Mathematical Engineering

Hung Nguyen-Schäfer
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Tensor Analysis and Elementary Differential Geometry for Physicists and Engineers

Second Edition

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Second Edition

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In memory of
Gregorio Ricci-Curbastro (1853–1925) and
Tullio Levi-Civita (1873–1941),
who invented Tensor Calculus, for which
Elwin Bruno Christoffel (1829–1900)
had prepared the ground;
Carl Friedrich Gauss (1777–1855) and
Bernhard Riemann (1826–1866),
who invented Differential Geometry

Preface to the Second Edition

In the second edition, all chapters are carefully revised in which typos are corrected and many additional sections are included. Furthermore, two new chapters that deal with Cartan differential forms and applications of tensors and Dirac notation to quantum mechanics are added in this book.

Contrary to tensors, Cartan differential forms based on exterior algebra in Chap. 4 using the wedge product are an approach to multivariable calculus that is independent of coordinates. Therefore, they are very useful methods for differential geometry, topology, and theoretical physics in multidimensional manifolds.

In Chap. 6, the quantum entanglement of a composite system that consists of two entangled subsystems has alternatively been interpreted by means of symmetries. Both mathematical approaches of Dirac matrix and wave formulations are used to analyze and calculate the expectation values, probability density operators, and wave functions for nonrelativistic and relativistic particles in a composite system using time-dependent Schrödinger equation (TDSE), Klein–Gordon equation, and Dirac equation as well.

I would like to thank Mrs. Eva Hestermann-Beyerle and Mrs. Birgit Kollmar-Thoni at Springer Heidelberg for their invaluable suggestions and excellent cooperation to publish this second edition successfully.

Finally, my special thanks go to my wife for her understanding, patience, and endless support as I wrote this book in my leisure and vacation time.

Ludwigsburg, Germany Hung Nguyen-Schäfer

Preface to the First Edition

This book represents a joint effort by a research engineer and a mathematician. The initial idea for it arose from our many years of experience in the automotive industry, from advanced research development, and of course from our common research interest in applied mathematics, physics, and engineering. The main reason for this cooperation is the fact that mathematicians generally approach problems using mathematical rigor, but which need not always be practically applicable; at the same time, engineers usually deal with problems involving applied mathematics, which must ultimately work in the "real world" of industry. Having recognized that what mathematicians consider rigor can be more like rigor mortis for engineers and physicists, this joint effort proposes a compromise between the mathematical rigors and less rigorous applied mathematics, incorporating different points of view.

Our main aim is to bridge the mathematical gap between where physics and engineering mathematics end and where tensor analysis begins, which we do with the help of a powerful and user-friendly tool often employed in computational methods for physical and engineering problems in any general curvilinear coordinate system. However, tensor analysis has certain strict rules and conventions that must unconditionally be adhered to. This book is intended to support research scientists and practicing engineers in various fields who use tensor analysis and differential geometry in the context of applied physics and electrical and mechanical engineering. Moreover, it can also be used as a textbook for graduate students in applied physics and engineering.

Tensor analysis and differential geometry were pioneered by great mathematicians in the late nineteenth century, chiefly Curbastro, Levi-Civita, Christoffel, Ricci, Gauss, Riemann, Weyl, and Minkowski, and later promoted by well-known theoretical physicists in the early twentieth century, mainly Einstein, Dirac, Heisenberg, and Fermi, working on relativity and quantum mechanics. Since then, tensor analysis and differential geometry have taken on an increasingly important role in the mathematical language used in the modern physics of quantum mechanics and general relativity and in many applied sciences fields. They have also been applied to computational mechanical and electrical engineering in classical mechanics, aero- and vibroacoustics, computational fluid dynamics (CFD), continuum mechanics, electrodynamics, and cybernetics.

Approaching the topics of tensors and differential geometry in a mathematically rigorous way would require an immense amount of effort, which would not be practical for working engineers and applied physicists. As such, we decided to present these topics in a comprehensive and approachable way that will show readers how to work with tensors and differential geometry and to apply them to modeling the physical and engineering world. This book also includes numerous examples with solutions and concrete calculations in order to guide readers through these complex topics step by step. For the sake of simplicity and keeping the target audience in mind, we deliberately neglect certain aspects of mathematical rigor in this book, discussing them informally instead. Therefore, those readers who are more mathematically interested should consult the recommended literature.

We would like to thank Mrs. Hestermann-Beyerle and Mrs. Kollmar-Thoni at Springer Heidelberg for their helpful suggestions and valued cooperation during the preparation of this book.

Ludwigsburg, Germany Hung Nguyen-Schäfer
Heidelberg, Germany Jan-Philip Schmidt

Contents

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Jan-Philip Schmidt is a mathematician. He studied mathematics, physics, and economics at the University of Heidelberg, Germany. He received a Ph.D. degree in mathematics from the University of Heidelberg in 2012. His doctoral thesis was funded by a research fellowship from the Heidelberg Academy of Sciences, in collaboration with the Interdisciplinary Center for Scientific Computing (IWR) at the University of Heidelberg. His academic working experience comprises several research visits in France and Israel, as well as project works at the Max-Planck-Institute for Mathematics in the Sciences (MPIMIS) in Leipzig and at the Max-Planck-Institute for Molecular Genetics (MPIMG) in Berlin. He also worked as a research associate in the AVACS program at Saarland University, Cluster of Excellence (MMCI).

Chapter 1
General Basis and Bra-Ket Notation

We begin this chapter by reviewing some mathematical backgrounds dealing with coordinate transformations and general basis vectors in general curvilinear coordinates. Some of these aspects will be informally discussed for the sake of simplicity. Therefore, those readers interested in more in-depth coverage should consult the literature recommended under Further Reading. To simplify notation, we will denote a basis vector simply as **basis** in the following section.

We assume that the reader has already had fundamental backgrounds about vector analysis in finite N-dimensional spaces with the general bases of curvilinear coordinates. However, this topic is briefly recapitulated in Appendix E.

1.1 Introduction to General Basis and Tensor Types

A physical state generally depending on N different variables is defined as a point $\mathrm{P}(u^1,\ldots,u^N)$ that has N independent coordinates of u^i. At changing the variables, such as time, locations, and physical characteristics, the physical state P moves from one position to other positions. All relating positions generate a set of points in an N-dimension space. This is the point space with N dimensions (N-point space). Additionally, the state change between two points could be described by a vector $\mathbf{r}$ connecting them that obviously consists of N vector components. All state changes are displayed by the vector field that belongs to the vector space with N dimensions (N-vector space). Generally, a differentiable hypersurface in an N-dimensional space with general curvilinear coordinates $\{u^i\}$ for $i = 1, 2,\ldots,N$ is defined as a differentiable $(N-1)$-dimensional subspace with a codimension of one. Subspaces with any codimension are called manifolds of an N-dimensional space (cf. Appendix E).

Physically, vectors are invariant under coordinate transformations and therefore do not change in any coordinate system. However, their components change and

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depend on the coordinate system. That means the vector components vary as the coordinate system changes. Generally, tensors are a very useful tool applied to the coordinate transformations between two general curvilinear coordinate systems in finite N-dimensional real spaces. The exemplary second-order tensor can be defined as a multilinear functional **T** that maps an arbitrary vector in a vector space into the image vector in another vector space. Like vectors, tensors are also invariant under coordinate transformations; however, the tensor components change and depend on the relating transformed coordinate system. Therefore, the tensor components change as the coordinate system varies.

Scalars, vectors, and matrices are special types of tensors:

- scalar (invariant) is a zero-order tensor,
- vector is a first-order tensor,
- matrix is arranged by a second-order tensor,
- bra and ket are first- and second-order tensors,
- Levi-Civita permutation symbols in a three-dimensional space are third-order pseudo-tensors (Table 1.1).

We consider two important spaces in tensor analysis: first, Euclidean N-spaces with orthogonal and curvilinear coordinate systems; second, general curvilinear Riemannian manifolds of dimension N (cf. Appendix E).

1.2 General Basis in Curvilinear Coordinates

We consider three covariant basis vectors $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ to the general curvilinear coordinates (u^1,u^2,u^3) at the point P in Euclidean space $\mathbf{E^3}$. The non-orthonormal basis $\mathbf{g}_i$ can be calculated from the orthonormal bases $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ in Cartesian coordinates $x^j = x^j(u^i)$ using Einstein summation convention (cf. Sect. 2.1).

Table 1.1 Different types of tensors

Tensors of	0-order T	1-order T_i	2-order T_{ij}	3-order T_{ijk}	Higher-order $T_{ij...pk}$
Scalar a $\in \mathbf{R}$	X				
Vector $\mathbf{v} \in \mathbf{R}^N$		X			
Matrix $\mathbf{M} \in \mathbf{R}^N \times \mathbf{R}^N$			X		
Bra $\langle \mathbf{B}|$ and ket $|\mathbf{A}\rangle \in \mathbf{R}^N$, $\mathbf{R}^N \times \mathbf{R}^N$		X	X		
Levi-Civita symbols $\in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N$				X	(X)
Higher-order tensors $\in \mathbf{R}^N \times \ldots \times \mathbf{R}^N$					X

$$\begin{aligned}\mathbf{g}_i &\equiv \frac{\partial \mathbf{r}}{\partial u^i} = \sum_{j=1}^{3} \frac{\partial \mathbf{r}}{\partial x^j} \cdot \frac{\partial x^j}{\partial u^i} \equiv \frac{\partial \mathbf{r}}{\partial x^j} \cdot \frac{\partial x^j}{\partial u^i} \\ &= \mathbf{e}_j \frac{\partial x^j}{\partial u^i} \text{ for } j = 1, 2, 3\end{aligned} \tag{1.1}$$

The metric coefficients can be calculated by the scalar products of the covariant and contravariant bases in general curvilinear coordinates with non-orthonormal bases (i.e. non-orthogonal and non-unitary). There are the covariant, contravariant, and mixed metric coefficients g_{ij}, g^{ij}, and g_i^j, respectively.

$$\begin{aligned}&g_{ij} = g_{ji} = \mathbf{g}_i \cdot \mathbf{g}_j = \mathbf{g}_j \cdot \mathbf{g}_i \neq \delta_i^j \\ &g^{ij} = g^{ji} = \mathbf{g}^i \cdot \mathbf{g}^j = \mathbf{g}^j \cdot \mathbf{g}^i \neq \delta_i^j \\ &g_i^j = \mathbf{g}_i \cdot \mathbf{g}^j = \mathbf{g}_j \cdot \mathbf{g}^i = \delta_i^j\end{aligned} \tag{1.2}$$

where the Kronecker delta δ_i^j is defined as

$$\delta_i^j \equiv \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j. \end{cases}$$

Similarly, the bases of the orthonormal coordinates can be written in the non-orthonormal bases of the curvilinear coordinates $u^i = u^i(x^j)$.

$$\begin{aligned}\mathbf{e}_j &\equiv \frac{\partial \mathbf{r}}{\partial x^j} = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial u^i}{\partial x^j} \equiv \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial u^i}{\partial x^j} \\ &= \mathbf{g}_i \frac{\partial u^i}{\partial x^j} \text{ for } i = 1, 2, 3\end{aligned} \tag{1.3}$$

The covariant and contravariant bases of the orthonormal coordinates (orthogonal and unitary bases) have the following properties:

$$\begin{aligned}\mathbf{e}_i \cdot \mathbf{e}_j &= \mathbf{e}_j \cdot \mathbf{e}_i = \delta_i^j; \\ \mathbf{e}^i \cdot \mathbf{e}^j &= \mathbf{e}^j \cdot \mathbf{e}^i = \delta_i^j; \\ \mathbf{e}^i \cdot \mathbf{e}_j &= \mathbf{e}_j \cdot \mathbf{e}^i = \delta_i^j.\end{aligned} \tag{1.4}$$

The contravariant basis $\mathbf{g}^k$ of the curvilinear coordinate u^k is perpendicular to the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$ at the given point P, as shown in Fig. 1.1. The contravariant basis $\mathbf{g}^k$ can be defined as

$$\alpha \mathbf{g}^k \equiv \mathbf{g}_i \times \mathbf{g}_j \equiv \frac{\partial \mathbf{r}}{\partial u^i} \times \frac{\partial \mathbf{r}}{\partial u^j} \tag{1.5}$$

where

α is a scalar factor;

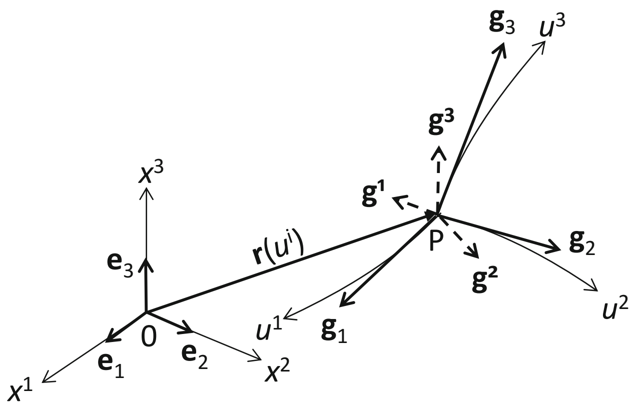

Fig. 1.1 Covariant and contravariant bases of curvilinear coordinates

$\mathbf{g}^k$ is the contravariant basis of the curvilinear coordinate of u^k.

Multiplying Eq. (1.5) by the covariant basis $\mathbf{g}_k$, the scalar factor α results in

$$\begin{aligned}\left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot \mathbf{g}_k &= \alpha \mathbf{g}^k \cdot \mathbf{g}_k = \alpha \delta_k^k = \alpha \\ &\equiv \left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right]\end{aligned} \tag{1.6}$$

The expression in the square brackets is called the scalar triple product.

Therefore, the contravariant bases of the curvilinear coordinates result from Eqs. (1.5) and (1.6).

$$\mathbf{g}^i = \frac{\mathbf{g}_j \times \mathbf{g}_k}{\left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right]}; \quad \mathbf{g}^j = \frac{\mathbf{g}_k \times \mathbf{g}_i}{\left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right]}; \quad \mathbf{g}^k = \frac{\mathbf{g}_i \times \mathbf{g}_j}{\left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right]} \tag{1.7}$$

Obviously, the relation of the covariant and contravariant bases results from Eq. (1.7).

$$\mathbf{g}^k \cdot \mathbf{g}_i = \frac{\left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot \mathbf{g}_i}{\left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right]} = \delta_i^k \tag{1.8}$$

where δ_i^k is the Kronecker delta.

The scalar triple product is an invariant under cyclic permutation; therefore, it has the following properties:

$$\left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot \mathbf{g}_k = (\mathbf{g}_k \times \mathbf{g}_i) \cdot \mathbf{g}_j = \left(\mathbf{g}_j \times \mathbf{g}_k\right) \cdot \mathbf{g}_i \tag{1.9}$$

Furthermore, the scalar triple product of the covariant bases of the curvilinear coordinates can be calculated [1].

$$[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = \varepsilon_{ijk} \frac{\partial x^i}{\partial u^1} \frac{\partial x^j}{\partial u^2} \frac{\partial x^k}{\partial u^3} = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{vmatrix} \equiv J \tag{1.10}$$

where J is the Jacobian, determinant of the covariant basis tensor; ε_{ijk} is the Levi-Civita symbols in Eq. (A.5), cf. Appendix A.

Squaring the scalar triple product in Eq. (1.10), one obtains

$$\begin{aligned} [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]^2 &= \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{vmatrix}^2 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} \\ &= \left| g_{ij} \right| \equiv g = J^2 \Rightarrow J = \pm\sqrt{g} \end{aligned} \tag{1.11}$$

where $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ are the covariant metric coefficients.

Thus, the scalar triple product of the covariant bases results in a right-handed rule coordinate system in which the Jacobian is always positive.

$$\begin{aligned} J &= [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 \\ &= \sqrt{g} > 0 \end{aligned} \tag{1.12}$$

The covariant and contravariant bases of the orthogonal cylindrical and spherical coordinates will be studied in the following subsections.

1.2.1 Orthogonal Cylindrical Coordinates

Cylindrical coordinates (r,θ,z) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary. Figure 1.2 shows a point P in the cylindrical coordinates (r,θ,z) which is embedded in the orthonormal Cartesian coordinates (x^1,x^2,x^3). However, the cylindrical coordinates change as the point P varies.

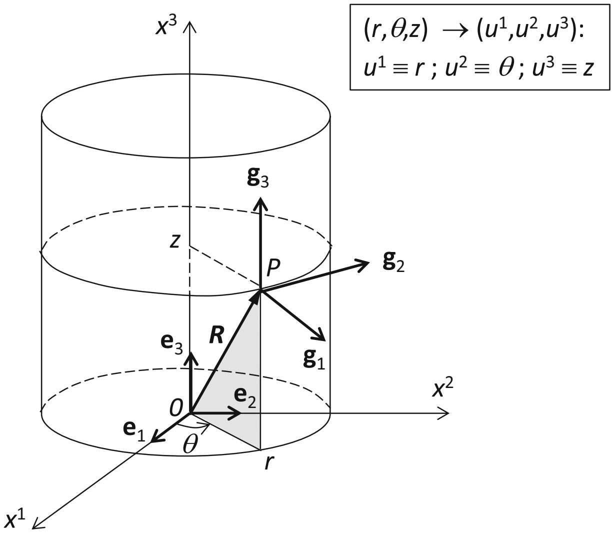

Fig. 1.2 Covariant bases of orthogonal cylindrical coordinates

The vector **OP** can be written in Cartesian coordinates (x^1,x^2,x^3):

$$\begin{aligned}\mathbf{R} &= (r\cos\theta)\,\mathbf{e}_1 + (r\sin\theta)\,\mathbf{e}_2 + z\,\mathbf{e}_3 \\ &\equiv x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3\end{aligned} \tag{1.13}$$

where

$\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the orthonormal bases of Cartesian coordinates;

θ is the polar angle.

To simplify the formulation with Einstein symbol, the coordinates of u^1, u^2, and u^3 are used for r, θ, and z, respectively. Therefore, the coordinates of $P(u^1,u^2,u^3)$ can be expressed in Cartesian coordinates:

$$P\left(u^1,u^2,u^3\right) = \left\{\begin{array}{l} x^1 = r\cos\theta \equiv u^1\cos u^2 \\ x^2 = r\sin\theta \equiv u^1\sin u^2 \\ x^3 = z \equiv u^3 \end{array}\right\} \tag{1.14}$$

The covariant bases of the curvilinear coordinates can be computed from

$$\mathbf{g}_i = \frac{\partial \mathbf{R}}{\partial u^i} = \frac{\partial \mathbf{R}}{\partial x^j}\cdot\frac{\partial x^j}{\partial u^i} = \mathbf{e}_j\frac{\partial x^j}{\partial u^i} \text{ for } j = 1,2,3 \tag{1.15}$$

The covariant basis matrix **G** can be calculated from Eq. (1.15).

$$\mathbf{G} = [\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3] = \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.16}$$

The determinant of the covariant basis matrix $\mathbf{G}$ is called the Jacobian J.

$$|\mathbf{G}| \equiv J = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \tag{1.17}$$

The inversion of the matrix $\mathbf{G}$ yields the contravariant basis matrix $\mathbf{G}^{-1}$. The relation between the covariant and contravariant bases results from Eq. (1.8).

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j \quad \text{(Kronecker delta)} \tag{1.18a}$$

At det $(\mathbf{G}) \neq 0$ given from Eq. (1.17), Eq. (1.18a) is equivalent to

$$\mathbf{G}^{-1}\mathbf{G} = \mathbf{I} \tag{1.18b}$$

Thus, the contravariant basis matrix $\mathbf{G}^{-1}$ can be calculated from the inversion of the covariant basis matrix $\mathbf{G}$, as given in Eq. (1.16).

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \begin{pmatrix} \frac{\partial u^1}{\partial x^1} & \frac{\partial u^1}{\partial x^2} & \frac{\partial u^1}{\partial x^3} \\ \frac{\partial u^2}{\partial x^1} & \frac{\partial u^2}{\partial x^2} & \frac{\partial u^2}{\partial x^3} \\ \frac{\partial u^3}{\partial x^1} & \frac{\partial u^3}{\partial x^2} & \frac{\partial u^3}{\partial x^3} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & r \end{pmatrix} \tag{1.19a}$$

The contravariant bases of the curvilinear coordinates can be written as

$$\mathbf{g}^i = \frac{\partial u^i}{\partial x^j}\mathbf{e}_j \text{ for } j = 1, 2, 3 \tag{1.19b}$$

The calculation of the determinant and inversion matrix of $\mathbf{G}$ will be discussed in the following section.

According to Eq. (1.16), the covariant bases can be rewritten as

$$\begin{cases} \mathbf{g}_1 = (\cos\theta)\,\mathbf{e}_1 + (\sin\theta)\,\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow |\mathbf{g}_1| = 1 \\ \mathbf{g}_2 = (-r\sin\theta)\,\mathbf{e}_1 + (r\cos\theta)\,\mathbf{e}_2 + 0\cdot\ \mathbf{e}_3 \Rightarrow |\mathbf{g}_2| = r \\ \mathbf{g}_3 = 0\cdot\mathbf{e}_1 + 0\cdot\mathbf{e}_2 + 1\cdot\mathbf{e}_3 \Rightarrow |\mathbf{g}_3| = 1 \end{cases} \tag{1.20}$$

The contravariant bases result from Eq. (1.19b).

$$\begin{cases} \mathbf{g}^1 = (\cos\theta)\,\mathbf{e}_1 + (\sin\theta)\,\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^1\right| = 1 \\ \mathbf{g}^2 = \left(-\dfrac{\sin\theta}{r}\right)\mathbf{e}_1 + \left(\dfrac{\cos\theta}{r}\right)\mathbf{e}_2 + 0\cdot\ \mathbf{e}_3 \Rightarrow \left|\mathbf{g}^2\right| = \dfrac{1}{r} \\ \mathbf{g}^3 = 0\ \cdot\mathbf{e}_1 + 0\cdot\ \mathbf{e}_2 + 1\cdot\ \mathbf{e}_3 \Rightarrow \left|\mathbf{g}^3\right| = 1 \end{cases} \tag{1.21}$$

Not only the covariant bases but also the contravariant bases of the cylindrical coordinates are orthogonal due to

$$\begin{aligned} &\mathbf{g}_i\cdot\mathbf{g}^j = \mathbf{g}^j\cdot\mathbf{g}_i = \delta_i^j; \\ &\mathbf{g}_i\cdot\mathbf{g}_j = 0 \text{ for } \mathrm{i}\neq\mathrm{j}; \\ &\mathbf{g}^i\cdot\ \mathbf{g}^j = 0 \text{ for } \mathrm{i}\neq\mathrm{j}. \end{aligned}$$

1.2.2 Orthogonal Spherical Coordinates

Spherical coordinates (ρ,φ,θ) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary. Figure 1.3 shows a point P in the spherical coordinates (ρ,φ,θ) which is embedded in the orthonormal Cartesian coordinates (x^1,x^2,x^3). However, the spherical coordinates change as the point P varies.

The vector **OP** can be rewritten in Cartesian coordinates (x^1,x^2,x^3):

$$\begin{aligned} \mathbf{R} = &\ (\rho\sin\varphi\cos\theta)\,\mathbf{e}_1 + \ (\rho\sin\varphi\sin\theta)\,\mathbf{e}_2 + \rho\cos\varphi\,\mathbf{e}_3 \\ &\equiv x^1\mathbf{e}_1 + \ x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \end{aligned} \tag{1.22}$$

where

$\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the orthonormal bases of Cartesian coordinates;
φ is the equatorial angle;
θ is the polar angle.

To simplify the formulation with Einstein symbol, the coordinates of u^1, u^2, and u^3 are used for ρ, φ, and θ, respectively. Therefore, the coordinates of $P(u^1,u^2,u^3)$ can be expressed in Cartesian coordinates:

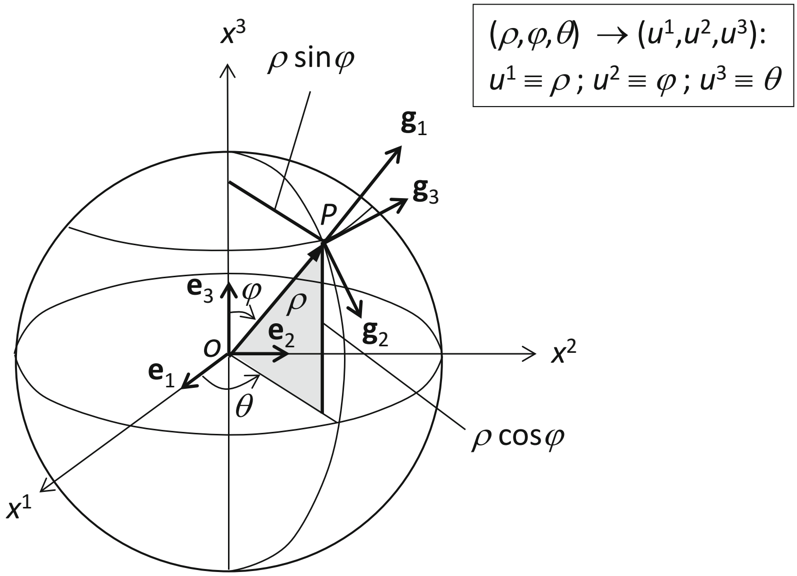

Fig. 1.3 Covariant bases of orthogonal spherical coordinates

$$P\left(u^1,u^2,u^3\right) = \left\{\begin{array}{l} x^1 = \rho\sin\varphi\cos\theta \equiv u^1\sin u^2\cos u^3 \\ x^2 = \rho\sin\varphi\sin\theta \equiv u^1\sin u^2\cos u^3 \\ x^3 = \rho\cos\varphi \equiv u^1\cos u^2 \end{array}\right\} \tag{1.23}$$

The covariant bases of the curvilinear coordinates can be computed by means of

$$\begin{aligned} \mathbf{g}_i &= \frac{\partial \mathbf{R}}{\partial u^i} = \frac{\partial \mathbf{R}}{\partial x^j}\cdot\frac{\partial x^j}{\partial u^i} \\ &= \mathbf{e}_j\frac{\partial x^j}{\partial u^i}\ \text{for } j = 1,2,3 \end{aligned} \tag{1.24}$$

Thus, the covariant basis matrix **G** can be calculated from Eq. (1.24).

$$\begin{aligned} \mathbf{G} = [\,\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3\,] &= \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{pmatrix} \\ &= \begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix} \end{aligned} \tag{1.25}$$

$$|\mathbf{G}| \equiv J = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix} = \rho^2\sin\varphi \tag{1.26}$$

The determinant of the covariant basis matrix **G** is called the Jacobian J.

Similarly, the contravariant basis matrix $\mathbf{G}^{-1}$ is the inversion of the covariant basis matrix.

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \begin{pmatrix} \dfrac{\partial u^1}{\partial x^1} & \dfrac{\partial u^1}{\partial x^2} & \dfrac{\partial u^1}{\partial x^3} \\ \dfrac{\partial u^2}{\partial x^1} & \dfrac{\partial u^2}{\partial x^2} & \dfrac{\partial u^2}{\partial x^3} \\ \dfrac{\partial u^3}{\partial x^1} & \dfrac{\partial u^3}{\partial x^2} & \dfrac{\partial u^3}{\partial x^3} \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} \rho\sin\varphi\cos\theta & \rho\sin\varphi\sin\theta & \rho\cos\varphi \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\left(\dfrac{\sin\theta}{\sin\varphi}\right) & \left(\dfrac{\cos\theta}{\sin\varphi}\right) & 0 \end{pmatrix} \tag{1.27a}$$

The contravariant bases of the curvilinear coordinates can be written as

$$\mathbf{g}^i = \frac{\partial u^i}{\partial x^j}\mathbf{e}_j \text{ for } j = 1, 2, 3 \tag{1.27b}$$

The matrix product $\mathbf{GG}^{-1}$ must be an identity matrix according to Eq. (1.18b).

$$\mathbf{G}^{-1}\mathbf{G} = \frac{1}{\rho}\begin{pmatrix} \rho\sin\varphi\cos\theta & \rho\sin\varphi\sin\theta & \rho\cos\varphi \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\left(\dfrac{\sin\theta}{\sin\varphi}\right) & \left(\dfrac{\cos\theta}{\sin\varphi}\right) & 0 \end{pmatrix} \cdot \begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \mathbf{I} \tag{1.28}$$

According to Eq. (1.25), the covariant bases can be written as

$$\begin{aligned} \mathbf{g}_1 &= (\sin\varphi\cos\theta)\,\mathbf{e}_1 + (\sin\varphi\sin\theta)\,\mathbf{e}_2 + \cos\varphi\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_1| = 1 \\ \mathbf{g}_2 &= (\rho\cos\varphi\cos\theta)\,\mathbf{e}_1 + (\rho\cos\varphi\sin\theta)\,\mathbf{e}_2 - (\rho\sin\varphi)\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_2| = \rho \\ \mathbf{g}_3 &= (-\rho\sin\varphi\sin\theta)\,\mathbf{e}_1 + (\rho\sin\varphi\cos\theta)\,\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow |\mathbf{g}_3| = \rho\sin\varphi \end{aligned} \tag{1.29}$$

The contravariant bases result from Eq. (1.27b).

$$
\begin{aligned}
\mathbf{g}^1 &= (\sin\varphi\cos\theta)\,\mathbf{e}_1 + (\sin\varphi\sin\theta)\,\mathbf{e}_2 + \cos\varphi\,\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^1\right| = 1 \\
\mathbf{g}^2 &= \left(\frac{1}{\rho}\cos\varphi\cos\theta\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\cos\varphi\sin\theta\right)\mathbf{e}_2 - \left(\frac{1}{\rho}\sin\varphi\right)\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^2\right| = \frac{1}{\rho} \\
\mathbf{g}^3 &= \left(-\frac{1}{\rho}\frac{\sin\theta}{\sin\varphi}\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\right)\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^3\right| = \frac{1}{\rho\sin\varphi}
\end{aligned}
\tag{1.30}
$$

Not only the covariant bases but also the contravariant bases of the spherical coordinates are orthogonal due to

$$
\begin{aligned}
&\mathbf{g}_i\cdot\mathbf{g}^j = \mathbf{g}^j\cdot\mathbf{g}_i = \delta_i^j; \\
&\mathbf{g}_i\cdot\ \mathbf{g}_j = 0 \text{ for } i\neq j; \\
&\mathbf{g}^i\cdot\ \mathbf{g}^j = 0 \text{ for } i\neq j.
\end{aligned}
$$

1.3 Eigenvalue Problem of a Linear Coupled Oscillator

In the following subsection, we will give an example of the application of vector and matrix analysis to the eigenvalue problems in mechanical vibration. Figure 1.4 shows the free vibrations without damping of a three-mass system with the masses m_1, m_2, and m_3 connected by the springs with the constant stiffness k_1, k_2, and k_3. In case of the small vibration amplitudes and constant spring stiffnesses, the vibrations can be considered linear. Otherwise, the vibrations are nonlinear for that the bifurcation theory must be used to compute the responses [2].

Using Newton's second law, the homogenous vibration equations (free vibration equations) of the three-mass system can be written as [2–6]:

$$
\begin{aligned}
&m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0 \\
&m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3(x_2 - x_3) = 0 \\
&m_3\ddot{x}_3 + k_3(x_3 - x_2) = 0
\end{aligned}
\tag{1.31}
$$

Thus,

$$
\begin{aligned}
&m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \\
&m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = 0 \\
&m_3\ddot{x}_3 - k_3x_2 + k_3x_3 = 0
\end{aligned}
$$

Using the abbreviations of $k_{12}\equiv k_1+k_2$ and $k_{23}\equiv k_2+k_3$, one obtains

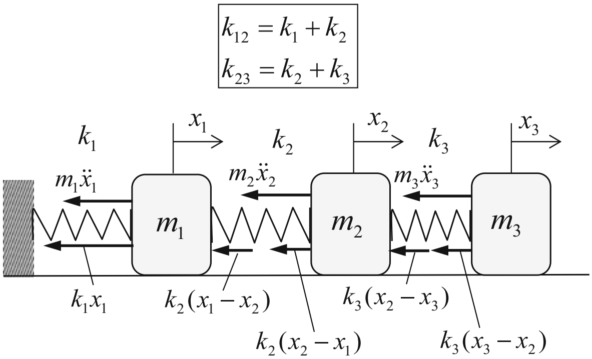

Fig. 1.4 Free vibrations of a three-mass system

$$\begin{aligned}
&m_1\ddot{x}_1 + k_{12}x_1 - k_2x_2 = 0\\
&m_2\ddot{x} - k_2x_1 + k_{23}x_2 - k_3x_3 = 0\\
&m_3\ddot{x} - k_3x_2 + k_3x_3 = 0
\end{aligned}$$

The vibration equations can be rewritten in the matrix formulation:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \cdot \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{bmatrix} k_{12} & -k_2 & 0 \\ -k_2 & k_{23} & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.32}$$

Thus,

$$\begin{aligned}
&\ddot{\mathbf{x}} + \left(\mathbf{M}^{-1}\mathbf{K}\right)\mathbf{x} = \mathbf{0}\\
&\Leftrightarrow \ddot{\mathbf{x}} + \mathbf{A}\mathbf{x} = \mathbf{0}
\end{aligned} \tag{1.33}$$

where

$$\mathbf{A} \equiv \mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} \frac{k_{12}}{m_1} & -\frac{k_2}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_{23}}{m_2} & -\frac{k_3}{m_2} \\ 0 & -\frac{k_3}{m_3} & \frac{k_3}{m_3} \end{bmatrix}$$

The free vibration response of Eq. (1.33) can be assumed as

$$\begin{aligned}
&\mathbf{x} = \mathbf{X}e^{\lambda t}\\
&\Rightarrow \dot{\mathbf{x}} = \lambda\left(\mathbf{X}e^{\lambda t}\right) = \lambda\mathbf{x}\\
&\Rightarrow \ddot{\mathbf{x}} = \lambda^2\left(\mathbf{X}e^{\lambda t}\right) = \lambda^2\mathbf{x}
\end{aligned} \tag{1.34}$$

where λ is the complex eigenvalue that is defined by

$$\lambda = \alpha \pm j\omega \quad \in \quad \mathbf{C} \tag{1.35}$$

in which ω is the eigenfrequency; α is the growth/decay rate [2].

Substituting Eq. (1.34) into Eq. (1.33) one obtains the eigenvalue problem

$$\left(\mathbf{A} + \lambda^2 \mathbf{I}\right)\mathbf{X}\, e^{\lambda t} = \mathbf{0} \tag{1.36}$$

where $\mathbf{X}$ is the eigenvector relating to its eigenvalue λ; $\mathbf{I}$ is the identity matrix.

For any non-trivial solution of $\mathbf{x}$, the determinant of $(\mathbf{A} + \lambda^2\mathbf{I})$ must vanish.

$$\det\left(\mathbf{A} + \lambda^2 \mathbf{I}\right) = 0 \tag{1.37}$$

Equation (1.37) is called the characteristic equation of the eigenvalue. Obviously, this characteristic equation is a polynomial of λ^{2N} where N is the degrees of freedom (DOF) of the vibration system. In this case, the DOF equals 3 for a three-mass system in the translational vibration.

Solving the characteristic equation (1.37), one obtains 6 eigenvalues (=2*DOF) for the vibration equations of the three-mass system. In the case without damping where the real parts are equal to zero, there are six different eigenvalues, three of them for forward whirls and three for backward whirls.

$$\lambda_{1,2} = \pm j\omega_1; \quad \lambda_{3,4} = \pm j\omega_2; \quad \lambda_{5,6} = \pm j\omega_3 \tag{1.38}$$

Substituting the eigenvalue λ_i into Eq. (1.36), one obtains the corresponding eigenvector $\mathbf{X}_i$.

$$\left(\mathbf{A} + \lambda_i^2 \mathbf{I}\right)\mathbf{X}_i = \mathbf{0} \ \text{ for } \ i = 1, 2, \ldots, 6. \tag{1.39}$$

The eigenvectors in Eq. (1.39) relating to the eigenvalues show the vibration modes of the system.

It is well known that N ordinary differential equations (ODEs) of second order can be transformed into $2N$ ODEs of first order using the simple trick of adding N identical ODEs of first order to the original ODEs.

$$\begin{aligned} &\begin{pmatrix} \dot{\mathbf{x}} = \dot{\mathbf{x}} \\ \mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \dot{\mathbf{x}} = \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{x} \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & 0 \end{bmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix} \end{aligned} \tag{1.40}$$

Substituting a new $(2\mathrm{N} \times 1)$ vector of

$$\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix} \Rightarrow \dot{\mathbf{z}} = \begin{pmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{pmatrix} \tag{1.41}$$

into Eq. (1.40), the vibration equations of first order can be rewritten down

$$\begin{aligned} &\begin{pmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & 0 \end{bmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix} \\ &\Leftrightarrow \dot{\mathbf{z}} = \mathbf{B}\mathbf{z} \end{aligned} \tag{1.42}$$

The free vibration response of Eq. (1.42) can be assumed as

$$\mathbf{z} = \mathbf{Z}e^{\lambda t} \Rightarrow \dot{\mathbf{z}} = \lambda\left(\mathbf{Z}e^{\lambda t}\right) = \lambda\mathbf{z} \tag{1.43}$$

where λ is the complex eigenvalue given in

$$\lambda = \alpha \pm j\omega \in \mathbf{C} \tag{1.44}$$

Within ω is the eigenfrequency; α is the growth/decay rate.

Substituting Eq. (1.43) into Eq. (1.42) one obtains the eigenvalue problem

$$(\mathbf{B} - \lambda\mathbf{I})\mathbf{Z}e^{\lambda t} = \mathbf{0} \tag{1.45}$$

where $\mathbf{Z}$ is the eigenvector relating to its eigenvalue λ; $\mathbf{I}$ is the identity matrix.

For any non-trivial solution of $\mathbf{z}$, the determinant of $(\mathbf{B} - \lambda\mathbf{I})$ must vanish:

$$\det(\mathbf{B} - \lambda\mathbf{I}) = 0 \tag{1.46}$$

Equation (1.46) is called the characteristic equation of the eigenvalue that is identical to Eq. (1.37). Obviously, this characteristic equation is a polynomial of λ^{2N} where N is the degrees of freedom (DOF) of the vibration system. In this case, the DOF equals 3 because of the three-mass system.

Solving the characteristic equation (1.46), one obtains 6 eigenvalues (= 2*DOF) for the vibration equations of the three-mass system. In the case without damping where the real parts are equal to zero, there are six different eigenvalues, three of them for forward whirls and three for backward whirls.

$$\lambda_{1,2} = \pm j\omega_1; \quad \lambda_{3,4} = \pm j\omega_2; \quad \lambda_{5,6} = \pm j\omega_3 \tag{1.47}$$

Substituting the eigenvalue λ_{i} into Eq. (1.45), it gives the corresponding eigenvector $\mathbf{Z}_{\mathrm{i}}$.

$$(\mathbf{B} - \lambda_i\mathbf{I})\mathbf{Z}_i = \mathbf{0} \text{ for } i = 1, 2, \ldots, 6. \tag{1.48}$$

The eigenvectors in Eq. (1.48) relating to the eigenvalues describe the vibration modes of the system.

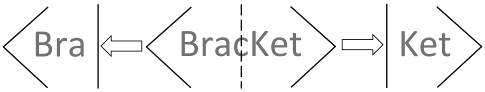

Fig. 1.5 Notation of bra and ket

1.4 Notation of Bra and Ket

The notation of bra and ket was defined by Dirac for applications in quantum mechanics and statistical thermodynamics [7]. Bra and ket are tuples of independent coordinates in a finite N-dimensional space in Riemannian manifold (cf. Appendix E). The name of bra and ket comes from the angle bracket $< >$, as shown in Fig. 1.5. Dividing the bracket into two parts, one obtains the left one called *bra* and the right one named *ket*.

In general, bra and ket can be considered as vectors, matrices, and high-order tensors. In contrast to vectors (first-order tensors), bra and ket have generally neither direction nor vector length in the point space. They are only a tuple of N coordinates (dimensions), such as of time, position, momentum, velocity, etc. Bra and ket are independent of any coordinate system, but their components depend on the relating basis of the coordinate system; i.e., they are changed at the new basis by coordinate transformations. Therefore, the bra and ket notation is a powerful tool mostly used in quantum mechanics and statistical thermodynamics in order to describe a physical state as a point of N dimensions in a finite N-dimensional complex space.

Some examples of bra and ket can be written in different types:

$$\text{Ket vector } |\mathbf{K}\rangle = \begin{bmatrix} 1+i \\ -1 \\ 2-i \\ 1 \end{bmatrix} \rightarrow \text{Bra vector } \langle\mathbf{K}| = [\,(1-i) \quad -1 \quad (2+i) \quad 1\,];$$

$$\text{Ket matrix } |\mathbf{M}\rangle = \begin{bmatrix} 1+i & -2 \\ 1 & 2-i \end{bmatrix} \rightarrow \text{Bra matrix } \langle\mathbf{M}| = \begin{bmatrix} 1-i & 1 \\ -2 & 2+i \end{bmatrix}.$$

1.5 Properties of Kets

We denote the finite N-dimensional complex vector space by $\mathbf{C}^N$. A ket $|\mathbf{K}\rangle$ can be defined as an N-tuple of the coordinates $u^1,\ldots,u^N$: $\mathbf{K}(u^1,\ldots,u^N) \in \mathbf{C}^\mathbf{N}$. Given three arbitrary kets $|\mathbf{A}\rangle$, $|\mathbf{B}\rangle$, and $|\mathbf{C}\rangle \in \mathbf{C}^N$ and two scalars α, $\beta \in \mathbf{C}$, the following properties of kets result in [8]:

– Commutative property of ket addition:

$$|\mathbf{A}\rangle + |\mathbf{B}\rangle = |\mathbf{B}\rangle + |\mathbf{A}\rangle$$

– Distributive property of ket multiplication by a scalar addition:

$$(\alpha + \beta)|\mathbf{A}\rangle = \alpha|\mathbf{A}\rangle + \beta|\mathbf{A}\rangle$$

– Distributive property of multiplication of ket addition by a scalar:

$$\alpha\big(|\mathbf{A}\rangle + |\mathbf{B}\rangle\big) = \alpha|\mathbf{A}\rangle + \alpha|\mathbf{B}\rangle$$

– Associative property of ket addition:

$$|\mathbf{A}\rangle + \big(|\mathbf{B}\rangle + |\mathbf{C}\rangle\big) = \big(|\mathbf{A}\rangle + |\mathbf{B}\rangle\big) + |\mathbf{C}\rangle$$

– Associative property of ket multiplication by scalars:

$$\alpha\big(\beta|\mathbf{A}\rangle\big) = \beta\big(\alpha|\mathbf{A}\rangle\big) = \alpha\beta|\mathbf{A}\rangle$$

– Property of ket addition to the null ket $|\mathbf{0}\rangle$:

$$|\mathbf{A}\rangle + |\mathbf{0}\rangle = |\mathbf{A}\rangle$$

– Property of ket multiplication by the null scalar:

$$0\cdot|\mathbf{A}\rangle = |\mathbf{0}\rangle$$

– Property of ket addition to an inverse ket $|-\mathbf{A}\rangle$:

$$|\mathbf{A}\rangle + |-\mathbf{A}\rangle = |\mathbf{A}\rangle - |\mathbf{A}\rangle = |\mathbf{0}\rangle.$$

1.6 Analysis of Bra and Ket

1.6.1 Bra and Ket Bases

Ket vector $|\mathbf{A}\rangle$ of the coordinates $(u^1,\ldots,u^N)$: $\mathbf{A}(a_1,\ldots,a_N) \in \mathbf{V}^N$ is the sum of its components in the orthonormal ket bases and can be written as

$$|\mathbf{A}\rangle \equiv \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ a_N \end{pmatrix} = \sum_{i=1}^{N} a_i|\mathbf{i}\rangle; \quad a_i \equiv (\alpha_i + \mathrm{j}\beta_i) \in \mathbf{C} \tag{1.49}$$

where

a_i is the ket component in the basis $|\,\mathbf{i}\,\rangle$; a_i is a complex number, $a_i \in \mathbf{C}$;

$|\,\mathbf{i}\,\rangle$ is the orthonormal basis of the coordinates $(u^1,\ldots,u^N)$.

The ket bases of $\{|\mathbf{1}\rangle, |\mathbf{2}\rangle,\ldots,|\mathbf{N}\rangle\}$ in the coordinates $(u^1,\ldots,u^N)$ are column vectors, as given in

$$|\mathbf{1}\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ . \\ 0 \end{pmatrix}; \quad |\mathbf{2}\rangle \equiv \begin{pmatrix} 0 \\ 1 \\ . \\ 0 \end{pmatrix}; \quad \ldots; \quad |\mathbf{i}\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad \ldots; \quad |\mathbf{N}\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ . \\ 1 \end{pmatrix} \tag{1.50}$$

Bra $\langle\mathbf{A}|$ is defined as the transpose conjugate (also adjoint) $|\mathbf{A}\rangle^*$ of the ket $|\mathbf{A}\rangle$. Therefore, the bra is a row vector and its elements are the conjugates of the ket elements.

To formulate bra of $|\mathbf{A}\rangle$, at first ket $|\mathbf{A}\rangle$ must be transposed; then, its complex elements are conjugated into bra elements.

$$\begin{aligned} |\mathbf{A}\rangle &\equiv \begin{pmatrix} \alpha_1 + \mathrm{j}\beta_1 \\ \alpha_2 + \mathrm{j}\beta_2 \\ \cdots \\ \alpha_N + \mathrm{j}\beta_N \end{pmatrix} \Rightarrow \\ \left|\mathbf{A}^T\right\rangle &= \left[\,(\alpha_1 + \mathrm{j}\beta_1) \quad (\alpha_2 + \mathrm{j}\beta_2) \quad \ldots \quad (\alpha_N + \mathrm{j}\beta_N)\,\right] \Rightarrow \\ |\mathbf{A}\rangle^* &\equiv \left[\,(\alpha_1 - \mathrm{j}\beta_1) \quad (\alpha_2 - \mathrm{j}\beta_2) \quad \ldots \quad (\alpha_N - \mathrm{j}\beta_N)\,\right] \equiv \langle\mathbf{A}| \end{aligned} \tag{1.51}$$

The ket vector $|\mathbf{A}\rangle^*$ is called the transpose conjugate (adjoint) of the ket vector $|\mathbf{A}\rangle$ and equals bra $\langle\mathbf{A}|$.

Thus, bra $\langle\mathbf{A}|$ can be written in the bra bases

$$\langle\mathbf{A}| \equiv |\mathbf{A}\rangle^* = \sum_{j=1}^{N} \langle\mathbf{j}|.\, a_j^*; \quad a_j^* \equiv \left(\alpha_j - \mathrm{j}\beta_j\right) \in \mathbf{C} \tag{1.52}$$

where the component a^* is the complex conjugate of its component a.

Analogously, the bra bases result from Eq. (1.50).

$$\begin{aligned} \langle\mathbf{1}| &\equiv [\,1 \quad 0 \quad . \quad 0\,]; \\ \langle\mathbf{2}| &\equiv [\,0 \quad 1 \quad . \quad 0\,]; \\ \langle\mathbf{j}| &\equiv [\,0 \quad 0 \quad 1 \quad 0\,]; \\ \langle\mathbf{N}| &\equiv [\,0 \quad 0 \quad 0 \quad 1\,]. \end{aligned} \tag{1.53}$$

Due to orthonormality, the product of bra and ket is a scalar and obviously equals the Kronecker delta.

$$\langle\mathbf{i}| \cdot |\mathbf{j}\rangle \equiv \langle\mathbf{i}|\mathbf{j}\rangle = \delta_i^j = \begin{cases} 0; & i \neq j \\ 1; & i = j \end{cases} \tag{1.54}$$

The combined symbol $\langle \mathbf{i} \mid \mathbf{j} \rangle$ of bra $\langle \mathbf{i}|$ and ket $|\mathbf{j}\rangle$ in Eq. (1.54) is defined as the inner product (scalar product) of bra $\langle \mathbf{i}|$ and ket $|\mathbf{j}\rangle$.

1.6.2 Gram-Schmidt Scheme of Basis Orthonormalization

The basis $\{|\mathbf{g}_i\rangle\}$ is non-orthonormal in the curvilinear coordinates in the space $\mathbf{R}^3$. Using the Gram-Schmidt scheme [8, 9], the orthonormal bases ($|\mathbf{e}_1\rangle,|\mathbf{e}_2\rangle,|\mathbf{e}_3\rangle$) are created from the non-orthogonal bases ($|\mathbf{g}_1\rangle,|\mathbf{g}_2\rangle,|\mathbf{g}_3\rangle$). The orthonormalization procedure of the basis is discussed in Appendix E.2.6.

The first orthonormal ket basis is generated by

$$|\mathbf{e}_1\rangle \equiv |\mathbf{1}\rangle = \frac{|\mathbf{g}_1\rangle}{|\mathbf{g}_1|}$$

The second orthonormal ket basis results from

$$|\mathbf{e}_2\rangle \equiv |\mathbf{2}\rangle = \frac{|\mathbf{g}_2\rangle - \langle \mathbf{e}_1|\mathbf{g}_2\rangle \cdot |\mathbf{e}_1\rangle}{\left||\mathbf{g}_2\rangle - \langle \mathbf{e}_1|\mathbf{g}_2\rangle \cdot |\mathbf{e}_1\rangle\right|}$$

The third orthonormal ket basis is similarly calculated in

$$|\mathbf{e}_3\rangle \equiv |\mathbf{3}\rangle = \frac{|\mathbf{g}_3\rangle - \langle \mathbf{e}_1|\mathbf{g}_3\rangle \cdot |\mathbf{e}_1\rangle - \langle \mathbf{e}_2|\mathbf{g}_3\rangle \cdot |\mathbf{e}_2\rangle}{\left||\mathbf{g}_3\rangle - \langle \mathbf{e}_1|\mathbf{g}_3\rangle \cdot |\mathbf{e}_1\rangle - \langle \mathbf{e}_2|\mathbf{g}_3\rangle \cdot |\mathbf{e}_2\rangle\right|}$$

Generally, the orthonormal ket basis $|\mathbf{e}_j\rangle$ can be rewritten in the N-dimensional space.

$$\left|\mathbf{e}_j\right\rangle \equiv |\mathbf{j}\rangle = \frac{\left|\mathbf{g}_j\right\rangle - \sum_{i=1}^{j-1} \left\langle \mathbf{e}_i|\mathbf{g}_j\right\rangle \cdot |\mathbf{e}_i\rangle}{\left|\left|\mathbf{g}_j\right\rangle - \sum_{i=1}^{j-1} \left\langle \mathbf{e}_i|\mathbf{g}_j\right\rangle \cdot |\mathbf{e}_i\rangle\right|} \quad \text{for } j = 1, 2, \ldots, N \tag{1.55}$$

Using the Gram-Schmidt procedure, the ket orthonormal bases of $\{|\mathbf{1}\rangle, |\mathbf{2}\rangle,\ldots,|\mathbf{N}\rangle\}$ in the coordinates $(u^1,\ldots,u^N)$ are generated from any non-orthonormal bases, as given in Eq. (1.50). The bra orthonormal bases of $\{\langle \mathbf{1}|, \langle \mathbf{2}|,\ldots,\langle \mathbf{N}|\}$ in the coordinates $(u^1,\ldots,u^N)$ are the adjoint of the ket orthonormal bases.

1.6.3 Cauchy-Schwarz and Triangle Inequalities

The Cauchy-Schwarz and triangle inequalities immediately apply to the Bra-Ket notation:

1) **Cauchy-Schwarz Inequality**

The well-known Cauchy-Schwarz inequality provides the relation between the inner product of bra and ket, and their norms.

$$\langle \mathbf{A}|\mathbf{B}\rangle \leq ||\mathbf{A}\rangle| \cdot ||\mathbf{B}\rangle|; \ \ |\mathbf{A}\rangle, \ |\mathbf{B}\rangle \in \mathbf{V}^N \tag{1.56}$$

2) **Triangle Inequality**

The triangle inequality formulates the inequality between the sum of two kets and the ket norms.

$$||\mathbf{A}\rangle + |\mathbf{B}\rangle| \leq ||\mathbf{A}\rangle| + ||\mathbf{B}\rangle|; \ \ |\mathbf{A}\rangle, \ |\mathbf{B}\rangle \in \mathbf{V}^{\mathrm{N}} \tag{1.57}$$

1.6.4 Computing Ket and Bra Components

The component of a ket results from multiplying the ket by a bra basis according to Eqs. (1.49) and (1.54):

$$\begin{aligned} \langle \mathbf{j}| \cdot |\mathbf{A}\rangle \equiv \langle \mathbf{j}|\mathbf{A}\rangle &= \sum_{i=1}^{N} \langle \mathbf{j}| \cdot \left(a_i|\mathbf{i}\rangle\right) \\ &= \sum_{i=1}^{N} \langle \mathbf{j}|\mathbf{i}\rangle \cdot a_i = \sum_{i=1}^{N} \delta_j^i \cdot a_i \\ &= a_j \equiv \left(\alpha_j + \mathrm{j}\beta_j\right) \end{aligned} \tag{1.58}$$

Equation (1.58) indicates that the ket component in the orthogonal bases equals the scalar product between the ket and its relating basis.

Similarly, the bra component can be computed by multiplying the bra by a ket basis.

$$\begin{aligned} \langle \mathbf{A}| \cdot |\mathbf{j}\rangle \equiv \langle \mathbf{A}|\mathbf{j}\rangle &= \sum_{i=1}^{N} \left(\langle \mathbf{i}|a_i^*\right)\cdot|\mathbf{j}\rangle \\ &= \sum_{i=1}^{N} \langle \mathbf{i}|\mathbf{j}\rangle \cdot a_i^* = \sum_{i=1}^{N} \delta_i^j \cdot a_i^* \\ &= a_j^* \equiv \left(\alpha_j - \mathrm{j}\beta_j\right) \end{aligned} \tag{1.59}$$

It is straightforward that the bra component is equal to the complex conjugate of the relating ket component a_j, as given in Eq. (1.52).

1.6.5 Inner Product of Bra and Ket

The inner product of bra $\langle \mathbf{A}|$ and ket $|\mathbf{B}\rangle$ is defined as

$$\begin{aligned} \langle \mathbf{A}|\mathbf{B}\rangle &= \left(\sum_{i=1}^{N} \langle \mathbf{i}|a_i^*\right) \cdot \left(\sum_{j=1}^{N} b_j|\mathbf{j}\rangle\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_i^* b_j \langle \mathbf{i}|\mathbf{j}\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i^* b_j \delta_i^j \\ &= \sum_{i=1}^{N} a_i^* b_i \end{aligned} \tag{1.60a}$$

It is obvious that the inner product of bra and ket is a complex number according to Eqs. (1.59) and (1.60a).

In case of $|\mathbf{A}\rangle = |\mathbf{B}\rangle$, the inner product in Eq. (1.60a) becomes

$$\begin{aligned} \langle \mathbf{A}|\mathbf{A}\rangle &= \left(\sum_{i=1}^{N} \langle \mathbf{i}|a_i^*\right) \cdot \left(\sum_{j=1}^{N} a_j|\mathbf{j}\rangle\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_i^* a_j \langle \mathbf{i}|\mathbf{j}\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i^* a_j \delta_i^j \\ &= \sum_{i=1}^{N} a_i^* a_i = \sum_{i=1}^{N} \left(\alpha_i^2 + \beta_i^2\right) = ||\mathbf{A}\rangle|^2 \end{aligned} \tag{1.60b}$$

Thus, the norm (length) of the ket $|\mathbf{A}\rangle$ is given in

$$||\mathbf{A}\rangle| = \sqrt{\langle \mathbf{A}|\mathbf{A}\rangle} \tag{1.60c}$$

The inner product in Eq. (1.60a) can be rewritten using Eq. (1.52).

$$
\begin{aligned}
\langle \mathbf{A}|\mathbf{B}\rangle &= \sum_{i=1}^{N} a_i^* b_i = \begin{bmatrix} a_1^* & a_2^* & a_i^* & a_N^* \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_i \\ b_N \end{bmatrix} \\
&= |\mathbf{A}\rangle^* \cdot |\mathbf{B}\rangle = \langle \mathbf{A}| \cdot |\mathbf{B}\rangle \equiv \langle \mathbf{A}|\mathbf{B}\rangle
\end{aligned}
\tag{1.61}
$$

Similarly, the inner product of bra $\langle \mathbf{B}|$ and ket $|\mathbf{A}\rangle$ can be calculated as follows:

$$
\begin{aligned}
\langle \mathbf{B}|\mathbf{A}\rangle &= \left(\sum_{j=1}^{N} \langle \mathbf{j}| b_j^* \right) \cdot \left(\sum_{i=1}^{N} a_i |\mathbf{i}\rangle \right) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} b_j^* a_i \langle \mathbf{j}|\mathbf{i}\rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} b_j^* a_i \delta_j^i \\
&= \sum_{i=1}^{N} b_i^* a_i
\end{aligned}
\tag{1.62}
$$

Conjugating Eq. (1.62), one obtains the transpose conjugate of $\langle \mathbf{B}|\mathbf{A}\rangle$:

$$
\begin{aligned}
\langle \mathbf{B}|\mathbf{A}\rangle^* &= \left(\sum_{i=1}^{N} b_i^* a_i \right)^* \\
&= \sum_{i=1}^{N} a_i^* \left(b_i^* \right)^* = \sum_{i=1}^{N} a_i^* b_i \\
&= \langle \mathbf{A}|\mathbf{B}\rangle
\end{aligned}
\tag{1.63}
$$

Thus, the inner product is skew-symmetric (anti-symmetric) contrary to the inner product of two regular vectors.

Some properties of the inner product (scalar product) are valid:

Skew-symmetry: $\langle \mathbf{A}|\mathbf{B}\rangle = \langle \mathbf{B}|\mathbf{A}\rangle^*$;
Positive definiteness: $\langle \mathbf{A}|\mathbf{A}\rangle = |\,|\mathbf{A}\rangle\,|^2 \geq 0$;
Distributive property: $\langle\, \mathbf{A}|(\alpha \mathbf{B} + \beta \mathbf{C})\,\rangle = \alpha \langle \mathbf{A}|\mathbf{B}\rangle + \beta \langle \mathbf{A}|\mathbf{C}\rangle$ for $\alpha, \beta \in \mathbf{C}$.

Furthermore, the linear adjoint operator has the following properties:

1. $(\alpha\beta\gamma)^* = [\alpha(\beta\gamma)]^* = (\beta\gamma)^* \alpha^* = \gamma^* \beta^* \alpha^*$ **for** $\alpha, \beta, \gamma \in \mathbf{C}$
   Note that the product order is changed in the adjoint operation of the scalar product.
2. $(\alpha|\mathbf{A}\rangle)^* = |\alpha \mathbf{A}\rangle^* = |\mathbf{A}\rangle^*\ \alpha^* = \langle \mathbf{A}|\ \alpha^*$ for $\alpha \in \mathbf{C}$
3. $(\langle \mathbf{A}|\alpha)^* = \alpha^* \langle \mathbf{A}|^* = \alpha^* |\mathbf{A}\rangle$for $\alpha \in \mathbf{C}$
4. $\langle \mathbf{A}|\alpha^{**} = \langle \mathbf{A}|\alpha$ for $\alpha \in \mathbf{C}$
5. $\langle \mathbf{B}|\mathbf{A}\rangle^* = |\mathbf{A}\rangle^*.\langle \mathbf{B}|^* = \langle \mathbf{A}|.|\mathbf{B}\rangle \equiv \langle \mathbf{A}|\mathbf{B}\rangle$: skew-symmetric (anti-symmetric)
6. $\langle \mathbf{A}|\alpha^* |\mathbf{B}\rangle^* = |\mathbf{B}\rangle^*.\alpha.\langle \mathbf{A}|^* = \langle \mathbf{B}|.\alpha.|\mathbf{A}\rangle \equiv \langle \mathbf{B}|\alpha|\mathbf{A}\rangle$ for $\alpha \in \mathbf{C}$
7. $(|\mathbf{A}\rangle\langle \mathbf{B}|)^* = \langle \mathbf{B}|^*.|\mathbf{A}\rangle^* = |\mathbf{B}\rangle\langle \mathbf{A}|$: the outer product of ket and bra.

1.6.6 Outer Product of Bra and Ket

The outer product of ket $|\mathbf{A}\rangle$ and bra $\langle\mathbf{B}|$ is defined as

$$\begin{aligned}|\mathbf{A}\rangle\langle\mathbf{B}| &= \left(\sum_{i=1}^{N} a_i|\mathbf{i}\rangle\right)\cdot\left(\sum_{j=1}^{N}\langle\mathbf{j}|b_j^*\right)\\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_i b_j^*|\mathbf{i}\rangle\langle\mathbf{j}|\end{aligned} \tag{1.64}$$

where the product term $|\mathbf{i}\rangle\langle\mathbf{j}|$ is called the outer product of the bases $|\mathbf{i}\rangle$ and $\langle\mathbf{j}|$.

Contrary to the inner product resulting a scalar of (1×1) matrix $\in \mathbf{V}$, the outer product is an operator of $(N \times N)$ matrix $\in \mathbf{V}^{N\times N}$ because the ket is an $(N \times 1)$ column vector $\in \mathbf{V}^N$ and the bra is a $(1 \times N)$ row vector $\in \mathbf{V}^N$.

Now, ket $|\mathbf{A}\rangle$ can be expressed in ket bases:

$$|\mathbf{A}\rangle = \sum_{i=1}^{N}|\mathbf{i}\rangle\, a_i \tag{1.65}$$

According to Eq. (1.58), the ket component is

$$a_i = \langle\mathbf{i}|\mathbf{A}\rangle$$

Substituting a_i into Eq. (1.65), one obtains the ket

$$|\mathbf{A}\rangle = \sum_{i=1}^{N}|\mathbf{i}\rangle\langle\mathbf{i}|\mathbf{A}\rangle \equiv \sum_{i=1}^{N}\mathbf{I}_i|\mathbf{A}\rangle \tag{1.66}$$

where $\mathbf{I}_i$ is the projection operator (outer product) according to [8], as defined by

$$\mathbf{I}_i \equiv |\mathbf{i}\rangle\langle\mathbf{i}| = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix} \tag{1.67}$$

The element I_{ij} of the projection operator (matrix) is 1 at the i row and j column, as shown in Eq. (1.67); otherwise, other elements are equal to zero.

Obviously, the sum of all projection operators is the identity matrix.

$$\mathbf{I} \equiv \sum_{i=1}^{N}\mathbf{I}_i = \sum_{i=1}^{N}|\mathbf{i}\rangle\langle\mathbf{i}| \tag{1.68}$$

According to Eq. (1.66), the identity property of the ket is proved by

$$|\mathbf{A}\rangle = \sum_{i=1}^{N} \mathbf{I}_i|\mathbf{A}\rangle = \mathbf{I}|\mathbf{A}\rangle. \tag{1.69}$$

1.6.7 Ket and Bra Projection Components on the Bases

The projection component of ket $|\mathbf{A}\rangle$ on the basis $|\ \mathbf{i}\ \rangle$ can be calculated as

$$\begin{aligned} |\mathbf{A}\rangle_i &= \mathbf{I}_i|\mathbf{A}\rangle \\ &= |\mathbf{i}\rangle\ \langle\mathbf{i}|\cdot\ |\mathbf{A}\rangle = |\mathbf{i}\rangle\ \langle\mathbf{i}|\mathbf{A}\rangle \\ &= |\mathbf{i}\rangle a_i \end{aligned} \tag{1.70}$$

Thus, ket $|\mathbf{A}\rangle$ can be expressed, as shown in Eq. (1.49):

$$|\mathbf{A}\rangle = \sum_{i=1}^{N} |\mathbf{A}\rangle_i = \sum_{i=1}^{N} |\mathbf{i}\rangle a_i \tag{1.71}$$

Similarly, the projection component of bra $\langle\mathbf{A}|$ on the basis $\langle\ \mathbf{i}\ |$ is computed as

$$\begin{aligned} \langle\mathbf{A}|_i &= \langle\mathbf{A}|\mathbf{I}_i \\ &= \langle\mathbf{A}|\cdot|\mathbf{i}\rangle\ \langle\mathbf{i}| = \langle\mathbf{A}|\mathbf{i}\rangle\ \langle\mathbf{i}| \\ &= \langle\mathbf{i}|a_i^* \end{aligned} \tag{1.72}$$

Bra $\langle\mathbf{A}|$ can be expressed in its projection components, as given in Eq. (1.52).

$$\begin{aligned} \langle\mathbf{A}| &= \sum_{i=1}^{N}\langle\mathbf{A}|_i = \sum_{i=1}^{N}\langle\mathbf{i}|a_i^* \\ &= |\mathbf{A}\rangle^* = \left(\sum_{i=1}^{N}|\mathbf{i}\rangle a_i\right)^* = \sum_{i=1}^{N} a_i^*\langle\mathbf{i}| \end{aligned} \tag{1.73}$$

1.6.8 Linear Transformation of Kets

We consider the complex vector spaces **V** and **V**'. Each of them belongs to the finite N-dimensional complex space $\mathbf{C}^N$. The linear transformation T maps the ket $|\mathbf{A}\rangle$ in **V** into the image ket $|\mathbf{A}'\rangle$ in **V**', as shown in Fig. 1.6.

