Factor: Vector Analysis

• Problem 1:

Consider the system described by the characteristic equation:

$$s^3 + 3s^2 + 2s + 6 = 0$$

a) Decide the root locus as K varies from $-\infty$ to $+\infty$

b) Use asymptotic approximations.

c) Locate intercepts, the real axis asymptote (if any), the breakpoints (if any), and all break-away and re-entry points.

d) Use actual calculations to find the exact loci for the breakpoints and re-entry points.

Solution: The original form:

$$1 + K\frac{1}{s^3 + 3s^2 + 2s + 6} = 0 \quad \text{(K is open-loop transfer function)}$$

$$L(s) = \frac{1}{s^3 + 3s^2 + 2s}$$

$$\frac{L(s)}{K} = \frac{1}{s(s^2 + 3s + 2)} = \frac{1}{s(s+1)(s+2)}$$

$$\boxed{\frac{L(s)}{K} = \frac{1}{s(s+1)(s+2)}}$$

Poles: $s = 0, -1, -2$

Zeros: ∞ (zero at infinity order 3)

a) Asymptotes:

∞ of asymptotes = (no of poles) − (no of zeros)

$$= 3 - 0 = 3$$

$$\text{Angle of asymptotes} = \frac{(2q+1)\pi}{n-m}$$

$$\text{Center of asymptotes} = \frac{\sum(\text{poles}) - \sum(\text{zeros})}{n-m}$$

$$= \frac{(0 + (-1) + (-2)) - 0}{3 - 0}$$

$$= \frac{-3}{3} = -1$$

$$\text{Angle of asymptotes: } \theta_{asymp} = \frac{(2q+1)\pi}{n-m}$$

$$q = 0: \quad \theta_{asymp} = \frac{(2 \times 0 + 1)\pi}{3 - 0} = \frac{\pi}{3}$$

$$q = 1: \quad \theta_{asymp} = \frac{(2 \times 1 + 1)\pi}{3 - 0} = \pi$$

$$q = 2: \quad \theta_{asymp} = \frac{(2 \times 2 + 1)\pi}{3 - 0} = \frac{5\pi}{3}$$

• Origin of asymptotes:
the complex poles are the origin of asymptotes

• Angle of departure:
the complex poles are the angle of departure

• Breakaway point:

$$\frac{dK}{ds} = \frac{L(s)}{K(s)} = 0$$

$$\frac{d}{ds}\left(\frac{s^3 + 3s^2 + 2s}{1}\right) = 0$$

$$3s^2 + 6s + 2 = 0$$

$$s = \frac{-6 \pm \sqrt{36 - 24}}{6}$$

$$s_1 = -1.5 + 0.29 = -1.21$$

$$s_2 = -1.5 - 0.29 = -1.79$$

$$\text{Since the poles are fixed, breakaway point should be one of both the real and imaginary roots}$$

• Remaining Angle:

$$\theta_{break} = \frac{(2q+1)\pi}{\text{no of breakaway point}}$$

$$= \frac{(2q+1)\pi}{2}$$

$$= (q + 1/2)\pi$$

$$\text{when } (1/2)\pi \rightarrow \text{Break} = (0 + 1/2)\pi = 90°$$

$$\text{when } (3/2)\pi \rightarrow \text{Break} = (1 + 1/2)\pi = 270°$$

• Crossing and imaginary axis:

Our characteristic equation is:

$$1 + K\frac{1}{s^3 + 3s^2 + 2s + 6} = 0$$

$$s^3 + 3s^2 + 2s + 6 = 0$$

s^3	1	2
s^2	3	6
s^1	$\frac{(3 \times 2) - 6}{3} = 0$	
s^0	6	

Conditions to cross the imaginary axis:

(1) Row should be an odd-degree to find the auxiliary equation.

(2) The entire row should contain all zeros.

$$\rightarrow \text{Row}_{s^1} = 0$$

$$\boxed{K = 6}$$

Let's consider the crossing poles:

Put $K = 6$ in characteristic equation:

$$s^3 + 3s^2 + 2s + 6 = 0$$

$$(s^3 + 3s^2) + (2s + 6) = 0$$

Remember: $s^3 - y^3 = (s - y)(s^2 + y^2 + s^2)$

$$s^2(s + 3) + 2(s + 3) = 0$$

$$(s + 3)(s^2 + 2) = (s - j\sqrt{2})(s + j\sqrt{2})$$

Crossing real and imaginary poles are at:

$$s = -3 \quad \text{and} \quad s = \pm j\sqrt{2}$$

$$\boxed{\frac{\omega_{cross}}{\omega_{cross}} = \sqrt{2}} \qquad \boxed{\frac{\omega_{cross}}{\omega_{cross}} = \sqrt{2}}$$

$$\boxed{\text{Crossing point} = \pm j\sqrt{2}}$$

• Remember the closed-loop poles will cross the crossing point with $s = \pm j\sqrt{2}$

• Problem 2:

a) Derive the closed-loop differential equations for the position x_1 and x_2 for mechanical system shown below.

Assume that the springs k_1, k_2, k_3 are linear springs, with $k_1 = k_2 = k_3 = 1$ corresponding to the unstretched state of the springs.

Assume that the dampers b_1 are two linear dampers.

