Assignment 1 - Question 12

Setup: Libraries and Configuration

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns

# Plotting settings
plt.style.use('seaborn-v0_8-darkgrid')
sns.set_palette("husl")

# Random seed
seed = 42
np.random.seed(seed)

print(f"NumPy: {np.__version__}")
print(f"Matplotlib: {plt.matplotlib.__version__}")
```

```
NumPy: 2.2.6
Matplotlib: 3.10.8
```

Question 12 (a)

Goal: Simulate N = {10, 100} lifetimes from Exponential(θ_0) with true rate $\theta_0 = 0.2$.

Prepare a Random Uniform Distribution

In [2]:
```
def uniform_dist(n_samples, low=0.0, high=1.0, seed=seed):
    rng = np.random.default_rng(seed)
    return rng.uniform(low, high, n_samples)

# Example
n_samples = 10
data = uniform_dist(n_samples, low=-3.0, high=3.0)
assert n_samples == len(data)
print(f"{len(data)} samples from Uniform distribution: {data}")
```

```
10 samples from Uniform distribution: [ 1.64373629 -0.36672936  2.15158752  1.18420817 -2.43493591  2.85373411
  1.56683821  1.71638583 -2.2313182  -0.29768437]
```

The Exponential Update

In [3]:
```
# Exponential update
def exp_update(theta, gradient, learning_rate=0.1):
    return theta * np.exp(-learning_rate * gradient)
```

Configurations

In [4]:
```
theta0 = -0.2 # initial weight
gradients = uniform_dist(100, low=-1.0, high=1.0)
```

The Simulation

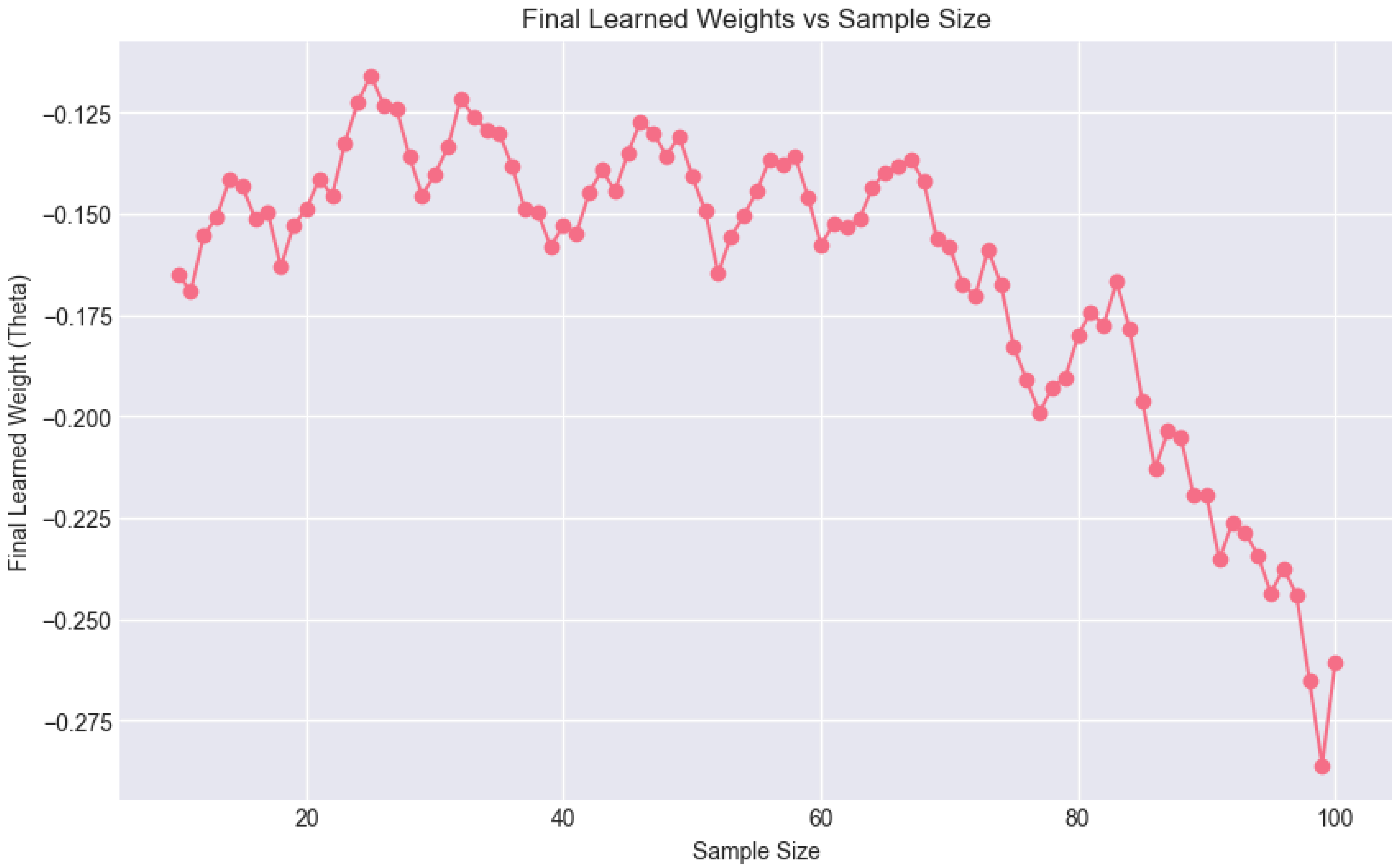
In [5]:

```
final_thetas = []
for n in range(10, 101): # sample sizes to try
    # Learning
    theta = theta0
    updated_thetas = []
    for gradient in gradients[:n]:
        theta = exp_update(theta, gradient, learning_rate=0.1)
        updated_thetas.append(theta)

    final_thetas.append(theta)
    print(f"Final theta after {n} updates: {theta}\n")

