# THE SUBCOMPLETENESS OF DIAGONAL PRIKRY FORCING

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Abstract.

## 1. Introduction

## 2. Preliminaries

Jensen [2, Section 3.3] shows that Prikry forcing and Namba forcing (under $\mathsf{CH}$) are subcomplete. Below we use an adaptation of Jensen's proof showing that Prikry forcing is subcomplete to see that some kinds of generalized diagonal Prikry forcing, in particular those Prikry forcings that we refer to here as generalized diagonal Prikry forcing, studied by Fuchs [1], are subcomplete.

Before defining subcomplete forcing and showing consequences of it, some preliminary information is necessary. This can all be found in Jensen's lecture notes from the 2012 AII Summer School in Singapore. For the published version refer to [2]. I will also refer to unpublished handwritten notes from Jensen's website.

First, a brief outline of some of the notation used in what follows:

- Forcing notions $\mathbb{P} = \langle \mathbb{P}, \leq \rangle$ are taken to be partial orders that are separative and contain a "top" element weaker than all elements of $\mathbb{P}$, denoted $\mathbb{1}$.
- We will be working with transitive models of $\mathsf{ZFC}^-$, the axioms of Zermelo-Fraenkel Set Theory without the axiom of $\mathsf{Powerset}$, and with the axiom of $\mathsf{Collection}$ instead of $\mathsf{Replacement}$. Usually we will name these models $N$, $M$, or $\overline{N}$.
- I will follow Jensen to use the notation $\sigma : N \prec M$ for when $\sigma$ is an elementary embedding, but I will add some notation not used by Jensen, letting $N \preccurlyeq M$ denote and emphasize that $N$ is an elementary substructure of $M$.
- Let $N$ be a transitive $\mathsf{ZFC}^-$ model. I write height$(N)$ to mean $\mathrm{Ord} \cap N$. Let $\alpha$ be an ordinal. Then I write $\alpha^N$ for $\alpha \cap N$.
- Let $A$ be a set of ordinals. Then $\lim(A)$ is the set of limit points in $A$.
- Let $\theta$ be a cardinal. $H_\theta$ refers to the collection of sets hereditarily of size less than $\theta$. Relativizing the concept to a particular model of set theory, $M$, I write $H_\theta^M$ to mean the collection of sets in $M$ that are hereditarily of size less than $\theta$ in $M$. In this case, if $\theta$ is determined by some computation, I mean for that computation to take place in $M$.
- Let $\tau$ be a cardinal. With abuse of notation we write $L_\tau[A]$ to refer to the structure $\langle L_\tau[A]; \in, A \cap L_\tau[A] \rangle$.
- If $\Gamma$ is a class of forcings that preserve stationary subsets of $\omega_1$, then the $\Gamma$-forcing axiom ($\mathsf{FA}_\Gamma$) posits that for all $\mathbb{P} \in \Gamma$, if $\mathcal{D}$ is a collection of $\omega_1$-many dense subsets of $\mathbb{P}$, then there is a $\mathcal{D}$-generic filter (a filter that intersects every element of $\mathcal{D}$ nontrivially). We write $\mathsf{MA}$ for Martin's Axiom, $\mathsf{PFA}$ for the Proper Forcing Axiom, and $\mathsf{MM}$ for Martin's Maximum, the forcing axiom for classes of forcing notions that preserve stationary subsets of $\omega_1$.

- We often make use of the following abbreviation: if a map $\sigma$ satisfies $\sigma(\overline{a}) = a$ and $\sigma(\overline{b}) = b$, we write $\sigma(\overline{a}, \overline{b}) = a, b$.

## 3. The Weight of a Forcing Notion

**Definition 3.1.** For a forcing notion $\mathbb{P}$, we write $\delta(\mathbb{P})$ to denote the least cardinality of a dense subset in $\mathbb{P}$. This is sometimes referred to as the ***weight*** of a poset.

Although Jensen defines $\delta(\mathbb{P})$ for Boolean algebras, it's also relevant for posets. As Jensen states, for forcing notions $\mathbb{P}$, the weight can be replaced with the cardinality of $\mathbb{P}$, or even $\mathbb{P}$, for the purpose of defining subcompleteness. However, $\delta(\mathbb{P})$ and $|\mathbb{P}|$ are not necessarily the same, since there could be a large set of points in the poset that all have a common strengthening.

More can be said about the weight of forcing notions, and I give some basic results below.

**Lemma 3.2.** *Let $\mathbb{P}$ be a poset. Then maximal antichains in $\mathbb{P}$ have size at most $\delta(\mathbb{P})$.*

*Proof.* Supposing $\mathbb{P}$ has a maximal antichain $A$ of size $\kappa$, any dense set $D$ (including one of smallest size) in $\mathbb{P}$ needs to have the property that for each element $a \in A$, there is an element $d \in D$ below $a$, satisfying $d \leq a$. Let's assume that $D$ has size $\delta(\mathbb{P})$; then $D$ must have size at least $\kappa$ since $A$ is a maximal antichain; indeed we have a function $f : A \to D$, where $f(a)$ is an element of $D$ below $a$. It must be that $f$ is an injection, since otherwise there would be an element $d \in D$ below two distinct elements of $A$, contradicting the fact that $A$ is an antichain. Thus $\kappa = |A| \leq |D| = \delta(\mathbb{P})$ as desired. $\square$

Below we define a homomorphism between partial orders, following Schindler [3, Definition 6.47].

**Definition 3.3.** Let $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle$ be posets. A map $\pi : \mathbb{P} \to \mathbb{Q}$ is a ***homomorphism*** so long as it preserves order and incompatibility:

1. For all $p, q \in \mathbb{P}$ we have that $p \leq_{\mathbb{P}} q \implies \pi(p) \leq_{\mathbb{Q}} \pi(q)$.
2. For all $p, q \in \mathbb{P}$ we have that $p \perp q \implies \pi(p) \perp \pi(q)$.

A homomorphism $\pi : \mathbb{P} \to \mathbb{Q}$ is said to be ***dense*** so long as for every $q \in \mathbb{Q}$ there is some $p \in \mathbb{P}$ such that $\pi(p) \leq q$, i.e., range$(\pi)$ is dense in $\mathbb{Q}$.

We give some immediate remarks on our above definition. First of all, we claim property **1** automatically entails that for all $p, q \in \mathbb{P}$, if $\pi(p) \perp \pi(q)$ then $p \perp q$, since clearly if $p$ is compatible with $q$ we have that $\pi(p)$ is compatible with $\pi(q)$, which is the contrapositive of the claim. Furthermore, if $\mathbb{P}$ is separative (which indeed we assume of all of our forcing notions) then homomorphisms are automatically *strong*, in that the implications of items **1** and **2** may both be reversed and we have the following:

1. For all $p, q \in \mathbb{P}$ we have that $p \leq_{\mathbb{P}} q \iff \pi(p) \leq_{\mathbb{Q}} \pi(q)$.
2. For all $p, q \in \mathbb{P}$ we have that $p \perp q \iff \pi(p) \perp \pi(q)$.

Item **1** holds since if otherwise, meaning if $\pi(p) \leq \pi(q)$ for $p, q \in \mathbb{P}$ and $p$ is not stronger than $q$, then there is a $p^* \leq p$ that is incompatible with $q$ by separativity, which means $\pi(p^*) \leq p$ and $\pi(p^*) \perp \pi(q)$, a contradiction. Thus, if $\mathbb{P}$ is a forcing notion, by property **1**, homomorphisms from $\mathbb{P}$ to $\mathbb{Q}$ are automatically embeddings.

**Lemma 3.4.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions. If $\pi : \mathbb{P} \to \mathbb{Q}$ is a dense homomorphism (which means $\pi$ is a dense embedding, as explained above) then $\delta(\mathbb{P}) = \delta(\mathbb{Q})$.*

*Proof.* Suppose that $D \subseteq \mathbb{P}$ is dense and $|D| = \delta(\mathbb{P})$. Then $\pi``\mathbb{P}$ is dense in $\mathbb{Q}$ and $\pi``D$ is dense in $\pi``\mathbb{P}$, since $\pi$ is a homomorphism and preserves the partial order. This means that $\pi``D$ is dense in $\mathbb{Q}$. Any other dense set in $\mathbb{Q}$ has to have size at least $\delta(\mathbb{Q})$, and thus $\delta(\mathbb{Q}) \leq |\pi``D| \leq |D| = \delta(\mathbb{P})$ as desired.

We now use the remarks showing that our homomorphisms are strong to find a dense subset of $\mathbb{P}$ that has size at most $\delta(\mathbb{Q})$. Let $\Delta \subseteq \mathbb{Q}$ be dense satisfying $|\Delta| = \delta(\mathbb{Q})$. Since $\pi``\mathbb{P}$ is dense in $\mathbb{Q}$, for each $q \in \Delta$ there is a $p_q \in \mathbb{P}$ such that $\pi(p_q) \leq q$. Let $D^*$ be the set of such $p_q$'s, where there is only one $p_q$ chosen for each $q \in \Delta$. To show that $D^*$ is dense, let $p \in \mathbb{P}$. Then as $\Delta$ is dense, there is $q \in \Delta$ satisfying $q \leq \pi(p)$. Thus there is $\pi(p_q) \leq q \leq \pi(p)$ where $p_q \in D^*$, so we have $p_q \leq p$ showing that $D^*$ is dense. This means that $\delta(\mathbb{P}) \leq \delta(\mathbb{Q})$ as desired. □

If $\mathbb{P}$ is a dense subset of $\mathbb{Q}$, then, of course, $\delta(\mathbb{P}) = \delta(\mathbb{Q})$.

## 4. Fullness

In the definition of subcompleteness, in lieu of working directly with $H_\theta$ and its well-order for "large enough" cardinals $\theta$ as our standard setup (as is often done for proper forcing, for example), I will follow Jensen and work with models $N$ of the form:

$$H_\theta \subseteq N = L_\tau[A] \models \mathsf{ZFC}^-,$$

where $\tau > \theta$ is a cardinal that is not necessarily regular. Such $H_\theta$ will need to be large enough so that $N$ has the correct $\omega_1$ and $H_{\omega_1}$. One justification for working with these models is that such $N$ will naturally contain a well order of $H_\theta$, along with its Skolem functions and other useful bits of information we would like to have at our disposal. Additionally a benefit of working with models of the form $L_\tau[A]$ is that $L_\tau[A]$ is easily definable in $L_\tau[A][G]$, if $G$ is generic, using $A$.

In the standard fashion we will look at countable elementary substructures $X$ of the $N$ as above; $X \preccurlyeq N$. We then take the countable transitive collapse of such an $X$, and write $\overline{N} \cong X$. We will refer to these embeddings by writing

$$\sigma : \overline{N} \cong X \preccurlyeq N.$$

Often we will write

$$\sigma : \overline{N} \prec N$$

to suppress mention of $X$, the range of $\sigma$.

In fact, for our purposes it will not be quite enough for such an $\overline{N}$ to be transitive, we need a bit more, exactly given by the property of *fullness*. Fullness of a model ensures that it is not pointwise definable, so that there can be many elementary maps between the smaller structure and the larger structure $N$ in our setup. Before defining fullness exactly, let us give some more of the concepts and definitions we will be working with.

For the embeddings above that we will be working with, it is not hard to see what the critical point is. Given $\sigma : \overline{N} \prec N$ where $\overline{N}$ is countable and transitive, $\mathrm{cp}(\sigma)$ is exactly $\omega_1^{\overline{N}} = \sigma^{-1}(\omega_1^N)$, since $\overline{N}$ is countable.

**Fact 4.1.** *Let $\overline{N}$, $N$ be transitive $\mathsf{ZFC}^-$ models, where $\overline{N}$ is countable and $H_{\omega_1} \subseteq N$, with $\sigma : \overline{N} \prec N$. Then $\mathrm{cp}(\sigma) = \omega_1^{\overline{N}}$ and $\sigma \restriction (H_{\omega_1})^{\overline{N}} = \mathrm{id}$.*

*Proof.* Let $\alpha = \mathrm{cp}(\sigma)$. Then $\alpha$ is a cardinal in $\overline{N}$, and $\alpha > \omega$ as both $\overline{N}$ and $N$ are models of $\mathsf{ZFC}^-$.

