Spring 2016

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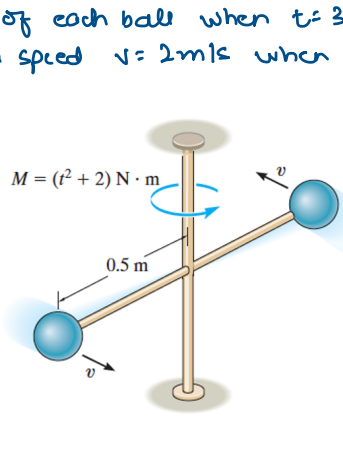
1. **Problem 01:**

Each ball has a negligible size and a mass of 10 kg and is attached to the end of a rod whose mass may be neglected. If the rod is subjected to a torque. $M = (t^2+2)$ Nm, where t is in seconds, determine the speed of each ball when t = 3 seconds. Each ball has a speed v = 2 m/s when t = 0.

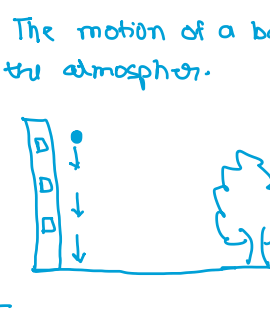


Basics:

1. **Newton's Laws of Motion:**

☐ **First Law**: An object at rest remains at rest, and an object in motion remains in motion at constant speed and in a straight line unless acted on by an unbalanced force. [Law of Inertia]

Examples: The motion of a ball falling down through the atmosphere.



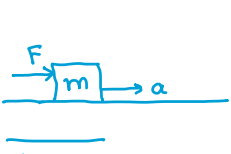
☐ **Second Law**: It states that an acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass. [Force]

$$\boxed{F = m.a}$$

F → net force

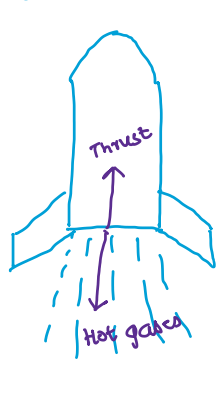
m → mass

a → acceleration



☐ **Third Law**:- It states that for every action, there is an equal and opposite reaction. Whenever one object exerts a force on another object, the second object exerts an equal and opposite force on the first. [Action & Reaction]

Example: The motion of rocket produces thrust and hot exhaust gases flow out the back of the engine, and a thrusting force is produced in the opposite direction.



Solution:

Given: Mass of each ball = 10 kg

Torque, M = (t^2+2) Nm

Radius from center to each ball, r = 0.5 m

Initial speed of each ball at t = 0, v_0 = 2 m/s

We know from Newton's second law of Rotation:

$$\boxed{\text{Torque} = \text{Inertia} \times \text{Angular Acceleration}}$$

- **Calculating the Moment of Inertial of System:**

$$I_{system} = I_{ball1} + I_{ball2}$$
$$= mr^2 + mr^2$$
$$= 2mr^2$$
$$= 2 \cdot 10 \cdot (0.5)^2$$
$$= \cancel{2} \cdot \cancel{10}^5 \cdot \frac{1}{\cancel{4}_2}$$
$$= 5 \text{ kg.m}^2$$

- **Angular Acceleration:**

We know that,

$$\overset{\text{Torque}}{M} = I_{system}\overset{\text{angular acceleration}}{\alpha}$$

$$t^2+2 = 5 \cdot \alpha$$

$$\boxed{\alpha = \frac{t^2+2}{5}} \rightarrow \text{unit: rad/sec}^2$$

- **Angular Velocity (ω):-**

We know that,

$$\alpha = \frac{d\omega}{dt}$$

$$\therefore d\omega = \alpha\, dt$$

$$\boxed{\int_{\omega_0}^{\omega} d\omega = \int \alpha \cdot dt} \quad \text{- - - - (1)}$$

let's calculate initial angular velocity:

$$v = r \cdot \omega$$
$$\therefore v_0 = r \cdot \omega_0$$
$$\Rightarrow \omega_0 = \frac{v_0}{r} = \frac{2}{0.5}$$
$$= 4 \text{ rad/sec}$$

