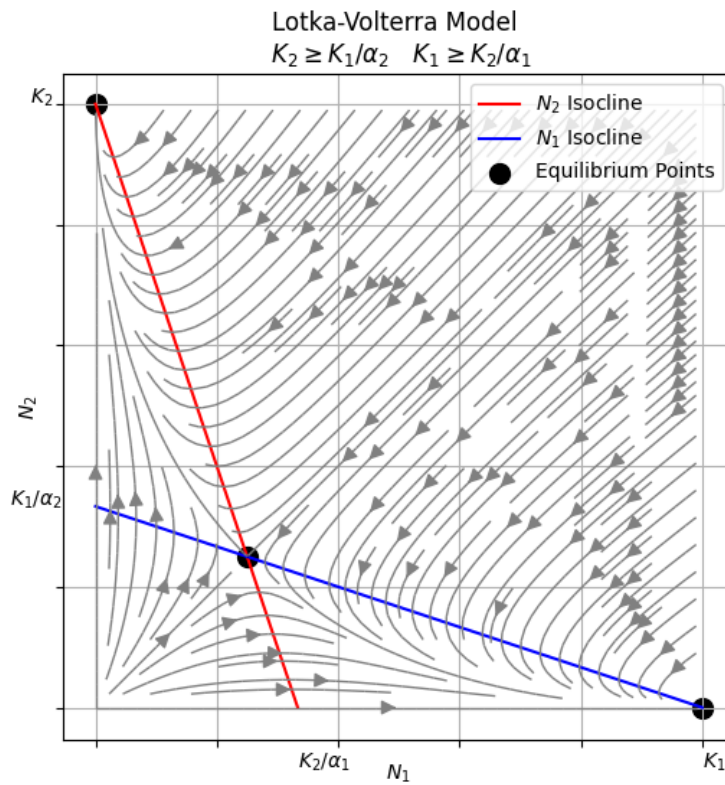


CSYS 6020: Problem Set 1

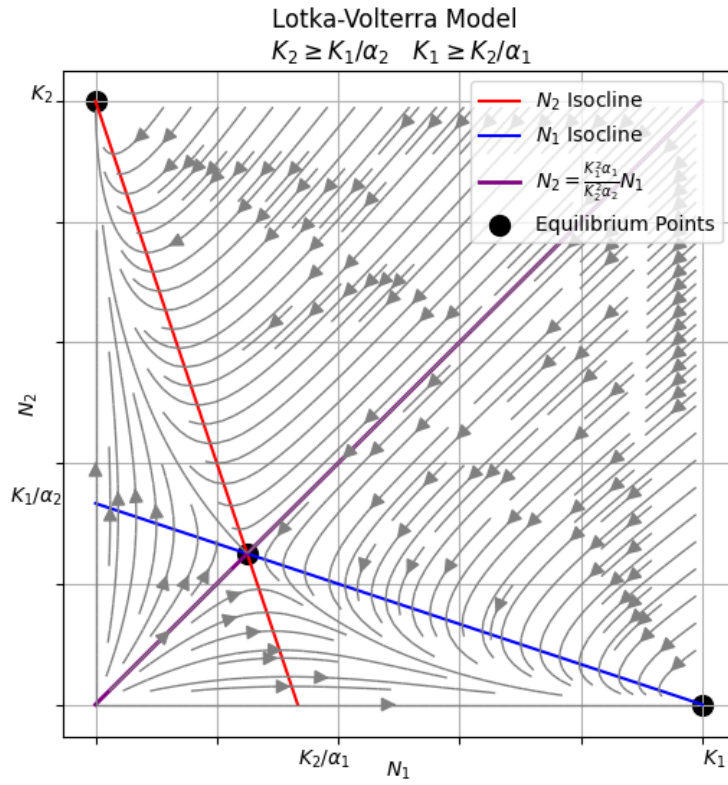
Kevin Motia

October 3, 2023

1



The phase space has stable equilibrium points at $(0, K_2)$ and $(K_1, 0)$. It has an unstable equilibrium point at the intersection of the two lines $(\frac{K_1 - \alpha_2 K_2}{1 - \alpha_1 \alpha_2}, \frac{K_2 - \alpha_1 K_1}{1 - \alpha_1 \alpha_2})$.



