

THE SUBCOMPLETENESS OF DIAGONAL PRIKRY FORCING

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ABSTRACT. It is shown that generalized Prikry forcing to add a countable sequence to each measurable in an infinite discrete set of measurable cardinals is subcomplete. To do this we show that a simplified version of generalized Prikry forcing which adds a point below each measurable cardinal in an infinite discrete set, which we call generalized diagonal Prikry forcing, is subcomplete. We also show that generalized diagonal Prikry forcing is subcomplete above μ where μ is a regular cardinal below the first limit point of the infinite discrete set of measurable cardinals used in the forcing.

1. INTRODUCTION

Subcomplete forcing is a class of forcing notions that we shall see do not add reals, but may potentially alter cofinalities to ω . Examples of subcomplete forcing include Prikry forcing and Namba forcing (under CH). This separates subcomplete forcings from proper forcings, that have countable covering and thus cannot change cofinalities to ω .

Jensen [3, Section 3.3] shows that Prikry forcing and Namba forcing (under CH) are subcomplete. Fuchs has shown that Magidor forcing [2] is subcomplete. Below we use an adaptation of Jensen's proof showing that Prikry forcing is subcomplete to see that those "Prikry-like forcings" which we will call generalized Prikry forcing after Fuchs ([1]), are subcomplete.

2. PRELIMINARIES

Before defining subcomplete forcing and generalized diagonal Prikry forcing, some preliminary information is necessary. This can all be found in Jensen's lecture notes from the 2012 AII Summer School in Singapore. For the published version refer to [3]. I will also refer to unpublished handwritten notes from Jensen's website.

First, a brief outline of some of the notation used in what follows: Forcing notions $\mathbb{P} = \langle \mathbb{P}, \leq \rangle$ are taken to be partial orders that are separative and contain a "top" element weaker than all elements of \mathbb{P} , denoted 1 . We will be working with transitive models of ZFC^- , the axioms of Zermelo-Fraenkel Set Theory without the axiom of **Powerset**, and with the axiom of **Collection** instead of **Replacement**. Usually we will name these models N , M , or \bar{N} . I will follow Jensen to use the notation $\sigma : N \prec M$ for when σ is an elementary embedding, but I will add some notation not used by Jensen, letting $N \preceq M$ denote and emphasize that N is an elementary substructure of M . Let N be a transitive ZFC^- model. I write $\text{height}(N)$ to mean $\text{Ord} \cap N$. Let α be an ordinal. Then I write α^N for $\alpha \cap N$. Let θ be a cardinal. H_θ refers to the collection of sets hereditarily of size less than θ . Relativizing the concept to a particular model of set theory, M , I write H_θ^M to mean the collection of sets in M that are hereditarily of size less than θ in M . In this case, if θ is determined by some computation, I mean for that computation to take place in M . Let τ be a cardinal. With abuse of notation we write $L_\tau[A]$ to refer to the structure $\langle L_\tau[A]; \in, A \cap L_\tau[A] \rangle$. We often make use of the following abbreviation: if a map σ satisfies $\sigma(\bar{a}) = a$ and $\sigma(\bar{b}) = b$, we write $\sigma(\bar{a}, \bar{b}) = a, b$.

2.1. The Weight of a Forcing Notion.

Definition 2.1. For a forcing notion \mathbb{P} , we write $\delta(\mathbb{P})$ to denote the least cardinality of a dense subset in \mathbb{P} . This is sometimes referred to as the *weight* of a poset.

Although Jensen defines $\delta(\mathbb{P})$ for Boolean algebras, it's also relevant for posets. As Jensen states, for forcing notions \mathbb{P} , the weight can be replaced with the cardinality of \mathbb{P} , or even \mathbb{P} , for the purpose of defining subcompleteness. However, $\delta(\mathbb{P})$ and $|\mathbb{P}|$ are not necessarily the same, since there could be a large set of points in the poset that all have a common strengthening.

If \mathbb{P} is a dense subset of \mathbb{Q} , then, of course, $\delta(\mathbb{P}) = \delta(\mathbb{Q})$.

2.2. Fullness. In the definition of subcompleteness, in lieu of working directly with H_θ and its well-order for “large enough” cardinals θ as our standard setup (as is often done for proper forcing, for example), I will follow Jensen and work with models N of the form:

$$H_\theta \subseteq N = L_\tau[A] \models \text{ZFC}^-,$$

where $\tau > \theta$ is a cardinal that is not necessarily regular. Such H_θ will need to be large enough so that N has the correct ω_1 and H_{ω_1} . One justification for working with these models is that such N will naturally contain a well order of H_θ , along with its Skolem functions and other useful bits of information we would like to have at our disposal. Additionally a benefit of working with models of the form $L_\tau[A]$ is that $L_\tau[A]$ is easily definable in $L_\tau[A][G]$, if G is generic, using A .

In the standard fashion we will look at countable elementary substructures X of the N as above; $X \prec N$. We then take the countable transitive collapse of such an X , and write $\bar{N} \cong X$. We will refer to these embeddings by writing $\sigma : \bar{N} \cong X \prec N$. Often we will write $\sigma : \bar{N} \prec N$ to suppress mention of X , the range of σ .

In fact, for our purposes it will not be quite enough for such an \bar{N} to be transitive, we need a bit more, exactly given by the property of *fullness*. Fullness of a model ensures that it is not pointwise definable, so that there can be many elementary maps between the smaller structure and the larger structure N in our setup. Before defining fullness, there are more preliminary points to clarify.

For the embeddings we are working with, it is not hard to see what the critical point is. Given $\sigma : \bar{N} \prec N$ where \bar{N} is countable and transitive, $\text{cp}(\sigma)$ is exactly $\omega_1^{\bar{N}} = \sigma^{-1}(\omega_1^N)$, since \bar{N} is countable.

Remark 2.2. Let \bar{N}, N be transitive ZFC^- models, where \bar{N} is countable and $H_{\omega_1} \subseteq N$, with $\sigma : \bar{N} \prec N$. Then $\text{cp}(\sigma) = \omega_1^{\bar{N}}$ and $\sigma \upharpoonright (H_{\omega_1})^{\bar{N}} = \text{id}$.

Proof. Let $\alpha = \text{cp}(\sigma)$. Then α is a cardinal in \bar{N} , and $\alpha > \omega$ as both \bar{N} and N are models of ZFC^- .

It must be that $\alpha = \omega_1^{\bar{N}}$, since $\omega_1^{\bar{N}}$ is a countable ordinal in V and thus in N , but $\sigma(\omega_1^{\bar{N}}) = \omega_1^N > \omega_1^{\bar{N}}$. Every $x \in H_\alpha^{\bar{N}} = (H_{\omega_1})^{\bar{N}}$ may be coded as a subset of ω and thus as an ordinal less than ω_1 by $\gamma = \text{TC}(\{x\})$, and $\sigma(\gamma) = \gamma$, and the desired result follows. \square

Let $N = L_\tau[A]$ for some cardinal τ and set A , be a transitive ZFC^- model, let X be a set, and let δ be a cardinal. Our notation for the *Skolem hull*, in N , closing under $\delta \cup X$, is $\mathcal{Hull}^N(\delta \cup X)$. It is defined to be the smallest $Y \prec N$ satisfying $X \cup \delta \subseteq Y$.

We gather two immediate, basic results that we will refer to later below.

Remark 2.3. Let $N = L_\tau[A]$ be a transitive ZFC^- model, δ and γ be cardinals, and X a set. If $\delta \leq \gamma$ then $\mathcal{Hull}^N(\delta \cup X) \subseteq \mathcal{Hull}^N(\gamma \cup X)$.

Proof. This is trivial; if $t \in \mathcal{Hull}^N(\delta \cup X)$ then t is N -definable from some $\xi < \delta \leq \gamma$ and $\vec{x} \in X$ so $t \in \mathcal{Hull}^N(\gamma \cup X)$ as well. \square

Remark 2.4. Let N be a transitive ZFC^- model, δ a cardinal, and X, Y sets. If $\mathcal{Hull}^N(\delta \cup X) = \mathcal{Hull}^N(\delta \cup Y)$ then $\mathcal{Hull}^N(\gamma \cup X) = \mathcal{Hull}^N(\gamma \cup Y)$ for all $\gamma \geq \delta$.

Proof. Let $t \in \mathcal{Hull}^N(\gamma \cup X)$. Then there is some formula φ where t unique such that $N \models \varphi(t, \xi, x)$ for $\xi < \gamma$, $x \in X$. But since then $x \in X \subseteq \mathcal{Hull}^N(\delta \cup X) = \mathcal{Hull}^N(\delta \cup Y) \subseteq \mathcal{Hull}^N(\gamma \cup Y)$, it must be the case that $t \in \mathcal{Hull}^N(\gamma \cup Y)$ as well. Likewise for the reverse inclusion. \square

Finally, still before defining fullness, we need one more definition.

Definition 2.5. We say that a transitive model N is *regular* in a transitive model M so long as for all functions $f : x \rightarrow N$, where x is an element of N and $f \in M$, we have that $f''x \in N$.

To elucidate the definition further, we immediately have the following lemma:

Lemma 2.6. Let $N, M \models \text{ZFC}^-$ be transitive. N is regular in M iff $N = H_\gamma^M$, where $\gamma = \text{height}(N)$ is a regular cardinal in M .

Proof. For the backward direction, suppose that $N = H_\gamma^M$ where $\gamma = \text{height}(N)$ is a regular cardinal in M . Then for all $f : x \rightarrow N$, with $x \in N$ and $f \in M$, certainly $f''x \in N$ as well.

For the forward direction, indeed γ has to be regular in M since otherwise M would contain a cofinal function $f : \alpha \rightarrow \gamma$ where $\alpha < \gamma$. By the transitivity of N , this implies that $\alpha \in N$. Thus $\cup f''\alpha$ is in $N \models \text{ZFC}^-$ by regularity, so $\gamma \in N$, a contradiction. We have that $N \subseteq H_\gamma^M$ since N is a transitive ZFC^- model, so the transitive closure of elements of N may be computed in N and thus have size less than γ , so they are in H_γ^M as $\gamma \in M$. To show that $H_\gamma^M \subseteq N$, let $x \in H_\gamma^M$. We assume by \in -induction that $x \subseteq N$. Then there is a surjection $f : \alpha \rightarrow x$ where $\alpha < \gamma$, in M . Hence by regularity, $x = f''\alpha \in N$ as desired. \square

We now define fullness.

Definition 2.7. A structure M is *full* so long as M is transitive, $\omega \in M$, and there is a γ such that M is regular in $L_\gamma(M)$ where $L_\gamma(M) \models \text{ZFC}^-$.

Perhaps this property seems rather mysterious, but the fullness of a countable structure guarantees that it is not pointwise definable, which we will see is a necessary property of some of the models that come up in the definition subcomplete forcing.

