Differentiation & Integration Basics

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① Integrate: $\int \frac{dx}{\sqrt{1-x^2}}$ Author: Sameer Bhardwaj

METHOD 1

Trig Substitution

$x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$\theta = \sin^{-1}\sqrt{x}$

$d\theta = \frac{dx}{2\sqrt{x}}$

$\int \frac{dx}{\sqrt{1-x^2}}$

Trig 2: $v = \sqrt{1-x^2} \Rightarrow v^2 = 1 - x^2$

$2v\,dv + 0 = 2x\,dx$

$v\,dv = -x\,dx$

$\frac{v\,dv}{-x} = dx$

$\frac{v\,dv}{-\sqrt{1-v^2}} = dx$

$\int \frac{dx}{\sqrt{1-x^2}}$

Trig 3 Euler Substitution

$v = \sqrt{1-x^2} = \sqrt{1+x^2}$

$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}+x}\right| + C$

METHOD 2

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$\int \frac{dx}{\sqrt{1-x^2}}$

$\int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$

$\int \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}}$

$\int \frac{\cos\theta\, d\theta}{\cos\theta}$

$\int \sec\theta\, d\theta$

$\ln|\sec\theta + \tan\theta| + C$

$\ln\left|\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right| + C$

$\ln\left|\frac{1+\sin\theta}{\cos\theta}\right| + C$

$\ln\left|\frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}}\right| + C$

$\ln\left|\frac{1+\sin\theta}{\sqrt{\cos^2\theta}}\right| + C$

$\ln\left|\frac{1+\sin\theta}{\cos\theta}\right| + C$

$\ln\left|\sec\theta + \frac{\sin\theta}{\cos\theta}\right| + C$

$\ln\left|\sec\theta + \tan\theta\right| + C$

Trig Sub:

① $\int \sin\theta\, d\theta = \ln|\sin\theta - \cos\theta| + C$

$\ln\left|\frac{1}{\sec\theta}\right| + C$

② $\int \frac{1}{\sec\theta}\, d\theta = \ln|\sec\theta| + C$

③ $\sec\theta = \sqrt{1+\tan^2\theta}$

④ $\sec\theta = \frac{1}{\cos\theta}$

⑤ $\sec\theta = \frac{1}{\sqrt{1-\sin^2\theta}}$

METHOD 3

$\int \frac{dx}{\sqrt{1-x^2}}$

$u = \sqrt{1-x^2}$

$u^2 = 1 - x^2$

$2u\,du = -2x\,dx$

$du = \frac{-x\,dx}{u}$

$dx = \frac{-u\,du}{x}$

$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dx}{u}$

$\int \frac{-u\,du}{x(\sqrt{1-x^2})}$

$\int \left[\frac{-u\,du}{x}\right] \frac{dx}{u}$

$\int \frac{-1}{x}\, dx$

$-\ln|x| + C$

$-\ln\left|\frac{\sqrt{1-x^2}}{x}\right| + C$

$-\ln\left|\frac{\sqrt{1-x^2}}{x}\right| + C$

② Show: $\int \frac{dx}{\sqrt{1-x^2}} = \ln\left|\sin^{-1}(x)\right| + C$

Solution

METHOD 1

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$\theta = \sin^{-1} x$

$d\theta = \frac{dx}{\sqrt{1-x^2}}$

$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dx}{\sqrt{1-\sin^2\theta}}$

$\int \frac{dx}{\sqrt{\cos^2\theta}}$

$\int \frac{dx}{\cos\theta}$

$\int \sec\theta\, d\theta$

$\ln|\sec\theta + \tan\theta| + C$

$\ln\left|\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right| + C$

$\ln\left|\frac{1+\sin\theta}{\cos\theta}\right| + C$

$\ln\left|\frac{1+\sin\theta}{\sqrt{\cos^2\theta}}\right| + C$

$\ln\left|\frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}}\right| + C$

$\ln\left|\frac{1+x}{\sqrt{1-x^2}}\right| + C$

METHOD 2

$\theta = \sin^{-1} x$

$x^2 + \cos^2\theta = 1$ $\quad \theta = \sin^{-1} x \rightarrow ③$

$\cos^2\theta = 1 - x^2$

$\cos\theta = \sqrt{1-x^2}$

$\sin\theta = x$

$d\theta = \frac{\cos\theta\, d\theta}{\cos\theta} \quad \cdots ①$

$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{d\theta}{\sqrt{1-\sin^2\theta}}$

$\int \frac{d\theta}{\sqrt{\cos^2\theta}}$

$\int \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}} \quad \frac{\cos\theta\, d\theta}{\cos\theta}$

$\int \frac{d\theta}{\cos\theta}$

$\int \sec\theta\, d\theta$

$\ln\left|\sec\theta + \tan\theta\right| + C$

$\ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C$

$\ln\left|\frac{1+x}{\sqrt{1-x^2}}\right| + C$

DERIVATIVE RULES

EXPONENTIAL FUNCTIONS

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^u) = e^u\, du(x)$
- $\frac{d}{dx}(e^{g(x)}) = e^{g(x)}\, g'(x)$
- $\frac{d}{dx}(e^{f(x)}) = e^{f(x)}\, de(x)\, g'(x)$

LOGARITHMIC FUNCTIONS

- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(\ln(g(x))) = \frac{1}{g(x)}\, g'(x)$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln(a)}$
- $\frac{d}{dx}(\log_a g(x)) = \frac{1}{g(x)\ln(a)}\, g'(x)$

TRIGONOMETRIC FUNCTIONS

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\csc^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\csc x) = -\csc x \cot x$

INVERSE TRIGONOMETRIC FUNCTIONS

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \quad -1 \le x \le 1$
- $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \quad -1 \le x \le 1$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2} \quad -\infty \le x \le \infty$
- $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}} \quad -\infty \le x \le \infty$

BASIC INTEGRATION RULES: