Qualifying Exam 2019 Solution

Davis Silverman
Yihao Sun
Mengyu Liu

1

Given the small-step semantics $E_1 \hookrightarrow E_2$:
formulate big-step semantics $\Downarrow \subseteq \mathbf{AEXP} \times \mathbb{Z}$

$$\textbf{BNum}\ \frac{}{\underline{n} \Downarrow n} \qquad \textbf{BMult}\ \frac{E_1 \Downarrow n_1 \quad E_2 \Downarrow n_2}{E_1 * E_2 \Downarrow n} n = mult(n1, n2)$$

$$\textbf{BCTrue}\ \frac{E_c \Downarrow n \quad E_t \Downarrow v}{E_c\ ?\ E_t : E_f \Downarrow v} n \neq 0 \qquad \textbf{BCFalse}\ \frac{E_c \Downarrow n \quad E_f \Downarrow v}{E_c\ ?\ E_t : E_f \Downarrow v} n = 0$$

2

$$\frac{\dfrac{\underline{5}\Downarrow 5 \quad \underline{0}\Downarrow 0}{\underline{5}*\underline{0}\Downarrow 0}\, 0{=}mult(5,0)\ \textbf{BMult} \qquad \dfrac{\underline{7}\Downarrow 7 \quad \dfrac{\underline{3}\Downarrow 3 \quad \underline{4}\Downarrow 4}{\underline{3}*\underline{4}\Downarrow 12}\, 12{=}mult(3,4)\ \textbf{BMult}}{\underline{7}*\underline{3}*\underline{4}\Downarrow 84}\, 84{=}mult(7,12)\ \textbf{BMult}}{\underline{5} * \underline{0}\ ?\ \underline{2} * \underline{6} : \underline{7} * \underline{3} * \underline{4} \Downarrow 84}\textbf{BCFalse}$$

3

Claim: If $E \hookrightarrow E' then |E| > |E'|$

Proof by induction on E.

E has 3 different possible states: $\underline{n}$, $E_1 * E_2$, and $E_0 \mathbin{?} E_1 : E_2$.

When E is $\underline{n}$, there are no rules such that $\underline{n} \hookrightarrow E'$.

When E is $E_1 * E_2$, there are 5 rules that may apply.

- **Mult**

$\underline{n_1} * \underline{n_2} \hookrightarrow \underline{n}$
$\Rightarrow |\underline{n_1}| + |\underline{n_2}| + 1 > |\underline{n}| \leftrightarrow 1 > 0. \checkmark$

- **$\mathbf{Left_0}$**

$\underline{0} * \underline{n_2} \hookrightarrow \underline{0}$
$\Rightarrow |0| + |\underline{n_2}| + 1 > |0| \leftrightarrow 1 > 0. \checkmark$

- **$\mathbf{Right_0}$**

$\underline{n_1} * \underline{0} \hookrightarrow \underline{0}$
$\Rightarrow |\underline{n_1}| + |0| + 1 > |0| \leftrightarrow 1 > 0. \checkmark$

- **Left**

$E_1 * E_2 \hookrightarrow E_1' * E_2$ given $E_1 \hookrightarrow E_1'$
and Inductive Hypothesis $|E_1| > |E_1'|$
$\Rightarrow |E_1| + |E_2| + 1 > |E_1'| + |E_2| + 1 \leftrightarrow |E_1| > |E_1'|$.
This is true by the Inductive Hypothesis $\checkmark$

- **Right**

$E_1 * E_2 \hookrightarrow E_1 * E_2'$ given $E_2 \hookrightarrow E_2'$
and Inductive Hypothesis $|E_2| > |E_2'|$
$\Rightarrow |E_1| + |E_2| + 1 > |E_1| + |E_2'| + 1 \leftrightarrow |E_2| > |E_2'|$.
This is true by the Inductive Hypothesis

When E is $E_0 \mathbin{?} E_1 : E_2$, there are 3 rules that may apply. ✓

- **Cond**

$E_0 \mathbin{?} E_1 : E_2 \hookrightarrow E_0' \mathbin{?} E_1 : E_2$ given $E_0 \hookrightarrow E_0'$
and Inductive Hypothesis $|E_0| > |E_0'|$
$\Rightarrow |E_0| + |E_1| + |E_2| + 1 > |E_0'| + |E_1| + |E_2'| + 1 \leftrightarrow |E_0| > |E_0'|$.
This is true by the Inductive Hypothesis ✓

- **Cond$_{\mathbf{Z}}$**

$\underline{n} \mathbin{?} E_1 : E_2 \hookrightarrow E_2$ where $n = 0$
$\Rightarrow |\underline{n}| + |E_1| + |E_2| + 1 > |E_2| \leftrightarrow |E_1| + |E_2| + 1 > |E_2|$
This is true mathematically. ✓

- **Cond$_{\mathbf{NZ}}$**

$\underline{n} \mathbin{?} E_1 : E_1 \hookrightarrow E_2$ where $n \neq 0$
$\Rightarrow |\underline{n}| + |E_1| + |E_2| + 1 > |E_1| \leftrightarrow |E_1| + |E_2| + 1 > |E_1|$
This is true mathematically. ✓

This shows that in all cases, $E \hookrightarrow E' \rightarrow |E| > |E'|$. □

4

The size of an expression provides an upper bound on the amount of computations required to evaluate the expression.

If evaluation of an expression is monotonically decreasing (as proven in question 3) then the size is finite. Therefore, there is always an amount of computations required to compute the expression, meaning that it is guaranteed to terminate.