Lemma 2.8. If M is countable and full, then M is not pointwise definable.

Proof. Suppose toward a contradiction that M is countable, full, and pointwise definable. By fullness there is some $L_\gamma(M) \models \text{ZFC}^-$ such that M is regular in $L_\gamma(M)$. By pointwise definability, for each element $x \in M$, we have attached to it some formula $\varphi(x)$ such that $M \models \varphi(x)$ uniquely, meaning that $\varphi(y)$ fails for every other element $y \in M$. Thus in $L_\gamma(M)$ we may define a function $f : \omega \cong M$, that takes the n th formula in the language of set theory to its unique witness in M . In particular we have that $L_\gamma(M)$ witnesses that M is countable. However, this would allow M to witness its own countability, since M must contain f as well by regularity. \square

Let N be a transitive ZFC^- model. Suppose $\bar{N} \cong X \prec N$ where X is countable and \bar{N} is full. Then as we will find, there may possibly be more than one elementary embedding $\sigma : \bar{N} \prec N$. If there was only one unique embedding, we would be able to define \bar{N} pointwise, by elementarity of the unique map and since \bar{N} is countable.

The following lemma will prove useful when showing that various forcing notions are subcomplete, since subcompleteness requires certain ground-model Skolem hulls to match those in forcing extensions.

Lemma 2.9 (Jensen). *Let \mathbb{P} be a forcing notion, and let $\delta = \delta(\mathbb{P})$ be the smallest size of a dense subset in \mathbb{P} . Suppose that $\mathbb{P} \in H_\theta \subseteq N$ where $N = L_\tau[A] \models \text{ZFC}^-$. Let $\sigma : \bar{N} \prec N$ where \bar{N} is countable and full, and let $\sigma(\bar{\mathbb{P}}) = \mathbb{P}$.*

Suppose $G \subseteq \mathbb{P}$ is N -generic and $\bar{G} \subseteq \bar{\mathbb{P}}$ is \bar{N} -generic, and that $\sigma''\bar{G} \subseteq G$, so σ lifts (or extends) in $V[G]$ to an embedding $\sigma^ : \bar{N}[\bar{G}] \prec N[G]$.*

Then

$$N \cap \mathcal{Hull}^{N[G]}(\delta \cup \text{range}(\sigma^*)) = \mathcal{Hull}^N(\delta \cup \text{range}(\sigma)).$$

Proof. First, we establish that $\mathcal{Hull}^N(\delta \cup \text{range}(\sigma)) \subseteq \mathcal{Hull}^{N[G]}(\delta \cup \text{range}(\sigma^*)) \cap N$. To see this, let $x \in \mathcal{Hull}^N(\delta \cup \text{range}(\sigma))$. Then x is N -definable from $\xi < \delta$ and $\sigma(\bar{z})$ where $\bar{z} \in \bar{N}$. Since σ^* extends σ , this means that $x \in N$ and that x is $N[G]$ -definable from ξ and $\sigma^*(\bar{z})$. This is because $N = L_\tau[A]$ is definable in $N[G]$ using A .

For the other direction, let $x \in \mathcal{Hull}^{N[G]}(\delta \cup \text{range}(\sigma^*)) \cap N$. Then x is $N[G]$ -definable from $\xi < \delta$ and $\sigma^*(\bar{z})$ where $\bar{z} \in \bar{N}[\bar{G}]$; and also $x \in N$. Letting $\dot{z} \in \bar{N}^{\bar{\mathbb{P}}}$ such that $\bar{z} = \dot{z}^{\bar{G}}$ we have

$$\sigma^*(\bar{z}) = \sigma^*(\dot{z}^{\bar{G}}) = \sigma(\dot{z})^G$$

so we have that there is some formula φ such that x is the unique witness:

$$x = \text{that } y \text{ where } N[G] \models \varphi(y, \xi, \sigma(\dot{z})^G).$$

Take $f \in \bar{N}$ mapping $\bar{\delta}$ onto a dense subset of $\bar{\mathbb{P}}$. Then $\sigma(f)$ maps δ onto a dense subset of \mathbb{P} . Thus there is $\nu < \delta$ such that $\sigma(f)(\nu) \in G$ and $\sigma(f)(\nu) \Vdash \varphi(\check{x}, \check{\xi}, \sigma(\dot{z}))$. Thus

$$x = \text{that } y \text{ where } \sigma(f)(\nu) \Vdash_{\mathbb{P}}^N \varphi(\check{y}, \check{\xi}, \sigma(\dot{z}))$$

so $x \in \mathcal{Hull}^N(\delta \cup \text{range}(\sigma))$. □

2.3. Subcomplete Forcing. Subcompleteness should be seen as a weakening of the class of complete (a.k.a. countably closed) forcing notions. We follow [3, Chapter 3] for the following definitions and the proofs that we cite as due to Jensen.

Definition 2.10. A forcing notion \mathbb{P} is *complete* so long as for sufficiently large θ we have; letting:

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \text{ZFC}^-$ where $\tau > \theta$ and $A \subseteq \tau$
- $\sigma : \bar{N} \cong X \prec N$ where X is countable and \bar{N} is transitive
- $\sigma(\bar{\theta}, \bar{\mathbb{P}}, \bar{s}) = \theta, \mathbb{P}, s$ for some $s \in N$;

if \bar{G} is $\bar{\mathbb{P}}$ -generic over \bar{N} then there is a *completeness condition* $p \in \mathbb{P}$ forcing that whenever $G \ni p$ is \mathbb{P} -generic, $\sigma''\bar{G} \subseteq G$.

In particular, below the condition p we have that σ lifts to an embedding $\sigma^* : \bar{N}[\bar{G}] \prec N[G]$. We say that such a θ as above *verifies* the completeness of \mathbb{P} .

The adjustment made to get subcomplete forcings is to not necessarily insist the the original embedding lifts in the forcing extension. Instead subcompleteness asks for there to be an embedding in the extension, an embedding that is sufficiently similar to the original embedding, that lifts. However, as discussed in Section 2.2, the domain of the embedding must be *full* to ensure that there can even consistently be more than one embedding. The definition is given below.

Definition 2.11. A forcing notion \mathbb{P} is *subcomplete* so long as, for sufficiently large θ we have that whenever we are in a situation where:

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \text{ZFC}^-$ where $\tau > \theta$ and $A \subseteq \tau$
- $\sigma : \bar{N} \cong X \prec N$ where X is countable and \bar{N} is full

- $\sigma(\bar{\theta}, \bar{\mathbb{P}}, \bar{s}) = \theta, \mathbb{P}, s$ for some $s \in N$;

then we have that if \bar{G} is $\bar{\mathbb{P}}$ -generic over \bar{N} then there is a *subcompleteness condition* $p \in \mathbb{P}$ such that whenever $G \ni p$ is \mathbb{P} -generic, there is $\sigma' \in V[G]$ satisfying:

1. $\sigma' : \bar{N} \prec N$
2. $\sigma'(\bar{\theta}, \bar{\mathbb{P}}, \bar{s}) = \theta, \mathbb{P}, s$
3. $\mathcal{Hull}^N(\delta(\mathbb{P}) \cup \text{range}(\sigma')) = \mathcal{Hull}^N(\delta(\mathbb{P}) \cup X)$
4. $\sigma' \text{ `` } \bar{G} \subseteq G \text{ ''}$.

In other words and in particular, the subcompleteness condition p forces that there is an embedding σ' in the extension $V[G]$ that lifts, by 4., to an embedding $\sigma'^* : N[G] \prec N[G]$ in $V[G]$.

We say that such a θ as above *verifies* the subcompleteness of \mathbb{P} .

Often we write δ instead of $\delta(\mathbb{P})$ for the weight of \mathbb{P} when there should be no confusion as to which poset \mathbb{P} we are working with.¹

Condition 3. in the definition above ensures that subcomplete forcing notions may be iterated, and is rarely used in the following proofs. Thus we will tend to leave out of the discussion if we are showing properties of subcomplete posets.

Remark 2.12. If \mathbb{P} is subcomplete as verified by θ , then \mathbb{P} is subcomplete as verified by $\theta' > \theta$ since $\mathbb{P} \in H_\theta$ obviously implies that $\mathbb{P} \in H_{\theta'}$. Thus if we are in a situation where $\mathbb{P} \in H_{\theta'} \subseteq N = L_\tau[A]$ for $\tau > \theta'$ then we may reduce down to the case where $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A]$ and $\tau > \theta' > \theta$ using the same τ and apply subcompleteness. In particular, \mathbb{P} is subcomplete as long as it is verified to be subcomplete by some θ . So we may replace “sufficiently large θ ” with “some θ ” in the definition of subcompleteness.²

2.4. Barwise Theory. In order to show that many posets are subcomplete, Jensen exploits Barwise Theory and techniques using countable admissible structures to obtain transitive models of infinitary languages. Barwise creates an M -finite predicate logic, a first order theory in which arbitrary, but M -finite, disjunctions and conjunctions are allowed. The following is an outline of [3, Chapter 1 & 2].

Definition 2.13. Let M be a transitive structure with potentially infinitely many predicates. A theory defined over M is *M-finite* so long as it is in M . A theory is $\Sigma_1(M)$, also known as *M-recursively enumerable* or *M-re*, if the theory is Σ_1 -definable, with parameters from M .

Of course we can generalize this to the entire usual Levy hierarchy of formulae, but for our purposes we only need to know what it means for a theory to be $\Sigma_1(M)$. If \mathcal{L} is a $\Sigma_1(M)$ -definable language or theory, the rough idea is that to check whether a sentence is in \mathcal{L} , one should imagine enumerating the formulae of \mathcal{L} to find a sentence and a witness to it in the structure M . Below we elaborate on what models we are working with.

Definition 2.14. A transitive structure M is *admissible* if it models the axioms of Kripke-Platek Set Theory (KP) which consists of the axioms of Empty Set, Pairing, Union, Σ_0 -Collection, and Σ_0 -Separation.

Jensen also makes use of models of ZF^- that are not necessarily well-founded.

¹See Section 2.1 for more on $\delta(\mathbb{P})$.

²See [3, Section 3.1 Lemma 2.4].

Definition 2.15. Let $\mathfrak{A} = \langle A, \in_{\mathfrak{A}}, B_1, B_2, \dots \rangle$ be a (possibly) ill-founded model of ZF^- , where \mathfrak{A} is allowed to have predicates other than \in . The **well-founded core** of \mathfrak{A} , denoted $\text{wfc}(\mathfrak{A})$, is the restriction of \mathfrak{A} to the set of all $x \in A$ such that $\in_{\mathfrak{A}} \cap \mathcal{C}(x)^2$ is well founded, where $\mathcal{C}(x)$ is the closure of $\{x\}$ under $\in_{\mathfrak{A}}$. A model \mathfrak{A} of ZF^- is **solid** so long as $\text{wfc}(\mathfrak{A})$ is transitive and $\in_{\text{wfc}(\mathfrak{A})} = \in \cap \text{wfc}(\mathfrak{A})^2$.

