PoCS What's The Story?

Principles of Complex Systems, Vols. 1, 2, & 3D
CSYS/MATH 6701, 6713, & a pretend number
University of Vermont, Fall 2023
Solutions to Assignment 17
So much universe, and so little time

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1. Derive Murray's law.

   Per lectures, find the minimum rate of energy expenditure working from the assertion that:
   $$P = P_{\mathrm{drag}} + P_{\mathrm{met}} = \Phi^2 \frac{8\eta\ell}{\pi r^4} + c_{\mathrm{met}} r^2 \ell,$$
   where met stands for metabolic.

   We are interested in how P varies with the tube radius r.

   Per lectures, we defined the 'parent' branch's radius as r_{parent}, and the 'offspring' branches as having radii $r_{\mathrm{offspring1}}$ and $r_{\mathrm{offspring2}}$ (which need not be the same).

   Show that minimizing energy expenditure leads to $r^3_{\mathrm{parent}} = r^3_{\mathrm{offspring1}} + r^3_{\mathrm{offspring2}}$.

   Note that in the LaTeX settings for assignments, various derivative notations are included.

   Here, you will want to use partial derivatives, and here's a start.

   Note the code-like formatting as expounded on here. Far easier to create, edit, debug, read.

$$\boxed{\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left(\Phi^2 \frac{8\eta\ell}{\pi r^4} + c_{\mathrm{met}} r^2 \ell\right)}$$

