A Terrace for $\mathbb{R}$

Definition. Let G be a group of order $2^{\aleph_0}$ that has no involutions and identity element e. For a bijection $\mathbf{a} : \mathbb{R} \longrightarrow G$ define a function $\mathbf{a}_{(d)} : \mathbb{R} \longrightarrow G \setminus \{e\}$ for each $d \in \mathbb{R}^+$ by

$$\mathbf{a}_{(d)}(i) = \mathbf{a}(i)^{-1}\mathbf{a}(i + d).$$

If each $\mathbf{a}_{(d)}$ is a bijection then $\mathbf{a}$ is a *directed T_∞-terrace for G*.

If instead we have that $A \subseteq \mathbb{R}$ is countable, and $\mathbf{a} : A \longrightarrow G$ and $\mathbf{a}_{(d)} : A_{(d)} \longrightarrow G$ are injections for each $d \in \mathbb{R}^+$, where $A_{(d)} = \{a \in A \mid a + d \in A\}$, we say that $\mathbf{a}$ a *countable partial directed T_∞-terrace on G* and we call each $\mathbf{a}_{(d)}$ the *partial $T_{(d)}$-sequencing corresponding to* $\mathbf{a}$.

Theorem 1. *Assume* CH. *The group* $(\mathbb{R}, +)$ *has a directed T_∞-terrace* $\mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}$.

Proof. Consider the poset $\mathbb{P}$ consisting of conditions which are countable partial directed T_∞-terraces on $\mathbb{R}$ partially ordered so that $\mathbf{a} \leq \mathbf{b}$ (following convention in set theory, we say $\mathbf{a}$ is *stronger* than $\mathbf{b}$) if and only if $\operatorname{dom}\mathbf{b} \subseteq \operatorname{dom}\mathbf{b}$ and $\mathbf{a} \restriction \operatorname{dom}\mathbf{b} = \mathbf{b}$.

It is not hard to see that $\mathbb{P}$ is countably closed. Suppose we have an decreasing chain of countable partial directed T_∞-terraces, $\mathbf{a}_n$ for $n \in \mathbb{N}$, on $\mathbb{R}$. Then the union of all of them, $\mathbf{a}$, is a countable partial directed T_∞-terrace on $\mathbb{R}$. Indeed, $\mathbf{a}$ is a bijection since each $\mathbf{a}_n$ in the chain is. For each $d \in \mathbb{R}^+$, we have that $\mathbf{a}_{(d)}$ is injective since $\operatorname{dom}\mathbf{a}_{(d)} \subseteq \operatorname{dom}\mathbf{a}$. Moreover $\mathbf{a}_{(d)}$ is a surjection since if $\mathbf{a}_{(d)}(i) = \mathbf{a}_{(d)}(j)$ then it must be that for some $n, m \in \mathbb{N}$, say $n \leq m$, we have that $\mathbf{a}_n(i + d) - \mathbf{a}_n(i) = \mathbf{a}_m(j + d) - \mathbf{a}_m(j)$, but this would imply that $i = j$ since then $\mathbf{a}_m \leq \mathbf{a}_n$ and $\mathbf{a}_m$ is surjective.

Need to establish:

1. *It is dense to add a real number r to the domain of a condition in $\mathbb{P}$:* i.e., for each $r \in \mathbb{R}$, the set $D_r = \{\mathbf{d} \in \mathbb{P} \mid r \in \operatorname{dom}\mathbf{d}\}$ is dense.

   To see this, let $\mathbf{a} \in \mathbb{P}$ with domain A. Choose $r \in \mathbb{R} \setminus A$. We need to find $\mathbf{d} \in D_r$ satisfying $\mathbf{d} \leq \mathbf{a}$. In order to find such a $\mathbf{d}$, first we must ensure that $\mathbf{d}(r) \neq \mathbf{a}(i)$ for each $i \in A$.

   Secondly, we must ensure the T_d-sequencings for $\mathbf{d}$ are bijections. This amounts to ensuring that for each pair $i, i + d \in A$,

$$\begin{aligned} \mathbf{a}(i + d) - \mathbf{a}(i) &\neq \mathbf{a}(r + d) - \mathbf{d}(r) \\ &\neq \mathbf{d}(r) - \mathbf{a}(r - d) \end{aligned}$$

   If $r - d$ and/or $r + d$ happen to be in A.

