NAME: Kevin Motia
DATE: 2024-03-25
Homework: Assignment 4
Data Science II

Problem 1

X	Y	C_0	C_1
0	0	0	0*
0	0	0	0*
0	0	0	0*
0	0	0	0*
1	1	1*	1
1	1	1*	1
1	1	1*	1
0	1	1	0*

$$\theta = E[C_1] - E[C_0] = \frac{3}{8} - \frac{4}{8} = \frac{-1}{8}$$
$$(\theta < 0)$$

$$\alpha = E[Y|x=1] - E[Y|x=0] = 1 - \frac{1}{5} = \frac{4}{5}$$
$$(\alpha > 0)$$

The intuition for this example, we can think of the situation of patients undergoing a certain therapy. X is the treatment, and Y outcome of the treatment. In this case, Y tells us if the patient survives (1) or dies (0). My example shows the treatment did not impact the patients' health outcomes. However, for one patient, the data indicates that they survived without treatment, and would have died with treatment. This situation described by the counterfactual could arise in a setting where the therapy being studied is not effective, yet can have serious potential side effects, including death.

Problem 2

2.1 Conditional Distributions

For Z=0:
For X=0:

$$\frac{0.405 + 0.045}{0.405 + 0.045 + 0.045 + 0.005} = 0.9$$

For X=1:

$$\frac{0.405 + 0.005}{0.405 + 0.045 + 0.045 + 0.005} = 0.1$$

For Y=0:

$$\frac{0.405 + 0.405}{0.405 + 0.045 + 0.045 + 0.005} = 0.9$$

For Y=1:

$$\frac{0.045 + 0.005}{0.405 + 0.045 + 0.045 + 0.005} = 0.1$$

For Z=1:
For X=0:

$$\frac{0.125 + 0.125}{0.125 + 0.125 + 0.125 + 0.125} = 0.25$$

For X=1:

$$\frac{0.125 + 0.125}{0.125 + 0.125 + 0.125 + 0.125} = 0.25$$

For Y=0:

$$\frac{0.125 + 0.125}{0.125 + 0.125 + 0.125 + 0.125} = 0.25$$

For Y=1:

$$\frac{0.125 + 0.125}{0.125 + 0.125 + 0.125 + 0.125} = 0.25$$

2.2 Independence

We want to show that X $\perp\!\!\!\perp$ Y | Z. This means that we must show that for every combination of values, Pr(X,Y|Z) = Pr(X|Z) Pr(Y|Z)
We find the below two tables by calculating the probabilities from each quadrant in the tables of joint distributions relative to the entries of the entire table.

Table 1: **Z=0**

	Y=0	**Y=1**	**Total**
X=0	0.81	0.09	0.9
X=1	0.09	0.01	0.10
Total	0.9	0.10	1

Table 2: **Z=1**

	Y=0	**Y=1**	**Total**
X=0	0.25	0.25	0.50
X=1	0.25	0.25	0.50
Total	0.25	0.25	1

Table 3: **Calculations**

Case	**Pr(X,Y|Z)**	**Pr(X|Z) Pr(Y|Z)**
X=0, Y=0, Z=0	0.81	0.9 * 0.9
X=0, Y=0, Z=1	0.25	0.5 * 0.5
X=0, Y=1, Z=0	0.09	0.9 * 0.1
X=1, Y=0, Z=0	0.01	0.1 * 0.1
X=0, Y=1, Z=1	0.25	0.5 * 0.5
X=1, Y=1, Z=0	0.01	0.1 * 0.1
X=1, Y=0, Z=1	0.25	0.5 * 0.5
X=1, Y=1, Z=1	0.25	0.5 * 0.5

From **Table 3**, we can see that for all cases, Pr(X,Y|Z) = Pr(X|Z) Pr(Y|Z) is true. Therefore, X $\perp\!\!\!\perp$ Y | Z.

2.3 Marginal Distributions

Using the tables of joint distributions, we find the following table:

Table 4: **Z=Marginal Distribution**

	Y=0	**Y=1**	**Total**
X=0	0.53	0.17	0.7
X=1	0.17	0.13	0.3
Total	0.7	0.3	1

To prove that X is not marginally independent of Y, we determine if for at least one case, Pr(X,Y) $\neq$ Pr(X)Pr(Y). Here, we can observe that Pr(X=0, Y=0) $\neq$ Pr(X=0)Pr(Y=0). The calculation is as follows:

$$\frac{0.53}{1} \neq (0.7)(0.7)$$

$$0.53 \neq 0.49$$

Therefore, X is not marginally independent of Y.

