THE SUBCOMPLETENESS OF DIAGONAL PRIKRY FORCING

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ABSTRACT. It is shown that generalized Prikry forcing to add a countable sequence to each measurable in an infinite discrete set of measurable cardinals is subcomplete. To do this it is shown that a simplified version of generalized Prikry forcing which adds a point below each measurable cardinal in an infinite discrete set, called generalized diagonal Prikry forcing, is subcomplete. It is also shown that generalized diagonal Prikry forcing is subcomplete above μ where μ is a regular cardinal below the first limit point of the infinite discrete set of measurable cardinals used in the forcing.

1. INTRODUCTION

Subcomplete forcing is a class of forcing notions that do not add reals, but may potentially alter cofinalities to ω. Examples of subcomplete forcing include Prikry forcing and Namba forcing (under CH). This separates subcomplete forcings from proper forcings, that have countable covering and thus cannot change cofinalities to ω.

Jensen [4, Section 3.3] shows that Prikry forcing and Namba forcing (under CH) are subcomplete. Fuchs has shown that Magidor forcing [2] is subcomplete. Here an adaptation of Jensen's proof showing that Prikry forcing is subcomplete is employed to see that many "Prikry-like forcings," which are referred to as generalized diagonal Prikry forcing after Fuchs ([1]), are subcomplete.

2. PRELIMINARIES

Before defining subcomplete forcing and generalized diagonal Prikry forcing, some preliminary information is necessary.

First, a brief outline of some of the notation used in what follows: Forcing notions $\mathbb{P} = \langle \mathbb{P}, \leq \rangle$ are taken to be partial orders that are separative and contain a "top" element weaker than all elements of $\mathbb{P}$, denoted $\mathbb{1}$. Write N, M, or $\overline{N}$ to denote models of ZFC^-, the axioms of Zermelo-Fraenkel Set Theory without the axiom of $\mathsf{Powerset}$, and with the axiom of $\mathsf{Collection}$ instead of $\mathsf{Replacement}$. Usually we will name these models. For σ an elementary embedding, write $\sigma : N \prec M$ and write $N \preccurlyeq M$ to denote and emphasize that N is an elementary substructure of M. Let N be a transitive ZFC^- model. Write $\text{height}(N)$ to mean $\text{Ord} \cap N$. Let α be an ordinal. Then write α^N for $\alpha \cap N$. Let θ be a cardinal. The collection of sets hereditarily of size less than θ will be referred to as H_θ. Relativizing the concept to a particular model of set theory, M, write H_θ^M to mean the collection of sets in M that are hereditarily of size less than θ in M. In this case, if θ is determined by some computation, the computation is meant to take place in M. Let τ be a cardinal. With abuse of notation, write $L_\tau[A]$ to refer to the structure $\langle L_\tau[A]; \in, A \cap L_\tau[A] \rangle$. If a map σ satisfies $\sigma(\overline{a}) = a$ and $\sigma(\overline{b}) = b$, this will be abbreviated as $\sigma(\overline{a}, \overline{b}) = a, b$.

2.1. **The Weight of a Forcing Notion.**

Definition 2.1. For a forcing notion $\mathbb{P}$, we write $\delta(\mathbb{P})$ to denote the least cardinality of a dense subset in $\mathbb{P}$. This is sometimes referred to as the *weight* of a poset.

Although Jensen defines $\delta(\mathbb{P})$ for Boolean algebras, it's also relevant for posets. As Jensen states, for forcing notions $\mathbb{P}$, the weight can be replaced with the cardinality of $\mathbb{P}$, or even $\mathbb{P}$, for the purpose of defining subcompleteness. However, $\delta(\mathbb{P})$ and $|\mathbb{P}|$ are not necessarily the same, since there could be a large set of points in the poset that all have a common strengthening.

If $\mathbb{P}$ is a dense subset of $\mathbb{Q}$, then, of course, $\delta(\mathbb{P}) = \delta(\mathbb{Q})$.

2.2. **Fullness.** In the definition of subcompleteness, in lieu of working directly with H_θ and its well-order for "large enough" cardinals θ (as is often done for proper forcing, for example), models N look as follows:

$$H_\theta \subseteq N = L_\tau[A] \models \mathsf{ZFC}^-,$$

where $\tau > \theta$ is a cardinal that is not necessarily regular. Such H_θ will need to be large enough so that N has the correct ω_1 and H_{ω_1}. One justification for working with these models is that such N will naturally contain a well order of H_θ, along with its Skolem functions and other useful bits of useful information. Additionally a benefit of working with such models is that $L_\tau[A]$ is easily definable in $L_\tau[A][G]$, if G is generic, using A. In the standard fashion, take countable elementary substructures X, where $X \preccurlyeq N$. Then take the countable transitive collapse of such an X, and write $\overline{N} \cong X$. Such embeddings will be referred to as

$$\sigma : \overline{N} \cong X \preccurlyeq N.$$

Often, to suppress the mention of range of σ, or X in the above, all that will be written is $\sigma : \overline{N} \prec N$.

In fact, it will not be quite enough for such an $\overline{N}$ to be transitive, what is needed is exactly granted by the property of *fullness*. Fullness of a model ensures that the model is not pointwise definable. Thus in the standard setup, there can be many elementary maps between the smaller structure and the larger structure N. Before defining fullness, there are more preliminary points to clarify.

It is not hard to see what the critical point of these embeddings is. Given $\sigma : \overline{N} \prec N$ where $\overline{N}$ is countable and transitive, $\mathrm{cp}(\sigma)$ is exactly $\omega_1^{\overline{N}} = \sigma^{-1}(\omega_1^N)$, since $\overline{N}$ is countable.

Remark 2.2. Let $\overline{N}$, N be transitive ZFC^- models, where $\overline{N}$ is countable and $H_{\omega_1} \subseteq N$, with $\sigma : \overline{N} \prec N$. Then $\mathrm{cp}(\sigma) = \omega_1^{\overline{N}}$ and $\sigma \upharpoonright (H_{\omega_1})^{\overline{N}} = \mathrm{id}$.

Let $N = L_\tau[A]$ for some cardinal τ and set A, be a transitive ZFC^- model, let X be a set, and let δ be a cardinal. Our notation for the *Skolem hull*, in N, closing under $\delta \cup X$, is $\mathcal{H}ull^N(\delta \cup X)$. It is defined to be the smallest $Y \preccurlyeq N$ satisfying $X \cup \delta \subseteq Y$.

The two basic results below follow immediately.

Remark 2.3. Let $N = L_\tau[A]$ be a transitive ZFC^- model, δ and γ be cardinals, and X a set. If $\delta \leq \gamma$ then $\mathcal{H}ull^N(\delta \cup X) \subseteq \mathcal{H}ull^N(\gamma \cup X)$.

Remark 2.4. Let N be a transitive ZFC^- model, δ a cardinal, and X, Y sets. If $\mathcal{H}ull^N(\delta \cup X) = \mathcal{H}ull^N(\delta \cup Y)$ then $\mathcal{H}ull^N(\gamma \cup X) = \mathcal{H}ull^N(\gamma \cup Y)$ for all $\gamma \geq \delta$.

Toward defining fullness, the notion of regularity of transitive models needs to be defined.

Definition 2.5. We say that a transitive model N is *regular* in a transitive model M so long as for all functions $f : x \longrightarrow N$, where x is an element of N and $f \in M$, we have that $f``x \in N$.

The following lemma is meant to elucidate the significance of regularity.

Lemma 2.6. *Let $N, M \models \mathsf{ZFC}^-$ be transitive. N is regular in M iff $N = H_\gamma^M$, where $\gamma = \mathrm{height}(N)$ is a regular cardinal in M.*

Proof. For the backward direction, suppose that $N = H_\gamma^M$ where $\gamma = \mathrm{height}(N)$ is a regular cardinal in M. Then for all $f : x \longrightarrow N$, with $x \in N$ and $f \in M$, certainly $f``x \in N$ as well.

For the forward direction, indeed γ has to be regular in M since otherwise M would contain a cofinal function $f : \alpha \longrightarrow \gamma$ where $\alpha < \gamma$. By the transitivity of N, this implies that $\alpha \in N$. Thus $\cup f``\alpha$ is in $N \models \mathsf{ZFC}^-$ by regularity, so $\gamma \in N$, a contradiction. We have that $N \subseteq H_\gamma^M$ since N is a transitive ZFC^- model, so the transitive closure of elements of N may be computed in N and thus have size less than γ, so they are in H_γ^M as $\gamma \in M$. To show that $H_\gamma^M \subseteq N$, let $x \in H_\gamma^M$. We assume by $\in$-induction that $x \subseteq N$. Then there is a surjection $f : \alpha \twoheadrightarrow x$ where $\alpha < \gamma$, in M. Hence by regularity, $x = f``\alpha \in N$ as desired. $\square$

Definition 2.7. A structure M is *full* so long as M is transitive, $\omega \in M$, and there is a γ such that M is regular in $L_\gamma(M)$ where $L_\gamma(M) \models \mathsf{ZFC}^-$.

Perhaps the property of fullness seems rather mysterious at first, but the fullness of a countable structure guarantees that it is not pointwise definable, a necessary property of some of the models that come up in the definition subcomplete forcing.

Lemma 2.8. *If M is countable and full, then M is not pointwise definable.*

Proof. Suppose toward a contradiction that M is countable, full, and pointwise definable. By fullness there is some $L_\gamma(M) \models \mathsf{ZFC}^-$ such that M is regular in $L_\gamma(M)$. By pointwise definability, for each element $x \in M$, we have attached to it some formula $\varphi(x)$ such that $M \models \varphi(x)$ uniquely, meaning that $\varphi(y)$ fails for every other element $y \in M$. Thus in $L_\gamma(M)$ we may define a function $f : \omega \cong M$, that takes the nth formula in the language of set theory to its unique witness in M. In particular we have that $L_\gamma(M)$ witnesses that M is countable. However, this would allow M to witness its own countability, since M must contain f as well by regularity. $\square$

Let N be a transitive ZFC^- model. Suppose $\overline{N} \cong X \preccurlyeq N$ where X is countable and $\overline{N}$ is full. Then there may possibly be more than one elementary embedding $\sigma : \overline{N} \prec N$. Indeed: if there was only one unique embedding, then $\overline{N}$ may be defined pointwise, by elementarity of the unique map, and since $\overline{N}$ is countable.

The following lemma will prove useful when showing that various forcing notions are subcomplete, since subcompleteness requires certain ground-model Skolem hulls to match those in forcing extensions.

Lemma 2.9 (Jensen)**.** *Let $\mathbb{P}$ be a forcing notion, and let $\delta = \delta(\mathbb{P})$ be the smallest size of a dense subset in $\mathbb{P}$. Suppose that $\mathbb{P} \in H_\theta \subseteq N$ where $N = L_\tau[A] \models \mathsf{ZFC}^-$. Let $\sigma : \overline{N} \prec N$ where $\overline{N}$ is countable and full, and let $\sigma(\overline{\mathbb{P}}) = \mathbb{P}$.*

Suppose $G \subseteq \mathbb{P}$ is N-generic and $\overline{G} \subseteq \overline{\mathbb{P}}$ is $\overline{N}$-generic, and that $\sigma``\overline{G} \subseteq G$, so σ lifts (or extends) in $V[G]$ to an embedding $\sigma^* : \overline{N}[\overline{G}] \prec N[G]$.

