PoCS What's The Story?

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number
University of Vermont, Fall 2023
Solutions to Assignment 16
Don't eat anything that glows

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1. Tokunaga's law implies Horton's laws:

   In lectures, we established the following:

$$n_\omega = \underbrace{2\, n_{\omega+1}}_{\text{generation}} + \sum_{\omega'=\omega+1}^{\Omega} \underbrace{T_{\omega'-\omega}\, n_{\omega'}}_{\text{absorption}}$$

   From here, derive Horton's law for stream numbers: $n_\omega / n_{\omega+1} = R_n$, where $R_n > 1$ and is independent of ω, and find R_n in terms of Tokunaga's two parameters T_1 and R_T.

   Solution:

$$T_k = T_1 R_T^{k-1} \quad \text{let } k = \omega' - \omega$$

$$T_{\omega'-\omega} = T_1 R_T^{\omega'-\omega-1}$$

$$n_\omega = 2n_{\omega+1} + \sum_{\omega'=\omega+1}^{\Omega} T_{\omega'-\omega} n_{\omega'}$$

$$= 2n_{\omega+1} + \sum_{\omega'=\omega+1}^{\Omega} T_1 R_T^{\omega'-\omega-1} n_{\omega'}$$

$$= 2n_{\omega+1} + \sum_{k=1}^{\Omega-\omega} T_1 R_T^{k-1} n_{k+\omega}$$

Note: Let $k = \omega' - \omega$ with the limits being $k = 1 \to \Omega - \omega$

And replace ω'... lower: $k = \omega + 1 - \omega = 1$ and upper: $k = \Omega - \omega$

$$= 2n_{\omega+1} + \sum_{k=1}^{\Omega-\omega} T_1 R_T^{k-1} n_{k+\omega}$$

Dividing by $\dfrac{n_\omega}{n_{\omega+1}} = \dfrac{2n_{\omega+1}}{n_{\omega+1}} + \displaystyle\sum_{k=1}^{\Omega-\omega} T_1 R_T^{k-1} \dfrac{n_{k+\omega}}{n_{\omega+1}}$

Let $\Omega \to \infty$ and note that $\dfrac{n_\omega}{n_{\omega+1}} = R_n \Rightarrow n_\omega = n_{\omega+1} R_n = n_{\omega+2} R_n^2 = n_{\omega+3} R_n^3 = n_{\omega+k} R_n^k$

$$\frac{n_{\omega+k} R_n^k}{n_{\omega+1}} = R_n \Rightarrow \frac{n_{\omega+k}}{n_{\omega+1}} = R_n^{1-k} = R_n^{-(k-1)}$$

So, now...

$$\frac{n_\omega}{n_{\omega+1}} = 2 + \sum_{k=1}^{\infty} T_1 R_T^{k-1} \frac{n_{k+\omega}}{n_{\omega+1}}$$

$$\frac{n_\omega}{n_{\omega+1}} = 2 + \sum_{k=1}^{\infty} \left(\frac{R_T}{R_n}\right)^{k-1} \quad \text{assuming } \frac{R_T}{R_n} < 1$$

$$= 2 + T_1 \left(\frac{1}{1 - \frac{R_T}{R_n}}\right) = R_n \quad \text{multiplying by } 1 - \frac{R_T}{R_n}$$

$$= (R_n - 2)\left(1 - \frac{R_T}{R_n}\right) = T_1 \quad \text{multiplying by } R_n$$

$$= (R_n - 2)(R_n - R_T) = T_1 R_n$$

$$\Rightarrow R_n^2 - 2R_n - R_T R_n + 2R_T = T_1 R_n \Rightarrow R_n^2 - (2 + R_T + T_1)R_n + 2R_T = 0$$

