A Terrace for $\mathbb{R}$

Definition. Let G be a group of order $2^{\aleph_0}$ that has no involutions and identity element e. For a bijection $\mathbf{a} : \mathbb{R} \longrightarrow G$ define a function $\mathbf{a}_{(d)} : \mathbb{R} \longrightarrow G \setminus \{e\}$ for each $d \in \mathbb{R}^+$ by

$$\mathbf{a}_{(d)}(i) = \mathbf{a}(i)^{-1}\mathbf{a}(i + d).$$

If each $\mathbf{a}_{(d)}$ is a bijection then $\mathbf{a}$ is a *directed T_∞-terrace for G.*

If instead we have that $A \subseteq \mathbb{R}$ is countable, and $\mathbf{a} : A \longrightarrow G$ and $\mathbf{a}_{(d)} : A_{(d)} \longrightarrow G$ are injections for each $d \in \mathbb{R}^+$, where $A_{(d)} = \{a \in A \mid a + d \in A\}$, we say that $\mathbf{a}$ a *countable partial directed T_∞-terrace on G* and we call each $\mathbf{a}_{(d)}$ the *partial $T_{(d)}$-sequencing corresponding to* $\mathbf{a}$.

Theorem 1. *Assume* CH*. The group* $(\mathbb{R}, +)$ *has a directed T_∞-terrace* $\mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}$*.*

Proof. Consider the poset $\mathbb{P}$ consisting of conditions which are countable partial directed T_∞-terraces on $\mathbb{R}$ partially ordered so that $\mathbf{a} \leq \mathbf{b}$ (following convention in set theory, we say $\mathbf{a}$ is *stronger* than $\mathbf{b}$) if and only if $\operatorname{dom}\mathbf{b} \subseteq \operatorname{dom}\mathbf{a}$ and $\mathbf{a} \restriction \operatorname{dom}\mathbf{b} = \mathbf{b}$.

It is not hard to see that $\mathbb{P}$ is countably closed. Suppose we have an decreasing chain of countable partial directed T_∞-terraces, $\mathbf{a}_n$ for $n \in \mathbb{N}$, on $\mathbb{R}$. Then the union of all of them, $\mathbf{a}$, is a countable partial directed T_∞-terrace on $\mathbb{R}$. Indeed, $\mathbf{a}$ is a bijection since each $\mathbf{a}_n$ in the chain is. For each $d \in \mathbb{R}^+$, we have that $\mathbf{a}_{(d)}$ is injective since $\operatorname{dom}\mathbf{a}_{(d)} \subseteq \operatorname{dom}\mathbf{a}$. Moreover $\mathbf{a}_{(d)}$ is a surjection since if $\mathbf{a}_{(d)}(i) = \mathbf{a}_{(d)}(j)$ then it must be that for some $n, m \in \mathbb{N}$, say $n \leq m$, we have that $\mathbf{a}_n(i + d) - \mathbf{a}_n(i) = \mathbf{a}_m(j + d) - \mathbf{a}_m(j)$, but this would imply that $i = j$ since then $\mathbf{a}_m \leq \mathbf{a}_n$ and $\mathbf{a}_m$ is surjective.

Need to establish:

(1) *It is dense to add a real number r to the domain of a condition in $\mathbb{P}$:* i.e., for each $r \in \mathbb{R}$, the set $D_r = \{\mathbf{d} \in \mathbb{P} \mid r \in \operatorname{dom}\mathbf{d}\}$ is dense.

To see this, let $\mathbf{a} \in \mathbb{P}$ with domain A. Choose $r \in \mathbb{R} \setminus A$. We need to find $\mathbf{d} \in D_r$ satisfying $\mathbf{d} \leq \mathbf{a}$. In order to find such a $\mathbf{d}$, first we must ensure that $\mathbf{d}(r) \neq \mathbf{a}(i)$ for each $i \in A$.

Secondly, we must ensure the T_d-sequencings for $\mathbf{d}$ are bijections. This amounts to ensuring that for each pair $i, i + d \in A$,

$$\begin{aligned} \mathbf{a}(i + d) - \mathbf{a}(i) &\neq \mathbf{a}(r + d) - \mathbf{d}(r) \\ &\neq \mathbf{d}(r) - \mathbf{a}(r - d) \end{aligned}$$

If $r - d$ and/or $r + d$ happen to be in A.