# Plot thetas
plt.figure(figsize=(10, 6))
plt.plot(range(10, 101), final_thetas, marker='o')
plt.title("Final Learned Weights vs Sample Size")
plt.xlabel("Sample Size")
plt.ylabel("Final Learned Weight (Theta)")
plt.grid(True)
plt.show()
```

```
Final theta after 10 updates: -0.16491935941584984

Final theta after 11 updates: -0.16923647864664154

Final theta after 12 updates: -0.15539092227421694

Final theta after 13 updates: -0.15098356608038488

Final theta after 14 updates: -0.14154513964928608

Final theta after 15 updates: -0.1431561273839617

Final theta after 16 updates: -0.15118255652783463

Final theta after 17 updates: -0.14954107926934

Final theta after 18 updates: -0.16317246053792028

Final theta after 19 updates: -0.15282316172092175

Final theta after 20 updates: -0.14885141109010788

Final theta after 21 updates: -0.14136299563890076

Final theta after 22 updates: -0.14553634157816145

Final theta after 23 updates: -0.13246072956320598

Final theta after 24 updates: -0.12244500575564157

Final theta after 25 updates: -0.11581398161234334

Final theta after 26 updates: -0.12310745053312237

Final theta after 27 updates: -0.12392956187972376

Final theta after 28 updates: -0.13576868630631345

Final theta after 29 updates: -0.14548815920648478

Final theta after 30 updates: -0.14025818638240412

Final theta after 31 updates: -0.13355755138286945

Final theta after 32 updates: -0.12163570279620681

Final theta after 33 updates: -0.12594753902813766

Final theta after 34 updates: -0.12925323241952325

Final theta after 35 updates: -0.13004263521363113

Final theta after 36 updates: -0.1383750964506651

Final theta after 37 updates: -0.14900558477958842

Final theta after 38 updates: -0.14973136697981446

Final theta after 39 updates: -0.15813687127645185

Final theta after 40 updates: -0.15285627991875403

Final theta after 41 updates: -0.1547897507315117
```

```
Final theta after 42 updates: -0.14482586794239616

Final theta after 43 updates: -0.13913978699232263

Final theta after 44 updates: -0.1444604490846242

Final theta after 45 updates: -0.1351727785230217

Final theta after 46 updates: -0.1271796850661864

Final theta after 47 updates: -0.13007422560985624

Final theta after 48 updates: -0.13569905907563415

Final theta after 49 updates: -0.13083546378973904

Final theta after 50 updates: -0.1406099908379113

Final theta after 51 updates: -0.1493075682689818

Final theta after 52 updates: -0.16476759090367366

Final theta after 53 updates: -0.1555785992758262

Final theta after 54 updates: -0.15053278397372702

Final theta after 55 updates: -0.14448097217814806

Final theta after 56 updates: -0.13659249610051427

Final theta after 57 updates: -0.13771947923641426

Final theta after 58 updates: -0.13583903491295315

Final theta after 59 updates: -0.14598607177053774

Final theta after 60 updates: -0.1576859195403534

Final theta after 61 updates: -0.15246340671105763

Final theta after 62 updates: -0.15334731325048648

Final theta after 63 updates: -0.15135955249451752

Final theta after 64 updates: -0.1435464070868267

Final theta after 65 updates: -0.139730380647369

Final theta after 66 updates: -0.1382410406835984

Final theta after 67 updates: -0.1366137226830698

Final theta after 68 updates: -0.14207674670576093

Final theta after 69 updates: -0.15605426735929287

Final theta after 70 updates: -0.15804192352535065

Final theta after 71 updates: -0.16732589904772077

Final theta after 72 updates: -0.17041517631146608

Final theta after 73 updates: -0.15878594838168053
```

```
Final theta after 74 updates: -0.16746412705015373

Final theta after 75 updates: -0.1829309232724213

Final theta after 76 updates: -0.19110668545899867

Final theta after 77 updates: -0.19916090844926873

Final theta after 78 updates: -0.19281473010593653

Final theta after 79 updates: -0.19062789797810017

Final theta after 80 updates: -0.18010566526985594

Final theta after 81 updates: -0.17428310197078242

Final theta after 82 updates: -0.17757687745633494

Final theta after 83 updates: -0.16676732053005627

Final theta after 84 updates: -0.17825319201732873

Final theta after 85 updates: -0.19610741641389853

Final theta after 86 updates: -0.21286389692113428

Final theta after 87 updates: -0.2036048508455203

Final theta after 88 updates: -0.20516318024922567

Final theta after 89 updates: -0.21954370139817128

Final theta after 90 updates: -0.21949783143205384

Final theta after 91 updates: -0.23530438523516098

Final theta after 92 updates: -0.22624440598079837

Final theta after 93 updates: -0.22869393980109348

Final theta after 94 updates: -0.23420114894818128

Final theta after 95 updates: -0.24368537320740438

Final theta after 96 updates: -0.2374177910343628

Final theta after 97 updates: -0.24407093455413903

Final theta after 98 updates: -0.26505276372586545

Final theta after 99 updates: -0.28609609020874877

Final theta after 100 updates: -0.2608506969305339
```



Question 12 (b)

Goal: Compute and print the MLE $\hat{\theta}$ from the simulated data.

MLE $= n / \sum_{i=1}^{N} x_i = 1/E[X]$

In [6]:
```
# Compute MLE for each sample size
mle_thetas = []

for n in range(10, 101):
    # Generate samples
    samples = uniform_dist(n, low=0.0, high=1.0, seed=seed)

    # Compute MLE: n / sum(X_i) = 1 / E[X]
    theta_mle = 1.0 / np.mean(samples)
    mle_thetas.append(theta_mle)

    print(f"Sample Size: {n}")
    print(f"True theta: {theta0}")
    print(f"MLE theta: {theta_mle:.4f}")
    print(f"Sample mean: {np.mean(samples):.4f}\n")

# Plot MLE thetas
plt.figure(figsize=(10, 6))
plt.plot(range(10, 101), mle_thetas, marker='o', color='orange')
plt.title("MLE Thetas vs Sample Size")
plt.xlabel("Sample Size")
```

```
plt.ylabel("MLE Theta")
plt.grid(True)
plt.show()
```

```
Sample Size: 10
True theta: -0.2
MLE theta: 1.6766
Sample mean: 0.5964

Sample Size: 11
True theta: -0.2
MLE theta: 1.7364
Sample mean: 0.5759

Sample Size: 12
True theta: -0.2
MLE theta: 1.6525
Sample mean: 0.6052

Sample Size: 13
True theta: -0.2
MLE theta: 1.6444
Sample mean: 0.6081

Sample Size: 14
True theta: -0.2
MLE theta: 1.6039
Sample mean: 0.6235

Sample Size: 15
True theta: -0.2
MLE theta: 1.6354
Sample mean: 0.6115

Sample Size: 16
True theta: -0.2
MLE theta: 1.7023
Sample mean: 0.5874

Sample Size: 17
True theta: -0.2
MLE theta: 1.7079
Sample mean: 0.5855

Sample Size: 18
True theta: -0.2
MLE theta: 1.7968
Sample mean: 0.5565

Sample Size: 19
True theta: -0.2
MLE theta: 1.7519
Sample mean: 0.5708

Sample Size: 20
True theta: -0.2
MLE theta: 1.7426
Sample mean: 0.5738

Sample Size: 21
True theta: -0.2
MLE theta: 1.7164
Sample mean: 0.5826

Sample Size: 22
True theta: -0.2
MLE theta: 1.7475
Sample mean: 0.5722
```

```
Sample Size: 23
True theta: -0.2
MLE theta: 1.6961
Sample mean: 0.5896

Sample Size: 24
True theta: -0.2
MLE theta: 1.6605
Sample mean: 0.6022

Sample Size: 25
True theta: -0.2
MLE theta: 1.6413
Sample mean: 0.6093

Sample Size: 26
True theta: -0.2
MLE theta: 1.6854
Sample mean: 0.5933

Sample Size: 27
True theta: -0.2
MLE theta: 1.6989
Sample mean: 0.5886

Sample Size: 28
True theta: -0.2
MLE theta: 1.7569
Sample mean: 0.5692

Sample Size: 29
True theta: -0.2
MLE theta: 1.8022
Sample mean: 0.5549

Sample Size: 30
True theta: -0.2
MLE theta: 1.7885
Sample mean: 0.5591

Sample Size: 31
True theta: -0.2
MLE theta: 1.7695
Sample mean: 0.5651

Sample Size: 32
True theta: -0.2
MLE theta: 1.7310
Sample mean: 0.5777

Sample Size: 33
True theta: -0.2
MLE theta: 1.7542
Sample mean: 0.5701

Sample Size: 34
True theta: -0.2
MLE theta: 1.7724
Sample mean: 0.5642

Sample Size: 35
True theta: -0.2
MLE theta: 1.7810
```

```
Sample mean: 0.5615

Sample Size: 36
True theta: -0.2
MLE theta: 1.8144
Sample mean: 0.5512

Sample Size: 37
True theta: -0.2
MLE theta: 1.8526
Sample mean: 0.5398

Sample Size: 38
True theta: -0.2
MLE theta: 1.8584
Sample mean: 0.5381

Sample Size: 39
True theta: -0.2
MLE theta: 1.8864
Sample mean: 0.5301

Sample Size: 40
True theta: -0.2
MLE theta: 1.8741
Sample mean: 0.5336

Sample Size: 41
True theta: -0.2
MLE theta: 1.8824
Sample mean: 0.5312

Sample Size: 42
True theta: -0.2
MLE theta: 1.8573
Sample mean: 0.5384

Sample Size: 43
True theta: -0.2
MLE theta: 1.8444
Sample mean: 0.5422

Sample Size: 44
True theta: -0.2
MLE theta: 1.8623
Sample mean: 0.5370

Sample Size: 45
True theta: -0.2
MLE theta: 1.8398
Sample mean: 0.5435

Sample Size: 46
True theta: -0.2
MLE theta: 1.8208
Sample mean: 0.5492

Sample Size: 47
True theta: -0.2
MLE theta: 1.8323
Sample mean: 0.5458

Sample Size: 48
True theta: -0.2
```

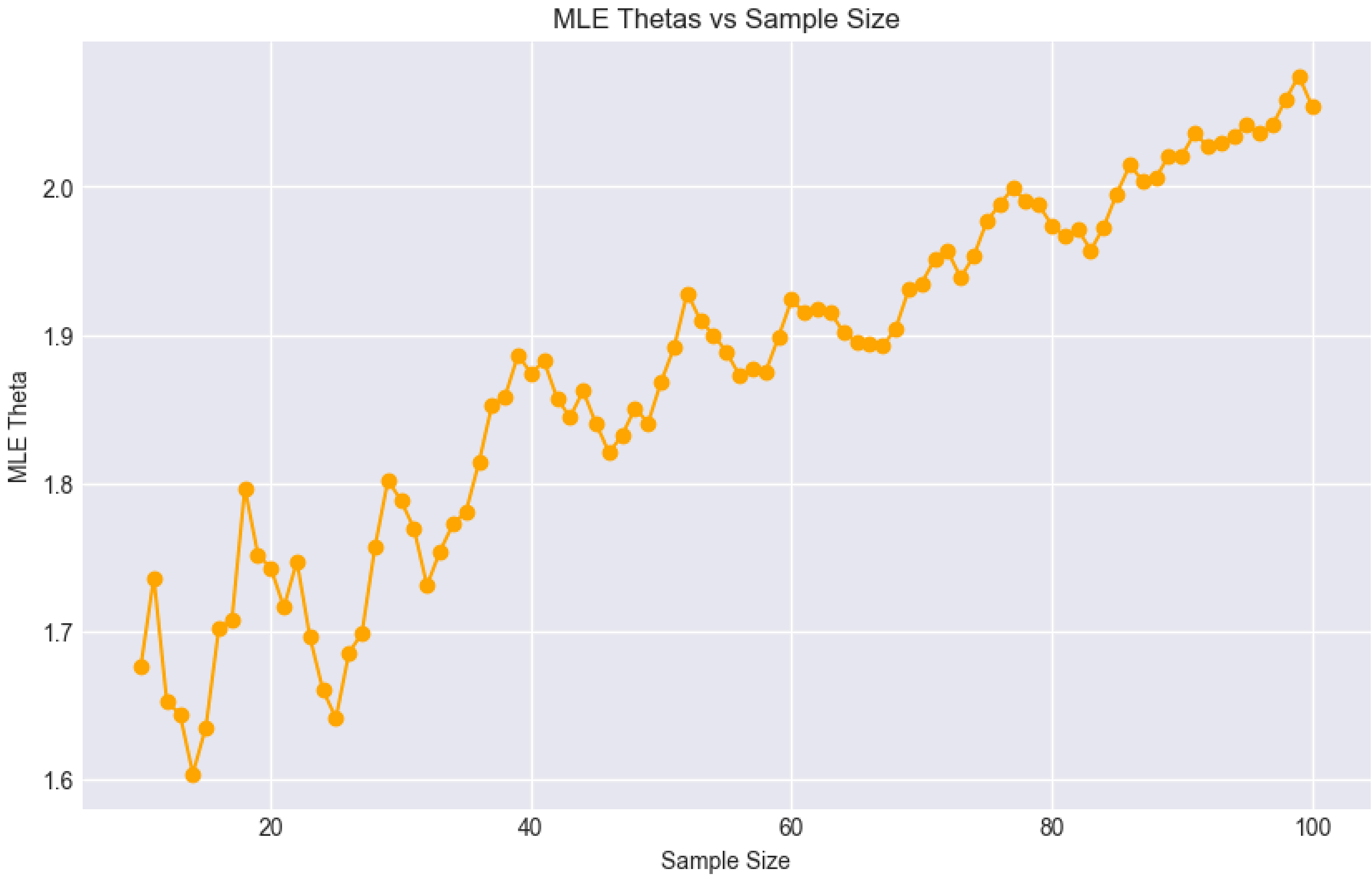
```
MLE theta: 1.8505
Sample mean: 0.5404

