Definition 1. Let $A \subseteq \mathbb{R}$ be countable and let $\mathbf{a} : A \longrightarrow \mathbb{R}$ be an injection. For each $d \in \mathbb{R}^+$ let $A_{(d)} = \{a \in A \mid a + d \in A\}$. Define functions $\mathbf{a}_{(d)} : A_{(d)} \longrightarrow \mathbb{R} \setminus \{0\}$ by

$$\mathbf{a}_{(d)}(a) = \mathbf{a}(a + d) - \mathbf{a}(a).$$

If each $\mathbf{a}_{(d)}$ is injective then call $\mathbf{a}$ a ***countable partial directed*** T_∞***-terrace on*** $\mathbb{R}$. We call each $\mathbf{a}_{(d)}$ the ***partial*** $T_{(d)}$***-sequencing corresponding to*** $\mathbf{a}$.

Let G be a group of order $\aleph_0$ or $2^{\aleph_0}$ that has no involutions and identity element e. For a bijection $\mathbf{a} : I \longrightarrow G$ define a function $\mathbf{a}_{(d)} : I \longrightarrow G \setminus \{e\}$ for each $d \in I^+$ by

$$\mathbf{a}_{(d)}(i) = \mathbf{a}(i)^{-1}\mathbf{a}(i + d).$$

If each $\mathbf{a}_{(d)}$ is a bijection then $\mathbf{a}$ is a ***directed*** T_∞***-terrace for*** G.

Theorem 2. *Assume* CH*. The group* $(\mathbb{R}, +)$ *has a directed* T_∞*-terrace* $\mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}$*.*

Proof. Consider the poset $\mathbb{P}$ of countable partial directed T_∞-terrace on $\mathbb{R}$ partially ordered so that $\mathbf{a} \leq \mathbf{c}$ if and only if $\operatorname{dom} \mathbf{a} \subseteq \operatorname{dom} \mathbf{c}$ and $\mathbf{c} \restriction \operatorname{dom} \mathbf{a} = \mathbf{a}$.

Need to establish:

1. $\mathbb{P}$ *is countably closed:* Suppose we have an decreasing chain of countable partial directed T_∞-terraces on $\mathbb{R}$. Then the union of all of them is a countable partial directed T_∞-terrace on $\mathbb{R}$.
2. *It is dense to add a real number* r *to the domain of a condition in* $\mathbb{P}$*:* i.e., for each $r \in \mathbb{R}$, the set $D_r = \{\mathbf{d} \in \mathbb{P} \mid r \in \operatorname{dom} \mathbf{d}\}$ is dense. To see this, let $\mathbf{a} \in \mathbb{P}$ with domain A. Choose $r \in \mathbb{R} \setminus A$. We need to find $\mathbf{d} \in D_r$ satisfying $\mathbf{d} \leq \mathbf{a}$. But in order to find such a $\mathbf{d}$, we just have two types of restrictions on what values of $\mathbf{d}(r)$ may take. We must ensure that $\mathbf{d}(r) \neq \mathbf{a}(a)$ for each $a \in A$, and $\mathbf{d}(r)$ must be chosen so as to not cause any repeats in the differences at distance $d = |a - r|$. This set of restrictions rules out finite or countably many values, and A is countable. So the set of values that $\mathbf{d}(r)$ cannot be is at most countable, but $\mathbb{R}$ is uncountable and so we are able to choose an allowable value.
3. *It is dense to add a real number* r *to the range of a condition in* $\mathbb{P}$*:* i.e., for each $r \in \mathbb{R}$, the set $E_r = \{\mathbf{e} \in \mathbb{P} \mid r \in \operatorname{range} \mathbf{e}\}$ is dense. Again the idea should be that we only have to avoid countably many scenarios, but we have room in $\mathbb{R}$ for that. Choose $r \in \mathbb{R} \setminus \operatorname{range} \mathbf{a}$. We need to find $\mathbf{e} \in E_r$ satisfying $\mathbf{e} \leq \mathbf{a}$. This amounts to finding $\overline{r} \notin A = \operatorname{dom} \mathbf{a}$ so that we can let $\mathbf{d}(\overline{r}) = r$, subject to the further restriction that if some element of A happens to have the form $\overline{r} + d$ for some $d \in \mathbb{R}^+$, then $\mathbf{e}(\overline{r} + d) - \mathbf{e}(\overline{r}) \neq \mathbf{a}(a + d) - \mathbf{a}(a)$ for all $a \in A$ with $a + d \in A$. All of our searches here involve checking against what is already in A or the range of $\mathbf{a}$, both of which are countable, and as $\mathbb{R}$ is uncountable we are able to find such values.
4. *For each* $d \in \mathbb{R}^+$ *it is dense to add a real number* r *to the domain of a condition's partial* $T_{(d)}$*-sequencing:* This is captured by D_r, since for a condition $\mathbf{a}$, the domain of $\mathbf{a}_{(d)}$ is contained in A.
5. *For each* $d \in \mathbb{R}^+$ *it is dense to add a real number* r *to the range of a condition's partial* $T_{(d)}$*-sequencing:* In other words, we would like to show that for each $r \in \mathbb{R}$ and each $d \in \mathbb{R}^+$, the set $F_r^d = \left\{\mathbf{f} \in \mathbb{P} \mid r \in \operatorname{range} \mathbf{f}_{(d)}\right\}$ is dense in $\mathbb{P}$. To see this, fix $d \in \mathbb{R}^+$ and let $r \in \mathbb{R}$. Let $\mathbf{a} \in \mathbb{P}$, and suppose that $r \notin \operatorname{range} \mathbf{a}_{(d)}$. We want to see that it is possible to extend $\mathbf{a}$ to a condition $\mathbf{f} \in F_r^d$ such that $r = \mathbf{f}(\overline{r} + d) - \mathbf{f}(\overline{r})$ for some $\overline{r} \in \mathbb{R}$. This amounts to finding a suitable $\overline{r}$, and indeed it is enough to choose so that neither $\overline{r}$ or $\overline{r} + d$ are in $A = \operatorname{dom} \mathbf{a}$. Of course then we need to ensure that $\mathbf{f}(\overline{r}), \mathbf{f}(\overline{r} + d) \notin \operatorname{range} \mathbf{a}$, and also that $\mathbf{f}(\overline{r}) + \mathbf{f}(\overline{r} + d) = r$.

