

# Bayesian Modeling: Foundations and Inference

ELG 5218 - Uncertainty Evaluation in Engineering Measurements and Machine Learning

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# Learning goals

By the end of this lecture, you should be able to:

- Define **uncertainty quantification (UQ)**.
- Specify a Bayesian model using **prior**, **likelihood**, and **posterior**.
- Explain and compute **MLE** and **MAP**.
- Derive and interpret a simple conjugate update (Beta–Binomial).
- Compute and interpret **credible intervals** (central and HPD/HDI).
- Use the posterior to form the **posterior predictive distribution**.

# Roadmap

- Introduction to UQ (what / why / how to represent uncertainty)
- Confidence intervals vs credible intervals (CI vs Crl)
- Bayesian inference framework (prior, likelihood, posterior, evidence)
- Conjugate priors and Beta–Binomial example
- Point estimation: MLE and MAP (and connection to regularization)
- Posterior summaries (mean/median/mode, intervals) and prediction (PPD)

# What is Uncertainty Quantification?

**Uncertainty Quantification (UQ)** develops rigorous methods to characterize the impact of “limited knowledge” on quantities of interest.

Two fundamental sources of uncertainty

- ① **Aleatoric Uncertainty:** inherent randomness in physical processes (irreducible)
- ② **Epistemic Uncertainty:** lack of knowledge that can be reduced with more data/modeling

Key questions in UQ

- What is the expected value of our quantity of interest?
- How much does it vary (variance / standard deviation)?
- What range of values is plausible (intervals)?
- How do we update beliefs when new data arrive?

**Example:** Heart rate measurement reported as  $(60 \pm 4)$  beats/minute.

# Representing uncertainty

## Common ways to express uncertainty:

- ① **Standard error**: variability of an estimate
- ② **Confidence intervals**: frequentist
- ③ **Credible intervals**: Bayesian
- ④ **Probability distributions**: full description of uncertainty
- ⑤ **Quantiles / moments**: extracted summaries

*A point estimate without uncertainty quantification is incomplete.*

# A visual language for uncertainty

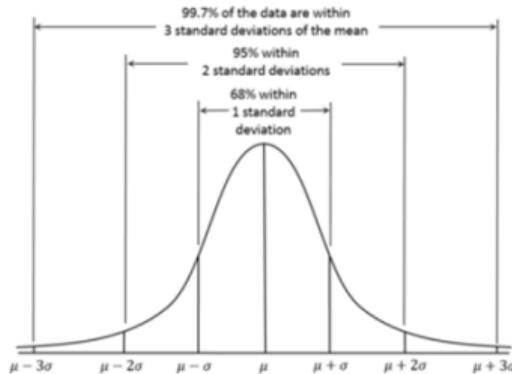


Figure: Normal distribution with interval bands

- Distributions communicate **shape** (skew, multimodality), not only spread.
- Intervals are summaries; distributions are the full story.

# Confidence intervals: frequentist perspective

**Definition:** A 95% confidence interval (CI) is constructed so that:

95% of similarly constructed CIs contain the true parameter value.

## Key properties

- The parameter  $\theta$  is **fixed but unknown**
- The interval is **random** (depends on the sample)
- Interpretation is about **long-run frequency** across repeated sampling

## What it does not mean:

$$\mathbb{P}(\theta \in [a, b]) = 0.95 \quad \text{WRONG}$$

# Credible intervals: Bayesian perspective

**Definition:** A 95% credible interval (Crl) satisfies:

$$\mathbb{P}(\theta \in [a, b] \mid \text{data}) = 0.95$$

## Key properties

- The parameter  $\theta$  is treated as a **random variable**
- The interval is **fixed** (given the posterior)
- Interpretation is **posterior probability** (direct probability statement)

## Two common credible intervals

- ① **Central (equal-tailed)**:  $\alpha/2$  mass in each tail
- ② **HPD / HDI**: region(s) with highest posterior density

## Credible intervals: central (equal-tailed)

**Central interval:** contains  $\alpha/2$  probability in each tail.

$$\text{Crl}_{1-\alpha} = [q_{\alpha/2}, q_{1-\alpha/2}]$$

**Example (95%):**

$$\text{Crl}_{0.95} = [q_{0.025}, q_{0.975}]$$

**Interpretation:**

$$\mathbb{P}(\theta \in [q_{0.025}, q_{0.975}] \mid D) = 0.95$$

- simple and widely used
- may exclude the mode for skewed posteriors

# Credible intervals: highest posterior density (HPD/HDI)

**HPD/HDI:** region with the highest posterior density containing  $(1 - \alpha)$  mass.

## Intuition

- all points inside are more credible than points outside
- typically shorter than equal-tailed intervals for skewed posteriors
- may be multi-interval for multimodal posteriors