The image ket $|\mathbf{A}'\rangle$ can be written in the bases $|\mathbf{i}\rangle$ [8, 9] as

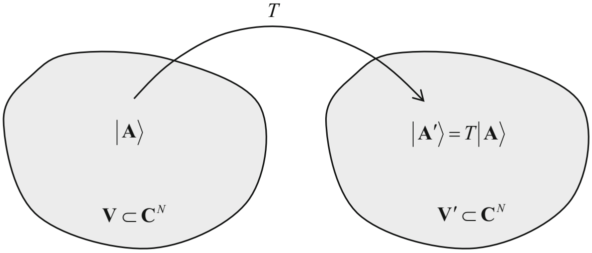

Fig. 1.6 Linear transformation T of a ket $|\mathbf{A}\rangle$

$$
\begin{aligned}
&T : |\mathbf{A}\rangle \rightarrow \left|\mathbf{A}'\right\rangle = T|\mathbf{A}\rangle \\
&\left|\mathbf{A}'\right\rangle = T\sum_{i=1}^{N}|\mathbf{i}\rangle a_i = \sum_{i=1}^{N} T|\mathbf{i}\rangle a_i
\end{aligned}
\tag{1.74}
$$

In this case, the ket basis $|\mathbf{i}\rangle$ is also mapped into the image ket basis $|\mathbf{i}'\rangle$. The transformation operator T for the basis can be written as

$$T : |\mathbf{i}\rangle \rightarrow \left|\mathbf{i}'\right\rangle \Rightarrow \quad \left|\mathbf{i}'\right\rangle = T|\mathbf{i}\rangle \tag{1.75}$$

The image ket basis $|\mathbf{i}'\rangle$ is formulated as a linear combination of the old ket bases $|\mathbf{i}\rangle$.

$$\left|\mathbf{i}'\right\rangle = T|\mathbf{i}\rangle = \sum_{j=1}^{N} T_{ji}|\mathbf{j}\rangle; \quad \mathbf{i} = 1, 2, \ldots, N \tag{1.76}$$

where the operator element T_{ji} is in the j row and i column of the transformation matrix $\mathbf{T}$ of the transformation operator T.

Multiplying both sides of Eq. (1.76) by the bra basis $\langle\mathbf{k}|$, one obtains

$$
\begin{aligned}
\left\langle \mathbf{k}|\mathbf{i}'\right\rangle &= \langle\mathbf{k}|T|\mathbf{i}\rangle = \langle\mathbf{k}| \cdot \sum_{j=1}^{N} T_{ji}|\mathbf{j}\rangle \\
&= \sum_{j=1}^{N} T_{ji}\langle\mathbf{k}|\mathbf{j}\rangle = \sum_{j=1}^{N} T_{ji}\delta_k^j = T_{ki}
\end{aligned}
\tag{1.77}
$$

Thus, the operator element results from Eq. (1.77):

$$T_{ki} = \left\langle \mathbf{k}|\mathbf{i}'\right\rangle = \langle\mathbf{k}|T|\mathbf{i}\rangle \tag{1.78}$$

Substituting Eq. (1.76) into Eq. (1.74), one obtains the image ket.

$$\begin{aligned}\left|\mathbf{A}'\right\rangle &= T|\mathbf{A}\rangle = T\sum_{i=1}^{N}|\mathbf{i}\rangle a_i \\ &= \sum_{i=1}^{N}T|\mathbf{i}\rangle a_i = \sum_{i=1}^{N}\left(\sum_{j=1}^{N}T_{ji}|\mathbf{j}\rangle\right)a_i \\ &= \sum_{j=1}^{N}\left(\sum_{i=1}^{N}T_{ji}a_i\right)|\mathbf{j}\rangle \equiv \sum_{j=1}^{N}a_j'|\mathbf{j}\rangle\end{aligned} \tag{1.79}$$

The component of the image ket in the basis $|\mathbf{j}\rangle$ is given from Eqs. (1.78) and (1.79).

$$\begin{aligned}& a_j' = \sum_{i=1}^{N}T_{ji}a_i = \sum_{i=1}^{N}\langle\mathbf{j}|T|\mathbf{i}\rangle\, a_i \Leftrightarrow \\ & \left|\mathbf{A}'\right\rangle = \mathbf{T}_{N\times N}|\mathbf{A}\rangle\end{aligned} \tag{1.80}$$

The image ket in Eq. (1.80) can be rewritten in the transformation matrix $\mathbf{T}_{N\times N}$.

$$\begin{bmatrix} a_1' \\ a_2' \\ . \\ a_j' \\ a_N' \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & . & T_{1i} & T_{1N} \\ T_{21} & T_{22} & . & T_{2i} & T_{2N} \\ . & . & . & . & . \\ T_{j1} & . & . & T_{ji} & T_{jN} \\ T_{N1} & T_{N2} & . & T_{Ni} & T_{NN} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ . \\ a_i \\ a_N \end{bmatrix} \tag{1.81}$$

where the matrix element T_{ji} is computed by the ket transformation T, as given in Eq. (1.78).

$$T_{ji} = \langle\mathbf{j}|T|\mathbf{i}\rangle$$

1.6.9 Coordinate Transformations

The ket basis $|\mathbf{e}_i\rangle$ is transformed into the new ket basis $|\mathbf{g}_i\rangle$ by the transformation S^{-1}, as shown in Fig. 1.7. The transformed ket basis can be written as [9].

$$\begin{aligned}& S^{-1} : |\mathbf{e}_i\rangle \to |\mathbf{g}_i\rangle \Rightarrow \quad |\mathbf{g}_i\rangle = S^{-1}|\mathbf{e}_i\rangle \Leftrightarrow \\ & S : |\mathbf{g}_i\rangle \to |\mathbf{e}_i\rangle \Rightarrow |\mathbf{e}_i\rangle = S|\mathbf{g}_i\rangle\end{aligned} \tag{1.82}$$

Analogous to the ket transformation, the old ket basis can be written in a linear combination of the new bases:

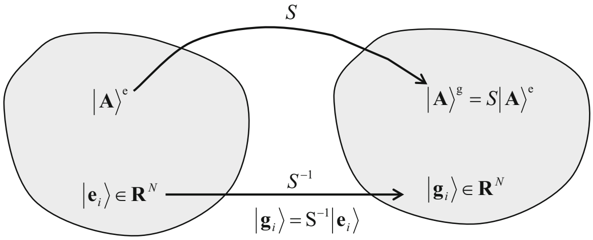

Fig. 1.7 Coordinate transformation of bases

$$|\mathbf{e}_i\rangle = \sum_{j=1}^{N} \left|\mathbf{g}_j\right\rangle S_{ji} \text{ for } i = 1,\ 2,\ \ldots, N \tag{1.83}$$

in which S_{ji} is the matrix element of the transformation matrix $\mathbf{S}$.

Multiplying both sides of Eq. (1.83) by the bra basis $\langle \mathbf{g}_k|$, one obtains the matrix element S_{ki}.

$$\begin{aligned}\langle \mathbf{g}_k|\mathbf{e}_i\rangle &= \langle \mathbf{g}_k| \cdot \sum_{j=1}^{N} \left|\mathbf{g}_j\right\rangle S_{ji} = \sum_{j=1}^{N} \left\langle \mathbf{g}_k|\mathbf{g}_j\right\rangle S_{ji} \\ &= \sum_{j=1}^{N} \delta_k^j S_{ji} = S_{ki}\end{aligned} \tag{1.84}$$

Thus, using Eq. (1.82) it gives

$$S_{ki} = \langle \mathbf{g}_k|\mathbf{e}_i\rangle = \langle \mathbf{g}_k|S|\mathbf{g}_i\rangle \tag{1.85}$$

An arbitrary ket $|\mathbf{A}\rangle$ can be expressed linearly in terms of the old ket basis:

$$|\mathbf{A}\rangle = \sum_{i=1}^{N} |\mathbf{e}_i\rangle\ a_i^e \tag{1.86}$$

where a_i^e is the ket component in the old basis $|\mathbf{e}_i\rangle$.

Substituting Eq. (1.83) into Eq. (1.86), one obtains

$$\begin{aligned}|\mathbf{A}\rangle &= \sum_{i=1}^{N} |\mathbf{e}_i\rangle\ a_i^e = \sum_{i=1}^{N} \left(\sum_{k=1}^{N} |\mathbf{g}_k\rangle S_{ki} \right) a_i^e \\ &= \sum_{k=1}^{N} \left(\sum_{i=1}^{N} S_{ki} a_i^e \right) |\mathbf{g}_k\rangle \equiv \sum_{k=1}^{N} a_k^g |\mathbf{g}_k\rangle\end{aligned} \tag{1.87}$$

Therefore, the ket component in the transformed basis $|\mathbf{g}_k\rangle$ can be calculated by Eq. (1.87).

$$a_k^g = \sum_{i=1}^{N} S_{ki} a_i^e = \sum_{i=1}^{N} \langle \mathbf{g}_\mathrm{k} | S | \mathbf{g}_\mathrm{i} \rangle \; a_i^e \Leftrightarrow |\mathbf{A}\rangle^g = S|\mathbf{A}\rangle^e \tag{1.88}$$

The transformed ket in Eq. (1.88) can be rewritten in the transformation matrix $\mathbf{S}_{N \times N}$.

$$\begin{bmatrix} a_1 \\ a_2 \\ . \\ a_j \\ a_N \end{bmatrix}^g = \begin{bmatrix} S_{11} & S_{12} & . & S_{1i} & S_{1N} \\ S_{21} & S_{22} & . & S_{2i} & S_{2N} \\ . & . & . & . & . \\ S_{j1} & S_{j2} & . & S_{ji} & S_{jN} \\ S_{N1} & S_{N2} & . & S_{Ni} & S_{NN} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ . \\ a_i \\ a_N \end{bmatrix}^e \tag{1.89}$$

where the matrix element is given in Eq. (1.89):

$$S_{ki} = \langle \mathbf{g}_k | \mathbf{e}_i \rangle = \langle \mathbf{g}_k | S | \mathbf{g}_i \rangle \tag{1.90}$$

The components of the transformed ket $a_k{}^g$ in the new basis $|\mathbf{g}_k\rangle$ are derived from Eq. (1.89).

In the following section, a combined transformation of kets consisting of three transformations is carried out [9, 10], as shown in Fig. 1.8:

1. Basis transformation S^{-1} from $|\mathbf{A}\rangle^g$ in the basis $|\mathbf{g}_i\rangle$ to $|\mathbf{A}\rangle^e$ in the basis $|\mathbf{e}_i\rangle$;
2. Ket transformation T from $|\mathbf{A}\rangle^e$ to $|\mathbf{A}'\rangle^e$ in the basis $|\mathbf{e}_i\rangle$;
3. Basis transformation S from $|\mathbf{A}'\rangle^e$ in the basis $|\mathbf{e}_i\rangle$ to $|\mathbf{A}'\rangle^g$ in the basis $|\mathbf{g}_i\rangle$.

The first transformation yields the first transformed ket:

$$|\mathbf{A}\rangle^e = S^{-1}|\mathbf{A}\rangle^g \tag{1.91}$$

The second transformation yields the second transformed ket:

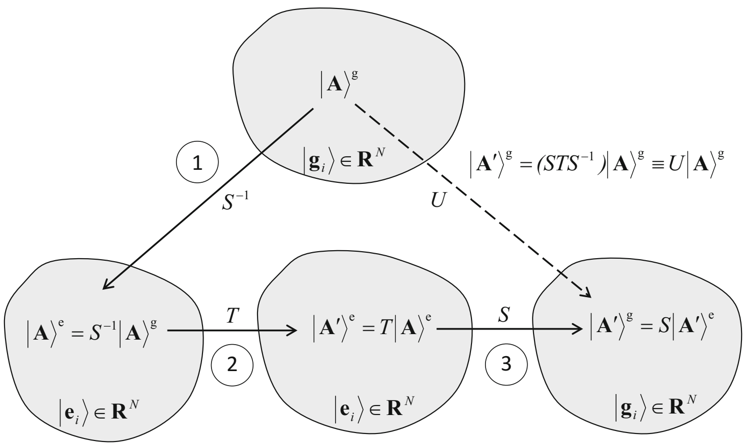

Fig. 1.8 The combined transformation of kets

$$\left|\mathbf{A}'\right\rangle^{e} = T|\mathbf{A}\rangle^{e} \tag{1.92}$$

The third transformation leads to the third transformed ket:

$$\left|\mathbf{A}'\right\rangle^{\mathrm{g}} = S\left|\mathbf{A}'\right\rangle^{\mathrm{e}} \tag{1.93}$$

Finally, the combined transformed ket of three transformations results from Eqs. (1.91), (1.92), and (1.93).

$$\left|\mathbf{A}'\right\rangle^{g} = S\left|\mathbf{A}'\right\rangle^{e} = ST|\mathbf{A}\rangle^{e} = \left(STS^{-1}\right)|\mathbf{A}\rangle^{g} \equiv U|\mathbf{A}\rangle^{g} \tag{1.94}$$

where the combined transformation U is defined as (STS^{-1}).

The component of the product of many operators is computed [8, 9] according to Eq. (1.78).

$$\begin{aligned} U_{ij} &= \langle \mathbf{i}|U|\mathbf{j}\rangle = \langle \mathbf{i}|\left(STS^{-1}\right)|\mathbf{j}\rangle \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} \langle \mathbf{i}|S|\mathbf{k}\rangle \cdot \langle \mathbf{k}|T|\mathbf{l}\rangle \cdot \langle \mathbf{l}|S^{-1}|\mathbf{j}\rangle \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} S_{ik}T_{kl}S_{lj}^{-1} \end{aligned} \tag{1.95}$$

The ket transformed component of $|\mathbf{A}'\rangle^{g}$ can be obtained from Eqs. (1.94) and (1.95).

$$\begin{aligned} &\left|\mathbf{A}'\right\rangle^{g} = U|\mathbf{A}\rangle^{g} \Leftrightarrow \\ &a_i'g = \sum_{j=1}^{N} U_{ij}a_j^{g} = \sum_{j=1}^{N}\left(\sum_{k=1}^{N}\sum_{l=1}^{N} S_{ik}T_{kl}S_{lj}^{-1}\right) a_j^{g} \end{aligned} \tag{1.96}$$

The transformed ket in Eq. (1.96) can be rewritten in the transformation matrix $\mathbf{U}_{N\times N}$.

$$\begin{bmatrix} a_1' \\ a_2' \\ . \\ a_j' \\ a_N' \end{bmatrix}^{g} = \begin{bmatrix} U_{11} & U_{12} & . & U_{1i} & U_{1N} \\ U_{21} & U_{22} & . & U_{2i} & U_{2N} \\ . & . & . & . & . \\ U_{j1} & . & . & U_{ji} & U_{jN} \\ U_{N1} & U_{N2} & . & U_{Ni} & U_{NN} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ . \\ a_i \\ a_N \end{bmatrix}^{g} \tag{1.97}$$

1.6.10 Hermitian Operator

Hermitian operator plays a key role in eigenvalue problems using in quantum mechanics. Let $\mathbf{T}$ be a matrix; the adjoint $\mathbf{T}^\dagger$ (spoken $\mathbf{T}$ dagger) is defined as the transpose conjugate (adjoint) of the matrix $\mathbf{T}$. In quantum mechanics, the operator $T^\dagger$ of the matrix $\mathbf{T}$ is called a Hermitian operator (or self-adjoint operator) if it equals the operator T of the matrix $\mathbf{T}$ (i.e., $T^\dagger = T$ or $\mathbf{T}^\dagger = \mathbf{T}$), cf. Chap. 6.

Let T be an *operator* (matrix or second-order tensor). Its *transpose* is written as

$$T = T_{ij} \Rightarrow T^T = T_{ji}$$

The *complex conjugate* of T is defined as

$$T^* = T^*_{ij}$$

The *transpose conjugate* of T is called the *adjoint* $T^\dagger$ that is calculated as

$$\begin{aligned}
&\left(T^T\right)^* = \left(T_{ji}\right)^* = T^*_{ji} \equiv T^\dagger = T^\dagger_{ij} \\
&\Leftrightarrow \left(T^*_{ji}\right)^* = T_{ji} = \left(T^\dagger_{ij}\right)^* \\
&\Leftrightarrow T^T = \left(T^\dagger\right)^*
\end{aligned}$$

If T is *hermitian*, then T is *self-adjoint*; i.e.,

$$T^\dagger_{ij} = T_{ij} \Leftrightarrow T^\dagger = T.$$

An arbitrary ket $|\mathbf{B}\rangle$ can be transformed by the linear operator T into a ket:

$$T : |\mathbf{B}\rangle \in \mathbf{R}^N \to T|\mathbf{B}\rangle \in \mathbf{R}^N \tag{1.98}$$

The inner product of bra $\langle \mathbf{A}|$ and transformed ket $T|\mathbf{B}\rangle$ can be written using the hermitian matrix $\mathbf{T}^\dagger = \mathbf{T}^*$:

$$\begin{aligned}
\langle \mathbf{A}|T\mathbf{B}\rangle &= \mathbf{A}^* \cdot \mathbf{TB} = \left(\mathbf{A}^*\mathbf{T}\right) \cdot \mathbf{B} \\
&= \left(\mathbf{T}^*\mathbf{A}\right)^* \cdot \mathbf{B} = \left(\mathbf{T}^\dagger\mathbf{A}\right)^* \cdot \mathbf{B} = \left\langle T^\dagger\mathbf{A}\right| \cdot |\mathbf{B}\rangle \\
&= \left\langle T^\dagger\mathbf{A}|\mathbf{B}\right\rangle
\end{aligned} \tag{1.99}$$

This result shows that the inner product between the transformed bra $\langle T^\dagger\mathbf{A}|$ and ket $|\mathbf{B}\rangle$ is the same inner product of the bra $\langle \mathbf{A}|$ and transformed ket $T|\mathbf{B}\rangle$. As a rule of thumb, the inner product does not change when moving the operator T from the second ket into the first bra and changing T into $T^\dagger$.

There are some properties of the inner product with a complex number α and its conjugate α^*:

$$\begin{aligned} \langle \mathbf{A}|\alpha\, \mathbf{B}\rangle &= \alpha\, \langle \mathbf{A}|\mathbf{B}\rangle = \left\langle \alpha^* \mathbf{A}|\mathbf{B}\right\rangle \ \text{ for } \ \alpha \in \mathbf{C} \\ \langle \alpha\, \mathbf{A}|\mathbf{B}\rangle &= \alpha^* \langle \mathbf{A}|\mathbf{B}\rangle = \left\langle \mathbf{A}|\alpha^* \mathbf{B}\right\rangle \ \text{ for } \ \alpha \in \mathbf{C} \end{aligned} \tag{1.100}$$

The eigenvalue problem derives from the characteristic equation:

$$T|\mathbf{A}\rangle = \lambda|\mathbf{A}\rangle; \quad |\mathbf{A}\rangle \neq \mathbf{0}, \ \text{ for } \lambda \in \mathbf{C} \tag{1.101}$$

The inner product between the bra $\langle \mathbf{A}|$ and its transformed ket $T|\mathbf{A}\rangle$ results as

$$\langle \mathbf{A}|T\mathbf{A}\rangle = \langle \mathbf{A}|\lambda\mathbf{A}\rangle = \lambda\langle \mathbf{A}|\mathbf{A}\rangle \tag{1.102}$$

Some characteristics of the eigenvalue problem are discussed in the following section [8, 9]:

1) *Eigenvalue of the Hermitian operator is a real number.*
   According to Eq. (1.99), the inner product in Eq. (1.102) with the Hermitian operator $T^\dagger$ $(= T)$ becomes

$$\begin{aligned} \langle \mathbf{A}|T\mathbf{A}\rangle &= \left\langle T^\dagger \mathbf{A}|\mathbf{A}\right\rangle \equiv \langle T\mathbf{A}|\mathbf{A}\rangle \\ &= \langle \lambda\mathbf{A}|\mathbf{A}\rangle = \lambda^* \langle \mathbf{A}|\mathbf{A}\rangle \end{aligned} \tag{1.103}$$

   Comparing Eq. (1.102) to Eq. (1.103), one obtains

$$\lambda^* = \lambda \tag{1.104}$$

   This result proves that the eigenvalue λ must be a real number.
2) *Eigenkets of the Hermitian operator are orthogonal.*
   Given two eigenkets with their different eigenvalues λ and μ, the eigenvalue problems can be formulated:

$$\begin{aligned} T|\mathbf{A}\rangle &= \lambda|\mathbf{A}\rangle; \quad |\mathbf{A}\rangle \neq \mathbf{0}, \\ T|\mathbf{B}\rangle &= \mu|\mathbf{B}\rangle; \quad |\mathbf{B}\rangle \neq \mathbf{0}, \end{aligned} \tag{1.105}$$

   The inner product between the kets $|\mathbf{A}\rangle$ and $T|\mathbf{B}\rangle$ can be given according to Eq. (1.100).

$$\langle \mathbf{A}|T\mathbf{B}\rangle = \langle \mathbf{A}|\mu\, \mathbf{B}\rangle = \mu\langle \mathbf{A}|\mathbf{B}\rangle \tag{1.106}$$

   Similarly, the inner product between the kets $T|\mathbf{A}\rangle$ and $|\mathbf{B}\rangle$ can be rewritten according to Eq. (1.100).

$$\langle T\mathbf{A}|\mathbf{B}\rangle = \langle \lambda\mathbf{A}|\mathbf{B}\rangle = \lambda^* \langle \mathbf{A}|\mathbf{B}\rangle \tag{1.107}$$

   Comparing Eq. (1.106) to Eq. (1.107), one obtains the Hermitian operator $T^\dagger$ $(= T)$ according to Eq. (1.99):

$$\begin{aligned}&\langle \mathbf{A}|T\mathbf{B}\rangle = \left\langle T^{\dagger}\mathbf{A}|\mathbf{B}\right\rangle = \langle T\mathbf{A}|\mathbf{B}\rangle \Leftrightarrow \\ &\mu\langle \mathbf{A}|\mathbf{B}\rangle = \lambda^{*}\langle \mathbf{A}|\mathbf{B}\rangle\end{aligned} \tag{1.108}$$

Therefore,

$$\left(\mu - \lambda^{*}\right)\langle \mathbf{A}|\mathbf{B}\rangle = 0 \tag{1.109}$$

Because the eigenvalues μ and λ^{*} are different, the inner product $\langle \mathbf{A}|\mathbf{B}\rangle$ must be zero according to Eq. (1.109). This result indicates that the eigenkets $|\mathbf{A}\rangle$ and $|\mathbf{B}\rangle$ are orthogonal.

3) *Hermitian matrix is diagonalizable in the normalized basis.*
For the eigenvalue problem in Eq. (1.101), there exists an eigenket (eigenvector) relating to its eigenvalue. Instead of the formulation given in Eq. (1.105), the Hermitian matrix can be easily written in the orthonormal basis $|\mathbf{e}_i\rangle$ for the eigenvalue λ_i:

$$\mathbf{T}|\mathbf{e}_i\rangle = \lambda_i|\mathbf{e}_i\rangle \Leftrightarrow \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_i & 0 \\ 0 & 0 & 0 & \lambda_N \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \lambda_i \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ for } i = 1,\ 2,\ \ldots, N \tag{1.110}$$

Therefore, the Hermitian matrix $\mathbf{T}$ is obviously diagonalizable at changing the eigenvector basis $|\mathbf{X}_i\rangle$ into the orthonormal basis $|\mathbf{e}_i\rangle$. In this case, the eigenvalues locate on the main diagonal and other matrix elements are zero in the Hermitian matrix, as shown in Eq. (1.110) [9].

1.7 Applying Bra and Ket Analysis to Eigenvalue Problems

Many problems in physics and engineering can be formulated similar to

$$\left|\dot{\mathbf{X}}\right\rangle = \mathbf{T}|\mathbf{X}\rangle \tag{1.111}$$

The solution of Eq. (1.111) can be assumed as

$$|\mathbf{X}\rangle = |\mathbf{E}\rangle\, e^{\lambda t} \tag{1.112}$$

where $|\mathbf{E}\rangle$ is the eigenvector, λ is the complex eigenvalue, $\lambda = \alpha + \mathrm{j}\omega \in \mathbf{C}$ in which ω is the eigenfrequency and α is the growth/decay rate.

Calculating the first derivative of the solution, one obtains

$$\left|\dot{\mathbf{X}}\right\rangle = \lambda|\mathbf{E}\rangle\, e^{\lambda t} = \lambda|\mathbf{X}\rangle \tag{1.113}$$

Substituting Eq. (1.113) into Eq. (1.111), the eigenvalue problem is given as

$$\mathbf{T}|\mathbf{X}\rangle = \lambda|\mathbf{X}\rangle \tag{1.114}$$

Equation (1.114) can be rewritten in the matrix form with the identity matrix **I**.

$$(\mathbf{T} - \lambda\mathbf{I})|\mathbf{X}\rangle = |\mathbf{0}\rangle \tag{1.115}$$

For nontrivial solutions of Eq. (1.115), the eigenvalue-related determinant must be zero.

$$\det(\mathbf{T} - \lambda\mathbf{I}) \equiv |\mathbf{T} - \lambda\mathbf{I}| = 0 \tag{1.116}$$

Equation (1.116) is called the characteristic equation whose solutions are the eigenvalues. The characteristic equation is the polynomial of λ^n; n equals two times of the degrees of freedom (DOF) of the system.

$$P(\lambda^n) \equiv a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots. + a_1\lambda + a_0 = 0 \tag{1.117}$$

There exists an eigenvector (eigenket) for each eigenvalue. The eigenvector results from Eq. (1.115).

$$\begin{aligned} &(\mathbf{T} - \lambda_{\mathrm{i}}\mathbf{I})|\mathbf{X}_{\mathrm{i}}\rangle = (\mathbf{T} - \lambda_{\mathrm{i}}\mathbf{I})|\mathbf{E}_{\mathrm{i}}\rangle e^{\lambda_i t} = |\mathbf{0}\rangle \\ &\forall \lambda_i \in \mathbf{C} \Rightarrow (\mathbf{T} - \lambda_{\mathrm{i}}\mathbf{I})|\mathbf{E}_{\mathrm{i}}\rangle = |\mathbf{0}\rangle \end{aligned} \tag{1.118}$$

An example for the eigenvalue problem will be given in the following subsection.

Let **T** be the system matrix; it can be written as

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

The characteristic equation of the eigenvalues λ yields

$$
\begin{aligned}
|\mathbf{T}-\lambda\mathbf{I}| &= \begin{vmatrix} (1-\lambda) & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & 1 & (1-\lambda) \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} -\lambda & -1 \\ 1 & (1-\lambda) \end{vmatrix} = (1-\lambda)[\lambda(\lambda-1)+1] \\
&= (1-\lambda)\left(\lambda^2-\lambda+1\right) \\
&= (1-\lambda)\left(\lambda-\frac{1+j\sqrt{3}}{2}\right)\cdot\left(\lambda-\frac{1-j\sqrt{3}}{2}\right) = 0
\end{aligned}
$$

Thus, there are three eigenvalues as follows:

$$
\begin{aligned}
\lambda_1 &= 1; \\
\lambda_2 &= \frac{1}{2}+j\frac{\sqrt{3}}{2}; \\
\lambda_3 &= \frac{1}{2}-j\frac{\sqrt{3}}{2}
\end{aligned}
\quad .
$$

Using Eq. (1.118), one obtains the eigenvectors of the eigenvalue problem:

- For $\lambda = \lambda_1 = 1$:

$$
\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Therefore,

$$
\begin{cases} 0x_1 = 0 \rightarrow x_1 \equiv 1 \\ -x_2 - x_3 = 0 \rightarrow x_3 = -x_2 = 0 \\ x_2 = 0 \rightarrow x_2 = 0 \end{cases} \Rightarrow |\mathbf{E}_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
$$

- For $\lambda = \lambda_2 = \frac{1}{2}+j\frac{\sqrt{3}}{2}$:

$$
\begin{bmatrix} \frac{1}{2}-j\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right) & -1 \\ 0 & 1 & \frac{1}{2}-j\frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Thus,

$$\begin{cases} \left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)x_1 = 0 \to x_1 = 0 \\ -\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)x_2 - x_3 = 0 \to x_2 \equiv 2j \\ x_2 + \left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)x_3 = 0 \to x_3 = \sqrt{3} - j \end{cases} \Rightarrow |\mathbf{E}_2\rangle = \begin{pmatrix} 0 \\ 2j \\ \sqrt{3} - j \end{pmatrix}$$

- For $\lambda = \lambda_3 = \frac{1}{2} - j\frac{\sqrt{3}}{2}$:

$$\begin{bmatrix} \frac{1}{2} + j\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) & -1 \\ 0 & 1 & \frac{1}{2} + j\frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

$$\begin{cases} \left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)x_1 = 0 \to x_1 = 0 \\ -\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)x_2 - x_3 = 0 \to x_2 \equiv -2j \\ x_2 + \left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)x_3 = 0 \to x_3 = \sqrt{3} + j. \end{cases} \Rightarrow |\mathbf{E}_3\rangle = \begin{pmatrix} 0 \\ -2j \\ \sqrt{3} + j \end{pmatrix}$$

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Chapter 2
Tensor Analysis

2.1 Introduction to Tensors

Tensors are a powerful mathematical tool that is used in many areas in engineering and physics including general relativity theory, quantum mechanics, statistical thermodynamics, classical mechanics, electrodynamics, solid mechanics, and fluid dynamics. Laws of physics and physical invariants must be independent of any arbitrarily chosen coordinate system. Tensors describing these characteristics are invariant under coordinate transformations; however, their tensor components heavily depend on the coordinate bases. Therefore, the tensor components change as the coordinate system varies in the considered spaces. Before going into details, we provide less experienced readers with some examples.

Different tensors are listed in Table 2.1, which can be expressed in different chosen bases for any curvilinear coordinate. Using Einstein summation convention, the notation can be shortened. Note that Einstein summation convention is only valid for the same indices in the lower and upper positions. The relating contravariant or covariant tensor components can be expressed in the covariant or contravariant bases (cf. Appendix E). The tensor order is determined by the number of the coordinate basis. Thus, the component of a first-order tensor has only one dummy index i relating to a single basis. In case of a second-order tensor, its component contains two dummy indices i and j that relate to the double bases. Similarly, the component of an N-order tensor has N dummy indices relating to N bases.

The dummy indices (inner indices) are the repeated indices running from the values from 1 to N in Einstein summation convention. The free index (outer index) can be independently chosen for any value from 1 to N; i.e., for any tensor component in the particular coordinate, as shown in the below example. Note that the dimensions of the dummy and free indices must be the same value of the space dimensions.

H. Nguyen-Schäfer, J.-P. Schmidt, *Tensor Analysis and Elementary Differential Geometry for Physicists and Engineers*, Mathematical Engineering,
DOI 10.1007/978-3-662-48497-5_2

Table 2.1 Tensors in general curvilinear coordinates

Type	Component	Basis	Tensor
First-order tensors $\in \mathbf{R}^N$	T^i, T_i	$\mathbf{g}_i, \mathbf{g}^i$	$\mathbf{T}^{(1)} = T^i\mathbf{g}_i, T_i\mathbf{g}^i$
Second-order tensors $\in \mathbf{R}^N \times \mathbf{R}^N$	$T^{ij}, T_{ij}, T^i{}_j$, $T^j{}_i$	$\mathbf{g}_i, \mathbf{g}^i$, $\mathbf{g}_j, \mathbf{g}^j$	$\mathbf{T}^{(2)} = T^{ij}\mathbf{g}_i\mathbf{g}_j$, $\mathrm{T}_{ij}\mathbf{g}^i\mathbf{g}^j$, $\mathrm{T}^i{}_j\mathbf{g}_i\mathbf{g}^j$, $T^j{}_i\mathbf{g}^i\mathbf{g}_j$
Third-order tensors $\in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N$	T^{ijk}, T_{ijk}, $T^{ik}{}_{.j}$, $T^j{}_{ik.}$	$\mathbf{g}_i, \mathbf{g}^i$, $\mathbf{g}_j, \mathbf{g}^j$, $\mathbf{g}_k, \mathbf{g}^k$	$\mathbf{T}^{(3)} = T^{ijk}\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k$, $T_{ijk}\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k$, $T^{ik}{}_{.j}\mathbf{g}_i\mathbf{g}_k\mathbf{g}^j$, $T^j{}_{ik.}\mathbf{g}^i\mathbf{g}^k\mathbf{g}_j$
N-order tensors $\in \mathbf{R}^N \times \ldots \times \mathbf{R}^N$	$T^{ijk\ldots n}$, $T_{ijk\ldots n}$, $T^{ik\ldots n}{}_{.j}$, $T^j{}_{ik\ldots n.}$	$\mathbf{g}_i, \mathbf{g}^i$, $\mathbf{g}_j, \mathbf{g}^j$, $\mathbf{g}_k, \mathbf{g}^k$, $,\ldots$ $\mathbf{g}_n, \mathbf{g}^n$	$\mathbf{T}^{(\mathrm{N})} = T^{ijk\ldots n}\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k\cdots\mathbf{g}_n$, $T^{ik\ldots n}{}_{.j}\mathbf{g}_i\mathbf{g}_k\cdots\mathbf{g}_n\mathbf{g}^j$

$$\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j \equiv \sum_{j=1}^{N}\sum_{i=1}^{N} T^{ij}\mathbf{g}_i\mathbf{g}_j;\ i,\ j:\ dummy\ indices$$

$$\mathbf{T}^j = T^{ij}\mathbf{g}_i \equiv \sum_{i=1}^{N} T^{ij}\mathbf{g}_i;\ i:\ dummy,\ j:\ free\ index$$

2.2 Definition of Tensors

The definition of tensors is based on multilinear algebra by a multilinear map. We consider the linear vector space L and its dual vector space L^*. Each of the vector spaces belongs to the finite N-dimensional space $\mathbf{R}^N$, the image space W, to the real space $\mathbf{R}$. A mixed tensor of type (m, n) is a multilinear functional $\mathbf{T}$ which maps an $(m+n)$ tuple of vectors of the vector spaces L and L^* into the real space W [1] (cf. Fig. 2.1):

$$\mathbf{T}_n^m \equiv \mathbf{T}:\ \underbrace{(L \times \cdots \times L)}_{n\ \ copies} \times \underbrace{(L^* \times \cdots \times L^*)}_{m\ \ copies} \ \rightarrow\ W$$

$$\underbrace{\mathbf{R}^N \times \cdots \times \mathbf{R}^N}_{n\ \ copies} \times \underbrace{\mathbf{R}^N \times \cdots \times \mathbf{R}^N}_{m\ \ copies} \ \rightarrow\ \mathbf{R}$$

$$\mathbf{T}: (\mathbf{u}_1, \ldots, \mathbf{u}_n; \mathbf{v}_1, \ldots, \mathbf{v}_m)\ \rightarrow\ \mathbf{T}(\mathbf{u}_1, \ldots, \mathbf{u}_n; \mathbf{v}_1, \ldots, \mathbf{v}_m) \in \mathbf{R} \tag{2.1}$$

Mapping the covariant basis $\{\mathbf{g}_{jn}\} \in L$ and contravariant basis $\{\mathbf{g}^{im}\} \in L^*$ by the multilinear functional $\mathbf{T}$ of the tensor type (m, n), one obtains its images in the real

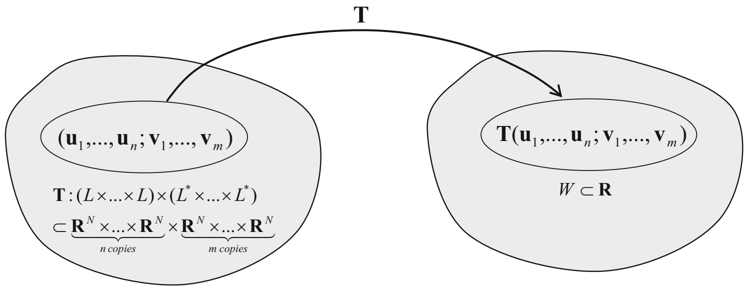

Fig. 2.1 Multilinear mapping functional **T**

space $W \subset \mathbf{R}$. The real images are called the components of the $(m+n)$-order mixed tensor **T** with respect to the relating bases:

$$\mathbf{T}\left(\mathbf{g}_{j1},\ldots,\mathbf{g}_{jn};\mathbf{g}^{i1},\ldots,\mathbf{g}^{im}\right) \equiv T^{i1\ldots im}_{j1\ldots jn} \in W \subset \mathbf{R} \tag{2.2}$$

Thus, the $(m+n)$-order mixed tensor **T** of type $(m, n) \in T_n{}^m(L)$ can be expressed in the covariant and contravariant bases. In total, the $(m+n)$-order tensor **T** has N^{m+n} components.

$$\mathbf{T} = T^{i1\ldots im}_{j1\ldots jn}\, \mathbf{g}_{i1}\ldots\, \mathbf{g}_{im}\mathbf{g}^{j1}\ldots\, \mathbf{g}^{jn} \in T^m_n(L) \tag{2.3}$$

where $T_n{}^m(L)$ is called the tensor space that consists of all tensors **T** of type (m, n):

$$T^m_n(L) = \underbrace{(L\times\cdots\times L)}_{m\ \ copies}\times\underbrace{\left(L^*\times\cdots\times L^*\right)}_{n\ \ copies}$$

In case of covariant and contravariant tensors **T**, only the basis of the dual vector space L^* or real vector space L are respectively considered in Eq. (2.3).

- n-order covariant tensors of type $(0, n)$ in the tensor space $T_n(L)$:

$$\mathbf{T} = T_{j1\ldots jn}\, \mathbf{g}^{j1}\ldots\, \mathbf{g}^{jn} \in T_n(L) = \underbrace{\left(L^*\times\cdots\times L^*\right)}_{n\ \ copies} \tag{2.4}$$

- m-order contravariant tensors of type $(m, 0)$ in the tensor space $T^m(L)$:

$$\mathbf{T} = T^{i1\ldots im}\, \mathbf{g}_{i1}\ldots\, \mathbf{g}_{im} \in T^m(L) = \underbrace{(L\times\cdots\times L)}_{m\ \ copies} \tag{2.5}$$

2.2.1 An Example of a Second-Order Covariant Tensor

An arbitrary vector $\mathbf{v}$ can be expressed in the covariant basis $\mathbf{g}_k$ in the N-dimensional vector space $\mathbf{V}$ as

$$\mathbf{v} = v^k \mathbf{g}_k \ \text{ for } k = 1, 2, \ldots, N \tag{2.6}$$

Applying the bilinear mapping $\mathbf{T}$ to the vector $\mathbf{v}$ and using the Kronecker delta, one obtains its mapping image $\mathbf{Tv}$. Straightforwardly, this is a tensor of one lower order compared to the mapping tensor $\mathbf{T}$.

$$\begin{aligned} \mathbf{Tv} &\equiv T_{ij}\mathbf{g}^i\mathbf{g}^j \cdot \left(v^k \mathbf{g}_k\right) \\ &= T_{ij}v^k(\mathbf{g}^i \cdot \mathbf{g}_k)\mathbf{g}^j \\ &= T_{ij}v^k\delta_k^i\mathbf{g}^j \text{ for } i = k \\ &= \left(T_{kj}v^k\right)\mathbf{g}^j \text{ for } j,\ k = 1,\ 2, \ldots, N \\ &\equiv T_j^*\mathbf{g}^j \text{ for } j = 1,\ 2, \ldots, N \end{aligned} \tag{2.7a}$$

whereas the second-order covariant tensor $\mathbf{T}$ can be expressed as

$$\begin{aligned} &\mathbf{T} = T_{ij}\mathbf{g}^i\mathbf{g}^j \text{ for } i, j = 1, 2, \ldots, N; \\ &\mathbf{T} \in \mathbf{R}^N \times \mathbf{R}^N \end{aligned}$$

Note that in case of a three-dimensional vector space $\mathbf{R}^3$ ($N = 3$), there are nine covariant components T_{ij}. The number of the tensor components can be calculated by N^n ($3^2 = 9$), in which n is the number of indices i and j ($n = 2$).

Obviously, that the mapping image $\mathbf{Tv}$ is also a tensor of one lower order compared to the tensor $\mathbf{T}$. The covariant tensor component T_j^* can be calculated by

$$T_j^* \equiv T_{kj}v^k = \mathbf{g}_j \cdot \mathbf{Tv}. \tag{2.7b}$$

2.3 Tensor Algebra

2.3.1 General Bases in General Curvilinear Coordinates

The vector $\mathbf{r}$ can be written in Cartesian coordinates of Euclidean space $\mathbf{E}^3$, as displayed in Fig. 2.2.

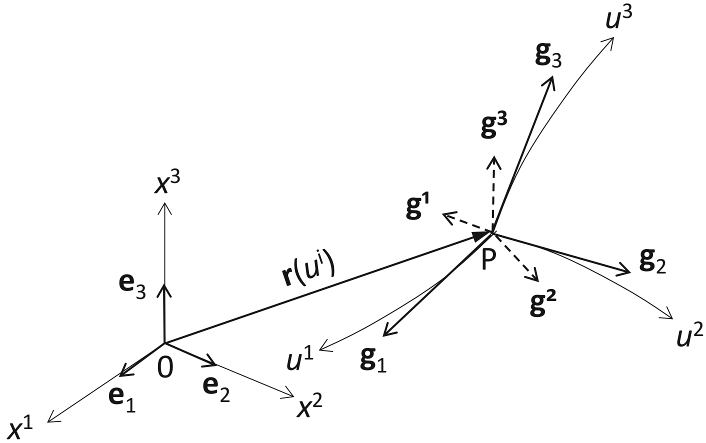

Fig. 2.2 Bases of general curvilinear coordinates in the space $\mathbf{E}^3$

$$\mathbf{r} = x^i \mathbf{e}_i \tag{2.8}$$

The differential $d\mathbf{r}$ results from Eq. (2.8) in

$$d\mathbf{r} = \mathbf{e}_i dx^i = \frac{\partial \mathbf{r}}{\partial x^i} dx^i \tag{2.9}$$

Using the chain rule of differentiation, the orthonormal bases $\mathbf{e}_i$ of the coordinates x^i are defined by

$$\begin{aligned} \mathbf{e}_i &\equiv \frac{\partial \mathbf{r}}{\partial x^i} = \frac{\partial \mathbf{r}}{\partial u^j} \frac{\partial u^j}{\partial x^i} \\ &\equiv \mathbf{g}_j \frac{\partial u^j}{\partial x^i} \text{ for } j = 1, 2, \ldots, N \end{aligned} \tag{2.10}$$

Analogously, the bases of the curvilinear coordinates u^i can be calculated in the curvilinear coordinate system of $\mathbf{E}^N$

$$\begin{aligned} \mathbf{g}_j &\equiv \frac{\partial \mathbf{r}}{\partial u^j} = \frac{\partial \mathbf{r}}{\partial x^k} \frac{\partial x^k}{\partial u^j} \\ &= \mathbf{e}_k \frac{\partial x^k}{\partial u^j} \text{ for } k = 1, 2, \ldots, N \end{aligned} \tag{2.11}$$

The curvilinear coordinates u^i are functions of the coordinates x^i, the covariant bases in Eq. (2.11) can be calculated using the chain rule of differentiation.

$$\begin{aligned}\mathbf{g}_j &= \frac{\partial \mathbf{r}}{\partial u^j} = \frac{\partial \mathbf{r}}{\partial x^i}\frac{\partial x^i}{\partial u^j} \\ &= \mathbf{e}_i \frac{\partial x^i}{\partial u^j} \\ &\equiv \mathbf{e}_i x^i_j \text{ for } i = 1, 2, \ldots, N\end{aligned} \tag{2.12}$$

Thus, the curvilinear basis $\mathbf{g}_j$ can be written in a linear combination of the orthonormal basis $\mathbf{e}_i$ according to Eq. (2.12). The derivative x^i_j is called the shift tensor between the orthonormal and curvilinear coordinates.

Generally, the basis $\mathbf{g}_i$ of the curvilinear coordinate u^i can be rewritten in a linear combination of the basis $\mathbf{g}'_j$ of other curvilinear coordinate u'^j. The derivative u'^j_i is defined as the shift tensor between both curvilinear coordinates.

$$\begin{aligned}\mathbf{g}_i &= \frac{\partial \mathbf{r}}{\partial u'^j}\frac{\partial u'^j}{\partial u^i} \\ &= \mathbf{g}'_j \frac{\partial u'^j}{\partial u^i} \equiv \mathbf{g}'_j u'^j_i \text{ for } j = 1, 2, \ldots, N\end{aligned} \tag{2.13}$$

In the curvilinear coordinate system (u^1,u^2,u^3) of Euclidean space $\mathbf{E}^3$, its basis is generally non-orthogonal and non-unitary (non-orthonormal basis); i.e., the bases are not mutually perpendicular and their vector lengths are not equal to one [2–4]. In this case, the curvilinear coordinate system (u^1,u^2,u^3) has three covariant bases $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ and three contravariant bases $\mathbf{g}^1$, $\mathbf{g}^2$, and $\mathbf{g}^3$ at the origin P, as shown in Fig. 2.2. Generally, the origin P of the curvilinear coordinates could move everywhere in Euclidean space. Therefore, the bases of the curvilinear coordinates only depend on the respective origin P. For this reason, the curvilinear bases are not fixed in the whole curvilinear coordinates like in Cartesian coordinates.

The vector $\mathbf{r}$ of the point $P(u^1,u^2,u^3)$ can be written in covariant and contravariant bases.

$$\begin{aligned}\mathbf{r} &= u^1\mathbf{g}_1 + u^2\mathbf{g}_2 + u^3\mathbf{g}_3 \\ &= u_1\mathbf{g}^1 + u_2\mathbf{g}^2 + u_3\mathbf{g}^3\end{aligned} \tag{2.14}$$

where

u^1, u^2, u^3 are the contravariant vector components of the coordinates (u^1,u^2,u^3);
$\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ are the covariant bases of the coordinate system (u^1,u^2,u^3);
u_1, u_2, u_3 are the covariant vector components of the coordinates (u^1,u^2,u^3);
$\mathbf{g}^1$, $\mathbf{g}^2$, $\mathbf{g}^3$ are the contravariant bases of the coordinate system (u^1,u^2,u^3).

The covariant base $\mathbf{g}_i$ is defined by the tangential vector to the corresponding curvilinear coordinate u^i for $i = 1, 2, 3$. Both bases $\mathbf{g}_1$ and $\mathbf{g}_2$ generates a tangential surface to the curvilinear surface (u^1u^2) at the considered origin P. Note that the basis $\mathbf{g}_1$ is not perpendicular to the bases $\mathbf{g}_2$ and $\mathbf{g}_3$. However, the contravariant basis $\mathbf{g}^3$ is perpendicular to the tangential surface $(\mathbf{g}_1\mathbf{g}_2)$ at the origin P. Generally, the contravariant basis ($\mathbf{g}^k$) results from the cross product of the other covariant bases ($\mathbf{g}_i \times \mathbf{g}_j$).

$$\alpha\, \mathbf{g}^k = \mathbf{g}_i \times\ \mathbf{g}_j \text{ for } i, j,\ k = 1,\ 2,\ 3 \tag{2.15}$$

where α is a scalar factor.

Multiplying Eq. (2.15) by the covariant basis $\mathbf{g}_k$, the scalar factor α can be calculated as

$$\begin{aligned} &\alpha\left(\mathbf{g}^k \cdot\ \mathbf{g}_k\right) = \alpha\, \delta_k^k = \alpha = \left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot\ \mathbf{g}_k \\ &\Rightarrow \alpha = \left(\mathbf{g}_i \times \mathbf{g}_j\right)\ \cdot \mathbf{g}_k \equiv \left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right] \end{aligned} \tag{2.16}$$

The scalar factor α equals the scalar triple product that is given in [3]:

$$\begin{aligned} \alpha &\equiv [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = \left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot \mathbf{g}_k = (\mathbf{g}_k \times \mathbf{g}_i) \cdot \mathbf{g}_j = \left(\mathbf{g}_j \times \mathbf{g}_k\right) \cdot\ \mathbf{g}_i \\ &= \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}^{\frac{1}{2}} = \begin{vmatrix} g_{31} & g_{32} & g_{33} \\ g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{vmatrix}^{\frac{1}{2}} = \begin{vmatrix} g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \\ g_{11} & g_{12} & g_{13} \end{vmatrix}^{\frac{1}{2}} \\ &= \sqrt{\det\left(g_{ij}\right)} \equiv \sqrt{g} = J > 0 \end{aligned} \tag{2.17}$$

where J is defined as the Jacobian, as given in

$$J \equiv \varepsilon_{ijk} \frac{\partial x^i}{\partial u^1} \frac{\partial x^j}{\partial u^2} \frac{\partial x^k}{\partial u^3} = \begin{vmatrix} \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^1}{\partial u^2} & \dfrac{\partial x^1}{\partial u^3} \\ \dfrac{\partial x^2}{\partial u^1} & \dfrac{\partial x^2}{\partial u^2} & \dfrac{\partial x^2}{\partial u^3} \\ \dfrac{\partial x^3}{\partial u^1} & \dfrac{\partial x^3}{\partial u^2} & \dfrac{\partial x^3}{\partial u^3} \end{vmatrix} \tag{2.18}$$

Thus,

$$\mathbf{g}^k = \frac{\varepsilon_{ijk}}{\sqrt{g}} \left(\mathbf{g}_i \times\ \mathbf{g}_j\right) = \frac{\varepsilon_{ijk}}{J} \left(\mathbf{g}_i \times\ \mathbf{g}_j\right) \tag{2.19}$$

where ε_{ijk} is the Levi-Civita permutation symbols in Eq. (A.5), cf. Appendix A.

According to Eq. (2.19), the contravariant basis $\mathbf{g}^k$ is perpendicular to both covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$. Additionally, the contravariant basis $\mathbf{g}^k$ is chosen such that the vector length of the contravariant basis equals the inversed vector length of its relating covariant basis.

Therefore,

$$\mathbf{g}^k \cdot \mathbf{g}_i = \frac{\left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot \mathbf{g}_i}{\left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right]} = \delta_i^k \tag{2.20}$$

As a result, the relation between the contravariant and covariant bases is given in the general curvilinear coordinate system ($u^1,\ldots,u^N$).

$$\begin{cases} \mathbf{g}_i \cdot \mathbf{g}^k = \mathbf{g}^k \cdot \mathbf{g}_i = \delta_i^k \text{ for } i,k = 1,\ 2,\ \ldots,N \\ \mathbf{g}_i \cdot \mathbf{g}_k = \mathbf{g}_k \cdot \mathbf{g}_i \neq \delta_i^k \text{ for } i,k = 1,\ 2,\ \ldots,N \end{cases} \tag{2.21}$$

The basis $\mathbf{g}_i$ is called dual to the basis $\mathbf{g}^j$ [5] if

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j \text{ for } i,j = 1,\ 2,\ \ldots,N \tag{2.22}$$

where δ_i^j is the Kronecker delta.

Let $\{\mathbf{g}_1, \mathbf{g}_2,\ldots,\mathbf{g}_N\}$ be a covariant basis of the curvilinear coordinates $\{u^i\}$. The contravariant basis $\{\mathbf{g}^1, \mathbf{g}^2,\ldots,\mathbf{g}^N\}$, the dual basis to the covariant basis, can be written in the matrix formulation as

$$\mathbf{G} = [\mathbf{g}_1 \quad \mathbf{g}_2 \quad . \quad \mathbf{g}_i \quad \mathbf{g}_N]\,;\ \ \mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ . \\ \mathbf{g}^j \\ \mathbf{g}^N \end{bmatrix} \quad \Rightarrow \quad \mathbf{G}^{-1}\mathbf{G} = \mathbf{I} \tag{2.23}$$

where $\mathbf{g}^j$ is the j row vector of $\mathbf{G}^{-1}$; $\mathbf{g}_i$ is the i column vector of $\mathbf{G}$.

The covariant and contravariant bases (dual bases) of the orthogonal cylindrical and spherical coordinates are computed in the following section.

2.3.1.1 Orthogonal Cylindrical Coordinates

The cylindrical coordinates (r,θ,z) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary. Figure 2.3 shows a point P in the cylindrical coordinates (r,θ,z) embedded in the orthonormal Cartesian coordinates (x^1,x^2,x^3). However, the cylindrical coordinates change as the point P varies.

The vector **OP** can be written in Cartesian coordinates (x^1,x^2,x^3):

$$\begin{aligned} \mathbf{R} &= (r\cos\theta)\,\mathbf{e}_1 + (r\sin\theta)\,\mathbf{e}_2 + z\,\mathbf{e}_3 \\ &\equiv x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \end{aligned} \tag{2.24}$$

where

$\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the orthonormal bases of Cartesian coordinates;
θ is the polar angle.

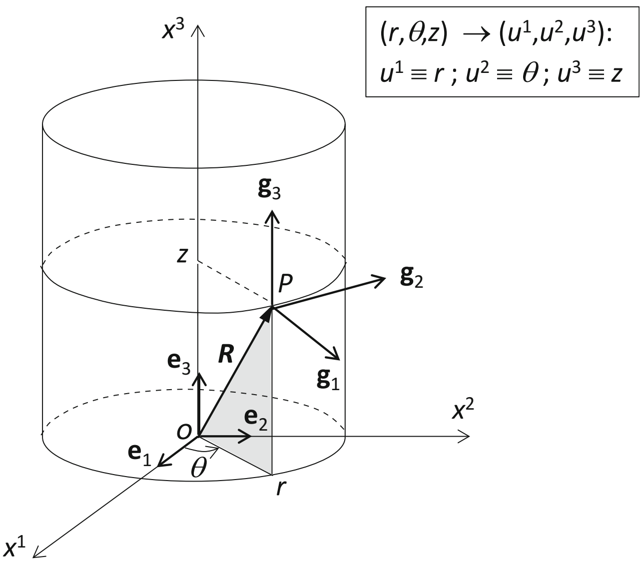

Fig. 2.3 Covariant bases of orthogonal cylindrical coordinates

To simplify the formulation with Einstein symbol, the coordinates of u^1, u^2, and u^3 are used for r, θ, and z, respectively. Therefore, the coordinates of $P(u^1,u^2,u^3)$ are given in Cartesian coordinates:

$$P\left(u^1,u^2,u^3\right) = \left\{ \begin{array}{l} x^1 = r\cos\theta \equiv u^1 \cos u^2 \\ x^2 = r\sin\theta \equiv u^1 \sin u^2 \\ x^3 = z \equiv u^3 \end{array} \right\} \tag{2.25}$$

The covariant bases of the curvilinear coordinates are computed from

$$\mathbf{g}_i = \frac{\partial \mathbf{R}}{\partial u^i} = \frac{\partial \mathbf{R}}{\partial x^j} \cdot \frac{\partial x^j}{\partial u^i} = \mathbf{e}_j \frac{\partial x^j}{\partial u^i} \text{ for } j = 1, 2, 3 \tag{2.26}$$

The covariant basis matrix **G** yields from Eq. (2.26):

$$\begin{aligned}\mathbf{G} &= [\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3] \\ &= \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\end{aligned} \tag{2.27}$$

The determinant of **G** is called the Jacobian J.

$$|\mathbf{G}| \equiv J = \begin{vmatrix} \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^1}{\partial u^2} & \dfrac{\partial x^1}{\partial u^3} \\ \dfrac{\partial x^2}{\partial u^1} & \dfrac{\partial x^2}{\partial u^2} & \dfrac{\partial x^2}{\partial u^3} \\ \dfrac{\partial x^3}{\partial u^1} & \dfrac{\partial x^3}{\partial u^2} & \dfrac{\partial x^3}{\partial u^3} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \tag{2.28}$$

The relation between the covariant and contravariant bases yields from Eq. (2.22):

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta_j^i \quad \text{(Kronecker delta)} \tag{2.29}$$

Thus, the contravariant basis matrix $\mathbf{G}^{-1}$ results from the inversion of the covariant basis matrix $\mathbf{G}$, as given in Eq. (2.27).

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & r\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & r \end{pmatrix} \tag{2.30}$$

The calculation of the determinant and inversion matrix of $\mathbf{G}$ will be discussed in the following section.

According to Eq. (2.27), the covariant bases can be denoted as

$$\begin{cases} \mathbf{g}_1 = (\cos\theta)\,\mathbf{e}_1 + (\sin\theta)\,\mathbf{e}_2 + 0\cdot\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_1| = 1 \\ \mathbf{g}_2 = -(r\sin\theta)\,\mathbf{e}_1 + (r\cos\theta)\,\mathbf{e}_2 + 0\cdot\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_2| = r \\ \mathbf{g}_3 = 0\cdot\,\mathbf{e}_1 + 0\cdot\mathbf{e}_2 + 1\cdot\mathbf{e}_3 \Rightarrow |\mathbf{g}_3| = 1 \end{cases} \tag{2.31}$$

The contravariant bases result from Eq. (2.30).

$$\begin{cases} \mathbf{g}^1 = (\cos\theta)\,\mathbf{e}_1 + (\sin\theta)\,\mathbf{e}_2 + 0\cdot\,\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^1\right| = 1 \\ \mathbf{g}^2 = -\left(\dfrac{\sin\theta}{r}\right)\mathbf{e}_1 + \left(\dfrac{\cos\theta}{r}\right)\mathbf{e}_2 + 0\cdot\,\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^2\right| = \dfrac{1}{r} \\ \mathbf{g}^3 = 0\cdot\mathbf{e}_1 + 0\cdot\,\mathbf{e}_2 + 1\cdot\,\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^3\right| = 1 \end{cases} \tag{2.32}$$

Not only the covariant bases but also the contravariant bases of the cylindrical coordinates are orthogonal due to

$$\begin{aligned} &\mathbf{g}_i \cdot \mathbf{g}^j = \mathbf{g}^j \cdot \mathbf{g}_i = \delta_i^j \\ &\mathbf{g}_i \cdot\, \mathbf{g}_j = 0 \text{ for } i \neq j; \\ &\mathbf{g}^i \cdot\, \mathbf{g}^j = 0 \text{ for } i \neq j. \end{aligned}$$

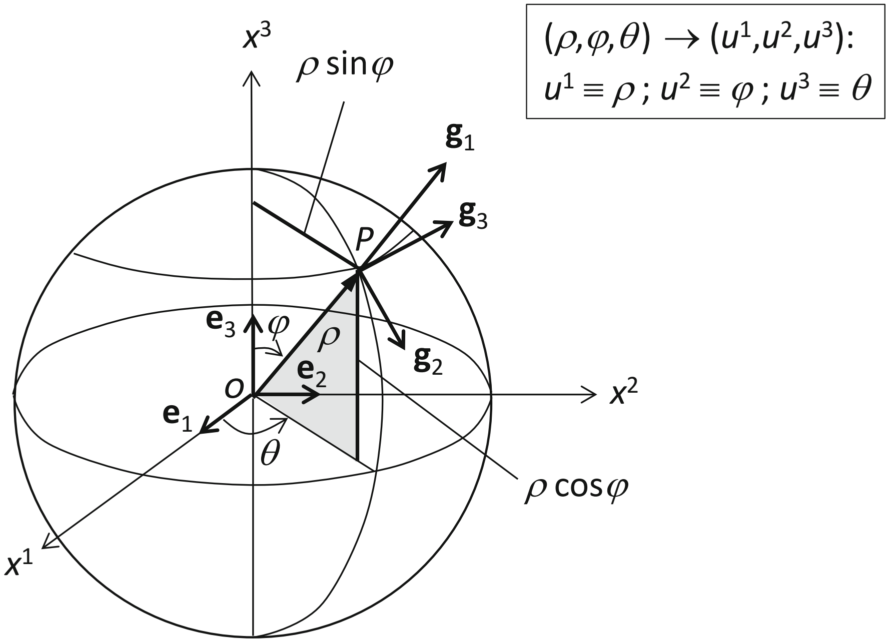

Fig. 2.4 Covariant bases of orthogonal spherical coordinates

2.3.1.2 Orthogonal Spherical Coordinates

The spherical coordinates (ρ,φ,θ) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary.

The spherical coordinates (ρ,φ,θ) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary. Figure 2.4 shows a point P in the spherical coordinates (r,θ,z) embedded in the orthonormal Cartesian coordinates (x^1,x^2,x^3). However, the spherical coordinates change as the point P varies.

The vector **OP** can be written in Cartesian coordinates (x^1,x^2,x^3):

$$\begin{aligned}\mathbf{R} =\; & (\rho \sin\varphi \cos\theta)\,\mathbf{e}_1 + (\rho \sin\varphi \sin\theta)\,\mathbf{e}_2 + \rho \cos\varphi\,\mathbf{e}_3 \\ \equiv\; & x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3\end{aligned} \tag{2.33}$$

where

$\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the orthonormal bases of Cartesian coordinates;
φ is the equatorial angle;
θ is the polar angle.

To simplify the formulation with Einstein symbol, the coordinates of u^1, u^2, and u^3 are used for ρ, φ, and θ, respectively. Therefore, the coordinates of $P(u^1,u^2,u^3)$ are given in Cartesian coordinates:

$$P\left(u^1,u^2,u^3\right) = \left\{ \begin{array}{l} x^1 = \rho \sin\varphi \cos\theta \equiv u^1 \sin u^2 \cos u^3 \\ x^2 = \rho \sin\varphi \sin\theta \equiv u^1 \sin u^2 \cos u^3 \\ x^3 = \rho \cos\varphi \equiv u^1 \cos u^2 \end{array} \right\} \tag{2.34}$$

The covariant bases of the curvilinear coordinates are computed from

$$\mathbf{g}_i = \frac{\partial \mathbf{R}}{\partial u^i} = \frac{\partial \mathbf{R}}{\partial x^j} \cdot \frac{\partial x^j}{\partial u^i} = \mathbf{e}_j \frac{\partial x^j}{\partial u^i} \text{ for } j = 1,2,3 \tag{2.35}$$

Thus, the covariant basis matrix **G** can be calculated from Eq. (2.35).

$$\begin{aligned} \mathbf{G} = [\,\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3\,] &= \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{pmatrix} \\ &= \begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix} \end{aligned} \tag{2.36}$$

The determinant of the covariant basis matrix **G** is called the Jacobian J.

$$\begin{aligned} |\mathbf{G}| \equiv J &= \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix} \\ &= \rho^2 \sin\varphi \end{aligned} \tag{2.37}$$

Similarly, the contravariant basis matrix $\mathbf{G}^{-1}$ is the inversion of the covariant basis matrix **G**.

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \frac{1}{\rho} \begin{pmatrix} \rho\sin\varphi\cos\theta & \rho\sin\varphi\sin\theta & \rho\cos\varphi \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\left(\frac{\sin\theta}{\sin\varphi}\right) & \left(\frac{\cos\theta}{\sin\varphi}\right) & 0 \end{pmatrix} \tag{2.38}$$

The matrix product $\mathbf{G}\mathbf{G}^{-1}$ must be an identity matrix according to Eq. (2.23).

$$
\mathbf{G}^{-1}\mathbf{G} = \frac{1}{\rho}\begin{pmatrix} \rho\sin\varphi\cos\theta & \rho\sin\varphi\sin\theta & \rho\cos\varphi \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\left(\dfrac{\sin\theta}{\sin\varphi}\right) & \left(\dfrac{\cos\theta}{\sin\varphi}\right) & 0 \end{pmatrix}
\cdot\begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \mathbf{I} \tag{2.39}
$$

According to Eq. (2.36), the covariant bases can be written as

$$
\begin{aligned}
\mathbf{g}_1 &= (\sin\varphi\cos\theta)\,\mathbf{e}_1 + (\sin\varphi\sin\theta)\,\mathbf{e}_2 + \cos\varphi\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_1| = 1 \\
\mathbf{g}_2 &= (\rho\cos\varphi\cos\theta)\,\mathbf{e}_1 + (\rho\cos\varphi\sin\theta)\,\mathbf{e}_2 - (\rho\sin\varphi)\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_2| = \rho \\
\mathbf{g}_3 &= (-\rho\sin\varphi\sin\theta)\,\mathbf{e}_1 + (\rho\sin\varphi\cos\theta)\,\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow |\mathbf{g}_3| = \rho\sin\varphi
\end{aligned} \tag{2.40}
$$

The contravariant bases result from Eq. (2.38).

$$
\begin{aligned}
\mathbf{g}^1 &= (\sin\varphi\cos\theta)\,\mathbf{e}_1 + (\sin\varphi\sin\theta)\,\mathbf{e}_2 + \cos\varphi\,\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^1\right| = 1 \\
\mathbf{g}^2 &= \left(\frac{1}{\rho}\cos\varphi\cos\theta\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\cos\varphi\sin\theta\right)\mathbf{e}_2 - \left(\frac{1}{\rho}\sin\varphi\right)\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^2\right| = \frac{1}{\rho} \\
\mathbf{g}^3 &= -\left(\frac{1}{\rho}\frac{\sin\theta}{\sin\varphi}\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\right)\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^3\right| = \frac{1}{\rho\sin\varphi}
\end{aligned} \tag{2.41}
$$

Not only the covariant bases but also the contravariant bases of the spherical coordinates are orthogonal due to

$$
\begin{aligned}
&\mathbf{g}_i\cdot\mathbf{g}^j = \mathbf{g}^j\cdot\mathbf{g}_i = \delta_i^j; \\
&\mathbf{g}_i\cdot\mathbf{g}_j = 0 \quad \text{for } i \neq j; \\
&\mathbf{g}^i\cdot\mathbf{g}^j = 0 \quad \text{for } i \neq j.
\end{aligned}
$$

2.3.2 Metric Coefficients in General Curvilinear Coordinates

The covariant basis vectors $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ to the general curvilinear coordinates (u^1, u^2,u^3) at the point P can be calculated from the orthonormal bases ($\mathbf{e}_1$,$\mathbf{e}_2$,$\mathbf{e}_3$) in Cartesian coordinates $x^j = x^j(u^i)$, as shown in Fig. 2.2.

$$\begin{aligned}\mathbf{g}_i &\equiv \frac{\partial \mathbf{r}}{\partial u^i} = \frac{\partial \mathbf{r}}{\partial x^k}\frac{\partial x^k}{\partial u^i} \\ &= \mathbf{e}_k \frac{\partial x^k}{\partial u^i} \text{ for } k = 1, 2, 3\end{aligned} \tag{2.42}$$

The covariant metric coefficients g_{ij} are defined as

$$\begin{aligned}g_{ij} &\equiv \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial u^i}\frac{\partial \mathbf{r}}{\partial u^j} = \mathbf{g}_j \cdot \mathbf{g}_i \equiv g_{ji} \\ &= \frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}\mathbf{e}_k \mathbf{e}_l = \frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}\delta_{kl} \\ &= \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j}\end{aligned} \tag{2.43}$$

Similarly, the contravariant metric coefficients g^{ij} can be denoted as

$$g^{ij} \equiv \mathbf{g}^i \cdot \mathbf{g}^j = \mathbf{g}^j \cdot \mathbf{g}^i = g^{ji} \tag{2.44}$$

Furthermore, the contravariant basis can be rewritten as a linear combination of the covariant bases.

$$\mathbf{g}^i = A^{ij}\mathbf{g}_j \tag{2.45}$$

According to Eq. (2.44) and using Eqs. (2.21) and (2.45), the contravariant metric coefficients can be expressed as

$$\begin{aligned}g^{ik} &\equiv \mathbf{g}^i \cdot \mathbf{g}^k = A^{ij}\mathbf{g}_j \cdot \mathbf{g}^k \\ &= A^{ij}\delta_j^k = A^{ik}\end{aligned} \tag{2.46}$$

Thus,

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j \text{ for } j = 1, 2, 3 \tag{2.47}$$

Analogously, one obtains the covariant basis

$$\mathbf{g}_k = g_{kl}\mathbf{g}^l \text{ for } l = 1, 2, 3 \tag{2.48}$$

The mixed metric coefficients can be defined by

$$\begin{aligned} g_k^i &\equiv \mathbf{g}^i \cdot \mathbf{g}_k = \left(g^{ij}\mathbf{g}_j\right) \cdot \mathbf{g}_k \\ &= \left(g^{ij}\mathbf{g}_j\right) \cdot \left(g_{kl}\mathbf{g}^l\right) = g^{ij} g_{kl}\left(\mathbf{g}_j \cdot \mathbf{g}^l\right) \\ &= g^{ij} g_{kl} \delta_j^l = g^{ij} g_{kj} \\ &= \delta_k^i \end{aligned} \tag{2.49a}$$

Thus,

$$g^{ij} g_{kj} = g_{kj} g^{ij} = \delta_k^i \tag{2.49b}$$

Therefore, the contravariant metric tensor is the inverse of the covariant metric tensor.

$$g^{ij} g_{kj} = g_{kj} g^{ij} = \delta_k^i \ \Leftrightarrow \mathbf{M}^{-1}\mathbf{M} = \mathbf{M}\mathbf{M}^{-1} = \mathbf{I} \tag{2.50}$$

where $\mathbf{M}^{-1}$ and $\mathbf{M}$ are the contravariant and covariant metric tensors.

Thus,

$$\begin{aligned} \mathbf{M}^{-1}\mathbf{M} &= \begin{bmatrix} g^{11} & g^{12} & . & g^{1N} \\ g^{21} & g^{22} & . & g^{2N} \\ . & . & g^{ij} & . \\ g^{N1} & g^{N2} & . & g^{NN} \end{bmatrix} \cdot \begin{bmatrix} g_{11} & g_{12} & \cdot & g_{1N} \\ g_{21} & g_{22} & \cdot & g_{2N} \\ \cdot & \cdot & g_{ij} & \cdot \\ g_{N1} & g_{N2} & \cdot & g_{NN} \end{bmatrix} \\ &= \left(g^{ij} g_{kj}\right) = \left(\delta_k^i\right) = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ 0 & . & 1 & 0 \\ 0 & 0 & . & 1 \end{bmatrix} = \mathbf{I} \end{aligned} \tag{2.51}$$

According to Eqs. (2.42) and (2.49a), the contravariant bases of the curvilinear coordinates can be derived as

$$\begin{aligned} &\delta_i^k \equiv \mathbf{g}^k \cdot \mathbf{g}_i = \mathbf{g}^k \cdot \frac{\partial x^j}{\partial u^i}\mathbf{e}_j \Rightarrow \\ &\delta_i^k \cdot \mathbf{e}_j = \left(\mathbf{g}^k \cdot \frac{\partial x^j}{\partial u^i}\mathbf{e}_j\right) \cdot \mathbf{e}_j = \mathbf{g}^k \cdot \frac{\partial x^j}{\partial u^i} \Rightarrow \\ &\mathbf{g}^k = \delta_i^k \frac{\partial u^i}{\partial x^j}\mathbf{e}_j = \frac{\partial u^k}{\partial x^j}\mathbf{e}_j \end{aligned} \tag{2.52}$$

Thus,

$$\mathbf{g}^j = \frac{\partial u^j}{\partial x^i}\mathbf{e}_i \text{ for } i = 1, 2, 3 \tag{2.53}$$

Generally, the covariant and contravariant metric coefficients of the general curvilinear coordinates have the following properties:

$$\begin{cases} g_{ji} = g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \neq \delta_i^j \text{ (cov. metric coefficient)} \\ g^{ji} = g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j \neq \delta_i^j \text{ (contrav. metric coefficient)} \\ g_i^j = \mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j \text{ (mixed metric coefficient)} \end{cases} \tag{2.54}$$

Using Eqs. (2.42) and (2.53), one obtains the Kronecker delta

$$\begin{aligned} \delta_i^j = \mathbf{g}_i \cdot \mathbf{g}^j &= \frac{\partial x^k}{\partial u^i}\frac{\partial u^j}{\partial x^l}\mathbf{e}_k \cdot \mathbf{e}_l = \frac{\partial x^k}{\partial u^i}\frac{\partial u^j}{\partial x^l}\delta_k^l \\ &= \frac{\partial x^k}{\partial u^i}\frac{\partial u^j}{\partial x^k} = \frac{\partial u^j}{\partial u^i} \end{aligned} \tag{2.55a}$$

The Kronecker delta is defined by

$$\delta_i^j \equiv \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases} \tag{2.55b}$$

As an example, the covariant metric tensor **M** in the cylindrical coordinates results from Eq. (2.31).

$$\mathbf{M} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.56a}$$

The contravariant metric coefficients in the contravariant metric tensor $\mathbf{M}^{-1}$ are calculated from inverting the covariant metric tensor **M**.

$$\mathbf{M}^{-1} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.56b}$$

Analogously, the covariant metric tensor **M** in the spherical coordinates results from Eq. (2.40).

$$\mathbf{M} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & (\rho \sin\varphi)^2 \end{bmatrix} \tag{2.57a}$$

The contravariant metric coefficients in the contravariant metric tensor $\mathbf{M}^{-1}$ are calculated from inverting the covariant metric tensor **M**.

$$\mathbf{M}^{-1} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^{-2} & 0 \\ 0 & 0 & (\rho \sin \varphi)^{-2} \end{bmatrix} \tag{2.57b}$$

2.3.3 Tensors of Second Order and Higher Orders

Mapping an arbitrary vector $\mathbf{x} \in \mathbf{R}^N$ by a linear functional $\mathbf{T}$, one obtains its image vector $\mathbf{y} = \mathbf{Tx}$ [2, 3].

$$\begin{aligned} &\mathbf{T} : \mathbf{R}^N \rightarrow \quad \mathbf{R}^N \\ &\mathbf{T} : \mathbf{x} \rightarrow \mathbf{y} = \mathbf{Tx} \ \equiv \mathbf{T} \cdot \left(x^j \mathbf{g}_j \right) = x^j \mathbf{T} \cdot \mathbf{g}_j \\ &\qquad\qquad\qquad\quad = x^j T_{ik} \mathbf{g}^i \left(\mathbf{g}^k \cdot \mathbf{g}_j \right) = x^j T_{ik} \delta_j^k \mathbf{g}^i = T_{ij} x^j \mathbf{g}^i \end{aligned} \tag{2.58}$$

where $\mathbf{T}$ is a second-order tensor $\in \mathbf{R}^N \times \mathbf{R}^N$.

It is obvious that the image vector $\mathbf{y} = \mathbf{Tx}$ is a tensor of one lower order compared to the tensor $\mathbf{T}$ that can be considered as a linear operator.

The second-order tensor $\mathbf{T}$ can be generated from the tensor product (dyadic product) of two vectors $\mathbf{u}$ and $\mathbf{v}$, as denoted in Eq. (2.60).

Let $\mathbf{u}$ and $\mathbf{v}$ be two arbitrary vectors. They can be written in the covariant and contravariant bases as

$$\begin{cases} \mathbf{u} = u^i \mathbf{g}_i = u_i \mathbf{g}^i \in \mathbf{R}^N; \\ \mathbf{v} = v^j \mathbf{g}_j = v_j \mathbf{g}^j \in \mathbf{R}^N \end{cases} \tag{2.59}$$

The tensor product of two vectors $\mathbf{u}$ and $\mathbf{v}$ results in the second-order tensor $\mathbf{T}$.

$$\begin{aligned} &\mathbf{T} : \mathbf{u}, \mathbf{v} \in \mathbf{R}^N \rightarrow \mathbf{T} \equiv \mathbf{u} \otimes \mathbf{v} \in \mathbf{R}^N \times \mathbf{R}^N \\ &\mathbf{T} = u^i v^j \mathbf{g}_i \otimes \mathbf{g}_j \equiv u^i v^j \mathbf{g}_i \mathbf{g}_j \equiv T^{ij} \mathbf{g}_i \mathbf{g}_j \\ &\mathbf{T} = u_i v_j \mathbf{g}^i \otimes \mathbf{g}^j \equiv u_i v_j \mathbf{g}^i \mathbf{g}^j \equiv T_{ij} \mathbf{g}^i \mathbf{g}^j \end{aligned} \tag{2.60}$$

Note that the terms $\mathbf{g}_i \mathbf{g}_j$ and $\mathbf{g}^i \mathbf{g}^j$ are called the covariant and contravariant basis tensors. Hence, they are not the same notations as the covariant and contravariant metric coefficients g_{ij} and g^{ij}, respectively.

$$\begin{aligned} \mathbf{g}_i \mathbf{g}_j &\neq \mathbf{g}_i \cdot \mathbf{g}_j \ \equiv g_{ij} \\ \mathbf{g}^i \mathbf{g}^j &\neq \mathbf{g}^i \cdot \mathbf{g}^j \ \equiv g^{ij} \end{aligned}$$

Similarly, one obtains the properties of the mixed basis tensors:

$$\mathbf{g}^i\mathbf{g}_j \neq \mathbf{g}^i \cdot \mathbf{g}_j \equiv g^i_j = \delta^i_j$$
$$\mathbf{g}_i\mathbf{g}^j \neq \mathbf{g}_i \cdot \mathbf{g}^j \equiv g^j_i = \delta^j_i$$

Each of the covariant and contravariant tensor components T_{ij} and T^{ij} contains nine independent elements $(N^2 = 9) \in \mathbf{R}^3 \times \mathbf{R}^3$ in a three-dimensional space $(N = 3)$.

$$\begin{aligned} T^{ij} &= u^i v^j; \\ T_{ij} &= u_i v_j \end{aligned} \tag{2.61}$$

An example of the tensor product of two contravariant vectors **u** and **v** in a three-dimensional space $\mathbf{R}^3$ is given.

$$\begin{aligned} \mathbf{T} = \mathbf{u} \otimes \mathbf{v} &= \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \otimes \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \\ &= \begin{pmatrix} u^1v^1 & u^1v^2 & u^1v^3 \\ u^2v^1 & u^2v^2 & u^2v^3 \\ u^3v^1 & u^3v^2 & u^3v^3 \end{pmatrix} = \begin{pmatrix} T^{11} & T^{12} & T^{13} \\ T^{21} & T^{22} & T^{23} \\ T^{31} & T^{32} & T^{33} \end{pmatrix}; \ \forall u^i, v^j, T^{ij} \in \mathbf{R} \end{aligned}$$

The second-order tensor **T** consists of nine tensor components $(N^2 = 3^2) \in \mathbf{R}^3 \times \mathbf{R}^3$ in a three-dimensional space $\mathbf{R}^3$ $(N = 3)$ with two indices i and j.

The basis $\mathbf{g}_j$ of the general curvilinear coordinates is mapped by the linear functional **T** in Eq. (2.58) into the image vector $\mathbf{T}_j$ that can be written according to Eq. (2.2) as

$$\mathbf{T}_j \equiv \mathbf{T} \cdot \mathbf{g}_j \tag{2.62}$$

Each vector $\mathbf{T}_j$ can be expressed in a linear combination of the contravariant basis $\mathbf{g}^i$ as

$$\mathbf{T}_j = T_{ij}\mathbf{g}^i \tag{2.63}$$

where T_{ij} is the covariant tensor component of the second-order tensor **T**.

Multiplying Eq. (2.63) by the covariant basis $\mathbf{g}_i$ and using Eq. (2.62), the covariant tensor component T_{ij} results in

$$\begin{aligned} &\left(T_{ij}\mathbf{g}^i\right) \cdot \mathbf{g}_i = \left(\mathbf{T} \cdot \mathbf{g}_j\right) \cdot \mathbf{g}_i = \mathbf{g}_i \cdot \mathbf{T} \cdot \mathbf{g}_j \\ &\Rightarrow T_{ij}\delta^i_i = T_{ij} = \mathbf{g}_i \cdot \mathbf{T} \cdot \mathbf{g}_j \end{aligned} \tag{2.64}$$

Equation (2.64) can be written in the contravariant bases $\mathbf{g}^i$ and $\mathbf{g}^j$ as follows:

$$
\begin{aligned}
T_{ij}(\mathbf{g}^i\mathbf{g}^j) &= \mathbf{g}_i \cdot \mathbf{T} \cdot \mathbf{g}_j(\mathbf{g}^i\mathbf{g}^j) \\
&= (\mathbf{g}_i \cdot \mathbf{g}^i)\mathbf{T}\left(\mathbf{g}^j \cdot \mathbf{g}_j\right) = \delta_i^i \mathbf{T} \delta_j^j = \mathbf{T} \\
&\Rightarrow \mathbf{T} = T_{ij}\mathbf{g}^i\mathbf{g}^j
\end{aligned}
\tag{2.65}
$$

Similarly, the vector $\mathbf{T}^j$ is formulated in a linear combination of the covariant basis $\mathbf{g}_i$.

$$
\mathbf{T}^j = \mathbf{T} \cdot \mathbf{g}^j = T^{ij}\mathbf{g}_i \tag{2.66}
$$

in which T^{ij} is the contravariant component of the second-order tensor $\mathbf{T}$.

Multiplying Eq. (2.66) by the contravariant basis $\mathbf{g}^i$, the contravariant tensor component T^{ij} can be computed as

$$
\begin{aligned}
&\left(T^{ij}\mathbf{g}_i\right) \cdot \mathbf{g}^{\mathrm{i}} = (\mathbf{T} \cdot \mathbf{g}^j) \cdot \mathbf{g}^i = \mathbf{g}^i \cdot \mathbf{T} \cdot \mathbf{g}^j \\
&\Rightarrow T^{ij}\delta_i^i = T^{ij} = \mathbf{g}^i \cdot \mathbf{T} \cdot \mathbf{g}^j
\end{aligned}
\tag{2.67}
$$

Similarly, Eq. (2.67) can be written in the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$

$$
\begin{aligned}
T^{ij}\left(\mathbf{g}_i\mathbf{g}_j\right) &= \mathbf{g}^i.\mathbf{T}.\mathbf{g}^j\left(\mathbf{g}_i\mathbf{g}_j\right) \\
&= (\mathbf{g}^i.\mathbf{g}_i)\mathbf{T}\left(\mathbf{g}_j.\mathbf{g}^j\right) = \delta_i^i \mathbf{T} \delta_j^j = \mathbf{T} \\
&\Rightarrow \mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j
\end{aligned}
\tag{2.68}
$$

Alternatively, the vector component can be rewritten in a linear combination of the mixed tensor component.

$$
\mathbf{T}^j = \mathbf{T} \cdot \mathbf{g}^j = T_{i.}^{\ j}\mathbf{g}^i \tag{2.69}
$$

Multiplying Eq. (2.69) by the covariant basis $\mathbf{g}_i$, the mixed tensor component $T_{i.}^{\ j}$ can be calculated as

$$
\begin{aligned}
&\left(T_{i.}^{\ j}\mathbf{g}^i\right) \cdot \mathbf{g}_i = (\mathbf{T} \cdot \mathbf{g}^j) \cdot \mathbf{g}_i = \mathbf{g}_i \cdot \mathbf{T} \cdot \mathbf{g}^j \\
&\Rightarrow T_{i.}^{\ j}\delta_i^i = T_{i.}^{\ j} = \mathbf{g}_i \cdot \mathbf{T} \cdot \mathbf{g}^j
\end{aligned}
\tag{2.70}
$$

Note that in Eq. (2.70), the dot after the lower index indicates the position of the basis of the upper index locating after the tensor $\mathbf{T}$. In this case, the tensor $\mathbf{T}$ is located between the lower basis $\mathbf{g}_i$ and upper basis $\mathbf{g}^j$ [4–6].

Equation (2.70) can be written in the covariant and contravariant bases $\mathbf{g}_j$ and $\mathbf{g}^i$ as follows:

$$\begin{aligned} & T_{i.}^{j}\left(\mathbf{g}^{i}\mathbf{g}_{j}\right) = \mathbf{g}_{i} \cdot \mathbf{T} \cdot \mathbf{g}^{j}\left(\mathbf{g}^{i}\mathbf{g}_{j}\right) \\ & = (\mathbf{g}_{i} \cdot \mathbf{g}^{i})\mathbf{T}\left(\mathbf{g}_{j} \cdot \mathbf{g}^{j}\right) = \delta_{i}^{i}\mathbf{T}\delta_{j}^{j} = \mathbf{T} \\ & \Rightarrow \mathbf{T} = T_{i.}^{j}\mathbf{g}^{i}\mathbf{g}_{j} \end{aligned} \tag{2.71}$$

Analogously, one obtains the mixed tensor component $T_{.j}^{\ i}$.

$$\begin{aligned} \left(T_{.j}^{\ i}\mathbf{g}^{j}\right) \cdot \mathbf{g}_{j} &= (\mathbf{T} \cdot \mathbf{g}^{i}) \cdot \mathbf{g}_{j} = \mathbf{T} \cdot \left(\mathbf{g}^{i} \cdot \mathbf{g}_{j}\right) \\ &= \mathbf{T} \cdot \left(\mathbf{g}_{j} \cdot \mathbf{g}^{i}\right) = \mathbf{g}^{i} \cdot \mathbf{T} \cdot \mathbf{g}_{j} \\ &\Rightarrow T_{.j}^{\ i}\delta_{j}^{j} = T_{.j}^{\ i} = \mathbf{g}^{i} \cdot \mathbf{T} \cdot \mathbf{g}_{j} \end{aligned} \tag{2.72}$$

Note that in Eq. (2.72), the dot before the lower index indicates the position of the basis of the upper index locating in front of the tensor **T**. In this case, the tensor **T** is located between the upper basis $\mathbf{g}^i$ and lower basis $\mathbf{g}_j$ [4–6].

Equation (2.72) can be written in the covariant and contravariant bases $\mathbf{g}_i$ and $\mathbf{g}^j$ as follows:

$$\begin{aligned} & T_{.j}^{\ i}(\mathbf{g}_{i}\mathbf{g}^{j}) = \mathbf{g}^{i} \cdot \mathbf{T} \cdot \mathbf{g}_{j}(\mathbf{g}_{i}\mathbf{g}^{j}) \\ & = (\mathbf{g}^{i} \cdot \mathbf{g}_{i})\mathbf{T}\left(\mathbf{g}^{j} \cdot \mathbf{g}_{j}\right) = \delta_{i}^{i}\mathbf{T}\delta_{j}^{j} = \mathbf{T} \\ & \Rightarrow \mathbf{T} = T_{.j}^{\ i}\mathbf{g}_{i}\mathbf{g}^{j} \end{aligned} \tag{2.73}$$

Briefly, the second-order tensor can be written in different expressions according to the covariant, contravariant, and mixed components.

$$\mathbf{T}^{(2)} = \begin{cases} T_{ij}\mathbf{g}^{i}\mathbf{g}^{j}; & T_{ij} = \mathbf{g}_{i} \cdot \mathbf{T} \cdot \mathbf{g}_{j} \\ T^{ij}\mathbf{g}_{i}\mathbf{g}_{j}; & T^{ij} = \mathbf{g}^{i} \cdot \mathbf{T} \cdot \mathbf{g}^{j} \\ T_{i.}^{j}\mathbf{g}^{i}\mathbf{g}_{j}; & T_{i.}^{j} = \mathbf{g}_{i} \cdot \mathbf{T} \cdot \mathbf{g}^{j} \\ T_{.j}^{\ i}\mathbf{g}_{i}\mathbf{g}^{j}; & T_{.j}^{\ i} = \mathbf{g}^{i} \cdot \mathbf{T} \cdot \mathbf{g}_{j} \end{cases} \tag{2.74}$$

Note that if the second-order tensor **T** is symmetric, then

$$T_{ij} = T_{ji}; \quad T^{ij} = T^{ji}; \quad T_{i.}^{j} = T_{.i}^{\ j}; \quad T_{.j}^{\ i} = T_{j.}^{i} \tag{2.75}$$

Compared to the second-order tensors, the first-order tensor $\mathbf{T}^{(1)}$ has only one dummy index, as shown in

$$\mathbf{T}^{(1)} = \begin{cases} T_{i}\mathbf{g}^{i}; & T_{i} = \mathbf{T} \cdot \mathbf{g}_{i} \\ T^{i}\mathbf{g}_{i}; & T^{i} = \mathbf{T} \cdot \mathbf{g}^{i} \end{cases} \tag{2.76}$$

An N-order tensor $\mathbf{T}^{(N)}$ is the tensor product of the N covariant, contravariant, and mixed bases of the coordinates:

$$\mathbf{T}^{(N)} = \begin{cases} T^{ij\ldots n}\mathbf{g}_i\mathbf{g}_j\ldots\mathbf{g}_n \\ T_{ij\ldots n}\mathbf{g}^i\mathbf{g}^j\ldots\mathbf{g}^n \\ T^{ij\cdot\cdot}_{\;\;\;l\ldots n}\mathbf{g}_i\mathbf{g}_j\ldots\mathbf{g}^l\ldots\mathbf{g}^n \end{cases} \tag{2.77}$$

The N-order tensors contain the 2^N expressions in total. Two of them are in respect of the covariant and contravariant tensor components; and $(2^N - 2)$ expressions, in respect of the mixed tensor components [3]. In case of a second-order tensor $\mathbf{T}^{(2)}$ for $N = 2$, there are four expressions: two with the covariant and contravariant tensor components and two with the mixed tensor components, as displayed in Eq. (2.74).

2.3.4 Tensor and Cross Products of Two Vectors in General Bases

2.3.4.1 Tensor Product

Let $\mathbf{u}$ and $\mathbf{v}$ be two arbitrary vectors in the finite N-dimensional vector space $\mathbf{R}^N$. They can be written in the covariant and contravariant bases as

$$\begin{cases} \mathbf{u} = u^i\mathbf{g}_i = u_i\mathbf{g}^i \in \mathbf{R}^N; \\ \mathbf{v} = v^j\mathbf{g}_j = v_j\mathbf{g}^j \in \mathbf{R}^N \end{cases} \tag{2.78}$$

The tensor product $\mathbf{T}$ of two vectors generates a second-order tensor that can be defined by the linear functional $\mathbf{T}$.

- In the covariant bases:

$$\begin{aligned} \mathbf{T}:\mathbf{u},\mathbf{v} \in \mathbf{R}^N \to \mathbf{T} \equiv \mathbf{u}\otimes\mathbf{v} = u^iv^j\mathbf{g}_i\mathbf{g}_j = u_iv_j\mathbf{g}^i\mathbf{g}^j \in \mathbf{R}^N \times \mathbf{R}^N \\ \mathbf{T} = u^iv^j\mathbf{g}_i\mathbf{g}_j \equiv T^{ij}\mathbf{g}_i\mathbf{g}_j \\ = u_iv_j\mathbf{g}^i\mathbf{g}^j \equiv T_{ij}\mathbf{g}^i\mathbf{g}^j \end{aligned} \tag{2.79}$$

where
  T^{ij} is the contravariant component of the second-order tensor $\mathbf{T}$;
  T_{ij} is the covariant component of the second-order tensor $\mathbf{T}$;

- In the covariant and contravariant bases:

$$\begin{aligned} \mathbf{T}:\mathbf{u},\mathbf{v} \in \mathbf{R}^N \to \mathbf{T} \equiv \mathbf{u}\otimes\mathbf{v} = u^iv_j\mathbf{g}_i\mathbf{g}^j = u_iv^j\mathbf{g}^i\mathbf{g}_j \in \mathbf{R}^N \times \mathbf{R}^N \\ \mathbf{T} = u^iv_j\mathbf{g}_i\mathbf{g}^j \equiv T^i_j\mathbf{g}_i\mathbf{g}^j \\ = u_iv^j\mathbf{g}^i\mathbf{g}_j \equiv T_i^j\mathbf{g}^i\mathbf{g}_j \end{aligned} \tag{2.80}$$

where T^i_j and T_i^j are the mixed components of the second-order tensor $\mathbf{T}$.

The tensor product $\mathbf{T}$ of two vectors in an orthonormal basis (e.g. Euclidean coordinate system) is an invariant (scalar). The invariant is independent of the coordinate system and has an intrinsic value in any coordinate transformations. In Newtonian mechanics, the mechanical work $\mathbf{W}$ that is created by the force vector $\mathbf{F}$ and path vector $\mathbf{x}$ does not change in any chosen coordinate system. This mechanical work $\mathbf{W} = \mathbf{F} \cdot \mathbf{x}$ is called an invariant and has an intrinsic value of energy.

Given three arbitrary vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbf{R}^N$ and a scalar α in $\mathbf{R}$, the tensor product of two vectors has the following properties [3]:

- Distributive property

$$\begin{aligned}\mathbf{u}(\mathbf{v}+\mathbf{w}) &= \mathbf{uv}+\mathbf{uw}\\ (\mathbf{u}+\mathbf{v})\mathbf{w} &= \mathbf{uw}+\mathbf{vw}\end{aligned}$$

- Associative property

$$(\alpha\,\mathbf{u})\mathbf{v} = \mathbf{u}(\alpha\,\mathbf{v}) = \alpha\,\mathbf{uv}$$

2.3.4.2 Cross Product

The cross product of two vectors $\mathbf{u}$ and $\mathbf{v}$ can be defined by a linear functional $\mathbf{T}$.

$$\mathbf{T}:\mathbf{u},\mathbf{v}\in\mathbf{R}^N \rightarrow \mathbf{T} = \mathbf{u}\times\mathbf{v} = u^i v^j \left(\mathbf{g}_i \times \mathbf{g}_j\right) \in \mathbf{R}^N \tag{2.81}$$

Obviously, the cross product $\mathbf{T}$ of two vectors is a vector (first-order tensor) of which the direction is perpendicular to the bases of $\mathbf{g}_i$ and $\mathbf{g}_j$.

Using the scalar triple product in Eq. (1.10), the cross product of the bases can be written as

$$\left(\mathbf{g}_i \times\ \mathbf{g}_j\right) = \varepsilon_{ijk}\sqrt{g}\,\mathbf{g}^k = \varepsilon_{ijk}J\,\mathbf{g}^k \tag{2.82}$$

where ε_{ijk} is the Levi-Civita permutation symbol; J is the Jacobian.

Thus, the cross product in Eq. (2.81) can be expressed as

$$\begin{aligned}\mathbf{T} &= \mathbf{u}\times\mathbf{v} \equiv T_k\mathbf{g}^k\\ &= \left(\varepsilon_{ijk}\sqrt{g}\,u^i v^j\right)\mathbf{g}^k = \left(\varepsilon_{ijk}J\,u^i v^j\right)\mathbf{g}^k\end{aligned} \tag{2.83}$$

The covariant component T_k of the first-order tensor $\mathbf{T}$ results from Eqs. (2.81) to (2.83).

$$
\begin{aligned}
&\mathbf{T} = u^i v^j \left(\mathbf{g}_i \times \mathbf{g}_j\right) = u^i v^j \left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot \mathbf{g}_k \, \mathbf{g}^k \equiv T_k \mathbf{g}^k \\
&\Rightarrow T_k = u^i v^j \left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot \mathbf{g}_k = \varepsilon_{ijk} \sqrt{g}\, u^i v^j = \varepsilon_{ijk} J \, u^i v^j
\end{aligned}
\tag{2.84}
$$

The Levi-Civita permutation symbol (pseudo-tensor) can be defined as

$$
\varepsilon_{ijk} = \begin{cases} +1 \text{ if } (i,\ j,\ k) \text{ is an even permutation;} \\ -1 \ \text{ if } (i,\ j,\ k) \text{ is an odd permutation;} \\ 0 \ \text{ if } \ i = j, \text{ or } \ i = k; \text{ or } \ j = k \end{cases}
\tag{2.85}
$$

Therefore,

$$
\varepsilon_{ijk} = \begin{cases} \quad \varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} \text{ (even permutation);} \\ -\varepsilon_{ikj} = -\varepsilon_{kji} = -\varepsilon_{jik} \ \text{ (odd permutation)} \end{cases}
\tag{2.86}
$$

The permutation symbol ε_{ijk} contains totally 27 elements ($N^n = 3^3$) for i, j, k ($n = 3$) in a 27-dimensional tensor space $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$.

Note that the permutation symbol is used in Eq. (2.83) because the direction of the cross-product vector is opposite if the dummy indices are interchanged with each other in Einstein summation convention (cf. Appendix A).

$$
\begin{aligned}
&\sqrt{g}\, \mathbf{g}^k = J\, \mathbf{g}^k = \left(\mathbf{g}_i \times \mathbf{g}_j\right) = -\left(\mathbf{g}_j \times \mathbf{g}_i\right) \\
&\Rightarrow \mathbf{g}^k = \frac{\varepsilon_{ijk}\left(\mathbf{g}_i \times \mathbf{g}_j\right)}{\sqrt{g}} = \frac{\varepsilon_{ijk}\left(\mathbf{g}_i \times \mathbf{g}_j\right)}{J} \\
&\Rightarrow \left(\mathbf{g}_i \times \mathbf{g}_j\right) = \varepsilon_{ijk} \sqrt{g} \mathbf{g}^k = \varepsilon_{ijk} J \mathbf{g}^k
\end{aligned}
\tag{2.87}
$$

where J denotes the Jacobian.

2.3.5 Rules of Tensor Calculations

In order to carry out the tensor calculations, some fundamental rules must be taken into account in tensor calculus [11–13].

2.3.5.1 Calculation of Tensor Components

Let $\mathbf{T}$ be a second-order tensor; it can be written in different tensor forms:

$$\begin{aligned}\mathbf{T} = &\ T^{ij}\mathbf{g}_i\mathbf{g}_j = T^i_{\ j}\mathbf{g}_i\mathbf{g}^j \\ &= T_{ij}\mathbf{g}^i\mathbf{g}^j = T_i^{\ j}\mathbf{g}_j\mathbf{g}^i\end{aligned} \tag{2.88}$$

Multiplying the first row of Eq. (2.88) by the covariant basis $\mathbf{g}_k$, one obtains

$$\begin{aligned}&T^{ij}\left(\mathbf{g}_k \cdot \mathbf{g}_j\right)\mathbf{g}_i = T^{ij}g_{kj}\mathbf{g}_i = T^i_{\ k}\mathbf{g}_i \\ &\Rightarrow T^i_{\ k} = T^{ij}g_{kj} \text{ for } j = 1,\ 2,\ \ldots, N\end{aligned} \tag{2.89a}$$

Analogously, multiplying the second row of Eq. (2.88) by the contravariant basis $\mathbf{g}^k$, one obtains

$$\begin{aligned}&T_{ij}\left(\mathbf{g}^k \cdot \mathbf{g}^j\right)\mathbf{g}^{\mathrm{i}} = T_{ij}g^{kj}\mathbf{g}^i = T_i^{\ k}\mathbf{g}^i \\ &\Rightarrow T_i^{\ k} = T_{ij}g^{kj} \text{ for } j = 1,\ 2,\ \ldots, N\end{aligned} \tag{2.89b}$$

Multiplying Eq. (2.89a) by g^{kj}, the contravariant tensor components result in

$$T^{ij} = T^i_{\ k}g^{kj} \text{ for } k = 1,\ 2,\ \ldots, \mathrm{N} \tag{2.90a}$$

Multiplying Eq. (2.89b) by g_{kj}, one obtains the covariant tensor components

$$T_{ij} = T_i^{\ k}g_{kj} \text{ for } k = 1,\ 2,\ \ldots, N \tag{2.90b}$$

Substituting Eqs. (2.89a) and (2.90a), one obtains the contraction rules between the contravariant tensor components.

$$T^{ij} = T^{ip}g_{pk}g^{kj} \text{ for } k,p = 1,\ 2,\ \ldots, N \tag{2.91a}$$

Similarly, the contraction rules between the covariant tensor components result from substituting Eqs. (2.89b) and (2.90b).

$$T_{ij} = T_{ip}g^{pk}g_{kj} \text{ for } \ k,p = 1,\ 2,\ \ldots, N \tag{2.91b}$$

Analogously, the contraction rules between the mixed tensor components can be derived as

$$T^i_{\ j} = T^i_{\ p}g^{pk}g_{kj} \text{ for } \ k,p = 1,\ 2,\ \ldots, N \tag{2.92a}$$

Similarly, one obtains

$$T_i^{\ j} = T_i^{\ p}g_{pk}g^{kj} \text{ for } \ k,p = 1,\ 2,\ \ldots, N \tag{2.92b}$$

2.3.5.2 Addition Law

Tensors of the same orders and types can be added together. The resulting tensor has the same order and type of the initial tensors. The tensor resulted from the addition of two covariant or contravariant tensors **A** and **B** can be calculated as

$$\begin{aligned}
&\mathbf{C} = \mathbf{A} + \mathbf{B} = \left(A_{ijk} + B_{ijk}\right) \mathbf{g}^i \mathbf{g}^j \mathbf{g}^k = C_{ijk} \mathbf{g}^i \mathbf{g}^j \mathbf{g}^k = \mathbf{B} + \mathbf{A} \\
&\Rightarrow C_{ijk} = A_{ijk} + B_{ijk} = B_{ijk} + A_{ijk}; \\
&\mathbf{C} = \mathbf{A} + \mathbf{B} = \left(A^{ijk} + B^{ijk}\right) \mathbf{g}_i \mathbf{g}_j \mathbf{g}_k = C^{ijk} \mathbf{g}_i \mathbf{g}_j \mathbf{g}_k = \mathbf{B} + \mathbf{A} \\
&\Rightarrow C^{ijk} = A^{ijk} + B^{ijk} = B^{ijk} + A^{ijk}
\end{aligned} \tag{2.93}$$

Similarly, the tensor resulted from the addition of two mixed tensors **A** and **B** can be written as

$$\begin{aligned}
\mathbf{C} = \mathbf{A} + \mathbf{B} &= \left(A_{ijk}^{pq} + B_{ijk}^{pq}\right) \mathbf{g}_p \mathbf{g}_q \mathbf{g}^i \mathbf{g}^j \mathbf{g}^k \\
&= C_{ijk}^{pq} \mathbf{g}_p \mathbf{g}_q \mathbf{g}^i \mathbf{g}^j \mathbf{g}^k = \mathbf{B} + \mathbf{A} \\
\Rightarrow C_{ijk}^{pq} &= A_{ijk}^{pq} + B_{ijk}^{pq} = B_{ijk}^{pq} + A_{ijk}^{pq}
\end{aligned} \tag{2.94}$$

Straightforwardly, the addition of tensors is commutative, as proved in Eqs. (2.93) and (2.94).

2.3.5.3 Outer Product

On the contrary, the outer product can be carried out at tensors of different orders and types. The tensor components resulted from the outer product of two mixed tensors **A** and **B** can be calculated as

$$\begin{aligned}
\mathbf{AB} &= \left(A_{ij}^{pq} \mathbf{g}_p \mathbf{g}_q \mathbf{g}^i \mathbf{g}^j\right) \left(B_{kl}^{rts} \mathbf{g}^k \mathbf{g}^l \mathbf{g}_r \mathbf{g}_t \mathbf{g}_s\right) \\
&= C_{ijkl}^{pqrst} \mathbf{g}_p \mathbf{g}_q \mathbf{g}^i \mathbf{g}^j \mathbf{g}^k \mathbf{g}^l \mathbf{g}_r \mathbf{g}_t \mathbf{g}_s \neq \mathbf{BA} \\
\Rightarrow C_{ijkl}^{pqrst} &= A_{ij}^{pq} B_{kl}^{rts} = B_{kl}^{rts} A_{ij}^{pq}
\end{aligned} \tag{2.95}$$

The outer product of two tensors results a tensor with the order that equals the sum of the covariant and contravariant indices. The outer product is not commutative, but their tensor components are commutative, as shown in Eq. (2.95). In this example, the resulting ninth-order tensor is generated from the outer product of the mixed fourth-order tensor **A** and mixed fifth-order tensor **B**. Obviously, the outer product of tensors **A**, **B**, and **C** is associative, i.e. **A**(**BC**) = (**AB**)**C**.

2.3.5.4 Contraction Law

The contraction operation can be only carried out at the mixed tensor types of different orders. The tensor contraction is operated in many contracting steps where the tensor order is shortened by eliminating the same covariant and contravariant indices of the tensor components.

We consider a mixed tensor of high orders. In this example, the mixed fifth-order tensor **A** of type (2, 3) can be transformed from the coordinates $\{u^i\}$ into the barred coordinates $\{\bar{u}^{\mathrm{i}}\}$. The transformed tensor components can be calculated according to the transformation law in Eq. (2.144).

$$\overline{A}^{ij}_{klm} = \frac{\partial \overline{u}^i}{\partial u^p}\frac{\partial \overline{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \overline{u}^k}\frac{\partial u^s}{\partial \overline{u}^l}\frac{\partial u^t}{\partial \overline{u}^m}A^{pq}_{rst} \tag{2.96}$$

Carrying out the first contraction of $\bar{A}$ in Eq. (2.96) at $l=i$, one obtains the tensor components

$$\begin{aligned}
\overline{A}^{ij}_{kim} &= \frac{\partial \overline{u}^i}{\partial u^p}\frac{\partial \overline{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \overline{u}^k}\frac{\partial u^s}{\partial \overline{u}^l}\frac{\partial u^t}{\partial \overline{u}^m}A^{pq}_{rst}\\
&= \frac{\partial \overline{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \overline{u}^k}\frac{\partial u^t}{\partial \overline{u}^m}\left(\frac{\partial \overline{u}^i}{\partial u^p}\frac{\partial u^s}{\partial \overline{u}^i}\right)A^{pq}_{rst}\\
&= \frac{\partial \overline{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \overline{u}^k}\frac{\partial u^t}{\partial \overline{u}^m}\delta^s_p A^{pq}_{rst}\\
&= \frac{\partial \overline{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \overline{u}^k}\frac{\partial u^t}{\partial \overline{u}^m}A^{pq}_{rpt}\\
\Leftrightarrow \overline{B}^{j}_{km} &= \frac{\partial \overline{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \overline{u}^k}\frac{\partial u^t}{\partial \overline{u}^m}B^{q}_{rt}
\end{aligned} \tag{2.97a}$$

As a result, the resulting tensor components B are the third-order tensor type after the first contraction of A at $s=p$:

$$A^{pq}_{rst}\delta^p_s = A^{pq}_{rpt} \equiv B^q_{rt} \tag{2.97b}$$

Further contracting the tensor components $\overline{B}$ in Eq. (2.97a) at $k=j$, one obtains

$$\begin{aligned}
\overline{B}^{j}_{jm} &= \frac{\partial u^t}{\partial \overline{u}^m}\left(\frac{\partial \overline{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \overline{u}^j}\right)B^q_{rt}\\
&= \frac{\partial u^t}{\partial \overline{u}^m}\delta^r_q B^q_{rt}\\
&= \frac{\partial u^t}{\partial \overline{u}^m}B^q_{qt}\\
\Leftrightarrow \overline{C}_m &= \frac{\partial u^t}{\partial \overline{u}^m}C_t
\end{aligned} \tag{2.98a}$$

As a result, the resulting tensor components C are the first-order tensor type after the second contraction of B in Eq. (2.98a) at $r = q$:

$$B_{rt}^{q}\delta_{r}^{q} = B_{qt}^{q} \equiv C_t \tag{2.98b}$$

2.3.5.5 Inner Product

The inner product of tensors comprises two basic operations of the outer product and at least one contraction of tensors. As an example, the outer product of two third-order tensors **A** and **B** results in a sixth-order tensor.

$$\begin{aligned}\mathbf{AB} &= \left(A_{q}^{mp}\mathbf{g}_m\mathbf{g}_p\mathbf{g}^q\right)\left(B_{st}^{r}\mathbf{g}_r\mathbf{g}^s\mathbf{g}^t\right)\\ &= A_{q}^{mp}B_{st}^{r}\mathbf{g}^q\mathbf{g}_r\mathbf{g}_m\mathbf{g}_p\mathbf{g}^s\mathbf{g}^t\end{aligned} \tag{2.99}$$

Using the first tensor contraction in Eq. (2.99) at $r = q$, one obtains the resulting fourth-order tensor components of the inner product.

$$A_{q}^{mp}B_{st}^{r}\delta_{q}^{r} = A_{q}^{mp}B_{st}^{q} \equiv C_{st}^{mp} \tag{2.100}$$

Similarly, using the second tensor contraction law in Eq. (2.100) at $p = s$, the resulting second-order tensor components result in

$$C_{st}^{mp}\delta_{s}^{p} = C_{st}^{ms} \equiv D_{t}^{m} \tag{2.101}$$

Finally, applying the third tensor contraction law to Eq. (2.101) at $m = t$, the resulting tensor component is an invariant (zeroth-order tensor).

$$D_{t}^{m}\delta_{t}^{m} = D_{t}^{t} \equiv D \tag{2.102}$$

In another approach, one can calculate the tensor components of the inner product of two contravariant tensors **A** and **B** multiplying by the metric tensor.

$$A^{ijk}B^{lm} \rightarrow A^{ijk}B^{lm}g_{lk} \tag{2.103}$$

Using Eq. (2.89a), one obtains the resulting tensor components

$$B^{lm}g_{lk} = B_{k}^{m} \tag{2.104}$$

Substituting Eq. (2.104) into Eq. (2.103) and using the tensor contraction law, one obtains the resulting tensor components

$$C^{ijm} \equiv A^{ijk} B_k^m = B_k^m A^{ijk} \tag{2.105}$$

Equation (2.105) denotes that the inner product of the tensor components is commutative.

2.3.5.6 Indices Law

Using the metric tensors, the operation of moving indices enables changing indices of the tensor components from the upper into lower positions and vice versa. Multiplying a tensor component by the metric tensor components, the lower index (covariant index) is moved into the upper index (contravariant index) and vice versa.

– Moving covariant indices i, j to the upper position:

$$A_{ij}^k \rightarrow A_{ij}^k g^{il} = A_j^{kl} \rightarrow A_j^{kl} g^{jm} = A^{klm} \tag{2.106a}$$

– Moving contravariant indices i, j to the lower position:

$$A_k^{ij} \rightarrow A_k^{ij} g_{jl} = A_{kl}^i \rightarrow A_{kl}^i g_{im} = A_{klm} \tag{2.106b}$$

2.3.5.7 Quotient Law

The quotient law of tensors postulates that if the tensor product of **AB** and **B** are tensors, **A** must be a tensor.

$$(\mathbf{AB} \;= \mathbf{C} \therefore \text{tensor}) \cap (\mathbf{B} \therefore \text{tensor}) \Rightarrow (\mathbf{A} \therefore \text{tensor}) \tag{2.107a}$$

Proof Using the contraction law, the barred components in the transformation coordinates $\{\bar{u}^i\}$ of the tensor product **AB** result in

$$A_{ij}^k B_k^{il} = C_j^l \Rightarrow \overline{A}_{ij}^k \overline{B}_k^{il} = \overline{C}_j^l \tag{2.107b}$$

According to the transformation law (2.144), the transformed components in the coordinates $\{\bar{u}^i\}$ of the tensors **B** and **C** can be calculated as

$$\begin{aligned}\overline{B}_k^{il} &= B_p^{mn} \frac{\partial \overline{u}^i}{\partial u^m} \frac{\partial \overline{u}^l}{\partial u^n} \frac{\partial u^p}{\partial \overline{u}^k}; \\ \overline{C}_j^l &= C_q^n \frac{\partial \overline{u}^l}{\partial u^n} \frac{\partial u^q}{\partial \overline{u}^j}\end{aligned} \tag{2.107c}$$

Substituting Eq. (2.107c) into Eq. (2.107b) and using the contraction law, one obtains

$$\begin{aligned}\overline{A}_{ij}^k \left(B_p^{mn} \frac{\partial \overline{u}^i}{\partial u^m} \frac{\partial \overline{u}^l}{\partial u^n} \frac{\partial u^p}{\partial \overline{u}^k} \right) &= C_q^n \frac{\partial \overline{u}^l}{\partial u^n} \frac{\partial u^q}{\partial \overline{u}^j} \\ &= \left(A_{mq}^p B_p^{mn} \right) \frac{\partial \overline{u}^l}{\partial u^n} \frac{\partial u^q}{\partial \overline{u}^j}\end{aligned} \tag{2.107d}$$

Rearranging the terms of Eq. (2.107d), one obtains

$$\begin{aligned}&\left(\overline{A}_{ij}^k \frac{\partial \overline{u}^i}{\partial u^m} \frac{\partial u^p}{\partial \overline{u}^k} - A_{mq}^p \frac{\partial u^q}{\partial \overline{u}^j} \right) \frac{\partial \overline{u}^l}{\partial u^n} B_p^{mn} = 0 \\ &\Rightarrow \forall B_p^{mn}; \frac{\partial \overline{u}^l}{\partial u^n} \neq 0 : \overline{A}_{ij}^k \frac{\partial \overline{u}^i}{\partial u^m} \frac{\partial u^p}{\partial \overline{u}^k} = A_{mq}^p \frac{\partial u^q}{\partial \overline{u}^j}\end{aligned} \tag{2.107e}$$

Applying the inner product by $\left(\frac{\partial u^r}{\partial \overline{u}^i} \frac{\partial \overline{u}^k}{\partial u^s} \right)$ to Eq. (2.107e), one obtains the barred components of **A** at $r = m$ and $s = p$.

$$\begin{aligned}\overline{A}_{ij}^k \delta_m^r \delta_s^p &= A_{mq}^p \frac{\partial u^q}{\partial \overline{u}^j} \left(\frac{\partial u^r}{\partial \overline{u}^i} \frac{\partial \overline{u}^k}{\partial u^s} \right) \\ \Rightarrow \overline{A}_{ij}^k &= A_{mq}^p \frac{\partial \overline{u}^k}{\partial u^p} \frac{\partial u^m}{\partial \overline{u}^i} \frac{\partial u^q}{\partial \overline{u}^j}\end{aligned} \tag{2.107f}$$

Equation (2.107f) proves that **A** is a mixed third-order tensor of type (1, 2), cf. Eq. (2.107c).

2.3.5.8 Symmetric Tensors

Tensor **T** is called symmetric in the given basis if two covariant or contravariant indices of the tensor component can be interchanged without changing the tensor component value.

$$
\begin{aligned}
&T_{ij}{}^{ijij} = T_{ji}{}^{jiji} : \text{ symmetric in } i \text{ and } j \\
&T^{ij} = T^{ji} : \text{ symmetric in } i \text{ and } j \\
&T^{ijk}_{pq} = T^{ikj}_{pq} : \text{symmetric in } j \text{ and } k \\
&T^{ijk}_{pq} = T^{ijk}_{qp} : \text{ symmetric in } p \text{ and } q
\end{aligned}
\tag{2.108}
$$

In case of a second-order tensor, the tensor **T** is symmetric if **T** equals its transpose.

$$\mathbf{T} = \mathbf{T}^T \tag{2.109}$$

2.3.5.9 Skew-Symmetric Tensors

The sign of the tensor component is opposite if a pair of the covariant or contravariant indices are interchanged with each other. In this case, the tensor is skew-symmetric (anti-symmetric).

Tensor **T** is defined as a skew-symmetric tensor (anti-symmetric) if

$$
\begin{aligned}
&T_{ij} = -T_{ji} : \text{skew-symmetric in } i \text{ and } j \\
&T^{ij} = -T^{ji} : \text{skew-symmetric in } i \text{ and } j \\
&T^{ijk}_{pq} = -T^{ikj}_{pq} : \text{ skew- symmetric in } j \text{ and } k \\
&T^{ijk}_{pq} = -T^{ijk}_{qp} : \text{ skew-symmetric in } p \text{ and } q
\end{aligned}
\tag{2.110}
$$

In case of a second-order tensor, the tensor **T** is skew-symmetric if **T** is opposite to its transpose.

$$\mathbf{T} = -\mathbf{T}^T \tag{2.111}$$

An arbitrary tensor **T** can be generally decomposed into the symmetric and skew-symmetric tensors:

$$\mathbf{T} = \frac{1}{2}\left(\mathbf{T} + \mathbf{T}^T\right) + \frac{1}{2}\left(\mathbf{T} - \mathbf{T}^T\right) \equiv \mathbf{T}_{\text{sym}} + \mathbf{T}_{\text{skew}} \tag{2.112}$$

Proof

- The first tensor $\mathbf{T}_{\text{sym}}$ is symmetric:

$$\mathbf{T}_{\text{sym}} \equiv \frac{1}{2}\left(\mathbf{T} + \mathbf{T}^T\right) = \mathbf{T}^T_{sym} \quad \text{(qed)}$$

- The second tensor $\mathbf{T}_{\text{skew}}$ is skew-symmetric:

$$\mathbf{T}_{\text{skew}} \equiv \frac{1}{2}\left(\mathbf{T} - \mathbf{T}^T\right) = -\mathbf{T}_{\text{skew}}^T \quad \text{(qed)}$$

2.4 Coordinate Transformations

Tensors are tuples of independent coordinates in a finite multifold N-dimensional tensor space ($\mathbf{R}^N \times \ldots \times \mathbf{R}^N$). The tensor describes physical states generally depending on different variables (dimensions). Each physical state can be defined as the point $P(u^1,\ldots,u^N)$ with N coordinates of u^i. By changing the variables, such as time, locations, and physical characteristics (e.g. pressure, temperature, density, velocity), the physical state point varies in the multifold N-dimensional space.

The tensor does not change itself and is invariant in any coordinate system. However, its components change in the new basis by the coordinate transformation since the basis changes as the coordinate system varies. In this case, applications of tensor analysis have been used to describe the transformation between two general curvilinear coordinate systems in the multifold N-dimensional spaces. Hence, tensors are a very useful tool applied to the coordinate transformations in the multifold N-dimensional tensor spaces. High-order tensors can be generated by a multilinear map between two multifold N-dimensional spaces (cf. Sect. 2.2). Their components change in the relating bases by the coordinate transformations, as displayed in Fig. 2.5.

In the following section, the relations between the tensor components in different curvilinear coordinates of the finite N-dimensional spaces will be discussed.

2.4.1 Transformation in the Orthonormal Coordinates

The simple coordinate transformation of rotation between the orthonormal coordinates x_i and u_j in Euclidean coordinate system is carried out.

An arbitrary vector $\mathbf{r}$ (first-order tensor) can be written in both coordinate systems:

$$\begin{aligned} \mathbf{r} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \equiv x_i\mathbf{e}_i \\ &= u_1\mathbf{g}_1 + u_2\mathbf{g}_2 \equiv u_j\mathbf{g}_\mathrm{j} \end{aligned} \tag{2.113}$$

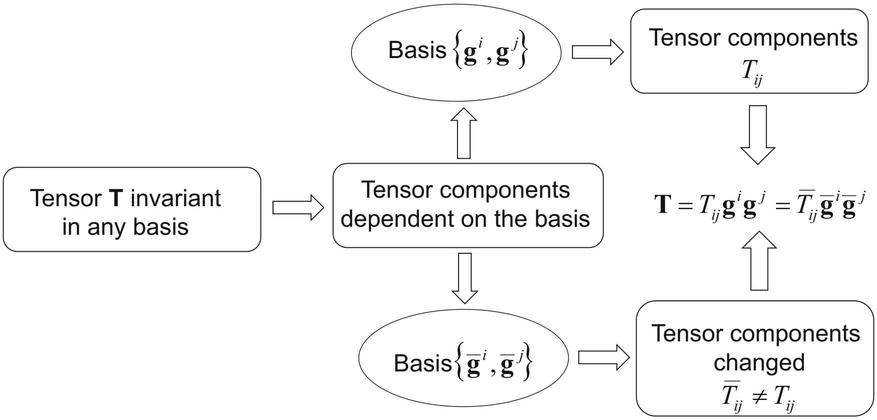

Fig. 2.5 Tensor and tensor components in different bases

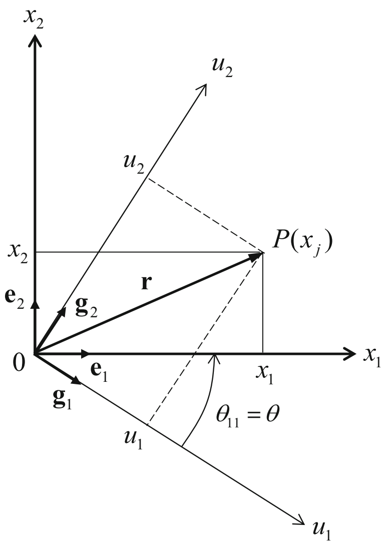

Fig. 2.6 Two-dimensional coordinate transformation of rotation

The vector components in the coordinate u_j can be calculated in (Fig. 2.6)

$$\begin{cases} u_1 = \cos\theta_{11}x_1 + \cos\theta_{12}x_2 \\ u_2 = \cos\theta_{21}x_1 + \cos\theta_{22}x_2 \end{cases} \tag{2.114}$$

Thus,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_{11} & \cos\theta_{12} \\ \cos\theta_{21} & \cos\theta_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow \mathbf{u} = \mathbf{Tx} \tag{2.115}$$

where $\mathbf{T}$ is the transformation matrix.

Setting $\theta_{11} = \theta$, one obtains

$$\begin{aligned}\cos\theta_{11} &= \cos\theta; \quad \cos\theta_{12} = \cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta \\ \cos\theta_{21} &= \cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta; \quad \cos\theta_{22} = \cos\theta\end{aligned}$$

Therefore, the transformation matrix $\mathbf{T}$ becomes

$$\mathbf{T} = \begin{bmatrix} \cos\theta_{11} & \cos\theta_{12} \\ \cos\theta_{21} & \cos\theta_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2.116}$$

The transformed coordinates can be computed by the transformation T:

$$T : \mathbf{x} \to \mathbf{u} = \mathbf{Tx} : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.117}$$

where θ is the rotation angle of the rotating coordinates u_j.

Transforming backward Eq. (2.117), one obtains the coordinates x_i

$$T^{-1} : \mathbf{u} \to \mathbf{x} = \mathbf{T}^{-1}\mathbf{u} : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.118}$$

The vector component on the basis is obtained multiplying Eq. (2.113) by the relating basis $\mathbf{e}_i$ or $\mathbf{g}_j$.

$$\begin{aligned} x_i &= \mathbf{r} \cdot \mathbf{e}_i; \quad i = 1,\ 2 \\ u_j &= \mathbf{r} \cdot \mathbf{g}_j; \quad j = 1,\ 2 \end{aligned} \tag{2.119}$$

Substituting Eq. (2.119) into Eq. (2.117), one obtains the transformation matrix between two coordinate systems.

$$\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} \Leftrightarrow \mathbf{g} = \mathbf{T} \cdot \mathbf{e} \tag{2.120}$$

Similarly,

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} \Leftrightarrow \mathbf{e} = \mathbf{T}^{-1} \cdot \mathbf{g} \tag{2.121}$$

2.4.2 Transformation of Curvilinear Coordinates in E^N

In the following section, second-order tensors **T** are used in the transformation of general curvilinear coordinates in Euclidean space $\mathbf{E}^N$, as shown in Fig. 2.7.

The basis $\mathbf{g}_i$ of the curvilinear coordinate $\{u^i\}$ can be transformed into the new basis $\overline{\mathbf{g}}_i$ of the curvilinear coordinate $\{\bar{u}^i\}$ using the linear transformation **S**. The new covariant basis can be rewritten as a linear combination of the old basis.

$$S : \mathbf{g}_i \to \overline{\mathbf{g}}_i = S_i^j \mathbf{g}_j \Leftrightarrow \overline{\mathbf{G}} = \mathbf{GS} \tag{2.122}$$

where S_i^j are the mixed transformation components of the second-order tensor **S**.

The old covariant basis results can be calculated as

$$\mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial u^j} = \frac{\partial \mathbf{r}}{\partial \bar{u}^i}\frac{\partial \bar{u}^i}{\partial u^j} \equiv \overline{\mathbf{g}}_i \left(S^{-1}\right)_j^i \Rightarrow \left(S^{-1}\right)_j^i \equiv \frac{\partial \bar{u}^i}{\partial u^j} \tag{2.123}$$

Inversing the basis matrix in Eq. (2.122), the new contravariant basis can be calculated as

$$\overline{\mathbf{G}}^{-1} = (\mathbf{GS})^{-1} = \mathbf{S}^{-1}\mathbf{G}^{-1} \Rightarrow \overline{\mathbf{g}}^i = \left(S^{-1}\right)_j^i \mathbf{g}^j \tag{2.124}$$

Multiplying Eq. (2.124) by the linear transformation **S**, the old contravariant basis results in

$$\mathbf{G}^{-1} = \mathbf{S}\overline{\mathbf{G}}^{-1} \Rightarrow \mathbf{g}^i = S_j^i \overline{\mathbf{g}}^j \tag{2.125}$$

According to Eqs. (2.11) and (2.122), the new covariant basis can be calculated as

$$\overline{\mathbf{g}}_j = \frac{\partial \mathbf{r}}{\partial \bar{u}^j} = \frac{\partial \mathbf{r}}{\partial u^k}\frac{\partial u^k}{\partial \bar{u}^j} \equiv \mathbf{g}_k S_j^k \Rightarrow S_j^k \equiv \frac{\partial u^k}{\partial \bar{u}^j} \tag{2.126a}$$

Combining Eqs. (2.123), (2.124) and (2.126a) and using the chain rule of differentiation, one obtains the relation of the mixed transformation components between two general curvilinear coordinates in Euclidean space $\mathbf{E}^N$.

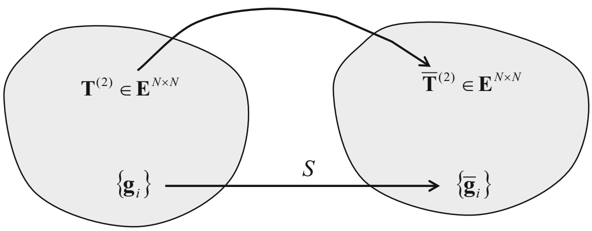

Fig. 2.7 Basis transformation of general curvilinear coordinates in $\mathbf{E}^N$

$$
\begin{aligned}
\overline{\mathbf{g}}_j \cdot \overline{\mathbf{g}}^i &= \overline{\mathbf{g}}^i \cdot \overline{\mathbf{g}}_j = \left(S^{-1}\right)^i_l S^k_j \mathbf{g}^l \cdot \mathbf{g}_k = \left(S^{-1}\right)^i_l S^k_j \delta^l_k \\
&= \left(S^{-1}\right)^i_k S^k_j = \frac{\partial \overline{u}^i}{\partial u^k}\frac{\partial u^k}{\partial \overline{u}^j} = \frac{\partial \overline{u}^i}{\partial \overline{u}^j} = \delta^i_j \\
\Leftrightarrow \quad & \mathbf{S}\mathbf{S}^{-1} = \mathbf{S}^{-1}\mathbf{S} = \mathbf{I}
\end{aligned}
\tag{2.126b}
$$

Therefore, the transformation tensor **S** can be written as

$$
\mathbf{S} = \begin{bmatrix} \dfrac{\partial u^1}{\partial \overline{u}^1} & \dfrac{\partial u^1}{\partial \overline{u}^2} & \cdot & \dfrac{\partial u^1}{\partial \overline{u}^N} \\ \dfrac{\partial u^2}{\partial \overline{u}^1} & \dfrac{\partial u^2}{\partial \overline{u}^2} & \cdot & \dfrac{\partial u^2}{\partial \overline{u}^N} \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial u^N}{\partial \overline{u}^1} & \dfrac{\partial u^N}{\partial \overline{u}^2} & \cdot & \dfrac{\partial u^N}{\partial \overline{u}^N} \end{bmatrix} \in \mathbf{R}^N \times \mathbf{R}^N \tag{2.127a}
$$

The transformation tensor **S** in Eq. (2.127a) is identical to the Jacobian matrix between two coordinate systems $\{u^i\}$ and $\{\overline{u}^i\}$.

