Second Order Differential Equations

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Author: Kumar Anurag

1. Introduction:

Have you ever wondered how to solve equations like below:

$$y'' - 5y' + 6y = 0$$

Let's learn today.

Any general 2nd order differential equation will be of form:

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = G(x)$$

When $G(x) = 0$ → we will have homogeneous linear equations.

When $G(x) \neq 0$ → then we will have non-homogeneous equation

Now, when $P(x)$, $Q(x)$ and $R(x)$ are some constants, like:

$$ay'' + by' + cy = 0 \qquad \text{-----} (1)$$

a typical solution will be of form:

$$y = e^{rx}$$

$$\rightarrow \dot{y} = re^{rx}$$
$$\rightarrow \ddot{y} = r^2 e^{rx}$$

Substituting these back into eq^n (1), we get

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$e^{rx}\left[ar^2 + br + c\right] = 0$$

$$ar^2 + br + c = 0$$

Now, this just become a quadratic in r:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1

$b^2 - 4ac > 0$

$r = r_1$ and r_2

General Solution:

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Case 2

$b^2 - 4ac = 0$

$r = \underbrace{r}_{\text{only one root}}$

General Solution:

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

Case 3

$b^2 - 4ac < 0$

$r = \alpha + \beta i, \ \alpha - \beta i$

General Solution:

$$y = e^{\alpha x}\left[c_1 \cos(\beta x) + c_2 \sin(\beta x)\right]$$

2. Examples:

1. $y'' - 5y' + 6y = 0$

$ar^2 + br + c = 0$

$a = 1 \quad b = -5 \quad c = 6$

$r^2 - 5r + 6 = 0$
$(r-3)(r-2) = 0$
$r = 3, 2$

$$\boxed{y = c_1 e^{3x} + c_2 e^{2x}}$$

2. $y'' - 6y' + 9y = 0$

$ar^2 + br + c = 0$
$a = 1 \quad b = -6 \quad c = 9$
$r^2 - 6r + 9 = 0$
$(r-3)(r-3) = 0$
$r = 3$

$$\boxed{y = c_1 e^{3x} + c_2 x e^{3x}}$$

3. $9\frac{d^2y}{dx^2} + 24\frac{dy}{dx} + 16y = 0$

$9r^2 + 24r + 16 = 0$

$$r = \frac{-24 \pm \sqrt{(24)^2 - 4(9)(16)}}{2(9)}$$
$$= \frac{-24 \pm \sqrt{576 - 576}}{18}$$
$$= \frac{-24}{18}$$
$$= -\frac{4}{3}$$

$$\boxed{y = c_1 e^{-4/3x} + c_2 x e^{-4/3x}}$$

4. $y'' + 8y' + 25y = 0$

$r^2 + 8r + 25 = 0$

$$r = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(25)}}{2(1)}$$
$$= \frac{-8 \pm \sqrt{64 - 100}}{2}$$
$$= \frac{-8 \pm 6i}{2}$$
$$= -4 \pm 3i$$

$$\boxed{y = e^{-4x}\left[c_1 \cos(3x) + c_2 \sin(3x)\right]}$$

5. **INITIAL VALUE PROBLEM:**

$y'' + 4y = 0$ and $y(0) = 4$, $y'(0) = 6$

Rewrite:

$y'' + 0 \cdot y' + 4y = 0$

$r^2 + 0 \cdot r + 4 = 0$
$r^2 = -4$
$r = \sqrt{-4}$

$r = \pm 2i$
$r = 0 \pm 2i$

$$y = e^{0x}\left[c_1 \cos(2x) + c_2 \sin(2x)\right]$$

$$\boxed{y(x) = c_1 \cos(2x) + c_2 \sin(2x)} \quad \text{-----} (1)$$

y is function of x

Given: $y(0) = 4$

$c_1 \cos 0 + c_2 \sin 0 = 4$
$\boxed{c_1 = 4}$

$y'(0) = 0$

Differentiating (1):

$y'(x) = -c_1 \sin(2x)(2) + c_2 \cos(2x)(2)$

$y'(0) = -c_1[\sin 0](2) + c_2(\cos 0)\, 2$
$6 = 2c_2$

$\boxed{c_2 = 3}$

Substituting c_1 and c_2 in eq (1):

$$\boxed{y(x) = 4\cos(2x) + 3\sin(2x)}$$

6. **Boundary Value Problem:**

$y'' - 2y' + y = 0 \quad ; \quad y(0) = 3$ and $y(1) = 7e$

Before we solve this, let's understand the difference b/w initial value problem and boundary value problem.

In initial value problem, we were given $y'(0) = 6$ but in boundary value problem, we won't having values at y-prime (or at $\frac{dy}{dx}$).

Okay, let's solve our given question:

$y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$
$(r-1)^2 = 0$
$r = 1$

$$\boxed{y(x) = c_1 e^x + c_2 x e^x}$$

Given:

$y(0) = 3$
$c_1 e^0 + c_2 (0) e^0 = 3$
$\boxed{c_1 = 3}$

$y(1) = 7e$
$c_1 e + c_2 (1) e = 7e$
$c_1 + c_2 = 7$
$3 + c_2 = 7$
$\boxed{c_2 = 4}$

$$\boxed{y(x) = 3e^x + 4xe^x}$$

— x — THE END — x —