Then

$$N \cap \mathcal{H}ull^{N[G]}(\delta \cup \mathrm{range}(\sigma^*)) = \mathcal{H}ull^N(\delta \cup \mathrm{range}(\sigma)).$$

Proof. First, we establish that $\mathcal{H}ull^N(\delta \cup \mathrm{range}(\sigma)) \subseteq \mathcal{H}ull^{N[G]}(\delta \cup \mathrm{range}(\sigma^*)) \cap N$. To see this, let $x \in \mathcal{H}ull^N(\delta \cup \mathrm{range}(\sigma))$. Then x is N-definable from $\xi < \delta$ and $\sigma(\overline{z})$ where $\overline{z} \in \overline{N}$. Since σ^* extends σ, this means that $x \in N$ and that x is $N[G]$-definable from ξ and $\sigma^*(\overline{z})$. This is because $N = L_\tau[A]$ is definable in $N[G]$ using A.

For the other direction, let $x \in \mathcal{H}ull^{N[G]}(\delta \cup \mathrm{range}(\sigma^*)) \cap N$. Then x is $N[G]$-definable from $\xi < \delta$ and $\sigma^*(\overline{z})$ where $\overline{z} \in \overline{N}[\overline{G}]$; and also $x \in N$. Letting $\dot{\overline{z}} \in \overline{N}^{\overline{\mathbb{P}}}$ such that $\overline{z} = \dot{\overline{z}}^{\overline{G}}$ we have

$$\sigma^*(\overline{z}) = \sigma^*(\dot{\overline{z}}^{\overline{G}}) = \sigma(\dot{\overline{z}})^G$$

so we have that there is some formula φ such that x is the unique witness:

$$x \text{ is that } y \text{ where } N[G] \models \varphi(y, \xi, \sigma(\dot{\overline{z}})^G).$$

Take $f \in \overline{N}$ mapping $\overline{\delta}$ onto a dense subset of $\overline{\mathbb{P}}$. Then $\sigma(f)$ maps δ onto a dense subset of $\mathbb{P}$. Thus there is $\nu < \delta$ such that $\sigma(f)(\nu) \in G$ and $\sigma(f)(\nu) \Vdash \varphi(\check{x}, \check{\xi}, \sigma(\dot{\overline{z}}))$. Thus

$$x \text{ is that } y \text{ where } \sigma(f)(\nu) \Vdash^N_{\mathbb{P}} \varphi(\check{y}, \check{\xi}, \sigma(\dot{\overline{z}}))$$

so $x \in \mathcal{H}ull^N(\delta \cup \mathrm{range}(\sigma))$. □

2.3. **Subcomplete Forcing.** Subcompleteness should be seen as a weakening of the class of complete (a.k.a. countably closed) forcing notions. The following definitions and proofs are due to Jensen [4, Chapter 3].

Definition 2.10. A forcing notion $\mathbb{P}$ is *complete* so long as for sufficiently large θ we have that once we are in the following situation:

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \mathsf{ZFC}^-$ where $\tau > \theta$ and $A \subseteq \tau$
- $\sigma : \overline{N} \cong X \preccurlyeq N$ where X is countable and $\overline{N}$ is transitive
- $\sigma(\overline{\theta}, \overline{\mathbb{P}}, \overline{s}) = \theta, \mathbb{P}, s$ for some $s \in N$;

then we have that if $\overline{G}$ is $\overline{\mathbb{P}}$-generic over $\overline{N}$ then there is a *completeness condition* $p \in \mathbb{P}$ forcing that whenever $G \ni p$ is $\mathbb{P}$-generic, $\sigma``\overline{G} \subseteq G$.

In particular, below the condition p we have that σ lifts to an embedding $\sigma^* : \overline{N}[\overline{G}] \prec N[G]$. We say that such a θ as above *verifies* the completeness of $\mathbb{P}$.

The adjustment made to get subcomplete forcings is to not necessarily insist the the original embedding lifts in the forcing extension. Instead subcompleteness asks for there to be an embedding in the extension, an embedding that is sufficiently similar to the original embedding, that lifts. However, as discussed in Section 2.2, the domain of the embedding must be *full* to ensure that there can even consistently be more than one embedding. The definition is given below.

Definition 2.11. A forcing notion $\mathbb{P}$ is *subcomplete* so long as, for sufficiently large θ we have that whenever we are in the *standard setup*, i.e.,

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \mathsf{ZFC}^-$ where $\tau > \theta$ and $A \subseteq \tau$
- $\sigma : \overline{N} \cong X \preccurlyeq N$ where X is countable and $\overline{N}$ is full
- $\sigma(\overline{\theta}, \overline{\mathbb{P}}, \overline{s}) = \theta, \mathbb{P}, s$ for some $s \in N$;

then we have that if $\overline{G}$ is $\overline{\mathbb{P}}$-generic over $\overline{N}$ then there is a *subcompleteness condition* $p \in \mathbb{P}$ such that whenever $G \ni p$ is $\mathbb{P}$-generic, there is $\sigma' \in V[G]$ satisfying:

(1) $\sigma' : \overline{N} \prec N$
(2) $\sigma'(\overline{\theta}, \overline{\mathbb{P}}, \overline{s}) = \theta, \mathbb{P}, s$
(3) $\mathcal{H}ull^N(\delta(\mathbb{P}) \cup \mathrm{range}(\sigma')) = \mathcal{H}ull^N(\delta(\mathbb{P}) \cup X)$
(4) $\sigma'``\overline{G} \subseteq G$.

In other words and in particular, the subcompleteness condition p forces that there is an embedding σ' in the extension $V[G]$ that lifts, by (4), to an embedding $\sigma'^* : \overline{N}[\overline{G}] \prec N[G]$ in $V[G]$.

We say that such a θ as above *verifies* the subcompleteness of $\mathbb{P}$.

Often we write δ instead of $\delta(\mathbb{P})$ for the weight of $\mathbb{P}$ when there should be no confusion as to which poset $\mathbb{P}$ we are working with.[1]

Remark 2.12. If $\mathbb{P}$ is subcomplete as verified by θ, then $\mathbb{P}$ is subcomplete as verifed by $\theta' > \theta$ since $\mathbb{P} \in H_\theta$ obviously implies that $\mathbb{P} \in H_{\theta'}$. Thus a standard setup such as $\mathbb{P} \in H_{\theta'} \subseteq N = L_\tau[A]$ for $\tau > \theta'$ may be reduced down to the case where $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A]$ and $\tau > \theta' > \theta$ using the same τ. In particular, $\mathbb{P}$ is subcomplete as long as it is verified to be subcomplete by some θ. So "sufficiently large θ" may be replaced with "some θ" in the definition of subcompleteness.[2]

2.4. **Subcomplete above μ.** The notion of subcompleteness above μ is an attempt to measure where exactly subcompleteness kicks in; in some sense it gives a level as to where forcing fails to be complete. The following definition is from Jensen [3, Chapter 2 p. 47].

Definition 2.13. Let μ be a cardinal. We say that a forcing notion $\mathbb{P}$ is *subcomplete above* μ so long as for sufficiently large $\theta > \mu$, whenever we are in the standard setup (as in the definition of subcomplete forcing), then, for any $\overline{G} \subseteq \overline{\mathbb{P}}$, there is a subcompleteness condition $p \in \mathbb{P}$ such that whenever $G \ni p$ is $\mathbb{P}$-generic, then there is $\sigma' \in V[G]$ satisfying in addition that $\sigma' \upharpoonright H^{\overline{N}}_{\overline{\mu}} = \sigma \upharpoonright H^{\overline{N}}_{\overline{\mu}}$.

As usual, this means that in particular below the condition p there is an embedding σ' that lifts by (4) to an embedding $\sigma'^* : \overline{N}[\overline{G}] \prec N[G]$ in $V[G]$. As usual we say that such θ as above *verifies the subcompleteness above μ of* $\mathbb{P}$, and we may say that $\mathbb{P}$ is subcomplete if there is a θ that verifies its subcompleteness above μ.

Remark 2.14. If $\mathbb{P}$ is subcomplete, then $\mathbb{P}$ is subcomplete above $\omega_1^{\overline{N}}$.

Proof. By Remark 2.2, we have that for any elementary embedding such as σ or σ' from countable $\overline{N}$ to N, $\sigma' \upharpoonright H^{\overline{N}}_{\omega_1} = \sigma \upharpoonright H^{\overline{N}}_{\omega_1} = \mathrm{id}$. $\square$

The meaning of subcompleteness above μ is somewhat given by the following theorem.

Theorem 2.15. *If $\mathbb{P}$ is subcomplete above μ then $\mathbb{P}$ does not add new countable subsets of μ.*

Proof. Suppose not. Let $\mathbb{P}$ be subcomplete above μ. Let $\dot{B} \subseteq \mu$ be countable with

$$p \Vdash \dot{f} : \check{\omega} \cong \dot{B} \ \wedge \ \dot{f} \notin V.$$

Take $\theta > \mu$ large enough to verify the subcompleteness of $\mathbb{P}$.

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \mathsf{ZFC}^-$ where $\tau > \theta$ and $A \subseteq \tau$
- $\sigma : \overline{N} \cong X \preccurlyeq N$ where $\overline{N}$ is countable and full
- $\sigma(\overline{\theta}, \overline{\mu}, \overline{\mathbb{P}}, \overline{p}, \overline{\dot{B}}, \overline{\dot{f}}) = \theta, \mu, \mathbb{P}, p, \dot{B}, \dot{f}$.

Let $\overline{G}$ be $\overline{\mathbb{P}}$-generic over $\overline{N}$ with $\overline{p} \in \overline{G}$. By the subcompleteness of $\mathbb{P}$ above μ there is $p \in \mathbb{P}$ such that whenever $p \in G$ where G is $\mathbb{P}$-generic, there is $\sigma' \in V[G]$ satisfying:

(1) $\sigma' : \overline{N} \prec N$
(2) $\sigma'(\overline{\theta}, \overline{\mu}, \overline{\mathbb{P}}, \overline{p}, \overline{\dot{B}}, \overline{\dot{f}}) = \theta, \mu, \mathbb{P}, p, \dot{B}, \dot{f}$
(3) $\mathcal{H}ull^N(\delta \cup \mathrm{range}(\sigma')) = \mathcal{H}ull^N(\delta \cup X)$
(4) $\sigma' ``\overline{G} \subseteq G$
(5) $\sigma' \upharpoonright \overline{\mu} = \sigma \upharpoonright \overline{\mu}$.

[1] See Section 2.1 for more on $\delta(\mathbb{P})$.
[2] See [4, Section 3.1 Lemma 2.4].

Let $\overline{f} = \overline{\dot{f}}^G$ and $f = \dot{f}^G$. By (4) and (5), for each n,

$$f(n) = \sigma'(\overline{f}(n)) = \sigma(\overline{f}(n)),$$

meaning that $f \in V$ since $\sigma \in V$, which is a contradiction. $\square$

Thus if $\mathbb{P}$ is subcomplete above μ, then μ's cardinality, and even its cofinality, cannot be altered to be ω via $\mathbb{P}$.

2.5. **Barwise Theory.** In order to show that many posets are subcomplete, Jensen exploits Barwise Theory and techniques using countable admissible structures to obtain transitive models of infinitary languages. Barwise creates an M-finite predicate logic, a first order theory in which arbitrary, but M-finite, disjunctions and conjunctions are allowed. The following is an outline of [4, Chapter 1 & 2].

