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1. Problem #1:

Goal: $\lim_{x\to 9} \frac{\sqrt{x+1}-\sqrt{4x+8}}{\sqrt{3x-3}} = ?$

Solution:

Preparation: $\sqrt{3x-3} = \sqrt{3\cdot 3x-3} \quad 1\cdot 3 = 3$

$= (\sqrt{3x-3}-1)\cdot(\sqrt{3x-3}+1)$

$= (3x-3)\cdot(\sqrt{3x-3}+\sqrt{3x-3})$

$\lim_{x\to 9} \frac{\sqrt{x+1}-(2)}{\sqrt{3x-3}} \cdot \frac{(\sqrt{x+1}+2)}{(\sqrt{x+1}+2)} = \frac{\lim_{x\to 9} \frac{x-3}{\sqrt{3x-3}}}{\lim_{x\to 9} \frac{\sqrt{x+1}+2}{\sqrt{3x-3}}}$

first limit second limit

first limit:

$\lim_{x\to 9} \frac{\sqrt{x+1}-2}{\sqrt{3x-3}} \cdot \frac{\sqrt{3x-3}}{\sqrt{3x-3}} = \frac{\sqrt{x+1}\cdot\sqrt{3x-3}-2\sqrt{3x-3}}{\sqrt{3x-3}\cdot\sqrt{3x-3}}$

$\lim_{x\to 9} \frac{(x+1-3)}{\sqrt{3x-3}\cdot\sqrt{3x-3}} - \frac{\sqrt{3x-3}\cdot 2}{\sqrt{3x-3}\cdot\sqrt{3x-3}} = \frac{\sqrt{x+1}\cdot\sqrt{3x-3}-2\sqrt{3x-3}}{\sqrt{3x-3}\cdot\sqrt{3x-3}}$

$\lim_{x\to 9} \frac{(x-2)}{(x+1-3)} \cdot \frac{(\sqrt{x+1}+\sqrt{3x})}{(\sqrt{x+1}+\sqrt{3x})} - \frac{2\sqrt{3x-3}}{\sqrt{3x-3}\cdot\sqrt{3x-3}}$

$\lim_{x\to 9} \frac{(x-2)}{(x-2)} \cdot \frac{(\sqrt{x+1}+\sqrt{3x})}{(\sqrt{x+1}+\sqrt{3x})} - \frac{2\sqrt{3x-3}}{\sqrt{3x-3}\cdot\sqrt{3x-3}}$

$= \frac{(\sqrt{x+1}+\sqrt{3x})}{(\sqrt{x+1}+\sqrt{3x})} - \frac{2\sqrt{3x-3}}{\sqrt{3x-3}\cdot\sqrt{3x-3}}$

$= \frac{(x+1)(2\cdot 2)}{(3x-3)} - \frac{2}{(3x)} \cdot \frac{\sqrt{x}}{\sqrt{x}}$

$= \frac{1}{3}$

Second limit:

$\lim_{x\to 9} \frac{\sqrt{4x+8}\cdot\sqrt{3x}}{\sqrt{3x-3}\cdot\sqrt{3x}}$ $\quad \sqrt{4x+8}\cdot\sqrt{3x}$

$(a+b)^2 = a^2+2ab+b^2$

$\lim_{x\to 9} \frac{(3x)(4x+8)}{(3x-3)\cdot\sqrt{3x}\cdot\sqrt{3x}}$ $\quad$ Dividing by '9'

$\lim_{x\to 9} \frac{(x-3)}{(x-3)} \cdot \frac{\sqrt{3x}\cdot\sqrt{4x+8}}{\sqrt{3x}\cdot\sqrt{3x}}$ $\quad \left(\frac{1}{9}\right)$

$\lim_{x\to 9} \frac{(x-3)}{\sqrt{3x}\cdot\sqrt{3x}} \cdot \frac{(\sqrt{3x}\cdot\sqrt{4x+8})}{(\sqrt{3x}\cdot\sqrt{3x})}$

$\lim_{x\to 9} \frac{1}{3} \cdot \frac{(x-3)}{(3x-3)} \cdot \frac{(\sqrt{3x}\cdot\sqrt{4x+8})}{(\sqrt{3x}\cdot\sqrt{3x})}$

