Assignment 1

Date 03 Feb 2026

Question 9

(a) Given probability density $p(x|\theta) = \theta e^{-\theta x}$, which is also a likelihood of data samples in $X = \{x_1, \dots, x_n\}$ in a learned model with parameters θ. To find the MLE, we simply find:

$$MLE = \arg\max_{\theta} \ln p(x|\theta)$$
$$= \arg\max_{\theta} \ln(\theta e^{-\theta x})$$

Since a probability density function must be concave, that shows a maxima, we can first find

$$\frac{d}{d\theta} \ln(\theta e^{-\theta x})$$
$$= \frac{d}{d\theta} \ln\theta + \ln e^{-\theta x}$$
$$= \frac{1}{\theta} - \frac{d}{d\theta}\theta x$$
$$= \frac{1}{\theta} - x$$

when $\frac{d}{d\theta} = 0$, it attains the maxima with an optimal θ.

$$\frac{d}{d\theta} = \frac{1}{\theta} - x = 0$$
$$\Rightarrow x = \frac{1}{\theta}$$
$$\Rightarrow \hat{\theta} = \frac{1}{x}$$

Since x here is a random variable representing all samples $\{x_1, \dots, x_n\}$, we can rewrite it as

$$\theta = \frac{1}{x_i}, \text{ where } i \in [1, N]$$

Likelihood often attains maximum in a probability density function in the middle of the curve, for a normal distribution with a nicely bell-shaped curve, so we know $MLE \approx \bar{x} \Rightarrow \hat{\theta} = \frac{1}{\bar{x}}$. □

Question 10

$MLE = \hat{\theta} = \frac{1}{\bar{x}}$, given $X = \{5, 6, 4\} \Rightarrow \bar{X} = \frac{5+6+4}{3}$

$\Rightarrow \bar{X} = 5 \Rightarrow \therefore MLE = \frac{1}{5} = 0.2$ //

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Question 11

(a) Based on Baye's theorem, $p(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)}$, where

X denotes the data $\{x_1, \dots, x_N\}$, we know:

$$P(\theta|X) \propto P(X|\theta) \cdot P(\theta)$$

Given $p(\theta) = \theta^3 e^{-3\theta}$ as the prior, we need likelihood:

$$p(X|\theta) = \prod_{n=1}^{N} p(x_n|\theta)$$

Based on the exponential family's distribution, we have:

$$p(X|\theta) = \frac{1}{Z(\theta)} h(x) \exp[\theta^T \phi(x)]$$

where $Z(\theta)$ is a partition function, $h(x)$ is a constant, $\phi(x)$ is another constant (sufficient statistics)

$$p(X|\theta) = h(x) \exp[\theta^T \phi(x) - A(\theta)]$$

$$A(\theta) = \log Z(\theta)$$

$$p(X|\theta) = h(x) \exp[\ln p(\theta)]$$

$$= h(x)\, p(\theta)$$

$$\ln p(X|\theta) = \ln[h(x)] + \ln[\theta^3 e^{-3\theta}]$$

$$= \ln[h(x)] + \ln\theta^3 - 3\theta$$

Combining prior, we can deduce posterior from the ratio:

$$p(\theta|X) \propto p(X|\theta) \cdot p(\theta)$$

$$\ln p(\theta|X) = \ln p(X|\theta) \cdot \ln p(\theta)$$

$$= (\ln[h(x)] + \ln\theta^3 - 3\theta) \cdot \ln(\theta^3 e^{-3\theta})$$

$$= (\ln\theta^3 - 3\theta + C_{likelihood})(\ln\theta^3 - 3\theta)$$

$$\frac{d}{d\theta} \ln p(\theta|X) = \frac{d}{d\theta}(\ln\theta^3 - 3\theta)^2$$

$$= (\ln\theta^3 - 3\theta)(\frac{3}{\theta} - 6\theta)$$

$$= \frac{3\ln\theta^3}{\theta} - 9 - 9 + 18\theta^2$$

$$= \frac{3\ln\theta^3}{\theta} - 18 + 18\theta^2$$

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Question 11 (contd.)

