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1. Problem 01:

Solve the differential equation below for $x(t)$:

$$\dot{x}(t) = 2\sqrt{x(t)+1}$$

where $x(0)=0$.

Solution:

$$\frac{dx}{dt} = 2\sqrt{x+1}$$

$$\frac{dx}{2\sqrt{x+1}} = dt$$

$$\int \frac{dx}{\sqrt{x+1}} = \int 2\,dt$$

Put $x+1 = u$

$dx = du$

$$\int \frac{du}{\sqrt{u}} = \int 2dt$$

$$2\sqrt{u} = 2t + c$$

$$2\sqrt{x+1} = 2t + c$$

Given:

When $t=0 \rightarrow x=0$

$$2\sqrt{0+1} = 2(0) + c$$

$$\boxed{c=2}$$

$$2\sqrt{x+1} = 2t+2$$

$$\sqrt{x+1} = t+1$$

$$x+1 = (t+1)^2$$

$$x(t) = (t+1)^2 - 1$$

$$\boxed{x(t) = t^2+2t}$$

2. Problem 02:

What is the maximum value of the function below for $x^Tx = x_1^2 + x_2^2 = 1$

$$f(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

for $x^Tx = 1$

HINT: Consider the eigenvalues of the matrix.

Solution:

$$f(x) = [x_1 \ x_2] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boxed{f(x) = x^TAx}$$

Let's find out the largest eigenvalue of A:

$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$(A-\lambda I)x = 0$$

$$\det(A-\lambda I) = 0 \quad \text{(Characteristic Equation)}$$

$$\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = 0$$

$$(2-\lambda)(2-\lambda) - 1 = 0$$

$$4 - 4\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda-3)(\lambda-1) = 0$$

$$\boxed{\lambda = 3, 1} \rightarrow \text{Eigenvalues}$$

Largest Eigenvalue, $\lambda_{max} = 3$

$$f(x) = x^TAx$$

$$f(x) = x^T(Ax)$$

$$f(x) = x^T(\lambda x)$$

$$f(x)_{max} = x^T(\lambda_{max})x$$

$$= [x_1 \ x_2](3)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3[x_1 \ x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 3(x_1^2 + x_2^2)$$

$$= 3(1)$$

$$\therefore \boxed{f(x)_{max} = 3}$$

3. Problem 03:

Evaluate: $\lim_{n\to\infty}\left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}\right)$

Solution:

$$S_n = \sum_{n=1}^{n} \frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{n}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \Big[\frac{1}{1} - \cancel{\frac{1}{2}}$$

$$+ \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$$

$$+ \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$$

$$\vdots$$

$$- \frac{1}{n+1}\Big]$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1}$$

$$= \boxed{1}$$

4. Problem 04:

Find: $\int \frac{dx}{x(x+1)(x^2+x+1)}$

Solution:

Partial Fraction Decomposition:

$$\frac{1}{x(x+1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2+x+1}$$

When $(x=0)$: $A = \frac{1}{(0+1)(0+0+1)} = 1$

When $(x+1=0)$: $B = \frac{1}{(-1)(1-1+1)} = -1$

$$\frac{1}{x(x+1)(x^2+x+1)} = \frac{A(x+1)(x^2+x+1) + B(x)(x^2+x+1) + (Cx+D)(x)(x+1)}{x(x+1)(x^2+x+1)}$$

$$1 = A[x^3+x^2+x+x^2+x+1] + B[x^3+x^2+x] + Cx^3 + Dx^2 + Cx^2 + Dx$$

$$1 = x^3(A+B+C) + x^2(A+A+B+D+C) + \cdots$$

Coefficients of x^3 & x^2 must be zero as there is no term of x^3 & x^2 is LHS.

$$A+B+C=0$$

$$1-1+C=0$$

$$\boxed{C=0}$$

$$A+A+B+D+C=0$$

$$2A+B+D+C=0$$

$$2(1)-1+D+0=0$$

$$1+D=0$$

$$\boxed{D=-1}$$

$$\int \frac{dx}{x(x+1)(x^2+x+1)} = \int\left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{x^2+x+1}\right)dx$$

$$= \ln|x| - \ln|x+1| - \left(\int \frac{1}{x^2+x+1}dx\right) + C$$

$$= \ln\left|\frac{x}{x+1}\right| - \int \frac{dx}{x^2+x+\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2+1} + C$$

$$= \ln\left|\frac{x}{x+1}\right| - \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + C$$

$$= \ln\left|\frac{x}{x+1}\right| - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right) + C$$

$$= \ln\left|\frac{x}{x+1}\right| - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

—— x —— THE END —— x ——