

Let $A \subseteq \mathbb{R}$ be countable and let $\mathbf{a} : A \rightarrow \mathbb{R}$ be an injection. For each $d \in \mathbb{R}^+$ let $A_{(d)} = \{a \in A \mid a + d \in A\}$. Define functions $\mathbf{a}_{(d)} : A_{(d)} \rightarrow \mathbb{R} \setminus \{0\}$ by

$$\mathbf{a}_{(d)}(a) = \mathbf{a}(a + d) - \mathbf{a}(a).$$

If each $\mathbf{a}_{(d)}$ is injective then call \mathbf{a} a *countable partial directed T_∞ -terrace on \mathbb{R}* . We call each $\mathbf{a}_{(d)}$ the *partial $T_{(d)}$ -sequencing corresponding to \mathbf{a}* .

Let G be a group of order 2^{\aleph_0} that has no involutions and identity element e . For a bijection $\mathbf{a} : \mathbb{R} \rightarrow G$ define a function $\mathbf{a}_{(d)} : \mathbb{R} \rightarrow G \setminus \{e\}$ for each $d \in \mathbb{R}^+$ by

$$\mathbf{a}_{(d)}(i) = \mathbf{a}(i)^{-1} \mathbf{a}(i + d).$$

If each $\mathbf{a}_{(d)}$ is a bijection then \mathbf{a} is a *directed T_∞ -terrace for G* .

Theorem 1. *Assume CH.¹ The group $(\mathbb{R}, +)$ has a directed T_∞ -terrace $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}$.*

Proof. Consider the poset \mathbb{P} consisting of conditions which are countable partial directed T_∞ -terraces on \mathbb{R} partially ordered so that $\mathbf{a} \leq \mathbf{c}$ (\mathbf{a} is stronger than \mathbf{c}) if and only if $\text{dom } \mathbf{a} \subseteq \text{dom } \mathbf{c}$ and $\mathbf{c} \upharpoonright \text{dom } \mathbf{a} = \mathbf{a}$.

Need to establish:

1. *\mathbb{P} is countably closed:* Suppose we have an decreasing chain of countable partial directed T_∞ -terraces on \mathbb{R} . Then the union of all of them is a countable partial directed T_∞ -terrace on \mathbb{R} .
2. *It is dense to add a real number r to the domain of a condition in \mathbb{P} :* i.e., for each $r \in \mathbb{R}$, the set $D_r = \{\mathbf{d} \in \mathbb{P} \mid r \in \text{dom } \mathbf{d}\}$ is dense. To see this, let $\mathbf{a} \in \mathbb{P}$ with domain A . Choose $r \in \mathbb{R} \setminus A$. We need to find $\mathbf{d} \in D_r$ satisfying $\mathbf{d} \leq \mathbf{a}$. But in order to find such a \mathbf{d} , first we must ensure that $\mathbf{d}(r) \neq \mathbf{a}(a)$ for each $a \in A$. Secondly, for each pair $a, a + d \in A$, we must ensure that $\mathbf{a}(a + d) - \mathbf{a}(a) \neq \mathbf{d}(r + d) - \mathbf{d}(r)$ if $r + d \in A$. As A and the ranges of \mathbf{a} and $\mathbf{a}_{(d)}$ are countable, the set of values to rule out for $\mathbf{d}(r)$ is at most countable, and we just need to make sure it's not one of those values. As \mathbb{R} is uncountable, this can be done.
3. *It is dense to add a real number r to the range of a condition in \mathbb{P} :* i.e., for each $r \in \mathbb{R}$, the set $E_r = \{\mathbf{e} \in \mathbb{P} \mid r \in \text{range } \mathbf{e}\}$ is dense. Again the idea should be that we only have to avoid countably many scenarios, but we have room in \mathbb{R} for that. Choose $r \in \mathbb{R} \setminus \text{range } \mathbf{a}$. We need to find $\mathbf{e} \in E_r$ satisfying $\mathbf{e} \leq \mathbf{a}$. This amounts to finding $\bar{r} \notin A = \text{dom } \mathbf{a}$ so that we can let $\mathbf{d}(\bar{r}) = r$, subject to the further restriction that if some element of A happens to have the form $\bar{r} + d$ for some $d \in \mathbb{R}^+$, then $\mathbf{e}(\bar{r} + d) - \mathbf{e}(\bar{r}) \neq \mathbf{a}(a + d) - \mathbf{a}(a)$ for all $a \in A$ with $a + d \in A$. All of our searches here involve checking against what is already in A or the range of \mathbf{a} , both of which are countable, and as \mathbb{R} is uncountable we are able to find such values.
4. *For each $d \in \mathbb{R}^+$ it is dense to add a real number r to the domain of a condition's partial $T_{(d)}$ -sequencing:* This is captured by 2., since we may add both r and $d + r$ to the domain of a condition.
5. *For each $d \in \mathbb{R}^+$ it is dense to add a real number r to the range of a condition's partial $T_{(d)}$ -sequencing:* In other words, we would like to show that for each $r \in \mathbb{R}$ and each $d \in \mathbb{R}^+$, the set $F_r^d = \{\mathbf{f} \in \mathbb{P} \mid r \in \text{range } \mathbf{f}_{(d)}\}$ is dense in \mathbb{P} . To see this, fix $d \in \mathbb{R}^+$ and let $r \in \mathbb{R}$. Let $\mathbf{a} \in \mathbb{P}$, and suppose that $r \notin \text{range } \mathbf{a}_{(d)}$. We want to see that it is possible to extend \mathbf{a} to a condition $\mathbf{f} \in F_r^d$ such that $r = \mathbf{f}(\bar{r} + d) - \mathbf{f}(\bar{r})$ for some $\bar{r} \in \mathbb{R}$. This amounts to finding a

