

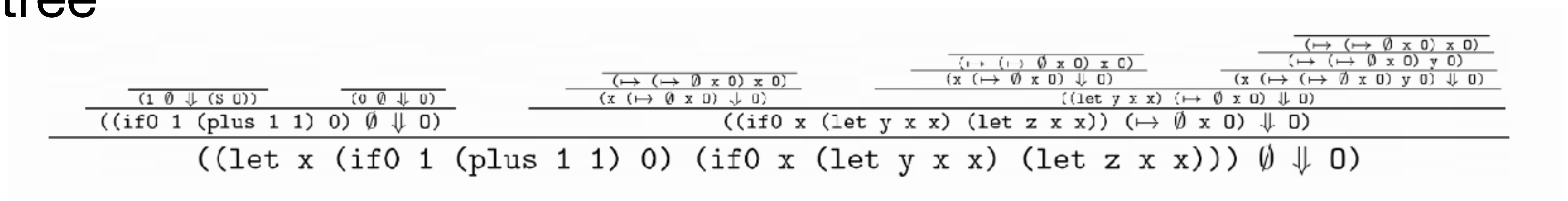
Kris Micinski

Why Semantics?

- Semantics let us define what a language **means**
- How to do this?
  - **Operationally** — define an “interpreter” for the language, semantics is whatever the interpreter produces.
    - Interpreter must be small to be trustworthy—otherwise hard to trust
  - **Denotationally** — define a mapping from programs to mathematical objects (e.g., functions from/to probability distributions)

Operational Semantics

- We will focus on two kinds of operational semantics
  - **“Big Step”** define a *recursive* interpreter, which produces *proofs* of its correctness by recursively walking over terms / judgements
    - Output is “a big tree”



  - **“Small Step”**
    - Output is a **sequence** of *states*; semantics is transitive closure of transition relation
  - We will discuss both of these for a dirt-simple language, IfArith

Natural Deduction for IfArith

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In this lecture, we'll introduce **natural deduction**

Natural deduction is a mathematical formalism that helps ground the ideas in metacircular interpreters

Natural deduction first used in mathematical logic, to specify **proofs** using inductive data

We will use natural deduction as a framework for specifying semantics of various languages throughout the course

Introduction Rules | **Elimination Rules**

$$\dfrac{\dfrac{}{\vdash^N A}\,u \quad \vdots \quad \vdash^N B}{\vdash^N A \supset B}\supset\mathrm{I}^u \qquad\qquad \dfrac{\vdash^N A \supset B \qquad \vdash^N A}{\vdash^N B}\supset\mathrm{E}$$

$$\dfrac{\dfrac{}{\vdash^N A}\,u \quad \vdots \quad \vdash^N p}{\vdash^N \neg A}\neg\mathrm{I}^{p,u} \qquad\qquad \dfrac{\vdash^N \neg A \qquad \vdash^N A}{\vdash^N C}\neg\mathrm{E}$$

$$\dfrac{\vdash^N [a/x]A}{\quad}\forall\mathrm{I}^a \qquad\qquad \dfrac{\vdash^N \forall x.\, A}{\quad}\forall\mathrm{E}$$

When we specify the semantics of a language using natural deduction, we give its semantics via a set of **inference rules**

Rules read: if the thing on the top is true, then the thing on the **bottom** is also true.

This rule says: “if c is an integer (mathematically: c ∈ $\mathbb{Q}$), then c evaluates to c.”

$$\textbf{Const}: \frac{c \in \mathbb{Q}}{c \Downarrow c}$$

Note: the notation e $\Downarrow$ v is read “e evaluates to v.”

Some rules will have more than one **antecedent** (thing on the top).

You read these: "if the first thing, and second thing, and … are **all** true, then the thing on the bottom is true."

$$\textbf{Plus}: \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 + n_1}{(\mathsf{plus}\ e_0\ e_1) \Downarrow n'}$$

"If $e_0 \Downarrow n_0$, and $e_1 \Downarrow n_1$, and $n' = n_0 + n1$, **then** I can say (plus e_0 e_1) $\Downarrow n'$."

