Rules - Bonus Reading

1 Notations

$G(s)$ - Open loop transfer function.
n - number of poles of $G(s)$
m - number of zeros of $G(s)$
p_i - $p_1, p_2, \ldots, p_n$ $(p_i \in \mathbb{C})$
q_i - order of repeated pole in zero
z_i - zeros from a pole
θ_{z_i} - angle from a zero

2 Rule 1: Symmetry

The root locus diagram is therefore symmetric about the real axis.

3 Rule 2: Zero & Pole

Poles are shown as $\times$'s and zeros are shown as $\bigcirc$'s.

$$\boxed{L(s) = \frac{(s+3)}{(s+1)}}$$

Zero: $(s+3)=0$
$\therefore s=-3$

Pole: $(s+1)=0$
$\therefore s=-1$

4 Rule: Left to Real Axis

"On RHS", but relal on the real axis to the left of odd numbered poles and zeros.

$$\boxed{L(s) = \frac{(s+3)}{(s+1)(s-1)}}$$

5 Rule: Branch Number

Branches extend from poles to zeros if q_i (ordering zeros and ordering) $\neq 1$

$$L = \frac{(s+3)}{(s+1)(s+2)}$$

$$\boxed{L(s) = \frac{(s+3)}{(s+1)(s+2)(s+3)}}$$

Zero: $(s+3)=0$, $(s+3)=0$
$\therefore s=-3, -3$ $(q_i = 2)$

Zero: $(s+3)=0$

Centroid, $\sigma_c = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$

$= \frac{(-1+(-2)+(-3)) - (-3)}{3-1}$

$= \frac{-3}{2}$

$= -2$

$\theta_{asymp} = \frac{180 + 360k}{n-m}$, $\quad k = 0, 1, (n-m-1)$
$= 0, 1, 2 \; (3-1-1)$
$= 0, 1, 2$
$= 180, 360$

$\theta_{asymp} = \frac{180 + 360(0)}{(3-1)} = 90°$

$\theta_{asymp} = \frac{180 + 360(1)}{(3-1)} = \frac{180 + 360}{2}$
$= 90 + 180°$
$= 270°$

Note: the root locus is symmetric (on real) the imaginary axis. They will always, let on poles, so it won't create a RHS root, while obeying a above the root locus region.

6 Rule: Angle of Departure (Bonus on exam)

Angle of departure $= \frac{1}{q_i}\left[\sum(\theta_{z_i}) - \sum(\theta_{p_i}) - 180 + 360k\right]$
from a pole.

$$\boxed{L(s) = \frac{(s+3)}{(s+1)(s+2)(s+3)}}$$

q_i - order of repeated pole from -1.

$q_i(\theta_{p_1}) = \theta_{z_1} + 180$

$q_i(\theta_{p_1}) = \theta_{z_1} + \theta_{p_1} = 90 + 90°$

$\therefore$ Angle of departure $= \frac{1}{q_i}\left[\sum(\theta_{z_i}) - \sum(\theta_{p_i}) - 180 + 360k\right]$

$= \frac{1}{2}\left[(-135° - 180° - 180° - 360°)\right]$

$= -315° - 360k$

$\theta_{z_1} \in \{-1\}$
$\theta_{p_1} \in \{-1, -2, -3\}$
$q_i \in \{-1\}$
$\therefore k = 1$

$\therefore$ when $k=1$, Angle of departure $= -315° - 360(1)$
$= -315°$ (clockwise)
$= +45°$ (anticlockwise)
$= 135°$ (Bonus)

Angle of departure $= -315° - 360(0)$
$= -180°$ (anticlockwise)
$= +180°$ (Angle of departure anticlockwise)
$= 180°$ (Bonus)

Which the angle between the root locus and the real axis.