Inverting the transformation tensor **S**, the back transformation results in

$$
\mathbf{S}^{-1} = \begin{bmatrix} \dfrac{\partial \overline{u}^1}{\partial u^1} & \dfrac{\partial \overline{u}^1}{\partial u^2} & \cdot & \dfrac{\partial \overline{u}^1}{\partial u^N} \\ \dfrac{\partial \overline{u}^2}{\partial u^1} & \dfrac{\partial \overline{u}^2}{\partial u^2} & \cdot & \dfrac{\partial \overline{u}^2}{\partial u^N} \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial \overline{u}^N}{\partial u^1} & \dfrac{\partial \overline{u}^N}{\partial u^2} & \cdot & \dfrac{\partial \overline{u}^N}{\partial u^N} \end{bmatrix} \in \mathbf{R}^N \times \mathbf{R}^N \tag{2.127b}
$$

The relation between the new and old components of an arbitrary vector **v** (first-order tensor) is similarly given in the coordinate transformation S according to Eqs. (2.122) and (2.124).

$$
\begin{aligned}
\overline{v}_i &= \overline{\mathbf{g}}_i \cdot \mathbf{v} = S^j_i \mathbf{g}_j \cdot \left(v_j \mathbf{g}^j\right) = S^j_i v_j \\
\overline{v}^i &= \overline{\mathbf{g}}^i \cdot \mathbf{v} = \left(S^{-1}\right)^i_j \mathbf{g}^j \cdot \left(v^j \mathbf{g}_j\right) = \left(S^{-1}\right)^i_j v^j
\end{aligned}
\tag{2.128}
$$

Using Eq. (2.74), the relation of the components of the second-order tensor **T** can be derived.

– Covariant metric tensor components:

$$
\begin{aligned}
\overline{T}_{ij} &= \overline{\mathbf{g}}_i \cdot \mathbf{T} \cdot \overline{\mathbf{g}}_j = \left(S^k_i \mathbf{g}_k\right) \cdot \mathbf{T} \cdot \left(S^l_j \mathbf{g}_l\right) \\
&= S^k_i S^l_j (\mathbf{g}_k \cdot \mathbf{T} \cdot \mathbf{g}_l) = S^k_i S^l_j T_{kl}
\end{aligned}
\tag{2.129}
$$

– Contravariant metric tensor components:

$$\begin{aligned}\overline{T}^{ij} &= \overline{\mathbf{g}}^i \cdot \mathbf{T} \cdot \overline{\mathbf{g}}^j = \left(S^{-1}\right)^i_k \mathbf{g}^k \cdot \mathbf{T} \cdot \left(S^{-1}\right)^j_l \mathbf{g}^l \\ &= \left(S^{-1}\right)^i_k \left(S^{-1}\right)^j_l \left(\mathbf{g}^k \cdot \mathbf{T} \cdot \mathbf{g}^l\right) = \left(S^{-1}\right)^i_k \left(S^{-1}\right)^j_l T^{kl}\end{aligned} \tag{2.130}$$

– Mixed metric tensor components:

$$\begin{aligned}\overline{T}_{j.}^{\ i} &= \overline{\mathbf{g}}_j \cdot \mathbf{T} \cdot \overline{\mathbf{g}}^i = \left(S_{\mathrm{j}}^{\mathrm{k}} \mathbf{g}_k\right) \cdot \mathbf{T} \cdot \left(S^{-1}\right)^i_l \mathbf{g}^l \\ &= S_j^k \left(S^{-1}\right)^i_l \left(\mathbf{g}_k \cdot \mathbf{T} \cdot \mathbf{g}^l\right) = S_j^k \left(S^{-1}\right)^i_l T_{k.}^{\ l}\end{aligned} \tag{2.131}$$

Note that in Eq. (2.131), the dot after the lower index indicates the position of the basis of the upper index locating after the tensor **T**. In this case, the tensor **T** is located between the upper basis $\mathbf{g}^l$ and lower basis $\mathbf{g}_k$.

$$\begin{aligned}\overline{T}^{\ i}_{.j} &= \overline{\mathbf{g}}^i \cdot \mathbf{T} \cdot \overline{\mathbf{g}}_j = \left(S^{-1}\right)^i_k \mathbf{g}^k \cdot \mathbf{T} \cdot \left(S_j^l \mathbf{g}_l\right) \\ &= \left(S^{-1}\right)^i_k S_j^l \left(\mathbf{g}^k \cdot \mathbf{T} \cdot \mathbf{g}_l\right) = \left(S^{-1}\right)^i_k S_j^l T^{\ k}_{.l}\end{aligned} \tag{2.132}$$

Note that in Eq. (2.132), the dot before the lower index indicates the position of the basis of the upper index locating in front of the tensor **T**. In this case, the tensor **T** is located between the upper basis $\mathbf{g}^k$ and lower basis $\mathbf{g}_l$.

2.4.3 Examples of Coordinate Transformations

2.4.3.1 Cylindrical Coordinates

The transformation S from Cartesian $\{u^i\}$ to cylindrical coordinates $\{\ \overline{u}^i\}$:

$$S : \begin{cases} u^1 = r\cos\theta \equiv \overline{u}^1 \cos\overline{u}^2 \\ u^2 = r\sin\theta \equiv \overline{u}^1 \sin\overline{u}^2 \\ u^3 = z = \overline{u}^3 \end{cases}$$

The covariant transformation matrix **S** can be calculated as

$$\mathbf{S} = \begin{bmatrix} \dfrac{\partial u^1}{\partial \overline{u}^1} & \dfrac{\partial u^1}{\partial \overline{u}^2} & \dfrac{\partial u^1}{\partial \overline{u}^3} \\ \dfrac{\partial u^2}{\partial \overline{u}^1} & \dfrac{\partial u^2}{\partial \overline{u}^2} & \dfrac{\partial u^2}{\partial \overline{u}^3} \\ \dfrac{\partial u^3}{\partial \overline{u}^1} & \dfrac{\partial u^3}{\partial \overline{u}^2} & \dfrac{\partial u^3}{\partial \overline{u}^3} \end{bmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The determinant of **S** is called the Jacobian J.

$$|\mathbf{S}| = \begin{vmatrix} \dfrac{\partial u^1}{\partial \overline{u}^1} & \dfrac{\partial u^1}{\partial \overline{u}^2} & \dfrac{\partial u^1}{\partial \overline{u}^3} \\ \dfrac{\partial u^2}{\partial \overline{u}^1} & \dfrac{\partial u^2}{\partial \overline{u}^2} & \dfrac{\partial u^2}{\partial \overline{u}^3} \\ \dfrac{\partial u^3}{\partial \overline{u}^1} & \dfrac{\partial u^3}{\partial \overline{u}^2} & \dfrac{\partial u^3}{\partial \overline{u}^3} \end{vmatrix} \equiv J = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

The contravariant transformation matrix $\mathbf{S}^{-1}$ results from the inversion of the covariant matrix **S**.

$$\mathbf{S}^{-1} = \begin{bmatrix} \dfrac{\partial \overline{u}^1}{\partial u^1} & \dfrac{\partial \overline{u}^1}{\partial u^2} & \dfrac{\partial \overline{u}^1}{\partial u^3} \\ \dfrac{\partial \overline{u}^2}{\partial u^1} & \dfrac{\partial \overline{u}^2}{\partial u^2} & \dfrac{\partial \overline{u}^2}{\partial u^3} \\ \dfrac{\partial \overline{u}^3}{\partial u^1} & \dfrac{\partial \overline{u}^3}{\partial u^2} & \dfrac{\partial \overline{u}^3}{\partial u^3} \end{bmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & r \end{pmatrix}$$

2.4.3.2 Spherical Coordinates

The transformation S from Cartesian $\{u^i\}$ to spherical coordinates $\{\overline{u}^i\}$:

$$S : \begin{cases} u^1 = \rho\sin\varphi\cos\theta \equiv \overline{u}^1 \sin\overline{u}^2 \cos\overline{u}^3 \\ u^2 = \rho\sin\varphi\sin\theta \equiv \overline{u}^1 \sin\overline{u}^2 \cos\overline{u}^3 \\ u^3 = \rho\cos\varphi = \overline{u}^1 \cos\overline{u}^2 \end{cases}$$

The covariant transformation matrix **S** can be calculated as

$$\mathbf{S} = \begin{bmatrix} \dfrac{\partial u^1}{\partial \overline{u}^1} & \dfrac{\partial u^1}{\partial \overline{u}^2} & \dfrac{\partial u^1}{\partial \overline{u}^3} \\ \dfrac{\partial u^2}{\partial \overline{u}^1} & \dfrac{\partial u^2}{\partial \overline{u}^2} & \dfrac{\partial u^2}{\partial \overline{u}^3} \\ \dfrac{\partial u^3}{\partial \overline{u}^1} & \dfrac{\partial u^3}{\partial \overline{u}^2} & \dfrac{\partial u^3}{\partial \overline{u}^3} \end{bmatrix} = \begin{pmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{pmatrix}$$

The determinant of **S** is called the Jacobian J.

$$|\mathbf{S}| = \begin{vmatrix} \dfrac{\partial u^1}{\partial \overline{u}^1} & \dfrac{\partial u^1}{\partial \overline{u}^2} & \dfrac{\partial u^1}{\partial \overline{u}^3} \\ \dfrac{\partial u^2}{\partial \overline{u}^1} & \dfrac{\partial u^2}{\partial \overline{u}^2} & \dfrac{\partial u^2}{\partial \overline{u}^3} \\ \dfrac{\partial u^3}{\partial \overline{u}^1} & \dfrac{\partial u^3}{\partial \overline{u}^2} & \dfrac{\partial u^3}{\partial \overline{u}^3} \end{vmatrix} \equiv J$$
$$= \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix} = \rho^2\sin\varphi$$

The contravariant transformation matrix results from the inversion of the matrix **S**.

$$\mathbf{S}^{-1} = \begin{bmatrix} \dfrac{\partial \overline{u}^1}{\partial u^1} & \dfrac{\partial \overline{u}^1}{\partial u^2} & \dfrac{\partial \overline{u}^1}{\partial u^3} \\ \dfrac{\partial \overline{u}^2}{\partial u^1} & \dfrac{\partial \overline{u}^2}{\partial u^2} & \dfrac{\partial \overline{u}^2}{\partial u^3} \\ \dfrac{\partial \overline{u}^3}{\partial u^1} & \dfrac{\partial \overline{u}^3}{\partial u^2} & \dfrac{\partial \overline{u}^3}{\partial u^3} \end{bmatrix} = \frac{1}{\rho}\begin{pmatrix} \rho\sin\varphi\cos\theta & \rho\sin\varphi\sin\theta & \rho\cos\varphi \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\left(\dfrac{\sin\theta}{\sin\varphi}\right) & \left(\dfrac{\cos\theta}{\sin\varphi}\right) & 0 \end{pmatrix}$$

2.4.4 Transformation of Curvilinear Coordinates in $\boldsymbol{R}^{\mathrm{N}}$

In the following section, second-order tensors **T** will be used in the transformation of general curvilinear coordinates in Riemannian manifold $\mathbf{R}^N$, as shown in Fig. 2.8. In Riemannian manifold, the bases $\mathbf{g}_i$ and $\overline{\mathbf{g}}_i$ of the curvilinear coordinates u^i and $\overline{u}^i$ do not exist any longer. Instead of the metric coefficients, the transformation coefficients that depend on the relating coordinates have been used in Riemannian manifold [3].

The new barred curvilinear coordinate $\overline{u}^i$ is a function of the old curvilinear coordinates u^j, $j = 1,2,\ldots,N$. Therefore, it can be written in a linear function of u^j.

$$\begin{aligned} &\overline{u}^i = \overline{u}^i(u^1,\ldots,u^N) \\ &\Rightarrow \overline{u}^i = \overline{a}^i_j u^j \text{ for } j = 1,2,\ldots,N \\ &\Rightarrow d\overline{u}^i = \overline{a}^i_j du^j \text{ for } j = 1,2,\ldots,N \end{aligned} \tag{2.133}$$

where $\bar{a}^i_j$ is the transformation coefficient in the coordinate transformation S.

Using the chain rule of differentiation, one obtains

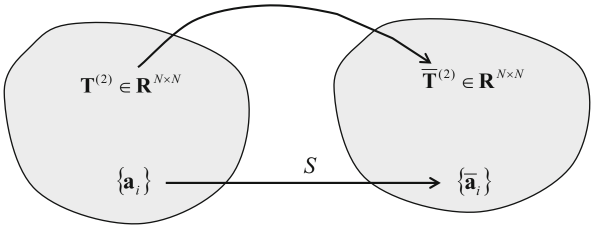

Fig. 2.8 Basis transformation of general curvilinear coordinates in $\mathbf{R}^N$

$$\begin{aligned} d\overline{u}^i &= \frac{\partial \overline{u}^i}{\partial u^j} du^j = \overline{a}^i_j du^j \\ \Rightarrow \overline{a}^i_j &= \frac{\partial \overline{u}^i}{\partial u^j} \end{aligned} \tag{2.134}$$

Analogously, the transformation coefficients of the back transformation result in

$$\begin{aligned} du^i &= \frac{\partial u^i}{\partial \overline{u}^j} d\overline{u}^j = \underline{a}^i_j d\overline{u}^j \\ \Rightarrow \underline{a}^i_j &= \frac{\partial u^i}{\partial \overline{u}^j} \end{aligned} \tag{2.135}$$

Combining Eqs. (2.134) and (2.135) and using the Kronecker delta, one obtains the relation between the transformation coefficients

$$\begin{aligned} &\underline{a}^i_j \overline{a}^j_k = \frac{\partial u^i}{\partial \overline{u}^j}\frac{\partial \overline{u}^j}{\partial u^k} = \frac{\partial u^i}{\partial u^k} = \delta^i_k \\ &\Leftrightarrow \left(\underline{a}^i_j\right)\left(\overline{a}^j_k\right) = \left(\overline{a}^j_k\right)\left(\underline{a}^i_j\right) = \mathbf{I} \end{aligned} \tag{2.136}$$

The relation of the second-order tensor components between the new and old curvilinear coordinates can be calculated using Eq. (2.133).

$$\begin{aligned} T^{(2)} &= \overline{T}_{kl} \overline{u}^k \overline{u}^l = \left(\overline{T}_{kl} \overline{a}^k_i \overline{a}^l_j\right) u^i u^j \\ &\equiv T_{ij} u^i u^j \end{aligned} \tag{2.137}$$

Thus,

$$\begin{aligned} T_{ij} &= \overline{a}^k_i \overline{a}^l_j \overline{T}_{kl} = \frac{\partial \overline{u}^k}{\partial u^i}\frac{\partial \overline{u}^l}{\partial u^j} \overline{T}_{kl}; \\ \overline{T}_{ij} &= \underline{a}^k_i \underline{a}^l_j T_{kl} = \frac{\partial u^k}{\partial \overline{u}^i}\frac{\partial u^l}{\partial \overline{u}^j} T_{kl} \end{aligned} \tag{2.138}$$

In the same way, the covariant, contravariant, and mixed components of the second-order tensor **T** between both coordinates in the transformation can be derived as:

- Covariant tensor components:

$$T_{ij} = \overline{a}_i^k \overline{a}_j^l \overline{T}_{kl} \Leftrightarrow \overline{T}_{ij} = \underline{a}_i^k \underline{a}_j^l T_{kl} \tag{2.139}$$

- Contravariant tensor components:

$$T^{ij} = \underline{a}_k^i \underline{a}_l^j \overline{T}^{kl} \Leftrightarrow \overline{T}^{ij} = \overline{a}_k^i \overline{a}_l^j T^{kl} \tag{2.140}$$

- Mixed tensor components:

$$T_{i.}^{j} = \overline{a}_i^k \underline{a}_l^j \overline{T}_{k.}^{l} \Leftrightarrow \overline{T}_{i.}^{j} = \underline{a}_i^k \overline{a}_l^j T_{k.}^{l} \tag{2.141}$$

$$T_{.j}^{i} = \underline{a}_k^i \overline{a}_j^l \overline{T}_{.l}^{k} \Leftrightarrow \overline{T}_{.j}^{i} = \overline{a}_k^i \underline{a}_j^l T_{.l}^{k} \tag{2.142}$$

Generally, the transformation coefficients of high-order tensors can be alternatively computed as

$$\begin{aligned} T_{ij}^{klm} &= \left(\frac{\partial u^k}{\partial \overline{u}^p} \frac{\partial u^l}{\partial \overline{u}^q} \frac{\partial u^m}{\partial \overline{u}^r} \right) \left(\frac{\partial \overline{u}^s}{\partial u^i} \frac{\partial \overline{u}^t}{\partial u^j} \right) \overline{T}_{st}^{pqr} \\ &= \left(\underline{a}_p^k \underline{a}_q^l \underline{a}_r^m \right) \left(\overline{a}_i^s \overline{a}_j^t \right) \overline{T}_{st}^{pqr} \end{aligned} \tag{2.143}$$

Therefore,

$$\begin{aligned} \overline{T}_{ij}^{klm} &= \left(\frac{\partial \overline{u}^k}{\partial u^p} \frac{\partial \overline{u}^l}{\partial u^q} \frac{\partial \overline{u}^m}{\partial u^r} \right) \left(\frac{\partial u^s}{\partial \overline{u}^i} \frac{\partial u^t}{\partial \overline{u}^j} \right) T_{st}^{pqr} \\ &= \left(\overline{a}_p^k \overline{a}_q^l \overline{a}_r^m \right) \left(\underline{a}_i^s \underline{a}_j^t \right) T_{st}^{pqr} \end{aligned} \tag{2.144}$$

2.5 Tensor Calculus in General Curvilinear Coordinates

In the following sections, some necessary symbols, such as the Christoffel symbols, Riemann-Christoffel tensor, and fundamental invariants of the Nabla operator have to be taken into account in the tensor applications to fluid mechanics and other working areas.

2.5.1 Physical Component of Tensors

Various types of the second-order tensors are shown in Eq. (2.74). The physical tensor component is defined as the tensor component on its covariant unitary basis

$\mathbf{g}_i^*$. Therefore, the basis of the general curvilinear coordinates must be normalized (cf. Appendix B).

Dividing the covariant basis by its vector length, the covariant unitary basis (covariant normalized basis) results in

$$\mathbf{g}_i^* = \frac{\mathbf{g}_i}{|\mathbf{g}_i|} = \frac{\mathbf{g}_i}{\sqrt{g_{(ii)}}} \Rightarrow |\mathbf{g}_i^*| = 1 \tag{2.145}$$

The covariant basis norm $|\mathbf{g}_i|$ can be considered as a scale factor h_i.

$$h_i = |\mathbf{g}_i| = \sqrt{g_{(ii)}} \text{ (no summation over } i)$$

Thus, the covariant basis can be related to its covariant unitary basis by the relation of

$$\mathbf{g}_i = \sqrt{g_{(ii)}}\mathbf{g}_i^* = h_i \mathbf{g}_i^* \tag{2.146}$$

The contravariant basis can be related to its covariant unitary basis using Eqs. (2.47) and (2.146).

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j = g^{ij}h_j\mathbf{g}_j^* \tag{2.147}$$

The contravariant second-order tensor can be written in the covariant unitary bases using Eq. (2.146).

$$\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j = \left(T^{ij}h_ih_j\right)\mathbf{g}_i^*\mathbf{g}_j^* \equiv T^{*ij}\mathbf{g}_i^*\mathbf{g}_j^* \tag{2.148}$$

Thus, the physical contravariant tensor components result in

$$T^{*ij} \equiv T^{ij}h_ih_j \tag{2.149}$$

The covariant second-order tensor can be written in the contravariant unitary bases using Eq. (2.147).

$$\mathbf{T} = T_{ij}\mathbf{g}^i\mathbf{g}^j = \left(T_{ij}g^{ik}g^{jl}h_kh_l\right)\mathbf{g}_k^*\mathbf{g}_l^* \equiv T_{ij}^*\mathbf{g}_k^*\mathbf{g}_l^* \tag{2.150}$$

Similarly, the physical covariant tensor component results in

$$T_{ij}^* = T_{ij}g^{ik}g^{jl}h_kh_l \tag{2.151}$$

The mixed tensors can be written in the covariant unitary bases using Eqs. (2.146) and (2.147)

$$\begin{aligned}\mathbf{T} &= T^i_j \mathbf{g}_i \mathbf{g}^j = T^i_j \mathbf{g}_i \left(g^{jk}\mathbf{g}_k\right) \\ &= T^i_j \left(h_i \mathbf{g}^*_i\right)\left(g^{jk} h_k \mathbf{g}^*_k\right) = \left(T^i_j g^{jk} h_i h_k\right)\mathbf{g}^*_i \mathbf{g}^*_k \\ &\equiv \left(T^i_j\right)^* \mathbf{g}^*_i \mathbf{g}^*_k\end{aligned} \tag{2.152}$$

Thus, the physical mixed tensor component results from Eq. (2.152):

$$\left(T^i_j\right)^* = T^i_j g^{jk} h_i h_k \tag{2.153}$$

2.5.2 Derivatives of Covariant Bases

Let $\mathbf{g}_i$ be a covariant basis in the curvilinear coordinates $\{u^i\}$. The derivative of the covariant basis with respect to the time variable t can be computed as

$$\dot{\mathbf{g}}_i = \frac{\partial \mathbf{g}_i}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}}{\partial u^i}\right) \equiv \dot{\mathbf{r}}_{,i} \tag{2.154}$$

Due to u^i is a differentiable function of t, Eq. (2.154) can be rewritten as

$$\dot{\mathbf{g}}_i = \frac{\partial \mathbf{g}_i}{\partial t} = \frac{\partial \mathbf{g}_i}{\partial u^j}\frac{\partial u^j}{\partial t} \equiv \mathbf{g}_{i,j}\dot{u}^j \tag{2.155}$$

where $\mathbf{g}_{i,j}$ is called the derivative of the covariant basis $\mathbf{g}_i$ of the curvilinear coordinates $\{u^i\}$.

Using the chain rule of differentiation, the covariant basis of the curvilinear coordinates $\{u^i\}$ can be calculated in Cartesian coordinates $\{x^i\}$.

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial u^i} = \frac{\partial \mathbf{r}}{\partial x^p}\frac{\partial x^p}{\partial u^i} = \mathbf{e}_p x^p_{,i} \tag{2.156}$$

Similarly, one obtains the covariant basis of the coordinates $\{x^i\}$.

$$\mathbf{e}_p = \frac{\partial \mathbf{r}}{\partial x^p} = \frac{\partial \mathbf{r}}{\partial u^k}\frac{\partial u^k}{\partial x^p} = \mathbf{g}_k u^k_{,p} \tag{2.157}$$

The derivative of the covariant basis of the coordinates $\{u^i\}$ can be obtained from Eqs. (2.156) and (2.157).

$$\begin{aligned}\mathbf{g}_{i,j} &= \frac{\partial \mathbf{g}_i}{\partial u^j} = \frac{\partial\left(\mathbf{e}_p x^p_{,i}\right)}{\partial u^j} = \mathbf{e}_p \frac{\partial x^p_{,i}}{\partial u^j} \\ &= u^k_{,p} \frac{\partial x^p_{,i}}{\partial u^j} \mathbf{g}_k = \left(\frac{\partial u^k}{\partial x^p} \frac{\partial^2 x^p}{\partial u^i \partial u^j}\right) \mathbf{g}_k \\ &\equiv \Gamma^k_{ij} \mathbf{g}_k \ \text{ for } k = 1, 2, \ldots, N\end{aligned} \tag{2.158}$$

The symbol $\Gamma^k{}_{ij}$ in Eq. (2.158) is defined as the second-kind Christoffel symbol, which has 27 (=3^3) components for a three-dimensional space ($N = 3$).

Thus, the second-order Christoffel symbols that only depend on both coordinates of $\{u^i\}$ and $\{x^i\}$ can be written as

$$\begin{aligned}\Gamma^k_{ij} &= \frac{\partial u^k}{\partial x^p} \frac{\partial^2 x^p}{\partial u^i \partial u^j} \\ &= \frac{\partial u^k}{\partial x^p} \frac{\partial^2 x^p}{\partial u^j \partial u^i} = \Gamma^k_{ji}\end{aligned} \tag{2.159}$$

The result of Eq. (2.159) proves that the second-kind Christoffel symbols are symmetric with respect to i and j.

The second-kind Christoffel symbols are given by multiplying both sides of Eq. (2.158) by the contravariant basis $\mathbf{g}^l$.

$$\begin{aligned}&\Gamma^k_{ij}\left(\mathbf{g}_k \cdot \mathbf{g}^l\right) = \Gamma^k_{ij} \delta^l_k = \mathbf{g}^l \cdot \mathbf{g}_{i,j} \\ &\Rightarrow \Gamma^l_{ij} = \mathbf{g}^l \cdot \mathbf{g}_{i,j}\end{aligned} \tag{2.160}$$

Substituting Eq. (2.158) into Eq. (2.155), one obtains the relation between the covariant basis time derivative and the Christoffel symbol.

$$\dot{\mathbf{g}}_i = \mathbf{g}_{i,j} \dot{u}^j = \Gamma^k_{ij} \dot{u}^j \mathbf{g}_k \tag{2.161}$$

Furthermore, the covariant basis derivative can be calculated in Cartesian coordinate $\{x^i\}$ using Eq. (2.156).

$$\mathbf{g}_{i,j} = \frac{\partial \mathbf{g}_i}{\partial u^j} = \frac{\partial\left(\mathbf{e}_p x^p_{,i}\right)}{\partial u^j} = \mathbf{e}_p \frac{\partial\left(x^p_{,i}\right)}{\partial u^j} = \mathbf{e}_p x^p_{,ij} \tag{2.162}$$

According to Eq. (2.159), the second-kind Christoffel symbols can be rewritten as

$$\begin{aligned}\Gamma^k_{ij} &= \frac{\partial u^k}{\partial x^p} \frac{\partial^2 x^p}{\partial u^i \partial u^j} = \frac{\partial u^k}{\partial x^p} \frac{\partial^2 x^p}{\partial u^j \partial u^i} \\ &= u^k_{,p} x^p_{,ij} = u^k_{,p} x^p_{,ji}\end{aligned} \tag{2.163}$$

2.5.3 Christoffel Symbols of First and Second Kind

According to Eq. (2.160), the second-kind Christoffel symbol can be defined as

$$\begin{aligned}\Gamma_{ij}^{k} \equiv \left\{ \begin{matrix} & k & \\ i & & j \end{matrix} \right\} &= \mathbf{g}^k \cdot \frac{\partial^2 \mathbf{r}}{\partial u^i \partial u^j} = \mathbf{g}^k \cdot \frac{\partial^2 \mathbf{r}}{\partial u^j \partial u^i} \\ &= \mathbf{g}^k \cdot \mathbf{g}_{i,j} = \mathbf{g}^k \cdot \mathbf{g}_{j,i} = \Gamma_{ji}^{k}\end{aligned} \tag{2.164}$$

Equation (2.164) reconfirms the symmetric property of the Christoffel symbols with respect to the indices i and j. Obviously, the Christoffel symbols are coordinate dependent; therefore, they are not tensors.

In order to compute the second-kind Christoffel symbols in the covariant metric coefficients, the derivative of g_{ij} with respect to u^k has to be taken into account.

$$\begin{aligned}& g_{ij} = \left(\mathbf{g}_i \cdot \mathbf{g}_j \right) \\ \Rightarrow\ & g_{ij,k} \equiv \frac{\partial g_{ij}}{\partial u^k} = \left(\mathbf{g}_i \cdot \mathbf{g}_j \right)_{,k} = \mathbf{g}_{i,k} \cdot \mathbf{g}_j + \mathbf{g}_i \cdot \mathbf{g}_{j,k}\end{aligned} \tag{2.165}$$

Using Eq. (2.158) at changing the index j into k; then, i into j, one obtains the following relations

$$\mathbf{g}_{i,k} = \Gamma_{ik}^{p} \mathbf{g}_p\ ;\ \ \mathbf{g}_{j,k} = \Gamma_{jk}^{p} \mathbf{g}_p \tag{2.166}$$

Substituting Eq. (2.166) into Eq. (2.165), one obtains the derivative of g_{ij} with respect to u^k.

$$\begin{aligned}g_{ij,k} &= \mathbf{g}_{i,k} \cdot \mathbf{g}_j + \mathbf{g}_{j,k} \cdot \mathbf{g}_i \\ &= \Gamma_{ik}^{p} \mathbf{g}_p \cdot \mathbf{g}_j + \Gamma_{jk}^{p} \mathbf{g}_p \cdot \mathbf{g}_i \\ &= \Gamma_{ik}^{p} g_{pj} + \Gamma_{jk}^{p} g_{pi}\end{aligned} \tag{2.167}$$

Interchanging k with i in Eq. (2.167), one obtains

$$g_{kj,i} = \Gamma_{ki}^{p} g_{pj} + \Gamma_{ji}^{p} g_{pk} \tag{2.168}$$

Analogously, one reaches the relation interchanging k with j in Eq. (2.167).

$$g_{ik,j} = \Gamma_{ij}^{p} g_{pk} + \Gamma_{kj}^{p} g_{pi} \tag{2.169}$$

Combining Eqs. (2.167)–(2.169), the Christoffel symbols can be written in the derivatives of the covariant metric coefficients.

$$g_{pj}\, \Gamma_{ik}^{p} = \frac{1}{2} \left(g_{ij,k} + g_{kj,i} - g_{ik,j} \right) \tag{2.170}$$

Multiplying Eq. (2.170) by g^{qj}, the Christoffel symbols result according to Eq. (2.50) in

$$\begin{aligned} &\Gamma_{ik}^{p} g_{pj}\, g^{qj} = \Gamma_{ik}^{p} \delta_{p}^{q} \\ &\Rightarrow \Gamma_{ik}^{q} = \frac{1}{2} g^{qj} \left(g_{ij,k} + g_{kj,i} - g_{ik,j} \right) \end{aligned} \tag{2.171}$$

Changing j into p, k into j, and q into k in Eq. (2.171), one obtains

$$\begin{aligned} \Gamma_{ij}^{k} &= \frac{1}{2} \left(g_{ip,j} + g_{jp,i} - g_{ij,p} \right) g^{kp} \\ &\equiv \Gamma_{ijp} g^{kp} \end{aligned} \tag{2.172}$$

Changing the index p into k, the first-kind Christoffel symbol Γ_{ijp} in Eq. (2.172) that has 27 ($=3^3$) components for a three-dimensional space ($N = 3$) is defined as

$$\begin{aligned} \Gamma_{ijk} &\equiv [i\ j,\ k] \\ &\equiv \frac{1}{2} \left(g_{ik,j} + g_{jk,i} - g_{ij,k} \right) \\ &= g_{pk} \Gamma_{ij}^{p} \ \text{ for } p = 1, 2, \ldots, N \end{aligned} \tag{2.173a}$$

Other expressions of the Christoffel symbols can be found in some literature.

$$\begin{aligned} \Gamma_{ijk} &\equiv [i\ j,\ k] \\ &= g_{pk} \left\{ \begin{matrix} p \\ i \quad j \end{matrix} \right\} \equiv g_{pk} \Gamma_{ij}^{p} \ \text{ for } p = 1, 2, \ldots, N \end{aligned} \tag{2.173b}$$

2.5.4 Prove That the Christoffel Symbols Are Symmetric

1. The first-kind Christoffel symbol is symmetric with respect to i, j
   According to Eq. (2.173a), the first-kind Christoffel symbol can be written as

$$\Gamma_{ijk} = \frac{1}{2} \left(g_{ik,j} + g_{jk,i} - g_{ij,k} \right)$$

   Interchanging i with j in the equation, one obtains

$$\Gamma_{jik} = \frac{1}{2} \left(g_{jk,i} + g_{ik,j} - g_{ji,k} \right) = \frac{1}{2} \left(g_{ik,j} + g_{jk,i} - g_{ij,k} \right) = \Gamma_{ijk} \ \ (q.e.d.)$$

2. The second-kind Christoffel symbol is symmetric with respect to i, j
   Using Eq. (2.172), the second-kind Christoffel symbol can be expressed as

$$\Gamma_{ij}^k = g^{kp}\, \Gamma_{ijp}$$

Due to the symmetry of the first-kind Christoffel symbol, the second-kind Christoffel symbol results in

$$\begin{aligned} \Gamma_{ij}^k &= g^{kp}\, \Gamma_{ijp} = g^{kp}\, \Gamma_{jip} \\ &= \Gamma_{ji}^k\ (q.e.d.) \end{aligned}$$

2.5.5 Examples of Computing the Christoffel Symbols

Given a curvilinear coordinate $\{u^i\}$ with $\mathrm{u}^1 = u$; $\mathrm{u}^2 = v$; $\mathrm{u}^3 = w$ in another coordinate $\{x^i\}$, the relation between two coordinate systems can be written as

$$\begin{cases} x^1 = uv \\ x^2 = w \\ x^3 = u^2 - v \end{cases}$$

The covariant basis matrix **G** can be calculated from

$$\mathbf{G} = [\,\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3\,] = \begin{pmatrix} \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^1}{\partial u^2} & \dfrac{\partial x^1}{\partial u^3} \\ \dfrac{\partial x^2}{\partial u^1} & \dfrac{\partial x^2}{\partial u^2} & \dfrac{\partial x^2}{\partial u^3} \\ \dfrac{\partial x^3}{\partial u^1} & \dfrac{\partial x^3}{\partial u^2} & \dfrac{\partial x^3}{\partial u^3} \end{pmatrix} = \begin{bmatrix} v & u & 0 \\ 0 & 0 & 1 \\ 2u & -1 & 0 \end{bmatrix}$$

Therefore, the covariant bases can be given in

$$\begin{cases} \mathbf{g}_1 = (v,\ 0,\ 2u) \\ \mathbf{g}_2 = (u,\ 0,\ -1) \\ \mathbf{g}_3 = (0,\ 1,\ 0) \end{cases}$$

The determinant of **G** that equals the Jacobian J of

$$|\mathbf{G}| = J = \begin{vmatrix} v & u & 0 \\ 0 & 0 & 1 \\ 2u & -1 & 0 \end{vmatrix} = 2u^2 + v \neq 0$$

The contravariant basis matrix $\mathbf{G}^{-1}$ is the inverse matrix of the covariant basis matrix **G**.

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \begin{pmatrix} \dfrac{\partial u^1}{\partial x^1} & \dfrac{\partial u^1}{\partial x^2} & \dfrac{\partial u^1}{\partial x^3} \\ \dfrac{\partial u^2}{\partial x^1} & \dfrac{\partial u^2}{\partial x^2} & \dfrac{\partial u^2}{\partial x^3} \\ \dfrac{\partial u^3}{\partial x^1} & \dfrac{\partial u^3}{\partial x^2} & \dfrac{\partial u^3}{\partial x^3} \end{pmatrix}$$

$$= \frac{1}{J}\begin{bmatrix} 1 & 0 & u \\ 2u & 0 & -v \\ 0 & (2u^2+v) & 0 \end{bmatrix} = \begin{bmatrix} J^{-\mathbf{1}} & 0 & uJ^{-\mathbf{1}} \\ 2uJ^{-\mathbf{1}} & 0 & -vJ^{-\mathbf{1}} \\ 0 & 1 & 0 \end{bmatrix}$$

Thus,

$$\left|\mathbf{G}^{-1}\right| = \frac{1}{(2u^2+v)} = \frac{1}{J}$$

Testing: $\left|\mathbf{G}^{-1}\right| \cdot |\mathbf{G}| = J^{-\mathbf{1}}J = |\mathbf{I}| = 1 \ \ (q.e.d.)$

Thus, the contravariant bases result in

$$\begin{cases} \mathbf{g}^1 = J^{-1}(1,\ 0,\ u) \\ \mathbf{g}^2 = J^{-1}(2u,\ 0,\ -v) \\ \mathbf{g}^3 = J^{-1}(0,\ 2u^2+v,\ 0) \end{cases}$$

Some examples of the second-kind Christoffel symbols of 27 components can be computed from Eq. (2.160).

$$\begin{aligned}
&\Gamma_{ij}^{k} = \mathbf{g}_{i,j}\,\mathbf{g}^k \Rightarrow \\
&\Gamma_{11}^{1} = \ \mathbf{g}_{1,1}\,\mathbf{g}^1 = J^{-1}(0\cdot 1 + 0\cdot 0 + 2\cdot u) = 2uJ^{-1} \\
&\Gamma_{12}^{1} = \ \mathbf{g}_{1,2}\,\mathbf{g}^1 = J^{-1}(1\cdot 1 + 0\cdot 0 + 0\cdot u) = J^{-1} \\
&\Gamma_{13}^{1} = \ \mathbf{g}_{1,3}\,\mathbf{g}^1 = J^{-1}(0\cdot 1 + 0\cdot 0 + 0\cdot u) = 0 \\
&\ldots \\
&\Gamma_{32}^{3} = \ \mathbf{g}_{3,2}\,\mathbf{g}^3 = J^{-1}(0\cdot 0 + 0\cdot (2u^2+v) + 0\cdot 0) = 0 \\
&\Gamma_{33}^{3} = \ \mathbf{g}_{3,3}\,\mathbf{g}^3 = J^{-1}(0\cdot 0 + 0\cdot (2u^2+v) + 0\cdot 0) = 0
\end{aligned}$$

The first-kind Christoffel symbols containing 27 components in $\mathbf{R}^3$ can be computed from Eq. (2.173a).

$$\Gamma_{ijk} = g_{pk}\Gamma_{ij}^{p} = \left(\mathbf{g}_p \cdot \mathbf{g}_k\right)\Gamma_{ij}^{p} \text{ for } p = 1,\ 2,\ 3$$

2.5.6 Coordinate Transformations of the Christoffel Symbols

The second-kind Christoffel symbols like tensor components strongly depend on the coordinates at the coordinate transformations. The curvilinear coordinates $\{u^i\}$ are transformed into the new barred curvilinear coordinates $\{\bar{u}^i\}$. Therefore, the old basis is also changed into the new basis.

The second-kind Christoffel symbols can be written in the new basis of the barred coordinates $\{\bar{u}^i\}$.

$$\bar{\Gamma}_{ij}^k = \bar{\mathbf{g}}^k \cdot \frac{\partial \bar{\mathbf{g}}_i}{\partial u^j} = \bar{\mathbf{g}}^k \cdot \bar{\mathbf{g}}_{i,j} \tag{2.174}$$

Using the chain rule of differentiation, the basis of the coordinates $\{u^i\}$ can be calculated as

$$\begin{aligned} \mathbf{g}_p &= \frac{\partial \mathbf{r}}{\partial u^p} = \frac{\partial \mathbf{r}}{\partial \bar{u}^k} \frac{\partial \bar{u}^k}{\partial u^p} \\ &= \bar{\mathbf{g}}_k \frac{\partial \bar{u}^k}{\partial u^p} \end{aligned} \tag{2.175}$$

Multiplying Eq. (2.175) by the new contravariant basis of the coordinates $\{\bar{u}^i\}$, one obtains changing the indices m into k, and p into l.

$$\begin{aligned} &\bar{\mathbf{g}}^m \cdot \mathbf{g}_p = \bar{\mathbf{g}}^m \cdot \bar{\mathbf{g}}_k \frac{\partial \bar{u}^k}{\partial u^p} = \delta_k^m \frac{\partial \bar{u}^k}{\partial u^p} = \frac{\partial \bar{u}^m}{\partial u^p} \\ &\Rightarrow \left(\bar{\mathbf{g}}^m \cdot \mathbf{g}_p\right)\mathbf{g}^p = \bar{\mathbf{g}}^m \left(\mathbf{g}_p \cdot \mathbf{g}^p\right) = \bar{\mathbf{g}}^m \delta_p^p = \frac{\partial \bar{u}^m}{\partial u^p} \mathbf{g}^p \\ &\Rightarrow \bar{\mathbf{g}}^k = \frac{\partial \bar{u}^k}{\partial u^l} \mathbf{g}^l \end{aligned} \tag{2.176}$$

The new covariant basis of the coordinates $\{\bar{u}^i\}$ can be calculated as

$$\bar{\mathbf{g}}_i = \frac{\partial \mathbf{r}}{\partial \bar{u}^i} = \frac{\partial \mathbf{r}}{\partial u^p} \frac{\partial u^p}{\partial \bar{u}^i} = \mathbf{g}_p \frac{\partial u^p}{\partial \bar{u}^i} \tag{2.177}$$

Thus, the new covariant basis derivative with respect to j of the coordinates $\{\bar{u}^i\}$ results in

$$\bar{\mathbf{g}}_{i,j} = \frac{\partial \bar{\mathbf{g}}_i}{\partial \bar{u}^j} = \frac{\partial}{\partial \bar{u}^j} \left(\frac{\partial u^p}{\partial \bar{u}^i} \mathbf{g}_p \right) = \frac{\partial^2 u^p}{\partial \bar{u}^j \partial \bar{u}^i} \mathbf{g}_p + \frac{\partial u^p}{\partial \bar{u}^i} \frac{\partial \mathbf{g}_p}{\partial \bar{u}^j} \tag{2.178}$$

Substituting Eqs. (2.176) and (2.178) into Eq. (2.174), one obtains the second-kind Christoffel symbols in the new basis.

$$\begin{aligned}\overline{\Gamma}_{ij}^{k} &= \overline{\mathbf{g}}^k \cdot \overline{\mathbf{g}}_{i,j} \\ &= \frac{\partial \overline{u}^k}{\partial u^l}\mathbf{g}^l \cdot \left(\frac{\partial^2 u^p}{\partial \overline{u}^j \partial \overline{u}^i}\mathbf{g}_p + \frac{\partial u^p}{\partial \overline{u}^i}\frac{\partial \mathbf{g}_p}{\partial \overline{u}^j}\right)\end{aligned} \tag{2.179}$$

Using Eq. (2.166) and the chain rule of differentiation, the second term in the parentheses of Eq. (2.179) can be computed as

$$\begin{aligned}\frac{\partial \mathbf{g}_p}{\partial \overline{u}^j} &= \frac{\partial \mathbf{g}_p}{\partial u^q}\frac{\partial u^q}{\partial \overline{u}^j} \\ &= \frac{\partial u^q}{\partial \overline{u}^j}\mathbf{g}_{p,q} = \frac{\partial u^q}{\partial \overline{u}^j}\left(\Gamma_{pq}^{r}\mathbf{g}_r\right)\end{aligned} \tag{2.180}$$

Inserting Eq. (2.180) into Eq. (2.179), the transformed Christoffel symbols in the new barred coordinates can be calculated as

$$\begin{aligned}\overline{\Gamma}_{ij}^{k} &= \overline{\mathbf{g}}^k \cdot \overline{\mathbf{g}}_{i,j} \\ &= \frac{\partial \overline{u}^k}{\partial u^l}\left(\frac{\partial^2 u^p}{\partial \overline{u}^j \partial \overline{u}^i}\left(\mathbf{g}^l \cdot \mathbf{g}_p\right) + \frac{\partial u^p}{\partial \overline{u}^i}\frac{\partial u^q}{\partial \overline{u}^j}\Gamma_{pq}^{r}\left(\mathbf{g}^l \cdot \mathbf{g}_r\right)\right) \\ &= \frac{\partial \overline{u}^k}{\partial u^l}\left(\frac{\partial^2 u^p}{\partial \overline{u}^j \partial \overline{u}^i}\delta_p^l + \frac{\partial u^p}{\partial \overline{u}^i}\frac{\partial u^q}{\partial \overline{u}^j}\Gamma_{pq}^{r}\delta_r^l\right) \\ &= \frac{\partial \overline{u}^k}{\partial u^l}\left(\frac{\partial^2 u^l}{\partial \overline{u}^i \partial \overline{u}^j} + \frac{\partial u^p}{\partial \overline{u}^i}\frac{\partial u^q}{\partial \overline{u}^j}\Gamma_{pq}^{l}\right)\end{aligned} \tag{2.181}$$

Therefore, the transformed Christoffel symbols in the new barred coordinates $\{\overline{u}^i\}$ result in

$$\overline{\Gamma}_{ij}^{k} = \Gamma_{pq}^{l}\frac{\partial \overline{u}^k}{\partial u^l}\frac{\partial u^p}{\partial \overline{u}^i}\frac{\partial u^q}{\partial \overline{u}^j} + \frac{\partial \overline{u}^k}{\partial u^l}\frac{\partial^2 u^l}{\partial \overline{u}^i \partial \overline{u}^j} \tag{2.182}$$

Rearranging the terms in Eq. (2.181), the second derivatives of u^l with respect to the new barred coordinates result in

$$\begin{aligned}\frac{\partial u^l}{\partial \overline{u}^k}\overline{\Gamma}_{ij}^{k} &= \left(\frac{\partial^2 u^l}{\partial \overline{u}^i \partial \overline{u}^j} + \frac{\partial u^p}{\partial \overline{u}^i}\frac{\partial u^q}{\partial \overline{u}^j}\Gamma_{pq}^{l}\right) \\ \Rightarrow \frac{\partial^2 u^l}{\partial \overline{u}^i \partial \overline{u}^j} &= \frac{\partial u^l}{\partial \overline{u}^k}\overline{\Gamma}_{ij}^{k} - \frac{\partial u^p}{\partial \overline{u}^i}\frac{\partial u^q}{\partial \overline{u}^j}\Gamma_{pq}^{l}\end{aligned} \tag{2.183}$$

Using Eq. (2.160), all Christoffel symbols in Cartesian coordinates $\{x^i\}$ vanish because the basis $\mathbf{e}_i$ does not change in any coordinate x^j.

$$\Gamma_{ij}^{k} = \mathbf{e}^k \cdot \mathbf{e}_{i,j} = \mathbf{e}^k \cdot \frac{\partial \mathbf{e}_i}{\partial x^j} = 0 \tag{2.184}$$

2.5.7 Derivatives of Contravariant Bases

Like Eq. (2.158), the derivative of the contravariant basis of the curvilinear coordinates $\{u^i\}$ with respect to u^j can be defined as

$$\mathbf{g}_{,j}^{i} = \frac{\partial \mathbf{g}^i}{\partial u^j} \equiv \hat{\Gamma}_{jk}^{\ i} \mathbf{g}^k \tag{2.185}$$

where $\hat{\Gamma}_{jk}^{\ i}$ are the second-kind Christoffel symbols in the contravariant bases $\mathbf{g}^k$.

In order to compute those Christoffel symbols, some calculating steps are carried out in the following section.

The derivative of the product between the covariant and contravariant bases with respect to u^j can be computed using Eqs. (2.156), (2.164) and (2.185).

$$\begin{aligned}
\left(\mathbf{g}^i \cdot \mathbf{g}_j\right)_{,k} &= \mathbf{g}_{,k}^{i} \cdot \mathbf{g}_j + \mathbf{g}^i \cdot \mathbf{g}_{j,k} = \left(\delta_j^i\right)_{,k} \\
&= \hat{\Gamma}_{kl}^{\ i}\left(\mathbf{g}^l \cdot \mathbf{g}_j\right) + \Gamma_{jk}^{l}(\mathbf{g}_l \cdot \mathbf{g}^i) \\
&= \hat{\Gamma}_{kl}^{\ i}\delta_j^l + \Gamma_{jk}^{l}\delta_l^i \\
&= \hat{\Gamma}_{kj}^{\ i} + \Gamma_{jk}^{i} = \hat{\Gamma}_{kj}^{\ i} + \Gamma_{kj}^{i} \\
&= 0
\end{aligned} \tag{2.186}$$

Thus, the relation between the Christoffel symbols of two coordinates results in

$$\hat{\Gamma}_{kj}^{\ i} = -\Gamma_{kj}^{i} = -\Gamma_{jk}^{i} \tag{2.187}$$

Using Eqs. (2.164) and (2.187), one obtains

$$\hat{\Gamma}_{kj}^{\ i} = -\Gamma_{kj}^{i} = -\Gamma_{jk}^{i} = \hat{\Gamma}_{jk}^{\ i} \tag{2.188}$$

It proves that the Christoffel symbol $\hat{\Gamma}_{jk}^{\ i}$ is symmetric with respect to j and k.

Finally, the derivatives of the contravariant basis $\mathbf{g}^i$ with respect to u^j result from Eqs. (2.185) and (2.187).

$$\begin{aligned}\mathbf{g}^i_{,j} &= \hat{\Gamma}^{\,i}_{jk}\mathbf{g}^k = -\Gamma^i_{jk}\mathbf{g}^k \\ &= \hat{\Gamma}^{\,i}_{kj}\mathbf{g}^k = -\Gamma^i_{kj}\mathbf{g}^k\end{aligned} \tag{2.189}$$

2.5.8 Derivatives of Covariant Metric Coefficients

The derivatives of the covariant metric coefficient can be derived from the first-kind Christoffel symbols written as

$$\Gamma_{ikj} = \frac{1}{2}\left(g_{ij,k} + g_{kj,i} - g_{ik,j}\right); \qquad \Gamma_{jki} = \frac{1}{2}\left(g_{ji,k} + g_{ki,j} - g_{jk,i}\right) \tag{2.190}$$

The derivative of the covariant metric coefficient results by adding both Christoffel symbols given in Eq. (2.190).

$$\begin{aligned}\Gamma_{ikj} + \Gamma_{jki} &= \frac{1}{2}\left(g_{ij,k} + g_{kj,i} - g_{ik,j}\right) + \frac{1}{2}\left(g_{ji,k} + g_{ki,j} - g_{jk,i}\right) \\ &= \frac{1}{2}\left(g_{ij,k} + \underline{g_{kj,i}} - \underline{g_{ik,j}}\right) + \frac{1}{2}\left(g_{ij,k} + \underline{g_{ik,j}} - \underline{g_{kj,i}}\right) \\ &= g_{ij,k}\end{aligned} \tag{2.191}$$

Therefore, the derivatives of the covariant metric coefficient g_{ij} with respect to u^k can be expressed in the first-kind Christoffel symbols.

$$g_{ij,k} \equiv \frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ikj} + \Gamma_{jki} \tag{2.192}$$

Using Eq. (2.173a), Eq. (2.192) can be rewritten in the second-kind Christoffel symbols.

$$\begin{aligned}g_{ij,k} &= \Gamma_{ikj} + \Gamma_{jki} \\ &= g_{jl}\Gamma^l_{ik} + g_{il}\Gamma^l_{jk}\end{aligned} \tag{2.193}$$

Similar to Eq. (2.192), one can write

$$g_{jk,i} = \Gamma_{jik} + \Gamma_{kij} \tag{2.194}$$

Subtracting Eq. (2.192) from Eq. (2.194), one obtains the relation

$$\begin{aligned} g_{ij,k} - g_{jk,i} &= \underline{\Gamma_{ikj}} + \Gamma_{jki} - \Gamma_{jik} - \underline{\Gamma_{kij}} \\ &= \Gamma_{jki} - \Gamma_{jik} \\ &= \Gamma_{kji} - \Gamma_{ijk} \end{aligned} \tag{2.195}$$

2.5.9 *Covariant Derivatives of Tensors*

2.5.9.1 Contravariant First-Order Tensors with Components T^i

The contravariant first-order tensor (vector) **T** can be written in the covariant basis.

$$\mathbf{T} = T^i \mathbf{g}_i \tag{2.196}$$

Using Eq. (2.158), the derivative of the contravariant tensor **T** with respect to u^j results in

$$\begin{aligned} \mathbf{T}_{,j} &= \left(T^i \mathbf{g}_i\right)_{,j} = T^i_{\ ,j} \mathbf{g}_i + T^i \mathbf{g}_{i,j} \\ &= T^i_{\ ,j} \mathbf{g}_i + T^i \Gamma^k_{ij} \mathbf{g}_k \end{aligned} \tag{2.197}$$

The derivative of the contravariant tensor component T^i with respect to u^j in Eq. (2.197) can be defined as

$$T^i_{\ ,j} \equiv \frac{\partial T^i}{\partial u^j} \tag{2.198}$$

Interchanging i with k in the second term on the RHS of Eq. (2.197), one obtains

$$\begin{aligned} T^i \Gamma^k_{ij} \mathbf{g}_k &= T^k \Gamma^i_{kj} \mathbf{g}_i \\ &= T^k \Gamma^i_{jk} \mathbf{g}_i \end{aligned} \tag{2.199}$$

Substituting Eq. (2.199) into Eq. (2.197), one obtains the derivative of **T** with respect to u^j.

$$\begin{aligned} \mathbf{T}_{,j} &= T^i_{\ ,j} \mathbf{g}_i + T^k \Gamma^i_{jk} \mathbf{g}_i \\ &= \left(T^i_{\ ,j} + \Gamma^i_{jk} T^k\right) \mathbf{g}_i \\ &\equiv T^i\big|_j \mathbf{g}_i \end{aligned} \tag{2.200}$$

Therefore, the covariant derivative with respect to u^j of the contravariant first-order tensor (vector) can be written as

$$T^i\big|_j = T^i_{,j} + \Gamma^i_{jk} T^k = \mathbf{T}_{,j} \cdot \mathbf{g}^i \tag{2.201}$$

The covariant derivative of the contravariant first-order tensor component is transformed in the new barred coordinates $\{\bar{u}^i\}$.

$$\begin{aligned} T^i\big|_k &= \bar{T}^j\Big|_n \frac{\partial u^i}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} \\ \Rightarrow \quad \bar{T}^j\Big|_n &= T^i\big|_k \frac{\partial \bar{u}^j}{\partial u^i} \frac{\partial u^k}{\partial \bar{u}^n} \end{aligned} \tag{2.202a}$$

Proof The first-order tensor component can be written as

$$T^i = \bar{T}^j \frac{\partial u^i}{\partial \bar{u}^j} \tag{2.202b}$$

Differentiating T^i with respect to u^k and using the chain rule of differentiation, one obtains

$$\begin{aligned} \frac{\partial T^i}{\partial u^k} &= \left(\bar{T}^j \frac{\partial u^i}{\partial \bar{u}^j} \right)_{,k} \\ &= \left(\frac{\partial \bar{T}^j}{\partial \bar{u}^n} \frac{\partial \bar{u}^n}{\partial u^k} \right) \frac{\partial u^i}{\partial \bar{u}^j} + \bar{T}^j \left(\frac{\partial^2 u^i}{\partial \bar{u}^j \partial \bar{u}^n} \right) \frac{\partial \bar{u}^n}{\partial u^k} \end{aligned} \tag{2.202c}$$

Using Eq. (2.183), we have

$$\frac{\partial^2 u^i}{\partial \bar{u}^n \partial \bar{u}^j} = \bar{\Gamma}^k_{nj} \frac{\partial u^i}{\partial \bar{u}^k} - \Gamma^i_{pq} \frac{\partial u^p}{\partial \bar{u}^n} \frac{\partial u^q}{\partial \bar{u}^j} \tag{2.202d}$$

Substituting Eq. (2.202d) into Eq. (2.202c) and interchanging the indices k with j and j with m, one obtains

$$\frac{\partial T^i}{\partial u^k} = \left(\frac{\partial \bar{T}^j}{\partial \bar{u}^n} \frac{\partial \bar{u}^n}{\partial u^k} \right) \frac{\partial u^i}{\partial \bar{u}^j} + \bar{T}^j \left(\bar{\Gamma}^k_{nj} \frac{\partial u^i}{\partial \bar{u}^k} - \Gamma^i_{pq} \frac{\partial u^p}{\partial \bar{u}^n} \frac{\partial u^q}{\partial \bar{u}^j} \right) \frac{\partial \bar{u}^n}{\partial u^k}$$

Thus,

$$\begin{aligned} \frac{\partial T^i}{\partial u^k} + \bar{T}^j \Gamma^i_{pq} \left(\frac{\partial u^p}{\partial \bar{u}^n} \frac{\partial u^q}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} \right) &= \frac{\partial \bar{T}^j}{\partial \bar{u}^n} \frac{\partial u^i}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} + \bar{T}^j \bar{\Gamma}^k_{nj} \frac{\partial u^i}{\partial \bar{u}^k} \frac{\partial \bar{u}^n}{\partial u^k} \\ &= \frac{\partial \bar{T}^j}{\partial \bar{u}^n} \left(\frac{\partial u^i}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} \right) + \bar{T}^m \bar{\Gamma}^j_{nm} \left(\frac{\partial u^i}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} \right) \end{aligned} \tag{2.202e}$$

The terms on the RHS of Eq. (2.202e) can be written as

$$\frac{\partial \overline{T}^j}{\partial \overline{u}^n}\left(\frac{\partial u^i}{\partial \overline{u}^j}\frac{\partial \overline{u}^n}{\partial u^k}\right)+\overline{T}^m\overline{\Gamma}^j_{nm}\left(\frac{\partial u^i}{\partial \overline{u}^j}\frac{\partial \overline{u}^n}{\partial u^k}\right)=\left[\overline{T}^j_{,n}+\overline{T}^m\overline{\Gamma}^j_{nm}\right]\frac{\partial u^i}{\partial \overline{u}^j}\frac{\partial \overline{u}^n}{\partial u^k}$$
$$=\overline{T}^j\Big|_n\frac{\partial u^i}{\partial \overline{u}^j}\frac{\partial \overline{u}^n}{\partial u^k} \tag{2.202f}$$

Using Eq. (2.202b) and interchanging the indices p with k and q with m, the terms on the LHS of Eq. (2.202e) are rearranged in

$$\begin{aligned}\frac{\partial T^i}{\partial u^k}+\overline{T}^j\Gamma^i_{pq}\left(\frac{\partial u^p}{\partial \overline{u}^n}\frac{\partial u^q}{\partial \overline{u}^j}\frac{\partial \overline{u}^n}{\partial u^k}\right)&=T^i_{,k}+\left(T^m\frac{\partial \overline{u}^j}{\partial u^m}\right)\Gamma^i_{pq}\left(\frac{\partial u^p}{\partial \overline{u}^n}\frac{\partial u^q}{\partial \overline{u}^j}\frac{\partial \overline{u}^n}{\partial u^k}\right)\\&=T^i_{,k}+T^m\Gamma^i_{km}\left(\frac{\partial u^k}{\partial \overline{u}^n}\frac{\partial u^m}{\partial \overline{u}^j}\frac{\partial \overline{u}^n}{\partial u^k}\right)\frac{\partial \overline{u}^j}{\partial u^m}\\&=\left[T^i_{,k}+T^m\Gamma^i_{km}\right]\equiv T^i\Big|_k\end{aligned} \tag{2.202g}$$

Substituting Eqs. (2.202f) and (2.202g) into Eq. (2.202e), one obtains Eq. (2.202a).

$$\begin{aligned}T^i\Big|_k&=\overline{T}^j\Big|_n\frac{\partial u^i}{\partial \overline{u}^j}\frac{\partial \overline{u}^n}{\partial u^k}\\\Rightarrow\ \overline{T}^j\Big|_n&=T^i\Big|_k\frac{\partial \overline{u}^j}{\partial u^i}\frac{\partial u^k}{\partial \overline{u}^n}\quad (q.e.d.)\end{aligned}$$

2.5.9.2 Covariant First-Order Tensors with Components T_i

The covariant first-order tensor (vector) **T** can be written in the contravariant basis.

$$\mathbf{T}=T_i\mathbf{g}^i \tag{2.203}$$

Using Eq. (2.189), the partial derivative of the tensor **T** results in

$$\begin{aligned}\mathbf{T}_{,j}&=\left(T_i\mathbf{g}^i\right)_{,j}=T_{i,j}\mathbf{g}^i+T_i\mathbf{g}^i_{,j}\\&=T_{i,j}\mathbf{g}^i-T_i\Gamma^i_{jk}\mathbf{g}^k\end{aligned} \tag{2.204}$$

The partial derivative of the covariant tensor component T_i with respect to u^j in Eq. (2.204) can be defined as

$$T_{i,j}\equiv\frac{\partial T_i}{\partial u^j} \tag{2.205}$$

Interchanging i with k in the second term on the RHS of Eq. (2.204), one obtains

$$T_i \Gamma^i_{jk} \mathbf{g}^k \equiv T_k \Gamma^k_{ji} \mathbf{g}^i = T_k \Gamma^k_{ij} \mathbf{g}^i \tag{2.206}$$

Substituting Eq. (2.206) into Eq. (2.204), one obtains the derivative of the first-order tensor component (vector) **T** with respect to u^j.

$$\begin{aligned} \mathbf{T}_{,j} &= T_{i,j}\mathbf{g}^i - T_k \Gamma^k_{ij} \mathbf{g}^i \\ &= \left(T_{i,j} - \Gamma^k_{ij} T_k\right) \mathbf{g}^i \\ &\equiv T_i\big|_j \mathbf{g}^i \end{aligned} \tag{2.207}$$

Therefore, the covariant derivative of the tensor component T_i with respect to u^j can be defined as

$$T_i\big|_j = T_{i,j} - \Gamma^k_{ij} T_k = \mathbf{T}_{,j} \cdot \mathbf{g}_i \tag{2.208}$$

The covariant derivative of the first-order tensor component T_i is transformed in the new barred coordinates $\{\bar{u}^i\}$, similarly to Eq. (2.202a), cf. [4, 7].

$$\begin{aligned} & T_k|_l = \overline{T}_i\big|_j \frac{\partial \bar{u}^i}{\partial u^k} \frac{\partial \bar{u}^j}{\partial u^l} \\ \Rightarrow \quad & \overline{T}_i\big|_j = T_k|_l \frac{\partial u^k}{\partial \bar{u}^i} \frac{\partial u^l}{\partial \bar{u}^j} \end{aligned} \tag{2.209}$$

2.5.9.3 Second-Order Tensors

Second-order tensors can be written in different expressions with covariant and contravariant bases.

$$\mathbf{T} = T_{ij}\mathbf{g}^i\mathbf{g}^j = T^{ij}\mathbf{g}_i\mathbf{g}_j = T_i^{\ j}\mathbf{g}^i\mathbf{g}_j = T^i_{\ j}\mathbf{g}_i\mathbf{g}^j \tag{2.210}$$

Similarly, the covariant derivatives with respect to u^k of the second-order tensor components of **T** can be calculated as, cf. [3, 4, 7]

$$\begin{aligned} T_{ij}|_k &= T_{ij,k} - \Gamma^m_{ik} T_{mj} - \Gamma^m_{jk} T_{im} \\ T^{ij}|_k &= T^{ij}_{\ ,k} + \Gamma^i_{km} T^{mj} + \Gamma^j_{km} T^{im} \\ T^i_{\ j}|_k &= T^i_{\ j,k} + \Gamma^i_{km} T^m_{\ j} - \Gamma^m_{jk} T^i_{\ m} \\ T_i^{\ j}|_k &= T_{i,k}^{\ j} - \Gamma^m_{ik} T_m^{\ j} + \Gamma^j_{km} T_i^{\ m} \end{aligned} \tag{2.211a}$$

where $T_{ij,k}$, $T^{ij}_{\ ,k}$, and $T^i_{\ j,k}$ are the partial derivatives with respect to u^k of the covariant, contravariant, and mixed tensor components. Note that they are different to the covariant derivatives of the tensor components, as defined in Eq. (2.211a).

In the coordinate transformation from the curvilinear coordinates $\{u^i\}$ to the new barred curvilinear coordinates $\{\bar{u}^\alpha\}$, the covariant derivative of the covariant second-order tensor with respect to $\bar{u}^\gamma$ can be calculated using the chain rule of differentiation, similarly to Eq. (2.202a), cf. [4, 7].

$$\overline{T}_{\alpha\beta}\big|_\gamma = T_{ij}\big|_k \frac{\partial u^i}{\partial \overline{u}^\alpha} \frac{\partial u^j}{\partial \overline{u}^\beta} \frac{\partial u^k}{\partial \overline{u}^\gamma} \tag{2.211b}$$

where the partial derivatives $\bar{u}^\alpha_{,i}$ are called the shift tensor between two coordinate systems. This relation in Eq. (2.211b) is the chain rule of the covariant derivatives of the second-order tensors in the coordinate transformation.

