TI Statistics 2019
TA 1
Appendix

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The file contains:

- Distribution graph by previous TAs
- TA solution for Ch4.36 (3rd edition)
- TA solution for Exam question 19

DISCRETE

HYPERBOLIC

BINOMIAL (n, p)

$X=k$ # Success out of n experiments

- $P(k)=\binom{n}{k}p^k(1-p)^{n-k}$, $k=0,1,\dots n$
- $E(X)=np$
- $Var(X)=np(1-p)$

$\sum_{i=1}^{n} X_i$

Ind. trials

$n=1$

$n\to\infty$, $\mu=np$, $\sigma^2=np(1-p)$

$n\to\infty$, $p\to 0$, $np=\lambda$

BERNOULLI (p)

$X\in\{0,1\}$ outcome of experiment

- $P(x)=\begin{cases} p^x(1-p)^{1-x} & \text{, if } x=0 \text{ or } x=1 \\ 0 & \text{, otherwise} \end{cases}$
- $E(X)=p$
- $Var(X)=p(1-p)$

Ind. trials

Ind. trials

POSSION (λ)

$X=k$ # events occur in a given interval of time

- $P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda}$, $k=0,1,2\dots$
- $e^{\lambda}=\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$
- $E(X)=\lambda$
- $Var(X)=\lambda$

$\lambda=\sigma^2$ $\mu=\lambda$

$\lambda\to\infty$

$n\to\infty$

$n(1-p)=\lambda$

generalize to rth success

GEOMETRIC (p)

$X=k$ # trials for [$\underbrace{\text{failures}}_{k-1}$ + $\underbrace{\text{first success}}_{k\text{th}}$]

infinite sequence

- $P(k)=P(X=k)=(1-p)^{k-1}p$, $k=1,2\dots$
- $\sum_{k=1}^{\infty}(1-p)^{k-1}p=1$
- $E(X)=\frac{1}{p}$
- $Var(X)=\frac{1-p}{p^2}$
- $F(x)=1-(1-p)^x$

$r=1$

NEG. BIN (r, p)

$X=k$ # trials under infinite sequence

[$\underbrace{(\text{failures} + r-1 \text{ success})}_{\text{mix } k-1}$ + rth success]

- $P(k)=P(X=k)=\binom{k-1}{r-1}p^r(1-p)^{k-r}$, $k=r, r+1\dots$
- $E(X)=\frac{r}{p}$
- $Var(X)=\frac{r(1-p)}{p^2}$

analogue

CONTINUOUS

STD. NORMAL

- $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, $x\in(-\infty,+\infty)$
- $E(X)=\mu=0$
- $Var(X)=\sigma^2=1$

$\frac{X-\mu}{\sigma}$

$\mu+\sigma X$

$n\geq 30$

$t(n)$

- $Z\sim N(0,1)$
  $U\sim\chi^2_n$
  Z, U independent
  $T_n=\frac{Z}{\sqrt{U/n}}\sim t_n$

$\sum_{i=1}^{n}X_i^2\sim\chi^2(n)$

$CHI^2(n)$

$\alpha=\frac{n}{2}$

$\lambda=\frac{1}{2}$

NORMAL (μ, σ)

model measurement error (CLT)

- $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$, $x\in(-\infty,+\infty)$
- $E(X)=\mu$
- $Var(X)=\sigma^2$
- CLT $Z_n=\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}\xrightarrow{d}N(0,1)$

the times between events of a Possion process are i.i.d follow EXP

$\sum_{i=1}^{n}X_i\sim Gamma(n,\lambda)$

EXPONENTIAL $(\lambda)\to\frac{1}{\theta}$

model Lifetime / waiting time (memoryless)

- $f(x)=\lambda e^{-\lambda x}=\frac{1}{\theta}e^{-\frac{x}{\theta}}$, $x\geq 0$
- $E(X)=\frac{1}{\lambda}=\theta$
- $Var(X)=\frac{1}{\lambda^2}=\theta^2$
- $F(x)=\begin{cases} 1-e^{-\lambda x} & , x\geq 0 \\ 0 & , x<0 \end{cases}$

$\alpha=1$

model lifetime (with memory)

GAMMA $(\alpha, \underset{n}{\lambda})$

- $f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$, $x\geq 0$
  $\Gamma(\alpha)=\int_0^\infty u^{\alpha-1}e^{-u}du$
- $E(X)=\frac{\alpha}{\lambda}$
- $Var(X)=\frac{\alpha}{\lambda^2}$

UNIFORM

CH 4.36 (3rd Edition)

- Halve original sample: g_A, g_B
- $N_j \equiv$ # of tests run in arm $j \in \{A, B\}$
- $N = N_A + N_B$ # all tests run
- Of interest $\mathbb{E}[N]$
  - ㄴ note, by symmetry, $\mathbb{E}[N] = 2\,\mathbb{E}[N_j]$
- $p \equiv \mathbb{P}$(test is negative for a single subject)

$\boxed{\text{Test } g_j}$

$\mathbb{P}(\boxed{-}) = p^{\frac{n}{2}}$ $\qquad$ $\mathbb{P}(\boxed{+}) = 1 - p^{\frac{n}{2}}$

1

test run in arm j if 'Test g_j' $= \boxed{-}$

Halve g_j: g_{j1}, g_{j2}, test both

		Test g_{j2}: $\ominus \mid \boxed{+}$, $p^{\frac{n}{4}}$	$\oplus \mid \boxed{+}$, $1 - p^{\frac{n}{4}}$
Test g_{j1}	$\ominus \mid \boxed{+}$, $p^{\frac{n}{4}}$	3	$3 + \frac{n}{4}$
	$\oplus \mid \boxed{+}$, $1 - p^{\frac{n}{4}}$	$3 + \frac{n}{4}$	$3 + 2 \cdot \frac{n}{4}$

test run in arm j if 'Test g_j' $= \boxed{+}$ and 'Test g_{j2}' $= \oplus$ and 'Test g_{j1}' $= \ominus$

$$\mathbb{E}[N_j] = 1 \cdot p^{\frac{n}{2}} + \left(1 - p^{\frac{n}{2}}\right)\Big[3 \cdot \left(p^{\frac{n}{4}}\right)^2 + \left(3 + \frac{n}{4}\right) p^{\frac{n}{4}} \left(1 - p^{\frac{n}{4}}\right) + \left(3 + \frac{n}{4}\right)\left(1 - p^{\frac{n}{4}}\right) p^{\frac{n}{4}} + \left(3 + 2 \cdot \frac{n}{4}\right)\left(1 - p^{\frac{n}{4}}\right)^2\Big]$$

EXAM Q19

(a)

Method 1

$$Y = AX = \begin{pmatrix} a & -1 \\ b & ab \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} aX_1 - X_2 \\ bX_1 + abX_2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$Cov(Y) = \begin{pmatrix} \overset{①}{Cov(Y_1, Y_1)} & \overset{②}{Cov(Y_1, Y_2)} \\ \overset{②}{Cov(Y_2, Y_1)} & \overset{③}{Cov(Y_2, Y_2)} \end{pmatrix}$$

① $Cov(Y_1, Y_1) = \mathbb{V}[Y_1] = \mathbb{V}[aX_1 - X_2]$

$= a^2\sigma^2 + \sigma^2 = (a^2+1)\sigma^2$

By $X_1, X_2 \overset{IDP}{\sim} N(0, \sigma^2)$

② $Cov(Y_2, Y_1) = Cov(Y_1, Y_2) = Cov(aX_1 - X_2, bX_1 + abX_2)$

$= ab\,Cov(X_1, X_1) + a^2b\,Cov(X_1, X_2)$

$- b\,Cov(X_2, X_1) - ab\,Cov(X_2, X_2)$

$= ab\sigma^2 + a^2b\cdot \underset{X_1 \perp\!\!\!\perp X_2}{0} - b\cdot \underset{X_1 \perp\!\!\!\perp X_2}{0} - ab\sigma^2$

$= 0$

③ $Cov(Y_2, Y_2) = \mathbb{V}[Y_2] = \mathbb{V}[bX_1 + abX_2]$

$= (b^2 + a^2b^2)\sigma^2 = b^2(a^2+1)\sigma^2$

Combine ①②③,

$$Cov(Y) = \begin{pmatrix} (a^2+1)\sigma^2 & 0 \\ 0 & b^2(a^2+1)\sigma^2 \end{pmatrix}$$

Method 2

Notice $X = (X_1, X_2)^T$ with $X_1, X_2 \overset{iid}{\sim} N(0, \sigma^2)$

Then $X \sim \underbrace{MN_2}(\vec{\mu}_x, \Sigma_X)$ where

Multivariate Normal Distribution

OR Bivariate Normal Distribution IN THIS CASE

(See proof in question b, but we just need to know the expectation vector and cov. matrix for X to obtain Σ_Y, by the proposition of cov. matrix)

$$\vec{\mu}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_X = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \sigma^2 I_2$$

$Y = AX$.

Then $\Sigma_Y = A\Sigma_X A'$

$$= \sigma^2 \begin{pmatrix} a & -1 \\ b & ab \end{pmatrix}\begin{pmatrix} a & b \\ -1 & ab \end{pmatrix}$$

$$= \sigma^2 \begin{pmatrix} a^2+1 & 0 \\ 0 & b^2(a^2+1) \end{pmatrix}$$

(b)

Method 1

Special property of the bivariate normal distribution

If X is normal, Y is normal, and $X \perp\!\!\!\perp Y$,

then (X, Y) is bivariate normal. (see slides)

$X_1 \sim N(0, \sigma^2)$, $X_2 \sim N(0, \sigma^2)$, $X_1 \perp\!\!\!\perp X_2$

$\Rightarrow X = (X_1, X_2)^T \sim MN_2(\vec{\mu}_X, \Sigma_X)$ where

$$\vec{\mu}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_X = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

