Theory and Implementation of Particle Filters

ELG 5218 - Uncertainty Evaluation in Engineering Measurements and Machine Learning

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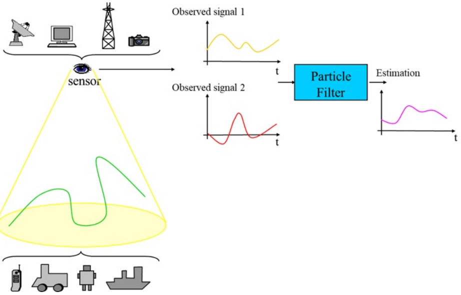
Outline

Notebook: bearings_only_particle_filter.ipynb

Big Picture

Goal: Estimate a latent stochastic process given some noisy observations.

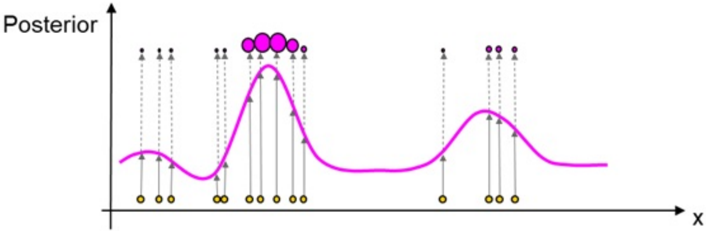
Concepts:
Bayesian filtering
Monte Carlo sampling



Sequential Monte Carlo Techniques

- Particle filter is a technique for implementing recursive Bayesian filter by Monte Carlo sampling
- The idea: represent the posterior density by a set of random particles with associated weights.
- Compute estimates based on these samples and weights

- Bootstrap filtering
- The condensation algorithm
- Particle filtering
- Interacting particle approximations
- Survival of the fittest



Historical Development of Particle Filters (SMC)

Foundations in Monte Carlo (1953): Sampling-based computation becomes practical

- Metropolis et al., "Equation of State Calculations by Fast Computing Machines" (Metropolis MCMC)

Sequential growth idea (1955): Early *sequential* Monte Carlo / "growth principle"

- Rosenbluth & Rosenbluth, "Monte Carlo Calculation of the Average Extension of Molecular Chains"

Modern particle filtering (1993): Recursive Bayesian estimation for nonlinear / non-Gaussian models

- Gordon, Salmond & Smith, "Novel Approach to Nonlinear/Non-Gaussian Bayesian State Estimation" (Bootstrap filter)

Statistical vision adoption (1996–1998): Generalization + practical breakthroughs

- Kitagawa (1996), "Monte Carlo Filter and Smoother for Non-Gaussian Nonlinear State Space Models"
- Isard & Blake (1998), "CONDENSATION—Conditional Density Propagation for Visual Tracking"
- Liu & Chen (1998), "Sequential Monte Carlo Methods for Dynamic Systems" (SMC framework)

Tutorials

- Arulampalam et al. (2002), "A Tutorial on Particle Filters for Online Nonlinear/Non-Gaussian Bayesian Tracking"

Applications Across Domains

Applied everywhere where
time series data is needed

Signal Processing & Communications

- Image processing and segmentation
- Model selection
- Tracking and navigation
- Channel estimation
- Blind equalization
- Positioning in wireless networks

Scientific Disciplines

- Biology & Biochemistry
- Chemistry
- Economics & Business
- Geosciences
- Immunology
- Materials Science
- Pharmacology & Toxicology
- Psychiatry/Psychology
- Social Sciences

Example: Bearings-Only Tracking

Problem: Estimate the position and velocity of a target using only bearing measurements.

Variable Definitions:

- $s_{x,k}, s_{y,k}$ — target position in 2D at time k
- $V_{x,k}, V_{y,k}$ — target velocity components
- Δt — sampling interval
- y_k — measured bearing (angle) to the target

Noise Terms:

- $\mathbf{u}_k \sim \mathcal{N}(0, Q)$ — process noise (model uncertainty)
- $v_k \sim \mathcal{N}(0, R)$ — measurement noise (bearing error)

Latent State Vector:
$\mathbf{z}_k = [s_{x,k}, V_{x,k}, s_{y,k}, V_{y,k}]^\top$

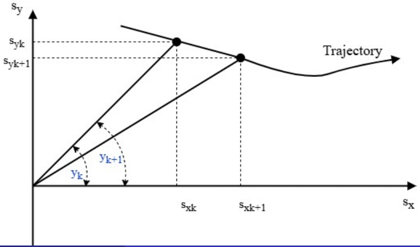
Typically used in passive sonar and EO tracking.

State Equation (Motion Model):

$$\mathbf{z}_{k+1} = \begin{bmatrix} s_{x,k} + V_{x,k}\Delta t \\ V_{x,k} \\ s_{y,k} + V_{y,k}\Delta t \\ V_{y,k} \end{bmatrix} + \mathbf{u}_k$$

Observation Equation (Bearing):

$$y_k = \arctan\left(\frac{s_{y,k}}{s_{x,k}}\right) + v_k$$

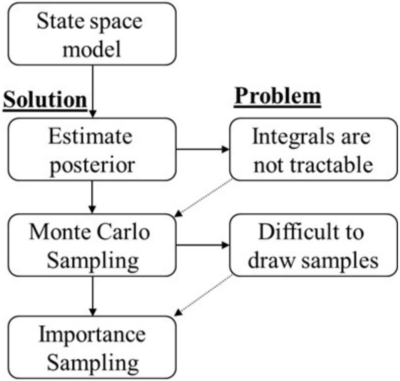


Example: Car Positioning

- Observations: velocity and turn information
- Car equipped with electronic roadmap
- Initial position accuracy: ≈ 1 km
- Particles initially spread evenly on roads
- As car moves, particles concentrate at actual location

Fundamental concepts

- State space representation
- Bayesian filtering
- Monte-Carlo sampling
- Importance sampling



State Space Representation

The latent state sequence $\{\mathbf{z}_k\}$ is a Markov random process.

Latent State Equation:

$$\mathbf{z}_k = f_z(\mathbf{z}_{k-1}, \mathbf{u}_k)$$

- $\mathbf{z}_k$: latent state vector at time k
- f_z: state transition function
- $\mathbf{u}_k$: process noise

Observation Equation:

$$\mathbf{y}_k = f_y(\mathbf{z}_k, \mathbf{v}_k)$$

- $\mathbf{y}_k$: observations
- f_y: observation function
- $\mathbf{v}_k$: observation noise

Density Representation

Dynamic systems can be represented using conditional densities:

Latent State Equation (density form):

$$p(\mathbf{z}_k \mid \mathbf{z}_{k-1})$$

Observation Equation (density form):

$$p(\mathbf{y}_k \mid \mathbf{z}_k)$$

These depend on:

- Functions $f_z(\cdot)$ and $f_y(\cdot)$
- Distributions of $\mathbf{u}_k$ and $\mathbf{v}_k$

Bayesian Filtering Objective

Estimate the unknown latent state $\mathbf{z}_k$ based on observations $\{\mathbf{y}_t\}_{k=1}^{k}$.

Goal: Find the posterior distribution

$$p(\mathbf{z}_{0:k} \mid \mathbf{y}_{1:k})$$

where:

- $\mathbf{z}_{0:k} = (\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_k)$
- $\mathbf{y}_{1:k} = (\mathbf{y}_1, \ldots, \mathbf{y}_k)$

Benefit: All kinds of estimates can be computed from the posterior.

$$E(g(\mathbf{z}_{0:k})) = \int g(\mathbf{z}_{0:k})\, p(\mathbf{z}_{0:k} \mid \mathbf{y}_{1:k})\, d\mathbf{z}_{0:k}.$$

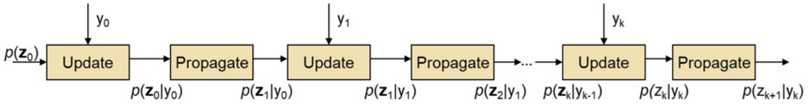
At time $k = 0$:

Update (Bayes' Theorem):

$$p(\mathbf{z}_0 \mid \mathbf{y}_0) = \frac{p(\mathbf{y}_0 \mid \mathbf{z}_0)p(\mathbf{z}_0)}{p(\mathbf{y}_0)}$$

Propagate:

$$p(\mathbf{z}_1 \mid \mathbf{y}_0) = \int p(\mathbf{z}_1 \mid \mathbf{z}_0)p(\mathbf{z}_0 \mid \mathbf{y}_0)\, d\mathbf{z}_0$$



Update and Propagate Steps: General Case

At time $k > 0$:
Filtering Density (Update):

$$p(\mathbf{z}_k \mid \mathbf{y}_{1:k}) = \frac{p(\mathbf{y}_k \mid \mathbf{z}_k)p(\mathbf{z}_k \mid \mathbf{y}_{1:k-1})}{p(\mathbf{y}_k \mid \mathbf{y}_{1:k-1})}$$

Predictive Density (Propagate):

$$p(\mathbf{z}_k \mid \mathbf{y}_{1:k-1}) = \int p(\mathbf{z}_k \mid \mathbf{z}_{k-1})p(\mathbf{z}_{k-1} \mid \mathbf{y}_{1:k-1})\, d\mathbf{z}_{k-1}$$

Meaning of the Densities
Posterior: $p(\mathbf{z}_k \mid \mathbf{y}_{1:k})$

- Probability of latent state given all observations

Prior: $p(\mathbf{z}_k \mid \mathbf{z}_{k-1})$

- Transition model: how latent state evolves over time

Likelihood: $p(\mathbf{y}_k \mid \mathbf{z}_k)$

- Probability of observation given latent state

Challenges in Bayesian Filtering

Why is direct solution difficult?

- Required integrals are *not tractable*
- Closed-form solutions exist only in limited cases

Example: Closed-form solution exists for

- Gaussian noise
- Linear state-space model
- $\Rightarrow$ Optimal solution via Kalman filter

Solution Strategy: Use Monte Carlo techniques to approximate integrals.

Classical Monte Carlo Integration

Goal: Estimate expectation

$$\mathbb{E}_Z[f(\mathbf{Z})] = \int f(\mathbf{z})p(\mathbf{z})\,d\mathbf{z}$$

Procedure:

1. Simulate M random samples $\{\mathbf{z}^{(m)}\}_{m=1}^{M}$ from $p(\mathbf{z})$
2. Compute empirical average:

$$\mathbb{E}_Z[f(\mathbf{Z})] \approx \hat{f} = \frac{1}{M}\sum_{m=1}^{M} f(\mathbf{z}^{(m)})$$

Example: Estimate variance of zero-mean Gaussian process.

Importance Sampling

Motivation: Sampling directly from $p(\mathbf{z})$ may be difficult.

Solution: Sample from proposal $q(\mathbf{z})$ and reweight samples.

Key Insight:

$$\mathbb{E}_Z[f(\mathbf{Z})] = \int f(\mathbf{z}) \frac{p(\mathbf{z})}{q(\mathbf{z})} q(\mathbf{z})\, d\mathbf{z}$$

Importance Sampling Procedure

Steps:

1. Simulate M samples from proposal: $\{\mathbf{z}^{(m)}\}_{m=1}^{M} \sim q(\mathbf{z})$
2. Compute importance weights:

$$w^{(m)} = \frac{p(\mathbf{z}^{(m)})}{q(\mathbf{z}^{(m)})}$$

3. Compute weighted average:

$$\mathbb{E}_Z[f(\mathbf{Z})] \approx \sum_{m=1}^{M} \tilde{w}^{(m)} f(\mathbf{z}^{(m)})$$

where $\tilde{w}^{(m)} = w^{(m)} / \sum_{j=1}^{M} w^{(j)}$

Design Considerations for Proposal

Choose proposal $q(\mathbf{z})$ such that:

- **Easy to sample from**
- **Resembles original density:** $q(\mathbf{z})$ should be close to $p(\mathbf{z})$
- **Good overlap:** Minimizes variance of weight estimates

Importance sampling is *free* to choose the proposal density!

