Let $A \subseteq \mathbb{R}$ be countable and let $\mathbf{a} : A \longrightarrow \mathbb{R}$ be an injection. For each $d \in \mathbb{R}^+$ let $A_{(d)} = \{a \in A \mid a + d \in A\}$. Define functions $\mathbf{a}_{(d)} : A_{(d)} \longrightarrow \mathbb{R} \setminus \{0\}$ by

$$\mathbf{a}_{(d)}(a) = \mathbf{a}(a + d) - \mathbf{a}(a).$$

If each $\mathbf{a}_{(d)}$ is injective then call $\mathbf{a}$ a *countable partial directed T_∞-terrace on* $\mathbb{R}$. We call each $\mathbf{a}_{(d)}$ the *partial $T_{(d)}$-sequencing corresponding to* $\mathbf{a}$.

Let G be a group of order $2^{\aleph_0}$ that has no involutions and identity element e. For a bijection $\mathbf{a} : \mathbb{R} \longrightarrow G$ define a function $\mathbf{a}_{(d)} : \mathbb{R} \longrightarrow G \setminus \{e\}$ for each $d \in \mathbb{R}^+$ by

$$\mathbf{a}_{(d)}(i) = \mathbf{a}(i)^{-1}\mathbf{a}(i + d).$$

If each $\mathbf{a}_{(d)}$ is a bijection then $\mathbf{a}$ is a *directed T_∞-terrace for G.*

Theorem 1. *Assume* CH.[1] *The group $(\mathbb{R}, +)$ has a directed T_∞-terrace $\mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}$.*

Proof. Consider the poset $\mathbb{P}$ consisting of conditions which are countable partial directed T_∞-terraces on $\mathbb{R}$ partially ordered so that $\mathbf{a} \leq \mathbf{c}$ (following convention in set theory, we say $\mathbf{a}$ is *stronger* than $\mathbf{c}$) if and only if $\operatorname{dom} \mathbf{c} \subseteq \operatorname{dom} \mathbf{a}$ and $\mathbf{a} \restriction \operatorname{dom} \mathbf{c} = \mathbf{c}$.

Need to establish:

1. $\mathbb{P}$ *is countably closed:* Suppose we have an decreasing chain of countable partial directed T_∞-terraces on $\mathbb{R}$. Then the union of all of them is a countable partial directed T_∞-terrace on $\mathbb{R}$.
2. *It is dense to add a real number r to the domain of a condition in $\mathbb{P}$:* i.e., for each $r \in \mathbb{R}$, the set $D_r = \{\mathbf{d} \in \mathbb{P} \mid r \in \operatorname{dom} \mathbf{d}\}$ is dense. To see this, let $\mathbf{a} \in \mathbb{P}$ with domain A. Choose $r \in \mathbb{R} \setminus A$. We need to find $\mathbf{d} \in D_r$ satisfying $\mathbf{d} \leq \mathbf{a}$. But in order to find such a $\mathbf{d}$, first we must ensure that $\mathbf{d}(r) \neq \mathbf{a}(a)$ for each $a \in A$. Secondly, for each pair $a, a + d \in A$, we must ensure that
$$\mathbf{a}(a + d) - \mathbf{a}(a) \neq \mathbf{a}(r + d) - \mathbf{d}(r)$$
if $r + d \in A$, that
$$\mathbf{a}(a + d) - \mathbf{a}(a) \neq \mathbf{d}(r) - \mathbf{a}(r - d)$$
if $r - d \in A$, and if both $r - d, r + d \in A$ then
$$\mathbf{a}(r + d) - \mathbf{d}(r) \neq \mathbf{d}(r) - \mathbf{a}(r - d).$$
As A and the ranges of $\mathbf{a}$ and $\mathbf{a}_{(d)}$ are countable, the set of values to rule out for $\mathbf{d}(r)$ is at most countable, and we just need to make sure it's not one of those values. As $\mathbb{R}$ is uncountable, this can be done.
3. *It is dense to add a real number r to the range of a condition in $\mathbb{P}$:* i.e., for each $r \in \mathbb{R}$, the set $E_r = \{\mathbf{e} \in \mathbb{P} \mid r \in \operatorname{range} \mathbf{e}\}$ is dense. Again the idea should be that we only have to avoid countably many scenarios, but we have room in $\mathbb{R}$ for that. Choose $r \in \mathbb{R} \setminus \operatorname{range} \mathbf{a}$. We need to find $\mathbf{e} \in E_r$ satisfying $\mathbf{e} \leq \mathbf{a}$. This amounts to finding $\overline{r} \notin A = \operatorname{dom} \mathbf{a}$ so that we can let $\mathbf{d}(\overline{r}) = r$, subject to the further restriction that if some element of A happens to have the form $\overline{r} + d$ (or $\overline{r} - d$) for some $d \in \mathbb{R}^+$, then $\mathbf{e}(\overline{r} + d) - \mathbf{e}(\overline{r}) \neq \mathbf{a}(a + d) - \mathbf{a}(a)$ (or $\mathbf{e}(\overline{r} - d) - \mathbf{e}(\overline{r}) \neq \mathbf{a}(a + d) - \mathbf{a}(a)$) for all $a \in A$ with $a + d \in A$. All of our searches here involve checking against what is already in A or the range of $\mathbf{a}$, both of which are countable, and as $\mathbb{R}$ is uncountable we are able to find such values.

