Kalman Filtering: Problems

ELG 5218 – Uncertainty Evaluation in Engineering Measurements and Machine Learning

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1 Joint Distributions and Correlation

Problem 1.1: 2D Joint Distribution

Consider two random time series with means $\mu_1 = 1$, $\mu_2 = 2$, variances $\sigma_1^2 = 1$, $\sigma_2^2 = 2$, and correlation coefficient $\rho = 0.5$.

1. Write the covariance matrix $\mathbf{\Sigma}$.
2. What is the covariance $\text{COV}(z_1, z_2)$?
3. Describe the shape of the 2D Gaussian distribution (e.g., orientation, spread).
4. How would the shape change if $\rho = -0.5$? If $\rho = 0$?

Problem 1.2: Covariance and Correlation

1. Define the sample covariance between two time series $z_1(t)$ and $z_2(t)$ with N samples.
2. Define the correlation coefficient $\rho(z_1, z_2)$ and explain its range and interpretation.
3. If $\text{COV}(z_1, z_2) = 3.0$, $\sigma_1 = 2$, and $\sigma_2 = 3$, compute the correlation coefficient.
4. What does $\rho = 0.9$ imply about the relationship between z_1 and z_2?

Problem 1.3: Multivariate Gaussian Transformation

Consider a multivariate Gaussian vector $\mathbf{z} = [z_1, z_2]^\top$ with mean $\boldsymbol{\mu}$ and covariance $\mathbf{\Sigma}_z$.
Let $\mathbf{y} = A\mathbf{z} + \mathbf{b}$ where A is a 2×2 matrix.

1. Derive the mean of $\mathbf{y}$.
2. Derive the covariance of $\mathbf{y}$.
3. If $A = \begin{bmatrix} 2 & 0.5 \\ 0 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\mathbf{\Sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, compute $\boldsymbol{\mu}_y$ and $\mathbf{\Sigma}_y$.

2 State Space Models and Dynamic Systems

Problem 2.1: Constant Velocity Model

An object moves in 2D with constant velocity. State: $\mathbf{z}_t = [z_{1,t}, \dot{z}_{1,t}, z_{2,t}, \dot{z}_{2,t}]^\top$.

1. Derive the state transition matrix F for sampling interval Δt.
2. If only position (z_1, z_2) is measured, what is H?

3 Kalman Filter Fundamentals

Problem 3.1: Prediction and Update Steps

Consider the scalar system:

$$z_{t+1} = z_t + q_t, \quad y_t = z_t + r_t$$

with $q_t \sim \mathcal{N}(0, Q)$, $r_t \sim \mathcal{N}(0, R)$, and initial estimate $\hat{z}_{0|0} = 5$, true value $z_0 = 10$.

1. Describe the prediction step: what does the filter predict for $\hat{z}_{1|0}$?
2. If measurement $y_1 = 11$ is observed, describe the update step.
3. Write the update equation: $\hat{z}_{1|1} = \hat{z}_{1|0} + K_1(y_1 - \hat{z}_{1|0})$.
4. Explain what the Kalman gain K_1 represents.

Problem 3.2: Kalman Gain Properties

The Kalman gain is defined as:

$$K_t = \frac{\Sigma_{t|t-1}}{\Sigma_{t|t-1} + R}$$

1. What happens to K_t if $R \to 0$ (very accurate measurements)?
2. What happens to K_t if $R \to \infty$ (very noisy measurements)?
3. What happens to K_t if $\Sigma_{t|t-1} \to 0$ (very confident prediction)?
4. Interpret the Kalman gain as a confidence-weighted average.

Problem 3.3: Covariance Evolution

Starting with $\Sigma_{0|0} = 100$ (very uncertain), $Q = 0.01$, $R = 4$:

1. Compute $\Sigma_{1|0}$ (predicted variance).
2. Compute K_1.
3. Compute $\Sigma_{1|1}$ (updated variance).
4. Repeat for $t = 2, 3, 4$ and describe the trend.
5. At what iteration does steady state occur?

Problem 3.4: 2D constant-velocity model: matrix shapes & one update

State $\mathbf{z}_t = [x_t, \dot{x}_t, y_t, \dot{y}_t]$ with $\Delta t = 1$ and

$$\mathbf{F} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Assume $\mathbf{Q} = 0.1\,\mathbf{I}_4$, $\mathbf{R} = \mathrm{diag}(1, 4)$, prior $\boldsymbol{\mu}_{t-1|t-1} = 0$, $\boldsymbol{\Sigma}_{t-1|t-1} = \mathbf{I}_4$, no control inputs, and measurement $\mathbf{y}_t = [1, -2]$.

1. Compute $\boldsymbol{\mu}_{t|t-1}$ and $\boldsymbol{\Sigma}_{t|t-1}$.
2. Compute $\hat{\mathbf{y}}_t$, $\mathbf{S}_t$, and $\mathbf{K}_t$.
3. Compute the updated mean $\boldsymbol{\mu}_{t|t}$ (you may leave $\boldsymbol{\Sigma}_{t|t}$ in symbolic form).
4. Intuition: which coordinate (x or y) should be corrected more strongly, and why?