Initial Value and Final Value Theorems

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Author: Kumar Anurag

1. Introduction:-

- **Initial Value of a Function:** This is the value of a function at $t = 0^+$.

- **Final Value of a Function:** This is the value of a function at $t = \infty$. This is also known as steady state value.

- **Example:**

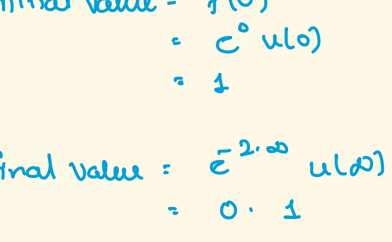
$$f(t) = e^{-2t} u(t)$$

Solution:

→ Initial Value $= f(0) = e^{0} u(0) = 1$

→ final value $= e^{-2 \cdot \infty} u(\infty) = 0 \cdot 1 = 0$

Now, let's plot:-



- **Example:**

$$F(s) = \frac{1}{s+2}$$

Solution: In order to find out initial and final value of given function, first we need to take inverse Laplace transform to obtain function in time domain.

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s+2}\right]$$

$$f(t) = e^{-2t} u(t)$$

Now, we already know initial and final value of this function as we already solved in previous example.

✱ Let's talk about one PROBLEM:

Have you noticed a problem in previous example. For calculating initial and final value first we took to time domain from s-domain.

But I don't wanna do that. What if I say, I wanna calculate initial and final value in s-domain.

Now, that's when INITIAL VALUE THEOREM & FINAL VALUE THEOREM comes into the picture.

2. Initial Value Theorem:-

The initial value theorem is a property of Laplace Transform by which we can find out the initial value of a function in the s-domain.

If, $f(t) \rightleftharpoons F(s)$

then initial value theorem says:

$$\boxed{\lim_{t \to 0^+} f(t) = f(0^+) = \lim_{s \to \infty} s \cdot F(s)}$$

Think of it like, time (t) and frequency (s) are reciprocal of each other, so when t approaches to 0 in time domain, the s approaches to ∞ in s-domain.

Example: $F(s) = \frac{1}{s+2}$; Find the value of this function at $t = 0^+$

Solution: From initial value theorem, we know that:

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} s \cdot F(s) = \lim_{s \to \infty} s \cdot \frac{1}{(s+2)} = \lim_{s \to \infty} \frac{1}{1 + \left(\frac{2}{s}\right)} = \underline{\underline{1}}$$

↳ In previous examples, we got this same result when we calculated in time-domain.

* Conditions for Initial Value Theorem:-

1. Applicable only when $f(t) = 0$, $t < 0$.
2. $f(t)$ must not contain any impulse or discontinuities at $t = 0^+$

3. Final Value Theorem:-

The final value theorem is the property of Laplace Transform by which we can find out the final value of a function in the s-domain.

If, $f(t) \rightleftharpoons F(s)$

then by final value theorem:

$$\boxed{\lim_{t \to \infty} f(t) = f(\infty) = \lim_{s \to 0} s \cdot F(s)}$$

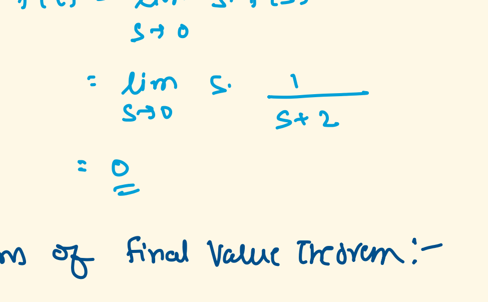
Example: $F(s) = \frac{1}{s+2}$; find the value of this function at $t = \infty$.

Solution: From final value theorem, we have:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s \cdot F(s) = \lim_{s \to 0} s \cdot \frac{1}{s+2} = \underline{\underline{0}}$$

* Conditions of Final Value Theorem:-

1. All the poles of F(s) must lie in the left half plane (LHP).



2. F(s) must not have more than 1 pole at origin.

Example: Given $F(s) = \frac{1}{s-2}$, find the steady state value.

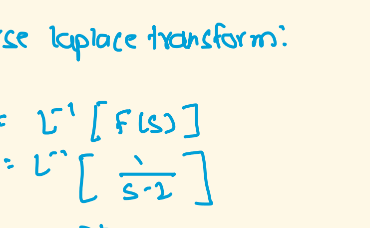
Solution: Pole of F(s) is at $s = 2$, which is in the right half plane (RHP). So, final value theorem is not applicable.

* **Reason:** Why final value theorem is not applicable?

Taking the inverse laplace transform:

$$f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s-2}\right] = e^{2t} \cdot u(t)$$

Plotting:



unbounded function

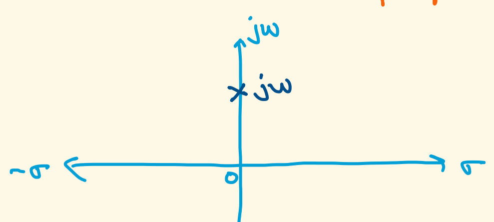
↳ we can see the final value is not defined

and that's why final value theorem is not applicable

Example: If $F(s) = \frac{\omega}{s^2 + \omega^2}$, find out $\lim_{t \to \infty} f(t)$.

Solution: Poles: $s = \pm j\omega$

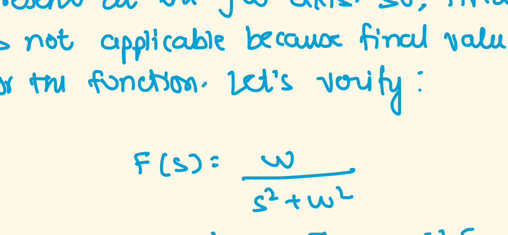
↳ because we are working in complex plane.



Since it has a pair of complex conjugate poles present at the jω axis. So, final value theorem is not applicable because final value is not defined for the function. Let's verify:

$$F(s) = \frac{\omega}{s^2 + \omega^2}$$

$$f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{\omega}{s^2 + \omega^2}\right] = \sin(\omega t)$$



↳ this is an oscillatory function which oscillates between 1 and -1 and it's final value is not defined. So, final value theorem is not applicable.

4. Final Value Theorem (Solved Problem):-

Problem: The system $G(s) = \frac{0.8}{s^2 + s - 2}$ is subjected to unit step input. The steady state output of the system is:

(a) 0.8 (b) 0 (c) -0.4 (d) Unbounded

[GATE IN 1999]

Solution: **Given:** $G(s) = \frac{0.8}{s^2 + s - 2}$

We are also given that, G(s) is subjected to unit step input, which means:

↳ unit step input in laplace domain

Therefore, $C(s) = G(s) \cdot R(s) = G(s) \cdot \frac{1}{s} = \frac{1}{s} \cdot \frac{0.8}{s^2 + s - 2}$

Applying final value theorem on C(s):

$$\lim_{s \to 0} s \cdot C(s) = \cancel{s} \cdot \frac{1}{\cancel{s}} \cdot \frac{0.8}{s^2 + s - 2} = \frac{0.8}{s^2 + s - 2} = \frac{0.8}{(s-1)(s+2)}$$

Poles: $s = 1, -2$

Since this pole is on right hand plane, so final value theorem is not applicable. final value of this function is not defined.

Since this pole lies in Right Hand plane (RHP). So system is UNSTABLE

So, option (d) unbounded is correct ✓

5. References:

1. Neso Academy

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