Sample Size: 49
True theta: -0.2
MLE theta: 1.8406
Sample mean: 0.5433

Sample Size: 50
True theta: -0.2
MLE theta: 1.8683
Sample mean: 0.5352

Sample Size: 51
True theta: -0.2
MLE theta: 1.8916
Sample mean: 0.5287

Sample Size: 52
True theta: -0.2
MLE theta: 1.9281
Sample mean: 0.5186

Sample Size: 53
True theta: -0.2
MLE theta: 1.9095
Sample mean: 0.5237

Sample Size: 54
True theta: -0.2
MLE theta: 1.9000
Sample mean: 0.5263

Sample Size: 55
True theta: -0.2
MLE theta: 1.8884
Sample mean: 0.5296

Sample Size: 56
True theta: -0.2
MLE theta: 1.8725
Sample mean: 0.5340

Sample Size: 57
True theta: -0.2
MLE theta: 1.8771
Sample mean: 0.5327

Sample Size: 58
True theta: -0.2
MLE theta: 1.8749
Sample mean: 0.5333

Sample Size: 59
True theta: -0.2
MLE theta: 1.8987
Sample mean: 0.5267

Sample Size: 60
True theta: -0.2
MLE theta: 1.9238
Sample mean: 0.5198

Sample Size: 61
```

```
True theta: -0.2
MLE theta: 1.9148
Sample mean: 0.5222

Sample Size: 62
True theta: -0.2
MLE theta: 1.9178
Sample mean: 0.5214

Sample Size: 63
True theta: -0.2
MLE theta: 1.9153
Sample mean: 0.5221

Sample Size: 64
True theta: -0.2
MLE theta: 1.9015
Sample mean: 0.5259

Sample Size: 65
True theta: -0.2
MLE theta: 1.8954
Sample mean: 0.5276

Sample Size: 66
True theta: -0.2
MLE theta: 1.8940
Sample mean: 0.5280

Sample Size: 67
True theta: -0.2
MLE theta: 1.8923
Sample mean: 0.5284

Sample Size: 68
True theta: -0.2
MLE theta: 1.9042
Sample mean: 0.5251

Sample Size: 69
True theta: -0.2
MLE theta: 1.9306
Sample mean: 0.5180

Sample Size: 70
True theta: -0.2
MLE theta: 1.9349
Sample mean: 0.5168

Sample Size: 71
True theta: -0.2
MLE theta: 1.9510
Sample mean: 0.5126

Sample Size: 72
True theta: -0.2
MLE theta: 1.9565
Sample mean: 0.5111

Sample Size: 73
True theta: -0.2
MLE theta: 1.9387
Sample mean: 0.5158
```

```
Sample Size: 74
True theta: -0.2
MLE theta: 1.9531
Sample mean: 0.5120

Sample Size: 75
True theta: -0.2
MLE theta: 1.9765
Sample mean: 0.5059

Sample Size: 76
True theta: -0.2
MLE theta: 1.9881
Sample mean: 0.5030

Sample Size: 77
True theta: -0.2
MLE theta: 1.9989
Sample mean: 0.5003

Sample Size: 78
True theta: -0.2
MLE theta: 1.9907
Sample mean: 0.5023

Sample Size: 79
True theta: -0.2
MLE theta: 1.9879
Sample mean: 0.5030

Sample Size: 80
True theta: -0.2
MLE theta: 1.9741
Sample mean: 0.5065

Sample Size: 81
True theta: -0.2
MLE theta: 1.9666
Sample mean: 0.5085

Sample Size: 82
True theta: -0.2
MLE theta: 1.9714
Sample mean: 0.5073

Sample Size: 83
True theta: -0.2
MLE theta: 1.9572
Sample mean: 0.5109

Sample Size: 84
True theta: -0.2
MLE theta: 1.9730
Sample mean: 0.5069

Sample Size: 85
True theta: -0.2
MLE theta: 1.9954
Sample mean: 0.5012

Sample Size: 86
True theta: -0.2
MLE theta: 2.0146
Sample mean: 0.4964
```

```
Sample Size: 87
True theta: -0.2
MLE theta: 2.0041
Sample mean: 0.4990

Sample Size: 88
True theta: -0.2
MLE theta: 2.0058
Sample mean: 0.4986

Sample Size: 89
True theta: -0.2
MLE theta: 2.0212
Sample mean: 0.4948

Sample Size: 90
True theta: -0.2
MLE theta: 2.0209
Sample mean: 0.4948

Sample Size: 91
True theta: -0.2
MLE theta: 2.0364
Sample mean: 0.4911

Sample Size: 92
True theta: -0.2
MLE theta: 2.0272
Sample mean: 0.4933

Sample Size: 93
True theta: -0.2
MLE theta: 2.0293
Sample mean: 0.4928

Sample Size: 94
True theta: -0.2
MLE theta: 2.0342
Sample mean: 0.4916

Sample Size: 95
True theta: -0.2
MLE theta: 2.0425
Sample mean: 0.4896

Sample Size: 96
True theta: -0.2
MLE theta: 2.0364
Sample mean: 0.4911

Sample Size: 97
True theta: -0.2
MLE theta: 2.0419
Sample mean: 0.4897

Sample Size: 98
True theta: -0.2
MLE theta: 2.0592
Sample mean: 0.4856

Sample Size: 99
True theta: -0.2
MLE theta: 2.0750
```

```
Sample mean: 0.4819

Sample Size: 100
True theta: -0.2
MLE theta: 2.0546
Sample mean: 0.4867
```



Observation: As the sample size grows, the Maximum Likelihood Estimation (MLE) is enlarged.

Question 12 (c) - Add a Prior

Goal: Using the given prior, compute the posterior distribution parameters and plot the posterior density. On the same plot, show the likelihood function (up to a constant). and the prior density for comparison.

Formulate the Prior

In [7]:
```python
def prior_dist(theta):
    # Exponential Dist
    coeff = theta ** 3
    exponent = -3 * theta
    return coeff * np.exp(exponent)