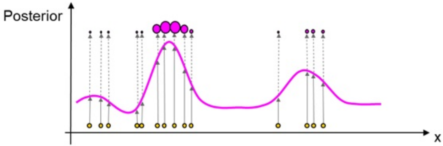
Core Idea:

- Update filtering density recursively using Bayesian filtering
- Compute integrals using importance sampling
- Represent posterior with particles and weights

Representation:

$$p(\mathbf{z}_k \mid \mathbf{y}_{1:k}) \approx \sum_{m=1}^{M} w_k^{(m)} \delta(\mathbf{z}_k - \mathbf{z}_k^{(m)})$$

where $\{\mathbf{z}_k^{(m)}\}$ are particles and $\{w_k^{(m)}\}$ are normalized weights.



Sequential Importance Sampling: Derivation (1/3)

Goal: posterior over the full trajectory

We consider the full posterior of states up to time k: $p(\mathbf{z}_{0:k} \mid \mathbf{y}_{1:k})$.

Bayesian recursion (factorization)

Using the state-space assumptions (Markov + conditional independence), we get:

$$
\begin{aligned}
p(\mathbf{z}_{0:k} \mid \mathbf{y}_{1:k}) &= \frac{p(\mathbf{y}_{1:k} \mid \mathbf{z}_{0:k}) p(\mathbf{z}_{0:k})}{p(\mathbf{y}_{1:k})} \propto p(\mathbf{y}_{1:k} \mid \mathbf{z}_{0:k}) p(\mathbf{z}_{0:k}) \\
&= p(\mathbf{y}_k \mid \mathbf{y}_{1:k-1}, \mathbf{z}_{0:k})\, p(\mathbf{y}_{1:k-1} \mid \mathbf{z}_{0:k})\, p(\mathbf{z}_{0:k}) \\
&\overset{\text{CI}}{=} p(\mathbf{y}_k \mid \mathbf{z}_k)\, p(\mathbf{y}_{1:k-1} \mid \mathbf{z}_{0:k-1})\, p(\mathbf{z}_k \mid \mathbf{z}_{0:k-1})\, p(\mathbf{z}_{0:k-1}) \\
&\overset{\text{Markov}}{=} p(\mathbf{y}_k \mid \mathbf{z}_k)\, p(\mathbf{z}_k \mid \mathbf{z}_{k-1})\, \underbrace{p(\mathbf{y}_{1:k-1} \mid \mathbf{z}_{0:k-1}) p(\mathbf{z}_{0:k-1})}_{\propto p(\mathbf{z}_{0:k-1} \mid \mathbf{y}_{1:k-1})} \\
&\propto p(\mathbf{y}_k \mid \mathbf{z}_k)\, p(\mathbf{z}_k \mid \mathbf{z}_{k-1})\, p(\mathbf{z}_{0:k-1} \mid \mathbf{y}_{1:k-1}).
\end{aligned}
$$

Sequential Importance Sampling: Derivation (2/3)

Start from the (unnormalized) IS weight definition

$$w_k^{(m)} \propto \frac{p(\mathbf{z}_{0:k}^{(m)} \mid \mathbf{y}_{1:k})}{q(\mathbf{z}_{0:k}^{(m)} \mid \mathbf{y}_{1:k})}.$$

Importance sampling idea (trajectory proposal)

Draw particles from a proposal $\mathbf{z}_{0:k}^{(m)} \sim q(\mathbf{z}_{0:k} \mid \mathbf{y}_{1:k}), m = 1, \ldots, M$,

and assign (unnormalized) importance weights proportional to target/proposal:

$$w_k^{(m)} \propto \frac{p(\mathbf{y}_k \mid \mathbf{z}_k^{(m)})\, p(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{k-1}^{(m)})\, p(\mathbf{z}_{0:k-1}^{(m)} \mid \mathbf{y}_{1:k-1})}{q(\mathbf{z}_{0:k}^{(m)} \mid \mathbf{y}_{1:k})}.$$