Assume that b_2 is a nonlinear damper.

b) Describe the system under differential equations in a state-space system in the form:

$$\dot{x} = Ax + Bu$$

Solution:

$$\sum F_{x_1} = \frac{d^2x_1(t)}{dt^2} + \frac{dx_1(t)}{dt} + x_1 - \frac{dx_2(t)}{dt} - x_2$$

$$\sum F_{x_2} = \frac{d^2x_2(t)}{dt^2} + \frac{dx_2(t)}{dt} + x_2 - \frac{dx_1(t)}{dt} - x_1 - u(t)$$

$$\because \sum(\text{Force}_{x_1}) = m_1 a_1$$

$$m_1 a_1 = \ddot{x}_1 + \dot{x}_1 - \dot{x}_2 + x_1 - x_2 + \dot{x}_1 + x_1 - \dot{x}_2 + x_2$$

$$m_1 a_1 = \ddot{x}_1 + 2\dot{x}_1 - 2\dot{x}_2 + 2x_1 - 2x_2 \quad \text{(1)}$$

$$\because \sum(\text{Force}_{x_2}) = m_2 a_2$$

$$m_2 a_2 = \ddot{x}_2 + \dot{x}_2 - \dot{x}_1 + x_2 - x_1 + \dot{x}_2 - \dot{x}_1 + x_2 - u(t)$$

$$m_2 a_2 = \ddot{x}_2 + 2\dot{x}_2 - 2\dot{x}_1 + 2x_2 - 2x_1 - u(t) \quad \text{(2)}$$

$$\because m_1 = m_2 = 1 \text{ kg}, \quad \ddot{x}_1 + 2\dot{x}_1 - 2\dot{x}_2 + 2x_1 - 2x_2 = \ddot{x}_2 + 2\dot{x}_2 - 2\dot{x}_1 + 2x_2 - 2x_1 - u(t)$$

$$\ddot{x}_1 + 2\dot{x}_1 - 2\dot{x}_2 + 2x_1 - 2x_2 - \ddot{x}_2 - 2\dot{x}_2 + 2\dot{x}_1 - 2x_2 + 2x_1 = -u(t)$$

$$\ddot{x}_1 + 4\dot{x}_1 - 4\dot{x}_2 + 4x_1 - 4x_2 - \ddot{x}_2 = -u(t)$$

$$\ddot{x}_1 = -4\dot{x}_1 + 4\dot{x}_2 - 4x_1 + 4x_2 + \ddot{x}_2 - u(t)$$

(a) Part (2):

$$\ddot{x}_1 = -4\dot{x}_1 + 4\dot{x}_2 - 4x_1 + 4x_2 + \ddot{x}_2 - u(t)$$

Now (1):

$$\ddot{x}_2 = -\dot{x}_1 + \dot{x}_2 - x_1 + x_2 - \frac{dx_2(t)}{dt} + \dot{x}_1 - x_1 + x_2$$

State space: $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = Ax + Bu$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -4 & 4 \\ -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} \underbrace{u}_{u}$$

• Problem 3:

A given system has transfer function:

$$y(s) = \frac{2s+3}{s^2 + 2s + 2} u(s)$$

When excited with input: $u(t) = \sin t$

$$y(s) = \frac{2s+3}{s^2 + 2s + 2} u(s) = \frac{2s+3}{s^2 + 2s + 2} \frac{1}{s^2 + 1}$$

(1) Use Laplace transformation techniques to obtain the time response $y(t)$ for the system. Use the Laplace transformation techniques to obtain the complete solution when the input is $u(t) = \sin t$. Find the transient response.

(2) Continued to Part (1). Use the system for input $u(t) = \cos(\omega t)$.

[Use at the end]

Solution:

$$\text{(a) } y(s) = \frac{2s+3}{s^2 + 2s + 2} \frac{1}{s^2 + 1}$$

$$\text{Let } \frac{2s+3}{s^2 + 2s + 2} = \frac{2s+3}{(s+1)^2 + 1}$$

$$\text{Let } s + 1 = y$$

$$\text{Let } s = y - 1$$

$$\text{Let } \omega = 1$$

$$\text{Initial conditions: } y(0^+) = 1, \quad y'(0^+) = 1, \quad y''(0^+) = 1$$

Take Laplace transformation:

$$L\{\ddot{y}\} = s^2 Y(s) + s^2 y(0^+) + s y'(0^+) + y''(0^+)$$

$$= (s^2 + 2s + 2)Y(s) + (s^2 + 2s + 2)y(0^+) + (2s + 3)y'(0^+)$$

$$= 0 + 0 + \frac{1}{s^2 + 1} \quad \text{[Laplace transform of } \sin t\text{]}$$

$$\boxed{Y(s) = \frac{1}{(s^2 + 1)(s^2 + 2s + 2)}}$$

$$\text{Let } \frac{1}{(s^2 + 1)(s^2 + 2s + 2)} = \frac{s}{s^2 + 1}$$

$$\text{Let } \omega = 1 \Rightarrow \text{Poles: } s_1 = \pm j$$

$$s_2 = -1 \pm j \text{ poles}$$

Partial fraction:

$$\frac{1}{(s^2 + 1)(s^2 + 2s + 2)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2s + 2}$$

$$\Rightarrow 1 = (As + B)(s^2 + 2s + 2) + (Cs + D)(s^2 + 1)$$

$$\text{If } s = j: \quad 1 = (Aj + B)(-1 + 2j + 2)$$

$$\text{If } s = -j: \quad 1 = (-Aj + B)(-1 - 2j + 2)$$

$$\Rightarrow 1 = (Aj + B)(1 + 2j) + (-Aj + B)(1 - 2j)$$

$$\Rightarrow 1 = Aj + 2Aj^2 + B + 2Bj - Aj + 2Aj^2 + B - 2Bj$$

$$\Rightarrow 1 = 4Aj^2 + 2B$$

$$\Rightarrow 1 = -4A + 2B$$

$$\Rightarrow 1 = 2B - 4A$$

$$\Rightarrow 1 = 2B - 4A$$

$$\Rightarrow 1 = 2B - 4A