TI Statistics 2019
TA 1
Appendix
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The file contains:

- Distribution graph by previous TAs
- TA solution for Ch4.36 (3rd edition)
- TA solution for Exam question 19
- TA solution for Exam question 1

DISCRETE

CONTINUOUS

HYPERBOLIC

BINOMIAL (n, p)

X=k #success out of n experiments

- $P(k)=\binom{n}{k}p^k(1-p)^{n-k}$, $k=0,1,\dots n$
- $E(X)=np$
- $Var(X)=np(1-p)$

BERNOULLI $\xrightarrow{\sum_{i=1}^n X_i \text{, ind. trials}}$ BINOMIAL

BINOMIAL $\xrightarrow{n=1}$ BERNOULLI

BINOMIAL $\xrightarrow{n\to\infty,\ p\to 0,\ np=\lambda}$ POSSION

BINOMIAL $\xrightarrow{n\to\infty,\ \mu=np,\ \sigma^2=np(1-p)}$ NORMAL

BERNOULLI (p)

$X\in\{0,1\}$ outcome of experiment

- $P(x)=\begin{cases}p^x(1-p)^{1-x}, & \text{if } x=0 \text{ or } x=1\\ 0, & \text{otherwise}\end{cases}$
- $E(X)=p$
- $Var(X)=p(1-p)$

BERNOULLI $\xrightarrow{\text{ind. trials}}$ GEOMETRIC

BERNOULLI $\xrightarrow{\text{ind. trials}}$ NEG. BIN

POSSION (λ)

X=k #events occur in a given interval of time

- $P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda}$, $k=0,1,2\dots$
- $e^{\lambda}=\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$
- $E(X)=\lambda$
- $Var(X)=\lambda$

POSSION $\xrightarrow{\lambda=\sigma^2,\ \mu=\lambda,\ \lambda\to\infty}$ NORMAL

NEG. BIN $\xrightarrow{n\to\infty,\ n(1-p)=\lambda}$ POSSION

POSSION $\to$ EXPONENTIAL: the times between events of a Possion process are i.i.d. follow Exp

GEOMETRIC (p)

X=k #trials for [failures + first success] infinite sequence

failures $\to$ k-1, first success $\to$ kth

- $P(k)=P(X=k)=(1-p)^{k-1}p$, $k=1,2\dots$
- $\sum_{k=1}^{\infty}(1-p)^{k-1}p=1$
- $E(X)=\frac{1}{p}$
- $Var(X)=\frac{1-p}{p^2}$
- $F(x)=1-(1-p)^x$

GEOMETRIC $\xrightarrow{\text{generalize to rth success}}$ NEG. BIN

NEG. BIN $\xrightarrow{r=1}$ GEOMETRIC

NEG. BIN (r, p)

X=k #trials under infinite sequence

$[\underbrace{(\text{failures}+r-1 \text{ success})}_{\text{mix } k-1}+r\text{th success}]$

- $P(k)=P(X=k)=\binom{k-1}{r-1}p^r(1-p)^{k-r}$, $k=r, r+1,\dots$
- $E(X)=\frac{r}{p}$
- $Var(X)=\frac{r(1-p)}{p^2}$

analogue

STD. NORMAL

- $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, $x\in(-\infty,+\infty)$
- $E(X)=\mu=0$
- $Var(X)=\sigma^2=1$

NORMAL $\xrightarrow{\frac{X-\mu}{\sigma}}$ STD. NORMAL

STD. NORMAL $\xrightarrow{\mu+\sigma X}$ NORMAL

t(n) $\xrightarrow{n\geq 30}$ STD. NORMAL

STD. NORMAL $\xrightarrow{\sum_{i=1}^n X_i^2\sim\chi^2(n)}$ CHI^2(n)

t(n)

- $Z\sim N(0,1)$

$U\sim\chi^2_n$

Z, U independent

$T_n=\frac{Z}{\sqrt{U/n}}\sim t_n$

CHI^2(n)

GAMMA $\xrightarrow{\alpha=\frac{n}{2},\ \lambda=\frac{1}{2}}$ CHI^2(n)

NORMAL (μ, σ)

model measurement error (CLT)

- $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$, $x\in(-\infty,+\infty)$
- $E(X)=\mu$
- $Var(X)=\sigma^2$
- CLT $Z_n=\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}\xrightarrow{d}N(0,1)$

EXPONENTIAL (λ) $\to\frac{1}{\theta}$

model lifetime/waiting time (memoryless)

- $f(x)=\lambda e^{-\lambda x}=\frac{1}{\theta}e^{-\frac{x}{\theta}}$, $x\geq 0$
- $E(X)=\frac{1}{\lambda}=\theta$
- $Var(X)=\frac{1}{\lambda^2}=\theta^2$
- $F(x)=\begin{cases}1-e^{-\lambda x}, & x\geq 0\\ 0, & x<0\end{cases}$

EXPONENTIAL $\xrightarrow{\sum_{i=1}^n X_i\sim Gamma(n,\lambda)}$ GAMMA

GAMMA $\xrightarrow{\alpha=1}$ EXPONENTIAL

GAMMA (α, λ)

$\alpha\to n$

model lifetime (with memory)

- $f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$, $x\geq 0$

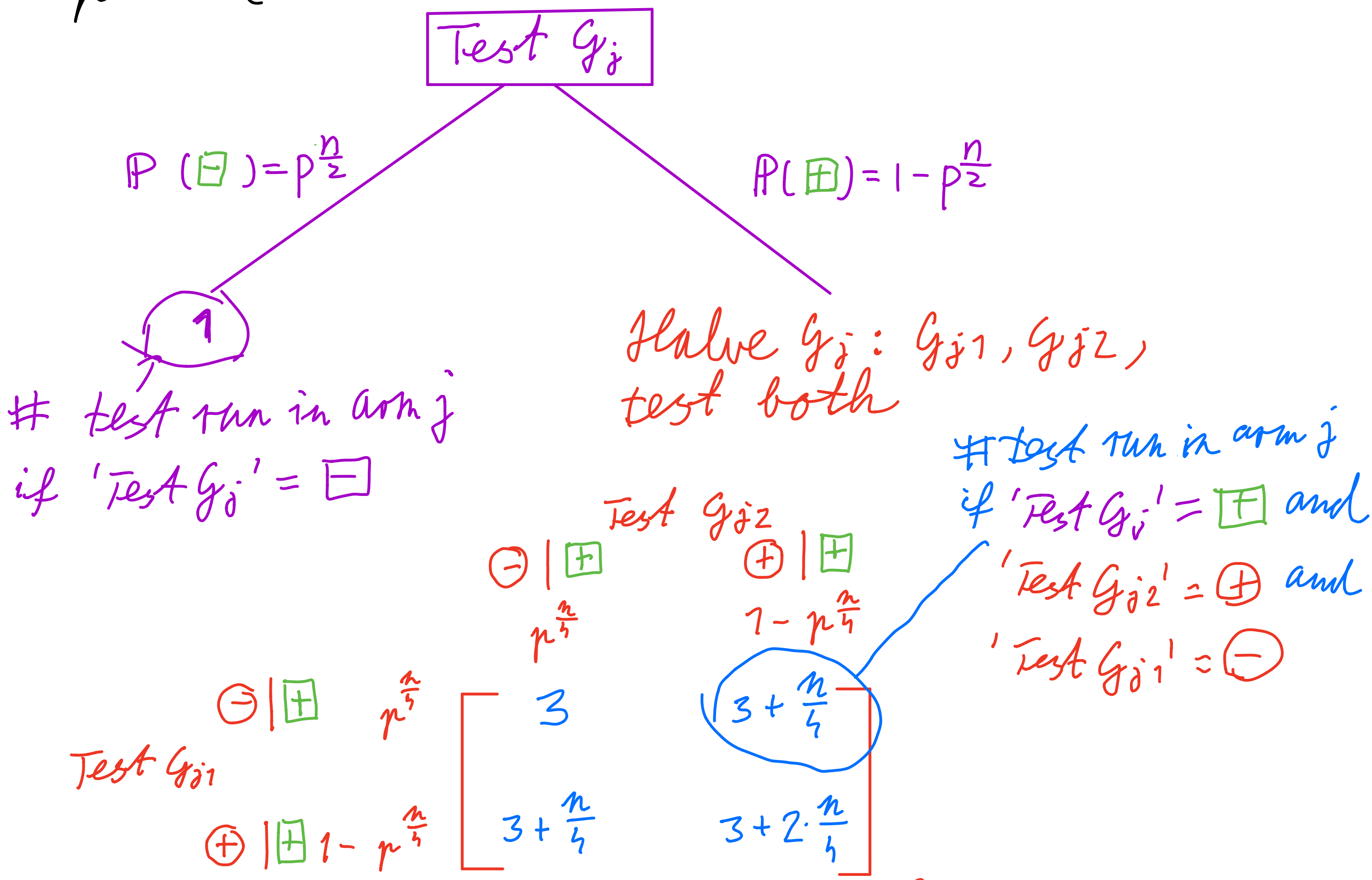
$\Gamma(\alpha)=\int_0^\infty u^{\alpha-1}e^{-u}du$

