

# THE SUBCOMPLETENESS OF DIAGONAL PRIKRY FORCING

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**ABSTRACT.** It is shown that generalized Prikry forcing to add a countable sequence to each measurable in an infinite discrete set of measurable cardinals is subcomplete. To do this it is shown that a simplified version of generalized Prikry forcing which adds a point below each measurable cardinal in an infinite discrete set, called generalized diagonal Prikry forcing, is subcomplete. It is also shown that generalized diagonal Prikry forcing is subcomplete above  $\mu$  where  $\mu$  is a regular cardinal below the first limit point of the infinite discrete set of measurable cardinals used in the forcing.

## 1. INTRODUCTION

Subcomplete forcing is a class of forcing notions that do not add reals, but may potentially alter cofinalities to  $\omega$ . Examples of subcomplete forcing include Prikry forcing and Namba forcing (under CH). This separates subcomplete forcings from proper forcings, that have countable covering and thus cannot change cofinalities to  $\omega$ .

Jensen [3, Section 3.3] shows that Prikry forcing and Namba forcing (under CH) are subcomplete. Fuchs has shown that Magidor forcing [2] is subcomplete. Here an adaptation of Jensen's proof showing that Prikry forcing is subcomplete is employed to see that many "Prikry-like forcings," which are referred to as generalized diagonal Prikry forcing after Fuchs ([1]), are subcomplete.

## 2. PRELIMINARIES

Before defining subcomplete forcing and generalized diagonal Prikry forcing, some preliminary information is necessary.

First, a brief outline of some of the notation used in what follows: Forcing notions  $\mathbb{P} = \langle \mathbb{P}, \leq \rangle$  are taken to be partial orders that are separative and contain a "top" element weaker than all elements of  $\mathbb{P}$ , denoted  $1$ . Write  $N$ ,  $M$ , or  $\overline{N}$  to denote models of  $\text{ZFC}^-$ , the axioms of Zermelo-Fraenkel Set Theory without the axiom of Powerset, and with the axiom of Collection instead of Replacement. Usually we will name these models. For  $\sigma$  an elementary embedding, write  $\sigma : N \prec M$  and write  $N \prec M$  to denote and emphasize that  $N$  is an elementary substructure of  $M$ . Let  $N$  be a transitive  $\text{ZFC}^-$  model. Write  $\text{height}(N)$  to mean  $\text{Ord} \cap N$ . Let  $\alpha$  be an ordinal. Then write  $\alpha^N$  for  $\alpha \cap N$ . Let  $\theta$  be a cardinal. The collection of sets hereditarily of size less than  $\theta$  will be referred to as  $H_\theta$ . Relativizing the concept to a particular model of set theory,  $M$ , write  $H_\theta^M$  to mean the collection of sets in  $M$  that are hereditarily of size less than  $\theta$  in  $M$ . In this case, if  $\theta$  is determined by some computation, the computation is meant to take place in  $M$ . Let  $\tau$  be a cardinal. With abuse of notation, write  $L_\tau[A]$  to refer to the structure  $\langle L_\tau[A]; \in, A \cap L_\tau[A] \rangle$ . If a map  $\sigma$  satisfies  $\sigma(\bar{a}) = a$  and  $\sigma(\bar{b}) = b$ , this will be abbreviated as  $\sigma(\bar{a}, \bar{b}) = a, b$ .

### 2.1. The Weight of a Forcing Notion.

**Definition 2.1.** For a forcing notion  $\mathbb{P}$ , we write  $\delta(\mathbb{P})$  to denote the least cardinality of a dense subset in  $\mathbb{P}$ . This is sometimes referred to as the *weight* of a poset.

Although Jensen defines  $\delta(\mathbb{P})$  for Boolean algebras, it's also relevant for posets. As Jensen states, for forcing notions  $\mathbb{P}$ , the weight can be replaced with the cardinality of  $\mathbb{P}$ , or even  $\mathbb{P}$ , for the purpose of defining subcompleteness. However,  $\delta(\mathbb{P})$  and  $|\mathbb{P}|$  are not necessarily the same, since there could be a large set of points in the poset that all have a common strengthening.

If  $\mathbb{P}$  is a dense subset of  $\mathbb{Q}$ , then, of course,  $\delta(\mathbb{P}) = \delta(\mathbb{Q})$ .

**2.2. Fullness.** In the definition of subcompleteness, in lieu of working directly with  $H_\theta$  and its well-order for “large enough” cardinals  $\theta$  (as is often done for proper forcing, for example), models  $N$  look as follows:

$$H_\theta \subseteq N = L_\tau[A] \models \text{ZFC}^-,$$

where  $\tau > \theta$  is a cardinal that is not necessarily regular. Such  $H_\theta$  will need to be large enough so that  $N$  has the correct  $\omega_1$  and  $H_{\omega_1}$ . One justification for working with these models is that such  $N$  will naturally contain a well order of  $H_\theta$ , along with its Skolem functions and other useful bits of useful information. Additionally a benefit of working with such models is that  $L_\tau[A]$  is easily definable in  $L_\tau[A][G]$ , if  $G$  is generic, using  $A$ . In the standard fashion, take countable elementary substructures  $X$ , where  $X \prec N$ . Then take the countable transitive collapse of such an  $X$ , and write  $\bar{N} \cong X$ . Such embeddings will be referred to as

$$\sigma : \bar{N} \cong X \prec N.$$

Often, to suppress the mention of range of  $\sigma$ , or  $X$  in the above, all that will be written is  $\sigma : \bar{N} \prec N$ .

In fact, it will not be quite enough for such an  $\bar{N}$  to be transitive, what is needed is exactly granted by the property of *fullness*. Fullness of a model ensures that the model is not pointwise definable. Thus in the standard setup, there can be many elementary maps between the smaller structure and the larger structure  $N$ . Before defining fullness, there are more preliminary points to clarify.

It is not hard to see what the critical point of these embeddings is. Given  $\sigma : \bar{N} \prec N$  where  $\bar{N}$  is countable and transitive,  $\text{cp}(\sigma)$  is exactly  $\omega_1^{\bar{N}} = \sigma^{-1}(\omega_1^N)$ , since  $\bar{N}$  is countable.

*Remark 2.2.* Let  $\bar{N}$ ,  $N$  be transitive  $\text{ZFC}^-$  models, where  $\bar{N}$  is countable and  $H_{\omega_1} \subseteq N$ , with  $\sigma : \bar{N} \prec N$ . Then  $\text{cp}(\sigma) = \omega_1^{\bar{N}}$  and  $\sigma \upharpoonright (H_{\omega_1})^{\bar{N}} = \text{id}$ .

Let  $N = L_\tau[A]$  for some cardinal  $\tau$  and set  $A$ , be a transitive  $\text{ZFC}^-$  model, let  $X$  be a set, and let  $\delta$  be a cardinal. Our notation for the *Skolem hull*, in  $N$ , closing under  $\delta \cup X$ , is  $\mathcal{Hull}^N(\delta \cup X)$ . It is defined to be the smallest  $Y \prec N$  satisfying  $X \cup \delta \subseteq Y$ .

The two basic results below follow immediately.

*Remark 2.3.* Let  $N = L_\tau[A]$  be a transitive  $\text{ZFC}^-$  model,  $\delta$  and  $\gamma$  be cardinals, and  $X$  a set. If  $\delta \leq \gamma$  then  $\mathcal{Hull}^N(\delta \cup X) \subseteq \mathcal{Hull}^N(\gamma \cup X)$ .

*Remark 2.4.* Let  $N$  be a transitive  $\text{ZFC}^-$  model,  $\delta$  a cardinal, and  $X, Y$  sets. If  $\mathcal{Hull}^N(\delta \cup X) = \mathcal{Hull}^N(\delta \cup Y)$  then  $\mathcal{Hull}^N(\gamma \cup X) = \mathcal{Hull}^N(\gamma \cup Y)$  for all  $\gamma \geq \delta$ .

Toward defining fullness, the notion of regularity of transitive models needs to be defined.

**Definition 2.5.** We say that a transitive model  $N$  is *regular* in a transitive model  $M$  so long as for all functions  $f : x \rightarrow N$ , where  $x$  is an element of  $N$  and  $f \in M$ , we have that  $f \text{ “} x \in N \text{”}$ .

The following lemma is meant to elucidate the significance of regularity.

**Lemma 2.6.** *Let  $N, M \models \text{ZFC}^-$  be transitive.  $N$  is regular in  $M$  iff  $N = H_\gamma^M$ , where  $\gamma = \text{height}(N)$  is a regular cardinal in  $M$ .*

*Proof.* For the backward direction, suppose that  $N = H_\gamma^M$  where  $\gamma = \text{height}(N)$  is a regular cardinal in  $M$ . Then for all  $f : x \rightarrow N$ , with  $x \in N$  and  $f \in M$ , certainly  $f''x \in N$  as well.

For the forward direction, indeed  $\gamma$  has to be regular in  $M$  since otherwise  $M$  would contain a cofinal function  $f : \alpha \rightarrow \gamma$  where  $\alpha < \gamma$ . By the transitivity of  $N$ , this implies that  $\alpha \in N$ . Thus  $\cup f''\alpha$  is in  $N \models \text{ZFC}^-$  by regularity, so  $\gamma \in N$ , a contradiction. We have that  $N \subseteq H_\gamma^M$  since  $N$  is a transitive  $\text{ZFC}^-$  model, so the transitive closure of elements of  $N$  may be computed in  $N$  and thus have size less than  $\gamma$ , so they are in  $H_\gamma^M$  as  $\gamma \in M$ . To show that  $H_\gamma^M \subseteq N$ , let  $x \in H_\gamma^M$ . We assume by  $\in$ -induction that  $x \subseteq N$ . Then there is a surjection  $f : \alpha \rightarrow x$  where  $\alpha < \gamma$ , in  $M$ . Hence by regularity,  $x = f''\alpha \in N$  as desired.  $\square$

**Definition 2.7.** A structure  $M$  is *full* so long as  $M$  is transitive,  $\omega \in M$ , and there is a  $\gamma$  such that  $M$  is regular in  $L_\gamma(M)$  where  $L_\gamma(M) \models \text{ZFC}^-$ .

Perhaps the property of fullness seems rather mysterious at first, but the fullness of a countable structure guarantees that it is not pointwise definable, a necessary property of some of the models that come up in the definition subcomplete forcing.

**Lemma 2.8.** *If  $M$  is countable and full, then  $M$  is not pointwise definable.*

*Proof.* Suppose toward a contradiction that  $M$  is countable, full, and pointwise definable. By fullness there is some  $L_\gamma(M) \models \text{ZFC}^-$  such that  $M$  is regular in  $L_\gamma(M)$ . By pointwise definability, for each element  $x \in M$ , we have attached to it some formula  $\varphi(x)$  such that  $M \models \varphi(x)$  uniquely, meaning that  $\varphi(y)$  fails for every other element  $y \in M$ . Thus in  $L_\gamma(M)$  we may define a function  $f : \omega \cong M$ , that takes the  $n$ th formula in the language of set theory to its unique witness in  $M$ . In particular we have that  $L_\gamma(M)$  witnesses that  $M$  is countable. However, this would allow  $M$  to witness its own countability, since  $M$  must contain  $f$  as well by regularity.  $\square$

Let  $N$  be a transitive  $\text{ZFC}^-$  model. Suppose  $\bar{N} \cong X \preccurlyeq N$  where  $X$  is countable and  $\bar{N}$  is full. Then there may possibly be more than one elementary embedding  $\sigma : \bar{N} \prec N$ . Indeed: if there was only one unique embedding, then  $\bar{N}$  may be defined pointwise, by elementarity of the unique map, and since  $\bar{N}$  is countable.

The following lemma will prove useful when showing that various forcing notions are subcomplete, since subcompleteness requires certain ground-model Skolem hulls to match those in forcing extensions.

**Lemma 2.9** (Jensen). *Let  $\mathbb{P}$  be a forcing notion, and let  $\delta = \delta(\mathbb{P})$  be the smallest size of a dense subset in  $\mathbb{P}$ . Suppose that  $\mathbb{P} \in H_\theta \subseteq N$  where  $N = L_\tau[A] \models \text{ZFC}^-$ . Let  $\sigma : \bar{N} \prec N$  where  $\bar{N}$  is countable and full, and let  $\sigma(\bar{\mathbb{P}}) = \mathbb{P}$ .*

*Suppose  $G \subseteq \mathbb{P}$  is  $N$ -generic and  $\bar{G} \subseteq \bar{\mathbb{P}}$  is  $\bar{N}$ -generic, and that  $\sigma''\bar{G} \subseteq G$ , so  $\sigma$  lifts (or extends) in  $V[G]$  to an embedding  $\sigma^* : \bar{N}[\bar{G}] \prec N[G]$ .*

*Then*

$$N \cap \text{Hull}^{N[G]}(\delta \cup \text{range}(\sigma^*)) = \text{Hull}^N(\delta \cup \text{range}(\sigma)).$$

*Proof.* First, we establish that  $\text{Hull}^N(\delta \cup \text{range}(\sigma)) \subseteq \text{Hull}^{N[G]}(\delta \cup \text{range}(\sigma^*)) \cap N$ . To see this, let  $x \in \text{Hull}^N(\delta \cup \text{range}(\sigma))$ . Then  $x$  is  $N$ -definable from  $\xi < \delta$  and  $\sigma(\bar{z})$  where  $\bar{z} \in \bar{N}$ . Since  $\sigma^*$  extends  $\sigma$ , this means that  $x \in N$  and that  $x$  is  $N[G]$ -definable from  $\xi$  and  $\sigma^*(\bar{z})$ . This is because  $N = L_\tau[A]$  is definable in  $N[G]$  using  $A$ .

For the other direction, let  $x \in \mathcal{Hull}^{N[G]}(\delta \cup \text{range}(\sigma^*)) \cap N$ . Then  $x$  is  $N[G]$ -definable from  $\xi < \delta$  and  $\sigma^*(\bar{z})$  where  $\bar{z} \in \bar{N}[\bar{G}]$ ; and also  $x \in N$ . Letting  $\dot{z} \in \bar{N}^{\bar{\mathbb{P}}}$  such that  $\bar{z} = \dot{z}^{\bar{G}}$  we have

$$\sigma^*(\bar{z}) = \sigma^*(\dot{z}^{\bar{G}}) = \sigma(\dot{z})^G$$

so we have that there is some formula  $\varphi$  such that  $x$  is the unique witness:

$$x = \text{that } y \text{ where } N[G] \models \varphi(y, \xi, \sigma(\dot{z})^G).$$

Take  $f \in \bar{N}$  mapping  $\bar{\delta}$  onto a dense subset of  $\bar{\mathbb{P}}$ . Then  $\sigma(f)$  maps  $\delta$  onto a dense subset of  $\mathbb{P}$ . Thus there is  $\nu < \delta$  such that  $\sigma(f)(\nu) \in G$  and  $\sigma(f)(\nu) \Vdash \varphi(\check{x}, \check{\xi}, \sigma(\dot{z}))$ . Thus

$$x = \text{that } y \text{ where } \sigma(f)(\nu) \Vdash_{\mathbb{P}}^N \varphi(\check{y}, \check{\xi}, \sigma(\dot{z}))$$

so  $x \in \mathcal{Hull}^N(\delta \cup \text{range}(\sigma))$ .  $\square$

**2.3. Subcomplete Forcing.** Subcompleteness should be seen as a weakening of the class of complete (a.k.a. countably closed) forcing notions. The following definitions and proofs are due to Jensen [3, Chapter 3].

**Definition 2.10.** A forcing notion  $\mathbb{P}$  is *complete* so long as for sufficiently large  $\theta$  we have that once we are in the following situation:

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \text{ZFC}^-$  where  $\tau > \theta$  and  $A \subseteq \tau$
- $\sigma : \bar{N} \cong X \prec N$  where  $X$  is countable and  $\bar{N}$  is transitive
- $\sigma(\bar{\theta}, \bar{\mathbb{P}}, \bar{s}) = \theta, \mathbb{P}, s$  for some  $s \in N$ ;

then we have that if  $\bar{G}$  is  $\bar{\mathbb{P}}$ -generic over  $\bar{N}$  then there is a *completeness condition*  $p \in \mathbb{P}$  forcing that whenever  $G \ni p$  is  $\mathbb{P}$ -generic,  $\sigma^{\text{``}}\bar{G} \subseteq G$ .

In particular, below the condition  $p$  we have that  $\sigma$  lifts to an embedding  $\sigma^* : \bar{N}[\bar{G}] \prec N[G]$ . We say that such a  $\theta$  as above *verifies* the completeness of  $\mathbb{P}$ .

The adjustment made to get subcomplete forcings is to not necessarily insist the the original embedding lifts in the forcing extension. Instead subcompleteness asks for there to be an embedding in the extension, an embedding that is sufficiently similar to the original embedding, that lifts. However, as discussed in Section 2.2, the domain of the embedding must be *full* to ensure that there can even consistently be more than one embedding. The definition is given below.

**Definition 2.11.** A forcing notion  $\mathbb{P}$  is *subcomplete* so long as, for sufficiently large  $\theta$  we have that whenever we are in the *standard setup*, i.e.,

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \text{ZFC}^-$  where  $\tau > \theta$  and  $A \subseteq \tau$
- $\sigma : \bar{N} \cong X \prec N$  where  $X$  is countable and  $\bar{N}$  is full
- $\sigma(\bar{\theta}, \bar{\mathbb{P}}, \bar{s}) = \theta, \mathbb{P}, s$  for some  $s \in N$ ;

then we have that if  $\bar{G}$  is  $\bar{\mathbb{P}}$ -generic over  $\bar{N}$  then there is a *subcompleteness condition*  $p \in \mathbb{P}$  such that whenever  $G \ni p$  is  $\mathbb{P}$ -generic, there is  $\sigma' \in V[G]$  satisfying:

- (1)  $\sigma' : \bar{N} \prec N$
- (2)  $\sigma'(\bar{\theta}, \bar{\mathbb{P}}, \bar{s}) = \theta, \mathbb{P}, s$
- (3)  $\mathcal{Hull}^N(\delta(\mathbb{P}) \cup \text{range}(\sigma')) = \mathcal{Hull}^N(\delta(\mathbb{P}) \cup X)$
- (4)  $\sigma'^{\text{``}}\bar{G} \subseteq G$ .

In other words and in particular, the subcompleteness condition  $p$  forces that there is an embedding  $\sigma'$  in the extension  $V[G]$  that lifts, by (4), to an embedding  $\sigma'^* : \bar{N}[\bar{G}] \prec N[G]$  in  $V[G]$ .

We say that such a  $\theta$  as above *verifies* the subcompleteness of  $\mathbb{P}$ .

Often we write  $\delta$  instead of  $\delta(\mathbb{P})$  for the weight of  $\mathbb{P}$  when there should be no confusion as to which poset  $\mathbb{P}$  we are working with.<sup>1</sup>

*Remark 2.12.* If  $\mathbb{P}$  is subcomplete as verified by  $\theta$ , then  $\mathbb{P}$  is subcomplete as verified by  $\theta' > \theta$  since  $\mathbb{P} \in H_\theta$  obviously implies that  $\mathbb{P} \in H_{\theta'}$ . Thus a standard setup such as  $\mathbb{P} \in H_{\theta'} \subseteq N = L_\tau[A]$  for  $\tau > \theta'$  may be reduced down to the case where  $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A]$  and  $\tau > \theta' > \theta$  using the same  $\tau$ . In particular,  $\mathbb{P}$  is subcomplete as long as it is verified to be subcomplete by some  $\theta$ . So “sufficiently large  $\theta$ ” may be replaced with “some  $\theta$ ” in the definition of subcompleteness.<sup>2</sup>

**2.4. Barwise Theory.** In order to show that many posets are subcomplete, Jensen exploits Barwise Theory and techniques using countable admissible structures to obtain transitive models of infinitary languages. Barwise creates an  $M$ -finite predicate logic, a first order theory in which arbitrary, but  $M$ -finite, disjunctions and conjunctions are allowed. The following is an outline of [3, Chapter 1 & 2].

**Definition 2.13.** Let  $M$  be a transitive structure with potentially infinitely many predicates. A theory defined over  $M$  is  $M$ -finite so long as it is in  $M$ . A theory is  $\Sigma_1(M)$ , also known as  $M$ -recursively enumerable or  $M$ -re, if the theory is  $\Sigma_1$ -definable, with parameters from  $M$ .

Of course this may be generalized to the entire usual Levy hierarchy of formulae, but only  $\Sigma_1(M)$  is needed in this paper. If  $\mathcal{L}$  is a  $\Sigma_1(M)$ -definable language or theory, the rough idea is that to check whether a sentence is in  $\mathcal{L}$ , one should imagine enumerating the formulae of  $\mathcal{L}$  to find a sentence and a witness to it in the structure  $M$ . The structures considered here are admissible, defined below.

**Definition 2.14.** A transitive structure  $M$  is *admissible* if it models the axioms of Kripke-Platek Set Theory (KP) which consists of the axioms of Empty Set, Pairing, Union,  $\Sigma_0$ -Collection, and  $\Sigma_0$ -Separation.

Jensen also makes use of models of  $\mathbf{ZF}^-$  that are not necessarily well-founded.

**Definition 2.15.** Let  $\mathfrak{A} = \langle A, \in_{\mathfrak{A}}, B_1, B_2, \dots \rangle$  be a (possibly) ill-founded model of  $\mathbf{ZF}^-$ , where  $\mathfrak{A}$  is allowed to have predicates other than  $\in$ . The *well-founded core* of  $\mathfrak{A}$ , denoted  $\text{wfc}(\mathfrak{A})$ , is the restriction of  $\mathfrak{A}$  to the set of all  $x \in A$  such that  $\in_{\mathfrak{A}} \cap \mathcal{C}(x)^2$  is well founded, where  $\mathcal{C}(x)$  is the closure of  $\{x\}$  under  $\in_{\mathfrak{A}}$ . A model  $\mathfrak{A}$  of  $\mathbf{ZF}^-$  is *solid* so long as  $\text{wfc}(\mathfrak{A})$  is transitive and  $\in_{\text{wfc}(\mathfrak{A})} = \in \cap \text{wfc}(\mathfrak{A})$ .<sup>2</sup>

Jensen [3, Section 1.2] notes that every consistent set of sentences in  $\mathbf{ZF}^-$  has a solid model, and if  $\mathfrak{A}$  is solid, then  $\omega \subseteq \text{wfc}(\mathfrak{A})$ . In addition,

**Fact 2.16** (Jensen). *If  $\mathfrak{A} \models \mathbf{ZF}^-$  is solid, then  $\text{wfc}(\mathfrak{A})$  is admissible.*

**Definition 2.17.** Let  $M$  be admissible, and let  $\mathcal{L} = \mathcal{L}(M)$  be an infinitary, axiomatized theory in  $M$ -finitary logic, called an  $\in$ -theory over  $M$ , with a fixed predicate  $\dot{\in}$  and *special constants* denoted  $\underline{x}$  for elements  $x \in M$ . The underlying axioms for these  $\in$ -theories will always involve  $\mathbf{ZFC}^-$  and some basic axioms ensuring that  $\dot{\in}$  behaves nicely; the **Basic Axioms** are:

- **Extensionality**
- A statement positing the extensionality of  $\dot{\in}$ , which is a scheme of formulae defined for each member of  $M$ . For each  $x \in M$ , include an  $M$ -re sentence (meaning it quantifies over  $M$ -finite sentence):

$$\forall v \left( v \dot{\in} \underline{x} \iff \bigvee_{z \in x} v = \underline{z} \right).$$

<sup>1</sup>See Section 2.1 for more on  $\delta(\mathbb{P})$ .

<sup>2</sup>See [3, Section 3.1 Lemma 2.4].

Here  $\bigvee$  denotes an infinite disjunction in the language.

An important fact ensured by our **Basic Axioms** is that the interpretations of these special constants in any solid model of the theory are the same as in  $M$ :

**Fact 2.18** (Jensen). *Let  $M$  be as in the above definition. Let  $\mathfrak{A}$  be a solid model of the  $\in$ -theory  $\mathcal{L}$ . Then for all  $x \in M$ , we have that  $\underline{x}^{\mathfrak{A}} = x \in \text{wfc}(\mathfrak{A})$ .*

Jensen uses the techniques of Barwise to come up with a proof system in this context, in which consistency of  $\in$ -theories can be discussed. In particular,  $\in$ -theories are correct: if there is a model of such a theory, then the theory is consistent.

**Fact 2.19** (Barwise Correctness). *Let  $\mathcal{L}$  be an  $\in$ -theory. If  $A$  is a set of  $\mathcal{L}$ -statements and  $\mathfrak{A} \models A$ , then  $A$  is consistent.*

Furthermore, compactness and completeness are shown, relativized to the  $M$ -finite predicate logics that are used here; solid models of consistent  $\Sigma_1(M)$   $\in$ -theories are produced for countable admissible structures  $M$ .

**Fact 2.20** (Barwise Completeness). *Let  $M$  be a countable admissible structure. Let  $\mathcal{L}$  be a consistent  $\Sigma_1(M)$   $\in$ -theory such that  $\mathcal{L} \vdash \text{ZF}^-$ . Then  $\mathcal{L}$  has a solid model  $\mathfrak{A}$  such that*

$$\text{Ord} \cap \text{wfc}(\mathfrak{A}) = \text{Ord} \cap M.$$

The following definition generalizes the concept of fullness.

**Definition 2.21.** A transitive  $\text{ZFC}^-$  model  $N$  is *almost full* so long as  $\omega \in N$  and there is a solid  $\mathfrak{A} \models \text{ZFC}^-$  with  $N \in \text{wfc}(\mathfrak{A})$  and  $N$  is regular in  $\mathfrak{A}$ .

Clearly if  $N$  is full, then  $N$  is almost full.

A useful technique when showing a particular forcing is subcomplete, once many different embeddings can be constructed that approximate the embedding required for subcompleteness, is to be able to transfer the consistency of  $\in$ -theories over one admissible structure to another.

**Definition 2.22.** If  $N$  is a transitive  $\text{ZFC}^-$  model, let  $\delta_N$  be the least  $\delta$  such that  $L_\delta(N)$  is admissible.

**Fact 2.23** (The Transfer Lemma). *Let  $N_1$  be almost full, and suppose that  $k : N_1 \prec N_0$  cofinally. If the  $\in$ -theory  $\mathcal{L}(L_{\delta_{N_1}}(N_1))$  is  $\Sigma_1$  over parameters  $N_1, p_1, \dots, p_n \in N_1$  and the same  $\in$ -theory over  $L_{\delta_{N_0}}(N_0)$ , namely  $\mathcal{L}(L_{\delta_{N_0}}(N_0))$ , is  $\Sigma_1$  over parameters  $N_0, k(p_1), \dots, k(p_n)$ , then if  $\mathcal{L}(L_{\delta_{N_1}}(N_1))$  is consistent, it follows that  $\mathcal{L}(L_{\delta_{N_0}}(N_0))$  is consistent as well.*

**2.5. Liftups.** The following definitions are meant to describe a method used to obtain useful embeddings, outlining a technique that is ostensibly the ultrapower construction. These embeddings facilitate the use of Barwise theory to obtain the consistency of the existence of desirable embeddings. Refer to [3, Chapter 1] for the definitions and theorems stated here.

**Definition 2.24.** Let  $\bar{N}$  and  $N$  be transitive  $\text{ZFC}^-$  models. We say that an elementary embedding  $\sigma : \bar{N} \prec N$  is *cofinal* so long as for each  $x \in N$  there is some  $u \in \bar{N}$  such that  $x \in \sigma(u)$ .

Let  $\alpha \in \bar{N}$ . We say that  $\sigma$  is  $\alpha$ -*cofinal* so long as every such  $u$  has size less than  $\alpha$  as computed in  $\bar{N}$ .

**Definition 2.25.** Let  $\alpha > \omega$  be a regular cardinal in  $\bar{N}$ , a transitive  $\text{ZFC}^-$  model. Let

$$\bar{\sigma} : H_{\alpha}^{\bar{N}} \prec H \text{ cofinally,}$$

where  $H$  is transitive. By a *transitive liftup* of  $\langle \bar{N}, \bar{\sigma} \rangle$  we mean a pair  $\langle N_*, \sigma_* \rangle$  such that

- $N_*$  is transitive
- $\sigma_* : \bar{N} \prec N_*$   $\alpha$ -cofinally
- $\sigma_* \upharpoonright H_\alpha^{\bar{N}} = \bar{\sigma}$ .

Reminiscent of ultrapowers, transitive liftings can be characterized in the following way:

**Lemma 2.26** (Jensen). *Let  $\bar{N}, N$  be transitive  $\text{ZFC}^-$  models with  $\sigma : \bar{N} \prec N$ . Then,  $\sigma$  is  $\alpha$ -cofinal if and only if elements of  $N$  are of the form  $\sigma(f)(\beta)$  for some  $f : \gamma \rightarrow \bar{N}$  where  $\gamma < \alpha$  and  $\beta < \sigma(\gamma)$ .*

*Proof.* We show each direction of the equivalence separately.

For the first direction, let  $x \in N$ , and take  $u \in \bar{N}$  with  $x \in \sigma(u)$  such that  $|u| < \alpha$  in  $\bar{N}$ . Let  $|u| = \gamma$ , and take  $f : \gamma \rightarrow u$  a bijection in  $\bar{N}$ . Then  $\sigma(f) : \sigma(\gamma) \rightarrow \sigma(u)$  is also a bijection in  $N$  by elementarity. Since  $x \in \sigma(u)$  we also have that  $x$  has a preimage under  $\sigma(f)$ , say  $\beta$ . So  $\sigma(f)(\beta) = x$  as desired.

For the backward direction, let  $x = \sigma(f)(\beta)$  be an element of  $N$ , for  $f : \gamma \rightarrow \bar{N}$  where  $\gamma < \alpha$  in  $\bar{N}$  and  $\beta < \sigma(\gamma)$ . Define  $u = f''\gamma$ . Then in  $\bar{N}$  we have that  $|u| < \alpha$ . In addition we have that  $x \in \sigma(u)$ , since  $\sigma(u)$  is in the range of  $\sigma(f)$ , where  $x$  lies.  $\square$

Furthermore, Jensen shows that transitive liftings exist so long as an embedding already exists, using an ultrapower-like construction, and have a uniqueness property.

**Fact 2.27** (Interpolation). *Let  $\sigma : \bar{N} \prec N$  with  $\bar{N} \models \text{ZFC}^-$  transitive, and let  $\alpha \in \bar{N}$  be a regular cardinal. Then:*

- (1) *The transitive lifting  $\langle N_*, \sigma_* \rangle$  of  $\langle \bar{N}, \sigma \upharpoonright H_\alpha^{\bar{N}} \rangle$  exists.*
- (2) *There is a unique  $k_* : N_* \prec N$  such that  $k_* \circ \sigma_* = \sigma$  and  $k_* \upharpoonright \bigcup \sigma'' H_\alpha^{\bar{N}} = \text{id}$ .*

For the next useful lemma, a more general notion of liftings needs to be defined. Of course the rich theory is established by Jensen, and is explored in detail in his notes.

**Definition 2.28.** Let  $\mathfrak{A}$  be a solid model of  $\text{ZFC}^-$  and let  $\tau \in \text{wfc}(\mathfrak{A})$  be an uncountable cardinal in  $\mathfrak{A}$ . Let

$$\sigma : H_\tau^{\mathfrak{A}} \prec H \text{ cofinally,}$$

where  $H$  is transitive. Then by a *lifting* of  $\langle \mathfrak{A}, \sigma \rangle$ , we mean a pair  $\langle \mathfrak{A}_*, \sigma_* \rangle$  such that

- $\sigma_* \supseteq \sigma$
- $\mathfrak{A}_*$  is solid
- $\sigma_* : \mathfrak{A} \rightarrow_{\Sigma_0} \mathfrak{A}_*$   $\tau$ -cofinally
- $H \in \text{wfc}(\mathfrak{A}_*)$

**Fact 2.29** (Jensen). *Let  $\mathfrak{A}$  be a solid model of  $\text{ZFC}^-$ . Let  $\tau > \omega$ ,  $\tau \in \text{wfc}(\mathfrak{A})$ , and let*

$$\sigma : H_\tau^{\mathfrak{A}} \prec H \text{ cofinally,}$$

*where  $H$  is transitive. Then  $\langle \mathfrak{A}, \sigma \rangle$  has a lifting  $\langle \mathfrak{A}_*, \sigma_* \rangle$ .*

The following lemma states that transitive liftings of full models are almost full.

**Lemma 2.30** (Jensen). *Let  $N = L_\tau[A] \models \text{ZFC}^-$  and  $\sigma : \bar{N} \prec N$  where  $\bar{N}$  is full. Suppose that  $\langle N_*, \sigma_* \rangle$  is a transitive lifting of  $\langle \bar{N}, \bar{\sigma} \rangle$ . Then  $N_*$  is almost full.*

*Proof.* Let  $L_\gamma(\bar{N})$  witness the fullness of  $\bar{N}$ . We will now apply Interpolation (Fact 2.27) to  $\mathfrak{A} = L_\gamma(\bar{N})$ , which makes sense since certainly  $\mathfrak{A}$  is a transitive model  $\text{ZFC}^-$ . Additionally, by Lemma 2.6 we have that  $\bar{N} = H_\tau^{\mathfrak{A}}$ , where  $\tau = \text{height}(\bar{N})$ . Since  $\langle N_*, \sigma_* \rangle$  is a transitive lifting, we have that

$$\sigma_* : H_\tau^{\mathfrak{A}} \prec N_* \text{ cofinally,}$$

where  $N_*$  is transitive. Thus since  $\mathfrak{A}$  is transitive,  $\langle \mathfrak{A}, \sigma_* \rangle$  has a liftup  $\langle \mathfrak{A}_*, \sigma_{**} \rangle$ , where  $\mathfrak{A}_* \models \text{ZFC}^-$  since  $\mathfrak{A}$  does,  $\mathfrak{A}_*$  is solid, where

$$\sigma_{**} : \mathfrak{A} \prec \mathfrak{A}_* \text{ } \tau\text{-cofinally.}$$

We have that  $N_* \subseteq \text{wfc}(\mathfrak{A}_*)$  and  $\tau_* = \sigma_{**}(\tau) = \text{height}(N_*)$  is regular since  $\tau$  is. Furthermore, we will show that  $N_* = H_{\tau_*}^{\mathfrak{A}_*}$ , completing the proof:

Certainly it is the case that  $N_* \subseteq H_{\tau_*}^{\mathfrak{A}_*}$ . But if  $x \in H_{\tau_*}$  in  $\mathfrak{A}_*$ , then by regularity we have that  $x \in \sigma_{**}(u)$  in  $\mathfrak{A}_*$ , where  $u \in \mathfrak{A}$ , and  $|u| < \tau$  in  $\mathfrak{A}$ . Let  $v = u \cap H_\tau$  in  $\mathfrak{A}$ . Then  $v \in \bar{N}$ , since  $\bar{N}$  is regular in  $\mathfrak{A}$ . But then  $x \in \sigma_*(v) \in N_*$ . So  $x \in N_*$ .  $\square$

### 3. GENERALIZED DIAGONAL PRIKRY FORCING

Generalized diagonal Prikry forcing is designed to add a point below every measurable cardinal in an infinite discrete set of measurable cardinals. In this section it is shown that such forcings are subcomplete.