Definition 2.16. Let M be a transitive structure with potentially infinitely many predicates. A theory defined over M is *M-finite* so long as it is in M. A theory is $\Sigma_1(M)$, also known as *M-recursively enumerable* or *M-re*, if the theory is Σ_1-definable, with parameters from M.

Of course this may be generalized to the entire usual Levy hierarchy of formulae, but only $\Sigma_1(M)$ is needed in this paper. If $\mathcal{L}$ is a Σ_1(M)-definable language or theory, the rough idea is that to check whether a sentence is in $\mathcal{L}$, one should imagine enumerating the formulae of $\mathcal{L}$ to find a sentence and a witness to it in the structure M. The structures considered here are admissible, defined below.

Definition 2.17. A transitive structure M is *admissible* if it models the axioms of $\mathsf{Kripke\text{-}Platek\ Set\ Theory}$ (KP) which consists of the axioms of $\mathsf{Empty\ Set}$, $\mathsf{Pairing}$, Union, Σ_0-$\mathsf{Collection}$, and Σ_0-$\mathsf{Separation}$.

Jensen also makes use of models of ZF^- that are not necessarily well-founded.

Definition 2.18. Let $\mathfrak{A} = \langle A, \in_{\mathfrak{A}}, B_1, B_2, \dots \rangle$ be a (possibly) ill-founded model of ZF^-, where $\mathfrak{A}$ is allowed to have predicates other than $\in$. The *well-founded core* of $\mathfrak{A}$, denoted $\mathrm{wfc}(\mathfrak{A})$, is the restriction of $\mathfrak{A}$ to the set of all $x \in A$ such that $\in_{\mathfrak{A}} \cap\, \mathcal{C}(x)^2$ is well founded, where $\mathcal{C}(x)$ is the closure of $\{x\}$ under $\in_{\mathfrak{A}}$. A model $\mathfrak{A}$ of ZF^- is *solid* so long as $\mathrm{wfc}(\mathfrak{A})$ is transitive and $\in_{\mathrm{wfc}(\mathfrak{A})} = \in \cap\, \mathrm{wfc}(\mathfrak{A})^2$.

Jensen [4, Section 1.2] notes that every consistent set of sentences in ZF^- has a solid model, and if $\mathfrak{A}$ is solid, then $\omega \subseteq \mathrm{wfc}(\mathfrak{A})$. In addition,

Fact 2.19 (Jensen)**.** *If $\mathfrak{A} \models \mathsf{ZF}^-$ is solid, then $\mathrm{wfc}(\mathfrak{A})$ is admissible.*

Definition 2.20. Let M be admissible. An infinitary axiomatized theory in M-finitary logic $\mathcal{L} = \mathcal{L}(M)$ is called an *$\in$-theory* over M, with a fixed predicate $\dot{\in}$ and *special constants* denoted $\underline{x}$ for elements $x \in M$. The underlying axioms for these $\in$-theories will always involve ZFC^- and some basic axioms ensuring that $\dot{\in}$ behaves nicely; the $\mathsf{Basic\ Axioms}$ are:

- $\mathsf{Extensionality}$
- A statement positing the extensionality of $\dot{\in}$, which is a scheme of formulae defined for each member of M. For each $x \in M$, include an *M-re* sentence (meaning it quantifies over M-finite sentence):

$$\forall v \left(v \mathrel{\dot{\in}} \underline{x} \iff \bigvee\!\!\!\bigvee_{z \in x} v = \underline{z} \right).$$

Here $\bigvee\!\!\!\bigvee$ denotes an infinite disjunction in the language.

In the above definition, it should be clarified that it is possible to consider the same $\in$-theory defined over different admissible structures; if M, M' are both admissible, then it is possible to have that $\mathcal{L}(M)$ and $\mathcal{L}(M')$: the distinction is only as to where the special constants come from.

An important fact ensured by our Basic Axioms is that the interpretations of these special constants in any solid model of the theory are the same as in M:

Fact 2.21 (Jensen)**.** *Let M be as in the above definition. Let $\mathfrak{A}$ be a solid model of the $\in$-theory $\mathcal{L}$. Then for all $x \in M$, we have that $\underline{x}^{\mathfrak{A}} = x \in \mathrm{wfc}(\mathfrak{A})$.*

Jensen uses the techniques of Barwise to come up with a proof system in this context, in which consistency of $\in$-theories can be discussed. In particular, $\in$-theories are correct: if there is a model of such a theory, then the theory is consistent.

Fact 2.22 (*Barwise Correctness*)**.** *Let $\mathcal{L}$ be an $\in$-theory. If A is a set of $\mathcal{L}$-statements and $\mathfrak{A} \models A$, then A is consistent.*

Furthermore, compactness and completeness are shown, relativized to the M-finite predicate logics that are used here; solid models of consistent $\Sigma_1(M)$ $\in$-theories are produced for countable admissible structures M.

Fact 2.23 (*Barwise Completeness*)**.** *Let M be a countable admissible structure. Let $\mathcal{L}$ be a consistent $\Sigma_1(M)$ $\in$-theory such that $\mathcal{L} \vdash \mathsf{ZF}^-$. Then $\mathcal{L}$ has a solid model $\mathfrak{A}$ such that*

$$\mathrm{Ord} \cap \mathrm{wfc}(\mathfrak{A}) = \mathrm{Ord} \cap M.$$

The following definition generalizes the concept of fullness.

Definition 2.24. A transitive ZFC^- model N is *almost full* so long as $\omega \in N$ and there is a solid $\mathfrak{A} \models \mathsf{ZFC}^-$ with $N \in \mathrm{wfc}(\mathfrak{A})$ and N is regular in $\mathfrak{A}$.

Clearly if N is full, then N is almost full.

A useful technique when showing a particular forcing is subcomplete, once many different embeddings can be constructed that approximate the embedding required for subcompleteness, is to be able to transfer the consistency of $\in$-theories over one admissible structure to another.

Definition 2.25. If N is a transitive ZFC^- model, let δ_N be the least δ such that $L_\delta(N)$ is admissible.

Fact 2.26 (*The Transfer Lemma*)**.** *Let N_1 be almost full, and suppose that $k : N_1 \prec N_0$ cofinally, for some N_0. Suppose that we have an $\Sigma_1(\langle N_1; p_1, \ldots, p_n\rangle)$ $\in$-theory $\mathcal{L}(L_{\delta_{N_1}}(N_1))$ for $p_1, \ldots, p_n \in N_1$.*

Moreover, suppose $\mathcal{L}(L_{\delta_{N_0}}(N_0))$ is $\Sigma_1(\langle N_1; k(p_1), \ldots, k(p_n)\rangle)$.

Then, if $\mathcal{L}(L_{\delta_{N_1}}(N_1))$ is consistent, it follows that $\mathcal{L}(L_{\delta_{N_0}}(N_0))$ is consistent as well.

2.6. **Liftups.** The following definitions are meant to describe a method used to obtain useful embeddings, outlining a technique that is ostensibly the ultrapower construction. These embeddings facilitate the use of Barwise theory to obtain the consistency of the existence of desirable embeddings. Refer to [4, Chapter 1] for the definitions and theorems stated here.

Definition 2.27. Let $\overline{N}$ and N be transitive ZFC^- models. We say that an elementary embedding $\sigma : \overline{N} \prec N$ is *cofinal* so long as for each $x \in N$ there is some $u \in \overline{N}$ such that $x \in \sigma(u)$.

Let $\alpha \in \overline{N}$. We say that σ is *α-cofinal* so long as every such u has size less than α as computed in $\overline{N}$.

Definition 2.28. Let $\alpha > \omega$ be a regular cardinal in $\overline{N}$, a transitive ZFC^- model. Let

$$\overline{\sigma} : H_\alpha^{\overline{N}} \prec H \text{ cofinally,}$$

where H is transitive. By a *transitive liftup* of $\langle \overline{N}, \overline{\sigma} \rangle$ we mean a pair $\langle N_*, \sigma_* \rangle$ such that

- N_* is transitive
- $\sigma_* : \overline{N} \prec N_*$ α-cofinally
- $\sigma_* \upharpoonright H_\alpha^{\overline{N}} = \overline{\sigma}$.

Reminiscent of ultrapowers, transitive liftups can be characterized in the following way:

Lemma 2.29 (Jensen)**.** *Let $\overline{N}$, N be transitive ZFC^- models with $\sigma : \overline{N} \prec N$. Then, σ is α-cofinal if and only if elements of N are of the form $\sigma(f)(\beta)$ for some $f : \gamma \to \overline{N}$ where $\gamma < \alpha$ and $\beta < \sigma(\gamma)$.*

Proof. We show each direction of the equivalence separately.

For the first direction, let $x \in N$, and take $u \in \overline{N}$ with $x \in \sigma(u)$ such that $|u| < \alpha$ in $\overline{N}$. Let $|u| = \gamma$, and take $f : \gamma \to u$ a bijection in $\overline{N}$. Then $\sigma(f) : \sigma(\gamma) \to \sigma(u)$ is also a bijection in N by elementarity. Since $x \in \sigma(u)$ we also have that x has a preimage under $\sigma(f)$, say β. So $\sigma(f)(\beta) = x$ as desired.

For the backward direction, let $x = \sigma(f)(\beta)$ be an element of N, for $f : \gamma \to \overline{N}$ where $\gamma < \alpha$ in $\overline{N}$ and $\beta < \sigma(\gamma)$. Define $u = f``\gamma$. Then in $\overline{N}$ we have that $|u| < \alpha$. In addition we have that $x \in \sigma(u)$, since $\sigma(u)$ is in the range of $\sigma(f)$, where x lies. $\square$

Furthermore, Jensen shows that transitive liftups exist so long as an embedding already exists, using an ultrapower-like construction, and have a uniqueness property.

Fact 2.30 (*Interpolation*)**.** *Let $\sigma : \overline{N} \prec N$ with $\overline{N} \models \mathsf{ZFC}^-$ transitive, and let $\alpha \in \overline{N}$ be a regular cardinal. Then:*

(1) The transitive liftup $\langle N_*, \sigma_* \rangle$ of $\langle \overline{N}, \sigma \upharpoonright H_\alpha^{\overline{N}} \rangle$ exists.
(2) There is a unique $k_* : N_* \prec N$ such that $k_* \circ \sigma_* = \sigma$ and $k_* \upharpoonright \bigcup \sigma`` H_\alpha^{\overline{N}} = \mathrm{id}$.

For the next useful lemma, a more general notion of liftups needs to be defined. Of course the rich theory is established by Jensen, and is explored in detail in his notes.

Definition 2.31. Let $\mathfrak{A}$ be a solid model of ZFC^- and let $\tau \in \mathrm{wfc}(\mathfrak{A})$ be an uncountable cardinal in $\mathfrak{A}$. Let

$$\sigma : H_\tau^{\mathfrak{A}} \prec H \text{ cofinally,}$$

where H is transitive. Then by a *liftup* of $\langle \mathfrak{A}, \sigma \rangle$, we mean a pair $\langle \mathfrak{A}_*, \sigma_* \rangle$ such that

- $\sigma_* \supseteq \sigma$
- $\mathfrak{A}_*$ is solid
- $\sigma_* : \mathfrak{A} \longrightarrow_{\Sigma_0} \mathfrak{A}_*$ τ-cofinally
- $H \in \mathrm{wfc}(\mathfrak{A}_*)$

Fact 2.32 (Jensen)**.** *Let $\mathfrak{A}$ be a solid model of ZFC^-. Let $\tau > \omega$, $\tau \in \mathrm{wfc}(\mathfrak{A})$, and let*

$$\sigma : H_\tau^{\mathfrak{A}} \prec H \text{ cofinally,}$$

where H is transitive. Then $\langle \mathfrak{A}, \sigma \rangle$ has a liftup $\langle \mathfrak{A}_*, \sigma_* \rangle$.

The following lemma states that transitive liftups of full models are almost full.

Lemma 2.33 (Jensen)**.** *Let $N = L_\tau[A] \models \mathsf{ZFC}^-$ and $\sigma : \overline{N} \prec N$ where $\overline{N}$ is full. Suppose that $\langle N_*, \sigma_* \rangle$ is a transitive liftup of $\langle \overline{N}, \overline{\sigma} \rangle$. Then N_* is almost full.*

Proof. Let $L_\gamma(\overline{N})$ witness the fullness of $\overline{N}$. We will now apply Interpolation (Fact 2.30) to $\mathfrak{A} = L_\gamma(\overline{N})$, which makes sense since certainly $\mathfrak{A}$ is a transitive model ZFC^-. Additionally, by Lemma 2.6 we have that $\overline{N} = H_\tau^{\mathfrak{A}}$, where $\tau = \text{height}(\overline{N})$. Since $\langle N_*, \sigma_* \rangle$ is a transitive liftup, we have that

$$\sigma_* : H_\tau^{\mathfrak{A}} \prec N_* \text{ cofinally,}$$

where N_* is transitive. Thus since $\mathfrak{A}$ is transitive, $\langle \mathfrak{A}, \sigma_* \rangle$ has a liftup $\langle \mathfrak{A}_*, \sigma_{**} \rangle$, where $\mathfrak{A}_* \models \mathsf{ZFC}^-$ since $\mathfrak{A}$ does, $\mathfrak{A}_*$ is solid, where

$$\sigma_{**} : \mathfrak{A} \prec \mathfrak{A}_* \ \tau\text{-cofinally.}$$

We have that $N_* \subseteq \text{wfc}(\mathfrak{A}_*)$ and $\tau_* = \sigma_{**}(\tau) = \text{height}(N_*)$ is regular since τ is. Furthermore, we will show that $N_* = H_{\tau_*}^{\mathfrak{A}_*}$, completing the proof:

Certainly it is the case that $N_* \subseteq H_{\tau_*}^{\mathfrak{A}^*}$. But if $x \in H_{\tau_*}$ in $\mathfrak{A}_*$, then by regularity we have that $x \in \sigma_{**}(u)$ in $\mathfrak{A}_*$, where $u \in \mathfrak{A}$, and $|u| < \tau$ in $\mathfrak{A}$. Let $v = u \cap H_\tau$ in $\mathfrak{A}$. Then $v \in \overline{N}$, since $\overline{N}$ is regular in $\mathfrak{A}$. But then $x \in \sigma_*(v) \in N_*$. So $x \in N_*$. □

3. Generalized Diagonal Prikry Forcing

Generalized diagonal Prikry forcing is designed to add a point below every measurable cardinal in an infinite discrete set of measurable cardinals. In this section it is shown that such forcings are subcomplete.

Definition 3.1. Let D be an infinite discrete set of measurable cardinals, meaning a set of measurable cardinals that does not contain any of its limit points. For $\kappa \in D$ let $U(\kappa)$ be a normal measure on κ, and let $\mathcal{U}$ denote the sequence of the $U(\kappa)$'s.

Define $\mathbb{D} = \mathbb{D}(\mathcal{U})$, *generalized diagonal Prikry forcing* from the list of measures $\mathcal{U}$, by taking conditions of the form (s, A) satisfying the following:

- The *stem* of the condition, s, is a function with domain in $[D]^{<\omega}$ taking each measurable cardinal $\kappa \in \text{dom}(s)$ to some ordinal $s(\kappa) < \kappa$.
- The *upper part* of the condition, A, is a function with domain $D \setminus \text{dom}(s)$ taking each measurable cardinal $\kappa \in \text{dom}(A)$ to some measure-one set $A(\kappa) \in U(\kappa)$.

The extension relation on conditions in $\mathbb{D}$ is defined so that $(s, A) \leq (t, B)$ so long as

- $s \supseteq t$.
- The points in s not in t come from B, i.e., for all $\kappa \in \text{dom}(s) \setminus \text{dom}(t)$, $s(\kappa) \in B(\kappa)$.
- For all $\kappa \in \text{dom}(A)$, $A(\kappa) \subseteq B(\kappa)$.

If G is a generic filter for $\mathbb{D}$, then its associated $\mathbb{D}$-generic sequence is

$$S = S_G = \bigcup \{ s \mid \exists A \ (s, A) \in G \}.$$

Here the definition of $\mathbb{D}(\mathcal{U})$ differs from that as in Fuchs of generalized Prikry foricing [1]. The main difference is that here only one point is added below each measurable cardinal $\kappa \in D$, which is done for simplicity's sake. It is not hard to see that the following theorem also shows that the forcing adding countably many points below each measurable cardinal in D is subcomplete. Adding countably many points below each measurable cardinal in D would collapse the cofinality of each $\kappa \in D$ to be ω, as one expects of a Prikry-like forcing.

Also in the above definition it hasn't been enforced that the stem a condition consists only of ordinals that are wedged between successive measurables in D; ie. for $\kappa \in D$, it is not explicitly insisted that $s(\kappa) \in [\sup(D \cap \kappa), \kappa)$. However, it is dense in $\mathbb{D}(\mathcal{U})$ for the conditions to be that way, since conditions may be strengthened by restricting their upper parts to a tail. Thus such a restriction may be freely added to the following genericity condition on $\mathbb{D}(\mathcal{U})$.

Fact 3.2 (Fuchs)**.** *Let D be an infinite discrete set of measurable cardinals, with $\mathcal{U}$ a corresponding list of measures $\langle U(\kappa) \mid \kappa \in D\rangle$. Then an increasing sequence of ordinals $S = \langle S(\kappa) \mid \kappa \in D\rangle$, where for each $\kappa \in D$, we have that*

$$\sup(D \cap \kappa) < S(\kappa) < \kappa,$$

is a $\mathbb{D}(\mathcal{U})$-generic sequence if and only if for all $\mathcal{X} = \langle X_\kappa \in U(\kappa) \mid \kappa \in D\rangle$, the set

$$\{\kappa \in D \mid S(\kappa) \notin X_\kappa\}$$

is finite.

The above genericity criterion on generalized diagonal Prikry forcing is similar to the Mathias criterion for Prikry forcing. It was shown in [1, Theorem 1].

Theorem 3.3. *Let D be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D\rangle$ be a list of measures associated to D. The generalized diagonal Prikry forcing $\mathbb{D} = \mathbb{D}(\mathcal{U})$ is subcomplete.*

Proof. Let $\theta > \delta(\mathbb{D}) = \delta$ be large enough, so that $[\delta]^{<\omega_1} \in H_\theta$.

First of all, it must be the case that δ, the density of $\mathbb{D}$, must be larger than all of the measurable cardinals in D.

Claim 1. $\delta \geq \sup D$.

Pf. Suppose instead that there is a dense $E \subseteq \mathbb{D}$ such that $\sup D \geq \kappa^* > |E|$ for some $\kappa^* \in D$. Then for each condition $(s, A) \in E$ either $\kappa^* \in \operatorname{dom} s$ or $\kappa^* \in \operatorname{dom} A$. So taking $E^* = \{(s, A) \in E \mid \kappa \in \operatorname{dom} s\} \subseteq E$, since $|E^*| < \kappa^*$ as well, there is an $\alpha < \kappa^*$ such that $\sup_{(s,A)\in E^*} s(\kappa^*)$. Let $p = (t, B) \in \mathbb{D}$ be defined so that $t(\kappa^*) = \alpha$ and $B(\kappa) = \kappa$ for all $\kappa \in D \setminus \{\kappa^*\}$. Then p cannot be strengthened by any condition in E since κ^* is not in any of the stems of conditions in E. So dense subsets of $\mathbb{D}$ must have size at least $\sup D$. □

Let $\nu = \delta^+$. Let $\kappa(0)$ be the first measurable cardinal in D.

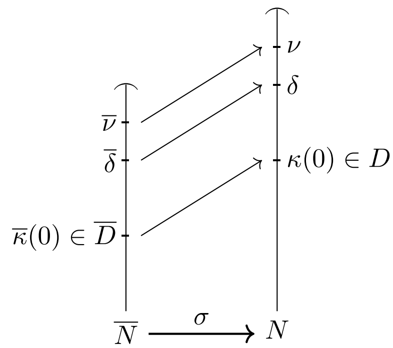
In order to show that $\mathbb{D}$ is subcomplete, suppose we are in the standard setup:

- $\mathbb{D} \in H_\theta \subseteq N = L_\tau[A] \models \mathsf{ZFC}^-$ where $\tau > \theta$ and $A \subseteq \tau$
- $\sigma : \overline{N} \cong X \preccurlyeq N$ where X is countable and $\overline{N}$ is full
- $\sigma(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta, \mathbb{D}, \mathcal{U}, c$ for some $c \in N$.

By our requirement on θ being large enough, we've also ensured that N is closed under countable sequences of ordinals less than δ.

In what follows, we will be taking a few different transitive liftups of restrictions of σ, and it will useful to keep track of embeddings between $\overline{N}$ and N pictorially. Although it's not extraordinarily illuminating at this point in our discussion, the following figure shows the situation we are currently in.

Here we place bars on everything that is relevant on the $\overline{N}$ side of the embedding. In particular, $\sigma(\overline{\delta}) = \delta$, $\overline{\nu} = \overline{\delta}^{+\overline{N}}$, $\overline{D}$ is the discrete set of measurables in $\overline{N}$ that each measure in $\overline{\mathcal{U}}$ comes from, and $\overline{\kappa}(0)$ is the first measurable in $\overline{D}$, in the sense of $\overline{N}$.



Toward showing that $\mathbb{D}$ is subcomplete, we are additionally given some $\overline{G} \subseteq \overline{\mathbb{D}}$ that is generic over $\overline{N}$. Rather than working with $\overline{G}$, we will work with $\overline{S} = \langle \overline{S}(\overline{\kappa}) \mid \overline{\kappa} \in \overline{D} \rangle$, its associated $\overline{\mathbb{D}}$-generic sequence. We must show following, where $C = \mathcal{H}ull^N(\delta \cup X)$:

Main Claim. There is a $\mathbb{D}$-generic sequence S and a map $\sigma' \in V[S]$ such that:

(1) $\sigma' : \overline{N} \prec N$
(2) $\sigma'(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta, \mathbb{D}, \mathcal{U}, c$
(3) $\mathcal{H}ull^N(\delta \cup \mathrm{range}(\sigma')) = C$
(4) $\sigma' `` \overline{S} \subseteq S$

Pf. This proof uses Barwise theory[3] heavily, and ultimately amounts to showing that a certain $\in$-theory, $\mathcal{T}$, which posits the existence of such a σ', is consistent. Such an embedding σ' could only possibly exist in a suitable generic extension, $V[S]$, where S is $\mathbb{D}$-generic sequence that we will find later. Once we have such a suitable $V[S]$, we will use Barwise theory to find an appropriate admissible structure in $V[S]$ for which the theory $\mathcal{T}$, positing the existence of such a suitable σ', defined below, has a model.

For an admissible structure M with $S, \overline{S}, \sigma, N, \theta, \mathbb{D}, \mathcal{U}, c \in M$ let the $\in$-theory $\mathcal{T}(M)$ be defined as follows:

predicate: $\in$
constants: $\dot{\sigma}, \underline{x}$ for $x \in M$
axioms:
- ZFC^- and $\mathsf{Basic\ Axioms}$.
- $\dot{\sigma} : \underline{\overline{N}} \prec \underline{N}$
- $\dot{\sigma}(\underline{\overline{\theta}}, \underline{\overline{\mathbb{D}}}, \underline{\overline{\mathcal{U}}}, \underline{\overline{c}}) = \underline{\theta}, \underline{\mathbb{D}}, \underline{\mathcal{U}}, \underline{c}$
- $\mathcal{H}ull^{\underline{N}}(\underline{\delta} \cup \mathrm{range}(\dot{\sigma})) = \mathcal{H}ull^{\underline{N}}(\underline{\delta} \cup \mathrm{range}(\underline{\sigma}))$
- $\dot{\sigma} `` \underline{\overline{S}} \subseteq \underline{S}$.

The $\in$-theory is $\Sigma_1(M)$, since all of the axioms are M-finite except for the $\mathsf{Basic\ Axioms}$, which altogether are M-*re*.

We need to find an appropriate $\mathbb{D}$-generic sequence S and a suitable admissible structure M containing S so that $\mathcal{T}(M)$ is consistent. To do this we use transitive liftups and Barwise theory. Transitive liftups will give us the consistency of certain embeddings that approximate the one we are looking for, and we will rely on Barwise Completeness (Fact 2.23) to obtain the existence of a model with our desired properties.

[3]Key definitions, facts, and theorems of Barwise theory are summarized Section 2.5.

Toward this end, let's take what will turn out to be our first transitive liftup, which is in some sense ensuring the consistency of (3) of our main claim.

Let $k_0 : N_0 \cong C$ where N_0 is transitive, and set $\sigma_0 = k_0^{-1} \circ \sigma$ and $\sigma_0(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta_0, \mathbb{D}_0, \mathcal{U}_0, c_0$. Since $\delta \subseteq C$ and N_0 is transitive, $\sigma_0(\overline{\delta}) = \delta$.

Indeed N_0 is actually a transitive liftup:

Claim 2. $\langle N_0, \sigma_0 \rangle$ is the transitive liftup of $\langle \overline{N}, \sigma \restriction H_{\overline{\nu}}^{\overline{N}} \rangle$.

Pf. Recall that $\nu = \delta^+$, and $\overline{\nu} = \overline{\delta}^{+\overline{N}}$. It must be shown that the embedding $\sigma_0 : \overline{N} \prec N_0$ is $\overline{\nu}$-cofinal and that $\sigma_0 \restriction H_{\overline{\nu}}^{\overline{N}} = \sigma \restriction H_{\overline{\nu}}^{\overline{N}}$.

To see that σ_0 is $\overline{\nu}$-cofinal, let $x \in N_0$. Then $k_0(x) \in C = \mathcal{H}ull^N(\delta \cup X)$ so $k_0(x)$ is uniquely N-definable from $\xi < \delta$ and $\sigma(\overline{z})$ where $\overline{z} \in \overline{N}$. In other words,

$$k_0(x) = \text{that } y \text{ such that } N \models \varphi(y, \xi, \sigma(\overline{z})).$$

Let $u \in \overline{N}$ be defined as

$$u = \left\{ w \in \overline{N} \mid w = \text{that } y \text{ such that } \overline{N} \models \varphi(y, \zeta, \overline{z}) \text{ for some } \zeta < \overline{\delta} \right\}.$$

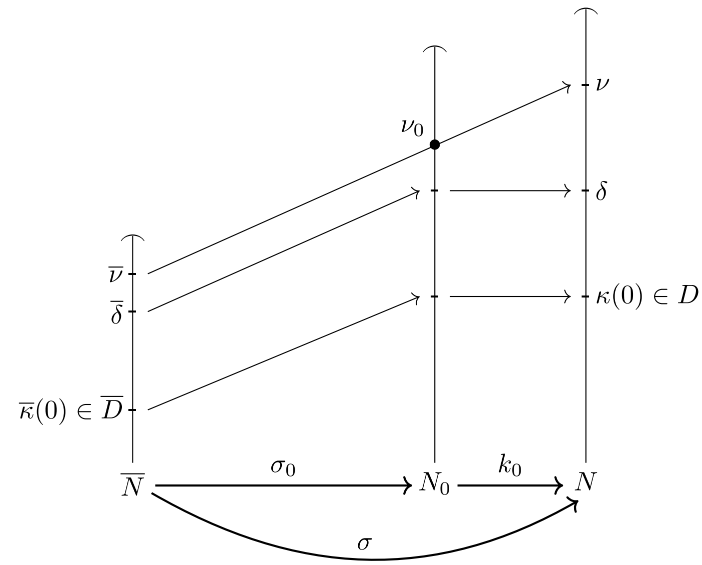
Certainly u is non-empty by elementarity, since $k_0(x) \in \sigma(u)$. Furthermore, $|u| \leq \overline{\delta} < \overline{\nu}$ since every $w \in u$ is unique, and needs a corresponding $\zeta < \overline{\delta}$ to satisfy the formula φ with. Thus $x \in k_0^{-1}(\sigma(u)) = \sigma_0(u)$ with $|u| < \overline{\nu}$ in $\overline{N}$, as desired.

Since $X \cup \delta \subseteq C$, the Skolem hull in N, we know that $\sigma``H_{\overline{\nu}}^{\overline{N}} \subseteq C$. Thus $k_0^{-1} \restriction \sigma``H_{\overline{\nu}}^{\overline{N}} = \text{id}$. Therefore $\sigma_0 \restriction H_{\overline{\nu}}^{\overline{N}} = \sigma \restriction H_{\overline{\nu}}^{\overline{N}}$, finishing the proof of the claim. $\square$

Since $\overline{\nu}$ is regular in $\overline{N}$, in N_0 so is $\nu_0 = \sigma_0(\overline{\nu}) = \sup \sigma_0``\overline{\nu}$. By Interpolation (Fact 2.30), we may say that k_0 is defined by

$$k_0 : N_0 \prec N \text{ where } k_0 \circ \sigma_0 = \sigma \text{ and } k_0 \restriction \nu_0 = \text{id}.$$

In particular, ν_0 is the critical point of k_0, which is continuous below ν_0. Thus we are in a situation that we will represent with the following diagram:



Already, we can say that σ_0 looks like it has one nice property: $\mathcal{H}ull^{N_0}(\delta \cup \mathrm{range}(\sigma_0)) = C$, which somewhat looks like (3) of the main claim. However, we have not yet performed forcing, σ_0 is definable in V, and we still need to find a way to extend the generic sequence $\overline{S}$ to a $\mathbb{D}$-generic sequence over N. We still have a lot more work to do before finding σ'.

We shall define another $\in$-theory, $\mathcal{L}_*$. This will assist us in obtaining the diagonal Prikry extension $V[S]$ we need to satisfy our main claim. In order to do this, we will take another transitive liftup and apply the Transfer Lemma (Fact 2.26), in order to see that this new $\in$-theory is consistent over an admissible structure on N_0.

Since we will be referring to the same $\in$-theory over two different transitive liftups, think of "$*$" in the subscript as a kind of placeholder for some transitive liftup in the following definition, since it would work for any suitable transitive liftup of $\overline{N}$.

Suppose that $\langle N_*, \sigma_* \rangle$ is a transitive liftup of $\overline{N}$ along with some reasonable restriction of σ, ie. the liftup of $\langle \overline{N}, \sigma \restriction H_\alpha^{\overline{N}} \rangle$, where $\alpha \geq \overline{\kappa}(0)$ is regular in $\overline{N}$, and say

$$\sigma_*(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta_*, \mathbb{D}_*, \mathcal{U}_*, c_*.$$

Recall that δ_{N_*} is least satisfying $L_{\delta_{N_*}}(N_*)$ is admissible.

Define the infinitary $\in$-theory $\mathcal{L}(N_*, \sigma_*) = \mathcal{L}_*$ as follows:

predicate: $\in$

constants: $\mathring{\sigma}, \mathring{S}, \underline{x}$ for $x \in L_{\delta_{N_*}}(N_*)$

axioms:
- ZFC^- and $\mathsf{Basic\ Axioms}$
- $\mathring{\sigma} : \underline{\overline{N}} \prec \underline{N_*}$ is $\underline{\overline{\kappa}(0)}$–cofinal
- $\mathring{\sigma}(\underline{\overline{\theta}}, \underline{\overline{\mathbb{D}}}, \underline{\overline{\mathcal{U}}}, \underline{\overline{c}}) = \underline{\theta_*}, \underline{\mathbb{D}_*}, \underline{\mathcal{U}_*}, \underline{c_*}$
- $\mathring{S}$ is a $\underline{\mathbb{D}_*}$-generic sequence over $\underline{N_*}$
- $\mathring{\sigma}``\underline{\overline{S}} \subseteq \mathring{S}$.

As defined, we have that $\mathcal{L}_*$ is a $\Sigma_1(L_{\delta_{N_*}}(N_*))$-theory, since altogether the $\mathsf{Basic\ Axioms}$ are $\Sigma_1(L_{\delta_{N_*}}(N_*))$.

We claim that the theory over the transitive liftup $\langle N_0, \sigma_0\rangle$ is consistent.

Claim 3. $\mathcal{L}_0 = \mathcal{L}(N_0, \sigma_0)$ is consistent.

Pf. Of course, it is not the case that σ_0 is $\overline{\kappa}(0)$-cofinal - all we know is that it is $\overline{\nu}$-cofinal. However, we know how to find an elementary embedding that is $\overline{\kappa}(0)$ cofinal: by taking a suitable transitive liftup.

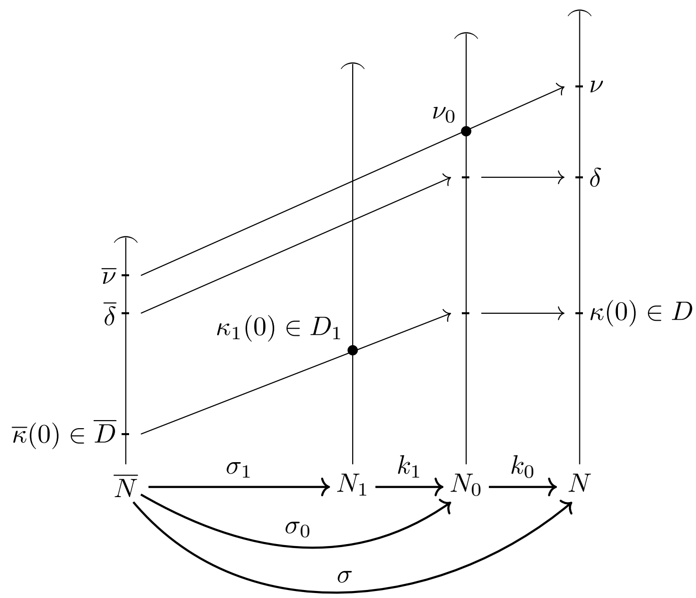
Let $\langle N_1, \sigma_1\rangle$ be the transitive liftup of $\langle \overline{N}, \sigma \restriction H^{\overline{N}}_{\overline{\kappa}(0)}\rangle$, which exists by Interpolation (Fact 2.30). So we have that $\sigma_1 \restriction H^{\overline{N}}_{\overline{\kappa}(0)} = \sigma \restriction H^{\overline{N}}_{\overline{\kappa}(0)}$. Let

$$\sigma_1(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1.$$

Since $\sigma_1 \restriction H^{\overline{N}}_{\overline{\kappa}(0)} = \sigma \restriction H^{\overline{N}}_{\overline{\kappa}(0)} = \sigma_0 \restriction H^{\overline{N}}_{\overline{\kappa}(0)}$, we also have a unique k_1 satisfying

$$k_1 : N_1 \prec N_0 \text{ where } k_1 \circ \sigma_1 = \sigma_0 \text{ and } k_1 \restriction \kappa_1(0) = \mathrm{id} \text{ where } \kappa_1(0) = k_1(\kappa(0)).$$

Indeed k_1 is continuous below $\kappa_1(0)$. We illustrate the final picture below, with σ and all of the relevant transitive liftups.



We first show that $\mathcal{L}_1 = \mathcal{L}(N_1, \sigma_1)$ is consistent, by seeing that it has a model. To do this, we will find a sequence extending σ_1"$\overline{S}$ that is $\mathbb{D}_1$-generic over N_1. Then we will use the Transfer Lemma to see that this transfers to the consistency of $\mathcal{L}_0$.

First, force with $\mathbb{D}_1 = \mathbb{D}_1(\mathcal{U}_1)$, which is a generalized diagonal Prikry forcing over $\mathcal{U}_1$, to obtain a diagonal Prikry sequence S_1'. Define, in $V[S_1']$, a new sequence S_1 as follows:

$$S_1(\kappa) = \begin{cases} S_1'(\kappa) & \text{if } \kappa \in D_1 \setminus \sigma_1 \text{“}\overline{D} \\ \sigma_1(\overline{S}(\overline{\kappa})) & \text{if } \kappa = \sigma_1(\overline{\kappa}) \in \sigma_1 \text{“}\overline{D}. \end{cases}$$

Claim 4. The sequence S_1 is a $\mathbb{D}_1$-generic sequence over N_1.

Pf. We will show that S_1 satisfies the generalized diagonal Prikry genericity criterion (Fact 3.2) over N_1. To do this, let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D_1\rangle$, with $\mathcal{X} \in N_1$, be a sequence of measure-one sets in the sequence of measures $\mathcal{U}_1$.

Note first that as S_1' is a generic sequence, it already satisfies the generalized diagonal Prikry genericity criterion, namely:

$$\{\kappa \in D_1 \mid S_1'(\kappa) \notin X_\kappa\} \text{ is finite.}$$

Recall that $\overline{S} = \langle \overline{S}(\overline{\kappa}) \mid \overline{\kappa} \in \overline{D}\rangle$ is a $\overline{\mathbb{D}}$-generic sequence as well. We need to see that in addition,

$$\left\{\overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin X_{\sigma_1(\overline{\kappa})}\right\} \text{ is finite,}$$

since then

$$\{\kappa \in D_1 \mid S_1(\kappa) \notin X_\kappa\} = \left\{\kappa \in D_1 \setminus \sigma_1``\overline{D} \mid S_1'(\kappa) \notin X_\kappa\right\} \cup \left\{\kappa = \sigma_1(\overline{\kappa}) \in \sigma_1``\overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin X_\kappa\right\}$$

is finite as well, completing the proof as desired.

By the $\overline{\kappa}(0)$-cofinality of σ_1, there is some $w \in \overline{N}$ such that $\mathcal{X} \in \sigma_1(w)$, where $|w| < \overline{\kappa}(0)$ in $\overline{N}$. Thus in N_1, we have that $|\sigma_1(w)| < \kappa_1(0)$. We may assume that w consists of functions $f \in \prod_{\overline{\kappa}\in\overline{D}} \overline{U}(\overline{\kappa})$. So for each $\kappa \in \sigma_1``\overline{D}$, we have that $X_\kappa \in \sigma_1(w)_\kappa = \left\{\sigma_1(f)(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa}\in\overline{D}} \overline{U}(\overline{\kappa}) \ \wedge\ f \in w\right\}$ and also $|\sigma_1(w)_\kappa| < \kappa_1(0)$. So all $\kappa \in \sigma_1``\overline{D}$ of course satisfy $\kappa \geq \kappa_1(0)$ and thus by the κ-completeness of $U_1(\kappa)$, we have that $W_\kappa := \cap\, \sigma_1(w)_\kappa \in U_1(\kappa)$. So we have established that $\mathcal{W}$, the sequence of W_κ for $\kappa \geq \kappa_1(0)$, is also a sequence of measure-one sets in N_1. Note in addition that for $\kappa \in \sigma_1``\overline{D}$, we have that $W_\kappa \subseteq X_\kappa$.

By elementarity, for each $\overline{\kappa} \in \overline{D}$, we have $\overline{W}_{\overline{\kappa}} = \cap\left\{f(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa}\in\overline{D}} \overline{U}(\overline{\kappa}) \ \wedge\ f \in w\right\}$ is a measure-one set in $\overline{U}(\overline{\kappa})$ and we also have that $\sigma_1(\overline{W}_{\overline{\kappa}}) = W_{\sigma_1(\overline{\kappa})}$. Moreover,

$$\left\{\overline{\kappa} \in \overline{D} \mid \overline{S}(\overline{\kappa}) \notin \overline{W}_{\overline{\kappa}}\right\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for $\overline{\mathbb{D}}$, which must be satisfied by $\overline{S}$. Thus by elementarity,

$$\left\{\overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin W_{\sigma_1(\overline{\kappa})}\right\} \supseteq \left\{\overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin X_{\sigma_1(\overline{\kappa})}\right\} \text{ is finite,}$$

as is desired, completing the proof of our claim. □

Moreover we have now shown that $\mathcal{L}(N_1, \sigma_1) = \mathcal{L}_1$ is consistent by Barwise Correctness (Fact 2.22), since we have just shown that $\langle H_\delta;\, \sigma_1, S_1\rangle$ is a model of $\mathcal{L}_1$.

Let's check that we may now apply the Transfer Lemma (Fact 2.26) to the embedding $k_1 : N_1 \prec N_0$. By Lemma 2.33 we have that N_1 is almost full. We also have that $\mathcal{L}_1 = \mathcal{L}(L_{\delta_{N_1}}(N_1))$ is

$$\Sigma_1(\langle N_1;\, \overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}, \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1\rangle)$$

while $\mathcal{L}_0 = \mathcal{L}(L_{\delta_{N_0}}(N_0))$ is

$$\Sigma_1(\langle N_0;\, k_1(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}, \theta_1, \mathbb{D}_1, \mathcal{U}_1, c_1)\rangle).$$

Furthermore k_1 is cofinal, since for each element $x \in N_0$, as σ_0 is cofinal, there is $u \in \overline{N}$ such that $x \in \sigma_0(u)$. Thus $\sigma_1(u) \in N_1$, and moreover $x \in k_1(\sigma_1(u)) = \sigma_0(u)$.

Therefore, we have that since $\mathcal{L}_1$ is consistent, $\mathcal{L}_0$ is consistent as desired. This completes the proof of *Claim* 2. □

From the consistency of $\mathcal{L}_0$, we would now like to use Barwise Completeness (Fact 2.23) to obtain a model of $\mathcal{L}_0$. To do this, we need the admissible structure the theory is defined over to be countable. So let's work in $V[F]$, a generic extension that collapses $L_{\delta_{N_0}}(N_0)$ to be countable. Then by Barwise Completeness, $\mathcal{L}_0$ has a solid model

$$\mathfrak{A} = \langle \mathfrak{A}; \mathring{S}^{\mathfrak{A}}, \mathring{\sigma}^{\mathfrak{A}} \rangle$$

such that

$$\mathrm{Ord} \cap \mathrm{wfc}(\mathfrak{A}) = \mathrm{Ord} \cap L_{\delta_{N_0}}(N_0).$$

Thus we have that $\mathring{\sigma}^{\mathfrak{A}} : \underline{\overline{N}}^{\mathfrak{A}} \prec \underline{N_0}^{\mathfrak{A}}$. By the $\mathsf{Basic\ Axioms}$ we have that $\underline{\overline{N}}^{\mathfrak{A}} = \overline{N}$ and $N_0 = \underline{N_0}^{\mathfrak{A}}$. Thus we may say that $\mathring{\sigma}^{\mathfrak{A}} : \overline{N} \prec N_0$.

Let $S = \mathring{S}^{\mathfrak{A}}$ and $\overset{*}{\sigma} = k_0 \circ \mathring{\sigma}^{\mathfrak{A}}$.

Then S is a $\mathbb{D}_0$-generic sequence over N_0, and as $k_0 : N_0 \cong C$ we also have that k_0"S is C-generic for $\mathbb{D}$. We need to see that S is $\mathbb{D}$-generic over V. To do this, let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D \rangle$ be a sequence of measure-one sets in the sequence of measures $\mathcal{U}$. We will verify the generalized diagonal Prikry genericity criterion. To do this, let $E \subseteq \mathbb{D}$ be dense and have size δ with $E \in C$. Since $\delta \subseteq C$, we have that $E \subseteq C$ as well. Find a condition $(s, A) \in E$ that strengthens $(\emptyset, \mathcal{X})$. Thus for $\kappa \in \mathrm{dom}\, A$, we have that $A(\kappa) \subseteq X_\kappa$. Define a sequence of measure-one sets B in C so that

$$B(\kappa) = \begin{cases} A(\kappa) & \text{if } \kappa \in \mathrm{dom}\, A \\ \kappa & \text{if } \kappa \in \mathrm{dom}\, s. \end{cases}$$

So we have that B is a sequence of measure-one sets in C. So $\{\kappa \in D \mid S(\kappa) \notin B(\kappa)\}$ is finite. Thus $\{\kappa \in D \mid S(\kappa) \notin X_\kappa\}$ is finite.

This will be the $\mathbb{D}$-generic sequence we need to satisfy our main claim. We will see in the following claim that $\overset{*}{\sigma}$ has all of the desired properties of our main claim, but it fails to be in $V[S]$, which is what we need. However, based on the following claim, $\overset{*}{\sigma}$ will at least enable us to see that our $\in$-theory $\mathcal{T}$ from long ago, defined to assist us in proving the main claim, is consistent over a suitable admissible structure.

Claim 5. The map $\overset{*}{\sigma}$ satisfies:

(1) $\overset{*}{\sigma} : \overline{N} \prec N$
(2) $\overset{*}{\sigma}(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \theta, \mathbb{D}, \mathcal{U}, c$
(3) $\mathcal{H}ull^N(\delta \cup \mathrm{range}(\overset{*}{\sigma})) = C$
(4) $\overset{*}{\sigma}$"$\overline{S} \subseteq S$

Pf. For (1), we have already seen above that $\mathring{\sigma}^{\mathfrak{A}} : \overline{N} \prec N_0$. Since $k_0 : N_0 \prec N$, the desired result follows.

For (2), $\overset{*}{\sigma}(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = k_0(\theta_0, \mathbb{D}_0, \mathcal{U}_0, c_0) = \theta, \mathbb{D}, \mathcal{U}, c$.