No. Given that the initial conditions are such that the organisms are in sites to the left of the purple line $N_2 = \frac{K_1^2 \alpha_1}{K_2^2 \alpha_2} N_1$, there is no such ρ value that would scale a given coordinate (N_1, N_2) such that it would land within a site where the populations would be changing in a way that would lead them to the unstable equilibrium point (given by the intersection of the isoclines in the figure). How the populations change over time is illustrated by the vector field.

3

Euler's Method Implementation:

```
def run_mathematical_SIS_euler(Si, Ii, beta, gamma, steps, h):
    S = Si
    I = Ii
    history = []
    for step in range(steps):
        dI_dt = ((beta * S * I) - (gamma * I))
        dS_dt = -((beta * S * I) - (gamma * I))
        next_I = I + dI_dt * h
        next_S = S + dS_dt * h
        I = next_I
        S = next_S
        history.append((S, I))
    St, It = map(np.array, zip(*history))
    return St, It
```

Heun's Method Implementation:

```
def run_mathematical_SIS_heun(Si, Ii, beta, gamma, steps, h):
    S = Si
    I = Ii
    history = []
    for step in range(steps):
        dI_dt = ((beta * S * I) - (gamma * I))
        dS_dt = -((beta * S * I) - (gamma * I))
        I_e = I + dI_dt * h
        S_e = S + dS_dt * h
        dI_dt_e = ((beta * S_e * I_e) - (gamma * I_e))
        dS_dt_e = -((beta * S_e * I_e) - (gamma * I_e))
        next_I = I + h * (dI_dt + dI_dt_e) / 2
        next_S = S + h * (dS_dt + dS_dt_e) / 2
        I = next_I
        S = next_S
        history.append((S, I))
    St, It = map(np.array, zip(*history))
    return St, It
```

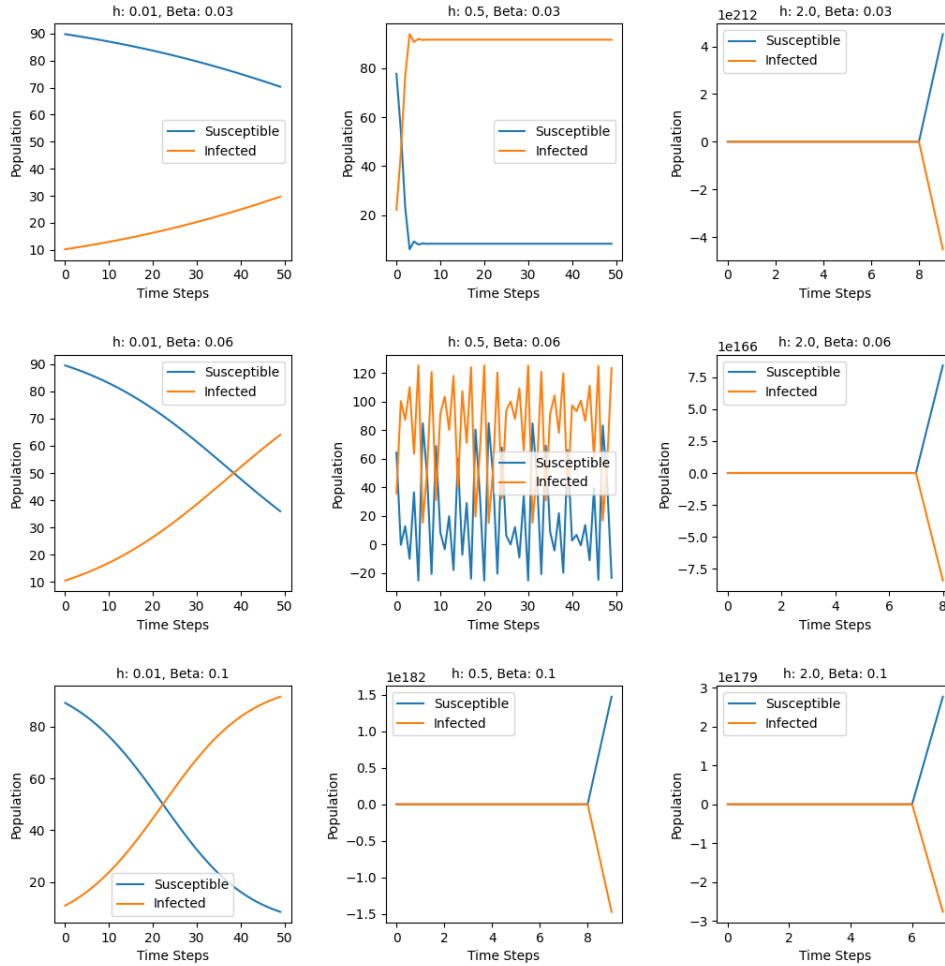
Euler's method and a discrete time model are very similar in that they both model systems in discrete time steps. An important distinction between the two is that Euler's method is used to discretize a continuous system in order to approximate differential equations. The discrete time steps can be of a desired size where smaller time steps are more accurate but require more computational resources.