(a)
$$\ln p(\theta|X) \propto \frac{3\ln\theta^3}{\theta} + 18\theta^2$$
$$\exp(\ln p(\theta|X)) \propto \exp\left(\frac{3\ln\theta^3}{\theta}\right)\exp(18\theta^2)$$
$$p(\theta|X) \propto (\exp(3) + \theta^3 - \exp(\theta)) \cdot \exp(18\theta^2)$$
$$\propto (\theta^3 - \exp\theta)(\exp 18 + \exp\theta^2)$$
$$p(\theta|X) \propto (\theta^3 - \exp\theta)(\exp\theta^2)$$

With a simplified ratio, we've come up with 2 terms, one is the likelihood term and another is the prior term, respectively.

(b) Conjugacy means normal posterior $\propto$ normal likelihood $\times$ normal prior. Apply into our case:
$$p(\theta|X) \propto (\theta^3 - \exp\theta)(\exp\theta^2) = p(X|\theta)\cdot p(\theta)$$
$$(\theta^3 - \exp\theta)(\exp\theta^2) = h(x)\,p(\theta)\cdot p(\theta)$$
$$\theta^3\exp\theta^2 - \exp\theta\cdot\exp\theta^2 = h(x)\,p(\theta)^2$$
$$\theta^3\exp\theta^2 - \exp\theta^3 = h(x)[\theta^3 e^{-3\theta}]^2$$
$$= h(x)(\theta^6\cdot\exp(-6\theta))$$
$$\exp\theta^2(\theta^3 - \exp\theta) = h(x)\,\theta^6\exp(-6\theta)$$
$$\ln[\exp\theta^2(\theta^3 - \exp\theta)] = \ln[h(x)] + \ln(\theta^6) - 6\theta$$
$$\theta^2(\theta^3 - \exp\theta) = \ln[h(x)] + \ln(\theta^6) - 6\theta$$
$$\theta^5 - \theta^2\exp\theta = \ln[h(x)] + \ln(\theta^6) - 6\theta$$
$$\frac{d}{d\theta}[\theta^5 - \theta^2\exp\theta] = \frac{6}{\theta} - 6$$
$$5\theta^4 - 2\theta e^\theta + \theta^2 e^\theta = \frac{6}{\theta} - 6$$
$$5\theta^5 - 2\theta^2 e^\theta + \theta^3 e^\theta = 6 - 6\theta$$
$$\Rightarrow 5\theta^5 + \theta^3 e^\theta - 2\theta^2 e^\theta + 6\theta - 6 = 0$$

To find a solution for θ, we need to solve $\frac{d}{d\theta} = 0$

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Question 11 (contd.)

(b)

$$25\theta^4 + 3\theta^2 e^\theta + \theta^3 e^\theta - 4\theta e^\theta - 2\theta^2 e^\theta + 6 = 0$$

$$\frac{d}{d\theta} = 0 \Rightarrow 100\theta^3 + 6\theta e^\theta + 3\theta^2 e^\theta + 3\theta^2 e^\theta + \theta^3 e^\theta$$
$$- 4e^\theta - 4\theta e^\theta - 4\theta e^\theta - 2\theta^2 e^\theta = 0$$

$$\Rightarrow 100\theta^3 + e^\theta[6\theta + 6\theta^2 + \theta^3 - 4 - 8\theta - 2\theta^2] = 0$$

$$\Rightarrow 100\theta^3 + e^\theta(\theta^3 + 4\theta^2 - 2\theta - 4) = 0$$

To eliminate e^θ, we need to take natural log on both sides. However, $\ln 0$ is undefined.

$\Rightarrow \therefore$ NO solution for θ in $p(\theta)$ in the relationship: $p(\theta / X) \propto p(X / \theta)\, p(\theta)$

$\Rightarrow \therefore$ Prior is not conjugate.