Subcomplete Forcing, Trees, and Generic Absoluteness

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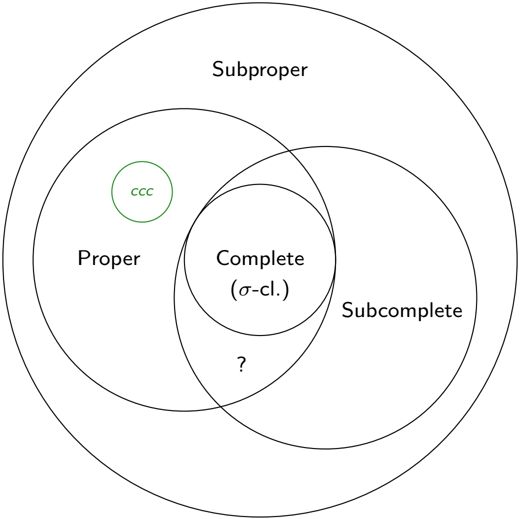
Subcomplete forcing is a class of forcing notions defined by Ronald B. Jensen. Subcomplete forcing does not add reals, but may potentially alter cofinalities to ω.

Examples of subcomplete forcing

- **Countably closed forcing**.
- **Namba forcing**, denoted by $\mathbb{N}$, a forcing notion consisting of subtrees $T \neq \emptyset$ of $\omega_2^{<\omega}$ ordered by inclusion, such that T is downward closed in $\omega_2^{<\omega}$ and where each node in T has ω_2-many eventual successors in T. Each condition in $\mathbb{N}$ has size ω_2. Namba forcing adds a cofinal sequence $S : \omega \longrightarrow \omega_2^V$ to the extension, a cofinal branch through $\omega_2^{<\omega}$. Under CH, Namba forcing adds no new reals and is subcomplete [**?**, Section 3.3].
- **Prikry forcing**, which forces a measurable cardinal to have cofinality ω while preserving cardinalities.
- **Generalized diagonal Prikry forcing**

- Revised countable support (*rcs*) iterations of subcomplete forcing notions.
- Lottery sums of subcomplete forcing notions.
- If $\mathbb{P}$ is subcomplete and $\pi : \mathbb{P} \longrightarrow \mathbb{Q}$ is a dense embedding, then $\mathbb{Q}$ is subcomplete.

How subcompleteness fits in with other forcing classes which preserve stationary subsets of ω_1:



Theorem

The following properties of an ω_1-tree T are preserved by subcomplete forcing:

1. *T is Aronszajn*
2. *T is not Kurepa*
3. *T is Suslin*
4. *T is Suslin and* UBP
5. *T is* *Suslin off the generic branch*
6. *T is n-fold* *Suslin off the generic branch* *(for $n \geq 2$)*
7. *T is $(n-1)$-fold* *Suslin off the generic branch* *and n-fold* UBP *(for $n \geq 2$)*

Observation

Subcomplete (or even countably closed) forcing may add a cofinal branch to an $(\omega_1, \leq 2^\omega)$*-tree.*

Proposition

Suppose that Friedman's Principle fails for ω_2*. Then subcomplete forcing may add a cofinal branch to an* $(\omega_1, \leq\omega_2 \cdot 2^\omega)$*-Aronszajn tree.*

Theorem

Subcomplete forcing cannot add (cofinal) branches to $(\omega_1, <2^\omega)$*-trees.*

Next, we will look at the preservation of wide Aronszajn trees from a different angle, and show that under CH, dropping the restriction to trees with countable levels amounts to making a statement about a certain form of generic absoluteness which we introduce in the following.

Definition

Let n be a natural number, let $\mathbb{P}$ be a notion of forcing, and let κ be a cardinal. Then $\mathbb{P}$*-generic* $\mathbf{\Sigma}^1_n(\kappa)$*-absoluteness* is the statement that for any model M of size κ for a countable first order language and every Σ^1_n-sentence φ over the language of M, the following holds:

$$(\langle M, \vec{A} \rangle \models \varphi)^V \iff 1_{\mathbb{P}} \Vdash_{\mathbb{P}} (\langle \check{M}, \check{\vec{A}} \rangle \models \varphi)$$

Here, we are referring to the second order satisfaction relation of M associated to its first order language. That is, the second order variables occurring in the body of ϕ are treated as some finitary relations.