[1] Even without assuming CH in the ground model, the forcing poset $\mathbb{P}$ in the construction will force CH and the existence of such a terrace in the forcing extension.

4. *For each $d \in \mathbb{R}^+$ it is dense to add a real number r to the domain of a condition's partial $T_{(d)}$-sequencing:* This is captured by 2., since we may add both r and $d + r$ to the domain of a condition.
5. *For each $d \in \mathbb{R}^+$ it is dense to add a real number r to the range of a condition's partial $T_{(d)}$-sequencing:* In other words, we would like to show that for each $r \in \mathbb{R}$ and each $d \in \mathbb{R}^+$, the set $F_r^d = \left\{\mathbf{f} \in \mathbb{P} \mid r \in \operatorname{range} \mathbf{f}_{(d)}\right\}$ is dense in $\mathbb{P}$. To see this, fix $d \in \mathbb{R}^+$ and let $r \in \mathbb{R}$. Let $\mathbf{a} \in \mathbb{P}$, and suppose that $r \notin \operatorname{range} \mathbf{a}_{(d)}$. We want to see that it is possible to extend $\mathbf{a}$ to a condition $\mathbf{f} \in F_r^d$ such that $r = \mathbf{f}(\overline{r} + d) - \mathbf{f}(\overline{r})$ for some $\overline{r} \in \mathbb{R}$. This amounts to finding a suitable $\overline{r}$. First we need $\overline{r}$ to be so that neither $\overline{r}$ nor $\overline{r} + d$ are in $A = \operatorname{dom} \mathbf{a}$. Then we need to ensure that $\mathbf{f}(\overline{r}), \mathbf{f}(\overline{r} + d) \notin \operatorname{range} \mathbf{a}$, and also that $r = \mathbf{f}(\overline{r} + d) - \mathbf{f}(\overline{r})$. It must also be the case that for any $a \in A$, we have that
$$\mathbf{f}(\overline{r}) - \mathbf{a}(a) \notin \operatorname{range} \mathbf{a}_{(\overline{r}-a)},\ \mathbf{a}(a) - \mathbf{f}(r) \notin \operatorname{range} \mathbf{a}_{(a-\overline{r})},$$
$$\mathbf{f}(\overline{r} + d) - \mathbf{a}(a) \notin \operatorname{range} \mathbf{a}_{(\overline{r}+d-a)},\ \mathbf{a}(a) - \mathbf{f}(\overline{r} + d) \notin \operatorname{range} \mathbf{a}_{(a-\overline{r}-d)}.$$
Moreover, we can't inadvertently mess up another sequencing. In particular, whenever we have that $\overline{r} + l, \overline{r} + d + l \in \operatorname{dom} A$, we must have that
$$\mathbf{a}(\overline{r} + l) - \mathbf{f}(\overline{r}) \neq \mathbf{a}(\overline{r} + d + l) - \mathbf{f}(\overline{r} + d),$$
meaning that
$$g = \mathbf{f}(\overline{r} + d) - \mathbf{f}(\overline{r}) \neq \mathbf{a}(\overline{r} + d + l) - \mathbf{a}(\overline{r} + l).$$
This contradicts $g \notin \operatorname{range} \mathbf{a}_{(d)}$. Dually, we need that whenever $\overline{r} - l, \overline{r} - d - l \in \operatorname{dom} A$, we must have that
$$\mathbf{f}(\overline{r}) - \mathbf{a}(\overline{r} - l) \neq \mathbf{f}(\overline{r} + d) - \mathbf{a}(\overline{r} + d - l),$$
which again contradicts $g \notin \operatorname{range} \mathbf{a}_{(d)}$. Since we have only eliminated countably many options, as we are restricted by A and its image under $\mathbf{a}$, we have plenty of room to choose such an $\overline{r}$ as desired.

