Author: Trevor Anning

▶ Problem #1:

The figure shown models a crankshaft and piston. The crankshaft has a radius, R, and is rotating at a constant angular velocity, $\dot{\theta}$, such that $\dot{\theta} = 2\pi$. The piston rod's length, L, is determined by the linear velocity of the piston, $\dot{x}$, in terms of the crankshaft angular velocity, $\dot{\theta}$, and time, t.

Solution:

From the figure:

$x = R\cos(90-\theta) + L\cos\phi$
$x = R\sin\theta + L\cos\phi$ ... (1)

From the above triangle:

$R\cos(90-\theta) = L\sin\phi$
$R\sin\theta = L\sin\phi$
$R^2\sin^2\theta = L^2(1-\cos^2\phi)$
$R^2\sin^2\theta = L^2(1-\cos^2\phi)$
$\frac{R^2}{L^2}\sin^2\theta = 1-\cos^2\phi$
$\sin\phi = \frac{R}{L}\sin\theta$

$\cos^2\phi = 1 - \frac{R^2}{L^2}\sin^2\theta$

$\cos\phi = \sqrt{1-\frac{R^2}{L^2}\sin^2\theta}$ ... (2)

From (1) and (2):

$x = R\sin\theta + L\sqrt{1-\frac{R^2}{L^2}\sin^2\theta}$

$x = R\left[\sin\theta + \frac{L}{R}\sqrt{1-\frac{R^2}{L^2}\sin^2\theta}\right]$

$x = R\left[\sin\theta + \frac{L}{R}\sqrt{1-\frac{R^2}{L^2}(1-\cos^2\theta)}\right]$

$x = R\left[\sin\theta + \sqrt{\frac{L^2}{R^2}-1+\cos^2\theta}\right]$

velocity of the piston, $v = \frac{dx}{dt}$

$v = R\left[\cos\theta\frac{d\theta}{dt} + \frac{1}{2\sqrt{\frac{L^2}{R^2}-1+\cos^2\theta}}(-2\cos\theta\sin\theta)\frac{d\theta}{dt}\right]$

$v = R\left[\cos\theta\frac{d\theta}{dt} - \frac{\sin\theta\cos\theta}{\sqrt{\frac{L^2}{R^2}-1+\cos^2\theta}}\frac{d\theta}{dt}\right]$

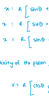
$v = R\left(\frac{d\theta}{dt}\right)\left[\cos\theta - \frac{\sin\theta\cos\theta}{\sqrt{\frac{L^2}{R^2}-1+\cos^2\theta}}\right]$

$v = R(\dot{\theta})\left[\cos\theta - \frac{\sin\theta\cos\theta}{\sqrt{\frac{L^2}{R^2}-1+\cos^2\theta}}\right]$

$\left[\because \dot{\theta} = \frac{d\theta}{dt}\right]$

▶ Problem #2:

The 2.5 kg slender rod, are pin connected and rotated from rest, as the position θ = 0°. If the spring has an unstretched length of 0.5 m, determine the angular velocity of rod BC when the system is at the position θ = 0°. Neglect the mass of the collar at C.



Solution:

Initial Condition: (θ=0°)

The stretched length of the spring is:

$x_1 = AB\cos\theta + BC\cos\theta$ → [Pythagorean theorem]

$= 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right) = 2$

$x_1 = 2.0\,m$

Initial Rotational Energy of the system:

$(E)_1 = mg\bar{y}_1 + mg\bar{y}_2 + \frac{1}{2}I\omega^2$

$= 2mg + \frac{1}{2}I\omega^2$

$= 2(2.5)(9.81)\left(\frac{1}{2}\sin 0°\right) + \frac{1}{2}(2.5)(2.0)^2$

$= 24.525\left(\frac{1}{2}\cdot 0\right) + 5(0)^2$

$= (24.525)(0)+0$

$= 0\ J$

Initial kinetic Energy of the system:

$(E)_2 = 0$ [because rods are at rest and spring is unstretched]

Final Condition: (θ=0°)

Stretched length of spring

$x_2 = AC = BC = 2$

$\Delta x = 2 - 0.5$

$= 1.5\,m$

Final Rotational Energy of the system:

$(E)_3 = \cancel{mg\bar{y}_1} + \cancel{mg\bar{y}_2} + \frac{1}{2}k\Delta x^2$ [∵ $\bar{y}=0$]

$= \frac{1}{2}(50)(1.5)^2$

$= 1.5(56.25)$

$= 84.375\ J$

Moment of inertia of rod about the end of rod:

$I = \frac{1}{12}ml^2 + ml^2$ → [Parallel Axis Theorem]

$= \frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2$

$= \frac{1}{12}ml^2 + \frac{1}{4}ml^2 = \frac{1}{3}ml^2 = \frac{1}{3}ml^2$

Final kinetic Energy of the system:

$(E)_4 = \frac{1}{2}I\omega_{AB}^2 + \frac{1}{2}I\omega_{BC}^2$

$= \frac{1}{2}I\omega_{AB}^2 + \frac{1}{2}I\omega_{BC}^2$ [$\because \omega_{AB} = \omega_{BC}$]

$= \frac{1}{2}\omega^2 + \frac{1}{2}\omega^2$

$= \frac{1}{2}ml^2\omega^2 + \frac{1}{2}ml^2\omega^2$

$= \frac{1}{2}(2.5)(2.0)^2\omega^2 + \frac{1}{2}(2.5)(2.0)^2\omega^2$

$= 10\omega^2$

Principle of Conservation of Energy:

$(E)_1 + (E)_2 = (E)_3 + (E)_4$

$24.525 + 0 = 84.375 + 10\omega^2$

$10\omega^2 = 24.525 - 84.375$

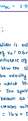
$10\omega^2 = -59.85$

$\omega = \sqrt{-5.985}\ rad/s$

▶ Problem #3:

The 2.5 kg slider is released from the position shown with a velocity of 3.6 m/s on the inclined rail and slides under the influence of gravity and friction. The coefficient of friction for the slider and the rail is 0.15. Calculate the velocity of the slider as it passes the position shown. The spring is compressed a distance of 100 mm. The spring action is compressive, resistance of and is known as "hardening" spring, and its different increases with deflection as given by:

$C = 1.4x + 0.6x^2$



Solution:

$\therefore \Sigma(F_{net})_y = 0$

$+F_N - mg\cos 30° = 0$

$F_N = mg\cos 30°$

$= (2.5)(9.81)\left(\frac{\sqrt{3}}{2}\right)$

$= 21.25\ N$

$\therefore \Sigma(F_{net})_x = ma$

$mg\sin 30° - f_k - ma$

$mg\sin 30° - \mu_k F_N = ma$

$(2.5)(9.81)\left(\frac{1}{2}\right) - (0.15)(21.25) = 2.5a$

$a = \frac{12.26 - 3.19}{2.5}$

$a = 3.63\ m/s^2$

Work Energy Statement:

Total work on non-conservative force (friction + spring) plus change in potential (gravity) equals change in kinetic energy

$\underbrace{-\mu_k mg\cos\theta}_{f_k} - \int_0^{0.1} mg\sin\theta\,dx + \int_0^{0.1}(1.4x+0.6x^2)\,dx$ → [Average force × distance]

$= \frac{1}{2}m(v_2^2 - v_1^2)$ ... (1)

Work by work done by gravity

W_g = work done by friction

- Initial KE = $\frac{1}{2}mv_1^2 = \frac{1}{2}(2.5)(3.6)^2 = 16.2$
- Gravity work done (Δh):

$W_g = mg\Delta h$
$= mg(0.53)\left(\frac{1}{2}\right)$
$= 6.63\ J$

- Friction work (approximate):

$W_f = \mu_k mg\cos\theta(L)$
$= (0.15)(2.5)(9.81)\cos 30°(0.1)$
$= 0.53\ J$

- Spring work from x=0 to x=0.1m

$\int_0^{0.1}(1.4x+0.6x^2)\,dx = \left[0.7x^2+0.2x^3\right]_0^{0.1}$

$= 0.0683\ J$

Substituting these values in eq (1):

$\frac{1}{2}(2.5)v_2^2 - 16.2 = 6.63 - 0.53 - 0.068$

$v_2 = 3.63\ m/s$

▶ Problem #4:

A particle of mass m starts from rest and moves in horizontal straight line under the action of a constant force, P. Resistance to motion is proportional to the square of velocity and is kv^2. Determine the total impulse I on the particle from the time it starts until it reaches its maximum velocity.

Solution:

F.B. diagram "m" box:

$\Sigma(F_{net}) = ma$

$+P - kv^2 = m\left(\frac{dv}{dt}\right)$

$m\left(\frac{dv}{dt}\right) = P - kv^2$

$\frac{dv}{dt} = \frac{1}{m}(P - kv^2)$

To find the time, we need to set $\frac{dv}{dt} = 0$

[∵ optimization problem]

$0 = \frac{1}{m}(P - kv_{max}^2)$

$P = kv_{max}^2$

$v_{max} = \sqrt{\frac{P}{k}}$

We know,

Impulse, $I = \int_0^{t_f}(\sum F_x)\,dt$

$= \int_0^{t_f} m\left(\frac{dv}{dt}\right)dt$

$= m\int_{v_{max}}^{v_{max}} dv$

$= m\int_0^{v_{max}} dv$

$= mv_{max}$

Impulse, $I = m\sqrt{\frac{P}{k}}$