**Definition 3.1.** Let  $D$  be an infinite discrete set of measurable cardinals, meaning a set of measurable cardinals that does not contain any of its limit points. For  $\kappa \in D$  let  $U(\kappa)$  be a normal measure on  $\kappa$ , and let  $\mathcal{U}$  denote the sequence of the  $U(\kappa)$ 's.

Define  $\mathbb{D} = \mathbb{D}(\mathcal{U})$ , *generalized diagonal Prikry forcing* from the list of measures  $\mathcal{U}$ , by taking conditions of the form  $(s, A)$  satisfying the following:

- The *stem* of the condition,  $s$ , is a function with domain in  $[D]^{<\omega}$  taking each measurable cardinal  $\kappa \in \text{dom}(s)$  to some ordinal  $s(\kappa) < \kappa$ .
- The *upper part* of the condition,  $A$ , is a function with domain  $D \setminus \text{dom}(s)$  taking each measurable cardinal  $\kappa \in \text{dom}(A)$  to some measure-one set  $A(\kappa) \in U(\kappa)$ .

The extension relation on conditions in  $\mathbb{D}$  is defined so that  $(s, A) \leq (t, B)$  so long as

- $s \supseteq t$ .
- The points in  $s$  not in  $t$  come from  $B$ , i.e., for all  $\kappa \in \text{dom}(s) \setminus \text{dom}(t)$ ,  $s(\kappa) \in B(\kappa)$ .
- For all  $\kappa \in \text{dom}(A)$ ,  $A(\kappa) \subseteq B(\kappa)$ .

If  $G$  is a generic filter for  $\mathbb{D}$ , then its associated  $\mathbb{D}$ -generic sequence is

$$S = S_G = \bigcup \{s \mid \exists A (s, A) \in G\}.$$

Here the definition of  $\mathbb{D}(\mathcal{U})$  differs from that as in Fuchs of generalized Prikry forcing [1]. The main difference is that here only one point is added below each measurable cardinal  $\kappa \in D$ , which is done for simplicity's sake. It is not hard to see that the following theorem also shows that the forcing adding countably many points below each measurable cardinal in  $D$  is subcomplete. Adding countably many points below each measurable cardinal in  $D$  would collapse the cofinality of each  $\kappa \in D$  to be  $\omega$ , as one expects of a Prikry-like forcing.

Also in the above definition it hasn't been enforced that the stem a condition consists only of ordinals that are wedged between successive measurables in  $D$ ; ie. for  $\kappa \in D$ , it is not explicitly insisted that  $s(\kappa) \in [\sup(D \cap \kappa), \kappa)$ . However, it is dense in  $\mathbb{D}(\mathcal{U})$  for the conditions to be that way, since conditions may be strengthened by restricting their upper parts to a tail. Thus such a restriction may be freely added to the following genericity condition on  $\mathbb{D}(\mathcal{U})$ .

**Fact 3.2** (Fuchs). *Let  $D$  be an infinite discrete set of measurable cardinals, with  $\mathcal{U}$  a corresponding list of measures  $\langle U(\kappa) \mid \kappa \in D \rangle$ . Then an increasing sequence of ordinals  $S = \langle S(\kappa) \mid \kappa \in D \rangle$ , where for each  $\kappa \in D$ , we have that*

$$\sup(D \cap \kappa) < S(\kappa) < \kappa,$$



is a  $\mathbb{D}(\mathcal{U})$ -generic sequence if and only if for all  $\mathcal{X} = \langle X_\kappa \in U(\kappa) \mid \kappa \in D \rangle$ , the set

$$\{\kappa \in D \mid S(\kappa) \notin X_\kappa\}$$

is finite.

The above genericity criterion on generalized diagonal Prikry forcing is similar to the Mathias criterion for Prikry forcing. It was shown in [1, Theorem 1].

**Theorem 3.3.** *Let  $D$  be an infinite discrete set of measurable cardinals. Let  $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D \rangle$  be a list of measures associated to  $D$ . The generalized diagonal Prikry forcing  $\mathbb{D} = \mathbb{D}(\mathcal{U})$  is subcomplete.*

*Proof.* Let  $\theta > \delta(\mathbb{D}) = \delta$  be large enough, so that  $[\delta]^{<\omega_1} \in H_\theta$ .

First of all, it must be the case that  $\delta$ , the density of  $\mathbb{D}$ , must be larger than all of the measurable cardinals in  $D$ .

*Claim 1.*  $\delta \geq \sup D$ .

*Pf.* Suppose instead that there is a dense  $E \subseteq \mathbb{D}$  such that  $\sup D \geq \kappa^* > |E|$  for some  $\kappa^* \in D$ . Then for each condition  $(s, A) \in E$  either  $\kappa^* \in \text{dom } s$  or  $\kappa^* \in \text{dom } A$ . So taking  $E^* = \{(s, A) \in E \mid \kappa^* \in \text{dom } s\} \subseteq E$ , since  $|E^*| < \kappa^*$  as well, there is an  $\alpha < \kappa^*$  such that  $\sup_{(s, A) \in E^*} s(\kappa^*) = \alpha$ . Let  $p = (t, B) \in \mathbb{D}$  be defined so that  $t(\kappa^*) = \alpha$  and  $B(\kappa) = \kappa$  for all  $\kappa \in D \setminus \{\kappa^*\}$ . Then  $p$  cannot be strengthened by any condition in  $E$  since  $\kappa^*$  is not in any of the stems of conditions in  $E$ . So dense subsets of  $\mathbb{D}$  must have size at least  $\sup D$ .  $\square$

Let  $\nu = \delta^+$ . Let  $\kappa(0)$  be the first measurable cardinal in  $D$ .

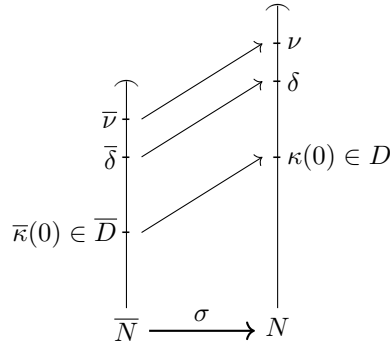
In order to show that  $\mathbb{D}$  is subcomplete, suppose we are in the standard setup:

- $\mathbb{D} \in H_\theta \subseteq N = L_\tau[A] \models \text{ZFC}^-$  where  $\tau > \theta$  and  $A \subseteq \tau$
- $\sigma : \bar{N} \cong X \preceq N$  where  $X$  is countable and  $\bar{N}$  is full
- $\sigma(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \theta, \mathbb{D}, \mathcal{U}, c$  for some  $c \in N$ .

By our requirement on  $\theta$  being large enough, we've also ensured that  $N$  is closed under countable sequences of ordinals less than  $\delta$ .

In what follows, we will be taking a few different transitive liftings of restrictions of  $\sigma$ , and it will be useful to keep track of embeddings between  $\bar{N}$  and  $N$  pictorially. Although it's not extraordinarily illuminating at this point in our discussion, the following figure shows the situation we are currently in.

Here we place bars on everything that is relevant on the  $\bar{N}$  side of the embedding. In particular,  $\sigma(\bar{\delta}) = \delta$ ,  $\bar{\nu} = \delta^{+\bar{N}}$ ,  $\bar{D}$  is the discrete set of measurables in  $\bar{N}$  that each measure in  $\bar{\mathcal{U}}$  comes from, and  $\bar{\kappa}(0)$  is the first measurable in  $\bar{D}$ , in the sense of  $\bar{N}$ .



Toward showing that  $\mathbb{D}$  is subcomplete, we are additionally given some  $\overline{G} \subseteq \overline{\mathbb{D}}$  that is generic over  $\overline{N}$ . Rather than working with  $\overline{G}$ , we will work with  $\overline{S} = \langle \overline{S}(\overline{\kappa}) \mid \overline{\kappa} \in \overline{D} \rangle$ , its associated  $\mathbb{D}$ -generic sequence. We must show following, where  $C = \mathcal{Hull}^N(\delta \cup X)$ :

*Main Claim.* There is a  $\mathbb{D}$ -generic sequence  $S$  and a map  $\sigma' \in V[S]$  such that:

- (1)  $\sigma' : \overline{N} \prec N$
- (2)  $\sigma'(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta, \mathbb{D}, \mathcal{U}, c$
- (3)  $\mathcal{Hull}^N(\delta \cup \text{range}(\sigma')) = C$
- (4)  $\sigma' \text{ `` } \overline{S} \subseteq S$

*Pf.* This proof uses Barwise theory<sup>3</sup> heavily, and ultimately amounts to showing that a certain  $\in$ -theory,  $\mathcal{T}$ , which posits the existence of such a  $\sigma'$ , is consistent. Such an embedding  $\sigma'$  could only possibly exist in a suitable generic extension,  $V[S]$ , where  $S$  is  $\mathbb{D}$ -generic sequence that we will find later. Once we have such a suitable  $V[S]$ , we will use Barwise theory to find an appropriate admissible structure in  $V[S]$  for which the theory  $\mathcal{T}$ , positing the existence of such a suitable  $\sigma'$ , defined below, has a model.

For an admissible structure  $M$  with  $S, \overline{S}, \sigma, N, \theta, \mathbb{D}, \mathcal{U}, c \in M$  let the  $\in$ -theory  $\mathcal{T}(M)$  be defined over  $M$  as follows:

**predicates:**  $\in$

**constants:**  $\dot{\sigma}, \underline{x}$  for  $x \in M$

**axioms:** • ZFC<sup>-</sup> and Basic Axioms.

- $\dot{\sigma} : \overline{N} \prec N$
- $\dot{\sigma}(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta, \mathbb{D}, \mathcal{U}, c$
- $\mathcal{Hull}^N(\underline{\delta} \cup \text{range}(\dot{\sigma})) = \mathcal{Hull}^N(\underline{\delta} \cup \text{range}(\underline{\sigma}))$
- $\dot{\sigma} \text{ `` } \overline{S} \subseteq S$ .

The  $\in$ -theory is  $\Sigma_1(M)$ , since all of the axioms are  $M$ -finite except for the Basic Axioms, which altogether are  $M$ -re as each of them are  $M$ -finite.

We need to find an appropriate  $\mathbb{D}$ -generic sequence  $S$  and a suitable admissible structure  $M$  containing  $S$  so that  $\mathcal{T}(M)$  is consistent. To do this we use transitive liftings and Barwise theory. Transitive liftings will give us the consistency of certain embeddings that approximate the one we are looking for, and we will rely on Barwise Completeness (*Fact 2.20*) to obtain the existence of a model with our desired properties.

