Matroid Intersection

Input: Two matroids $\mathcal{M}_1 = (X, \mathcal{I}_1)$ and $\mathcal{M}_2 = (X, \mathcal{I}_2)$.

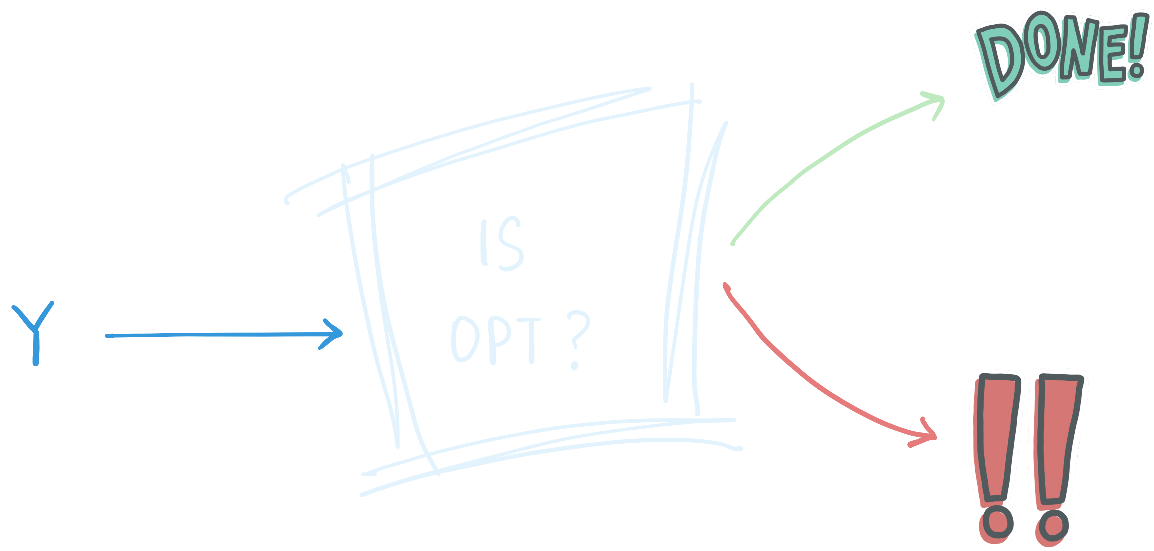
note that the ground set is the same for both $\mathcal{M}_1$ & $\mathcal{M}_2$

GOAL. Find a subset $Y \subseteq X$ such that

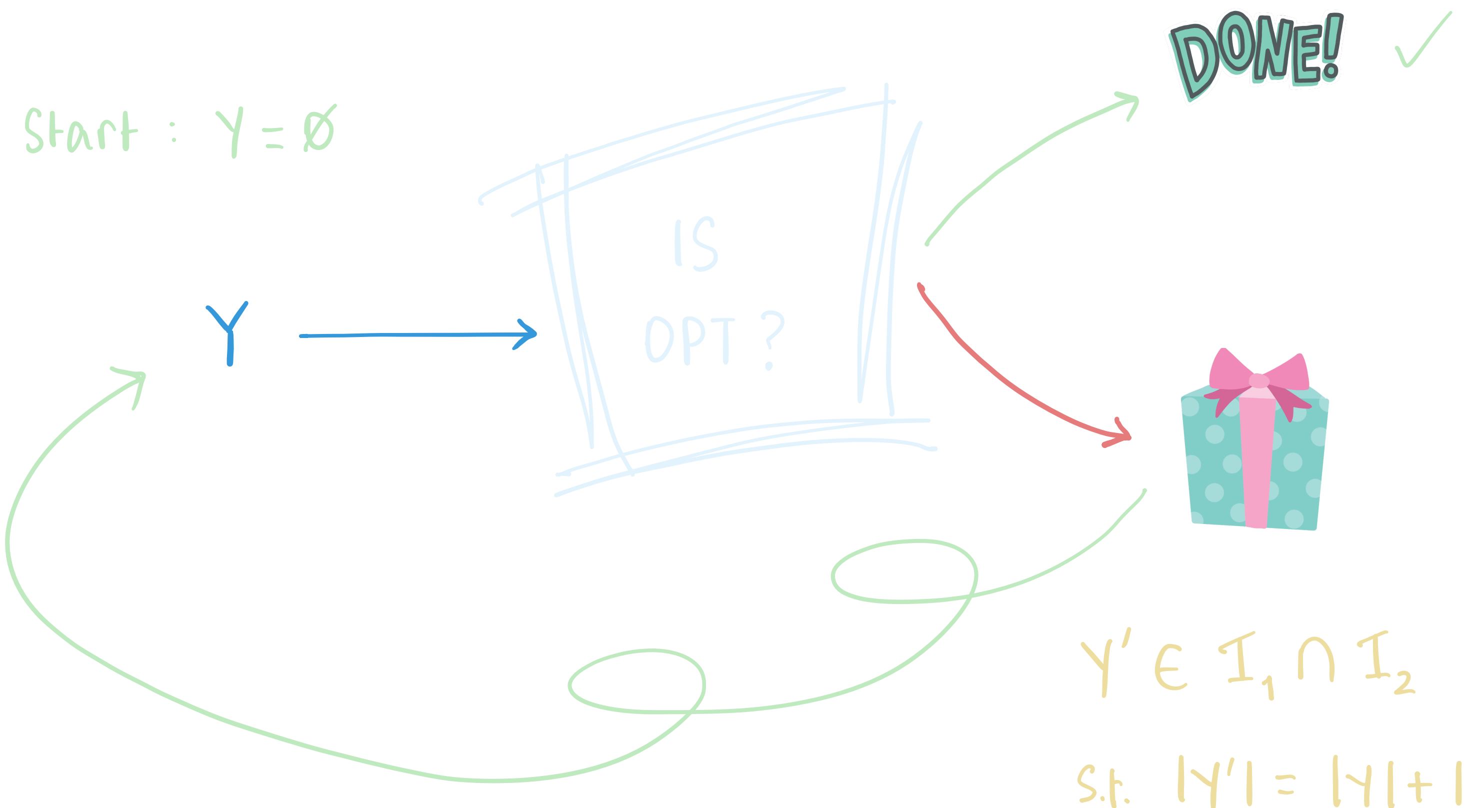
$$Y \in \mathcal{I}_1 \cap \mathcal{I}_2$$

and $|Y|$ is maximized.

Idea: Develop an optimality detector.

