Bayesian Learning: Gaussian Models

ELG 5218 - Uncertainty Evaluation in Engineering Measurements and Machine Learning

Miodrag Bolic

University of Ottawa

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Roadmap (40 slides, 3 hours)

1. **Gaussian models fundamentals:** Why Gaussians? Central Limit Theorem, conjugacy.
2. **Known variance case:** Precision additivity, conjugate priors, weighted averaging.
3. **Online learning (sequential updates):** Recursive Bayesian filtering, streaming data.
4. **Unknown variance:** Normal-Gamma conjugate family, Student-t marginal, robustness.
5. **Monte Carlo methods:** Posterior sampling, functions of parameters, practical inference.
6. **Multivariate case:** Matrix algebra, sensor fusion, conditional independence.
7. **Discussion:** Advanced topics, extensions, exercises.

The Gaussian family: central to Bayesian inference

Four reasons Gaussians dominate:

- **Universality (CLT):** Sums of iid random variables $\to$ Gaussian (approximately).
- **Analytical tractability:** Closed-form posterior, predictive, marginals, conditionals.
- **Conjugacy:** Normal prior $\times$ Normal likelihood $\Rightarrow$ Normal posterior.
- **Optimization:** Gaussian assumption $\to$ convex inference, efficient algorithms.

Univariate density:

$$p(x \mid \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Precision notation (cleaner for Bayesian updates):

$$\lambda = \frac{1}{\sigma^2} \quad \Rightarrow \quad p(x \mid \mu, \lambda) \propto \sqrt{\lambda} \exp\left(-\frac{\lambda(x-\mu)^2}{2}\right).$$

Multivariate Gaussian: covariance and precision

For $\boldsymbol{x} \in \mathbb{R}^d$:
Covariance form:

$$p(\boldsymbol{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-d/2} |\boldsymbol{\Sigma}|^{-1/2} \exp\left(-\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \right).$$

Precision form (more convenient for updates):

$$\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1} \quad \Rightarrow \quad p(\boldsymbol{x} \mid \boldsymbol{\mu}, \boldsymbol{\Lambda}) \propto |\boldsymbol{\Lambda}|^{1/2} \exp\left(-\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Lambda} (\boldsymbol{x} - \boldsymbol{\mu}) \right).$$

Key insight: Precision combines additively

In Bayesian updates, posterior precision = prior precision + data precision.
This additive structure is the **engine** driving conjugate family simplicity.

Covariance matrix properties (brief review)

Covariance between variables X_i and X_j:

$$\mathrm{Cov}(X_i, X_j) = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])].$$

Covariance matrix:

$$\boldsymbol{\Sigma} = \mathbb{E}[(\boldsymbol{x} - \boldsymbol{\mu})(\boldsymbol{x} - \boldsymbol{\mu})^T] = \begin{bmatrix} \sigma_1^2 & \mathrm{Cov}(X_1, X_2) & \cdots \\ \mathrm{Cov}(X_2, X_1) & \sigma_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Correlation: $\rho_{ij} = \mathrm{Cov}(X_i, X_j)/(\sigma_i \sigma_j) \in [-1, 1]$.

Why precision for Bayesian inference?

- Precisions in likelihood and prior **add directly**, not reciprocals.
- Posterior precision is always non-negative (automatic numerical stability).
- Computational benefits for high-dimensional problems.

Setup: Univariate Gaussian, known σ^2

Model: $x_1, \dots, x_n \mid \mu, \sigma^2 \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with σ^2 known.
Likelihood (in precision form):

$$p(x_1, \dots, x_n \mid \mu) \propto \exp\left(-\frac{\lambda}{2} \sum_{i=1}^{n} (x_i - \mu)^2\right), \quad \lambda = \frac{1}{\sigma^2}.$$

Using the sum-of-squares identity:

$$\sum_{i=1}^{n} (x_i - \mu)^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2.$$

First term is data-independent, so:

$$p(x_1, \dots, x_n \mid \mu) \propto \exp\left(-\frac{n\lambda}{2} (\bar{x} - \mu)^2\right).$$

Interpretation: Likelihood is Gaussian in μ with precision $n\lambda = n/\sigma^2$.