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$
\ partialdiff {P}{r}
=
\ partialdiff {}{r}
\left(
\Phi^{2}
\frac{
  8 \eta \ell
}{
  \pi r^{4}
}
+
c_{\textnormal{met}}
r^{2}
\ell
\right)
$
```

Solution:

$$P = P_{\mathsf{drag}} + P_{\mathsf{met}}$$

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left[\Phi^2 \frac{8\eta l}{\pi r^4} + c_{\mathsf{met}} r^2 l\right]$$

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left[\Phi^2 \frac{8\eta l}{\pi r^4} + c_{\mathsf{met}} r^2 l\right]$$

$$= \Phi^2 8\eta l \pi r^{-4} + c_{\mathsf{met}} r^2 l$$

$$= \Phi^2 8\eta l \pi(-4) r^{-5} + c_{\mathsf{met}}(2) r l$$

$$= \Phi^2 8\eta l \pi(-32) r^{-5} + c_{\mathsf{met}}(2) r l = 0$$

$$\Phi^2 8\eta l \pi(-32) r^{-5} = c_{\mathsf{met}}(2) r l$$

$$\Phi^2 = \frac{1}{16}\frac{1}{\eta}\frac{1}{\pi} c_{\mathsf{met}} r^6$$

$$\Phi^2 = \frac{c_{\text{met}}}{16\eta\pi} r^6$$

$$\Phi = \sqrt{\frac{c_{\text{met}}}{16\eta\pi}} r^3$$

Using $\Phi_0 = \Phi_1 + \Phi_2$

$$\sqrt{\frac{c_{\text{met}}}{16\eta\pi}} r_0^3 = \sqrt{\frac{c_{\text{met}}}{16\eta\pi}} r_1^3 + \sqrt{\frac{c_{\text{met}}}{16\eta\pi}} r_2^3$$

$$r_0^3 = r_1^3 + r_2^3$$

□

2. Derive the equivalent of Murray's law for branching networks where material moves by diffusion. Perhaps surprisingly, this connects the inner workings of insects, electrical networks, and search on networks.

   For diffusion, the impedance of a vessel is now $Z = c_{\mathrm{diff}} \ell r^{-2}$ where c_{diff} is a constant, ℓ is vessel length, and r is vessel radius.

   In terms of general impedance, the expression for the rate of energy expenditure is:

   $$P = P_{\mathrm{drag}} + P_{\mathrm{met}} = \Phi^2 Z + c_{\mathrm{met}} r^2 \ell.$$

   Solution:

   $$P = P_{\text{drag}} + P_{\text{met}} = \Phi^2 z + c_{\text{met}} r^2 l$$

   $$P = \Phi^2 c_{\text{diff}} l r^{-2} + c_{\text{met}} r^2 l$$

   $$\frac{\partial P}{\partial r} = \Phi^2 c_{\text{diff}} l (-2) r^{-3} + c_{\text{met}} (2) r l = 0$$

   $$\Phi^2 c_{\text{diff}} l (2) r^{-3} = c_{\text{met}} (2) r l$$

   $$\Phi^2 = \frac{c_{\text{met}}}{c_{\text{diff}}} r^4$$

   $$\Phi = \sqrt{\frac{c_{\text{met}}}{c_{\text{diff}}}} r^2$$

Using $\Phi_0 = \Phi_1 + \Phi_2$

$$\sqrt{\frac{c_{\textsf{met}}}{c_{\textsf{diff}}}} r_0^2 = \sqrt{\frac{c_{\textsf{met}}}{c_{\textsf{diff}}}} r_1^2 + \sqrt{\frac{c_{\textsf{met}}}{c_{\textsf{diff}}}} r_2^2$$

$$r_0^2 = r_1^2 + r_2^2$$

$\square$

3. Now derive the generalized version of Murray's law for a generalized impedance $Z = c_{\text{imp}} \ell r^{-2\alpha}$, where c_{imp} is a general impedance constant, ℓ is vessel length, and r is vessel radius.

   We can assume $\alpha > 0$ as impedance should decrease with wider vessels.

   We choose $r^{-2\alpha}$ because cross sectional area πr^2 can be considered the essential parameter here, and because we skipped to the end of the book and decided to rewrite the start.

   Solution:

$$P = \Phi^2 z + c_{\textsf{met}} r^2 l$$

$$P = \Phi^2 c_{\textsf{im}} r^{-2\alpha} + c_{\textsf{met}} r^2 l$$

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r} \left[\Phi^2 c_{\textsf{im}} l r^{-2\alpha} + c_{\textsf{met}} r^2 l \right]$$

$$= \Phi^2 c_{\textsf{im}} l (-2\alpha) r^{-2\alpha - 1} + c_{\textsf{met}} (2) r l$$