# Example
prior_dist(0.2)
```

Out[7]:
```
np.float64(0.004390493088752212)
```

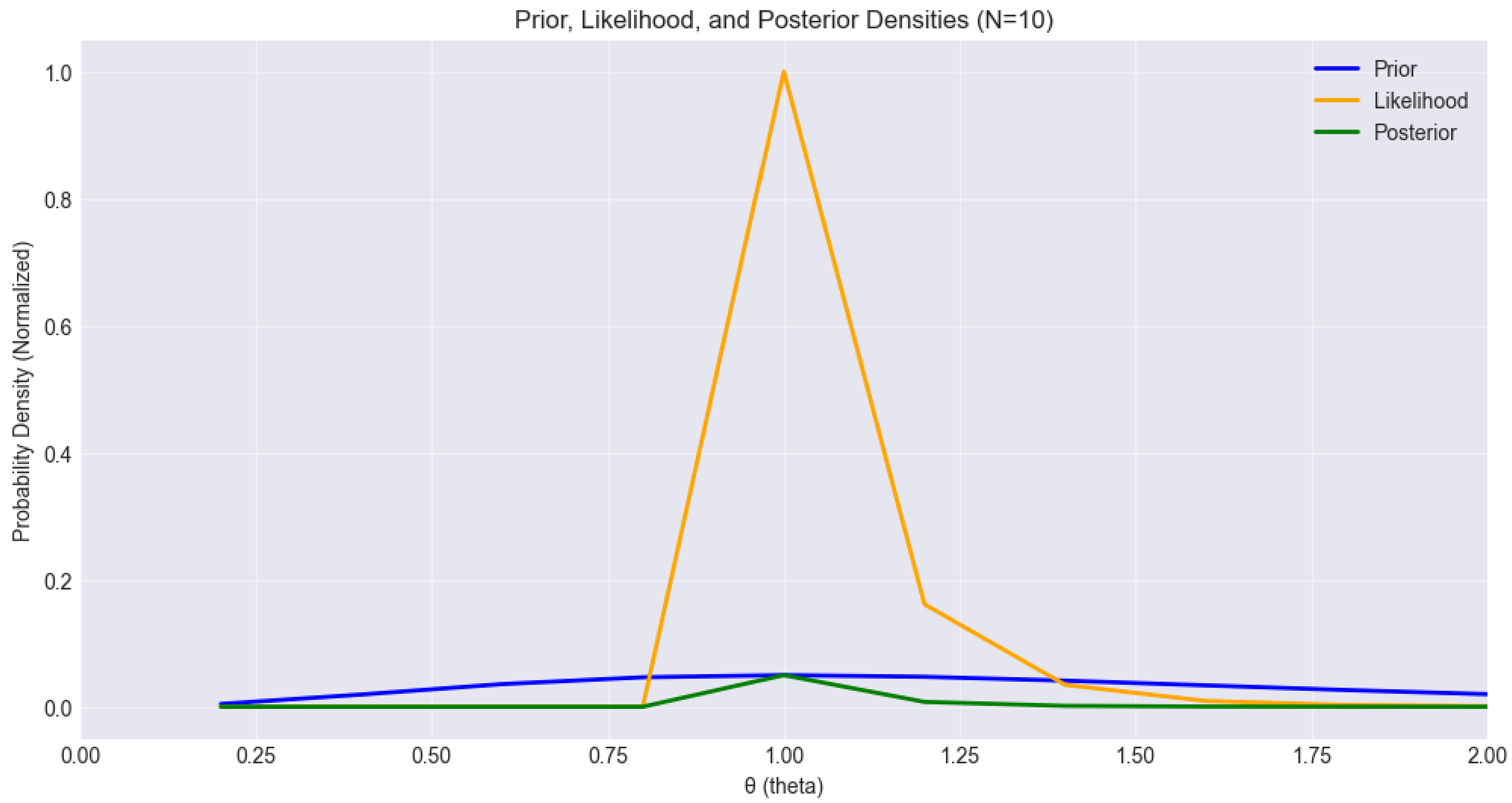
Formulate the Likelihood

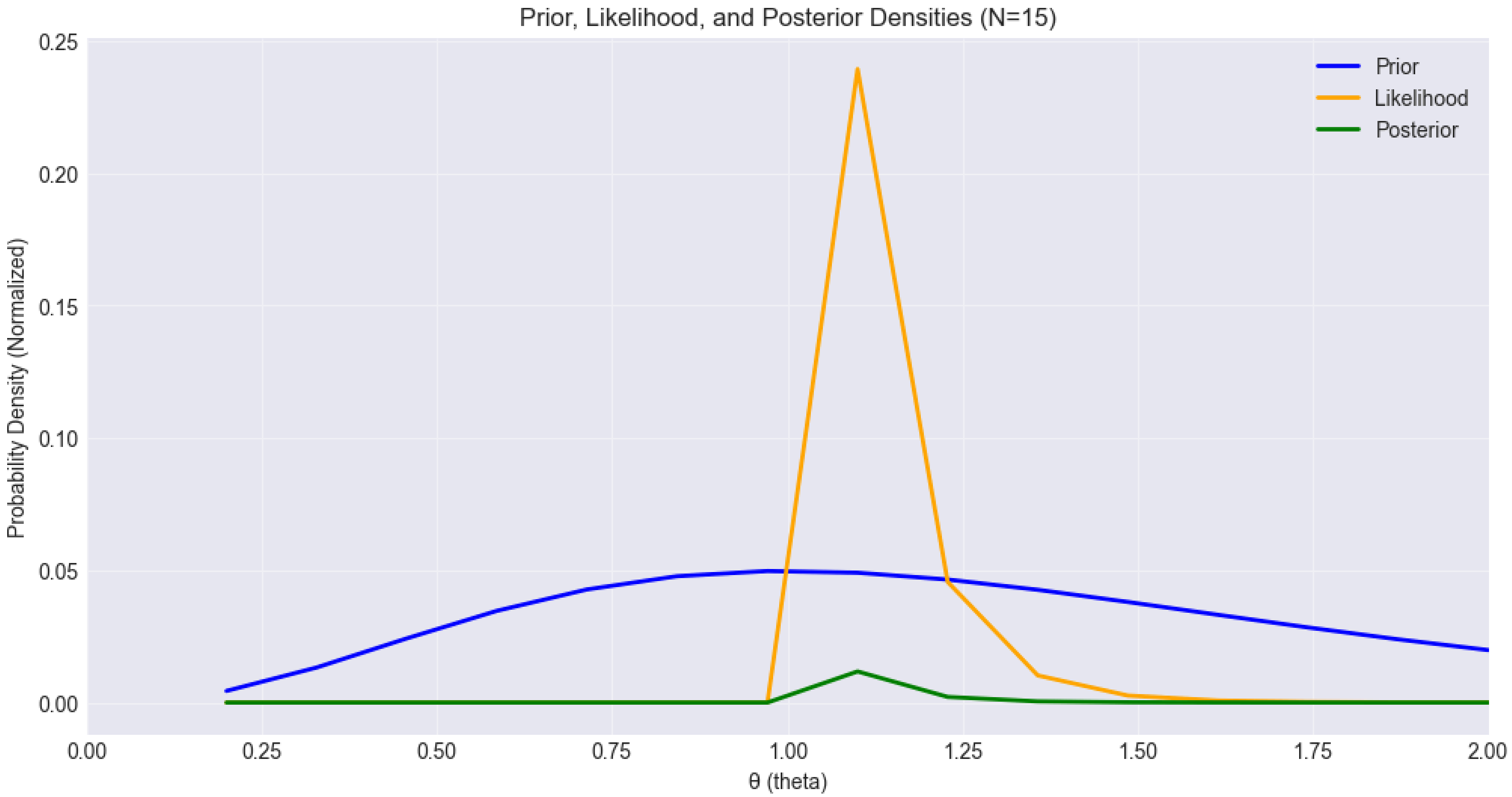
In [8]:
```
# Define likelihood function for uniform distribution U(0, theta)
def likelihood(theta_vals, samples):
    n = len(samples)
    max_sample = np.max(samples)

    likelihood_vals = np.where(
        theta_vals >= max_sample, # Condition: see if weights controlled by theta can explain data (take max sample)
        (1.0 / theta_vals) ** n, # Sucess case
        0.0 # Else
    )
    return likelihood_vals