Uniform (non-informative) prior

Prior: $p(\mu) \propto 1$ for $\mu \in \mathbb{R}$ (improper, but posterior is proper).

Posterior (via Bayes):

$$p(\mu \mid x_1, \ldots, x_n) \propto p(x_1, \ldots, x_n \mid \mu) \cdot p(\mu) \propto \exp\left(-\frac{n\lambda}{2}(\bar{x} - \mu)^2\right).$$

Recognizing this as Gaussian:

$$\mu \mid x_1, \ldots, x_n \sim \mathcal{N}\left(\bar{x}, \frac{\sigma^2}{n}\right).$$

In precision form:

$$\lambda_n = \frac{n}{\sigma^2} = n\lambda, \quad \mu_n = \bar{x}.$$

Key observations

- Posterior mean $=$ sample mean $\bar{x}$ (frequentist MLE).
- Posterior variance shrinks as $1/n$.
- Posterior precision grows linearly with sample size.

Normal prior (conjugate case)

Prior: $\mu \sim \mathcal{N}(\mu_0, \sigma_0^2)$ with precision $\lambda_0 = 1/\sigma_0^2$.

Posterior: Also Gaussian with

$$\lambda_n = \lambda_0 + n\lambda, \quad \mu_n = \frac{\lambda_0\mu_0 + n\lambda\bar{x}}{\lambda_n}.$$

Rewritten (cleaner form):

$$\mu_n = \frac{\lambda_0\mu_0 + (n/\sigma^2)\bar{x}}{\lambda_0 + n/\sigma^2} = w\bar{x} + (1-w)\mu_0,$$

where

$$w = \frac{\lambda_{\text{data}}}{\lambda_0 + \lambda_{\text{data}}} = \frac{n/\sigma^2}{\lambda_0 + n/\sigma^2}.$$

The golden rule: Precision Additivity

$$\boxed{\lambda_{\text{posterior}} = \lambda_{\text{prior}} + \lambda_{\text{data}}}$$

Posterior mean is precision-weighted average of prior and data.

Intuition: Precision-weighted averaging

- **High data precision** (n/σ^2 large): $w \approx 1$, posterior tracks data.
- **High prior precision** (λ_0 large): $w \approx 0$, posterior tracks prior.
- **Competing uncertainties:** Weight determined by relative precisions.
- **Posterior variance:** $\sigma_n^2 = 1/\lambda_n$, monotonically shrinks.

Example interpretation:

- Prior: expert opinion (uncertain, $\sigma_0 = 5$).
- Data: 10 precise measurements (noisy, $\sigma = 3$).
- Data precision $0.2/\sigma^2 > 0.04 =$ prior precision $\Rightarrow$ data dominates.

Worked example: Internet speed measurement

Context: ISP claims average speed is 20 Mbps. You measure 5 times.
Data: $x = \{15.77, 20.5, 8.26, 14.37, 21.09\}$, $\bar{x} = 16.0$ Mbps.
Known measurement noise: $\sigma = 5$ Mbps $\Rightarrow \lambda = 0.04$.
Prior: $\mu_0 = 20$ Mbps, $\sigma_0 = 5$ Mbps $\Rightarrow \lambda_0 = 0.04$.
Posterior calculation:

$$\lambda_n = 0.04 + 5 \cdot 0.04 = 0.24 \quad \Rightarrow \quad \sigma_n^2 = 4.17.$$

$$\mu_n = \frac{0.04 \cdot 20 + 0.2 \cdot 16}{0.24} = \frac{0.8 + 3.2}{0.24} = 16.67 \text{ Mbps}.$$

$$w = \frac{0.2}{0.24} = 0.833 \quad \Rightarrow \quad \text{Data dominates } (5\times \text{ more precise}).$$

Conclusion: Posterior $\Pr(\mu > 20 \mid \text{data}) \approx 0.05$ (ISP claim is dubious).