Using the quadratic formula yields the following:

$$R_n = \frac{2 + R_T + T_1 \pm \sqrt{(2 + R_T + T_1)^2 - 4(2R_T)}}{2}$$

From last homework... $R_T = 2; \quad T_1 = 2$

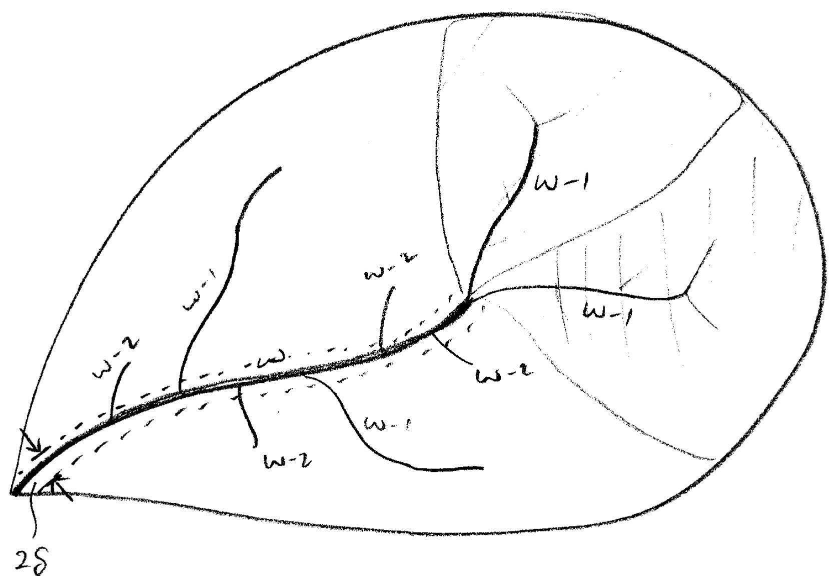
$$R_n = \frac{6 \pm \sqrt{6^2 - 16}}{2} = 3 \pm \sqrt{5}$$

$\square$

2. Show $R_n = R_a$ by using Tokunaga's law to find the average area of an order ω basin, $\langle a \rangle_\omega$, in terms of the average area of basins of order 1 to $\omega - 1$.

   (In lectures, we use Horton's laws to roughly demonstrate this result.)

   Here's the set up:



   Using the Tokunaga picture, we see a basin of order ω can be broken down into non-overlapping sub-basins.

   Connect $\langle a \rangle_\omega$ to the average areas of basins of lower orders as follows:

$$\langle a \rangle_\omega = 2 \langle a \rangle_{\omega-1} + \sum_{\omega'=1}^{\omega-1} T_{\omega,\omega'} \langle a \rangle_{\omega'} + 2\delta \langle s \rangle_\omega .$$

   The first term on the right hand side corresponds to the two 'generating' streams of order $\omega - 1$. The second term (the sum) accounts for side streams entering the sole order ω stream segment in the basin. And the last term gives the contribution of 'overland flow,' i.e., flow that does not arrive in the main stream segment through a stream. The length scale δ is the typical distance from stream to ridge.

   Solution:

$$\overline{a}_\omega = 2\overline{a}_{\omega-1} + \sum_{\omega'=1}^{\omega-1} T_{\omega,\omega'}\overline{a}_{\omega'} + 2\delta\overline{s}_\omega$$

$$\overline{a}_1 R_a^{\omega-1} = 2\overline{a}_1 R_a^{\omega-2} + 2\delta\overline{s}_1 R_s^{\omega-1} + T_1\overline{a}_1 \sum_{\omega'=1}^{\omega-1} R_T^{\omega-\omega'-1} R_a^{\omega'-1}$$

dividing $\overline{a}_1 R_a^{\omega-1}$

$$1 = \frac{2\overline{a}_1 R_a^{\omega-2}}{\overline{a}_1 R_a^{\omega-1}} + \frac{2\delta\overline{s}_1 R_s^{\omega-1}}{\overline{a}_1 R_a^{\omega-1}} + \frac{T_1\overline{a}_1}{\overline{a}_1} \sum_{\omega'=1}^{\omega-1} \frac{R_T^{\omega-\omega'-1} R_a^{\omega'-1}}{R_a^{\omega-1}}$$

$$= \frac{2}{R_a} + \frac{2\delta\overline{s}_1}{\overline{a}_1}\left(\frac{R_s}{R_a}\right)^{\omega-1} + T_1 \sum_{\omega'=1}^{\omega-1} R_T^{\omega-\omega'-1} R_a^{\omega'-\omega}$$ assuming $R_s < R_a$ and ω is large, goes to 0

$$= \frac{2}{R_a} + T_1 \sum_{\omega'=1}^{\omega-1} R_T^{\omega-\omega'-1} R_a^{\omega'-\omega}$$