Using ω_0 in (1):

$$[\omega]_{\omega_0}^{\omega} = \int_0^t \left(\frac{t^2+2}{5}\right) dt$$

$$\omega - \omega_0 = \frac{1}{5}\left[\frac{t^3}{3} + 2t\right]_0^t$$

$$\omega - 4 = \frac{1}{5}\left[\frac{t^3}{3} + 2t\right]$$

$$\boxed{\omega = 4 + \frac{t^3}{15} + \frac{2t}{5} \quad \text{rad/sec}}$$

- **Linear Velocity v:-**

We know that,

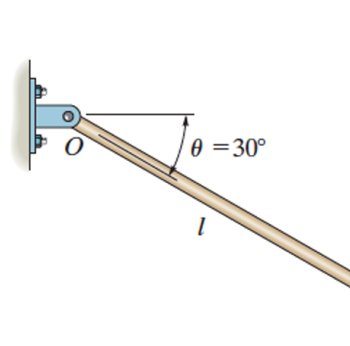
$$v = r \cdot \omega$$
$$= (0.5)\left(4 + \frac{t^3}{15} + \frac{2t}{5}\right) \quad \text{m/sec}$$

At t = 3 sec:

$$v_{t=3} = \frac{1}{2}\left(4 + \frac{3 \cdot 3 \cdot 3}{\cancel{3} \cdot 5} + \frac{2 \cdot 3}{5}\right)$$
$$= 2 + \frac{9}{10} + \frac{6}{10}$$
$$= 2 + 0.9 + 0.6$$
$$= 2.9 + 0.6$$
$$= 3.5 \text{ m/sec}$$

2. **Problem 02:**

The bar has a mass of 1 kg and length of 2 m. If it is released from rest to the position θ = 30°, determine its angular acceleration. $I_G = 1/12\, ml^2$. $I_O = 1/3\, ml^2$.

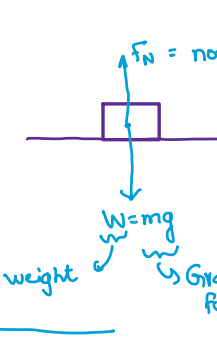


Basics:

1. **Free Body Diagram (FBD)**:-

FBD is a diagram that shows all of the forces acting on an object.

Example (a): A box is at rest on a table.

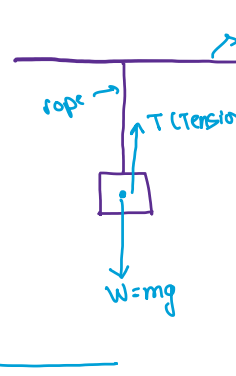


F_N = normal (perpendicular to surface) force exerting from surface.

∴ Since the object is at rest. So, $F_N = mg$

[On Earth, $g = 9.8 m/s^2$]

Example (b): A block attached to a rope hangs from a ceiling.



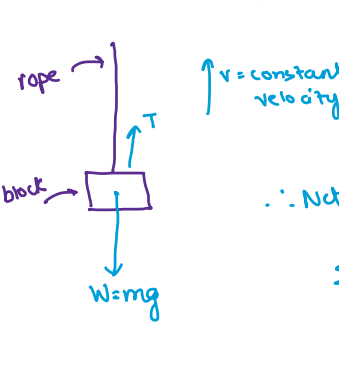
∴ Since block is at rest.

⇒ F_{Net} in y-direction = 0

$+T - mg = 0$

$\underline{T = mg}$

Example (c): A block is pulled upward at constant velocity.



∴ Net force in y-direction:

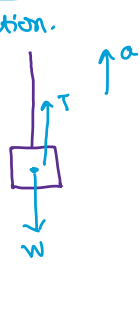
$$\underset{\substack{\downarrow \\ ma}}{\Sigma F_y} = +T - mg$$

$$ma = +T - mg$$

$$m \cdot 0 = T - mg \quad [\because \underset{\substack{\downarrow \\ \text{constant velocity}}}{a = 0}]$$

$$\Rightarrow \underline{T = mg}$$

Example (d): A block is pulled upward with a constant acceleration.



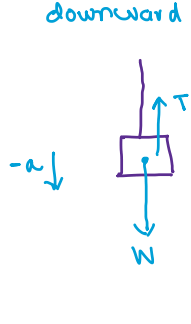
$$\Sigma F_y = T - mg$$

$$ma = T - mg$$

$$\boxed{ma + mg = T}$$

↳ Tension Expression

Example (e): A block descends lower with a downward acceleration using a rope.

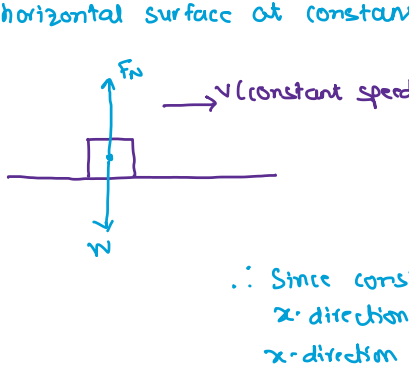


$$\Sigma F_y = +T - mg$$

$$-ma = +T - mg$$

$$\boxed{T = mg - ma}$$

Example (f): A block slides across a frictionless horizontal surface at constant speed.



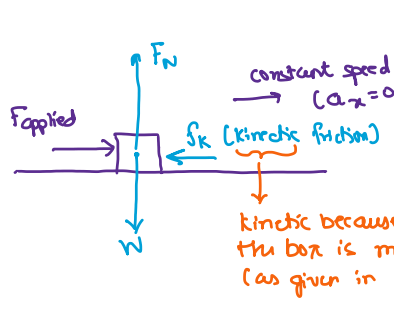
∴ Since constant velocity in x-direction, so net force in x-direction is zero. [No Friction]

$$\Sigma F_x = 0$$

∴ block is not moving in y-direction. So

$$\Sigma F_y = 0$$
$$+T - mg = 0$$
$$\boxed{T = mg}$$

Example (g): An applied force is used to push a box to the right at constant speed.



kinetic because the box is moving (as given in question)

FRICTION → Static friction (f_s), Kinetic friction (f_k)

$$\Sigma F_x = +F_{applied} - f_k$$

$$m\cancel{a_x}^0 = F_{applied} - f_k \quad [\because a_x = 0]$$

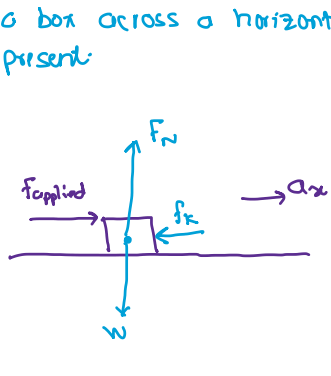
$$\boxed{F_{applied} = f_k}$$

$$\Sigma F_y = 0 \quad [\because \text{box not moving vertically}]$$

$$+F_N - W = 0$$

$$\boxed{F_N = W = mg}$$

Example (h): An applied force is used to accelerate a box across a horizontal surface with friction present.



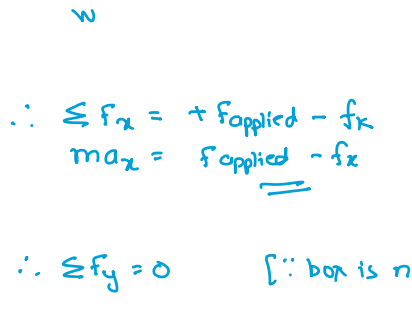
$$\therefore \Sigma F_x = +F_{applied} - f_k$$

$$ma_x = \underline{F_{applied} - f_k}$$

$$\therefore \Sigma F_y = 0 \quad [\because \text{box is not moving vertically}]$$

$$F_N = W = mg$$

Example (i): A rope is used to pull a block to the right across a horizontal surface with friction present at constant velocity.



F_N
constant velocity ($a_x = 0$)
T
rope
f_k
friction always opposes the motion
W

$$\therefore \Sigma F_x = +T - f_k$$

$$m\cancel{a_x}^0 = T - f_k \quad [\because a_x = 0]$$

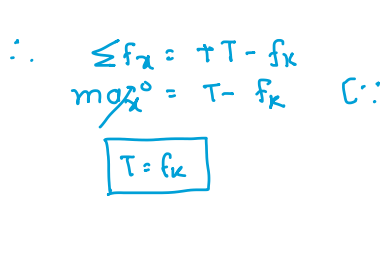
$$\boxed{T = f_k}$$

$$\therefore \Sigma F_y = +F_N - W$$

$$m\cancel{a_y}^0 = F_N - W \quad [\because a_y = 0]$$

$$\boxed{F_N = W}$$

Example (j): A rope, directed at 30° above horizontal, is used to pull block to the right across a horizontal surface with friction at constant acceleration.



$$\therefore \Sigma F_x = +T\cos\theta - f_k$$

$$ma_x = T\cos 30° - f_k$$

$$ma_x = \frac{\sqrt{3}}{2}T - f_k$$

$$\boxed{f_k = \frac{\sqrt{3}}{2}T - ma_x}$$

KNOWLEDGE: $f_k = \mu_k F_N$ → Kinetic friction Expression

$$\therefore \Sigma F_y = 0 \quad [\because \text{box is not moving vertically, it is only moving right}]$$

$$+F_N + T\sin\theta - W = 0$$

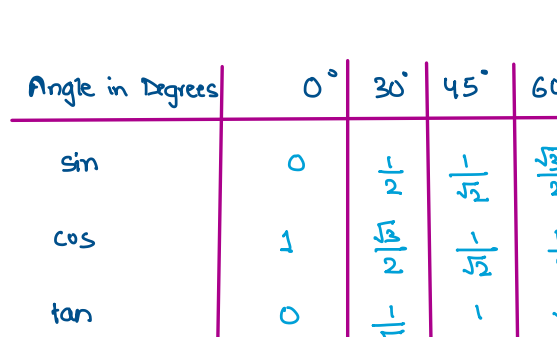
$$T\sin 30° = W - F_N$$

$$\boxed{T = 2(W - F_N)}$$

IMPORTANT TRIGONOMETRIC TABLE:

Angle in Degrees	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cosec	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not Defined
cot	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Example (k): A block slides down a frictionless incline.



$$\therefore \Sigma F_x = +W\cos(90-\theta)$$

$$ma_x = W\cos(90-\theta)$$

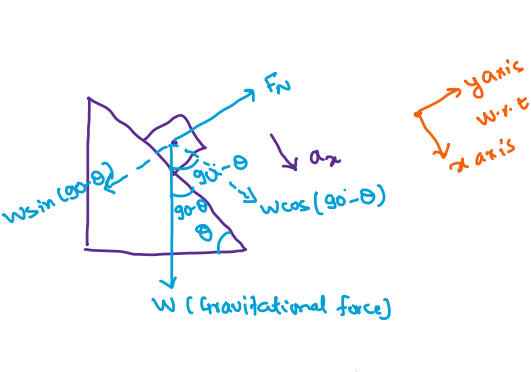
$$\boxed{ma_x = mg\sin\theta} \quad [\because \cos(\tfrac{\pi}{2}-\theta) = \sin\theta]$$

$$\therefore \Sigma F_y = 0$$

$$+F_N - W\sin(90-\theta) = 0$$

$$\boxed{F_N = mg\cos\theta} \quad [\because \sin(\tfrac{\pi}{2}-\theta) = \cos\theta]$$

Example (l): A block remains at rest on incline.



∴ Since, the box remains at rest, therefore.

$$\Sigma F_x = 0$$
$$+W\sin\theta - f_s = 0$$
$$\boxed{f_s = W\sin\theta}$$

$$\Sigma F_y = 0$$
$$+F_N - W\cos\theta = 0$$
$$\boxed{F_N = W\cos\theta}$$

Q. When static friction becomes kinetic friction?

Solution:

$$\boxed{f_s \leq \mu_s F_N}$$

let's say: $(f_s)_{max} = 100$ → after this static friction will become kinetic

F_A (Applied force)	f_s (static friction)
when, $F_A = 0$	then $f_s = 0$
when, $F_A = 10$	then f_s will be 10
when, $F_A = 20$ ⋮	then f_s will increase 20
when, $F_A = 90$	then f_s will increase 90 to oppose the motion.
when, $F_A = 100$	then, f_s will increase to 100 to oppose the motion. After this point f_s will become f_k (kinetic frictional force) and object will start moving in direction of Applied force.
when, $F_A = 110$	then $f_k = \mu_k F_N$ and object will be in motion.