- $E(X)=\frac{\alpha}{\lambda}$
- $Var(X)=\frac{\alpha}{\lambda^2}$

UNIFORM

CH 4.36 (3rd Edition)

- Halve original sample: G_A, G_B
- $N_j \equiv$ # of tests run in arm $j \in \{A, B\}$
- $N = N_A + N_B$ # all tests run
  - Of interest $\mathbb{E}[N]$
    - note, by symmetry, $\mathbb{E}[N] = 2\,\mathbb{E}[N_j]$
- $p \equiv \mathbb{P}$(test is negative for a single subject)



$$\mathbb{E}[N_j] = 1 \cdot p^{\frac{n}{2}} + \left(1 - p^{\frac{n}{2}}\right)\Big[3 \cdot \left(p^{\frac{n}{4}}\right)^2 + \left(3 + \tfrac{n}{4}\right) p^{\frac{n}{4}} \left(1 - p^{\frac{n}{4}}\right) + \left(3 + \tfrac{n}{4}\right)\left(1 - p^{\frac{n}{4}}\right) p^{\frac{n}{4}} + \left(3 + 2 \cdot \tfrac{n}{4}\right)\left(1 - p^{\frac{n}{4}}\right)^2\Big]$$

EXAM Q19

(a)

Method 1

$$Y = AX = \begin{pmatrix} a & -1 \\ b & ab \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} aX_1 - X_2 \\ bX_1 + abX_2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$\mathrm{Cov}(Y) = \begin{pmatrix} \overset{①}{\mathrm{Cov}(Y_1, Y_1)} & \overset{②}{\mathrm{Cov}(Y_1, Y_2)} \\ \overset{②}{\mathrm{Cov}(Y_2, Y_1)} & \overset{③}{\mathrm{Cov}(Y_2, Y_2)} \end{pmatrix}$$

① $\mathrm{Cov}(Y_1, Y_1) = \mathbb{V}[Y_1] = \mathbb{V}[aX_1 - X_2]$

$= a^2\sigma^2 + \sigma^2 = (a^2+1)\sigma^2$

By $X_1, X_2 \overset{IDP}{\sim} N(0, \sigma^2)$

② $\mathrm{Cov}(Y_2, Y_1) = \mathrm{Cov}(Y_1, Y_2) = \mathrm{Cov}(aX_1 - X_2, bX_1 + abX_2)$

$= ab\,\mathrm{Cov}(X_1, X_1) + a^2b\,\mathrm{Cov}(X_1, X_2)$

$- b\,\mathrm{Cov}(X_2, X_1) - ab\,\mathrm{Cov}(X_2, X_2)$

$= ab\sigma^2 + a^2b\cdot \underset{X_1 \perp\!\!\!\perp X_2}{0} - b\cdot \underset{X_1 \perp\!\!\!\perp X_2}{0} - ab\sigma^2$

$= 0$

③ $\mathrm{Cov}(Y_2, Y_2) = \mathbb{V}[Y_2] = \mathbb{V}[bX_1 + abX_2]$

$= (b^2 + a^2b^2)\sigma^2 = b^2(a^2+1)\sigma^2$

Combine ①②③,

$$\mathrm{Cov}(Y) = \begin{pmatrix} (a^2+1)\sigma^2 & 0 \\ 0 & b^2(a^2+1)\sigma^2 \end{pmatrix}$$

Method 2

Notice $X = (X_1, X_2)^T$ with $X_1, X_2 \overset{iid}{\sim} N(0, \sigma^2)$

Then $X \sim \text{Distribution}(\vec{\mu}_X, \Sigma_X)$ where

Actually $X \sim MN_2$, Multivariate Normal Distribution

OR Bivariate Normal Distribution IN THIS CASE

Why multivariate normal?

See proof in question b, but we just need to know the expectation vector and cov. matrix for X to obtain Σ_Y, by the proposition of cov. matrix, not necessarily the type of distribution.

$$\vec{\mu}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_X = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \sigma^2 I_2$$

$Y = AX$.

