Split Principles

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κ-Split

Split principles are "anti-large cardinal axioms" which characterize the failure of a regular cardinal to satisfy certain large cardinal properties.

Let κ be a cardinal. We shall refer to a sequence of the form $\langle d_\alpha \mid \alpha < \kappa \rangle$ as a κ-***list*** if for all $\alpha < \kappa$, $d_\alpha \subseteq \alpha$.

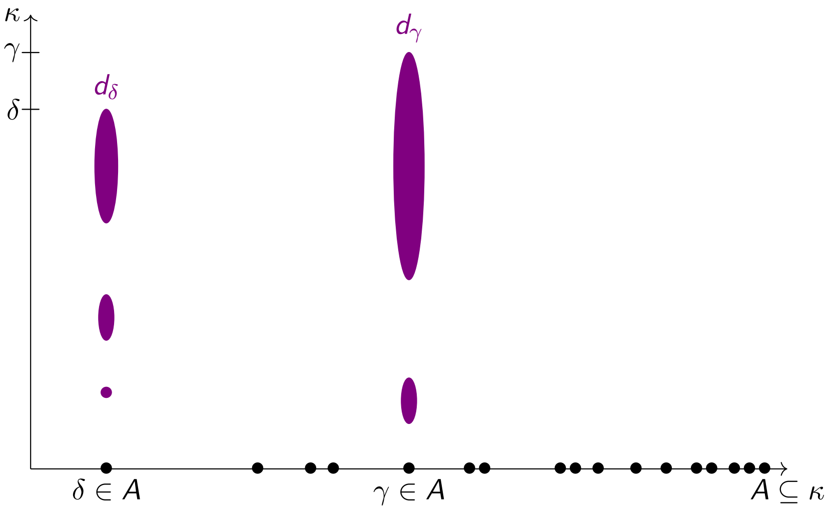
Definition (The original split principle)

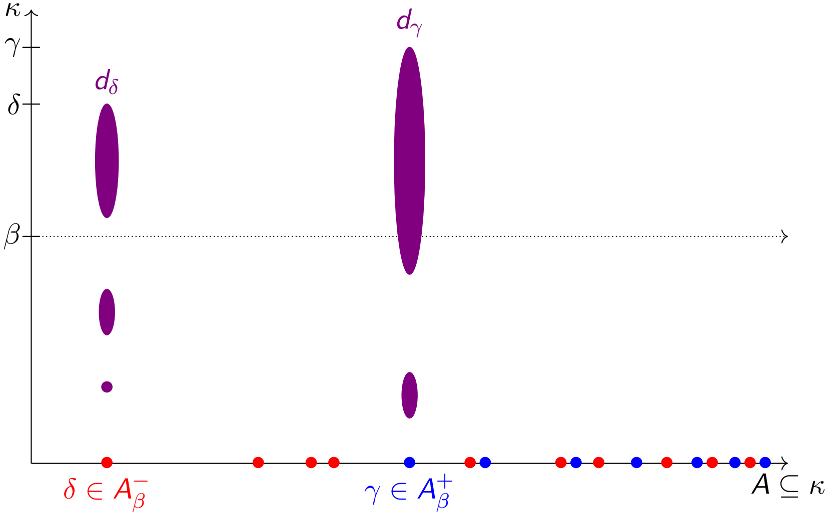
A κ-list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ ***splits unbounded sets*** so long as for every unbounded $A \subseteq \kappa$ there is a $\beta < \kappa$ which *splits* A (with respect to $\vec{d}$); i.e. both

$$A_\beta^+ := \{\alpha \in A \mid \beta \in d_\alpha\} \text{ and } A_\beta^- := \{\alpha \in A \mid \beta \notin d_\alpha\}$$

are unbounded in κ.

If there is a κ-list that splits unbounded sets, then we say that κ-Split holds.


κ
γ
δ
d_δ
d_γ
$\delta \in A$
$\gamma \in A$
$A \subseteq \kappa$


κ
γ
δ
β
d_γ
d_δ
$\delta \in A_\beta^-$
$\gamma \in A_\beta^+$
$A \subseteq \kappa$

Weakly compact cardinals

It turns out that for regular κ, the properties of a κ-list splitting unbounded sets and having a cofinal branch are complementary.

Definition

A κ-list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ has a ***cofinal branch*** so long as there is a $b \subseteq \kappa$ such that for all $\gamma < \kappa$ there is an $\alpha \geq \gamma$ such that

$$d_\alpha \cap \gamma = b \cap \gamma.$$

We say that the branch property $\mathrm{BP}(\kappa)$ holds if every κ-list has a cofinal branch.

Theorem

*Let κ be regular. Then κ-*Split $\iff \neg\,\mathrm{BP}(\kappa) \iff$ *κ is not weakly compact.*

Ineffable cardinals

Definition

A κ-list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ has an ***ineffable branch*** so long as there is $b \subseteq \kappa$ and a stationary $S \subseteq \kappa$ such that for all $\alpha \in S$,

$$d_\alpha = b \cap \alpha.$$

We say that κ has the ineffable branch property, or $\mathrm{IBP}(\kappa)$ holds, if every κ-list has an ineffable branch. A regular cardinal κ is ***ineffable*** if every κ-list has an ineffable branch ($\mathrm{IBP}(\kappa)$ holds).