Recursive (sequential) proposal

Choose the proposal to factor sequentially:

$$q(\mathbf{z}_{0:k} \mid \mathbf{y}_{1:k}) = q(\mathbf{z}_k \mid \mathbf{z}_{0:k-1}, \mathbf{y}_{1:k})\, q(\mathbf{z}_{0:k-1} \mid \mathbf{y}_{1:k-1}).$$

Sequential Importance Sampling: Derivation (3/3)

Derive weight recursion

Using the posterior recursion and the proposal factorization above, terms cancel and we obtain:

$$
\begin{aligned}
w_k^{(m)} &\propto \frac{p(\mathbf{z}_{0:k}^{(m)} \mid \mathbf{y}_{1:k})}{q(\mathbf{z}_{0:k}^{(m)} \mid \mathbf{y}_{1:k})} \\
&\propto \frac{p(\mathbf{y}_k \mid \mathbf{z}_k^{(m)})\, p(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{k-1}^{(m)})\, p(\mathbf{z}_{0:k-1}^{(m)} \mid \mathbf{y}_{1:k-1})}{q(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{0:k-1}^{(m)}, \mathbf{y}_{1:k})\, q(\mathbf{z}_{0:k-1}^{(m)} \mid \mathbf{y}_{1:k-1})} \\
&\propto \underbrace{\frac{p(\mathbf{z}_{0:k-1}^{(m)} \mid \mathbf{y}_{1:k-1})}{q(\mathbf{z}_{0:k-1}^{(m)} \mid \mathbf{y}_{1:k-1})}}_{\propto\, w_{k-1}^{(m)}} \; \frac{p(\mathbf{y}_k \mid \mathbf{z}_k^{(m)})\, p(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{k-1}^{(m)})}{q(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{0:k-1}^{(m)}, \mathbf{y}_{1:k})}.
\end{aligned}
$$

$$
\boxed{w_k^{(m)} \propto w_{k-1}^{(m)} \, \frac{p(\mathbf{y}_k \mid \mathbf{z}_k^{(m)})\; p(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{k-1}^{(m)})}{q(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{0:k-1}^{(m)}, \mathbf{y}_{1:k})}.}
$$

Sequential Importance Sampling (SIS): Algorithm

Initialization

Draw particles from the prior and set uniform weights:

$$\mathbf{z}_0^{(m)} \sim p(\mathbf{z}_0), \qquad w_0^{(m)} = \frac{1}{M}, \qquad m = 1, \ldots, M.$$

For $k = 1, 2, \ldots$ (Prediction + Update)

1. **Prediction / Proposal sampling:** draw new particles

$$\mathbf{z}_k^{(m)} \sim q(\mathbf{z}_k \mid \mathbf{z}_{0:k-1}^{(m)}, \mathbf{y}_{1:k}).$$

2. **Weight update:** update unnormalized weights using the recursion

$$w_k^{(m)} \propto w_{k-1}^{(m)} \frac{p(\mathbf{y}_k \mid \mathbf{z}_k^{(m)})\, p(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{k-1}^{(m)})}{q(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{0:k-1}^{(m)}, \mathbf{y}_{1:k})}.$$

3. **Normalize:** $\tilde{w}_k^{(m)} = \dfrac{w_k^{(m)}}{\sum_{j=1}^{M} w_k^{(j)}}.$

Prior Proposal

Assumption: prior proposal

We choose the proposal equal to the transition model:

$$\boxed{q(\mathbf{z}_k \mid \mathbf{z}_{k-1}^{(m)}, \mathbf{y}_k) = p(\mathbf{z}_k \mid \mathbf{z}_{k-1}^{(m)}).}$$

Why the weight computation simplifies

In the general SIS weight recursion, the incremental factor is

$$\frac{p(\mathbf{y}_k \mid \mathbf{z}_k^{(m)})\, p(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{k-1}^{(m)})}{q(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{k-1}^{(m)}, \mathbf{y}_k)}.$$

If we choose the **prior proposal** $q = p(\mathbf{z}_k \mid \mathbf{z}_{k-1})$, then the transition term *cancels*:

$$\frac{p(\mathbf{z}_k^{(m)} \mid \mathbf{z}_{k-1}^{(m)})}{q(\cdot)} = 1 \quad \Rightarrow \quad w_k^{*(m)} \propto w_{k-1}^{(m)}\, p(\mathbf{y}_k \mid \mathbf{z}_k^{(m)}).$$

