Subcomplete Forcing, Trees, and Generic Absoluteness

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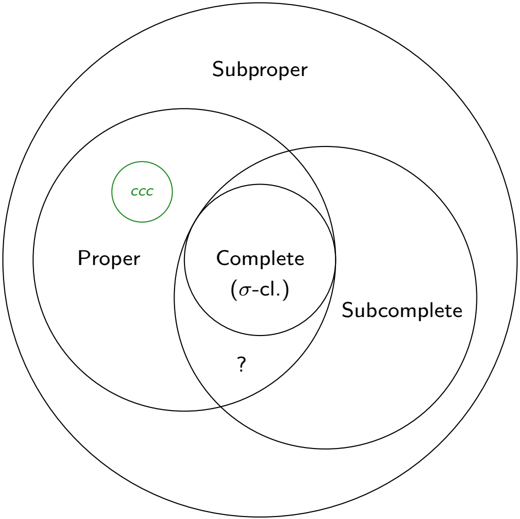
Subcomplete forcing is a class of forcing notions defined by Ronald B. Jensen. Subcomplete forcing does not add reals, but may potentially alter cofinalities to ω.

Examples of subcomplete forcing

- **Countably closed forcing**.
- **Namba forcing**, denoted by $\mathbb{N}$, a forcing notion consisting of subtrees $T \neq \emptyset$ of $\omega_2^{<\omega}$ ordered by inclusion, such that T is downward closed in $\omega_2^{<\omega}$ and where each node in T has ω_2-many eventual successors in T. Each condition in $\mathbb{N}$ has size ω_2. Namba forcing adds a cofinal sequence $S : \omega \longrightarrow \omega_2^V$ to the extension, a cofinal branch through $\omega_2^{<\omega}$. Under CH, Namba forcing adds no new reals and is subcomplete [**?**, Section 3.3].
- **Prikry forcing**, which forces a measurable cardinal to have cofinality ω while preserving cardinalities.
- **Generalized diagonal Prikry forcing**

- Revised countable support (*rcs*) iterations of subcomplete forcing notions.
- Lottery sums of subcomplete forcing notions.
- If $\mathbb{P}$ is subcomplete and $\pi : \mathbb{P} \longrightarrow \mathbb{Q}$ is a dense embedding, then $\mathbb{Q}$ is subcomplete.

How subcompleteness fits in with other forcing classes which preserve stationary subsets of ω_1:



Subcomplete forcing doesn't add branches to ω_1-trees

Theorem

Let T be an ω_1-tree. If $\mathbb{P}$ is subcomplete then $\mathbb{P}$ does not add new branches to T.

Proof sketch.

Assume not. Let q be a condition forcing that $\dot{b}$ is a new cofinal branch through $\check{T}$. Let θ verify the subcompleteness of $\mathbb{P}$ and find N, σ so that:

- $\mathbb{P} \in H_\theta \subseteq N$
- $\sigma : \overline{N} \cong X \prec N$ where X is countable and $\overline{N}$ is full
- $\sigma(\overline{\theta}, \overline{\mathbb{P}}, \overline{T}, \overline{q}, \overline{\dot{b}}) = \theta, \mathbb{P}, T, q, \dot{b}$.

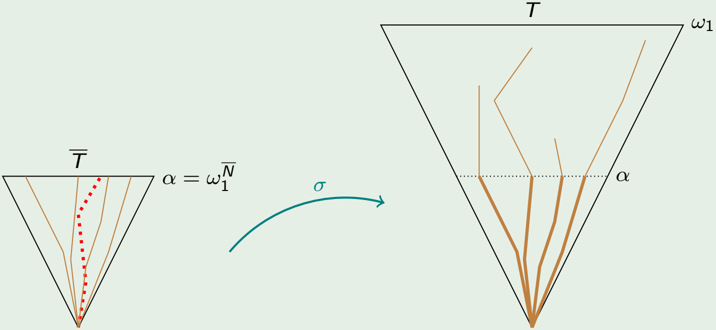
By elementarity, $\overline{q}$ forces $\overline{\dot{b}}$ to be a new cofinal branch through $\check{\overline{T}}$.
Let $\alpha = \omega_1^{\overline{N}}$. Note that $\mathrm{cp}(\sigma) = \alpha$.

Subcomplete forcing doesn't add branches to ω_1-trees

Proof sketch continued.

The idea is to construct a generic $\overline{G}$ for $\overline{\mathbb{P}}$ over $\overline{N}$, using the countability of $\overline{N}$ to diagonalize against all branches through T as seen on level α of the tree in N.

Inductively define a decreasing, chain of conditions $\overline{q}_n$, where $\overline{q}_0 = \overline{q}$, deciding values of $\dot{\overline{b}}$ in $\overline{T}$ differently than the nth "branch" on level α in T.



$\overline{T} = T \restriction \alpha$

Subcomplete forcing doesn't add branches to ω_1-trees

Proof sketch continued.

