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- Problem #1:

The figure shown a mechanical vibrating system where the of force (only liquid) is applied to the system, the mass moves, as shown in the figure on the right. Determine an and b of the system from this response curve. The displacement is a measured from the equilibrium position.

The formula for peak time, the time at which will reached its maximum value is equal to

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

Solution: The transfer function for this system is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2+bs+k}$$

Since, $F(s) = \frac{1}{s}$ (as of force step input)

we get,

$$X(s) = \frac{1}{s(ms^2+bs+k)}$$

From the given graph (1), we can see that the steady state value of y is 0.0058

$$y(\infty) = \lim_{s\to 0} s\cdot X(s) = 0 \cdot F(s)$$

$$\lim_{s\to 0} \frac{1}{ms^2+bs+k} = 0$$

$$\frac{1}{k} = 0$$

$$k = 10.8571$$

From the graph (1), $t_p = 0.0252$

$$t_p = \frac{\pi}{\omega_d} = 0.0252$$

$$\omega_d = \frac{\pi}{0.0252} = 124.6$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad (\text{or } \zeta = 0.0252)$$

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 124.6$$

$$\omega_n = 124.6$$

$$\omega_n^2 = \frac{k}{m} = 15.517$$

$$m = \frac{k}{\omega_n^2} = \frac{10.8571}{(124.6)^2} = 0.0699$$

$$m = 0.0699$$

From given graph, $t_p = 1$ seconds

$$t_p = \frac{\pi}{\omega_d} = 1$$

$$\omega_d = \frac{\pi}{1} = 3$$

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 3$$

$$\omega_n = 3$$

$$\omega_n^2 = \frac{k}{m} = 9$$

$$m = \frac{k}{\omega_n^2} = \frac{10.8571}{(3)^2} = 1.2063$$

$$m = 1.2063$$

$$\zeta = \frac{b}{2\sqrt{km}} \quad \cdots (1)$$

$$\omega_n = \sqrt{\frac{k}{m}} \quad \cdots (2)$$

$$\omega_n = \sqrt{\frac{10.8571}{1.2063}} = 3$$

$$\omega_n = 3$$

$$\zeta = \frac{b}{2\sqrt{km}} \quad \cdots (1)$$

$$b = 2\zeta\sqrt{km}$$

$$= 2(0.0252)\sqrt{(10.8571)(0.0699)}$$

$$= 0.0252 \times 0.8(0.2)$$

$$b = 0.0252$$

Answer: $b = 0.0252$ lbf/ft

$$b = 0.0252 \text{ lbf/ft}$$

- Problem #2:

A mechanical system is modeled by the following state equations:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The matrix $A(t)$ is determined from state transition matrix

$$\Phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

where x_1 and x_2 are state variables displacements and velocity respectively, and x_1 is the output. Determine the eigen λ_1 and λ_2 of the system, λ_1 is equal to the smaller eigen value, λ_2 is equal to the larger eigen value.

Solution: Given the state variables equations:

$$\dot{x}_1(t) = 0x_1(t) + 1x_2(t) \quad \cdots (1)$$

$$\dot{x}_2(t) = -2x_1(t) - 3x_2(t) \quad \cdots (2)$$

Considering the state variable equations:

$$\dot{x}_1(t) = 0x_1(t) + 1x_2(t)$$

$$\dot{x}_2(t) = -2x_1(t) - 3x_2(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Comparing with general formula for 1st order transfer function:

$$TF = \frac{\text{output}}{\text{input}} = \frac{Y(s)}{U(s)} = \frac{C}{sI - A}B + D$$

we get,

$$\frac{Y(s)}{U(s)} = \frac{1}{s} \qquad \text{and} \qquad \frac{Y(s)}{U(s)} = \frac{1}{s}$$

- Problem #3:

A controlled process is modeled by the following state equations:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The matrix $\Phi(t)$ is determined from state transition matrix

$$\Phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

where x_1 and x_2 are state variables displacements and velocity respectively, and x_1 is the output. Determine the eigen λ_1 and λ_2 of the system, λ_1 is equal to the smaller eigen value, λ_2 is equal to the larger eigen value.

Solution: Given the state variables equations:

$$\dot{x}_1(t) = 0x_1(t) + 1x_2(t) \quad \cdots (1)$$

$$\dot{x}_2(t) = -2x_1(t) - 3x_2(t) \quad \cdots (2)$$

Considering the state variable equations:

$$\dot{x}_1(t) = 0x_1(t) + 1x_2(t)$$

$$\dot{x}_2(t) = -2x_1(t) - 3x_2(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Comparing with general formula for 1st order transfer function:

$$TF = \frac{\text{output}}{\text{input}} = \frac{Y(s)}{U(s)} = \frac{C}{sI - A}B + D$$

we get,

$$\frac{Y(s)}{U(s)} = \frac{1}{s} \qquad \text{and} \qquad \frac{Y(s)}{U(s)} = \frac{1}{s}$$

- Problem #4:

Use the Routh-Hurwitz Criterion, $G(s) = \frac{K(s+3)}{s^3+4s^2+5s+2}$

(a) Indicate the range of values for K so that $G(s)$ is stable.