Q: μ_s is always greater than equal to μ_k

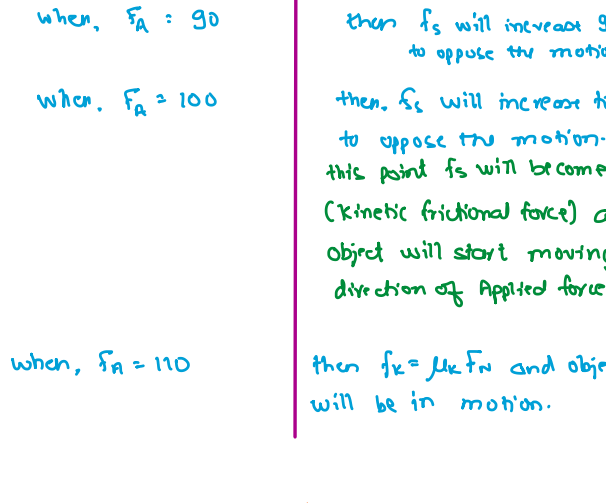
Solution:

$[\mu_s \geq \mu_k]$

It's very RARE that $\mu_s < \mu_k$ → Rare situation

Surfaces	μ_s	μ_k
Glass on Glass	0.94	0.4
Steel on Steel	0.74	0.57
Copper on Steel	0.53	0.36
Ice on ice	0.1	0.03
Teflon on steel	0.04	0.04

* **Important Stuff**:-



Ratio Changes:

sin → cos

cos → sin

tan → cot

cot → tan

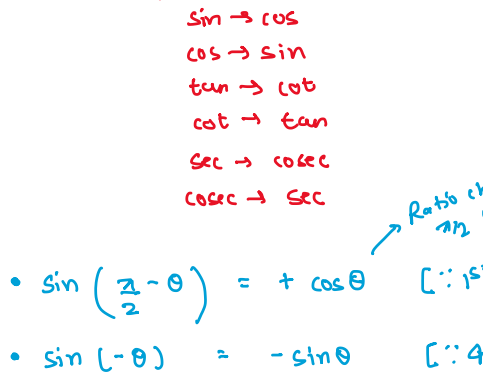
sec → cosec

cosec → sec

Ratio changes at π/2 or 3π/2

- $\sin\left(\frac{\pi}{2}-\theta\right) = +\cos\theta$ [∵ 1st Quad.]
- $\sin(-\theta) = -\sin\theta$ [∵ 4th Quad.]
- $\sec(\pi-\theta) = -\sec\theta$ [∵ 2nd Quad.]
- $\tan\left(\frac{3\pi}{2}+\theta\right) = -\cot\theta$ [∵ 4th Quad.]
- important * $\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$ [∵ 4th Quad]
- $\operatorname{cosec}(\pi+\theta) = -\operatorname{cosec}\theta$ [∵ 3rd Quad.]
- $\cos\left(\frac{3\pi}{2}+\theta\right) = +\sin\theta$ [∵ 4th Quad.]

Example (m): A block slides down an incline with friction present at constant acceleration.



$$\therefore \Sigma F_x = +W\sin\theta - f_k$$

$$+ma_x = W\sin\theta - f_k$$

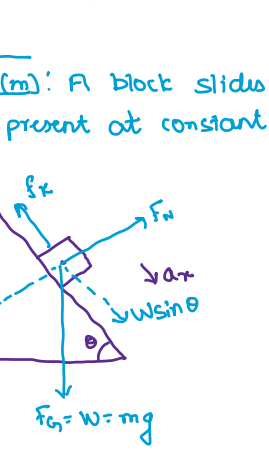
$$\boxed{ma_x = mg\sin\theta - \mu_k F_N} \quad [\because f_k = \mu_k F_N]$$

$$\therefore \Sigma F_y = 0 \quad [\because \text{block is only sliding in x-direction}]$$

$$+F_N - W\cos\theta = 0$$

$$\boxed{F_N = mg\cos\theta}$$

Example (n): A rope is used to pull a block up an incline against friction at constant velocity.



$$\therefore \Sigma F_x = +W\sin\theta + f_k - T$$

$$-m\cancel{a_x}^0 = W\sin\theta + \mu_k F_N - T \quad [\because a_x = 0]$$

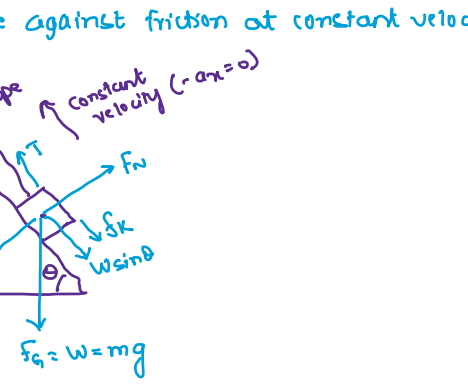
$$\boxed{T = mg\sin\theta + \mu_k F_N}$$

$$\therefore \Sigma F_y = 0 \quad [\because \text{block is not moving in y-direction}]$$

$$+F_N - W\cos\theta = 0$$

$$\boxed{F_N = mg\cos\theta}$$

SOLUTION OF PROBLEM 02:



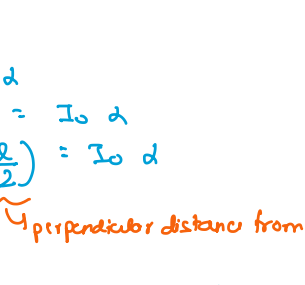
$$\tau_o = I_o\alpha$$
$$F \cdot r = I_o\alpha$$
$$(W\sin 60°) \cdot r = I_o\alpha$$
$$\left(mg\frac{\sqrt{3}}{2}\right) \cdot \underbrace{\left(\frac{2}{2}\right)}_{\text{perpendicular distance from pivot to center of mass of rod}} = I_o\alpha$$

$$1(9.8)\frac{\sqrt{3}}{2} \cdot \left(\frac{2}{2}\right) = \frac{1}{3}ml^2 \cdot \alpha$$
$$= \frac{1}{3}(1)(2)^2 \cdot \alpha$$

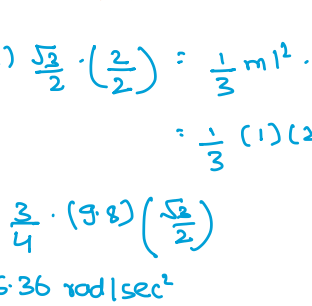
$$\alpha = \frac{3}{4} \cdot (9.8)\left(\frac{\sqrt{3}}{2}\right)$$
$$= \underline{6.36 \text{ rad/sec}^2}$$

3. **Problem 03:**

Problem: A constant force F is applied in the vertical direction to the symmetrical linkage starting from rest position shown. Determine the angular velocity ω which the links acquire as they reach the position θ = 0. Each link has a mass m_0. The wheel is a solid circular disk of mass m and rolls on the horizontal surface without slip. For the wheel $I_{cm} = \frac{mr^2}{2}$ and the linkage $I_{cm} = \frac{mb^2}{12}$



Solution:



Let each link have length b, mass m_0 and carry the top pin through a vertical drop of:

$$\Delta y = b\sin\theta$$

as the link sways from its initial angle θ down to the horizontal (θ = 0).

→ **Work done by the force:**

Since F is vertical and acts through a distance Δy, the work in bringing the links to θ = 0 is:

$$W = F\Delta y = Fb\sin\theta$$

→ **Kinetic Energy of the two links at θ = 0:**

At the instant they reach horizontal, each link is rotating about its lower pivot at angular speed ω. The moment of Inertia of one link about its end is:

$$I_{pivot} = I_{cm} + Md^2$$
$$= \frac{m_0b^2}{12} + m_0\left(\frac{b}{2}\right)^2$$
$$= \frac{m_0b^2}{12} + \frac{m_0b^2}{4}$$
$$= \frac{1}{3}m_0b^2.$$

Since there are 2 identical links, the total kinetic energy is:

$$\text{Total Kinetic Energy, } (KE)_{total} = \frac{1}{2}\overbrace{(2 \cdot I_{pivot})}^{\text{because 2 identical links}}\omega^2$$
$$= \frac{1}{3}m_0b^2\omega^2$$

(Notice that at the horizontal position the wheel is instantaneously at rest, so it carries zero kinetic energy at that instant.)

→ **Equate work to kinetic Energy**:-

By work-energy theorem:

$$Fb\sin\theta = \frac{1}{3}m_0b^2\omega^2$$

$$\omega^2 = \frac{3Fb\sin\theta}{m_0b^2}$$

$$\boxed{\omega = \sqrt{\frac{3F\sin\theta}{m_0b}}}$$

NOTE: We drop the disk entirely because in our work-energy balance its net change of kinetic energy is zero.

- At the start, everything (links + wheel) is at rest → $K_{wheel,i} = 0$
- At the instant the links first hit the horizontal (θ = 0), we can check that wheel is momentarily at rest again (pure rolling, but with zero instantaneous center-of-mass velocity) → $K_{wheel,f} = 0$
- Static friction does no work in pure rolling (the contact point is instantaneously stationary), so no energy is lost into the wheel b/w those two instants.