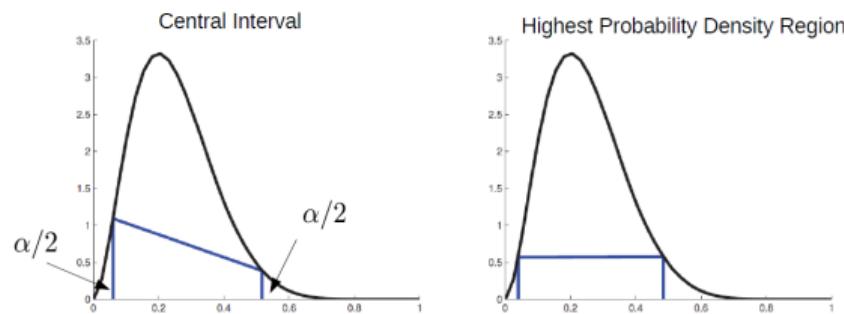


Figure: Central and HPD intervals

# Comparison: CI vs Crl

Aspect	Confidence interval	Credible interval
Parameter	Fixed	Random variable
Interval	Random	Fixed (given posterior)
Uses prior?	No	Yes
Meaning	Long-run coverage	Posterior probability
Computation	Often analytic	Often via sampling / numeric

# Modeling Data Probabilistically: A Simplistic View

- Assume a dataset  $X = \{x_1, \dots, x_N\}$  is generated from a probabilistic model with unknown parameters  $\theta$ .
- For i.i.d. observations:  $x_1, \dots, x_N \sim p(x | \theta)$ .
- Plate notation: shaded nodes = observed; unshaded nodes = unknown / unobserved.
- Goal: estimate the unknowns (here,  $\theta$ ) given the observed data  $X$ .
- Use the learned model for prediction:

$$p(x^* | \theta) \quad \text{or} \quad p(x^* | X).$$

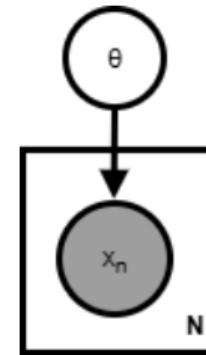
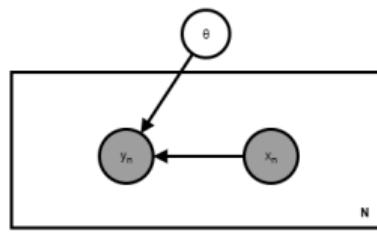


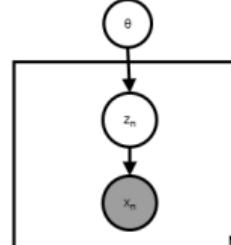
Figure: Simplified plate model for i.i.d. data

# Modeling Data Probabilistically

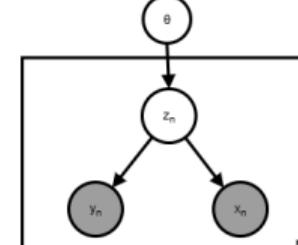
- This basic setup generalizes in many ways.
- Any node (even if observed) that we are uncertain about is modeled by a probability distribution.
- These nodes become the **random variables** of the model.
- The full model is specified via a **joint probability distribution** over all random variables.
- The goal is to **infer unknowns** of the model given the observed data.



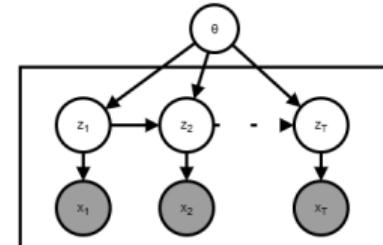
A Simple Supervised Learning Model



A Latent Variable Model for Unsupervised Learning



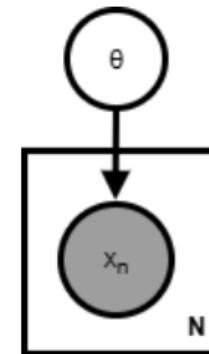
A Latent Variable Model for Supervised Learning



A Latent Variable Model for Sequential Data

# Model Specification: Likelihood and Prior

- Probabilistic models require two key ingredients: **likelihood** and **prior**.
- **Likelihood**  $p(x | \theta)$  (“observation model”): specifies how data is generated and measures data fit (loss) for a given  $\theta$ .
- **Prior**  $p(\theta)$ : specifies how plausible parameter values are *a priori*; it often acts like a regularizer.
- Domain knowledge can guide both likelihood and prior choices.



$$p(\theta | X) \propto p(X | \theta) p(\theta)$$

# Parameter Estimation vs. Bayesian Inference

- A simplest approach is **point estimation**: find  $\theta$  that makes the observed data most likely.

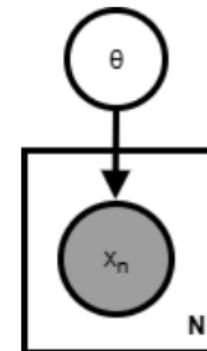
$$\hat{\theta} = \arg \max_{\theta} \log p(X | \theta).$$

- But a single point estimate does **not** quantify uncertainty in  $\theta$ .
- **Bayesian inference** estimates the **full posterior**:

$$p(\theta | X) = \frac{p(X | \theta) p(\theta)}{p(X)} \propto \underbrace{p(X | \theta)}_{\text{Likelihood}} \times \underbrace{p(\theta)}_{\text{Prior}}.$$

$$\text{Posterior} = \text{Likelihood} \times \text{Prior} \\ (\text{normalized})$$

- The posterior captures uncertainty in  $\theta$ ; we will study point estimation, Bayesian inference, and hybrids.



# The Bayesian approach: overview

**Core philosophy:** Probability represents a *degree of belief*, updated with evidence.

## Three key components

- ① **Prior**  $p(\theta)$ : initial beliefs about parameters
- ② **Likelihood**  $p(D | \theta)$ : probability of data given parameters
- ③ **Posterior**  $p(\theta | D)$ : updated beliefs after observing data

## Goal

- compute posterior, summarize it, and make predictions for new observations  $x^*$

# Bayes' theorem

$$p(\theta | D) = \frac{p(D | \theta) p(\theta)}{p(D)}$$

## Components

- $p(\theta | D)$  posterior (what we want)
- $p(D | \theta)$  likelihood
- $p(\theta)$  prior
- $p(D)$  evidence / marginal likelihood (normalization)

$$p(\theta | D) \propto p(D | \theta) p(\theta)$$

$$p(D) = \int p(D | \theta) p(\theta) d\theta$$

# Bayesian inference pipeline (big picture)

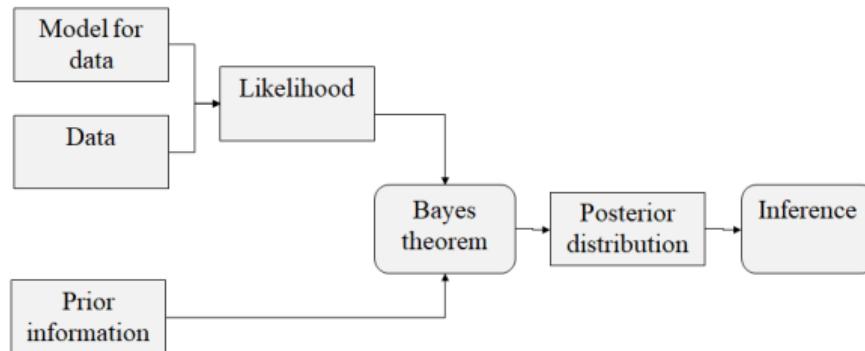


Figure: Bayesian inference

- Modeling: choose  $p(D | \theta)$  and  $p(\theta)$
- Inference: compute/approximate  $p(\theta | D)$
- Decision/prediction: use posterior and posterior predictive

# Prior distribution: encoding beliefs

**Purpose:** incorporate prior knowledge, stabilize inference, and regularize.

## Types of priors

- ① **Uninformative/flat**: minimal information (but still encodes assumptions)
- ② **Weakly informative**: gentle regularization (prevents extreme values)
- ③ **Informative**: genuine prior knowledge (historical data, expert belief)
- ④ **Conjugate**: chosen for analytic convenience

**Key principle:** Posterior is a compromise between prior and likelihood; strong data can overwhelm weak priors.

## How to choose a prior in practice

- **Domain knowledge:** expert judgment, historical datasets, physics constraints
- **Sensitivity analysis:** change priors and check how conclusions change
- **Prior predictive check:** sample from prior and see if it generates reasonable synthetic data
- **Weak vs strong:** with few observations, the prior matters more

# Likelihood function: probability of data given parameters

$$p(D | \theta) = \prod_{i=1}^N p(x_i | \theta)$$

## Likelihood is a model of the data-generating process



- The likelihood encodes sensor physics + imperfections.
- It is **not** “how likely  $x$  is”; it is “ $y$  is, if  $x$  were true”.
- Choosing the likelihood is often the most important modeling decision.