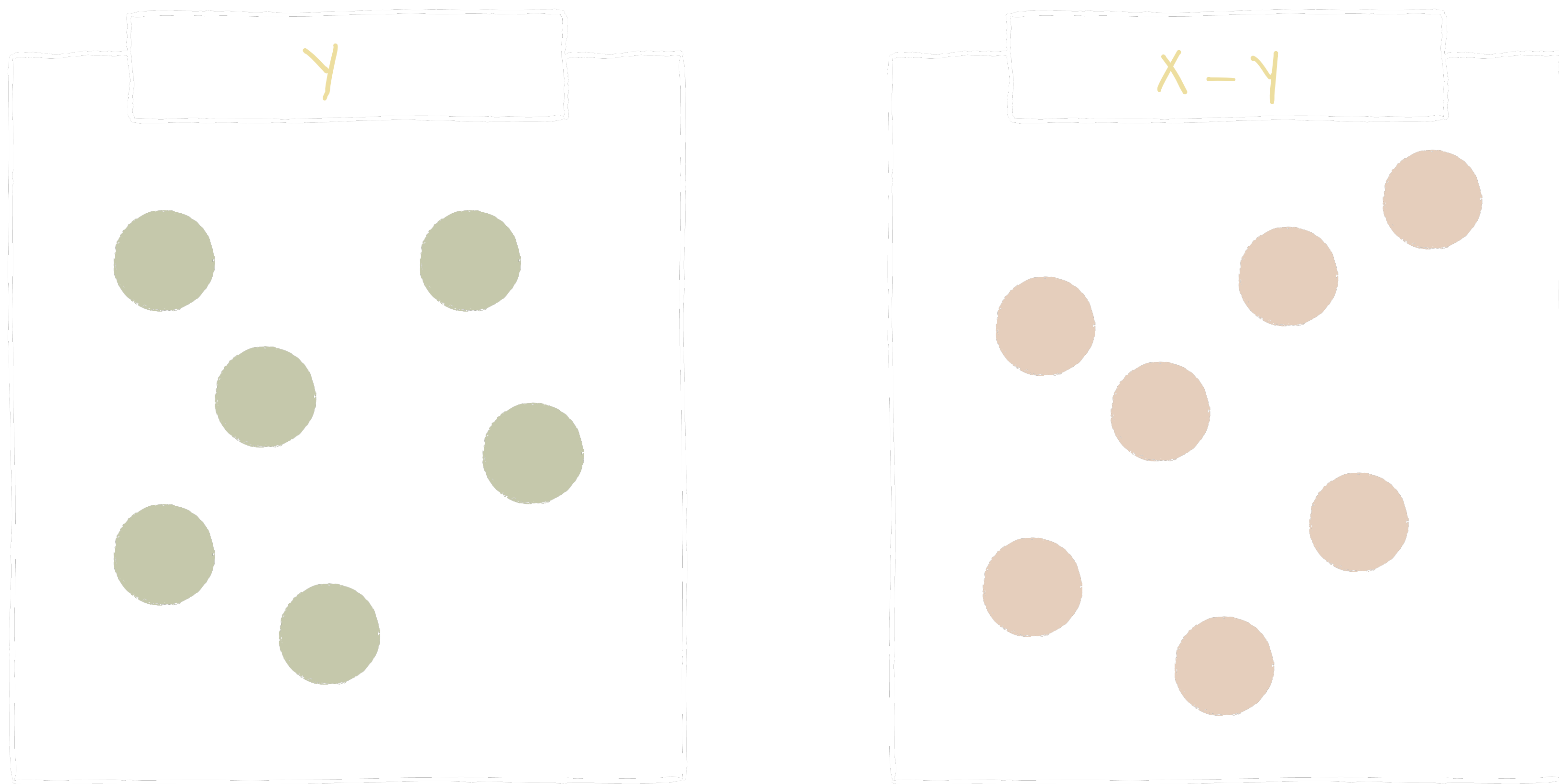


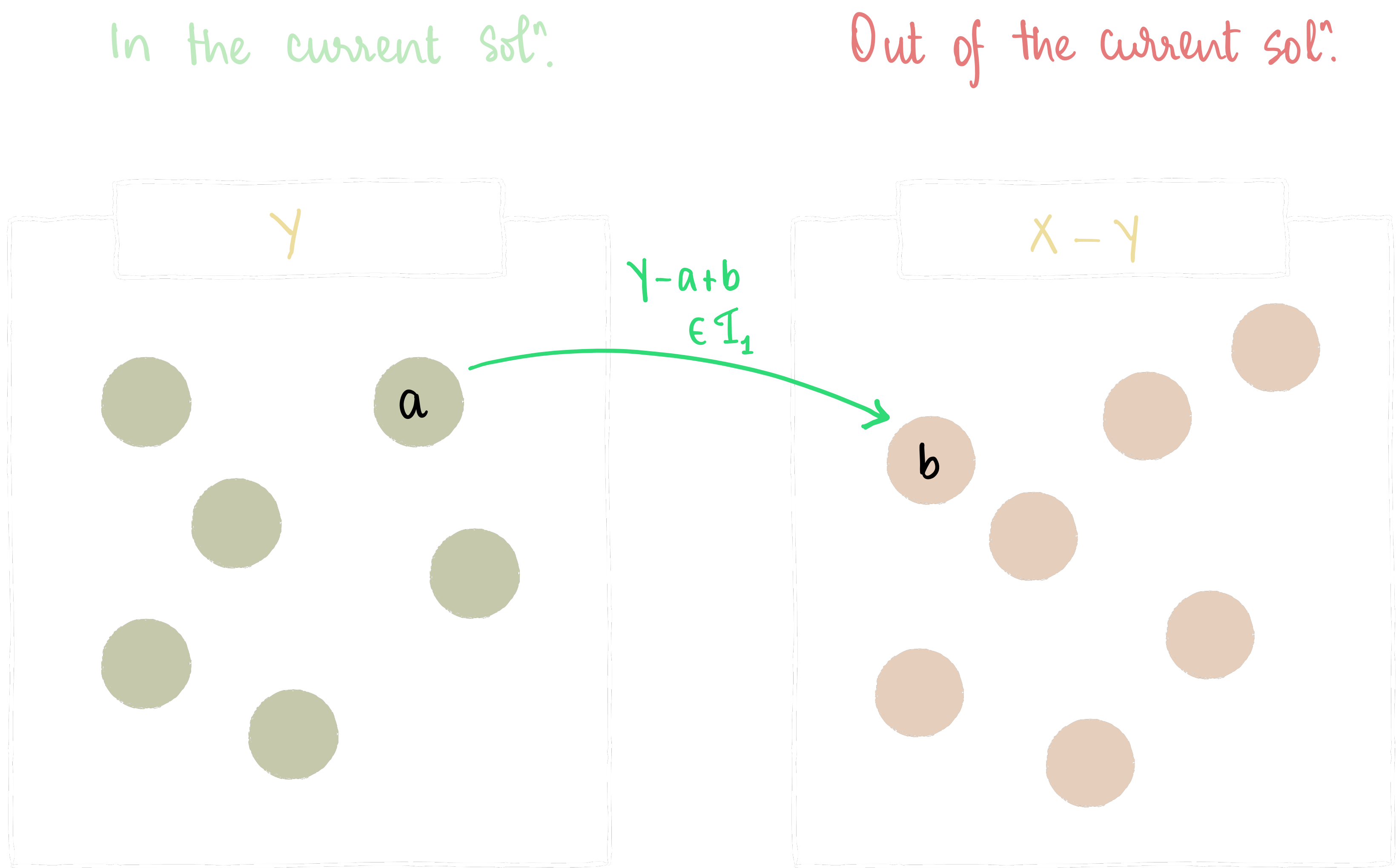
Idea: Develop an optimality detector.




y
x - y

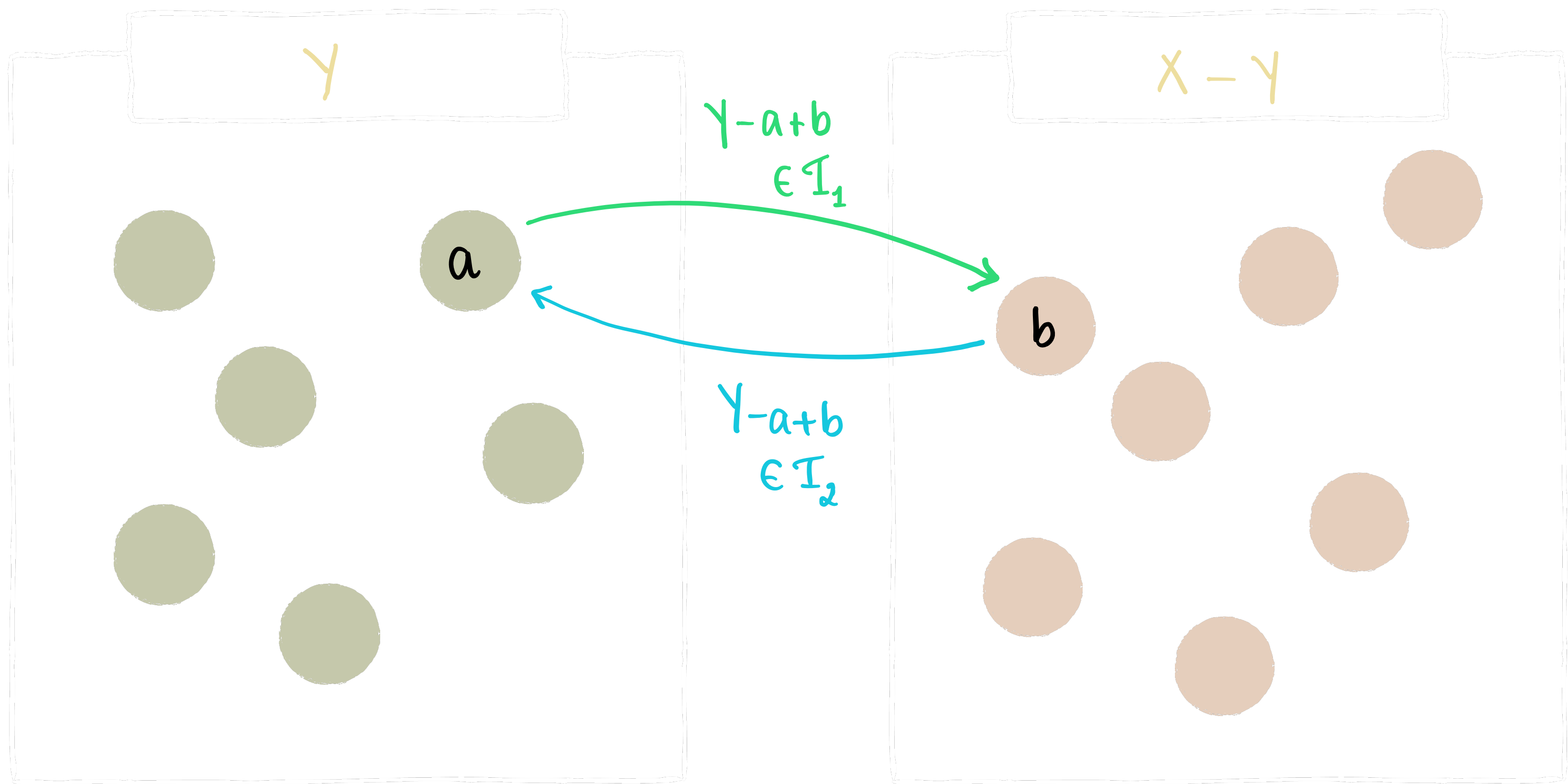
In the current sol^n.   Out of the current sol^n.



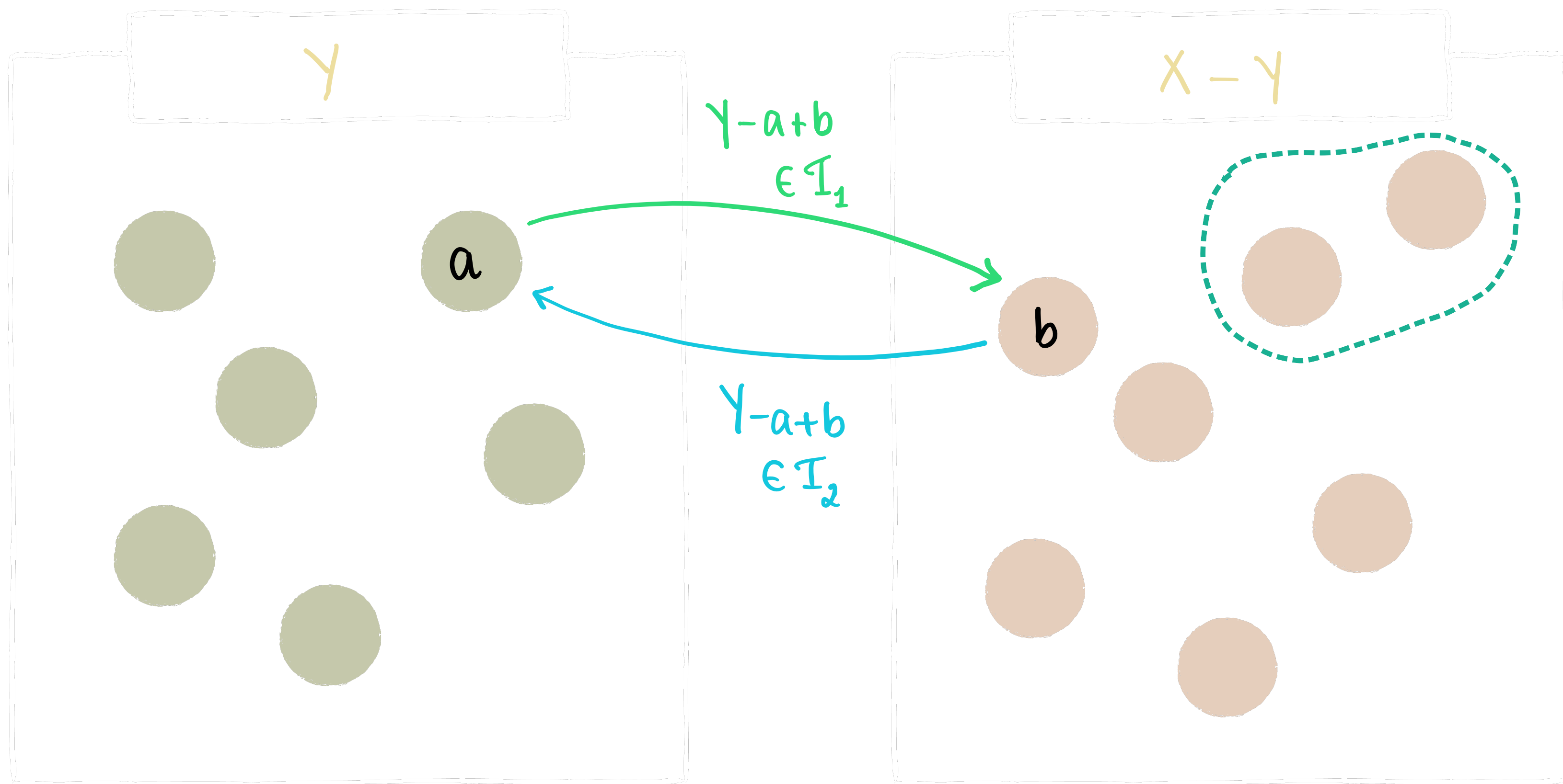

In the current sol^n.
Out of the current sol^n.
Y
$X - Y$
$Y - a + b$
$\in \mathcal{I}_1$
a
b

In the current solⁿ.

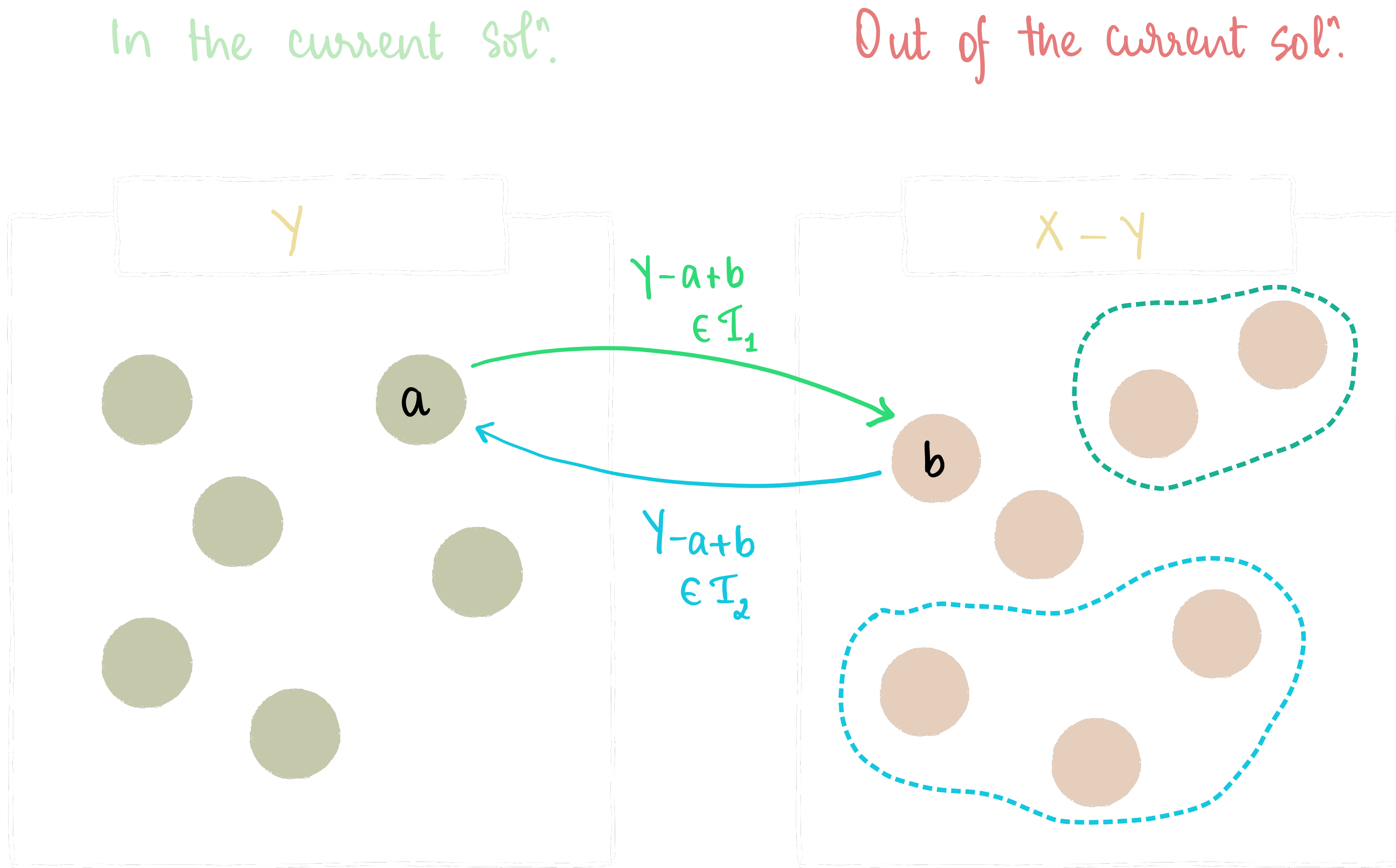
Out of the current solⁿ.



In the current solⁿ. Out of the current solⁿ.