Problem 3

3.1 X $\perp\!\!\!\perp$ Z.

We will prove that $f(X, Z) = f(X)f(Z)$ to show that X $\perp\!\!\!\perp$ Z.

$$f(X, Z) = f(X)f(Z)$$

Determining the probability function for this DAG:

$$f(X, Y, Z) = f(X)f(Z)f(Y|X, Z)$$

$$f(X, Z) = \sum_y f(X, Y, Z)$$

$$= \sum_y f(X)f(Z)f(Y|X, Z)$$

$$= f(X)f(Z)\sum_y f(Y|X, Z)$$

$$= f(X)f(Z)(1)$$

So, now we can see that:

$$f(X, Z) = f(X)f(Z)$$

3.2 X and Z are dependent given Y

We will prove that $f(X, Z|Y) = f(X|Y)f(Z|Y)$.

$$f(X, Y, Z) = f(X)f(Z)f(Y|X, Z)$$

From Bayes Rule:

$$f(X, Y, Z) = f(X)f(Z)\frac{f(X|Z, Y)f(Y)}{f(X, Z)}$$

Canceling $f(X)f(Z)$ with $f(X, Z)$

$$f(X, Y, Z) = f(X, Z|Y)f(Y)$$

$$\frac{f(X, Y, Z)}{f(Y)} = f(X, Z|Y)$$

Rearranging...

$$f(X, Z|Y) = f(X, Y, Z)\frac{1}{f(Y)}$$

$$f(X, Z|Y) = f(X)f(Z)f(Y|X, Z)\frac{1}{f(Y)}$$

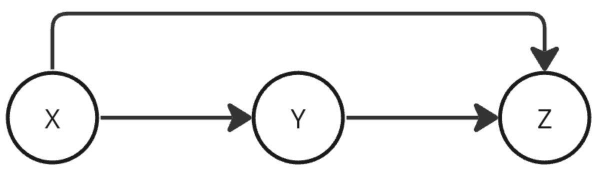
And now, we can see that

$$f(X, Z|Y) = f(X)f(Z)f(Y|X, Z)\frac{1}{f(Y)} \neq f(X|Y)f(Z|Y)$$

Here, X and Z both point to Y. In other words, X and Z both have direct causal influence on Y. We call this situation a "collider", where Y is the collider node.

Problem 4

4.1 Diagram



4.2 Derivation of Pr(Z=z | Y=y)

$$\begin{aligned} Pr(Z = z|Y = y) &= \frac{Pr(Y = y)Pr(Z = z)}{Pr(Y = y)} \\ &= \frac{Pr(Y = y, Z = z)}{Pr(Y = y)} \\ &= \frac{\sum_x Pr(X = x, Y = y, Z = z)}{Pr(Y = y)} \end{aligned}$$

Now, let's compute $Pr(X = x, Y = y, Z = z)$ from the diagram in order to plug it into the expression above.

$$Pr(X = x, Y = y, Z = z) = Pr(X = x)Pr(Y = y|X = x)Pr(Z = z|X = x, Y = y)$$

Plugging this in...

$$Pr(Z = z|Y = y) = \frac{\sum_x Pr(X = x)Pr(Y = y|X = x)Pr(Z = z|X = x, Y = y)}{Pr(Y = y)}$$

Now, we know everything in this expression except for Pr(Y=y). Let's find that in terms of the values we know.

$$Pr(Y = y) = \sum_x Pr(Y = y|X = x)Pr(X = x)$$

So finally, we can plug in $Pr(Y = y)$, and we get the following:

$$Pr(Z = z|Y = y) = \sum_x \frac{Pr(X = x)Pr(Y = y|X = x)Pr(Z = z|X = x, Y = y)}{\sum_x Pr(Y = y|X = x)Pr(X = x)}$$

We can now plug in Z=1 and Y=1...