We know (3) holds since $N_0 = \mathcal{H}ull^{N_0}(\delta \cup \mathrm{range}(\mathring{\sigma}^{\mathfrak{A}}))$. To see this, clearly we have that $\mathcal{H}ull^{N_0}(\delta \cup \mathrm{range}(\mathring{\sigma}^{\mathfrak{A}})) \subseteq N_0$, since $\delta \in N_0$ as $N_0 \cong C$, and certainly $\mathrm{range}(\mathring{\sigma}^{\mathfrak{A}})) \subseteq N_0$ as well. Then because $\mathring{\sigma}^{\mathfrak{A}}$ is $\overline{\kappa}'$-cofinal, by Lemma 2.29, we have, since $\mathring{\sigma}^{\mathfrak{A}}(\overline{\kappa}') < \delta$, that:

$$N_0 = \left\{ \mathring{\sigma}^{\mathfrak{A}}(f)(\beta) \mid f : \gamma \longrightarrow N_0,\ \gamma < \overline{\kappa}(0) \text{ and } \beta < \mathring{\sigma}^{\mathfrak{A}}(\gamma) \right\} \subseteq \mathcal{H}ull^N(\delta \cup \mathrm{range}(\mathring{\sigma}^{\mathfrak{A}})).$$

Thus $C = k_0$"$N_0 = \mathcal{H}ull^{N_0}(\delta \cup \mathrm{range}(k_0 \circ \mathring{\sigma}^{\mathfrak{A}}))$ as desired.

To see (4), note that $\overset{*}{\sigma} \restriction \overline{\kappa}(0) = \mathring{\sigma}^{\mathfrak{A}} \restriction \overline{\kappa}(0)$ since $k_0 \restriction \nu_0 = \mathrm{id}$.

This completes the proof of Claim 3. $\square$

We are almost done, but like we stated above, $\mathring{\sigma}^{\mathfrak{A}}$ is in $V[F]$, the generic extension needed to obtain a countable admissible structure to apply Barwise Completeness to. But $V[F]$ is not in $V[S]$, and so $\overset{*}{\sigma}$ is not necessarily in $V[S]$. We will use Barwise Completeness one last time, to finally find an embedding σ' with which to satisfy the main claim along with the S we found above.

Let λ be regular in $V[S]$ with $N \in H_\lambda^{V[S]}$. Then

$$M = \langle H_\lambda^{V[S]};\ N, \sigma, S;\ \theta, \delta, \mathbb{D}, \mathcal{U}, c\rangle \text{ is admissible.}$$

In order to satisfy our main claim, we need a model of $\mathcal{T}(M)$ in $V[S]$. By *Claim* 3, we have that $\langle M, \overset{*}{\sigma}\rangle$ is a model of $\mathcal{T}(M)$, but in $V[F]$, which is not $V[S]$. Still this means that by Barwise Correctness, $\mathcal{T}(M)$ is consistent.

Consider the Mostowski collapse of M; let

$$\pi : \tilde{M} \prec M \text{ where } \tilde{M} \text{ is countable and transitive.}$$

Note that $\tilde{M} \in H_{\omega_1}^{V[S]} = H_{\omega_1}^{V}$ since it is countable and diagonal Prikry forcing doesn't add bounded subsets to any $\kappa \in D$.[4] We also have that $\underline{\overline{N}}^{\tilde{\mathfrak{A}}} = \overline{N}$, since M sees that $\overline{N}$ is countable so $\tilde{M}$ sees that $\pi^{-1}(\overline{N})$ is, and it follows that $\pi^{-1}(\overline{N}) = \overline{N}$.

Plus, $\mathcal{T}(\tilde{M})$ is consistent, since otherwise its inconsistency could be pushed up via π to one in $\mathcal{T}(M)$, contradicting the model witnessing its consistency that we found in $V[F]$.

So by Barwise Completeness, $\mathcal{T}(\tilde{M})$ has a solid model

$$\tilde{\mathfrak{A}} = \langle \tilde{\mathfrak{A}}; \dot{\sigma}^{\tilde{\mathfrak{A}}}\rangle$$

such that

$$\mathrm{Ord} \cap \mathrm{wfc}(\tilde{\mathfrak{A}}) = \mathrm{Ord} \cap \tilde{M}.$$

Letting $\sigma' = \pi \circ \dot{\sigma}^{\tilde{\mathfrak{A}}}$, the main claim is now satisfied with σ' and our λ-diagonal Prikry sequence S.

Let us verify each of the properties of σ' required by the main claim. The verification of these properties shall use the agreement between $\tilde{\mathfrak{A}}$ and $\tilde{M}$ on the special constants of $\tilde{M}$ and on the ordinals. The fact that π does not affect $\overline{N}$ will be greatly taken advantage of.

First we show (1) of the main claim. Let's say that $\varphi[\sigma'(\overline{a})]$ holds in N. So $\varphi[\pi(\dot{\sigma}^{\tilde{\mathfrak{A}}}(\overline{a}))]^N$ holds in M. Thus $\varphi[\dot{\sigma}^{\tilde{\mathfrak{A}}}(\overline{a})]^{\pi^{-1}(N)}$ holds in $\tilde{M}$, and thus also in $\tilde{\mathfrak{A}}$. Indeed we know that $\underline{\overline{N}}^{\tilde{\mathfrak{A}}} = \overline{N}$, since $\tilde{\mathfrak{A}}$ agrees with $\tilde{M}$ about countable ordinals, which we may use to code $\overline{N}$. This means that $\varphi[\overline{a}]$ holds in $\overline{N}$, as desired.

To see (2), we have $\sigma'(\overline{\theta}, \overline{\mathbb{D}}, \overline{\mathcal{U}}, \overline{c}) = \pi(\underline{\theta}^{\tilde{\mathfrak{A}}}, \underline{\mathbb{D}}^{\tilde{\mathfrak{A}}}, \underline{\mathcal{U}}^{\tilde{\mathfrak{A}}}, \underline{c}^{\tilde{\mathfrak{A}}}) = \theta, \mathbb{D}, \mathcal{U}, c$.

For item (3), let $\tilde{N} = \underline{N}^{\tilde{\mathfrak{A}}}$, $\tilde{\sigma} = \underline{\sigma}^{\tilde{\mathfrak{A}}}$ and $\tilde{\delta} = \underline{\delta}^{\tilde{\mathfrak{A}}}$. So $\pi(\tilde{\delta}) = \delta$ and $\pi(\tilde{\sigma}) = \sigma$. Because of the way the $\in$-theory $\mathcal{T}$ was defined, we already have:

$$(1) \qquad \mathcal{H}ull^{\tilde{N}}(\tilde{\delta} \cup \mathrm{range}(\dot{\sigma}^{\tilde{\mathfrak{A}}})) = \mathcal{H}ull^{\tilde{N}}(\tilde{\delta} \cup \mathrm{range}(\tilde{\sigma})).$$

To see $\mathcal{H}ull^{N}(\delta \cup \mathrm{range}(\sigma')) \subseteq \mathcal{H}ull^{N}(\delta \cup X)$, suppose $x \in \mathcal{H}ull^{N}(\delta \cup \mathrm{range}(\sigma'))$. Then we have that N sees that there is some formula φ, $\overline{z} \in \overline{N}$, and $\xi < \delta$ where x is unique such that $\varphi(x, \pi(\dot{\sigma}^{\tilde{\mathfrak{A}}}(\overline{z})), \xi)$. In particular, x is in the range of π. Thus $\tilde{x} = \pi^{-1}(x) \in \tilde{N}$ and $\tilde{\xi} < \tilde{\delta}$ such that $\varphi(\tilde{x}, \dot{\sigma}^{\tilde{\mathfrak{A}}}(\overline{z}), \tilde{\xi})$ holds. Thus by (1), we have that $\tilde{x}$ is unique such that $\varphi(\tilde{x}, \tilde{\sigma}(\overline{y}), \tilde{\zeta})$ for some $\overline{y} \in \overline{N}$

[4] As Fuchs [1] points out, this result is a modification to the proof that generalized diagonal Prikry forcing preserves cardinalities.

and $\tilde{\zeta} < \tilde{\delta}$. Thus pushing back up through π, letting $\zeta = \pi(\tilde{\zeta})$, we have that $x = \pi(\tilde{x})$ is unique such that $\varphi(x, \sigma(\overline{y}), \zeta)$ holds, so $x \in \mathcal{H}ull^N(\delta \cup X)$.

To see that $\mathcal{H}ull^N(\delta \cup X) \subseteq \mathcal{H}ull^N(\delta \cup \text{range}(\sigma'))$ works similarly. Let $x \in \mathcal{H}ull^N(\delta \cup X)$. Then there is $\overline{z} \in \overline{N}$ and $\xi < \delta$ such that $\varphi(x, \sigma(\overline{z}), \xi)$ holds. So in particular, x is in the domain of π. So we may find $\tilde{\xi} < \tilde{\delta}$ such that $\tilde{x} = \pi^{-1}(x)$ is unique such that $\varphi(\tilde{x}, \tilde{\sigma}(\overline{z}), \tilde{\xi})$. So by (1), we have that there is $\overline{y} \in \overline{N}$ and $\tilde{\zeta} < \tilde{\delta}$ such that $\tilde{x}$ is unique satisfying $\varphi(\tilde{x}, \dot{\sigma}^{\mathfrak{A}}(\overline{y}), \tilde{\zeta})$. Finally, by pushing back up through π, letting $\pi(\tilde{\zeta}) = \zeta$, we have that $x = \pi(\tilde{x})$ is unique such that $\varphi(x, \sigma'(\overline{y}), \zeta)$, so $x \in \mathcal{H}ull^N(\delta \cup \text{range}(\sigma'))$ as desired.

To see item (4), note that $\underline{\overline{S}}^{\tilde{\mathfrak{A}}} = \overline{S}$ since $\overline{S} \subseteq \overline{N}$. So we have already by the definition of $\mathcal{T}$ that $\dot{\sigma}^{\tilde{\mathfrak{A}}}``\overline{S} \subseteq \underline{S}^{\tilde{\mathfrak{A}}}$. Thus $\pi \circ \dot{\sigma}^{\tilde{\mathfrak{A}}}``\overline{S} \subseteq \pi``\underline{S}^{\tilde{\mathfrak{A}}} \subseteq S$ as desired.

This completes the proof of the main claim. □

We have satisfied the main claim, so we are done, we have shown that $\mathbb{D}$ is subcomplete. □

Some slight modifications to the above proof give the following two corollaries.

The first point is that the above proof also shows that generalized diagonal Prikry forcing that adds a countable sequence to each measurable cardinal is subcomplete. Before stating the corolloary let's define the forcing. Again let D be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D\rangle$ be a list of measures associated to D. Let $\mathbb{D}^*(\mathcal{U}) = \mathbb{D}^*$ be defined the same as $\mathbb{D}(\mathcal{U})$ except the stem of a condition, s, in $\mathbb{D}^*(\mathcal{U})$ is a function with domain in $[D]^{<\omega}$ taking each measurable cardinal $\kappa \in \text{dom}(s)$ to finitely many ordinals $s(\kappa) \subseteq \kappa$. The upper part and extension relation is defined in the same way; the only slight modification is that again we have $(s, A) \leq (t, B)$ so long as points in s not in t come from B, which in the case means that for $\kappa \in \text{dom}(s)$ we have that each element of $s(\kappa)$ not in $t(\kappa)$ is in $B(\kappa)$. We may again form a $\mathbb{D}^*$-generic sequence $S = S_G$ for a generic $G \subseteq \mathbb{D}^*$, and we may write $S = \langle S(\kappa) \mid \kappa \in D\rangle$ where $S(\kappa)$ is a countable sequence of ordinals less than κ. The genericity criterion for generic diagonal Prikry sequences is as that for $\mathbb{D}$, which is given in [1, Theorem 1], as stated in Fact 3.2, with the modification that S is $\mathbb{D}^*$ generic if and only if for all $\mathcal{X}$, the set $\{\alpha \mid \exists \kappa \in D\ \alpha \in S(\kappa) \setminus X_\kappa\}$ is finite.