   As A and the ranges of $\mathbf{a}$ and $\mathbf{a}_{(d)}$ are countable, the set of values to rule out for $\mathbf{d}(r)$ is at most countable, and we just need to make sure it's not one of those values. As $\mathbb{R}$ is uncountable, this can be done.

2. *It is dense to add a real number r to the range of a condition in $\mathbb{P}$:* i.e., for each $r \in \mathbb{R}$, the set $E_r = \{\mathbf{e} \in \mathbb{P} \mid r \in \operatorname{range}\mathbf{e}\}$ is dense.

   Again the idea should be that we only have to avoid countably many scenarios, but we have room in $\mathbb{R}$ for that. Suppose $r \in \mathbb{R} \setminus \operatorname{range}\mathbf{a}$. We need to find $\mathbf{e} \in E_r$

satisfying $\mathbf{e} \leq \mathbf{a}$. This amounts to finding $\overline{r} \notin A = \operatorname{dom} \mathbf{a}$ so that we can let $\mathbf{d}(\overline{r}) = r$, satisfying $\overline{r} \notin A_{(d)}$ for whenever $A_{(d)}$ is nonempty.

Both A and $A_{(d)} \subseteq A$ are countable, so this can be done.

3. *For each $d \in \mathbb{R}^+$ it is dense to add a real number r to the domain of a condition's partial $T_{(d)}$-sequencing:* This is captured by 1., since we may add both r and $d+r$ to the domain of a condition.

4. *For each $d \in \mathbb{R}^+$ it is dense to add a real number r to the range of a condition's partial $T_{(d)}$-sequencing:* In other words, we would like to show that for each $r \in \mathbb{R}$ and each $d \in \mathbb{R}^+$, the set $F_r^d = \left\{\mathbf{f} \in \mathbb{P} \mid r \in \operatorname{range} \mathbf{f}_{(d)}\right\}$ is dense in $\mathbb{P}$.

   To see this, fix $d \in \mathbb{R}^+$ and let $r \in \mathbb{R}$. Let $\mathbf{a} \in \mathbb{P}$, and suppose that $r \notin \operatorname{range} \mathbf{a}_{(d)}$. We want to see that it is possible to extend $\mathbf{a}$ to a condition $\mathbf{f} \in F_r^d$ such that $r = \mathbf{f}(\overline{r}+d) - \mathbf{f}(\overline{r})$ for some $\overline{r} \in \mathbb{R}$. This amounts to finding a suitable $\overline{r}$. First we need $\overline{r}$ to be so that neither $\overline{r}$ nor $\overline{r}+d$ are in $A = \operatorname{dom} \mathbf{a}$. Then we need to ensure that $\mathbf{f}(\overline{r}), \mathbf{f}(\overline{r}+d) \notin \operatorname{range} \mathbf{a}$, and also that $r = \mathbf{f}(\overline{r}+d) - \mathbf{f}(\overline{r})$.

   It must also be the case that for any $i \in A$, we have that

$$\mathbf{f}(\overline{r}) - \mathbf{a}(i) \notin \operatorname{range} \mathbf{a}_{(\overline{r}-i)}, \quad \mathbf{a}(i) - \mathbf{f}(r) \notin \operatorname{range} \mathbf{a}_{(i-\overline{r})},$$

$$\mathbf{f}(\overline{r}+d) - \mathbf{a}(i) \notin \operatorname{range} \mathbf{a}_{(\overline{r}+d-i)}, \quad \mathbf{a}(i) - \mathbf{f}(\overline{r}+d) \notin \operatorname{range} \mathbf{a}_{(i-\overline{r}-d)}.$$

   Moreover, we can't inadvertently mess up another sequencing. In particular, whenever we have that $\overline{r}+l, \overline{r}+d+l \in \operatorname{dom} A$, we must have that

$$\mathbf{a}(\overline{r}+l) - \mathbf{f}(\overline{r}) \neq \mathbf{a}(\overline{r}+d+l) - \mathbf{f}(\overline{r}+d),$$

   meaning that

$$r = \mathbf{f}(\overline{r}+d) - \mathbf{f}(\overline{r}) \neq \mathbf{a}(\overline{r}+d+l) - \mathbf{a}(\overline{r}+l).$$

   This contradicts $r \notin \operatorname{range} \mathbf{a}_{(d)}$. Dually, we need that whenever $\overline{r}-l, \overline{r}-d-l \in \operatorname{dom} A$, we must have that

$$\mathbf{f}(\overline{r}) - \mathbf{a}(\overline{r}-l) \neq \mathbf{f}(\overline{r}+d) - \mathbf{a}(\overline{r}+d-l),$$

   which again contradicts $r \notin \operatorname{range} \mathbf{a}_{(d)}$.

   Since we have only eliminated countably many options, as we are restricted by A and its image under $\mathbf{a}$, we have plenty of room to choose such an $\overline{r}$ as desired.

Now that we have verified these collections sets are dense, we may find a filter $\mathcal{G} \subseteq \mathbb{P}$ which meets the family of dense sets

$$\mathcal{D} = \{D_r \mid r \in \mathbb{R}\} \cup \{E_r \mid r \in \mathbb{R}\} \cup \left\{F_r^d \mid d \in \mathbb{R}^+, r \in \mathbb{R}\right\}$$

because $|\mathcal{D}| = \aleph_1$ as CH holds, and since the forcing axiom for countably closed forcing is true.

To see why such a filter can be built, simply construct a sequence of $\mathbf{g}_\alpha$'s by transfinite induction, enabling the filter to be defined by

$$\mathcal{G} = \{\mathbf{f} \in \mathbb{P} \mid \mathbf{f} \geq \mathbf{g}_\alpha \text{ for some } \alpha < \omega_1\}.$$

The idea is to start meeting each of the dense sets in $\mathcal{D}$ one-by-one, ensuring that the filter is closed downward. Enumerate the dense sets as $\mathcal{D} = \langle \mathcal{D}_\alpha \mid \alpha < \omega_1 \rangle$. Let $\mathbf{g}_0 \in \mathcal{D}_0$. Then at stage $n \leq \omega$, let $\mathbf{g}_n \leq \mathbf{g}_{n-1}$ satisfy $\mathbf{g}_n \in \mathcal{D}_n$. Density allows us to continue the construction through all successor stages. At limit stages, say $\lambda < \omega_1$, we use the fact that $\mathbb{P}$ is countably closed to find a condition strengthening the chain of our constructed $\mathbf{g}_\alpha$s for $\alpha < \gamma$, and then strengthen this condition to obtain $\mathbf{g}_\lambda \in \mathcal{D}_\lambda$.

By construction, $\cup G$ defines a function $\mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}$ with the desired properties.

1. $\mathbf{g}$ *is a bijection:* This is ensured by meeting, for each $r \in \mathbb{R}$, the dense sets D_r for injectivity and for meeting E_r for each $r \in \mathbb{R}$ for surjectivity.

2. *For each $d \in \mathbb{R}^+$, $\mathbf{g}_{(d)}$ is a bijection:* This is ensured by item 3. above and the dense sets F_r^d for each $r \in \mathbb{R}$. □

Corollary 1. *If* CH *holds, there is a Vatican square on $\mathbb{R}$.*

Corollary 2. *There is a real-preserving forcing which adds a Vatican square on $\mathbb{R}$.*

Question 1. Let G be an abelian group of size continuum with continuum-many non-involutions. Does G have a directed T_∞-terrace?

The answer is yes. See Matt's note.

Theorem 2. *Let G be an abelian group of size $\aleph_1$ with $\aleph_1$-many non-involutions. Then G has a directed T_∞-terrace.*

Question 2. Can we force to add a terrace of size $\aleph_{n+1}$ given that there is a partial terrace of size $\aleph_n$?

Poset would consist of partial terraces, as usual, and should be $< \aleph_{n+1}$-closed, so that no new elements are added to the group of size $\aleph_{n+1}$.