Posterior for Internet speed measurement

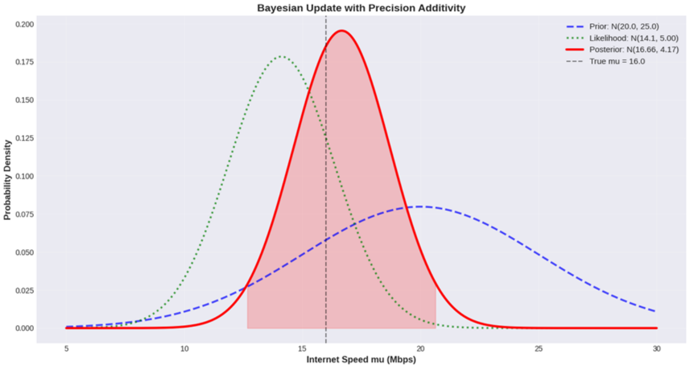


Figure: Prior-Likelihood-Posterior. The shaded region marks out $Pr(\theta > 20|x_1, ..., x_n)$.

Sensor fusion example: Temperature + Humidity

Scenario: Two correlated measurements (temperature, humidity).

True state: $\boldsymbol{\mu} = \begin{bmatrix} 22C \\ 55\% \end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix} 2 & 1.5 \\ 1.5 & 4 \end{bmatrix}$.

Data: $n = 15$ bivariate readings.

Prior (weak): $\boldsymbol{\mu}_0 = \begin{bmatrix} 20 \\ 50 \end{bmatrix}$, independent prior on correlation.

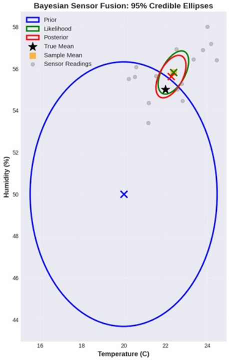
Matrix precision additivity:

$$\boldsymbol{\Lambda}_n = \boldsymbol{\Lambda}_0 + n\boldsymbol{\Sigma}^{-1} = \text{prior precision} + \text{data precision}.$$

Posterior:

$$\boldsymbol{\Sigma}_n = \boldsymbol{\Lambda}_n^{-1}, \quad \boldsymbol{\mu}_n = \boldsymbol{\Sigma}_n(\boldsymbol{\Lambda}_0\boldsymbol{\mu}_0 + n\boldsymbol{\Sigma}^{-1}\bar{\boldsymbol{x}}).$$

Posterior for the fusion example



2D visualization:

1. Prior: Ellipse centered at $[20, 50]$ (weak, large).
2. Likelihood: Ellipse centered at sample mean (from 15 readings).
3. Posterior: Ellipse centered at μ_n (tighter).
4. Correlation improvement (ellipse axes align with data).

Sequential vs. batch updating

Batch form:

$$p(\mu \mid x_1, \ldots, x_n) = \frac{p(x_1, \ldots, x_n \mid \mu)p(\mu)}{p(x_1, \ldots, x_n)}.$$

Sequential form:

$$p(\mu \mid x_1, \ldots, x_n) = \frac{p(x_n \mid \mu) \cdot p(\mu \mid x_1, \ldots, x_{n-1})}{p(x_n \mid x_1, \ldots, x_{n-1})}.$$

Key insight: Posterior at step $n-1$ becomes the prior at step n.

Streaming advantage

- Never store all past data, only current posterior parameters.
- Each new observation x_n triggers one simple update.
- Perfect for sensor networks, real-time monitoring, online learning.
- Memory: $O(d)$ vs. $O(nd)$ for batch processing.

Recursive updating: precision and mean

Starting from $\mu \mid x_1, \ldots, x_{n-1} \sim \mathcal{N}(\mu_{n-1}, \sigma^2_{n-1})$, when x_n arrives:

Precision update:

$$\lambda_n = \lambda_{n-1} + \lambda = \lambda_{n-1} + \frac{1}{\sigma^2}.$$

Mean update (in learning rate form):

$$\mu_n = \mu_{n-1} + w_n(x_n - \mu_{n-1}),$$

where the learning rate is

$$w_n = \frac{\lambda}{\lambda_n} = \frac{1/\sigma^2}{\lambda_{n-1} + 1/\sigma^2}.$$

Interpretation

- w_n is the fractional move toward new observation (shrinks over time).
- As n increases, λ_n grows, so w_n shrinks (diminishing returns).
- Posterior variance $\sigma_n^2 = 1/\lambda_n$ monotonically decreases.

Example: Temperature sensor over time

Scenario: Noisy temperature sensor, true temperature $\mu = 22$C.

Setup:

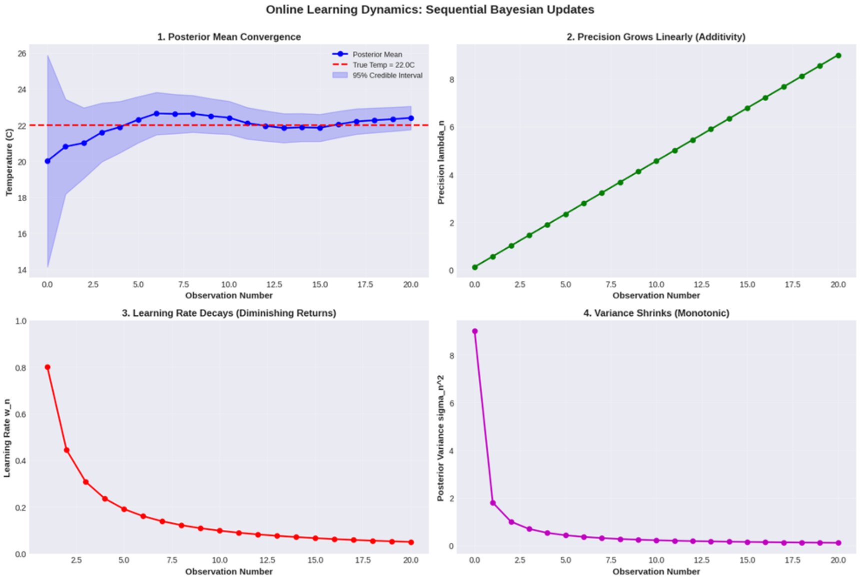
- Measurement noise: $\sigma = 1.5$C $\Rightarrow \lambda = 0.444$.
- Prior: $\mu_0 = 20$C, $\sigma_0 = 3$C $\Rightarrow \lambda_0 = 0.111$.
- 20 sequential readings, typical values: $\{20.1, 22.3, 21.8, 23.1, \ldots\}$.

Dynamics:

n	λ_n	σ_n	w_n
1	0.556	1.342	0.800
5	2.333	0.655	0.190
10	4.556	0.468	0.097
20	8.889	0.335	0.050

Trend: Precision grows linearly, learning rate decays, variance shrinks monotonically.

Online learning dynamics



Why Bayesian online learning beats alternatives

Compare three approaches for 1M sensor readings:

	Storage	Updates	Recompute
Batch (store all)	$O(n)$	Once at end	Full posterior
Bayes online	$O(1)$	Each reading	Full posterior

Key advantage: Bayes online gives full posterior & uncertainty, not just point estimate.

Streaming Bayesian inference

Sequential updates are exact (not approximations). Posterior is proper at every step. No batch reprocessing needed. Natural uncertainty quantification.

Chi-square, Gamma, and why we care

Definition (sum of squares). If $Z_1, \dots, Z_k \overset{iid}{\sim} \mathcal{N}(0,1)$, then

$$X = \sum_{i=1}^{k} Z_i^2 \ \sim \ \chi_k^2.$$

This makes χ^2 the natural distribution for *energy*, *squared error*, and *quadratic forms*.

Key relationship: Chi-square is a special case of the Gamma distribution.

$$X \sim \chi_k^2 \quad \Longleftrightarrow \quad X \sim \mathrm{Gamma}\left(\alpha = \frac{k}{2},\ \theta = 2\right)$$

(using Gamma(shape α, scale θ)). Equivalently, with rate $\beta = 1/\theta$:

$$X \sim \mathrm{Gamma}\left(\alpha = \frac{k}{2},\ \beta = \frac{1}{2}\right).$$

Why it matters (canonical results). If $x_1, \dots, x_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, then

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2,$$

which drives: (i) confidence intervals for σ^2, (ii) chi-square tests, (iii) Gaussian-conjugate updates (via Gamma on precision).
Visualization: probstats.org (interactive PDFs/CDFs for χ^2 and Gamma),

Inverse-χ^2, scaled inverse-χ^2, and Gamma priors on precision

Inverse-χ^2 (basic transform). If $X \sim \chi^2_\nu$, then

$$V = \frac{1}{X} \sim \text{Inv-}\chi^2_\nu.$$

Scaled inverse-χ^2 (Bayesian variance prior). A very common conjugate prior for a Normal variance σ^2 is:

$$\sigma^2 \sim \text{Scale-inv-}\chi^2(\nu, \tau^2), \qquad \text{equivalently} \qquad \sigma^2 \sim \mathrm{Inv\text{-}Gamma}\left(\frac{\nu}{2}, \frac{\nu\tau^2}{2}\right),$$

(Inv-Gamma(shape, scale) parameterization).
Precision view (often cleaner). Let $\lambda = 1/\sigma^2$ (precision). Then:

$$\sigma^2 \sim \mathrm{Inv\text{-}Gamma}\left(\frac{\nu}{2}, \frac{\nu\tau^2}{2}\right) \quad \Longleftrightarrow \quad \lambda \sim \Gamma\left(\frac{\nu}{2}, \frac{\nu\tau^2}{2}\right)$$

(Gamma(shape, *rate*) parameterization for λ).

Visualizationn

Inverse-χ^2 / scaled inverse-χ^2 have heavy right tails (variance can be large). Interactive plots: probstats.org (Gamma, χ^2); Wolfram: InverseChiSquareDistribution.

Student-t distribution and its relationship to the Gaussian

Definition (location–scale Student-t). A random variable T has a Student-t distribution with ν degrees of freedom, location μ, and scale s if

$$T \sim t_\nu(\mu, s) \quad \Longleftrightarrow \quad \frac{T-\mu}{s} \sim t_\nu.$$

Its density has heavier tails than a Gaussian (more mass far from the mean), especially for small ν. **Relationship to Gaussian.**

- As $\nu \to \infty$, $t_\nu(\mu, s) \Rightarrow \mathcal{N}(\mu, s^2)$ (tails become Gaussian).
- For $\nu > 1$, $\mathbb{E}[T] = \mu$; for $\nu > 2$, $\mathrm{Var}(T) = \dfrac{\nu}{\nu-2}\, s^2$ (larger than s^2).
- Small ν (e.g., $\nu \in [3, 10]$) yields strong robustness: outliers are less influential than under a Gaussian likelihood.

Where it appears in Bayesian inference. In the Normal model with unknown variance, integrating out σ^2 (with an Inv-Gamma / scaled-inv-χ^2 prior) yields a Student-t marginal for μ and a Student-t posterior predictive for new observations.

Quick visualization (interactive)

Overlay t_ν vs Gaussian for different ν: probstats.org. (Use as a screenshot figure or live demo.)

Challenge: Both μ and σ^2 unknown

Model: $x_1, \ldots, x_n \mid \mu, \sigma^2 \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ with both unknown.
Problem: Prior must specify joint $p(\mu, \sigma^2)$ with possible dependence.
Solution: Use conjugate factorization

$$p(\mu, \sigma^2) = p(\mu \mid \sigma^2) \cdot p(\sigma^2).$$

Conjugate choice:

- **Conditional:** $\mu \mid \sigma^2 \sim \mathcal{N}(\mu_0, \sigma^2/\kappa_0)$ (precision scales with σ^2).
- **Marginal:** $\sigma^2 \sim \mathrm{Inv}\text{-}\chi^2(\nu_0, \sigma_0^2)$ (Inverse-Chi-squared prior).

Hyperparameters

μ_0: prior mean for μ. κ_0: prior "effective sample size" (scales prior variance). ν_0: degrees of freedom for σ^2 prior. σ_0^2: prior location guess for variance.

