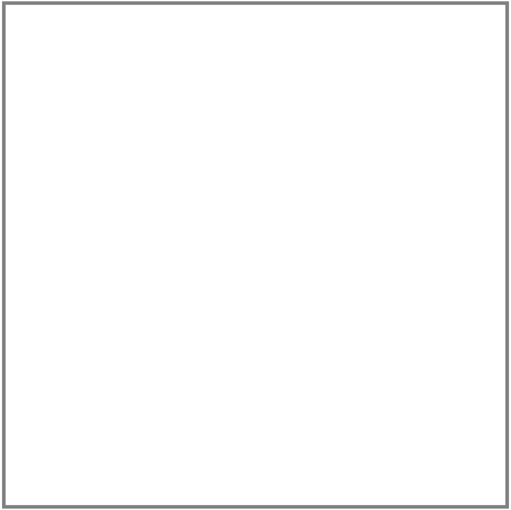
Infinite Latin Squares

Kaethe Minden
(based on joint work with Anthony Evans, Gage Martin, and Matt Ollis)

Bard College at Simon's Rock
The Early College

March 3, 2020

A square



0	1
2	3

A Latin square

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

A Roman square

3	2	0	1
0	3	1	2
1	0	2	3
2	1	3	0

3	2	0	1
0	3	1	2
1	0	2	3
2	1	3	0

A Vatican square

0	3	1	2
1	0	2	3
3	2	0	1
2	1	3	0

0	3	1	2
1	0	2	3
3	2	0	1
2	1	3	0

A Vatican square?

0	1	2	3

A Vatican square?

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

A Vatican square? No!

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

How can we tell if a square is group-based?

0	5	6	1
7	0	1	8
9	2	3	10
2	11	12	4

The infinite case

Definition

Say that an infinite group G is *squareful* if the set $\{g^2 : g \in G\}$ has the same cardinality as G.

- If G is an abelian squareful group and I is an index set with $|I| = |G|$, then we can construct an infinite Vatican square on I using the Cayley table of G.
- We show that there is a Vatican square of each infinite order that cannot be produced by permuting the rows and columns of a Cayley table. Whether a finite Vatican square with this property exists is an open question.
- We also show that there is a Latin square of each infinite order such that no permutation of its rows and columns gives a Vatican (or even Roman) square.

Infinite Sudoku?

Definition

If a partial latin square P is a subset of a latin square L then P is *completable to L*. If there is exactly one such L, then P is *uniquely completable*. If P is uniquely completable but no proper subset of P is uniquely completable then P is a *critical set*.

Lots of work has been done on uniquely completable partial Latin squares and critical sets in the finite case. The goal here is to ask the same (or similar) questions in the infinite case and see if any fun math happens.

Question

Is this a critical set for the integer square?

				4	5	6	7	8
				3	4	5	6	7
				2	3	4	5	6
				1	2	3	4	5
-4	-3	-2	-1	$\cdot$				
-5	-4	-3	-2					
-6	-5	-4	-3					
-7	-6	-5	-4					
-8	-7	-6	-5					