Jensen [3, Section 1.2] notes that every consistent set of sentences in ZF^- has a solid model, and if \mathfrak{A} is solid, then $\omega \subseteq \text{wfc}(\mathfrak{A})$. In addition,

Fact 2.16 (Jensen). *If $\mathfrak{A} \models \text{ZF}^-$ is solid, then $\text{wfc}(\mathfrak{A})$ is admissible.*

Definition 2.17. The context for Barwise theory is countable admissible structures. If M is admissible, we work with infinitary, axiomatized theories \mathcal{L} in M -finitary logic, called \in -**theories**, with a fixed predicate $\dot{\in}$ and **special constants** denoted \underline{x} for elements $x \in M$. Our underlying axioms for these \in -theories will always involve ZFC^- and some basic axioms ensuring that $\dot{\in}$ behaves nicely; the Basic Axioms are:

- Extensionality
- A statement positing the extensionality of $\dot{\in}$, which is a scheme of formulae defined for each member of M . For each $x \in M$, include an M -re sentence (meaning it quantifies over M -finite sentence):

$$\forall v \left(v \dot{\in} \underline{x} \iff \bigvee_{z \in x} v = \underline{z} \right).$$

Here \bigvee denotes an infinite disjunction in the language.

An important fact ensured by our Basic Axioms is that the interpretations of these special constants in any solid model of the theory are the same as in M :

Fact 2.18 (Jensen). *Let M be as in the above definition. Let \mathfrak{A} be a solid model of the \in -theory \mathcal{L} . Then for all $x \in M$, we have that $\underline{x}^{\mathfrak{A}} = x \in \text{wfc}(\mathfrak{A})$.*

Pf. Shown by \in -induction. □

Jensen uses the techniques of Barwise to come up with a proof system in this context, in which consistency of \in -theories can be discussed. In particular, \in -theories are correct: if we have a model of such a theory, then it is consistent.

Fact 2.19 (Barwise Correctness). *Let \mathcal{L} be an \in -theory. If A is a set of \mathcal{L} -statements and $\mathfrak{A} \models A$, then A is consistent.*

Furthermore, compactness and completeness are shown, relativized to the M -finite predicate logics that are used here. In our context, for countable admissible structures M , we will obtain solid models of consistent $\Sigma_1(M)$ \in -theories. In particular, the form of Barwise Completeness that we make use of here is stated below.

Fact 2.20 (Barwise Completeness). *Let M be a countable admissible structure. Let \mathcal{L} be a consistent $\Sigma_1(M)$ \in -theory such that $\mathcal{L} \vdash \text{ZF}^-$. Then \mathcal{L} has a solid model \mathfrak{A} such that*

$$\text{Ord} \cap \text{wfc}(\mathfrak{A}) = \text{Ord} \cap M.$$

We will need the following definition, which is a generalization of fullness.

Definition 2.21. A transitive ZFC^- model N is **almost full** so long as $\omega \in N$ and there is a solid $\mathfrak{A} \models \text{ZFC}^-$ with $N \in \text{wfc}(\mathfrak{A})$ and N is regular in \mathfrak{A} .

Clearly if N is full, then N is almost full.

A useful technique when showing a particular forcing is subcomplete, once many different embeddings can be constructed that approximate the embedding required for subcompleteness, is to be able to transfer the consistency of \in -theories over one admissible structure to another.

Definition 2.22. If N is a transitive ZFC^- model, let δ_N be the least δ such that $L_\delta(N)$ is admissible.

Fact 2.23 (Transfer). Suppose that for M admissible, $\mathcal{L}(M)$ is a $\Sigma_1(M)$ infinitary \in -theory. Let N_1 be almost full, and suppose that $k : N_1 \prec N_0$ cofinally. If both $\mathcal{L}(L_{\delta_{N_1}}(N_1))$ is Σ_1 over parameters $N_1, p_1, \dots, p_n \in N_1$ and $\mathcal{L}(L_{\delta_{N_0}}(N_0))$ is Σ_1 over parameters $N_0, k(p_1), \dots, k(p_n)$, then if $\mathcal{L}(L_{\delta_{N_1}}(N_1))$ is consistent, it follows that $\mathcal{L}(L_{\delta_{N_0}}(N_0))$ is consistent as well.

2.5. Liftups. The following definitions are to describe a method to obtain embeddings, a technique that is ostensibly the ultrapower construction. These embeddings facilitate the use of Barwise theory to obtain the consistency of the existence of desirable embeddings. We follow [3, Chapter 1] here.

Definition 2.24. Let \bar{N} and N be transitive ZFC^- models. We say that an elementary embedding $\sigma : \bar{N} \prec N$ is **cofinal** so long as for each $x \in N$ there is some $u \in \bar{N}$ such that $x \in \sigma(u)$.

Let $\alpha \in \bar{N}$. We say that σ is **α -cofinal** so long as every such u has size less than α as computed in \bar{N} .

Definition 2.25. Let $\alpha > \omega$ be a regular cardinal in \bar{N} , a transitive ZFC^- model. Let

$$\bar{\sigma} : H_\alpha^{\bar{N}} \prec H \text{ cofinally,}$$

where H is transitive. By a **transitive liftup** of $\langle \bar{N}, \bar{\sigma} \rangle$ we mean a pair $\langle N_*, \sigma_* \rangle$ such that

- N_* is transitive
- $\sigma_* : \bar{N} \prec N_*$ α -cofinally
- $\sigma_* \upharpoonright H_\alpha^{\bar{N}} = \bar{\sigma}$

Reminiscent of ultrapowers, transitive liftups can be characterized in the following way:

Lemma 2.26 (Jensen). Let \bar{N}, N be transitive ZFC^- models with $\sigma : \bar{N} \prec N$. Then, σ is α -cofinal \iff elements of N are of the form $\sigma(f)(\beta)$ for some $f : \gamma \rightarrow \bar{N}$ where $\gamma < \alpha$ and $\beta < \sigma(\gamma)$.

Proof. We show each direction of the equivalence separately.

\implies : Let $x \in N$, and take $u \in \bar{N}$ with $x \in \sigma(u)$ such that $|u| < \alpha$ in \bar{N} . Let $|u| = \gamma$, and take $f : \gamma \rightarrow u$ a bijection in \bar{N} . Then $\sigma(f) : \sigma(\gamma) \rightarrow \sigma(u)$ is also a bijection in N by elementarity. Since $x \in \sigma(u)$ we also have that x has a preimage under $\sigma(f)$, say β . So $\sigma(f)(\beta) = x$ as desired.

\impliedby : Let $x = \sigma(f)(\beta)$ be an element of N , for $f : \gamma \rightarrow \bar{N}$ where $\gamma < \alpha$ in \bar{N} and $\beta < \sigma(\gamma)$. Define $u = f''\gamma$. Then in \bar{N} we have that $|u| < \alpha$. In addition we have that $x \in \sigma(u)$, since $\sigma(u)$ is in the range of $\sigma(f)$, where x lies. \square

Furthermore, Jensen shows that transitive liftups exist so long as an embedding already exists, using an ultrapower-like construction, and have a uniqueness property.

Fact 2.27 (Interpolation). Let $\sigma : \bar{N} \prec N$ with $\bar{N} \models \text{ZFC}^-$ transitive, and let $\alpha \in \bar{N}$ be a regular cardinal. Then:

1. The transitive liftup $\langle N_*, \sigma_* \rangle$ of $\langle \bar{N}, \sigma \upharpoonright H_\alpha^{\bar{N}} \rangle$ exists.
2. There is a unique $k_* : N_* \prec N$ such that $k_* \circ \sigma_* = \sigma$ and $k_* \upharpoonright \bigcup \sigma'' H_\alpha^{\bar{N}} = \text{id}$.

For the following useful lemma, we will need to define the following, more general, notion of liftups. Of course the rich theory is established by Jensen, and is explored in detail in his notes. We will only use this more general definition for the following lemma, which is why we did not introduce liftups in this way from the beginning.

Definition 2.28. Let \mathfrak{A} be a solid model of ZFC^- and let $\tau \in \text{wfc}(\mathfrak{A})$ be an uncountable cardinal in \mathfrak{A} . Let

$$\sigma : H_\tau^\mathfrak{A} \prec H \text{ cofinally,}$$

where H is transitive. Then by a **liftup** of $\langle \mathfrak{A}, \sigma \rangle$, we mean a pair $\langle \mathfrak{A}_*, \sigma_* \rangle$ such that

- $\sigma_* \supseteq \sigma$
- \mathfrak{A}_* is solid
- $\sigma_* : \mathfrak{A} \rightarrow_{\Sigma_0} \mathfrak{A}_*$ τ -cofinally
- $H \in \text{wfc}(\mathfrak{A}_*)$

Fact 2.29 (Jensen). *Let \mathfrak{A} be a solid model of ZFC^- . Let $\tau > \omega$, $\tau \in \text{wfc}(\mathfrak{A})$, and let*

$$\sigma : H_\tau^\mathfrak{A} \prec H \text{ cofinally,}$$

where H is transitive. Then $\langle \mathfrak{A}, \sigma \rangle$ has a liftup $\langle \mathfrak{A}_, \sigma_* \rangle$.*

The following lemma states that transitive liftups of full models are almost full.

Lemma 2.30 (Jensen). *Let $N = L_\tau[A] \models \text{ZFC}^-$ and $\sigma : \bar{N} \prec N$ where \bar{N} is full. Suppose that $\langle N_*, \sigma_* \rangle$ is a transitive liftup of $\langle \bar{N}, \bar{\sigma} \rangle$. Then N_* is almost full.*

Proof. Let $L_\gamma(\bar{N})$ witness the fullness of \bar{N} . We will now apply Interpolation (Fact 2.27) to $\mathfrak{A} = L_\gamma(\bar{N})$, which makes sense since certainly \mathfrak{A} is a transitive model ZFC^- . Additionally, by **Lemma 2.6** we have that $\bar{N} = H_\tau^\mathfrak{A}$, where $\tau = \text{height}(\bar{N})$. Since $\langle N_*, \sigma_* \rangle$ is a transitive liftup, we have that

$$\sigma_* : H_\tau^\mathfrak{A} \prec N_* \text{ cofinally,}$$

where N_* is transitive. Thus since \mathfrak{A} is transitive, $\langle \mathfrak{A}, \sigma_* \rangle$ has a liftup $\langle \mathfrak{A}_*, \sigma_{**} \rangle$, where $\mathfrak{A}_* \models \text{ZFC}^-$ since \mathfrak{A} does, \mathfrak{A}_* is solid, where

$$\sigma_{**} : \mathfrak{A} \prec \mathfrak{A}_* \text{ } \tau\text{-cofinally.}$$

We have that $N_* \subseteq \text{wfc}(\mathfrak{A}_*)$ and $\tau_* = \sigma_{**}(\tau) = \text{height}(N_*)$ is regular since τ is. Furthermore, we will show that $N_* = H_{\tau_*}^{\mathfrak{A}_*}$, completing the proof:

Certainly it is the case that $N_* \subseteq H_{\tau_*}^{\mathfrak{A}_*}$. But if $x \in H_{\tau_*}^{\mathfrak{A}_*}$ in \mathfrak{A}_* , then by regularity we have that $x \in \sigma_{**}(u)$ in \mathfrak{A}_* , where $u \in \mathfrak{A}$, and $|u| < \tau$ in \mathfrak{A} . Let $v = u \cap H_\tau$ in \mathfrak{A} . Then $v \in \bar{N}$, since \bar{N} is regular in \mathfrak{A} . But then $x \in \sigma_*(v) \in N_*$. So $x \in N_*$. \square

3. GENERALIZED DIAGONAL PRIKRY FORCING

Definition 3.1. Let D be an infinite discrete set of measurable cardinals, meaning a set of measurable cardinals that does not contain any of its limit points. For $\kappa \in D$ let $U(\kappa)$ be a normal measure on κ , and let \mathcal{U} denote the sequence of the $U(\kappa)$'s.