$$= \Phi^2 c_{\textsf{im}} l 2\alpha r^{-2\alpha - 1} = c_{\textsf{met}} 2 r l$$

$$\Phi^2 = \frac{c_{\textsf{met}}}{c_{\textsf{im}}} \frac{r^{1+2\alpha+1}}{\alpha} = \frac{c_{\textsf{met}}}{c_{\textsf{im}}} \alpha^{-1} r^{2+2\alpha}$$

Letting $k^2 = \frac{c_{\textsf{met}}}{c_{\textsf{im}}}$

$$\Phi^2 = k^2 \alpha^{-1} r^{2+2\alpha} \implies \Phi = k \alpha^{-1/2} r^{1+\alpha}$$

Using $\Phi_0 = \Phi_1 + \Phi_2$

$$k\alpha^{-1/2} r_0^{1+\alpha} = k\alpha^{-1/2} r_1^{1+\alpha} + k\alpha^{-1/2} r_2^{1+\alpha}$$

$$r_0^{1+\alpha} = r_1^{1+\alpha} + r_2^{1+\alpha}$$

□

4. Murray's law for real data.

   See if you can track down a data set for real branching networks where Murray's law might reasonably apply, and then test how well Murray's law holds up.

   Could be blood vessels, trees, ... [?, ?, ?].

   As always, you are welcome to collaborate. Feel free to share data sets on Teams.

   Solution:

   From Table III in "On connecting large vessels to small. The meaning of Murray's law" by T. F. Sherman:

$$1752919488 = 608 \times (96)^3 + 45 \times (300)^3$$

$$1875000000 = 15 \times (500)^3$$

$$1.753 \times 10^9 \approx 1.875 \times 10^9$$

   It looks like Murray's law is holding up for this test. It appears that the predicted value from Murray's law is slightly lower than the actual value, but these values are approximately equivalent in the context of their magnitudes. □

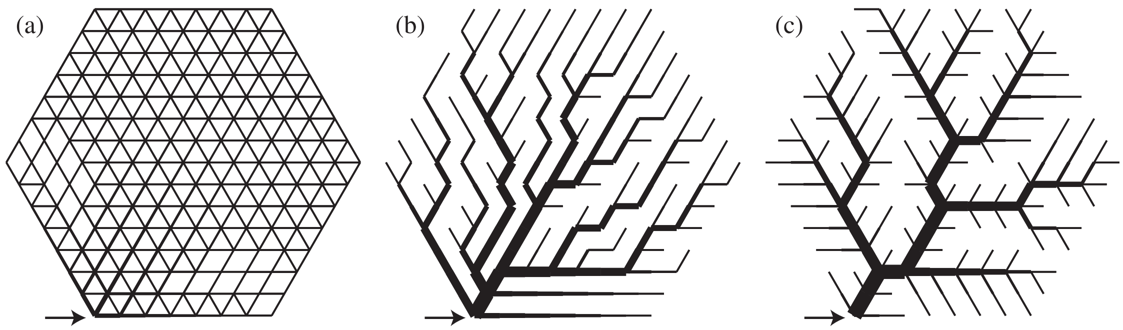
5. (3 + 3)

   Let's start on trying to reproduce reproduce Bohn and Magnasco's Figs. 2a and 2b in [?].

   A profound physical result. For movement of stuff, when should networks exist?

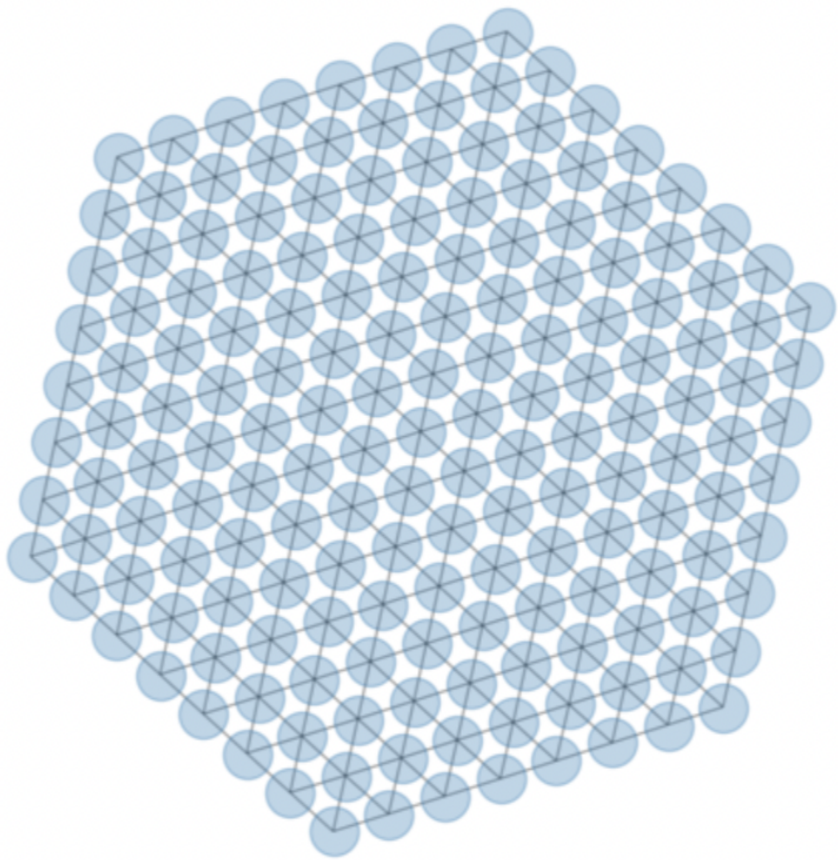
   Preliminary work:

   - Construct an adjacency matrix for the underlying hexagonal lattice where the side number of nodes is a variable n.
   - Plot the $n = 8$ version to match with the grids underlying the figures below.



Solution:

The following figure and table are obtained from the attached code.



	(0, 0)	(0, -1)	(-1, 0)	(-1, 1)	(0, 1)	(1, -1)	(1, 0)	(0, -2)	(-1, -1)	(1, -2)	...	(5, 2)	(6, 1)	(7, -7)	(7, -6)	(7, -5)	(7, -4)	(7, -3)	(7, -2)	(7, -1)	(7, 0)
(0, 0)	0.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0	0.0	0.0	...	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
(0, -1)	1.0	0.0	1.0	0.0	0.0	1.0	0.0	1.0	1.0	1.0	...	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
(-1, 0)	1.0	1.0	0.0	1.0	0.0	0.0	0.0	0.0	1.0	0.0	...	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
(-1, 1)	1.0	0.0	1.0	0.0	1.0	0.0	0.0	0.0	0.0	0.0	...	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
(0, 1)	1.0	0.0	0.0	1.0	0.0	0.0	1.0	0.0	0.0	0.0	...	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
(7, -4)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	...	0.0	0.0	0.0	0.0	1.0	0.0	1.0	0.0	0.0	0.0
(7, -3)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	...	0.0	0.0	0.0	0.0	0.0	1.0	0.0	1.0	0.0	0.0
(7, -2)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	...	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	1.0	0.0
(7, -1)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	...	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0	1.0
(7, 0)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	...	0.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.0

169 rows × 169 columns

□