Subcomplete Forcing, Trees, and Generic Absoluteness

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Generic Absoluteness

Definition

Let $\mathbb{P}$ be a forcing notion and κ be a cardinal. Then $\mathbb{P}$-**generic** $\mathbf{\Sigma}^1_1(\kappa)$-**absoluteness** states that for any model M of size κ for a countable first order language and every Σ^1_1-sentence φ over the language of M, for any finite list of finitary predicates $\vec{A}$,

$$(\langle M, \vec{A}\rangle \models \varphi)^V \iff 1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\langle \check{M}, \check{\vec{A}}\rangle \models \varphi).$$

- For a class Γ of forcing notions, Γ-generic absoluteness is the statement that $\mathbb{P}$-generic absoluteness holds for every $\mathbb{P} \in \Gamma$.
- One might wish to work with canonical models such as H_{ω_1} in the above - write $\mathbf{\Sigma}^1_1(H_{\omega_1})$ instead.
- Γ-generic $\mathbf{\Sigma}^1_1(H_{\omega_1})$-absoluteness is equivalent to Γ-generic $\mathbf{\Sigma}^1_1(2^\omega)$-absoluteness, that is, as far as the classes of c.c.c, proper, semi-proper, stationary set preserving or subcomplete forcing are concerned.

Background on generic absoluteness

Observation

$\mathbb{P}$-generic $\mathbf{\Sigma}^1_1(\omega)$-absoluteness holds for any poset $\mathbb{P}$.

Proof.

Upward $\mathbb{P}$-generic $\mathbf{\Sigma}^1_1(\kappa)$-absoluteness is true, for any κ.
To show downward, let M be a countable model and suppose $\mathbb{P}$ forces $M \models \varphi$ for some Σ^1_1-sentence $\varphi = \exists X\ \psi(X)$. Let $M, \mathbb{P} \in X \preccurlyeq H_\theta$ for some large enough H_θ, and let $\overline{N} \cong X$ be transitive. $\overline{N}$ sees that $\mathbb{P}$ forces $M \models \varphi$. We may build a generic for $\overline{N}$ in V, and in $\overline{N}[\overline{G}]$, choosing a witness A for φ, we have that

$$\langle M, A \rangle \models \psi.$$

Again by upward absoluteness, this means $M \models \varphi$ in V. $\square$

Background on generic absoluteness

Observation

1. $\mathcal{C}oll(\omega_1, \omega_2)$-generic $\mathbf{\Sigma}^1_1(\omega_2)$-absoluteness fails.
2. If $\mathbb{P}$ is a forcing that adds a real, then $\mathbb{P}$-generic $\mathbf{\Sigma}^1_1(H_{\omega_1})$-absoluteness fails.

Theorem (Fuchs, 2008)

- *Countably closed-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness is provable in* ZFC.
- *The countably closed maximality principle implies countably closed-generic $\mathbf{\Sigma}^1_2(H_{\omega_1})$-absoluteness.*

Dually to the situation with countably closed forcing, the underlying main question is whether subcomplete-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness is provable in ZFC.

Generic absoluteness and trees

Lemma

Assume CH. *Let* Γ *be a natural class of forcing notions. Then the following are equivalent.*

1. *Every* $\mathbb{P} \in \Gamma$ *preserves* $(\omega_1, \leq\omega_1)$*-Aronszajn trees and does not add reals.*
2. Γ*-generic* $\mathbf{\Sigma}^1_1(\omega_1)$*-absoluteness holds.*

Thus we have a convenient rephrasing of our main question about whether subcomplete-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness is provable in ZFC.

Main Question

Can subcomplete forcing add cofinal branches to $(\omega_1, \leq\omega_1)$-Aronszajn trees?

Subcomplete forcing

Subcomplete forcing is a class of forcing notions defined by Ronald B. Jensen. Subcomplete forcing does not add reals, but may potentially alter cofinalities to ω.

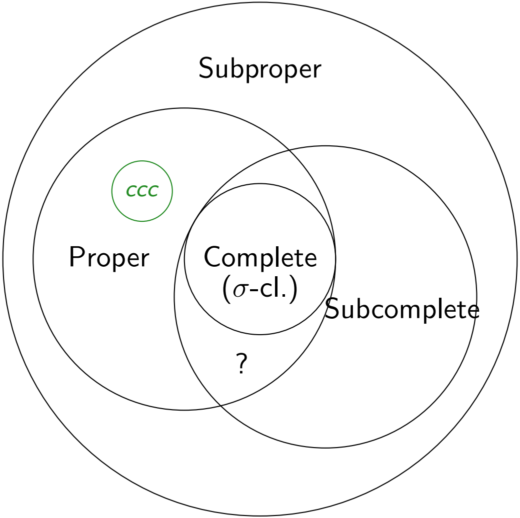
Examples of subcomplete forcing

- (Jensen) Countably closed forcing.
- (Jensen) Namba forcing under CH.
- (Jensen) Prikry forcing.
- (M.) Generalized diagonal Prikry forcing.
- (Fuchs) Magidor Forcing.

