Invert the Matrix

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We know about functions, $f(x)$
and we also know that they have inverse $f^{-1}(x)$

Same concepts can be applied to matrices as well.

Let's calculate inverse of a function, first:

$$f(x) = 2x + 3$$

Solution: $\underbrace{y}_{} = 2\underbrace{x}_{} + 3$ ← swap

after swapping

$$x = 2y + 3$$

Now solve for y

$$y = \frac{(x-3)}{2}$$

This is $f^{-1}(x)$

$$f^{-1}(x) = \frac{(x-3)}{2}$$

Let's verify:

$f(x) = 2x + 3$ $\quad\quad$ $f^{-1}(x) = \frac{(x-3)}{2}$

INPUT 1 → $f(x)$: $2\cdot(1)+3$ → OUTPUT 5 $\quad$ INPUT 5 → $f^{-1}(x)$: $\frac{5-3}{2} = \frac{2}{2}$ → OUTPUT 1

Hence, verified :)

Now, let's talk about MATRICES

let A be a matrix,

Normally, for a variable! $x^{-1} = \frac{1}{x}$

But this is not the case for matrices:

$$A^{-1} \neq \frac{1}{A}$$

{there is no concept of matrix division}

Moving on, we know that

$$x \cdot x^{-1} = 1$$

In case of matrices,

$$A \cdot A^{-1} = \underbrace{I}_{\text{Identity Matrix}}$$

* Finding the inverse of a Two-by-Two Matrix:-

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{\underbrace{ad-bc}_{\text{determinant}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

let's verify:

$$A \cdot A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad-bc}\right) \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\underbrace{ad-bc}_{\text{Scalar}}} \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{2\times2} \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{2\times2}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Hence proved.

Q: Why do we need inverse matrix?

Since there is no concept of matrix division but there will be lot of situations where we need division from algebraic point of view for example,

$$XA = B$$

and we are tasked to solve for X.
As we cannot divide both sides by A but we can multiply both sides by A^{-1}.

$$XAA^{-1} = BA^{-1}$$
$$XI = BA^{-1}$$
$$\boxed{X = BA^{-1}}$$

(∵ provided B & A^{-1} are compatible for multiplication)

Notice:

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$\boxed{X = A^{-1}B}$$

(∵ order matters)

PROPERTY: $(BA)^{-1} = A^{-1}B^{-1}$

Remember: Not every matrix will have an INVERSE

if $\det(A) = 0$ then inverse does not exist

Such matrix is called SINGULAR MATRIX.

* Finding the inverse of a 3x3 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}_{3\times3}$$

STEP-01: find the matrix of minors

$$\text{minor matrix of } A = \begin{bmatrix} -24 & -20 & -5 \\ -18 & -15 & -4 \\ 5 & +4 & 1 \end{bmatrix}$$

block lines

STEP-02: Matrix of co-factors

$$\text{cofactor matrix} = \begin{bmatrix} -24 & -20 & -5 \\ -18 & -15 & -4 \\ 5 & 4 & 1 \end{bmatrix} \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= \begin{bmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{bmatrix}$$

Trick: Corner and center entries will remain as they are and other entries will have sign inverted.

STEP-03: find the adjugate / adjoint

$$\text{adj}(A) = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

Keep diagonal same and rest are swapped.

STEP-04: find determinant $\det(A)$

$$\det(A) = 1\begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix}$$

$$= -24 - 2(-20) + 3(0-5)$$

$$= -24 + 40 - 15$$
$$= -39 + 40$$
$$= 1$$

STEP-05: find the inverse

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

$$= \frac{1}{1} \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & 4 \\ -5 & 4 & 1 \end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & 4 \\ -5 & 4 & 1 \end{bmatrix}}$$