On the other hand, a discrete time model is particularly useful for modeling systems that naturally evolve over time in discrete time steps. They do not involve the approximation of differential equations, which makes them not very suitable for modeling continuous systems.

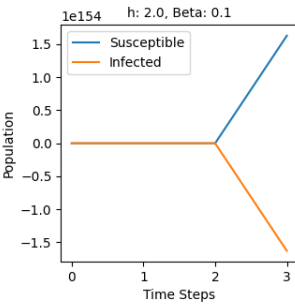
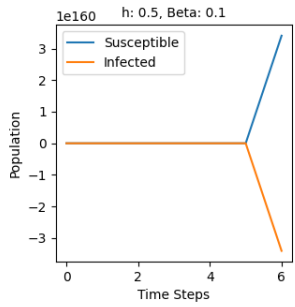
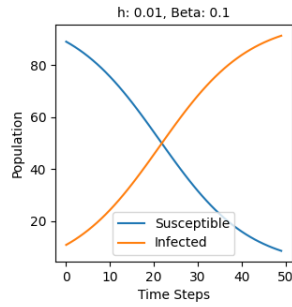
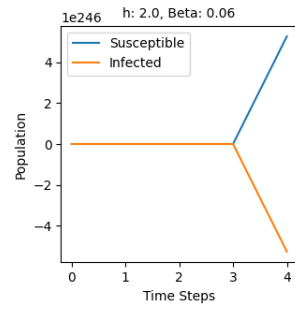
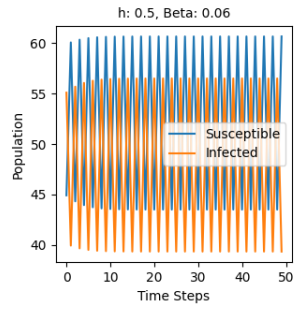
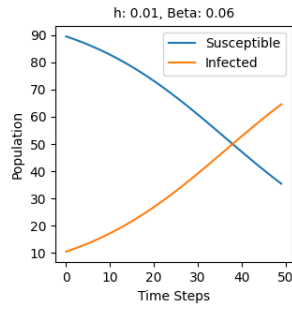
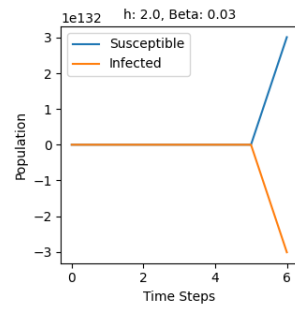
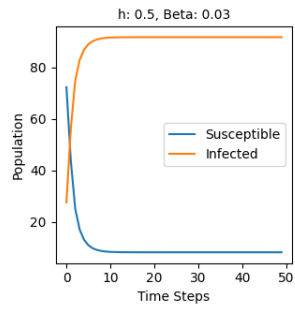
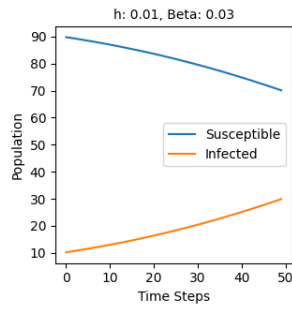
As expected, Euler's and Heun's method yield different plots, where population evolves over time. It appears that the plots illustrating the implementation of Euler's method to numerically solve the continuous SIS model have less smoothness relative to the plots generated via Heun's method. The differences are more apparent when comparing the plots of larger time step sizes (h). This is particularly visible in the plot where $h=0.5$ and $\text{Beta}=0.03$, where we can see that the populations have sharp changes in Euler's method before they begin to level out, while Heun's method changes the populations gradually over time.