Subcomplete forcing can be iterated without adding reals, and SCFA may be forced from a supercompact by the usual argument. Unlike other forcing axioms, however, SCFA is compatible with CH.

Subcomplete forcing

How subcompleteness fits in with other forcing classes which preserve stationary subsets of ω_1:



Subcomplete forcing's effect on trees

Theorem

The following properties of an ω_1-tree T are preserved by subcomplete forcing:

1. *T is Aronszajn*
2. *T is not Kurepa*
3. *T is Suslin*
4. *T is Suslin and* UBP
5. *T is $\textit{Suslin off the generic branch}$*
6. *T is n-fold $\textit{Suslin off the generic branch}$ (for $n \geq 2$)*
7. *T is $(n-1)$-fold $\textit{Suslin off the generic branch}$ and n-fold* UBP *(for $n \geq 2$)*

Subcomplete forcing's effect on wider trees

Observation

- Subcomplete (or even countably closed) forcing may add a cofinal branch to an $(\omega_1, {\leq}2^\omega)$-tree.
- Subcomplete forcing cannot add (cofinal) branches to $(\omega_1, {<}2^\omega)$-trees.

Again we turn to the question stated earlier:

Main Question

Can subcomplete forcing add cofinal branches to $(\omega_1, {\leq}\omega_1)$-Aronszajn trees?

By the second point of the above observation, if *CH* fails, then the answer to the main question is no.

Generic absoluteness and bounded forcing axioms

Theorem (Bagaria, 2000)

Let Γ be a natural forcing class. Then the following are equivalent:

1. *The bounded forcing axiom for Γ.*
2. **Γ-generic $\Sigma_1(H_{\omega_2})$-absoluteness**: *for all $\mathbb{P} \in \Gamma$ and $G \subseteq \mathbb{P}$ generic over V,*
$$\langle H_{\omega_2}, \in \rangle \prec_{\Sigma_1} \langle H_{\omega_2}, \in \rangle^{\mathrm{V}[G]}.$$

Using codes, Σ_1-statements over H_{ω_2} can be translated into Σ^1_1-statements over H_{ω_1}.

Lemma

Let $\mathbb{P}$ be a forcing that does not add reals. Consider the following:

1. *$\mathbb{P}$-generic $\Sigma_1(H_{\omega_2})$-absoluteness holds.*
2. *$\mathbb{P}$-generic $\boldsymbol{\Sigma}^1_1(H_{\omega_1})$-absoluteness holds.*

We have that $2 \Longrightarrow 1$, and if CH holds, then $1 \Longrightarrow 2$.

Answer to the main question

Theorem

Assuming CH, the following are equivalent.

1. BSCFA.
2. *Subcomplete generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness.*
3. *Subcomplete forcing preserves $(\omega_1, {\leq}\omega_1)$-Aronszajn trees.*

This puts us in a position to answer the main question completely.

Theorem

Splitting in two cases, we have:

1. *If CH fails, then subcomplete forcing preserves $(\omega_1, {\leq}\omega_1)$-Aronszajn trees.*
2. *If CH holds, then subcomplete forcing preserves $(\omega_1, {\leq}\omega_1)$-Aronszajn trees iff BSCFA holds.*

Other forcing classes

Observation

Let Γ be a natural class of forcing notions. Then 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3.:

1. BFA_Γ.
2. Γ-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness.
3. Forcing notions in Γ preserve $(\omega_1, {\leq}\omega_1)$-Aronszajn trees.

Theorem

Consider the following statements.

1. MA.
2. *ccc-generic* $\mathbf{\Sigma}^1_1(\omega_1)$*-absoluteness.*
3. *ccc forcing preserves* $(\omega_1, \leq \omega_1)$*-Aronszajn trees.*

Then 1 $\iff$ 2 $\implies$ 3 *but 3 does not imply 2. In fact, 3 is consistent with* CH*, while 1/2 imply the failure of* CH*.*

Final questions

The general relationship between the pertinent properties is unclear.

Question

Let Γ be the class of proper, semi-proper, stationary set preserving or subcomplete forcings. Which implications hold between the following properties?

1. BFA_Γ.
2. Γ-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness.
3. Forcings in Γ preserve $(\omega_1, \leq\omega_1)$-Aronszajn trees.

There are some interesting questions about subcomplete-generic absoluteness when CH fails. In this case, BSCFA may still hold.

Question

What is the consistency strength of $\neg$CH together with subcomplete-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness?

Thank you.

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