It must be that $\alpha = \omega_1^{\overline{N}}$, since $\omega_1^{\overline{N}}$ is a countable ordinal in $V$ and thus in $N$, but $\sigma(\omega_1^{\overline{N}}) = \omega_1^N > \omega_1^{\overline{N}}$. Every $x \in H_\alpha^{\overline{N}} = (H_{\omega_1})^{\overline{N}}$ may be coded as a subset of $\omega$ and thus as an ordinal less than $\omega_1$ by $\gamma = \mathrm{TC}(\{x\})$, and $\sigma(\gamma) = \gamma$, and the desired result follows. □

Let $N = L_\tau[A]$ for some cardinal $\tau$ and set $A$, be a transitive $\mathsf{ZFC}^-$ model, let $X$ be a set, and let $\delta$ be a cardinal. Our notation for the ***Skolem hull***, in $N$, closing under $\delta \cup X$, is the following:

$$\mathcal{S}k^N(\delta \cup X) = \text{the smallest } Y \preccurlyeq N \text{ satisfying } X \cup \delta \subseteq Y.$$

We gather two immediate, basic results that we will refer to later below.

**Lemma 4.2.** *Let $N = L_\tau[A]$ be a transitive $\mathsf{ZFC}^-$ model, $\delta$ and $\gamma$ be cardinals, and $X$ a set. If $\delta \leq \gamma$ then $\mathcal{S}k^N(\delta \cup X) \subseteq \mathcal{S}k^N(\gamma \cup X)$.*

*Proof.* This is trivial; if $t \in \mathcal{S}k^N(\delta \cup X)$ then $t$ is $N$-definable from some $\xi < \delta \leq \gamma$ and $\vec{x} \in X$ so $t \in \mathcal{S}k^N(\gamma \cup X)$ as well. $\square$

**Lemma 4.3.** *Let $N$ be a transitive $\mathsf{ZFC}^-$ model, $\delta$ a cardinal, and $X$, $Y$ sets. If $\mathcal{S}k^N(\delta \cup X) = \mathcal{S}k^N(\delta \cup Y)$ then $\mathcal{S}k^N(\gamma \cup X) = \mathcal{S}k^N(\gamma \cup Y)$ for all $\gamma \geq \delta$.*

*Proof.* Let $t \in \mathcal{S}k^N(\gamma \cup X)$. Then there is some formula $\varphi$ where $t$ unique such that $N \models \varphi(t, \xi, x)$ for $\xi < \gamma$, $x \in X$. But since then $x \in X \subseteq \mathcal{S}k^N(\delta \cup X) = \mathcal{S}k^N(\delta \cup Y) \subseteq \mathcal{S}k^N(\gamma \cup Y)$, it must be the case that $t \in \mathcal{S}k^N(\gamma \cup Y)$ as well. Likewise for the reverse inclusion. $\square$

Before defining fullness, we need one more definition.

**Definition 4.4.** We say that a transitive model $N$ is ***regular*** in a transitive model $M$ so long as for all functions $f : x \longrightarrow N$, where $x$ is an element of $N$ and $f \in M$, we have that $f``x \in N$.

To elucidate the definition further, we immediately have the following lemma:

**Lemma 4.5.** *Let $N, M \models \mathsf{ZFC}^-$ be transitive. $N$ is regular in $M$ iff $N = H_\gamma^M$, where $\gamma = \text{height}(N)$ is a regular cardinal in $M$.*

*Proof.* For the backward direction, suppose that $N = H_\gamma^M$ where $\gamma = \text{height}(N)$ is a regular cardinal in $M$. Then for all $f : x \longrightarrow N$, with $x \in N$ and $f \in M$, certainly $f``x \in N$ as well.

For the forward direction, indeed $\gamma$ has to be regular in $M$ since otherwise $M$ would contain a cofinal function $f : \alpha \longrightarrow \gamma$ where $\alpha < \gamma$. By the transitivity of $N$, this implies that $\alpha \in N$. Thus $\cup f``\alpha$ is in $N \models \mathsf{ZFC}^-$ by regularity, so $\gamma \in N$, a contradiction. We have that $N \subseteq H_\gamma^M$ since $N$ is a transitive $\mathsf{ZFC}^-$ model, so the transitive closure of elements of $N$ may be computed in $N$ and thus have size less than $\gamma$, so they are in $H_\gamma^M$ as $\gamma \in M$. To show that $H_\gamma^M \subseteq N$, let $x \in H_\gamma^M$. We assume by $\in$-induction that $x \subseteq N$. Then there is a surjection $f : \alpha \twoheadrightarrow x$ where $\alpha < \gamma$, in $M$. Hence by regularity, $x = f``\alpha \in N$ as desired. $\square$

We now define fullness.

**Definition 4.6.** A structure $M$ is ***full*** so long as $M$ is transitive, $\omega \in M$, and there is a $\gamma$ such that $M$ is regular in $L_\gamma(M)$ where $L_\gamma(M) \models \mathsf{ZFC}^-$.

Perhaps this property seems rather mysterious, but the fullness of a countable structure guarantees that it is not pointwise definable, which we will see is a necessary property of some of the models that come up in the definition subcomplete forcing.

**Lemma 4.7.** *If $M$ is countable and full, then $M$ is not pointwise definable.*

*Proof.* Suppose toward a contradiction that $M$ is countable, full, and pointwise definable. By fullness there is some $L_\gamma(M) \models \mathsf{ZFC}^-$ such that $M$ is regular in $L_\gamma(M)$. By pointwise definability, for each element $x \in M$, we have attached to it some formula $\varphi(x)$ such that $M \models \varphi(x)$ uniquely, meaning that $\varphi(y)$ fails for every other element $y \in M$. Thus in $L_\gamma(M)$ we may define a function

$f : \omega \cong M$, that takes the $n$th formula in the language of set theory to its unique witness in $M$. In particular we have that $L_\gamma(M)$ witnesses that $M$ is countable. However, this would allow $M$ to witness its own countability, since $M$ must contain $f$ as well by regularity. $\square$

Let $N$ be a transitive $\mathsf{ZFC}^-$ model. Suppose $\overline{N} \cong X \preccurlyeq N$ where $X$ is countable and $\overline{N}$ is full. Then as we will find, there may possibly be more than one elementary embedding $\sigma : \overline{N} \prec N$. If there was only one unique embedding, we would be able to define $\overline{N}$ pointwise, by elementarity of the unique map and since $\overline{N}$ is countable.

The following lemma shows that fullness is in some sense not much harder than transitivity to satisfy.

**Lemma 4.8.** *Let $\theta > \omega_1$ satisfy $H_\theta \subseteq N = L_\tau[A] \models \mathsf{ZFC}^-$ with $\tau > \theta$ regular and $A \subseteq \tau$, and let $s \in N$. Then*

$$\left\{\omega_1^{\overline{N}} \mid \textit{there is } \sigma \textit{ such that } \sigma : \overline{N} \prec N \textit{ where } \overline{N} \textit{ is countable and full, and } s \in \mathrm{range}(\sigma)\right\} \subseteq \omega_1$$

*contains a club.*

*Proof.* Let $\tau' = (\tau^+)^{L[A]}$. Let $\sigma' : \overline{L_{\tau'}[A]} \cong X \preccurlyeq L_{\tau'}[A]$ where $X$ is countable and $L_{\overline{\tau'}}[\overline{A}]$ is the Mostowski collapse of $X$. Let $\overline{\tau}$ be the largest cardinal of $L_{\overline{\tau'}}[\overline{A}]$. We would like to show that $\overline{N} = L_{\overline{\tau}}[\overline{A}]$ is full, by showing it is regular in $L_{\overline{\tau'}}[\overline{A}] = L_{\overline{\tau'}}(\overline{N})$.

*Claim.* $\overline{N}$ is regular in $L_{\overline{\tau'}}[\overline{A}]$.

*Pf.* Let $f : x \longrightarrow \overline{N}$, where $x \in \overline{N}$ and $f \in L_{\overline{\tau'}}[\overline{A}]$, and indeed $\mathrm{range}(f) = f``x \in L_{\overline{\tau'}}[\overline{A}]$. Firstly, $f``x \subseteq L_\gamma[\overline{A}]$, for some $\gamma < \overline{\tau}$, because $\overline{\tau}$ is a regular cardinal in $L_{\overline{\tau'}}[A]$. We inductively define a sequence of Skolem hulls in order to see that ultimately $f``x$ must be an element of $\overline{N}$. Let

$$X_0 = \mathcal{S}k^{L_{\overline{\tau'}}[\overline{A}]}(\gamma \cup \{f``x\}).$$

In $L_{\overline{\tau'}}[\overline{A}]$, $X_0$ has size $\gamma$, so in particular it has size less than $\overline{\tau}$. Also we have that $L_\gamma[\overline{A}] \subseteq X_0$. As an aside, we point out that one might already want to take the transitive collapse of this structure to obtain something that looks like $L_{\overline{\overline{\tau'}}}[\overline{\overline{A}}]$. The $\overline{\overline{A}}$ arising from this is not very easy to work with; it is not necessarily true that in this case, $\overline{\overline{A}} = \overline{A} \cap \overline{\overline{\tau'}}$ as one would hope. To remedy this we will go on to define a hull whose transitive collapsed version of $\overline{A}$ is an initial segment of $\overline{A}$, ultimately making the collapsed structure definable in $L_{\overline{\tau}}[\overline{A}]$. To do this, inductively assume $X_n$ is defined with size less than $\overline{\tau}$, and set $\gamma_n = \sup(X_n) \cap \overline{\tau}$. We have that $|\gamma_n| < \overline{\tau}$, as $\overline{\tau}$ is regular in $L_{\overline{\tau'}}[\overline{A}]$. We let

$$X_{n+1} = \mathcal{S}k^{L_{\overline{\tau'}}[\overline{A}]}(\gamma_n \cup X_n).$$

This defines an elementary chain $\langle X_n \mid n < \omega\rangle$ in $L_{\overline{\tau'}}[\overline{A}]$. So $X_\omega = \bigcup_{n<\omega} X_n$ is an elementary substructure of $L_{\overline{\tau'}}[\overline{A}]$. Additionally, let $\gamma_\omega := X_\omega \cap \overline{\tau} = \sup_{n<\omega} \gamma_n$.

Let $k : \overline{L_{\overline{\tau'}}[\overline{A}]} \cong X_\omega \preccurlyeq L_{\overline{\tau'}}[\overline{A}]$. Then $\overline{L_{\overline{\tau'}}[\overline{A}]} = L_{\overline{\overline{\tau'}}}[\overline{\overline{A}}]$ and by construction, $k \restriction \gamma_\omega = \mathrm{id}$. Additionally we now have

$$\overline{\overline{A}} = k^{-1}``\overline{A} = k^{-1}``(\overline{A} \cap \gamma_\omega) = \overline{A} \cap \gamma_\omega.$$

Since $L_\gamma[\overline{A}] \subseteq X_0$, we have that $k^{-1}(f``x) = f``x$. Thus $f``x \in L_{\overline{\overline{\tau'}}}[\overline{A} \cap \gamma_\omega] \in L_{\overline{\tau}}[\overline{A}] = \overline{N}$, since $\overline{\overline{\tau'}} < \tau$ and $\overline{A} \cap \gamma_\omega < \overline{\tau}$. Thus $\overline{N}$ is regular in $L_{\overline{\tau'}}[\overline{A}]$, proving the *Claim.* $\square$

Thus $\overline{N}$ is full. To show the claim, we need to show that there are club-many such $\overline{N}$s. But this is true because there are club-many relevant Skolem hulls from which the $\overline{N}$ arise.

In particular, let

$$\Gamma = \Big\{ \overline{N} = L_{\overline{\tau}}[\overline{A}] \mid \text{ for some } \gamma < \omega_1,\, L_{\overline{\tau}'}[\overline{A}] \cong \mathcal{S}k^{L_{\tau'}[A]}(\gamma \cup \{s\}) \preccurlyeq L_{\tau'}[A] \\ \text{where } \overline{\tau} \text{ is the largest cardinal in } L_{\overline{\tau}'}[\overline{A}] \Big\}.$$

Then $C = \Big\{ \omega_1^{\overline{N}} \mid \overline{N} \in \Gamma \Big\} \subseteq \omega_1$ is club.

