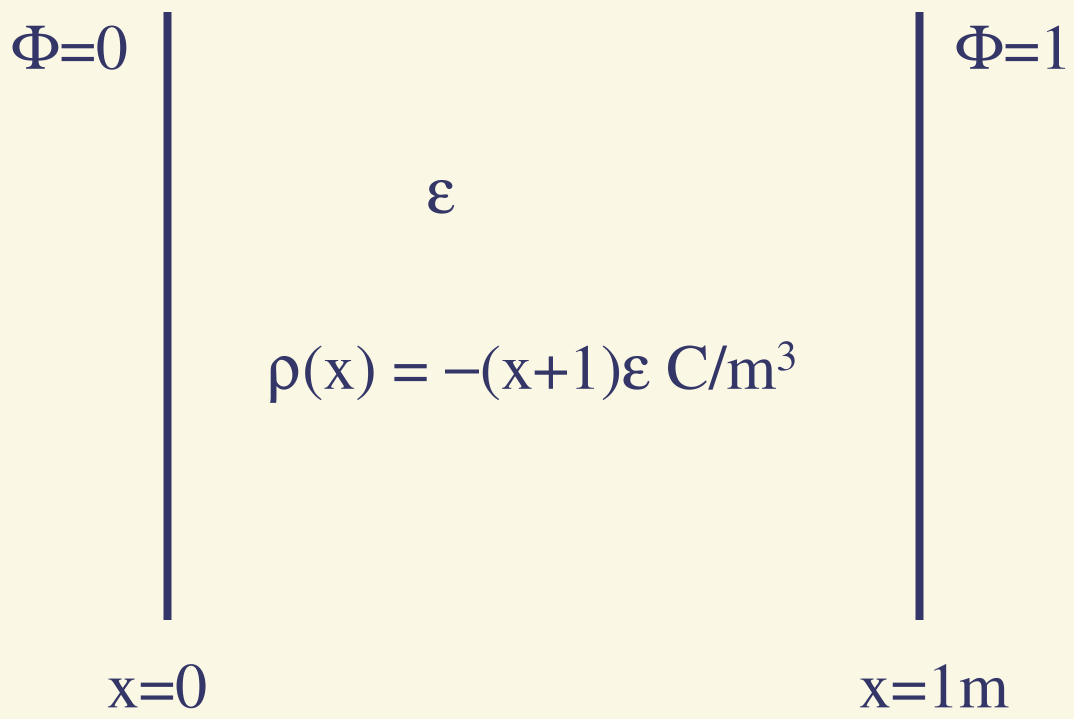
Outline

- A Simple Example
  - The Ritz Method
  - Galerkin’s Method
  - The Finite-Element Method
- FEM Definition
- Basic FEM Steps

Example

Problem Statement



Example

Differential equation:

$$\frac{d^2\phi}{dx^2} = x + 1 \qquad 0 < x < 1$$

Boundary condition:

$$\phi\big|_{x=0} = 0$$
$$\phi\big|_{x=1} = 1$$

Solution:

$$\phi(x) = \tfrac{1}{6}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{3}x$$

Example

A. The Ritz method

Vartiational solution: $\delta F(\phi) = 0$

$$F(\tilde{\phi}) = \frac{1}{2}\int_0^1 \left(\frac{d\tilde{\phi}}{dx}\right)^2 dx + \int_0^1 (x+1)\tilde{\phi}\, dx$$

Proof:

$$\delta F(\phi) = \int_0^1 \frac{d\phi}{dx}\frac{d\delta\phi}{dx}\, dx + \int_0^1 (x+1)\delta\phi\, dx = 0$$

$$\delta\phi \frac{d\phi}{dx}\bigg|_{x=0}^{x=1} - \int_0^1 \left(\frac{d^2\phi}{dx^2} - x - 1\right)\delta\phi\, dx = 0$$