Analogously, the covariant derivatives of the second-order tensors of different types in the new barred curvilinear coordinates are calculated using the shift tensors.

$$\begin{aligned}
\overline{T}_{\alpha\beta}\big|_\gamma &= T_{ij}\big|_k \frac{\partial u^i}{\partial \overline{u}^\alpha} \frac{\partial u^j}{\partial \overline{u}^\beta} \frac{\partial u^k}{\partial \overline{u}^\gamma}\ ; \\
\overline{T}^{\alpha\beta}\Big|_\gamma &= T^{ij}\big|_k \frac{\partial \overline{u}^\alpha}{\partial u^i} \frac{\partial \overline{u}^\beta}{\partial u^j} \frac{\partial u^k}{\partial \overline{u}^\gamma}; \\
\overline{T}^\alpha_\beta\Big|_\gamma &= T^i_j\Big|_k \frac{\partial \overline{u}^\alpha}{\partial u^i} \frac{\partial u^j}{\partial \overline{u}^\beta} \frac{\partial u^k}{\partial \overline{u}^\gamma}.
\end{aligned} \tag{2.211c}$$

2.5.10 Riemann-Christoffel Tensor

The Riemann-Christoffel tensor is closely related to the Gaussian curvature of the surface in differential geometry that will be discussed in Chap. 3.

At first, let us look into the second covariant derivative of an arbitrary first-order tensor. The covariant derivative of the tensor with respect to u^j has been derived in Eq. (2.208).

$$T_i\big|_j = T_{i,j} - \Gamma^k_{ij} T_k \tag{2.212}$$

Obviously, the covariant derivative $T_i\big|_j$ is a second-order tensor component.

Differentiating $T_i\big|_j$ with respect to u^k, the covariant derivative of the second-order tensor (component) $T_i\big|_j$ is the second covariant derivative of an arbitrary first-order tensor (component) T_i. This second covariant derivative has been given from Eq. (2.211a) [3].

$$
\begin{aligned}
T_i\big|_{jk} &\equiv \left(T_i\big|_j\right)\big|_k \\
&= \left(T_i\big|_j\right)_{,k} - \Gamma^m_{ik} T_m\big|_j - \Gamma^m_{jk} T_i\big|_m \\
&= T_i\big|_{j,\;k} - \Gamma^m_{ik} T_m\big|_j - \Gamma^m_{jk} T_i\big|_m
\end{aligned}
\tag{2.213}
$$

Equation (2.212) delivers the relations of

$$
T_i\big|_{j,k} = T_{i,jk} - \left(\Gamma^m_{ij,k} T_m + \Gamma^m_{ij} T_{m,k}\right) \tag{2.214a}
$$

$$
\Gamma^m_{ik} T_m\big|_j = \Gamma^m_{ik}\left(T_{m,j} - \Gamma^n_{mj} T_n\right) \tag{2.214b}
$$

$$
\Gamma^m_{jk} T_i\big|_m = \Gamma^m_{jk}\left(T_{i,m} - \Gamma^n_{im} T_n\right) \tag{2.214c}
$$

Inserting Eqs. (2.214a)–(2.214c) into Eq. (2.213), one obtains the second covariant derivative of T_i.

$$
\begin{aligned}
T_i\big|_{jk} &= T_i\big|_{j,\;k} - \Gamma^m_{ik} T_m\big|_j - \Gamma^m_{jk} T_i\big|_m \\
&= T_{i,jk} - \left(\Gamma^m_{ij,k} T_m + \Gamma^m_{ij} T_{m,k}\right) \\
&\quad - \Gamma^m_{ik}\left(T_{m,j} - \Gamma^n_{mj} T_n\right) - \Gamma^m_{jk}\left(T_{i,m} - \Gamma^n_{im} T_n\right) \\
&= T_{i,jk} - \Gamma^m_{ij,k} T_m - \Gamma^m_{ij} T_{m,k} \\
&\quad - \Gamma^m_{ik} T_{m,j} + \Gamma^m_{ik}\Gamma^n_{mj} T_n - \Gamma^m_{jk} T_{i,m} + \Gamma^m_{jk}\Gamma^n_{im} T_n
\end{aligned}
\tag{2.215}
$$

where the second partial derivative of T_i is symmetric with respect to j and k:

$$
T_{i,jk} \equiv \frac{\partial^2 T_i}{\partial u^j \partial u^k} = \frac{\partial^2 T_i}{\partial u^k \partial u^j} \equiv T_{i,kj} \tag{2.216}
$$

Interchanging the indices j with k in Eq. (2.215), one obtains

$$
\begin{aligned}
T_i\big|_{kj} &= T_{i,kj} - \Gamma^m_{ik,j} T_m - \Gamma^m_{ik} T_{m,j} \\
&\quad - \Gamma^m_{ij} T_{m,k} + \Gamma^m_{ij}\Gamma^n_{mk} T_n - \Gamma^m_{kj} T_{i,m} + \Gamma^m_{kj}\Gamma^n_{im} T_n
\end{aligned}
\tag{2.217}
$$

Using the symmetry properties given in Eqs. (2.164) and (2.216), Eq. (2.217) can be rewritten as

$$
\begin{aligned}
T_i\big|_{kj} &= T_{i,jk} - \Gamma^m_{ik,j} T_m - \Gamma^m_{ik} T_{m,j} \\
&\quad - \Gamma^m_{ij} T_{m,k} + \Gamma^m_{ij}\Gamma^n_{mk} T_n - \Gamma^m_{jk} T_{i,m} + \Gamma^m_{jk}\Gamma^n_{im} T_n
\end{aligned}
\tag{2.218}
$$

In a flat space, the second covariant derivatives in Eqs. (2.215) and (2.218) are identical. However, they are not equal in a curved space because of its surface curvature. The difference of both second covariant derivatives is proportional to the curvature tensor. Subtracting Eq. (2.215) from Eq. (2.218), the curvature tensor results in

$$\begin{aligned} T_i\big|_{jk} - T_i\big|_{kj} &= \left(\Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik}\Gamma^n_{mj} - \Gamma^m_{ij}\Gamma^n_{mk}\right)T_n \\ &\equiv R^n_{ijk}T_n \end{aligned} \tag{2.219}$$

The Riemann-Christoffel tensor (Riemann curvature tensor) can be expressed as

$$R^n_{ijk} \equiv \Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik}\Gamma^n_{mj} - \Gamma^m_{ij}\Gamma^n_{mk} \tag{2.220}$$

Straightforwardly, the Riemann-Christoffel tensor is a fourth-order tensor with respect to the indices of i, j, k, and n. They contain 81 ($=3^4$) components in a three-dimensional space.

In Eq. (2.220), the partial derivatives of the Christoffel symbols are defined by

$$\Gamma^n_{ik,j} = \frac{\partial \Gamma^n_{ik}}{\partial u^j}; \quad \Gamma^n_{ij,k} = \frac{\partial \Gamma^n_{ij}}{\partial u^k} \tag{2.221}$$

Furthermore, the covariant Riemann curvature tensor of fourth order is defined by the Riemann-Christoffel tensor and covariant metric coefficients.

$$R_{lijk} = g_{ln}R^n_{ijk} \Leftrightarrow R^n_{ijk} = g^{ln}R_{lijk} \tag{2.222}$$

The Riemann curvature tensor has four following properties using the relation given in Eq. (2.222) [4]:

- First skew-symmetry with respect to l and i:

$$R_{lijk} = -R_{iljk} \tag{2.223}$$

- Second skew-symmetry with respect to j and k:

$$\begin{aligned} R_{lijk} &= -R_{likj}; \\ R^n_{ijk} &= -R^n_{ikj} \end{aligned} \tag{2.224}$$

- Block symmetry with respect to two pairs (l, i) and (j, k):

$$R_{lijk} = R_{jkli} \tag{2.225}$$

- Cyclic property in i, j, k:

$$\begin{aligned} R_{lijk} + R_{ljki} + R_{lkij} = 0; \\ R^n_{ijk} + R^n_{jki} + R^n_{kij} = 0 \end{aligned} \tag{2.226}$$

Equation (2.226) is called the Bianchi first identity.

Resulting from these properties, there are six components of R_{lijk} in the three-dimensional space as follows [3]:

$$R_{lijk} = R_{3131},\ \ R_{3132},\ \ R_{3232},\ \ R_{1212},\ \ R_{3112},\ \ R_{3212} \tag{2.227}$$

In Cartesian coordinates, all Christoffel symbols equal zero according to Eq. (2.184). Therefore, the Riemann-Christoffel tensor, as given in Eq. (2.220) must be equal to zero.

$$\begin{aligned} R^n_{ijk} &= \Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik}\Gamma^n_{mj} - \Gamma^m_{ij}\Gamma^n_{mk} \\ &= 0 \end{aligned} \tag{2.228}$$

Thus, the Riemann surface curvature tensor can be written as

$$R_{lijk} = g_{ln}R^n_{ijk} = 0 \tag{2.229}$$

In this case, Euclidean N-space with orthonormal Cartesian coordinates is considered as a flat space because the Riemann curvature tensor there equals zero.

In the following section, the Riemann curvature tensor can be calculated from the Christoffel symbols of first and second kinds.

From Eq. (2.222) the Riemann curvature tensor can be rewritten as

$$R_{hijk} = g_{hl}R^l_{ijk}$$

Using Eq. (2.220), the Riemann curvature tensor results in

$$\begin{aligned} R_{hijk} &= g_{hn}R^n_{ijk} \\ &= g_{hn}\left(\Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik}\Gamma^n_{mj} - \Gamma^m_{ij}\Gamma^n_{mk}\right) \\ &= \frac{\partial\left(g_{hn}\Gamma^n_{ik}\right)}{\partial u^j} - \Gamma^n_{ik}g_{hn,j} - \frac{\partial\left(g_{hn}\Gamma^n_{ij}\right)}{\partial u^k} + \Gamma^n_{ij}g_{hn,k} + g_{hn}\Gamma^m_{ik}\Gamma^n_{mj} - g_{hn}\Gamma^m_{ij}\Gamma^n_{mk} \\ &= \frac{\partial\left(g_{hn}\Gamma^n_{ik}\right)}{\partial u^j} - \Gamma^n_{ik}g_{hn,j} - \frac{\partial\left(g_{hn}\Gamma^n_{ij}\right)}{\partial u^k} + \Gamma^n_{ij}g_{hn,k} + \Gamma^m_{ik}\Gamma_{mjh} - \Gamma^m_{ij}\Gamma_{mkh} \end{aligned} \tag{2.230a}$$

Changing the index m into n in both last terms on the RHS of Eq. (2.230a), one obtains

$$R_{hijk} = \frac{\partial\left(g_{hn}\Gamma^n_{ik}\right)}{\partial u^j} - \Gamma^n_{ik} g_{hn,j} - \frac{\partial\left(g_{hn}\Gamma^n_{ij}\right)}{\partial u^k} + \Gamma^n_{ij} g_{hn,k} + \Gamma^n_{ik}\Gamma_{njh} - \Gamma^n_{ij}\Gamma_{nkh}$$
$$= \frac{\partial\left(g_{hn}\Gamma^n_{ik}\right)}{\partial u^j} - \frac{\partial\left(g_{hn}\Gamma^n_{ij}\right)}{\partial u^k} - \Gamma^n_{ik}\left(g_{hn,j} - \Gamma_{njh}\right) + \Gamma^n_{ij}\left(g_{hn,k} - \Gamma_{nkh}\right) \tag{2.230b}$$

Using Eq. (2.192) for the first-kind Christoffel symbols in Eq. (2.230b), the Riemann curvature tensor becomes

$$\begin{aligned} R_{hijk} = g_{hn}R^n_{ijk} &= \frac{\partial \Gamma_{ikh}}{\partial u^j} - \frac{\partial \Gamma_{ijh}}{\partial u^k} \\ &-\Gamma^n_{ik}\left(\Gamma_{hjn} + \underline{\Gamma_{njh}} - \underline{\Gamma_{njh}}\right) + \Gamma^n_{ij}\left(\Gamma_{hkn} + \underline{\Gamma_{nkh}} - \underline{\Gamma_{nkh}}\right) \\ &= \Gamma_{ikh,j} - \Gamma_{ijh,k} - \Gamma^n_{ik}\Gamma_{hjn} + \Gamma^n_{ij}\Gamma_{hkn} \end{aligned} \tag{2.230c}$$

2.5.11 Ricci's Lemma

The covariant derivative of the metric covariant coefficient g_{ij} with respect to u^k results from Eq. (2.211a) changing T_{ij} into g_{ij}. Then, using Eq. (2.193), one obtains

$$\begin{aligned} g_{ij}|_k &= \frac{\partial g_{ij}}{\partial u^k} - \left(g_{mj}\Gamma^m_{ik} + g_{in}\Gamma^n_{jk}\right) \\ &= g_{ij,k} - g_{ij,k} = 0 \;\rightarrow\; (q.e.d.) \end{aligned} \tag{2.231a}$$

Therefore,

$$g_{ij,k} \equiv \frac{\partial g_{ij}}{\partial u^k} = g_{mj}\Gamma^m_{ik} + g_{in}\Gamma^n_{jk} \tag{2.231b}$$

The Kronecker delta is the product of the covariant and contravariant metric coefficients:

$$\delta^j_l = g_{li}g^{ij}$$

The partial derivative with respect to u^k of the Kronecker delta (invariant) equals zero and can be written as

$$\begin{aligned}\delta^j_{l,k} &= (g_{li}g^{ij})_{,k} = \frac{\partial g_{li}}{\partial u^k}g^{ij} + g_{li}\frac{\partial g^{ij}}{\partial u^k} \\ &= g_{li,k}g^{ij} + g_{li}g^{ij}_{,k} \\ &= 0\end{aligned} \tag{2.232a}$$

Multiplying Eq. (2.232a) by g^{lm}, one obtains

$$\begin{aligned}& g^{ij}g^{lm}g_{li,k} + \delta^m_i g^{ij}_{,k} = 0 \\ & \Rightarrow g^{mj}_{,k} = -g^{ij}g^{lm}g_{li,k}\end{aligned} \tag{2.232b}$$

Interchanging the indices, it results in

$$\begin{aligned}& g^{mj}_{,k} = -g^{ij}g^{lm}g_{li,k} \\ & \Rightarrow g^{ij}_{,k} = -g^{mi}g^{nj}g_{mn,k}\end{aligned} \tag{2.232c}$$

Using Eq. (2.211a), the covariant derivative of the metric contravariant coefficient g^{ij} with respect to u^k can be written as

$$\begin{aligned}g^{ij}|_k &= \frac{\partial g^{ij}}{\partial u^k} + g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km} \\ &= g^{ij}_{,k} + g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km}\end{aligned} \tag{2.233a}$$

Substituting Eqs. (2.231b), (2.232c) and (2.233a), one obtains after interchanging the indices

$$\begin{aligned}g^{ij}|_k &= g^{ij}_{,k} + \left(g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km}\right) \\ &= -g^{mi}g^{nj}g_{mn,k} + \left(g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km}\right) \\ &= -g^{mi}\Gamma^j_{mk} - g^{nj}\Gamma^i_{nk} + \left(g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km}\right) \\ &= -g^{im}\Gamma^j_{km} - g^{mj}\Gamma^i_{km} + \left(g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km}\right) \\ &= 0 \ \ (q.e.d.)\end{aligned} \tag{2.233b}$$

Note that Eqs. (2.231a) and (2.233b) are known as *Ricci's lemma*.

2.5.12 Derivative of the Jacobian

In the following section, the derivative of the Jacobian J that is always positive in a right-handed-rule coordinate system can be calculated and its result is very useful in the Nabla operator (cf. [4, 7]).

The determinant of the metric coefficient tensor is given from Eq. (2.17):

$$\det\left(g_{ij}\right) = \begin{vmatrix} g_{11} & g_{12} & \cdot & g_{1N} \\ g_{21} & g_{22} & \cdot & g_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ g_{N1} & g_{N2} & \cdot & g_{NN} \end{vmatrix} = g = J^2 > 0 \tag{2.234}$$

The contravariant metric coefficient g^{ij} results from the cofactor G^{ij} of the covariant metric coefficient g_{ij} und the determinant g.

$$g^{ij} = \frac{G^{ij}}{g} \Rightarrow G^{ij} = gg^{ij} \tag{2.235}$$

Differentiating both sides of Eq. (2.234) with respect to u^k, one obtains

$$\frac{\partial g}{\partial u^k} = \frac{\partial g_{ij}}{\partial u^k} G^{ij} \text{ for } i,j = 1, 2, \ldots, N \tag{2.236}$$

Prove Eq. (2.236):

$$\begin{aligned} \frac{\partial g}{\partial u^k} &= \begin{vmatrix} \frac{\partial g_{11}}{\partial u^k} & \frac{\partial g_{12}}{\partial u^k} & \cdot & \frac{\partial g_{1N}}{\partial u^k} \\ g_{21} & g_{22} & \cdot & g_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ g_{N1} & g_{N2} & \cdot & g_{NN} \end{vmatrix} + \ldots + \begin{vmatrix} g_{11} & g_{12} & \cdot & g_{1N} \\ g_{21} & g_{22} & \cdot & g_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial g_{N1}}{\partial u^k} & \frac{\partial g_{N2}}{\partial u^k} & \cdot & \frac{\partial g_{NN}}{\partial u^k} \end{vmatrix} \\ &= \frac{\partial g_{11}}{\partial u^k} G^{11} + \frac{\partial g_{12}}{\partial u^k} G^{12} + \ldots + \ldots + \frac{\partial g_{NN}}{\partial u^k} G^{NN} \\ &= \frac{\partial g_{ij}}{\partial u^k} G^{ij} \text{ for } i,j = 1, 2, \ldots, N \ \ (q.e.d.) \end{aligned} \tag{2.237}$$

Substituting Eq. (2.235) into Eq. (2.236), it gives

$$\begin{aligned} \frac{\partial g}{\partial u^k} &= \frac{\partial g_{ij}}{\partial u^k} G^{ij} \\ &= \frac{\partial g_{ij}}{\partial u^k} gg^{ij} \\ &\equiv g_{ij,k} gg^{ij} \end{aligned} \tag{2.238}$$

Inserting Eq. (2.231b) into Eq. (2.238), one obtains

$$\begin{aligned}\frac{\partial g}{\partial u^k} &= g_{ij,k} g g^{ij} \\ &= \left(g_{mj} \Gamma_{ik}^{m} + g_{in} \Gamma_{jk}^{n} \right) g g^{ij} \\ &= g \left(\delta_m^i \Gamma_{ik}^{m} + \delta_n^j \Gamma_{jk}^{n} \right) \\ &= g \left(\Gamma_{ik}^{i} + \Gamma_{jk}^{j} \right) = 2g \Gamma_{ik}^{i}\end{aligned} \tag{2.239}$$

Using the chain rule of differentiation, the Christoffel symbol in Eq. (2.239) can be expressed in the Jacobian $J > 0$.

$$\begin{aligned}\Gamma_{ik}^{i} &= \frac{1}{2g} \frac{\partial g}{\partial u^k} = \frac{\partial \left(\ln \sqrt{g} \right)}{\partial u^k} \\ &= \frac{\partial (\ln J)}{\partial u^k} = \frac{1}{J} \frac{\partial J}{\partial u^k}\end{aligned} \tag{2.240}$$

Prove that $R_{ijk}^{i} = 0$

Using Eq. (2.220) for $n = i$, the Riemann-Christoffel tensor can be written as

$$R_{ijk}^{i} = \Gamma_{ik,j}^{i} - \Gamma_{ij,k}^{i} + \Gamma_{ik}^{m} \Gamma_{mj}^{i} - \Gamma_{ij}^{m} \Gamma_{mk}^{i}$$

Interchanging j with k in the last term on the RHS of the equation, one obtains

$$\begin{aligned}R_{ijk}^{i} &= \Gamma_{ik,j}^{i} - \Gamma_{ij,k}^{i} + \Gamma_{ik}^{m} \Gamma_{mj}^{i} - \Gamma_{ik}^{m} \Gamma_{mj}^{i} \\ &= \Gamma_{ik,j}^{i} - \Gamma_{ij,k}^{i}\end{aligned}$$

Using Eq. (2.240), the above Riemann-Christoffel tensor can be rewritten as

$$\begin{aligned}R_{ijk}^{i} &= \Gamma_{ik,j}^{i} - \Gamma_{ij,k}^{i} \\ &= \frac{\partial^2 \left(\ln \sqrt{g} \right)}{\partial u^j \partial u^k} - \frac{\partial^2 \left(\ln \sqrt{g} \right)}{\partial u^k \partial u^j} \\ &= 0 \quad (q.e.d.)\end{aligned} \tag{2.241}$$

2.5.13 Ricci Tensor

Both Ricci and Einstein tensors are very useful mathematical tools in the relativity theory. Note that tensors using in the relativity fields have been mostly written in the abstract index notation defined by Penrose [8]. This index notation uses the indices to express the tensor types, rather than their covariant components in the basis $\{\mathbf{g}^i\}$. The first-kind Ricci tensor results from the index contraction of k and n for $n = k$ of the Riemann-Christoffel tensor, as given in Eq. (2.220).

$$\begin{aligned} R_{ij} &\equiv R^k_{ijk} \\ &= \frac{\partial \Gamma^k_{ik}}{\partial u^j} - \frac{\partial \Gamma^k_{ij}}{\partial u^k} - \Gamma^r_{ij}\Gamma^k_{rk} + \Gamma^r_{ik}\Gamma^k_{rj} \end{aligned} \tag{2.242}$$

The second-kind Ricci tensor can be defined as

$$\begin{aligned} R^i_j &\equiv g^{ik} R_{kj} \\ &= g^{ik}\left(\frac{\partial \Gamma^m_{km}}{\partial u^j} - \frac{\partial \Gamma^m_{kj}}{\partial u^m} - \Gamma^m_{kj}\Gamma^n_{mn} + \Gamma^m_{kn}\Gamma^n_{mj}\right) \end{aligned} \tag{2.243}$$

Using the Christoffel symbol in Eq. (2.240)

$$\begin{aligned} \Gamma^j_{ij} &= \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial u^i} = \frac{\partial\left(\ln\sqrt{g}\right)}{\partial u^i} \\ &= \frac{1}{J}\frac{\partial J}{\partial u^i} = \frac{\partial(\ln J)}{\partial u^i}, \end{aligned} \tag{2.244}$$

the first-kind Ricci tensor can be rewritten as

$$\begin{aligned} R_{ij} &= \frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^i \partial u^j} - \frac{\partial \Gamma^k_{ij}}{\partial u^k} - \Gamma^k_{ij}\frac{\partial\left(\ln\sqrt{g}\right)}{\partial u^k} + \Gamma^r_{ik}\Gamma^k_{rj} \\ &= \frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^i \partial u^j} - \left(\frac{\partial \Gamma^k_{ij}}{\partial u^k} + \Gamma^k_{ij}\frac{\partial\left(\ln\sqrt{g}\right)}{\partial u^k}\right) + \Gamma^r_{ik}\Gamma^k_{rj} \\ &= \frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^i \partial u^j} - \frac{1}{\sqrt{g}}\left(\sqrt{g}\frac{\partial \Gamma^k_{ij}}{\partial u^k} + \Gamma^k_{ij}\frac{\partial\sqrt{g}}{\partial u^k}\right) + \Gamma^r_{ik}\Gamma^k_{rj} \\ &= \frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^i \partial u^j} - \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\Gamma^k_{ij}\right)}{\partial u^k} + \Gamma^r_{ik}\Gamma^k_{rj} \end{aligned} \tag{2.245}$$

Interchanging i with j in Eq. (2.245), one obtains

$$\begin{aligned} R_{ji} &= \frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^j \partial u^i} - \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\Gamma^k_{ji}\right)}{\partial u^k} + \Gamma^r_{jk}\Gamma^k_{ri} \\ &= \frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^i \partial u^j} - \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\Gamma^k_{ij}\right)}{\partial u^k} + \Gamma^k_{ir}\Gamma^r_{kj} \\ &= R_{ij} \end{aligned} \tag{2.246}$$

This result states that the first-kind Ricci tensor is symmetric with respect to i and j.

Substituting Eq. (2.245) with $k = m$, $r = n$, and $i = k$ into Eq. (2.243), the second-kind Ricci tensor results in

$$
\begin{aligned}
R_j^i &= g^{ik} R_{kj} \\
&= g^{ik}\left(\frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^k \partial u^j} - \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\Gamma_{kj}^m\right)}{\partial u^m} + \Gamma_{km}^n \Gamma_{nj}^m\right)
\end{aligned}
\tag{2.247}
$$

The Ricci curvature R can be defined as

$$R \equiv R_i^i = g^{ij} R_{ij} \tag{2.248}$$

Substituting Eq. (2.245) into Eq. (2.248), the Ricci curvature results in

$$R = g^{ij}\left(\frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^i \partial u^j} - \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\Gamma_{ij}^k\right)}{\partial u^k} + \Gamma_{ik}^r \Gamma_{rj}^k\right) \tag{2.249}$$

2.5.14 Einstein Tensor

The Einstein tensor is defined by the second-kind Ricci tensor, Kronecker delta, and the Ricci curvature.

$$G_j^i \equiv R_j^i - \tfrac{1}{2}\delta_j^i R \tag{2.250a}$$

The Einstein tensor is a mixed second-order tensor and can be written as

$$G_j^i = g^{ik} G_{kj} \tag{2.250b}$$

Using the tensor contraction rules, the covariant Einstein tensor results in

$$
\begin{aligned}
G_{ij} = g_{ik} G_j^k &= g_{ik}\left(R_j^k - \frac{1}{2}\delta_j^k R\right) \\
&= R_{ij} - \frac{1}{2} g_{ij} R \\
&= R_{ji} - \frac{1}{2} g_{ji} R \\
&= G_{ji}
\end{aligned}
\tag{2.251}
$$

This result proves that the covariant Einstein tensor is symmetric due to the symmetry of the Ricci tensor.

The Bianchi first identity in Eq. (2.226) gives

$$R_{lijk} + R_{ljki} + R_{lkij} = 0;\\ R^n_{ijk} + R^n_{jki} + R^n_{kij} = 0 \tag{2.252}$$

Differentiating covariantly Eq. (2.252) with respect to u^m, u^k, and u^l and then multiplying it by the covariant metric coefficients g_{in}, one obtains the *Bianchi second identity*, cf. [4, 7, 9, 10].

$$\begin{aligned}\left(R^n_{jkl}\Big|_m + R^n_{jlm}\Big|_k + R^n_{jmk}\Big|_l\right)\cdot g_{in} &= 0 \cdot g_{in}\\ \Rightarrow R_{ijkl}\big|_m + R_{ijlm}\big|_k + R_{ijmk}\big|_l &= 0\end{aligned} \tag{2.253}$$

Due to skew-symmetry of the covariant Riemann curvature tensors, as discussed in Eqs. (2.223) and (2.224), Eq. (2.253) can be rewritten as

$$R_{ijkl}\big|_m - R_{ijml}\big|_k - R_{jimk}\big|_l = 0 \tag{2.254}$$

Multiplying Eq. (2.254) by $g^{il}g^{jk}$ and using the tensor contraction rules (cf. Sect. 2.3.5), one obtains

$$\begin{aligned}&R_{ijkl}\big|_m - R_{ijml}\big|_k - R_{jimk}\big|_l = 0 \quad \Leftrightarrow\\ &g^{il}g^{jk}R_{ijkl}\big|_m - g^{il}g^{jk}R_{ijml}\big|_k - g^{il}g^{jk}R_{jimk}\big|_l\\ &= g^{jk}R_{jk}\big|_m - g^{jk}R_{jm}\big|_k - g^{il}R_{im}\big|_l\\ &= R\big|_m - R^k_m\big|_k - R^l_m\big|_l \quad (l \to k)\\ &= R\big|_m - 2R^k_m\big|_k\\ &= 0\end{aligned}$$

Thus,

$$R^k_m\big|_k = \frac{1}{2}R\big|_m \tag{2.255}$$

Using Eq. (2.232a) and the symmetry of the Christoffel symbols, the covariant derivative of the Kronecker delta with respect to u^k is equal to zero.

$$\begin{aligned}\delta^i_j\Big|_k &= \delta^i_{j,k} + \Gamma^i_{km}\delta^m_j - \Gamma^m_{jk}\delta^i_m\\ &= \delta^i_{j,k} + \left(\Gamma^i_{kj} - \Gamma^i_{jk}\right)\\ &= 0\end{aligned} \tag{2.256}$$

Differentiating covariantly the Einstein tensor in Eq. (2.250a) with respect to u^k and using Eq. (2.256), one obtains the covariant derivative

$$G^i_j\Big|_k = \left(R^i_j - \tfrac{1}{2}\delta^i_j R\right)\Big|_k = R^i_j\Big|_k - \frac{1}{2}\left(\delta^i_j\Big|_k R + \delta^i_j R\Big|_k\right) = R^i_j\Big|_k - \frac{1}{2}\delta^i_j R\Big|_k \tag{2.257}$$

Changing the index i into k in Eq. (2.257) and using Eq. (2.255), the divergence of the Einstein tensor equals zero.

$$G^k_j\Big|_k = R^k_j\Big|_k - \frac{1}{2}\,\delta^k_j R\Big|_k = R^k_j\Big|_k - \tfrac{1}{2}R|_j = 0$$
$$\Rightarrow Div\,\mathbf{G} \equiv \nabla\ \cdot \mathbf{G} = G^k_j\Big|_k \mathbf{g}^j = \mathbf{0} \quad (q.e.d.) \tag{2.258}$$

This result is very important and has been often used in the general relativity theories and other relativity fields.

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Chapter 3
Elementary Differential Geometry

3.1 Introduction

We consider an N-dimensional Riemannian manifold **M** and let $\mathbf{g}_i$ be a basis at the point $P_i(u^1,\ldots,u^N)$ and $\mathbf{g}_j$ be another basis at the other point $P_j(u^1,\ldots,u^N)$. Note that each such basis may only exist in a local neighborhood of the respective points, and not necessarily for the whole space. For each such point we may construct an embedded affine tangential manifold. The N-tuple of coordinates are invariant in any chosen basis; however, its components on the coordinates change as the coordinate system varies. Therefore, the relating components have to be taken into account by the coordinate transformations.

3.2 Arc Length and Surface in Curvilinear Coordinates

Consider two points $P(u^1,\ldots,u^N)$ and $Q(u^1,\ldots,u^N)$ of an N-tuple of the coordinates $(u^1,\ldots,u^N)$ in the parameterized curve $C \in \mathbf{R}^N$. The coordinates $(u^1,\ldots,u^N)$ can be assumed to be a function of the parameter λ that varies from $P(\lambda_1)$ to $Q(\lambda_2)$, as shown in Fig. 3.1.

The ***arc length ds*** between the points P and Q results from

$$\left(\frac{ds}{d\lambda}\right)^2 = \frac{d\mathbf{r}}{d\lambda} \cdot \frac{d\mathbf{r}}{d\lambda} \tag{3.1}$$

where the derivative of the vector $\mathbf{r}(u,v)$ can be calculated by

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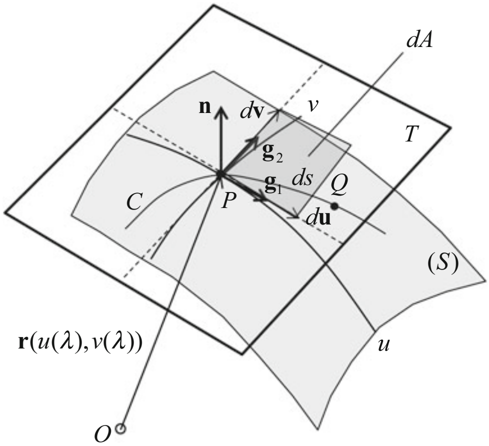

Fig. 3.1 Arc length and surface on the surface (S)

$$
\begin{aligned}
\frac{d\mathbf{r}}{d\lambda} &= \frac{d(\mathbf{g}_i u^i)}{d\lambda} = \mathbf{g}_i \left(\frac{\partial u^i}{\partial \lambda} \right) \\
&\equiv \mathbf{g}_i \dot{u}^i(\lambda); \quad \forall i = 1,\ 2
\end{aligned}
\tag{3.2}
$$

Substituting Eq. (3.2) into Eq. (3.1), one obtains the arc length PQ.

$$
ds = \sqrt{\varepsilon(\mathbf{g}_i \dot{u}) \cdot \left(\mathbf{g}_j \dot{v}\right)}\, d\lambda = \sqrt{\varepsilon\, g_{ij} \dot{u} \cdot \dot{v}}\, d\lambda \tag{3.3}
$$

where $\varepsilon(=\pm 1)$ is the functional indicator, which ensures that the square root always exists.

Therefore, the arc length of PQ is given by integrating Eq. (3.3) from the parameter λ_1 to the parameter λ_2.

$$
s = \int_{\lambda_1}^{\lambda_2} \sqrt{\varepsilon\, g_{ij}\, \dot{u}\,(\lambda) \cdot \dot{v}\,(\lambda)}\, d\lambda \tag{3.4}
$$

where the covariant metric coefficients g_{ij} are defined by

$$
\begin{aligned}
g_{ij} &= g_{ji} = \mathbf{g}_i \cdot \mathbf{g}_j \\
&= \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v}; \quad i \equiv u, \quad j \equiv \ v
\end{aligned}
\tag{3.5}
$$

Thus, the metric coefficient tensor of the parameterized surface S is given in

$$\begin{aligned}\mathbf{M} = \left(g_{ij}\right) &= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \\ &= \begin{bmatrix} r_u r_u & r_u r_v \\ r_v r_u & r_v r_v \end{bmatrix} \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix}\end{aligned} \tag{3.6}$$

The area differential of the tangent plane T at the point P can be calculated by

$$\begin{aligned} dA &= |d\mathbf{u} \times d\mathbf{v}| \\ &= |\mathbf{g}_1 du \times \mathbf{g}_2 dv| = |\mathbf{g}_1 \times \mathbf{g}_2| du dv \end{aligned} \tag{3.7}$$

Using the Lagrange's identity, Eq. (3.7) becomes

$$\begin{aligned} dA &= |\mathbf{g}_1 \times \mathbf{g}_2| du dv \\ &= \sqrt{g_{11}g_{22} - (g_{12})^2} du dv = \sqrt{\det\left(g_{ij}\right)} du dv \\ &= \sqrt{EG - F^2} du dv \end{aligned} \tag{3.8}$$

Integrating Eq. (3.8), the area of the surface S results in

$$A = \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{\lambda_2} \sqrt{EG - F^2} du dv \tag{3.9}$$

In the following section, the circumference at the equator and surface area of a sphere with a radius R are calculated (see Fig. 3.2).

The location vector of a given point $P(u(\lambda),v(\lambda))$ in the parameterized surface of the sphere (S) can be written as

$$\begin{aligned} &(S) : x^2 + y^2 + z^2 = R^2 \Rightarrow \\ &\mathbf{r}(\varphi, \theta) = \begin{pmatrix} R \sin\varphi \cos\theta \\ R \sin\varphi \sin\theta \\ R \cos\varphi \end{pmatrix}; \quad u \equiv \varphi \in [0, \pi[; \; v \equiv \theta \in [0, 2\pi[\end{aligned} \tag{3.10}$$

The covariant bases can be calculated in

$$\begin{aligned} \mathbf{g}_u &= \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{pmatrix} R \cos\varphi \cos\theta \\ R \cos\varphi \sin\theta \\ -R \sin\varphi \end{pmatrix}; \\ \mathbf{g}_v &= \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} -R \sin\varphi \sin\theta \\ R \sin\varphi \cos\theta \\ 0 \end{pmatrix} \end{aligned} \tag{3.11}$$

Thus, the metric coefficient tensor results from Eq. (3.11).

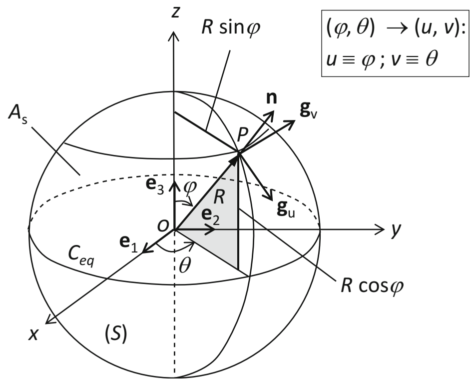

Fig. 3.2 Arc length and surface of a sphere (S)

$$\mathbf{M} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\varphi \end{bmatrix} \tag{3.12}$$

The circumference at the equator is given at $u \equiv \varphi = \pi/2$ and $v \equiv \theta(\lambda) = \lambda$.

$$C_{eq} = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{ij}\,\dot{u}(\lambda) \cdot \dot{v}(\lambda)}\, d\lambda = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{11}\dot{u}^2 + 2g_{12}\dot{u}\,\dot{v} + g_{22}\dot{v}^2}\, d\lambda \tag{3.13}$$

where $\dot{u} = 0$; $\dot{v} = 1$.

Therefore,

$$C_{eq} = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{22}}\, d\lambda = \int_0^{2\pi} R \sin\frac{\pi}{2} d\lambda = 2\pi R \tag{3.14}$$

The surface area of the sphere can be computed according to Eq. (3.9).

$$\begin{aligned} A_S &= \int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2} \sqrt{EG - F^2}\, du\, dv = \int_0^{2\pi}\int_0^{\pi} \sqrt{EG - F^2}\, d\varphi\, d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} R^2 \sin\varphi\, d\varphi\, d\theta = -\int_0^{2\pi} R^2 \cos\varphi\Big|_0^{\pi} d\theta = 4\pi R^2 \end{aligned} \tag{3.15}$$

3.3 Unit Tangent and Normal Vector to Surface

The unit tangent vectors to the parameterized curves u and v at the point P have the same direction as the covariant bases $\mathbf{g}_1$ and $\mathbf{g}_2$. Both tangent vectors generate the tangent plane T to the differentiable Riemannian surface (S) at the point P. The unit normal vector $\mathbf{n}$ is perpendicular to the tangent plane T at the point P, as shown in Fig. 3.1.

The unit tangent vector $\mathbf{t}$ is defined, as given in Eq. (B.1a).

$$\begin{aligned} &\mathbf{t}_i = \mathbf{g}_i^* \equiv \frac{\mathbf{g}_i}{\sqrt{g_{(ii)}}} = \frac{1}{\sqrt{g_{(ii)}}}\left(\frac{\partial \mathbf{r}}{\partial u^i}\right); \quad \forall i = 1,\ 2 \\ &\Rightarrow \begin{cases} \mathbf{t}_1 = \mathbf{g}_1^* = \dfrac{1}{\sqrt{g_{11}}}\left(\dfrac{\partial \mathbf{r}}{\partial u}\right); \quad (1 \equiv u) \\ \mathbf{t}_2 = \mathbf{g}_2^* = \dfrac{1}{\sqrt{g_{22}}}\left(\dfrac{\partial \mathbf{r}}{\partial v}\right); \quad (2 \equiv v) \end{cases} \end{aligned} \tag{3.16}$$

The unit normal vector is perpendicular to the unit tangent vectors at the point P and can be written as

$$\mathbf{n} = (\mathbf{t}_1 \times \mathbf{t}_2) = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{|\mathbf{g}_1 \times \mathbf{g}_2|} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|} \tag{3.17}$$

Using Eq. (3.8), the unit normal vector can be rewritten as

$$\begin{aligned} \mathbf{n} = (\mathbf{t}_1 \times \mathbf{t}_2) &= \frac{\mathbf{g}_1 \times \mathbf{g}_2}{|\mathbf{g}_1 \times \mathbf{g}_2|} \\ &= \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{\det\left(g_{ij}\right)}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{EG - F^2}} \end{aligned} \tag{3.18}$$

in which the cross product of $\mathbf{g}_1$ and $\mathbf{g}_2$ can be calculated by

$$\mathbf{g}_1 \times \mathbf{g}_2 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \left(\frac{\partial \mathbf{r}}{\partial u}\right)_1 & \left(\frac{\partial \mathbf{r}}{\partial u}\right)_2 & \left(\frac{\partial \mathbf{r}}{\partial u}\right)_3 \\ \left(\frac{\partial \mathbf{r}}{\partial v}\right)_1 & \left(\frac{\partial \mathbf{r}}{\partial v}\right)_2 & \left(\frac{\partial \mathbf{r}}{\partial v}\right)_3 \end{vmatrix} = \varepsilon_{ijk}\left(\frac{\partial \mathbf{r}}{\partial u}\right)_i \left(\frac{\partial \mathbf{r}}{\partial v}\right)_j \mathbf{e}_k \tag{3.19}$$

where ε_{ijk} is the permutation symbol given in Eq. (A.5) in Appendix A.

The unit normal vector to the differentiable spherical surface (S) at the point P in Fig. 3.2 can be computed from Eq. (3.11).

$$\mathbf{g}_1 = \left(\frac{\partial \mathbf{r}}{\partial \varphi}\right) = \begin{pmatrix} R\cos\varphi\cos\theta \\ R\cos\varphi\sin\theta \\ -R\sin\varphi \end{pmatrix};\ \mathbf{g}_2 = \left(\frac{\partial \mathbf{r}}{\partial \theta}\right) = \begin{pmatrix} -R\sin\varphi\sin\theta \\ R\sin\varphi\cos\theta \\ 0 \end{pmatrix} \tag{3.20}$$

Therefore,

$$\begin{aligned} \mathbf{g}_1 \times \mathbf{g}_2 &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ R\cos\varphi\cos\theta & R\cos\varphi\sin\theta & -R\sin\varphi \\ -R\sin\varphi\sin\theta & R\sin\varphi\cos\theta & 0 \end{vmatrix} \\ &= \begin{pmatrix} R^2\sin^2\varphi\cos\theta \\ R^2\sin^2\varphi\sin\theta \\ R^2\sin\varphi\cos\varphi \end{pmatrix} \end{aligned} \tag{3.21}$$

Thus, the unit normal vector results from Eqs. (3.12), (3.18), and (3.21).

$$\mathbf{n} = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{\sqrt{EG - F^2}} = \frac{1}{R^2\sin\varphi}\begin{pmatrix} R^2\sin^2\varphi\cos\theta \\ R^2\sin^2\varphi\sin\theta \\ R^2\sin\varphi\cos\varphi \end{pmatrix} = \begin{pmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{pmatrix} \tag{3.22}$$

Straightforwardly, the unit normal vector depends on each point $P(\varphi, \theta)$ on the spherical surface (S) and has a vector length of 1.

3.4 The First Fundamental Form

The first and second fundamental forms of surfaces are two important characteristics in differential geometry as they are used to measure arc lengths and areas of surfaces, to identify isometric surfaces, and to find the extrema of surfaces. The Gaussian and mean curvatures of surfaces are based on both fundamental forms. Initially, the first fundamental form is examined in the following section.

Figure 3.3 displays the unit tangent vector **t** to the parameterized curve C at the point P in the differentiable surface (S). The unit normal vector **n** to the surface (S) at the point P is perpendicular to **t** and (S) at the point P.

Both unit tangent and normal vectors generate a Frenet orthonormal frame $\{\mathbf{t}, \mathbf{n}, (\mathbf{n} \times \mathbf{t})\}$ in which three unit vectors are orthogonal to each other, as shown in Fig. 3.3. The curvature vector **k** to the curve C can be rewritten as a linear combination of the normal curvature vector $\mathbf{k}_n$ and the geodesic curvature vector $\mathbf{k}_g$ at the point P in the Frenet frame.

$$\begin{aligned} &\mathbf{k} = \mathbf{k}_n + \mathbf{k}_g \Leftrightarrow \kappa\mathbf{n}_C = \kappa_n\mathbf{n} + \kappa_g(\mathbf{n} \times \mathbf{t}) \\ &\Rightarrow \kappa = \sqrt{\kappa_n^2(\lambda) + \kappa_g^2(\lambda)} \end{aligned} \tag{3.23}$$

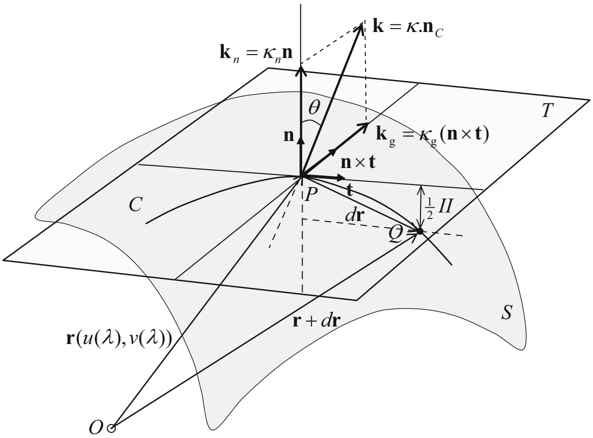

Fig. 3.3 Normal and geodesic curvatures of the surface S

where

κ is the curvature of the curve C at P;

κ_n is the normal curvature of the surface (S) at P in the direction $\mathbf{t}$;

κ_g is the geodesic curvature of the surface (S) at P.

The first fundamental form I of the surface (S) is defined by the arc length on the curve C in the surface (S).

$$I \equiv ds^2 = d\mathbf{r} \cdot d\mathbf{r} = \left(\frac{\partial \mathbf{r}}{\partial u^i} du^j\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial u^j} du^j\right) \tag{3.24}$$

The first term on the RHS of Eq. (3.24) can be rewritten in the parameterized coordinate $u^i(\lambda)$.

$$\frac{\partial \mathbf{r}}{\partial u^i} du^i = \mathbf{g}_i du^i = \mathbf{g}_i \frac{du^i(\lambda)}{d\lambda} d\lambda = \mathbf{g}_i \dot{u}^i(\lambda) d\lambda \tag{3.25}$$

in which $\mathbf{g}_i$ is the covariant basis of the curvilinear coordinate u^i, as shown in Fig. 3.1.

Inserting Eq. (3.25) into Eq. (3.24), the first fundamental form results in

$$\begin{aligned} I &= \mathbf{g}_i \cdot \mathbf{g}_j \dot{u}^i(\lambda) \dot{u}^j(\lambda) d\lambda^2 \\ &= g_{ij} du^i du^j \\ &= g_{11} du^2 + 2 g_{12} du dv + g_{22} dv^2 \end{aligned} \tag{3.26}$$

Using Eq. (3.6), the first fundamental form of the surface (S) can be rewritten as

$$I = Edu^2 + 2Fdudv + Gdv^2 \tag{3.27a}$$

Therefore, the arc length ds can be rewritten as

$$ds = \sqrt{E\dot{u}^2 + 2F\dot{u}\,\dot{v} + G\dot{v}^2}d\lambda \tag{3.27b}$$

where E, F, and G are the covariant metric coefficients of the metric tensor $\mathbf{M}$, as given in

$$\mathbf{M} = \left(g_{ij}\right) \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \mathbf{r}_u \cdot \mathbf{r}_u & \mathbf{r}_u \cdot \mathbf{r}_v \\ \mathbf{r}_v \cdot \mathbf{r}_u & \mathbf{r}_v \cdot \mathbf{r}_v \end{bmatrix} \tag{3.28}$$

3.5 The Second Fundamental Form

The second fundamental form II is defined as twice of the projection of the arc length vector $d\mathbf{r}$ on the unit normal vector $\mathbf{n}$ of the parameterized surface (S) at the point P, as demonstrated in Fig. 3.3.

$$\frac{1}{2}II \equiv d\mathbf{r} \cdot \mathbf{n} \tag{3.29}$$

Using the Taylor's series for a vectorial function with two variables u and v, the differential of the arc length vector $d\mathbf{r}(u,v)$ can be written in the second order.

$$\begin{aligned} d\mathbf{r} &= \left(\frac{\partial \mathbf{r}}{\partial u}\right)du + \left(\frac{\partial \mathbf{r}}{\partial v}\right)dv \\ &\quad + \frac{1}{2}\left(\frac{\partial^2 \mathbf{r}}{\partial u^2}du^2 + 2\frac{\partial^2 \mathbf{r}}{\partial u \partial v}dudv + \frac{\partial^2 \mathbf{r}}{\partial v^2}dv^2\right) + O\left(d\mathbf{r}^3\right) \\ &\equiv \mathbf{r}_u du + \mathbf{r}_v dv + \frac{1}{2}\left(\mathbf{r}_{uu}du^2 + 2\mathbf{r}_{uv}dudv + \mathbf{r}_{vv}dv^2\right) + O\left(d\mathbf{r}^3\right) \end{aligned} \tag{3.30}$$

Therefore, the second fundamental form can be computed as

$$II \approx 2(\mathbf{r}_u \cdot \mathbf{n}du + \mathbf{r}_v \cdot \mathbf{n}dv) + \left(\mathbf{r}_{uu} \cdot \mathbf{n}du^2 + 2\mathbf{r}_{uv} \cdot \mathbf{n}dudv + \mathbf{r}_{vv} \cdot \mathbf{n}dv^2\right) \tag{3.31}$$

Due to the orthogonality of ($\mathbf{r}_u$, $\mathbf{r}_v$) and $\mathbf{n}$, one obtains the inner products

$$\mathbf{r}_u \cdot \mathbf{n} = \mathbf{r}_v \cdot \mathbf{n} = 0 \tag{3.32}$$

where the covariant bases of the curvilinear coordinate (u,v) are shown in Fig. 3.1.

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \mathbf{g}_1; \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \mathbf{g}_2 \tag{3.33}$$

Substituting Eqs. (3.31) and (3.32), the second fundamental form results in

$$\begin{aligned} II &= \mathbf{r}_{uu} \cdot \mathbf{n} du^2 + 2\mathbf{r}_{uv} \cdot \mathbf{n} dudv + \mathbf{r}_{vv} \cdot \mathbf{n} dv^2 \\ &\equiv Ldu^2 + 2Mdudv + Ndv^2 \end{aligned} \tag{3.34}$$

in which L, M, and N are the elements of the Hessian tensor [1, 2]

$$\mathbf{H} = (h_{ij}) \equiv \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{uu} \cdot \mathbf{n} & \mathbf{r}_{uv} \cdot \mathbf{n} \\ \mathbf{r}_{uv} \cdot \mathbf{n} & \mathbf{r}_{vv} \cdot \mathbf{n} \end{bmatrix} \tag{3.35a}$$

with

$$\mathbf{n} = (\mathbf{t}_1 \times \mathbf{t}_2) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{\det(g_{ij})}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{EG - F^2}} \tag{3.35b}$$

In case of the projection equals zero, the second fundamental form II is also equal to zero. It gives the quadratic equation of du according to Eq. (3.34).

$$Ldu^2 + 2Mdudv + Ndv^2 = 0 \tag{3.36}$$

Resolving Eq. (3.36) for du, one obtains the solution

$$du = \left(\frac{-M \pm \sqrt{(M^2 - LN)}}{L} \right) dv \tag{3.37}$$

There are three cases for Eq. (3.37) with $L \neq 0$ [3, 4]:

$(M^2 - LN) > 0$: two different solutions of du.

The surface (S) cuts the tangent plane T with two lines that intersect each other at the point P (hyperbolic point);

$(M^2 - LN) = 0$: two identical solutions of du.

The surface (S) cuts the tangent plane T with one line that passes through the point P (parabolic point);

$(M^2 - LN) < 0$: no solution of du.

The surface (S) does not cut the tangent plane T except at the point P (elliptic point).

In another way, the second fundamental form II can be derived by the change rate of the differential of the arc length ds when the surface (S) moves along the unit normal vector $\mathbf{n}$ with a parameterized variable α according to [2].

The location vector of the point P can be written in the parameterized variable α.

$$\mathbf{R}_u(u, v, \alpha) = \mathbf{r}(u, v) - \alpha\mathbf{n}(u, v) \tag{3.38}$$

The second fundamental form II can be calculated at $\alpha = 0$ using Eq. (3.27a):

$$\begin{aligned} II &= ds.\frac{\partial(ds)}{\partial\alpha}|_{\alpha=0} = \frac{1}{2}\frac{\partial(ds^2)}{\partial\alpha}|_{\alpha=0} = \frac{1}{2}\frac{\partial I}{\partial\alpha}|_{\alpha=0} \\ &= \frac{1}{2}\frac{\partial}{\partial\alpha}\left(Edu^2 + 2Fdudv + Gdv^2\right)|_{\alpha=0} \\ &= \frac{1}{2}\frac{\partial E}{\partial\alpha}|_{\alpha=0}du^2 + \frac{\partial F}{\partial\alpha}|_{\alpha=0}dudv + \frac{1}{2}\frac{\partial G}{\partial\alpha}|_{\alpha=0}dv^2 \end{aligned} \tag{3.39}$$

in which the first term on the RHS of Eq. (3.39) can be calculated as

$$\begin{aligned} E &= \mathbf{R}_u(u, v, \alpha) \cdot \mathbf{R}_u(u, v, \alpha) \\ &= (\mathbf{r}_u - \alpha\,\mathbf{n}_u) \cdot (\mathbf{r}_u - \alpha\,\mathbf{n}_u) \\ &= \mathbf{n}_u^2\alpha^2 - 2\mathbf{r}_u \cdot \mathbf{n}_u\alpha + \mathbf{r}_u^2 \end{aligned} \tag{3.40}$$

Thus,

$$\frac{1}{2}\frac{\partial E}{\partial\alpha}|_{\alpha=0} = \left(\mathbf{n}_u^2\alpha - \mathbf{r}_u\mathbf{n}_u\right)|_{\alpha=0} = -\mathbf{r}_u \cdot \mathbf{n}_u \tag{3.41}$$

Further calculations deliver the second and third terms on the RHS of Eq. (3.39):

$$\frac{\partial F}{\partial\alpha}|_{\alpha=0} = -(\mathbf{r}_u \cdot \mathbf{n}_v + \mathbf{r}_v \cdot \mathbf{n}_u); \tag{3.42}$$

$$\frac{1}{2}\frac{\partial G}{\partial\alpha}|_{\alpha=0} = -\mathbf{r}_v \cdot \mathbf{n}_v \tag{3.43}$$

Using the orthogonality of $\mathbf{r}_u$ and $\mathbf{n}$, one obtains

$$\frac{\partial(\mathbf{r}_u \cdot \mathbf{n})}{\partial u} = \mathbf{r}_{uu} \cdot \mathbf{n} + \mathbf{r}_u \cdot \mathbf{n}_u = 0 \Rightarrow \mathbf{r}_{uu} \cdot \mathbf{n} = -\mathbf{r}_u \cdot \mathbf{n}_u \tag{3.44}$$

Similarly, one obtains using the orthogonality of $\mathbf{r}_u$ and $\mathbf{n}$; $\mathbf{r}_v$ and $\mathbf{n}$

$$\begin{aligned} &\frac{\partial(\mathbf{r}_u \cdot \mathbf{n})}{\partial v} = \frac{\partial(\mathbf{r}_v \cdot \mathbf{n})}{\partial u} = 0 \Rightarrow 2\mathbf{r}_{uv} \cdot \mathbf{n} = -(\mathbf{r}_u \cdot \mathbf{n}_v + \mathbf{r}_v \cdot \mathbf{n}_u); \\ &\frac{\partial(\mathbf{r}_v \cdot \mathbf{n})}{\partial v} = 0 \Rightarrow \mathbf{r}_{vv} \cdot \mathbf{n} = -\mathbf{r}_v \cdot \mathbf{n}_v \end{aligned} \tag{3.45}$$

Substituting Eqs. (3.41)–(3.45) into Eq. (3.39), the second fundamental form II can be written as

$$
\begin{aligned}
II &= \frac{1}{2}\frac{\partial E}{\partial \alpha}|_{\alpha=0}du^2 + \frac{\partial F}{\partial \alpha}|_{\alpha=0}dudv + \frac{1}{2}\frac{\partial G}{\partial \alpha}|_{\alpha=0}dv^2 \\
&= -\mathbf{r}_u \cdot \mathbf{n}_u du^2 - (\mathbf{r}_u \cdot \mathbf{n}_v + \mathbf{r}_v \cdot \mathbf{n}_u)dudv - \mathbf{r}_v \cdot \mathbf{n}_v dv^2 \\
&= \mathbf{r}_{uu} \cdot \mathbf{n} du^2 + 2\mathbf{r}_{uv} \cdot \mathbf{n} dudv + \mathbf{r}_{vv} \cdot \mathbf{n} dv^2 \\
&\equiv Ldu^2 + 2Mdudv + Ndv^2
\end{aligned}
\tag{3.46}
$$

where L, M, and N are the components of the Hessian tensor

$$
\mathbf{H} = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{uu} \cdot \mathbf{n} & \mathbf{r}_{uv} \cdot \mathbf{n} \\ \mathbf{r}_{uv} \cdot \mathbf{n} & \mathbf{r}_{vv} \cdot \mathbf{n} \end{bmatrix}
\tag{3.47}
$$

3.6 Gaussian and Mean Curvatures

The Gaussian and mean curvatures are based on the principal normal curvatures κ_1 and κ_2 in the direction $\mathbf{t}_1$ and $\mathbf{t}_2$ of the surface (S) at a given point P, as shown in Fig. 3.4. The unit tangent vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ and the unit normal vector $\mathbf{n}$ at the point P generate two principal curvature planes that are perpendicular to each other. The normal curvature κ_1 of the surface (S) in the principal direction $\mathbf{t}_1$ at the point P is defined as the maximum normal curvature in the curvature plane P_1; the normal curvature κ_2 in the principal direction $\mathbf{t}_2$ is the minimum normal curvature in the curvature plane P_2.

The maximum and minimum normal curvatures κ_1 and κ_2 of the surface (S) at the point P are the eigenvalues of the corresponding eigenvectors $\mathbf{t}_1$ and $\mathbf{t}_2$ [1, 2, 4, 5]. These eigenvalues are given from the characteristic equation that can be derived from the first and second fundamental forms in Eqs. (3.27a) and (3.46).

The Gaussian curvature of the surface (S) at the point P is defined by

$$
K = \kappa_1 \kappa_2 \tag{3.48}
$$

The mean curvature of the surface (S) at the point P is defined by

$$
H = \frac{1}{2}(\kappa_1 + \kappa_2) \tag{3.49}
$$

The covariant metric tensor related to the first fundamental form can be written as

$$
\begin{aligned}
& I = Edu^2 + 2Fdudv + Gdv^2; \\
& \mathbf{M} = \left(g_{ij}\right) \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \mathbf{r}_u \cdot \mathbf{r}_u & \mathbf{r}_u \cdot \mathbf{r}_v \\ \mathbf{r}_v \cdot \mathbf{r}_u & \mathbf{r}_v \cdot \mathbf{r}_v \end{bmatrix}
\end{aligned}
\tag{3.50}
$$

The Hessian tensor related to the second fundamental form can be written as

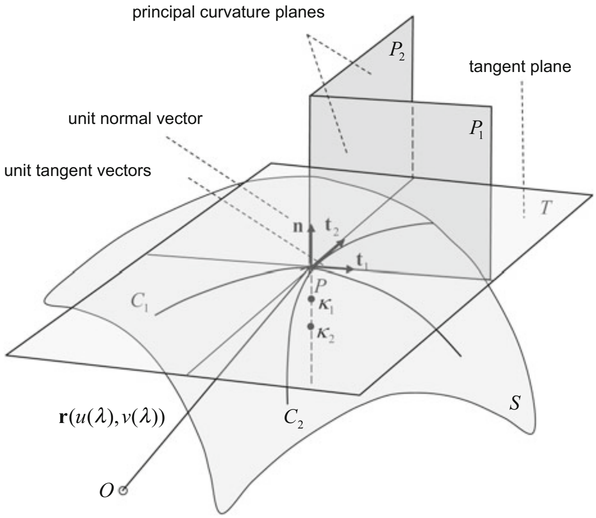

Fig. 3.4 Gaussian and mean curvatures of the surface S

$$\begin{aligned} II &= Ldu^2 + 2Mdudv + Ndv^2; \\ \mathbf{H} &= \left(h_{ij}\right) \equiv \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{uu} \cdot \mathbf{n} & \mathbf{r}_{uv} \cdot \mathbf{n} \\ \mathbf{r}_{uv} \cdot \mathbf{n} & \mathbf{r}_{vv} \cdot \mathbf{n} \end{bmatrix} \end{aligned} \tag{3.51}$$

The characteristic equation of the principal curvatures results in

$$\det(\mathbf{H} - \kappa\mathbf{M}) = \begin{vmatrix} (L - \kappa E) & (M - \kappa F) \\ (M - \kappa F) & (N - \kappa G) \end{vmatrix} = 0 \tag{3.52}$$

Therefore,

$$\begin{aligned} &(L - \kappa E).(N - \kappa G) - (M - \kappa F)^2 = 0 \Leftrightarrow \\ &\left(EG - F^2\right)\kappa^2 + (EN - 2MF + LG)\kappa + \left(LN - M^2\right) = 0 \end{aligned} \tag{3.53}$$

The Gaussian curvature results from Eq. (3.53)

$$K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2} = \frac{\det\left(h_{ij}\right)}{\det\left(g_{ij}\right)} \tag{3.54}$$

Note that the Gaussian curvature K at a point in the surface is the product of two principal curvatures at this point. According to Gauss's Theorema Egregium (remarkable theorem) in [1, 2, 4, 5], the Gaussian curvature depends only on the first fundamental form I.

Similarly, the mean curvature results in

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = -\frac{(EN - 2MF + LG)}{2(EG - F^2)} \tag{3.55}$$

The maximum and minimum principal curvatures κ_1 and κ_2 of the surface S at the point P result from Eqs. (3.54) and (3.55).

$$\begin{cases} \kappa_1 = \kappa_{\max} = H + \sqrt{H^2 - K} \\ \kappa_2 = \kappa_{\min} = H - \sqrt{H^2 - K} \end{cases} \tag{3.56}$$

In the following section, the Gaussian and mean curvatures of the rotational paraboloid surface (S) in $\mathbf{R}^3$ are computed as follows:

$$\begin{aligned} &(S): z = x^2 + y^2; \\ &h(t) = \sqrt{t};\ t > 0 \end{aligned} \tag{3.57}$$

The location vector of the curvilinear surface (S) can be written as

$$\mathbf{r}(t, h) = \begin{pmatrix} h(t)\cos\phi \\ h(t)\sin\phi \\ h^2(t) \end{pmatrix} \tag{3.58}$$

The bases of the curvilinear coordinate (t, ϕ) result from Eq. (3.58) in

$$\mathbf{r}_t = \frac{\partial \mathbf{r}}{\partial t} = \begin{pmatrix} \dot{h}(t)\cos\phi \\ \dot{h}(t)\sin\phi \\ 2h\dot{h}(t) \end{pmatrix};\ \mathbf{r}_\phi = \frac{\partial \mathbf{r}}{\partial \phi} = \begin{pmatrix} -h\sin\phi \\ h\cos\phi \\ 0 \end{pmatrix} \tag{3.59}$$

Thus, the covariant metric tensor can be further computed as

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} \mathbf{r}_t \cdot \mathbf{r}_t & \mathbf{r}_t \cdot \mathbf{r}_\phi \\ \mathbf{r}_\phi \cdot \mathbf{r}_t & \mathbf{r}_\phi \cdot \mathbf{r}_\phi \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \\ &= \begin{bmatrix} \dot{h}^2(1 + 4h^2) & 0 \\ 0 & h^2 \end{bmatrix} \end{aligned} \tag{3.60}$$

The unit normal vector can be calculated as

$$\begin{aligned} \mathbf{n} &= \frac{\mathbf{g}_1 \times \mathbf{g}_2}{\sqrt{EG - F^2}} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dot{h}\ \cos\phi & \dot{h}\ \sin\phi & 2h\dot{h} \\ -h\sin\phi & h\cos\phi & 0 \end{vmatrix} \\ &= \frac{1}{h\dot{h}\sqrt{1 + 4h^2}} \begin{pmatrix} -2\dot{h}h^2\cos\phi \\ -2\dot{h}h^2\sin\phi \\ \dot{h}h \end{pmatrix} = \frac{-1}{\sqrt{1 + 4h^2}} \begin{pmatrix} 2h\cos\phi \\ 2h\sin\phi \\ -1 \end{pmatrix} \end{aligned} \tag{3.61}$$

The components of the Hessian tensor are calculated by differentiating Eq. (3.59) with respect to t and ϕ.

$$\mathbf{r}_{tt} = \frac{\partial^2 \mathbf{r}}{\partial t^2} = \begin{pmatrix} \ddot{h}\cos\phi \\ \ddot{h}\sin\phi \\ 2\ddot{h}h + 2\dot{h}^2 \end{pmatrix}; \quad \mathbf{r}_{t\phi} = \frac{\partial^2 \mathbf{r}}{\partial t \partial \phi} = \begin{pmatrix} -\dot{h}\sin\phi \\ \dot{h}\cos\phi \\ 0 \end{pmatrix};$$
$$\mathbf{r}_{\phi\phi} = \frac{\partial^2 \mathbf{r}}{\partial \phi^2} = \begin{pmatrix} -h\cos\phi \\ -h\sin\phi \\ 0 \end{pmatrix} \tag{3.62}$$

The Hessian tensor results from Eqs. (3.61) and (3.62) in

$$\mathbf{H} = \begin{bmatrix} \mathbf{r}_{tt}\cdot\mathbf{n} & \mathbf{r}_{t\phi}\cdot\mathbf{n} \\ \mathbf{r}_{\phi t}\cdot\mathbf{n} & \mathbf{r}_{\phi\phi}\cdot\mathbf{n} \end{bmatrix} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$
$$= \frac{2}{\sqrt{1+4h^2}} \begin{pmatrix} \dot{h}^2 & 0 \\ 0 & h^2 \end{pmatrix} \tag{3.63}$$

Therefore, the Gaussian curvature can be calculated from Eqs. (3.60) and (3.63).

$$K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2} = \frac{\left(\frac{2}{\sqrt{1+4h^2}}\right)^2 \dot{h}^2 h^2}{\dot{h}^2 h^2 \left(1+4h^2\right)}$$
$$= \frac{4}{\left(1+4h^2\right)^2} = \frac{4}{(1+4t)^2} > 0 \tag{3.64}$$

In case of $(M^2 - LN) < 0$, no solution of du exists. Thus, the surface (S) does not cut the tangent plane T except at the point P that is called the elliptic point.

Analogously, the mean curvature results from Eqs. (3.60) and (3.63) in

$$H = \frac{1}{2}\left(\kappa_1 + \kappa_2\right) = \frac{EN - 2MF + LG}{2\left(EG - F^2\right)}$$
$$= \frac{\left(\frac{2\dot{h}^2 h^2}{\sqrt{1+4h^2}}\right)\cdot\left[1 + \left(1+4h^2\right)\right]}{2\dot{h}^2 h^2\left(1+4h^2\right)} = \frac{2(1+2t)}{(1+4t)^{\frac{3}{2}}} \tag{3.65}$$

in which $h^2 = t$.

3.7 Riemann Curvature

The Riemann curvature (also Riemann curvature tensor) is closely related to the Gaussian curvature of the surface in differential geometry [1–3]. At first, let us look into the second covariant derivative of an arbitrary first-order tensor. The covariant derivative of the tensor with respect to u^j was derived in Eq. (2.208).

$$T_i\big|_j = T_{i,j} - \Gamma_{ij}^k T_k \tag{3.66}$$

Obviously, the covariant derivative $T_i|_j$ is a second-order tensor component.

Differentiating $T_i|_j$ with respect to u^k, the covariant derivative of the second-order tensor (component) $T_i|_j$ is the second covariant derivative of an arbitrary first-order tensor (component) T_i. This second covariant derivative has been given from Eq. (2.211a) [3].

$$\begin{aligned} T_i\big|_{jk} &\equiv \left(T_i\big|_j\right)\big|_k \\ &= \left(T_i\big|_j\right)_{,k} - \Gamma_{ik}^m T_m\big|_j - \Gamma_{jk}^m T_i\big|_m \\ &= T_i\big|_{j,\,k} - \Gamma_{ik}^m T_m\big|_j - \Gamma_{jk}^m T_i\big|_m \end{aligned} \tag{3.67}$$

Equation (3.66) delivers the relations of

$$T_i\big|_{j,k} = T_{i,jk} - \left(\Gamma_{ij,k}^m T_m + \Gamma_{ij}^m T_{m,k}\right) \tag{3.68a}$$

$$\Gamma_{ik}^m T_m\big|_j = \Gamma_{ik}^m\left(T_{m,j} - \Gamma_{mj}^n T_n\right) \tag{3.68b}$$

$$\Gamma_{jk}^m T_i\big|_m = \Gamma_{jk}^m\left(T_{i,m} - \Gamma_{im}^n T_{\mathrm{n}}\right) \tag{3.68c}$$

Inserting Eqs. (3.68a)–(3.68c) into Eq. (3.67), one obtains the second covariant derivative of T_i.

$$\begin{aligned} T_i\big|_{jk} &= T_i\big|_{j,\,k} - \Gamma_{ik}^m T_m\big|_j - \Gamma_{jk}^m T_i\big|_m \\ &= T_{i,jk} - \left(\Gamma_{ij,k}^m T_m + \Gamma_{ij}^m T_{m,k}\right) \\ &\quad - \Gamma_{ik}^m\left(T_{m,j} - \Gamma_{mj}^n T_n\right) - \Gamma_{jk}^m\left(T_{i,m} - \Gamma_{im}^n T_n\right) \\ &= T_{i,jk} - \Gamma_{ij,k}^m T_m - \Gamma_{ij}^m T_{m,k} \\ &\quad - \Gamma_{ik}^m T_{m,j} + \Gamma_{ik}^m \Gamma_{mj}^n T_n - \Gamma_{jk}^m T_{i,m} + \Gamma_{jk}^m \Gamma_{im}^n T_{\mathrm{n}} \end{aligned} \tag{3.69}$$

where the second partial derivative of T_i is symmetric with respect to j and k:

$$T_{i,jk} \equiv \frac{\partial^2 T_i}{\partial u^j \partial u^k} = \frac{\partial^2 T_i}{\partial u^k \partial u^j} \equiv T_{i,kj} \tag{3.70}$$

Interchanging the indices j with k in Eq. (3.69), one obtains

$$\begin{aligned} T_i\big|_{kj} = T_{i,kj} - \Gamma^m_{ik,j} T_m - \Gamma^m_{ik} T_{m,j} \\ - \Gamma^m_{ij} T_{m,k} + \Gamma^m_{ij} \Gamma^n_{mk} T_n - \Gamma^m_{kj} T_{i,m} + \Gamma^m_{kj} \Gamma^n_{im} T_{\mathrm{n}} \end{aligned} \tag{3.71}$$

Using the symmetry properties given in Eq. (3.70), Eq. (3.71) can be rewritten as

$$\begin{aligned} T_i\big|_{kj} = T_{i,jk} - \Gamma^m_{ik,j} T_m - \Gamma^m_{ik} T_{m,j} \\ - \Gamma^m_{ij} T_{m,k} + \Gamma^m_{ij} \Gamma^n_{mk} T_n - \Gamma^m_{jk} T_{i,m} + \Gamma^m_{jk} \Gamma^n_{im} T_n \end{aligned} \tag{3.72}$$

In a flat space, the second covariant derivatives in Eqs. (3.69) and (3.72) are identical. On the contrary, they are not equal in a curved space because of its surface curvature. The difference of both second covariant derivatives is proportional to the curvature tensor. Subtracting Eq. (3.69) from Eq. (3.72), the curvature tensor results in

$$\begin{aligned} T_i\big|_{jk} - T_i\big|_{kj} &= \left(\Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik} \Gamma^n_{mj} - \Gamma^m_{ij} \Gamma^n_{mk} \right) T_n \\ &\equiv R^n_{ijk} T_n \end{aligned} \tag{3.73}$$

Thus, the Riemann curvature (also Riemann-Christoffel tensor) can be expressed as

$$R^n_{ijk} \equiv \Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik} \Gamma^n_{mj} - \Gamma^m_{ij} \Gamma^n_{mk} \tag{3.74}$$

It is straightforward that the Riemann-Christoffel tensor is a fourth-order tensor with respect to the indices of i, j, k, and n. They contain 81 ($=3^4$) components in a three-dimensional space.

In Eq. (3.74), the partial derivatives of the Christoffel symbols are defined by

$$\Gamma^n_{ik,j} = \frac{\partial \Gamma^n_{ik}}{\partial u^j}; \quad \Gamma^n_{ij,k} = \frac{\partial \Gamma^n_{ij}}{\partial u^k} \tag{3.75}$$

According to Eq. (2.172), the second-kind Christoffel symbol is given

$$\Gamma^k_{ij} = \frac{1}{2} \left(g_{ip,j} + g_{jp,i} - g_{ij,p} \right) g^{kp} \tag{3.76}$$

Therefore, the Riemann curvature tensor in Eq. (3.74) only depends on the covariant and contravariant metric coefficients of the metric tensor **M**, as given in Eq. (3.28).

Furthermore, the covariant Riemann curvature tensor of fourth order is defined by the Riemann-Christoffel tensor and covariant metric coefficients.

$$R_{lijk} \equiv g_{ln} R^n_{ijk} \Leftrightarrow R^n_{ijk} = g^{ln} R_{lijk} \tag{3.77}$$

For a differentiable two-dimensional manifold of the curvilinear coordinates (u,v), the Bianchi first identity gives the relation between the Riemann curvature tensors R and Gaussian curvature K, cf. Eq. (3.117b).

$$R_{lijk} \equiv K \cdot \left(g_{lj} g_{ki} - g_{lk} g_{ji} \right) \tag{3.78}$$

Equation (3.78) indicates that the Gaussian curvature K of the two-dimensional surface only depends on the metric coefficients of E, F, and G. Therefore, the Gaussian curvature is only a function of the first fundamental form I. This result was proved by *Gauss Theorema Egregium* [1, 4–6].

The Riemann curvature tensor has the following properties:

- First skew-symmetry with respect to l and i:

$$R_{lijk} = -R_{iljk} \tag{3.79}$$

- Second skew-symmetry with respect to j and k:

$$R_{lijk} = -R_{likj}; \quad R^n_{ijk} = -R^n_{ikj} \tag{3.80}$$

- Block symmetry with respect to two pairs (l, i) and (j, k):

$$R_{lijk} = R_{jkli} \tag{3.81}$$

- Cyclic property in i, j, k:

$$\begin{aligned} &R_{lijk} + R_{ljki} + R_{lkij} = 0; \\ &R^n_{ijk} + R^n_{jki} + R^n_{kij} = 0 \end{aligned} \tag{3.82}$$

Resulting from these properties, there are six components of R_{lijk} in the three-dimensional space as follows [2]:

$$R_{lijk} = R_{3131}, \quad R_{3132}, \quad R_{3232}, \quad R_{1212}, \quad R_{3112}, \quad R_{3212} \tag{3.83}$$

In Cartesian coordinates, all second-kind Christoffel symbols equal zero according to Eq. (2.184). Therefore, the Riemann-Christoffel tensor, as given in Eq. (3.74) must be equal to zero.

$$R^n_{ijk} = \Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik}\Gamma^n_{mj} - \Gamma^m_{ij}\Gamma^n_{mk} \equiv 0 \tag{3.84}$$

Therefore, the Riemann curvature tensor in Cartesian coordinates becomes

$$R_{lijk} \equiv g_{ln} R^{n}_{ijk} = 0 \tag{3.85}$$

3.8 Gauss-Bonnet Theorem

The Gauss-Bonnet theorem in differential geometry connects the Gaussian and geodesic curvatures of the surface to the surface topology by means of the Euler's characteristic.

Figure 3.5 displays a differentiable Riemannian surface (S) surrounded by a closed boundary curve Γ. The Gaussian curvature vector **K** is perpendicular to the manifold surface at the point P lying in the curve C and has the direction of the unit normal vector **n**. The geodesic curvature vector $\mathbf{k}_g$ has the amplitude of the geodesic curvature κ_g; its direction of ($\mathbf{n} \times \mathbf{t}$) is perpendicular to the unit normal and tangent vectors in the Frenet orthonormal frame.

The Gauss-Bonnet theorem is formulated for a simple closed boundary curve Γ [1, 2, 4].

$$\iint_S K dA + \oint_\Gamma \kappa_g d\Gamma = 2\pi \tag{3.86}$$

The compact curvilinear surface S is triangulated into a finite number of curvilinear triangles. Each triangle contains a point P on the surface. This procedure is called the surface triangulation where two neighboring curvilinear triangles have one common vertex and one common edge (see Fig. 3.6).

Therefore, the integral of the geodesic curvature over all triangles on the compact curvilinear surface S equals zero [2].

$$\oint_\Gamma \kappa_g d\Gamma = 0 \tag{3.87}$$

In case of a compact triangulated surface, the Gauss-Bonnet theorem can be written in Euler's characteristic χ of the compact triangulated surface S_T [1, 2, 4].

$$\iint_{S_T} K dA = 2\pi\chi(S_T) \tag{3.88}$$

The Euler's characteristic of the compact triangulated surface can be defined by

$$\chi(S_T) \equiv V - E + F \tag{3.89}$$

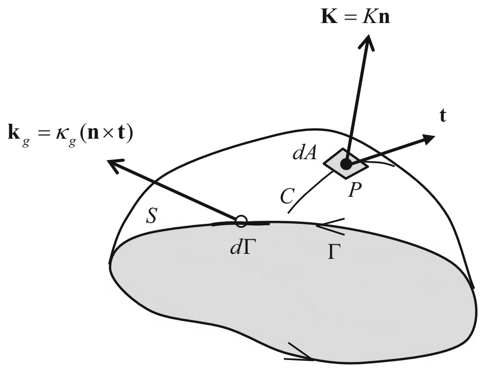

Fig. 3.5 Gaussian and geodesic curvatures for a simple closed curve Γ

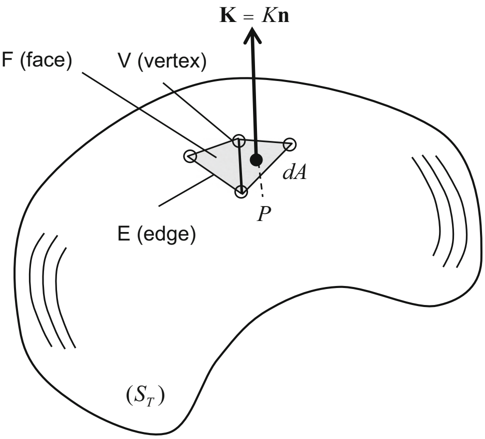

Fig. 3.6 Gaussian and geodesic curvatures for a compact triangulated surface

in which V, E, and F are the numbers of vertices, edges, and faces of the considered compact triangulated surface, respectively.

Substituting Eqs. (3.8), (3.54), and (3.89) into Eq. (3.88), the Gauss-Bonnet theorem can be written for a compact triangulated surface.

$$\iint\limits_{S_T} \frac{LN - M^2}{\sqrt{EG - F^2}} dudv = 2\pi(V - E + F) \tag{3.90}$$

3.9 Gauss Derivative Equations

Gauss derivative equations were derived from the second-kind Christoffel symbols and the basis $\mathbf{g}_3$ that denotes the normal unit vector $\mathbf{n}$ ($=\mathbf{g}_3$) of the curvilinear surface with the covariant bases ($\mathbf{g}_i$, $\mathbf{g}_j$) for $i, j = 1, 2$ at any point $P(u,v)$ in the surface. Note that all indices i, j, k in the curvature surface S vary from 1 to 2.

The partial derivatives of the covariant basis $\mathbf{g}_i$ with respect to u^j can be written according to Eq. (2.166) as

$$\mathbf{g}_{i,j} = \Gamma_{ij}^{k}\mathbf{g}_k + \Gamma_{ij}^{3}\mathbf{g}_3 \text{ for } i,j,k = 1,\ 2 \tag{3.91}$$

The Christoffel symbols in the normal direction $\mathbf{n}$ result from Eq. (2.164).

$$\Gamma_{ij}^{3} = \mathbf{n}\cdot\mathbf{g}_{i,j} = \mathbf{g}_3\cdot\mathbf{g}_{i,j} \tag{3.92}$$

Differentiating $\mathbf{g}_i.\mathbf{g}_3$ and using the orthogonality of $\mathbf{g}_i$ and $\mathbf{g}_3$, the partial derivative of the covariant basis $\mathbf{g}_i$ with respect to u^j can be calculated as

$$\begin{aligned}&(\mathbf{g}_i\cdot\mathbf{g}_3)_{,j} = \mathbf{g}_{i,j}\cdot\mathbf{g}_3 + \mathbf{g}_i\cdot\mathbf{g}_{3,j} = 0\\ &\Rightarrow \mathbf{g}_{i,j} = -\mathbf{g}_3\left(\mathbf{g}_i\cdot\mathbf{g}_{3,j}\right) = -\left(\mathbf{g}_3\cdot\mathbf{g}_{3,j}\right)\mathbf{g}_i\end{aligned} \tag{3.93}$$

Substituting Eq. (3.93) into Eq. (3.92), the Christoffel symbols in the normal direction $\mathbf{n}$ can be expressed as

$$\begin{aligned}\Gamma_{ij}^{3} &= \mathbf{g}_3\cdot\mathbf{g}_{i,j} = -(\mathbf{g}_3\cdot\mathbf{g}_3)\mathbf{g}_{3,j}\cdot\mathbf{g}_i\\ &= -\mathbf{g}_{3,j}\cdot\mathbf{g}_i = -\mathbf{n}_j\cdot\mathbf{g}_i\\ &\equiv h_{ij} = h_{ji} = \Gamma_{ji}^{3} \text{ for } i,j = 1,2\end{aligned} \tag{3.94a}$$

Thus,

$$\mathbf{n}_j = -h_{ij}\mathbf{g}^i \text{ for } i,j = 1,2 \tag{3.94b}$$

where h_{ij} is the symmetric covariant components of the Hessian tensor $\mathbf{H}$, as given in Eq. (3.104b).

Inserting Eq. (3.94a) into Eq. (3.91), the covariant derivative of the basis $\mathbf{g}_i$ with respect to u^j results in

$$\begin{aligned}\mathbf{g}_{i,j} &= \Gamma_{ij}^{k}\mathbf{g}_k + h_{ij}\mathbf{g}_3\\ &= \Gamma_{ij}^{k}\mathbf{g}_k + h_{ij}\mathbf{n} \text{ for } i,j,k = 1,\ 2\end{aligned} \tag{3.95}$$

This equation is called Gauss derivative equations in which the second-kind Christoffel symbol is defined as

$$\Gamma_{ij}^{k} = \tfrac{1}{2} g^{kp} \left(g_{jp,i} + g_{pi,j} - g_{ij,p} \right) \tag{3.96}$$

3.10 Weingarten's Equations

The Weingarten's equations deal with the derivatives of the normal unit vector $\mathbf{n}$ ($=\mathbf{g}_3$) of the surface at the point *P(u,v)* in the curvilinear coordinates $\{u^i\}$.

The partial derivative of the normal unit vector $\mathbf{n}$ results from Eq. (3.91).

$$\mathbf{n}_i \equiv \mathbf{g}_{3,i} = \Gamma_{3i}^{k} \mathbf{g}_k + \Gamma_{3i}^{3} \mathbf{g}_3 \text{ for } i,k = 1,2 \tag{3.97}$$

Differentiating $\mathbf{g}_3 . \mathbf{g}_3 = 1$ with respect to u^i, one obtains

$$(\mathbf{g}_3 \cdot \mathbf{g}_3)_{,i} = 2\mathbf{g}_3 \cdot \mathbf{g}_{3,i} = 0 \Rightarrow \mathbf{g}_3 \cdot \mathbf{g}_{3,i} = 0 \tag{3.98}$$

Using Eqs. (3.92) and (3.98), one obtains

$$\Gamma_{ji}^{3} = \mathbf{g}_3 \cdot \mathbf{g}_{j,i} \ \Rightarrow \Gamma_{3i}^{3} = \mathbf{g}_3 \cdot \mathbf{g}_{3,i} = 0 \ \text{ for } \ i = 1,2 \tag{3.99}$$

Inserting Eq. (3.99) into Eq. (3.97) and using Eq. (3.94b), it gives

$$\begin{aligned} \mathbf{n}_i &= \Gamma_{3i}^{k} \mathbf{g}_k = -h_{ik} \mathbf{g}^k = -\left(h_{ik} g^{kj} \right) \mathbf{g}_j = -h_i^j \mathbf{g}_j \\ \Rightarrow \frac{\partial \mathbf{n}}{\partial u^i} &\equiv \mathbf{n}_i = -h_i^j \mathbf{g}_\mathrm{j} \text{ for } i,j \ = 1,2 \end{aligned} \tag{3.100}$$

The mixed components h_i^j are calculated from the Hessian tensor $\mathbf{H}$ and metric tensor $\mathbf{M}$, as shown in Eq. (3.105).

$$\left(h_i^j \right) = \left(h_{ik} g^{kj} \right) = \mathbf{H}\mathbf{M}^{-1} \tag{3.101}$$

Using Eq. (3.101) for $i, j = u, v$, the Weingarten's equations (3.100) can be written as

$$\begin{aligned} \mathbf{n}_u &= \left(\frac{FM - LG}{EG - F^2} \right) \mathbf{r}_u + \left(\frac{FL - EM}{EG - F^2} \right) \mathbf{r}_v \\ \Leftrightarrow \frac{\partial \mathbf{n}}{\partial u} &= \left(\frac{FM - LG}{EG - F^2} \right) \frac{\partial \mathbf{r}}{\partial u} + \left(\frac{FL - EM}{EG - F^2} \right) \frac{\partial \mathbf{r}}{\partial v} \end{aligned} \tag{3.102}$$

$$\begin{aligned} \mathbf{n}_v &= \left(\frac{FN-GM}{EG-F^2}\right)\mathbf{r}_u + \left(\frac{FM-EN}{EG-F^2}\right)\mathbf{r}_v \\ \Leftrightarrow \frac{\partial \mathbf{n}}{\partial v} &= \left(\frac{FN-GM}{EG-F^2}\right)\frac{\partial \mathbf{r}}{\partial u} + \left(\frac{FM-EN}{EG-F^2}\right)\frac{\partial \mathbf{r}}{\partial v} \end{aligned} \tag{3.103}$$

where the covariant metric and Hessian tensors result from the coefficients of the first and second fundamental forms *I* and *II*.

$$\begin{aligned} \mathbf{M} &= \left(g_{ij}\right) \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \mathbf{r}_u \cdot \mathbf{r}_u & \mathbf{r}_u \cdot \mathbf{r}_v \\ \mathbf{r}_v \cdot \mathbf{r}_u & \mathbf{r}_v \cdot \mathbf{r}_v \end{bmatrix} \\ \Rightarrow \mathbf{M}^{-1} &= (g^{ij}) = \frac{1}{EG-F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \end{aligned} \tag{3.104a}$$

$$\mathbf{H} = \left(h_{ij}\right) \equiv \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{uu} \cdot \mathbf{n} & \mathbf{r}_{uv} \cdot \mathbf{n} \\ \mathbf{r}_{uv} \cdot \mathbf{n} & \mathbf{r}_{vv} \cdot \mathbf{n} \end{bmatrix} \tag{3.104b}$$

Therefore,

$$\mathbf{HM}^{-1} = \left(h_i^j\right) = \frac{-1}{EG-F^2}\begin{bmatrix} (FM-LG) & (FL-EM) \\ (FN-GM) & (FM-EN) \end{bmatrix} \tag{3.105}$$

3.11 Gauss-Codazzi Equations

The Gauss-Codazzi equations are based on the Gauss derivative and Weingarten's equations. The Gauss derivative equation (3.95) can be written as

$$\begin{aligned} \mathbf{g}_{i,j} &= \Gamma_{ij}^k \mathbf{g}_k + \Gamma_{ij}^3 \mathbf{n} \\ &= \Gamma_{ij}^k \mathbf{g}_k + K_{ij}\mathbf{n} \;\text{ for }\; i,j,k \;=\; 1,2 \end{aligned} \tag{3.106}$$

in which the symmetric covariant components K_{ij} ($=h_{ij}$) of the surface curvature tensor $\mathbf{K}$ are given in Eq. (3.94a).

$$\begin{aligned} \Gamma_{ij}^3 &= K_{ij} \in \mathbf{K}\ (\equiv \mathbf{H}) \\ &= K_{ji} = \Gamma_{ji}^3 \end{aligned} \tag{3.107}$$

The covariant derivative of $\mathbf{g}_i$ with respect to u^j results from Eq. (3.106).

$$\begin{aligned} \mathbf{g}_i\big|_j &= \mathbf{g}_{i,j} - \Gamma_{ij}^k \mathbf{g}_k \\ &= K_{ij}\mathbf{n} \;\text{ for }\; i,j \;=\; 1,2 \end{aligned} \tag{3.108}$$

The Weingarten's equation (3.100) can also be written in the mixed components of the surface curvature tensor **K**.

$$\mathbf{n}_i \equiv \mathbf{g}_{3,i} = -K_i^j \mathbf{g}_j \text{ for } i,j = 1,2 \tag{3.109}$$

in which the mixed components of the surface curvature tensor **K** is defined according to Eq. (3.101) as

$$K_i^j = K_{ik} g^{kj} \in \mathbf{KM}^{-1} \text{ for } i,j,k = 1,2 \tag{3.110}$$

Differentiating Eq. (3.108) with respect to u^k and using Eq. (3.109), the covariant second derivatives of the basis $\mathbf{g}_i$ can be calculated as

$$\mathbf{g}_i\big|_{jk} = K_{ij,k}\mathbf{n} + K_{ij}\mathbf{n}_k = K_{ij,k}\mathbf{n} - K_{ij}K_k^l \mathbf{g}_l \tag{3.111}$$

Similarly, the covariant second derivatives of the basis $\mathbf{g}_i$ with respect to u^k and u^j result from interchanging k with j in Eq. (3.111).

$$\mathbf{g}_i\big|_{kj} = K_{ik,j}\mathbf{n} + K_{ik}\mathbf{n}_j = K_{ik,j}\mathbf{n} - K_{ik}K_j^l \mathbf{g}_l \tag{3.112}$$

The Riemann curvature tensor R results from the difference of the covariant second derivatives of Eqs. (3.111) and (3.112) according to Eq. (3.73).

$$\begin{aligned} &\mathbf{g}_i\big|_{jk} - \mathbf{g}_i\big|_{kj} = R^l_{ijk}\mathbf{g}_l \\ &= \left(K_{ij,k} - K_{ik,j}\right)\mathbf{n} + \left(K_{ik}K_j^l - K_{ij}K_k^l\right)\mathbf{g}_l \end{aligned} \tag{3.113}$$

Multiplying Eq. (3.113) by the normal unit vector **n** and using the orthogonality between $\mathbf{g}_l$ and **n**, one obtains

$$\begin{aligned} &\left(K_{ij,k} - K_{ik,j}\right)\mathbf{n}\cdot\mathbf{n} + \left(K_{ik}K_j^l - K_{ij}K_k^l\right)\mathbf{g}_l\cdot\mathbf{n} = R^l_{ijk}\mathbf{g}_l\cdot\mathbf{n} \\ \Rightarrow\ &\left(K_{ij,k} - K_{ik,j}\right)\cdot 1 + \left(K_{ik}K_j^l - K_{ij}K_k^l\right)\cdot 0 = R^l_{ijk}\cdot 0 \end{aligned}$$

Thus,

$$K_{ij,k} - K_{ik,j} = 0 \text{ for } i,j,k = 1,2 \tag{3.114}$$

Equation (3.114) is called the *Codazzi's equation*.

Multiplying both sides of Eq. (3.113) by $\boldsymbol{g}_m$, using the Codazzi's equation, and employing the tensor contraction rules, one obtains

$$R^l_{ijk}\mathbf{g}_l \cdot \mathbf{g}_m = \left(K_{ik}K^l_j - K_{ij}K^l_k\right)\mathbf{g}_l \cdot \mathbf{g}_m$$
$$\Rightarrow R^m_{ijk}g_{lm} = \left(K_{ik}K^m_j - K_{ij}K^m_k\right)g_{lm}$$

Thus, the Riemann curvature tensors can be calculated as

$$R_{lijk} = K_{ik}K_{lj} - K_{ij}K_{lk} \tag{3.115}$$

Equation (3.115) is called the *Gauss equation*. As a result, both Eqs. (3.114) and (3.115) are defined as the *Gauss-Codazzi equations*.

The Codazzi's equation (3.114) gives only two independent nontrivial terms [3, 6]:

$$K_{ij,k} = K_{ik,j} \Rightarrow \quad (K_{11,2} = K_{12,1}\ ;\ \ K_{21,2} = K_{22,1}) \tag{3.116}$$

On the contrary, the Gauss equation delivers only one independent nontrivial term [3, 6]:

$$R_{1212} = K_{11}K_{22} - K^2_{12} \tag{3.117a}$$

Therefore, the Gaussian or total curvature K in Eq. (3.54) can be rewritten as

$$\begin{aligned} K &= \det\left(K^j_i\right) = \left(K^1_1K^2_2 - K^1_2K^2_1\right) \\ &= \frac{K_{11}K_{22} - K^2_{12}}{g_{11}g_{22} - g^2_{12}} = \frac{R_{1212}}{g} \end{aligned} \tag{3.117b}$$

in which

K^j_i is the mixed components of the mixed tensor $\mathbf{KM}^{-1}$, as shown in Eq. (3.105);
K_{ij} is the covariant components of the surface curvature tensor $\mathbf{K}$;
R_{1212} is the covariant component of the Riemann curvature tensor;
g is the determinant of the covariant metric tensor (g_{ij}).

3.12 Lie Derivatives

The Lie derivatives (pronouncing/Li:/) named after the Norwegian mathematician Sophus Lie (1842–1899) are very useful geometrical tools in Lie algebras and Lie groups in differential geometry of curved manifolds. The Lie derivatives are based on vector fields that are tangent to the set of curves (also called congruence) of the curved manifold.

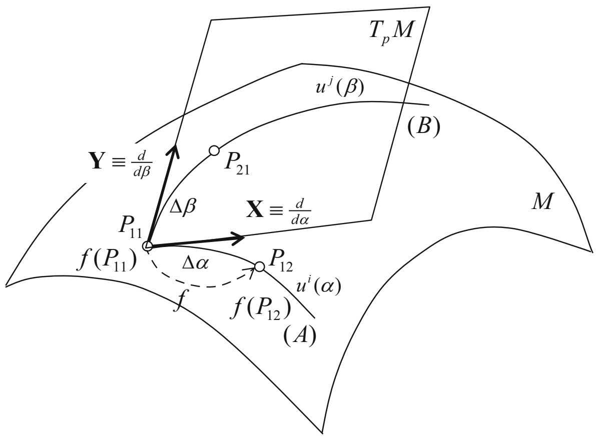

Fig. 3.7 Vector field on the tangent space T_pM of the manifold M

3.12.1 Vector Fields in Riemannian Manifold

The vector field tangent to the curve (A) parameterized by the geodesic parameter $\boldsymbol{\alpha}$ can be written in the coordinate $u^{\boldsymbol{i}}$ of the N-dimensional manifold M, as displayed in Fig. 3.7. The vector field lies on the tangent space T_pM (tangent surface) that is tangent to the manifold M at the contact point P. As the contact point moves along the curve on the manifold M, a tangent bundle TM of the manifold M is generated. Therefore, the tangent bundle consists of all tangent spaces of the manifold M.

Using Einstein summation convention (cf. Sect. 2.2.2), the vector field $\mathbf{X}$ on the tangent space T_pM can be written with the basis vectors $\frac{\partial}{\partial u^i}$ of the coordinates u^i.

$$\begin{aligned}\mathbf{X} &\equiv \frac{d}{d\alpha} = \frac{du^i}{d\alpha}\frac{\partial}{\partial u^i} \\ &\equiv X^i \frac{\partial}{\partial u^i} \text{ for } i = 1, 2, \ldots, N\end{aligned} \tag{3.118}$$

where $\boldsymbol{X}^i$ is the vector component in the coordinate u^i.

Similarly, the vector field tangent to the curve (B) parameterized by another geodesic parameter β can be written in the coordinate u^j.

$$\begin{aligned}\mathbf{Y} &\equiv \frac{d}{d\beta} = \frac{du^j}{d\beta}\frac{\partial}{\partial u^j} \\ &\equiv Y^j \frac{\partial}{\partial u^j} \text{ for } j = 1, 2, \ldots, N\end{aligned} \tag{3.119}$$

where $\boldsymbol{Y}^j$ is the vector component in the coordinate u^j.

3.12.2 Lie Bracket

Let **X** and **Y** be the vector fields of the congruence in the N-dimensional manifold M and f be a mapping function of the coordinate u^i in the curve. The commutator of a vector field is called the Lie bracket and can be defined by

$$[\mathbf{X},\mathbf{Y}] \equiv \mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f)) \tag{3.120}$$

The first mapping operator on the RHS of Eq. (3.120) can be calculated using the chain rule of differentiation.

$$\begin{aligned}\mathbf{X}(\mathbf{Y}(f)) &= X^j \frac{\partial}{\partial u^j}\left(Y^i \frac{\partial}{\partial u^i}\right)\\ &= X^j \frac{\partial Y^i}{\partial u^j}\frac{\partial}{\partial u^i} + X^j Y^i \frac{\partial^2}{\partial u^i \partial u^j}\end{aligned} \tag{3.121}$$

Analogously, the second mapping operator on the RHS of Eq. (3.120) results in

$$\begin{aligned}\mathbf{Y}(\mathbf{X}(f)) &= Y^j \frac{\partial}{\partial u^j}\left(X^i \frac{\partial}{\partial u^i}\right)\\ &= Y^j \frac{\partial X^i}{\partial u^j}\frac{\partial}{\partial u^i} + X^i Y^j \frac{\partial^2}{\partial u^i \partial u^j}\end{aligned} \tag{3.122}$$

Interchanging the indices i with j in the second term on the RHS of Eq. (3.122), one obtains

$$\mathbf{Y}(\mathbf{X}(f)) = Y^j \frac{\partial X^i}{\partial u^j}\frac{\partial}{\partial u^i} + X^j Y^i \frac{\partial^2}{\partial u^j \partial u^i} \tag{3.123}$$

Subtracting Eq. (3.121) from Eq. (3.123), the Lie bracket is given.

$$\begin{aligned}[\mathbf{X},\mathbf{Y}] &= \left(X^j \frac{\partial Y^i}{\partial u^j} - Y^j \frac{\partial X^i}{\partial u^j}\right)\frac{\partial}{\partial u^i}\\ &\equiv [\mathbf{X},\mathbf{Y}]^i \frac{\partial}{\partial u^i} \quad \text{for} \quad i = 1, 2, \ldots, N\end{aligned} \tag{3.124}$$

Thus, the component i in the coordinate u^i of the Lie bracket is defined by

$$[\mathbf{X},\mathbf{Y}]^i \equiv X^j \frac{\partial Y^i}{\partial u^j} - Y^j \frac{\partial X^i}{\partial u^j} \text{ for } i,j = 1, 2, \ldots, N \tag{3.125}$$

The vectors **X** and **Y** commute if its Lie bracket equals zero.

$$[\mathbf{X},\mathbf{Y}] = 0 \tag{3.126}$$

According to Eq. (3.125), the Lie bracket is skew-symmetric (anti-symmetric).

$$\begin{aligned}[\mathbf{X},\mathbf{Y}] &= -[\mathbf{Y},\mathbf{X}] \\ &= -\left(Y^j\frac{\partial X^i}{\partial u^j} - X^j\frac{\partial Y^i}{\partial u^j}\right)\frac{\partial}{\partial u^i}\end{aligned} \tag{3.127}$$

The Lie bracket (commutator) of the vector field (**X**, **Y**) can be expressed in another way as

$$\begin{aligned}[\mathbf{X},\mathbf{Y}] &= X^j\frac{\partial}{\partial u^j}\left(Y^i\frac{\partial}{\partial u^i}\right) - Y^j\frac{\partial}{\partial u^j}\left(X^i\frac{\partial}{\partial u^i}\right) \\ &= \mathbf{X}\frac{d}{d\beta} - \mathbf{Y}\frac{d}{d\alpha}\end{aligned} \tag{3.128}$$

Therefore, the Lie bracket of the vector field can be written as

$$\begin{aligned}[\mathbf{X},\mathbf{Y}] &\equiv \left[\frac{d}{d\alpha},\frac{d}{d\beta}\right] \\ &= \frac{d}{d\alpha}\frac{d}{d\beta} - \frac{d}{d\beta}\frac{d}{d\alpha}\end{aligned} \tag{3.129}$$

The Lie bracket of a vector field is generally not equal to zero in a curved manifold due to space torsions and Riemann surface curvatures that will be discussed later in Sect. 3.12.5.

The Lie bracket has some cyclic permutation properties of the vector field (**X**, **Y**, and **Z**) in the N-dimensional manifold M.

$$\begin{aligned}[\mathbf{X},[\mathbf{Y},\mathbf{Z}]] &= \mathbf{XYZ} - \mathbf{XZY} - \mathbf{YZX} + \mathbf{ZYX} \\ [\mathbf{Y},[\mathbf{Z},\mathbf{X}]] &= \mathbf{YZX} - \mathbf{YXZ} - \mathbf{ZXY} + \mathbf{XZY} \\ [\mathbf{Z},[\mathbf{X},\mathbf{Y}]] &= \mathbf{ZXY} - \mathbf{ZYZ} - \mathbf{XYZ} + \mathbf{YXZ}\end{aligned} \tag{3.130}$$

The Jacobi identity written in the Lie brackets results from substituting the properties of Eq. (3.130):

$$[\mathbf{X},[\mathbf{Y},\mathbf{Z}]] + [\mathbf{Y},[\mathbf{Z},\mathbf{X}]] + [\mathbf{Z},[\mathbf{X},\mathbf{Y}]] = 0 \tag{3.131}$$

3.12.3 Lie Dragging

3.12.3.1 Lie Dragging of a Function

Let f be a function that maps a point P_{11} to another point P_{12} on the curve (A) by a geodesic parameter distance $\Delta\alpha$. The function $f(P_{12})$ at the point P_{12} is called the image of $f(P_{11})$ at the point P_{11} carrying by the mapping function f at $\Delta\alpha$ on the same curve $(A) \in M$ (see Fig. 3.7).

If the image function $f(P_{12})$ equals the function $f(P_{11})$, the function f is invariant under the mapping. Furthermore, the mapping function f can be defined as Lie dragged if f is invariant for any geodesic parameter distance $\Delta\alpha$ along any congruence on the manifold M.

$$\frac{df}{d\alpha} = 0 \Leftrightarrow f \text{ is Lie dragged.} \tag{3.132}$$

3.12.3.2 Lie Dragging of a Vector Field

Let **X** and **Y** be vector fields on the tangent space T_pM of the tangent bundle TM in an N-dimensional manifold M, as shown in Fig. 3.8. The congruence consists of α- and β-curves with the coordinates u^i and u^j. The tangent vector **X** to the curve (A_1) at the point P_{11} is dragged to the curve (A_2) at the point P_{21} by a geodesic parameter distance $\Delta\beta$.

The image vector **X*** of the original vector **X** is dragged by $\Delta\beta$ from (A_1) to (A_2) and tangent to the curve (A_2) at the point P_{21}. Generally, both tangent vectors **X** and **X*** are different to each other under the Lie dragging by $\Delta\beta$. However, if they are equal for every geodesic parameter distance $\Delta\beta$, the Lie dragging is invariant. In this case, the vector field is called Lie dragged on the manifold M [7, 8].

Similarly, the tangent vector **Y** to the curve (B_1) at the point P_{11} is dragged to the image vector **Y*** tangent the curve (B_2) at the point P_{12} by a geodesic parameter distance $\Delta\alpha$ in the tangent bundle TM. Thus, the vector fields of **X** and **Y** along the congruence are generated on the manifold M. The vector field is defined as Lie dragged if its Lie bracket or the commutator given in Eq. (3.129) equals zero.

$$\begin{aligned}[\mathbf{X}, \mathbf{Y}] &\equiv \left[\frac{d}{d\alpha}, \frac{d}{d\beta}\right] \\ &= \frac{d}{d\alpha}\frac{d}{d\beta} - \frac{d}{d\beta}\frac{d}{d\alpha} = 0 \\ &\Rightarrow \frac{d}{d\alpha}\frac{d}{d\beta} = \frac{d}{d\beta}\frac{d}{d\alpha}\end{aligned} \tag{3.133}$$

In this case, the vector field is commute under the Lie-dragged procedure on the manifold M. In general, the Lie bracket of a vector field is not always equal to zero due to space torsions besides the Riemann surface curvature on the manifold M.

3.12.4 Lie Derivatives

The Lie derivatives of a function with respect to the vector field **X** are defined by the change rate of the function f between two different points P_{11} and P_{12} in the same curve under the Lie dragging by a geodesic parameter distance $\Delta\alpha$.

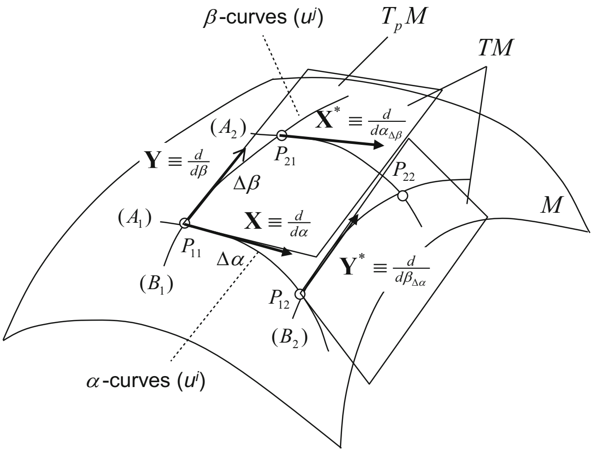

Fig. 3.8 Lie dragging vector fields on the manifold M

$$\begin{aligned}
\pounds_{\mathbf{X}} f &\equiv \lim_{\Delta\alpha \to 0} \frac{f(\alpha_0 + \Delta\alpha) - f(\alpha_0)}{\Delta\alpha} \\
&= \left.\frac{df(u^i)}{d\alpha}\right|_{\alpha_0} = \left.\frac{du^i}{d\alpha}\frac{\partial f}{\partial u^i}\right|_{\alpha_0} \\
&= X^i \frac{\partial f}{\partial u^i}
\end{aligned} \tag{3.134}$$

Thus, the Lie derivative of a function f with respect to the vector field $\mathbf{X}$ can be simply expressed in

$$\pounds_{\mathbf{X}} f = \left(X^i \frac{\partial}{\partial u^i}\right) f = \mathbf{X} f \tag{3.135}$$

Analogously, the Lie derivative of a vector $\mathbf{Y}$ with respect to the vector field $\mathbf{X}$ results from the Lie bracket with the basis vectors $\frac{\partial}{\partial u^i}$ of the coordinates u^i [8, 9].

$$\begin{aligned}
\pounds_{\mathbf{X}} \mathbf{Y} = [\mathbf{X}, \mathbf{Y}] &= [\mathbf{X}, \mathbf{Y}]^i \frac{\partial}{\partial u^i} \\
&= (\pounds_{\mathbf{X}} \mathbf{Y})^i \frac{\partial}{\partial u^i} \quad \text{for } i = 1, 2, \ldots, N
\end{aligned} \tag{3.136}$$

According to Eq. (3.125), the component i in the coordinate u^i of the Lie derivative of the vector $\mathbf{Y}$ with respect to the vector field $\mathbf{X}$ can be calculated as

$$\begin{aligned}
(\pounds_{\mathbf{X}} \mathbf{Y})^i &= [\mathbf{X}, \mathbf{Y}]^i \\
&= \left(X^j \frac{\partial}{\partial u^j}\right) Y^i - \left(Y^j \frac{\partial}{\partial u^j}\right) X^i \\
&= \frac{d}{d\alpha} Y^i - \frac{d}{d\beta} X^i \text{ for } i = 1, 2, \ldots, N
\end{aligned} \tag{3.137}$$

Using Eqs. (3.136) and (3.137) the Lie derivative of a vector field is skew-symmetric because of the skew-symmetry of the Lie bracket, as shown in Eq. (3.127).

$$\begin{aligned}\pounds_{\mathbf{X}}\mathbf{Y} &= (\pounds_{\mathbf{X}}\mathbf{Y})^i \frac{\partial}{\partial u^i} = \left(X^j \frac{\partial Y^i}{\partial u^j} - Y^j \frac{\partial X^i}{\partial u^j}\right) \frac{\partial}{\partial u^i} \\ &= -\pounds_{\mathbf{Y}}\mathbf{X}\end{aligned} \tag{3.138}$$

In the following section, the properties of the Lie derivatives are proved.

3.12.4.1 Lie Derivative of a Function Product

Let f and g be functions. The Lie derivative of product of two functions is given as

$$\pounds_{\mathbf{X}}(fg) = (\pounds_{\mathbf{X}} f)g + f(\pounds_{\mathbf{X}} g) \tag{3.139}$$

Proof

$$\begin{aligned}\pounds_{\mathbf{X}}(fg) = \mathbf{X}(fg) &= X^i \frac{\partial}{\partial u^i}(fg) \\ &= \left(X^i \frac{\partial f}{\partial u^i}\right) g + f\left(X^i \frac{\partial g}{\partial u^i}\right) \\ &= (\pounds_{\mathbf{X}} f)g + f(\pounds_{\mathbf{X}} g) \quad (q.e.d.)\end{aligned}$$

3.12.4.2 Lie Derivative of a Tensor Product

Let $\mathbf{S}$ and $\mathbf{T}$ be tensor fields and $\mathbf{X}$ a vector field. The Lie derivative of tensor product is given as

$$\pounds_{\mathbf{X}}(\mathbf{S} \otimes \mathbf{T}) = (\pounds_{\mathbf{X}}\mathbf{S}) \otimes \mathbf{T} + \mathbf{S} \otimes (\pounds_{\mathbf{X}}\mathbf{T}) \tag{3.140}$$

Proof

$$\begin{aligned}\pounds_{\mathbf{X}}(\mathbf{S} \otimes \mathbf{T}) &= [\mathbf{X}, \mathbf{S} \otimes \mathbf{T}] \\ &= [\mathbf{X}, \mathbf{S}] \otimes \mathbf{T} + \mathbf{S} \otimes [\mathbf{X}, \mathbf{T}] \\ &= (\pounds_{\mathbf{X}}\mathbf{S}) \otimes \mathbf{T} + \mathbf{S} \otimes (\pounds_{\mathbf{X}}\mathbf{T})\end{aligned}$$

The Lie derivative of mixed tensors $\mathbf{T}$ with respect to a vector field $\mathbf{X}$ can be written in the Lie derivative components and their bases of the coordinates.

$$\pounds_{\mathbf{X}}\mathbf{T} \equiv [\mathbf{X}, \mathbf{T}] = (\pounds_{\mathbf{X}} T)^{i \ldots k}_{l \ldots n} \partial_i \otimes \ldots \otimes \partial_k \otimes du^l \otimes \ldots \otimes du^n$$

where $\partial_i,\ldots,\partial_k$ and $du^l,\ldots,du^n$ are the covariant and contravariant bases of the local coordinates $\{u^i\}$, respectively (cf. Sect. 3.14), $\otimes$ is the tensor product, and $T^{i\ldots k}_{l\ldots n}$ are the components of the mixed tensor $\mathbf{T}$.

The Lie derivative components of the tensor field $\mathbf{T}$ with respect to a vector field $\mathbf{X}$ are computed in [9]:

$$(\pounds_{\mathbf{X}}T)^{i\ldots k}_{l\ldots n} = X^p T^{i\ldots k}_{l\ldots n,p} + X^p_{,l} T^{i\ldots k}_{p\ldots n} + \ldots - X^k_{,p} T^{i\ldots p}_{l\ldots n} - \ldots \tag{3.141}$$

in which the partial derivatives are defined as

$$T^{i\ldots k}_{l\ldots n,p} \equiv \frac{\partial T^{i\ldots k}_{l\ldots n}}{\partial u^p};\; X^p_{,l} \equiv \frac{\partial X^p}{\partial u^l};\, X^k_{,p} \equiv \frac{\partial X^k}{\partial u^p}$$

3.12.4.3 Lie Derivative of a Differential Form

Let $d\omega$ be a differential k-form that is dual to its k-order covariant tensor field on a differentiable manifold M and $\mathbf{X}$ be a vector field. The Lie derivative of the differential k-form $d\omega$ with respect to $\mathbf{X}$ is given as

$$\pounds_{\mathbf{X}}(d\omega) = d(\pounds_{\mathbf{X}}\omega) \tag{3.142}$$

Proof
At first, Eq. (3.142) is derived for a differential zero-form on a differentiable N-dimensional manifold M.

The zero-form f is defined as a smooth function on the manifold M. Its differential zero-form df can be written in the dual bases du^j using Einstein's summation convention.

$$df = \frac{\partial f(u^j)}{\partial u^j} du^j \text{ for } j = 1, 2, \ldots, N$$

Using the chain rule of differentiation, the LHS of Eq. (3.142) for $\omega \equiv f$ can be calculated as

$$\begin{aligned}\pounds_{\mathbf{X}}(df) &= \pounds_{\mathbf{X}}\left(\frac{\partial f}{\partial u^j} du^j\right)\\ &= \pounds_{\mathbf{X}}\left(\frac{\partial f}{\partial u^j}\right) du^j + \frac{\partial f}{\partial u^j} \pounds_{\mathbf{X}}\left(du^j\right)\end{aligned}$$

Interchanging the index i with j in the second term in Eq. (3.143a), one obtains

$$\begin{aligned}
\pounds_{\mathbf{X}}(df) &= X^i \frac{\partial}{\partial u^i}\left(\frac{\partial f}{\partial u^j}\right) du^j + \frac{\partial f}{\partial u^j}\left(\frac{\partial X^j}{\partial u^i} du^i\right) \\
&= X^i \frac{\partial}{\partial u^i}\left(\frac{\partial f}{\partial u^j}\right) du^j + \frac{\partial f}{\partial u^i}\left(\frac{\partial X^i}{\partial u^j} du^j\right) \\
&= \left(X^i \frac{\partial^2 f}{\partial u^i \partial u^j} + \frac{\partial f}{\partial u^i}\frac{\partial X^i}{\partial u^j}\right) du^j
\end{aligned} \tag{3.143a}$$

Using the chain rule of differentiation, the RHS of Eq. (3.142) for $\omega \equiv f$ is computed as

$$\begin{aligned}
d(\pounds_{\mathbf{X}} f) = d(\mathbf{X}f) &= d\left(X^i \frac{\partial f}{\partial u^i}\right) \\
&= X^i d\left(\frac{\partial f}{\partial u^i}\right) + \frac{\partial f}{\partial u^i} dX^i \\
&= \left(X^i \frac{\partial^2 f}{\partial u^i \partial u^j} + \frac{\partial f}{\partial u^i}\frac{\partial X^i}{\partial u^j}\right) du^j
\end{aligned} \tag{3.143b}$$

Subtracting Eq. (3.143a) from Eq. (3.143b), Eq. (3.142) is proved for a differential zero-form.

$$\begin{aligned}
&\pounds_{\mathbf{X}}(df) - d(\pounds_{\mathbf{X}} f) = 0 \\
&\Rightarrow \pounds_{\mathbf{X}}(df) = d(\pounds_{\mathbf{X}} f)
\end{aligned}$$

This equation is called the Cartan's formula in the special case for a differential zero-form *df*.

In the following step, Eq. (3.142) is derived for a differential one-form on a differentiable *N*-dimensional manifold *M*.

The one-form ω on the manifold *M* can be generally written in terms of the dual bases du^j using Einstein summation convention, cf. Eq. (3.179).

$$\omega = \omega_j du^j$$

Thus, the differential one-form results in

$$d\omega = \frac{\partial \omega}{\partial u^j} du^j$$

The LHS of Eq. (3.142) can be calculated as

$$\begin{aligned}
\pounds_{\mathbf{X}}(d\omega) &= \pounds_{\mathbf{X}}\left(\frac{\partial \omega}{\partial u^j} du^j\right) \\
&= \pounds_{\mathbf{X}}\left(\frac{\partial \omega}{\partial u^j}\right) du^j + \frac{\partial \omega}{\partial u^j} \pounds_{\mathbf{X}}(du^j)
\end{aligned}$$

within

$$\begin{aligned}
\pounds_{\mathbf{X}}(du^j) &= d(\pounds_{\mathbf{X}} u^j) = d\left(X^i \frac{\partial u^j}{\partial u^i}\right) = d\left(X^i \delta_i^j\right) \\
&= dX^j = \frac{\partial X^j}{\partial u^i} du^i \text{for } i = 1, 2, \ldots, N
\end{aligned}$$

Further calculating the LHS of Eq. (3.142), one obtains interchanging i with j in the second term on the RHS of Eq. (3.143c)

$$\begin{aligned}
\pounds_{\mathbf{X}}(d\omega) &= X^i \frac{\partial}{\partial u^i}\left(\frac{\partial \omega}{\partial u^j}\right) du^j + \frac{\partial \omega}{\partial u^j}\left(\frac{\partial X^j}{\partial u^i} du^i\right) \\
&= X^i \frac{\partial}{\partial u^i}\left(\frac{\partial \omega}{\partial u^j}\right) du^j + \frac{\partial \omega}{\partial u^i}\left(\frac{\partial X^i}{\partial u^j} du^j\right) \\
&= \left(X^i \frac{\partial^2 \omega}{\partial u^i \partial u^j} + \frac{\partial \omega}{\partial u^i}\frac{\partial X^i}{\partial u^j}\right) du^j
\end{aligned} \tag{3.143c}$$

Next, the RHS of Eq. (3.142) is computed using the chain rule of differentiation.

$$\begin{aligned}
d(\pounds_{\mathbf{X}}\omega) &= d\left(X^i \frac{\partial}{\partial u^i}\omega\right) \\
&= X^i d\left(\frac{\partial \omega}{\partial u^i}\right) + \frac{\partial \omega}{\partial u^i} dX^i \\
&= \left(X^i \frac{\partial^2 \omega}{\partial u^i \partial u^j} + \frac{\partial \omega}{\partial u^i}\frac{\partial X^i}{\partial u^j}\right) du^j
\end{aligned} \tag{3.143d}$$

Comparing Eqs. (3.143c) and (3.143d), Eq. (3.142) is proved for a differential one-form.

$$\pounds_{\mathbf{X}}(d\omega) = d(\pounds_{\mathbf{X}}\omega)$$

Finally, Eq. (3.142) is derived for an arbitrary differential k-form ($k > 0$) on a differentiable N-dimensional manifold M using the Cartan's formula.

Let C be a fiber bundle of the manifold M and a point $p \in C$. A k-form at the point p on $C \in M$ is an element of the cotangent bundle $T^*M \subset M$ consisting of k cotangent spaces (cf. Sect. 3.14).

Generally, the k-form on the manifold M is defined in local coordinates with the dual bases du^i as

$$\omega = \sum_{i_1 < i_2 < \cdots < i_k}^{N} \omega_{i_1 i_2 \ldots i_k}\left(u^i\right) du^{i_1} \wedge du^{i_2} \wedge \cdots \wedge du^{i_k}$$

where $\wedge$ is the wedge (exterior) product; ω_i are smooth functions of the coordinates u^i.

The **Cartan's formula** of a k-form ω ($k > 0$) is derived in [8].

$$£_{\mathbf{X}}\omega = i_{\mathbf{X}} d\omega + d(i_{\mathbf{X}}\omega) \tag{3.144a}$$

where the notation $i_{\mathbf{X}}\omega$ is called the interior product of the k-form ω with respect to the vector field $\mathbf{X}$ on the differentiable N-dimensional manifold M and defined as

$$\begin{aligned} i_{\mathbf{X}}\omega &\equiv (\omega, \mathbf{X}) = \omega(\mathbf{X}) \\ &= X^j \frac{\partial \omega}{\partial u^j} \end{aligned}$$

From the Cartan's formula, one obtains the interior product of the differential k-form.

$$i_{\mathbf{X}} d\omega = £_{\mathbf{X}}\omega - d(i_{\mathbf{X}}\omega) \tag{3.144b}$$

Changing the k-form ω into its exterior derivative $d\omega$ in the Cartan's formula in Eq. (3.144a), using Eq. (3.144b) and $dd = 0$, Eq. (3.142) has been derived.

$$\begin{aligned} £_{\mathbf{X}}(d\omega) &= i_{\mathbf{X}} d(d\omega) + d(i_{\mathbf{X}} d\omega) \\ &= i_{\mathbf{X}} d(d\omega) + d(£_{\mathbf{X}}\omega - d(i_{\mathbf{X}}\omega)) \\ &= i_{\mathbf{X}} dd\omega + d(£_{\mathbf{X}}\omega) - dd(i_{\mathbf{X}}\omega) \\ &= d(£_{\mathbf{X}}\omega) : q.e.d. \end{aligned}$$

3.12.4.4 Lie Derivative of a One-Form and Vector Product

Let $\mathbf{X}$ and $\mathbf{Y}$ be vector fields, and ω a one-form. The Lie derivative of product between the one-form and vector is given as

$$£_{\mathbf{X}}(\omega\mathbf{Y}) - (£_{\mathbf{X}}\omega)\mathbf{Y} = \omega[\mathbf{X}, \mathbf{Y}] \tag{3.145}$$

Proof

$$\begin{aligned} £_{\mathbf{X}}(\omega\mathbf{Y}) &= [\mathbf{X}, \omega\mathbf{Y}] \\ &= [\mathbf{X}, \omega]\mathbf{Y} + \omega[\mathbf{X}, \mathbf{Y}] \\ &= (\mathbf{X}\omega)\mathbf{Y} + \omega[\mathbf{X}, \mathbf{Y}] \\ &= (£_{\mathbf{X}}\omega)\mathbf{Y} + \omega(£_{\mathbf{X}}\mathbf{Y}) \end{aligned}$$

Therefore,

$$\pounds_{\mathbf{X}}(\omega\mathbf{Y}) - \left(\pounds_{\mathbf{X}}\omega\right)\mathbf{Y} = \omega[\mathbf{X},\mathbf{Y}] \quad (q.e.d.)$$

3.12.4.5 Lie Derivative of a One-Form

Let ω be a one-form and $\mathbf{X}$ a vector field. The Lie derivative of one-form is given as

$$\pounds_{\mathbf{X}}\omega = (\mathbf{X}\omega_i)du^i + \omega_i dX^i \tag{3.146}$$

Proof From Eqs. (3.137) and (3.145) one obtains interchanging the index i with j.

$$\begin{aligned}
(\pounds_{\mathbf{X}}\omega)_i Y^i &= \pounds_{\mathbf{X}}\left(\omega_i Y^i\right) - \omega_i(\pounds_{\mathbf{X}}\mathbf{Y})^i \\
&= X^j \frac{\partial\left(\omega_i Y^i\right)}{\partial u^j} - \omega_i\left(X^j\frac{\partial Y^i}{\partial u^j} - Y^j\frac{\partial X^i}{\partial u^j}\right) \\
&= X^j\omega_i\frac{\partial Y^i}{\partial u^j} + X^jY^i\frac{\partial \omega_i}{\partial u^j} - \omega_i X^j\frac{\partial Y^i}{\partial u^j} + \omega_i Y^j\frac{\partial X^i}{\partial u^j} \\
&= X^jY^i\frac{\partial \omega_i}{\partial u^j} + \omega_i Y^j\frac{\partial X^i}{\partial u^j} \quad (i \leftrightarrow j) \\
&= \left(X^j\frac{\partial \omega_i}{\partial u^j} + \omega_j\frac{\partial X^j}{\partial u^i}\right)Y^i
\end{aligned} \tag{3.147}$$

Equation (3.147) gives

$$(\pounds_{\mathbf{X}}\omega)_i = X^j\frac{\partial \omega_i}{\partial u^j} + \omega_j\frac{\partial X^j}{\partial u^i} \tag{3.148}$$

Multiplying Eq. (3.148) by du^i and interchanging i with j in the second term on the RHS of Eq. (3.149), one obtains

$$\begin{aligned}
(\pounds_{\mathbf{X}}\omega)_i du^i &= X^j\frac{\partial \omega_i}{\partial u^j}du^i + \omega_j\frac{\partial X^j}{\partial u^i}du^i \\
&= (\mathbf{X}\omega_i)du^i + \omega_i dX^i
\end{aligned} \tag{3.149}$$

Using the chain rule of differentiation, the RHS of Eq. (3.149) can be written as

$$\begin{aligned}
(\mathbf{X}\omega_i)du^i + \omega_i dX^i &= (\pounds_{\mathbf{X}}\omega_i)du^i + \omega_i\pounds_{\mathbf{X}}\left(du^i\right) \\
&= \pounds_{\mathbf{X}}\left(\omega_i du^i\right) \\
&\equiv \pounds_{\mathbf{X}}\omega
\end{aligned} \tag{3.150}$$

where the one-form ω can be expressed in the contravariant basis vector du^i using Einstein summation convention, cf. Eq. (3.179).

$$\omega = \omega_i du^i$$

Thus, one obtains the Lie derivative of the one-form in the direction of the vector field **X**.

$$\begin{aligned} \pounds_{\mathbf{X}}\omega &= (\pounds_{\mathbf{X}}\omega)_i du^i \\ &= (\mathbf{X}\omega_i)du^i + \omega_i dX^i \quad (q.e.d.) \end{aligned}$$

3.12.5 Torsion and Curvature in a Distorted and Curved Manifold

The normal vector field **N** perpendicular to the surface of the manifold M is dragged in two different paths from the same point P_{11} via P_{21} to Q in the one path; and via P_{12} to S in the other path, as shown in Fig. 3.9. Due to the effect of space torsions and surface curvatures, the vector field **N** does not close the connection loop at the path ends Q and S of the dragging paths. The gap of the path ends is O(ε^2) in a distorted and curved manifold and is reduced to the order of O(ε^3) in an only curved manifold [7].

The Lie derivative of the vector **Y** with respect to the vector field **X** is induced by the space torsion of the distorted manifold. As a consequence, the torsion tensor ε^2 [**X,Y**] generates the open connection gap QR (see Fig. 3.9). The Riemann surface curvature is to blame for the other open connection gap RS on the order of O(ε^3) in the curved manifold.

Therefore, the connection loop is always closed in a torsion-free and flat space.

$$[\mathbf{Y},\mathbf{X}] - [\mathbf{X},\mathbf{Y}] = 0 \Leftrightarrow \pounds_{\mathbf{Y}}\mathbf{X} = \pounds_{\mathbf{X}}\mathbf{Y} \tag{3.151}$$

The curvature equation of a distorted and curved manifold can be written by means of the Lie formulations, Riemann curvature tensors, and covariant metric coefficients [7].

$$\begin{aligned} &[\mathbf{Y},\mathbf{X}] - [\mathbf{X},\mathbf{Y}] = \varepsilon^2[\mathbf{X},\mathbf{Y}] + \varepsilon^2 R_{lijk} \\ &\Leftrightarrow \pounds_{\mathbf{Y}}\mathbf{X} - \pounds_{\mathbf{X}}\mathbf{Y} = \varepsilon^2\pounds_{\mathbf{X}}\mathbf{Y} + \varepsilon^2 g_{nl}R^n_{ijk} \end{aligned} \tag{3.152}$$

where R_{lijk} is the Riemann curvature tensors of the curved manifold M.

3.12.6 Killing Vector Fields

The Killing vector field **K** is defined as a vector field in an N-dimensional manifold in which the Lie derivative of the metric tensor **g** with respect to the vector field **K** along the congruence equals zero.