Clearly $C$ is unbounded, since for any $\gamma < \omega_1$, there is $\sigma' : L_{\overline{\tau}'}[\overline{A}] \cong \mathcal{S}k^{L_{\tau'}[A]}(\gamma \cup \{s\}) \preccurlyeq L_{\tau'}[A]$. Let $\sigma'(\overline{s}) = s$. Since $\overline{\tau} > \omega_1^{\overline{N}}$ by elementarity, we have that $\sigma = \sigma' \restriction L_{\overline{\tau}}[\overline{A}]$ is an elementary embedding as well; $\sigma : \overline{N} \prec N$, and $\sigma(\overline{s}) = s$. Then the critical point of $\sigma$, namely $\omega_1^{\overline{N}}$, is above $\gamma$.

To see that $C$ is closed, suppose we have an unbounded set of of $\overline{N} \in \Gamma$ and an associated sequence of $\alpha = \omega_1^{\overline{N}}$. As we saw above, each of these $\overline{N}$ can be thought of as the domain of some elementary $\sigma : \overline{N} \cong X \preccurlyeq N$, where $X$ is countable. Then we can take the union of these $X$ to obtain $X \preccurlyeq N$. To form $\sigma : \overline{N} \prec N$ take the Mostowski collapse. The critical point of $\sigma$ will be the supremum of all of the $\alpha$'s. □

The following lemma will prove useful when showing that various forcing notions are subcomplete, since subcompleteness requires certain ground-model Skolem hulls to match those in forcing extensions.

**Lemma 4.9** (Jensen)**.** *Let $\mathbb{P}$ be a forcing notion, and let $\delta = \delta(\mathbb{P})$ be the smallest size of a dense subset in $\mathbb{P}$. Suppose that $\mathbb{P} \in H_\theta \subseteq N$ where $N = L_\tau[A] \models \mathsf{ZFC}^-$. Let $\sigma : \overline{N} \prec N$ where $\overline{N}$ is countable and full, and let $\sigma(\overline{\mathbb{P}}) = \mathbb{P}$.*

*Suppose $G \subseteq \mathbb{P}$ is $N$-generic and $\overline{G} \subseteq \overline{\mathbb{P}}$ is $\overline{N}$-generic, and that $\sigma``\overline{G} \subseteq G$, so $\sigma$ lifts (or extends) in $V[G]$ to an embedding $\sigma^* : \overline{N}[\overline{G}] \prec N[G]$.*

*Then*

$$N \cap \mathcal{S}k^{N[G]}(\delta \cup \operatorname{range}(\sigma^*)) = \mathcal{S}k^N(\delta \cup \operatorname{range}(\sigma)).$$

*Proof.* First, we establish that $\mathcal{S}k^N(\delta \cup \operatorname{range}(\sigma)) \subseteq \mathcal{S}k^{N[G]}(\delta \cup \operatorname{range}(\sigma^*)) \cap N$. To see this, let $x \in \mathcal{S}k^N(\delta \cup \operatorname{range}(\sigma))$. Then $x$ is $N$-definable from $\xi < \delta$ and $\sigma(\overline{z})$ where $\overline{z} \in \overline{N}$. Since $\sigma^*$ extends $\sigma$, this means that $x \in N$ and that $x$ is $N[G]$-definable from $\xi$ and $\sigma^*(\overline{z})$. This is because $N = L_\tau[A]$ is definable in $N[G]$ using $A$.

For the other direction, let $x \in \mathcal{S}k^{N[G]}(\delta \cup \operatorname{range}(\sigma^*)) \cap N$. Then $x$ is $N[G]$-definable from $\xi < \delta$ and $\sigma^*(\overline{z})$ where $\overline{z} \in \overline{N}[\overline{G}]$; and also $x \in N$. Letting $\dot{\overline{z}} \in \overline{N}^{\overline{\mathbb{P}}}$ such that $\overline{z} = \dot{\overline{z}}^{\overline{G}}$ we have

$$\sigma^*(\overline{z}) = \sigma^*(\dot{\overline{z}}^{\overline{G}}) = \sigma(\dot{\overline{z}})^G$$

so we have that there is some formula $\varphi$ such that $x$ is the unique witness:

$$x = \text{that } y \text{ where } N[G] \models \varphi(y, \xi, \sigma(\dot{\overline{z}})^G).$$

Take $f \in \overline{N}$ mapping $\overline{\delta}$ onto a dense subset of $\overline{\mathbb{P}}$. Then $\sigma(f)$ maps $\delta$ onto a dense subset of $\mathbb{P}$. Thus there is $\nu < \delta$ such that $\sigma(f)(\nu) \in G$ and $\sigma(f)(\nu) \Vdash \varphi(\check{x}, \check{\xi}, \sigma(\dot{\overline{z}}))$. Thus

$$x = \text{that } y \text{ where } \sigma(f)(\nu) \Vdash_{\mathbb{P}}^N \varphi(\check{y}, \check{\xi}, \sigma(\dot{\overline{z}}))$$

so $x \in \mathcal{S}k^N(\delta \cup \operatorname{range}(\sigma))$. □

## 5. Barwise Theory

In order to show that many posets are subcomplete, Jensen takes advantage of Barwise Theory and techniques using countable admissible structures to obtain transitive models of infinitary languages. Barwise creates an $M$-finite predicate logic, a first order theory in which arbitrary, but $M$-finite, disjunctions and conjunctions are allowed. The following is an outline of [2, Chapter 1 & 2].

**Definition 5.1.** Let $M$ be a transitive structure with potentially infinitely many predicates. A theory defined over $M$ is ***$M$-finite*** so long as it is in $M$. A theory is $\Sigma_1(M)$, also known as ***$M$-recursively enumerable*** or $M$-*re*, if the theory is $\Sigma_1$-definable, with parameters from $M$.

Of course we can generalize this to the entire usual Levy hierarchy of formulae, but for our purposes we only need to know what it means for a theory to be $\Sigma_1(M)$. If $\mathcal{L}$ is a $\Sigma_1$(M)-definable language or theory, the rough idea is that to check whether a sentence is in $\mathcal{L}$, one should imagine enumerating the formulae of $\mathcal{L}$ to find the sentence and a witness to it in the structure $M$. Below we elaborate on what models we are working with.

**Definition 5.2.** A transitive structure $M$ is ***admissible*** if it models the axioms of Kripke-Platek Set Theory (KP) which consists of the axioms of Empty Set, Pairing, Union, $\Sigma_0$-Collection, and $\Sigma_0$-Separation.

Jensen also makes use of models of $\mathsf{ZF}^-$ that are not necessarily well-founded.

**Definition 5.3.** Let $\mathfrak{A} = \langle A, \in_{\mathfrak{A}}, B_1, B_2, \dots \rangle$ be a (possibly) ill-founded model of $\mathsf{ZF}^-$, where $\mathfrak{A}$ is allowed to have predicates other than $\in$. The ***well-founded core*** of $\mathfrak{A}$, denoted $\mathrm{wfc}(\mathfrak{A})$, is the restriction of $\mathfrak{A}$ to the set of all $x \in A$ such that $\in_{\mathfrak{A}} \cap \mathcal{C}(x)^2$ is well founded, where $\mathcal{C}(x)$ is the closure of $\{x\}$ under $\in_{\mathfrak{A}}$. A model $\mathfrak{A}$ of $\mathsf{ZF}^-$ is ***solid*** so long as $\mathrm{wfc}(\mathfrak{A})$ is transitive and $\in_{\mathrm{wfc}(\mathfrak{A})} = \in \cap \, \mathrm{wfc}(\mathfrak{A})^2$.

Jensen [2, Section 1.2] notes that every consistent set of sentences in $\mathsf{ZF}^-$ has a solid model, and if $\mathfrak{A}$ is solid, then $\omega \subseteq \mathrm{wfc}(\mathfrak{A})$. In addition,

**Fact 5.4** (Jensen)**.** *If $\mathfrak{A} \models \mathsf{ZF}^-$ is solid, then* $\mathrm{wfc}(\mathfrak{A})$ *is admissible.*

**Definition 5.5.** The context for Barwise theory is countable admissible structures. If $M$ is admissible, we work with infinitary, axiomatized theories $\mathcal{L}$ in $M$-finitary logic, called ***$\in$-theories***, with a fixed predicate $\dot\in$ and ***special constants*** denoted $\underline{x}$ for elements $x \in M$. Our underlying axioms for these $\in$-theories will always involve $\mathsf{ZFC}^-$ and some basic axioms ensuring that $\dot\in$ behaves nicely; the Basic Axioms are:

- Extensionality
- A statement positing the extensionality of $\dot\in$, which is a scheme of formulae defined for each member of $M$. For each $x \in M$, include an $M$-*re* sentence (meaning it quantifies over $M$-finite sentence):
$$\forall v \left( v \mathrel{\dot\in} \underline{x} \iff \bigvee\!\!\!\bigvee_{z \in x} v = \underline{z} \right).$$

Here $\bigvee\!\!\!\bigvee$ denotes an infinite disjunction in the language.

An important fact ensured by our Basic Axioms is that the interpretations of these special constants in any solid model of the theory are the same as in $M$:

**Fact 5.6** (Jensen)**.** *Let $M$ be as in the above definition. Let $\mathfrak{A}$ be a solid model of the $\in$-theory $\mathcal{L}$. Then for all $x \in M$, we have that $\underline{x}^{\mathfrak{A}} = x \in \mathrm{wfc}(\mathfrak{A})$.*

*Pf.* Shown by $\in$-induction. □

Jensen uses the techniques of Barwise to come up with a proof system in this context, in which consistency of $\in$-theories can be discussed. In particular, $\in$-theories are correct: if we have a model of such a theory, then it is consistent.

**Fact 5.7** (***Barwise Correctness***)**.** *Let $\mathcal{L}$ be an $\in$-theory. If $A$ is a set of $\mathcal{L}$-statements and $\mathfrak{A} \models A$, then $A$ is consistent.*

Furthermore, compactness and completeness are shown, relativized to the $M$-finite predicate logics that are used here. In our context, for countable admissible structures $M$, we will obtain solid models of consistent $\Sigma_1(M)$ $\in$-theories. In particular, the form of Barwise Completeness that we make use of here is stated below.

**Fact 5.8** (***Barwise Completeness***)**.** *Let $M$ be a countable admissible structure. Let $\mathcal{L}$ be a consistent $\Sigma_1(M)$ $\in$-theory such that $\mathcal{L} \vdash \mathsf{ZF}^-$. Then $\mathcal{L}$ has a solid model $\mathfrak{A}$ such that*

$$\mathrm{Ord} \cap \mathrm{wfc}(\mathfrak{A}) = \mathrm{Ord} \cap M.$$

We will need the following definition, which is a generalization of fullness.

**Definition 5.9.** A transitive $\mathsf{ZFC}^-$ model $N$ is ***almost full*** so long as $\omega \in N$ and there is a solid $\mathfrak{A} \models \mathsf{ZFC}^-$ with $N \in \mathrm{wfc}(\mathfrak{A})$ and $N$ is regular in $\mathfrak{A}$.

Clearly if $N$ is full, then $N$ is almost full.

A useful technique when showing a particular forcing is subcomplete, once many different embeddings can be constructed that approximate the embedding required for subcompleteness, is to be able to transfer the consistency of $\in$-theories over one admissible structure to another.

**Definition 5.10.** If $N$ is a transitive $\mathsf{ZFC}^-$ model, let $\delta_N$ be the least $\delta$ such that $L_\delta(N)$ is admissible.

**Fact 5.11** (***Transfer***)**.** *Suppose that for $M$ admissible, $\mathcal{L}(M)$ is a $\Sigma_1(M)$ infinitary $\in$-theory. Let $N_1$ be almost full, and suppose that $k : N_1 \prec N_0$ cofinally. If both $\mathcal{L}(L_{\delta_{N_1}}(N_1))$ is $\Sigma_1$ over parameters $N_1, p_1, \ldots, p_n \in N_1$ and $\mathcal{L}(L_{\delta_{N_0}}(N_0))$ is $\Sigma_1$ over parameters $N_0, k(p_1), \ldots, k(p_n)$, then if $\mathcal{L}(L_{\delta_{N_1}}(N_1))$ is consistent, it follows that $\mathcal{L}(L_{\delta_{N_0}}(N_0))$ is consistent as well.*

## 6. Liftups

The following definitions are to describe a method to obtain emeddings, a technique that is ostensibly the ultrapower construction. These embeddings facilitate the use of Barwise theory to obtain the consistency of the existence of desirable embeddings. We follow [2, Chapter 1] here.