(b) Determine the values of K and the pole and the fact that the system's gain.

Solution: Let

$$1 + G(s)H(s) = 0$$

$$s^3 + 4s^2 + 5s + 2 + K(s+3) = 0$$

$$s^3 + 4s^2 + 5s + 2 + Ks + 3K = 0$$

$$s^3 + 4s^2 + (5+K)s + (2+3K) = 0$$

Creating the Routh table:

s^3	1	$5+K$
s^2	4	$2+3K$
s^1	$\frac{20-2K-3K}{4}$	
s^0		

Now, the Routh-Hurwitz Criterion. According to the column, all terms in first column should be of same sign. (or non-zero or may not should be positive).

$$\frac{20-2K-3K}{4} > 0 \qquad \text{and} \qquad 2+3K > 0$$

$$\frac{20-5K}{4} > 0 \qquad\qquad 2+3K > 0$$

- Problem #5:

Use the Vibrating Transfer Function, $G(s) = \frac{\omega_n^2 s}{s^2+2\zeta\omega_n s+\omega_n^2}$

(a) Indicate the peak time for $\zeta = 0.5$ and $\omega_n = 10$

(b) Determine the values of ζ and ω_n such that the overshoot gives percent.

Solution: Let

$$\zeta = 0.5 \qquad \omega_n = 10$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

Now: $\zeta = 0.5$ $\omega_n = 10$

$$t_p = \frac{\pi}{\sqrt{1-(0.5)^2}\,(10)}$$

Peak time

$$t_p = 0.3628$$

Now: $\zeta = 0.5$

- Definition Non-dimensionalize:

The ref magnitude = (max ref point) - (min ref point)

$$= y_{ss} - y_0$$

Height of overshoot Δy_{max} $\frac{|y_{max} - y_{ss}|}{|y_{ss} - y_0|}$

$$= \frac{|y_{max} - y_{ss}|}{|y_{ss} - y_0|}$$

$$= \frac{|y_{max} - y_{ss}|}{|y_{ss} - y_0|} \times 100\%$$

Contact of Magnitude, $M_p = \frac{y_{max} - y_{ss}}{y_{ss} - y_0} \times 100\%$

$$= \frac{|y_{max} - y_{ss}|}{|y_{ss} - y_0|} \times 100\%$$

- Rise time, t_r

By the Standard Rise time definition, $t_r = \frac{\pi - \beta}{\omega_d}$

(a) Indicate the rise time for $\zeta = 0.5$ and $\omega_n = 10$

(b) Determine the values of ζ and ω_n such that the rise time gives.

Solution: Let

$$\zeta = 0.5 \qquad \omega_n = 10$$

$$t_r = \frac{\pi - \beta}{\omega_d}$$

$$\beta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\beta = \tan^{-1}\left(\frac{\sqrt{1-(0.5)^2}}{0.5}\right)$$

$$\beta = \tan^{-1}(1.73)$$

$$\beta = 60^\circ$$

Now: $\zeta = 0.5$ $\omega_n = 10$

$$t_r = \frac{\pi - 60^\circ}{\sqrt{1-(0.5)^2}\,(10)}$$

Rise time

$$t_r = 0.2094$$

- Settling time, t_s

The settling time is the time required for the response to reach and stay within a specified percentage of the final value.

(a) Indicate the settling time for $\zeta = 0.5$ and $\omega_n = 10$

(b) Determine the values of ζ and ω_n such that the settling time indicated equals 1 sec.

Solution: Since, Δ is given that $\Delta \le y_{ss}$

$$\Delta = 0.02$$

$$\Delta = y_{ss}(1 \pm 2\zeta\,\Phi(t)) \quad \text{(Overshoot)}$$

substituting values of y_{ss} & Δ and $\zeta = 0.5$

$$\Delta = y_{ss}(1 \pm 2\zeta\,\Phi(t)) \quad \text{(Overshoot)}$$

$$0.02 = y_{ss}(1 \pm 2(0.5)\,\Phi(t))$$

$$0.02 = y_{ss}(1 \pm \Phi(t))$$

Taking square transform both sides of Delaplace and final value theorem, we get

$$\lim_{s\to 0} s\,\Phi(s) = \lim_{s\to 0} s\,\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\,\frac{1}{s}(s)$$

$$\lim_{s\to 0} s\,\Phi(s) = \lim_{s\to 0} \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\Phi(t) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad \cdots (1)$$

(a) Unit step input, $U(s) = \frac{1}{s}$

$$\Phi(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\,\frac{1}{s}$$

$$\Phi(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$\Phi(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$\Phi(s) = \frac{\omega_n^2}{s(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$\Phi(s) = \frac{\omega_n^2}{s(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

Taking inverse laplace transform both sides,

$$\Phi(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)$$

Partial fraction decomposition:

$$\Phi(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$\Phi(s) = \frac{\omega_n^2}{s(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta