Infinite Latin Squares: Neighbor Balance and Orthogonality

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Abstract

Regarding neighbor balance, we consider natural generalizations of D-complete Latin squares and Vatican squares from the finite to the infinite. We show that if G is an infinite abelian group such that the number of square elements is equinumerous with the whole group then it is possible to permute the rows and columns of the Cayley table to give an infinite Vatican square. We also construct an Vatican square of every infinite order that is not obtainable by permuting the rows and columns of a Cayley table. Regarding orthogonality, we show that if G is as above then G has a set of $|G|$ mutually orthogonal orthomorphisms and hence there is a set of $|G|$ mutually orthogonal Latin squares based on G. `[And we can maybe do a bit better than this last sentence.]`

Keywords: complete Latin square; complete mapping; directed terrace; infinite design; infinite Latin square; mutually orthogonal Latin squares; orthomorphism; R-sequencing; sequencing; Vatican square.

1 Introduction

A finite Latin square is *row complete* or *Roman* if any two distinct symbols appear in adjacent cells within rows once in each order. If the transpose of a Latin square is row complete then the square is *column complete*; a square that is row complete and column complete is *complete*. Finite row complete squares exist for all composite orders [13] and finite complete squares are known to exist for all even orders [12] and many odd composite orders at which a nonabelian group exists; see, for example, [15].

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Vatican and D-complete squares strengthen this notion of completeness. A Latin square is is *row D-complete* if any two distinct symbols appear in cells that are distance d apart in rows at most once in each order for each $d \leq D$. *Column D-completeness* is defined analogously and a square that is both row and column D-complete is *D-complete*. The 1-completeness property is the same as completeness.

An $(n-1)$-complete square of order n is called *Vatican*; that is, Vatican squares have the pair-occurrence restriction at every possible distance.

Vatican squares are known to exist for all orders that are one less than a prime. In addition to this, 2-complete squares are known to exist at orders $2p$ where p is a prime congruent to 5, 7 or 19 modulo 24, orders $2m$ where $5 \leq m \leq 25$, and order 21 [6, 16].

In this paper we extend these notions to the infinite and prove various existence results. As in [4], we use Zermelo-Fraenkel set theory with the axiom of choice. In order to work with infinite sets, we use the set-theoretic machinery of ordinals and transfinite induction. An ordinal is an isomorphism type of well-ordered sets. The finite ordinals correspond to the natural numbers (or rather the unique well-ordered sets with 0 or 1 or 2 etc. as elements), but there are also infinite ordinals. The first infinite ordinal, denoted ω, corresponds to the well-ordered set of natural numbers. The ordinal corresponding to the well-order of the natural numbers with an added maximal element is denoted $\omega + 1$, and we can keep going after this to obtain $\omega + 2, \omega + 3$, and so on, reaching the limit $\omega + \omega$ (aka $\omega \cdot 2$), and beyond.

As ordinals represent canonical well-orderings, they each support a notion of induction similar to the usual one on the natural numbers. Fixing an ordinal λ, transfinite induction up to λ allows us to prove that a property P holds for all ordinals below λ by showing, from the hypothesis that P holds for all ordinals below some $\alpha < \lambda$, that P holds for α (in this formulation the induction principle is analogous to strong induction on the natural numbers, and in fact reduces to it if we set $\lambda = \omega$).

It follows from the axiom of choice that for any set x, there is a bijection between x and some ordinal. In the usual set-theoretic practice, the cardinality of x is defined as the least such ordinal.

We require a definition of an infinite Latin square that allows us to talk about spatial relationships. This is accomplished by using a subset of an ordered field to index the rows and columns. When that field is $\mathbb{Q}$ or $\mathbb{R}$, the infinite Latin squares we obtain are natually embedded in $\mathbb{R}^2$.

Let $\mathbb{F}$ be an ordered field and let $I \subseteq \mathbb{F}$. For each $d \in \mathbb{F}^+$ let $I_{(d)} = \{i \in I \mid i + d \in I\}$. If $|I| = |\mathbb{F}|$ and for each d we have either $|I_{(d)}| = |\mathbb{F}|$ or $|I_{(d)}| = 0$, then I is an *index set*. For our purposes, we may assume that $I_{(1)} \neq \emptyset$ without loss of generality.

Given an index set I, a *Latin square* on I with symbol set X is a function $L : I \times I \to X$ such that for each $i \in I$ the restriction of L to $I \times \{i\}$ is a bijection with X, as is the restriction to $\{i\} \times I$. In other words, each symbol appears once in each "row" and once in each "column."

This definition is compatible with the definition for Latin squares of arbitrary cardinality of Hilton and Wojciechowski [14]. In the countable case with $I = \mathbb{N} \subseteq \mathbb{Q}$ or $I = \mathbb{Z} \subseteq \mathbb{Q}$ we get the "quarter-plane Latin squares" and "full-plane Latin squares" respectively of Caulfield [5].

The definition of completeness for infinite squares is obtained by identifying the ideas of adjacency and being at distance 1. This is perfectly natural when $I \in \{\mathbb{N}, \mathbb{Z}\}$ and again matches the definition of Caulfield [5]. It does not seem to capture a property of particular combinatorial interest otherwise, but when we move to generalizing Vatican squares we get the very natural notion of pairs appearing once at all distances. Indeed, the definition of an infinite Vatican square is arguably more natural than the finite version as it allows every pair to appear exactly once at every distance rather than merely at most once.