Define $\mathbb{D} = \mathbb{D}(\mathcal{U})$, **generalized diagonal Prikry forcing** from the list of measures \mathcal{U} , by taking conditions of the form (s, A) satisfying the following:

- The *stem* of the condition, s , is a function with domain in $[D]^{<\omega}$ taking each measurable cardinal $\kappa \in \text{dom}(s)$ to some ordinal $s(\kappa) < \kappa$.
- The *upper part* of the condition, A , is a function with domain $D \setminus \text{dom}(s)$ taking each measurable cardinal $\kappa \in \text{dom}(A)$ to some measure-one set $A(\kappa) \in U(\kappa)$.

The extension relation on conditions in \mathbb{D} is defined so that $(s, A) \leq (t, B)$ so long as

- $s \supseteq t$.
- The points in s not in t come from B , i.e., for all $\kappa \in \text{dom}(s) \setminus \text{dom}(t)$, $s(\kappa) \in B(\kappa)$.
- For all $\kappa \in \text{dom}(A)$, $A(\kappa) \subseteq B(\kappa)$.

If G is a generic filter for \mathbb{D} , then its associated \mathbb{D} -generic sequence is

$$S = S_G = \bigcup \{s \mid \exists A (s, A) \in G\}.$$

Note that our definition of $\mathbb{D}(\mathcal{U})$ differs from that as in Fuchs of generalized Prikry forcing [1]. The main difference is that here we only add one point below each measurable cardinal $\kappa \in D$, which is done for simplicity's sake. It is not hard to see that the following theorem showing generalized diagonal Prikry forcing is subcomplete also shows that the forcing adding countably many points below each measurable cardinal in D (where the conditions consist of finite stems) is subcomplete. Adding countably many points below each measurable cardinal in D would collapse the cofinality of each $\kappa \in D$ to be ω , as one expects of a Prikry-like forcing.

Also in the above definition we haven't enforced that the stem of our conditions only consist of ordinals that are wedged between successive measurables in D ; ie. for $\kappa \in D$, we do not explicitly insist that $s(\kappa) \in [\sup(D \cap \kappa), \kappa)$. However, it is dense in $\mathbb{D}(\mathcal{U})$ for the conditions to be that way, since we can always strengthen conditions by restricting their upper parts to a tail. Thus we may freely add this restriction to the following genericity condition on $\mathbb{D}(\mathcal{U})$. The following is a genericity criterion on generalized diagonal Prikry forcing similar to the Mathias criterion for Prikry forcing. It was shown in [1, Theorem 1].

Fact 3.2 (Fuchs). *Let D be an infinite discrete set of measurable cardinals, with \mathcal{U} a corresponding list of measures $\langle U(\kappa) \mid \kappa \in D \rangle$. Then an increasing sequence of ordinals $S = \langle S(\kappa) \mid \kappa \in D \rangle$, where for each $\kappa \in D$*

$$\sup(D \cap \kappa) < S(\kappa) < \kappa,$$

is a $\mathbb{D}(\mathcal{U})$ -generic sequence if and only if for all $\mathcal{X} = \langle X_\kappa \in U(\kappa) \mid \kappa \in D \rangle$, the set

$$\{\kappa \in D \mid S(\kappa) \notin X_\kappa\} \text{ is finite.}$$

Theorem 3.3. *Let D be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D \rangle$ be a list of measures associated to D . Then $\mathbb{D} = \mathbb{D}(\mathcal{U})$ is subcomplete.*

Proof. Let $\theta \gg \delta(\mathbb{D}) = \delta$ be large enough, so that $[\delta]^{<\omega_1} \in H_\theta$.

It must be the case that $\delta \geq \sup D$: to see this, suppose instead that there is a dense $E \subseteq \mathbb{D}$ such that $\sup D \geq \kappa^* > |E|$ for some $\kappa^* \in D$. Then for each condition $(s, A) \in E$ either $\kappa^* \in \text{dom } s$ or $\kappa^* \in \text{dom } A$. So taking $E^* = \{(s, A) \in E \mid \kappa^* \in \text{dom } s\} \subseteq E$, since $|E^*| < \kappa^*$ as well, there is an $\alpha < \kappa^*$ such that $\sup_{(s, A) \in E^*} s(\kappa^*) = \alpha$. Let $p = (t, B) \in \mathbb{D}$ be defined so that $t(\kappa^*) = \alpha$ and $B(\kappa) = \kappa$ for all $\kappa \in D \setminus \{\kappa^*\}$. Then p cannot be strengthened by any condition in E since κ^* is not in any of the stems of conditions in E . So dense subsets of \mathbb{D} must have size at least $\sup D$.

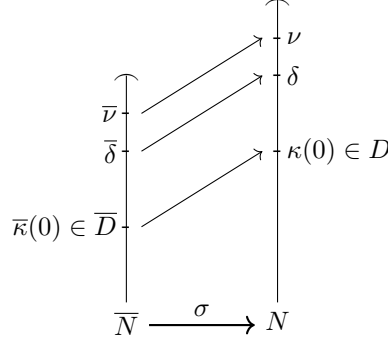
Let $\nu = \delta^+$. Let $\kappa(0)$ be the first measurable cardinal in D .

In order to show that \mathbb{D} is subcomplete, suppose we are in the following situation:

- $\mathbb{D} \in H_\theta \subseteq N = L_\tau[A] \models \text{ZFC}^-$ where $\tau > \theta$ and $A \subseteq \tau$
- $\sigma : \bar{N} \cong X \prec N$ where X is countable and \bar{N} is full
- $\sigma(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \theta, \mathbb{D}, \mathcal{U}, c$ for some $c \in N$.

By our requirement on θ , we've ensured that N is closed under countable sequences of ordinals less than δ .

In what follows, we will be taking a few different transitive liftings of restrictions of σ , and it will be useful to keep track of embeddings between \bar{N} and N pictorially. Although it's not extraordinarily illuminating at this point in our discussion, the following figure shows the situation we are currently in, where $\sigma(\bar{\delta}) = \delta$, $\bar{\nu} = \bar{\delta}^{+\bar{N}}$, \bar{D} is the discrete set of measurables in \bar{N} that each measure in $\bar{\mathcal{U}}$ comes from, and $\bar{\kappa}(0)$ is the first measurable in \bar{D} , in the sense of \bar{N} .



Toward showing that \mathbb{D} is subcomplete, we are additionally given some $\bar{G} \subseteq \bar{\mathbb{D}}$ that is generic over \bar{N} . Rather than working with \bar{G} , we will work with $\bar{S} = \langle \bar{S}(\bar{\kappa}) \mid \bar{\kappa} \in \bar{D} \rangle$, its associated \mathbb{D} -generic sequence. We must show following, where $C = \mathcal{Hull}^N(\delta \cup X)$:

Claim (Main). There is a \mathbb{D} -generic sequence S and a map $\sigma' \in V[S]$ such that:

1. $\sigma' : \bar{N} \prec N$
2. $\sigma'(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \theta, \mathbb{D}, \mathcal{U}, c$
3. $\mathcal{Hull}^N(\delta \cup \text{range}(\sigma')) = C$
4. $\sigma' \text{ `` } \bar{S} \subseteq S$

Pf. This proof uses Barwise theory (key definitions, facts, and theorems are summarized Section ??) heavily, and ultimately amounts to showing that a certain \in -theory, \mathcal{T} , which posits the existence of such a σ' , is consistent. Such an embedding σ' can only possibly exist in a suitable generic extension, $V[S]$, where S is \mathbb{D} -generic sequence that we will find later. Once we have such a suitable $V[S]$, we will use Barwise theory to find an appropriate admissible structure in $V[S]$ for which the theory \mathcal{T} , positing the existence of such a suitable σ' , defined below, has a model.

For an admissible structure \mathfrak{M} with $S, \bar{S}, \sigma, N, \theta, \mathbb{D}, \mathcal{U}, c \in \mathfrak{M}$ let the infinitary \in -theory $\mathcal{T}(\mathfrak{M})$ be defined over \mathfrak{M} as follows:

predicates: \in

constants: $\dot{\sigma}, \underline{x}$ for $x \in \mathfrak{M}$

axioms: • ZFC⁻ and Basic Axioms.

- $\dot{\sigma} : \bar{N} \prec N$
- $\dot{\sigma}(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \underline{\theta}, \underline{\mathbb{D}}, \underline{\mathcal{U}}, \underline{c}$
- $\mathcal{Hull}^N(\underline{\delta} \cup \text{range}(\dot{\sigma})) = \mathcal{Hull}^N(\underline{\delta} \cup \text{range}(\underline{\sigma}))$
- $\dot{\sigma} \text{ `` } \bar{S} \subseteq \underline{S}$.

The \in -theory is $\Sigma_1(\mathfrak{M})$, since all of the axioms are \mathfrak{M} -finite except for the Basic Axioms, which altogether are \mathfrak{M} -re as each of them are \mathfrak{M} -finite.

We need to find an appropriate \mathbb{D} -generic sequence S and a suitable admissible structure \mathfrak{M} containing S so that $\mathcal{T}(\mathfrak{M})$ is consistent. To do this we use transitive liftings and Barwise theory.

Transitive liftings will give us the consistency of certain embeddings that approximate the one we are looking for, and we will rely on Barwise Completeness (*Fact 2.20*) to obtain the existence of a model with our desired properties.

Toward this end, let's take what will turn out to be our first transitive lifting, which is in some sense ensuring the consistency of having property **3** of our main claim.

