Spring 2019

Wednesday, June 8, 2020 12:51 PM

Author: Kumar Anurag

1. Problem 01:

a) Find the integral: $\int_1^e x\ln^2 x\,dx$

b) Find the derivative $\frac{dy}{dx}$: $y = \frac{x^2+x^3-2}{\sqrt{1-x^2}}$

Solution:

a) Let's first solve the indefinite integral:

$$\int x\ln^2 x\,dx = \int \underbrace{\ln^2 x}_{u} \cdot \underbrace{x}_{v}\,dx$$

Applying integration by parts:

$$\int u v\,dx = u\int v\,dx - \int\left(u'\int v\right)dx$$

u → function of x

$$\int \ln^2 x \cdot x\,dx = \ln^2 x\int x\,dx - \int\left(\frac{2\ln x}{x}\int x\right)dx$$

$$= \ln^2 x\left(\frac{x^2}{2}\right) - \int \frac{2\ln x}{x}\cdot\frac{x^2}{2}\,dx$$

$$= \ln^2 x\left(\frac{x^2}{2}\right) - \int \ln x \cdot x\,dx$$

Applying integration by parts again:

$$= \frac12 x^2\ln^2 x - \left[\ln x\int x\,dx - \int\left(\frac1x\int x\right)dx\right]$$

$$= \frac12 x^2\ln^2 x - \ln x\,\frac{x^2}{2} + \int\frac1x\cdot\frac{x^2}{2}\,dx$$

$$= \frac12 x^2\ln^2 x - \frac12 x^2\ln x + \frac12\int x\,dx$$

$$= \frac12 x^2\ln^2 x - \frac12 x^2\ln x + \frac12\left(\frac{x^2}{2}\right)$$

$$\int_1^e x\ln^2 x\,dx = \left[\frac12 x^2\ln^2 x - \frac12 x^2\ln x + \frac14 x^2\right]_1^e$$

$$= \frac12 e^2 - \frac12 e^2 + \frac14 e^2 - \frac14$$

$$\boxed{\int_1^e x\ln^2 x\,dx = \frac14 e^2 - \frac14}$$

b) $y = \frac{x^2+x^3-2}{\sqrt{1-x^2}}$

Apply quotient rule:

$$\frac{dy}{dx} = \frac{\left(\sqrt{1-x^2}\right)(2x+3x^2) - (x^2+x^3-2)\left(\frac{-2x}{2\sqrt{1-x^2}}\right)}{(1-x^2)}$$

[illegible]

$$\boxed{\frac{dy}{dx} = \frac{[illegible]}{2(1-x^2)^{3/2}}}$$

2. Problem 02:

Find a general solution of the differential equation:

$$y''' - 7y'' + 15y' - 9y = e^x(8x-12)$$

Solution:

→ Complementary Function (CF):

Auxiliary equation:

$$r^3 - 7r^2 + 15r - 9 = 0$$

Since, r = 1 satisfies the equation, so it is one of the roots

$$(r-1)(r^2-6r+9) = 0$$

$$(r-1)(r-3)^2 = 0$$

$$r = 1, 3, 3$$

$$CF = c_1e^x + c_2e^{3x} + c_3xe^{3x}$$

→ Particular Integral (PI):

$$PI = e^x \underbrace{Q(x)}_{\text{a general polynomial in } x}$$

let's say. $Q(x) = ax^2 + bx$

$$PI = e^x(ax^2+bx)$$

Now we need to find out a and b.

let $L(y) = y''' - 7y'' + 15y' - 9y$

$$L[e^xQ(x)] = [D^3 - 7D^2 + 15D - 9]\,e^xQ(x)$$

when you shift e^x to left D will become (D+1)

$$= e^x\underbrace{[(D+1)^3 - 7(D+1)^2 + 15(D+1) - 9]}_{\text{after solving this term, we get}}Q(x)$$

$$L[e^xQ(x)] = e^x[D^3 - 4D^2 + 4D]\,Q(x) \quad \text{--- (1)}$$

D means $\frac{d}{dx}$

$$\boxed{\begin{aligned}&\text{Properties:}\\ &(a+b)^3 = a^3+b^3+3ab(a+b)\\ &(a-b)^3 = a^3+b^3-3ab(a-b)\\ &a^3+b^3 = (a+b)(a^2-ab+b^2)\\ &a^3-b^3 = (a-b)(a^2+ab+b^2)\end{aligned}}$$

Since,

$$Q(x) = ax^2+bx$$
$$Q'(x) = 2ax+b \quad = D$$
$$Q''(x) = 2a \quad = D^2$$
$$Q'''(x) = 0 \quad = D^3$$

Using these values in eqⁿ (1):

$$L[e^xQ(x)] = e^x[0 - 4(2a) + 4(2ax+b)]$$
$$= e^x[0 - 8a + 8ax + b]$$
$$= e^x[8ax + (b-8a)]$$

RHS given in our question $= e^x[8x-12]$

By comparing, we get:

$8ax = 8x$, $\boxed{a=1}$

$b - 8a = -12$
$b - 8(1) = -12$
$\boxed{b=-4}$

$$PI = e^x(ax^2+bx) = e^x(x^2-4x)$$

$$\boxed{\begin{aligned}y &= CF + PI\\ &= c_1e^x + c_2e^{3x} + c_3xe^{3x} + e^x(x^2-4x)\end{aligned}}$$

3. Problem 03:

Invert the matrix:

$$A = \begin{bmatrix}1 & 0 & 2\\ 2 & -1 & 1\\ 1 & 3 & -1\end{bmatrix}$$

Solution:

Matrix of Minors:

$$\begin{bmatrix}-2 & -3 & 7\\ -6 & -3 & 3\\ 2 & -3 & -1\end{bmatrix}$$

Matrix of co-factors:

$$\begin{bmatrix}-2 & 3 & 7\\ 6 & -3 & -3\\ 2 & 3 & -1\end{bmatrix}$$

reverse sign

Adjugate:

$$adj(A) = \begin{bmatrix}-2 & 6 & 2\\ 3 & -3 & 3\\ 7 & -3 & -1\end{bmatrix}$$

Keep the diagonal same and swap the rest

Determinant:

$$det(A) = 1(-2) - 0(-3) + 2(7) = -2 + 14 = 12$$

Inverse:

$$A^{-1} = \frac{1}{det(A)}\,adj(A)$$

$$= \frac{1}{12}\begin{bmatrix}-2 & 6 & 2\\ 3 & -3 & 3\\ 7 & -3 & -1\end{bmatrix}$$

$$= \begin{bmatrix}-1/6 & 1/2 & 1/6\\ 1/4 & -1/4 & 1/4\\ 7/12 & -1/4 & -1/12\end{bmatrix}$$

4. Problem 04:

a) Evaluate the limit: $\lim_{x\to0}\frac{\tan x - x}{x - \sin x}$

b) Find the sum of the series:

$$\sum_{n=2}^{\infty}\frac{6}{36n^2+12n-35}$$

Solution:

a) $\lim_{x\to0}\frac{\tan x - x}{x-\sin x}$

$$\boxed{\begin{aligned}&\text{Knowledge Cloud:}\\ &\text{L'Hôpital' Cases:}\\ &\frac00 \quad \frac{\infty}{\infty} \quad 0\times\infty \quad 1^\infty \quad 0^0 \quad \infty^0 \quad \infty-\infty\end{aligned}}$$

Since the given limit is of the form $\frac00$

Apply L'Hôpital's Rule:

$$\lim_{x\to0}\frac{\sec^2x - 1}{1-\cos x}$$

It is still of the form $\frac00$, so

Apply L'Hôpital Rule again

$$\lim_{x\to0}\frac{2\sec x(\sec x\tan x)}{\sin x}$$

$$\lim_{x\to0}\frac{2\sec^2x(\sin x)}{\sin x(\cos x)}$$

$$\lim_{x\to0} 2\sec^3x$$

$$= \boxed{2}$$

(b) $\sum_{n=2}^{\infty}\frac{6}{36n^2+12n-35}$

$$\frac{6}{36n^2+12-35} = \frac{6}{(6n-5)(6n+7)}$$

Partial fraction Decomposition:

$$\frac{6}{(6n-5)(6n+7)} = \frac{A}{6n-5} + \frac{B}{6n+7}$$

When $(6n-5=0)$ ⇒ $A = \frac{6}{6\left(\frac56\right)+7} = \frac12$

When $(6n+7=0)$ ⇒ $B = \frac{6}{6\left(\frac{-7}{6}\right)-5} = -\frac12$

$$\frac{6}{36n^2+12-35} = \frac12\left[\frac{1}{6n-5} - \frac{1}{6n+7}\right]$$

Writing out first few terms:

$(n=2)$: $\frac17 - \cancel{\frac1{19}}$

$(n=3)$: $\frac1{13} - \cancel{\frac1{25}}$

$(n=4)$: $\cancel{\frac1{19}} - \cancel{\frac1{31}}$

$(n=5)$: $\cancel{\frac1{25}} - \cancel{\frac1{37}}$

⋮

Surviving Positive Terms: $\underset{(n=2)}{\frac17} + \underset{(n=3)}{\frac1{13}}$

Surviving Negative Terms: $\underset{(n=n-1)}{\frac{-1}{6(n-1)+7}} \quad \underset{(n=n)}{\frac{-1}{6n+7}}$

$$\sum_{n=2}^{\infty}\frac{6}{36n^2+12n-35} = \frac12\sum_{n=2}^{\infty}\left(\frac1{6n-5} - \frac1{6n+7}\right)$$

$$= \frac12\sum_{n=2}^{\infty}\left(\frac17 + \frac1{13}\underbrace{-\frac1{6(n-1)+7} - \frac1{6n+7}}_{\text{since } n\to\infty \text{ so, they will be zero}}\right)$$

$$= \frac12\left(\frac17 + \frac1{13}\right)$$

$$= \frac{1}{2\times7\times13}(20)$$

$$= \boxed{\frac{10}{91}}$$

—— x —— THE END —— x ——