Formally, an infinite Latin square on an index set I is *row complete* or *Roman* if each pair of distinct symbols appears exactly once in each order at distance 1 in rows. The square is *complete* if the corresponding property also holds in columns. An infinite Latin square with indexing set I is *row D-complete* if each pair of symbols appear exactly once in each order at distance d in rows for each d such that $I_{(d)} \neq \emptyset$ and $0 < d \leq D$. The square is *D-complete* if the corresponding property also holds in columns. Further, the square is *Vatican* if for each d with $I_{(d)} \neq \emptyset$ we have that each pair of distinct symbols appears at distance d exactly once in each order in rows and once in each order in columns.

Our first method for constructing squares uses Cayley tables of groups. In the finite case all known constructions for complete squares–and hence D-complete and Vatican squares—use the notion of "sequenceability" of a group and generalisations of it. In the next section we show that similar notions are sufficient to construct infinite D-complete and Vatican squares. Say that an infinite group G is *squareful* if the set $\{g^2 : g \in G\}$ has the same cardinality as G. If G is an abelian squareful group and I is an index set with $|I| = |G|$, then we can construct an infinite Vatican square using the Cayley table of G.

In Section 3 we explore non-group-based methods. We show that there is a Vatican square of each infinite order that cannot be produced from the rows and columns of a Cayley table. Whether a finite Vatican square with this property exists is an open question. We also show that there is a Latin square of each infinite order such that with no orderings of its rows and columns gives a Vatican (or even row-complete) square.

As infinite sets can be bijective with proper subsets of themselves, we can define a variation on Vatican squares that only makes sense for infinite orders. Say that an infinite Latin square on index set I is *semi-Vatican* if for each d with $I_{(d)} \neq \emptyset$ we have that each pair of distinct symbols appears at distance d exactly once in rows and once in columns. Although this does not have a finite analogue, all known constructions for Vatican squares of finite order n have $n/2$ rows that together form a "row semi-Vatican rectangle" and the remaining $n/2$ rows are the reverse of these ones.

All of the results for Vatican squares transfer to the semi-Vatican case with little modification. In addition to this, looking at the semi-Vatican case allows for an explicit construction of one in the case $I = \mathbb{R}$ using only the tools of undergraduate Calculus.

Moving to orthogonality, two finite Latin squares on a symbol set X are orthogonal if for each pair $(x_1, x_2) \in X \times X$ there is exactly one position such that x_1 is in that position in the first square and x_2 is in that position in the second pair. This definition carries over without modification to the infinite case (the countable version of which is given in [9, p. 116]).

In Section 5 we see that the methods from Section 2 may be quickly adapted to produce sets of κ mutually orthogonal Latin squares of order κ for all infinite orders κ via Cayley tables of abelian squareful groups. This is analgous to the finite construction of "orthomorphisms" via "R-sequencings". `[And we maybe do even more stuff with` `orthomorphisms]`

2 Vatican squares from groups

Let I be an index set in an ordered field $\mathbb{F}$. Let G be a group of order $|I|$. For a bijection $\mathbf{a} : I \to G$ define a function $\mathbf{a}_{(d)} : I_{(d)} \to G \setminus \{e\}$ for each $d \in \mathbb{F}^+$ with $I_{(d)} \neq \emptyset$ by

$$\mathbf{a}_{(d)}(i) = \mathbf{a}(i)^{-1}\mathbf{a}(i+d).$$

Such a function is called a T_d-sequencing for $\mathbf{a}$, if it is a bijection. If there is a D such that for all $d < D$ with $I_{(d)} \neq \emptyset$ we have that each $\mathbf{a}_{(d)}$ is a bijection, then $\mathbf{a}$ is a *directed T_D-terrace for G*. If $\mathbf{a}_{(d)}$ is a bijection for all d with $I_{(d)} \neq \emptyset$ then $\mathbf{a}$ is a *directed T_∞-terrace* for G.

These definitions closely mimic the versions for finite groups [1]. They can be used to produce Latin squares with neighbor balance properties in much the same way. Theorem 2.1 generalizes Gordon's result [12] for finite complete squares and Anderson's [1] and Etzion, Golomb and Taylor's results [10] for finite Vatican squares to the infinite.

For any bijection $\mathbf{a} : I \to G$ define a square $L(\mathbf{a}) = (\ell_{ij})$ by $\ell_{ij} = a(i)^{-1}a(j)$. As $\mathbf{a}$ is a bijection, each row and column contains each symbol exactly once and so L is a Latin square. Call a Latin square created in this way *based on G*, or simply *group-based*.

Theorem 2.1. *Let G be a group of infinite order κ. If G has a directed T_D-terrace for an index set I then there is a D-complete Latin square of order $|G|$ on I. Further, if G has a directed T_∞-terrace then there is a Vatican square of order $|G|$.*

Proof. Let $\mathbf{a}$ be a directed T_D-terrace for G on I and consider $L(\mathbf{a})$.

Take x and y to be distinct elements of G. As $\mathbf{a}_{(d)}$ is a bijection, there is a unique j with $\mathbf{a}(j)^{-1}\mathbf{a}(j+d) = x^{-1}y$ and a unqiue i with $\mathbf{a}(i)^{-1}\mathbf{a}(j) = x$. We therefore have that x appears in row i and column j of $L(\mathbf{a})$ and that y appears in row i and column $j+d$ of $L(\mathbf{a})$ and that x and y do not appear anywhere else with y exactly distance d to the right of x.

There is also a unique i with $xy^{-1} = \mathbf{a}(i)^{-1}\mathbf{a}(i+d)$ and then a unique j with $\mathbf{a}(i)^{-1}\mathbf{a}(j) = x$. This identifies a unique place where y appears at exactly distance d above x in the square. Therefore $L(\mathbf{a})$ is a D-complete square on I.

If we replace $\mathbf{a}$ with a directed T_∞-terrace in the above argument we see that $L(\mathbf{a})$ is a Vatican square on I. $\square$

The 1-complete case with $I \in \{\mathbb{N}, \mathbb{Z}\}$ of Theorem 2.1 is equivalent to results of Caulfield [5].

We wish to know which infinite groups have directed T_D- and T_∞-terraces.

We use transfinite induction to build such terraces, and later to build the squares more generally. The construction will proceed by building better and better approximations to the object in transfinitely many steps, using each step to satisfy a requirement. Our approximations will be coherent and, at the end of the construction, we will end up with an object that satisfies all the requirements for the object we are trying to build.

We will organize these constructions as follows. In each case we will consider the partially ordered set $\mathbb{P} = \langle \mathbb{P}, \leq \rangle$, consisting of the approximations under consideration.

We call elements of $\mathbb{P}$ conditions, and they are ordered so that $p \leq q$ if p is a better approximation than q (we say that p is a stronger condition than q, or that it extends q).[1] Our inductive construction will thus amount to building a descending chain of conditions, at each step meeting a requirement. A major part of our proofs will be showing that, given a requirement, any condition can be extended to meet it. This is usually phrased in terms of the set of conditions which satisfy the requirement being dense (in the sense of the order topology, i.e., for a set to be dense, any condition has an extensions in this set). The key is is to judiciously pick out the dense sets in the poset which will enable the object we are trying to build to clearly satisfy our desired properties.

Theorem 2.2. *Let G be an abelian squareful group. Then G has a directed T_∞-terrace.*

Proof. Let I be an index set in an ordered field $\mathbb{F}$, where $|I| = \kappa$ is the order of G. We build a T_∞-terrace for G by transfinite induction on κ. Consider the poset $\mathbb{P}$ consisting of partial directed T_∞-terraces on G. These partial terraces are as in the definition of a directed T_∞-terrace, except the functions $\mathbf{a}$ and its T_d-sequencings $\mathbf{a}_{(d)}$ for each $d \in \mathbb{F}^+$ with $I_{(d)} \neq \emptyset$ are only required to be injective partial functions from I to G with domains of cardinality less than κ. Here $\mathbb{P}$ should be partially ordered so that $\mathbf{a} \leq \mathbf{b}$ if and only if $\mathbf{a}$ extends $\mathbf{b}$ as a function, i.e. $\operatorname{dom} \mathbf{b} \subseteq \operatorname{dom} \mathbf{a}$ and $\mathbf{a} \restriction \operatorname{dom} \mathbf{b} = \mathbf{b}$.

Below we identify the requirements the approximations have to meet and establish that the set of conditions satisfying each of them is dense.

1. For each $i \in I$, the set $D_i = \{\mathbf{d} \in \mathbb{P} \mid i \in \operatorname{dom} \mathbf{d}\}$ is dense.

   To see this, let $\mathbf{a} \in \mathbb{P}$ with domain A and $i \in I \setminus A$. We need to find $\mathbf{d} \in D_i$ satisfying $\mathbf{d} \leq \mathbf{a}$. In order to find such a $\mathbf{d}$, first we must ensure that the value we assign to i is not equal to anything in the range of $\mathbf{a}$, namely $\mathbf{d}(i) \neq \mathbf{a}(a)$ for each $a \in A$. Since the range of $\mathbf{a}$ is smaller than κ, this forbids fewer than κ many possible values for $\mathbf{d}(i)$.

   Secondly, we must ensure the T_d-sequencings for $\mathbf{d}$ are injections. This amounts to ensuring that for each d and each $a \in A_{(d)}$,

   $$\mathbf{a}(a)^{-1}\mathbf{a}(a+d) \neq \mathbf{d}(i)^{-1}\mathbf{a}(i+d)$$

   and/or

   $$\mathbf{a}(a)^{-1}\mathbf{a}(a+d) \neq \mathbf{a}(i-d)^{-1}\mathbf{d}(i)$$

   if $i-d$ and/or $i+d$ happen to be in A. Since there are strictly fewer than κ many elements in the range of $\mathbf{a}$, this leaves fewer than κ many elements of G to avoid assigning $\mathbf{d}(i)$. In the case where $i-d$ and $i+d$ are both in A we need to also make sure that $\mathbf{d}(i)^{-1}\mathbf{a}(i+d) \neq \mathbf{a}(i-d)^{-1}\mathbf{d}(i)$. As G is abelian, this is the same as making sure that

   $$(\mathbf{d}(i))^2 \neq \mathbf{a}(i+d)\mathbf{a}(i-d).$$

   Again, since the range of $\mathbf{a}$ is small, there are fewer than κ many forbidden values for $\mathbf{d}(i)^2$ and, since G is squareful, this gives fewer than κ new forbidden values for $\mathbf{d}(i)$.

[1] Both the term condition and the direction of the order $\leq$, while seemingly ad hoc, are well established in set-theoretic practice.

This means that altogether the set of values to rule out for $\mathbf{d}(i)$ has size less than κ, and we can just pick an element of G that has not been forbidden and assign it to $\mathbf{d}(i)$. Then $\mathbf{d}$ is a partial T_∞-terrace with i in its domain.

2. For each $g \in G$, the set $D_g = \{\mathbf{d} \in \mathbb{P} \mid g \in \operatorname{range} \mathbf{d}\}$ is dense.

   Again, as in the above case, the idea should be that we only have to avoid fewer than κ many cases, but we have room in I for that.

   Let $\mathbf{a}$ be a condition with $g \in G \setminus \operatorname{range} \mathbf{a}$. We need to find $\mathbf{d} \in D_g$ satisfying $\mathbf{d} \leq \mathbf{a}$. This amounts to finding $\overline{g}$ so that we can let $\mathbf{d}(\overline{g}) = g$. This $\overline{g}$ of course cannot be in $A = \operatorname{dom} \mathbf{a}$, but also $\overline{g} \pm d$ should not be in A for any d such that $A_{(d)}$ is nonempty, and $\overline{g}$ should not equal $\frac{a+a'}{2}$ for any pair $a, a' \in A$. This again forbids only fewer than κ many values for $\overline{g}$, so we can make a suitable choice.

3. For each $g \in G$ and each $d \in \mathbb{F}^+$ with $I_{(d)} \neq \emptyset$, the set $D_g^d = \left\{\mathbf{d} \in \mathbb{P} \mid g \in \operatorname{range} \mathbf{d}_{(d)}\right\}$ is dense.

   To see this, fix $d \in \mathbb{F}^+$ such that $I_{(d)} \neq \emptyset$ and let $g \in G$. Let $\mathbf{a} \in \mathbb{P}$, and suppose that $g \notin \operatorname{range} \mathbf{a}_{(d)}$. We want to see that it is possible to extend $\mathbf{a}$ to a condition $\mathbf{d} \in D_g^d$ such that $g = \mathbf{d}(\overline{g})^{-1}\mathbf{d}(\overline{g} + d)$ for some $\overline{g} \in I$. This amounts to finding a suitable $\overline{g}$. First we need $\overline{g}$ to be so that $\overline{g} \notin A_{(d)}$ where $A = \operatorname{dom} \mathbf{a}$. Then we need to ensure that $\mathbf{d}(\overline{g}), \mathbf{d}(\overline{g}+d) \notin \operatorname{range} \mathbf{a}$, and also obviously that $g = \mathbf{d}(\overline{g})^{-1}\mathbf{d}(\overline{g}+d)$. Also make sure that neither $\overline{g}$ nor $\overline{g} + d$ fall halfway between elements of A.

   It must also be the case that for any $a \in A$, we have that

   $$\mathbf{a}(a)^{-1}\mathbf{d}(\overline{g}) \notin \operatorname{range} \mathbf{a}_{(\overline{g}-a)}, \quad \mathbf{d}(\overline{g})^{-1}\mathbf{a}(a) \notin \operatorname{range} \mathbf{a}_{(a-\overline{g})},$$

   $$\mathbf{a}(a)^{-1}\mathbf{d}(\overline{g} + d) \notin \operatorname{range} \mathbf{a}_{(\overline{g}+d-a)}, \quad \mathbf{d}(\overline{g} + d)^{-1}\mathbf{a}(a) \notin \operatorname{range} \mathbf{a}_{(a-\overline{g}-d)}.$$

   Since we have only eliminated less than κ many options, as we are restricted by A and its image under $\mathbf{a}$, we have plenty of room to choose a $\overline{g}$ as desired.

   We should note that we will not inadvertently disrupt another sequencing. In particular, if we have that both $\overline{g} + d', \overline{g} + d + d' \in A$ for some $d' \in I^+$ and

   $$\mathbf{d}(\overline{g})^{-1}\mathbf{a}(\overline{g} + d') = \mathbf{d}(\overline{g} + d)^{-1}\mathbf{a}(\overline{g} + d + d'),$$

   then, as G is abelian, we must satisfy

   $$g = \mathbf{d}(\overline{g})^{-1}\mathbf{d}(\overline{g} + d) = \mathbf{a}(\overline{g} + d')^{-1}\mathbf{a}(\overline{g} + d + d').$$

   But this contradicts the requirement that $g \notin \operatorname{range} \mathbf{a}_{(d)}$. Dually, whenever there is some $d' \in I^+$ such that $\overline{g} - d', \overline{g} - d - d' \in A$, and

   $$\mathbf{a}(\overline{g} - d')^{-1}\mathbf{d}(\overline{g}) = \mathbf{a}(\overline{g} + d - d')^{-1}\mathbf{d}(\overline{g} + d),$$

   and this again contradicts $g \notin \operatorname{range} \mathbf{a}_{(d)}$.

Let

$$\mathcal{D} = \{D_i \mid i \in I\} \cup \{D_g \mid g \in G\} \cup \left\{D_g^d \mid d \in \mathbb{F}^+ \text{ with } I_{(d)} \neq \emptyset, g \in G\right\},$$

and note that $|\mathcal{D}| = \kappa$, so we may enumerate all of the dense sets as $\mathcal{D} = \langle \mathcal{D}_\alpha \mid \alpha < \kappa \rangle$.

We will define a descending sequence of conditions $\langle \mathbf{b}_\alpha \mid \alpha < \kappa \rangle$ by transfinite induction, ensuring that $\mathbf{b}_\alpha \in D_\alpha$ and $|b_\alpha| < |\alpha + \omega|$, so that the sequence isn't growing at too fast a rate, at each step. (We saw it is possible to meet the dense sets without growing the conditions too fast while showing that each set in $\mathcal{D}$ is dense.) Assume that we have built an initial segment of this sequence $\langle \mathbf{b}_\alpha \mid \alpha < \lambda \rangle$, and we wish to construct the condition $\mathbf{b}_\lambda$ in the next step. First notice that $\mathbf{b}'_\lambda = \bigcup_{\alpha<\lambda} \mathbf{b}_\alpha$ is itself a condition. This is because the only requirements are that $\mathbf{b}'_\lambda$ and $(\mathbf{b}'_\lambda)_{(d)}$ be injective functions, and this will be true if it was true for every earlier $\mathbf{b}_\alpha$. Furthermore, the domain of $\mathbf{b}'_\lambda$ is just the union of the domains of the earlier $\mathbf{b}_\alpha$ which were smaller than κ, so it itself is smaller than κ.