$$Y_1 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_1 \}$$

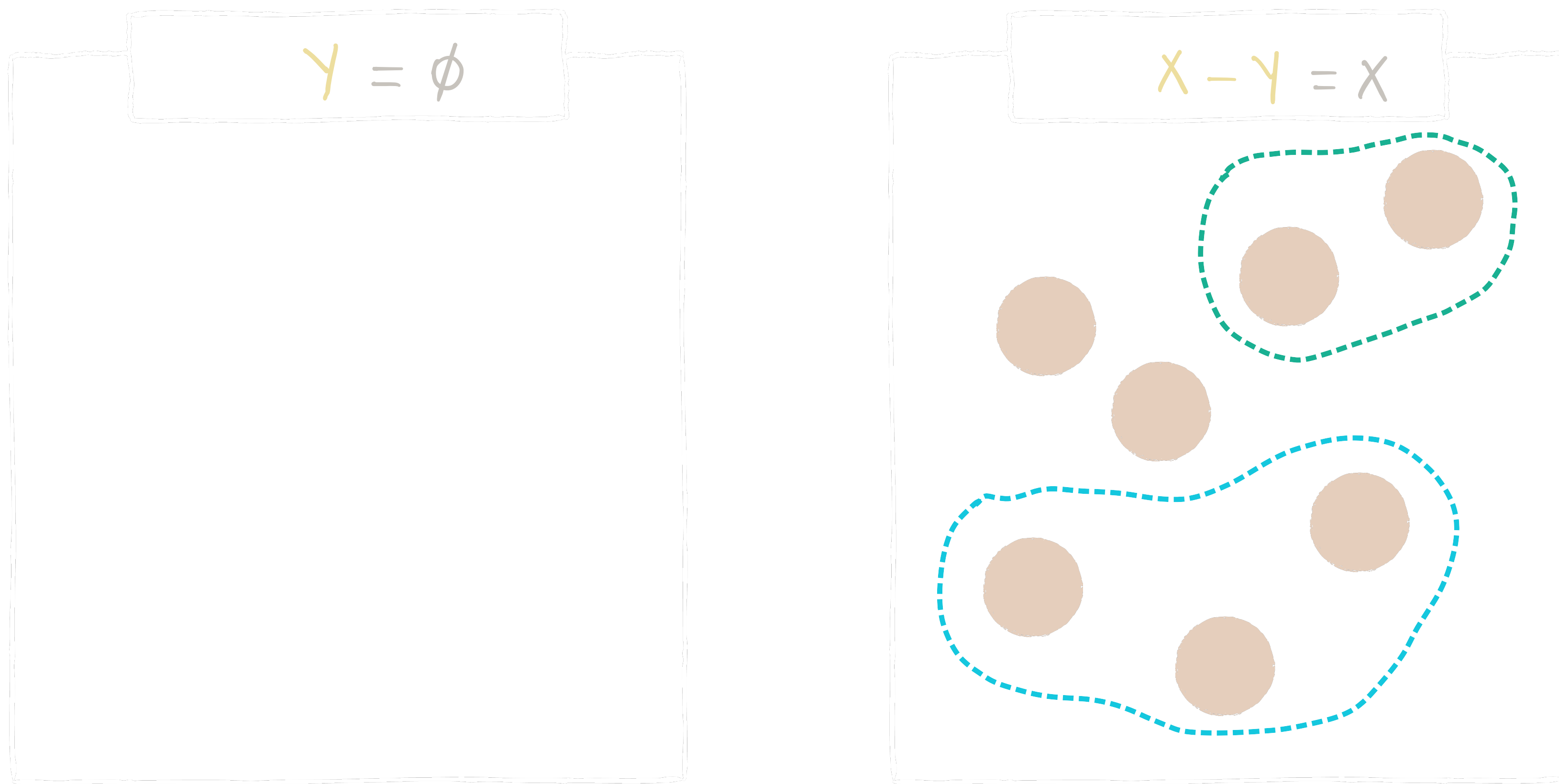


$$Y_1 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_1 \}$$

$$Y_2 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_2 \}$$

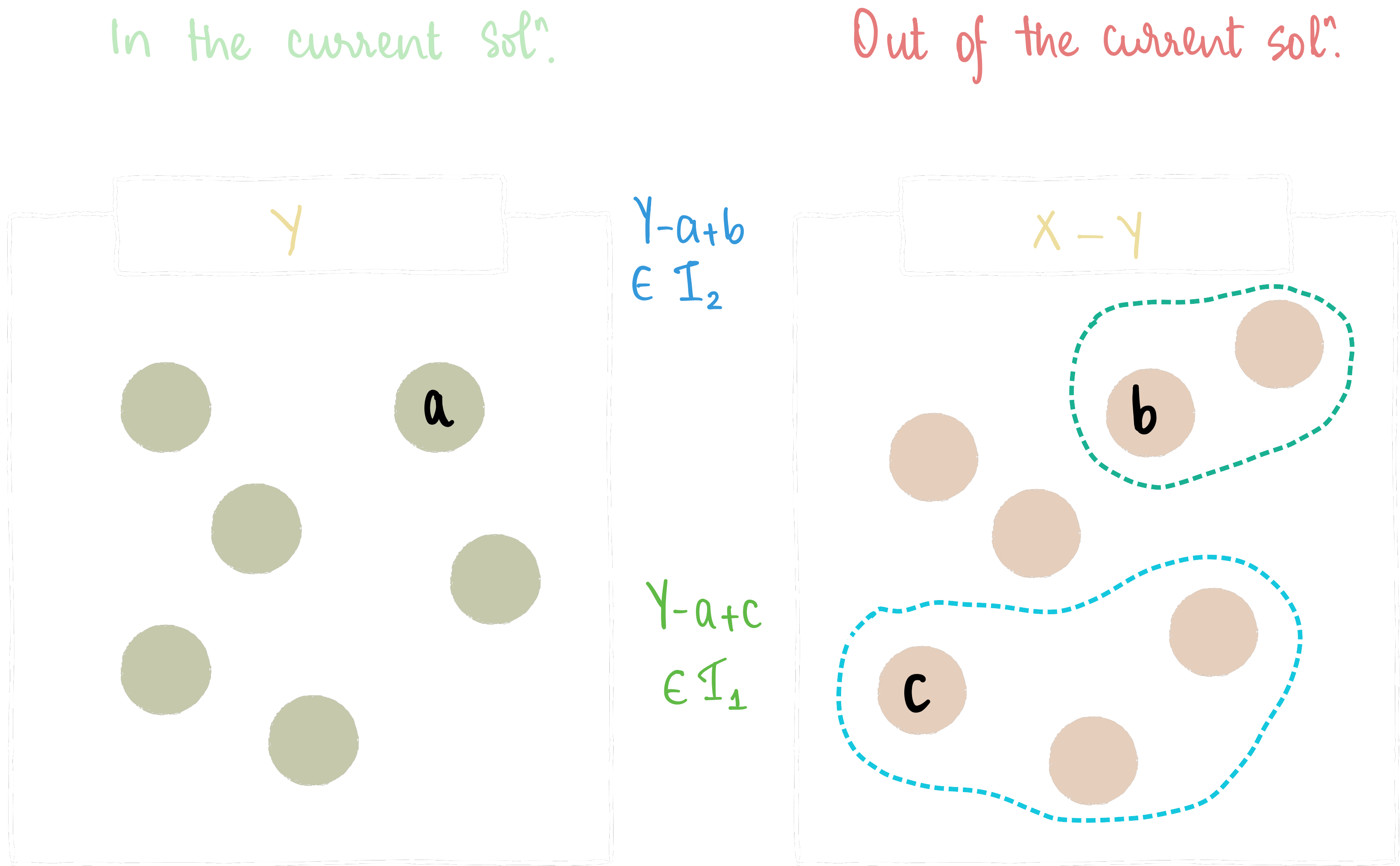
In the current sol^n.

Out of the current sol^n.



$$Y_1 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_1 \}$$

$$Y_2 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_2 \}$$



$$Y_1 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_1 \}$$

$$Y_2 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_2 \}$$

$extras = \{a, b\}$

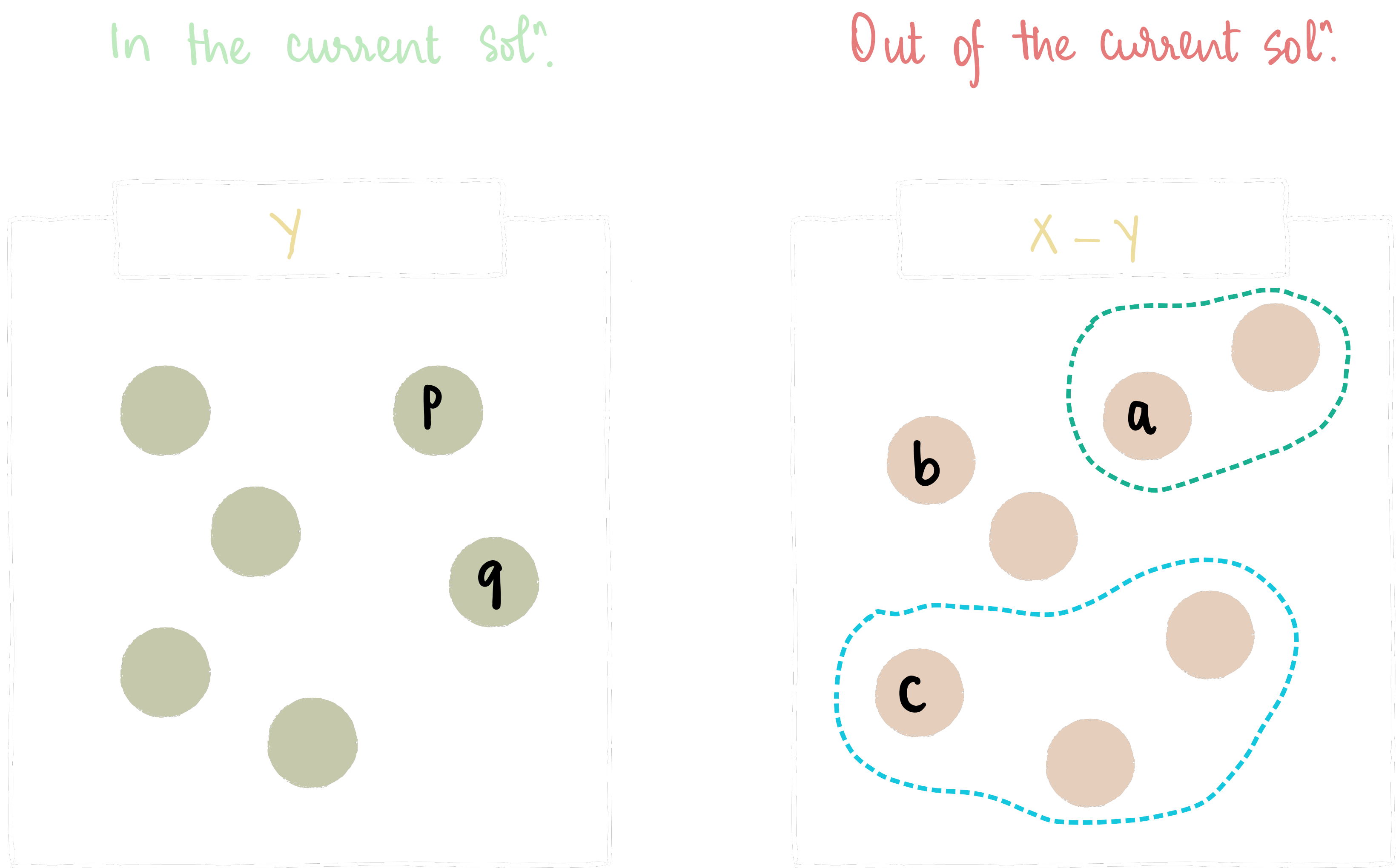
EA

$Y \cup \{b\} \in I_1$

$Y \cup \{c\} - \{a\} \in I_1$

$Y \cup \{c\} \notin I_1$

$\Rightarrow Y \cup \{c\} - \{a\} \cup \{b\} \in I_1$



$$Y_1 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_1 \}$$

$$Y_2 = \{ e \in X - Y \mid Y \cup \{e\} \in \mathcal{I}_2 \}$$

* reasonable

$- \{p, q\} \in I_1 \cap I_2$

More generally,

we have ...