What may be causing such differences is that Euler's method assumes that the rate of change remains constant over each time step. Heun's method addresses this by calculating a weighted average of the rate of change at the current point in the simulation to the rate of change at the predicted point. This accounts for changes in the rate of change over the course of a given time step. This is why smaller time steps yield better agreement in the plots, and why we see Heun's method yields smoother plots than Euler's method.

Euler's Method:



Heun's Method:



$$x(t+h) \approx x(t) + hf(x,t) + \frac{h^2}{2}f'(x,t) + \frac{h^3}{3!}f''(x,t) \quad (\text{Taylor series})$$

$$x_e(t+h) = x(t) + hf(x,t) \quad (\text{Euler's Method})$$

$$x_h(t+h) = x(t) + \frac{h}{2}(f(x,t) + f(x_e(t+h))) \quad (\text{Heun's method})$$

$$= x(t) + \frac{h}{2}(f(x,t) + f(x_e(t+h))) \quad (\text{Rearranged Heun's Method})$$

$$= x(t) + \frac{h}{2}f(x,t) + \frac{h}{2}f(x_e(t+h))$$

$$f(x(t)) = \frac{dx}{dt} \quad (\text{Hint})$$

$$x_h(t+h) = x(t) + \frac{h}{2}f(x,t) + \frac{h}{2}x'_e(t+h)$$

$$= x(t) + \frac{h}{2}f(x,t) + \frac{h}{2}(x'(t) + hf'(x,t))$$

$$= x(t) + \frac{h}{2}f(x,t) + \frac{h}{2}(f(x,t) + hf'(x,t))$$

$$= x(t) + \frac{h}{2}f(x,t) + \frac{h}{2}f(x,t) + \frac{h^2}{2}f'(x,t)$$

$$= x(t) + hf(x,t) + \frac{h^2}{2}f'(x,t)$$

Calculating global truncation error:

$$\begin{aligned} & \sum (Taylor - Heun) \\ &= \sum \left((x(t) + hf(x,t) + \frac{h^2}{2}f'(x,t) + \frac{h^3}{3!}f''(x,t)) - (x(t) + hf(x,t) + \frac{h^2}{2}f'(x,t)) \right) \\ &= \frac{h^3}{3!}f''(x,t) \\ GlobalErr &= \frac{T}{h}O(h^3) = O(h^2) \end{aligned}$$

6

At least two degrees of freedom are required for cycles to occur, which involves crossing given point in phase space in different directions. The Lotka-Volterra model accomplishes this by its unfixed population size and predator-prey interactions, which create a feedback loop because the population change of each species depends on the populations of both species. As the predator population increases in size, it leads to a decrease in prey population, which leads to a decrease in predator population, which leads to an increase in prey population, and so on.

