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1. **Problem 01:**

a) Find: $\lim_{x\to\infty} \ln(1+2^x)\ln\left(1+\frac{3}{x}\right)$

b) find: $\lim_{x\to 0} \frac{\sinh^2 x}{\ln(\cosh 3x)}$

Solution:

a) $\lim_{x\to\infty} \ln(1+2^x)\cdot\ln\left(1+\frac{3}{x}\right)$

when $u\to$ small $\Rightarrow$ $\ln(1+u) \approx u$

$\lim_{x\to\infty} \ln(1+2^x)\cdot\left(\frac{3}{x}\right)$

$3\lim_{x\to\infty} \frac{\ln(1+2^x)}{x}$

$\frac{\infty}{\infty}$ form (indeterminant form)

Applying L'Hôpital's Rule:

$3\lim_{x\to\infty} \frac{2^x\ln(2)}{(1+2^x)}$

Still $\frac{\infty}{\infty}$

Applying L'Hopital's Rule:

$3\ln(2)\lim_{x\to\infty} \frac{\cancel{2^x}\ln(2)}{\cancel{2^x\ln(2)}}$

$= \boxed{3\ln(2)}$

b) $\lim_{x\to 0} \frac{\sinh^2 x}{\ln(\cosh 3x)}$

As $x\to 0 \Rightarrow \sinh x \to 0$

As $x\to 0 \Rightarrow \cosh x\to 1 \Rightarrow \ln(\cosh x)\to 0$

$\frac{0}{0}$ form (indeterminant form)

Applying L'Hôpital's Rule:

$\lim_{x\to 0} \frac{2\sinh x.\cosh x}{\frac{3\sinh 3x}{\cosh(3x)}}$

$\lim_{x\to 0} \frac{2}{3}\frac{\sinh x\cosh x}{\tanh(3x)}$

when x = small $\Rightarrow \sinh\theta \approx \theta$
$\Rightarrow \cosh\theta \approx 1$
$\Rightarrow \tanh\theta \approx \theta$

$\lim_{x\to 0} \frac{2}{3}\frac{x.1}{(3x)}$

$= \boxed{\frac{2}{3}}$

2. **Problem 02:**

Solve: $y'' + 6y' + 13y = 18e^{-x}$

if at $x=0$, $y=\frac{2}{3}$ and $y'=2$

Solution:

Homogeneous Solution:

Auxiliary equation: $r^2+6r+13=0$

$r = \frac{-6\pm\sqrt{36-52}}{2}$

$r = -3\pm 2i$

$$\boxed{y_h(x) = e^{-3x}\left[c_1\cos(2x) + c_2\sin(2x)\right]}$$

Particular Solution:

$y_p(x) = Ae^{-x}$

$y_p' = -Ae^{-x}$ $\qquad y_p'' = Ae^{-x}$

Substituting in LHS of given equation:

$y_p'' + 6y_p' + 13y_p = Ae^{-x} + 6(-Ae^{-x}) + 13(Ae^{-x})$

$= e^{-x}[A-6A+13A]$

$= e^{-x}[8A]$

comparing this with RHS of given problem, we get

$\Rightarrow A=1$

$\boxed{y_p(x) = e^{-x}}$

$y(x) = y_h(x) + y_p(x)$

$$\boxed{y(x) = e^{-3x}\left(c_1\cos(2x)+c_2\sin(2x)\right] + e^{-x}}$$

Applying initial conditions:

At $x=0$:

$y(0) = \frac{2}{3}$

$e^0(c_1\cos 0 + c_2\sin 0) + e^0 = \frac{2}{3}$

$c_1 + 1 = \frac{2}{3}$

$\boxed{c_1 = -\frac{1}{3}}$

$y'(x) = e^{-3x}\left[-c_1(\sin 2x)(2) + c_2\cos(2x)(2)\right]$
$+ \left[c_1\cos(2x)+c_2\sin(2x)\right]e^{-3x}(-3)$
$+ e^{-x}(-1)$

$y'(0) = 2$

$e^0[-c_1\sin 0(2) + c_2\cos 0(2)]$
$+ [c_1\cos 0 + c_2\sin 0]e^0(-3) + e^0(-1)$
$= 2$

$2c_2 + (-3)(c_1) - 1 = 2$

$2c_2 + (-3)\left(-\frac{1}{3}\right) = 3$

$2c_2 + 1 = 3$

$c_2 = 1$

Final solution:

$$\boxed{y(x) = e^{-3x}\left(-\frac{1}{3}\cos 2x + \sin 2x\right) + e^{-x}}$$

3. **Problem 03:**

a) Compute: $\frac{d}{dx}\sin(\text{arc}\sin dx + \text{arc}\cos dx)$

b) compute: $\int x\frac{dx}{x^4-2x^2-1}$

Solution:

a) $\frac{d}{dx}\sin\left[\sin^{-1}dx + \cos^{-1}dx\right]$

$= \frac{d}{dx}\sin\left(\frac{\pi}{2}\right)$ $\quad\left[\because \sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}\right]$

$= \frac{d}{dx}1$

$= \underline{\underline{0}}$

b) $\int x.\frac{dx}{x^4-2x^2-1}$

Put $x^2 = t \Rightarrow 2x\,dx = dt$
$x\,dx = \frac{dt}{2}$

$\frac{1}{2}\int\frac{dt}{t^2-2t-1} = \frac{1}{2}\int\frac{dt}{t^2-2t+1^2-1^2-1}$

$= \frac{1}{2}\int\frac{dt}{(t-1)^2-2}$

$= \frac{1}{2}\int\frac{dt}{(t-1)^2-(\sqrt{2})^2}$

$= \frac{1}{2}\left(\frac{1}{2\sqrt{2}}\right)\ln\left|\frac{t-1-\sqrt{2}}{t-1+\sqrt{2}}\right| + C$

$\left[\because \int\frac{1}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C\right]$

$= \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-1-\sqrt{2}}{x^2-1+\sqrt{2}}\right| + C$

4. **Problem 04:**

Invert the Matrix:

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Solution:

Matrix of minors:

$$\begin{bmatrix} -3 & -7 & 2 \\ -2 & 0 & -1 \\ -2 & -7 & -1 \end{bmatrix}$$

Cofactor matrix:

$$\begin{bmatrix} -3 & 7 & 2 \\ 2 & 0 & 1 \\ -2 & 7 & -1 \end{bmatrix}$$

reverse the sign

Adjugate:

$$\text{adj}(A) = \begin{bmatrix} -3 & 2 & -2 \\ 7 & 0 & 7 \\ 2 & 1 & -1 \end{bmatrix}$$

Keep diagonal same and swap rest

Determinant:

$\det(A) = -1(-3) - (0)(-7) + 2(2)$
$= 3 + 0 + 4$
$= 7$

Inverse:

$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$

$= \frac{1}{7}\begin{bmatrix} -3 & 2 & -2 \\ 7 & 0 & 7 \\ 2 & 1 & -1 \end{bmatrix}$

$$= \boxed{\begin{bmatrix} -3/7 & 2/7 & -2/7 \\ 1 & 0 & 1 \\ 2/7 & 1/7 & -1/7 \end{bmatrix}}$$

——— x ——— THE END ——— x ———