$$\textbf{Plus}: \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 + n_1}{(\text{plus } e_0\ e_1) \Downarrow n'}$$

$$\textbf{Const} : \frac{c \in \mathbb{Q}}{c \Downarrow c} \qquad \textbf{Plus} : \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 + n_1}{(\mathsf{plus}\ e_0\ e_1) \Downarrow n'}$$

$$\textbf{Div} : \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 / n_1}{(\mathsf{div}\ e_0\ e_1) \Downarrow n'}$$

The natural deduction rule for **div** is similar

$$\textbf{Const} : \frac{c \in \mathbb{Q}}{c \Downarrow c} \qquad \textbf{Plus} : \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 + n_1}{(\text{plus } e_0\ e_1) \Downarrow n'}$$

$$\textbf{Div} : \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 / n_1}{(\text{div } e_0\ e_1) \Downarrow n'}$$

$$\textbf{Not}_\mathbf{0} : \frac{e \Downarrow 0}{(\text{not } e) \Downarrow 1} \qquad \textbf{Not}_\mathbf{1} : \frac{e \Downarrow n \quad n \neq 0}{(\text{not } e) \Downarrow 0}$$

We have **two** rules for not

Natural Deduction Rules for IfArith

$$\textbf{Const} : \frac{c \in \mathbb{Q}}{c \Downarrow c} \qquad \textbf{Plus} : \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 + n_1}{(\mathsf{plus}\ e_0\ e_1) \Downarrow n'}$$

$$\textbf{Div} : \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 / n_1}{(\mathsf{div}\ e_0\ e_1) \Downarrow n'}$$

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$$\textbf{If}_\mathbf{T} : \frac{e_0 \Downarrow 0 \quad e_1 \Downarrow n'}{(\mathsf{if}\ e_0\ e_1\ e_2) \Downarrow n'} \qquad \textbf{If}_\mathbf{F} : \frac{e_0 \Downarrow n \quad n = 0 \quad e_2 \Downarrow n'}{(\mathsf{if}\ e_0\ e_1\ e_2) \Downarrow n'}$$

Question: Now that we have the rules, what can we do with them?

Answer: Use them to **formally prove** that some program calculates some result

Let’s say I want to prove that the following program evaluates to 4:

```
(if (plus 1 -1) 3 4)
```

What rule could go here..?

$$\frac{???}{\left(\text{if } (\text{plus } 1\ \ -1)\ 3\ 4\right) \Downarrow 4}$$

$$\mathbf{If_T} : \frac{e_0 \Downarrow n \quad n \neq 0 \quad e_1 \Downarrow n'}{(\mathsf{if}\ e_0\ e_1\ e_2) \Downarrow n'} \quad \mathbf{If_F} : \frac{e_0 \Downarrow 0 \quad e_2 \Downarrow n'}{(\mathsf{if}\ e_0\ e_1\ e_2) \Downarrow n'}$$

$$\frac{???}{\left(\mathsf{if}\ (\mathsf{plus}\ 1\ \ -1)\ 3\ 4\right) \Downarrow 4}$$

$$\textbf{If}_\textbf{T} : \frac{e_0 \Downarrow n \quad n \neq 0 \quad e_1 \Downarrow n'}{(\text{if } e_0\ e_1\ e_2) \Downarrow n'} \quad \textbf{If}_\textbf{F} : \frac{e_0 \Downarrow 0 \quad e_2 \Downarrow n'}{(\text{if } e_0\ e_1\ e_2) \Downarrow n'}$$

$$\frac{???}{\left(\text{if } (\text{plus } 1\ \ -1)\ 3\ 4\right) \Downarrow 4}$$

To apply a natural-deduction rule,
we must perform **unification**

There can be no variables in the resulting unification!

$$\mathbf{If_F} : \frac{e_0 \Downarrow 0 \quad e_2 \Downarrow n'}{(\text{if } e_0\ e_1\ e_2) \Downarrow n'}$$

$$\frac{(\text{plus } 1\ \ -1) \Downarrow 0 \qquad\qquad 4 \Downarrow 4}{\left(\text{if } (\text{plus } 1\ \ -1)\ 3\ 4\right) \Downarrow 4}$$

We perform unification:

e_0: (plus 1 -1), e_1: 3

e_2: 4, n': 4

Not done yet, now we have to prove **these** things

$$\frac{(\text{plus } 1\ \ -1) \Downarrow 0 \qquad 4 \Downarrow 4}{\left(\text{if } (\text{plus } 1\ \ -1)\ 3\ 4\right) \Downarrow 4}$$

Why can we say $4 \Downarrow 4$? Because of the **Const** rule

$$\frac{(\text{plus}\ 1\ \ -1) \Downarrow 0 \qquad \dfrac{4 \in \mathbb{Q}}{4 \Downarrow 4}}{\left(\text{if}\ (\text{plus}\ 1\ \ -1)\ 3\ 4\right) \Downarrow 4}$$

We're not done yet, because **plus** requires an antecedent:

$$\textbf{Plus}: \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 + n_1}{(\text{plus } e_0\ e_1) \Downarrow n'}$$

$$\frac{(\text{plus } 1\ \ -1) \Downarrow 0 \qquad \dfrac{4 \in \mathbb{Q}}{4 \Downarrow 4}}{\left(\text{if } (\text{plus } 1\ \ -1)\ 3\ 4\right) \Downarrow 4}$$

But we're **still** not done, because we need to finish these three

$$\dfrac{\dfrac{1 \Downarrow 1 \quad -1 \Downarrow -1 \quad 1 + -1 = 0}{(\mathsf{plus}\ 1\ -1) \Downarrow 0} \qquad \dfrac{4 \in \mathbb{Q}}{4 \Downarrow 4}}{\left(\mathsf{if}\ (\mathsf{plus}\ 1\ -1)\ 3\ 4\right) \Downarrow 4}$$

Things that are simply true from algebra require no antecedents, we take them as "axioms."

$$\dfrac{\dfrac{\dfrac{1 \in \mathbb{Q}}{1 \Downarrow 1} \quad \dfrac{-1 \in \mathbb{Q}}{-1 \Downarrow -1} \quad \dfrac{}{1 + -1 = 0}}{(\text{plus } 1\ -1) \Downarrow 0} \qquad \dfrac{4 \in \mathbb{Q}}{4 \Downarrow 4}}{\left(\text{if } (\text{plus } 1\ -1)\ 3\ 4\right) \Downarrow 4}$$

This is a complete proof that the program computes 4

$$\dfrac{\dfrac{\dfrac{1 \in \mathbb{Q}}{1 \Downarrow 1} \quad \dfrac{-1 \in \mathbb{Q}}{-1 \Downarrow -1} \quad \dfrac{}{1 + -1 = 0}}{(\mathsf{plus}\ 1\ -1) \Downarrow 0} \qquad \dfrac{4 \in \mathbb{Q}}{4 \Downarrow 4}}{\left(\mathsf{if}\ (\mathsf{plus}\ 1\ -1)\ 3\ 4\right) \Downarrow 4}$$

Question: could you write this proof..? What would happen if you tried…?

$$\frac{???}{\left(\text{if } (\text{plus } 1\ \ -1)\ 3\ 4 \Downarrow 3\right)}$$

$$\mathbf{If_T} : \frac{e_0 \Downarrow n \quad n \neq 0 \quad e_1 \Downarrow n'}{(\text{if } e_0\ e_1\ e_2) \Downarrow n'} \quad \mathbf{If_F} : \frac{e_0 \Downarrow 0 \quad e_2 \Downarrow n'}{(\text{if } e_0\ e_1\ e_2) \Downarrow n'}$$

$$\frac{:(}{\left(\text{if } (\text{plus } 1\ \ -1)\ 3\ 4\right) \Downarrow 3}$$

Answer: you **can't** write this proof, because IfT will only let you evaluate e1 when e0 is non-0!

$$\frac{???}{\big(\text{plus (plus } 0\ 1)\ 2\big) \Downarrow 3} \qquad \frac{???}{(\text{if } 1\ (\text{div } 1\ 1)\ 2) \Downarrow 1}$$

$$\textbf{Const}: \frac{c \in \mathbb{Q}}{c \Downarrow c} \qquad \textbf{Plus}: \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 + n_1}{(\text{plus } e_0\ e_1) \Downarrow n'}$$

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Small-Step Semantics of IfArith

CIS352 — Spring 2021

Kris Micinski

Code in the description!