Toward this end, let's take what will turn out to be our first transitive lifting, which is in some sense ensuring the consistency of (3) of our main claim.

Let  $k_0 : N_0 \cong C$  where  $N_0$  is transitive, and set  $\sigma_0 = k_0^{-1} \circ \sigma$  and  $\sigma_0(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta_0, \mathbb{D}_0, \mathcal{U}_0, c_0$ . Since  $\delta \subseteq C$  and  $N_0$  is transitive,  $\sigma_0(\overline{\delta}) = \delta$ .

Indeed  $N_0$  is actually a transitive lifting:

*Claim 2.*  $\langle N_0, \sigma_0 \rangle$  is the transitive lifting of  $\langle \overline{N}, \sigma \restriction H_{\overline{\nu}}^{\overline{N}} \rangle$ .

*Pf.* Recall that  $\nu = \delta^+$ , and  $\overline{\nu} = \overline{\delta}^+$ . It must be shown that the embedding  $\sigma_0 : \overline{N} \prec N_0$  is  $\overline{\nu}$ -cofinal and that  $\sigma_0 \restriction H_{\overline{\nu}}^{\overline{N}} = \sigma \restriction H_{\overline{\nu}}^{\overline{N}}$ .

To see that  $\sigma_0$  is  $\overline{\nu}$ -cofinal, let  $x \in N_0$ . Then  $k_0(x) \in C = \mathcal{Hull}^N(\delta \cup X)$  so  $k_0(x)$  is uniquely  $N$ -definable from  $\xi < \delta$  and  $\sigma(\overline{z})$  where  $\overline{z} \in \overline{N}$ . In other words,

$$k_0(x) = \text{that } y \text{ such that } N \models \varphi(y, \xi, \sigma(\overline{z})).$$

<sup>3</sup>Key definitions, facts, and theorems of Barwise theory are summarized Section 2.4.

Let  $u \in \bar{N}$  be defined as

$$u = \{w \in \bar{N} \mid w = \text{that } y \text{ such that } \bar{N} \models \varphi(y, \zeta, \bar{z}) \text{ for some } \zeta < \bar{\delta}\}.$$

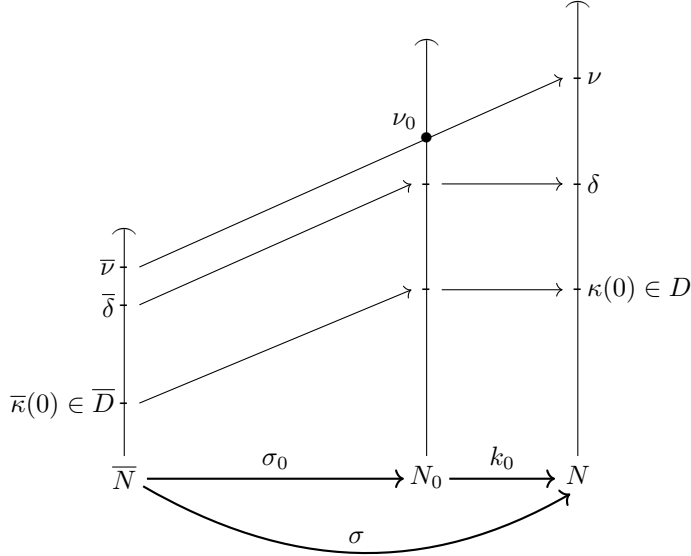
Certainly  $u$  is non-empty by elementarity, since  $k_0(x) \in \sigma(u)$ . Furthermore,  $|u| \leq \bar{\delta} < \bar{\nu}$  since every  $w \in u$  is unique, and needs a corresponding  $\zeta < \bar{\delta}$  to satisfy the formula  $\varphi$  with. Thus  $x \in k_0^{-1}(\sigma(u)) = \sigma_0(u)$  with  $|u| < \bar{\nu}$  in  $\bar{N}$ , as desired.

Since  $X \cup \delta \subseteq C$ , the Skolem hull in  $N$ , we know that  $\sigma^{\ast} H_{\bar{\nu}}^{\bar{N}} \subseteq C$ . Thus  $k_0^{-1} \upharpoonright \sigma^{\ast} H_{\bar{\nu}}^{\bar{N}} = \text{id}$ . Therefore  $\sigma_0 \upharpoonright H_{\bar{\nu}}^{\bar{N}} = \sigma \upharpoonright H_{\bar{\nu}}^{\bar{N}}$ , finishing the proof of the claim.  $\square$

Since  $\bar{\nu}$  is regular in  $\bar{N}$ , in  $N_0$  so is  $\nu_0 = \sigma_0(\bar{\nu}) = \sup \sigma_0^{\ast} \bar{\nu}$ . By Interpolation (*Fact 2.27*), we may say that  $k_0$  is defined by

$$k_0 : N_0 \prec N \text{ where } k_0 \circ \sigma_0 = \sigma \text{ and } k_0 \upharpoonright \nu_0 = \text{id}.$$

In particular,  $\nu_0$  is the critical point of  $k_0$ , which is continuous below  $\nu_0$ . Thus we are in a situation that we will represent with the following diagram:



Already, we can say that  $\sigma_0$  looks like it has one nice property:  $\mathcal{Hull}^{N_0}(\delta \cup \text{range}(\sigma_0)) = C$ , which somewhat looks like (3) of the main claim. However, we have not yet performed forcing,  $\sigma_0$  is definable in  $V$ , and we still need to find a way to extend the generic sequence  $\bar{S}$  to a  $\mathbb{D}$ -generic sequence over  $N$ . We still have a lot more work to do before finding  $\sigma'$ .

We shall define another  $\in$ -theory,  $\mathcal{L}_*$ . This will assist us in obtaining the diagonal Prikry extension  $V[S]$  we need to satisfy our main claim. In order to do this, we will take another transitive liftup and apply Transfer (*Fact 2.23*), in order to see that this new  $\in$ -theory is consistent over an admissible structure on  $N_0$ .

Since we will be referring to the same  $\in$ -theory over two different transitive liftups, I would like to think of “ $*$ ” as a kind of placeholder for a transitive liftup in the following definition.

Suppose that  $\langle N_*, \sigma_* \rangle$  is a transitive liftup of  $\bar{N}$  along with some reasonable restriction of  $\sigma$ , ie. the liftup of  $\langle \bar{N}, \sigma \upharpoonright H_{\alpha}^{\bar{N}} \rangle$ , where  $\alpha \geq \bar{\kappa}(0)$  is regular in  $\bar{N}$ , and say

$$\sigma_*(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \theta_*, \mathbb{D}_*, \mathcal{U}_*, c_*.$$

Recall that  $\delta_{N_*}$  is the least such that  $L_{\delta_{N_*}}(N_*)$  is admissible.

Define the infinitary  $\in$ -theory  $\mathcal{L}(N_*, \sigma_*) = \mathcal{L}_*$  as follows:

- predicates:**  $\in$   
**constants:**  $\dot{\sigma}, \dot{S}, \underline{x}$  for  $x \in L_{\delta_{N_*}}(N_*)$   
**axioms:**
- ZFC<sup>-</sup> and Basic Axioms
  - $\dot{\sigma} : \bar{N} \prec N_*$  is  $\bar{\kappa}(0)$ -cofinal
  - $\dot{\sigma}(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \theta_*, \mathbb{D}_*, \mathcal{U}_*, c_*$
  - $\dot{S}$  is a  $\mathbb{D}_*$ -generic sequence over  $N_*$
  - $\dot{\sigma} \text{ `` } \bar{S} \subseteq \dot{S} \text{ ''}$ .

As defined, we have that  $\mathcal{L}_*$  is a  $\Sigma_1(L_{\delta_{N_*}}(N_*))$ -theory, since altogether the Basic Axioms are  $\Sigma_1(L_{\delta_{N_*}}(N_*))$ .

We claim that the theory is consistent.

*Claim 3.*  $\mathcal{L}_0 = \mathcal{L}(N_0, \sigma_0)$  is consistent.

*Pf.* Of course, it is not the case that  $\sigma_0$  is  $\bar{\kappa}(0)$ -cofinal - all we know is that it is  $\bar{\nu}$ -cofinal. However, we know how to find an elementary embedding that is  $\bar{\kappa}(0)$  cofinal: by taking a suitable transitive liftup.

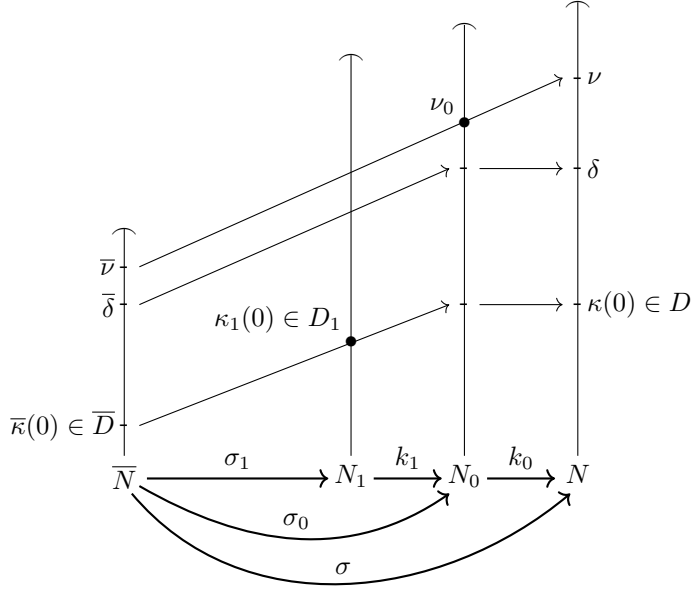
Let  $\langle N_1, \sigma_1 \rangle$  be the transitive liftup of  $\langle \bar{N}, \sigma \upharpoonright H_{\bar{\kappa}(0)}^{\bar{N}} \rangle$ , which exists by Interpolation (*Fact 2.27*). So we have that  $\sigma_1 \upharpoonright H_{\bar{\kappa}(0)}^{\bar{N}} = \sigma \upharpoonright H_{\bar{\kappa}(0)}^{\bar{N}}$ . Let

$$\sigma_1(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1.$$

Since  $\sigma_1 \upharpoonright H_{\bar{\kappa}(0)}^{\bar{N}} = \sigma \upharpoonright H_{\bar{\kappa}(0)}^{\bar{N}} = \sigma_0 \upharpoonright H_{\bar{\kappa}(0)}^{\bar{N}}$ , we also have a unique

$$k_1 : N_1 \prec N_0 \text{ where } k_1 \circ \sigma_1 = \sigma_0 \text{ and } k_1 \upharpoonright \kappa_1(0) = \text{id where } k_1(0) = k_1(\kappa(0)).$$

Indeed  $k_1$  is continuous below  $\kappa_1(0)$ . We illustrate the final picture below, with  $\sigma$  and all of the relevant transitive liftups.



We first show that  $\mathcal{L}_1 = \mathcal{L}(N_1, \sigma_1)$  is consistent, by seeing that it has a model. To do this, we will find a sequence extending  $\sigma_1 \text{``}\bar{S}$  that is  $\mathbb{D}_1$ -generic over  $N_1$ . Then we will use the Transfer Lemma to see that this transfers to the consistency of  $\mathcal{L}_0$ .