Since we have only eliminated countably many options, as we are restricted by A and its image under $\mathbf{a}$, we have plenty of room to choose such an $\bar{r}$ as desired.

We may find a filter $G \subseteq \mathbb{P}$ which meets the family of dense sets

$$\mathcal{D} = \{D_r \mid r \in \mathbb{R}\} \cup \{E_r \mid r \in \mathbb{R}\} \cup \left\{F_r^d \mid d \in \mathbb{R}^+, r \in \mathbb{R}\right\}$$

because $|\mathcal{D}| = \aleph_1$, and since the forcing axiom for countably closed forcing is true. To see this, we may construct a sequence of $\mathbf{g}_\alpha$'s enabling the generic filter to be defined by

$$G = \{\mathbf{f} \in \mathbb{P} \mid \mathbf{f} \geq \mathbf{g}_\alpha \text{ for some } \alpha < \omega_1\}$$

by transfinite induction. The idea is to start meeting each of the dense sets in $\mathcal{D}$ one-by-one, ensuring that the filter is closed downward. Enumerate the dense sets as $\mathcal{D} = \langle \mathcal{D}_\alpha \mid \alpha < \omega_1 \rangle$. Let $\mathbf{g}_0 \in \mathcal{D}_0$. Then at stage $n \leq \omega$, let $\mathbf{g}_n \leq \mathbf{g}_{n-1}$ satisfy $\mathbf{g}_n \in \mathcal{D}_n$. Density allows us to continue the construction through all successor stages. At limit stages, say $\lambda < \omega_1$, we use the fact that $\mathbb{P}$ is countably closed to find a condition strengthening the chain of our constructed $\mathbf{g}_\alpha$s for $\alpha < \gamma$, and then strengthen this condition to obtain $\mathbf{g}_\lambda \in D_\lambda$.

By construction, $\cup G$ defines a function $\mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}$ with the desired properties.

1. $\mathbf{g}$ *is a bijection:* This is ensured by meeting, for each $r \in \mathbb{R}$, the dense sets D_r for injectivity and E_r for surjectivity.
2. *For each* $d \in \mathbb{R}^+$*,* $\mathbf{g}_{(d)}$ *is a bijection:* This is ensured by item 4 above and the dense sets F_r^d.

□