Last Week: Defined **Big-Step** semantics for IfArith

Last Week: Defined **Big-Step** semantics for IfArith

Two different, but similar, formulations:

- Metacircular Interpreter in Racket
- Natural Deduction

The metacircular interpreter is our
"implementation" of natural deduction

```
(define (evaluate e)
  (match e
    [(? integer? n) n]
    [`(plus ,(? expr? e0) ,(? expr? e1))
     (+ (evaluate e0) (evaluate e1))]
    [`(div ,(? expr? e0) ,(? expr? e1))
     (/ (evaluate e0) (evaluate e1))]
    [`(not ,(? expr? e-guard))
    (if (= (evaluate e-guard) 0) 1 0)]
    [`(if ,(? expr? e0) ,(? expr? e1) ,(? expr? e2))
     (if (equal? 0 (evaluate e0)) (evaluate e2) (evaluate e1))]
    [_ "unexpected input"]))
```

$$\textbf{Const} : \frac{c \in \mathbb{Q}}{c \Downarrow c} \qquad \textbf{Plus} : \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 + n_1}{(\text{plus } e_0\ e_1) \Downarrow n'}$$

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$$\textbf{If}_\mathbf{T} : \frac{e_0 \Downarrow 0 \quad e_1 \Downarrow n'}{(\text{if } e_0\ e_1\ e_2) \Downarrow n'} \qquad \textbf{If}_\mathbf{F} : \frac{e_0 \Downarrow n \quad n = 0 \quad e_2 \Downarrow n'}{(\text{if } e_0\ e_1\ e_2) \Downarrow n'}$$

```
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    [`(div ,(? expr? e0) ,(? expr? e1))
     (/ (evaluate e0) (evaluate e1))]
    [`(not ,(? expr? e-guard))
    (if (= (evaluate e-guard) 0) 1 0)]
    [`(if ,(? expr? e0) ,(? expr? e1) ,(? expr? e2))
     (if (equal? 0 (evaluate e0)) (evaluate e2) (evaluate e1))]
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$$\textbf{Const}: \frac{c \in \mathbb{Q}}{c \Downarrow c} \qquad \textbf{Plus}: \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 + n_1}{(\text{plus } e_0\ e_1) \Downarrow n'}$$

$$\textbf{Div}: \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 / n_1}{(\text{div } e_0\ e_1) \Downarrow n'}$$

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```
(define (evaluate e)
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     (/ (evaluate e0) (evaluate e1))]
    [`(not ,(? expr? e-guard))
    (if (= (evaluate e-guard) 0) 1 0)]
    [`(if ,(? expr? e0) ,(? expr? e1) ,(? expr? e2))
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$$\textbf{Const}:\frac{c \in \mathbb{Q}}{c \Downarrow c} \qquad \textbf{Plus}:\frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1 \quad n' = n_0 + n_1}{(\text{plus } e_0\ e_1) \Downarrow n'}$$

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$$\textbf{Not}_0:\frac{e \Downarrow 0}{(\text{not } e) \Downarrow 1} \qquad \textbf{Not}_1:\frac{e \Downarrow n \quad n \neq 0}{(\text{not } e) \Downarrow 0}$$

$$\textbf{If}_\textbf{T}:\frac{e_0 \Downarrow 0 \quad e_1 \Downarrow n'}{(\text{if } e_0\ e_1\ e_2) \Downarrow n'} \qquad \textbf{If}_\textbf{F}:\frac{e_0 \Downarrow n \quad n = 0 \quad e_2 \Downarrow n'}{(\text{if } e_0\ e_1\ e_2) \Downarrow n'}$$

This week we'll be looking at **small-step** interpreters

Implement and formalize **textual reduction**

Small-step interpreters specify execution as a sequence of **steps**, where each step makes only a small, local computation

```
  (div (plus 2 2) (plus 3 -1))
→ (div 4 (plus 3 -1))
→ (div 4 2)
→ 2
```

We will define the rules precisely in a few slides…

This allows us to reason about, and implement, control over execution in a fine-grained way at each step.

```
  (div (plus 2 2) (plus 3 -1))
→ (div 4 (plus 3 -1))
→ (div 4 2)
→ 2
```

Allows us to reason about traces of the program more easily. Useful for things like…

- Reasoning about finite prefix of infinitely-looping programs (servers)
- Temporal properties of the program (data-race freedom, etc…)

Our job is to define this step function / operator, written mathematically as $e_0 \rightarrow e_1$

```
  (div (plus 2 2) (plus 3 -1))
→ (div 4 (plus 3 -1))
→ (div 4 2)
→ 2
```

First observation: can only take a step when both arguments to plus / div are **values**

```
  (div (plus 2 2) (plus 3 -1))
→ (div 4 (plus 3 -1))
→ (div 4 2)
→ 2
```

We can immediately evaluate `(plus 2 2)` to 4, and then to step the whole expression, we substitute 4 in place of `(plus 2 2)`

```
  (div (plus 2 2) (plus 3 -1))
→ (div 4 (plus 3 -1))
→ (div 4 2)
→ 2
```

We first identify a **redex** (“reducible expression”)

Now two rules (so far)

- Immediately reduce plus/div when args are values
- When e_0 or e_1 is **not** a value, reduce one of them and replace it

```
  (div (plus 2 2) (plus 3 -1))
→ (div 4 (plus 3 -1))
→ (div 4 2)
→ 2
```

- Immediately reduce plus/div when args are values

Let's translate this into the natural deduction style..

By the way, in this lecture we are defining a **new set of rules** for the small-step semantics, which I will call **SmallIfArith**

These rules are separate from the rules for **IfArith**

"Immediately reduce plus/div when args are values"

"Immediately reduce plus/div when args are values"

$$\textbf{StepPlus}\ \frac{n_0 \in \mathbb{Q} \quad n_1 \in \mathbb{Q} \quad n' = n_0 + n_1}{(\text{plus } n_0\ n_1) \to n'}$$

"When e_0 or e_1 is **not** a value, reduce one of them and replace it"

$$\textbf{PlusLeft}\ \frac{e_0 \rightarrow e'}{(\text{plus}\ e_0\ e_1) \rightarrow (\text{plus}\ e'\ e_1)}$$

$$\textbf{PlusRight}\ \frac{e_1 \rightarrow e'}{(\text{plus}\ n\ e_1) \rightarrow (\text{plus}\ n\ e')}$$

The n here is a bit crucial: it adds determinism to our semantics!

"When e_0 or e_1 is **not** a value, reduce one of them and replace it"

$$\textbf{StepPlus}\ \frac{n_0 \in \mathbb{Q} \quad n_1 \in \mathbb{Q} \quad n' = n_0 + n_1}{(\text{plus } n_0\ n_1) \to n'}$$

$$\textbf{PlusRight}\ \frac{n \in \mathbb{Q} \quad {\color{red}e_1} \to {\color{blue}e'}}{(\text{plus } n\ {\color{red}e_1}) \to (\text{plus } n\ {\color{blue}e'})}$$

$$\textbf{PlusLeft}\ \frac{{\color{red}e_0} \to {\color{blue}e'}}{(\text{plus } {\color{red}e_0}\ e_1) \to (\text{plus } {\color{blue}e'}\ e_1)}$$

"To process (plus e_0 e_1), first check if is a value. If it is, then check if e_1 is a value. If both are, perform the addition."

"When e_0 or e_1 is **not** a value, reduce one of them and replace it"

$$\textbf{StepPlus}\ \frac{n_0 \in \mathbb{Q} \quad n_1 \in \mathbb{Q} \quad n' = n_0 + n_1}{(\text{plus } n_0\ n_1) \to n'}$$

$$\textbf{PlusRight}\ \frac{n \in \mathbb{Q} \quad e_1 \to e'}{(\text{plus } n\ e_1) \to (\text{plus } n\ e')}$$

$$\textbf{PlusLeft}\ \frac{e_0 \to e'}{(\text{plus } e_0\ e_1) \to (\text{plus } e'\ e_1)}$$

These are the three cases you need to consider for +

Very similar operation for division…

$$\textbf{StepDiv}\ \frac{n_0 \in \mathbb{Q} \quad n_1 \in \mathbb{Q} \quad n' = n_0/n_1}{(\mathsf{div}\ n_0\ n_1) \rightarrow n'}$$

$$\textbf{DivRight}\ \frac{n \in \mathbb{Q} \quad e_1 \rightarrow e'}{(\mathsf{div}\ n\ e_1) \rightarrow (\mathsf{div}\ n\ e')}$$

$$\textbf{DivLeft}\ \frac{e_0 \rightarrow e'}{(\mathsf{div}\ e_0\ e_1) \rightarrow (\mathsf{div}\ e'\ e_1)}$$

$$\textbf{PlusLeft}\ \frac{e_0 \rightarrow e'}{(\text{plus}\ e_0\ e_1) \rightarrow (\text{plus}\ e'\ e_1)}$$

$$\textbf{PlusRight}\ \frac{e_1 \rightarrow e'}{(\text{plus}\ e_0\ e_1) \rightarrow (\text{plus}\ e_0\ e')}$$

What would happen if we did this instead…?

Semantics would be **nondeterministic**

```
((plus 1 2) (plus 2 2)) -> (plus (plus 1 2) 4)
((plus 1 2) (plus 2 2)) -> (plus 3 (plus 2 2))
```

$$\textbf{PlusLeft}\ \frac{e_0 \rightarrow e'}{(\text{plus}\ e_0\ e_1) \rightarrow (\text{plus}\ e'\ e_1)}$$

$$\textbf{PlusRight}\ \frac{e_1 \rightarrow e'}{(\text{plus}\ e_0\ e_1) \rightarrow (\text{plus}\ e_0\ e')}$$

This will manifest by complicating our definition of step

```
(define/contract (step e)
  (expr? -> expr?)
  …)
```

We would need instead…

```
(define/contract (step e)
  (expr? -> (listof expr?))
  …)
```

What about not..?

$$\textbf{StepNot}_\mathbf{0}\ \frac{n \neq 0}{(\mathsf{not}\ n) \rightarrow 0}$$

$$\textbf{StepNot}_\mathbf{1}\ \frac{n = 0}{(\mathsf{not}\ n) \rightarrow 1}$$

$$\textbf{StepNot}\ \frac{e \rightarrow e'}{(\mathsf{not}\ e) \rightarrow (\mathsf{not}\ e')}$$

Finally, if…

$$\mathbf{If_T}\ \frac{n \neq 0}{(\mathsf{if}\ n\ e_1\ e_2) \to e_1}$$

$$\mathbf{If_F}\ \frac{n = 0}{(\mathsf{if}\ n\ e_1\ e_2) \to e_2}$$

$$\mathbf{If}\ \frac{e_0 \to e'}{(\mathsf{if}\ e_0\ e_1\ e_2) \to (\mathsf{if}\ e'\ e_1\ e_2)}$$

So many rules! Rules are overly complicated: next lecture we will refactor them to be more attractive…

$$\textbf{StepPlus}\ \frac{n_0 \in \mathbb{Q} \quad n_1 \in \mathbb{Q} \quad n' = n_0 + n_1}{(\text{plus } n_0\ n_1) \to n'}$$

$$\textbf{StepDiv}\ \frac{n_0 \in \mathbb{Q} \quad n_1 \in \mathbb{Q} \quad n' = n_0/n_1}{(\text{div } n_0\ n_1) \to n'}$$

$$\textbf{PlusRight}\ \frac{n \in \mathbb{Q} \quad e_1 \to e'}{(\text{plus } n\ e_1) \to (\text{plus } n\ e')}$$

$$\textbf{DivRight}\ \frac{n \in \mathbb{Q} \quad e_1 \to e'}{(\text{div } n\ e_1) \to (\text{div } n\ e')}$$

$$\textbf{PlusLeft}\ \frac{e_0 \to e'}{(\text{plus } e_0\ e_1) \to (\text{plus } e'\ e_1)}$$

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$$\textbf{StepNot}_0\ \frac{n \neq 0}{(\text{not } n) \to 0}$$

$$\textbf{StepNot}_1\ \frac{n = 0}{(\text{not } n) \to 1}$$

$$\textbf{StepNot}\ \frac{e \to e'}{(\text{not } e) \to (\text{not } e')}$$

$$\textbf{If}_\textbf{T}\ \frac{n \neq 0}{(\text{if } n\ e_1\ e_2) \to e_1} \qquad \textbf{If}_\textbf{F}\ \frac{n = 0}{(\text{if } n\ e_1\ e_2) \to e_2}$$

$$\textbf{If}\ \frac{e_0 \to e'}{(\text{if } e_0\ e_1\ e_2) \to (\text{if } e'\ e_1\ e_2)}$$

One very important omission: there is **no defined step** for values!

These rules only tell us how to step expressions. We need to keep doing that (in a loop) until we reach a value.

Now that we have the rules, let's code them up as a small-step interpreter

```
(define/contract (step e)
  (-> (lambda (x) (and (expr? x) (not (value? x))) expr?)
  'todo)
```