First, force with  $\mathbb{D}_1$ , which is diagonal Prikry over  $N_1$ , to obtain a diagonal Prikry sequence  $S'_1$ . Define, in  $V[S'_1]$ , a new sequence  $S_1$  as follows:

$$S_1(\kappa) = \begin{cases} S'_1(\kappa) & \text{if } \kappa \in D_1 \setminus \sigma_1 \text{``}\bar{D} \\ \sigma_1(\bar{S}(\bar{\kappa})) & \text{if } \kappa = \sigma_1(\bar{\kappa}) \in \sigma_1 \text{``}\bar{D}. \end{cases}$$

*Main Claim.* The sequence  $S_1$  is a  $\mathbb{D}_1$ -generic sequence over  $N_1$ .

*Pf.* We will show that  $S_1$  satisfies the generalized diagonal Prikry genericity criterion (*Fact 3.2*) over  $N_1$ . To do this, let  $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D_1 \rangle$ , with  $\mathcal{X} \in N_1$ , be a sequence of measure-one sets in the sequence of measures  $\mathcal{U}_1$ .

Note first that  $S'_1$  is a generic sequence, it already satisfies the generalized diagonal Prikry genericity criterion, namely:

$$\{\kappa \in D_1 \mid S'_1(\kappa) \notin X_\kappa\} \text{ is finite.}$$

Recall that  $\bar{S} = \langle \bar{S}(\bar{\kappa}) \mid \bar{\kappa} \in \bar{D} \rangle$  is a  $\mathbb{D}$ -generic sequence as well. We need to see that in addition,

$$\{\bar{\kappa} \in \bar{D} \mid \sigma_1(\bar{S}(\bar{\kappa})) \notin X_{\sigma_1(\bar{\kappa})}\} \text{ is finite,}$$

since then

$$\{\kappa \in D_1 \mid S_1(\kappa) \notin X_\kappa\} = \{\kappa \in D_1 \setminus \sigma_1 \text{``}\bar{D} \mid S'_1(\kappa) \notin X_\kappa\} \cup \{\kappa = \sigma_1(\bar{\kappa}) \in \sigma_1 \text{``}\bar{D} \mid \sigma_1(\bar{S}(\bar{\kappa})) \notin X_\kappa\}$$

is finite as well, completing the proof as desired.

By the  $\bar{\kappa}(0)$ -cofinality of  $\sigma_1$ , there is some  $w \in \bar{N}$  such that  $\mathcal{X} \in \sigma_1(w)$ , where  $|w| < \bar{\kappa}(0)$  in  $\bar{N}$ . Thus in  $N_1$ , we have that  $|\sigma_1(w)| < \kappa_1(0)$ . We may assume that  $w$  consists of functions  $f \in \prod_{\bar{\kappa} \in \bar{D}} \bar{U}(\bar{\kappa})$ . So for each  $\kappa \in \sigma_1 \text{``}\bar{D}$ , we have that  $X_\kappa \in \sigma_1(w)_\kappa = \{\sigma_1(f)(\bar{\kappa}) \mid f \in \prod_{\bar{\kappa} \in \bar{D}} \bar{U}(\bar{\kappa}) \wedge f \in w\}$

and also  $|\sigma_1(w)_\kappa| < \kappa_1(0)$ . So all  $\kappa \in \sigma_1 \text{“}\overline{D}$  of course satisfy  $\kappa \geq \kappa_1(0)$  and thus by the  $\kappa$ -completeness of  $U_1(\kappa)$ , we have that  $W_\kappa := \cap \sigma_1(w)_\kappa \in U_1(\kappa)$ . So we have established that  $\mathcal{W}$ , the sequence of  $W_\kappa$  for  $\kappa \geq \kappa_1(0)$ , is also a sequence of measure-one sets in  $N_1$ . Note in addition that for  $\kappa \in \sigma_1 \text{“}\overline{D}$ , we have that  $W_\kappa \subseteq X_\kappa$ .

By elementarity, for each  $\bar{\kappa} \in \overline{D}$ , we have  $\overline{W}_{\bar{\kappa}} = \cap \{f(\bar{\kappa}) \mid f \in \prod_{\bar{\kappa} \in \overline{D}} \overline{U}(\bar{\kappa}) \wedge f \in w\}$  is a measure-one set in  $\overline{U}(\bar{\kappa})$  and we also have that  $\sigma_1(\overline{W}_{\bar{\kappa}}) = W_{\sigma_1(\bar{\kappa})}$ . Moreover,

$$\{\bar{\kappa} \in \overline{D} \mid \overline{S}(\bar{\kappa}) \notin \overline{W}_{\bar{\kappa}}\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for  $\overline{\mathbb{D}}$ , which must be satisfied by  $\overline{S}$ . Thus by elementarity,

$$\{\bar{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\bar{\kappa})) \notin W_{\sigma_1(\bar{\kappa})}\} \supseteq \{\bar{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\bar{\kappa})) \notin X_{\sigma_1(\bar{\kappa})}\} \text{ is finite,}$$

as is desired, completing the proof of our claim.  $\square$

Moreover we have now shown that  $\mathcal{L}(N_1, \sigma_1) = \mathcal{L}_1$  is consistent by Barwise Correctness (*Fact 2.19*), since we have just shown that  $\langle H_\delta; \sigma_1, S_1 \rangle$  is a model of  $\mathcal{L}_1$ .

Let's check that we may now apply Transfer (*Fact 2.23*) to the embedding  $k_1 : N_1 \prec N_0$ . By **Lemma 2.30** we have that  $N_1$  is almost full. We also have that  $\mathcal{L}_1 = \mathcal{L}(L_{\delta_{N_1}}(N_1))$  is  $\Sigma_1$  over parameters

$$N_1 \text{ and } \bar{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \bar{c}, \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1 \in N_1$$

while  $\mathcal{L}_0 = \mathcal{L}(L_{\delta_{N_0}}(N_0))$  is  $\Sigma_1$  over parameters

$$N_0 \text{ and } k_1(\bar{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \bar{c}, \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1) \in N_0.$$

Furthermore  $k_1$  is cofinal, since for each element  $x \in N_0$ , as  $\sigma_0$  is cofinal, there is  $u \in \overline{N}$  such that  $x \in \sigma_0(u)$ . Thus  $\sigma_1(u) \in N_1$ , and moreover  $x \in k_1(\sigma_1(u)) = \sigma_0(u)$ . Therefore, we have that since  $\mathcal{L}_1$  is consistent,  $\mathcal{L}_0$  is consistent as desired. This completes the proof of *Claim 2*.  $\square$

From the consistency of  $\mathcal{L}_0$ , we would now like to use Barwise Completeness (*Fact 2.20*) to obtain a model of  $\mathcal{L}_0$ . To do this, we need the admissible structure the theory is defined over to be countable. So let's work in  $V[F]$ , a generic extension that collapses  $L_{\delta_{N_0}}(N_0)$  to be countable. Then by Barwise Completeness,  $\mathcal{L}_0$  has a solid model

$$\mathfrak{A} = \langle \mathfrak{A}; \dot{S}^\mathfrak{A}, \dot{\sigma}^\mathfrak{A} \rangle$$

such that

$$\text{Ord} \cap \text{wfc}(\mathfrak{A}) = \text{Ord} \cap L_{\delta_{N_0}}(N_0).$$

Thus we have that  $\dot{\sigma}^\mathfrak{A} : \overline{N}^\mathfrak{A} \prec \underline{N}_0^\mathfrak{A}$ . By the Basic Axioms we have that  $\overline{N}^\mathfrak{A} = \overline{N}$  and  $N_0 = \underline{N}_0^\mathfrak{A}$ . Thus we may say that  $\dot{\sigma}^\mathfrak{A} : \overline{N} \prec N_0$ .

Let  $S = \dot{S}^\mathfrak{A}$  and  $\dot{\sigma} = k_0 \circ \dot{\sigma}^\mathfrak{A}$ .

Then  $S$  is a  $\mathbb{D}_0$ -generic sequence over  $N_0$ , and as  $k_0 : N_0 \cong C$  we also have that  $k_0 \text{“} S \text{ is } C\text{-generic for } \mathbb{D}$ . We need to see that  $S$  is  $\mathbb{D}$ -generic over  $V$ . To do this, let  $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D \rangle$  be a sequence of measure-one sets in the sequence of measures  $\mathcal{U}$ . We will verify the generalized diagonal Prikry genericity criterion. To do this, let  $E \subseteq \mathbb{D}$  be dense and have size  $\delta$  with  $E \in C$ . Since  $\delta \subseteq C$ , we have that  $E \subseteq C$  as well. Find a condition  $(s, A) \in E$  that strengthens  $(\emptyset, \mathcal{X})$ .

Thus for  $\kappa \in \text{dom } A$ , we have that  $A(\kappa) \subseteq X_\kappa$ . Define a sequence of measure-one sets  $B$  in  $C$  so that

$$B(\kappa) = \begin{cases} A(\kappa) & \text{if } \kappa \in \text{dom } A \\ \kappa & \text{if } \kappa \in \text{dom } s. \end{cases}$$

So we have that  $B$  is a sequence of measure-one sets in  $C$ . So  $\{\kappa \in D \mid S(\kappa) \notin B(\kappa)\}$  is finite. Thus  $\{\kappa \in D \mid S(\kappa) \notin X_\kappa\}$  is finite.

This will be the  $\mathbb{D}$ -generic sequence we need to satisfy our main claim. We will see in the following claim that  $\sigma^*$  has all of the desired properties of our main claim, but it fails to be in  $V[S]$ , which is what we need. However, based on the following claim,  $\sigma^*$  will at least enable us to see that our  $\in$ -theory  $\mathcal{T}$  from long ago, defined to assist us in proving the main claim, is consistent over a suitable admissible structure.

*Claim 4.* The map  $\sigma^*$  satisfies:

- (1)  $\sigma^* : \bar{N} \prec N$
- (2)  $\sigma^*(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \theta, \mathbb{D}, \mathcal{U}, c$
- (3)  $\mathcal{Hull}^N(\delta \cup \text{range}(\sigma^*)) = C$
- (4)  $\sigma^* \text{ `` } \bar{S} \subseteq S$

*Pf.* For item **1**, we have already seen above that  $\dot{\sigma}^{\mathfrak{A}} : \bar{N} \prec N_0$ . Since  $k_0 : N_0 \prec N$ , the desired result follows.

For item **2**,  $\sigma^*(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = k_0(\theta_0, \mathbb{D}_0, \mathcal{U}_0, c_0) = \theta, \mathbb{D}, \mathcal{U}, c$ .

Item **3** holds since  $N_0 = \mathcal{Hull}^{N_0}(\delta \cup \text{range}(\dot{\sigma}^{\mathfrak{A}}))$ . To see this, clearly we have that  $\mathcal{Hull}^{N_0}(\delta \cup \text{range}(\dot{\sigma}^{\mathfrak{A}})) \subseteq N_0$ , since  $\delta \in N_0$  as  $N_0 \cong C$ , and certainly  $\text{range}(\dot{\sigma}^{\mathfrak{A}}) \subseteq N_0$  as well. Then because  $\dot{\sigma}^{\mathfrak{A}}$  is  $\bar{\kappa}'$ -cofinal, by **Lemma 2.26**, we have, since  $\dot{\sigma}^{\mathfrak{A}}(\bar{\kappa}') < \delta$ , that:

$$N_0 = \{\dot{\sigma}^{\mathfrak{A}}(f)(\beta) \mid f : \gamma \longrightarrow N_0, \gamma < \bar{\kappa}(0) \text{ and } \beta < \dot{\sigma}^{\mathfrak{A}}(\gamma)\} \subseteq \mathcal{Hull}^N(\delta \cup \text{range}(\dot{\sigma}^{\mathfrak{A}})).$$

Thus  $C = k_0 \text{ `` } N_0 = \mathcal{Hull}^{N_0}(\delta \cup \text{range}(k_0 \circ \dot{\sigma}^{\mathfrak{A}}))$  as desired.

