Fall 2018

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1. Problem 01:

Given the following linear system of equations, solve for the variables using the Gauss Elimination (row reductions) and demonstrate if the solution is unique, no solution exists, or if infinitely many solutions exist.

$$8x_2 + 2x_3 = -7$$
$$3x_1 + 5x_2 + 2x_3 = 8$$
$$6x_1 + 2x_2 + 8x_3 = 26$$

Solution:

KNOWLEDGE CLOUD:

A matrix is said to be Row Reduced Echelon form (or Row canonical form) if it satisfy the following properties:-

1. The first non-zero element of every non-zero row is 1
2. If a column contains 1st non zero entry of any row then every entry in that column is zero.
3. All non-zero rows of a matrix lies above zero rows.
4. First non-zero entry (leading coefficient) of a row is always strictly to the right of the leading coefficient of the row above it

Writing the given set of equations in Matrix form:

$$\begin{bmatrix} 0 & 8 & 2 \\ 3 & 5 & 2 \\ 6 & 2 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 8 \\ 26 \end{bmatrix}$$

$$\boxed{AX = B}$$

Augmented matrix:

$$C = [A \mid B]$$

$$C = \left[\begin{array}{ccc|c} 0 & 8 & 2 & -7 \\ 3 & 5 & 2 & 8 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

Elementry operations to obtain Row Reduced Echelon form:

(i) Swap R_1 and R_2

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

(ii) $R_1 = R_1/3$

$$\left[\begin{array}{ccc|c} 1 & 5/3 & 2/3 & 8/3 \\ 0 & 8 & 2 & -7 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

(iii) $R_2 = R_3 - 6R_1$

$$\left[\begin{array}{ccc|c} 1 & 5/3 & 2/3 & 8/3 \\ 0 & 8 & 2 & -7 \\ 0 & -8 & 4 & 10 \end{array}\right]$$

(iv) $R_2 = R_2/8$

$$\left[\begin{array}{ccc|c} 1 & 5/3 & 2/3 & 8/3 \\ 0 & 1 & 1/4 & -7/8 \\ 0 & -8 & 4 & 10 \end{array}\right]$$

(v) $R_3 = R_3 + 8R_2$ and $R_1 = R_1 - \frac{5}{3}R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 99/24 \\ 0 & 1 & 1/4 & -7/8 \\ 0 & 0 & 6 & 3 \end{array}\right]$$

(vi) $R_3 = R_3/6$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 33/8 \\ 0 & 1 & 1/4 & -7/8 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

(vii) $R_1 = R_1 - \frac{1}{4}R_3$
$R_2 = R_2 - \frac{1}{4}R_3$

$$C = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

→ Reduced row echelon form

Required Solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 12 \end{bmatrix}$$

$$\boxed{x_1 = 4, \quad x_2 = -1 \quad \text{and} \quad x_3 = 12}$$

$\text{rank}(A) = 3$ $\text{rank}(C) = 3$

We know that,

(n: no. of rows in X)

$\text{rank}(A) = \text{rank}(C) = n$	⇒ Unique Solution
$\text{rank}(A) = \text{rank}(C) < n$	⇒ Infinitely many Solution
$\text{rank}(A) < \text{rank}(C)$	⇒ No solution

$\therefore \text{rank}(A) = \text{rank}(C) = 3 = n$ ⇒ So unique solution

Note:

$\text{rank}(A)$ = no. of non-zero rows in A

2. Problem 02:

Solve the following differential equation:

$$\ddot{y} + 0.2\dot{y} + 4.01y = 0$$

given that $y(0) = 0$ and $\dot{y}(0) = 2$

Solution:

$$\ddot{y} + 0.2\dot{y} + 4.01y = 0$$

Taking laplace transform both sides

$$L\{\ddot{y}\} + 0.2\,L\{\dot{y}\} + 4.01\,L\{y\} = 0$$

$$s^2Y(s) - s\,y(0) - y'(0) + 0.2[sY(s) - y(0)] + 4.01\,Y(s) = 0$$

Substituting $y(0) = 0$ and $\dot{y}(0) = 2$, we get

$$s^2Y(s) - s(0) - (2) + 0.2[sY(s) - 0] + 4.01\,Y(s) = 0$$

$$Y(s)[s^2 + 0.2s + 4.01] = 2$$

$$\boxed{Y(s) = \frac{2}{s^2 + 0.2s + 4.01}}$$

Completing the square of denominator:

