PoCS What's The Story?

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number
University of Vermont, Fall 2023
Solutions to Assignment 18
Huge raving maniac with national borders and an anthem

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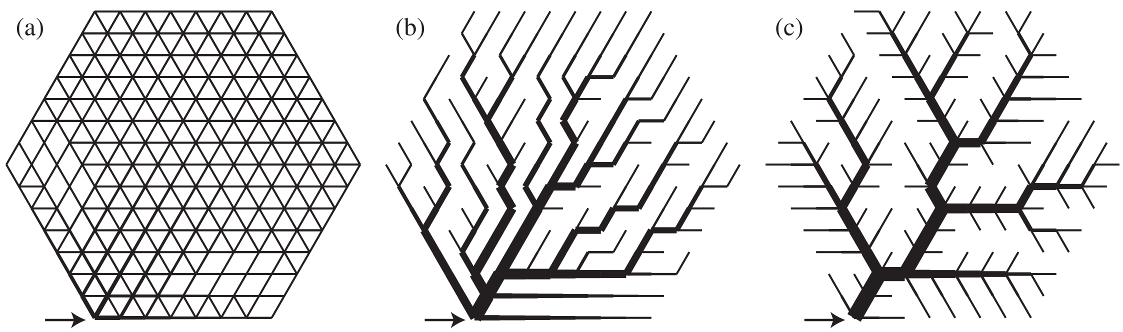
1. (3 + 3) Reproduce Bohn and Magnasco's Figs. 2a and 2b in [?]:



Steps are given below but please read through the paper to understand how they set things up.

The full team is encouraged to work together on Teams.

(a) Done (previous assignment): Construct an adjacency matrix $\mathbf{A}$ representing the hexagonal lattice used in [?]. Plot this adjacency matrix.

(b) Run a minimization procedure to construct Figs. 2a and 2b which correspond to $\gamma = 2$ and $\gamma = 1/2$. Steps:

i. Set each link's length to 1 (the d_{kl}). The goal then reduces to minimizing the cost

$$C = \sum_{k,l} |I_{kl}|^{\Gamma}$$

where I_{kl} is the current on link kl and $\Gamma = 2\gamma/(\gamma+1)$.

ii. Place a current source of nominal size i_0 at one node (as indicated in Fig. 2 above).

iii. All other nodes are sinks, drawing a current of

$$i_k = -\frac{i_0}{N_{\mathrm{nodes}-1}}.$$

iv. Suggest setting $i_0 = 1000$ (arbitrary but useful value given the size of the network).

v. Generate an initial set of random conductances for each link, the $\{\kappa_{kl}\}$. From the paper, these must sum to some global constraint as

$$K = \left(\sum_{k,l} \kappa_{kl}^{\gamma} \right)^{1/\gamma} .$$

This constraint is meant to represent a limitation on the amount of material that can be used to build the network.
Note: There seems to be no reason not to set $K = 1$. However, taking the initial value of K determined by the initial set of random conductances would work.
To our notational peril, we now have a lot of k types on deck.

vi. Solve the following to determine the potential U at each node, and hence the current on each link using:

$$i_k = \sum_l \kappa_{kl}(U_k - U_l),$$

and then

$$I_{kl} = \kappa_{kl}(U_l - U_k).$$

Note: the paper erroneously has $I_{kl} = R_{kl}(U_l - U_k)$ below equation 4; there are a few other instances of similar miswritings of R_{kl} instead of κ_{kl}.

vii. Now, use scaling in equation (10) to compute a new set of $\{\kappa_{kl}\}$ from the I_{kl}. Everything boils down to

$$\kappa_{kl} \propto |I_{kl}|^{-(\Gamma-2)},$$

where the constant of proportionality is determined by again making sure $K^{\gamma} = \sum_{k,l} \kappa_{kl}^{\gamma}$.

Some help—Let's sort out the key equation:

$$i_k = \sum_l \kappa_{kl}(U_k - U_l).$$

Each time we loop around through this equation, we know the i_k and the κ_{kl} and must determine the U_k. In matrixology, we love $A\vec{x} = \vec{b}$ problems so let's see if we can fashion one:

$$i_k = \sum_l \kappa_{kl}(U_k - U_l)$$

$$= \sum_l \kappa_{kl} U_k - \sum_l \kappa_{kl} U_l$$

$$= U_k \sum_l \kappa_{kl} - \sum_l \mathbf{K}_{kl} U_l$$

$$= \lambda_k U_k - [\mathbf{K}\vec{U}]_k$$

where we have set $\lambda_k = \sum_l \kappa_{kl}$, the sum of the kth row of the matrix $\mathbf{K}$. We now construct a diagonal matrix Λ with the λ_k on the diagonal, and obtain:

$$\vec{i} = (\Lambda - \mathbf{K})\,\vec{U}.$$

The above is in the form $A\vec{x} = \vec{b}$ so we can solve for $\vec{U}$ using standard features of R, Matlab, Python, ... (hopefully).

Solution:

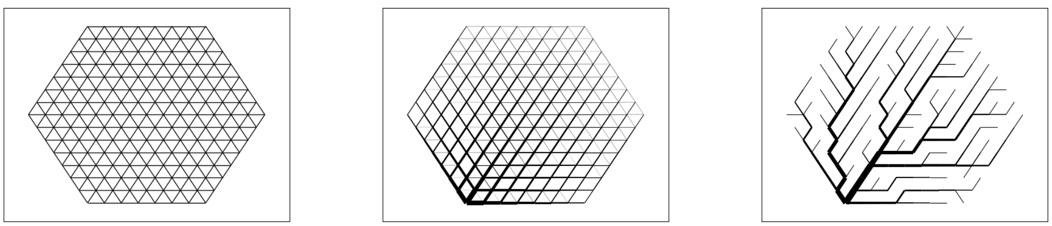


Figure 1: Replication of underlying lattice, and Figs. 2a and 2b from Structure, scaling, and phase transition in the optimal transport network by S. Bohn and M. O. Magnasco

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