We may find a filter $\mathcal{G} \subseteq \mathbb{P}$ which meets the family of dense sets
$$\mathcal{D} = \{D_r \mid r \in \mathbb{R}\} \cup \{E_r \mid r \in \mathbb{R}\} \cup \left\{F_r^d \mid d \in \mathbb{R}^+, r \in \mathbb{R}\right\}$$
because $|\mathcal{D}| = \aleph_1$ as CH holds, and since the forcing axiom for countably closed forcing is true. To see this, we may construct a sequence of $\mathbf{g}_\alpha$'s enabling the filter to be defined by
$$\mathcal{G} = \{\mathbf{f} \in \mathbb{P} \mid \mathbf{f} \geq \mathbf{g}_\alpha \text{ for some } \alpha < \omega_1\}$$
by transfinite induction. The idea is to start meeting each of the dense sets in $\mathcal{D}$ one-by-one, ensuring that the filter is closed downward. Enumerate the dense sets as $\mathcal{D} = \langle \mathcal{D}_\alpha \mid \alpha < \omega_1 \rangle$. Let $\mathbf{g}_0 \in \mathcal{D}_0$. Then at stage $n \leq \omega$, let $\mathbf{g}_n \leq \mathbf{g}_{n-1}$ satisfy $\mathbf{g}_n \in \mathcal{D}_n$. Density allows us to continue the construction through all successor stages. At limit stages, say $\lambda < \omega_1$, we use the fact that $\mathbb{P}$ is countably closed to find a condition strengthening the chain of our constructed $\mathbf{g}_\alpha$s for $\alpha < \gamma$, and then strengthen this condition to obtain $\mathbf{g}_\lambda \in \mathcal{D}_\lambda$.

By construction, $\cup G$ defines a function $\mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}$ with the desired properties.

1. $\mathbf{g}$ *is a bijection:* This is ensured by meeting, for each $r \in \mathbb{R}$, the dense sets D_r for injectivity and for meeting E_r for each $r \in \mathbb{R}$ for surjectivity.
2. *For each $d \in \mathbb{R}^+$, $\mathbf{g}_{(d)}$ is a bijection:* This is ensured by item 4. above and the dense sets F_r^d for each $r \in \mathbb{R}$.

□

Question 2. Is it possible to have a directed T_∞-terrace on $\mathbb{R}$ and $\neg$CH?