Then $\Sigma_Y = A\Sigma_X A'$

$$= \sigma^2 \begin{pmatrix} a & -1 \\ b & ab \end{pmatrix}\begin{pmatrix} a & b \\ -1 & ab \end{pmatrix}$$

$$= \sigma^2 \begin{pmatrix} a^2+1 & 0 \\ 0 & b^2(a^2+1) \end{pmatrix}$$

(b)

Method 1

Special property of the bivariate normal distribution

If X is normal, Y is normal, and $X \perp\!\!\!\perp Y$, then (X, Y) is bivariate normal. (see slides)

$X_1 \sim N(0, \sigma^2)$, $X_2 \sim N(0, \sigma^2)$, $X_1 \perp\!\!\!\perp X_2$

$\Rightarrow X = (X_1, X_2)^T \sim MN_2(\vec{\mu}_X, \Sigma_X)$ where

$$\vec{\mu}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_X = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

By proposition of the multivariate normal distribution

If $X \sim MN_2(\vec{\mu}_X, \Sigma_X)$,

then $Y = \underset{2\times 2}{A}X \sim MN_2(\vec{\mu}_Y, \Sigma_Y)$ where

$$\vec{\mu}_Y = A\vec{\mu}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_Y = A\Sigma_X A' = \sigma^2 \begin{pmatrix} a^2+1 & 0 \\ 0 & b^2(a^2+1) \end{pmatrix}$$

Method 2

By the definition of multivariate normal distribution

$X \sim MN_2(\vec{\mu}, \Sigma)$ if it has the same distribution as the vector $\vec{\mu} + LZ$ for a matrix L with $\Sigma = LL'$ and $Z = (Z_1, Z_2)^T$ a vector of iid. r.v. following standard normal distribution

Let $\vec{\mu} = \vec{0}$

$$L = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

$$\Rightarrow \vec{\mu} + LZ = \begin{pmatrix} \sigma Z_1 \\ \sigma Z_2 \end{pmatrix} \sim \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\right)$$

$\sigma Z_1 = X_1$, $\sigma Z_2 = X_2$

Hence $X \sim MN_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\right)$

By corollary, if $X \sim MN_2(\vec{\mu}_X, \Sigma_X)$,

then $Y = \underset{2\times 2}{A}X \sim MN_2(A\vec{\mu}_X, A\Sigma_X A')$

(c) Combine answers for question (a) and (b).

$$\left.\begin{array}{l}\Sigma_Y \text{ is diagnal} \\ Y \sim MN_2(\vec{\mu}_y, \Sigma_Y)\end{array}\right\} \Rightarrow Y_1 \perp\!\!\!\perp Y_2$$

OR refer to special property of the bivariate normal distribution (see slides)

For bivariate normal (Y_1, Y_2), independence is equivalent to being uncorrelated

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(d) Method 1:

$$Y_1^2 \perp\!\!\!\perp Y_2^2 \Leftrightarrow \mathbb{E}[g(Y_1^2)h(Y_2^2)] = \mathbb{E}[g(Y_1^2)]\,\mathbb{E}[h(Y_2^2)] \quad \forall\, g, h$$

$$\mathbb{E}[g(Y_1^2)h(Y_2^2)] = \int_{D[Y_1]}\int_{D[Y_2]} g(y_1^2)h(y_2^2) f_{Y_1,Y_2}(y_1,y_2)\,dy_1\,dy_2$$

$$= \int_{D[Y_1]} g(y_1^2) f_{Y_1}(y_1)\,dy_1 \int_{D[Y_2]} h(y_2^2) f_{Y_2}(y_2)\,dy_2$$

By $Y_1 \perp\!\!\!\perp Y_2$, $f_{Y_1Y_2}(y_1,y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2)$

$$= \mathbb{E}[g(Y_1^2)]\,\mathbb{E}[h(Y_1^2)] \quad \blacksquare$$

Method 2:

$$\circ \quad Y_2^2 \perp\!\!\!\perp Y_2^2 \Leftrightarrow F_{Y_1^2 Y_2^2}(z_1, z_2) = F_{Y_1^2}(z_1) F_{Y_2^2}(z_2)$$

$$F_{Y_1^2Y_2^2}(z_1, z_2) = \mathbb{P}(Y_1^2 \le z_1, Y_2^2 \le z_2)$$

$$= \mathbb{P}(Y_1 \in [-\sqrt{z_1}, \sqrt{z_1}], \; Y_2 \in [-\sqrt{z_2}, \sqrt{z_2}])$$

