Sequent Calculus

CIS700 — Fall 2024

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Note: my presentation mostly follows the book of Sara Negri and Jan von Plato (Chapters 1/2)

Hilbert-Style Systems

- "Axiomatic systems,"
  - Many basic axioms taken for granted
  - But only a single inference rule!
- Downsides: lots of axioms assumed, and proofs tend to be very ugly

Axioms (Freely-Admittable at any Time)

$$\vdash A \to (B \to A)$$

$$\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$$

$$\vdash \neg\neg P \to P$$

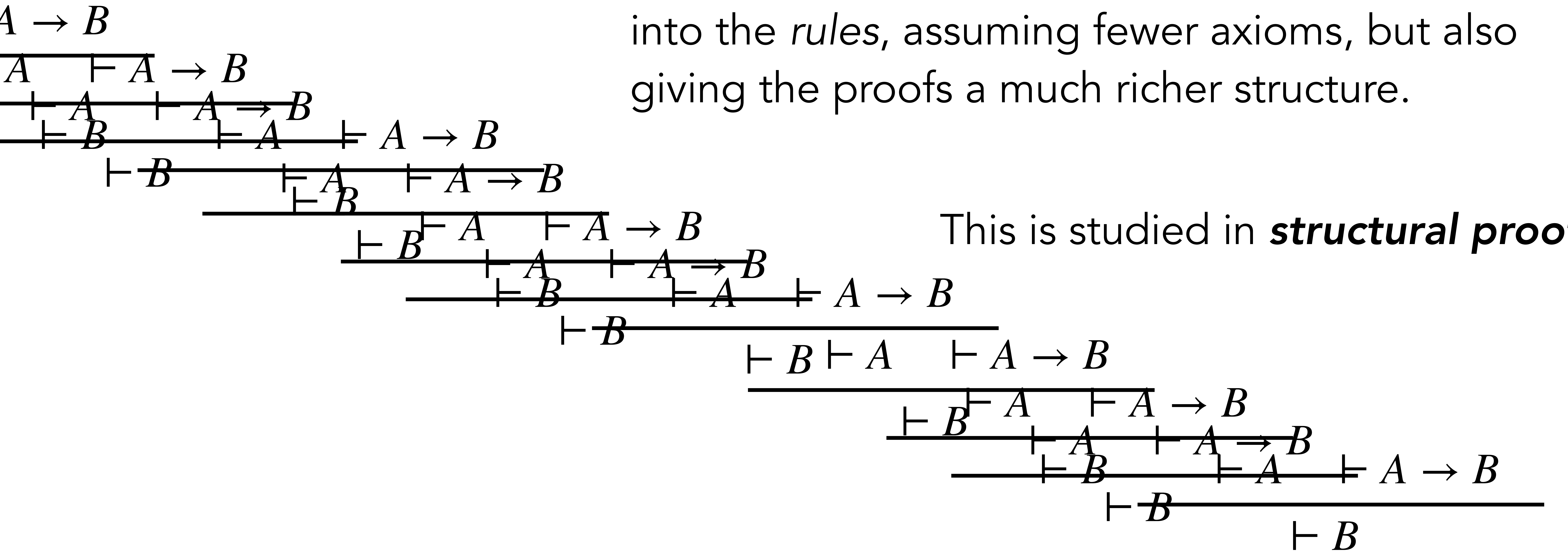
Inference Rule (Modus Ponens)

$$\frac{\vdash A \qquad \vdash A \to B}{\vdash B}$$

Generates ugly proofs because the proofs have been “linearized” into a long chain of assertions.

Assumptions must be “pushed up” to their uses.

By contrast, natural deduction puts more structure into the *rules*, assuming fewer axioms, but also giving the proofs a much richer structure.