$$\overset{\in Y_1}{b_0}\ a_1\ b_1\ a_2\ b_2\ a_3\ b_3 \cdots a_{k-1}\ b_{k-1}\ a_k\ \overset{\in Y_2}{b_k}$$

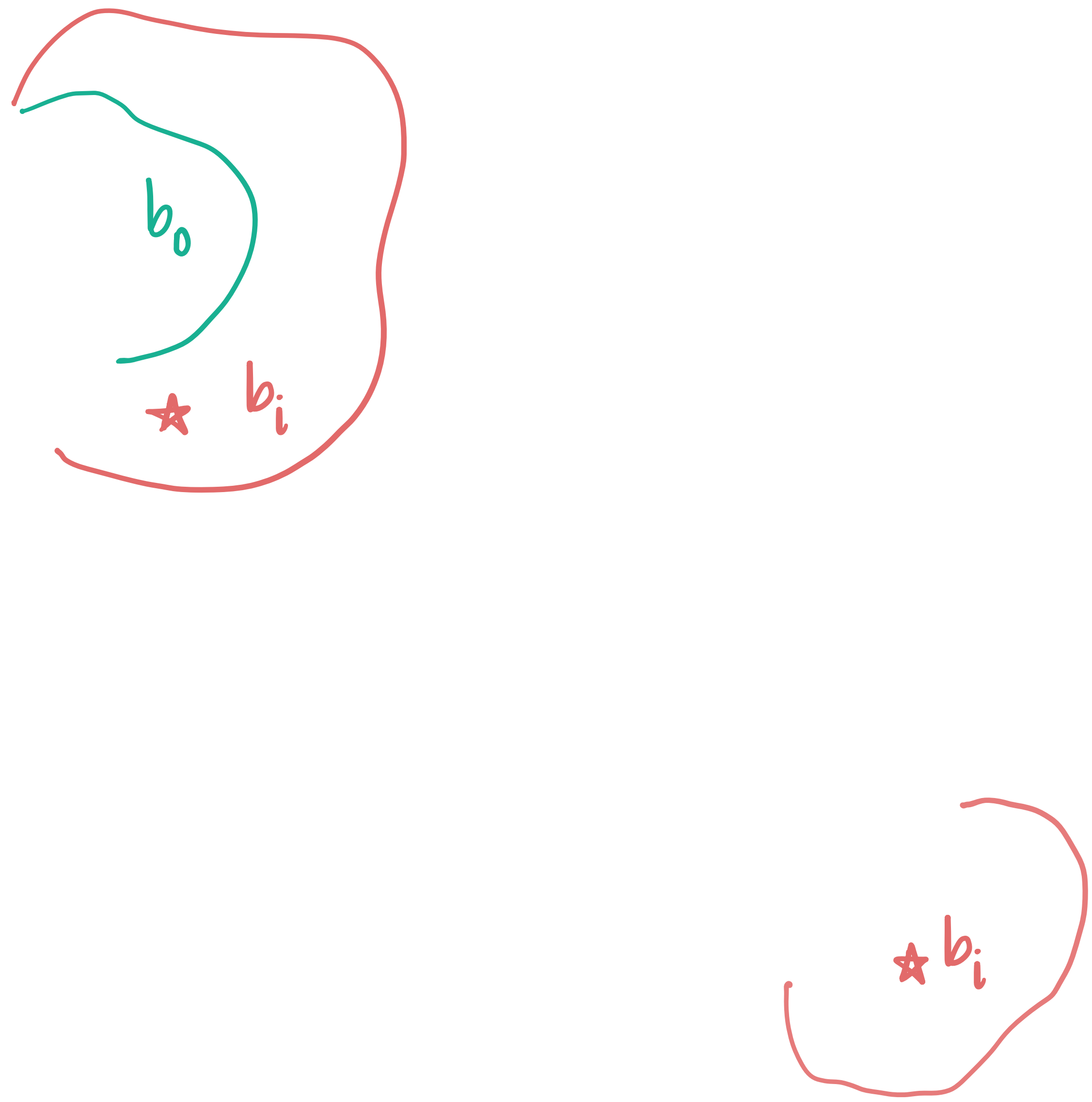
<u>shortest $Y_1 - Y_2$ path.</u>

$a_1, a_2, \ldots, a_k$

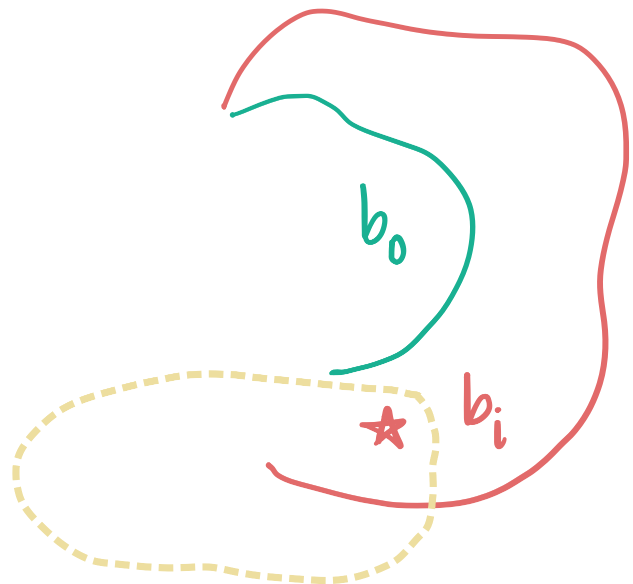
$b_0, b_1, \ldots, b_k$

$\longrightarrow$ INDUCTION ON k

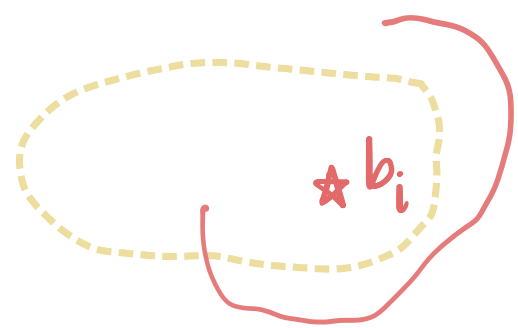
$$\underbrace{\overset{\in Y_1}{b_0}\, a_1\, b_1\, a_2\, b_2\, a_3\, b_3 \cdots a_{k-1}\, b_{k-1}\, a_k\, \overset{\in Y_2}{b_k}}_{\underline{\text{shortest } Y_1 - Y_2 \text{ path.}}}$$