¹Even without assuming CH in the ground model, the forcing poset \mathbb{P} in the construction will force CH and the existence of such a terrace in the forcing extension.

suitable \bar{r} , and indeed it is enough to choose \bar{r} so that neither \bar{r} or $\bar{r}+d$ are in $A = \text{dom } \mathbf{a}$. Of course then we need to ensure that $\mathbf{f}(\bar{r}), \mathbf{f}(\bar{r}+d) \notin \text{range } \mathbf{a}$, and also that $\mathbf{f}(\bar{r}) + \mathbf{f}(\bar{r}+d) = r$. Since we have only eliminated countably many options, as we are restricted by A and its image under \mathbf{a} , we have plenty of room to choose such an \bar{r} as desired.

We may find a filter $G \subseteq \mathbb{P}$ which meets the family of dense sets

$$\mathcal{D} = \{D_r \mid r \in \mathbb{R}\} \cup \{E_r \mid r \in \mathbb{R}\} \cup \{F_r^d \mid d \in \mathbb{R}^+, r \in \mathbb{R}\}$$

because $|\mathcal{D}| = \aleph_1$ as CH holds, and since the forcing axiom for countably closed forcing is true. To see this, we may construct a sequence of \mathbf{g}_α 's enabling the generic filter to be defined by

$$G = \{\mathbf{f} \in \mathbb{P} \mid \mathbf{f} \geq \mathbf{g}_\alpha \text{ for some } \alpha < \omega_1\}$$

by transfinite induction. The idea is to start meeting each of the dense sets in \mathcal{D} one-by-one, ensuring that the filter is closed downward. Enumerate the dense sets as $\mathcal{D} = \langle \mathcal{D}_\alpha \mid \alpha < \omega_1 \rangle$. Let $\mathbf{g}_0 \in \mathcal{D}_0$. Then at stage $n \leq \omega$, let $\mathbf{g}_n \leq \mathbf{g}_{n-1}$ satisfy $\mathbf{g}_n \in \mathcal{D}_n$. Density allows us to continue the construction through all successor stages. At limit stages, say $\lambda < \omega_1$, we use the fact that \mathbb{P} is countably closed to find a condition strengthening the chain of our constructed \mathbf{g}_α s for $\alpha < \gamma$, and then strengthen this condition to obtain $\mathbf{g}_\lambda \in \mathcal{D}_\lambda$.

By construction, $\cup G$ defines a function $\mathbf{g} : \mathbb{R} \rightarrow \mathbb{R}$ with the desired properties.

1. *\mathbf{g} is a bijection:* This is ensured by meeting, for each $r \in \mathbb{R}$, the dense sets D_r for injectivity and E_r for surjectivity.
2. *For each $d \in \mathbb{R}^+$, $\mathbf{g}_{(d)}$ is a bijection:* This is ensured by item 4. above and the dense sets F_r^d .

□

Question 2. Is it possible to have a directed T_∞ -terrace on \mathbb{R} and $\neg\text{CH}$?

Ramblings: From the footnote, just using the above forcing won't answer this question. After forcing with \mathbb{P} , it is possible to add a bunch of reals and make CH fail, but then we have reals not accounted for in the terrace, so it's not a full terrace on \mathbb{R} anymore. So then the question is, if you force, say, $2^{\aleph_0} = \aleph_2$, is there a way to build up the terrace we had on ω_1 (when this had the same size as \mathbb{R}) to ω_2 ? Without forcing CH to hold again? In this case the indexing set for the old terrace, which is now a partial terrace, would be a subset of \mathbb{R} , presumably. And looks like $\langle \mathbb{R}, + \rangle$ I guess?