7 Rule: Break from the Real Axis

"Break breaks into or away from the real axis at the same angle theta."

$$\frac{dK}{ds}\Big|_{s=s_b} = 0$$

$$\theta_{break} = \frac{180 + 360k}{\text{number of branches meeting}}$$

$$\boxed{L(s) = \frac{(s+3)}{(s+1)(s+2)}}$$

Zero: $(s+3)=0$
$\therefore s=-3$

Pole: $(s+1)(s+2)=0$
$\therefore s=-1, -2$

$$\frac{dK}{ds}\Big|_{s=s_b} = \frac{d}{ds}\left(\frac{-(s+3)}{(s+1)(s+2)}\right)\Big|_{s=s_b} = 0$$

also differentiating the final expression

$$\frac{-1(s+3)(s+2)}{(s+1)(s+2)}\Big|_{s=s_b} = 0$$

$$\frac{-1(s^2+5s+6)}{(s^2+3s+2)}\Big|_{s=s_b} = 0$$

$\frac{-1}{1}$ is only for this function to be zero, the numerator must be zero.

$$\frac{-1}{1}(s^2+5s+6) = 0$$

$$s^2 + 5s + 6 = 0$$

$$s = \frac{-5 \pm \sqrt{25-24}}{2}$$

These are the possible break away points.

$s = -2.5$ (Bonus)
$s = -3$ (the break in)

Then we can see that -2.5 is a break away point but -3 is a break in, only when $-2.5 < s < -1$ is satisfying root locus.

Therefore, -2.5 is the break away point.

$$\theta_{break} = \frac{180 + 360k}{\text{2 of branches meeting}}$$

$\therefore$ we have 2 branches meeting, one coming from -1 to -2.5 and another coming from -2 to -2.5

$\therefore k = 0, 1$

$$= \frac{180 + 360(0)}{2}$$

when $(k=0)$ $\theta_{break} = 90°$ $(\angle 180)$
when $(k=1)$ $\theta_{break} = 180 + 360$
$= 270°$ (the bonus)

8 Rule: Angle of Arrival

Angle of Arrival $= \frac{1}{q_i}\left[\sum(\theta_{z_i}) - \sum(\theta_{p_i}) + 180 + 360k\right]$
to a zero.

$$L(s) = \frac{s^2 + 4s + 13}{(s+1)}$$

q_i - order of repeated pole/zero $= 1$

$\sum(\theta_{z_i}) + \theta_{p_1} + \theta_{z_1} = 180 + 90° + 180°$

$\sum(\theta_{z_i}) + \theta_{p_1} = 90°$

Angle of Arrival $= \frac{1}{q_i}\left[\sum(\theta_{z_i}) - \sum(\theta_{p_i}) + 180 + 360k\right]$

$= \frac{1}{1}\left[90° - 90 + 180 + 360k\right]$

$= 90° + 360k$

when $(k=0)$ Angle of Arrival $= 90° + 360(0)$
$= 90°$

9 Summary:

1. L - Open loop transfer function, centroid, $\sum(L(s))$
2. n - no of poles, m - no of zeros
3. n - no of zeros, m - n
4. $\sum(L(s)) = -\sigma$
5. q_i - order of repeated pole or zero
6. q_i - order of zero pole
7. θ_{z_i} - angle from zero
8. The root locus diagram is therefore symmetric about the real axis.
9. Poles are shown as $\times$'s and zeros are shown as $\bigcirc$'s.
10. Branches extend from poles to zeros if q_i (ordering zeros and ordering) $\neq 1$
11. On RHS, left side on real axis, to the left of odd numbered poles and zeros.
12. Asymptotic intercept, the real axis and σ_c $\sum$ and σ_c $\sum$ zeros, the angle diagram
$\sigma_c = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}$, $\quad \theta_{asymp} = \frac{180 + 360k}{n-m}$
13. Angle of departure from a pole.
$\theta_{departure} = \frac{1}{q_i}\left[\sum(\theta_{z_i}) - \sum(\theta_{p_i}) + 180 + 360k\right]$
14. Breaks breaks into or away from and arrive at to with angle theta.
$\frac{dK}{ds}\Big|_{s=s_b} = 0$, $\quad \theta_{break} = \frac{180 + 360k}{\text{no of branches meeting}}$
15. Arrivals.
16. Data Engineering

THE END