Setting the value of '9'

$\lim_{x\to 9} \frac{(\sqrt{3x}\cdot\sqrt{4x+8})}{(\sqrt{3x}\cdot\sqrt{3x})} = \frac{(\sqrt{3x}\cdot\sqrt{4x+8})}{(\sqrt{3x}\cdot\sqrt{3x})}$

$= \frac{(\sqrt{3x}\cdot\sqrt{4x+8})}{(\sqrt{3x}\cdot\sqrt{3x})} = \frac{(\sqrt{3\cdot 9}\cdot\sqrt{4\cdot 9+8})}{(\sqrt{3\cdot 9}\cdot\sqrt{3\cdot 9})}$

$= \frac{(\sqrt{27}\cdot\sqrt{36+8})}{(\sqrt{27}\cdot\sqrt{27})} = \frac{(\sqrt{27}\cdot\sqrt{44})}{(\sqrt{27}\cdot\sqrt{27})}$

$= \frac{(3\sqrt{3}\cdot 2\sqrt{11})}{3\sqrt{3}\cdot 3\sqrt{3}}$

$= \frac{4\sqrt{11}}{3\cdot 3}$

$= \frac{2\sqrt{11}}{3\cdot 3}$

Final limit: first limit + second limit

$= \frac{1}{3} + \frac{2\sqrt{11}}{3\cdot 3}$

$= \frac{3\cdot 3 + 2\sqrt{11}}{3\cdot 3}$

$= \boxed{\frac{11+2\sqrt{11}}{9}}$

2. Problem #2:

a) Solve $\frac{dy}{dx} = -3xy$, $y(0) = 1$

b) Solve $\frac{d^2y}{dx^2} - 3y' - 4y = 10e^{2x}$

where $y(0) = 1$, $y'(0) = 2$

Solution:

1) $\frac{dy}{dx} = -3xy$

$\frac{dy}{y} = \int -3x\,dx$

$\ln|y| = -x^2 + C$

$y = e^{-x^2+C}$

$y = e^{-x^2}\cdot e^C$

$\boxed{y = Ce^{-x^2}}$

$y(0) = 1$

$Ce^0 = 1$

$\boxed{C=1}$

$\boxed{y(x) = e^{-x^2}}$

2) $y'' - 3y' + 3y = 10e^{2x}$

Homogeneous Solution:

$r^2 - 3r + 3 = 0$

$r = \frac{3 \pm \sqrt{9-12}}{2}$

$r = (-3) \pm (-1)$

$(-3)/(-1) = 0$

$r = 3$

$y_c(x) = C_1e^{-x} + C_2e^{2x}$

Particular Solution:

$y_p(x) = Ae^{2x}$

$y_p'(x) = Ae^{2x}(-2) = -2Ae^{2x}$

$y_p''(x) = -2A\,e^{2x}(-2)$

$= 4Ae^{2x}$

Substituting these values in into of given equation:

$y'' - 3y' + 3y = 4Ae^{2x} - 3(-2Ae^{2x}) + 3Ae^{2x}$

$= e^{2x}(4A + 6A + 3A)$

$= e^{2x}(13A)$

Comparing it with this eq given equation:

$13A = 10$

$\boxed{A = \frac{10}{13}}$

$y_p(x) = \frac{10}{13}e^{2x}$

Final solution

$y(x) = y_c(x) + y_p(x)$

$\boxed{y(x) = C_1e^{-x} + C_2e^{2x} + \frac{10}{13}e^{2x}}$

Applying the initial condition:

$\therefore y(0) = 1$

$C_1e^0 + C_2e^0 + \frac{10}{13}e^0 = 1$

$C_1 + C_2 + \frac{10}{13} = 1 \Rightarrow C_1 + C_2 = \frac{3}{13} \quad -(1)$

$y'(x) = -C_1e^{-x} + 2C_2e^{2x} + \frac{10}{13}e^{2x}(-2)$

$y'(0) = -3$

$-C_1e^0 + 2C_2e^0 - \frac{20}{13}e^0 = -3$

$-C_1 + 2C_2 = \frac{20}{13} - 3$

$-C_1 + 2C_2 = \frac{-8}{13} \quad \text{——} (2)$

Eq (1) - Eq(2):

$-2C_1 = \frac{3}{13} + \frac{8}{13}$

$-2C_1 = 2$

$\boxed{C_1 = -1}$

$C_1 + C_2 = \frac{3}{13}$

$-1 + C_2 = \frac{3}{13}$

$\boxed{C_2 = \frac{16}{13}}$

$\boxed{y(x) = \frac{3}{13}e^{-x} - e^{2x} + \frac{10}{13}e^{2x}}$

3. Problem #3:

a) Compute $\frac{d}{dx}\left[\sec^{-1}\left(\sqrt{\cos 2x}\right) - \left(\sqrt{\cos 2x}\right)\right]$

b) Compute: $\int \frac{dx}{\sqrt{x+1}+\sqrt{x}}$

Solution:

a) $\frac{d}{dx}\left[\sec^{-1}\left(\sqrt{\cos 2x}\right) - \left(\sqrt{\cos 2x}\right)\right]$

$= \frac{1}{|x|\sqrt{x^2-1}} \cdot \frac{1}{2\sqrt{\cos 2x}} \cdot (-\sin 2x)(2) - \frac{-\sin 2x\,(2)}{2\sqrt{\cos 2x}}$

$= \frac{-2\sin 2x}{2\sqrt{\cos 2x}} \cdot \left[\frac{1}{|\cos 2x|} - 1\right]$

$= \frac{-\sin 2x}{\sqrt{\cos 2x}} \cdot \left[\frac{1 - |-1 - \cos 2x|}{|\cos 2x|}\right]$

$= \frac{2}{\sqrt{\cos 2x}} \cdot \left[\frac{\sin 2x + \sin 2x}{|\cos 2x|\sqrt{\cos 2x}}\right]$

$= \frac{4\sin 2x}{2|\cos 2x|\sqrt{\cos 2x}}$

b) $\int \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int \frac{\sqrt{x+1}-\sqrt{x}}{(x+1)-(x)}\,dx$

$= \int \frac{\sqrt{x+1}-\sqrt{x}}{1}\,dx$

$= \frac{1}{2}\int \left[\sqrt{x+1} - \sqrt{x}\right]\,dx$

$= \frac{1}{2}\left[\frac{2}{3}(x+1)^{3/2} - \frac{2}{3}(x)^{3/2}\right] + C$

$= \frac{1}{3}\left[(x+1)^{3/2} - (x)^{3/2}\right] + C$

4. Problem #4:

Find the area fully enclosed by the parametric curve:

$x = 3t - t^3$

$y = 3t^2 - t^3$

Solution:

$x(t) = 3t - t^3$

$y(t) = 3t^2 - t^3$

Area enclosed by a parametric curve,

$A = \int_a^b y(t)\,x'(t)\,dt$

$x'(t) = 3 - 3t$

Intersection Points:

Set $x = 0 \rightarrow 3t - t^3 = 0$

$3t = t^3$

$\boxed{t = 0, \pm 1}$

Set $y = 0 \rightarrow 3t^2 - t^3 = 0$

$3t^2 - t^3 = 0$

$3 = t^2$

$\boxed{t = \pm 1}$

Curve passes through origin and meets at 2

$A = \int_{-1}^{0} y(t)\,x'(t)\,dt$

$+ \int_{0}^{1} (3t^2 - t^3)(3 - 3t)\,dt$

$= \int_{-1}^{0} (-3t^3 - 3t^4 + t^4)\,dt$

$+ \int_{0}^{1} (9t^2 - 12t^3 + 3t^4)\,dt$

$= \left[-\frac{3t^4}{4} - \frac{3t^5}{5} + \frac{t^5}{5}\right]_{-1}^{0}$

$+ \left[\frac{9t^3}{3} - \frac{12t^4}{4} + \frac{3t^5}{5}\right]_{0}^{1}$

$= \frac{3}{4} - \frac{3}{5} + \frac{1}{5} + \frac{9}{3} - \frac{12}{4} + \frac{3}{5}$

$= \frac{3}{4} - \frac{4}{5} + \frac{9}{3} - 3$ $\quad$ negative, we will consider

$= 0.583$

Note: If you want positive area then

i.e. $A = \int_a^b |y(t)\,x'(t)|\,dt$

—— X —— THE END —— X ——