# Subcomplete Forcing and Trees

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## Abstract

In this talk I will introduce subcompleteness, and we will familiarize ourselves with the definition by seeing it does not add reals. We will then see that subcomplete forcing does not add branches to $\omega_1$-trees, one of a handful of recent results examining the interaction between subcomplete forcing and $\omega_1$–trees.

# Subcomplete forcing

Subcomplete forcing is a class of forcing notions defined by Ronald B. Jensen. Subcomplete forcing does not add reals, but may potentially alter cofinalities to $\omega$.

## Known examples of subcomplete forcing

- **Countably closed forcing**.
- **Namba forcing**, under CH, which forces the cofinality of $\omega_2$ to be $\omega$.
- **Prikry forcing**, which forces a measurable cardinal to have cofinality $\omega$ while preserving cardinalities.
- Certain kinds of **generalized diagonal Prikry forcing** which force a sequence of measurables to have cofinality $\omega$.
- The forcing $\mathbb{P}_{\mathbf{A}}$, which shoots a closed set of order type $\omega_1$ through a stationary set $A \subseteq \kappa$ of $\omega$-cofinal ordinals, and ultimately forces $\neg\Box_\kappa$.

# The Standard Setup

We follow Jensen in his definitions of these classes of forcings, which takes some getting used to. In his standard setup, given a poset $\mathbb{P}$, we work with models $N$ of the form:

$$H_\theta \subseteq N = \langle L_\tau[A], \in, A \cap L_\tau[A] \rangle \models \mathrm{ZFC}^-$$

for some $\tau > \theta$ and $A \subseteq \tau$, where $\theta$ is large enough so that $\mathbb{P} \in H_\theta$.
The reason for working with such models is that such $N$ will naturally contain well orders of $H_\theta$, among other useful bits of information like the Skolem functions of $H_\theta$.
Furthermore, we will often take a countable substructure $X$ of $N$, and then the transitive collapse $\overline{N}$ of this structure. Embeddings arising from this procedure will be denoted

$$\sigma : \overline{N} \cong X \prec N.$$

For elements $s \in N$ we often write $\overline{s}$ for its preimage in $\overline{N}$, ie. $\sigma(\overline{s}) = s$.

# Complete forcing

In order to define subcomplete forcing, we will first need to define Shelah's notion of complete forcings. Roughly speaking, complete forcings posit that below some condition, generics for countable substructures extend to generics for the larger structure.

## Definition

$\mathbb{P}$ is ***complete*** $\iff$ for sufficiently large $\theta$ we have; letting:

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \text{ZFC}^-, \tau > \theta$
- $\sigma : \overline{N} \cong X \prec N$ where $X$ is countable and $\overline{N}$ is transitive

if $\overline{G}$ is $\overline{\mathbb{P}}$-generic over $\overline{N}$
then there is $p \in \mathbb{P}$ such that whenever $G \ni p$ is $\mathbb{P}$-generic, $\sigma``\overline{G} \subseteq G$.
In particular, this means $p$ forces that $\sigma$ lifts to an elementary embedding

$$\sigma^* : \overline{N}[\overline{G}] \to N[G].$$

# Complete forcing

## Proposition

*If $\mathbb{P}$ is countably closed then $\mathbb{P}$ is complete.*

## Proof.

Let $\mathbb{P}$ be countably closed, and suppose we have $\theta >> |\mathbb{P}|$ satisfying the standard setup:

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \mathsf{ZFC}^-, \tau > \theta$
- $\sigma : \overline{N} \cong X \prec N$ where $X$ is countable and $\overline{N}$ is transitive

and let $\overline{G} \subseteq \overline{\mathbb{P}}$ be $\overline{N}$-generic. We can find such a $\overline{G}$ since $\overline{N}$ is countable. Take $p \leq \bigcup \sigma``\overline{G}$, which works since $\mathbb{P}$ is in fact countably directed closed. □

# Complete forcing

## Proposition (Jensen)

*If $\mathbb{P}$ is complete, then $\mathbb{P}$ is equivalent to a countably closed poset.*