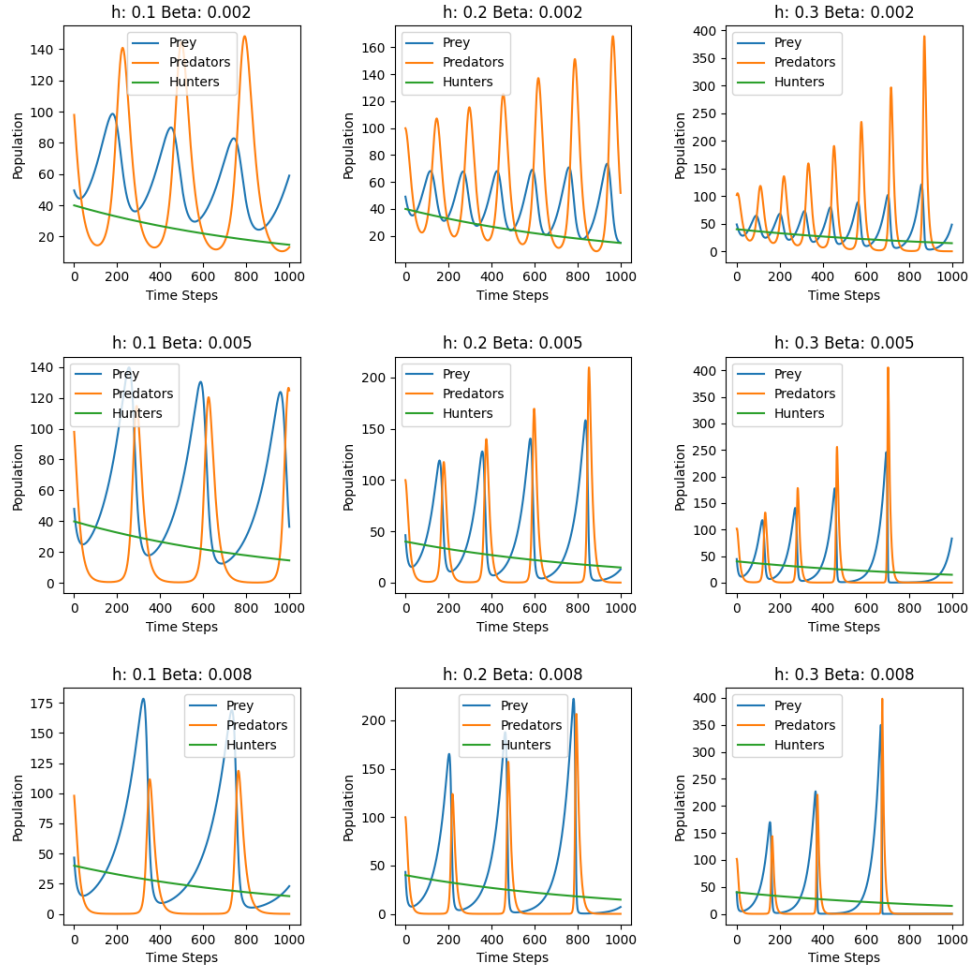
On the other hand, the SIS model cannot feature cyclic behavior. There is a continuous influx of new susceptible individuals into the infectious state. As a result, the number of infected individuals remains relatively constant, and no feedback loop occurs. There are no inter-species dependencies with unfixed populations that cause a feedback loop like in the Lotka-Volterra model.

Similarly, the SIR model cannot lead to cycles. It only contains a single degree of freedom. Once an individual reaches the recovery state, they can no longer become susceptible. All individuals in the system reach the recovery state after sufficient time has passed.

However, the SIRS model can lead to cycles. As the number of infectious individuals increases, it leads to more recoveries, which in turn reduces the number of susceptible individuals and in turn, infectious individuals. This causes a feedback loop where the number of individuals in each state fluctuate periodically.

Turning our focus to chaos, at least three degrees of freedom are required for it to be observed in a system. Classic epidemic models like SIS, SIR, and SIRS typically do not exhibit chaotic behavior under standard conditions. They are not highly sensitive to initial conditions because these models are deterministic and rely on well-defined sets of equations and parameters.

Euler's Method used in Hunter-Predator-Prey Model:



State Variables:

- P: Prey population
- D: Predator Population
- H: Hunter Population

Parameters:

- P_i : Initial prey population
- D_i : Initial predator population
- H_i : Initial hunter population
- α : Prey birth rate
- β : Predation rate
- γ : Predator birth rate
- δ : Predator death rate
- steps: number of steps
- h: step size

Some interesting assumptions:

- Only 3 types of species present
- Prey cannot die of natural causes, only by being consumed by a predator
- Predators can die of natural causes
- Predators can be killed by hunters
- Hunters die of natural causes, not by predators
- Hunters cannot reproduce
- Assumed that $\alpha, \beta, \gamma, \delta$ do not change over time
- All prey are identical
- All predators are identical
- All hunters are identical
- The system is closed (No outside causes of death or addition of any of the 3 species)

In this model, a system contains a species of prey, predator, and hunter. The system is meant to model a human hunting environment. The model can contain cycles, as seen by the plots, and I believe that they could feature unstable equilibrium points and stable equilibrium points. However, this would depend on the parameters of the system.

Using this hunter-predator-prey model, researchers might see how the presence and activities of human hunters impact the dynamics of wolf and reindeer populations. It could help gain some insight on sustainable hunting practices, and help lawmakers make more informed decisions on hunting regulations.

Optimal strategies for sustainable pest management could also be investigated, including when and how much hunting is needed and when to encourage the presence of natural predators.