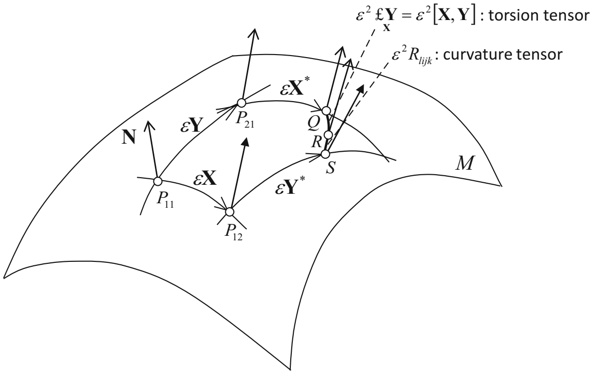

Fig. 3.9 Connection loop of vector fields in a distorted and curved manifold M

$$\pounds_{\mathbf{K}} \mathbf{g} = 0 \tag{3.153}$$

Equation (3.153) shows that the metric tensor $\mathbf{g}$ is invariant on the manifold with respect to the Killing vector field $\mathbf{K}$.

The covariant tensor components of the Lie derivative of the metric tensor with respect to the Killing vector field $\mathbf{K}$ can be expressed in [8].

$$(\pounds_{\mathbf{K}} \mathbf{g})_{ij} = K^k \frac{\partial g_{ij}}{\partial u^k} + g_{ik} \frac{\partial K^k}{\partial u^j} + g_{kj} \frac{\partial K^k}{\partial u^i} = 0 \tag{3.154}$$

Equation (3.154) can be written in one-dimensional coordinate u^k with respect to the Killing vector field $\mathbf{K}$.

$$(\pounds_{\mathbf{K}} \mathbf{g})_{ij} = \frac{\partial g_{ij}}{\partial u^k} = 0 \tag{3.155}$$

Therefore, if the covariant metric coefficient is independent of any coordinate, the basis of the coordinate is a Killing vector.

As an example for the Killing vector field, the covariant metric coefficients of the spherical coordinates (r,φ,θ) are given in

$$\begin{aligned}
g_{rr} &= \mathbf{g}_r \cdot \mathbf{g}_r \equiv \frac{\partial}{\partial r} \cdot \frac{\partial}{\partial r} = 1 \\
g_{\varphi\varphi} &= \mathbf{g}_\varphi \cdot \mathbf{g}_\varphi \equiv \frac{\partial}{\partial \varphi} \cdot \frac{\partial}{\partial \varphi} = r^2 \\
g_{\theta\theta} &= \mathbf{g}_\theta \cdot \mathbf{g}_\theta \equiv \frac{\partial}{\partial \theta} \cdot \frac{\partial}{\partial \theta} = r^2 \sin^2\varphi
\end{aligned} \tag{3.156}$$

Equation (3.156) shows that the metric coefficients are independent of the coordinates (r,φ,θ). Hence, the basis vectors $\mathbf{g}_r$, $\mathbf{g}_\varphi$, and $\mathbf{g}_\theta$ are the Killing vectors.

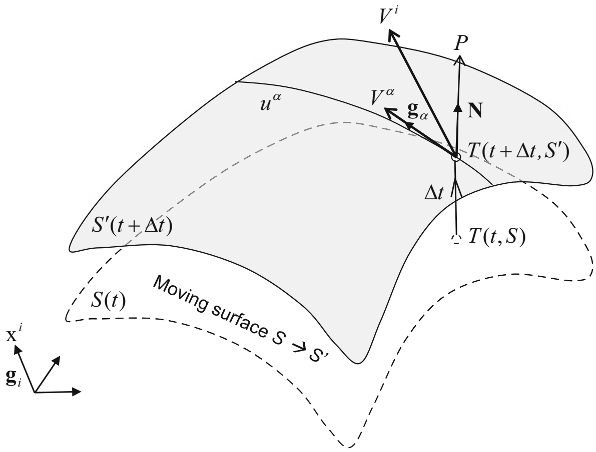

Fig. 3.10 Invariant fields on a moving surface $S(t)$

3.13 Invariant Time Derivatives on Moving Surfaces

In the following section, the invariant time derivatives of tensors are applied to a surface $S(t)$ moving with a velocity vector **V** in the ambient coordinate system. For this case, the invariant time derivative of an invariant field $T(t, S)$ parameterized by the time t and moving surface S can be calculated in the surface coordinate. Generally, two coordinates of the *unchanged* ambient coordinate x^i with the covariant basis $\mathbf{g}_i$ and the *moving* surface coordinate u^α with the covariant basis $\mathbf{g}_\alpha$ are used in the moving surface $S(t)$, as shown in Fig. 3.10.

The surface S at the time t moves to the new surface position S' at the time $t + \Delta t$ at which the invariant field $T(t, S)$ at the time t is changed into $T(t + \Delta t, S')$ in a very short time interval Δt. The time-dependent surface S moves with a coordinate velocity V^i in the ambient coordinate x^i [10].

The ambient coordinate velocity of the moving surface $S(t)$ in the coordinate x^{i} can be defined by

$$V^i \equiv \frac{\partial x^i(t,S)}{\partial t} \tag{3.157}$$

The tangential coordinate velocity V^α results from projecting the ambient coordinate velocity V^i onto the surface along the surface coordinate u^α. To calculate the tangential coordinate velocity, the surface velocity vector **V** can be formulated in both ambient and surface coordinates using the chain rule of differentiation.

$$\begin{aligned}\mathbf{V} &= V^{\alpha}\mathbf{g}_{\alpha} \\ &= V^{i}g_{i} = V^{i}\frac{\partial \mathbf{r}}{\partial x^{i}} \\ &= V^{i}\frac{\partial \mathbf{r}}{\partial u^{\alpha}}\frac{\partial u^{\alpha}}{\partial x^{i}} = V^{i}\frac{\partial u^{\alpha}}{\partial x^{i}}\mathbf{g}_{\alpha} \\ &\equiv \left(V^{i}x^{\alpha}_{,i}\right)\mathbf{g}_{\alpha}\end{aligned} \tag{3.158}$$

where the derivative $x_{,i}{}^{\alpha}$ is called the shift tensor between the ambient and surface coordinates.

Thus, the tangential coordinate velocity V^{α} results from the ambient coordinate velocity and shift tensor.

$$V^{\alpha} = V^{i}\frac{\partial u^{\alpha}}{\partial x^{i}} = V^{i}x^{\alpha}_{,i} \tag{3.159a}$$

Analogously, one obtains

$$V^{i} = V^{\alpha}\frac{\partial x^{i}}{\partial u^{\alpha}} = V^{\alpha}x^{i}_{,\alpha} \tag{3.159b}$$

3.13.1 Invariant Time Derivative of an Invariant Field

The invariant time derivative of an invariant field $T(t, S)$ can be defined as the time change rate of the invariant field itself and its change rates along the surface coordinates between the old and new surface positions [10].

$$\dot{\nabla}\, T \equiv \frac{\partial T(t,S)}{\partial t} - V^{\alpha}\nabla_{\alpha}T \tag{3.160}$$

At first, the covariant surface derivative of a first-order tensor $\mathbf{T}$ can be written as

$$\begin{aligned}\nabla_{\alpha}\mathbf{T} = \frac{\partial \mathbf{T}}{\partial u^{\alpha}} &= \frac{\partial\left(T^{i}\mathbf{g}_{i}\right)}{\partial u^{\alpha}} \\ &= \frac{\partial T^{i}}{\partial u^{\alpha}}\mathbf{g}_{i} + T^{i}\frac{\partial \mathbf{g}_{i}}{\partial u^{\alpha}} \\ &= \frac{\partial T^{i}}{\partial u^{\alpha}}\mathbf{g}_{i} + T^{i}\frac{\partial \mathbf{g}_{i}}{\partial x^{j}}\frac{\partial x^{j}}{\partial u^{\alpha}}\end{aligned} \tag{3.161a}$$

Using Eq. (2.158), the partial derivative of the basis $\mathbf{g}_i$ results in

$$\frac{\partial \mathbf{g}_i}{\partial x^j} = \Gamma_{ij}^k \mathbf{g}_k \tag{3.161b}$$

Inserting Eq. (3.161b) into Eq. (3.161a), one obtains

$$\begin{aligned}\nabla_\alpha \mathbf{T} &= \frac{\partial T^i}{\partial u^\alpha}\mathbf{g}_i + \frac{\partial x^j}{\partial u^\alpha}\Gamma_{ij}^k T^i \mathbf{g}_k \\ &= \left(\frac{\partial T^k}{\partial u^\alpha} + \frac{\partial x^j}{\partial u^\alpha}\Gamma_{ij}^k T^i\right)\mathbf{g}_k \\ &\equiv \left(\nabla_\alpha T^k\right)\mathbf{g}_k\end{aligned} \tag{3.161c}$$

Thus,

$$\begin{aligned}&\nabla_\alpha T^k = \frac{\partial T^k}{\partial u^\alpha} + \frac{\partial x^j}{\partial u^\alpha}\Gamma_{ij}^k T^i \\ &\Leftrightarrow \nabla_\alpha T^i = \frac{\partial T^i}{\partial u^\alpha} + \frac{\partial x^j}{\partial u^\alpha}\Gamma_{jm}^i T^m\end{aligned} \tag{3.161d}$$

The covariant surface derivative of a contravariant first-order tensor results in using Eq. (3.161d) and chain rule of coordinates.

$$\begin{aligned}\nabla_\alpha T^i \equiv T^i\big|_\alpha &= \frac{\partial T^i}{\partial u^\alpha} + \frac{\partial x^j}{\partial u^\alpha}\Gamma_{jm}^i T^m \\ &= \frac{\partial T^i}{\partial x^j}\frac{\partial x^j}{\partial u^\alpha} + \frac{\partial x^j}{\partial u^\alpha}\Gamma_{jm}^i T^m \\ &= \frac{\partial x^j}{\partial u^\alpha}\left(\frac{\partial T^i}{\partial x^j} + \Gamma_{jm}^i T^m\right) \\ &= x^j_{,\alpha} T^i\big|_j \equiv x^j_{,\alpha}\nabla_j T^i\end{aligned} \tag{3.161e}$$

Analogously, the covariant surface derivative of a mixed second-order tensor results using Eq. (2.211a) and chain rule of coordinates in

$$\begin{aligned}\nabla_\alpha T^i_j \equiv T^i_j\Big|_\alpha &= \frac{\partial T^i_j}{\partial u^\alpha} + \frac{\partial x^k}{\partial u^\alpha}\left(\Gamma_{km}^i T^m_j - \Gamma_{kj}^n T^i_n\right) \\ &= \frac{\partial T^i_j}{\partial x^k}\frac{\partial x^k}{\partial u^\alpha} + \frac{\partial x^k}{\partial u^\alpha}\left(\Gamma_{km}^i T^m_j - \Gamma_{kj}^n T^i_n\right) \\ &= \frac{\partial x^k}{\partial u^\alpha}\left(\frac{\partial T^i_j}{\partial x^k} + \Gamma_{km}^i T^m_j - \Gamma_{kj}^n T^i_n\right) \\ &= x^k_{,\alpha} T^i_j\Big|_k \equiv x^k_{,\alpha}\nabla_k T^i_j\end{aligned} \tag{3.161f}$$

The ambient coordinate is dependent on time and the surface coordinate of the moving surface S. Thus, the invariant field T can be expressed as

$$T(t,S) = T(t,\ x(t,S)\ ) \tag{3.162}$$

Using the Taylor series and the chain rule of differentiation, the partial time derivative of T with two independent variables of t and S can be calculated as

$$\begin{aligned}\frac{\partial T(t,S)}{\partial t} &= \frac{\partial T(t,x)}{\partial t} + \frac{\partial T(t,x)}{\partial x^i}\frac{\partial x^i(t,S)}{\partial t}\\ &= \frac{\partial T(t,x)}{\partial t} + (\nabla_i T)\, V^i\end{aligned} \tag{3.163}$$

because

$$\nabla_i T = \frac{\partial T(t,x)}{\partial x^i}; \quad V^i = \frac{\partial x^i(t,S)}{\partial t} \tag{3.164}$$

The invariant time derivative in Eq. (3.160) can be rewritten using Eqs. (3.161e) and (3.163).

$$\begin{aligned}\dot{\nabla} T &\equiv \frac{\partial T(t,S)}{\partial t} - V^\alpha \nabla_\alpha T\\ &= \frac{\partial T(t,x)}{\partial t} + V^i \nabla_i T - V^\alpha x^k_{,\alpha} \nabla_k T\\ &= \frac{\partial T(t,x)}{\partial t} + V^i \nabla_i T - V^\alpha x^i_{,\alpha} \nabla_i T\\ &= \frac{\partial T(t,x)}{\partial t} + \left(V^i - x^i_{,\alpha} V^\alpha\right) \nabla_i T\end{aligned} \tag{3.165a}$$

The second term on the RHS in Eq. (3.165a) can be further calculated using Eq. (3.159a).

$$\begin{aligned}V^i - x^i_{,\alpha} V^\alpha &= V^i - \frac{\partial x^i}{\partial u^\alpha}\left(\frac{\partial u^\alpha}{\partial x^j} V^j\right)\\ &= V^i - x^i_{,\alpha} x^\alpha_{,j} V^j\\ &= V^j\left(\delta^i_j - x^i_{,\alpha} x^\alpha_{,j}\right)\end{aligned} \tag{3.165b}$$

The useful relation between the contra- and covariant normal vector components and shift tensors of coordinates is derived by [10]:

$$N^i N_j + x^i_{,\alpha} x^\alpha_{,j} = \delta^i_j \tag{3.165c}$$

in which δ^i_j is the Kronecker delta.

Substituting Eq. (3.165c) into Eq. (3.165b), the invariant time derivative of T given in Eq. (3.165a) can be rewritten as

$$\begin{aligned}\dot{\nabla} T &= \frac{\partial T(t,x)}{\partial t} + \left(V^j N_j\right) N^i \nabla_i T \\ &= \frac{\partial T(t,x)}{\partial t} + P N^i \nabla_i T\end{aligned} \tag{3.165d}$$

where P is the normal velocity at a given point on the moving surface S, as displayed in Fig. 3.10. In fact, the normal velocity is the projection of the ambient coordinate velocity V^{i} on the surface normal N_i.

$$\begin{aligned}&P = V^i N_i \Rightarrow \\ &\mathbf{P} = P\mathbf{N} \\ &\quad = \left(V^i N_i\right)\mathbf{N} = V^i N_i N^j \mathbf{g}_j\end{aligned} \tag{3.165e}$$

3.13.2 Invariant Time Derivative of Tensors

Analogously, the invariant time derivative of tensors can be derived from the invariant field. The contravariant tensor can be written in the covariant basis $\mathbf{g}_i$.

$$\mathbf{T} = T^i \mathbf{g}_i \tag{3.166}$$

The invariant time derivative of the tensor $\mathbf{T}$ can be expressed on the moving surface S according to Eq. (3.160) [10].

$$\dot{\nabla}\, \mathbf{T} = \frac{\partial \mathbf{T}(t,S)}{\partial t} - V^\alpha \nabla_\alpha \mathbf{T} \tag{3.167}$$

Substituting Eq. (3.166) into Eq. (3.167), one obtains

$$\begin{aligned}\dot{\nabla}\, \mathbf{T} &= \frac{\partial \left(T^i \mathbf{g}_i\right)}{\partial t} - V^\alpha \nabla_\alpha \left(T^i \mathbf{g}_i\right) \\ &= \frac{\partial T^i}{\partial t}\mathbf{g}_i + T^i \frac{\partial \mathbf{g}_i}{\partial t} - V^\alpha \left(\nabla_\alpha T^i\right)\mathbf{g}_i\end{aligned} \tag{3.168}$$

Using Eqs. (2.158) and (3.157), the time derivative of the coordinate basis $\mathbf{g}_i$ in Eq. (3.168) can be calculated.

$$\begin{aligned}\dot{\mathbf{g}}_i &\equiv \frac{\partial \mathbf{g}_i}{\partial t} = \frac{\partial \mathbf{g}_i}{\partial x^j}\frac{\partial x^j}{\partial t} = \mathbf{g}_{i,j}\frac{\partial x^j}{\partial t} \\ &= \Gamma_{ij}^k \frac{\partial x^j}{\partial t}\mathbf{g}_k = \Gamma_{ij}^k V^j \mathbf{g}_k\end{aligned} \tag{3.169}$$

Therefore, the invariant time derivative of the tensor $\mathbf{T}$ can be rewritten as

$$
\begin{aligned}
\dot{\nabla}\,\mathbf{T} &= \frac{\partial T^i}{\partial t}\mathbf{g}_i + V^j \Gamma_{ij}^k T^i \mathbf{g}_k - V^\alpha \left(\nabla_\alpha T^i\right)\mathbf{g}_i \\
&= \left(\frac{\partial T^k}{\partial t} + V^j \Gamma_{ij}^k T^i - V^\alpha \nabla_\alpha T^k\right)\mathbf{g}_k \\
&\equiv \dot{\nabla}\, T^k \mathbf{g}_k
\end{aligned}
\tag{3.170}
$$

The invariant time derivative of a contravariant tensor T^i is given from Eq. (3.170).

$$
\dot{\nabla}\, T^i = \frac{\partial T^i}{\partial t} + V^j \Gamma_{jk}^i T^k - V^\alpha \nabla_\alpha T^i \tag{3.171}
$$

Similarly, one obtains the invariant time derivative of a covariant tensor T_i

$$
\dot{\nabla}\, T_i = \frac{\partial T_i}{\partial t} - V^j \Gamma_{ji}^k T_k - V^\alpha \nabla_\alpha T_i \tag{3.172}
$$

The invariant time derivative of a mixed tensor $T_j^{\ i}$ can be derived in

$$
\begin{aligned}
\dot{\nabla}\, T_j^i &= \frac{\partial T_j^i}{\partial t} + V^k \Gamma_{km}^i T_j^m - V^k \Gamma_{kj}^n T_n^i - V^\alpha \nabla_\alpha T_j^i \\
&= \frac{\partial T_j^i}{\partial t} + V^k \left(\Gamma_{km}^i T_j^m - \Gamma_{kj}^n T_n^i\right) - V^\alpha \nabla_\alpha T_j^i.
\end{aligned}
\tag{3.173}
$$

The general invariant time derivative of a mixed fourth-order tensor can be derived in [10].

$$
\begin{aligned}
\dot{\nabla}\, T_{j\beta}^{i\alpha} = \frac{\partial T_{j\beta}^{i\alpha}}{\partial t} &- V^\gamma \nabla_\gamma T_{j\beta}^{i\alpha} + V^p \Gamma_{pq}^i T_{j\beta}^{q\alpha} - V^p \Gamma_{pj}^q T_{q\beta}^{i\alpha} \\
&+ \dot{\Gamma}_{\ \delta}^{\alpha} T_{j\beta}^{i\delta} - \dot{\Gamma}_{\ \beta}^{\delta} T_{j\delta}^{i\alpha}
\end{aligned}
\tag{3.174}
$$

where the time derivative of the Christoffel symbols for a moving surface is defined as

$$
\dot{\Gamma}_{\ \beta}^{\alpha} = \nabla_\beta V^\alpha - P R_\beta^\alpha \tag{3.175}
$$

in which P is the normal velocity in Eq. (3.165e) and $R_\beta^{\ \alpha}$ is the mean curvature of the moving surface.

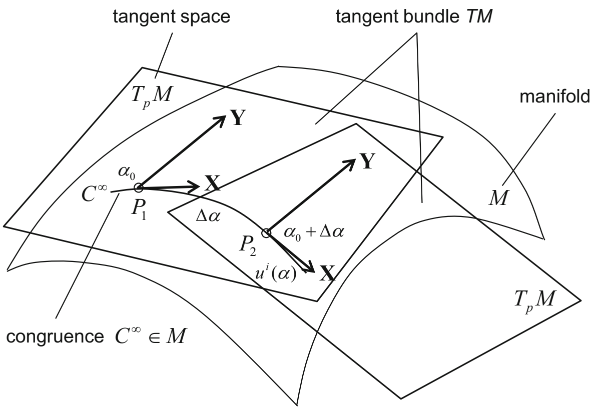

Fig. 3.11 Transport of a vector field **Y** in the tangent bundle TM

3.14 Tangent, Cotangent Bundles and Manifolds

Some definitions of spaces and manifolds are discussed before dealing with the Levi-Civita connection. Let C^∞ be a set of infinitely differentiable curves (congruence) on the manifold M. The vector $\mathbf{X}$ is tangent to the curve C^∞ at the point P_1 and moves along it as the parameter α varies in the local coordinate $u^i(\alpha)$, as shown in Fig. 3.11.

The tangent vector $\mathbf{X}$ lies on the tangent space T_pM (tangent surface) that is tangent to the curve C^∞ at the point P_1 on the manifold M. As the vector $\mathbf{X}$ moves along C^∞, the tangent space also changes from P_1 to P_2. Thus, the tangent spaces T_pM generate a tangent bundle TM that consists of all tangent spaces T_pM of the manifold M.

The covariant basis of the local coordinate $u^i(\alpha)$ on the tangent space T_pM is written as

$$\partial_i \equiv \frac{\partial}{\partial u^i} \tag{3.176}$$

Therefore, the tangent vector $\mathbf{X}$ on the tangent space $\mathrm{T_pM}$ is expressed in the covariant basis.

$$\begin{aligned}\mathbf{X} &= X^i \frac{\partial}{\partial u^i} \\ &= \frac{\partial u^i(\alpha)}{\partial \alpha}\frac{\partial}{\partial u^i} = \frac{d}{d\alpha}\end{aligned} \tag{3.177}$$

Furthermore, the dual space of the tangent space T_pM is called the cotangent space $T_p{}^*M$ in which the contravariant basis vectors du^i of the local coordinates $u^i(\alpha)$ are used for covariant tensors, covariant vectors, and differential forms. Analogously, the cotangent bundle T^*M covers all cotangent spaces $T_p{}^*M$ of the manifold M.

Due to orthonormality of the covariant and contravariant bases, their inner product (dot or scalar product) vanishes for $i \neq j$. Using the Kronecker delta, the contravariant basis (dual basis) du^j on the cotangent space $T_p{}^*M$ is defined as

$$\left(\partial_i, du^j\right) = \delta_i^j \tag{3.178}$$

In differential geometry, a differential one-form $d\omega$ on a differentiable manifold M is defined as a smooth cross section of the cotangent bundle T^*M of the manifold M. As a result, the differential one-form $d\omega$ is naturally dual to its first-order covariant tensor field of type (0,1) in the dual cotangent space. Sometimes, differential one-forms are called covariant vector fields or dual vector fields within physics.

Using Einstein summation convention, the one-form ω_p at the point P is generally used on the dual space (cotangent space) with the contravariant basis du^i.

$$\omega_p = \omega_i du^i \tag{3.179}$$

where ω_i is a function of u^i.

The differential one-form results from Eq. (3.179) in

$$d\omega_p = \left.\frac{\partial \omega}{\partial u^i}\right|_p du^i = \frac{\partial \omega_p}{\partial u^i} du^i \tag{3.180}$$

Using the Leibniz's rule, the product of two differential one-forms f_p and g_p is calculated.

$$d(fg)_p = f_p dg_p + g_p df_p \tag{3.181}$$

Differential 0-forms, 1-forms, and 2-forms are the special cases of the k-form ω that can be generally expressed in terms of the dual bases on a differentiable N-dimensional manifold M as (cf. Chap. 4)

$$\omega = \sum_{i_1 < i_2 < \cdots < i_k}^{N} h\left(u^{i_1}, \ldots, u^{i_k}\right) du^{i_1} \wedge du^{i_2} \wedge \cdots \wedge du^{i_k}$$

where $\wedge$ is the wedge (exterior) product; h is a smooth function of the coordinates u^i.

The differential k-form $d\omega$, that results from the k-form ω, is called the exterior derivative of ω. Note that differential forms, wedge products, and exterior derivatives are independent of any coordinates on the manifold M.

3.15 Levi-Civita Connection on Manifolds

Levi-Civita connection (LC connection) describes the process of transporting (dragging) a vector field **Y** with respect to another vector field **X** on a smooth differentiable N-dimensional manifold M (*affine connection*). This connection is related to the Riemann connection; therefore, it is sometime called the Riemann Levi-Civita connection (RLC connection).

The covariant derivative of the vector field **Y** with respect to the vector **X** moving along to a differentiable curve C^{∞} on the manifold M is generally used to formulate the transport of the vector field **Y** on the N-dimensional manifold M, as displayed in Fig. 3.12. As a result, the Levi-Civita connection on (M, g) associates the covariant derivative (absolute derivative) with respect to the geodesic parameter α of the coordinates $u^i(\alpha)$ on the manifold M.

The Levi-Civita connection ∇ is defined as the covariant derivative of **Y** with respect to the vector field **X.**

$$\begin{aligned} \nabla_{\mathbf{X}}\mathbf{Y} &\equiv \frac{D\mathbf{Y}(\alpha)}{D\alpha} = \left(\dot{Y}^i + Y^j \dot{u}^k \Gamma^i_{jk}\right)\partial_i \\ &\equiv (\nabla_{\mathbf{X}}\mathbf{Y})^i \partial_i \end{aligned} \tag{3.182}$$

where

$$\mathbf{X} = \frac{d}{d\alpha}; \quad \dot{Y}^i \equiv \frac{dY^i(\alpha)}{d\alpha}; \quad \dot{u}^k \equiv \frac{du^k(\alpha)}{d\alpha}$$

The component i of the Levi-Civita connection of the vector field **Y** with respect to the vector field **X** results from Eq. (3.182) in

$$(\nabla_{\mathbf{X}}\mathbf{Y})^i = \dot{Y}^i + Y^j \dot{u}^k \Gamma^i_{jk} \tag{3.183}$$

Proof
Using Leibniz rule of differentiation, one obtains

$$\begin{aligned} \nabla_{\mathbf{X}}\mathbf{Y} &= \nabla_{\mathbf{X}}\left(Y^i \partial_i\right) \\ &= \left(\nabla_{\mathbf{X}} Y^i\right)\partial_i + Y^i(\nabla_{\mathbf{X}}\partial_i) \end{aligned} \tag{3.184}$$

The covariant derivative of a function $f = Y^i(\alpha)$ is calculated as

$$\begin{aligned} &\nabla_{\mathbf{X}} f = \mathbf{X} f \\ &\Rightarrow \nabla_{\mathbf{X}} Y^i(\alpha) = \mathbf{X} Y^i(\alpha) \ = \frac{d}{d\alpha} Y^i(\alpha) \ \equiv \dot{Y}^i(\alpha) \end{aligned} \tag{3.185}$$

The covariant derivative of the coordinate basis with respect to vector **X** is expressed as

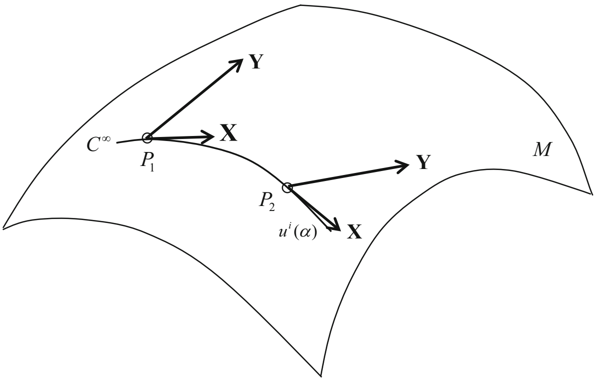

Fig. 3.12 Levi-Civita connection of vector field **Y** with respect to **X**

$$\begin{aligned}\nabla_{\mathbf{X}}\partial_i &= X^j \nabla_{\partial_j}(\partial_i) \\ &= X^j \Gamma^k_{ij}\partial_k\end{aligned} \tag{3.186}$$

in which the covariant derivative of the basis ∂_j of the coordinate u^j with respect to ∂_i is defined as

$$\nabla_{\partial_j}(\partial_i) = \Gamma^k_{ji}\partial_k = \Gamma^k_{ij}\partial_k \tag{3.187}$$

Substituting Eqs. (3.185) and (3.186) into Eq. (3.184), one obtains interchanging k with i in the second term on the RHS.

$$\begin{aligned}\nabla_{\mathbf{X}}\mathbf{Y} &= \dot{Y}^i\partial_i + Y^i\left(X^j\Gamma^k_{ij}\partial_k\right) \\ &= \dot{Y}^i\partial_i + Y^k X^j \Gamma^i_{kj}\partial_i \\ &= \left(\dot{Y}^i + Y^k X^j \Gamma^i_{kj}\right)\partial_i\end{aligned} \tag{3.188}$$

The vector **X** is written in the bases of the coordinates u^i.

$$\begin{aligned}\mathbf{X} &= \frac{d}{d\alpha} = \frac{du^i}{d\alpha}\frac{\partial}{\partial u^i} \\ &= X^i\frac{\partial}{\partial u^i} = X^i\partial_i\end{aligned}$$

Thus,

$$X^j = \frac{du^j}{d\alpha} \equiv \dot{u}^j(\alpha) \tag{3.189}$$

Inserting Eq. (3.189) into Eq. (3.188) and interchanging k with j, the Levi-Civita connection ∇ in Eq. (3.182) is derived.

$$\nabla_{\mathbf{X}}\mathbf{Y} = \left(\dot{Y}^i + Y^k \dot{u}^j \Gamma^i_{kj}\right)\partial_i = \left(\dot{Y}^i + Y^j \dot{u}^k \Gamma^i_{jk}\right)\partial_i$$

An affine connection ∇ is called the *Levi-Civita connection* on a smooth differentiable geometric manifold (M, g) if

It preserves the metric:

$\nabla_{\mathbf{X}} g = 0$ for all vector fields on M (*Riemann connection*)

and is torsion-free:

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}]$$

where [**X**,**Y**] is defined as the Lie bracket.

Generally, the covariant derivative of a vector field **Y** with respect to **X** at a non-parallel transport does not equal zero (cf. Fig. 3.12). In the case of a parallel transport, the direction of the vector field **Y** does not change as the vector **X** moves along the curve C^∞ on M. Then, the covariant derivative of **Y** with respect to **X** vanishes:

$$\nabla_{\mathbf{X}}\mathbf{Y} = \left(\dot{Y}^i + Y^j \dot{u}^k \Gamma^i_{jk}\right)\partial_i = 0 \tag{3.190}$$

Therefore, the first-order differential equation of **Y** for a parallel transport results in

$$\begin{aligned} &\frac{dY^i(\alpha)}{d\alpha} + \Gamma^i_{jk}\frac{du^k(\alpha)}{d\alpha}Y^j(\alpha) = 0 \\ \Leftrightarrow\ &\frac{du^k}{d\alpha}\left(\frac{\partial Y^i}{\partial u^k} + \Gamma^i_{jk}Y^j\right) = 0 \end{aligned} \tag{3.191}$$

Thus,

$$\dot{Y}^i(\alpha) = -\Gamma^i_{jk}\dot{u}^k Y^j \equiv S^i_j(\alpha)Y^j(\alpha)$$

A geodesic is the shortest distance between two any points on the manifold M. The curve C^∞ is geodesic (or auto-parallel) if the covariant derivative of vector fields **X** with respect to itself vanishes:

$$\nabla_{\mathbf{X}}\mathbf{X} = 0 \tag{3.192}$$

Substituting **X** into **Y** in Eq. (3.191), the second-order differential equation of a geodesic results in

$$\begin{aligned} X^i(\alpha) &= \frac{du^i}{d\alpha} \to Y^i(\alpha) \\ \Rightarrow \frac{d^2u^i}{d\alpha^2} &+ \Gamma^i_{jk}\frac{du^j}{d\alpha}\frac{du^k}{d\alpha} = 0 \text{ for } \forall i = 1,2,..,N \end{aligned} \tag{3.193}$$

Furthermore, let **X**, **Y**, **Z** be vector fields, f and h be linear functions. The Levi-Civita connection ∇ on a smooth differentiable geometric manifold (M, g) satisfies the following properties:

– ***Linearity:***

$$\nabla_{\mathbf{X}}(\mathbf{Y} + \mathbf{Z}) = \nabla_{\mathbf{X}}\mathbf{Y} + \nabla_{\mathbf{X}}\mathbf{Z} \tag{3.194a}$$

– ***Leibniz rule of differentiation:***

$$\begin{aligned} \nabla_{\mathbf{X}}(f\mathbf{Y}) &= f\nabla_{\mathbf{X}}\mathbf{Y} + (\nabla_{\mathbf{X}}f)\mathbf{Y} \\ &= f\nabla_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}f)\mathbf{Y} \end{aligned} \tag{3.194b}$$

– ***Function linearity:***

$$\nabla_{(f\mathbf{X}+h\mathbf{Y})}\mathbf{Z} = f\nabla_{\mathbf{X}}\mathbf{Z} + h\nabla_{\mathbf{Y}}\mathbf{Z} \tag{3.194c}$$

– ***Free torsion:***

$$\begin{aligned} &\mathbf{T}(\mathbf{X},\mathbf{Y}) \equiv \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}] = 0 \\ &\Rightarrow \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X},\mathbf{Y}] \end{aligned} \tag{3.194d}$$

where **T**(**X**,**Y**) is the torsion tensor; [**X**,**Y**] is the Lie bracket.

– ***Metric compatibility:***

$$\nabla_{\mathbf{X}}g(\mathbf{Y},\mathbf{Z}) = g(\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}) + g(\mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}) \tag{3.194e}$$

where g is the bilinear function $g(.,.)$ relating to the metric tensor on the geometric manifold (M, g).

The metric tensor **g**, a covariant second-order tensor of type (0,2) on the manifold M is defined as

$$\begin{aligned} \mathbf{g} &= g_{ij}du^i \otimes du^j; \\ g_{ij} &\equiv g(\partial_i, \partial_j) = \partial_i \cdot \partial_j \end{aligned} \tag{3.195}$$

The differential distance between two points can be written in a two-form of the metric tensor **g**:

$$ds = \sqrt{g_{ij}du^i du^j}$$

The metric tensor **g** generates a bilinear function $g(.,.)$ on the tangent space T_pM at any point p on the manifold M. In fact, the metric operator $g(.,.)$ is a bilinear mapping of two arbitrary vectors at any point p to a real function.

$$\begin{aligned} & g : T_pM \times T_pM \to \mathbf{R} \\ & g(\mathbf{X},\mathbf{Y})_p = g_p\left(\mathbf{X}_p,\mathbf{Y}_p\right) = \left(g_{ij}X^iY^j\right)_p \end{aligned} \tag{3.196}$$

where $\mathbf{X}_p$ and $\mathbf{Y}_p$ are tangent vectors at the point p on T_pM.

The vector fields **X** and **Y** are written on the manifold M as

$$\mathbf{X} = X^i\partial_i; \quad \mathbf{Y} = Y^j\partial_j.$$

Using Eq. (3.196), one obtains for any point p on the manifold M

$$\begin{aligned} & g : T_pM \times T_pM \to \mathbf{R} \\ & g(\mathbf{X},\mathbf{Y}) = g\left(X^i\partial_i, Y^j\partial_j\right) = g\left(\partial_i, \partial_j\right)X^iY^j \\ & \Rightarrow g(\mathbf{X},\mathbf{Y}) = g_{ij}X^iY^j \text{ for } \forall i,j = 1,2,\ldots,N \end{aligned} \tag{3.197}$$

The metric operator $g(.,.)$ has the following properties:

$$\begin{aligned} & g(\mathbf{X},\mathbf{Y}) = g(\mathbf{Y},\mathbf{X}) : \textit{symmetric}; \\ & g(\alpha\mathbf{X} + \beta\mathbf{Y},\mathbf{Z}) = \alpha g(\mathbf{X},\mathbf{Z}) + \beta g(\mathbf{Y},\mathbf{Z}) : \textit{bilinear} \end{aligned} \tag{3.198}$$

in which α and β are scalars.

The Levi-Civita connection satisfies the **Koszul formula** for vector fields **X**, **Y**, and **Z** on the tangent space T_pM.

$$\begin{aligned} 2g(\nabla_\mathbf{X}\mathbf{Y},\mathbf{Z}) = & \nabla_\mathbf{X} g(\mathbf{Y},\mathbf{Z}) + \nabla_\mathbf{Y} g(\mathbf{X},\mathbf{Z}) - \nabla_\mathbf{Z} g(\mathbf{X},\mathbf{Y}) \\ & + g([\mathbf{X},\mathbf{Y}],\mathbf{Z}) - g([\mathbf{X},\mathbf{Z}],\mathbf{Y}) - g([\mathbf{Y},\mathbf{Z}],\mathbf{X}) \end{aligned} \tag{3.199}$$

Proof

Using the metric compatibility in Eq. (3.194e), one obtains

$$\begin{aligned} \nabla_\mathbf{X} g(\mathbf{Y},\mathbf{Z}) &= g(\nabla_\mathbf{X}\mathbf{Y},\mathbf{Z}) + g(\mathbf{Y},\nabla_\mathbf{X}\mathbf{Z}); \\ \nabla_\mathbf{Y} g(\mathbf{X},\mathbf{Z}) &= g(\nabla_\mathbf{Y}\mathbf{X},\mathbf{Z}) + g(\mathbf{X},\nabla_\mathbf{Y}\mathbf{Z}); \\ \nabla_\mathbf{Z} g(\mathbf{X},\mathbf{Y}) &= g(\nabla_\mathbf{Z}\mathbf{X},\mathbf{Y}) + g(\mathbf{X},\nabla_\mathbf{Z}\mathbf{Y}). \end{aligned} \tag{3.200}$$

Substituting Eq. (3.200) into the RHS of Eq. (3.199), its new RHS becomes

$$\begin{aligned} RHS \equiv\ & g(\nabla_{\mathbf{X}}\mathbf{Y} + \nabla_{\mathbf{Y}}\mathbf{X} + [\mathbf{X},\mathbf{Y}],\mathbf{Z}) \\ & + g(\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{\mathbf{Z}}\mathbf{X} - [\mathbf{X},\mathbf{Z}],\mathbf{Y}) \\ & + g(\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Z}}\mathbf{Y} - [\mathbf{Y},\mathbf{Z}],\mathbf{X}) \end{aligned} \tag{3.201}$$

Levi-Civita connection is torsion free and therefore satisfies using Eq. (3.194d)

$$\begin{aligned} \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} &= [\mathbf{X},\mathbf{Y}]; \\ \nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{\mathbf{Z}}\mathbf{X} &= [\mathbf{X},\mathbf{Z}]; \\ \nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Z}}\mathbf{Y} &= [\mathbf{Y},\mathbf{Z}]. \end{aligned} \tag{3.202}$$

Inserting Eq. (3.202) into Eq. (3.201), the Koszul formula has been proved.

$$RHS \quad \equiv \quad g(2\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}) = 2g(\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}) \equiv LHS$$

In general, the Levi-Civita connection of tensor fields $\mathbf{T}$ with respect to a vector field $\mathbf{X}$ on the manifold M is calculated in [9]

$$\nabla_{\mathbf{X}}\mathbf{T} = (\nabla_{\mathbf{X}}T)^{i\ldots k}_{l\ldots n}\partial_i \otimes \ldots \otimes \partial_k \otimes du^l \otimes \ldots \otimes du^n \tag{3.203}$$

in which

$$(\nabla_{\mathbf{X}}T)^{i\ldots k}_{l\ldots n} = X^p T^{i\ldots k}_{l\ldots n,p} + X^p \Gamma^k_{pq} T^{i\ldots q}_{l\ldots n} + \ldots - X^p \Gamma^q_{lp} T^{i\ldots k}_{q\ldots n} - \ldots \tag{3.204}$$

Let $\mathbf{S}$ and $\mathbf{T}$ be arbitrary tensor fields, and $\mathbf{X}$ be a vector field. The Levi-Civita connection ∇ on a smooth differentiable geometric manifold (M, g) fulfills the following properties:

– ***Linearity:***

$$\nabla_{\mathbf{X}}(\mathbf{S} + \mathbf{T}) = \nabla_{\mathbf{X}}\mathbf{S} + \nabla_{\mathbf{X}}\mathbf{T} \tag{3.205a}$$

– ***Leibniz rule for tensor product:***

$$\nabla_{\mathbf{X}}(\mathbf{S} \otimes \mathbf{T}) = (\nabla_{\mathbf{X}}\mathbf{S}) \otimes \mathbf{T} + \mathbf{S} \otimes (\nabla_{\mathbf{X}}\mathbf{T}) \tag{3.205b}$$

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Chapter 4
Differential Forms

4.1 Introduction

Alternative to tensors, differential forms are very useful in differential geometry without considering the coordinates compared to tensors. Differential forms are based on exterior algebra in which the coordinates are not taken into account. The exterior algebra was developed by Élie Cartan (1869–1951) and Henri Poincaré (1854–1912).

Before going into details in differential forms in exterior algebra, we look into multi-dimensional spaces and shapes of the objects under points of view between topology and differential geometry. Geometers are concerned with the exact shape, size, and curvature of the object in all possible dimensions. On the contrary, topologists generally look into the overall shape of the object as a whole without considering it in detail as the geometers did.

In topology, there are usually two basic kinds of spaces with one-dimensional and two-dimensional shapes of any object [1]. The one-dimensional spaces are a straight line or a circle; however, both are fundamentally different from each other. The difference is that the circle can be transformed into any shape of loops, but a circle cannot be made in a line without cutting it. The two-dimensional spaces are classified into two basic types: either a sphere or a donut (torus) with two-dimensional surfaces. The big difference between them is that the sphere has no hole in it and the donut with a hole through it. Whatever what you do, it is definitely that a donut cannot be transformed into a sphere without cutting a hole through the middle. Cubes, hexahedrons, pyramids, and tetrahedrons are topologically homeomorphic (similar shape) to a sphere with genus 0. Note that the genus denotes the number of holes in the object. Obviously, a donut (torus) has genus 1, which denotes one hole in it. Pretzels with two holes have genus 2; pretzels with three holes, genus 3 (see Fig. 4.1).

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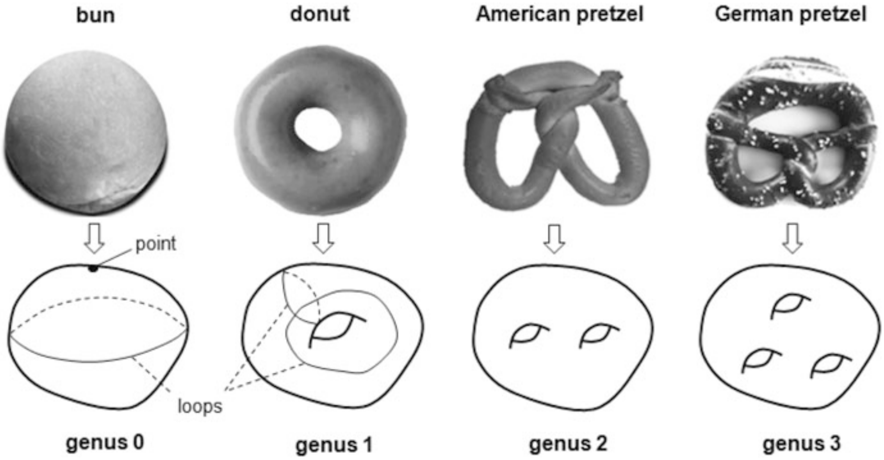

Fig. 4.1 Topological manifolds with various genera of 0, 1, 2, and 3

If different objects have the same genus, they are homeomorphisms or topological isomorphisms in the mathematical field of topology. Any cube with a hole through it can be somehow reshaped into the shape of a torus (genus 1) by squeezing and stretching. However, an object with any genus other than one (genus $\neq 1$); e.g., bun (genus 0), American pretzel (genus 2), and German pretzel (genus 3) cannot be molded into a donut (genus 1) without punching a hole through it or cutting the holes through the middles.

Objects with surfaces at topological dimensions equal to or higher than three are quite difficult to display. They are called hypersurfaces of N-manifolds ($N \geq 3$). In fact, a sphere has a topological dimension of two because only its 2-D surface is considered. Therefore, it is called 2-sphere $\mathbf{S}^2$ or a sphere.

Any surface on which every loop can be shrinked to a point is defined as a sphere (genus 0). On the contrary, loops on the torus (genus 1) surface cannot be tightened to a point, as shown in Fig. 4.1.

In differential geometry, the dimension of a space (manifold) is the necessary numbers of coordinates to determine the characteristic of a given point (location and physical states) on the manifold. Therefore, the manifold of the object in differential geometry is studied in more detail in higher-dimensional Riemannian spaces, different to topology in which the object is considered as a whole. Hence, the sphere $\mathbf{S}^2$ is considered as a three-dimensional space in differential geometry.

The computations in N-dimensional spaces are carried out using tensors and differential forms without drawing them on the paper. In order to maintain that the maximum velocity of any object must be less than or equal to the light speed ($v \leq c$), the four-dimensional Minkowski spacetime (t, x, y, z) is generally used in the Maxwell's equations, special relativity theory, and cosmology (cf. Chap. 5).

4.2 Definitions of Spaces on the Manifold

Differential forms deal with calculations in differential geometry without considering any coordinate contrary to vector and tensor calculus. However, they use many mapping operators in various spaces, vector and differential form bundles using exterior algebra that is quite different to linear algebra. Unfortunately, they are very confused in the applications of differential forms. In order to comprehend the differential forms, some necessary spaces and bundles on the manifold are defined in the following section [2].

Let M be a smooth and differentiable N-dimensional manifold, M^* be a dual manifold of M, and p be a point on the manifold M.

- The tangent space T_pM at the point p on M is defined as the space that contains the tangent vector to the manifold M at the point p. The basis vectors in the tangent space T_pM have been discussed in Sect. 3.14.
- The cotangent space $T_p{}^*M$ at the point p on M is defined as the dual space of the manifold M. The dual bases dx^i for $i = i_1,\ldots,i_k$ are used in the cotangent space $T_p{}^*M$.
- The tangent bundle TM is a set of all tangent spaces T_pM as the point p moves on the manifold M.
- The cotangent (dual) bundle T^*M is a set of all dual spaces $T_p{}^*M$ as the point p moves on the manifold M.
- $\wedge^k(T_pM)$ is defined as the subset of the k-tangent-vector space at the point p on the tangent space T_pM.
- $\wedge^k(T_p{}^*M)$ is defined as the subset of the k-form space at the point p on the cotangent (dual) space $T_p{}^*M$.
- $\wedge^k(M)$ is defined as the exterior k-vector-bundle space on M that consists of all subsets of the tangent vector spaces $\wedge^k(T_pM)$.
- $\wedge^k(M^*)$ is defined as the exterior k-form-bundle space on M that consists of all subsets of the k-form spaces $\wedge^k\,(T_p{}^*M)$.
- $A^k(M)$ denotes the space of smooth sections of the exterior k-form bundle of $\wedge^k$ (M^*).

4.3 Differential k-Forms

The differential k-form ω_p at the point p on M is an alternating (skew-symmetric) multilinear map from p into the subset of the k-form space $\wedge^k(T_p{}^*M)$.

$$\begin{aligned} &\omega_p : \underbrace{T_p^*M \times \ldots \times T_p^*M}_{k\ times} \to \Lambda^k\left(T_p^*M\right) \subseteq \Lambda^k(M) \\ &\Leftrightarrow \omega_p : \left(dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) \to \omega_p = \sum_I f_I\left(x_{p_1}, \ldots, x_{p_n}\right) dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\ &\qquad\qquad\qquad\qquad\qquad\quad \equiv \sum_I f_I\left(x_{p_1}, \ldots, x_{p_n}\right) dx^I \in \Lambda^k\left(T_p^*M\right) \end{aligned} \tag{4.1}$$

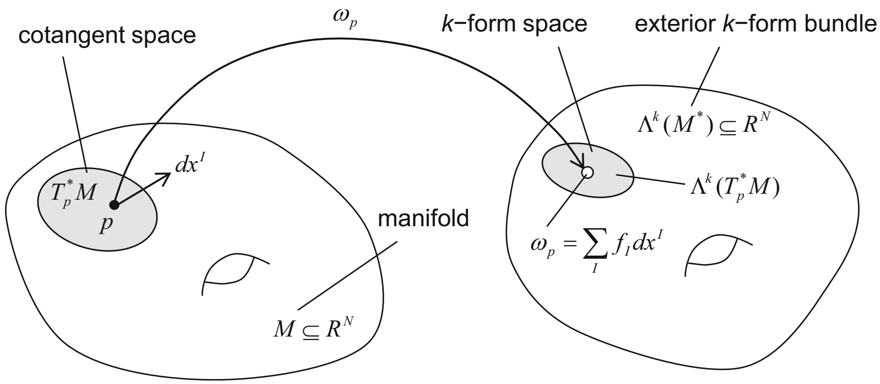

Fig. 4.2 A differential k-form ω_p of a point p on M

The elementary k-form of ω_p is defined as

$$\begin{aligned} & dx^I \equiv dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \\ & \text{for } I = (i_1, \ldots, i_k);\ \ i_1 < \ldots < i_k \end{aligned} \tag{4.2}$$

in which the dual bases dx^i for $i = i_1, \ldots, i_k$ are used in the cotangent (dual) space $T_p{}^*M$ on the manifold M.

The k-form ω_p is a subset of the exterior form bundle on M (see Fig. 4.2).

$$\omega_p \in \Lambda^k\left(T_p^*M\right) \subset \Lambda^k\left(M^*\right) \equiv \bigcup_{p \in M} \Lambda^k\left(T_p^*M\right) \tag{4.3}$$

Some examples of k-forms are given as

- Zero-form ($k = 0$): $\omega = 1$
- One-form ($k = 1$): $\omega = yzdx + xydy - x^2zdz$
- Two-form ($k = 2$): $\omega = x^2y^3(dx \wedge dy) + xy^3(dy \wedge dz) - xyz^2(dx \wedge dz)$
- Three-form ($k = 3$): $\omega = xy^2z^3(dx \wedge dy \wedge dz)$

The *wedge product* (*exterior product*) of two any differential one-forms ω and η is defined as

$$\omega \wedge \eta \equiv \omega \otimes \eta - \eta \otimes \omega \tag{4.4}$$

where $\otimes$ is the tensor product (cf. Chaps. 2 and 6).

Let ω and η be one-forms for $k = 1$. If the order of two dual bases is interchanged, one permutation is carried out. As a result, the sign of the k-form is changed. In the case of an even number of permutations, the sign of the k-form is +1 and the sign is −1 for an odd number of permutations (see Table 4.1).

Table 4.1 Permutations of the dual bases in 3-forms

i_1	i_2	i_3	3-forms	dx^{I}	σ
x	y	z	$dx \wedge dy \wedge dz$	$= +dx \wedge dy \wedge dz$	0
x	z	y	$dx \wedge dz \wedge dy$	$= -dx \wedge dy \wedge dz$	1
y	x	z	$dy \wedge dx \wedge dz$	$= -dx \wedge dy \wedge dz$	1
y	z	x	$dy \wedge dz \wedge dx$	$= +dx \wedge dy \wedge dz$	2
z	x	y	$dz \wedge dx \wedge dy$	$= +dx \wedge dy \wedge dz$	2
z	y	x	$dz \wedge dy \wedge dx$	$= -dx \wedge dy \wedge dz$	3

Generally, the elementary k-forms result after σ permutations as

$$dx^{\sigma(i_1)} \wedge dx^{\sigma(i_2)} \wedge \ldots \wedge dx^{\sigma(i_k)} = (-1)^{m_\sigma} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \tag{4.5}$$

where m_σ is the modulo 2 of σ that is the residual of the permutation number σ divided by 2.

$$m_\sigma = \sigma \text{mod} 2 \tag{4.6}$$

Some examples of the permutations of k-forms are given in the following section.

– If no permutation ($\sigma = 0$) is carried out, one obtains

$$m_0 = 0\text{mod}2 = 0 + 0 \Rightarrow m_0 = 0$$

Thus,

$$dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} = (+1) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

– If σ equals one permutation ($\sigma = 1$), one obtains

$$m_1 = 1\text{mod}2 = 0 + 1 \Rightarrow m_1 = 1$$

Thus,

$$dx^{i_2} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} = (-1) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

– If σ permutations are carried out, one obtains

$$\begin{aligned} m_\sigma &= \sigma \text{mod}2; \quad \sigma > 1 \\ &= \begin{cases} 1 \ \textit{if } \sigma \textit{ is an odd number} \\ 0 \ \textit{if } \sigma \textit{ is an even number} \end{cases} \end{aligned}$$

Therefore,

$$dx^{i_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_{k-1}} = (-1)^{m_\sigma} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$

Table 4.1 displays the permutations of the dual bases in six 3-forms ($k! = 3! = 6$) with the number of permutations σ. The first 3-form of the dual bases is called *the elementary form* without any permutation ($\sigma = 0$). Moving the dual bases in the first 3-form, there are fives possibilities of the 3-forms that are transformed into the elementary form using Eq. (4.5). Interchanging dz with dy in the second 3-form, one obtains the elementary form at one permutation ($\sigma = 1$); thus, its sign is changed. The fourth 3-form needs two permutations ($\sigma = 2$) of dz with dx and dy with dx to become the elementary form; therefore, its sign is unchanged. Similarly, other remaining 3-forms are easily transformed into the elementary form using Table 4.1.

In general, let ω be a k-form, η be an l-form, and v be a m-form. The wedge product of two differential forms has the following properties:

$$\begin{aligned} &\omega \wedge (\eta + v) = (\omega \wedge \eta) + (\omega \wedge v): \quad \textit{left} - \textit{distributive} \\ &(\eta + v) \wedge \omega = (\eta \wedge \omega) + (v \wedge \omega): \quad \textit{right} - \textit{distributive} \\ &\omega \wedge (\eta \wedge v) = (\omega \wedge \eta) \wedge v: \quad \textit{associative} \\ &\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega: \quad \textit{graded} - \textit{anticommutative} \end{aligned} \tag{4.7}$$

Using the graded-anticommutative law in Eq. (4.7), one obtains for $\omega = \eta$

$$\omega \wedge \omega = 0 \Rightarrow d\omega \wedge d\omega = 0 \tag{4.8}$$

Moving k dual bases of the k-form behind the l dual bases of the l-form, the wedge product of two differential forms in Eq. (4.7) is proved.

$$\begin{aligned} \omega \wedge \eta &= \sum_{i,j} \omega_{i_1 \ldots i_k} \eta_{j_1 \ldots j_l} \left(dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) \wedge \left(dx^{j_1} \wedge \ldots \wedge dx^{j_l}\right) \\ &= \sum_{i,j} (-1)^l \omega_{i_1 \ldots i_k} \eta_{j_1 \ldots j_l} \left(dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}\right) \wedge \\ &\qquad \left(dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_l}\right) \wedge dx^{i_k} \\ &= \sum_{i,j} (-1)^{lk} \eta_{j_1 \ldots j_l} \omega_{i_1 \ldots i_k} \left(dx^{j_1} \wedge \ldots \wedge dx^{j_l}\right) \wedge \left(dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right) \\ &= (-1)^{kl} \eta \wedge \omega \quad \text{(q.e.d.)} \end{aligned} \tag{4.9}$$

According to Eq. (4.9), the wedge product of two differential k- and l-forms has the order of $(k + l)$ and belongs to the exterior $(k + l)$-form space in exterior algebra.

$$\begin{aligned} &\omega \in \Lambda^k\left(T_p^* M\right) \subset \Lambda^k\left(M^*\right); \;\; \eta \in \Lambda^l\left(T_p^* M\right) \subset \Lambda^l\left(M^*\right) \\ &\Rightarrow \omega \wedge \eta = (-1)^{kl} \eta \wedge \omega \in \Lambda^{k+l}\left(T_p^* M\right) \subset \Lambda^{k+l}\left(M^*\right) \end{aligned}$$

The differential forms of ω and η are written in their dual bases as

$$\omega = \sum_{i_1<\ldots<i_k} \omega_{i_1\ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \equiv \sum_I \omega_I dx^I;$$
$$\eta = \sum_{j_1<\ldots<j_l} \eta_{j_1\ldots j_l} dx^{j_1} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_l} \equiv \sum_J \eta_J dx^J.$$

where ω_I and η_J are smooth differentiable functions of x^I and x^J, respectively.

The dimension of the k-form space $\wedge^k(T_p{}^*M)$ in an N-dimensional space $\mathbf{R}^N$ is defined as the number of choosing k-element subsets of an N-element set that consists of N distinct elements. The dimension of the k-form space is defined as the possible number of k permutations from N dimensions disregarding the permutation order. As a result, the k-form dimension is calculated as the binomial coefficient in combinatorics.

$$\binom{N}{k} = \frac{N!}{k!\,(N-k)!} = \dim \Lambda^k\left(T_p^* M\right);\quad 0 \le k \le N \tag{4.10}$$

The factorial of N is defined by

$$N! = \prod_{k=1}^{N} k = 1 \cdot 2 \cdots (N-1) \cdot N \, for \; \forall N > 0;\quad 0! \equiv 1$$

Thus, the factorial of N can be written in the recurrence relation as

$$N! = \begin{cases} 1 & if\ N = 0; \\ (N-1)!N & if\ N > 0. \end{cases}$$

Using Eq. (4.10), the dimensions of k-form spaces in a three-dimensional space $\mathbf{R}^3$ are calculated and shown in Table 4.2.

4.4 The Notation $\omega \cdot X$

Let ω be a one-form in the 1-form space $\wedge^1\ (T_p{}^*M)$; X be a vector field in the tangent space $T_pM \in \mathbf{R}^N$.

A linear mapping of $y \in M$ to $\omega{\cdot}X(y)$ by the mapping function $\omega{\cdot}X$ is written as

$$\omega \cdot X : y \in M \to \omega \cdot X(y) \in \mathbf{R}$$

Table 4.2 Dimensions of the k-form spaces in $\mathbf{R}^3$

ω $(0 \le k \le 3)$	Form space	Dual bases of k-forms	Dimensions of form space
0–form	$\wedge^0(\boldsymbol{R}^3) = \boldsymbol{R}$	1	1
1–form	$\wedge^1(\boldsymbol{R}^3)$	dx^1, dx^2, dx^3	3
2–form	$\wedge^2(\boldsymbol{R}^3)$	$dx^1 \wedge dx^2,\ dx^2 \wedge dx^3, dx^1 \wedge dx^3$	3
3–form	$\wedge^3(\boldsymbol{R}^3)$	$dx^1 \wedge dx^2 \wedge dx^3$	1

The notation $\omega{\cdot}X$ linking the differential forms to vector fields is defined as

$$y \in M \to \omega \cdot X(y) \equiv \omega(y) \cdot X(y) \in \mathbf{R}$$

Using Einstein's summation convention, the one-form ω in the 1-form space $\wedge^1(T_p{}^*M)$ and the vector field X in the tangent space T_pM are written as

$$\omega = h_i dx^i \in \Lambda^1\left(T_p^*M\right);\ i = 1, \ldots, N$$
$$X = \xi^j \frac{\partial}{\partial x^j} \in T_pM;\ j = 1, \ldots, N$$

Due to orthonormality of the bases of the tangent and cotangent spaces according to Eq. (3.178), the notation $\omega{\cdot}X(y)$ is written using Kronecker delta as

$$\begin{aligned}\omega \cdot X(y) &= \omega(y) \cdot X(y) = h_i(y) \cdot \xi^j(y)\delta_i^j \\ &= h_i(y) \cdot \xi^i(y) \in \mathbf{R};\quad i = 1, \ldots, N\end{aligned}$$

Analogously, the notation is extended to p-forms ω in the p-form space $\wedge^p(T_p{}^*M)$ and vector bundles $(X_1, \ldots, X_p)$ in the p-vector-bundle space $\wedge^p(M)$ as

$$y \in M \to \omega \cdot \left(X_1 \wedge \ldots \wedge X_p\right)(y) = \omega(y) \cdot \left(X_1(y) \wedge \ldots \wedge X_p(y)\right) \in \mathbf{R}$$

Due to linearity of the p-forms, an element (monomial) of the p-form is used in this case. The monomial ω of the p-form for $y \in M$ is written as

$$\omega(y) = h(y) dx^{i_1} \wedge \ldots \wedge dx^{i_p} \in \Lambda^p\left(T_p^*M\right)$$

Using orthonormality of the bases, the extended notation $\omega{\cdot}X$ results as [3]

$$\omega \cdot \left(X_1 \wedge \ldots \wedge X_p\right) = \sum_{\sigma \in \Sigma_p} (-1)^{m_\sigma} h \cdot X_{\sigma(i_1)}\left(x^{i_1}\right) \cdots X_{\sigma\left(i_p\right)}\left(x^{i_p}\right)$$

where

σ is the number of permutations;

Σ_p is a set of permutations $\{1, 2, \ldots, p\}$;

$m_\sigma = \sigma$ mod 2, cf. Eq. (4.6);
x^i are the local coordinates in the cotangent space $T_p{}^*M$.

4.5 Exterior Derivatives

Let ω be any k-form; η be any l-form in any N-dimensional space $\mathbf{R}^N$. The exterior derivative $d\omega$ of the k-form ω is a $(k+1)$-form [3–5].

$$\omega \in \Lambda^k\left(\mathbf{R}^N\right) \subset \Lambda^k\left(M^*\right) \Rightarrow d\omega \in \Lambda^{k+1}\left(\mathbf{R}^N\right)$$

The k-form ω is written in the dual bases as

$$\omega = \sum_{i_1<\ldots<i_k} f_{i_1\ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \equiv \sum_I f_I dx^I$$

where f_I is a smooth function $\in \mathbf{R}$.

The *exterior derivative* $d\omega$ of the k-form ω is defined as

$$\begin{aligned} d\omega &= d\left(\sum_I f_I dx^I\right) = \sum_I df_I \wedge dx^I \in \Lambda^{k+1}\left(\mathbf{R}^N\right) \\ &= \sum_I df_I \wedge \left(dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}\right) \\ &= \sum_I \left(\sum_{j=1}^{k} \frac{\partial f_I}{\partial x^j} dx^j\right) \wedge \left(dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}\right) \end{aligned} \tag{4.11}$$

Generally, the exterior derivative upgrades the differential form by one order. The exterior derivative has the following properties:

$$d(\omega + \eta) = d\omega + d\eta \tag{4.12a}$$

$$df_I = \sum_i \frac{\partial f_I}{\partial x^i} dx^i \tag{4.12b}$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \tag{4.12c}$$

$$d(d\omega) = 0 : \text{ The Poincaré Lemma} \tag{4.12d}$$

Proof of Eq. (4.12c) Using Eq. (4.11) and the chain rule of differentiation, one obtains the exterior derivative of the wedge product of two differential forms.

$$
\begin{aligned}
d(\omega \wedge \eta) &= d\sum_{I,J} (\omega_I \eta_J) dx^I \wedge dx^J = \sum_{I,J} d\left(\omega_I \eta_J dx^I \wedge dx^J\right) \\
&= \sum_{I,J} \sum_i \left(\frac{\partial(\omega_I \eta_J)}{\partial x^i} dx^i\right) \wedge dx^I \wedge dx^J \\
&= \sum_{I,J} \sum_i \left(\frac{\partial \omega_I}{\partial x^i} \eta_J + \frac{\partial \eta_J}{\partial x^i} \omega_I\right) dx^i \wedge (dx^I \wedge dx^J) \\
&= \sum_I \sum_i \frac{\partial \omega_I}{\partial x^i} dx^i \wedge dx^I \wedge \sum_J \eta_J dx^J + \\
&\quad (-1)^k \sum_I \omega_I dx^I \wedge \sum_J \sum_i \left(\frac{\partial \eta_J}{\partial x^i} dx^i\right) \wedge dx^J \\
&= (d\omega \wedge \eta) + (-1)^k (\omega \wedge d\eta) \quad (q.e.d.)
\end{aligned}
$$

Proof of the Poincaré Lemma Eq. (4.12d) Using Eqs. (4.11) and (4.12a–c), the exterior derivative of any differential k-form ω is written as

$$
\begin{aligned}
d\omega &= d\left(\sum_I \omega_I dx^I\right) = \sum_I d\left(\omega_I dx^I\right) = \sum_I d\omega_I \wedge dx^I \\
&= \sum_I \left(\sum_i \frac{\partial \omega_I}{\partial x^i} dx^i\right) \wedge (dx^{i_1} \wedge \ldots \wedge dx^{i_k})
\end{aligned} \tag{4.13}
$$

Using Eq. (4.13) and the chain rule of differentiation, the exterior derivative of $d\omega$ is calculated as

$$
\begin{aligned}
d(d\omega) &= d\sum_I \left(\sum_i \frac{\partial \omega_I}{\partial x^i} dx^i\right) \wedge dx^I = \sum_I \left(\sum_{i,j} \frac{\partial^2 \omega_I}{\partial x^j \partial x^i} dx^j \wedge dx^i\right) \wedge dx^I \\
&= \sum_I \sum_{i<j} \underbrace{\left(\frac{\partial^2 \omega_I}{\partial x^j \partial x^i} - \frac{\partial^2 \omega_I}{\partial x^i \partial x^j}\right)}_{=\ 0} \left(dx^j \wedge dx^i\right) \wedge dx^I = 0 \quad (q.e.d.)
\end{aligned}
$$

This equation is called the Poincaré Lemma.