Particle Filter (Prior Proposal / Bootstrap PF)

Initialization

Draw particles from the prior and set uniform weights:
$\mathbf{z}_0^{(m)} \sim p(\mathbf{z}_0), \qquad w_0^{(m)} = \frac{1}{M}, \qquad m = 1, \ldots, M.$

For $k = 1, 2, \ldots$ (Prediction + Update)

1. Step 1: Propagate (particle generation) For each particle $m = 1, \ldots, M$:

$$\mathbf{z}_k^{(m)} \sim p(\mathbf{z}_k \mid \mathbf{z}_{k-1}^{(m)}).$$

2. Step 2: Weight update (simplified) + normalization

$$w_k^{*(m)} = w_{k-1}^{(m)}\, p(\mathbf{y}_k \mid \mathbf{z}_k^{(m)}), \qquad w_k^{(m)} = \frac{w_k^{*(m)}}{\sum_{j=1}^{M} w_k^{*(j)}}.$$

3. Step 3: Compute estimate Posterior expectation (for any test function g):

$$\mathbb{E}[g(\mathbf{z}_k) \mid \mathbf{y}_{1:k}] \approx \sum_{m=1}^{M} w_k^{(m)}\, g(\mathbf{z}_k^{(m)}).$$

The Resampling Problem

Weight Degeneracy:

- Over time, most particles have negligible weights
- Computational resources wasted on particles that don't contribute

Solution: Resampling

- Replicate particles in proportion to their weights
- High-weight particles sampled multiple times
- Low-weight particles eliminated

Resampling Step

Standard Resampling:
Sample new particles and reset weights:

$$\{\tilde{\mathbf{z}}_k^{(m)}\}_{m=1}^{M}, \quad \{\tilde{w}_k^{(m)} = 1/M\}_{m=1}^{M}$$

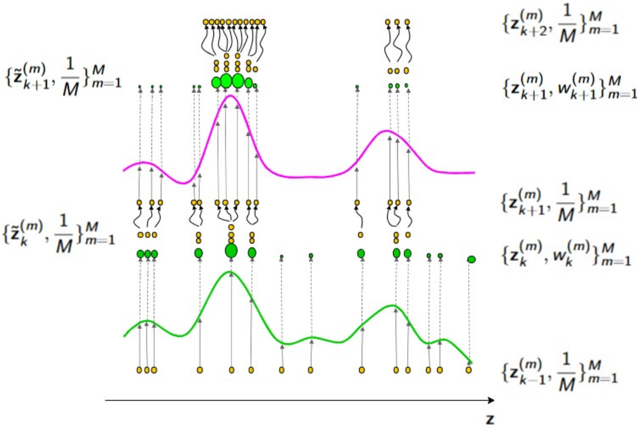
from the empirical distribution:

$$\sum_{j=1}^{M} w_k^{(j)} \delta(\mathbf{z}_k - \mathbf{z}_k^{(j)})$$

Effect:

- Particles with high weights are duplicated
- Particles with low weights disappear
- All weights reset to uniform: $w_k^{(m)} = 1/M$

Visualization of resampling



Complete Particle Filter Algorithm

Initialization

Draw particles from the prior and set uniform weights:
$\mathbf{z}_0^{(m)} \sim p(\mathbf{z}_0), \qquad w_0^{(m)} = \frac{1}{M}, \qquad m = 1, \ldots, M.$

For $k = 1, 2, \ldots$ (Prediction + Update)

1. Step 1: Propagate (particle generation) For each particle $m = 1, \ldots, M$:

$$\mathbf{z}_k^{(m)} \sim p(\mathbf{z}_k \mid \mathbf{z}_{k-1}^{(m)}).$$

2. Step 2: Weight update (simplified) + normalization

$$w_k^{*(m)} = w_{k-1}^{(m)} \, p(\mathbf{y}_k \mid \mathbf{z}_k^{(m)}), \qquad w_k^{(m)} = \frac{w_k^{*(m)}}{\sum_{j=1}^{M} w_k^{*(j)}}.$$