Let $k_0 : N_0 \cong C$ where N_0 is transitive, and set $\sigma_0 = k_0^{-1} \circ \sigma$ and $\sigma_0(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \theta_0, \mathbb{D}_0, \mathcal{U}_0, c_0$. Since $\delta \subseteq C$ and N_0 is transitive, $\sigma_0(\bar{\delta}) = \delta$.

Indeed N_0 is actually a transitive lifting:

Claim 1. $\langle N_0, \sigma_0 \rangle$ is the transitive lifting of $\langle \bar{N}, \sigma \restriction H_{\bar{\nu}}^{\bar{N}} \rangle$.

Pf. Recall that $\nu = \delta^+$, and $\bar{\nu} = \bar{\delta}^{+\bar{N}}$. It must be shown that the embedding $\sigma_0 : \bar{N} \prec N_0$ is $\bar{\nu}$ -cofinal and that $\sigma_0 \restriction H_{\bar{\nu}}^{\bar{N}} = \sigma \restriction H_{\bar{\nu}}^{\bar{N}}$.

To see that σ_0 is $\bar{\nu}$ -cofinal, let $x \in N_0$. Then $k_0(x) \in C = \mathcal{Hull}^N(\delta \cup X)$ so $k_0(x)$ is uniquely N -definable from $\xi < \delta$ and $\sigma(\bar{z})$ where $\bar{z} \in \bar{N}$. In other words,

$$k_0(x) = \text{that } y \text{ such that } N \models \varphi(y, \xi, \sigma(\bar{z})).$$

Let $u \in \bar{N}$ be defined as

$$u = \{w \in \bar{N} \mid w = \text{that } y \text{ such that } \bar{N} \models \varphi(y, \zeta, \bar{z}) \text{ for some } \zeta < \bar{\delta}\}.$$

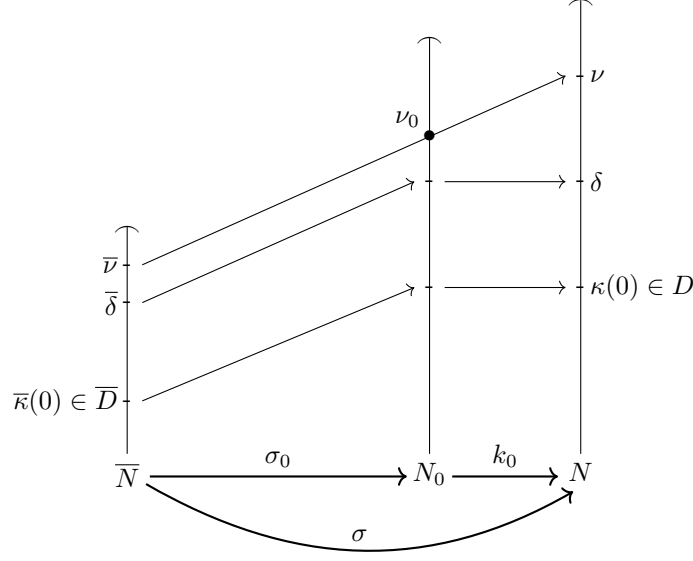
Certainly u is non-empty by elementarity, since $k_0(x) \in \sigma(u)$. Furthermore, $|u| \leq \bar{\delta} < \bar{\nu}$ since every $w \in u$ is unique, and needs a corresponding $\zeta < \bar{\delta}$ to satisfy the formula φ with. Thus $x \in k_0^{-1}(\sigma(u)) = \sigma_0(u)$ with $|u| < \bar{\nu}$ in \bar{N} , as desired.

Since $X \cup \delta \subseteq C$, the Skolem hull in N , we know that $\sigma^{\text{``}}H_{\bar{\nu}}^{\bar{N}} \subseteq C$. Thus $k_0^{-1} \restriction \sigma^{\text{``}}H_{\bar{\nu}}^{\bar{N}} = \text{id}$. Therefore $\sigma_0 \restriction H_{\bar{\nu}}^{\bar{N}} = \sigma \restriction H_{\bar{\nu}}^{\bar{N}}$, finishing the proof of the claim. \square

Since $\bar{\nu}$ is regular in \bar{N} , in N_0 so is $\nu_0 = \sigma_0(\bar{\nu}) = \sup \sigma_0^{\text{``}}\bar{\nu}$. By Interpolation (*Fact 2.27*), we may say that k_0 is defined by

$$k_0 : N_0 \prec N \text{ where } k_0 \circ \sigma_0 = \sigma \text{ and } k_0 \restriction \nu_0 = \text{id}.$$

In particular, ν_0 is the critical point of k_0 , which is continuous below ν_0 . Thus we are in a situation that we will represent with the following diagram:



Already, we can say that σ_0 looks like it has one nice property: $\mathcal{Hull}^{N_0}(\delta \cup \text{range}(\sigma_0)) = C$, which somewhat looks like **3** of the main claim. However, we have not yet performed forcing, σ_0 is definable in V , and we still need to find a way to extend the generic sequence \bar{S} to a \mathbb{D} -generic sequence over N . We still have a lot more work to do before finding σ' .

We shall define another \in -theory, \mathcal{L}_* that will assist us in obtaining the diagonal Prikry extension $V[S]$ we need to satisfy our main claim. In order to do this, we will take another transitive liftup and apply Transfer (*Fact 2.23*), in order to see that this new \in -theory is consistent over an admissible structure on N_0 .

Since we will be referring to the same \in -theory over two different transitive liftups, I would like to think of “ $*$ ” as a kind of placeholder for a transitive liftup in the following definition.

Suppose that $\langle N_*, \sigma_* \rangle$ is a transitive liftup of \bar{N} along with some reasonable restriction of σ , ie. the liftup of $\langle \bar{N}, \sigma \restriction H_\alpha^{\bar{N}} \rangle$, where $\alpha \geq \bar{\kappa}(0)$ is regular in \bar{N} , and say

$$\sigma_*(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \theta_*, \mathbb{D}_*, \mathcal{U}_*, c_*.$$

Recall that δ_{N_*} is the least such that $L_{\delta_{N_*}}(N_*)$ is admissible.

Define the infinitary \in -theory $\mathcal{L}(N_*, \sigma_*) = \mathcal{L}_*$ as follows:

predicates: \in

constants: $\dot{\sigma}, \dot{S}, \underline{x}$ for $x \in L_{\delta_{N_*}}(N_*)$

axioms: • ZFC⁻ and Basic Axioms

- $\dot{\sigma} : \bar{N} \prec N_*$ is $\bar{\kappa}(0)$ -cofinal
- $\dot{\sigma}(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \theta_*, \mathbb{D}_*, \mathcal{U}_*, c_*$
- \dot{S} is a \mathbb{D}_* -generic sequence over N_*
- $\dot{\sigma} \text{ “ } \bar{S} \subseteq \dot{S} \text{ ”}$.

As defined, we have that \mathcal{L}_* is a $\Sigma_1(L_{\delta_{N_*}}(N_*))$ -theory, since altogether the Basic Axioms are $\Sigma_1(L_{\delta_{N_*}}(N_*))$.

We claim that the theory is consistent.

Claim 2. $\mathcal{L}_0 = \mathcal{L}(N_0, \sigma_0)$ is consistent.

Pf. Of course, it is not the case that σ_0 is $\bar{\kappa}(0)$ -cofinal - all we know is that it is $\bar{\nu}$ -cofinal. However, we know how to find an elementary embedding that is $\bar{\kappa}(0)$ cofinal: by taking a suitable transitive liftup.

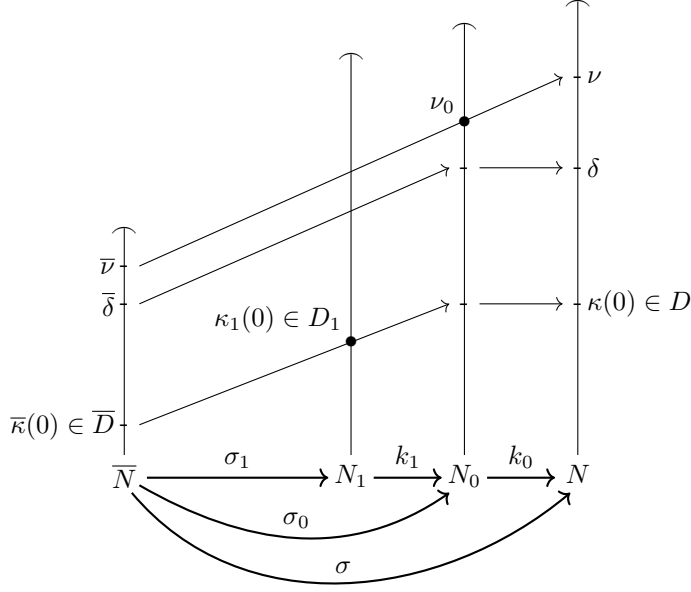
Let $\langle N_1, \sigma_1 \rangle$ be the transitive liftup of $\langle \bar{N}, \sigma \restriction H_{\bar{\kappa}(0)}^{\bar{N}} \rangle$, which exists by Interpolation (*Fact 2.27*). So we have that $\sigma_1 \restriction H_{\bar{\kappa}(0)}^{\bar{N}} = \sigma \restriction H_{\bar{\kappa}(0)}^{\bar{N}}$. Let

$$\sigma_1(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1.$$

Since $\sigma_1 \restriction H_{\bar{\kappa}(0)}^{\bar{N}} = \sigma \restriction H_{\bar{\kappa}(0)}^{\bar{N}} = \sigma_0 \restriction H_{\bar{\kappa}(0)}^{\bar{N}}$, we also have a unique

$$k_1 : N_1 \prec N_0 \text{ where } k_1 \circ \sigma_1 = \sigma_0 \text{ and } k_1 \restriction \kappa_1(0) = \text{id where } k_1(0) = k_1(\kappa(0)).$$

Indeed k_1 is continuous below $\kappa_1(0)$. We illustrate the final picture below, with σ and all of the relevant transitive liftups.



We first show that $\mathcal{L}_1 = \mathcal{L}(N_1, \sigma_1)$ is consistent, by seeing that it has a model. To do this, we will find a sequence extending σ_1 “ \bar{S} ” that is \mathbb{D}_1 -generic over N_1 . Then we will use the Transfer Lemma to see that this transfers to the consistency of \mathcal{L}_0 .

First, force with \mathbb{D}_1 , which is diagonal Prikry over N_1 , to obtain a diagonal Prikry sequence S'_1 . Define, in $V[S'_1]$, a new sequence S_1 as follows:

$$S_1(\kappa) = \begin{cases} S'_1(\kappa) & \text{if } \kappa \in D_1 \setminus \sigma_1 \text{ “}\bar{D}\text{”} \\ \sigma_1(\bar{S}(\bar{\kappa})) & \text{if } \kappa = \sigma_1(\bar{\kappa}) \in \sigma_1 \text{ “}\bar{D}\text{”}. \end{cases}$$

Claim. The sequence S_1 is a \mathbb{D}_1 -generic sequence over N_1 .