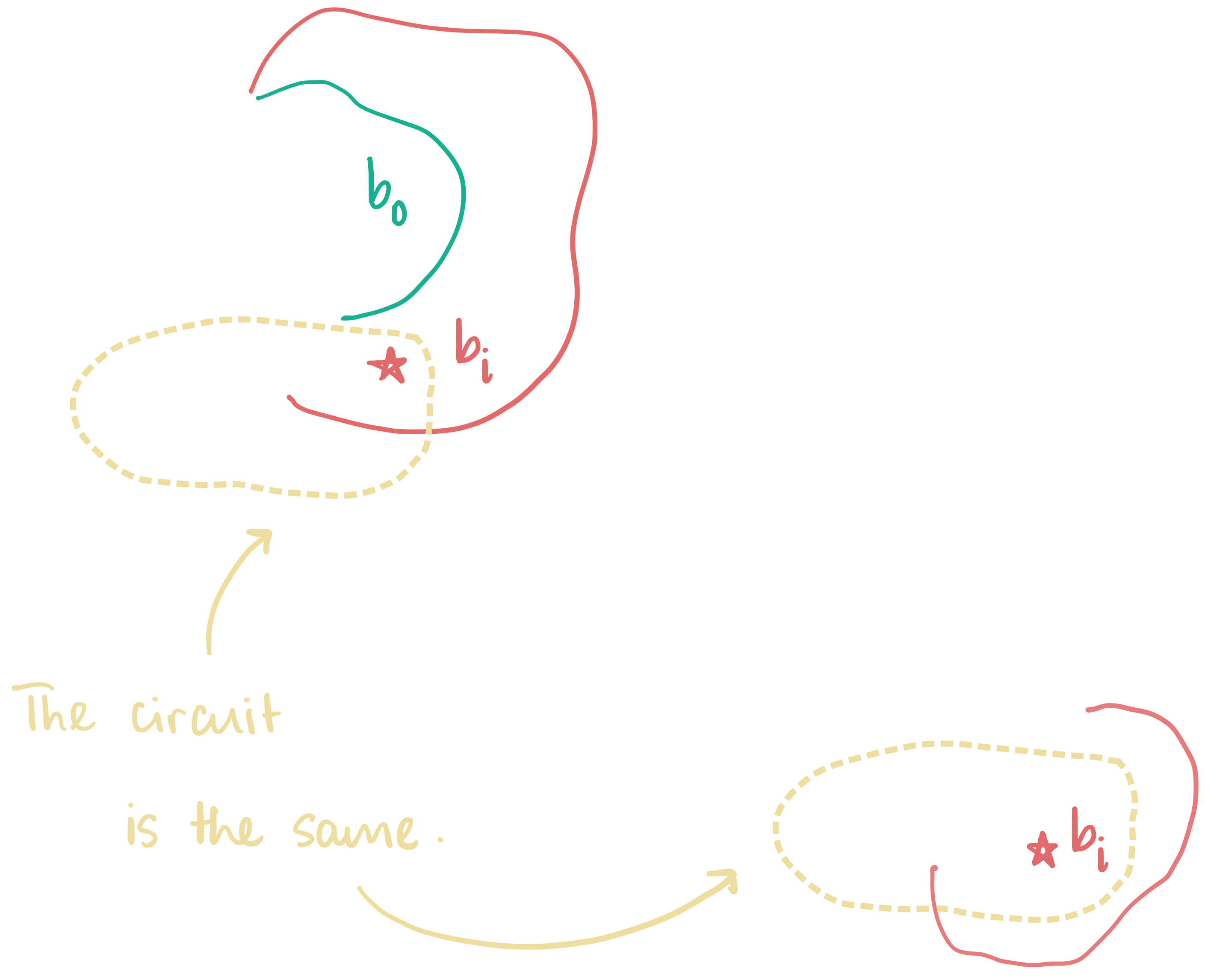
b_0
b_i
b_i



b_0
b_i

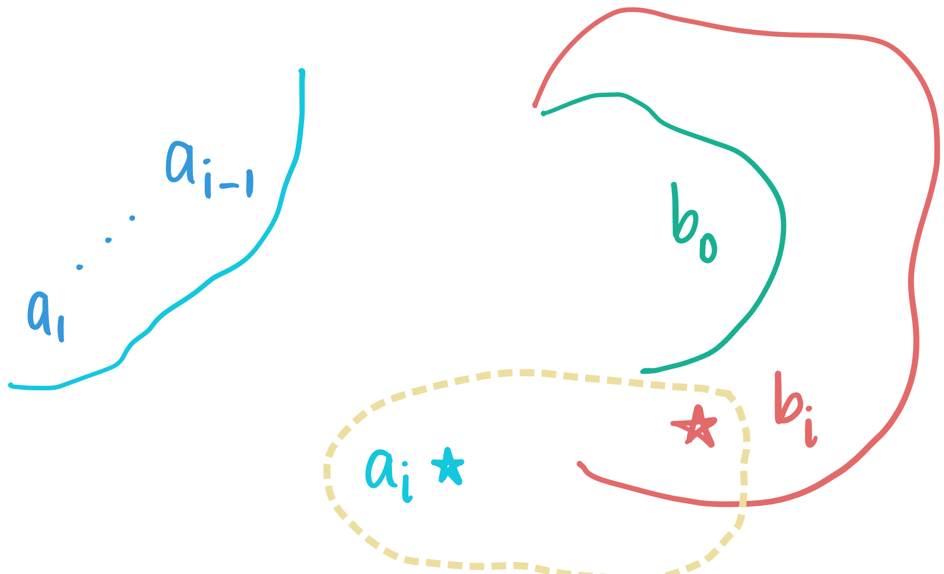


b_i

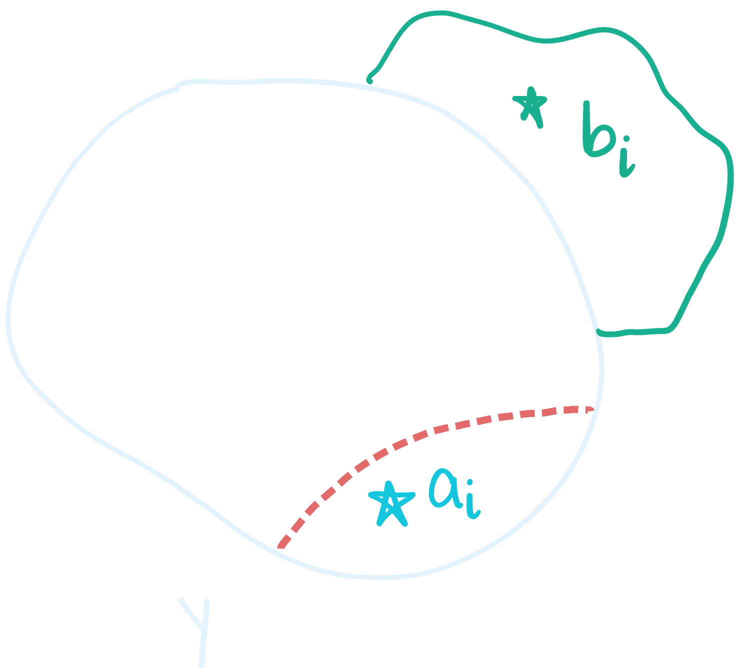


b_0
b_i
The circuit
is the same.
b_i

Claim Let C be the unique circuit in $Y \cup \{b_0, b_i\}$.



$a_i \in C$ & $a_j \notin C \quad \forall\ j \leq k$



$$\underbrace{Y - a_i + b_i \in \mathcal{I}_1}_{a_i b_i \in E}$$

$a_i \in$ any circuit of $Y \cup \{b_i\}$

$\Rightarrow$ $a_i \in$ any circuit of $Y \cup \{b_0, b_i\}$

Induction Step

$$Y^{(i)} := \quad a_1, a_2, \ldots, a_i$$

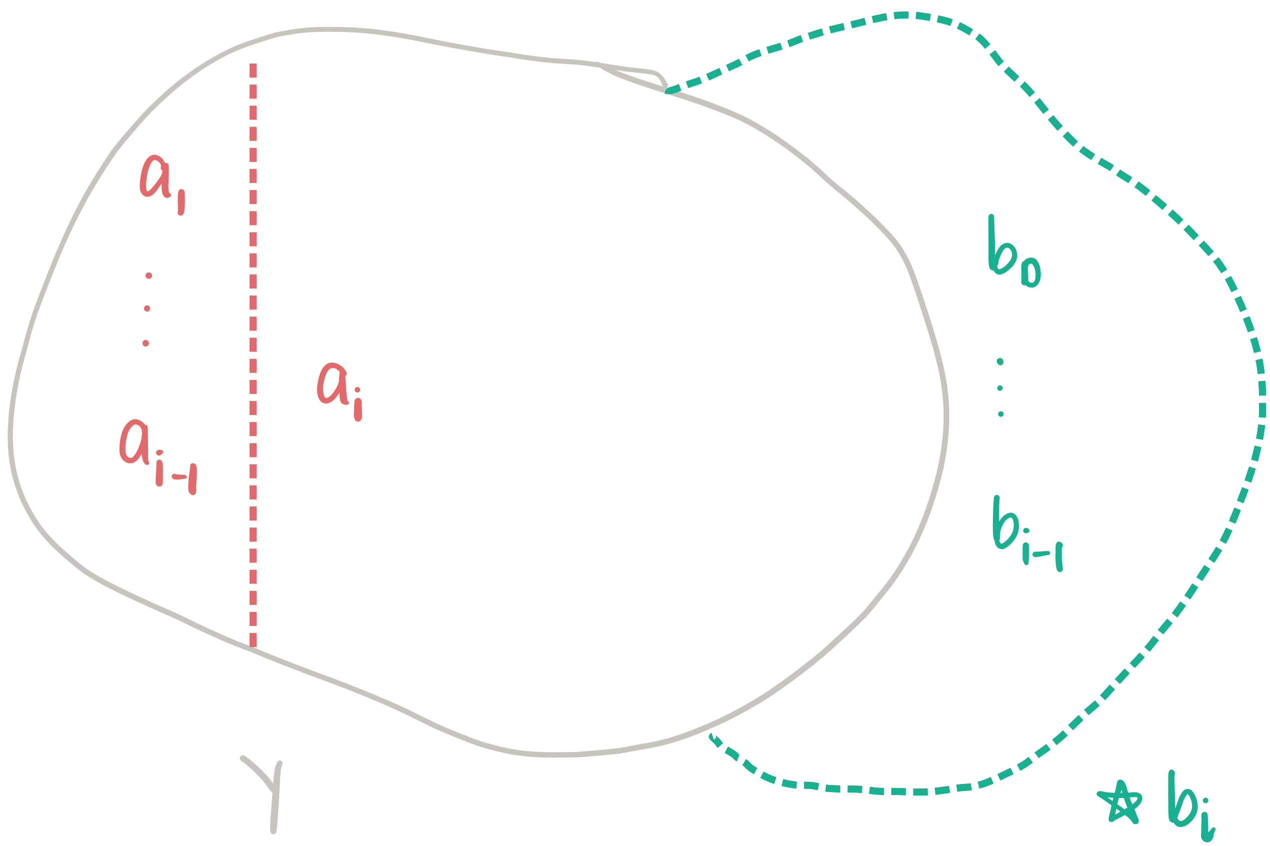
$$b_0, b_1, \ldots, b_i$$

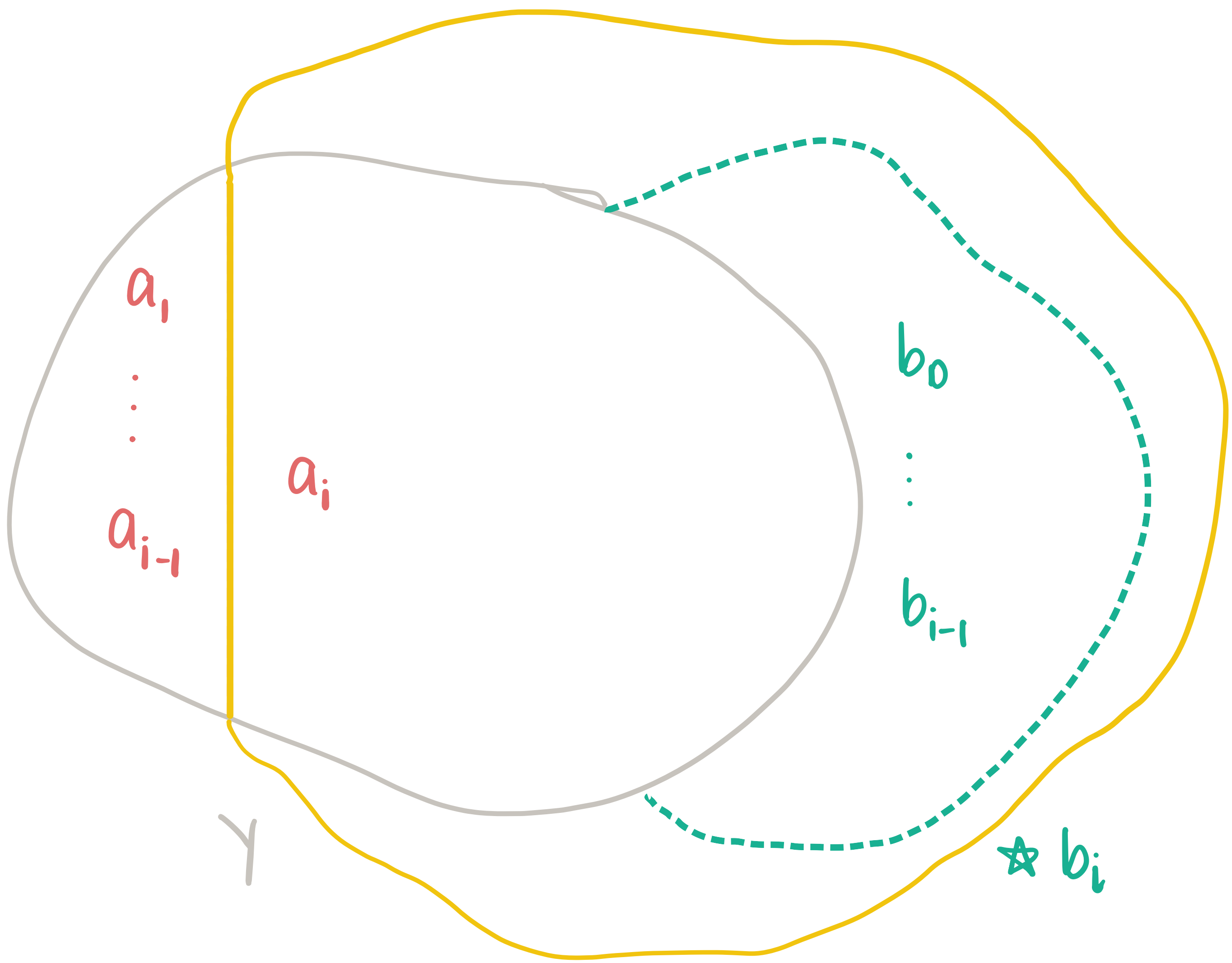
focus here.