3. Step 3: Resample if needed (weight variance high)
4. Step 4: Compute estimate

$$\hat{\mathbf{z}}_k = \sum_m w_k^{(m)} \mathbf{z}_k^{(m)}$$

Bearings-Only Tracking: Model

Latent State Vector:

$$\mathbf{z}_k = [s_{x,k}, V_{x,k}, s_{y,k}, V_{y,k}]^\top$$

Latent State Equation:

$$\mathbf{z}_k = \mathbf{F}\mathbf{z}_{k-1} + \mathbf{u}_k$$

where $\mathbf{u}_k \sim \mathcal{N}(0, \sigma_u^2)$ and

$$F = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Observation Equation:

$$y_k = \arctan(s_{y,k}/s_{x,k}) + v_k$$

where $v_k \sim \mathcal{N}(0, \sigma_v^2)$

Bearings-Only Tracking: Algorithm

Particle Generation:

$$\mathbf{u}_k^{(m)} \sim \mathcal{N}(0, \sigma_u^2)$$

$$\mathbf{z}_k^{(m)} = \mathbf{F}\mathbf{z}_{k-1}^{(m)} + \mathbf{u}_k^{(m)}$$

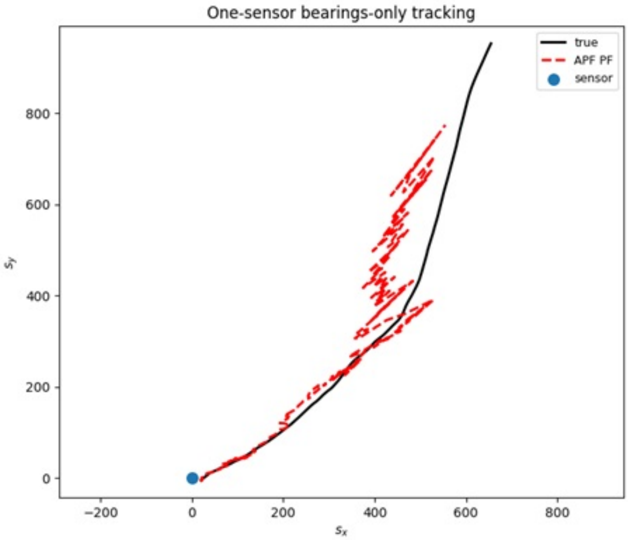
Weight Computation:

$$w_k^{*(m)} = w_{k-1}^{(m)} \mathcal{N}\left(y_k; \arctan(s_{y,k}^{(m)}/s_{x,k}^{(m)}), \sigma_v^2\right)$$

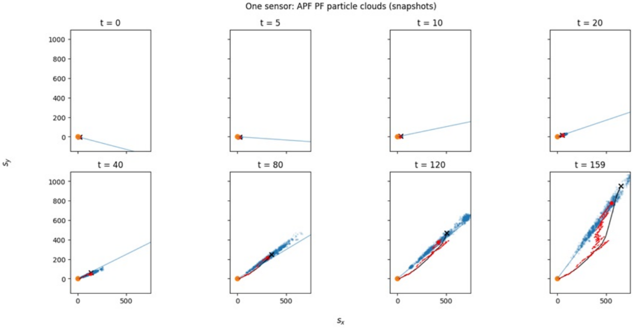
Normalization & Resampling:

$$w_k^{(m)} = \frac{w_k^{*(m)}}{\sum_{j=1}^{M} w_k^{*(j)}}, \quad \text{then resample}$$

Bearings-Only Tracking: Results



Bearings-Only Tracking: Results



Pay attention to the angles!
Measurements vs True

Particle Filter with Observation-Dependent Proposal (1/2)

Motivation: why include the observation in the proposal?

If we propose z_k using only the transition $p(z_k \mid z_{k-1})$, many particles land where $p(y_k \mid z_k)$ is tiny $\Rightarrow$ a few particles get almost all the weight (degeneracy).

General proposal

Choose any proposal that can use the new observation: $q(z_k \mid z_{k-1}, y_k)$.