$Y_1 \perp\!\!\!\perp Y_2$

$$= \mathbb{P}(Y_1 \in [-\sqrt{z_1}, \sqrt{z_1}])\,\mathbb{P}(Y_2 \in [-\sqrt{z_2}, \sqrt{z_2}])$$

$$= \mathbb{P}(Y_1^2 \le z_1)\,\mathbb{P}(Y_2^2 \le z_2)$$

$$= F_{Y_1^2}(z_1) F_{Y_2^2}(z_2) \quad \blacksquare$$

(e) $\left(\Sigma_{Y(1,1)}\right)^{-\frac{1}{2}} Y_1 \sim N(0,1)$

$\left(\Sigma_{Y(2,2)}\right)^{-\frac{1}{2}} Y_2 \sim N(0,1)$

$$\Rightarrow \begin{cases} \lambda_1 = \left(\Sigma_{Y(1,1)}\right)^{-1} = \dfrac{1}{a^2+1} \\ \lambda_2 = \left(\Sigma_{Y(1,1)}\right)^{-1} = \dfrac{1}{b^2(a^2+1)} \end{cases}$$

(f) Notice $U = \lambda_1 Y_1^2 + \lambda_2 Y_2^2 \sim \chi^2_{(2)}$

$\Leftrightarrow U \sim \text{Gamma}(1, \frac{1}{2})$

$\Leftrightarrow U \sim EXP(\frac{1}{2})$

Also recall the scale of exponential distribution: if $U \sim EXP(\lambda)$, then $cU \sim EXP(\frac{\lambda}{c})$

Let $c = \frac{1}{2}$

$$\Rightarrow \begin{cases} \lambda_1 = \dfrac{1}{2(a^2+1)} \\ \lambda_2 = \dfrac{1}{2b^2(a^2+1)} \end{cases}$$

EXAM Q1

(a) $X = g(U) = a + (b-a)U$

$$U = g^{-1}(x) = \frac{x-a}{b-a}$$

$$\frac{\partial g^{-1}(x)}{\partial x} = \frac{1}{b-a}$$

Note: $0 \le U \le 1 \Rightarrow a \le x \le b$

$$f_X(x) = f_U(g^{-1}(x)) \left| \frac{\partial g^{-1}(x)}{\partial x} \right|$$

$$= \frac{1}{b-a}$$ By $f_U(u) = 1$ for $u \in [0,1]$ and $a < b$

Hence, $X \sim U[a,b]$

(b) $\mathbb{E}[X] = \mathbb{E}[a+(b-a)U] = a + (b-a)\mathbb{E}[U] = \frac{1}{2}(a+b)$ ($\mathbb{E}$ is linear)

$\mathbb{V}[X] = \mathbb{V}[a+(b-a)U] = (b-a)^2\mathbb{V}[U] = \frac{1}{12}(b-a)^2$

(c) $\mathbb{P}(Y > y) = \mathbb{P}(-\theta \log U > y) = \mathbb{P}(U < e^{-\frac{y}{\theta}})$

$= e^{-\frac{y}{\theta}}$

$\Rightarrow F_Y(y) = \mathbb{P}(Y \le y) = 1 - \mathbb{P}(Y > y) = 1 - e^{-\frac{y}{\theta}}$

Method 1:

$f_Y(y) = F_Y'(y) = \frac{1}{\theta} e^{-\frac{y}{\theta}}$

Method 2:

$Y \sim EXP(\frac{1}{\theta})$

$\Rightarrow f_Y(y) = \frac{1}{\theta} e^{-\frac{y}{\theta}}$

(d) $\mathbb{E}[Y] = \mathbb{E}[-\theta \log U] = -\theta\, \mathbb{E}[\log U]$

$= -\theta \int_0^1 \log u \, f_U(u)\, du$ By $f_U(u) = 1$

$= -\theta \left(\underbrace{u \log u - u \Big|_0^1}_{(1\log 1 - 1) - (0 \underbrace{\lim_{u \to 0^+} \log u}_{\lim_{x \to 0^+} \frac{\log u}{\frac{1}{u}} \overset{H}{=} \lim_{x \to 0^+} \frac{\frac{1}{u}}{-\frac{1}{u^2}} = \lim_{x \to 0^+} -u = 0} - 0)} \right)$ By "Hint" in the question

$= \theta$