This is studied in ***structural proof theory***

By contrast, there is only **one** form of assumption in natural deduction!
That is, the primitive assumption that assuming P allows you to conclude P

$$\wedge\mathbf{E1}\ \frac{P \wedge Q \text{ True}}{P \text{ True}} \qquad \wedge\mathbf{E2}\ \frac{P \wedge Q \text{ True}}{Q \text{ True}} \qquad \wedge\mathbf{I}\ \frac{P \text{ True} \quad Q \text{ True}}{P \wedge Q \text{ True}}$$

$$\vee\mathbf{I1}\ \frac{P \text{ True}}{P \vee Q \text{ True}} \qquad \vee\mathbf{I2}\ \frac{Q \text{ True}}{P \vee Q \text{ True}} \qquad \vee\mathbf{E}^{\mathbf{uw}}\ \frac{A \vee B \text{ True} \quad \begin{matrix} \overline{A \text{ True}}^{\,u} \\ \vdots \\ C \text{ True} \end{matrix} \quad \begin{matrix} \overline{B \text{ True}}^{\,w} \\ \vdots \\ C \text{ True} \end{matrix}}{C \text{ True}}$$

$$\frac{A \Rightarrow B \text{ True} \qquad A \text{ True}}{B \text{ True}}\ \Rightarrow\mathbf{E} \qquad \frac{\begin{matrix} \overline{A \text{ True}}^{\,u} \\ \cdots \\ B \text{ True} \end{matrix}}{A \Rightarrow B \text{ True}}\ \Rightarrow\mathbf{I}^{\mathbf{u}}$$

Recall that in elementary natural deduction system, we did not use sequents—instead, assumptions were introduced via a name u. This presentation is due to Pfenning, other treatments of elementary natural deduction elide the binder (u) and assume some other process is done to check the relevant "scoping"

$$\dfrac{\dfrac{}{A \text{ True}}\,u \quad \cdots \quad B \text{ True}}{A \Rightarrow B \text{ True}} \Rightarrow\!\mathrm{I}^{u}$$

$$\dfrac{\dfrac{}{A \text{ True}} \quad \cdots \quad B \text{ True}}{A \Rightarrow B \text{ True}} \Rightarrow\!\mathrm{I}$$

In the **sequent calculus**, the usage of the sequent style allows us to explicate assumptions via an environment (which we can understand symbolically)

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow\!\mathbf{I}$$

The intros rule here allows us to pull A inside the set of assumptions. These conditionally-true statements allow a much crisper understanding of the **structure** of the assumptions in the current environment.

(We will see the final version of the rule later in G3ip)

The Negri and von Plato book introduces sequents as

$$\Gamma \Rightarrow \phi$$

If the **bag** Γ of formulas is assumed, then ϕ holds

Sequent calculus is a formal theory of the **derivability relation**
There is an important difference between $\Gamma \Rightarrow \phi$ and $\Gamma \vdash \phi$,
In the second, $\vdash$ is a “meta-level” statement, where $\Rightarrow$ is part of the syntax of **formulas**

The rules of natural deduction really must be read as **rule schemas**, and show active formulas, but leave the set of open assumptions as something that must be implicitly understood

$$
\dfrac{\begin{array}{cc} \Gamma & \Delta \\ \vdots & \vdots \\ A & B \end{array}}{A \wedge B} \wedge I
$$

Implicitly, the open assumptions are $\Gamma \cup \Delta$

Instead of working in terms of introduction and elimination forms, sequent calculus works in terms of **left** and **right** rules

Left rules transform the left side (assumptions), and right rules transform the conclusions

Introduction forms become **right** rules

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \land B}$$

"If I can use Γ to prove A, and I can use Δ to prove B, I can use $\Gamma \cup \Delta$ to prove $A \land B$"

Gentzen's original formulation had assumptions as **lists**, meaning that these two statements would be different sequents:

$$A, B \Rightarrow A \qquad\qquad B, A \Rightarrow A$$

This requires various **structural** rules to be able to say things like:

$$\frac{\Gamma, A \Rightarrow A}{A, \Gamma \Rightarrow A}$$

Things get simpler, however, if we just treat the context as a finite multiset (bag)—we have fewer structural rules

Elimination rules become **left** rules, which tell you how you may transform assumed knowledge

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \; L \wedge$$

"If I can use both A and B to prove C, then I can use $A \wedge B$ to prove C"

In every one of the rules, all formulas in a sequent calculus derivation are **subformulas** of the endsequent of the derivation

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \, L\wedge$$

This **subformula** property is useful in observing decidability

There is only one **logical axiom**

$$\frac{}{A \Rightarrow A}$$

The **structural rules** allow us to manipulate the environment

For example, **weakening** allows us to drop assumptions from the environment

$$\frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C}$$

"If we can prove C using Γ, we can also prove C using Γ and also assuming A"

