Subcomplete Forcing, Trees, and Generic Absoluteness

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Generic Absoluteness

Definition

Let $\mathbb{P}$ be a forcing notion, let n be a natural number, and let κ be a cardinal. Then $\mathbb{P}$-**generic** $\mathbf{\Sigma}^1_n(\kappa)$**-absoluteness** states that for any model M of size κ for a countable first order language and every Σ^1_n-sentence φ over the language of M, for any finite list of finitary predicates $\vec{A}$,

$$(\langle M, \vec{A}\rangle \models \varphi)^V \iff 1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\langle \check{M}, \check{\vec{A}}\rangle \models \varphi).$$

For a class Γ of forcing notions, Γ-generic absoluteness is the statement that $\mathbb{P}$-generic absoluteness holds for every $\mathbb{P} \in \Gamma$.

One might wish to work with canonical models such as H_{ω_1} in the above. As far as we are concerned it turns out that Γ-generic $\mathbf{\Sigma}^1_1(H_{\omega_1})$-absoluteness is equivalent to Γ-generic $\mathbf{\Sigma}^1_1(\kappa)$-absoluteness, where $\kappa = 2^\omega$, that is, as far as the classes of c.c.c, proper, semi-proper, stationary set preserving or subcomplete forcing are concerned.

Background on generic absoluteness

Observation

- For any poset $\mathbb{P}$, $\mathbb{P}$-generic $\mathbf{\Sigma}^1_1(\omega)$-absoluteness holds.
- If CH fails then $\mathcal{A}dd(\omega_1, 1)$-generic $\mathbf{\Sigma}^1_1(\omega_2)$-absoluteness fails.
- $\mathrm{Col}(\omega_1, \omega_2)$-generic $\mathbf{\Sigma}^1_1(\omega_2)$-absoluteness fails.
- If $\mathbb{P}$ is a forcing that adds a real, then $\mathbb{P}$-generic $\mathbf{\Sigma}^1_1(2^\omega)$-absoluteness fails.

Theorem ([**?**])

- *Countably closed-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness is provable in* ZFC.
- *The countably closed maximality principle implies countably closed-generic $\mathbf{\Sigma}^1_2(H_{\omega_1})$-absoluteness.*

Dually to the situation with countably closed forcing, the underlying main question is whether subcomplete-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness is provable in ZFC.

Generic absoluteness and trees

Lemma

Assume CH. *Let* $\mathbb{P}$ *be a forcing notion. Then the following are equivalent.*

1. *Whenever* T *is an* $(\omega_1, \leq\omega_1)$*-Aronszajn tree, then it is not the case that* $1_{\mathbb{P}}$ *forces that* $\check{T}$ *has a cofinal branch, and it is not the case that* $1_{\mathbb{P}} \Vdash_{\mathbb{P}}$ *"there is a new real".*
2. $\mathbb{P}$*-generic* $\mathbf{\Sigma}^1_1(\omega_1)$*-absoluteness holds.*

Thus we have a convenient rephrasing of our main question about whether subcomplete-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness is provable in ZFC.

Question (Main)

Can subcomplete forcing add cofinal branches to $(\omega_1, \leq\omega_1)$-Aronszajn trees?

Subcomplete forcing

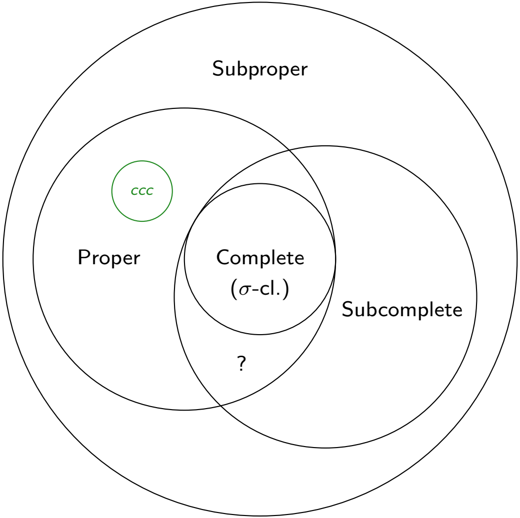
Subcomplete forcing is a class of forcing notions defined by Ronald B. Jensen. Subcomplete forcing does not add reals, but may potentially alter cofinalities to ω.