Pf. We will show that S_1 satisfies the generalized diagonal Prikry genericity criterion (*Fact 3.2*) over N_1 . To do this, let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D_1 \rangle$, with $\mathcal{X} \in N_1$, be a sequence of measure-one sets in the sequence of measures \mathcal{U}_1 .

Note first that S'_1 is a generic sequence, it already satisfies the generalized diagonal Prikry genericity criterion, namely:

$$\{\kappa \in D_1 \mid S'_1(\kappa) \notin X_\kappa\} \text{ is finite.}$$

Recall that $\bar{S} = \langle \bar{S}(\bar{\kappa}) \mid \bar{\kappa} \in \bar{D} \rangle$ is a $\bar{\mathbb{D}}$ -generic sequence as well. We need to see that in addition,

$$\{\bar{\kappa} \in \bar{D} \mid \sigma_1(\bar{S}(\bar{\kappa})) \notin X_{\sigma_1(\bar{\kappa})}\} \text{ is finite,}$$

since then

$$\{\kappa \in D_1 \mid S_1(\kappa) \notin X_\kappa\} = \{\kappa \in D_1 \setminus \sigma_1 \text{``}\bar{D}\text{"} \mid S'_1(\kappa) \notin X_\kappa\} \cup \{\kappa = \sigma_1(\bar{\kappa}) \in \sigma_1 \text{``}\bar{D}\text{"} \mid \sigma_1(\bar{S}(\bar{\kappa})) \notin X_\kappa\}$$

is finite as well, completing the proof as desired.

By the $\bar{\kappa}(0)$ -cofinality of σ_1 , there is some $w \in \bar{N}$ such that $\mathcal{X} \in \sigma_1(w)$, where $|w| < \bar{\kappa}(0)$ in \bar{N} . Thus in N_1 , we have that $|\sigma_1(w)| < \kappa_1(0)$. We may assume that w consists of functions $f \in \prod_{\bar{\kappa} \in \bar{D}} \bar{U}(\bar{\kappa})$. So for each $\kappa \in \sigma_1 \text{``}\bar{D}\text{"}$, we have that $X_\kappa \in \sigma_1(w)_\kappa = \{\sigma_1(f)(\bar{\kappa}) \mid f \in \prod_{\bar{\kappa} \in \bar{D}} \bar{U}(\bar{\kappa}) \wedge f \in w\}$ and also $|\sigma_1(w)_\kappa| < \kappa_1(0)$. So all $\kappa \in \sigma_1 \text{``}\bar{D}\text{"}$ of course satisfy $\kappa \geq \kappa_1(0)$ and thus by the κ -completeness of $U_1(\kappa)$, we have that $W_\kappa := \cap \sigma_1(w)_\kappa \in U_1(\kappa)$. So we have established that \mathcal{W} , the sequence of W_κ for $\kappa \geq \kappa_1(0)$, is also a sequence of measure-one sets in N_1 . Note in addition that for $\kappa \in \sigma_1 \text{``}\bar{D}\text{"}$, we have that $W_\kappa \subseteq X_\kappa$.

By elementarity, for each $\bar{\kappa} \in \bar{D}$, we have $\bar{W}_{\bar{\kappa}} = \cap \{f(\bar{\kappa}) \mid f \in \prod_{\bar{\kappa} \in \bar{D}} \bar{U}(\bar{\kappa}) \wedge f \in w\}$ is a measure-one set in $\bar{U}(\bar{\kappa})$ and we also have that $\sigma_1(\bar{W}_{\bar{\kappa}}) = W_{\sigma_1(\bar{\kappa})}$. Moreover,

$$\{\bar{\kappa} \in \bar{D} \mid \bar{S}(\bar{\kappa}) \notin \bar{W}_{\bar{\kappa}}\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for $\bar{\mathbb{D}}$, which must be satisfied by \bar{S} . Thus by elementarity,

$$\{\bar{\kappa} \in \bar{D} \mid \sigma_1(\bar{S}(\bar{\kappa})) \notin W_{\sigma_1(\bar{\kappa})}\} \supseteq \{\bar{\kappa} \in \bar{D} \mid \sigma_1(\bar{S}(\bar{\kappa})) \notin X_{\sigma_1(\bar{\kappa})}\} \text{ is finite,}$$

as is desired, completing the proof of our claim. \square

Moreover we have now shown that $\mathcal{L}(N_1, \sigma_1) = \mathcal{L}_1$ is consistent by Barwise Correctness (*Fact 2.19*), since we have just shown that $\langle H_\delta; \sigma_1, S_1 \rangle$ is a model of \mathcal{L}_1 .

Let's check that we may now apply Transfer (*Fact 2.23*) to the embedding $k_1 : N_1 \prec N_0$. By **Lemma 2.30** we have that N_1 is almost full. We also have that $\mathcal{L}_1 = \mathcal{L}(L_{\delta_{N_1}}(N_1))$ is Σ_1 over parameters

$$N_1 \text{ and } \bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}, \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1 \in N_1$$

while $\mathcal{L}_0 = \mathcal{L}(L_{\delta_{N_0}}(N_0))$ is Σ_1 over parameters

$$N_0 \text{ and } k_1(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}, \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1) \in N_0.$$

Furthermore k_1 is cofinal, since for each element $x \in N_0$, as σ_0 is cofinal, there is $u \in \bar{N}$ such that $x \in \sigma_0(u)$. Thus $\sigma_1(u) \in N_1$, and moreover $x \in k_1(\sigma_1(u)) = \sigma_0(u)$. Therefore, we have that since \mathcal{L}_1 is consistent, \mathcal{L}_0 is consistent as desired. This completes the proof of *Claim 2*. \square

From the consistency of \mathcal{L}_0 , we would now like to use Barwise Completeness (*Fact 2.20*) to obtain a model of \mathcal{L}_0 . To do this, we need the admissible structure the theory is defined over to

be countable. So let's work in $V[F]$, a generic extension that collapses $L_{\delta_{N_0}}(N_0)$ to be countable. Then by Barwise Completeness, \mathcal{L}_0 has a solid model

$$\mathfrak{A} = \langle \mathfrak{A}; \dot{S}^{\mathfrak{A}}, \dot{\sigma}^{\mathfrak{A}} \rangle$$

such that

$$\text{Ord} \cap \text{wfc}(\mathfrak{A}) = \text{Ord} \cap L_{\delta_{N_0}}(N_0).$$

Thus we have that $\dot{\sigma}^{\mathfrak{A}} : \overline{N}^{\mathfrak{A}} \prec N_0^{\mathfrak{A}}$. By the Basic Axioms we have that $\overline{N}^{\mathfrak{A}} = \overline{N}$ and $N_0 = N_0^{\mathfrak{A}}$. Thus we may say that $\dot{\sigma}^{\mathfrak{A}} : \overline{N} \prec N_0$.

Let $S = \dot{S}^{\mathfrak{A}}$ and $\dot{\sigma}^* = k_0 \circ \dot{\sigma}^{\mathfrak{A}}$.

Then S is a \mathbb{D}_0 -generic sequence over N_0 , and as $k_0 : N_0 \cong C$ we also have that k_0 “ S is C -generic for \mathbb{D} ”. We need to see that S is \mathbb{D} -generic over V . To do this, let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D \rangle$ be a sequence of measure-one sets in the sequence of measures \mathcal{U} . We will verify the generalized diagonal Prikry genericity criterion. To do this, let $E \subseteq \mathbb{D}$ be dense and have size δ with $E \in C$. Since $\delta \subseteq C$, we have that $E \subseteq C$ as well. Find a condition $(s, A) \in E$ that strengthens (\emptyset, \mathcal{X}) . Thus for $\kappa \in \text{dom } A$, we have that $A(\kappa) \subseteq X_\kappa$. Define a sequence of measure-one sets B in C so that

$$B(\kappa) = \begin{cases} A(\kappa) & \text{if } \kappa \in \text{dom } A \\ \kappa & \text{if } \kappa \in \text{dom } s. \end{cases}$$

So we have that B is a sequence of measure-one sets in C . So $\{\kappa \in D \mid S(\kappa) \notin B(\kappa)\}$ is finite. Thus $\{\kappa \in D \mid S(\kappa) \notin X_\kappa\}$ is finite.

This will be the \mathbb{D} -generic sequence we need to satisfy our main claim. We will see in the following claim that $\dot{\sigma}^*$ has all of the desired properties of our main claim, but it fails to be in $V[S]$, which is what we need. However, based on the following claim, $\dot{\sigma}^*$ will at least enable us to see that our \in -theory \mathcal{T} from long ago, defined to assist us in proving the main claim, is consistent over a suitable admissible structure.

Claim 3. The map $\dot{\sigma}^*$ satisfies:

1. $\dot{\sigma}^* : \overline{N} \prec N$
2. $\dot{\sigma}^*(\theta, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta, \mathbb{D}, \mathcal{U}, c$
3. $\mathcal{Hull}^N(\delta \cup \text{range}(\dot{\sigma}^*)) = C$
4. $\dot{\sigma}^* \restriction \overline{S} \subseteq S$

Pf. For item **1**, we have already seen above that $\dot{\sigma}^{\mathfrak{A}} : \overline{N} \prec N_0$. Since $k_0 : N_0 \prec N$, the desired result follows.

For item **2**, $\dot{\sigma}^*(\theta, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = k_0(\theta_0, \mathbb{D}_0, \mathcal{U}_0, c_0) = \theta, \mathbb{D}, \mathcal{U}, c$.

Item **3** holds since $N_0 = \mathcal{Hull}^{N_0}(\delta \cup \text{range}(\dot{\sigma}^{\mathfrak{A}}))$. To see this, clearly we have that $\mathcal{Hull}^{N_0}(\delta \cup \text{range}(\dot{\sigma}^{\mathfrak{A}})) \subseteq N_0$, since $\delta \in N_0$ as $N_0 \cong C$, and certainly $\text{range}(\dot{\sigma}^{\mathfrak{A}}) \subseteq N_0$ as well. Then because $\dot{\sigma}^{\mathfrak{A}}$ is $\overline{\kappa}'$ -cofinal, by **Lemma 2.26**, we have, since $\dot{\sigma}^{\mathfrak{A}}(\overline{\kappa}') < \delta$, that:

$$N_0 = \{\dot{\sigma}^{\mathfrak{A}}(f)(\beta) \mid f : \gamma \longrightarrow N_0, \gamma < \overline{\kappa}(0) \text{ and } \beta < \dot{\sigma}^{\mathfrak{A}}(\gamma)\} \subseteq \mathcal{Hull}^N(\delta \cup \text{range}(\dot{\sigma}^{\mathfrak{A}})).$$

Thus $C = k_0 \restriction N_0 = \mathcal{Hull}^{N_0}(\delta \cup \text{range}(k_0 \circ \dot{\sigma}^{\mathfrak{A}}))$ as desired.