For a class Γ of forcings, Γ-generic $\mathbf{\Sigma}^1_n(\kappa)$-absoluteness is the statement that $\mathbb{P}$-generic $\Sigma^1_n(M)$-absoluteness holds for every $\mathbb{P} \in \Gamma$. The classes of interest to us are the classes of c.c.c, proper, semi-proper, stationary set preserving or subcomplete forcings.

Observation

For any poset $\mathbb{P}$, $\mathbb{P}$-generic $\mathbf{\Sigma}^1_1(\omega)$-absoluteness holds.

There is a natural version of the absoluteness properties introduced in the previous definition where one talks about a certain canonical structure M, defined by a formula to be re-interpreted in $\mathrm{V}[G]$. For example, let's define $\mathbb{P}$-generic $\mathbf{\Sigma}^1_1(H_{\omega_1})$-absoluteness to mean

$$(\langle H_{\omega_1}, \vec{A}\rangle \models \varphi)^V \iff (\langle H_{\omega_1}, \vec{A}\rangle \models \varphi)^{V^{\mathbb{P}}}$$

for any finite set of finitary predicates $\vec{A}$, where H_{ω_1} is *re-interpreted* in $\mathrm{V}^{\mathbb{P}}$ on the right hand side, rather than working with the same model on both sides. Further, Γ-generic $\mathbf{\Sigma}^1_1(H_{\omega_1})$-absoluteness means that this holds for every $\mathbb{P} \in \Gamma$.
It turns out that Γ-generic $\mathbf{\Sigma}^1_1(H_{\omega_1})$-absoluteness is equivalent to Γ-generic $\mathbf{\Sigma}^1_1(\kappa)$-absoluteness, where $\kappa = 2^\omega$, as far as we are concerned, that is, as far as the classes of forcings listed in the definition are concerned.

Lemma

Assume CH. *Let* $\mathbb{P}$ *be a forcing notion. Then the following are equivalent.*

1. *Whenever* T *is an* $(\omega_1, \leq\omega_1)$*-Aronszajn tree, then it is not the case that* $1_{\mathbb{P}}$ *forces that* $\check{T}$ *has a cofinal branch, and it is not the case that* $1_{\mathbb{P}} \Vdash_{\mathbb{P}}$ *"there is a new real".*
2. $\mathbb{P}$*-generic* $\mathbf{\Sigma}^1_1(\omega_1)$*-absoluteness holds.*

Lemma

Assume CH. *Let* $\mathbb{P}$ *be a forcing notion. Then the following are equivalent.*

1. $\mathbb{P}$ *preserves* $(\omega_1, \leq\omega_1)$*-Aronszajn trees and does not add reals.*
2. *Strong* $\mathbb{P}$*-generic* $\mathbf{\Sigma}^1_1(\omega_1)$*-absoluteness holds.*

Theorem ([**?**, Theorem 5])

Let $\mathbb{P}$ be a forcing notion. Then the following are equivalent:

1. *The bounded forcing axiom holds for $\mathbb{P}$.*
2. *$\mathbb{P}$-generic $\Sigma_1(H_{\omega_2})$-absoluteness holds, meaning: if $\varphi(\vec{x})$ is a Σ_1-formula in the language of set theory and $\vec{a} \in H_{\omega_2}$, then $\langle H_{\omega_2}, \in \rangle \models \varphi(\vec{a})$ iff $\langle H_{\omega_2}, \in \rangle^{\mathrm{V}^{\mathbb{P}}} \models \varphi(\vec{a})$.*

A very popular way of reformulating this theorem is as follows.

Theorem

Let $\mathbb{P}$ be a forcing notion. Then the following are equivalent:

1. *The strong bounded forcing axiom holds for $\mathbb{P}$, meaning that the bounded forcing axiom holds for $\{\mathbb{P}_{\leq p} \mid p \in \mathbb{P}\}$.*
2. *Strong $\mathbb{P}$-generic $\Sigma_1(H_{\omega_2})$-absoluteness holds: if G is $\mathbb{P}$-generic over V, then*

$$\langle H_{\omega_2}, \in \rangle \prec_{\Sigma_1} \langle H_{\omega_2}, \in \rangle^{\mathrm{V}[G]}$$

Theorem

Assuming CH*, the following are equivalent.*

1. BSCFA.
2. *Subcomplete generic* $\mathbf{\Sigma}^1_1(\omega_1)$*-absoluteness.*
3. *Subcomplete forcing preserves* $(\omega_1, {\leq}\omega_1)$*-Aronszajn trees.*

This puts us in a position to answer Question **??**, asking whether subcomplete forcing may add a cofinal branch to an $(\omega_1, {\leq}\omega_1)$-Aronszajn tree, completely. Recall Theorem 2, which gives us part 1. of the following theorem.

Theorem

Splitting in two cases, we have:

1. *If* CH *fails, then subcomplete forcing preserves* $(\omega_1, {\leq}\omega_1)$*-Aronszajn trees.*
2. *If* CH *holds, then subcomplete forcing preserves* $(\omega_1, {\leq}\omega_1)$*-Aronszajn trees iff* BSCFA *holds.*

Observation

Let Γ be a class of forcings such that if $\mathbb{P} \in \Gamma$ and $p \in \mathbb{P}$, then $\mathbb{P}_{\leq p} \in \Gamma$. Then 1. $\Longrightarrow$ 2. $\Longrightarrow$ 3.:

1. BFA_Γ.
2. Γ*-generic* $\mathbf{\Sigma}^1_1(\omega_1)$*-absoluteness.*
3. *Forcings in* Γ *preserve* $(\omega_1, {\leq}\omega_1)$*-Aronszajn trees.*

The bounded forcing axiom for any of the canonical classes of forcing other than subcompleteness implies the failure of CH. This is why we slowly shift the focus to $\neg$CH in the following. First, let's observe some limitations on Γ-generic $\Sigma^1_1(\kappa)$-absoluteness.

Observation

We have the following limitations on generic $\mathbf{\Sigma}^1_1(\kappa)$*-absoluteness, for* κ *an arbitrary cardinal not necessarily* ω_1*.*

1. *If* CH *fails then* $\mathcal{A}dd(\omega_1, 1)$*-generic* $\mathbf{\Sigma}^1_1(\omega_2)$*-absoluteness fails.*
2. $\mathrm{Col}(\omega_1, \omega_2)$*-generic* $\mathbf{\Sigma}^1_1(\omega_2)$*-absoluteness fails.*
3. *If* $\mathbb{P}$ *is a forcing that adds a real, then* $\mathbb{P}$*-generic* $\mathbf{\Sigma}^1_1(2^\omega)$*-absoluteness fails.*

Corollary

The following are equivalent:

1. MA.
2. *for every cardinal $\kappa < 2^\omega$, c.c.c.-generic $\mathbf{\Sigma}^1_1(\kappa)$-absoluteness holds.*

For the other classes of forcing, we can answer a related question easily.

Observation

Let Γ contain all proper forcing notions or all subcomplete ones. Then the assertion that forcings in Γ preserve $(\omega_1, {\leq}\omega_1)$-trees does not imply BFA_Γ.

But the general relationship between the pertinent properties is unclear.

Question

Let Γ be the class of proper, semi-proper, stationary set preserving or subcomplete forcings. Which implications hold between the following properties?

1. BFA_Γ.
2. *Γ-generic $\mathbf{\Sigma}^1_1(\omega_1)$-absoluteness.*
3. *Forcings in Γ preserve $(\omega_1, {\leq}\omega_1)$-Aronszajn trees.*

There are some interesting questions about subcomplete-generic absoluteness when CH fails. In this case, BSCFA may still hold – for example, BMM, the bounded forcing axiom for stationary set preserving forcings, implies $\mathrm{BSCFA} + 2^\omega = \omega_2$. In a model of $\mathrm{BSCFA} + \neg\mathrm{CH}$, we clearly have subcomplete-generic $\mathbf{\Sigma}^1_1(\omega_1)$ absoluteness, by Observation 3. The consistency strength of BMM is known to be much higher than a reflecting cardinal, see [**?**]. We have seen in Theorem 2 that under $\neg$CH, subcomplete forcing preserves $(\omega_1, {\leq}\omega_1)$-Aronszajn trees, so CH certainly does not imply BSCFA. We are thus left with the following additional question:

Question

What is the consistency strength of $\neg$CH *together with subcomplete-generic* $\mathbf{\Sigma}^1_1(\omega_1)$*-absoluteness?*

Thank you.