`From the footnote, just using the above forcing won't answer this question. After forcing with` $\mathbb{P}$`, it is possible to add a bunch of reals and make CH fail, but then we have reals not accounted for in the terrace, so it's not a full terrace on` $\mathbb{R}$ `anymore. So then the question is, if you force, say,` $2^{\aleph_0} = \aleph_2$`, is there a way to build up the terrace we had on` ω_1 `(when this had the same size as` $\mathbb{R}$`) to` ω_2`? Without forcing CH to hold again? In this case the indexing set for the old terrace, which is now a partial terrace, would be a subset of` $\mathbb{R}$`, presumably. And looks like` $\langle \mathbb{R}, + \rangle$ `I guess?`

Question 3. Let G be an abelian group of size continuum with infinitely many non-involutions. Does G have a directed T_∞-terrace?

The better way to phrase this might be to go ahead and ask about groups of size $\aleph_1$. Just to make the statement as general as possible. In that case we need a generalized notion of index sets so we can define terraces. I am going to assume it makes some sense to have an index set be an ordered field (or maybe group but field seems easier).

By an index set I for G we mean an ordered field that has the same size as G, and I^+ is all of the elements of the ordered field that are greater than 0.

Let G be a group of order κ with identity element e. For a bijection $\mathbf{a} : I \longrightarrow G$ define a function $\mathbf{a}_{(d)} : I \longrightarrow G \setminus \{e\}$ for each $d \in I^+$ by

$$\mathbf{a}_{(d)}(i) = \mathbf{a}(i)^{-1}\mathbf{a}(i+d).$$

If each $\mathbf{a}_{(d)}$ is a bijection then $\mathbf{a}$ is a *directed T_∞-terrace for G.*

Let G be a group of size $\aleph_1$ and let I be an index set for G. Let $A \subseteq I$ be countable, with $\mathbf{a} : A \longrightarrow G$ an injection. For each $d \in I^+$ let $A_{(d)} = \{a \in A \mid a + d \in A\}$. Define functions $\mathbf{a}_{(d)} : A_{(d)} \longrightarrow G \setminus \{e\}$ by

$$\mathbf{a}_{(d)}(a) = \mathbf{a}(a)^{-1}\mathbf{a}(a+d).$$

If each $\mathbf{a}_{(d)}$ is injective then call $\mathbf{a}$ a *countable partial directed T_∞-terrace on* $\mathbb{R}$. We call each $\mathbf{a}_{(d)}$ the *partial $T_{(d)}$-sequencing corresponding to* $\mathbf{a}$.

Theorem 4. *Let G be an abelian group of size $\aleph_1$ with $\aleph_1$-many non-involutions. Then G has a directed T_∞-terrace.*

Proof. Let I be an index set for G. Consider the poset $\mathbb{P}$ consisting of conditions which are partial directed T_∞-terraces on G partially ordered so that $\mathbf{a} \leq \mathbf{c}$ if and only if $\operatorname{dom} \mathbf{c} \subseteq \operatorname{dom} \mathbf{a}$ and $\mathbf{a} \restriction \operatorname{dom} \mathbf{c} = \mathbf{c}$.

Need to establish:

1. $\mathbb{P}$ *is countably closed:* Suppose we have an decreasing chain of countable partial directed T_∞-terraces on G. Then the union of all of them is a countable partial directed T_∞-terrace on G.
2. *It is dense to add a value to the domain of a condition in* $\mathbb{P}$*:* i.e., for each $i \in I$, the set $D_i = \{\mathbf{d} \in \mathbb{P} \mid i \in \operatorname{dom} \mathbf{d}\}$ is dense. To see this, let $\mathbf{a} \in \mathbb{P}$ with domain A. Choose $i \in I \setminus A$. We need to find $\mathbf{d} \in D_i$ satisfying $\mathbf{d} \leq \mathbf{a}$. But in order to find such a $\mathbf{d}$, first we must ensure that $\mathbf{d}(i) \neq \mathbf{a}(a)$ for each $a \in A$. Secondly, for each pair $a, a+d \in A$, where $d \in I^+$, we must ensure that each of the following are not equal to each other:

$$\begin{aligned} \mathbf{a}(a)^{-1}\mathbf{a}(a+d) &\neq \mathbf{d}(i)^{-1}\mathbf{a}(i+d) \\ &\neq \mathbf{a}(i-d)^{-1}\mathbf{d}(i) \end{aligned}$$

if $i+d$ and/or $i-d \in A$.

