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1. Problem 01:

a) Integrate: $\int \frac{dx}{\sqrt{1+e^{2x}}}$ [hint: variable substitution?]

b) find derivative $f'(x)$: $f(x) = \frac{xt^2}{t+x^2}$

Solution:

a) Put $e^x = \tan\theta$

$$e^x\,dx = \sec^2\theta\,d\theta$$

$$dx = \frac{\sec^2\theta\,d\theta}{e^x} = \frac{\sec^2\theta\,d\theta}{\tan\theta} = \frac{1}{\cos^2\theta}\frac{\cos\theta}{\sin\theta}\,d\theta = \frac{1}{\sin\theta\cos\theta}\,d\theta$$

$$\int \frac{dx}{\sqrt{1+e^{2x}}}dx = \int \frac{1}{\sqrt{1+\tan^2\theta}}\frac{d\theta}{\sin\theta\cos\theta} = \int \frac{\cos\theta}{\sin\theta\cos\theta}\,d\theta = \int \operatorname{cosec}\theta\,d\theta = \ln|\operatorname{cosec}\theta - \cot\theta| + C$$

Since, $\tan\theta = e^x$

$\cot\theta = e^{-x}$ ...... (1)

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\sin\theta}\times\frac{\cos\theta}{\cos\theta} = \frac{1/\cos\theta}{\sin\theta/\cos\theta} = \frac{\sec\theta}{\tan\theta} = \frac{\sqrt{\sec^2\theta}}{\tan\theta} = \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$$

$$\operatorname{cosec}\theta = \frac{\sqrt{1+e^{2x}}}{e^x} \quad \text{.... (2)}$$

$$\int \frac{dx}{\sqrt{1+e^{2x}}} = \ln\left|\frac{\sqrt{1+e^{2x}}}{e^x} - e^{-x}\right| + C$$

$$= \ln\left|\frac{\sqrt{1+e^{2x}} - e^0}{e^x}\right| + C$$

$$= \boxed{\ln\left|\frac{\sqrt{1+e^{2x}}-1}{e^x}\right| + C}$$

b) $f(x) = \frac{xt^2}{t+x^2}$

Applying quotient Rule:

$$f'(x) = \frac{(t+x^2)(t^2) - (xt^2)(2x)}{(t+x^2)^2}$$

$$= \frac{t^3 + t^2x^2 - 2t^2x^2}{(t+x^2)^2}$$

$$= \frac{t^3 + x^2(t^2 - 2t^2)}{(t+x^2)^2}$$

$$= \frac{t^3 - x^2t^2}{(t+x^2)^2}$$

$$\boxed{f'(x) = \frac{t^2(t-x^2)}{(t+x^2)^2}}$$

2. Problem 02:

Invert the matrix: $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 3 \end{bmatrix}$

Solution:

Matrix of minors:

$$\begin{bmatrix} -5 & 3 & 6 \\ 5 & 0 & -5 \\ 5 & -1 & -7 \end{bmatrix}$$

Matrix of cofactors:

$$\begin{bmatrix} -5 & -3 & 6 \\ -5 & 0 & 5 \\ 5 & 1 & -7 \end{bmatrix}$$

reverse the sign

Adjugate:

$$\operatorname{adj}(A) = \begin{bmatrix} -5 & -5 & 5 \\ -3 & 0 & 1 \\ 6 & 5 & -7 \end{bmatrix}$$

Keep diagonal elements same and swap the rest

Determinant:

$$\det(A) = +1(-5) - 2(3) + 1(6) = -5 - 6 + 6 = -5$$

Inverse:

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-5}\begin{bmatrix} -5 & -5 & 5 \\ -3 & 0 & 1 \\ 6 & 5 & -7 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & 1 & -1 \\ 3/5 & 0 & -1/5 \\ -6/5 & -1 & 7/5 \end{bmatrix}}$$

3. Problem 03:

Solve the differential equation below for $y(t)$

$$\ddot{y}(t) + 2\dot{y}(t) + y(t) = 0$$

where $y(0) = 1$, $\dot{y}(0) = 0$

Solution:

Auxiliary Equation:

$$r^2 + 2r + 1 = 0$$
$$(r+1)^2 = 0$$
$$r = -1$$

$$\boxed{y(t) = c_1 e^{-t} + c_2 t e^{-t}}$$

Given:

$y(0) = 1$

$c_1 e^0 + c_2(0)e^0 = 1$

$\boxed{c_1 = 1}$

$$\dot{y}(t) = c_1 e^{-t}(-1) + c_2[t(-e^{-t}) + e^{-t}]$$

$\dot{y}(0) = 0$ ($\because$ given)

$$c_1 e^0(-1) + c_2[0 + e^0] = 0$$

$$-c_1 + c_2 = 0$$
$$-1 + c_2 = 0$$
$$\boxed{c_2 = 1}$$

$$\boxed{\therefore\ y(t) = e^{-t} + te^{-t}}$$

4. Problem 04:

Calculate: $\lim_{x\to 2} \sin(\pi x)\sqrt{\left|\frac{x+2}{x-2}\right|}$

Solution:

Knowledge cloud:

- $\sin(n\pi) = 0$
- $\sin\left(\frac{\pi}{2}\right) = \sin\left(\frac{5\pi}{2}\right) = \sin\left(\frac{9\pi}{2}\right) = +1$
- $\sin\left(\frac{3\pi}{2}\right) = \sin\left(\frac{7\pi}{2}\right) = \sin\left(\frac{11\pi}{2}\right) = -1$
- $\cos\left((2n+1)\frac{\pi}{2}\right) = 0$
- $\cos(2\pi) = \cos(4\pi) = \cos(6\pi) = +1$
- $\cos(\pi) = \cos(3\pi) = \cos(5\pi) = -1$

$$\lim_{x\to 2} \underbrace{\sin(\pi x)}_{\to 0} \underbrace{\sqrt{\left|\frac{x+2}{x-2}\right|}}_{\to\infty}$$

$0\cdot\infty$ indeterminant form

So, let's rewrite the expression

Put $y = x - 2 \Rightarrow x = y+2$

when $x \to 2 \Rightarrow y \to 0$

$$\lim_{y\to 0} \sin(\pi(y+2))\sqrt{\left|\frac{y+4}{y}\right|}$$

$$\lim_{y\to 0} \sin(2\pi + \pi y)\sqrt{\left|\frac{y+4}{y}\right|}$$

$$\lim_{y\to 0} \sin(\pi y)\sqrt{\left|\frac{y+4}{y}\right|}$$

when θ = small $\Rightarrow \sin\theta \approx \theta$

$$\lim_{y\to 0} \pi y\sqrt{\frac{|y+4|}{|y|}}$$

$$\lim_{y\to 0} \pi\frac{y}{\sqrt{|y|}}\sqrt{|y+4|}$$

$$\lim_{y\to 0} \pi\,\operatorname{sgn}(y)\cdot\sqrt{|y|}\,\sqrt{|y+4|}$$

$$= \boxed{0}$$

——— x ——— THE END ——— x ———