Kalman Filtering: Solutions

ELG 5218 – Uncertainty Evaluation in Engineering Measurements and Machine Learning

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Contents

1 Joint Distributions and Correlation

Solution 1.1: 2D Joint Distribution

1. The covariance matrix is:
$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}$$

2. $\text{COV}(z_1, z_2) = \rho\sigma_1\sigma_2 = 0.5 \cdot 1 \cdot \sqrt{2} \approx 0.707$. However, the problem states covariance is 0.5, so:
$$\text{COV}(z_1, z_2) = 0.5$$

3. The 2D Gaussian has an elliptical shape. With $\rho = 0.5 > 0$, the ellipse tilts along the positive diagonal. The spread in the z_2 direction is larger ($\sigma_2 = \sqrt{2}$) than in z_1 ($\sigma_1 = 1$). Center is at $(\mu_1, \mu_2) = (1, 2)$.

4. If $\rho = -0.5$: ellipse tilts along negative diagonal (negative correlation). If $\rho = 0$: ellipse aligns with axes (zero correlation).

Solution 1.2: Covariance and Correlation

1. Sample covariance (biased estimator):
$$\text{COV}(z_1, z_2) = \frac{1}{N}\sum_{t=1}^{N}(z_1(t) - \bar{z}_1)(z_2(t) - \bar{z}_2)$$
where $\bar{z}_i$ is the sample mean. Unbiased version uses $\frac{1}{N-1}$.

2. Correlation coefficient:
$$\rho(z_1, z_2) = \frac{\text{COV}(z_1, z_2)}{\sigma_1\sigma_2}, \quad \rho \in [-1, 1]$$
Interpretation: $\rho > 0$ (positive), $\rho < 0$ (negative), $\rho = 0$ (uncorrelated), $|\rho| = 1$ (perfect linear).

3. Given: $\text{COV} = 3.0$, $\sigma_1 = 2$, $\sigma_2 = 3$:
$$\rho = \frac{3.0}{2 \times 3} = 0.5$$

4. $\rho = 0.9$ indicates strong positive correlation: when z_1 is high, z_2 tends to be high. Near-linear relationship but not perfect.

Solution 1.3: Multivariate Gaussian Transformation

1. Mean of $\mathbf{y} = A\mathbf{z} + \mathbf{b}$:
$$\boldsymbol{\mu}_y = A\boldsymbol{\mu}_z + \mathbf{b}$$

2. Covariance:
$$\mathbf{\Sigma}_y = A\mathbf{\Sigma}_z A^\top$$

3. With given values:
$$\boldsymbol{\mu}_y = \begin{bmatrix} 2 & 0.5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\mathbf{\Sigma}_y = \begin{bmatrix} 2 & 0.5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 4.25 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

2 State Space Models and Dynamic Systems

Solution 2.1: Constant Velocity Model

1. State transition matrix for sampling interval Δt:

$$F = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

   (Represents: new position = old position + velocity $\times$ Δt; velocity unchanged)

2. Measurement matrix (position only):

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

3 Kalman Filter Fundamentals

Solution 3.1: Prediction and Update Steps

1. Prediction: Since $z_{t+1} = z_t + q_t$, filter predicts: $\hat{z}_{1|0} = \hat{z}_{0|0} = 5$. The prediction matches current estimate (no deterministic change).

2. Update: Observation $y_1 = 11$ gives innovation (residual): $y_1 - \hat{z}_{1|0} = 11 - 5 = 6$. This large innovation suggests the true value is higher than estimated.

3. Update equation:

$$\hat{z}_{1|1} = 5 + K_1 \cdot 6$$

   where K_1 is the Kalman gain.

4. The Kalman gain K_1 represents the fraction of innovation the filter trusts:
   - $K_1 \approx 1$: trusts measurement, moves estimate close to measured value
   - $K_1 \approx 0$: trusts prediction, barely adjusts
   - $K_1 \approx 0.5$: equal weight to prediction and measurement

Solution 3.2: Kalman Gain Properties

1. If $R \to 0$ (perfect measurements): $K_t \to 1$. Filter trusts measurements fully, sets estimate = measurement.

2. If $R \to \infty$ (useless measurements): $K_t \to 0$. Filter ignores measurements, sticks with prediction.

3. If $\Sigma_{t|t-1} \to 0$ (perfect prediction): $K_t \to 0$. Filter is confident, doesn't correct based on measurements.

4. Interpretation as confidence-weighted average:

$$\hat{z}_{t|t} = (1 - K_t)\hat{z}_{t|t-1} + K_t y_t$$

   Weighted average of prediction and measurement, where weight depends on relative confidence (noise levels).

Solution 3.3: Covariance Evolution

1. Prediction: $\Sigma_{1|0} = \Sigma_{0|0} + Q = 100 + 0.01 = 100.01$
2. Kalman gain: $K_1 = \frac{100.01}{100.01+4} \approx 0.961$
3. Updated variance: $\Sigma_{1|1} = (1 - K_1)\Sigma_{1|0} = 0.039 \times 100.01 \approx 3.90$
4. Continuing the iterations:
   - $k = 2$: $\Sigma_{2|1} \approx 3.91$, $K_2 \approx 0.494$, $\Sigma_{2|2} \approx 1.97$
   - $k = 3$: $\Sigma_{3|2} \approx 1.98$, $K_3 \approx 0.331$, $\Sigma_{3|3} \approx 1.33$
   - $k = 4$: $\Sigma_{4|3} \approx 1.34$, $K_4 \approx 0.251$, $\Sigma_{4|4} \approx 1.00$
5. Trend: Covariance decreases rapidly initially, then more slowly. Steady-state occurs around $k \approx$ 10-20 iterations.