Note: $T_{\omega,\omega'} = T_1 R_T^{\omega-\omega'-1}$

$$\overline{a}_\omega = R_a^{\omega-1}\overline{a}_1$$

$$\overline{s}_\omega = R_s^{\omega-1}\overline{s}_1$$

$$1 = \frac{2}{R_a} + T_1 \sum \omega' = 1^{\omega-1} R_T^{\omega-\omega'-1} R_a^{-(\omega-\omega')}$$ letting $i = \omega - \omega'$

$$= \frac{2}{R_a} + \frac{T_1}{R_a} \sum_{i=1}^{\omega-1} \left(\frac{R_T}{R_a}\right)^i$$ geometric sum

$$1 - \frac{2}{R_a} + \frac{T_1}{R_a} \frac{1}{1 - \frac{R_T}{R_a}}$$ multiplying by $1 - \frac{R_T}{R_a}$

$$1 - \frac{R_T}{R_a} = \frac{2}{R_a}\left(1 - \frac{R_T}{R_a}\right) + \frac{T_1}{R_a}\frac{1}{1 - \frac{R_T}{R_a}}\left(1 - \frac{R_T}{R_a}\right)$$ multiplying by R_a

$$R_a - R_T = 2\left(1 - \frac{R_T}{R_a}\right) + T_1 \quad \text{multiplying by } R_a$$

$$R_a^2 - R_T R_a = 2R_a\left(1 - \frac{R_T}{R_a}\right) + T_1 R_a$$

$$R_a^2 - R_T R_a = 2R_a - 2R_T + T_1 R_a$$

$$0 = R_a^2 - R_T R_a - 2R_a + 2R_T - T_1 R_a$$

$$R_a^2 - R_a(R_T + 2 + T_1) + 2R_T$$

Using the quadratic formula...

$$R_a = \frac{(R_T + 2 + T_1) \pm \sqrt{(R_T + 2 + T_1)^2 - 8R_T}}{2}$$

$\square$

3. For river networks, basin areas are distributed according to $P(a) \propto a^{-\tau}$.

   Determine the exponent τ in terms of the Horton ratios R_n and R_s.

   Guide:

   Follow the same procedure shown in lectures for $P(\ell) \propto \ell^{-\gamma}$.

   In class, we derived $P(\ell) \propto \ell^{-\gamma}$ starting from Horton's laws (see the section of scaling relations in the slides on Branching Networks II. In doing so, we started with the following observation:

   $$P_>(\ell_\omega) = \frac{N_>(\ell_\omega; \Delta)}{N_>(0; \Delta)}$$

   where $N_>(\ell_\omega; \Delta)$ was the number of sites with main stream length $> \ell_\omega$.

   Now, we can equally well use the right hand side to count the number of sites with drainage area exceeding a_ω. So,

   $$P_>(a_\omega) \propto \left(\frac{R_n}{R_s}\right)^{-\omega} = e^{-\omega \ln(R_n/R_s)}.$$

   Our task is now to wrangle the right hand side so that we see it in terms of a_ω.

Solution:

$$P_>(a_\omega) \propto \left(\frac{R_n}{R_s}\right)^{-\omega} = e^{-\omega \ln\left(\frac{R_n}{R_s}\right)}$$

$$a_\omega \propto R_a^\omega = R_n^\omega = e^{\omega \ln R_n}$$

$$P_>(a_\omega) = e^{-\omega \ln\left(\frac{R_n}{R_s}\right)\cdot 1} = P_>(a_\omega) = e^{-\omega \ln\left(\frac{R_n}{R_s}\right)\cdot \frac{\ln R_n}{\ln R_n}}$$

$$P_>(a_\omega) = \left(e^{-\omega \ln R_n}\right)^{-\ln\left(\frac{R_n}{R_s}\right)/\ln R_n}$$

$$= a_\omega^{-\ln\left(\frac{R_n}{R_s}\right)/\ln R_n}$$

$$= a_\omega^{-\frac{\ln R_n - \ln R_s}{\ln R_n}}$$

$$= a_\omega^{-(1-\frac{\ln R_s}{\ln R_n})+1-1}$$

$$= a_\omega^{-(2-\frac{\ln R_s}{\ln R_n})+1}$$

$$P_>(a_\omega) = a_\omega^{-\tau+1} \text{ where } \tau = 2 - \frac{\ln R_s}{\ln R_n}$$

□