**Definition 6.1.** Let $\overline{N}$ and $N$ be transitive $\mathsf{ZFC}^-$ models. We say that an elementary embedding $\sigma : \overline{N} \prec N$ is ***cofinal*** so long as for each $x \in N$ there is some $u \in \overline{N}$ such that $x \in \sigma(u)$.

Let $\alpha \in \overline{N}$. We say that $\sigma$ is ***$\alpha$-cofinal*** so long as every such $u$ has size less than $\alpha$ as computed in $\overline{N}$.

**Definition 6.2.** Let $\alpha > \omega$ be a regular cardinal in $\overline{N}$, a transitive $\mathsf{ZFC}^-$ model. Let

$$\overline{\sigma} : H_\alpha^{\overline{N}} \prec H \text{ cofinally,}$$

where $H$ is transitive. By a ***transitive liftup*** of $\langle \overline{N}, \overline{\sigma} \rangle$ we mean a pair $\langle N_*, \sigma_* \rangle$ such that

- $N_*$ is transitive
- $\sigma_* : \overline{N} \prec N_*$ $\alpha$-cofinally
- $\sigma_* \upharpoonright H_\alpha^{\overline{N}} = \overline{\sigma}$

Reminiscent of ultrapowers, transitive liftups can be characterized in the following way:

**Lemma 6.3** (Jensen). *Let $\overline{N}$, $N$ be transitive* $\mathsf{ZFC}^-$ *models with $\sigma : \overline{N} \prec N$. Then, $\sigma$ is $\alpha$-cofinal $\iff$ elements of $N$ are of the form $\sigma(f)(\beta)$ for some $f : \gamma \to \overline{N}$ where $\gamma < \alpha$ and $\beta < \sigma(\gamma)$.*

*Proof.* We show each direction of the equivalence separately.
"$\Longrightarrow$": Let $x \in N$, and take $u \in \overline{N}$ with $x \in \sigma(u)$ such that $|u| < \alpha$ in $\overline{N}$. Let $|u| = \gamma$, and take $f : \gamma \to u$ a bijection in $\overline{N}$. Then $\sigma(f) : \sigma(\gamma) \to \sigma(u)$ is also a bijection in $N$ by elementarity. Since $x \in \sigma(u)$ we also have that $x$ has a preimage under $\sigma(f)$, say $\beta$. So $\sigma(f)(\beta) = x$ as desired.
"$\Longleftarrow$": Let $x = \sigma(f)(\beta)$ be an element of $N$, for $f : \gamma \to \overline{N}$ where $\gamma < \alpha$ in $\overline{N}$ and $\beta < \sigma(\gamma)$. Define $u = f``\gamma$. Then in $\overline{N}$ we have that $|u| < \alpha$. In addition we have that $x \in \sigma(u)$, since $\sigma(u)$ is in the range of $\sigma(f)$, where $x$ lies. $\square$

Furthermore, Jensen shows that transitive liftups exist so long as an embedding already exists, using an ultrapower-like construction, and have a uniqueness property.

**Fact 6.4** (***Interpolation***). *Let $\sigma : \overline{N} \prec N$ with $\overline{N} \models$ $\mathsf{ZFC}^-$ transitive, and let $\alpha \in \overline{N}$ be a regular cardinal. Then:*

1. *The transitive liftup $\langle N_*, \sigma_* \rangle$ of $\langle \overline{N}, \sigma \upharpoonright H_\alpha^{\overline{N}} \rangle$ exists.*
2. *There is a unique $k_* : N_* \prec N$ such that $k_* \circ \sigma_* = \sigma$ and $k_* \upharpoonright \bigcup \sigma``H_\alpha^{\overline{N}} = \mathrm{id}$.*

For the following useful lemma, we will need to define the following, more general, notion of liftups. Of course the rich theory is established by Jensen, and is explored in detail in his notes. We will only use this more general definition for the following lemma, which is why we did not introduce liftups in this way from the beginning.

**Definition 6.5.** Let $\mathfrak{A}$ be a solid model of $\mathsf{ZFC}^-$ and let $\tau \in \mathrm{wfc}(\mathfrak{A})$ be an uncountable cardinal in $\mathfrak{A}$. Let

$$\sigma : H_\tau^{\mathfrak{A}} \prec H \text{ cofinally,}$$

where $H$ is transitive. Then by a ***liftup*** of $\langle \mathfrak{A}, \sigma \rangle$, we mean a pair $\langle \mathfrak{A}_*, \sigma_* \rangle$ such that

- $\sigma_* \supseteq \sigma$
- $\mathfrak{A}_*$ is solid
- $\sigma_* : \mathfrak{A} \to_{\Sigma_0} \mathfrak{A}_*$ $\tau$-cofinally
- $H \in \mathrm{wfc}(\mathfrak{A}_*)$

**Fact 6.6** (Jensen). *Let $\mathfrak{A}$ be a solid model of* $\mathsf{ZFC}^-$. *Let $\tau > \omega$, $\tau \in \mathrm{wfc}(\mathfrak{A})$, and let*

$$\sigma : H_\tau^{\mathfrak{A}} \prec H \textit{ cofinally,}$$

*where $H$ is transitive. Then $\langle \mathfrak{A}, \sigma \rangle$ has a liftup $\langle \mathfrak{A}_*, \sigma_* \rangle$.*

The following lemma states that transitive liftups of full models are almost full.

**Lemma 6.7** (Jensen). *Let $N = L_\tau[A] \models$ $\mathsf{ZFC}^-$ and $\sigma : \overline{N} \prec N$ where $\overline{N}$ is full. Suppose that $\langle N_*, \sigma_* \rangle$ is a transitive liftup of $\langle \overline{N}, \overline{\sigma} \rangle$. Then $N_*$ is almost full.*

*Proof.* Let $L_\gamma(\overline{N})$ witness the fullness of $\overline{N}$. We will now apply Interpolation (*Fact* 6.4) to $\mathfrak{A} = L_\gamma(\overline{N})$, which makes sense since certainly $\mathfrak{A}$ is a transitive model $\mathsf{ZFC}^-$. Additionally, by **Lemma 4.5** we have that $\overline{N} = H_\tau^{\mathfrak{A}}$, where $\tau = \text{height}(\overline{N})$. Since $\langle N_*, \sigma_* \rangle$ is a transitive liftup, we have that

$$\sigma_* : H_\tau^{\mathfrak{A}} \prec N_* \text{ cofinally},$$

where $N_*$ is transitive. Thus since $\mathfrak{A}$ is transitive, $\langle \mathfrak{A}, \sigma_* \rangle$ has a liftup $\langle \mathfrak{A}_*, \sigma_{**} \rangle$, where $\mathfrak{A}_* \models \mathsf{ZFC}^-$ since $\mathfrak{A}$ does, $\mathfrak{A}_*$ is solid, where

$$\sigma_{**} : \mathfrak{A} \prec \mathfrak{A}_* \ \tau\text{-cofinally}.$$

We have that $N_* \subseteq \text{wfc}(\mathfrak{A}_*)$ and $\tau_* = \sigma_{**}(\tau) = \text{height}(N_*)$ is regular since $\tau$ is. Furthermore, we will show that $N_* = H_{\tau_*}^{\mathfrak{A}_*}$, completing the proof:

Certainly it is the case that $N_* \subseteq H_{\tau_*}^{\mathfrak{A}^*}$. But if $x \in H_{\tau_*}$ in $\mathfrak{A}_*$, then by regularity we have that $x \in \sigma_{**}(u)$ in $\mathfrak{A}_*$, where $u \in \mathfrak{A}$, and $|u| < \tau$ in $\mathfrak{A}$. Let $v = u \cap H_\tau$ in $\mathfrak{A}$. Then $v \in \overline{N}$, since $\overline{N}$ is regular in $\mathfrak{A}$. But then $x \in \sigma_*(v) \in N_*$. So $x \in N_*$. □

## 7. Generalized Diagonal Prikry Forcing

**Definition 7.1.** Let $D$ be an infinite discrete set of measurable cardinals, meaning a set of measurable cardinals that does not contain any of its limit points. For $\kappa \in D$ let $U(\kappa)$ be a normal measure on $\kappa$, and let $\mathcal{U}$ denote the sequence of the $U(\kappa)$'s.

Define $\mathbb{D} = \mathbb{D}(\mathcal{U})$, ***generalized diagonal Prikry forcing*** from the list of measures $\mathcal{U}$, by taking conditions of the form $(s, A)$ satisfying the following:

- The *stem* of the condition, $s$, is a function with domain in $[D]^{<\omega}$ taking each measurable cardinal $\kappa \in \text{dom}(s)$ to some ordinal $s(\kappa) < \kappa$.
- The *upper part* of the condition, $A$, is a function with domain $D \setminus \text{dom}(s)$ taking each measurable cardinal $\kappa \in \text{dom}(A)$ to some measure-one set $A(\kappa) \in U(\kappa)$.

The extension relation on conditions in $\mathbb{D}$ is defined so that $(s, A) \leq (t, B)$ so long as

- $s \supseteq t$.
- The points in $s$ not in $t$ come from $B$, i.e., for all $\kappa \in \text{dom}(s) \setminus \text{dom}(t)$, $s(\kappa) \in B(\kappa)$.
- For all $\kappa \in \text{dom}(A)$, $A(\kappa) \subseteq B(\kappa)$.

If $G$ is a generic filter for $\mathbb{D}$, then its associated $\mathbb{D}$-generic sequence is

$$S = S_G = \bigcup \{s \mid \exists A \ (s, A) \in G\}.$$

Note that our definition of $\mathbb{D}(\mathcal{U})$ differs from that as in Fuchs [1]. The main difference is that here we only add one point below each measurable cardinal $\kappa \in D$, which is done for simplicity's sake. It is not hard to see that the following theorem showing generalized diagonal Prikry forcing is subcomplete also shows that the forcing adding countably many points below each measurable cardinal in $D$ (where the conditions consist of finite stems) is subcomplete. Adding countably many points below each measurable cardinal in $D$ would collapse the cofinality of each $\kappa \in D$ to be $\omega$, as one expects of a Prikry-like forcing.

Also in the above definition we haven't enforced that the stem of our conditions only consist of ordinals that are wedged between successive measurables in $D$; ie. for $\kappa \in D$, we do not explicitly insist that $s(\kappa) \in [\sup(D \cap \kappa), \kappa)$. However, it is dense in $\mathbb{D}(\mathcal{U})$ for the conditions to be that way, since we can always strengthen conditions by restricting their upper parts to a tail. Thus in the following characterization, we may freely add the condition to the following genericity condition on $\mathbb{D}(\mathcal{U})$. Thie following is a genericity criterion on generalized diagonal Prikry forcing similar to the Mathias criterion for Prikry forcing. It was shown in [1, Theorem 1].

**Fact 7.2** (Fuchs)**.** *Let $D$ be an infinite discrete set of measurable cardinals, with $\mathcal{U}$ a corresponding list of measures $\langle U(\kappa) \mid \kappa \in D\rangle$. Then an increasing sequence of ordinals $S = \langle S(\kappa) \mid \kappa \in D\rangle$, where for each $\kappa \in D$*

$$\sup(D \cap \kappa) < S(\kappa) < \kappa,$$

*is a $\mathbb{D}(\mathcal{U})$-generic sequence if and only if for all $\mathcal{X} = \langle X_\kappa \in U(\kappa) \mid \kappa \in D\rangle$, the set*

$$\{\kappa \in D \mid S(\kappa) \notin X_\kappa\} \text{ is finite.}$$

**Theorem 7.3.** *Let $D$ be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D\rangle$ be a list of measures associated to $D$. Then $\mathbb{D} = \mathbb{D}(\mathcal{U})$ is subcomplete.*

*Proof.* Let $\theta >> \delta(\mathbb{D}) = \delta$ be large enough, so that $[\delta]^{<\omega_1} \in H_\theta$.

It must be the case that $\delta \geq \sup D$: to see this, suppose instead that there is a dense $E \subseteq \mathbb{D}$ such that $\sup D \geq \kappa^* > |E|$ for some $\kappa^* \in D$. Then for each condition $(s, A) \in E$ either $\kappa^* \in \operatorname{dom} s$ or $\kappa^* \in \operatorname{dom} A$. So taking $E^* = \{(s, A) \in E \mid \kappa \in \operatorname{dom} s\} \subseteq E$, since $|E^*| < \kappa^*$ as well, there is an $\alpha < \kappa^*$ such that $\sup_{(s,A)\in E^*} s(\kappa^*)$. Let $p = (t, B) \in \mathbb{D}$ be defined so that $t(\kappa^*) = \alpha$ and $B(\kappa) = \kappa$ for all $\kappa \in D \setminus \{\kappa^*\}$. Then $p$ cannot be strengthened by any condition in $E$ since $\kappa^*$ is not in any of the stems of conditions in $E$. So dense subsets of $\mathbb{D}$ must have size at least $\sup D$.