As A and the ranges of $\mathbf{a}$ and $\mathbf{a}_{(d)}$ are countable, the set of values to rule out for $\mathbf{d}(r)$ is at most countable, and we just need to make sure it's not one of those values. As $\mathbb{R}$ is uncountable, this can be done.

(2) *It is dense to add a real number r to the range of a condition in $\mathbb{P}$:* i.e., for each $r \in \mathbb{R}$, the set $E_r = \{\mathbf{e} \in \mathbb{P} \mid r \in \operatorname{range}\mathbf{e}\}$ is dense.

Again the idea should be that we only have to avoid countably many scenarios, but we have room in $\mathbb{R}$ for that. Suppose $r \in \mathbb{R} \setminus \operatorname{range}\mathbf{a}$. We need to find $\mathbf{e} \in E_r$

satisfying $\mathbf{e} \leq \mathbf{a}$. This amounts to finding $\overline{r} \notin A = \operatorname{dom} \mathbf{a}$ so that we can let $\mathbf{d}(\overline{r}) = r$, satisfying $\overline{r} \notin A_{(d)}$ for whenever $A_{(d)}$ is nonempty.

Both A and $A_{(d)} \subseteq A$ are countable, so this can be done.

(3) *For each $d \in \mathbb{R}^+$ it is dense to add a real number r to the domain of a condition's partial $T_{(d)}$-sequencing:* This is captured by (1), since we may add both r and $d+r$ to the domain of a condition.

(4) *For each $d \in \mathbb{R}^+$ it is dense to add a real number r to the range of a condition's partial $T_{(d)}$-sequencing:* In other words, we would like to show that for each $r \in \mathbb{R}$ and each $d \in \mathbb{R}^+$, the set $F_r^d = \left\{\mathbf{f} \in \mathbb{P} \mid r \in \operatorname{range} \mathbf{f}_{(d)}\right\}$ is dense in $\mathbb{P}$.

To see this, fix $d \in \mathbb{R}^+$ and let $r \in \mathbb{R}$. Let $\mathbf{a} \in \mathbb{P}$, and suppose that $r \notin \operatorname{range} \mathbf{a}_{(d)}$. We want to see that it is possible to extend $\mathbf{a}$ to a condition $\mathbf{f} \in F_r^d$ such that $r = \mathbf{f}(\overline{r} + d) - \mathbf{f}(\overline{r})$ for some $\overline{r} \in \mathbb{R}$. This amounts to finding a suitable $\overline{r}$. First we need $\overline{r}$ to be so that neither $\overline{r}$ nor $\overline{r} + d$ are in $A = \operatorname{dom} \mathbf{a}$. Then we need to ensure that $\mathbf{f}(\overline{r}), \mathbf{f}(\overline{r} + d) \notin \operatorname{range} \mathbf{a}$, and also that $r = \mathbf{f}(\overline{r} + d) - \mathbf{f}(\overline{r})$.

It must also be the case that for any $i \in A$, we have that

$$\mathbf{f}(\overline{r}) - \mathbf{a}(i) \notin \operatorname{range} \mathbf{a}_{(\overline{r}-i)}, \quad \mathbf{a}(i) - \mathbf{f}(r) \notin \operatorname{range} \mathbf{a}_{(i-\overline{r})},$$

$$\mathbf{f}(\overline{r} + d) - \mathbf{a}(i) \notin \operatorname{range} \mathbf{a}_{(\overline{r}+d-i)}, \quad \mathbf{a}(i) - \mathbf{f}(\overline{r} + d) \notin \operatorname{range} \mathbf{a}_{(i-\overline{r}-d)}.$$

Moreover, we can't inadvertently mess up another sequencing. In particular, whenever we have that $\overline{r} + l, \overline{r} + d + l \in \operatorname{dom} A$, we must have that

$$\mathbf{a}(\overline{r} + l) - \mathbf{f}(\overline{r}) \neq \mathbf{a}(\overline{r} + d + l) - \mathbf{f}(\overline{r} + d),$$

meaning that

$$r = \mathbf{f}(\overline{r} + d) - \mathbf{f}(\overline{r}) \neq \mathbf{a}(\overline{r} + d + l) - \mathbf{a}(\overline{r} + l).$$

This contradicts $r \notin \operatorname{range} \mathbf{a}_{(d)}$. Dually, we need that whenever $\overline{r} - l, \overline{r} - d - l \in \operatorname{dom} A$, we must have that

$$\mathbf{f}(\overline{r}) - \mathbf{a}(\overline{r} - l) \neq \mathbf{f}(\overline{r} + d) - \mathbf{a}(\overline{r} + d - l),$$

which again contradicts $r \notin \operatorname{range} \mathbf{a}_{(d)}$.