$$s^2 + 0.2s + 4.01 = [s^2 + 2(0.1)s + (0.1)^2] - (0.1)^2 + 4.01$$
$$= (s+0.1)^2 - 0.01 + 4.01$$
$$= (s+0.1)^2 + 4$$
$$= (s+0.1)^2 + 2^2$$

$$\therefore Y(s) = \frac{2}{(s+0.1)^2 + 2^2} \quad \text{.....} (1)$$

We know that,

$$L\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}$$

Taking inverse Laplace transform in eq (1), we get

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{2}{(s+0.1)^2 + 2^2}\right]$$

$$\boxed{y(t) = e^{-0.1t}\sin(2t)}$$

3. Problem 03:

Find the first and second derivatives of the following expression:

$$y = -\frac{\cos x}{2\sin^2 x} + \ln\left(\sqrt{\frac{1+\cos x}{\sin x}}\right)$$

Solution: $y = \frac{-\cos x}{2\sin^2 x} + \frac{1}{2}\ln(1+\cos x) - \frac{1}{2}\ln\sin x$

$$\dot{y}_1 = \frac{2\sin^2 x\,\sin x + \cos x\,(4\sin x)(\cos x)}{4\sin^4 x}$$
$$= \frac{\sin^2 x + 2\cos^2 x}{2\sin^3 x}$$
$$= \frac{1 - \cos^2 x + 2\cos^2 x}{2\sin^3 x}$$
$$= \frac{1 + \cos^2 x}{2\sin^3 x}$$

$$\dot{y}_2 = \frac{1}{2}\,\frac{1}{1+\cos x}(-\sin x) = \frac{-\sin x}{2(1+\cos x)} \times \frac{1-\cos x}{1-\cos x}$$
$$= \frac{\cos x - 1}{2\sin x}$$

$$\dot{y}_3 = -\frac{1}{2}\,\frac{1}{\sin x}(\cos x) = \frac{-\cos x}{2\sin x}$$

$$\dot{y} = \dot{y}_1 + \dot{y}_2 + \dot{y}_3$$
$$= \frac{1+\cos^2 x}{2\sin^3 x} + \frac{\cos x - 1}{2\sin x} - \frac{\cos x}{2\sin x}$$
$$= \frac{1+\cos^2 x}{2\sin^3 x} + \frac{\cos x}{2\sin x} - \frac{1}{2\sin x} - \frac{\cos x}{2\sin x}$$
$$= \frac{1 + \cos^2 x - \sin^2 x}{2\sin^3 x}$$
$$= \frac{(1 - \sin^2 x) + \cos^2 x}{2\sin^3 x}$$
$$= \frac{2\cos^2 x}{2\sin^3 x}$$

$$\boxed{\dot{y} = \frac{\cos^2 x}{\sin^3 x}}$$

$$\ddot{y} = \frac{\sin^3 x(-2\cos x\sin x) - (\cos^2 x)(3\sin^2 x\cos x)}{\sin^6 x}$$
$$\ddot{y} = \frac{\sin x(-2\cos x\sin x) - (\cos^2 x)(3\cos x)}{\sin^4 x}$$
$$\ddot{y} = \frac{-2\cos x\sin^2 x - 3\cos^3 x}{\sin^4 x}$$

$$\boxed{\ddot{y} = -2\frac{\cos x}{\sin^2 x} - 3\frac{\cos^3 x}{\sin^4 x}}$$

4. Problem 04:

Find the indefinite integral:

$$\int \frac{\ln x\,dx}{x\sqrt{1+\ln x}}$$

Solution: Put $1 + \ln x = t \Rightarrow \ln x = t - 1$
$\frac{1}{x}dx = dt$

$$\therefore \int \frac{(t-1)}{\sqrt{t}}dt = \int\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)dt$$
$$= \frac{2}{3}t^{3/2} + \frac{1}{2}\sqrt{t} + C$$

$$\boxed{\int \frac{\ln x\,dx}{x\sqrt{1+\ln x}} = \frac{2}{3}(1+\ln x)^{3/2} - 2(1+\ln x)^{1/2} + C}$$

—— x —— THE END —— x ——