To see **4**, note that  $\sigma^* \upharpoonright \bar{\kappa}(0) = \dot{\sigma}^{\mathfrak{A}} \upharpoonright \bar{\kappa}(0)$  since  $k_0 \upharpoonright \nu_0 = \text{id}$ .

This completes the proof of *Claim 3*. □

We are almost done, but like we stated above,  $\dot{\sigma}^{\mathfrak{A}}$  is in  $V[F]$ , the generic extension needed to obtain a countable admissible structure to apply Barwise Completeness to. But  $V[F]$  is not in  $V[S]$ , and so  $\sigma^*$  is not necessarily in  $V[S]$ . We will use Barwise Completeness one last time, to finally find an embedding  $\sigma'$  with which to satisfy the main claim along with the  $S$  we found above.

Let  $\lambda$  be regular in  $V[S]$  with  $N \in H_\lambda^{V[S]}$ . Then

$$M = \langle H_\lambda^{V[S]}; N, \sigma, S; \theta, \delta, \mathbb{D}, \mathcal{U}, c \rangle \text{ is admissible.}$$

In order to satisfy our main claim, we need a model of  $\mathcal{T}(M)$  in  $V[S]$ . By *Claim 3*, we have that  $\langle M, \sigma^* \rangle$  is a model of  $\mathcal{T}(M)$ , but in  $V[F]$ , which is not  $V[S]$ . Still this means that by Barwise Correctness,  $\mathcal{T}(M)$  is consistent.

Consider the Mostowski collapse of  $M$ ; let

$$\pi : \tilde{M} \prec M \text{ where } \tilde{M} \text{ is countable and transitive.}$$

Note that  $\tilde{M} \in H_{\omega_1}^{V[S]} = H_{\omega_1}^V$  since it is countable and diagonal Prikry forcing doesn't add bounded subsets to any  $\kappa \in D$ .<sup>4</sup> We also have that  $\overline{N}^{\tilde{\mathfrak{A}}} = \overline{N}$ , since  $M$  sees that  $\overline{N}$  is countable so  $\tilde{M}$  sees that  $\pi^{-1}(\overline{N})$  is, and it follows that  $\pi^{-1}(\overline{N}) = \overline{N}$ .

Plus,  $\mathcal{T}(\tilde{M})$  is consistent, since otherwise its inconsistency could be pushed up via  $\pi$  to one in  $\mathcal{T}(M)$ , contradicting the model witnessing its consistency that we found in  $V[F]$ .

So by Barwise Completeness,  $\mathcal{T}(\tilde{M})$  has a solid model

$$\tilde{\mathfrak{A}} = \langle \tilde{\mathfrak{A}}; \dot{\sigma}^{\tilde{\mathfrak{A}}} \rangle$$

such that

$$\text{Ord} \cap \text{wfc}(\tilde{\mathfrak{A}}) = \text{Ord} \cap \tilde{M}.$$

Letting  $\sigma' = \pi \circ \dot{\sigma}^{\tilde{\mathfrak{A}}}$ , the main claim is now satisfied with  $\sigma'$  and our  $\lambda$ -diagonal Prikry sequence  $S$ .

Let us verify each of the properties of  $\sigma'$  required by the main claim. The verification of these properties shall use the agreement between  $\tilde{\mathfrak{A}}$  and  $\tilde{M}$  on the special constants of  $\tilde{M}$  and on the ordinals. The fact that  $\pi$  does not affect  $\overline{N}$  will be greatly taken advantage of.

First we show **1** of the main claim. Let's say that  $\varphi[\sigma'(\bar{a})]$  holds in  $N$ . So  $\varphi[\pi(\dot{\sigma}^{\tilde{\mathfrak{A}}}(\bar{a}))]^N$  holds in  $M$ . Thus  $\varphi[\dot{\sigma}^{\tilde{\mathfrak{A}}}(\bar{a})]^{\pi^{-1}(N)}$  holds in  $\tilde{M}$ , and thus also in  $\tilde{\mathfrak{A}}$ . Indeed we know that  $\overline{N}^{\tilde{\mathfrak{A}}} = \overline{N}$ , since  $\tilde{\mathfrak{A}}$  agrees with  $\tilde{M}$  about countable ordinals, which we may use to code  $\overline{N}$ . This means that  $\varphi[\bar{a}]$  holds in  $\overline{N}$ , as desired.

To see **2**, we have  $\sigma'(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \pi(\underline{\theta}^{\tilde{\mathfrak{A}}}, \underline{\mathbb{D}}^{\tilde{\mathfrak{A}}}, \underline{\mathcal{U}}^{\tilde{\mathfrak{A}}}, \underline{c}^{\tilde{\mathfrak{A}}}) = \theta, \mathbb{D}, \mathcal{U}, c$ .

For item **3**, let  $\tilde{N} = \underline{N}^{\tilde{\mathfrak{A}}}$ ,  $\tilde{\sigma} = \underline{\sigma}^{\tilde{\mathfrak{A}}}$  and  $\tilde{\delta} = \underline{\delta}^{\tilde{\mathfrak{A}}}$ . So  $\pi(\tilde{\delta}) = \delta$  and  $\pi(\tilde{\sigma}) = \sigma$ . Because of the way the  $\in$ -theory  $\mathcal{T}$  was defined, we already have:

$$(1) \quad \mathcal{Hull}^{\tilde{N}}(\tilde{\delta} \cup \text{range}(\dot{\sigma}^{\tilde{\mathfrak{A}}})) = \mathcal{Hull}^{\tilde{N}}(\tilde{\delta} \cup \text{range}(\tilde{\sigma})).$$

To see  $\mathcal{Hull}^N(\delta \cup \text{range}(\sigma')) \subseteq \mathcal{Hull}^N(\delta \cup X)$ , suppose  $x \in \mathcal{Hull}^N(\delta \cup \text{range}(\sigma'))$ . Then we have that  $N$  sees that there is some formula  $\varphi$ ,  $\bar{z} \in \overline{N}$ , and  $\xi < \delta$  where  $x$  is unique such that  $\varphi(x, \pi(\dot{\sigma}^{\tilde{\mathfrak{A}}}(\bar{z})), \xi)$ . In particular,  $x$  is in the range of  $\pi$ . Thus  $\tilde{x} = \pi^{-1}(x) \in \tilde{N}$  and  $\tilde{\xi} < \tilde{\delta}$  such that  $\varphi(\tilde{x}, \dot{\sigma}^{\tilde{\mathfrak{A}}}(\bar{z}), \tilde{\xi})$  holds. Thus by (1), we have that  $\tilde{x}$  is unique such that  $\varphi(\tilde{x}, \tilde{\sigma}(\bar{y}), \tilde{\zeta})$  for some  $\bar{y} \in \overline{N}$  and  $\tilde{\zeta} < \tilde{\delta}$ . Thus pushing back up through  $\pi$ , letting  $\zeta = \pi(\tilde{\zeta})$ , we have that  $x = \pi(\tilde{x})$  is unique such that  $\varphi(x, \sigma(\bar{y}), \zeta)$  holds, so  $x \in \mathcal{Hull}^N(\delta \cup X)$ .

To see that  $\mathcal{Hull}^N(\delta \cup X) \subseteq \mathcal{Hull}^N(\delta \cup \text{range}(\sigma'))$  works similarly. Let  $x \in \mathcal{Hull}^N(\delta \cup X)$ . Then there is  $\bar{z} \in \overline{N}$  and  $\xi < \delta$  such that  $\varphi(x, \sigma(\bar{z}), \xi)$  holds. So in particular,  $x$  is in the domain of  $\pi$ . So we may find  $\tilde{\xi} < \tilde{\delta}$  such that  $\tilde{x} = \pi^{-1}(x)$  is unique such that  $\varphi(\tilde{x}, \tilde{\sigma}(\bar{z}), \tilde{\xi})$ . So by (1), we have that there is  $\bar{y} \in \overline{N}$  and  $\tilde{\zeta} < \tilde{\delta}$  such that  $\tilde{x}$  is unique satisfying  $\varphi(\tilde{x}, \dot{\sigma}^{\tilde{\mathfrak{A}}}(\bar{y}), \tilde{\zeta})$ . Finally, by pushing back up through  $\pi$ , letting  $\pi(\tilde{\zeta}) = \zeta$ , we have that  $x = \pi(\tilde{x})$  is unique such that  $\varphi(x, \sigma'(\bar{y}), \zeta)$ , so  $x \in \mathcal{Hull}^N(\delta \cup \text{range}(\sigma'))$  as desired.

To see item **4**, note that  $\overline{S}^{\tilde{\mathfrak{A}}} = \overline{S}$  since  $\overline{S} \subseteq \overline{N}$ . So we have already by the definition of  $\mathcal{T}$  that  $\dot{\sigma}^{\tilde{\mathfrak{A}}}[\overline{S} \subseteq \underline{S}^{\tilde{\mathfrak{A}}}]$ . Thus  $\pi \circ \dot{\sigma}^{\tilde{\mathfrak{A}}}[\overline{S} \subseteq \pi[\underline{S}^{\tilde{\mathfrak{A}}}] \subseteq S$  as desired.

This completes the proof of the main claim. □

We have satisfied the main claim, so we are done, we have shown that  $\mathbb{D}$  is subcomplete. □

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<sup>4</sup>As Fuchs [1] points out, this result is a modification to the proof that generalized diagonal Prikry forcing preserves cardinalities.



Some slight modifications to the above proof give the following two corollaries.

The first point is that the above proof also shows that generalized diagonal Prikry forcing that adds a countable sequence to each measurable cardinal is subcomplete. Before stating the corollary let's define the forcing. Again let  $D$  be an infinite discrete set of measurable cardinals. Let  $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D \rangle$  be a list of measures associated to  $D$ . Let  $\mathbb{D}^*(\mathcal{U}) = \mathbb{D}^*$  be defined the same as  $\mathbb{D}(\mathcal{U})$  except the stem of a condition,  $s$ , in  $\mathbb{D}^*(\mathcal{U})$  is a function with domain in  $[D]^{<\omega}$  taking each measurable cardinal  $\kappa \in \text{dom}(s)$  to finitely many ordinals  $s(\kappa) \subseteq \kappa$ . The upper part and extension relation is defined in the same way; the only slight modification is that again we have  $(s, A) \leq (t, B)$  so long as points in  $s$  not in  $t$  come from  $B$ , which in the case means that for  $\kappa \in \text{dom}(s)$  we have that each element of  $s(\kappa)$  not in  $t(\kappa)$  is in  $B(\kappa)$ . We may again form a  $\mathbb{D}^*$ -generic sequence  $S = S_G$  for a generic  $G \subseteq \mathbb{D}^*$ , and we may write  $S = \langle S(\kappa) \mid \kappa \in D \rangle$  where  $S(\kappa)$  is a countable sequence of ordinals less than  $\kappa$ . The genericity criterion for generic diagonal Prikry sequences is as that for  $\mathbb{D}$ , which is given in [1, Theorem 1], as stated in **Fact 3.2**, with the modification that  $S$  is  $\mathbb{D}^*$  generic if and only if for all  $\mathcal{X}$ , the set  $\{\alpha \mid \exists \kappa \in D \ \alpha \in S(\kappa) \setminus X_\kappa\}$  is finite.