# Example
likelihood(np.array([0.5, 1.0, 1.5]), np.array([0.2, 0.4, 0.1]))
```

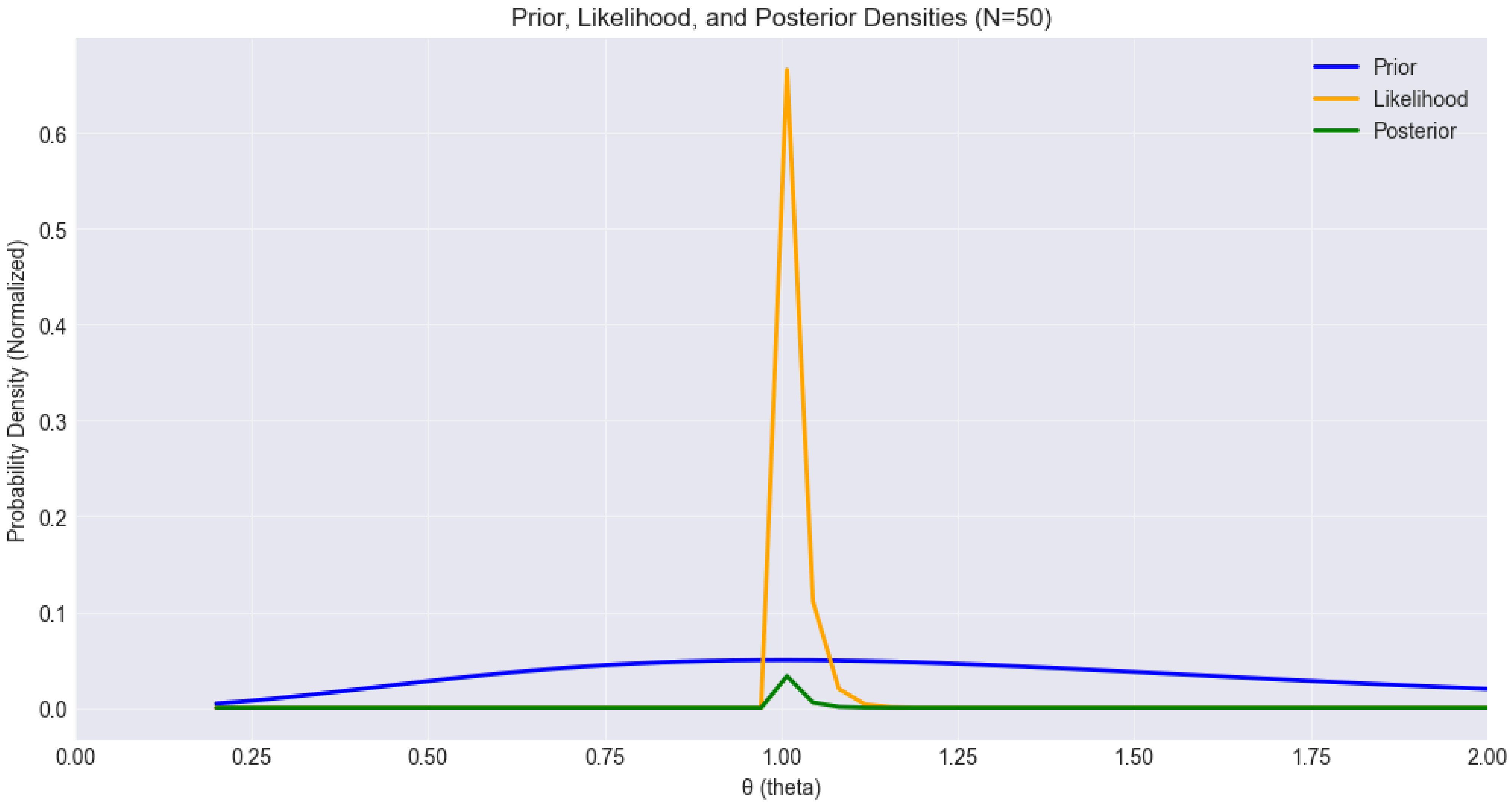
Out[8]:
```
array([8.        , 1.        , 0.2962963])
```

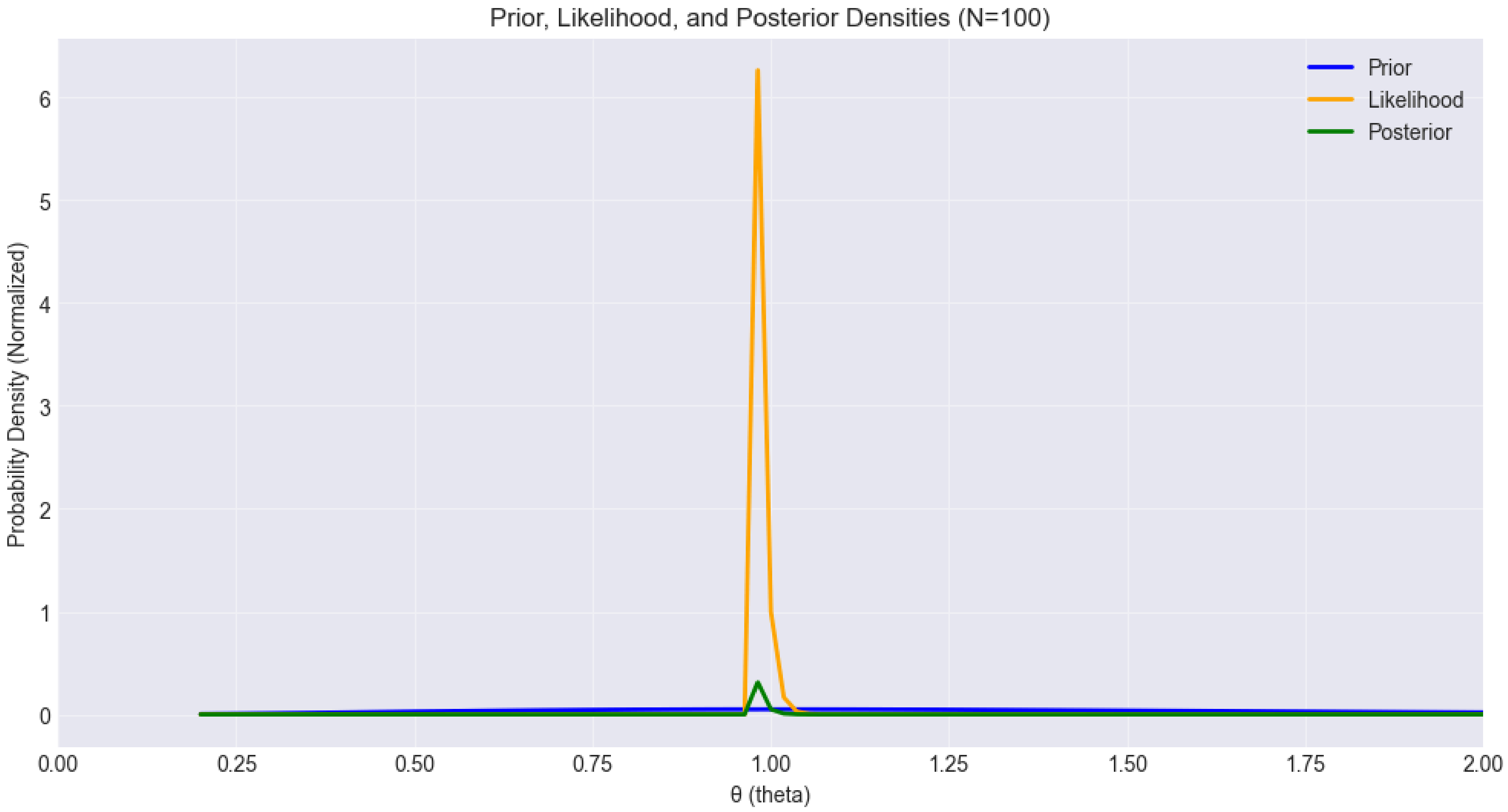
Run Simulations

In [9]:
```
# Plot all three on the same figure
def plot_distribution(theta_range, prior, likelihood, posterior, n):
    plt.figure(figsize=(12, 6))
    plt.plot(theta_range, prior, label='Prior', color='blue', linewidth=2)
    plt.plot(theta_range, likelihood, label='Likelihood', color='orange', linewidth=2)
    plt.plot(theta_range, posterior, label='Posterior', color='green', linewidth=2)
    plt.title(f"Prior, Likelihood, and Posterior Densities (N={n})")
    plt.xlabel("θ (theta)")
    plt.ylabel("Probability Density (Normalized)")
    plt.legend(fontsize=10, loc='upper right')
    plt.grid(True, alpha=0.3)
    plt.xlim(0, 2.0)
    plt.show()