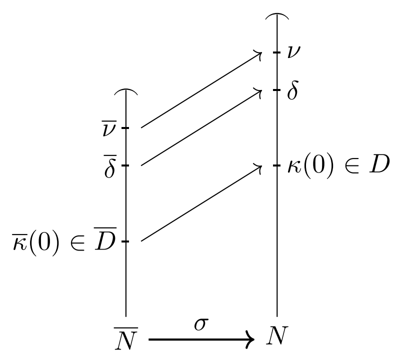
Let $\nu = \delta^+$. Let $\kappa(0)$ be the first measurable cardinal in $D$.

In order to show that $\mathbb{D}$ is subcomplete, suppose we are in the following situation:

- $\mathbb{D} \in H_\theta \subseteq N = L_\tau[A] \models \mathsf{ZFC}^-$ where $\tau > \theta$ and $A \subseteq \tau$
- $\sigma : \overline{N} \cong X \preccurlyeq N$ where $X$ is countable and $\overline{N}$ is full
- $\sigma(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta, \mathbb{D}, \mathcal{U}, c$ for some $c \in N$.

By our requirement on $\theta$, we've ensured that $N$ is closed under countable sequences of ordinals less than $\delta$.

In what follows, we will be taking a few different transitive liftups of restrictions of $\sigma$, and it will useful to keep track of embeddings between $\overline{N}$ and $N$ pictorially. Although it's not extraordinarily illuminating at this point in our discussion, the following figure shows the situation we are currently in, where $\sigma(\overline{\delta}) = \delta$, $\overline{\nu} = \overline{\delta}^{+\overline{N}}$, $\overline{D}$ is the discrete set of measurables in $\overline{N}$ that each measure in $\overline{\mathcal{U}}$ comes from, and $\overline{\kappa}(0)$ is the first measurable in $\overline{D}$, in the sense of $\overline{N}$.




Toward showing that $\mathbb{D}$ is subcomplete, we are additionally given some $\overline{G} \subseteq \overline{\mathbb{D}}$ that is generic over $\overline{N}$. Rather than working with $\overline{G}$, we will work with $\overline{S} = \langle \overline{S}(\overline{\kappa}) \mid \overline{\kappa} \in \overline{D}\rangle$, its associated $\overline{\mathbb{D}}$-generic sequence. We must show following, where $C = \mathcal{S}k^N(\delta \cup X)$:

*Claim* (Main). There is a $\mathbb{D}$-generic sequence $S$ and a map $\sigma' \in V[S]$ such that:

1. $\sigma' : \overline{N} \prec N$
2. $\sigma'(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta, \mathbb{D}, \mathcal{U}, c$
3. $\mathcal{S}k^N(\delta \cup \mathrm{range}(\sigma')) = C$
4. $\sigma' “\overline{S} \subseteq S$

*Pf.* This proof uses Barwise theory (key definitions, facts, and theorems are summarized Section 5) heavily, and ultimately amounts to showing that a certain $\in$-theory, $\mathcal{T}$, which posits the existence of such a $\sigma'$, is consistent. Such an embedding $\sigma'$ can only possibly exist in a suitable generic extension, $V[S]$, where $S$ is $\mathbb{D}$-generic sequence that we will find later. Once we have such a suitable $V[S]$, we will use Barwise theory to find an appropriate admissible structure in $V[S]$ for which the theory $\mathcal{T}$, positing the existence of such a suitable $\sigma'$, defined below, has a model.

For an admissible structure $\mathfrak{M}$ with $S, \overline{S}, \sigma, N, \theta, \mathbb{D}, \mathcal{U}, c \in \mathfrak{M}$ let the infinitary $\in$-theory $\mathcal{T}(\mathfrak{M})$ be defined over $\mathfrak{M}$ as follows:

**predicates:** $\in$

**constants:** $\dot{\sigma}, \underline{x}$ for $x \in \mathfrak{M}$

**axioms:**
- $\mathsf{ZFC}^-$ and $\mathsf{Basic\ Axioms}$.
- $\dot{\sigma} : \underline{\overline{N}} \prec \underline{N}$
- $\dot{\sigma}(\underline{\overline{\theta}}, \underline{\overline{\mathbb{D}}}, \underline{\overline{\mathcal{U}}}, \underline{\overline{c}}) = \underline{\theta}, \underline{\mathbb{D}}, \underline{\mathcal{U}}, \underline{c}$
- $\mathcal{S}k^{\underline{N}}(\underline{\delta} \cup \mathrm{range}(\dot{\sigma})) = \mathcal{S}k^{\underline{N}}(\underline{\delta} \cup \mathrm{range}(\underline{\sigma}))$
- $\dot{\sigma} “\underline{\overline{S}} \subseteq \underline{S}$.

The $\in$-theory is $\Sigma_1(\mathfrak{M})$, since all of the axioms are $\mathfrak{M}$-finite except for the $\mathsf{Basic\ Axioms}$, which altogether are $\mathfrak{M}$-*re* as each of them are $\mathfrak{M}$-finite.

We need to find an appropriate $\mathbb{D}$-generic sequence $S$ and a suitable admissible structure $\mathfrak{M}$ containing $S$ so that $\mathcal{T}(\mathfrak{M})$ is consistent. To do this we use transitive liftups and Barwise theory. Transitive liftups will give us the consistency of certain embeddings that approximate the one we are looking for, and we will rely on Barwise Completeness (*Fact* 5.8) to obtain the existence of a model with our desired properties.

Toward this end, let's take what will turn out to be our first transitive liftup, which is in some sense ensuring the consistency of having property ***3*** of our main claim.

Let $k_0 : N_0 \cong C$ where $N_0$ is transitive, and set $\sigma_0 = k_0^{-1} \circ \sigma$ and $\sigma_0(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta_0, \mathbb{D}_0, \mathcal{U}_0, c_0$. Since $\delta \subseteq C$ and $N_0$ is transitive, $\sigma_0(\overline{\delta}) = \delta$.

Indeed $N_0$ is actually a transitive liftup:

*Claim* 1. $\langle N_0, \sigma_0 \rangle$ is the transitive liftup of $\langle \overline{N}, \sigma \restriction H_{\overline{\nu}}^{\overline{N}} \rangle$.

*Pf.* Recall that $\nu = \delta^+$, and $\overline{\nu} = \overline{\delta}^{+\overline{N}}$. It must be shown that the embedding $\sigma_0 : \overline{N} \prec N_0$ is $\overline{\nu}$-cofinal and that $\sigma_0 \restriction H_{\overline{\nu}}^{\overline{N}} = \sigma \restriction H_{\overline{\nu}}^{\overline{N}}$.

To see that $\sigma_0$ is $\overline{\nu}$-cofinal, let $x \in N_0$. Then $k_0(x) \in C = \mathcal{S}k^N(\delta \cup X)$ so $k_0(x)$ is uniquely $N$-definable from $\xi < \delta$ and $\sigma(\overline{z})$ where $\overline{z} \in \overline{N}$. In other words,

$$k_0(x) = \text{that } y \text{ such that } N \models \varphi(y, \xi, \sigma(\overline{z})).$$

Let $u \in \overline{N}$ be defined as

$$u = \left\{ w \in \overline{N} \mid w = \text{that } y \text{ such that } \overline{N} \models \varphi(y, \zeta, \overline{z}) \text{ for some } \zeta < \overline{\delta} \right\}.$$

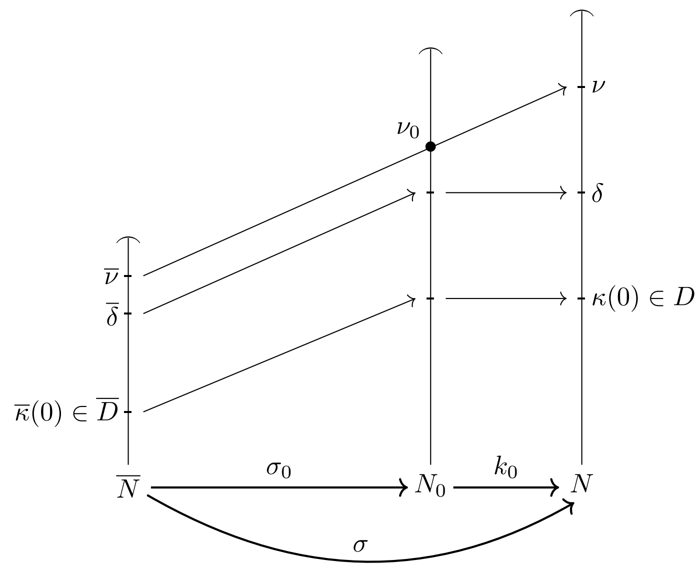
Certainly $u$ is non-empty by elementarity, since $k_0(x) \in \sigma(u)$. Furthermore, $|u| \leq \overline{\delta} < \overline{\nu}$ since every $w \in u$ is unique, and needs a corresponding $\zeta < \overline{\delta}$ to satisfy the formula $\varphi$ with. Thus $x \in k_0^{-1}(\sigma(u)) = \sigma_0(u)$ with $|u| < \overline{\nu}$ in $\overline{N}$, as desired.

Since $X \cup \delta \subseteq C$, the Skolem hull in $N$, we know that $\sigma``H^{\overline{N}}_{\overline{\nu}} \subseteq C$. Thus $k_0^{-1} \restriction \sigma``H^{\overline{N}}_{\overline{\nu}} = \mathrm{id}$. Therefore $\sigma_0 \restriction H^{\overline{N}}_{\overline{\nu}} = \sigma \restriction H^{\overline{N}}_{\overline{\nu}}$, finishing the proof of the claim. $\square$

Since $\overline{\nu}$ is regular in $\overline{N}$, in $N_0$ so is $\nu_0 = \sigma_0(\overline{\nu}) = \sup \sigma_0``\overline{\nu}$. By Interpolation (*Fact* 6.4), we may say that $k_0$ is defined by

$$k_0 : N_0 \prec N \text{ where } k_0 \circ \sigma_0 = \sigma \text{ and } k_0 \restriction \nu_0 = \mathrm{id}.$$

In particular, $\nu_0$ is the critical point of $k_0$, which is continuous below $\nu_0$. Thus we are in a situation that we will represent with the following diagram:




Already, we can say that $\sigma_0$ looks like it has one nice property: $\mathcal{S}k^{N_0}(\delta \cup \mathrm{range}(\sigma_0)) = C$, which somewhat looks like ***3*** of the main claim. However, we have not yet performed forcing, $\sigma_0$ is definable in $V$, and we still need to find a way to extend the generic sequence $\overline{S}$ to a $\mathbb{D}$-generic sequence over $N$. We still have a lot more work to do before finding $\sigma'$.

We shall define another $\in$-theory, $\mathcal{L}_*$ that will assist us in obtaining the diagonal Prikry extension $V[S]$ we need to satisfy our main claim. In order to do this, we will take another transitive liftup and apply Transfer (*Fact* 5.11), in order to see that this new $\in$-theory is consistent over an admissible structure on $N_0$.

Since we will be referring to the same $\in$-theory over two different transitive liftups, I would like to think of "$*$" as a kind of placeholder for a transitive liftup in the following definition.

Suppose that $\langle N_*, \sigma_* \rangle$ is a transitive liftup of $\overline{N}$ along with some reasonable restriction of $\sigma$, ie. the liftup of $\langle \overline{N}, \sigma \restriction H^{\overline{N}}_{\alpha} \rangle$, where $\alpha \geq \overline{\kappa}(0)$ is regular in $\overline{N}$, and say

$$\sigma_*(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta_*, \mathbb{D}_*, \mathcal{U}_*, c_*.$$

Recall that $\delta_{N_*}$ is the least such that $L_{\delta_{N_*}}(N_*)$ is admissible.