## Proof sketch.

Let $\theta$ verify the completeness of $\mathbb{P}$. Define a poset $\mathbb{Q}$ of conditions $q = \langle X_q, G_q \rangle$ such that $\mathbb{P}, \theta \in X_q \prec N$ where $X_q$ is countable and $G_q$ is $\mathbb{P}$–generic over $X_q$. Let

$$q \leq r \iff X_q \supseteq X_r \text{ and } G_r = G_q \cap X_r.$$

Clearly $\mathbb{Q}$ is countably closed. Define $\pi : \mathbb{Q} \to \mathrm{BA}(\mathbb{P})$ by taking $\pi(q) = \cap G_q$. That $\cap G_q \neq \emptyset$ follows as $\mathbb{P}$ is complete. Also verify:

- $q \leq r \implies \cap G_q \leq \cap G_r$
- $q || r \iff \cap G_q \bigcap \cap G_r \neq \emptyset$
- $D = \{\cap G_q \mid q \in \mathbb{Q}\}$ is dense in $\mathbb{P}$

$\square$

# Definition of subcompleteness

We now define subcomplete forcing. Generally speaking, instead of requiring the original embedding to lift we only ask that below some condition, there is an embedding sufficiently similar to the original one which lifts.

## Definition

$\mathbb{P}$ is ***subcomplete*** $\iff$ for sufficiently large $\theta$ we have; letting:

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \mathrm{ZFC}^-, \tau > \theta$
- $\sigma : \overline{N} \cong X \prec N$ where $X$ is countable and $\overline{N}$ is full [a]
- $\sigma(\overline{\theta}, \overline{\mathbb{P}}, \overline{s}) = \theta, \mathbb{P}, s$ for some $s \in N$;

if $\overline{G}$ is $\overline{\mathbb{P}}$-generic over $\overline{N}$ then there is $p \in \mathbb{P}$ such that whenever $G \ni p$ is $\mathbb{P}$-generic, there is $\sigma_0 \in V[G]$ satisfying:

1. $\sigma_0 : \overline{N} \to N$ elementarily
2. $\sigma_0(\overline{\theta}, \overline{\mathbb{P}}, \overline{s}) = \theta, \mathbb{P}, s$
3. $\mathcal{S}k^N(X \cup \mathbb{P}) = \mathcal{S}k^N(\mathrm{range}(\sigma_0) \cup \mathbb{P})$ [b]
4. $\sigma_0 `` \overline{G} \subseteq G$.

In particular, $p$ forces that there is a $\sigma_0$ which lifts by **4** to an embedding

$$\sigma_0^* : \overline{N}[\overline{G}] \to N[G].$$

[a] Fullness is a strengthens transitivity and is explained on the next slide.

[b] This is a technical condition that is mostly only used in iteration theorems.

# Fullness

What is fullness, and why is it required?

**Definition**

$\overline{N}$ is ***full*** $\iff$ $\overline{N}$ is transitive, $\omega \in \overline{N}$, and there is a $\gamma$ such that $L_\gamma(\overline{N}) \models \mathrm{ZFC}^-$ and $\overline{N}$ has regular height in $L_\gamma(\overline{N})$.

- If $\overline{N}$ is full, then $\overline{N}$ doesn't have definable Skolem functions; in particular it is not pointwise definable.
- Thus, if $\overline{N} \cong X \prec N$, where $X$ is countable, $\overline{N}$ is full, and $N = L_\tau[A]$, there may be more than one elementary embedding $\sigma : \overline{N} \to N$.

# Subcomplete forcing doesn't add reals.

**Proposition (Jensen)**

*If $\mathbb{P}$ is subcomplete, then $\mathbb{P}$ doesn't add any new reals.*

**Proof.**

Suppose toward a contradiction that $\mathbb{P}$ adds a new real. Let $q \in \mathbb{P}$ force that $\dot{r} : \check{\omega} \to 2$ is new. Let $\theta$ be large enough so that:

- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \mathrm{ZFC}^-, \tau > \theta$
- $\sigma : \overline{N} \cong X \prec N$ where $X$ is countable and $\overline{N}$ is full
- $\sigma(\overline{\theta}, \overline{\mathbb{P}}, \overline{q}, \overline{\dot{r}}) = \theta, \mathbb{P}, q, \dot{r}$.