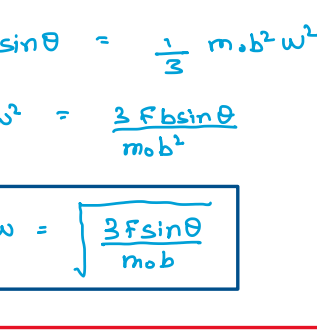
Since $\Delta K_{wheel} = K_{wheel,f} - K_{wheel,i}$
$= 0$

So, the wheel never appears in the energy equation

$$F b\sin\theta = \Delta K_{links} + \cancel{\Delta K_{wheel}}^0$$
$$= \Delta K_{links}$$

4. **Problem 04:**

The uniform disk of mass m is rotating with an angular velocity ω_0 when it is placed on the horizontal surface. Determine the time before it starts to roll without slip. The coefficient of kinetic friction is μ_k. For the wheel $I_{cm} = \frac{mr^2}{2}$.



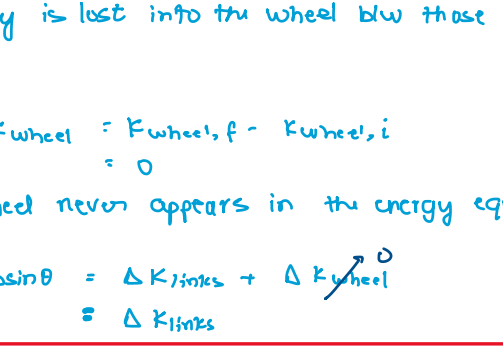
Solution:

Let's try to understand the situation first:

A uniform disk (mass m, radius r, moment of Inertia $I_{cm} = 1/2 \cdot mr^2$) is spinning with initial angular speed ω_0.

At that instant it is placed on a rough horizontal surface (coefficient of kinetic friction μ_k), its center of mass has zero velocity but the rim is slipping.

Now, we need to find out time T until it "locks in" to pure rolling without slip.



* **At t=0:**

- velocity at center of mass (com), $V_{com} = 0$
- angular velocity, ω_0
- The point of contact is slipping: its instantaneous velocity relative to ground is:

$$V_{contact} = V_{com} - r\omega$$
$$= 0 - r\omega$$
$$= -r\omega$$

↳ since it is non-zero so, kinetic friction applies.

- direction of kinetic friction:
  - Since the rim is slipping "backwards" (relative to com), friction acts forward, accelerating the center and applying a retarding torque to the spin.

* **Equations of motion:**

(i) frictional force (constant magnitude)

$$f_k = \mu_k mg \quad (\text{horizontal, forward})$$

(ii) Translation motion of com:

$$ma = +f_k$$
$$a = \frac{f_k}{m}$$
$$= \frac{\mu_k mg}{m}$$
$$\boxed{a = \mu_k \cdot g}$$

$$V_{com} = \cancel{u}^0 + at \quad [\because \text{no linear speed at } t=0]$$
$$= at$$
$$\boxed{V_{com} = \mu_k g t}$$

(iii) Rotational motion about com:

$$\tau_{friction} = -f_k r = I_{com} \cdot \alpha$$

$$\alpha = -\frac{f_k \cdot r}{I_{com}}$$
$$= -\frac{f_k \cdot r}{\frac{1}{2}mr^2}$$
$$= -\frac{2f_k}{mr}$$
$$= -\frac{2\mu_k mg}{mr}$$
$$= -\frac{2\mu_k \cdot g}{r}$$

$$\omega(t) = \omega_0 + \alpha t$$

$$\boxed{\omega(t) = \omega_0 - \frac{2\mu_k \cdot g}{r}t}$$

* **Pure Rolling Condition**:

Pure rolling begins at time T, when

$$V_{com}(T) = r \cdot \omega(T)$$

$$\mu_k \cdot g \cdot T = r \cdot \left(\omega_0 - \frac{2\mu_k \cdot g}{r}T\right)$$

$$T(\mu_k g + 2\mu_k \cdot g) = r\omega_0$$

$$\boxed{T = \frac{r\omega_0}{3\mu_k \cdot g}}$$

—— x —— THE END —— x ——