selected_samples_plot = [10, 15, 50, 100]

for n in range(10, 101):
    # Generate samples
    samples = uniform_dist(n, low=0.0, high=1.0, seed=seed)

    # For simplicity, MLE for uniform U(0, theta) is max(samples)
    max_sample = np.max(samples)
    theta_mle_viz = max_sample

    # Prior
    theta0 = 0.2
    theta_range = np.linspace(theta0, 2.0, n)

    # Compute prior, likelihood, and posterior
    prior_vals = prior_dist(theta_range)
    likelihood_vals = likelihood(theta_range, samples)

    # Posterior = likelihood * prior
    posterior = likelihood_vals * prior_vals

    if n in selected_samples_plot:
        plot_distribution(theta_range, prior_vals, likelihood_vals, posterior, n)
```


Prior, Likelihood, and Posterior Densities (N=10)
Prior
Likelihood
Posterior
Probability Density (Normalized)
θ (theta)
0.0
0.2
0.4
0.6
0.8
1.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00


Prior, Likelihood, and Posterior Densities (N=15)
Probability Density (Normalized)
θ (theta)
Prior
Likelihood
Posterior
0.00
0.05
0.10
0.15
0.20
0.25
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00


Prior, Likelihood, and Posterior Densities (N=50)
Prior
Likelihood
Posterior
Probability Density (Normalized)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
θ (theta)



Question 12 (d)

We've found a theoretically conjugate **posterior** in question 11 (a).

Goal: Discuss in 3–4 sentences whether the posterior is consistent with the theoretical conjugacy behavior derived.

Observations

Touching Spikes

When the sample size is small enough ($N = 10$), the posterior with the highest probability density could touch the peak of the prior's distribution, but not the likelihood. In fact, the maximum likelihood spikes to a point close to 1, whereas the rest only attains a point below 0.2.

Expectation

Theoretically, conjugacy is defined when $posterior = likelihood * prior$.

Explaination

From the chosen graph where sample size $N = 10$, looking at the highest density points, if posterior = prior, with a positively large likelihood, the product will never be identical to posterior.

Conclusion

Posterior is **NOT** conjugate to the theoretical derivation as well.