## Typical likelihood choices

- Binomial:** successes in  $n$  trials
- Gaussian:** continuous measurements with additive noise
- Poisson:** counts / event arrivals

## Posterior distribution: updated beliefs

$$p(\theta | D) = \frac{p(D | \theta) p(\theta)}{p(D)}$$

### Posterior interpretation

- **Mode** (MAP), **mean**, **median**
- **Spread**: uncertainty about  $\theta$
- **Quantiles**: credible intervals

# Posterior predictive distribution (PPD)

**Goal:** predict a new observation  $x^*$ .

$$p(x^* | D) = \int p(x^* | \theta) p(\theta | D) d\theta$$

## Interpretation

- averages predictions over all plausible parameter values
- accounts for parameter uncertainty and observation noise

**Plug-in approximation (often used, but weaker)**

$$p(x^* | D) \approx p(x^* | \hat{\theta})$$

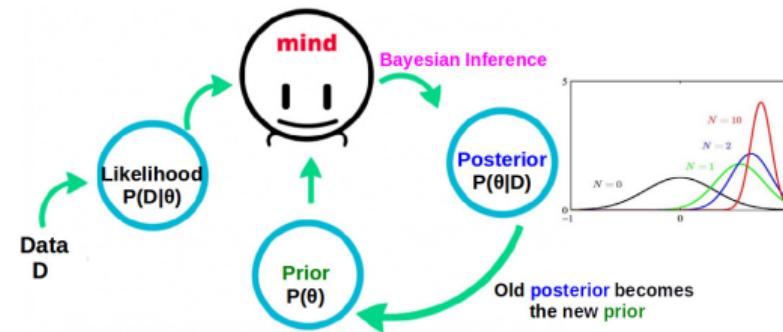


Figure: Bayesian Update

# Conjugate priors: computational convenience

**Definition:** A prior is **conjugate** to a likelihood if the posterior has the same functional form as the prior.

## Why it matters

- closed-form posterior (no sampling needed)
- easy to interpret updates
- sequential updating is straightforward

Likelihood	Prior	Posterior
Binomial / Bernoulli	Beta	Beta
Poisson	Gamma	Gamma
Gaussian (known $\sigma$ )	Gaussian	Gaussian
Multinomial	Dirichlet	Dirichlet

# When conjugacy breaks: optimization and approximation

- Many useful models do **not** have closed-form posteriors.
- Example: logistic regression likelihood (classification).
- Then we rely on:
  - **Optimization:** MAP via Newton / quasi-Newton.
  - **Approximation:** Laplace approximation, variational inference.
  - **Sampling:** MCMC (e.g., NUTS / HMC).

## Lecture 2

Gaussian and linear models, Bayesian linear regression, and MAP logistic regression with Newton's method.

# Beta–Binomial conjugacy: the canonical example

## Model

- Prior:  $\theta \sim \text{Beta}(\alpha, \beta)$
- Likelihood:  $k \sim \text{Binom}(n, \theta)$
- Posterior:  $\theta | k \sim \text{Beta}(\alpha + k, \beta + n - k)$

## Update rule

$$\alpha^* = \alpha + k, \quad \beta^* = \beta + (n - k)$$

## Pseudo-count interpretation

- prior contributes  $\alpha - 1$  “successes” and  $\beta - 1$  “failures”
- data contributes  $k$  successes and  $n - k$  failures

## Beta distribution: prior intuition

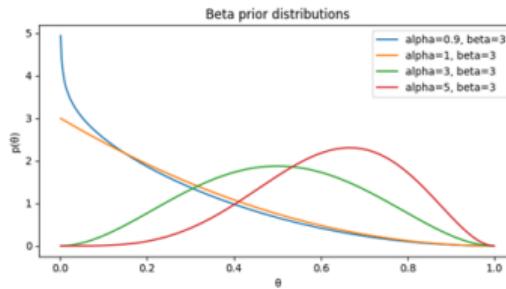


Figure: Beta prior shapes for different  $\alpha$  with  $\beta = 3$

- larger  $\alpha + \beta \Rightarrow$  stronger prior (more concentrated)
- $\alpha > \beta$  biases belief toward larger  $\theta$ ;  $\alpha < \beta$  toward smaller  $\theta$