Let $\overline{G} \subseteq \overline{\mathbb{P}}$ be an $\overline{N}$-generic filter with $\overline{q} \in \overline{G}$. Again, there is such a generic since $\overline{N}$ is countable.

## Proof continued.

By subcompleteness of $\mathbb{P}$, there is $p \in \mathbb{P}$ which forces that whenever $G \ni p$ is $\mathbb{P}$-generic, there is $\sigma_0 \in V[G]$ such that

- $\sigma_0 : \overline{N} \to N$ elementarily
- $\sigma_0(\overline{\theta}, \overline{\mathbb{P}}, \overline{q}, \overline{\dot{r}}) = \theta, \mathbb{P}, q, \dot{r}$
- $\sigma_0$ " $\overline{G} \subseteq G$.

Thus $\sigma_0$ lifts to $\sigma_0^* : \overline{N}[\overline{G}] \prec N[G]$ in $V[G]$. Let $r = \dot{r}^G$, $\overline{r} = \overline{\dot{r}}^{\overline{G}}$. Then for each $n < \omega$ we have that

$$r(n) = \sigma_0^*(\overline{r}(n)) = \overline{r}(n) \in V,$$

contradicting our original assumption that $q \in G$ forces $\dot{r}$ to be new. □

The idea here is that $\overline{N}$ is countable, so the critical point of $\sigma_0^*$, and indeed of any such $\sigma$ or $\sigma_0$, is $\omega_1^{\overline{N}}$.

# Subcomplete forcing doesn't add branches

### Theorem (Fuchs, M.)

*Let $T$ be an $\omega_1$-tree. If $\mathbb{P}$ is subcomplete then $\mathbb{P}$ does not add new cofinal branches to $T$.*

Some results immediately follow.

### Corollary

Subcomplete forcing preserves Aronszajn trees.

### Corollary

If an $\omega_1$-tree is not Kurepa, it cannot become Kurepa in a subcomplete forcing extension.

# Subcomplete forcing doesn't add branches

This proof is a forcing argument which uses a standard lifting technique in the context of small embeddings.

## Proof sketch (of Theorem)

Assume not. Let $q$ be a condition forcing that $\dot{b}$ is a new cofinal branch through $\check{T}$. Let $\theta$ be large enough so that:

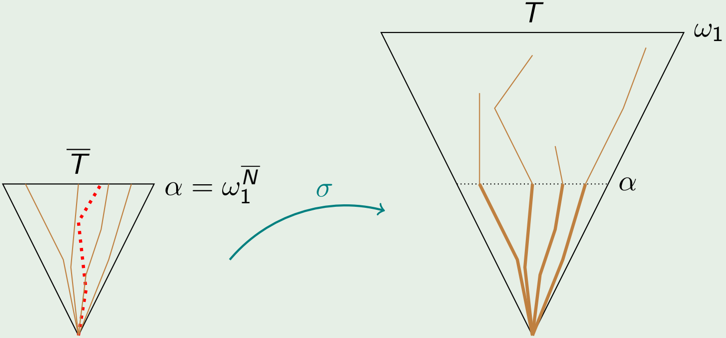
- $\mathbb{P} \in H_\theta \subseteq N = L_\tau[A] \models \mathrm{ZFC}^-, \tau > \theta$
- $\sigma : \overline{N} \cong X \prec N$ where $X$ is countable and $\overline{N}$ is full
- $\sigma(\overline{\theta}, \overline{\mathbb{P}}, \overline{T}, \overline{q}, \overline{\dot{b}}) = \theta, \mathbb{P}, T, q, \dot{b}$.