(Insert more of a justification here about G being such-and-such.) As A and the ranges of $\mathbf{a}$ and $\mathbf{a}_{(d)}$ are countable, the set of values to rule out for $\mathbf{d}(i)$ is at most countable, and we just need to make sure it's not one of those values. As I is uncountable, this can be done.

3. *It is dense to add a group member the range of a condition in $\mathbb{P}$:* i.e., for each $g \in G$, the set $E_g = \{\mathbf{e} \in \mathbb{P} \mid g \in \operatorname{range} \mathbf{e}\}$ is dense. Again the idea should be that we only have to avoid countably many scenarios, but we have room in G for that. Choose $g \in G \setminus \operatorname{range} \mathbf{a}$. We need to find $\mathbf{e} \in E_g$ satisfying $\mathbf{e} \leq \mathbf{a}$. This amounts to finding $i \notin A = \operatorname{dom} \mathbf{a}$ so that we can let $\mathbf{d}(i) = g$, subject to the further restriction that if some element of A happens to have the form $i+d$ (or $i-d$) for some $d \in I^+$, then $\mathbf{e}(i)^{-1}\mathbf{a}(i+d) \neq \mathbf{a}(a)^{-1}\mathbf{a}(a+d)$ (or $\mathbf{a}(i-d)^{-1}\mathbf{e}(i) \neq \mathbf{a}(a)^{-1}\mathbf{a}(a+d)$) for all $a \in A$ with $a+d \in A$. If both $i+d$ and $i+d$ are in A for $d \in I^+$, we also must have that $\mathbf{a}(i-d)^{-1}\mathbf{e}(i) \neq \mathbf{e}(i)^{-1}\mathbf{a}(i+d)$. All of our searches here involve checking against what is already in A or the range of $\mathbf{a}$, both of which are countable, and as G is uncountable we are able to find such values.
4. *For each $d \in I^+$ it is dense to add an element $i \in I$ to the domain of a condition's partial $T_{(d)}$-sequencing:* This is captured by 2., since we may add both i and $d+i$ to the domain of a condition.
5. *For each $d \in I^+$ it is dense to add a group member to the range of a condition's partial $T_{(d)}$-sequencing:* In other words, we would like to show that for each $g \in G$ and each $d \in I^+$, the set $F_g^d = \left\{\mathbf{f} \in \mathbb{P} \mid g \in \operatorname{range} \mathbf{f}_{(d)}\right\}$ is dense in $\mathbb{P}$.

To see this, fix $d \in I^+$ and let $g \in G$. Let $\mathbf{a} \in \mathbb{P}$, and suppose that $g \notin \operatorname{range} \mathbf{a}_{(d)}$. We want to see that it is possible to extend $\mathbf{a}$ to a condition $\mathbf{f} \in F_g^d$ such that $g = \mathbf{f}(i)^{-1}\mathbf{f}(i+d)$ for some $i \in I$. This amounts to finding a suitable i, so first of all we choose i so that neither i, $i+d$, nor $i-d$ are in $A = \operatorname{dom} \mathbf{a}$. Of course then we need to ensure that $\mathbf{f}(i), \mathbf{f}(i+d) \notin \operatorname{range} \mathbf{a}$, and $g = \mathbf{f}(i)^{-1}\mathbf{f}(i+d)$.