Examples of subcomplete forcing

- **Countably closed forcing**.
- **Namba forcing**, denoted by $\mathbb{N}$, a forcing notion consisting of subtrees $T \neq \emptyset$ of $\omega_2^{<\omega}$ ordered by inclusion, such that T is downward closed in $\omega_2^{<\omega}$ and where each node in T has ω_2-many eventual successors in T. Namba forcing adds a cofinal sequence $S : \omega \longrightarrow \omega_2^V$ to the extension. Under CH, Namba forcing adds no new reals and is subcomplete [**?**, Section 3.3].
- **Prikry forcing**, which forces a measurable cardinal to have cofinality ω while preserving cardinalities.
- **Generalized diagonal Prikry forcing**

Subcomplete forcing

How subcompleteness fits in with other forcing classes which preserve stationary subsets of ω_1:



Subcomplete forcing's effect on trees

Theorem

The following properties of an ω_1-tree T are preserved by subcomplete forcing:

1. *T is Aronszajn*
2. *T is not Kurepa*
3. *T is Suslin*
4. *T is Suslin and* UBP
5. *T is* $\textit{Suslin off the generic branch}$
6. *T is n-fold* $\textit{Suslin off the generic branch}$ *(for $n \geq 2$)*
7. *T is $(n-1)$-fold* $\textit{Suslin off the generic branch}$ *and n-fold* UBP *(for $n \geq 2$)*

Subcomplete forcing's effect on wider trees

Observation

- Subcomplete (or even countably closed) forcing may add a cofinal branch to an $(\omega_1, \leq 2^\omega)$-tree.
- Suppose that Friedman's Principle fails for ω_2. Then subcomplete forcing may add a cofinal branch to an $(\omega_1, \leq \omega_2 \cdot 2^\omega)$-Aronszajn tree.
- Subcomplete forcing cannot add (cofinal) branches to $(\omega_1, <2^\omega)$-trees.

Again we turn to the question stated earlier:

Question (Main)

Can subcomplete forcing add cofinal branches to $(\omega_1, \leq \omega_1)$-Aronszajn trees?

By the third bullet point in the above observation, if *CH* fails, then the answer to the main question is no.

Generic absoluteness and bounded forcing axioms

Theorem ([?, Theorem 5])

Let $\mathbb{P}$ be a forcing notion. Then the following are equivalent:

1. *The bounded forcing axiom holds for $\mathbb{P}$.*
2. *$\mathbb{P}$-generic $\Sigma_1(H_{\omega_2})$-absoluteness holds, meaning: if $\varphi(\vec{x})$ is a Σ_1-formula in the language of set theory and $\vec{a} \in H_{\omega_2}$, then $\langle H_{\omega_2}, \in \rangle \models \varphi(\vec{a})$ iff $\langle H_{\omega_2}, \in \rangle^{V^{\mathbb{P}}} \models \varphi(\vec{a})$.*

Observation

Let $\mathbb{P}$ be a notion of forcing that does not add reals. Consider the following statements.

1. $\mathbb{P}$-generic $\Sigma_1(H_{\omega_2})$-absoluteness holds.
2. $\mathbb{P}$-generic $\boldsymbol{\Sigma}^1_1(2^\omega)$-absoluteness holds.

We have that $2 \implies 1$, and if CH holds, then $1 \implies 2$. In particular, under CH, the two are equivalent.

Answer to the main question

Theorem

Assuming CH*, the following are equivalent.*

1. BSCFA*.*
2. *Subcomplete generic* $\mathbf{\Sigma}^1_1(\omega_1)$*-absoluteness.*
3. *Subcomplete forcing preserves* $(\omega_1, \leq\omega_1)$*-Aronszajn trees.*

This puts us in a position to answer the main question asking whether subcomplete forcing may add a cofinal branch to an $(\omega_1, \leq\omega_1)$-Aronszajn tree, completely.

Theorem

Splitting in two cases, we have:

1. *If* CH *fails, then subcomplete forcing preserves* $(\omega_1, \leq\omega_1)$*-Aronszajn trees.*
2. *If* CH *holds, then subcomplete forcing preserves* $(\omega_1, \leq\omega_1)$*-Aronszajn trees iff* BSCFA *holds.*

Other forcing classes

Observation

Let Γ be a class of forcings such that if $\mathbb{P} \in \Gamma$ and $p \in \mathbb{P}$, then $\mathbb{P}_{\leq p} \in \Gamma$. Then 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3.:

1. BFA_Γ.
2. Γ-generic $\boldsymbol{\Sigma}^1_1(\omega_1)$-absoluteness.
3. Forcings in Γ preserve $(\omega_1, \leq\omega_1)$-Aronszajn trees.

Observation

Let Γ contain all proper forcing notions or all subcomplete ones. Then 3. does not imply 1..

Proof.

It follows from MA_{ω_1} that every $(\omega_1, \leq\omega_1)$-tree is special, and hence that every such tree is preserved by every ω_1-preserving forcing. The consistency strength of MA_{ω_1} is the same as that of ZFC, while the consistency strength of BFA_Γ is at least a reflecting cardinal, which is strictly higher. □

Final remarks and questions

The general relationship between the pertinent properties is unclear.

Question

Let Γ be the class of proper, semi-proper, stationary set preserving or subcomplete forcings. Which implications hold between the following properties?

1. BFA_Γ.
2. Γ-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness.
3. Forcings in Γ preserve $(\omega_1, \leq\omega_1)$-Aronszajn trees.

There are some interesting questions about subcomplete-generic absoluteness when CH fails. In this case, BSCFA may still hold.

Question

What is the consistency strength of $\neg$CH together with subcomplete-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness?

Thank you.