We can now let $\mathbf{b}_\lambda$ be any extension of $\mathbf{b}'_\lambda$ in $\mathcal{D}_\lambda$; such an extension exists since we showed that $\mathcal{D}_\lambda$ is dense.

By construction, $\bigcup_{\alpha<\lambda} \mathbf{b}_\alpha$ defines a T_∞-terrace $\mathbf{b} : I \longrightarrow G$ as desired:

1. $\mathbf{b}$ *is a bijection:* This is ensured by meeting, for each $i \in I$, the dense sets D_i for injectivity and for meeting D_g for each $g \in G$ for surjectivity.

2. *For each $d \in I^+$, $\mathbf{b}_{(d)}$ is a bijection:* The fact that the sequencing is injective is ensured by item 1 as well, since at some point we will add both $i \in I$ and $i + d$ to the domain of the partial terrace we are constructing. The dense sets D_g^d for each $g \in G$ guarantee surjectivity. □

Corollary 2.3. *For every index set I there is a Vatican square on I. In particular, there is a Vatican square of every infinite order.*

Proof. For every infinite order κ there is an abelian squareful group G of order κ. Use G in Theorems 2.1 and 2.2 to produce the required Vatican square. □

The question of which infinite groups admit directed T_D- and T_∞-terraces arises. Vanden Eynden shows that all countably infinite groups have a directed 1-terrace on $\mathbb{N}$ [18] and the proof is easily adapted to apply to index set $\mathbb{Z}$.

The method of Theorem 2.2 can be applied to an additional family of groups:

Proposition 2.4. *If every non-identity element of an infinite abelian group G is an involution then G has a directed T_∞-terrace for any index set of size $|G|$.*

Proof. □

3 Squares not based on groups

The results of the previous section raise the question about what is and is not possible for infinite squares more generally. Perhaps *all* countably infinite squares may be made complete, or even Vatican, with a suitable permutation of their rows and columns? In a similar vein, it is known that all infinite Steiner triple systems are resolvable [8], an uncommon property among finite systems. However, Theorem 3.1 scuppers this possibility, showing that for every index set (and hence every infinite order) there is a square that cannot be made row-complete via permuting columns.

In the other direction, a question asked (and answered positively) about finite squares was whether there exist row-complete Latin squares that are not based on groups, see [7, 9, 17]. We answer the infinite version of this question, also positively, in Theorem 3.2. Indeed, this result gives a Vatican square that is not group-based for every index set. All known finite Vatican squares are based on groups.

Theorem 3.1. *For every index set I there is a Latin square on I that cannot be made row-complete by permuting columns.*

Proof. We build a Latin square on $\kappa = |I|$ by approximations to it, by transfinite induction on κ. At the end this will give us one on I via any bijection between I and κ, with symbol set κ as well.

Consider the poset $\mathbb{P}$ consisting of injective partial functions from $\kappa \times \kappa$ to κ whose domains are initial segments of $\kappa \times \kappa$ of size strictly less than κ. We think of these as populating a κ-by-κ grid with ordinals less than κ– each condition fills in a lower left block of the full grid. These rectangles must satisfy that each ordinal appears at most once in each row/column (they are Latin). Moreover we require that the Latin rectangles in $\mathbb{P}$ are *immune*, meaning that they cannot be made row-complete by permuting its columns. Again, partially order the immune Latin rectangles by $l \leq m$ if l extends m as functions.

We will start our transfinite induction with the following immune Latin rectangle, a condition which we name l_{-1}:

$$\begin{matrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{matrix}$$

We again list the requirements that the approximations will have to meet and show that densely many conditions satisfy these requirements.

1. For each $\alpha, \beta < \kappa$, the set $D_{\alpha,\beta} = \{l \in \mathbb{P} \mid \alpha \text{ appears in row } \beta \text{ of } l\}$ is dense.

   In other words, we need to see how to add a desired number α to row β of a condition l, if it isn't already there. We can simply add rows to l, filling them arbitrarily without violating the Latin constraints, until we get to the β-th row. Then we just place α into the β-th row, again without violating the Latin constraint (this might require us to add a new column as well) and fill in until we have filled in a whole block of the grid (note that at any given stage only fewer than κ many ordinals appear in the grid, so we can always fill in these blank spaces without violating the Latin constraint).

   We have produced a larger Latin rectangle l' which contains l and has α in the β-th row but may not be immune. We now perform the immunization procedure to extend l' to an immune Latin rectangle in $D_{\alpha,\beta}$.

   Immunization: For each combination of three different columns from the columns in our new rectangle l', pick the least number γ larger than the largest number used in l', and add the following three columns on top of the three different columns:

$$\begin{array}{ccc} \gamma+2 & \gamma & \gamma+1 \\ \gamma+1 & \gamma+2 & \gamma \\ \gamma & \gamma+1 & \gamma+2 \end{array}$$

in whatever order you would like. Keep doing this with every possible combination of three different columns until all of the possible combinations of 3 different columns have been exhausted. Then fill in the gaps with whatever you would like without contravening the Latin constraints.

To illustrate the technique, let's say that we would like to add the number 3 to row 0 in our initial immune Latin rectangle l_{-1}. Following the described algorithm, we first produce the following rectangle l'_{-1}

$$\begin{array}{cccc} 2 & 0 & 1 & 5 \\ 1 & 2 & 0 & 4 \\ 0 & 1 & 2 & 3 \end{array}.$$

We added 3 to the first row of a new column, and then filled up the column with the minimum values possible while making sure it is Latin. In this case the immunization isn't required, but we'll go ahead and do it to illustrate the process, and ensure it. In this example, there are 4 possible combinations of 3 columns available, so we end up adding 12 new rows to the top of our condition.

$$\begin{array}{cccc} & 17 & 15 & 16 \\ & 16 & 17 & 15 \\ & 15 & 16 & 17 \\ 14 & & 12 & 13 \\ 13 & & 14 & 12 \\ 12 & & 13 & 14 \\ 11 & 9 & & 10 \\ 10 & 11 & & 9 \\ 9 & 10 & & 11 \\ 8 & 6 & 7 & \\ 7 & 8 & 6 & \\ 6 & 7 & 8 & \\ 2 & 0 & 1 & 5 \\ 1 & 2 & 0 & 4 \\ 0 & 1 & 2 & 3 \end{array}$$

Then we can fill in the above with whatever values we would like, so long as the rectangle l''_{-1} we end up with is Latin.

2. For each $\alpha, \beta < \kappa$, the set $D^{\alpha,\beta} = \{l \in \mathbb{P} \mid \alpha \text{ appears in column } \beta \text{ of } l\}$ is dense.

   Here we need to add a specified number α to the column number β of a condition l, if it isn't already there. The strategy is the same as before: we just add columns to l until we get to the β-th column, possibly adding a new row in order to place α

in column β as required. Ensure along the way here that the rectangle l' produced is still Latin (don't add any numbers that are already in that row or column). Then run through the immunization procedure as outlined above, to produce a new condition extending l' that is in $D^{\alpha,\beta}$.

We enumerate the listed dense sets into a sequence $\{D_\alpha \mid \alpha < \kappa\}$ and build a descending chain of conditions $\langle l_\alpha \mid \alpha < \kappa \rangle$ by transfinite induction, ensuring that $l_\alpha \in D_\alpha$ at each step and that $|l_\alpha| < |\alpha + \omega|$ so that the sequence isn't growing at too fast a rate. This goes the same way as in the proof of theorem 2.2. Given an initial segment of this sequence $\langle l_\alpha \mid \alpha < \lambda \rangle$, notice that $l'_\lambda = \bigcup_{\alpha<\lambda} l_\alpha$ is a condition. We then let l_λ be any extension of l'_λ in D_λ.

We have produced $L = \cup \{l_\alpha \mid \alpha < \kappa\}$, which defines a Latin square with the desired properties. Indeed, by meeting the dense sets we have ensured that the square is total and that every column and row contain all ordinals below κ. Moreover, if L failed to be Latin, this failure would show up in some approximation l_α, but these are all Latin, so L must be as well. Finally, it isn't possible to permute columns in L to obtain a row-complete square, since then it would be possible to permute the columns in some large enough l_α, but each l_α is immune so that can't happen. $\square$

Prior to giving Theorem 3.2 we need a result that lets us be sure that a square is not group-based. The *quadrangle criterion* states that in a square based on a group if the three equations

$$a_{i_1 j_1} = a_{i_2 j_2},\ a_{k_1 j_1} = a_{k_2 j_2},\ a_{i_1 l_1} = a_{i_2 l_2}$$

are satisfied then $a_{k_1 l_1} = a_{k_2 l_2}$ [9, Theorem 1.2.1]. That is, if two "quadrangles" in a group-based square agree on three points then they agree on the fourth.

Theorem 3.2. *For every index set I there is a Vatican square on I that is not based on a group.*

Proof. Let $\mathbb{F}_\kappa$ be an ordered field on $\kappa = |I|$. We build a Vatican square L on $\mathbb{F}_\kappa$ with symbol set κ by transfinite induction. The strategy to build one on an arbitrary index set is exactly the same as outlined below, potentially starting with a different starting sub-square we call p_{-1} depending on your index set and symbol set.

Again we find it useful to define an appropriate poset, to build the Vatican square by growing a finite one via meeting dense sets. The conditions in our poset $\mathbb{P}$ consist of partial Vatican squares which are not group based. These are injective partial functions of the form $p : \mathbb{F}_\kappa \times \mathbb{F}_\kappa \longrightarrow \kappa$ whose domains restrict on any coordinate to initial segments of $\mathbb{F}_\kappa \times \mathbb{F}_\kappa$ of size strictly less than κ. In particular, for each $\alpha < \kappa$, we have that $p \restriction (\{\alpha\} \times \kappa)$ and $p \restriction (\kappa \times \{\alpha\})$ form an initial segment of κ on its relevant coordinate. Again we think of these populating $\kappa \times \kappa$ grid from the bottom left corner outward, although distances between columns and rows are computed by the ordered field $\mathbb{F}_\kappa$. Moreover, these squares should be Vatican in the sense of a finite square, in that for each $d \in \mathbb{F}_\kappa^+$ such that $(\mathbb{F}_\kappa)_{(d)} \neq \emptyset$ we have that each ordered pair of distinct symbols coming from κ appear at distance d at most once in each order in rows and at most once in each order in columns. We partially order $\mathbb{P}$ by letting $p \leq q$ so long as p extends q as functions.

We start the induction with the following partial Vatican square on κ, call it p_{-1}:

$$\begin{array}{cccc} 0 & 5 & 6 & 1 \\ 7 & 0 & 1 & 8 \\ 9 & 2 & 3 & 10 \\ 2 & 11 & 12 & 4 \end{array}$$

No square containing this as a subsquare can be group-based since it fails to satisfy the quadrangle criterion. We then begin meeting dense sets in some enumeration of the following four families of dense sets. We build a descending chain of conditions p_α, starting with p_{-1}, in κ many stages exactly as before. At each stage we use the fact that the union of a descending chain of fewer than κ many conditions is itself a condition, making sure that the sequence doesn't grow at too fast a rate, and then extend into the dense set under consideration at that stage.

1. For each $\alpha, \beta < \kappa$, the set $D_{\alpha,\beta} = \{p \in \mathbb{P} \mid \alpha \text{ appears in row } \beta \text{ of } p\}$ is dense.

   The trick here is not to mess up the Vatican property on the square. If α does not appear in row β of a condition p yet, it is not necessarily enough to simply add it to the end of row β, since it is possible that then α appears more than once at some distance from another element in that row or column. We are only guaranteed to be safe with that method if α doesn't already appear anywhere in the partial square. Thus the idea is to fill up the necessary entries of the square with new symbols that haven't appeared anywhere in the partial square yet. We can keep adding such entries until we are safe to add α in row β.

   If p doesn't even have a row β yet, then add new rows which have the same length as the longest row in p, with symbols not already in p. On the βth row, do the same thing, but add α at the end, creating a new column with just α in it.

   Otherwise, p already has entries in row β, this can again at worst amount to essentially doubling p in size, by adding enough new elements (at worst the length of its longest row plus β in the field addition of $\mathbb{F}_\kappa$) to row β so that no column or row distance between an element of row β of p and α could even have occurred in p initially.

   For example, if $\mathbb{F}_\kappa$ has the usual ordering on the natural numbers, in the above square p_{-1}, this is how we would add 0 to the bottom row:

$$\begin{array}{cccccccc} 0 & 5 & 6 & 1 \\ 7 & 0 & 1 & 8 \\ 9 & 2 & 3 & 10 \\ 2 & 11 & 12 & 4 & 13 & 14 & 15 & 0 \end{array}$$

   and if we would like to add 0 to the sixth row:

$$\begin{array}{ccccc} 17 & 18 & 19 & 20 & 0 \\ 13 & 14 & 15 & 16 \\ 0 & 5 & 6 & 1 \\ 7 & 0 & 1 & 8 \\ 9 & 2 & 3 & 10 \\ 2 & 11 & 12 & 4 \end{array}$$

2. For each $\alpha, \beta < \kappa$, the set $D^{\alpha,\beta} = \{p \in \mathbb{P} \mid \alpha \text{ appears in column } \beta \text{ of } p\}$ is dense.

   Same procedure as with rows.

3. For each $\alpha, \beta < \kappa$ and $d \in \mathbb{F}_\kappa^+$, the set

$$E^d_{\alpha,\beta} = \{p \in \mathbb{P} \mid \alpha,\ \beta \text{ appear at distance } d \text{ apart in some row}\}$$

   is dense.

   If α and β already do not appear distance d apart anywhere in a partial Vatican square p, what we should do to absolutely guarantee we have no conflicts is start a new row. Add new entries to this row with symbols different than all of the symbols appearing in p so far, up to the length of the longest row. Then add α, followed by enough symbols so as to be able to add β at distance d. This indeed will still be a condition, since β and d are less than κ.

   For example, with our starting square p_{-1}, this is how we would extend it to have 0 and 1 be distance 2 apart (if the field $\mathbb{F}_\kappa$ has the same ordering on finite numbers that the natural numbers do):

13	14	15	16	0	17	1
0	5	6	1			
7	0	1	8			
9	2	3	10			
2	11	12	4			

4. For each $\alpha, \beta < \kappa$ and $d \in \mathbb{F}_\kappa^+$, the set

$$E_d^{\alpha,\beta,\delta} = \{p \in \mathbb{P} \mid \alpha,\ \beta \text{ appear at distance } d \text{ apart in some column}\}$$

   is dense.

   Same procedure as with rows.

We produce a square L at the end of this construction by taking the union of all of the p_α's in the chain we described building above. Clearly L is Latin, since at some stage every ordinal less than κ was added to every row and every column as guaranteed by our first two families of dense sets. It must be that L is Vatican as well. First of all, we know that the pair occurrence for each row and column must be satisfied at least once in each row and column by meeting the last two families of dense sets described above. Moreover if this happened somewhere more than once, it would have to happen in some condition p_α, but conditions in $\mathbb{P}$ are not allowed to have this property. And L is not group-based since it includes the square p_{-1}. $\square$

4 Semi-Vatican squares

Infinite Semi-Vatican squares—recall that these are squares in which each pair of symbols appears exactly once at each distance d in rows and columns, rather than exactly once in each order—behave very similarly to Vatican squares.

The required generalization of directed T_D- and T_∞-terraces are directed S_D- and S_∞-terraces. As before, let I be an index set in an ordered field $\mathbb{F}$. Let G be a group of order $|I|$. For a bijection $\mathbf{a} : I \to G$ define a function $\mathbf{a}_{(d)} : I \to G \setminus \{e\}$ for each $d \in \mathbb{F}^+$ with $I_{(d)} \neq \emptyset$ by

$$\mathbf{a}_{(d)}(i) = \mathbf{a}(i)^{-1}\mathbf{a}(i+d).$$

If G has no involutions, and if there is a D such that for all $d < D$ with $I_{(d)} \neq \emptyset$ we have that the image of $\mathbf{a}_{(d)}$ contains exactly one occurrence from each set $\{x, x^{-1} : x \in G\backslash\{e\}\}$, then $\mathbf{a}$ is a *directed S_D-terrace for G*. If $\mathbf{a}_{(d)}$ has this property for all d with $I_{(d)} \neq \emptyset$ then $\mathbf{a}$ is a *directed S_∞-terrace* for G.

The requirement that G has no involutions comes into play when we consider constructing semi-Vatican squares using the method of Theorem 2.1. Suppose $z \in G$ is an involution and $\mathbf{a}_{(d)} = z$ for some bijection $\mathbf{a}$ with the usual definition for $\mathbf{a}_{(d)}$. If $\mathbf{a}(i) = x$, then $\mathbf{a}(i+d) = xz$. There is a j such that $\mathbf{a}(j) = xz$ and now $\mathbf{a}(i+d) = xz^2 = x$. Thus the pair $\{x, xz\}$ occurs twice at distance d in $L(\mathbf{a})$, once in each order. Hence a square constructed with this method using a group with an involution cannot be semi-Vatican.

The proofs of the previous two sections require only minor modifications to give the following slate of results:

Theorem 4.1. *Let G be a group of infinite order κ with no involutions. If G has a directed S_∞-terrace for an index set I then there is a semi-Vatican square of order $|G|$.*

Theorem 4.2. *Let G be an involution-free abelian squareful group of infinite order κ. Then G has a directed S_∞-terrace.*

Corollary 4.3. *For every index set I there is a semi-Vatican square on I. In particular, there is a semi-Vatican square of every infinite order.*

Theorem 4.4. *For every index set I there is a semi-Vatican square on I that is not based on a group.*

All of the existence results presented so far are non-constructive and rely on transfinite induction. Perhaps surprisingly, in the case when the group is $(\mathbb{R}, +)$, the tools of undergraduate calculus are sufficient to construct to a semi-Vatican square.

Theorem 4.5. *There is a semi-Vatican square on index set $\mathbb{R}$ based on $(\mathbb{R}, +)$.*

Proof. We give a direct definition for a directed S_∞-terrace $\mathbf{a}$:

$$\mathbf{a}(x) = \begin{cases} e^x - 1 & x \geq 0 \\ -\ln(1-x) & \text{otherwise} \end{cases}$$

This is a continuous, strictly increasing bijection from $\mathbb{R}$ to $\mathbb{R}$. Its derivative is:

$$\mathbf{a}'(x) = \begin{cases} e^x & x \geq 0 \\ \frac{1}{1-x} & \text{otherwise} \end{cases}$$

which is a continuous, strictly increasing bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$.

Therefore, for each $d \in \mathbb{R}^+$, we have that $\mathbf{a}_{(\mathbf{d})}$ is a bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$. Hence $\mathbf{a}$ is a directed S_∞-terrace and Theorem 4.1 gives a semi-Vatican square based on $I = \mathbb{R}$. $\square$

Similar approaches for Vatican squares quickly run into difficulties.

5 Orthogonality

Let G be a group and $\theta : G \to G$ a bijection. If $g \mapsto g^{-1}\theta(g)$ is a bijection then θ is an *orthomorphism*; if $g \mapsto g\theta(g)$ is a bijection then θ is a *complete mapping*. Two orthomorphisms, θ, ϕ are *orthogonal* if $g \mapsto \theta(g)^{-1}\phi(g)$ is a bijection.

`[orthomorphisms` $\to$ `orthogonal latin squares.]`

It's known that every infinite group has an orthomorphism [2], so there is a pair of orthogonal Latin squares at every infinite order.

The work of Section 2 may be adapted to give families of mutually orthogonal orthomorphisms. `[`R_∞ `stuff.]`

Theorem 5.1. *Let G be a group of infinite order κ. If G has a directed R_∞-terrace then G has a set of κ mutually orthogonal orthomorphisms.*

Proof. Let $\mathbf{a}$ be a directed T_∞-terrace for G over some index set I. For each d such that $I_d \neq \emptyset$ (of which there are κ) define $\theta_d(e) = e$ and $\theta_d(\mathbf{a}(i)) = \mathbf{a}(i+d)$. This gives us the orthogonal orthomorphisms we're looking for:

First, they are orthomorphisms: given $g \in G \setminus \{e\}$ with $\mathbf{a}(i) = g$ we get

$$g^{-1}\theta_d(g) = \mathbf{a}(i)^{-1}\mathbf{a}(i+d) = \mathbf{a}_{(d)}(i)$$

which, when we also consider that $\theta_d(e) = e$, gives us a bijection on G.

Second, they are orthogonal: again taking $g \in G \setminus \{e\}$ with $\mathbf{a}(i) = g$, if $d_2 > d_1$ we get:

$$\theta_{d_1}^{-1}(g)\theta_{d_2}(g) = \mathbf{a}(i+d_1)^{-1}\mathbf{a}(i+d_2) = \mathbf{a}_{(d_2-d_1)}(i+d_1).$$

If $d_1 < d_2$ we get:

$$\theta_{d_1}^{-1}(g)\theta_{d_2}(g) = \mathbf{a}(i+d_1)^{-1}\mathbf{a}(i+d_2) = \mathbf{a}_{(d_1-d_2)}(i+d_2)^{-1}.$$

Also $\theta_{d_1}(e)^{-1}\theta_{d_2}(e) = e$ in each case, giving bijections on G. □

If we have a directed R_D-terrace, with the obvious definition, then the same argument gives $|\{d \leq D : I_d \neq \emptyset\}|$ mutually orthogonal orthomorphisms for G.

Theorem 5.2. *Let G be an abelian squareful group of infinite order. Then G has a directed R_∞-terrace.*

Proof. The only difference between a directed R_∞-terrace and a directed T_∞-terrace is that the identity is not in the domain of a directed R_∞-terrace. The presence of the identity is not relied upon in the Theorem 2.2's proof that abelian squareful groups of infinite order have a directed T_∞ terrace; a simple adjustment of the argument produces the required directed R_∞-terrace. □

This immediately gives:

Corollary 5.3. *There is a set of κ mututally orthogonal squares of order κ for all infinite cardinalities κ.*

`[Prove stronger result about orthomorphisms that does not go via directed` R_∞`-terraces?]`

Another embellishment of the complete mapping concept is the strong complete mapping: If θ is both an orthomorphism and a complete mapping then it is a *strong complete mapping*. Every countably infinite group has a strong complete mapping [11].

`[Prove that all infinite groups have strong complete mappings?]`

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