# Posterior derivation (up to proportionality)

**Likelihood**

$$p(k \mid \theta) \propto \theta^k (1 - \theta)^{n-k}$$

**Prior**

$$p(\theta) \propto \theta^{\alpha-1} (1 - \theta)^{\beta-1}$$

**Posterior (unnormalized)**

$$p(\theta \mid k) \propto \theta^{k+\alpha-1} (1 - \theta)^{n-k+\beta-1}$$

## Posterior in closed form

$$\theta \mid k, n \sim \text{Beta}(\alpha + k, \beta + n - k)$$

### Posterior moments

$$\mathbb{E}[\theta \mid k] = \frac{\alpha + k}{\alpha + \beta + n}$$

$$\text{Var}[\theta \mid k] = \frac{(\alpha + k)(\beta + n - k)}{(\alpha + \beta + n)^2(\alpha + \beta + n + 1)}$$

## Prior vs posterior (visual)

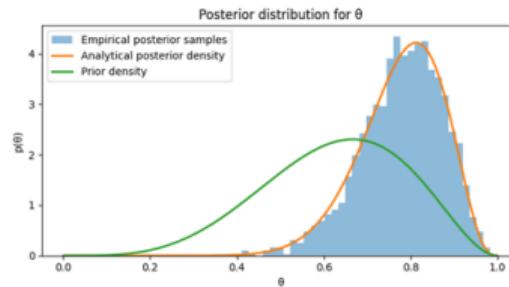
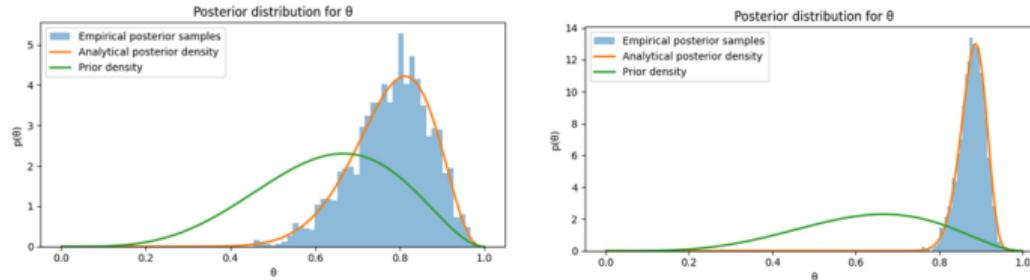


Figure: Prior and posterior density of  $\theta$

- Posterior shifts toward parameter values supported by data.
- Posterior becomes more concentrated as  $n$  increases.

# How dataset size changes uncertainty



**Figure:** Posterior densities for  $(k, n) = (9, 10)$  on the left and  $(k, n) = (90, 100)$  on the right under the same prior.

- Same ratio  $k/n$  can imply very different uncertainty depending on  $n$ .

## Why point estimates still matter

- Many applications require a single parameter value (for deployment simplicity).
- Point estimates are useful **summaries** of the posterior.
- MLE and MAP are the main point-estimation baselines.

# Maximum Likelihood Estimation (MLE)

**Definition:** parameter value maximizing the likelihood.

$$\hat{\theta}_{\text{MLE}} = \arg \max_{\theta} \log p(D | \theta)$$

## Binomial model

$$\log p(k | \theta, n) = k \log \theta + (n - k) \log(1 - \theta) + C$$

$$\frac{\partial}{\partial \theta} = \frac{k}{\theta} - \frac{n - k}{1 - \theta} = 0 \quad \Rightarrow \quad \hat{\theta}_{\text{MLE}} = \frac{k}{n}$$

**When to use:** large sample sizes, purely data-driven estimation.

# Maximum A Posteriori (MAP) estimation

**Definition:** parameter value maximizing the posterior.

$$\hat{\theta}_{\text{MAP}} = \arg \max_{\theta} \log [p(D | \theta)p(\theta)]$$

## Beta–Binomial MAP

$$\hat{\theta}_{\text{MAP}} = \frac{k + \alpha - 1}{n + \alpha + \beta - 2}$$

- incorporates prior information
- acts like regularization (shrinkage toward the prior)
- still a point estimate (does not capture posterior uncertainty)

