Infinite Latin Squares

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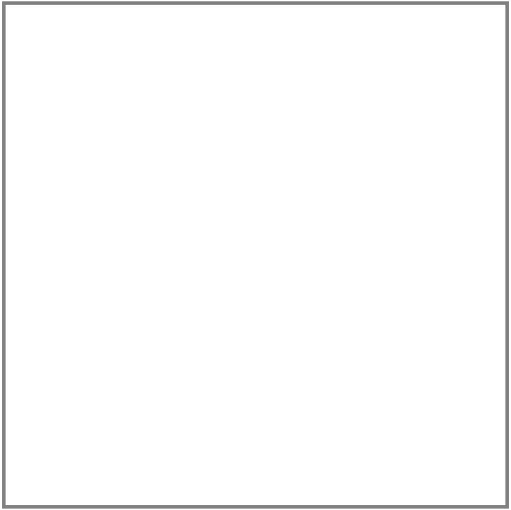
Bard College at Simon's Rock
The Early College

2 October 2020

A square



A square

0	1
2	3

A Latin square

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

A Latin square

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

A Roman square

3	2	0	1
0	3	1	2
1	0	2	3
2	1	3	0

A Roman square

3	2	0	1
0	3	1	2
1	0	2	3
2	1	3	0

A Vatican square

0	3	1	2
1	0	2	3
3	2	0	1
2	1	3	0

A Vatican square

0	3	1	2
1	0	2	3
3	2	0	1
2	1	3	0

A Vatican square?

0	1	2	3

A Vatican square?

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

A Vatican square? No!

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

An infinite Latin square

$$
\begin{array}{ccccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\
\cdots & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
\cdots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots \\
\cdots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\
\cdots & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & \cdots \\
\cdots & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & \cdots \\
\cdots & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Group-based Vatican squares

Vatican squares are known to exist for all orders $p - 1$, where p is prime (using terraces and Cayley tables).

Definition

Say that an infinite group G is *squareful* if the set $\{g^2 : g \in G\}$ has the same cardinality as G.

Result

If G is an infinite abelian squareful group, then we can construct an infinite Vatican square (using a "Cayley table").

How can we tell if a square is group-based?

0	5	6	1
7	0	1	8
9	2	3	10
2	11	12	4

Latin squares not based on groups

Results

- There is a Vatican square of each infinite order that cannot be produced by permuting the rows and columns of a Cayley table.
  Whether a finite Vatican square with this property exists is an open question!
- There is a Latin square of each infinite order such that no permutation of its rows and columns gives a Vatican (or even Roman) square.
- ...

Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Infinite Sudoku?

Definition

If a partial latin square P is a subset of a latin square L then P is *completable to* L. If there is exactly one such L, then P is *uniquely completable*. If P is uniquely completable but no proper subset of P is uniquely completable then P is a *critical set*.

Lots of work has been done on uniquely completable partial Latin squares and critical sets in the finite case. The goal here is to ask the same (or similar) questions in the infinite case and see if any fun math happens.

Infinite Sudoku? Maybe?

Question

Is this a critical set for the integer square?

$$
\begin{array}{cccccccccccc}
 & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & & & & & 4 & 5 & 6 & 7 & 8 & \cdots \\
 & & & & & 3 & 4 & 5 & 6 & 7 & \cdots \\
 & & & & & 2 & 3 & 4 & 5 & 6 & \cdots \\
 & & & & & 1 & 2 & 3 & 4 & 5 & \cdots \\
\cdots & -4 & -3 & -2 & -1 & \cdot & & & & & \\
\cdots & -5 & -4 & -3 & -2 & & & & & & \\
\cdots & -6 & -5 & -4 & -3 & & & & & & \\
\cdots & -7 & -6 & -5 & -4 & & & & & & \\
\cdots & -8 & -7 & -6 & -5 & & & & & & \\
 & \vdots & \vdots & \vdots & \vdots & & & & & &
\end{array}
$$

Thank you!