Furthermore list out the (countably many) dense sets of $\overline{\mathbb{P}}$, $\overline{\vec{D}}$, and ensure that each $\overline{q}_n \in \overline{D}_n$.

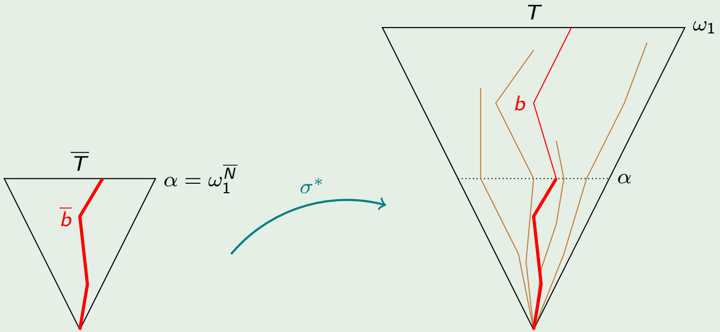
Let $\overline{G}$ be the generic filter generated by the $\overline{q}_n$, let $\overline{\dot{b}}^{\overline{G}} = \overline{b}$. Since $\mathbb{P}$ is subcomplete, there is a condition $p \in \mathbb{P}$ such that whenever G is $\mathbb{P}$-generic with $p \in G$, we have $\sigma' \in V[G]$ such that:

- $\sigma' : \overline{N} \longrightarrow N$ elementarily
- $\sigma'(\overline{\theta}, \overline{\mathbb{P}}, \overline{T}, \overline{q}, \overline{\dot{b}}) = \theta, \mathbb{P}, T, q, \dot{b}$
- $\sigma' `` \overline{G} \subseteq G$.

So there is a lift $\sigma^* : \overline{N}[\overline{G}] \longrightarrow N[G]$ elementary, a lift of σ', with $\sigma^*(\overline{b}) = \sigma'(\overline{\dot{b}})^G = \dot{b}^G = b$, and $\sigma^*(\overline{T}) = \sigma'(\overline{T})^G = T$. Now we have $N[G] \models q \in G$, so b is a cofinal branch through T.

Subcomplete forcing doesn't add branches to ω_1-trees

Proof sketch continued.



Since α is the critical point of the embedding, in $N[G]$, $b \restriction \alpha = \overline{b}$. However, $\overline{b}$ was constructed so as to not be equal to any branch restricted to level α, the ones we listed out initially, a contradiction. □

Subcomplete forcing doesn't add branches to ω_1-trees

Corollary

Subcomplete forcing preserves Aronszajn trees.

Corollary

If an ω_1-tree is not Kurepa, it cannot become Kurepa in a subcomplete forcing extension.

Moreover, subcomplete forcing does not add branches to potentially "wider" trees with levels of size less than $\mathfrak{c}$:

Theorem

Subcomplete forcing cannot add branches to $(\omega_1, < 2^\omega)$-trees.

Suslin tree preservation

Theorem (Jensen)

Subcomplete forcing preserves the property of being Suslin of ω_1-trees.

This proof of the above is necessarily different from the proof that subcomplete forcing doesn't add branches to ω_1-trees, as it is possible for maximal antichains to be added by subcomplete forcing.

Proposition

If T is a non-Suslin ω_1-tree, then $\mathcal{A}dd(\omega_1, 1)$ adds a new maximal antichain to T.

Proof.

Let $A = \{a_\alpha \mid \alpha < \omega_1\}$ be a maximal antichain in T. Let $G \subseteq \omega_1$ be $\mathcal{A}dd(\omega_1, 1)$-generic. Let $A' = \{a_\alpha \mid \alpha \notin G\} \cup \{t \in T \mid \exists \alpha \in G \ \ t \in \mathrm{succ}_T(a_\alpha)\}$. Then A' is a maximal antichain in T and $A' \notin V$ since $G = \{\alpha < \omega_1 \mid a_\alpha \notin A'\}$. $\square$

Corollary

Nontrivial ccc forcings are not subcomplete.

Proof sketch.

If $\mathbb{P}$ is subcomplete and *ccc* then $\mathbb{P}$ is countably distributive (since it can't add a real), and *ccc*

The unique branch property of Suslin trees

Definition

A normal ω_1-tree T has the **unique branch property** (*ubp*) so long as $\mathbb{1} \Vdash_T$ "$\check{T}$ has exactly one new cofinal branch." That is, after forcing with the tree, T has has exactly one cofinal branch which was not in the ground model.

Theorem

If T is a Suslin tree and $\mathbb{P}$ is subcomplete, then $[T]^{V^{\mathbb{P}\times T}} = [T]^{V^T}$. In other words, subcomplete forcing doesn't add to the collection of T-generic branches.

Corollary

Subcomplete forcing preserves the unique branch property of Suslin trees.

Proof of Theorem.