To see **4**, note that $\dot{\sigma}^* \restriction \overline{\kappa}(0) = \dot{\sigma}^{\mathfrak{A}} \restriction \overline{\kappa}(0)$ since $k_0 \restriction \nu_0 = \text{id}$.

This completes the proof of *Claim 3*. □

We are almost done, but like we stated above, $\dot{\sigma}^{\mathfrak{A}}$ is in $V[F]$, the generic extension needed to obtain a countable admissible structure to apply Barwise Completeness to. But $V[F]$ is not in $V[S]$, and so $\dot{\sigma}^*$ is not necessarily in $V[S]$. We will use Barwise Completeness one last time, to finally find an embedding σ' with which to satisfy the main claim along with the S we found above.

Let λ be regular in $V[S]$ with $N \in H_\lambda^{V[S]}$. Then

$$M = \langle H_\lambda^{V[S]}, N, \sigma, S; \theta, \delta, \mathbb{D}, \mathcal{U}, c \rangle \text{ is admissible.}$$

In order to satisfy our main claim, we need a model of $\mathcal{T}(M)$ in $V[S]$. By *Claim 3*, we have that $\langle M, \sigma^* \rangle$ is a model of $\mathcal{T}(M)$, but in $V[F]$, which is not $V[S]$. Still this means that by Barwise Correctness, $\mathcal{T}(M)$ is consistent.

Consider the Mostowski collapse of M ; let

$$\pi : \tilde{M} \prec M \text{ where } \tilde{M} \text{ is countable and transitive.}$$

Note that $\tilde{M} \in H_{\omega_1}^{V[S]} = H_{\omega_1}^V$ since it is countable and diagonal Prikry forcing doesn't add bounded subsets to any $\kappa \in D$.³ We also have that $\overline{N}^{\tilde{\mathfrak{A}}} = \overline{N}$, since M sees that \overline{N} is countable so \tilde{M} sees that $\pi^{-1}(\overline{N})$ is, and it follows that $\pi^{-1}(\overline{N}) = \overline{N}$.

Plus, $\mathcal{T}(\tilde{M})$ is consistent, since otherwise its inconsistency could be pushed up via π to one in $\mathcal{T}(M)$, contradicting the model witnessing its consistency that we found in $V[F]$.

So by Barwise Completeness, $\mathcal{T}(\tilde{M})$ has a solid model

$$\tilde{\mathfrak{A}} = \langle \tilde{\mathfrak{A}}; \dot{\sigma}^{\tilde{\mathfrak{A}}} \rangle$$

such that

$$\text{Ord} \cap \text{wfc}(\tilde{\mathfrak{A}}) = \text{Ord} \cap \tilde{M}.$$

Letting $\sigma' = \pi \circ \dot{\sigma}^{\tilde{\mathfrak{A}}}$, the main claim is now satisfied with σ' and our λ -diagonal Prikry sequence S .

Let us verify each of the properties of σ' required by the main claim. The verification of these properties shall use the agreement between $\tilde{\mathfrak{A}}$ and \tilde{M} on the special constants of \tilde{M} and on the ordinals. The fact that π does not affect \overline{N} will be greatly taken advantage of.

First we show **1** of the main claim. Let's say that $\varphi[\sigma'(\bar{a})]$ holds in N . So $\varphi[\pi(\dot{\sigma}^{\tilde{\mathfrak{A}}}(\bar{a}))]^N$ holds in M . Thus $\varphi[\dot{\sigma}^{\tilde{\mathfrak{A}}}(\bar{a})]^{\pi^{-1}(N)}$ holds in \tilde{M} , and thus also in $\tilde{\mathfrak{A}}$. Indeed we know that $\overline{N}^{\tilde{\mathfrak{A}}} = \overline{N}$, since $\tilde{\mathfrak{A}}$ agrees with \tilde{M} about countable ordinals, which we may use to code \overline{N} . This means that $\varphi[\bar{a}]$ holds in \overline{N} , as desired.

To see **2**, we have $\sigma'(\bar{\theta}, \bar{\mathbb{D}}, \bar{\mathcal{U}}, \bar{c}) = \pi(\theta^{\tilde{\mathfrak{A}}}, \mathbb{D}^{\tilde{\mathfrak{A}}}, \mathcal{U}^{\tilde{\mathfrak{A}}}, c^{\tilde{\mathfrak{A}}}) = \theta, \mathbb{D}, \mathcal{U}, c$.

For item **3**, let $\tilde{N} = \overline{N}^{\tilde{\mathfrak{A}}}$, $\tilde{\sigma} = \underline{\sigma}^{\tilde{\mathfrak{A}}}$ and $\tilde{\delta} = \underline{\delta}^{\tilde{\mathfrak{A}}}$. So $\pi(\tilde{\delta}) = \delta$ and $\pi(\tilde{\sigma}) = \sigma$. Because of the way the \in -theory \mathcal{T} was defined, we already have:

$$(1) \quad \text{Hull}^{\tilde{N}}(\tilde{\delta} \cup \text{range}(\dot{\sigma}^{\tilde{\mathfrak{A}}})) = \text{Hull}^{\tilde{N}}(\tilde{\delta} \cup \text{range}(\tilde{\sigma})).$$

To see $\text{Hull}^N(\delta \cup \text{range}(\sigma')) \subseteq \text{Hull}^N(\delta \cup X)$, suppose $x \in \text{Hull}^N(\delta \cup \text{range}(\sigma'))$. Then we have that N sees that there is some formula φ , $\bar{z} \in \overline{N}$, and $\xi < \delta$ where x is unique such that $\varphi(x, \pi(\dot{\sigma}^{\tilde{\mathfrak{A}}}(\bar{z})), \xi)$. In particular, x is in the range of π . Thus $\tilde{x} = \pi^{-1}(x) \in \tilde{N}$ and $\tilde{\xi} < \tilde{\delta}$ such that $\varphi(\tilde{x}, \dot{\sigma}^{\tilde{\mathfrak{A}}}(\bar{z}), \tilde{\xi})$ holds. Thus by (1), we have that \tilde{x} is unique such that $\varphi(\tilde{x}, \tilde{\sigma}(\bar{y}), \tilde{\zeta})$ for some $\bar{y} \in \overline{N}$

³As Fuchs [1] points out, this result is a modification to the proof that generalized diagonal Prikry forcing preserves cardinalities.

and $\tilde{\zeta} < \tilde{\delta}$. Thus pushing back up through π , letting $\zeta = \pi(\tilde{\zeta})$, we have that $x = \pi(\tilde{x})$ is unique such that $\varphi(x, \sigma(\bar{y}), \zeta)$ holds, so $x \in \mathcal{Hull}^N(\delta \cup X)$.

To see that $\mathcal{Hull}^N(\delta \cup X) \subseteq \mathcal{Hull}^N(\delta \cup \text{range}(\sigma'))$ works similarly. Let $x \in \mathcal{Hull}^N(\delta \cup X)$. Then there is $\bar{z} \in \bar{N}$ and $\xi < \delta$ such that $\varphi(x, \sigma(\bar{z}), \xi)$ holds. So in particular, x is in the domain of π . So we may find $\tilde{\xi} < \tilde{\delta}$ such that $\tilde{x} = \pi^{-1}(x)$ is unique such that $\varphi(\tilde{x}, \tilde{\sigma}(\bar{z}), \tilde{\xi})$. So by (1), we have that there is $\bar{y} \in \bar{N}$ and $\tilde{\zeta} < \tilde{\delta}$ such that \tilde{x} is unique satisfying $\varphi(\tilde{x}, \tilde{\sigma}^{\mathfrak{A}}(\bar{y}), \tilde{\zeta})$. Finally, by pushing back up through π , letting $\pi(\tilde{\zeta}) = \zeta$, we have that $x = \pi(\tilde{x})$ is unique such that $\varphi(x, \sigma'(\bar{y}), \zeta)$, so $x \in \mathcal{Hull}^N(\delta \cup \text{range}(\sigma'))$ as desired.

To see item 4, note that $\bar{S}^{\mathfrak{A}} = \bar{S}$ since $\bar{S} \subseteq \bar{N}$. So we have already by the definition of \mathcal{T} that $\dot{\sigma}^{\mathfrak{A}} \text{``} \bar{S} \subseteq \underline{S}^{\mathfrak{A}}$. Thus $\pi \circ \dot{\sigma}^{\mathfrak{A}} \text{``} \bar{S} \subseteq \pi \text{``} \underline{S}^{\mathfrak{A}} \subseteq S$ as desired.

This completes the proof of the main claim. \square

We have satisfied the main claim, so we are done, we have shown that \mathbb{D} is subcomplete. \square

Some slight modifications to the above proof give the following two corollaries.

The first point is that the above proof also shows that generalized diagonal Prikry forcing that adds a countable sequence to each measurable cardinal is subcomplete. Before stating the corollary let's define the forcing. Again let D be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D \rangle$ be a list of measures associated to D . Let $\mathbb{D}^*(\mathcal{U}) = \mathbb{D}^*$ be defined the same as $\mathbb{D}(\mathcal{U})$ except the stem of a condition, s , in $\mathbb{D}^*(\mathcal{U})$ is a function with domain in $[D]^{<\omega}$ taking each measurable cardinal $\kappa \in \text{dom}(s)$ to finitely many ordinals $s(\kappa) \subseteq \kappa$. The upper part and extension relation is defined in the same way; the only slight modification is that again we have $(s, A) \leq (t, B)$ so long as points in s not in t come from B , which in the case means that for $\kappa \in \text{dom}(s)$ we have that each element of $s(\kappa)$ not in $t(\kappa)$ is in $B(\kappa)$. We may again form a \mathbb{D}^* -generic sequence $S = S_G$ for a generic $G \subseteq \mathbb{D}^*$, and we may write $S = \langle S(\kappa) \mid \kappa \in D \rangle$ where $S(\kappa)$ is a countable sequence of ordinals less than κ . The genericity criterion for generic diagonal Prikry sequences is as that for \mathbb{D} , which is given in [1, Theorem 1], as stated in **Fact 3.2**, with the modification that S is \mathbb{D}^* generic if and only if for all \mathcal{X} , the set $\{\alpha \mid \exists \kappa \in D \ \alpha \in S(\kappa) \setminus X_\kappa\}$ is finite.