$$Pr(Z = z|Y = y) = \frac{(0.5)(0.88)(0.88) + (0.5)(.119)(0.5)}{0.5} = 0.83$$

4.3 Simulation

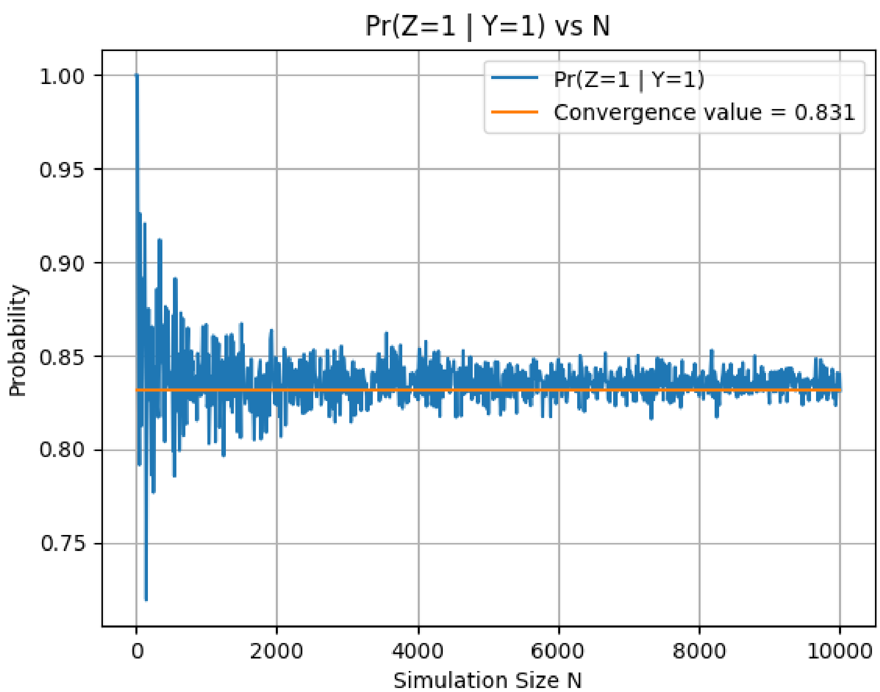


Figure 1: Simulation Plot of Pr(Z=1 | Y=1) as a function of simulation size N, where N ranges from 1 to 10000 in steps of 10.

4.4 Interventions

$$Pr(Z = z|Y := y) = \frac{Pr(Y = y)Pr(Z = 1)}{Pr(Y := y)}$$

$$= \frac{Pr(Y := y, Z = 1)}{Pr(Y := y)}$$

$$= \frac{\sum_x Pr(X = x, Y := y, Z = 1)}{Pr(Y = y)}$$

Now, let's compute $Pr(X = x, Y := y, Z = z)$ from the diagram in order to plug it into the expression above. Here, I follow the steps to write out a new DAG diagram by omitting the arrow pointing to Y. Next, we can re-write the DAG expression like so:

$$Pr(X = x, Y := y, Z = z) = Pr(X = x)Pr(Z = z|X = x, Y = y)$$

Plugging this into our expression for $Pr(Z = 1|Y := y)$, we get

$$Pr(Z = 1|Y := y) = \frac{\sum_x Pr(X = x)Pr(Z = 1|X = x, Y := y)}{Pr(Y := y)}$$

Continuing on, we get

$$Pr(Z = 1|Y := y) = \frac{\sum_x Pr(X = x)Pr(Z = 1|X = x, Y := y)}{\sum_x Pr(X = x)}$$

Now, we can calculate $Pr(Z = 1|Y := 1)$ as follows:

$$Pr(Z = 1|Y := 1) = \frac{(0.5)(0.5) + (0.5)(0.88)}{(1)} = 0.69$$

4.5 Fixed Y=1

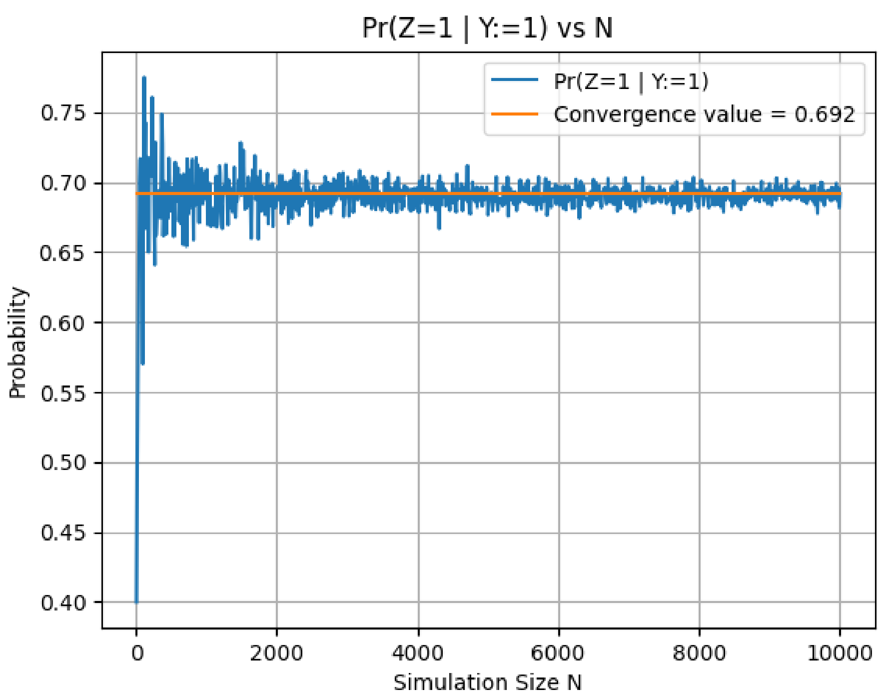


Figure 2: Simulation Plot of Pr(Z=1 | Y:=1) as a function of simulation size N, where N ranges from 1 to 10000 in steps of 10.