We need to have that for each pair $a, a+l \in A$, then:

$$\mathbf{f}(i)^{-1}\mathbf{a}(i+l) \neq \mathbf{a}(a)^{-1}\mathbf{a}(a+l) \neq \mathbf{a}(l-i)^{-1}\mathbf{f}(i)$$

whenever $i+l$ or $l-i$ happen to be in A, and

$$\mathbf{f}(i+d)^{-1}\mathbf{a}(i+d+l) \neq \mathbf{a}(a)^{-1}\mathbf{a}(a+l) \neq \mathbf{a}(l-i-d)^{-1}\mathbf{f}(i+d)$$

if $l+i+d$ and/or $l-i-d$ happen to be in A.

Moreover, we can't inadvertently mess up another sequencing. In particular, whenever we have that $i+l, i+d+l \in \operatorname{dom} A$, we must have that

$$\mathbf{f}(\overline{r})^{-1}\mathbf{a}(i+l) \neq \mathbf{f}(i+d)^{-1}\mathbf{a}(i+d+l),$$

meaning that

$$g = \mathbf{f}(i+d)^{-1}\mathbf{f}(i) \neq \mathbf{a}(i+l)^{-1}\mathbf{a}(i+d+l).$$

Since G is abelian, this contradicts $g \notin \operatorname{range} \mathbf{a}_{(d)}$. Dually, we need that whenever $i-l, i-d-l \in \operatorname{dom} A$, we must have that

$$\mathbf{a}(i-l)^{-1}\mathbf{f}(i) \neq \mathbf{a}(i+d-l)^{-1}\mathbf{f}(i+d),$$

which again contradicts $g \notin \operatorname{range} \mathbf{a}_{(d)}$.

Since we have only eliminated countably many options, as we are restricted by A and its image under $\mathbf{a}$, we have plenty of room in I to choose such an i as desired.

We may find a filter $\mathcal{G} \subseteq \mathbb{P}$ which meets the family of dense sets

$$\mathcal{D} = \{D_i \mid i \in I\} \cup \{E_g \mid g \in G\} \cup \left\{F_g^d \mid d \in I^+, g \in G\right\}$$

because $|\mathcal{D}| = \aleph_1$ as CH holds, and since the forcing axiom for countably closed forcing is true. To see this, we may construct a sequence of $\mathbf{g}_\alpha$'s enabling the filter to be defined by

$$G = \{\mathbf{f} \in \mathbb{P} \mid \mathbf{f} \geq \mathbf{g}_\alpha \text{ for some } \alpha < \omega_1\}$$

by transfinite induction. The idea is to start meeting each of the dense sets in $\mathcal{D}$ one-by-one, ensuring that the filter is closed downward. Enumerate the dense sets as $\mathcal{D} = \langle \mathcal{D}_\alpha \mid \alpha < \omega_1 \rangle$. Let $\mathbf{g}_0 \in \mathcal{D}_0$. Then at stage $n \leq \omega$, let $\mathbf{g}_n \leq \mathbf{g}_{n-1}$ satisfy $\mathbf{g}_n \in \mathcal{D}_n$. Density allows us to continue the construction through all successor stages. At limit stages, say $\lambda < \omega_1$, we use the fact that $\mathbb{P}$ is countably closed to find a condition strengthening the chain of our constructed $\mathbf{g}_\alpha$s for $\alpha < \gamma$, and then strengthen this condition to obtain $\mathbf{g}_\lambda \in D_\lambda$.

By construction, $\cup G$ defines a function $\mathbf{g} : I \longrightarrow G$ with the desired properties.

1. $\mathbf{g}$ *is a bijection:* This is ensured by meeting, for each $i \in I$, the dense sets D_i for injectivity and E_g for each $g \in G$ for surjectivity.
2. *For each $d \in I^+$, $\mathbf{g}_{(d)}$ is a bijection:* This is ensured by item 4. above and the dense sets F_g^d for each $g \in G$.

□

Question 5. Can we extend the definition of T_∞-terraces and so on to groups that have size bigger than $2^{\aleph_0}$?

`We would need to write down exactly what we should mean by an index set. Graphs with distinguished nodes (so that distances are unique)? Clearly I should look at the Hilton-Wojciechowski paper.`