Corollary 3.4. *Let D be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D\rangle$ be a list of measures associated to D. Then $\mathbb{D}^*(\mathcal{U})$ is subcomplete.*

Proof Sketch. The modifications are mostly notational, and the main one that needs to be made is to adjust the proof of the *Claim* within the proof of *Claim* 2. Here we have $\mathbb{D}_1^*$, the generalized diagonal Prikry forcing as computed in N_1, as well as $\overline{\mathbb{D}}$ of $\overline{N}$, and S_1, which we would like to show is a $\mathbb{D}_1^*$-generic sequence over N_1 in this case. S_1 is defined as $\sigma_1``\overline{S}$, using a diagonal Prikry sequence S_1' to fill in the missing coordinates, where S_1' is obtained by forcing with $\mathbb{D}_1$ over V.

We will show that S_1 satisfies the generalized diagonal Prikry genericity criterion over N_1 and follow the above proof. To do this, let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in D_1\rangle$, with $\mathcal{X} \in N_1$, be a sequence of measure-one sets in the sequence of measures $\mathcal{U}_1$.

Note first that S_1' is a generic sequence, it already satisfies the generalized diagonal Prikry genericity criterion, namely: $\{\alpha \mid \exists \kappa \in D_1\ \alpha \in S_1'(\kappa) \setminus X_\kappa\}$ is finite. Recall that $\overline{S} = \langle \overline{S}(\overline{\kappa}) \mid \overline{\kappa} \in \overline{D}\rangle$ is a $\overline{\mathbb{D}}$-generic sequence over $\overline{N}$ as well. We need to see that in addition, $\left\{\alpha \mid \exists \overline{\kappa} \in \overline{D}\ \alpha \in \sigma_1(\overline{S}(\overline{\kappa})) \setminus X_{\sigma_1(\overline{\kappa})}\right\}$ is finite.

By the $\overline{\kappa}(0)$-cofinality of σ_1, there is some $w \in \overline{N}$ such that $\mathcal{X} \in \sigma_1(w)$, where $|w| < \overline{\kappa}(0)$ in $\overline{N}$. Thus in N_1, $|\sigma_1(w)| < \kappa_1(0)$. For each $\kappa \in \sigma_1``\overline{D}$, we have that $X_\kappa \in \sigma_1(w)_\kappa =$

$\{\sigma_1(f)(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa}\in\overline{D}} \overline{U}(\overline{\kappa}) \ \wedge\ f \in w\}$ and also $|\sigma_1(w)_\kappa| < \kappa_1(0)$. All $\kappa \in \sigma_1``\overline{D}$ of course satisfy $\kappa \geq \kappa_1(0)$ so by the κ-completeness of $U_1(\kappa)$, we have that $W_\kappa := \cap\sigma_1(w)_\kappa \in U_1(\kappa)$ since $X_\kappa \in \sigma_1(w)_\kappa$. So we have established that $\mathcal{W}$, the sequence of W_κ for $\kappa \geq \kappa_1(0)$, is also a sequence of measure-one sets in N_1. Note in addition that for $\kappa \in \sigma_1``\overline{D}$, we have that $W_\kappa \subseteq X_\kappa$.

By elementarity, for each $\overline{\kappa} \in \overline{D}$, we have $\overline{W}_{\overline{\kappa}} = \cap\{f(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa}\in\overline{D}} \overline{U}(\overline{\kappa}) \ \wedge\ f \in w\}$ is a measure-one set in $\overline{U}(\overline{\kappa})$ and we also have that $\sigma_1(\overline{W}_{\overline{\kappa}}) = W_{\sigma_1(\overline{\kappa})}$. Moreover,

$$\{\alpha \mid \exists\overline{\kappa} \in \mathbb{D}\ \alpha \in \overline{S}(\overline{\kappa}) \setminus \overline{W}_{\overline{\kappa}}\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for $\overline{\mathbb{D}}$, which must be satisfied by $\overline{S}$. Thus by elementarity,

$$\{\alpha \mid \exists\overline{\kappa} \in \mathbb{D}\ \sigma_1(\overline{S}(\overline{\kappa})) \setminus W_{\sigma_1(\overline{\kappa})}\} \supseteq \{\alpha \mid \exists\overline{\kappa} \in \mathbb{D}\ \sigma_1(\overline{S}(\overline{\kappa})) \setminus X_{\sigma_1(\overline{\kappa})}\} \text{ is finite,}$$

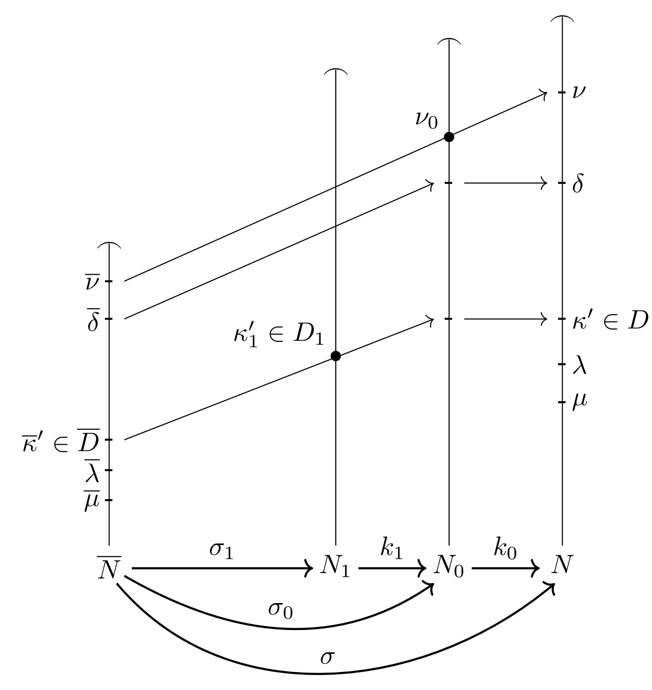
as is desired, completing the proof of the claim. □

One might consider a forcing like $\mathbb{D}$ and $\mathbb{D}^*$ that adds one point below each measurable cardinal sometimes, and other times adds a cofinal ω-sequence below the measurable cardinal. This forcing is clearly subcomplete as well.

Corollary 3.5. *Let D be an infinite discrete set of measurable cardinals. Let $\mathcal{U} = \langle U(\kappa) \mid \kappa \in D\rangle$ be a list of measures associated to D.*

Furthermore, let $\mu < \lambda$ be a regular cardinal, where $\lambda = \sup_{n<\omega} \kappa_n$, the first limit point of D. Then $\mathbb{D} = \mathbb{D}(\mathcal{U})$ is subcomplete above μ.

Proof Sketch. The idea is to follow the same exact proof as in the above theorem, except we achieve the following diagram:



Here we replace $\kappa(0)$ with some $\kappa' \in D$ such that $\lambda < \kappa'$, where there are finitely many measurables of D below κ'. So in particular, we let $\langle N_1, \sigma_1 \rangle$ be the liftup of $\langle \overline{N}, \sigma \restriction H^{\overline{N}}_{\kappa'} \rangle$ in *Claim* 2. In order to show the *Claim* that we have a generic sequence over $\mathbb{D}_1$, we follow the same argument as follows:

Let $\mathcal{X} = \langle X_\kappa \in U_1(\kappa) \mid \kappa \in \sigma_1``\overline{D} \rangle$, with $\mathcal{X} \in N_1$, be a sequence of measure one sets in the sequence of measures $\mathcal{U}_1$ with only coordinates coming from $\sigma_1``\overline{D}$. We need to see that

$$\left\{ \overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin X_{\sigma_1(\overline{\kappa})} \right\} \text{ is finite.}$$

By the $\overline{\kappa}'$-cofinality of σ_1, there is some $w \in \overline{N}$ such that $\mathcal{X} \in \sigma_1(w)$, where $|w| < \overline{\kappa}'$ in $\overline{N}$. Thus in N_1, $|\sigma_1(w)| < \kappa_1'$. For each $\kappa \in \sigma_1``\overline{D}$, we have that $X_\kappa \in \sigma_1(w)_\kappa = \left\{ \sigma_1(f)(\overline{\kappa}) \mid f \in \prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa}) \ \wedge \ f \in w \right\}$ and also

$$|\sigma_1(w)| < \kappa_1'.$$

So for all but finitely many $\kappa \in \sigma_1``\overline{D}$, namely for $\kappa \geq \kappa_1'$, by the κ-completeness of $U_1(\kappa)$, we have that

$$\cap \, \sigma_1(w) = W_\kappa \in U_1(\kappa).$$

So we have established that $\mathcal{W}$, the sequence of W_κ for $\kappa > \kappa_1'$, is also a sequence of measure-one sets in N_1. Note in addition that for $\kappa \in \sigma_1``\overline{D}$, $\kappa > \kappa_1'$, we have that $W_\kappa \subseteq X_\kappa$.

By elementarity, for each $\overline{\kappa} \in \overline{D}$, where $\overline{\kappa} > \overline{\kappa}'$, we have

$$\overline{W}_{\overline{\kappa}} = \cap \left\{ f(\overline{\kappa}) \mid f \in \textstyle\prod_{\overline{\kappa} \in \overline{D}} \overline{U}(\overline{\kappa}) \ \wedge \ f \in w \right\}$$

is a measure-one set in $\overline{U}(\overline{\kappa})$ and we also have that $\sigma_1(\overline{W}_{\overline{\kappa}}) = W_{\sigma_1(\overline{\kappa})}$. Moreover,

$$\left\{ \overline{\kappa} \in \overline{D} \mid \overline{S}(\overline{\kappa}) \notin \overline{W}_{\overline{\kappa}} \right\} \text{ is finite}$$

by the generalized diagonal Prikry genericity criterion for $\overline{\mathbb{D}}$, which must be satisfied by $\overline{S}$, and since there are only finitely many measurables in $\overline{D}$ less than $\overline{\kappa}'$ in $\overline{N}$. Thus by elementarity,

$$\left\{ \overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa})) \notin W_{\sigma_1(\overline{\kappa})} \right\} \supseteq \left\{ \overline{\kappa} \in \overline{D} \mid \sigma_1(\overline{S}(\overline{\kappa}) \notin X_{\sigma_1(\overline{\kappa})} \right\} \text{ is finite.}$$

Additionally the $\in$-theories $\mathcal{L}$ and $\mathcal{T}$ would have to be defined so as to include as an axiom that $\mathring{\sigma} \restriction \underline{\overline{\mu}} = \underline{\sigma} \restriction \underline{\overline{\mu}}$ and $\dot{\sigma} \restriction \underline{\overline{\mu}} = \underline{\sigma} \restriction \underline{\overline{\mu}}$ respectively. We would then need to show that $\overset{*}{\sigma} \restriction \overline{\mu} = \sigma \restriction \overline{\mu}$, but this would follow since k_0 is the identity on ν_0. Furthermore, it would need to be shown that $\sigma' \restriction \overline{\mu} = \sigma \restriction \overline{\mu}$, but this would follow from the requirement that $\dot{\sigma}^{\tilde{\mathfrak{A}}} \restriction \underline{\overline{\mu}} = \underline{\sigma}^{\tilde{\mathfrak{A}}} \restriction \overline{\mu}$, and since ordinals are computed properly by $\tilde{\mathfrak{A}}$. □

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