Gamma distribution for precision

Key shift: Instead of Inverse-χ^2 for variance, use Gamma for precision.

Gamma distribution (shape-rate parametrization):

$$\lambda \sim \mathrm{Gamma}(\alpha, \beta) \quad \Rightarrow \quad p(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}.$$

Properties:

- Mean: $\mathbb{E}[\lambda] = \alpha/\beta$.
- Variance: $\mathrm{Var}[\lambda] = \alpha/\beta^2$.
- Support: $\lambda > 0$ (necessary for precision).
- Conjugate: Gamma is conjugate to Normal likelihood.

Relationship to Inverse-Gamma:

$$\sigma^2 = 1/\lambda \sim \mathrm{Inv}\text{-}\Gamma, \quad \text{if} \quad \lambda \sim \mathrm{Gamma}.$$

Why Gamma for precision?

More numerically stable, directly expresses prior beliefs about precision (not variance). Simpler update equations.

Normal-Gamma conjugate prior

Joint prior (factorized):

$$p(\mu, \lambda) = p(\mu \mid \lambda) \cdot p(\lambda),$$

where

$$\mu \mid \lambda \sim \mathcal{N}(\mu_0, (\kappa_0 \lambda)^{-1}),$$
$$\lambda \sim \mathrm{Gamma}(\alpha_0, \beta_0).$$

Hyperparameters:

- μ_0, κ_0: prior location and strength for μ.
- α_0, β_0: shape and rate for precision.

Why this works: The likelihood factors as

$$p(x_1, \ldots, x_n \mid \mu, \lambda) = (\text{term in } \mu) \times (\text{term in } \lambda),$$

so posterior is also Normal-Gamma with updated hyperparameters.

Posterior: Normal-Gamma family

Result (conjugate update):

$$p(\mu, \lambda \mid \text{data}) = \mathcal{N}(\mu \mid \mu_n, (\kappa_n \lambda)^{-1}) \cdot \mathrm{Gamma}(\lambda \mid \alpha_n, \beta_n).$$

Update equations:

$$\begin{aligned}
\kappa_n &= \kappa_0 + n, \\
\mu_n &= \frac{\kappa_0 \mu_0 + n\bar{x}}{\kappa_n}, \\
\alpha_n &= \alpha_0 + \frac{n}{2}, \\
\beta_n &= \beta_0 + \frac{1}{2}\left[\sum (x_i - \bar{x})^2 + \frac{\kappa_0 n}{kappa_0 + n}(\bar{x} - \mu_0)^2\right].
\end{aligned}$$

Interpretation:

- Posterior precision κ_n is prior + data (mixture weight).
- Posterior α_n, β_n encode both data variance and mean discrepancy.

Marginal posterior for μ: Student-t distribution

When we marginalize out the unknown λ:

$$p(\mu \mid \text{data}) = \int p(\mu \mid \lambda, \text{data})\, p(\lambda \mid \text{data})\, d\lambda \sim \mathrm{t}_{\nu_n}(\mu_n, \sigma_n^2).$$

Degrees of freedom: $\nu_n = 2\alpha_n$.
Scale: $\sigma_n^2 = \frac{\beta_n}{\alpha_n \kappa_n}$.
Student-t density:

$$p(\mu \mid \text{data}) \propto \left(1 + \frac{\kappa_n(\mu - \mu_n)^2}{\nu_n \sigma_n^2}\right)^{-(\nu_n+1)/2}.$$

Why Student-t, not Gaussian?

Uncertainty in σ^2 creates heavier tails. Extreme values of μ are less surprising when variance is also unknown. Robustness to outliers.

Intuition: Heavy tails from variance uncertainty

Known σ^2:
Posterior for μ is Gaussian with variance σ^2/n (tight).
Unknown σ^2:
Posterior marginalizes over plausible σ^2 values:

$$p(\mu \mid \text{data}) = \int p(\mu \mid \sigma^2, \text{data}) p(\sigma^2 \mid \text{data}) d\sigma^2.$$

Some posterior draws have $\sigma^2 > \hat{\sigma}^2$ (true value underestimated) $\Rightarrow$ wider credible intervals.
This averaging manifests as Student-t tails (heavier than Gaussian).