Define the infinitary $\in$-theory $\mathcal{L}(N_*, \sigma_*) = \mathcal{L}_*$ as follows:

**predicates:** $\in$
**constants:** $\mathring{\sigma}, \mathring{S}, \underline{x}$ for $x \in L_{\delta_{N_*}}(N_*)$

**axioms:**
- ZFC$^-$ and Basic Axioms
- $\mathring{\sigma} : \underline{\overline{N}} \prec \underline{N_*}$ is $\underline{\overline{\kappa}(0)}$–cofinal
- $\mathring{\sigma}(\underline{\overline{\theta}}, \underline{\overline{\mathbb{D}}}, \underline{\overline{\mathcal{U}}}, \underline{\overline{c}}) = \underline{\theta_*}, \underline{\mathbb{D}_*}, \underline{\mathcal{U}_*}, \underline{c_*}$
- $\mathring{S}$ is a $\underline{\mathbb{D}_*}$-generic sequence over $\underline{N_*}$
- $\mathring{\sigma}``\underline{\overline{S}} \subseteq \mathring{S}$.

As defined, we have that $\mathcal{L}_*$ is a $\Sigma_1(L_{\delta_{N_*}}(N_*))$-theory, since altogether the Basic Axioms are $\Sigma_1(L_{\delta_{N_*}}(N_*))$.

We claim that the theory is consistent.

*Claim* 2. $\mathcal{L}_0 = \mathcal{L}(N_0, \sigma_0)$ is consistent.

*Pf.* Of course, it is not the case that $\sigma_0$ is $\overline{\kappa}(0)$-cofinal - all we know is that it is $\overline{\nu}$-cofinal. However, we know how to find an elementary embedding that is $\overline{\kappa}(0)$ cofinal: by taking a suitable transitive liftup.

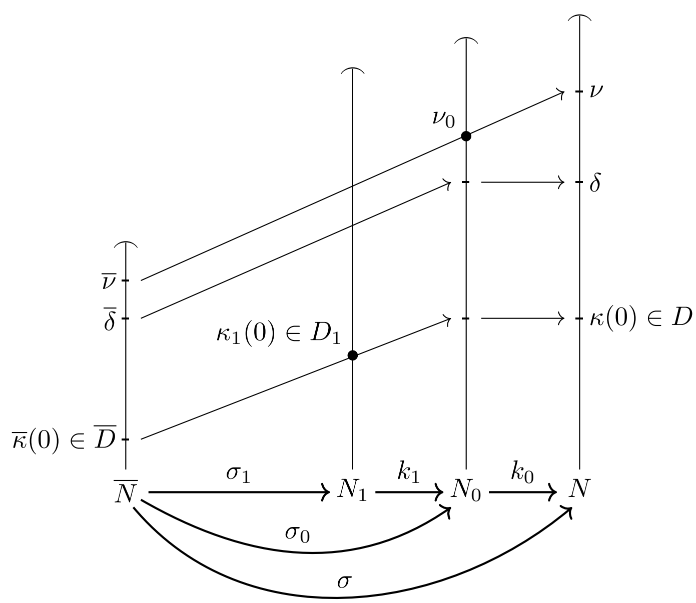
Let $\langle N_1, \sigma_1 \rangle$ be the transitive liftup of $\langle \overline{N}, \sigma \restriction H^{\overline{N}}_{\overline{\kappa}(0)} \rangle$, which exists by Interpolation (*Fact* 6.4). So we have that $\sigma_1 \restriction H^{\overline{N}}_{\overline{\kappa}(0)} = \sigma \restriction H^{\overline{N}}_{\overline{\kappa}(0)}$. Let

$$\sigma_1(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1.$$

Since $\sigma_1 \restriction H^{\overline{N}}_{\overline{\kappa}(0)} = \sigma \restriction H^{\overline{N}}_{\overline{\kappa}(0)} = \sigma_0 \restriction H^{\overline{N}}_{\overline{\kappa}(0)}$, we also have a unique

$$k_1 : N_1 \prec N_0 \text{ where } k_1 \circ \sigma_1 = \sigma_0 \text{ and } k_1 \restriction \kappa_1(0) = \mathrm{id} \text{ where } k_1(0) = k_1(\kappa(0)).$$

Indeed $k_1$ is continuous below $\kappa_1(0)$. We illustrate the final picture below, with $\sigma$ and all of the relevant transitive liftups.



We first show that $\mathcal{L}_1 = \mathcal{L}(N_1, \sigma_1)$ is consistent, by seeing that it has a model. To do this, we will find a sequence extending $\sigma_1``\overline{S}$ that is $\mathbb{D}_1$-generic over $N_1$. Then we will use the Transfer Lemma to see that this transfers to the consistency of $\mathcal{L}_0$.

First, force with $\mathbb{D}_1$, which is diagonal Prikry over $N_1$, to obtain a diagonal Prikry sequence $S_1'$. Define, in $V[S_1']$, a new sequence $S_1$ as follows:

$$S_1(\kappa) = \begin{cases} S_1'(\kappa) & \text{if } \kappa \in D_1 \setminus \sigma_1``\overline{D} \\ \sigma_1(\overline{S}(\overline{\kappa})) & \text{if } \kappa = \sigma_1(\overline{\kappa}) \in \sigma_1``\overline{D}. \end{cases}$$

*Claim.* The sequence $S_1$ is a $\mathbb{D}_1$-generic sequence over $N_1$.

*Pf.* We will show that $S_1$ satisfies the generalized diagonal Prikry genericity criterion (*Fact* 7.2) over $N_1$. To do this, let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D_1 \rangle$, with $\mathcal{X} \in N_1$, be a sequence of measure-one sets in the sequence of measures $\mathcal{U}_1$.

Note first that $S_1'$ is a generic sequence, it already satisfies the generalized diagonal Prikry genericity criterion, namely:

$$\{\kappa \in D_1 \mid S_1'(\kappa) \notin X_\kappa\} \text{ is finite.}$$

Recall that $\overline{S} = \langle \overline{S}(\overline{\kappa}) \mid \overline{\kappa} \in \overline{D} \rangle$ is a $\overline{\mathbb{D}}$-generic sequence as well. We need to see that in addition,

$$\left\{\overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin X_{\sigma_1(\overline{\kappa})}\right\} \text{ is finite,}$$

since then

$$\{\kappa \in D_1 \mid S_1(\kappa) \notin X_\kappa\} = \left\{\kappa \in D_1 \setminus \sigma_1``\overline{D} \mid S_1'(\kappa) \notin X_\kappa\right\} \cup \left\{\kappa = \sigma_1(\overline{\kappa}) \in \sigma_1``\overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin X_\kappa\right\}$$

is finite as well, completing the proof as desired.

By the $\overline{\kappa}(0)$-cofinality of $\sigma_1$, there is some $w \in \overline{N}$ such that $\mathcal{X} \in \sigma_1(w)$, where $|w| < \overline{\kappa}(0)$ in $\overline{N}$. Thus in $N_1$, we have that $|\sigma_1(w)| < \kappa_1(0)$. We may assume that $w$ consists of functions $f \in \prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa})$. So for each $\kappa \in \sigma_1``\overline{D}$, we have that $X_\kappa \in \sigma_1(w)_\kappa = \left\{\sigma_1(f)(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa}) \ \wedge \ f \in w\right\}$ and also $|\sigma_1(w)_\kappa| < \kappa_1(0)$. So all $\kappa \in \sigma_1``\overline{D}$ of course satisfy $\kappa \geq \kappa_1(0)$ and thus by the $\kappa$-completeness of $U_1(\kappa)$, we have that $W_\kappa := \cap \sigma_1(w)_\kappa \in U_1(\kappa)$. So we have established that $\mathcal{W}$, the sequence of $W_\kappa$ for $\kappa \geq \kappa_1(0)$, is also a sequence of measure-one sets in $N_1$. Note in addition that for $\kappa \in \sigma_1``\overline{D}$, we have that $W_\kappa \subseteq X_\kappa$.

By elementarity, for each $\overline{\kappa} \in \overline{D}$, we have $\overline{W}_{\overline{\kappa}} = \cap\left\{f(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa}) \ \wedge \ f \in w\right\}$ is a measure-one set in $\overline{U}(\overline{\kappa})$ and we also have that $\sigma_1(\overline{W}_{\overline{\kappa}}) = W_{\sigma_1(\overline{\kappa})}$. Moreover,

$$\left\{\overline{\kappa} \in \overline{D} \mid \overline{S}(\overline{\kappa}) \notin \overline{W}_{\overline{\kappa}}\right\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for $\overline{\mathbb{D}}$, which must be satisfied by $\overline{S}$. Thus by elementarity,

$$\left\{\overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin W_{\sigma_1(\overline{\kappa})}\right\} \supseteq \left\{\overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin X_{\sigma_1(\overline{\kappa})}\right\} \text{ is finite,}$$

as is desired, completing the proof of our claim. □

Moreover we have now shown that $\mathcal{L}(N_1, \sigma_1) = \mathcal{L}_1$ is consistent by Barwise Correctness (*Fact* 5.7), since we have just shown that $\langle H_\delta; \sigma_1, S_1 \rangle$ is a model of $\mathcal{L}_1$.

Let's check that we may now apply Transfer (*Fact* 5.11) to the embedding $k_1 : N_1 \prec N_0$. By **Lemma 6.7** we have that $N_1$ is almost full. We also have that $\mathcal{L}_1 = \mathcal{L}(L_{\delta_{N_1}}(N_1))$ is $\Sigma_1$ over parameters

$$N_1 \text{ and } \overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}, \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1 \in N_1$$

while $\mathcal{L}_0 = \mathcal{L}(L_{\delta_{N_0}}(N_0))$ is $\Sigma_1$ over parameters

$$N_0 \text{ and } k_1(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}, \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1) \in N_0.$$

Furthermore $k_1$ is cofinal, since for each element $x \in N_0$, as $\sigma_0$ is cofinal, there is $u \in \overline{N}$ such that $x \in \sigma_0(u)$. Thus $\sigma_1(u) \in N_1$, and moreover $x \in k_1(\sigma_1(u)) = \sigma_0(u)$. Therefore, we have that since $\mathcal{L}_1$ is consistent, $\mathcal{L}_0$ is consistent as desired. This completes the proof of *Claim* 2. □

From the consistency of $\mathcal{L}_0$, we would now like to use Barwise Completeness (*Fact* 5.8) to obtain a model of $\mathcal{L}_0$. To do this, we need the admissible structure the theory is defined over to be countable. So let's work in $V[F]$, a generic extension that collapses $L_{\delta_{N_0}}(N_0)$ to be countable. Then by Barwise Completeness, $\mathcal{L}_0$ has a solid model

$$\mathfrak{A} = \langle \mathfrak{A}; \mathring{S}^{\mathfrak{A}}, \mathring{\sigma}^{\mathfrak{A}} \rangle$$

such that

$$\mathrm{Ord} \cap \mathrm{wfc}(\mathfrak{A}) = \mathrm{Ord} \cap L_{\delta_{N_0}}(N_0).$$

Thus we have that $\mathring{\sigma}^{\mathfrak{A}} : \underline{\overline{N}}^{\mathfrak{A}} \prec \underline{N_0}^{\mathfrak{A}}$. By the Basic Axioms we have that $\underline{\overline{N}}^{\mathfrak{A}} = \overline{N}$ and $N_0 = \underline{N_0}^{\mathfrak{A}}$. Thus we may say that $\mathring{\sigma}^{\mathfrak{A}} : \overline{N} \prec N_0$.

Let $S = \mathring{S}^{\mathfrak{A}}$ and $\overset{*}{\sigma} = k_0 \circ \mathring{\sigma}^{\mathfrak{A}}$.

Then $S$ is a $\mathbb{D}_0$-generic sequence over $N_0$, and as $k_0 : N_0 \cong C$ we also have that $k_0$"$S$ is $C$-generic for $\mathbb{D}$. We need to see that $S$ is $\mathbb{D}$-generic over $V$. To do this, let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D \rangle$ be a sequence of measure-one sets in the sequence of measures $\mathcal{U}$. We will verify the generalized diagonal Prikry genericity criterion. To do this, let $E \subseteq \mathbb{D}$ be dense and have size $\delta$ with $E \in C$. Since $\delta \subseteq C$, we have that $E \subseteq C$ as well. Find a condition $(s, A) \in E$ that strengthens $(\emptyset, \mathcal{X})$. Thus for $\kappa \in \operatorname{dom} A$, we have that $A(\kappa) \subseteq X_\kappa$. Define a sequence of measure-one sets $B$ in $C$ so that

$$B(\kappa) = \begin{cases} A(\kappa) & \text{if } \kappa \in \operatorname{dom} A \\ \kappa & \text{if } \kappa \in \operatorname{dom} s. \end{cases}$$

So we have that $B$ is a sequence of measure-one sets in $C$. So $\{\kappa \in D \mid S(\kappa) \notin B(\kappa)\}$ is finite. Thus $\{\kappa \in D \mid S(\kappa) \notin X_\kappa\}$ is finite.