Corollary 3.4. *Let D be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D \rangle$ be a list of measures associated to D . Then $\mathbb{D}^*(\mathcal{U})$ is subcomplete.*

Proof Sketch. The modifications are mostly notational, and the main one that needs to be made is to adjust the proof of the *Claim* within the proof of *Claim 2*. Here we have \mathbb{D}_1^* , the generalized diagonal Prikry forcing as computed in N_1 , as well as $\bar{\mathbb{D}}$ of \bar{N} , and S_1 , which we would like to show is a \mathbb{D}_1^* -generic sequence over N_1 in this case. S_1 is defined as $\sigma_1 \text{``} \bar{S}$, using a diagonal Prikry sequence S'_1 to fill in the missing coordinates, where S'_1 is obtained by forcing with \mathbb{D}_1 over V .

We will show that S_1 satisfies the generalized diagonal Prikry genericity criterion over N_1 and follow the above proof. To do this, let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D_1 \rangle$, with $\mathcal{X} \in N_1$, be a sequence of measure-one sets in the sequence of measures \mathcal{U}_1 .

Note first that S'_1 is a generic sequence, it already satisfies the generalized diagonal Prikry genericity criterion, namely: $\{\alpha \mid \exists \kappa \in D_1 \ \alpha \in S'_1(\kappa) \setminus X_\kappa\}$ is finite. Recall that $\bar{S} = \langle \bar{S}(\bar{\kappa}) \mid \bar{\kappa} \in \bar{D} \rangle$ is a $\bar{\mathbb{D}}$ -generic sequence over \bar{N} as well. We need to see that in addition, $\{\alpha \mid \exists \bar{\kappa} \in \bar{D} \ \alpha \in \sigma_1(\bar{S}(\bar{\kappa})) \setminus X_{\sigma_1(\bar{\kappa})}\}$ is finite.

By the $\bar{\kappa}(0)$ -cofinality of σ_1 , there is some $w \in \bar{N}$ such that $\mathcal{X} \in \sigma_1(w)$, where $|w| < \bar{\kappa}(0)$ in \bar{N} . Thus in N_1 , $|\sigma_1(w)| < \kappa_1(0)$. For each $\kappa \in \sigma_1 \text{``} \bar{D}$, we have that $X_\kappa \in \sigma_1(w)_\kappa = \{\sigma_1(f)(\bar{\kappa}) \mid f \in \prod_{\bar{\kappa} \in \bar{D}} \bar{U}(\bar{\kappa}) \wedge f \in w\}$ and also $|\sigma_1(w)_\kappa| < \kappa_1(0)$. All $\kappa \in \sigma_1 \text{``} \bar{D}$ of course satisfy $\kappa \geq \kappa_1(0)$ so by the κ -completeness of $U_1(\kappa)$, we have that $W_\kappa := \cap \sigma_1(w)_\kappa \in U_1(\kappa)$ since

$X_\kappa \in \sigma_1(w)_\kappa$. So we have established that \mathcal{W} , the sequence of W_κ for $\kappa \geq \kappa_1(0)$, is also a sequence of measure-one sets in N_1 . Note in addition that for $\kappa \in \sigma_1(\overline{D})$, we have that $W_\kappa \subseteq X_\kappa$.

By elementarity, for each $\overline{\kappa} \in \overline{D}$, we have $\overline{W}_{\overline{\kappa}} = \cap \{f(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa}) \wedge f \in w\}$ is a measure-one set in $\overline{U}(\overline{\kappa})$ and we also have that $\sigma_1(\overline{W}_{\overline{\kappa}}) = W_{\sigma_1(\overline{\kappa})}$. Moreover,

$$\{\alpha \mid \exists \overline{\kappa} \in \mathbb{D} \alpha \in \overline{S}(\overline{\kappa}) \setminus \overline{W}_{\overline{\kappa}}\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for \mathbb{D} , which must be satisfied by \overline{S} . Thus by elementarity,

$$\{\alpha \mid \exists \overline{\kappa} \in \mathbb{D} \sigma_1(\overline{S}(\overline{\kappa})) \setminus W_{\sigma_1(\overline{\kappa})}\} \supseteq \{\alpha \mid \exists \overline{\kappa} \in \mathbb{D} \sigma_1(\overline{S}(\overline{\kappa})) \setminus X_{\sigma_1(\overline{\kappa})}\} \text{ is finite,}$$

as is desired, completing the proof of the claim. \square

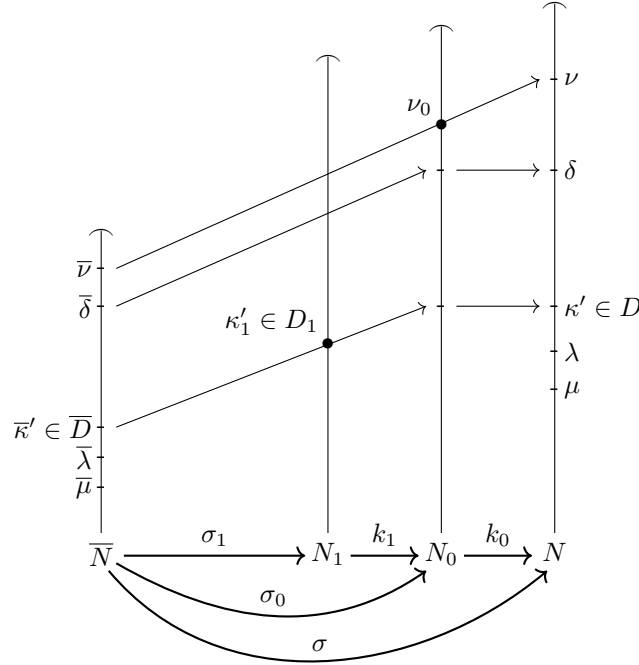
One might consider a forcing like \mathbb{D} and \mathbb{D}^* that adds one point below each measurable cardinal sometimes, and other times adds a cofinal ω -sequence below the measurable cardinal. This forcing is clearly subcomplete as well.

Below we refer to the concept of subcompleteness above μ , which was introduced in ??.

Corollary 3.5. *Let D be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D \rangle$ be a list of measures associated to D .*

Furthermore, let $\mu < \lambda$ be a regular cardinal, where $\lambda = \sup_{n < \omega} \kappa_n$, the first limit point of D . Then $\mathbb{D} = \mathbb{D}(\mathcal{U})$ is subcomplete above μ .

Proof Sketch. The idea is to follow the same exact proof as in the above theorem, except we achieve the following diagram:



Here we replace $\kappa(0)$ with some $\kappa' \in D$ such that $\lambda < \kappa'$, where there are finitely many measurables of D below κ' . So in particular, we let $\langle N_1, \sigma_1 \rangle$ be the liftup of $\langle \bar{N}, \sigma \restriction H_{\kappa'}^{\bar{N}} \rangle$ in *Claim 2*. In order to show the *Claim* that we have a generic sequence over \mathbb{D}_1 , we follow the same argument as follows:

Let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in \sigma_1 \text{``}\bar{D}\text{''} \rangle$, with $\mathcal{X} \in N_1$, be a sequence of measure one sets in the sequence of measures \mathcal{U}_1 with only coordinates coming from $\sigma_1 \text{``}\bar{D}\text{'}$. We need to see that

$$\{\bar{\kappa} \in \bar{D} \mid \sigma_1(\bar{S}(\bar{\kappa})) \notin X_{\sigma_1(\bar{\kappa})}\} \text{ is finite.}$$

By the $\bar{\kappa}'$ -cofinality of σ_1 , there is some $w \in \bar{N}$ such that $\mathcal{X} \in \sigma_1(w)$, where $|w| < \bar{\kappa}'$ in \bar{N} . Thus in N_1 , $|\sigma_1(w)| < \kappa'_1$. For each $\kappa \in \sigma_1 \text{``}\bar{D}\text{'}$, we have that $X_\kappa \in \sigma_1(w)_\kappa = \{\sigma_1(f)(\bar{\kappa}) \mid f \in \prod_{\bar{\kappa} \in \bar{D}} \bar{U}(\bar{\kappa}) \wedge f \in w\}$ and also

$$|\sigma_1(w)| < \kappa'_1.$$

So for all but finitely many $\kappa \in \sigma_1 \text{``}\bar{D}\text{'}$, namely for $\kappa \geq \kappa'_1$, by the κ -completeness of $U_1(\kappa)$, we have that

$$\cap \sigma_1(w) = W_\kappa \in U_1(\kappa).$$

So we have established that \mathcal{W} , the sequence of W_κ for $\kappa > \kappa'_1$, is also a sequence of measure-one sets in N_1 . Note in addition that for $\kappa \in \sigma_1 \text{``}\bar{D}\text{'}$, $\kappa > \kappa'_1$, we have that $W_\kappa \subseteq X_\kappa$.

By elementarity, for each $\bar{\kappa} \in \bar{D}$, where $\bar{\kappa} > \bar{\kappa}'$, we have

$$\bar{W}_{\bar{\kappa}} = \cap \{f(\bar{\kappa}) \mid f \in \prod_{\bar{\kappa} \in \bar{D}} \bar{U}(\bar{\kappa}) \wedge f \in w\}$$

is a measure-one set in $\bar{U}(\bar{\kappa})$ and we also have that $\sigma_1(\bar{W}_{\bar{\kappa}}) = W_{\sigma_1(\bar{\kappa})}$. Moreover,

$$\{\bar{\kappa} \in \bar{D} \mid \bar{S}(\bar{\kappa}) \notin \bar{W}_{\bar{\kappa}}\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for $\bar{\mathbb{D}}$, which must be satisfied by \bar{S} , and since there are only finitely many measurables in \bar{D} less than $\bar{\kappa}'$ in \bar{N} . Thus by elementarity,

$$\{\bar{\kappa} \in \bar{D} \mid \sigma_1(\bar{S}(\bar{\kappa})) \notin W_{\sigma_1(\bar{\kappa})}\} \supseteq \{\bar{\kappa} \in \bar{D} \mid \sigma_1(\bar{S}(\bar{\kappa})) \notin X_{\sigma_1(\bar{\kappa})}\} \text{ is finite.}$$

Additionally the \in -theories \mathcal{L} and \mathcal{T} would have to be defined so as to include as an axiom that $\dot{\sigma} \restriction \bar{\mu} = \underline{\sigma} \restriction \bar{\mu}$ and $\dot{\sigma} \restriction \bar{\mu} = \underline{\sigma} \restriction \bar{\mu}$ respectively. We would then need to show that $\dot{\sigma}^* \restriction \bar{\mu} = \sigma \restriction \bar{\mu}$, but this would follow since k_0 is the identity on ν_0 . Furthermore, it would need to be shown that $\sigma' \restriction \bar{\mu} = \sigma \restriction \bar{\mu}$, but this would follow from the requirement that $\dot{\sigma}^{\bar{\mathfrak{A}}} \restriction \bar{\mu} = \underline{\sigma}^{\bar{\mathfrak{A}}} \restriction \bar{\mu}$, and since ordinals are computed properly by $\bar{\mathfrak{A}}$. \square

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