By proposition of the multivariate normal distribution

If $X \sim MN_2(\vec{\mu}_X, \Sigma_X)$,

then $Y = \underset{2\times 2}{\underbrace{A}}X \sim MN_2(\vec{\mu}_Y, \Sigma_Y)$ where

$$\vec{\mu}_Y = A\vec{\mu}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_Y = A\Sigma_X A' = \sigma^2\begin{pmatrix} a^2+1 & 0 \\ 0 & b^2(a^2+1) \end{pmatrix}$$

Method 2

By the definition of multivariate normal distribution

$X \sim MN_2(\vec{\mu}, \Sigma)$ if it has the same distribution as the vector $\vec{\mu} + LZ$ for a matrix L with $\Sigma = LL'$ and $Z = (Z_1, Z_2)^T$ a vector of i.i.d. r.v. following standard normal distribution

Let $\vec{\mu} = \vec{0}$

$$L = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

$$\Rightarrow \vec{\mu} + LZ = \begin{pmatrix} \sigma Z_1 \\ \sigma Z_2 \end{pmatrix} \sim \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\right)$$

$\sigma Z_1 = X_1$, $\sigma Z_2 = X_2$

Hence $X \sim MN_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\right)$

By corollary, if $X \sim MN_2(\vec{\mu}_X, \Sigma_X)$,

then $Y = \underset{2\times 2}{\underbrace{A}}X \sim MN_2(A\vec{\mu}_X, A\Sigma_X A')$

(c) Combine answers for question (a) and (b).

$$\left.\begin{array}{l}\Sigma_Y \text{ is diagnal} \\ Y \sim MN_2(\vec{\mu}_y, \Sigma_Y)\end{array}\right\} \Rightarrow Y_1 \perp\!\!\!\perp Y_2$$

OR refer to special property of the bivariate normal distribution (see slides)

For bivariate normal (Y_1, Y_2), independence is equivalent to being uncorrelated

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(d) Method 1:

$$Y_1^2 \perp\!\!\!\perp Y_2^2 \iff \mathbb{E}[g(Y_1^2)h(Y_2^2)] = \mathbb{E}[g(Y_1^2)]\,\mathbb{E}[h(Y_2^2)] \quad \forall\, g, h$$

$$\mathbb{E}[g(Y_1^2)h(Y_2^2)] = \int_{D[Y_1]}\int_{D[Y_2]} g(y_1^2)h(y_2^2) f_{Y_1,Y_2}(y_1,y_2)\,dy_1\,dy_2$$

$$= \int_{D[Y_1]} g(y_1^2) f_{Y_1}(y_1)\,dy_1 \int_{D[Y_2]} h(y_2^2) f_{Y_2}(y_2)\,dy_2$$

By $Y_1 \perp\!\!\!\perp Y_2$, $f_{Y_1Y_2}(y_1,y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2)$

$$= \mathbb{E}[g(Y_1^2)]\,\mathbb{E}[h(Y_1^2)]$$

Method 2:

∘ $Y_2^2 \perp\!\!\!\perp Y_2^2 \iff F_{Y_1^2 Y_2^2}(z_1, z_2) = F_{Y_1^2}(z_1) F_{Y_2^2}(z_2)$

$$F_{Y_1^2 Y_2^2}(z_1, z_2) = \mathbb{P}(Y_1^2 \le z_1, Y_2^2 \le z_2)$$

$$= \mathbb{P}(Y_1 \in [-\sqrt{z_1}, \sqrt{z_1}], Y_2 \in [-\sqrt{z_2}, \sqrt{z_2}])$$

$Y_1 \perp\!\!\!\perp Y_2$

$$= \mathbb{P}(Y_1 \in [-\sqrt{z_1}, \sqrt{z_1}])\,\mathbb{P}(Y_2 \in [-\sqrt{z_2}, \sqrt{z_2}])$$

$$= \mathbb{P}(Y_1^2 \le z_1)\,\mathbb{P}(Y_2^2 \le z_2)$$

$$= F_{Y_1^2}(z_1) F_{Y_2^2}(z_2)$$

(e) $(\Sigma_{Y(1,1)})^{-\frac{1}{2}} Y_1 \sim N(0,1)$

$(\Sigma_{Y(2,2)})^{-\frac{1}{2}} Y_2 \sim N(0,1)$

$$\Rightarrow \begin{cases} \lambda_1 = (\Sigma_{Y(1,1)})^{-1} = \dfrac{1}{a^2+1} \\ \lambda_2 = (\Sigma_{Y(1,1)})^{-1} = \dfrac{1}{b^2(a^2+1)} \end{cases}$$

(f) Notice $U = \lambda_1 Y_1^2 + \lambda_2 Y_2^2 \sim \chi^2_{(2)}$

$\Leftrightarrow U \sim \text{Gamma}(1, \frac{1}{2})$

$\Leftrightarrow U \sim \text{EXP}(\frac{1}{2})$

Also recall the scale of exponential distribution: if $U \sim \text{EXP}(\lambda)$, then $cU \sim \text{EXP}(\frac{\lambda}{c})$

Let $c = 2$

$$\Rightarrow \begin{cases} \lambda_1 = \dfrac{2}{a^2+1} \\ \lambda_2 = \dfrac{2}{b^2(a^2+1)} \end{cases}$$