Particle Filtering: Problems

ELG 5218 – Uncertainty Evaluation in Engineering Measurements and Machine Learning

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1 Resampling Techniques

Problem 1: Multinomial Resampling

Given M particles $\{\mathbf{z}^{(i)}, \tilde{w}^{(i)}\}_{i=1}^{M}$ where $\sum_i \tilde{w}^{(i)} = 1$:

1. Explain how multinomial resampling works.
2. If a particle has normalized weight $\tilde{w}^{(i)} = 0.2$ and we draw $M = 100$ resampled particles, what is the expected number of copies of this particle?
3. Show that the resampled distribution:

$$\bar{p}(\mathbf{z}) = \sum_{i=1}^{M} \frac{N^{(i)}}{M} \delta(\mathbf{z} - \mathbf{z}^{(i)})$$

   is an unbiased approximation of $\hat{p}(\mathbf{z}) = \sum_{i=1}^{M} \tilde{w}^{(i)} \delta(\mathbf{z} - \mathbf{z}^{(i)})$.

2 Particle Filtering for Nonlinear State-Space Models

Problem 2: State-Space Models and Densities

For a nonlinear state-space model with:

Latent State Equation:

$$\mathbf{z}_k = f_z(\mathbf{z}_{k-1}, \mathbf{u}_k)$$

Observation Equation:

$$\mathbf{y}_k = f_y(\mathbf{z}_k, \mathbf{v}_k)$$

where the densities are:

- $p(\mathbf{z}_k|\mathbf{z}_{k-1})$: latent state transition probability
- $p(\mathbf{y}_k|\mathbf{z}_k)$: observation likelihood

1. What is the optimal proposal distribution $q^{\text{opt}}(\mathbf{z}_k|\mathbf{z}_{1:k-1}, \mathbf{y}_{1:k})$?
2. Show that using this optimal proposal leads to weights:

$$w_k(\mathbf{z}_{1:k}) = w_{k-1}(\mathbf{z}_{1:k-1}) p(\mathbf{y}_k|\mathbf{z}_{k-1})$$

Problem 3: Particle Filter with Prior Proposal

For the state-space model with proposal $q(\mathbf{z}_k|\mathbf{z}_{k-1}, \mathbf{y}_k) = p(\mathbf{z}_k|\mathbf{z}_{k-1})$:

1. Derive the weight update in this case.
2. For a simple 1D random walk: $\mathbf{z}_k = \mathbf{z}_{k-1} + \mathbf{u}_k$ where $\mathbf{u}_k \sim \mathcal{N}(0, \sigma_u^2 = 1)$, with observation $\mathbf{y}_k = \mathbf{z}_k + \mathbf{v}_k$ where $\mathbf{v}_k \sim \mathcal{N}(0, \sigma_v^2 = 1)$, compute densities $p(\mathbf{z}_k|\mathbf{z}_{k-1})$ and $p(\mathbf{y}_k|\mathbf{z}_k)$.

Problem 4: Effective Sample Size

The effective number of samples is approximated by:

$$N_{\text{eff}} \approx \frac{1}{\sum_{i=1}^{M} (\tilde{w}^{(i)})^2}$$

1. For what weight distributions is N_{eff} maximal?
2. If all particles have equal weight $\tilde{w}^{(i)} = 1/M$, what is N_{eff}?
3. If one particle has weight 1 and the rest have weight 0, what is N_{eff}?
4. Why is the resampling threshold typically set at $N_T = M/2$?

3 Practical Applications

Problem 5: Bearings-Only Tracking

Consider the bearings-only tracking problem from the course slides:

Latent State Vector: $\mathbf{z}_k = [s_{x,k}, V_{x,k}, s_{y,k}, V_{y,k}]^\top$ (position and velocity in 2D)

Latent State Equation:

$$\mathbf{z}_{k+1} = \begin{bmatrix} 1 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{z}_k + \mathbf{u}_k$$

where $\mathbf{u}_k \sim \mathcal{N}(0, Q)$ represents process noise.

Observation Equation:

$$\mathbf{y}_k = \arctan\left(\frac{s_{y,k}}{s_{x,k}}\right) + \mathbf{v}_k$$

where $\mathbf{v}_k \sim \mathcal{N}(0, R)$ is measurement noise (bearing error).

1. Design a particle filter for this nonlinear, non-Gaussian problem.
2. Specify the state equation $\mathbf{z}_k = f_z(\mathbf{z}_{k-1}, \mathbf{u}_k)$ and observation equation $\mathbf{y}_k = f_y(\mathbf{z}_k, \mathbf{v}_k)$ explicitly.
3. With $M = 100$ particles, describe the complete algorithm.
4. Discuss why the Kalman filter would fail for this problem.

Problem 6: Visual Object Tracking

Consider tracking an object in 2D with uniform process noise.

Latent State Equation:

$$\mathbf{z}_k = \mathbf{z}_{k-1} + \mathbf{u}_k, \quad \mathbf{u}_k \sim \mathcal{U}([-10, 10]^2)$$

Observation Equation:

$$\mathbf{y}_k = f_y(\mathbf{z}_k, \mathbf{v}_k) = \mathbf{z}_k + \mathbf{v}_k, \quad \mathbf{v}_k \sim \mathcal{N}(0, \Sigma_v)$$

where $\Sigma_v = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix}$.

1. Why is the Kalman filter not optimal here?
2. Design a particle filter for this non-Gaussian process noise problem.
3. Track an object moving in a circular trajectory and compare tracking quality with and without resampling.

Problem 7: Degeneracy vs Sample Impoverishment

1. Define and distinguish between:
   - Weight degeneracy
   - Sample impoverishment
2. Explain the mechanism by which resampling can cause sample impoverishment.
3. When is sample impoverishment most severe?