Robustness

Models estimating σ^2 are robust to outliers. Heavy tails allow occasional large deviations without dominating posterior.

Worked example: Internet speed (unknown variance)

Data: $n = 5$, $\bar{x} = 16.0$, $\sum(x_i - \bar{x})^2 = 67.6$ Mbps^2.
Prior: $\mu_0 = 20$, $\kappa_0 = 1$, $\alpha_0 = 2.5$, $\beta_0 = 12.5$ (informative).
Posterior:

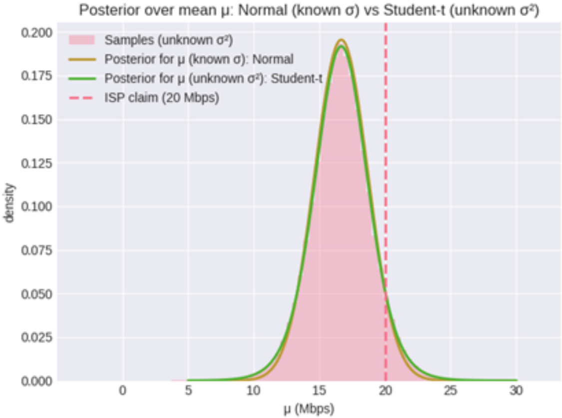
$$\begin{aligned}
\kappa_n &= 1 + 5 = 6, \\
\mu_n &= \frac{1 \cdot 20 + 5 \cdot 16}{6} = 16.67, \\
\alpha_n &= 2.5 + 2.5 = 5, \\
\beta_n &= 12.5 + 33.8 + 3.33 = 49.6.
\end{aligned}$$

Marginal posterior for μ:

$$\mu \mid \text{data} \sim \mathrm{t}_{10}(16.67, 1.65).$$

Posterior std: $\sqrt{1.65} \approx 1.28$ Mbps (wider than known-σ case).
Heavier tails: $\Pr(\mu > 20) \approx 0.09$ (vs. 0.05 in known case).

Student-t vs Gaussian



Summary: The Gaussian toolbox

1. **Known variance:** Posterior is Gaussian. Precision additivity.
2. **Online learning:** Sequential updates decay in influence. No batch reprocessing.
3. **Unknown variance:** Normal-Gamma posterior. Marginal μ is Student-t (heavier tails, robustness).
4. **Monte Carlo:** Simulate to compute expectations and functions of parameters.
5. **Multivariate:** Matrix precision additivity, conditioning, marginalization.
6. **Linear Gaussian:** Conditional, marginal, and predictive distributions all Gaussian.

Central principle

Precisions (inverse variances) combine additively. This algebraic property drives conjugacy, simplicity, and analytical tractability.

Exercise ideas for students

1. Derive posterior for μ when prior is Exponential and likelihood is Gaussian (hint: not conjugate).
2. Extend internet speed example to $n = 50$ measurements. Plot convergence of μ_n and learning rate w_n.
3. Prove: For Normal-Gamma posterior, $\mathbb{E}[\mu \mid \text{data}] = \mu_n$ and $\mathrm{Var}[\mu \mid \text{data}] = \mathbb{E}_\lambda[\sigma_n^2] + \mathrm{Var}_\lambda[\mu]$.
4. Implement online Bayesian filter for temperature sensor. Compare to exponential moving average (EMA).

References and further reading

- **Gelman et al. (2013):** "Bayesian Data Analysis" (3rd ed.). Chapters 2, 3.
- **Murphy (2012):** "Machine Learning: A Probabilistic Perspective". Chapters 2, 3.
- **Bernardo & Smith (2009):** "Bayesian Theory". Foundational reference.
- **Lecture notebooks:** "Gaussian_Models_Extended-1.ipynb" contains worked examples, interactive plots, and exercises.
- **Software:** PyMC3, Stan, NumPyro for Bayesian inference.

Acknowledgements/ Document preparation

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