By elementarity, $\overline{q}$ forces $\overline{\dot{b}}$ to be a new cofinal branch through $\overline{\check{T}}$.
Let $\alpha = \omega_1^{\overline{N}}$. Note that $\mathrm{cp}(\sigma) = \alpha$.
The idea is to construct a generic $\overline{G}$ for $\overline{\mathbb{P}}$ over $\overline{N}$, using the countability of $\overline{N}$ to diagonalize against all branches through $T$ as seen on level $\alpha$ of the tree in $N$.

## Proof sketch continued.

In particular since $\alpha$ is countable we may enumerate the branches on level $\alpha$ of $T$ in $N$. Also list the dense sets of $\overline{\mathbb{P}}$ in $\overline{N}$.

Then inductively define a sequence of conditions of the form $\overline{q}_n$, where $\overline{q}_0 = \overline{q}$, deciding values of $\dot{\overline{b}}$ in $\overline{T}$ differently than the $n$th branch we named. Ensure along the way that $\overline{q}_n$ gets in the $n$th dense set from our list, and extends the previous condition.




$\overline{T} = T \restriction \alpha$

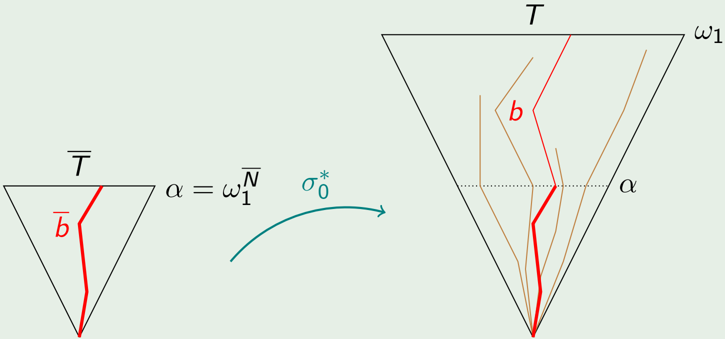
## Proof sketch continued.

Let $\overline{G}$ be the generic filter generated by the $\overline{q}_n$, let $\dot{\overline{b}}^{\overline{G}} = \overline{b}$. Since $\mathbb{P}$ is subcomplete, there is a condition $p \in \mathbb{P}$ such that whenever $G$ is $\mathbb{P}$-generic with $p \in G$, we have $\sigma_0 \in V[G]$ such that:

- $\sigma_0 : \overline{N} \to N$ elementarily
- $\sigma_0(\overline{\theta}, \overline{\mathbb{P}}, \overline{T}, \overline{q}, \dot{\overline{b}}) = \theta, \mathbb{P}, T, q, \dot{b}$
- $\sigma_0``\overline{G} \subseteq G$.

So there is a lift $\sigma_0^* : \overline{N}[\overline{G}] \to N[G]$ elementary, extending $\sigma_0$, with $\sigma_0^*(\overline{b}) = \sigma_0(\dot{\overline{b}})^G = \dot{b}^G = b$, and $\sigma_0^*(\overline{T}) = \sigma_0(\overline{T})^G = T$. Now we have $N[G] \models q \in G$, so $b$ is a cofinal branch through $T$.

## Proof sketch continued.




Since $\alpha$ is the critical point of the embedding, in $N[G]$, $b \upharpoonright \alpha = \overline{b}$. However, $\overline{b}$ was constructed so as to not be equal to any branch restricted to level $\alpha$, the ones we listed out initially, a contradiction. $\square$

# Further results

## Theorem (Jensen)

*If $\mathbb{P}$ is subcomplete and $T$ is a Suslin tree, then $T$ is Suslin in $V[G]$.*

## Some related results (Fuchs, M.)

- *ccc* forcings are not subcomplete.
- Subcomplete forcing may add maximal antichains to $\omega_1$–trees.
- Subcomplete forcing preserves the unique branch, or the Suslin-off-a-generic-branch property, of a Suslin tree.
- If $\mathrm{cof}\,(\kappa) = \omega_1$ then subcomplete forcing cannot add a thread to a $\Box(\kappa, \omega)$ sequence.

## Open Question

Can subcomplete forcing add branches to wide trees of height $\omega_1$?

Thank you.