I.H. $\quad a_1, a_2, \ldots, a_{i-1}$

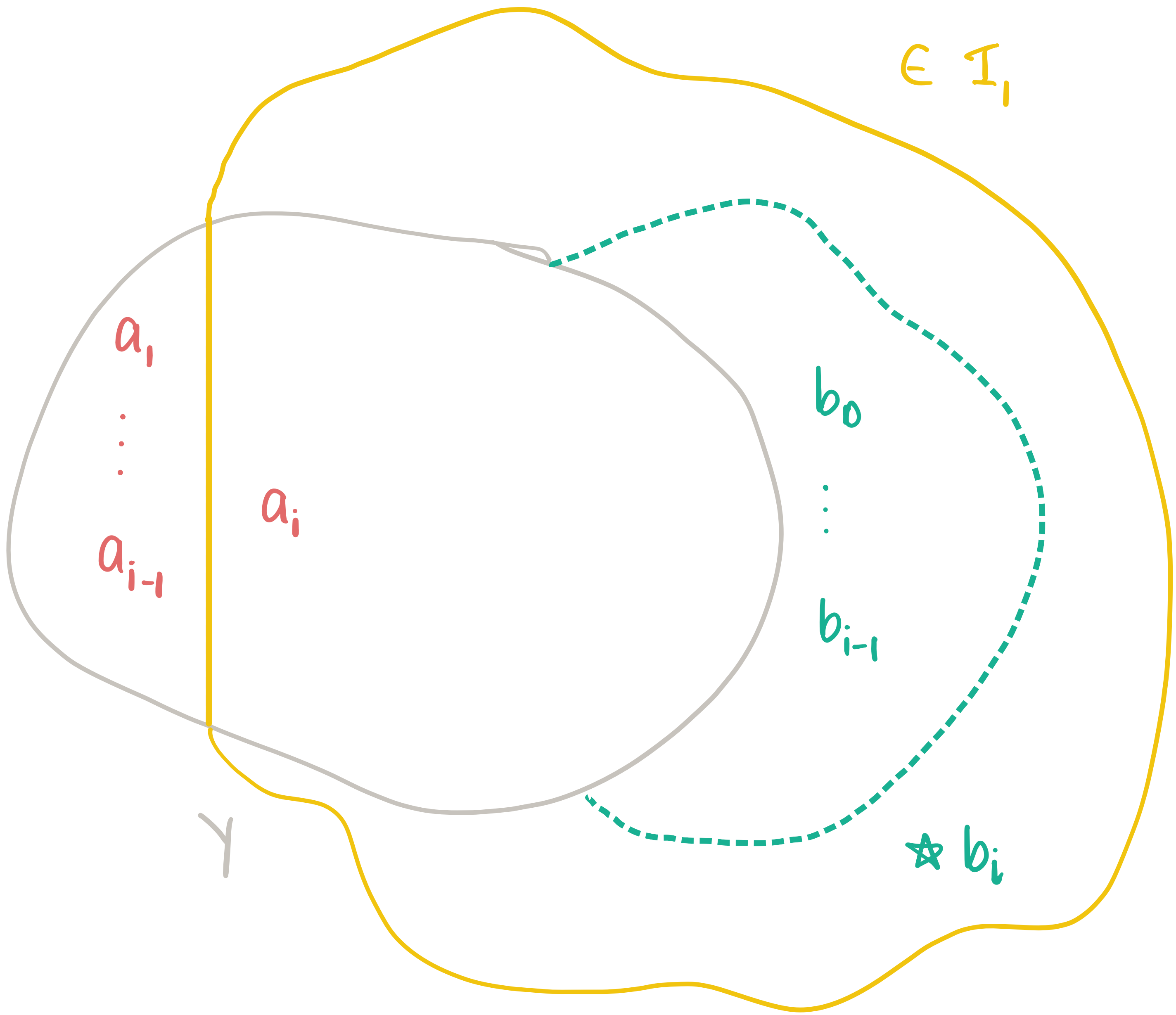
$$b_0, b_1, \ldots, b_{i-1}$$

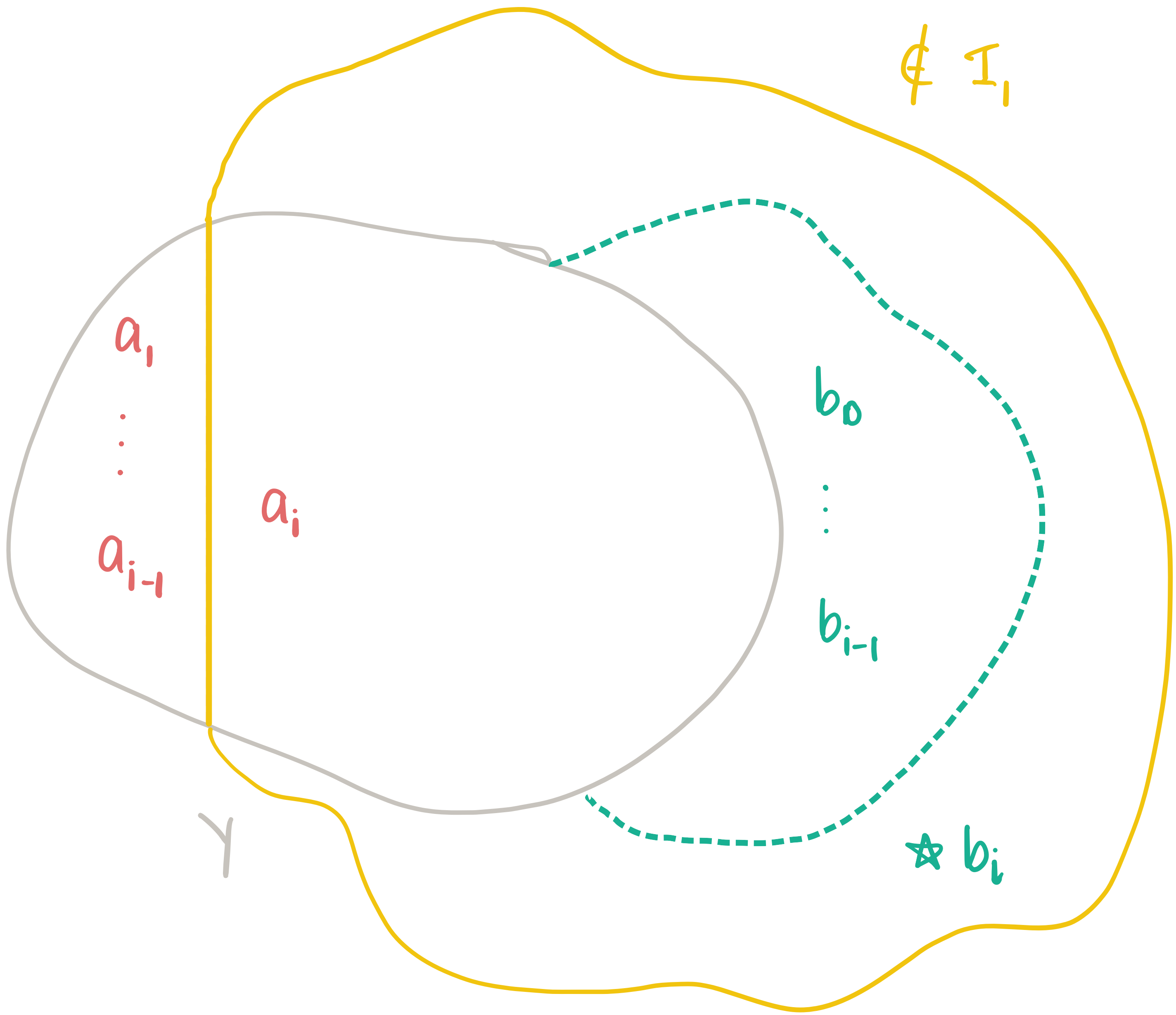
$Y^{(i-1)}$


a_1
a_{i-1}
a_i
b_0
b_{i-1}
$\star\ b_i$
Y

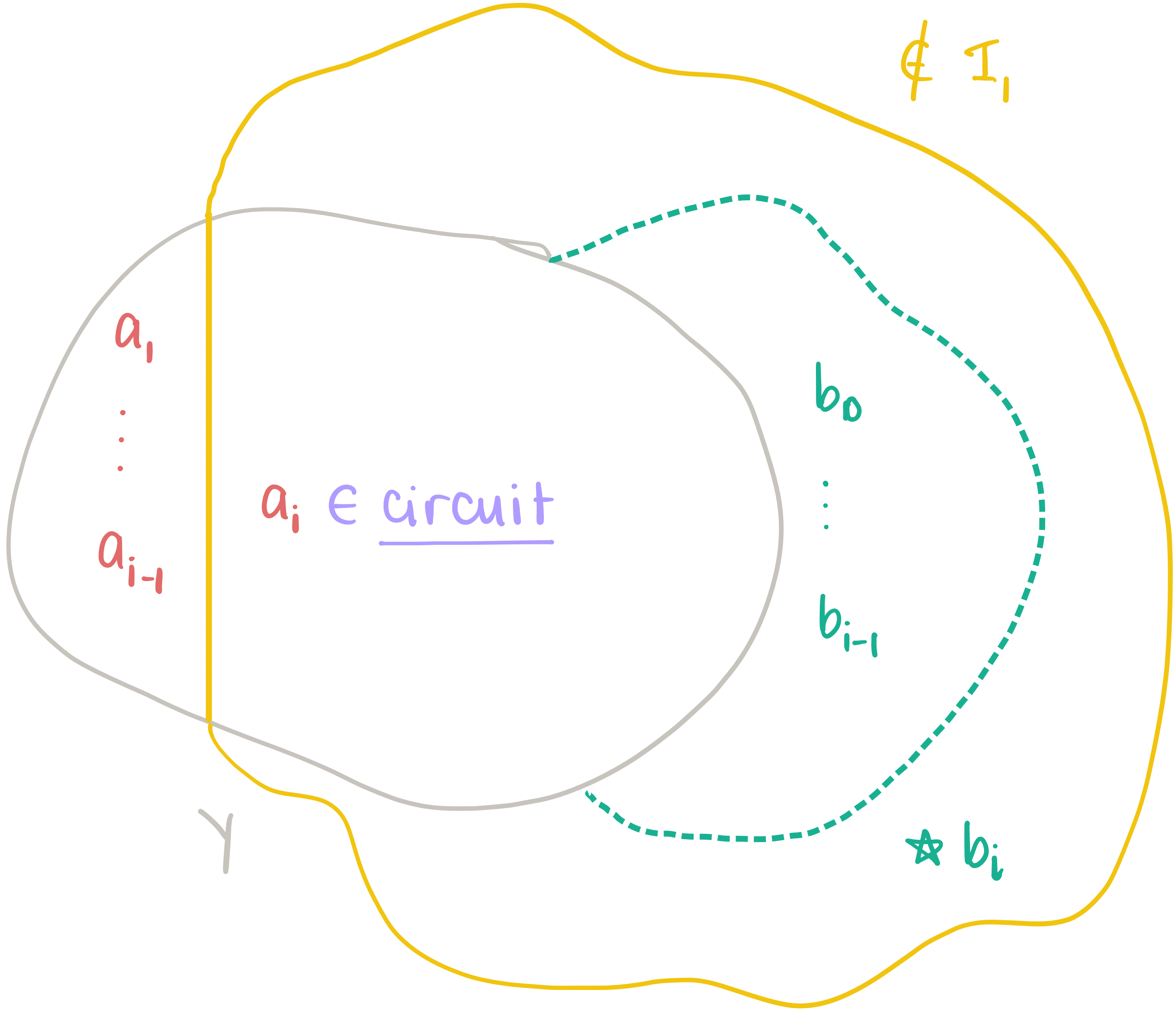


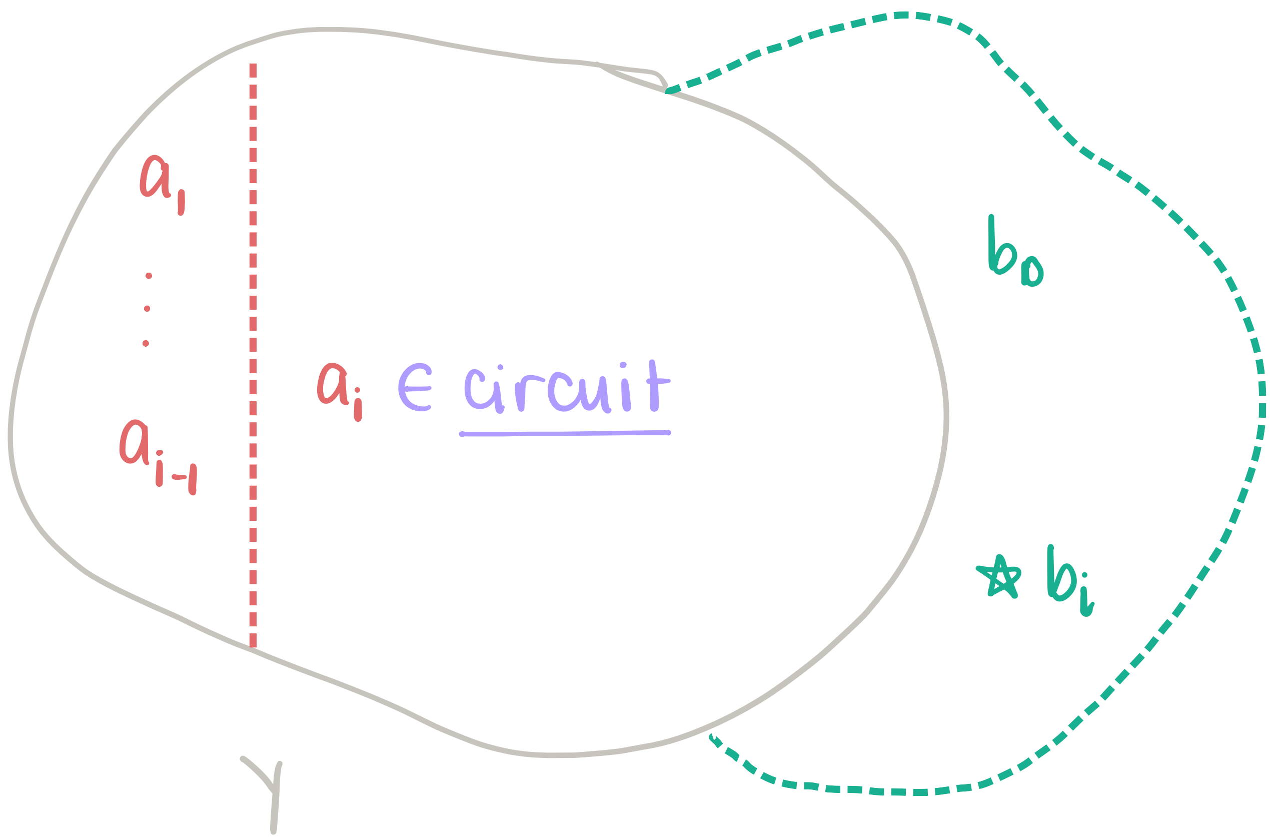
a_1
$\vdots$
a_{i-1}
a_i
b_0
$\vdots$
b_{i-1}
$\star\ b_i$
Y


$\in \mathcal{I}_1$
a_1
a_{i-1}
a_i
b_0
b_{i-1}
$\star\ b_i$
Y



$\notin I_1$
a_1
a_{i-1}
a_i
b_0
b_{i-1}
b_i
Y


$\notin I_1$
a_1
$\vdots$
a_{i-1}
$a_i \in$ circuit
b_0
$\vdots$
b_{i-1}
$\star\ b_i$
Y



a_1
$\vdots$
a_{i-1}
$a_i \in$ circuit
b_0
$\star\ b_i$
Y