Homework 3

1. Write the equation of a 1-D and multi-variate Gaussian. Take a log and simplify the equation

Ans: 1D Gaussian:

μ - mean

σ - std

x - r.v (vector)

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\Rightarrow \ln f(x) = \ln\left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right)$$

$$= -\ln\left(\sqrt{2\pi}\,\sigma\right) - \frac{(x-\mu)^2}{2\sigma^2}$$

$$\therefore \ln f(x) = -\ln\sqrt{2\pi} - \ln\sigma - \frac{(x-\mu)^2}{2\sigma^2}$$

Multi-variate Gaussian:

X - r.v - $[x_1, x_2, x_3, \ldots x_n]$
↓
vector

μ - mean of X

Σ - covariance matrix of X

$$f(x) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} e^{-\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}}$$

$$\Rightarrow \ln f(x) = \ln\left(\frac{1}{\sqrt{(2\pi)^n |\Sigma|}}\right) - \frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}$$

$$\therefore \ln f(x) = -\frac{n}{2} \ln 2\pi - \frac{1}{2} \ln |\Sigma| - \frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}$$

ii) Write the Bayes decision rule for minimum error and minimum risk. Under what condition both will be same.

Ans: ~~if x is some observation belonging to class either ω_1 or ω_2, then the bayesian decision rule is for selecting class ω_1 or ω_2 for x is~~

The bayesian decision rule for selecting class ω_1 or ω_2 for any observation x using minimum error,

$$P(error|x) = \min [P(x|\omega_1), P(x|\omega_2)]$$

If the risk associated with deciding α_i when its actually ω_j are as of below,

$$\lambda(\alpha_i|\omega_j) = \begin{cases} \lambda_{11} & \text{for } i=j=1 \\ \lambda_{21} & \text{for } i=2, j=1 \\ \lambda_{12} & \text{for, } i=1, j=2 \\ \lambda_{22} & \text{for, } i=2, j=2 \end{cases}$$

then, the Bayesian decision rule for minimum risk,

$$\frac{p(x|w_1)}{p(x|w_2)} > \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \cdot \frac{P(w_2)}{P(w_1)}$$

the minimum error & minimum risk rules are equal when,

$$\lambda(\alpha_i|\omega_j) = \begin{cases} 0 & i=j \\ 1 & i \neq j \end{cases}$$

the minimum risk rule could be written,

$$(\lambda_{21}-\lambda_{11}) P(x|w_1) > (\lambda_{12}-\lambda_{22}) P(x|w_2)$$

$$\Rightarrow (1-0) P(x|w_1) > \cancel{p(x|w_2)} (1-0) P(x|w_2)$$

$$\Rightarrow P(x|w_1) > P(x|w_2)$$

which is essentially the rule for minimum error.

3. ω_1:

$$\text{mean} = \frac{1}{4} \times \left(\begin{bmatrix} 12 \\ 4 \end{bmatrix} + \begin{bmatrix} 12 \\ 8 \end{bmatrix} + \begin{bmatrix} 10 \\ 6 \end{bmatrix} + \begin{bmatrix} 14 \\ 6 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

~~cov(x) :~~

cov:

$$X = \begin{bmatrix} 12 & 4 \\ 12 & 8 \\ 10 & 6 \\ 14 & 6 \end{bmatrix}$$

$$X - \mu = \begin{bmatrix} 12-12 & 4-6 \\ 12-12 & 8-6 \\ 10-12 & 6-6 \\ 14-12 & 6-6 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 2 \\ -2 & 0 \\ 2 & 0 \end{bmatrix}$$

$$\text{Cov}(x) = \frac{(X-\mu)^T . (X-\mu)}{4} = \frac{1}{4} \begin{bmatrix} 0 & 0 & -2 & 2 \\ -2 & 2 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & -2 \\ 0 & 2 \\ -2 & 0 \\ 2 & 0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

ω_2:

$$\text{mean} = \frac{1}{4}\left(\begin{bmatrix}9\\10\end{bmatrix} + \begin{bmatrix}9\\14\end{bmatrix} + \begin{bmatrix}7\\12\end{bmatrix} + \begin{bmatrix}11\\12\end{bmatrix}\right)$$

$$= \frac{1}{4} \times \begin{bmatrix}\cancel{6}\,36\\48\end{bmatrix}$$

$$= \begin{bmatrix}9\\12\end{bmatrix}$$

~~cov(x) x:~~

cov:

$$X = \begin{bmatrix}9 & 10\\9 & 14\\7 & 12\\11 & 12\end{bmatrix}$$

$$X - \mu = \begin{bmatrix}9-9 & 10-12\\9-9 & 14-12\\7-9 & 12-12\\11-9 & 12-12\end{bmatrix}$$

$$= \begin{bmatrix}0 & -2\\0 & 2\\-2 & 0\\2 & 0\end{bmatrix}$$

~~(x-~~

~~x-μ~~

$$\text{cov}(X) = \frac{(X-\mu)^T \cdot (X-\mu)}{4}$$

$$= \frac{1}{4}\begin{bmatrix}0 & 0 & -2 & 2\\-2 & 2 & 0 & 0\end{bmatrix} \times \begin{bmatrix}0 & -2\\0 & 2\\-2 & 0\\2 & 0\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}8 & 0\\0 & 8\end{bmatrix}$$

$$= \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$$

$\Rightarrow$ for both ω_1 an ω_2 the covariance matrix is diagonal.

This implies there is no correlation between features of ω_1 an ω_2.