**Corollary 3.4.** *Let  $D$  be an infinite discrete set of measurable cardinals. Let  $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D \rangle$  be a list of measures associated to  $D$ . Then  $\mathbb{D}^*(\mathcal{U})$  is subcomplete.*

*Proof Sketch.* The modifications are mostly notational, and the main one that needs to be made is to adjust the proof of the *Claim* within the proof of *Claim 2*. Here we have  $\mathbb{D}_1^*$ , the generalized diagonal Prikry forcing as computed in  $N_1$ , as well as  $\overline{\mathbb{D}}$  of  $\overline{N}$ , and  $S_1$ , which we would like to show is a  $\mathbb{D}_1^*$ -generic sequence over  $N_1$  in this case.  $S_1$  is defined as  $\sigma_1 \overline{S}$ , using a diagonal Prikry sequence  $S'_1$  to fill in the missing coordinates, where  $S'_1$  is obtained by forcing with  $\mathbb{D}_1$  over  $V$ .

We will show that  $S_1$  satisfies the generalized diagonal Prikry genericity criterion over  $N_1$  and follow the above proof. To do this, let  $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D_1 \rangle$ , with  $\mathcal{X} \in N_1$ , be a sequence of measure-one sets in the sequence of measures  $\mathcal{U}_1$ .

Note first that  $S'_1$  is a generic sequence, it already satisfies the generalized diagonal Prikry genericity criterion, namely:  $\{\alpha \mid \exists \kappa \in D_1 \ \alpha \in S'_1(\kappa) \setminus X_\kappa\}$  is finite. Recall that  $\overline{S} = \langle \overline{S}(\overline{\kappa}) \mid \overline{\kappa} \in \overline{D} \rangle$  is a  $\overline{\mathbb{D}}$ -generic sequence over  $\overline{N}$  as well. We need to see that in addition,  $\{\alpha \mid \exists \overline{\kappa} \in \overline{D} \ \alpha \in \sigma_1(\overline{S}(\overline{\kappa})) \setminus X_{\sigma_1(\overline{\kappa})}\}$  is finite.

By the  $\overline{\kappa}(0)$ -cofinality of  $\sigma_1$ , there is some  $w \in \overline{N}$  such that  $\mathcal{X} \in \sigma_1(w)$ , where  $|w| < \overline{\kappa}(0)$  in  $\overline{N}$ . Thus in  $N_1$ ,  $|\sigma_1(w)| < \kappa_1(0)$ . For each  $\kappa \in \sigma_1 \overline{D}$ , we have that  $X_\kappa \in \sigma_1(w)_\kappa = \{\sigma_1(f)(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa}) \wedge f \in w\}$  and also  $|\sigma_1(w)_\kappa| < \kappa_1(0)$ . All  $\kappa \in \sigma_1 \overline{D}$  of course satisfy  $\kappa \geq \kappa_1(0)$  so by the  $\kappa$ -completeness of  $U_1(\kappa)$ , we have that  $W_\kappa := \cap \sigma_1(w)_\kappa \in U_1(\kappa)$  since  $X_\kappa \in \sigma_1(w)_\kappa$ . So we have established that  $\mathcal{W}$ , the sequence of  $W_\kappa$  for  $\kappa \geq \kappa_1(0)$ , is also a sequence of measure-one sets in  $N_1$ . Note in addition that for  $\kappa \in \sigma_1 \overline{D}$ , we have that  $W_\kappa \subseteq X_\kappa$ .

By elementarity, for each  $\overline{\kappa} \in \overline{D}$ , we have  $\overline{W}_{\overline{\kappa}} = \cap \{f(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa}) \wedge f \in w\}$  is a measure-one set in  $\overline{U}(\overline{\kappa})$  and we also have that  $\sigma_1(\overline{W}_{\overline{\kappa}}) = W_{\sigma_1(\overline{\kappa})}$ . Moreover,

$$\{\alpha \mid \exists \overline{\kappa} \in \overline{D} \ \alpha \in \overline{S}(\overline{\kappa}) \setminus \overline{W}_{\overline{\kappa}}\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for  $\overline{\mathbb{D}}$ , which must be satisfied by  $\overline{S}$ . Thus by elementarity,

$$\{\alpha \mid \exists \overline{\kappa} \in \overline{D} \ \sigma_1(\overline{S}(\overline{\kappa})) \setminus W_{\sigma_1(\overline{\kappa})}\} \supseteq \{\alpha \mid \exists \overline{\kappa} \in \overline{D} \ \sigma_1(\overline{S}(\overline{\kappa})) \setminus X_{\sigma_1(\overline{\kappa})}\} \text{ is finite,}$$

as is desired, completing the proof of the claim.  $\square$

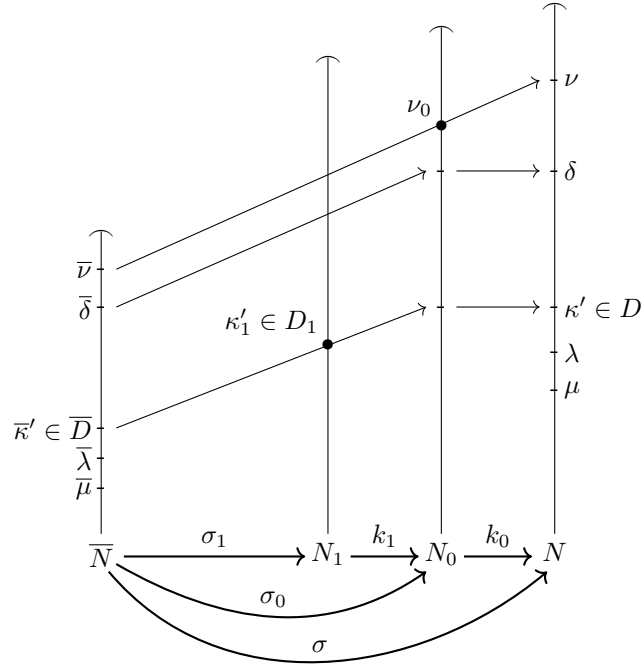
One might consider a forcing like  $\mathbb{D}$  and  $\mathbb{D}^*$  that adds one point below each measurable cardinal sometimes, and other times adds a cofinal  $\omega$ -sequence below the measurable cardinal. This forcing is clearly subcomplete as well.

Below we refer to the concept of subcompleteness above  $\mu$ , which was introduced in ??.

**Corollary 3.5.** *Let  $D$  be an infinite discrete set of measurable cardinals. Let  $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D \rangle$  be a list of measures associated to  $D$ .*

*Furthermore, let  $\mu < \lambda$  be a regular cardinal, where  $\lambda = \sup_{n < \omega} \kappa_n$ , the first limit point of  $D$ . Then  $\mathbb{D} = \mathbb{D}(\mathcal{U})$  is subcomplete above  $\mu$ .*

*Proof Sketch.* The idea is to follow the same exact proof as in the above theorem, except we achieve the following diagram:



Here we replace  $\kappa(0)$  with some  $\kappa' \in D$  such that  $\lambda < \kappa'$ , where there are finitely many measurables of  $D$  below  $\kappa'$ . So in particular, we let  $\langle N_1, \sigma_1 \rangle$  be the liftup of  $\langle \bar{N}, \sigma \restriction H_{\bar{\kappa}'}^{\bar{N}} \rangle$  in *Claim 2*. In order to show the *Claim* that we have a generic sequence over  $\mathbb{D}_1$ , we follow the same argument as follows:

Let  $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in \sigma_1(\bar{D}) \rangle$ , with  $\mathcal{X} \in N_1$ , be a sequence of measure one sets in the sequence of measures  $\mathcal{U}_1$  with only coordinates coming from  $\sigma_1(\bar{D})$ . We need to see that

$$\{\bar{\kappa} \in \bar{D} \mid \sigma_1(\bar{S}(\bar{\kappa})) \notin X_{\sigma_1(\bar{\kappa})}\} \text{ is finite.}$$

By the  $\bar{\kappa}'$ -cofinality of  $\sigma_1$ , there is some  $w \in \bar{N}$  such that  $\mathcal{X} \in \sigma_1(w)$ , where  $|w| < \bar{\kappa}'$  in  $\bar{N}$ . Thus in  $N_1$ ,  $|\sigma_1(w)| < \kappa'_1$ . For each  $\kappa \in \sigma_1(\bar{D})$ , we have that  $X_\kappa \in \sigma_1(w)_\kappa = \{\sigma_1(f)(\bar{\kappa}) \mid f \in \prod_{\bar{\kappa} \in \bar{D}} \bar{U}(\bar{\kappa}) \wedge f \in w\}$  and also

$$|\sigma_1(w)| < \kappa'_1.$$

So for all but finitely many  $\kappa \in \sigma_1 \text{“}\overline{D}\text{”}$ , namely for  $\kappa \geq \kappa'_1$ , by the  $\kappa$ -completeness of  $U_1(\kappa)$ , we have that

$$\cap \sigma_1(w) = W_\kappa \in U_1(\kappa).$$

So we have established that  $\mathcal{W}$ , the sequence of  $W_\kappa$  for  $\kappa > \kappa'_1$ , is also a sequence of measure-one sets in  $N_1$ . Note in addition that for  $\kappa \in \sigma_1 \text{“}\overline{D}\text{”}$ ,  $\kappa > \kappa'_1$ , we have that  $W_\kappa \subseteq X_\kappa$ .

By elementarity, for each  $\bar{\kappa} \in \overline{D}$ , where  $\bar{\kappa} > \bar{\kappa}'$ , we have

$$\overline{W}_{\bar{\kappa}} = \cap \{f(\bar{\kappa}) \mid f \in \prod_{\bar{\kappa} \in \overline{D}} \overline{U}(\bar{\kappa}) \wedge f \in w\}$$

is a measure-one set in  $\overline{U}(\bar{\kappa})$  and we also have that  $\sigma_1(\overline{W}_{\bar{\kappa}}) = W_{\sigma_1(\bar{\kappa})}$ . Moreover,

$$\{\bar{\kappa} \in \overline{D} \mid \overline{S}(\bar{\kappa}) \notin \overline{W}_{\bar{\kappa}}\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for  $\overline{\mathbb{D}}$ , which must be satisfied by  $\overline{S}$ , and since there are only finitely many measurables in  $\overline{D}$  less than  $\bar{\kappa}'$  in  $\overline{N}$ . Thus by elementarity,

$$\{\bar{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\bar{\kappa})) \notin W_{\sigma_1(\bar{\kappa})}\} \supseteq \{\bar{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\bar{\kappa})) \notin X_{\sigma_1(\bar{\kappa})}\} \text{ is finite.}$$

Additionally the  $\in$ -theories  $\mathcal{L}$  and  $\mathcal{T}$  would have to be defined so as to include as an axiom that  $\dot{\sigma} \restriction \underline{\mu} = \underline{\sigma} \restriction \underline{\mu}$  and  $\dot{\sigma} \restriction \underline{\mu} = \underline{\sigma} \restriction \underline{\mu}$  respectively. We would then need to show that  $\dot{\sigma}^* \restriction \underline{\mu} = \sigma \restriction \underline{\mu}$ , but this would follow since  $k_0$  is the identity on  $\nu_0$ . Furthermore, it would need to be shown that  $\sigma' \restriction \underline{\mu} = \sigma \restriction \underline{\mu}$ , but this would follow from the requirement that  $\dot{\sigma}^{\tilde{\mathfrak{A}}} \restriction \underline{\mu} = \underline{\sigma}^{\tilde{\mathfrak{A}}} \restriction \underline{\mu}$ , and since ordinals are computed properly by  $\tilde{\mathfrak{A}}$ .  $\square$

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