Suppose not. Let $\ddot{b}$ be a $\mathbb{P}$-name for a $\check{T}$-name for a new branch through T and suppose we have $p \in \mathbb{P}$, $t \in T$ satisfying that whenever $G \times b \subseteq \mathbb{P} \times T$ is generic with $\langle p, t\rangle \in G \times b$ we have that $(\ddot{b}^G)^b \in [T]^{V[G][b]} \setminus [T]^{V[b]}$.

Subcomplete forcing doesn't add generic branches to Suslin trees

Proof of Theorem continued.

Let θ verify the subcompleteness of $\mathbb{P}$, and let's get ourselves into the standard setup:

- $\mathbb{P} \in H_\theta \subseteq N \models \mathrm{ZFC}^-$
- $\sigma : \overline{N} \cong X \prec N$ where X is countable and $\overline{N}$ is full
- $\sigma(\overline{\theta}, \overline{\mathbb{P}}, \overline{T}, \overline{p}, \overline{\ddot{b}}, \overline{t}) = \theta, \mathbb{P}, T, p, \ddot{b}, t$.

Let $\alpha = \omega_1^{\overline{N}}$, the critical point of σ. We have $\overline{T} = T \restriction \alpha$ as usual.

Enumerate with $\vec{\overline{D}}$ the dense sets of $\overline{\mathbb{P}}$ in $\overline{N}$. Again the idea is to carefully construct a generic $\overline{G} \subseteq \overline{\mathbb{P}}$ over $\overline{N}$ by diagonalizing against branches $\vec{b}$ on level α of T. We may ensure that $\overline{t} \in b_0$. We construct a $\leq_{\overline{\mathbb{P}}}$-sequence $\langle \overline{p}_n \mid n < \omega \rangle$ satisfying, for each n:

1. $\overline{p}_n \in \overline{D}_n$
2. In $\overline{N}$, $\overline{p}_n \Vdash_{\overline{\mathbb{P}}} \left(\check{\overline{t'}} \Vdash_{\check{\overline{T}}} \overline{\ddot{b}}(\check{\gamma}) \neq (b_n\check{(}\gamma)) \right)$, for some $\gamma < \alpha$ and $\overline{t'} \in b_0$, $\sigma(\overline{t'}) = t' \geq_T t$. In other words, $\overline{p}_n$ forces that the canonical name for $\overline{t'}$ forces the value of the generic branch to be different from the nth "branch" in our list in N.

If we can satisfy these two conditions, then we are done.

Subcomplete forcing doesn't add generic branches to Suslin trees

Proof of Theorem continued.

Suppose $\overline{p}_m$ have been defined for $m < n$. To get $\overline{p}_n$, choose $\overline{q}_n$ below each $\overline{p}_m$ for all $m < n$, satisfying $\overline{q}_n \in \overline{D}_n$.

As $\overline{T}$ is Suslin in $\overline{N}$ and cofinal branches are generic for Suslin trees, we have that $\overline{N}[b_0]$ is a generic extension. Let $\overline{G}^0, \overline{G}^1$ be mutually $\overline{\mathbb{P}}$-generic over $\overline{N}[b_0]$ so that $\overline{p}, \overline{q}_n \in \overline{G}^0 \cap \overline{G}^1$. For $i = 0, 1$ let $\overline{c}^i = (\dot{\overline{b}}^{\overline{G}^i})^{b_0}$.

Since $\overline{p} \in \overline{G}^0, \overline{G}^1$ and $\overline{t} \in b_0$, both of the $\overline{c}^i$ are cofinal branches through $\overline{T}$. It follows from the mutual genericity of $\overline{G}^0$ and $\overline{G}^1$ that $\overline{c}^0 \neq \overline{c}^1$; otherwise, suppose that $\overline{c} = \overline{c}^0 = \overline{c}^1$. Then we'd have

$$\overline{c} \in \overline{N}[\overline{G}^0][b_0] \cap \overline{N}[\overline{G}^1][b_0] = \overline{N}[b_0][\overline{G}^0] \cap \overline{N}[b_0][\overline{G}^1] = \overline{N}[b_0]$$

so $\overline{c} \in [\overline{T}]^{V[b_0]}$, a contradiction.

So let $\overline{c} \in \{\overline{c}^0, \overline{c}^1\}$ be such that $\overline{c} \neq b_n$. Thus we may choose $\gamma < \alpha$ so that the value of $\overline{c}$ on level α is not the same as $b_n(\gamma)$. Then this holds in some $\overline{N}[\overline{G}^i][b_0]$, and we can obtain a condition $\overline{p}_n \leq \overline{q}_n$ forcing this. □

Suslin off the generic branch

Definition

A Suslin tree T is **Suslin off the generic branch** so long as after forcing with T to add a generic branch b, for any node t not in b, the tree T_t remains Suslin.

Theorem

If T is a Suslin tree which is also Suslin off the generic branch, then T is still Suslin off the generic branch after subcomplete forcing.

Thank you.