= 0 = 0

Example

Expand $$\tilde{\phi}(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

Apply BC:

$$\phi|_{x=0} = 0$$
$$\phi|_{x=1} = 1$$

$$\Rightarrow \quad c_1 = 0$$
$$c_2 = 1 - c_3 - c_4$$

$$\Rightarrow \quad \tilde{\phi}(x) = x + c_3(x^2 - x) + c_4(x^3 - x)$$

Functional:

$$F = \tfrac{2}{5}c_4^2 + \tfrac{1}{6}c_3^2 + \tfrac{1}{2}c_3 c_4 - \tfrac{23}{60}c_4 - \tfrac{1}{4}c_3 + \tfrac{4}{3}$$

$$\frac{\partial F}{\partial c_3} = \tfrac{1}{3}c_3 + \tfrac{1}{2}c_4 - \tfrac{1}{4}$$
$$\frac{\partial F}{\partial c_4} = \tfrac{1}{2}c_3 + \tfrac{4}{5}c_4 - \tfrac{23}{60}$$

$$\Rightarrow \quad \boxed{c_3 = \frac{1}{2}, \quad c_4 = \frac{1}{6}}$$

Example

B. Galerkin's method

Weighted residual: $$\int_0^1 w_i \left(\frac{d^2\tilde{\phi}}{dx^2} - x - 1 \right) dx = 0$$

Expansion: $$\tilde{\phi}(x) = x + c_3(x^2 - x) + c_4(x^3 - x)$$

Weighting: $w_1 = x^2 - x$ $w_2 = x^3 - x$

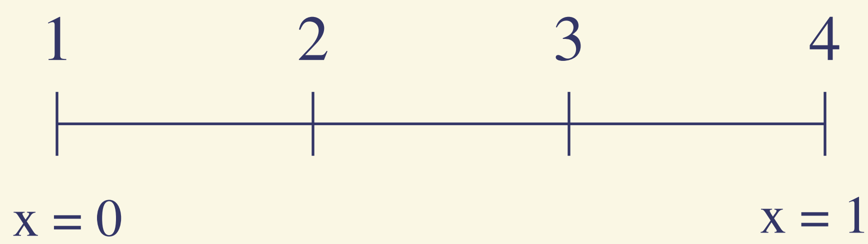
Solution:

$$\frac{1}{3}c_3 + \frac{1}{2}c_4 - \frac{1}{4} = 0$$
$$\frac{1}{2}c_3 + \frac{4}{5}c_4 - \frac{23}{60} = 0$$

$$\Rightarrow \boxed{c_3 = \frac{1}{2}, \quad c_4 = \frac{1}{6}}$$

Example

C. Another method



Linear interpolation:

$$\tilde{\phi}(x) = \phi_i \frac{x_{i+1} - x}{x_{i+1} - x_i} + \phi_{i+1} \frac{x - x_i}{x_{i+1} - x_i}$$

Boundary condition:

$$\phi_1 = 0 \text{ and } \phi_4 = 1$$

Example

Apply the Ritz method:

$$F = \sum_{i=1}^{3} \left[\frac{1}{2} \int_{x_i}^{x_{i+1}} \left(\frac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i} \right)^2 dx \right.$$
$$\left. + \int_{x_i}^{x_{i+1}} (x+1) \left(\phi_i \frac{x_{i+1} - x}{x_{i+1} - x_i} + \phi_{i+1} \frac{x - x_i}{x_{i+1} - x_i} \right) dx \right]$$

Integration:

$$F = 3\phi_2^2 + 3\phi_3^2 - 3\phi_2\phi_3 + \tfrac{4}{9}\phi_2 - \tfrac{22}{9}\phi_3 + \tfrac{49}{27}$$

$$\frac{\partial F}{\partial \phi_2} = 6\phi_2 - 3\phi_3 + \tfrac{4}{9} = 0$$
$$\frac{\partial F}{\partial \phi_3} = -3\phi_2 + 6\phi_3 - \tfrac{22}{9} = 0$$

$$\Rightarrow$$

$$\phi_2 = \tfrac{14}{81}$$
$$\phi_3 = \tfrac{40}{81}$$

Example

Apply Galerkin's method:

Weighting:

$$w_i = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{for} \quad x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{for} \quad x_i \le x \le x_{i+1} \end{cases}$$

Integration by parts:

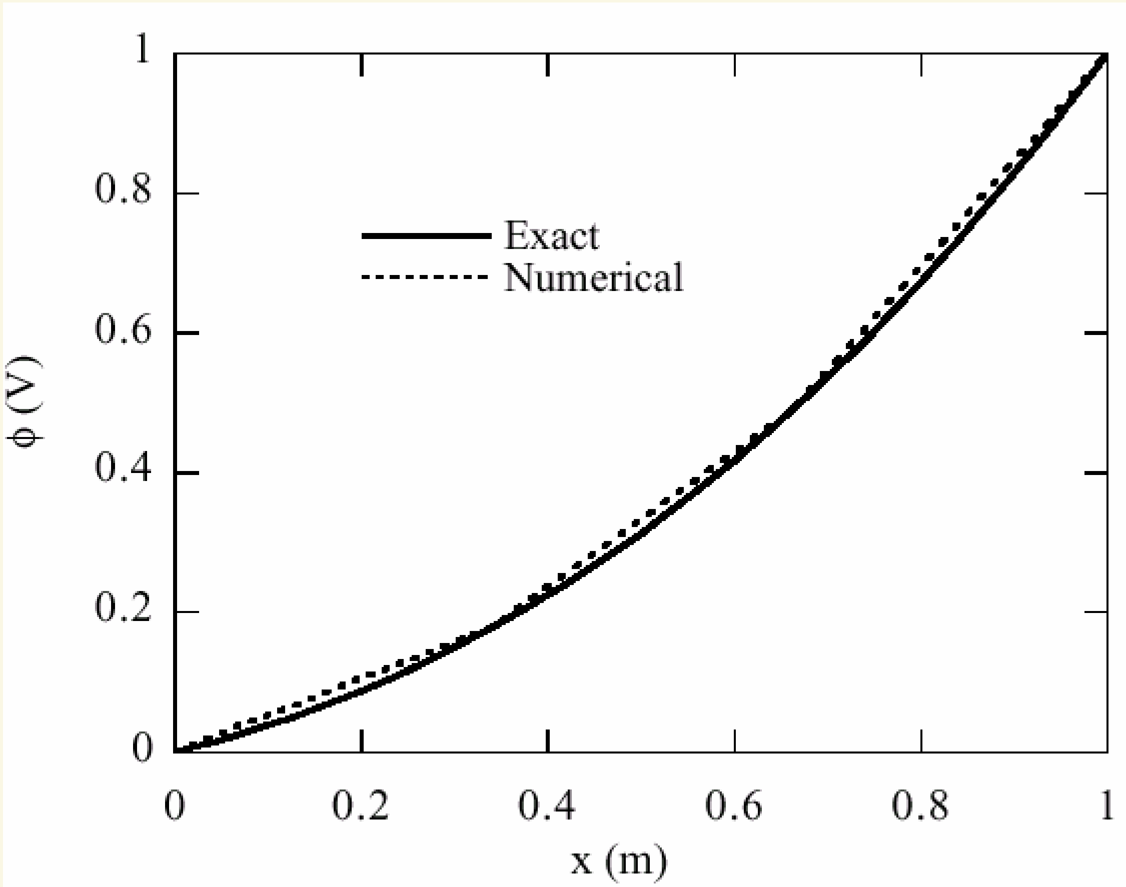
$$\int_{x_{i-1}}^{x_{i+1}} w_i \left(\frac{d^2 \tilde{\phi}}{dx^2} \right) dx = w_i \frac{d\tilde{\phi}}{dx} \bigg|_{x_{i-1}}^{x_{i+1}} - \int_{x_{i-1}}^{x_{i+1}} \frac{dw_i}{dx} \frac{d\tilde{\phi}}{dx}\, dx$$

$$\Rightarrow \int_{x_{i-1}}^{x_{i+1}} \frac{dw_i}{dx} \frac{d\tilde{\phi}}{dx}\, dx + \int_{x_{i-1}}^{x_{i+1}} (x+1) w_i \, dx = 0$$

$$\begin{aligned} 6\phi_2 - 3\phi_3 + \tfrac{4}{9} &= 0 \\ -3\phi_2 + 6\phi_3 - \tfrac{22}{9} &= 0 \end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned} \phi_2 &= \tfrac{14}{81} \\ \phi_3 &= \tfrac{40}{81} \end{aligned}}$$

Example

Numerical solution:



FEM Definition

The above solution procedure

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The finite element method

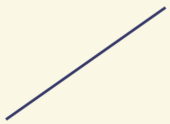
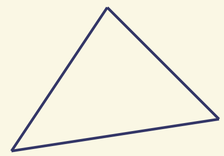
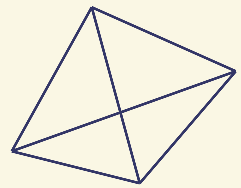
The Ritz variational FEM

The Galerkin FEM

Equivalent for self-adjoint problems

Basic FEM Steps

1. Discretization/subdivision of solution domain

1-D:  2-D:  3-D: 

2. Selection of interpolation schemes

   Linear or higher-order polynomials

3. Formulation of the system of equations

   Using either the Ritz or Galerkin method

4. Solution of the system of equations

   Using either a direct or iterative method