This will be the $\mathbb{D}$-generic sequence we need to satisfy our main claim. We will see in the following claim that $\overset{*}{\sigma}$ has all of the desired properties of our main claim, but it fails to be in $V[S]$, which is what we need. However, based on the following claim, $\overset{*}{\sigma}$ will at least enable us to see that our $\in$-theory $\mathcal{T}$ from long ago, defined to assist us in proving the main claim, is consistent over a suitable admissible structure.

*Claim* 3. The map $\overset{*}{\sigma}$ satisfies:

1. $\overset{*}{\sigma} : \overline{N} \prec N$
2. $\overset{*}{\sigma}(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta, \mathbb{D}, \mathcal{U}, c$
3. $\mathcal{S}k^N(\delta \cup \mathrm{range}(\overset{*}{\sigma})) = C$
4. $\overset{*}{\sigma}$"$\overline{S} \subseteq S$

*Pf.* For item ***1***, we have already seen above that $\mathring{\sigma}^{\mathfrak{A}} : \overline{N} \prec N_0$. Since $k_0 : N_0 \prec N$, the desired result follows.

For item ***2***, $\overset{*}{\sigma}(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = k_0(\theta_0, \mathbb{D}_0, \mathcal{U}_0, c_0) = \theta, \mathbb{D}, \mathcal{U}, c$.

Item ***3*** holds since $N_0 = \mathcal{S}k^{N_0}(\delta \cup \mathrm{range}(\mathring{\sigma}^{\mathfrak{A}}))$. To see this, clearly we have that $\mathcal{S}k^{N_0}(\delta \cup \mathrm{range}(\mathring{\sigma}^{\mathfrak{A}})) \subseteq N_0$, since $\delta \in N_0$ as $N_0 \cong C$, and certainly $\mathrm{range}(\mathring{\sigma}^{\mathfrak{A}})) \subseteq N_0$ as well. Then because $\mathring{\sigma}^{\mathfrak{A}}$ is $\overline{\kappa}'$-cofinal, by **Lemma 6.3**, we have, since $\mathring{\sigma}^{\mathfrak{A}}(\overline{\kappa}') < \delta$, that:

$$N_0 = \left\{\mathring{\sigma}^{\mathfrak{A}}(f)(\beta) \mid f : \gamma \longrightarrow N_0,\ \gamma < \overline{\kappa}(0) \text{ and } \beta < \mathring{\sigma}^{\mathfrak{A}}(\gamma)\right\} \subseteq \mathcal{S}k^{N}(\delta \cup \mathrm{range}(\mathring{\sigma}^{\mathfrak{A}})).$$

Thus $C = k_0 ``N_0 = \mathcal{S}k^{N_0}(\delta \cup \mathrm{range}(k_0 \circ \mathring{\sigma}^{\mathfrak{A}}))$ as desired.

To see ***4***, note that $\overset{*}{\sigma} \restriction \overline{\kappa}(0) = \mathring{\sigma}^{\mathfrak{A}} \restriction \overline{\kappa}(0)$ since $k_0 \restriction \nu_0 = \mathrm{id}$.

This completes the proof of *Claim* 3. $\square$

We are almost done, but like we stated above, $\mathring{\sigma}^{\mathfrak{A}}$ is in $V[F]$, the generic extension needed to obtain a countable admissible structure to apply Barwise Completeness to. But $V[F]$ is not in $V[S]$, and so $\overset{*}{\sigma}$ is not necessarily in $V[S]$. We will use Barwise Completeness one last time, to finally find an embedding $\sigma'$ with which to satisfy the main claim along with the $S$ we found above.

Let $\lambda$ be regular in $V[S]$ with $N \in H_{\lambda}^{V[S]}$. Then

$$M = \langle H_{\lambda}^{V[S]};\ N, \sigma, S;\ \theta, \delta, \mathbb{D}, \mathcal{U}, c\rangle \text{ is admissible.}$$

In order to satisfy our main claim, we need a model of $\mathcal{T}(M)$ in $V[S]$. By *Claim* 3, we have that $\langle M, \overset{*}{\sigma}\rangle$ is a model of $\mathcal{T}(M)$, but in $V[F]$, which is not $V[S]$. Still this means that by Barwise Correctness, $\mathcal{T}(M)$ is consistent.

Consider the Mostowski collapse of $M$; let

$$\pi : \tilde{M} \prec M \text{ where } \tilde{M} \text{ is countable and transitive.}$$

Note that $\tilde{M} \in H_{\omega_1}^{V[S]} = H_{\omega_1}^{V}$ since it is countable and diagonal Prikry forcing doesn't add bounded subsets to any $\kappa \in D$.[1] We also have that $\underline{\overline{N}}^{\tilde{\mathfrak{A}}} = \overline{N}$, since $M$ sees that $\overline{N}$ is countable so $\tilde{M}$ sees that $\pi^{-1}(\overline{N})$ is, and it follows that $\pi^{-1}(\overline{N}) = \overline{N}$.

Plus, $\mathcal{T}(\tilde{M})$ is consistent, since otherwise its inconsistency could be pushed up via $\pi$ to one in $\mathcal{T}(M)$, contradicting the model witnessing its consistency that we found in $V[F]$.

So by Barwise Completeness, $\mathcal{T}(\tilde{M})$ has a solid model

$$\tilde{\mathfrak{A}} = \langle \tilde{\mathfrak{A}}; \dot{\sigma}^{\tilde{\mathfrak{A}}}\rangle$$

such that

$$\mathrm{Ord} \cap \mathrm{wfc}(\tilde{\mathfrak{A}}) = \mathrm{Ord} \cap \tilde{M}.$$

Letting $\sigma' = \pi \circ \dot{\sigma}^{\tilde{\mathfrak{A}}}$, the main claim is now satisfied with $\sigma'$ and our $\lambda$-diagonal Prikry sequence $S$.

Let us verify each of the properties of $\sigma'$ required by the main claim. The verification of these properties shall use the agreement between $\tilde{\mathfrak{A}}$ and $\tilde{M}$ on the special constants of $\tilde{M}$ and on the ordinals. The fact that $\pi$ does not affect $\overline{N}$ will be greatly taken advantage of.

First we show ***1*** of the main claim. Let's say that $\varphi[\sigma'(\overline{a})]$ holds in $N$. So $\varphi[\pi(\dot{\sigma}^{\tilde{\mathfrak{A}}}(\overline{a}))]^N$ holds in $M$. Thus $\varphi[\dot{\sigma}^{\tilde{\mathfrak{A}}}(\overline{a})]^{\pi^{-1}(N)}$ holds in $\tilde{M}$, and thus also in $\tilde{\mathfrak{A}}$. Indeed we know that $\underline{\overline{N}}^{\tilde{\mathfrak{A}}} = \overline{N}$, since $\tilde{\mathfrak{A}}$ agrees with $\tilde{M}$ about countable ordinals, which we may use to code $\overline{N}$. This means that $\varphi[\overline{a}]$ holds in $\overline{N}$, as desired.

To see ***2***, we have $\sigma'(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \pi(\underline{\theta}^{\tilde{\mathfrak{A}}}, \underline{\mathbb{D}}^{\tilde{\mathfrak{A}}}, \underline{\mathcal{U}}^{\tilde{\mathfrak{A}}}, \underline{c}^{\tilde{\mathfrak{A}}}) = \theta, \mathbb{D}, \mathcal{U}, c$.

[1]As Fuchs [1] points out, this result is a modification to the proof that generalized diagonal Prikry forcing preserves cardinalities.

For item ***3***, let $\tilde{N} = \underline{N}^{\tilde{\mathfrak{A}}}$, $\tilde{\sigma} = \underline{\sigma}^{\tilde{\mathfrak{A}}}$ and $\tilde{\delta} = \underline{\delta}^{\tilde{\mathfrak{A}}}$. So $\pi(\tilde{\delta}) = \delta$ and $\pi(\tilde{\sigma}) = \sigma$. Because of the way the $\in$-theory $\mathcal{T}$ was defined, we already have:

$$\mathcal{S}k^{\tilde{N}}(\tilde{\delta} \cup \mathrm{range}(\dot{\sigma}^{\tilde{\mathfrak{A}}})) = \mathcal{S}k^{\tilde{N}}(\tilde{\delta} \cup \mathrm{range}(\tilde{\sigma})). \tag{1}$$

To see $\mathcal{S}k^N(\delta \cup \mathrm{range}(\sigma')) \subseteq \mathcal{S}k^N(\delta \cup X)$, suppose $x \in \mathcal{S}k^N(\delta \cup \mathrm{range}(\sigma'))$. Then we have that $N$ sees that there is some formula $\varphi$, $\overline{z} \in \overline{N}$, and $\xi < \delta$ where $x$ is unique such that $\varphi(x, \pi(\dot{\sigma}^{\tilde{\mathfrak{A}}}(\overline{z})), \xi)$. In particular, $x$ is in the range of $\pi$. Thus $\tilde{x} = \pi^{-1}(x) \in \tilde{N}$ and $\tilde{\xi} < \tilde{\delta}$ such that $\varphi(\tilde{x}, \dot{\sigma}^{\tilde{\mathfrak{A}}}(\overline{z}), \tilde{\xi})$ holds. Thus by (1), we have that $\tilde{x}$ is unique such that $\varphi(\tilde{x}, \tilde{\sigma}(\overline{y}), \tilde{\zeta})$ for some $\overline{y} \in \overline{N}$ and $\tilde{\zeta} < \tilde{\delta}$. Thus pushing back up through $\pi$, letting $\zeta = \pi(\tilde{\zeta})$, we have that $x = \pi(\tilde{x})$ is unique such that $\varphi(x, \sigma(\overline{y}), \zeta)$ holds, so $x \in \mathcal{S}k^N(\delta \cup X)$.

To see that $\mathcal{S}k^N(\delta \cup X) \subseteq \mathcal{S}k^N(\delta \cup \mathrm{range}(\sigma'))$ works similarly. Let $x \in \mathcal{S}k^N(\delta \cup X)$. Then there is $\overline{z} \in \overline{N}$ and $\xi < \delta$ such that $\varphi(x, \sigma(\overline{z}), \xi)$ holds. So in particular, $x$ is in the domain of $\pi$. So we may find $\tilde{\xi} < \tilde{\delta}$ such that $\tilde{x} = \pi^{-1}(x)$ is unique such that $\varphi(\tilde{x}, \tilde{\sigma}(\overline{z}), \tilde{\xi})$. So by (1), we have that there is $\overline{y} \in \overline{N}$ and $\tilde{\zeta} < \tilde{\delta}$ such that $\tilde{x}$ is unique satisfying $\varphi(\tilde{x}, \dot{\sigma}^{\mathfrak{A}}(\overline{y}), \tilde{\zeta})$. Finally, by pushing back up through $\pi$, letting $\pi(\tilde{\zeta}) = \zeta$, we have that $x = \pi(\tilde{x})$ is unique such that $\varphi(x, \sigma'(\overline{y}), \zeta)$, so $x \in \mathcal{S}k^N(\delta \cup \mathrm{range}(\sigma'))$ as desired.

To see item ***4***, note that $\overline{\underline{S}}^{\tilde{\mathfrak{A}}} = \overline{S}$ since $\overline{S} \subseteq \overline{N}$. So we have already by the definition of $\mathcal{T}$ that $\dot{\sigma}^{\tilde{\mathfrak{A}}}``\overline{S} \subseteq \underline{S}^{\tilde{\mathfrak{A}}}$. Thus $\pi \circ \dot{\sigma}^{\tilde{\mathfrak{A}}}``\overline{S} \subseteq \pi``\underline{S}^{\tilde{\mathfrak{A}}} \subseteq S$ as desired.

This completes the proof of the main claim. □

We have satisfied the main claim, so we are done, we have shown that $\mathbb{D}$ is subcomplete. □

Some slight modifications to the above proof give the following two corollaries.

