Homework Statystical Physics - Set 2

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SET 2

Probelm 3

Prove Carnot's theorem: No engine operating between two heat reservoirs can be more efficient than a Carnot engine operating between the same reservoirs.

(1)
$$\eta_I = \frac{W}{Q_H} = 1 - \frac{T_C}{T_H}$$

W is the work done by the system (energy exiting the system as work), Q_H is the heat put into the system (heat energy entering the system), T_C is the absolute temperature of the cold reservoir, and T_H is the absolute temperature of the hot reservoir.

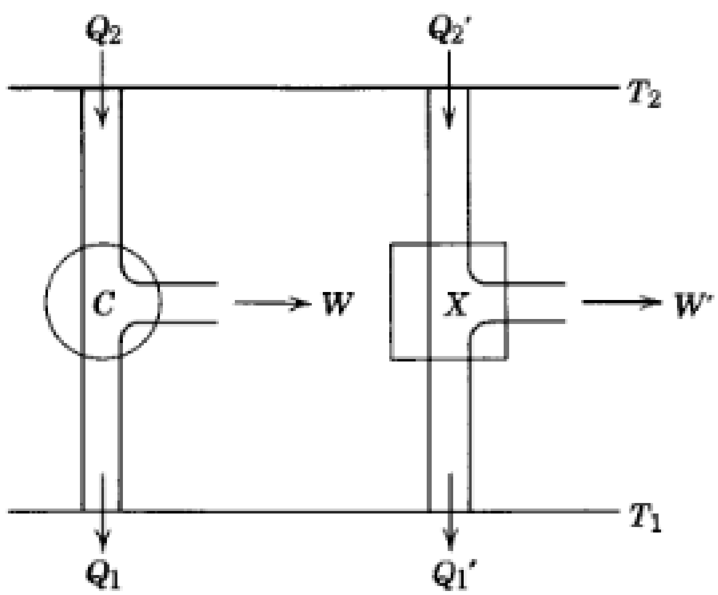


Figure.1: source: Kerson Huang - Statistical mechanics - Wiley (1987), pg. 12, Fig 1.5.

II LAW OF THERMODYNAMICS:

Kelvin Statement. *There exists no thermodynamic transformation whose sole effect is to extract a quantity of heat from a given heat reservoir and to convert it entirely into work.*

First let us assume:

(2)
$$\eta_I = \frac{W}{Q_2} > \eta_R = \frac{W^{'}}{Q_2^{'}},$$

where η_R represents the efficiency of a reversable carnot engine, η_I is the efficiency of an irreversible engine.

Let us first examine what happens when both engines recieve the same amount of energy $Q_2 = Q_2^{'}$. We can see from (2) that the aforementioned condition results in $W > W^{'} \Rightarrow W - W^{'} > 0$.

We will be able to prove Carnot's theorem by contradiction if we replace engine X from Figure.1 with a heat pomp recieving $Q_1^{'}$ from the sink and releasing $Q_2^{'}$ to the source.

From the first law of thermodynamics we have the expression for work in each cycle (for both engines C, X):

(2)
$$W = Q_2 - Q_1,$$

(3)
$$W^{'} = Q_2^{'} - Q_1^{'}.$$

Consequently, after N cycles of engine C and $N^{'}$ cycles (in reverse - we reverse the cycle of X replacing it with a heat pump) of engine X. The total work done by both engines can be written as:

(4)
$$W_{\text{total}} = W - W^{'},$$

(5)
$$W_{\text{total}} = N(Q_2 - Q_1) - N^{'}(Q_2^{'} - Q_1^{'}),$$

It is convenient to group quantities (with and without prime) together in the following way:

(6)
$$(Q_2)_{\text{total}} = NQ_2 - N^{'}Q_2^{'} = 0,$$

(we chose engines such that $Q_2 = Q_2^{'}$)

(7)
$$(Q_1)_{\text{total}} = NQ_1 - N^{'}Q_1^{'}.$$

Then we will be able to write:

(8)
$$W_{\text{total}} = (Q_2)_{\text{total}} - (Q_1)_{\text{total}} = -(Q_1)_{\text{total}}.$$

For $W_{\text{total}} > 0$ the system of these two engines behaves like an engine whose sole purpose is to convert the quantity of heat absorbed from heat reservoir (T_1) entirely into work. Which stands in clear contradiction to the Kelvin's statement. Hence, our assumption that $\eta_I > \eta_R$ must be wrong. This formulation is synonymous with the statement that no engine operating between two heat reservoirs can be more efficient than a Carnot engine operating between these two reservoirs.