Example: proving $A \to (B \to A)$ unconditionally

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}\,\text{Wk}}{A \Rightarrow (B \to A)}\,R\Rightarrow}{\Rightarrow A \to (B \to A)}\,R\Rightarrow$$

(Note: here I use → to avoid confusion with ⇒, I avoid ⊃ and instead use →)

Contraction allows us to throw away duplicated assumptions

$$\frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\ \text{Ctr}$$

The last rule is the most important rule, because it corresponds to *computation*. This rule is called the **cut**

$$\frac{\Gamma \Rightarrow A \qquad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\ \text{Cut}$$

"If we can assume Γ and prove A, and also—assuming A and Δ we can prove C, then using Γ and Δ, we can prove C."

Note that A is substituted into the second derivation, "cutting out" the assumption A in the conclusion

A central result of the sequent calculus is the **cut elimination theorem**, which says that whenever we can prove a statement using the cut, we *could* prove it without using the cut

Such cut-free sequent calculi enjoy a number of nice properties. For example, notice that the cut rule **breaks** the subformula rule: the formula A "pops out of nowhere," meaning that—if we applied backwards reasoning—we could always apply the cut with an *arbitrary* way, leading to nontermination of proof search

$$\frac{\Gamma \Rightarrow A \qquad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\ \text{Cut}$$

A central task in structural proof theory is to identify cut-free systems

Proving a cut elimination theorem amounts to showing that the cut is admissible in such systems, but that the cut may be excluded when we do metatheoretic proofs (proofs about the system)

G3ip (Chapter 2.2) is an intuitionistic sequent calculus in which all structural rules (weakening, contraction, and cut) are admissible

Sequents are of the form $\Gamma \Rightarrow C$ where Γ is a finite multiset

Axiom $\dfrac{}{P, \Gamma \Rightarrow P}$

Logical rules:

$$L\wedge\ \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad R\wedge\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \qquad L\bot\ \frac{}{\bot, \Gamma \Rightarrow C}$$

$$L\vee\ \frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \quad R\vee_1 \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \quad R\vee_2 \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}$$

$$L\!\rightarrow\ \frac{A \rightarrow B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \rightarrow B, \Gamma \Rightarrow C} \qquad R\!\rightarrow\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

Each rule has a **context** that is ***shared***

$$R\wedge \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

Rather than…

$$R \wedge \text{alt} \quad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma \cup \Delta \Rightarrow A \wedge B}$$

This rule is reasonable, but complicates aspects such as proof search, etc.…

The $L\rightarrow$ rule is worth mentioning. This rule plays the role of modus-ponens

$$\frac{A \Rightarrow B \text{ True} \qquad A \text{ True}}{B \text{ True}} \Rightarrow E \qquad\qquad L\rightarrow \frac{A \rightarrow B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \rightarrow B, \Gamma \Rightarrow C}$$

"If you can use $A \rightarrow B$ to prove A, and you can use B to prove C, then you can use $A \rightarrow B$ to prove C"

The formula $A \rightarrow B$ is repeated in the left subgoal, which is useful in proving contraction for this G3ip. In fact, Gentzen's original sequent calculus contained the rule… (notice the separate contexts and no repeated A)

$$L\rightarrow \frac{\Gamma \Rightarrow A \qquad B, \Delta \Rightarrow C}{A \rightarrow B, \Delta, \Gamma \Rightarrow C}$$

The shared contexts are needed to get a contraction-free calculus

No structural rules need to be *assumed* in G3ip, but each of them is admissible

Proving the admissibility of the structural rules is a good way to get a handle on the core mechanics of structural proof theory

The structural rules are…

$$\frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C}\ \text{Wk} \qquad \frac{A, A, \Gamma \Rightarrow C}{\Gamma, A \Rightarrow C}\ \text{Ctr}$$

$$\frac{\Gamma \Rightarrow A \qquad A, \Gamma \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\ \text{Cut}$$

Weakening is admissible because it is built into the axiom and $L\bot$ rules

$$\dfrac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C}\ \text{Wk}$$

$$\textbf{Axiom}\ \dfrac{}{P, \Gamma \Rightarrow C} \qquad\qquad L\bot\ \dfrac{}{\bot\ , \Gamma \Rightarrow C}$$

For working the proofs themselves, we'll follow Chapters 2.3 and 2.4 of the book, so these slides end here for now…