We have the following: $\mathrm{IBP}(\kappa) \implies \mathrm{BP}(\kappa)$. In particular, ineffable cardinals are weakly compact.

κ-SSplit

Definition

A κ-list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ ***splits stationary sets*** so long as for every stationary $S \subseteq \kappa$ there is $\beta < \kappa$ which splits S, i.e. both

$$S^+_\beta := \{\alpha \in S \mid \beta \in d_\alpha\} \text{ and } S^-_\beta := \{\alpha \in S \mid \beta \notin d_\alpha\}$$

are unbounded in κ.

We say that κ-SSplit holds so long as there is a κ-list which splits stationary sets.

Theorem

Let κ be regular. Then κ-SSplit $\iff$ κ is not ineffable.

Splitting subsets of $\mathcal{P}_\kappa\lambda$

Definition

Let κ and λ be regular cardinals. A $\mathcal{P}_\kappa\lambda$-list $\vec{d} = \langle d_x \mid x \in \mathcal{P}_\kappa\lambda \rangle$ (satisfying $d_x \subseteq x$) is said to ***split unbounded sets*** if for every unbounded $A \subseteq \mathcal{P}_\kappa\lambda$ there is $\beta < \lambda$ which splits A, ie. both

$$A^+_\beta := \{x \in A \mid \beta \in d_x\} \text{ and } A^-_\beta := \{x \in A \mid \beta \notin d_x\}$$

are unbounded in $\mathcal{P}_\kappa\lambda$.
If there is a $\mathcal{P}_\kappa\lambda$-list which splits unbounded sets, then $\mathcal{P}_\kappa\lambda$-Split holds.

A $\mathcal{P}_\kappa\lambda$-list $\vec{d} = \langle d_x \mid x \in \mathcal{P}_\kappa\lambda \rangle$ is said to ***split stationary sets*** if for every stationary $S \subseteq \mathcal{P}_\kappa\lambda$ there is $\beta < \lambda$ which splits S, ie. both

$$S^+_\beta := \{x \in S \mid \beta \in d_x\} \text{ and } S^-_\beta := \{x \in S \mid \beta \notin d_x\}$$

are unbounded in $\mathcal{P}_\kappa\lambda$.
If there is a $\mathcal{P}_\kappa\lambda$-list which splits stationary sets, then $\mathcal{P}_\kappa\lambda$-SSplit holds.

Large cardinal characterizations

Theorem

Let κ be regular. Then

- κ *is not weakly compact* $\iff$ κ-Split *holds.*
- κ *is not ineffable* $\iff$ κ-SSplit *holds.*
- κ *is not* λ*-ineffable* $\iff$ $\mathcal{P}_\kappa\lambda$-SSplit *holds.*
- κ *is not supercompact* $\iff$ *for unboundedly many* λ, $\mathcal{P}_\kappa\lambda$-SSplit *holds.*

Question

What about $\mathcal{P}_\kappa\lambda$-Split?

Strong/Cofinal Branches

Definition

Let $\vec{d} = \langle d_x \mid x \in \mathcal{P}_\kappa\lambda\rangle$ be a $\mathcal{P}_\kappa\lambda$-list. A set $B \subseteq \lambda$ is a ***cofinal branch*** through $\vec{d}$ so long as for all $x \in \mathcal{P}_\kappa\lambda$, there is some $y \in \mathcal{P}_\kappa\lambda$ with $y \supseteq x$ such that

$$d_y \cap x = B \cap x.$$

The ***branch property*** $\mathrm{BP}(\kappa, \lambda)$ holds iff every $\mathcal{P}_\kappa\lambda$-list has a cofinal branch.

A cofinal branch B is ***strong*** if there is an unbounded set $U \subseteq \mathcal{P}_\kappa\lambda$ satisfying that for all $x \in \mathcal{P}_\kappa\lambda$ there is a $y \supseteq x$ such that for all $z \supseteq y$ with $z \in U$,

$$d_z \cap x = B \cap x.$$

If every $\mathcal{P}_\kappa\lambda$-list has a strong branch, then the ***strong branch property*** $\mathrm{SBP}(\kappa, \lambda)$ holds.

$\mathcal{P}_\kappa\lambda$-Split

Theorem

Let κ be regular. then $\mathcal{P}_\kappa\lambda$-Split $\iff \neg\,\mathsf{SBP}(\kappa, \lambda)$.

Observation

If $\mathcal{P}_\kappa\lambda$-Split fails for every λ, then κ is strongly compact (i.e. $\mathsf{BP}(\kappa, \lambda)$ holds).

Questions

- Can there be a $\mathcal{P}_\kappa\lambda$-list that has a cofinal branch but no strong branch?
- How can the failure of $\mathsf{BP}(\kappa, \lambda)$ be characterized via split principles?

Thank you.