Bayesian Modeling Problems

ELG 5218 – Uncertainty Evaluation in Engineering Measurements and Machine Learning

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PART A: CONCEPTUAL QUESTIONS

A1. What is the difference between a 95% frequentist *confidence interval* and a 95% Bayesian *credible interval* for a parameter?

A2. Name and briefly describe the two types of uncertainty often characterized in Bayesian modeling.

A3. What is a *conjugate prior*? Give an example of a conjugate prior-likelihood pair.

A4. Explain the distinction between Maximum Likelihood Estimation (MLE) and Maximum A Posteriori (MAP) estimation for a model parameter.

PART B: MATHEMATICAL DERIVATIONS

B1. Consider a Binomial likelihood with k successes out of N trials for an unknown probability θ, and a Beta(α, β) prior for θ.

(a) Derive the posterior distribution for θ (show that the Beta prior is conjugate to the Binomial likelihood).

(b) Find the MAP (Maximum A Posteriori) estimate of θ from this posterior, and compare it to the MLE (Maximum Likelihood Estimate).

PART C: PARAMETRIC ANALYSIS

C1. You consider two Bayesian models for a probability θ (e.g. the fraction of defective items in manufacturing) with different prior strengths:

- Model A: $\theta \sim$ Beta$(1, 1)$ (a very weak, uniform prior).
- Model B: $\theta \sim$ Beta$(10, 10)$ (a stronger prior peaked around 0.5).

Both models are updated on the same dataset. Answer the following:

(a) Which model's posterior distribution for θ will have the larger variance? Explain why.

(b) Which model is likely to yield more extreme posterior *predictive* probabilities for new observations (i.e. predictions closer to 0 or 1)? Why?

(c) In a scenario where θ is actually very small (a rare event case), which prior (Model A or Model B) would be more appropriate to use, and why?

PART D: IMPLEMENTATION AND PRACTICAL QUESTIONS

D1. In a Beta-Binomial model, you have updated the posterior for θ given observed data. How can you compute the probability that the next trial will be a success? Provide:

(a) an analytic expression for this posterior predictive probability in terms of the posterior Beta parameters, and

(b) a brief description of how you could approximate this probability via Monte Carlo simulation.

PART E: OTHER PROBLEMS

E1. What is an *equal-tailed* 95% credible interval, and how does it differ from a 95% *highest posterior density* (HPD) credible interval? In what situations would these two types of intervals give notably different results?

E2. **Problem: Manufacturing Defect Detection**. A manufacturing line produces electronic components, and the historical defect rate is around 5%. You decide to use Bayesian modeling to monitor the defect rate in production. You assume a Beta prior for the defect probability θ. In a random sample of $N = 10$ components from a new batch, you observe $X = 2$ defective units.

(a) Why might a Bayesian approach be preferable to a frequentist approach (e.g. using just the sample proportion) for estimating θ in this scenario?

(b) Assume a prior $\theta \sim \text{Beta}(2, 38)$ reflecting the historical belief (mean $\approx$ 5%). Update this prior with the data. Give the posterior distribution for θ, and calculate the posterior mean and a 95% credible interval for θ.