## MLE vs MAP: practical example

**Scenario:** coin flip with  $k = 7$  successes in  $n = 10$  trials.

Prior: Beta( $\alpha = 5, \beta = 3$ ).

$$\hat{\theta}_{\text{MLE}} = \frac{7}{10} = 0.700 \quad \hat{\theta}_{\text{MAP}} = \frac{7 + 5 - 1}{10 + 5 + 3 - 2} = \frac{11}{16} = 0.6875$$

### Interpretation

- MLE uses only data.
- MAP balances data and prior (slight pull toward prior mean).
- With much more data, MLE and MAP converge.

## MAP as regularization (ML viewpoint)

$$\hat{\theta}_{\text{MAP}} = \arg \min_{\theta} \left( \underbrace{-\log p(D | \theta)}_{\text{data fit}} + \underbrace{-\log p(\theta)}_{\text{regularizer}} \right)$$

- Gaussian prior on weights  $\Rightarrow$  L2 regularization.
- Laplace prior  $\Rightarrow$  L1 regularization.
- Prior encodes what parameter values are plausible *before* seeing data.

# Summarizing the posterior from samples

If we have posterior samples  $\{\theta^{(1)}, \dots, \theta^{(S)}\}$ , we can compute:

## Central tendency

- mean:  $\bar{\theta} = \frac{1}{S} \sum_{s=1}^S \theta^{(s)}$
- median: 50th percentile
- mode: most probable value (e.g., MAP)

## Dispersion

- variance / standard deviation
- quantiles (IQR, 95% range)

# Posterior predictive vs plug-in prediction

## Posterior predictive

$$p(x^* | D) = \int p(x^* | \theta) p(\theta | D) d\theta$$

- accounts for parameter uncertainty
- often wider (more honest)

## Plug-in

$$p(x^* | D) \approx p(x^* | \hat{\theta})$$

- simpler
- can be overconfident

# Analytic vs Monte Carlo inference

- Conjugate models: compute posterior in closed form.
- General models: posterior may be intractable  $\Rightarrow$  approximate inference.

## Two views

- **Analytic:** derive posterior formula directly.
- **Sampling:** approximate posterior with samples (MC, MCMC).

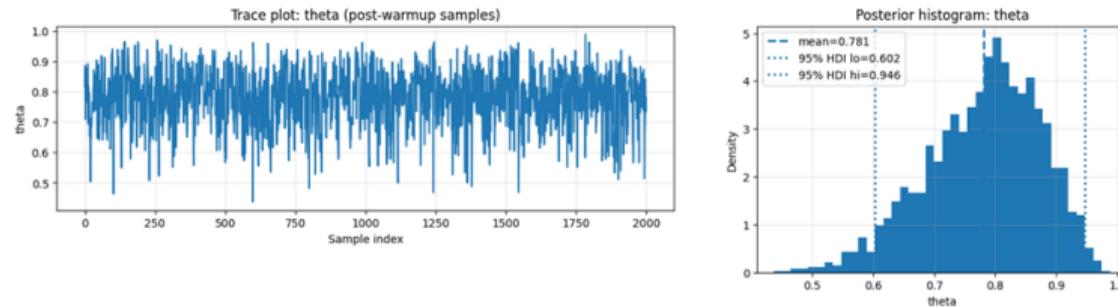


Figure: Posterior samples and their histogram

# Evidence and Bayes factors

## Marginal likelihood (evidence)

$$p(\mathbf{X}) = \int p(\mathbf{X} | \theta) p(\theta) d\theta$$

- Penalizes overly flexible models automatically (“Occam factor”).
- Enables model comparison:  $\text{BF}_{10} = \frac{p(\mathbf{X} | M_1)}{p(\mathbf{X} | M_0)}$ .

## In conjugate models

You can compute evidence in closed form (e.g., Beta-Binomial). For complex models we approximate.

## Key takeaways

- UQ asks: what is plausible, how variable, and how beliefs update with data.
- Bayesian inference: posterior  $\propto$  likelihood  $\times$  prior.
- MLE and MAP are point estimates; full Bayes keeps a distribution.
- Credible intervals (central/HPD) summarize posterior uncertainty.
- PPD averages over parameter uncertainty to avoid overconfidence.

## Acknowledgements/ Document preparation

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