We already know that W_{total} cannot be positive and instead must be $W_{total} \leq 0$. From where we derive that: (9)

$$\frac{Q_1'}{Q_2'} \leq \frac{Q_1}{Q_2},$$

which is to say that

(9)

$$\eta_I = \left(1 - \frac{Q_1}{Q_2}\right) \leq \eta_R = \left(1 - \frac{Q_1'}{Q_2'}\right).$$

Consequently, all possible engines have lower efficiency the Carnot engine and all Carnot engines operating between the same two temperatures have the same efficiency.

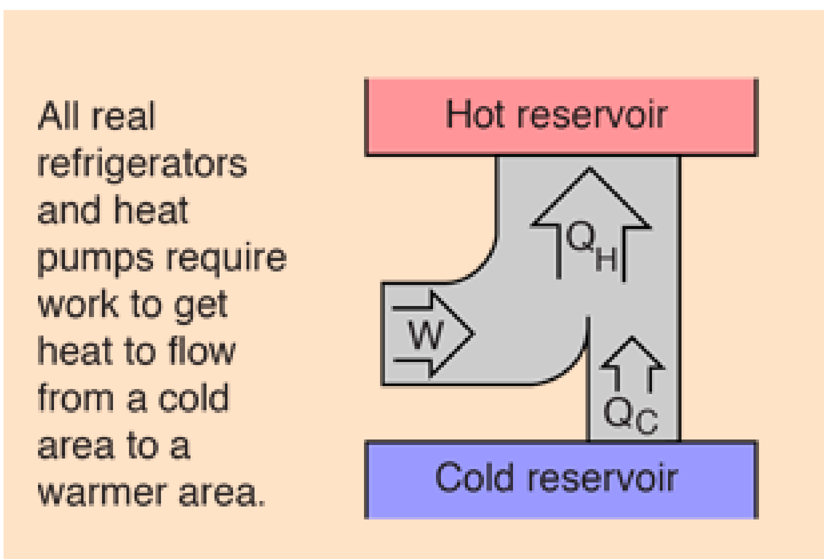
Problem 5 b).

Find the efficiency of reversed Carnot's cycle as exemplified by a heat pump where the efficiency is defined as:

(10)

$$\eta_I = \frac{Q_H}{W},$$

where Q_H is the amount of heat transferred to the room and W is the work. Calculate the efficiency as a function of minimal and maximal temperatures of the cycle: T_{min} i T_{max}.



source: http://hyperphysics.phy-astr.gsu.edu/hbase/thermo/heatpump.html

First of all let us notice that:

(11)
$$W = Q_H - Q_C,$$

where Q_C is the energy transferred from the cold reservoir.

Consquently, for a heat pump we have:

(12)
$$\eta_I = \frac{Q_H}{Q_H - Q_C}.$$

Now if we analyse the change of entropy for a reversable engine we get:

(13)
$$\Delta S = \Delta S_C + \Delta S_H = \frac{|Q_C|}{T_C} - \frac{|Q_H|}{T_H} = 0,$$

where T_C is T_{min}, T_H is T_{max}.

From which, we infer that:

(14)
$$\frac{|Q_C|}{|Q_H|} = \frac{T_C}{T_H}.$$

If we combine (13) and (14) we will end up with:

(15)
$$\eta_I = \frac{1}{1 - \frac{T_C}{T_H}} = \frac{T_H}{T_H - T_C} = \frac{T_{max}}{T_{max} - T_{min}}.$$