Correct importance-weight update

For particle m:

$$\boxed{w_k^{(m)} \propto w_{k-1}^{(m)} \frac{p(y_k \mid z_k^{(m)})\, p(z_k^{(m)} \mid z_{k-1}^{(m)})}{q(z_k^{(m)} \mid z_{k-1}^{(m)}, y_k)}}$$

Normalize: $w_k^{(m)} = \dfrac{w_k^{(m)}}{\sum_{j=1}^{M} w_k^{(j)}}$.

Quick mental example

- Suppose y_k strongly suggests $z_k \approx 10$.
- Prior proposal draws z_k near where $p(z_k \mid z_{k-1})$ is large (maybe far from 10).
- Observation-dependent proposal shifts draws toward 10 $\Rightarrow$ fewer wasted samples.

Intuition

A good q puts particles in regions of high likelihood (where the data says the state should be), so weights are less variable.

Particle Filter with Optimal Proposal (2/2)

Optimal proposal (variance-minimizing)

$$q_k^{\mathrm{opt}}(z_k \mid z_{k-1}, y_k) = p(z_k \mid z_{k-1}, y_k)$$

This choice minimizes the conditional variance of the incremental weights.

Key simplification of the weights

Start from the general update:

$$w_k \propto w_{k-1} \, \frac{p(y_k \mid z_k) \, p(z_k \mid z_{k-1})}{q(z_k \mid z_{k-1}, y_k)}.$$

With $q = q^{\mathrm{opt}} = p(z_k \mid z_{k-1}, y_k)$ and Bayes' rule:

$$p(z_k \mid z_{k-1}, y_k) = \frac{p(y_k \mid z_k) p(z_k \mid z_{k-1})}{p(y_k \mid z_{k-1})},$$

so the ratio collapses to the **predictive likelihood**:

$$\boxed{w_k^{(m)} \propto w_{k-1}^{(m)} \, p(y_k \mid z_{k-1}^{(m)})} \qquad p(y_k \mid z_{k-1}) = \int p(y_k \mid z_k) p(z_k \mid z_{k-1}) \, dz_k.$$

Advantages of Particle Filters

- **Arbitrary densities:** Represent multimodal, non-Gaussian distributions
- **Adaptive:** Focus on probable regions of state-space
- **Non-Gaussian noise:** Handle directly without approximation
- **Multiple models:** Framework supports switching models
- **Nonlinear flexibility:** No requirement for linearity

Disadvantages of Particle Filters

- **High complexity:** Computational cost scales with number of particles
- **Particle selection:** Difficult to determine optimal number M a priori
- **Curse of dimensionality:** M must increase with state dimension
- **Weight degeneracy:** Many particles accumulate negligible weights
- **Loss of diversity:** Resampling can cause particle collapse
- **Proposal choice:** Optimal density is problem-dependent

Computational Complexity: Bearings-Only Example

Scenario: $M = 1000$ particles

Per iteration:

- Four random number generations per particle
- Propagation through state equation (matrix multiplication)
- M evaluations of nonlinear functions:
  - Exponential
  - Arctangent
- Weight normalization
- Resampling (if needed)

Overall: $\mathcal{O}(M)$ per time step

Complexity Trade-offs

Component	Cost per Particle	Total
Random number generation	$\mathcal{O}(1)$	$\mathcal{O}(M)$
State propagation	$\mathcal{O}(d^2)$	$\mathcal{O}(Md^2)$
Likelihood evaluation	$\mathcal{O}(1)$	$\mathcal{O}(M)$
Weight normalization	-	$\mathcal{O}(M)$
Resampling	-	$\mathcal{O}(M \log M)$

where d is state dimension.

Summary

Particle Filters: Powerful framework for nonlinear, non-Gaussian estimation.

Key Features:

- Represent arbitrary posterior densities
- Recursive Bayesian filtering via Monte Carlo
- Flexible enough for diverse applications
- Practical algorithm with clear steps

Trade-offs:

- Computational cost vs. accuracy
- Curse of dimensionality
- Algorithm design choices (resampling, proposal)

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