Test of renormalisation group transformation for Ising model in 1D

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This is short theoretical explanation of the test: **IsingTestRenormGroup1D.h**.

1 Idea of renormalisation group

When one is near a critical point it is very difficult to compute the partition function numerically or by various expansion techniques. This is because of the presence of many length scales in the system. At high T, there is only short-range order, the spins form small clusters. The correlation length (approximately equal to the linear size of the largest cluster) is small. Close T_C somewhat larger patches in which most of the spins are lined up in the same direction begin to develop. When the system reaches T_C , these patches expand to infinite size, but fluctuations of smaller scale persist. At the critical temperature, spins form clusters at all lengthscales, including one infinite-size cluster (which cannot be seen on a finite system, but we know that the correlation length has to diverge at T_C). As a result, all scales of length must be included in a theoretical description .

In the theory of phase transitions , one is interested in the large distance behaviour or macroscopic properties of physical observables near the transition temperature $T = T_C$. At the critical temperature, the correlation length, which defines the scale on which correlations above T_C decay exponentially, diverges and the correlation functions decay only algebraically. This gives rise to non-trivial large distance properties that are, to a large extent, independent of the short distance structure, a property called universality.

Renormalisation group (RG) is a mathematical method that allows systematic investigation of the changes of a physical system as viewed at different distance scales. As the scale varies, it is as if we are changing magnifying glass we observe the system through. The system at one scale will generally be seen to consist of self-similar copies of itself when viewed at a smaller scale, with different parameters describing the components of the system. The components, or fundamental variables, may relate to atoms, elementary particles, atomic spins, etc. The parameters of the theory typically describe the interactions of the components. These may be variable "couplings" which measure the strength of various forces.

The method is based on averaging the components of the system over some blocks, e.g. in the 1D Ising model we build blocks of few neighbouring spins and assign a spin to them (following specific rule for that - like majority rule). That operation changes the correlation length allowing us to look at the system from “further away”. We might apply the operation recursively.

Above T_C, there is no long-range order, spins form random ”up” and ”down” clusters. Under each transformation, the correlation length (expressed in units of the new unit cell) decreases, the clusters become smaller and smaller as if the temperature were higher, but they never disappear. Decrease of the correlation length under successive transformations is an extremely useful property because it makes the fluctuations uncorrelated and we can solve the system using an approximate theory to calculate the properties in that region and then transform back to the original lattice.

2 Method

We consider 1-dimensional Ising model: chain of spins interacting only with their nearest neighbours at some set temperature T. We investigate system near the critical point, so in 1-dimensional case$T$ should be near 0 K.

Now we want to recursively average over short distance degrees of freedom. We divide the chain into blocks of 3 spins and assign each block a spin resulting from majority rule (block spin ”up” if the majority of the spins in the block is ”up”, and vice versa).:

$$S = sgn(\sum_{i=1}^{3} S_i).$$

In this way the length scale of the lattice is changed by a factor 3 each time. This is a ”real-space block-spin renormalization-group transformation”.

Quantity which will show us the results of renormalisation group transformation is the correlation length, which is:

$$\langle S_i; S_j\rangle = \langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle.$$

It shows us how much the two spins S_i and S_j are correlated. If the spins are independent then this quantity will be zero. Typically this quantity decays exponentially as $\mid$ i $-$ j $\mid\rightarrow\infty$ for all temperatures but critical temperature:

$$\langle S_i; S_j\rangle \propto \exp(-l/\xi)$$

where $l = |i - j|$ and ξ is a correlation length. Infinite chain limit of the correlation function exists and is given by:

$$\langle S_i; S_j\rangle = (\tanh(\beta))^l$$

where $\beta = 1/k_B T$. So the correlation length is given by:

$$\xi = \frac{-1}{\ln(\tanh(\beta))}.$$

3 Tests

In *IsingTestRenormGroup1D.h* we test renormalisation group transformation od 1D Ising chain. We apply recursively operation of averaging on blocks of 3 spins. We calculate spin-spin correlations and using formula $\langle S_i; S_j \rangle = (\tanh(\beta'))^l$ we get new interaction constants β' for shorter chains and out temperature $T' = 1/\beta'$. We plot dependence of calculated T' on length of chain (Fig. 1). We can see T' increase with decrease of chain length. That means that with each renormalisation step we wander off further from the critical point which is at 0 K. The initial temperature of the simulation was 1 K.

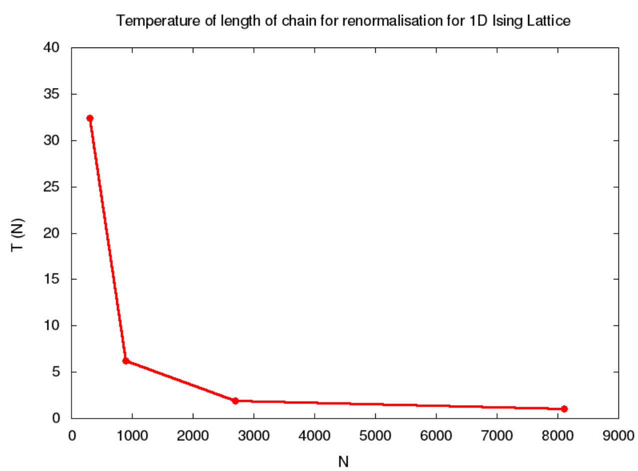

Figure 1: Temperature in function of chain length.

Additionally we plot (Fig.2) interaction constant $\beta' = 1/T'$ after one step of renormalisation in function of initial $\beta = 1/T$ (for change of chain length $900 \to 300$). For comparison we add $\beta' = \beta$ line. We can see β' after renormalisation is always smaller than β and it is getting nearer $\beta' = \beta$ line for higher temperatures. This result is consistent with similar results appearing in the literature.

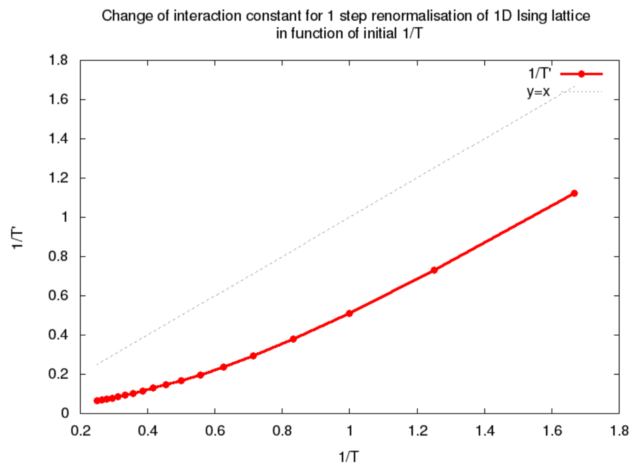

Figure 2: Change of interaction constant $\beta' = 1/T'$ after 1 step of renormalisation in function of initial $1/T$.

4 Comment on 2D model

We do not test renormalisation group transformation on 2-dimensional Ising model, however the procedure in that case would be similar. We might divide 2D lattice into blocks 3x3 and apply majority rule again. Our expectation would not be entirely the same as in 1-dimensional case. We know that further from critical point correlations between more distant spins become less important. As in 2D we have critical behaviour at some non-zero temperature T_c, the temperature calculated from spin correlation function will wander off T_c:

- in direction of ∞ if we start from $T > T_c$,
- in direction of 0 if we start from $T < T_c$.