Since we have only eliminated countably many options, as we are restricted by A and its image under $\mathbf{a}$, we have plenty of room to choose such an $\overline{r}$ as desired.

Now that we have verified these collections sets are dense, we may find a filter $\mathcal{G} \subseteq \mathbb{P}$ which meets the family of dense sets

$$\mathcal{D} = \{D_r \mid r \in \mathbb{R}\} \cup \{E_r \mid r \in \mathbb{R}\} \cup \left\{F_r^d \mid d \in \mathbb{R}^+, r \in \mathbb{R}\right\}$$

because $|\mathcal{D}| = \aleph_1$ as CH holds, and since the forcing axiom for countably closed forcing is true.

To see why such a filter can be built, simply construct a sequence of $\mathbf{g}_\alpha$'s by transfinite induction, enabling the filter to be defined by

$$\mathcal{G} = \{\mathbf{f} \in \mathbb{P} \mid \mathbf{f} \geq \mathbf{g}_\alpha \text{ for some } \alpha < \omega_1\}\,.$$

The idea is to start meeting each of the dense sets in $\mathcal{D}$ one-by-one, ensuring that the filter is closed downward. Enumerate the dense sets as $\mathcal{D} = \langle \mathcal{D}_\alpha \mid \alpha < \omega_1 \rangle$. Let $\mathbf{g}_0 \in \mathcal{D}_0$. Then at stage $n \leq \omega$, let $\mathbf{g}_n \leq \mathbf{g}_{n-1}$ satisfy $\mathbf{g}_n \in \mathcal{D}_n$. Density allows us to continue the construction through all successor stages. At limit stages, say $\lambda < \omega_1$, we use the fact that $\mathbb{P}$ is countably closed to find a condition strengthening the chain of our constructed $\mathbf{g}_\alpha$s for $\alpha < \gamma$, and then strengthen this condition to obtain $\mathbf{g}_\lambda \in \mathcal{D}_\lambda$.

By construction, $\cup G$ defines a function $\mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}$ with the desired properties.

(1) $\mathbf{g}$ *is a bijection:* This is ensured by meeting, for each $r \in \mathbb{R}$, the dense sets D_r for injectivity and for meeting E_r for each $r \in \mathbb{R}$ for surjectivity.

(2) *For each* $d \in \mathbb{R}^+$, $\mathbf{g}_{(d)}$ *is a bijection:* This is ensured by (3) and the dense sets F_r^d for each $r \in \mathbb{R}$. □

Corollary 1. *If* CH *holds, there is a Vatican square on* $\mathbb{R}$.

Corollary 2. *There is a real-preserving forcing which adds a Vatican square on* $\mathbb{R}$ *and forces* CH.

We could also force with conditions of size less than $2^{\aleph_0}$. This should not affect the size of $\mathbb{R}$, but should also add a terrace. Should be $< 2^{\aleph_0}$-closed. So in this case we can have a terrace on $\mathbb{R}$ without CH.

Question 1. Let G be an abelian group of size continuum with continuum-many non-involutions. Does G have a directed T_∞-terrace?

The answer is yes. See Matt's note.

Theorem 2. *Let G be an abelian group of size $\aleph_1$ with $\aleph_1$-many non-involutions. Then G has a directed T_∞-terrace.*

Question 2. Can we force to add a terrace of size $\aleph_{n+1}$ given that there is a partial terrace of size $\aleph_n$?

Yes. Poset would consist of partial terraces, as usual, and should be $< \aleph_{n+1}$-closed, so that no new elements are added to the group of size $\aleph_{n+1}$.

Can find a Vatican square of any order in this way, since all we need is an ordered field of the same size to have an index set. Ordered fields exist of any size by Löwenheim-Skolem?