The first point is that the above proof also shows that generalized diagonal Prikry forcing that adds a countable sequence to each measurable cardinal is subcomplete. Before stating the corolloary let's define the forcing. Again let $D$ be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D\rangle$ be a list of measures associated to $D$. Let $\mathbb{D}^*(\mathcal{U}) = \mathbb{D}^*$ be defined the same as $\mathbb{D}(\mathcal{U})$ except the stem of a condition, $s$, in $\mathbb{D}^*(\mathcal{U})$ is a function with domain in $[D]^{<\omega}$ taking each measurable cardinal $\kappa \in \mathrm{dom}(s)$ to finitely many ordinals $s(\kappa) \subseteq \kappa$. The upper part and extension relation is defined in the same way; the only slight modification is that again we have $(s, A) \leq (t, B)$ so long as points in $s$ not in $t$ come from $B$, which in the case means that for $\kappa \in \mathrm{dom}(s)$ we have that each element of $s(\kappa)$ not in $t(\kappa)$ is in $B(\kappa)$. We may again form a $\mathbb{D}^*$-generic sequence $S = S_G$ for a generic $G \subseteq \mathbb{D}^*$, and we may write $S = \langle S(\kappa) \mid \kappa \in D\rangle$ where $S(\kappa)$ is a countable sequence of ordinals less than $\kappa$. The genericity criterion for generic diagonal Prikry sequences is as that for $\mathbb{D}$, which is given in [1, Theorem 1], as stated in ***Fact* 7.2**, with the modification that $S$ is $\mathbb{D}^*$ generic if and only if for all $\mathcal{X}$, the set $\{\alpha \mid \exists \kappa \in D\ \alpha \in S(\kappa) \setminus X_\kappa\}$ is finite.

**Corollary 7.4.** *Let $D$ be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D\rangle$ be a list of measures associated to $D$. Then $\mathbb{D}^*(\mathcal{U})$ is subcomplete.*

*Proof Sketch.* The modifications are mostly notational, and the main one that needs to be made is to adjust the proof of the *Claim* within the proof of *Claim* 2. Here we have $\mathbb{D}_1^*$, the generalized diagonal Prikry forcing as computed in $N_1$, as well as $\overline{\mathbb{D}}$ of $\overline{N}$, and $S_1$, which we would like to show is a $\mathbb{D}_1^*$-generic sequence over $N_1$ in this case. $S_1$ is defined as $\sigma_1``\overline{S}$, using a diagonal Prikry sequence $S_1'$ to fill in the missing coordinates, where $S_1'$ is obtained by forcing with $\mathbb{D}_1$ over $V$.

We will show that $S_1$ satisfies the generalized diagonal Prikry genericity criterion over $N_1$ and follow the above proof. To do this, let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D_1 \rangle$, with $\mathcal{X} \in N_1$, be a sequence of measure-one sets in the sequence of measures $\mathcal{U}_1$.

Note first that $S_1'$ is a generic sequence, it already satisfies the generalized diagonal Prikry genericity criterion, namely: $\{\alpha \mid \exists \kappa \in D_1\ \alpha \in S_1'(\kappa) \setminus X_\kappa\}$ is finite. Recall that $\overline{S} = \langle \overline{S}(\overline{\kappa}) \mid \overline{\kappa} \in \overline{D} \rangle$ is a $\overline{\mathbb{D}}$-generic sequence over $\overline{N}$ as well. We need to see that in addition, $\left\{\alpha \mid \exists \overline{\kappa} \in \overline{D}\ \alpha \in \sigma_1(\overline{S}(\overline{\kappa})) \setminus X_{\sigma_1(\overline{\kappa})}\right\}$ is finite.

By the $\overline{\kappa}(0)$-cofinality of $\sigma_1$, there is some $w \in \overline{N}$ such that $\mathcal{X} \in \sigma_1(w)$, where $|w| < \overline{\kappa}(0)$ in $\overline{N}$. Thus in $N_1$, $|\sigma_1(w)| < \kappa_1(0)$. For each $\kappa \in \sigma_1``\overline{D}$, we have that $X_\kappa \in \sigma_1(w)_\kappa = \left\{\sigma_1(f)(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa})\ \wedge\ f \in w\right\}$ and also $|\sigma_1(w)_\kappa| < \kappa_1(0)$. All $\kappa \in \sigma_1``\overline{D}$ of course satisfy $\kappa \geq \kappa_1(0)$ so by the $\kappa$-completeness of $U_1(\kappa)$, we have that $W_\kappa := \cap\, \sigma_1(w)_\kappa \in U_1(\kappa)$ since $X_\kappa \in \sigma_1(w)_\kappa$. So we have established that $\mathcal{W}$, the sequence of $W_\kappa$ for $\kappa \geq \kappa_1(0)$, is also a sequence of measure-one sets in $N_1$. Note in addition that for $\kappa \in \sigma_1``\overline{D}$, we have that $W_\kappa \subseteq X_\kappa$.

By elementarity, for each $\overline{\kappa} \in \overline{D}$, we have $\overline{W}_{\overline{\kappa}} = \cap \left\{f(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa})\ \wedge\ f \in w\right\}$ is a measure-one set in $\overline{U}(\overline{\kappa})$ and we also have that $\sigma_1(\overline{W}_{\overline{\kappa}}) = W_{\sigma_1(\overline{\kappa})}$. Moreover,

$$\left\{\alpha \mid \exists \overline{\kappa} \in \mathbb{D}\ \alpha \in \overline{S}(\overline{\kappa}) \setminus \overline{W}_{\overline{\kappa}}\right\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for $\overline{\mathbb{D}}$, which must be satisfied by $\overline{S}$. Thus by elementarity,

$$\left\{\alpha \mid \exists \overline{\kappa} \in \mathbb{D}\ \sigma_1(\overline{S}(\overline{\kappa})) \setminus W_{\sigma_1(\overline{\kappa})}\right\} \supseteq \left\{\alpha \mid \exists \overline{\kappa} \in \mathbb{D}\ \sigma_1(\overline{S}(\overline{\kappa})) \setminus X_{\sigma_1(\overline{\kappa})}\right\} \text{ is finite,}$$

as is desired, completing the proof of the claim. □

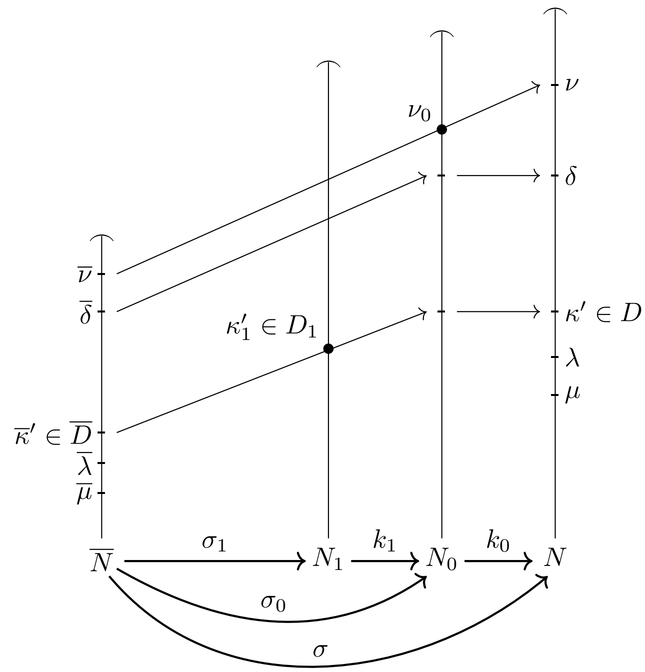
One might consider a forcing like $\mathbb{D}$ and $\mathbb{D}^*$ that adds one point below each measurable cardinal sometimes, and other times adds a cofinal $\omega$-sequence below the measurable cardinal. This forcing is clearly subcomplete as well.

Below we refer to the concept of subcompleteness above $\mu$, which was introduced in **??**.

**Corollary 7.5.** *Let $D$ be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D \rangle$ be a list of measures associated to $D$.*

*Furthermore, let $\mu < \lambda$ be a regular cardinal, where $\lambda = \sup_{n<\omega} \kappa_n$, the first limit point of $D$. Then $\mathbb{D} = \mathbb{D}(\mathcal{U})$ is subcomplete above $\mu$.*

*Proof Sketch.* The idea is to follow the same exact proof as in the above theorem, except we achieve the following diagram:




Here we replace $\kappa(0)$ with some $\kappa' \in D$ such that $\lambda < \kappa'$, where there are finitely many measurables of $D$ below $\kappa'$. So in particular, we let $\langle N_1, \sigma_1 \rangle$ be the liftup of $\langle \overline{N}, \sigma \restriction H^{\overline{N}}_{\overline{\kappa}'} \rangle$ in *Claim* 2. In order to show the *Claim* that we have a generic sequence over $\mathbb{D}_1$, we follow the same argument as follows:

Let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in \sigma_1``\overline{D} \rangle$, with $\mathcal{X} \in N_1$, be a sequence of measure one sets in the sequence of measures $\mathcal{U}_1$ with only coordinates coming from $\sigma_1``\overline{D}$. We need to see that

$$\left\{ \overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin X_{\sigma_1(\overline{\kappa})} \right\} \text{ is finite.}$$

By the $\overline{\kappa}'$-cofinality of $\sigma_1$, there is some $w \in \overline{N}$ such that $\mathcal{X} \in \sigma_1(w)$, where $|w| < \overline{\kappa}'$ in $\overline{N}$. Thus in $N_1$, $|\sigma_1(w)| < \kappa'_1$. For each $\kappa \in \sigma_1``\overline{D}$, we have that $X_\kappa \in \sigma_1(w)_\kappa = \left\{ \sigma_1(f)(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa}) \ \wedge \ f \in w \right\}$ and also

$$|\sigma_1(w)| < \kappa'_1.$$

So for all but finitely many $\kappa \in \sigma_1``\overline{D}$, namely for $\kappa \geq \kappa'_1$, by the $\kappa$-completeness of $U_1(\kappa)$, we have that

$$\cap \sigma_1(w) = W_\kappa \in U_1(\kappa).$$

So we have established that $\mathcal{W}$, the sequence of $W_\kappa$ for $\kappa > \kappa'_1$, is also a sequence of measure-one sets in $N_1$. Note in addition that for $\kappa \in \sigma_1``\overline{D}$, $\kappa > \kappa'_1$, we have that $W_\kappa \subseteq X_\kappa$.

By elementarity, for each $\overline{\kappa} \in \overline{D}$, where $\overline{\kappa} > \overline{\kappa}'$, we have

$$\overline{W}_{\overline{\kappa}} = \cap \left\{ f(\overline{\kappa}) \mid f \in \textstyle\prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa}) \ \wedge \ f \in w \right\}$$

is a measure-one set in $\overline{U}(\overline{\kappa})$ and we also have that $\sigma_1(\overline{W}_{\overline{\kappa}}) = W_{\sigma_1(\overline{\kappa})}$. Moreover,

$$\left\{ \overline{\kappa} \in \overline{D} \mid \overline{S}(\overline{\kappa}) \notin \overline{W}_{\overline{\kappa}} \right\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for $\overline{\mathbb{D}}$, which must be satisfied by $\overline{S}$, and since there are only finitely many measurables in $\overline{D}$ less than $\overline{\kappa}'$ in $\overline{N}$. Thus by elementarity,

$$\left\{\overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin W_{\sigma_1(\overline{\kappa})}\right\} \supseteq \left\{\overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa}) \notin X_{\sigma_1(\overline{\kappa})}\right\} \text{ is finite.}$$

Additionally the $\in$-theories $\mathcal{L}$ and $\mathcal{T}$ would have to be defined so as to include as an axiom that $\mathring{\sigma} \restriction \underline{\overline{\mu}} = \underline{\sigma} \restriction \underline{\overline{\mu}}$ and $\dot{\sigma} \restriction \underline{\overline{\mu}} = \underline{\sigma} \restriction \underline{\overline{\mu}}$ respectively. We would then need to show that $\overset{*}{\sigma} \restriction \overline{\mu} = \sigma \restriction \overline{\mu}$, but this would follow since $k_0$ is the identity on $\nu_0$. Furthermore, it would need to be shown that $\sigma' \restriction \overline{\mu} = \sigma \restriction \overline{\mu}$, but this would follow from the requirement that $\dot{\sigma}^{\tilde{\mathfrak{A}}} \restriction \overline{\mu} = \underline{\sigma}^{\tilde{\mathfrak{A}}} \restriction \overline{\mu}$, and since ordinals are computed properly by $\tilde{\mathfrak{A}}$. □

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