If f is any zero-form (a smooth function with $k=0$) and ω is any one-form, one obtains using Eqs. (4.12c) and (4.12d)

$$
\begin{aligned}
d(f \wedge d\omega) &\equiv d(f d\omega) \\
&= df \wedge d\omega + (-1)^0 f \wedge \underbrace{d(d\omega)}_{=0} = df \wedge d\omega
\end{aligned} \tag{4.14}
$$

Using Eq. (4.12b), Eq. (4.14) can be expressed as

$$d(f \wedge d\omega) = df \wedge d\omega = \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge d\omega \tag{4.15}$$

Some examples of differential forms and their exterior derivatives are given in the following section.

1. **One-form ($k = 1$)**
   A one-form ω is given in $\mathbf{R}^3$ as

$$\omega = xydx + xzdy + xyzdz \in \Lambda^1\left(\mathbf{R}^3\right)$$

Using Eq. (4.13), the exterior derivative of ω is calculated as

$$\begin{aligned} d\omega &= d(xy) \wedge dx + d(xz) \wedge dy + d(xyz) \wedge dz \\ &= xdy \wedge dx + (zdx \wedge dy + xdz \wedge dy) + (yzdx \wedge dz + xzdy \wedge dz) \\ &= -xdx \wedge dy + zdx \wedge dy - xdy \wedge dz + yzdx \wedge dz + xzdy \wedge dz \\ &= (-x + z)dx \wedge dy + x(z-1)dy \wedge dz + yzdx \wedge dz \in \Lambda^2\left(\mathbf{R}^3\right) \end{aligned}$$

The exterior derivative of $d\omega$ is calculated further using Eq. (4.5) as

$$\begin{aligned} d(d\omega) &= dd\omega \equiv d^2\omega \\ &= dz \wedge dx \wedge dy + (z-1)dx \wedge dy \wedge dz + zdy \wedge dx \wedge dz \\ &= (1 + z - 1 - z)dx \wedge dy \wedge dz \in \Lambda^3\left(\mathbf{R}^3\right) \\ &= 0 \ \ (q.e.d.) \end{aligned}$$

2. **Two-form ($k = 2$)**
   A two-form ω is given in $\mathbf{R}^3$ as

$$\omega = x^2(y+z)dx \wedge dy + y\left(x^2 + 2z\right)dy \wedge dz + 2xy^2 dx \wedge dz \in \Lambda^2\left(\mathbf{R}^3\right)$$

Using Eq. (4.13), the exterior derivative of ω is calculated as

$$\begin{aligned} d\omega &= d\left(x^2(y+z)\right)dx \wedge dy + d\left(y\left(x^2 + 2z\right)\right)dy \wedge dz + d\left(2xy^2\right)dx \wedge dz \\ &= x^2 dz \wedge dx \wedge dy + 2xydx \wedge dy \wedge dz + 4xydy \wedge dx \wedge dz \\ &= \left(x^2 + 2xy - 4xy\right)dx \wedge dy \wedge dz \\ &= x(x - 2y)dx \wedge dy \wedge dz \in \Lambda^3\left(\mathbf{R}^3\right) \end{aligned}$$

Due to $(dx^i \wedge dx^i) = 0$ and using Eq. (4.13), one obtains Eq. (4.12d).

$$\begin{aligned} d(d\omega) &= d[x(x-2y)dx \wedge dy \wedge dz] \\ &= d[x(x-2y)] \wedge dx \wedge dy \wedge dz \\ &= \sum_i \frac{\partial [x(x-2y)]}{\partial x^i} dx^i \wedge dx \wedge dy \wedge dz \\ &= 0 \ (q.e.d.) \end{aligned}$$

3. **Three-form (*k* =3)**
   A three-form ω is given in $\mathbf{R}^3$ as

$$\begin{aligned} \omega &= f(x,y,z)dx \wedge dy \wedge dz \\ &= x^2yz^3 dx \wedge dy \wedge dz \in \Lambda^3\left(\mathbf{R}^3\right) \end{aligned}$$

Using $(dx^i \wedge dx^i) = 0$ and Eq. (4.13), the exterior derivative of ω is calculated as

$$\begin{aligned} d\omega &= d\left(x^2yz^3 dx \wedge dy \wedge dz\right) \\ &= d\left(x^2yz^3\right) \wedge dx \wedge dy \wedge dz \\ &= \sum_i \frac{\partial (x^2yz^3)}{\partial x^i} dx^i \wedge dx \wedge dy \wedge dz \\ &= 0 \, for \ dx^i = dx, dy, dz \end{aligned}$$

Thus,

$$dd\omega = d(0) = 0 \ \ (q.e.d.)$$

4.6 Interior Product

The *interior product* or *interior derivative* degrades the differential form by one order. It is usually applied to exterior algebra of *k*-differential forms on a smooth *N*-dimensional manifold *M*.

In fact, the interior product is a linear map of a *k*-form ω to the $(k-1)$-form $i_X\omega$ in the vector fields $(X_1, \ldots, X_{k-1})$ on the manifold *M*. In exterior algebra, the interior product is an operator for contracting vectors and differential forms to one-order lower forms.

The interior product of any differential *k*-form ω on the manifold *M* is defined as

$$\begin{aligned} i_X : \omega \in \Lambda^k\left(T_p^*M\right) &\rightarrow i_X\omega \in \Lambda^{k-1}\left(T_p^*M\right) \\ (i_X\omega)(X_1, \ldots, X_{k-1}) &= \omega(X, X_1, \ldots, X_{k-1}) \equiv i_{X_{k-1}} \ldots \ i_{X_1} i_X \omega \\ &= \sum_{i_1 < \ldots . < i_{k-1}} \sum_i X^i \omega_{i i_1 \cdots i_{k-1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}} \end{aligned} \tag{4.16}$$

Using Einstein summation convention, the vector field X is written as

$$X = X^i \frac{\partial}{\partial x^i} \equiv X^i \partial_i$$

Let f be a smooth function (a zero-form), X be a vector field on M, ω be a one-form, η and ξ be k- and l-forms, respectively. The interior product has the following properties:

$$\begin{aligned} & i_X f = 0 \\ & i_X \omega = \omega(X) \equiv \langle \omega, X \rangle : \textit{inner product} \\ & i_X(\eta \wedge \xi) = (i_X \eta) \wedge \xi + (-1)^k \eta \wedge (i_X \xi) \\ & i_X i_Y \eta = -i_Y i_X \eta \end{aligned} \tag{4.17}$$

Furthermore, if α and β are one-forms; X and Y are the vector fields on the manifold M, one obtains from Eqs. (4.16) and (4.17)

$$\alpha(X, Y) = i_Y i_X \alpha = i_Y \underbrace{(\alpha(X))}_{0-form} = 0 \tag{4.18a}$$

and

$$\begin{aligned} (\alpha \wedge \beta)(X, Y) = i_Y[i_X(\alpha \wedge \beta)] &= i_Y \left[i_X \alpha \wedge \beta + (-1)^1 \alpha \wedge i_X \beta \right] \\ &= i_Y[\alpha(X) \wedge \beta - \alpha \wedge \beta(X)] \\ &= i_Y[\alpha(X)\beta - \beta(X)\alpha] \\ &= \alpha(X)(i_Y \beta) - \beta(X)(i_Y \alpha) \\ &= \alpha(X)\beta(Y) - \beta(X)\alpha(Y) \end{aligned} \tag{4.18b}$$

Therefore,

$$\begin{aligned} (\alpha \wedge \beta)(X, Y) &= \alpha(X)\beta(Y) - \beta(X)\alpha(Y) \\ &= \langle \alpha, X \rangle \cdot \langle \beta, Y \rangle - \langle \beta, X \rangle \cdot \langle \alpha, Y \rangle \end{aligned} \tag{4.19}$$

The Cartan's formula for any differential form ω is given in [4, 6]

$$£_X \omega = i_X d\omega + d(i_X \omega) \tag{4.20}$$

where ω is the k-form on M; $£_X \omega$ is the Lie derivative of ω, cf. Eq. (3.144a).

Changing ω into a zero-form f and using the property of $i_X f = 0$, one obtains the relation between the Lie derivative and interior product of any function f

$$\begin{aligned} £_X f &\equiv Xf \\ &= i_X df + d(i_X f) = i_X df \end{aligned} \tag{4.21}$$

In general, the Cartan's formula (known as Weil's formula) in differential calculus is written as

$$\pounds_X = di_X + i_X d = [d, i_X] \tag{4.22}$$

4.7 Pullback Operator of Differential Forms

The pullback operator of differential forms is used in the transformation of coordinate variables from one manifold to another manifold. It is a smooth linear map of two mapping functions f and g.

$$\begin{aligned}
&f : X(x) \in M \subset \mathbf{R}^N \to f(X) = y(x) \in N \subset \mathbf{R}^N \\
&g : f(X) \in N \subset \mathbf{R}^N \to g(f(X)) = g(y(x)) \in P \subset \mathbf{R} \\
&\Rightarrow f^* g : X(x) \in M \subset \mathbf{R}^N \to f^* g(X) = g(f(X)) \in P \subset \mathbf{R}
\end{aligned}$$

Let X (x) be a vector field on the manifold M. The pullback operator f^*g of g by f in the vector field $X(x) \in M$ is defined as

$$f^* g : X(x) \in M \to f^* g(X) = g(f(X)) = g(y(x)) \in P \tag{4.23}$$

The operator f^*g is called the pullback g by f since the mapping function g is pulled from N backwards to M (see Fig. 4.3).

The function f maps the vector field X to $f(X) = y(x)$ that is further mapped by the function g to $g(f(X))$. In fact, the pullback operator f^*g maps the vector field $X \in M$ to $g_o f(X) \in P$ directly.

An example of the pullback is given in the following section. Let ω be a one-form of y that is written as

$$\omega = y_1 y_2 dy_1 + y_2^2 y_3 dy_2 + y_1 y_2 y_3 dy_3 \in \Lambda^1 \left(\mathbf{R}^3\right)$$

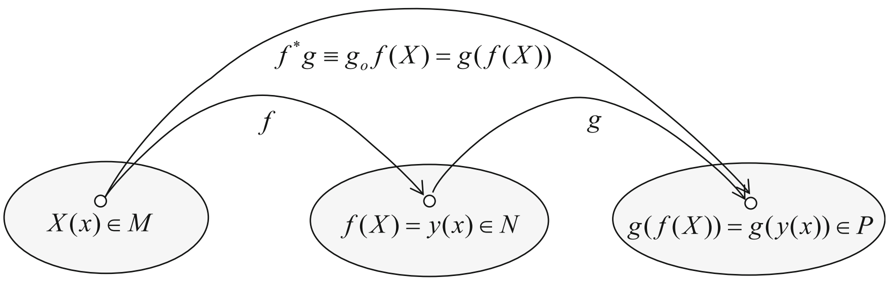

Fig. 4.3 Pullback operator f^*g of g by f over M

Let $X(t)$ be a vector field $(t, 2t, 3t) \in \mathbf{R}^3$. The linear function f maps X to $f(X)$ as

$$f : X(t) \to f(X) = y(t) = \begin{cases} y_1 = t \\ y_2 = 2t \\ y_3 = 3t \end{cases}$$
$$\Rightarrow df(X) = dy(t) = \begin{cases} dy_1 = dt \\ dy_2 = 2dt \\ dy_3 = 3dt \end{cases}$$

Using the above equation, the pullback of the one-form ω by f results as a one-form

$$\begin{aligned} f^* \omega(X) &= \omega(f(X)) = \omega(y(t)) \\ &= t(2t)(dt) + (2t)^2(3t)(2dt) + t(2t)(3t)(3dt) \\ &= 2t^2(21t + 1)dt \in \Lambda^1\left(\mathbf{R}^3\right) \end{aligned}$$

The pullback of differential forms has the following properties:

$$\begin{aligned} &f^*(\omega + \eta) = f^*\omega + f^*\eta; \\ &f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta; \\ &d\left(f^*\omega\right) = f^*(d\omega). \end{aligned} \tag{4.24}$$

where ω and η are the arbitrary k-forms; and f is the linear mapping function.

The function g in the pullback mapping is replaced by a k-form ω that is written in $\mathbf{R}^N$ as

$$\omega = \sum_J a_J dy^J = \sum_J a_J dy^{j_1} \wedge \ldots \wedge dy^{j_k} \in \Lambda^k\left(\mathbf{R}^N\right)$$

The pullback of the k-form ω by the mapping function f results as

$$\begin{aligned} f^*\omega &= f^*\left(\sum_J a_J dy^J\right) = \sum_J f^* a_J \cdot f^*\left(dy^{j_1} \wedge \ldots \wedge dy^{j_k}\right) \\ &= \sum_J \left(a_J \circ f\right) \cdot f^*\left(dy^{j_1}\right) \wedge \ldots \wedge f^*\left(dy^{j_k}\right) \\ &= \sum_J \left(a_J \circ f\right) \cdot df^{j_1}\left(x^i\right) \wedge \ldots \wedge df^{j_k}\left(x^i\right) \\ &= \sum_J \left(a_J \circ f\right)\left(\sum_{i_1} \frac{\partial f^{j_1}}{\partial x^{i_1}} dx^{i_1}\right) \wedge \ldots \wedge \left(\sum_{i_k} \frac{\partial f^{j_k}}{\partial x^{i_k}} dx^{i_k}\right) \\ &= \sum_J \left(a_J \circ f\right) \sum_{i_1, \ldots, i_k} \left(\frac{\partial f^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{j_k}}{\partial x^{i_k}}\right) dx^{i_1} \wedge \ldots \wedge dx^{i_k} \end{aligned}$$

The second $\sum$ term on the RHS of the above equation is in fact the Jacobian **J**.

$$\mathbf{J} \equiv \sum_{i_1,\ldots,i_k} \left(\frac{\partial f^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial f^{j_k}}{\partial x^{i_k}} \right) = \frac{\partial (y^{j_1}, \ldots, y^{j_k})}{\partial (x^{i_1}, \ldots, x^{i_k})}$$

Thus,

$$\begin{aligned} f^* \omega = \omega_{\mathrm{o}} f &= \mathbf{J} \sum_J (a_{J\,\mathrm{o}} f) dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\ &\equiv \mathbf{J} \sum_I a_I dx^I \in \Lambda^k(\mathbf{R}^N) \end{aligned} \tag{4.25}$$

4.8 Pushforward Operator of Differential Forms

Figure 4.4 shows the mapping scheme of the pushforward operator in an N-dimensional space $\mathbf{R}^N$. The linear smooth mapping function f creates the vector field $y \in V$ from any vector field $x \in U$:

$$\begin{aligned} &f : x \in U \subset \mathbf{R}^N \to y = f(x) \in V \subset \mathbf{R}^N \\ &\Rightarrow x = f^{-1}(y) \end{aligned} \tag{4.26}$$

The linear function f_* maps the dual space $T_X{}^* \mathbf{R}^N$ of X to the dual space $T_y{}^* \mathbf{R}^N$ of the vector field y:

$$df(x) \equiv f_* : X(x) \in T_X^* \mathbf{R}^N \to f_* X \in T_y^* \mathbf{R}^N \tag{4.27}$$

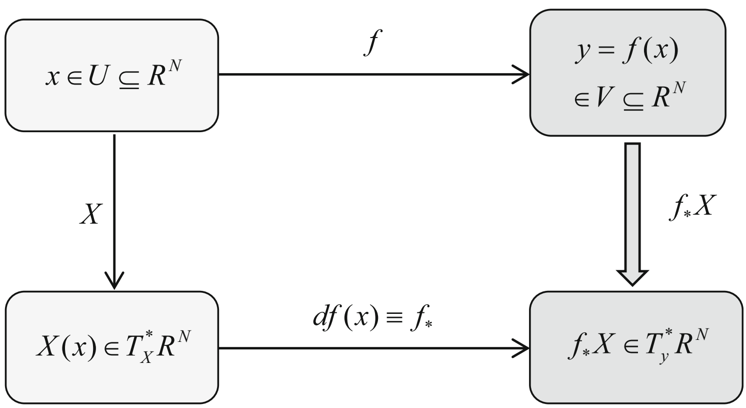

Fig. 4.4 Pushforward operator $f_* X$ of X by f

The operator f_*X is called the pushforward operator of X by the mapping function f that is defined as

$$f_*X(x) \equiv df(x)X\left(f^{-1}(y)\right) \in T_y^*\mathbf{R}^N \tag{4.28}$$

The pullback of a one-form ω by f for any vector field $X \in T_X{}^*\mathbf{R}^N$ is defined as [3]

$$\left(f^*\omega\right) \cdot X(x) = \omega \cdot f_*X(y) \tag{4.29a}$$

Therefore, the pullback of a k-form ω ($k > 1$) by f for the bundle of vector fields $(X_{i1},\ldots, X_{ik}) \in T^*M$ results from Eq. (4.29a).

$$f^*\omega \cdot (X_{i1} \wedge \ldots \wedge X_{ik}) = \omega \cdot (f_*X_{i1} \wedge \ldots \wedge f_*X_{ik}) \tag{4.29b}$$

4.9 The Hodge Star Operator

The Hodge star operator (star operator) is used to carry out various operations, such as gradient, div, and curl in exterior algebra, in which the coordinates are not taken into account. The Hodge * operator is a counterpart of Nabla operator in vector calculus in linear algebra that considers the coordinates in calculations.

The star operator is a linear map from an exterior k-form bundle to another exterior $(N - k)$-form bundle on the N-dimensional manifold M [5, 6]

$$* : \Lambda^k\left(\mathbf{R}^N\right) \to \Lambda^{N-k}\left(\mathbf{R}^N\right); \;\; 0 \le k \le N \tag{4.30}$$

where $\wedge^k(\mathbf{R}^N)$ is the differential k-form bundle that consists of all differential k-form spaces.

Let ω be any k-form that is written for $I = (i_1,\ldots,i_k)$ with $1 \le i_1 < \ldots < i_k \le N$ as

$$\omega = \sum_I f_I dx^I = \sum_I f_I dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \Lambda^k\left(\mathbf{R}^N\right) \tag{4.31}$$

The Hodge * operator maps the k-form ω to the Hodge dual $*\omega$ in the exterior $(N - k)$-form bundle.

The Hodge dual $*\omega$ is a pseudo $(N - k)$-form that is written as

$$\begin{aligned} &* : dx^I \in \Lambda^k\left(\mathbf{R}^N\right) \to *dx^I \in \Lambda^{N-k}\left(\mathbf{R}^N\right) \\ &\Rightarrow *\omega = \sum_I f_I *dx^I \in \Lambda^{N-k}\left(\mathbf{R}^N\right) \end{aligned} \tag{4.32}$$

Similarly, the Hodge dual elementary $(N - k)$-form is defined as the Hodge * operator of the elementary form dx^I as

$$*dx^I = \underbrace{dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \ldots \wedge dx^{i_N}}_{(N-k)\ times} \tag{4.33}$$

Substituting Eq. (4.33) into Eq. (4.32), the Hodge dual of the k-form ω is written as

$$\begin{aligned} *\omega &= \sum_I f_I {*dx^I} = \sum_I f_I dx^{i_{k+1}} \wedge dx^{i_{k+2}} \wedge \ldots \wedge dx^{i_N} \\ &\equiv \sum_{I,J} f_I dx^J \in \Lambda^{N-k}(\mathbf{R}^N) \end{aligned} \tag{4.34}$$

where $I=(i_1,\ldots,i_k)$ with $1 \le i_1 < \ldots < i_k \le N$ and $J=(i_{k+1},\ldots,i_N)$ with $1 \le i_{k+1} < \ldots < i_N \le N$.

Obviously, the wedge product of the elementary k-form and its Hodge dual is the elementary N-form.

$$\begin{aligned} dx^I \wedge *dx^I &= (dx^{i_1} \wedge \ldots \wedge dx^{i_k}) \wedge (dx^{i_{k+1}} \wedge \ldots \wedge dx^{i_N}) \\ &= dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_N} \equiv dx^N \end{aligned} \tag{4.35}$$

Some examples of the Hodge * operators of the dual bases (dx,dy,dz) in $\mathbf{R}^3$ are given in the following section.

$$\begin{aligned} &*dx = dy \wedge dz \Rightarrow *(dy \wedge dz) = dx \\ &*dy = dz \wedge dx \Rightarrow *(dz \wedge dx) = dy \\ &*dz = dx \wedge dy \Rightarrow *(dx \wedge dy) = dz \\ &*1 = dx \wedge dy \wedge dz \Rightarrow *(dx \wedge dy \wedge dz) = 1 \end{aligned} \tag{4.36a}$$

Proof Using Eqs. (4.5) and (4.35), one writes

$$\begin{aligned} &dx \wedge (*dx) = dx \wedge (dy \wedge dz) \in \Lambda^3(\mathbf{R}^3) \\ &\Rightarrow *dx = dy \wedge dz; \\ &dy \wedge (*dy) = dx \wedge dy \wedge dz = -dy \wedge (dx \wedge dz) \in \Lambda^3(\mathbf{R}^3) \\ &\Rightarrow *dy = -dx \wedge dz = dz \wedge dx; \\ &dz \wedge (*dz) = dx \wedge dy \wedge dz = dz \wedge (dx \wedge dy) \in \Lambda^3(\mathbf{R}^3) \\ &\Rightarrow *dz = dx \wedge dy; \\ &(dx \wedge dy \wedge dz) \wedge *(dx \wedge dy \wedge dz) = dx \wedge dy \wedge dz \\ &\Rightarrow *(dx \wedge dy \wedge dz) = 1. \end{aligned}$$

In general, Eq. (4.36a) can be expressed in an N-dimensional space $\mathbf{R}^N$.

$$\begin{aligned} &*dx^{i_1} = dx^{i_2} \wedge \ldots \wedge dx^{i_N}; \\ &*(dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_N}) = 1; \\ &*1 = dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_N} \end{aligned} \tag{4.36b}$$

4.9.1 Star Operator in Vector Calculus and Differential Forms

Let f be a function of x, y, and z in $\mathbf{R}^3$. The gradient of a function $f(x,y,z)$ is written in a one-form as

$$df = \frac{\partial f}{\partial x^i} dx^i = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \nabla f \cdot d\mathbf{r} \tag{4.37a}$$

Using the Hodge duals, the Laplacian of $f(x,y,z)$ is calculated as

$$\Delta f = \nabla \cdot \nabla f = *d(*df) \tag{4.37b}$$

Similarly, let $\mathbf{v}$ be any vector (one-form) in $\mathbf{R}^3$. The vector $\mathbf{v}$ (v_1,v_2,v_3) corresponds to a one-form ω_v that can be expressed as

$$\mathbf{v} \leftrightarrow \omega_v \equiv v_1 dx + v_2 dy + v_3 dz = \mathbf{v} \cdot d\mathbf{r} \tag{4.38}$$

Using Eq. (4.14), the exterior derivative of the one-form ω_v is written as

$$\begin{aligned} d\omega_v &= d(v_1 dx + v_2 dy + v_3 dz) \\ &= (dv_1 \wedge dx) + (dv_2 \wedge dy) + (dv_3 \wedge dz) \end{aligned}$$

The first term on the RHS of $d\omega_v$ is calculated as

$$\begin{aligned} dv_1 \wedge dx &= \left(\frac{\partial v_1}{\partial x} dx + \frac{\partial v_1}{\partial y} dy + \frac{\partial v_1}{\partial z} dz \right) \wedge dx \\ &= \frac{\partial v_1}{\partial y} dy \wedge dx + \frac{\partial v_1}{\partial z} dz \wedge dx \\ &= -\left(\frac{\partial v_1}{\partial y} \right) dx \wedge dy + \left(\frac{\partial v_1}{\partial z} \right) dz \wedge dx \end{aligned}$$

Analogously, the second and third terms on the RHS of $d\omega_v$ result as

$$\begin{aligned} dv_2 \wedge dy &= \left(\frac{\partial v_2}{\partial x} \right) dx \wedge dy - \left(\frac{\partial v_2}{\partial z} \right) dy \wedge dz \\ dv_3 \wedge dz &= -\left(\frac{\partial v_3}{\partial x} \right) dz \wedge dx + \left(\frac{\partial v_3}{\partial y} \right) dy \wedge dz \end{aligned}$$

Hence, the exterior derivative of the one-form ω_v is calculated as

$$\begin{aligned} d\omega_v &= d(v_1 dx + v_2 dy + v_3 dz) \\ &= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) dy \wedge dz + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) dz \wedge dx + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) dx \wedge dy \end{aligned}$$

Thus, the Hodge dual of the exterior derivative $d\omega_v$ results as

$$*d\omega_v = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)dx + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)dy + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)dz \tag{4.39a}$$

The curl of $\mathbf{v}$ is calculated in $\mathbf{R}^3$ as

$$\nabla \times \mathbf{v} = \left[\left(\tfrac{\partial v_3}{\partial y} - \tfrac{\partial v_2}{\partial z}\right) \quad \left(\tfrac{\partial v_1}{\partial z} - \tfrac{\partial v_3}{\partial x}\right) \quad \left(\tfrac{\partial v_2}{\partial x} - \tfrac{\partial v_1}{\partial y}\right) \right]^T \tag{4.39b}$$

Using Eqs. (4.39a) and (4.39b), one obtains the relation between the Hodge dual $*d\omega_v$ and curl $\mathbf{v}$

$$*d\omega_v = (\nabla \times \mathbf{v}) \cdot d\mathbf{r} \tag{4.40}$$

To calculate the divergence of a vector, a two-form η_v is defined using the vector components as

$$\eta_v \equiv v_1 dy \wedge dz + v_2 dz \wedge dx + v_3 dx \wedge dy$$

The Hodge dual of the one-form ω_v in Eq. (4.38) results as

$$*\omega_v \equiv v_1(*dx) + v_2(*dy) + v_3(*dz)$$

Using Eq. (4.36a), the two-form η_v equals the Hodge dual $*\omega_v$.

$$\eta_v = *\omega_v \tag{4.41}$$

The exterior derivative $d\eta_v$ is calculated as

$$d\eta_v = d(v_1 dy \wedge dz + v_2 dz \wedge dx + v_3 dx \wedge dy) \tag{4.42}$$

Using Eq. (4.14), the RHS of Eq. (4.42) is rewritten as

$$\begin{aligned} \text{RHS} &\equiv d(v_1 dy \wedge dz + v_2 dz \wedge dx + v_3 dx \wedge dy) \\ &= dv_1 \wedge (dy \wedge dz) + dv_2 \wedge (dz \wedge dx) + dv_3 \wedge (dx \wedge dy) \end{aligned} \tag{4.43}$$

Using the property of the wedge product $(dx \wedge dx) = 0$, the first term on the RHS in Eq. (4.43) is calculated as

$$\begin{aligned} dv_1 \wedge (dy \wedge dz) &= \left(\frac{\partial v_1}{\partial x}dx + \frac{\partial v_1}{\partial y}dy + \frac{\partial v_1}{\partial z}dz\right) \wedge (dy \wedge dz) \\ &= \frac{\partial v_1}{\partial x} dx \wedge dy \wedge dz \end{aligned}$$

Analogously, the second and third terms on the RHS in Eq. (4.43) result as

$$dv_2 \wedge (dz \wedge dx) = \frac{\partial v_2}{\partial y} dx \wedge dy \wedge dz;$$
$$dv_3 \wedge (dx \wedge dy) = \frac{\partial v_3}{\partial z} dx \wedge dy \wedge dz.$$

Therefore, the exterior derivative $d\eta_v$ results as

$$d\eta_v = \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) dx \wedge dy \wedge dz$$

Using Eqs. (4.36a) and (4.41), the Hodge dual $d\eta_v$ is written as

$$\begin{aligned} *d\eta_v &= \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) *(dx \wedge dy \wedge dz) \\ &= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = *d(*\omega_v) \end{aligned}$$

Furthermore, the divergence of $\mathbf{v}$ is calculated as

$$\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Therefore, the relation between $*d\omega_v$ and div $\mathbf{v}$ results as

$$*d(*\omega_v) = \nabla \cdot \mathbf{v} \tag{4.44}$$

Table 4.3 shows the overview of gradient, curl, and divergence operators that are displayed in vector calculus and differential forms.

4.9.2 Star Operator and Inner Product

Let ω be any k-form $\in \wedge^k(\mathbf{R}^N)$ in an N-dimensional space $\mathbf{R}^N$. The Hodge * operator $*\omega$ is the $(N-k)$-form $\in \wedge^{N-k}(\mathbf{R}^N)$.

Table 4.3 Gradient, curl, and divergence in differential forms

Operators	Variable	Grad	Curl	Divergence
Vector calculus	f	$\xrightarrow{grad}$ ∇f	$\xrightarrow{Laplacian}$ Δf	
	$\mathbf{v}$		$\xrightarrow{curl}$ $\nabla \times \mathbf{v}$	$\xrightarrow{div}$ $\nabla \cdot \mathbf{v}$
Differential forms	f	$\xrightarrow{d}$ df	$\xrightarrow{*d}$ $\Delta f = *d(*df)$	
	ω_v		$\xrightarrow{*d}$ $*d\omega_v$	$\xrightarrow{*d}$ $*d(*\omega_v)$

If η is any k-form $\in \wedge^k(\mathbf{R}^N)$, the inner product of two k-forms ω and η is defined so that [6, 7]

$$\begin{aligned}\eta \wedge {}^*\omega &= \langle \omega, \eta \rangle dx^N \\ &= \langle \omega, \eta \rangle dx^{i_1} \wedge \ldots \wedge dx^{i_N} \in \Lambda^N\left(\mathbf{R}^N\right)\end{aligned} \tag{4.45}$$

The inner product of two orthonormal dual bases is written in Kronecker delta as

$$\left\langle dx^i, dx^j \right\rangle = \delta_i^j = \begin{cases} 0 \, for \; i \neq j \\ 1 \, for \; i = j \end{cases} \tag{4.46}$$

Let dx^I and dx^J be two k-forms of the dual bases. The inner product of dx^I and dx^J is defined as

$$\begin{aligned}\langle dx^I, dx^J \rangle = \sum_\sigma (-1)^{m_\sigma} \left\langle dx^{i_1}, dx^{\sigma(j_1)} \right\rangle \left\langle dx^{i_2}, dx^{\sigma(j_2)} \right\rangle \cdots \\ \left\langle dx^{i_k}, dx^{\sigma(j_k)} \right\rangle \in \mathbf{R}\end{aligned} \tag{4.47}$$

where $m_\sigma = \sigma$ mod2; σ is the number of permutations on k elements.

For any $i \neq j$ in an N-dimensional space $\mathbf{R}^N$, one obtains using Eqs. (4.45) and (4.46)

$$\begin{aligned}dx^i \wedge {}^*dx^j &= \langle dx^j, dx^i \rangle dx^N \\ &= \delta_i^j \left(dx^{i_1} \wedge \ldots \wedge dx^{i_N}\right) \\ &= 0\end{aligned}$$

Some examples of elementary two- and three-forms are given here to demonstrate the inner product of the elementary k-forms.

For elementary two-forms ($k = 2$), there are two terms of the inner product; they result from $k!$. Using Eq. (4.47), the inner product is written as

$$\begin{aligned}\left\langle dx^{i_1} \wedge dx^{i_2}, dx^{j_1} \wedge dx^{j_2} \right\rangle &= (-1)^0 \left\langle dx^{i_1}, dx^{j_1} \right\rangle \left\langle dx^{i_2}, dx^{j_2} \right\rangle \\ &\quad + \; (-1)^1 \left\langle dx^{i_1}, dx^{j_2} \right\rangle \left\langle dx^{i_2}, dx^{j_1} \right\rangle \\ &= \left\langle dx^{i_1}, dx^{j_1} \right\rangle \left\langle dx^{i_2}, dx^{j_2} \right\rangle \\ &\quad - \left\langle dx^{i_1}, dx^{j_2} \right\rangle \left\langle dx^{i_2}, dx^{j_1} \right\rangle\end{aligned} \tag{4.48}$$

The terms of the inner product of two elementary 2-forms result from Table 4.4.

For elementary three-forms ($k = 3$), there are six terms of the inner product; they result from $k!$. These terms with their signs are given in Table 4.5. The contravariant indices of any term of the elementary forms are displayed in each line, in which the same indices of i and j (=1, 2, 3) could occur only two times. The number of permutations σ of the following terms results from the order of the first term of the

Table 4.4 Inner product of the elementary 2-forms

i	j	i	j	$2! = 2$ terms	Permutation σ
1	1	**2**	2	$+ < i_1, j_1 >< i_2, j_2 >$	$\sigma = 0 \to m_\sigma = 0$
1	2	**2**	1	$- < i_1, j_2 >< i_2, j_1 >$	$\sigma = 1 \to m_\sigma = 1$

Table 4.5 Inner product of the elementary 3-forms

i	j	i	j	i	j	$3! = 6$ terms	Permutation σ
1	1	**2**	2	**3**	3	$+ < i_1, j_1 >< i_2, j_2 >< i_3, j_3 >$	$\sigma = 0 \to m_\sigma = 0$
1	2	**2**	1	**3**	3	$- < i_1, j_2 >< i_2, j_1 >< i_3, j_3 >$	$\sigma = 1 \to m_\sigma = 1$
1	3	**2**	1	**3**	2	$+ < i_1, j_3 >< i_2, j_1 >< i_3, j_2 >$	$\sigma = 2 \to m_\sigma = 0$
1	3	**2**	2	**3**	1	$- < i_1, j_3 >< i_2, j_2 >< i_3, j_1 >$	$\sigma = 1 \to m_\sigma = 1$
1	2	**2**	3	**3**	1	$+ < i_1, j_2 >< i_2, j_3 >< i_3, j_1 >$	$\sigma = 2 \to m_\sigma = 0$
1	1	**2**	3	**3**	2	$- < i_1, j_1 >< i_2, j_3 >< i_3, j_2 >$	$\sigma = 1 \to m_\sigma = 1$

Table 4.6 Summary of the operators using in differential forms

Operators in $\mathbf{R}^N$	Symbols	Forms	Results	Orders
Wedge product (exterior product)	$\wedge$	k-form ω; p-form η	$\omega \wedge \eta$	$k + p$
Interior product (Interior derivative)	i_x	k-form ω	$i_x\omega$	$k - 1$
Exterior derivative	d	k-form ω	$d\omega$	$k + 1$
Pullback operator	f^*	function f; k-form ω	$f^* \omega$; $\omega_0 f$	k
Hodge star operator	$*$	k-form ω	$* \omega$	$N - k$

elementary forms. Namely, the second term of the elementary forms is given by one interchange of the indices 1 and 2; therefore, the number of permutation $\sigma = 1$.

Using Eq. (4.47), the inner product of two elementary 3-forms is written from Table 4.5 as

$$\begin{aligned}
&\langle dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3}, dx^{j_1} \wedge dx^{j_2} \wedge dx^{j_3} \rangle = \\
&+\langle dx^{i_1}, dx^{j_1} \rangle \langle dx^{i_2}, dx^{j_2} \rangle \langle dx^{i_3}, dx^{j_3} \rangle \\
&-\langle dx^{i_1}, dx^{j_2} \rangle \langle dx^{i_2}, dx^{j_1} \rangle \langle dx^{i_3}, dx^{j_3} \rangle + \langle dx^{i_1}, dx^{j_3} \rangle \langle dx^{i_2}, dx^{j_1} \rangle \langle dx^{i_3}, dx^{j_2} \rangle \\
&-\langle dx^{i_1}, dx^{j_3} \rangle \langle dx^{i_2}, dx^{j_2} \rangle \langle dx^{i_3}, dx^{j_1} \rangle + \langle dx^{i_1}, dx^{j_2} \rangle \langle dx^{i_2}, dx^{j_3} \rangle \langle dx^{i_3}, dx^{j_1} \rangle \\
&-\langle dx^{i_1}, dx^{j_1} \rangle \langle dx^{i_2}, dx^{j_3} \rangle \langle dx^{i_3}, dx^{j_2} \rangle
\end{aligned} \tag{4.49}$$

Table 4.6 gives the overview of the discussed operators in differential forms of exterior algebra.

4.9.3 *Star Operator in the Minkowski Spacetime*

The Minkowski spacetime is applied to the special relativity, in which time t is an extra dimension besides the space coordinates of x, y, and z. As a result, the Minkowski is called the four-dimensional spacetime $(t, x, y, z) \in \mathbf{R}^4$ (cf. Chap. 5).

In the following section, the Hodge * operator will be used in the Minkowski spacetime [6, 7].

In gravitation, relativity theory, and cosmology, the light speed c is defined as $c^2 \equiv 1$, in which c is considered as a constant. The distance ds between two arbitrary points in the Minkowski spacetime for special relativity is written as

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= dt^2 - dx^2 - dy^2 - dz^2 \end{aligned} \tag{4.50}$$

The Minkowski metric with four spacetime coordinates (t, x, y, z) is expressed as

$$g = \left(g_{\mu\nu} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.51}$$

The elementary form of the Minkowski spacetime is defined as

$$dx^I = dt \wedge dx \wedge dy \wedge dz \in \Lambda^4 \left(\mathbf{R}^4 \right) \tag{4.52}$$

Note that the inner products of the orthonormal dual bases of the four-dimensional Minkowski spacetime result from Eqs. (4.50) and (4.51)

$$\begin{aligned} &\langle dt, dt \rangle = +1; \\ &\langle dx, dx \rangle = \langle dy, dy \rangle = \langle dz, dz \rangle = -1; \\ &\langle dt, dx^i \rangle = \langle dx^i, dx^j \rangle = 0 \ \mathit{if} \ i \neq j. \end{aligned} \tag{4.53a}$$

Using Eqs. (4.48) and (4.53a), the inner products of 2-forms of the dual bases result as

$$\begin{aligned} &\langle dt \wedge dx, dt \wedge dx \rangle = \langle dt \wedge dt \rangle \langle dx \wedge dx \rangle - \langle dt \wedge dx \rangle \langle dx \wedge dx \rangle \\ &\qquad = (+1) \cdot (-1) - (0) \cdot (0) = -1 \\ &\Rightarrow \langle dt \wedge dy, dt \wedge dy \rangle = \langle dt \wedge dz, dt \wedge dz \rangle = -1 \\ &\langle dx \wedge dy, dx \wedge dy \rangle = \langle dx \wedge dx \rangle \langle dy \wedge dy \rangle - \langle dx \wedge dy \rangle \langle dy \wedge dx \rangle \\ &\qquad = (-1) \cdot (-1) - (0) \cdot (0) = +1 \\ &\Rightarrow \langle dx \wedge dz, dx \wedge dz \rangle = \langle dy \wedge dz, dy \wedge dz \rangle = +1 \end{aligned} \tag{4.53b}$$

Using Eqs. (4.49), (4.53a) and (4.53b), the inner products of 3-forms of the dual bases result as

$$\begin{aligned} &\langle dt \wedge dx \wedge dy, dt \wedge dx \wedge dy \rangle = +1; \\ &\langle dt \wedge dy \wedge dz, dt \wedge dy \wedge dz \rangle = +1; \\ &\langle dt \wedge dx \wedge dz, dt \wedge dx \wedge dz \rangle = +1; \\ &\langle dx \wedge dy \wedge dz, dx \wedge dy \wedge dz \rangle = -1. \end{aligned} \tag{4.53c}$$

Using the inner product of two k-forms in Eq. (4.45), the Hodge dual $*dt$ is calculated as

$$\begin{aligned} & dt \wedge *dt = \langle dt, dt \rangle dx^N = \langle dt, dt \rangle dt \wedge dx \wedge dy \wedge dz \\ & \Rightarrow dt \wedge \frac{*(dt)}{\langle dt, dt \rangle} = dt \wedge dx \wedge dy \wedge dz \\ & \Rightarrow dt \wedge \frac{*(dt)}{(+1)} = dt \wedge (dx \wedge dy \wedge dz) \end{aligned} \tag{4.54}$$

Therefore, the Hodge dual of the 1-form dt results as

$$*dt = dx \wedge dy \wedge dz \tag{4.55}$$

Using Eq. (4.54), one obtains three Hodge duals of 1-forms of the dual bases.

$$\begin{aligned} *dx &= dt \wedge dy \wedge dz \\ *dy &= dt \wedge dz \wedge dx \\ *dz &= dt \wedge dx \wedge dy \end{aligned} \tag{4.56}$$

Six Hodge duals of 2-forms of the dual bases are computed as [7, 8]

$$\begin{aligned} *(dt \wedge dx) &= -dy \wedge dz; \\ *(dt \wedge dy) &= -dz \wedge dx; \\ *(dt \wedge dz) &= -dx \wedge dy; \\ *(dx \wedge dy) &= -dz \wedge dt; \\ *(dx \wedge dz) &= -dt \wedge dy; \\ *(dy \wedge dz) &= -dx \wedge dt. \end{aligned} \tag{4.57}$$

Four Hodge duals of 3-forms of the dual bases are computed as [7, 8]

$$\begin{aligned} *(dx \wedge dy \wedge dz) &= dt; \\ *(dt \wedge dx \wedge dy) &= dz; \\ *(dt \wedge dz \wedge dx) &= dy; \\ *(dt \wedge dy \wedge dz) &= dx. \end{aligned} \tag{4.58}$$

According to Eq. (4.45), the Hodge dual of the 4-form of the dual bases is computed as

$$\begin{aligned} & *(dt \wedge dx \wedge dy \wedge dz) = 1 \\ & \Rightarrow *1 = dt \wedge dx \wedge dy \wedge dz \end{aligned} \tag{4.59}$$

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Chapter 5
Applications of Tensors and Differential Geometry

5.1 Nabla Operator in Curvilinear Coordinates

Nabla operator is a linear map of an arbitrary tensor into an image tensor in N-dimensional curvilinear coordinates. The Nabla operator can be usually defined in N-dimensional Cartesian coordinates $\{x^i\}$ using Einstein summation convention as

$$\nabla \equiv \mathbf{e}^i \frac{\partial}{\partial x^i} \text{ for } i = 1, 2, \ldots, N \tag{5.1}$$

According to Eq. (2.12), the relation between the bases of Cartesian and general curvilinear coordinates can be written as

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial u^i} = \frac{\partial \mathbf{r}}{\partial x^j} \frac{\partial x^j}{\partial u^i} = \mathbf{e}_j \frac{\partial x^j}{\partial u^i} \tag{5.2}$$

Multiplying Eq. (5.2) by $\mathbf{g}^i \mathbf{e}^j$, one obtains the basis of Cartesian coordinates expressed in the curvilinear coordinate basis.

$$\mathbf{e}^j = \mathbf{g}^i \frac{\partial x^j}{\partial u^i} \tag{5.3}$$

Using chain rule of coordinate transformation, the Nabla operator in the general curvilinear coordinates $\{u^i\}$ results from Eq. (5.3) [1, 2].

H. Nguyen-Schäfer, J.-P. Schmidt, *Tensor Analysis and Elementary Differential Geometry for Physicists and Engineers*, Mathematical Engineering,
DOI 10.1007/978-3-662-48497-5_5

$$\begin{aligned}\nabla \equiv \mathbf{e}^i\left(\frac{\partial}{\partial u^j}\frac{\partial u^j}{\partial x^i}\right) &= \mathbf{g}^k\frac{\partial x^i}{\partial u^k}\left(\frac{\partial}{\partial u^j}\frac{\partial u^j}{\partial x^i}\right)\\ &= \mathbf{g}^k\frac{\partial}{\partial u^j}\left(\frac{\partial x^i}{\partial u^k}\frac{\partial u^j}{\partial x^i}\right) = \mathbf{g}^k\frac{\partial}{\partial u^j}\left(\frac{\partial u^j}{\partial u^k}\right)\\ &= \mathbf{g}^k\frac{\partial}{\partial u^j}\left(\delta^j_k\right) = \mathbf{g}^k\frac{\partial}{\partial u^k}\end{aligned} \tag{5.4}$$

Thus, the Nabla operator can be written in the curvilinear coordinates $\{u^i\}$using Einstein summation convention.

$$\nabla \equiv \mathbf{g}^i\frac{\partial}{\partial u^i} = \mathbf{g}^i\nabla_i \text{ for } i = 1, 2, \ldots, N \tag{5.5}$$

5.2 Gradient, Divergence, and Curl

Let φ be a velocity potential that exists only in a vortex-free flow. The velocity potential can be defined as

$$\varphi = \int v\,dx \tag{5.6}$$

Differentiating Eq. (5.6) with respect to x, the velocity component results in

$$v = \frac{\partial \varphi}{\partial x} \tag{5.7}$$

The velocity vector $\mathbf{v}$ can be written in the general curvilinear coordinates $\{u^i\}$with the contravariant basis.

$$\mathbf{v} = v_i\mathbf{g}^i = \frac{\partial \varphi}{\partial u^i}\mathbf{g}^i \tag{5.8}$$

5.2.1 Gradient of an Invariant

The gradient of an invariant φ (function, zero-order tensor) can be defined by

Table 5.1 Essential Nabla operators

Operand ()	Operator Grad $\nabla()$	Div $\nabla \cdot ()$	Curl $\nabla \times ()$	Laplacian $\Delta()$
Function $f \in \mathbf{R}$ (0th order tensor)	Vector $\in \mathbf{R}^N$	–	–	Scalar $\in \mathbf{R}$
Vector $\mathbf{v} \in \mathbf{R}^N$ (1st order tensor)	2nd order tensor $\in \mathbf{R}^N \times \mathbf{R}^N$	Scalar $\in \mathbf{R}$	Vector $\in \mathbf{R}^N$	Vector $\in \mathbf{R}^N$
Tensor $\mathbf{T} \in \mathbf{R}^N \times \mathbf{R}^N$ (2nd order tensor)	3rd order tensor $\in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N$	Vector $\in \mathbf{R}^N$	–	–
Order of results	One order higher	One order lower	Order unchanged	Order unchanged

$$
\begin{aligned}
&Grad\ \varphi = \nabla\varphi = \mathbf{g}^i \frac{\partial \varphi}{\partial u^i} \equiv \varphi_{,i}\mathbf{g}^i = v_i\mathbf{g}^i = \mathbf{v} \\
&\Rightarrow \nabla\varphi = \mathbf{v} = \left(\frac{\partial \varphi}{\partial u^1}\mathbf{g}^1 + \frac{\partial \varphi}{\partial u^2}\mathbf{g}^2 + \frac{\partial \varphi}{\partial u^3}\mathbf{g}^3 \right)
\end{aligned}
\tag{5.9}
$$

Obviously, gradient of a function is a vector (cf. Table 5.1).

5.2.2 Gradient of a Vector

The gradient of a contravariant vector **v** can be calculated using the derivative of the covariant basis $\mathbf{g}_j$, as given in Eq. (2.158).

$$
\begin{aligned}
Grad\ \mathbf{v} = \nabla\mathbf{v} &= \left(\mathbf{g}^i \frac{\partial}{\partial u^i} \right) \left(v^j \mathbf{g}_j \right) = \mathbf{g}^i \frac{\partial \left(v^j \mathbf{g}_j \right)}{\partial u^i} \\
&= \mathbf{g}^i \left(v^j_{,i}\mathbf{g}_j + v^j \mathbf{g}_{j,i} \right) = \left(v^k_{,i}\mathbf{g}_k + v^j \Gamma^k_{ij}\mathbf{g}_k \right)\mathbf{g}^i \\
&= \left(v^k_{,i} + v^j \Gamma^k_{ij} \right) \mathbf{g}_k\mathbf{g}^i \\
&\equiv v^k\big|_i \mathbf{g}_k\mathbf{g}^i
\end{aligned}
\tag{5.10}
$$

Analogously, the gradient of a covariant vector **v** can be written using the derivative of the contravariant basis $\mathbf{g}^j$ in Eq. (2.189).

$$
\begin{aligned}
Grad\ \mathbf{v} = \nabla\mathbf{v} &= \mathbf{g}^i \frac{\partial\left(v_j \mathbf{g}^j\right)}{\partial u^{\mathrm{i}}} \\
&= \mathbf{g}^i\left(v_{j,i}\mathbf{g}^j + v_j \mathbf{g}^j_{,i}\right) = \left(v_{k,i}\mathbf{g}^k - v_j \Gamma^j_{ik}\mathbf{g}^k\right)\mathbf{g}^i \\
&= \left(v_{k,i} - v_j \Gamma^j_{ik}\right)\mathbf{g}^k \mathbf{g}^i \\
&\equiv v_k|_i \mathbf{g}^k \mathbf{g}^i
\end{aligned}
\tag{5.11}
$$

Obviously, grad of a vector is a second-order tensor; grad of a second-order tensor is a third-order tensor (cf. Table 5.1).

5.2.3 Divergence of a Vector

Let $\mathbf{v}$ be a vector in the curvilinear coordinates $\{u^i\}$; it can be written in the covariant basis $\mathbf{g}_j$.

$$
\mathbf{v} = v^j \mathbf{g}_j \tag{5.12}
$$

The divergence of $\mathbf{v}$ can be defined by

$$
\begin{aligned}
Div\ \mathbf{v} = \nabla\cdot\mathbf{v} &= \left(\mathbf{g}^i \frac{\partial}{\partial u^i}\right)\cdot\mathbf{v} = \mathbf{g}^i \cdot \frac{\partial\left(v^j \mathbf{g}_j\right)}{\partial u^i} \\
&= \mathbf{g}^i \cdot \left(v^j_{,i}\mathbf{g}_j + v^j \mathbf{g}_{j,i}\right) \equiv \mathbf{g}^i \cdot \nabla_i \mathbf{v}
\end{aligned}
\tag{5.13}
$$

Using Eq. (2.158), the derivative of the covariant basis $\mathbf{g}_j$ results in

$$
\mathbf{g}_{j,i} = \Gamma^k_{ji}\mathbf{g}_k = \Gamma^k_{ij}\mathbf{g}_k \tag{5.14}
$$

Substituting Eq. (5.14) into Eq. (5.13), one obtains the divergence of $\mathbf{v}$.

$$
\begin{aligned}
\nabla\cdot\mathbf{v} &= \mathbf{g}^i \cdot \left(v^j_{,i}\mathbf{g}_j + v^j \Gamma^k_{ij}\mathbf{g}_k\right) \\
&= \mathbf{g}^i \cdot \left(v^k_{,i}\mathbf{g}_k + v^j \Gamma^k_{ij}\mathbf{g}_k\right) = \mathbf{g}^i \cdot \left(v^k_{,i} + v^j \Gamma^k_{ij}\right)\mathbf{g}_k \\
&= \left(v^k_{,i} + v^j \Gamma^k_{ij}\right)\mathbf{g}^i \cdot \mathbf{g}_k \equiv v^k|_i \delta^i_k \\
&= \left(v^i_{,i} + v^j \Gamma^i_{ij}\right) = v^i|_i
\end{aligned}
\tag{5.15a}
$$

According to Eq. (2.240), the second-kind Christoffel symbol can be rewritten as

$$\Gamma^i_{ij} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial u^j} = \frac{\partial \left(\ln \sqrt{g} \right)}{\partial u^j} \tag{5.15b}$$

Substituting Eq. (5.15b) into Eq. (5.15a), the divergence of **v** can be expressed in

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{1}{\sqrt{g}} \left(\sqrt{g} \frac{\partial v^i}{\partial u^i} + v^i \frac{\partial J}{\partial u^i} \right) \\ &= \frac{1}{\sqrt{g}} \frac{\partial \left(\sqrt{g} v^i \right)}{\partial u^i} = \frac{1}{\sqrt{g}} \left(\sqrt{g} v^i \right)_{,i}\end{aligned} \tag{5.15c}$$

Analogously, the covariant vector **v** can be written in the contravariant basis $\mathbf{g}^j$.

$$\mathbf{v} = v_j \mathbf{g}^j \tag{5.16}$$

The divergence of **v** can be derived using the contraction law and derivative of the basis $\mathbf{g}^j$ in Eq. (2.189).

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \mathbf{g}^i \cdot \frac{\partial \left(v_j \mathbf{g}^j \right)}{\partial u^i} = \mathbf{g}^i \cdot \left(v_{j,i} \mathbf{g}^j + v_j \mathbf{g}^j_{,i} \right) \\ &= \mathbf{g}^i \cdot \left(v_{k,i} \mathbf{g}^k - v_j \Gamma^j_{ik} \mathbf{g}^k \right) \\ &= \left(v_{k,i} - v_j \Gamma^j_{ik} \right) \mathbf{g}^i \cdot \mathbf{g}^k \\ &\equiv v_k|_i \mathbf{g}^i \cdot \mathbf{g}^k = v_k|_i g^{ik} = v^m|_i g_{mk} g^{ik} \\ &= v^m|_i \delta^i_m = v^i|_i\end{aligned} \tag{5.17}$$

Obviously, divergence of a vector is a scalar (cf. Table 5.1).

Some useful abbreviations are listed as follows:

- Divergence of the contravariant vector **v**:

$$\begin{aligned}\nabla \cdot \mathbf{v} &\equiv \mathbf{g}^i \cdot \nabla_i \mathbf{v} = \mathbf{g}^i \cdot \nabla_i \left(v^j \mathbf{g}_j \right) \\ &= v^i_{,i} + v^j \Gamma^i_{ij} \equiv v^i|_i \\ &= \frac{1}{\sqrt{g}} \frac{\partial \left(\sqrt{g} v^i \right)}{\partial u^i} = \frac{1}{\sqrt{g}} \left(\sqrt{g} v^i \right)_{,i}\end{aligned} \tag{5.18a}$$

- Divergence of the covariant vector **v**:

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \mathbf{g}^i \cdot \nabla_i \mathbf{v} = \mathbf{g}^{\mathrm{i}} \cdot \nabla_i \left(v_j \mathbf{g}^j \right) \\ &= \left(v_{k,i} - v_j \Gamma^j_{ik} \right) g^{ki} = v_k|_i g^{ki}\end{aligned} \tag{5.18b}$$

- Covariant derivative of the contravariant vector component:

$$v^k|_i \equiv v^k_{,i} + v^j \Gamma^k_{ij} = \frac{\partial v^k}{\partial u^i} + v^j \Gamma^k_{ij} \tag{5.19a}$$

- Covariant derivative of the covariant vector component:

$$v_k|_i \equiv \left(v_{k,i} - v_j \Gamma^j_{ki} \right) = \frac{\partial v_k}{\partial u^i} - v_j \Gamma^j_{ki} \tag{5.19b}$$

- Covariant derivative of the contravariant vector $\mathbf{v}$ with respect to u^i:

$$\nabla_i \mathbf{v} = \left(v^k_{,i} + v^j \Gamma^k_{ij} \right) \mathbf{g}_k \ = v^k|_i \mathbf{g}_k \tag{5.20a}$$

- Covariant derivative of the covariant vector $\mathbf{v}$ with respect to u^i:

$$\nabla_i \mathbf{v} = \left(v_{k,i} - v_j \Gamma^j_{ik} \right) \mathbf{g}^k = v_k|_i \ \mathbf{g}^k = v^m|_i g_{mk} \ \mathbf{g}^k \tag{5.20b}$$

5.2.4 Divergence of a Second-Order Tensor

Let $\mathbf{T}$ be a contravariant tensor in the curvilinear coordinates $\{u^i\}$; it can be written in the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$.

$$\mathbf{T} \ = T^{ij} \mathbf{g}_i \mathbf{g}_j \tag{5.21}$$

The divergence of $\mathbf{T}$ can be calculated from

$$\begin{aligned} \nabla \cdot \mathbf{T} = \mathbf{g}^k \frac{\partial}{\partial u^k} \cdot \mathbf{T} = \mathbf{g}^k \cdot \frac{\partial \left(T^{ij} \mathbf{g}_i \mathbf{g}_j \right)}{\partial u^k} \\ = \mathbf{g}^k \cdot \left(T^{ij}_{,k} \mathbf{g}_i \mathbf{g}_j + T^{ij} \mathbf{g}_{i,k} \mathbf{g}_j + T^{ij} \mathbf{g}_i \mathbf{g}_{j,k} \right) \end{aligned} \tag{5.22}$$

Using Eq. (2.158), the derivative of the covariant basis $\mathbf{g}_i$ results in

$$\begin{aligned} \mathbf{g}_{i,k} &= \Gamma^m_{ik} \mathbf{g}_m = \Gamma^m_{ki} \mathbf{g}_m; \\ \mathbf{g}_{j,k} &= \Gamma^n_{jk} \mathbf{g}_n = \Gamma^n_{kj} \mathbf{g}_n \end{aligned}$$

Interchanging the indices, the divergence of a contravariant second-order tensor $\mathbf{T}$ becomes

$$\begin{aligned}\nabla\cdot\mathbf{T} &= \left(T^{ij}_{,k}\mathbf{g}_i\mathbf{g}_j + \Gamma^i_{km}T^{mj}\mathbf{g}_i\mathbf{g}_j + \Gamma^j_{km}T^{im}\mathbf{g}_i\mathbf{g}_j\right)\cdot\mathbf{g}^{\mathrm{k}} \\ &= \left(T^{ij}_{,k}\delta^k_i + \Gamma^i_{km}T^{mj}\delta^k_i + \Gamma^j_{km}T^{im}\delta^k_i\right)\mathbf{g}_j\end{aligned} \tag{5.23}$$

Equation (5.23) can be written in the covariant basis $\mathbf{g}_j$ at $k=i$.

$$\begin{aligned}\nabla\cdot\mathbf{T} &= \left(T^{ij}_{,k} + \Gamma^i_{km}T^{mj} + \Gamma^j_{km}T^{im}\right)\delta^k_i\mathbf{g}_j \\ &= \left(T^{ij}_{,i} + \Gamma^i_{im}T^{mj} + \Gamma^j_{im}T^{im}\right)\mathbf{g}_j \\ &\equiv T^{ij}|_i\mathbf{g}_j\end{aligned} \tag{5.24a}$$

Using Eq. (2.240), the covariant derivative of the tensor component T^{ij} with respect u^i on the RHS of Eq. (5.24a) can be expressed in

$$\begin{aligned}T^{ij}|_i &= T^{ij}_{,i} + \Gamma^i_{im}T^{mj} + \Gamma^j_{im}T^{im} \\ &= \frac{\partial T^{ij}}{\partial u^i} + T^{mj}\left(\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial u^m}\right) + T^{im}\Gamma^j_{im} \\ &= T^{im}\Gamma^j_{im} + \frac{1}{\sqrt{g}}\left(\sqrt{g}\frac{\partial T^{ij}}{\partial u^i} + T^{ij}\frac{\partial\sqrt{g}}{\partial u^i}\right) \\ &= T^{ik}\Gamma^j_{ik} + \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}T^{ij}\right)}{\partial u^i} = T^{ik}\Gamma^j_{ik} + \frac{1}{\sqrt{g}}\left(\sqrt{g}T^{ij}\right)_{,i}\end{aligned} \tag{5.24b}$$

Therefore,

$$\nabla\cdot\mathbf{T} = T^{ij}|_i\mathbf{g}_j = \left(T^{ik}\Gamma^j_{ik} + \frac{1}{\sqrt{g}}\left(\sqrt{g}T^{ij}\right)_{,i}\right)\mathbf{g}_j \tag{5.24c}$$

Interchanging the indices, the divergence of a covariant second-order tensor $\mathbf{T}$ can be written as

$$\begin{aligned}\nabla\cdot\mathbf{T} &= \mathbf{g}^k\cdot\frac{\partial\left(T_{ij}\mathbf{g}^i\mathbf{g}^j\right)}{\partial u^k} \\ &= \left(T_{ij,k}\mathbf{g}^i\mathbf{g}^j + T_{ij}\mathbf{g}^i_{,k}\mathbf{g}^j + T_{ij}\mathbf{g}^i\mathbf{g}^j_{,k}\right)\cdot\mathbf{g}^k \\ &= \left(T_{ij,k}\mathbf{g}^i\mathbf{g}^j - T_{ij}\Gamma^i_{km}\mathbf{g}^m\mathbf{g}^j - T_{ij}\Gamma^j_{km}\mathbf{g}^i\mathbf{g}^m\right)\cdot\mathbf{g}^k \\ &= \left(T_{ij,k} - T_{mj}\Gamma^m_{ki} - T_{im}\Gamma^m_{kj}\right)\mathbf{g}^i\left(\mathbf{g}^j\cdot\mathbf{g}^k\right) \\ &= T_{ij}|_k g^{jk}\mathbf{g}^i\end{aligned} \tag{5.25}$$

Furthermore, the divergence of a mixed second-order tensor $\mathbf{T}$ results as the same way at $k=i$.

$$
\begin{aligned}
\nabla \cdot \mathbf{T} &= \mathbf{g}^k \cdot \frac{\partial \left(T_j^i \mathbf{g}^j \mathbf{g}_i \right)}{\partial u^k} \\
&= \left(T_{j,k}^i + \Gamma_{km}^i T_j^m - \Gamma_{jk}^m T_m^i \right) \delta_i^k \mathbf{g}^j \\
&= \left(T_{j,i}^i + \Gamma_{im}^i T_j^m - \Gamma_{ji}^m T_m^i \right) \mathbf{g}^j \\
&\equiv T_j^i|_i \mathbf{g}^j \\
&= T_j^i|_i g^{kj} \mathbf{g}_k
\end{aligned}
\tag{5.26a}
$$

Using Eq. (2.240), the covariant derivative of the mixed tensor component with respect to u^i on the RHS of Eq. (5.26a) can be written in

$$
\begin{aligned}
T_j^i|_i &= T_{j,i}^i + \Gamma_{im}^i T_j^m - \Gamma_{ij}^m T_m^i \\
&= \frac{1}{\sqrt{g}} \left(\sqrt{g} \frac{\partial T_j^i}{\partial u^i} + T_j^i \frac{\partial \sqrt{g}}{\partial u^i} \right) - \Gamma_{ij}^m T_m^i \\
&= \frac{1}{\sqrt{g}} \left(\sqrt{g} T_j^i \right)_{,i} - T_k^i \Gamma_{ij}^k
\end{aligned}
\tag{5.26b}
$$

Therefore,

$$
\begin{aligned}
\nabla \cdot \mathbf{T} &= T_j^i|_i \mathbf{g}^j = T_j^i|_i g^{kj} \mathbf{g}_k \\
&= \left(\frac{1}{\sqrt{g}} \left(\sqrt{g} T_j^i \right)_{,i} - T_k^i \Gamma_{ij}^k \right) g^{kj} \mathbf{g}_k
\end{aligned}
\tag{5.26c}
$$

These results prove that the divergence of a second-order tensor **T**, such as the stress tensor **Π** or deformation tensor **D** results in a first-order tensor which is a vector in the curvilinear coordinates $\{u^i\}$.

Obviously, divergence of a second-order tensor is a vector (cf. Table 5.1).

5.2.5 Curl of a Covariant Vector

Let **v** be a covariant vector in the curvilinear coordinates $\{u^i\}$; it can be written in the contravariant basis $\mathbf{g}^j$.

$$\mathbf{v} = v_j \mathbf{g}^j \tag{5.27}$$

The curl (rotation) of **v** can be defined by

$$
\begin{aligned}
Curl\ \mathbf{v} \equiv Rot\ \mathbf{v} &\equiv \nabla \times \mathbf{v} \\
&= \mathbf{g}^i \frac{\partial}{\partial u^i} \times \left(v_j \mathbf{g}^j\right) = \mathbf{g}^i \times \frac{\partial\left(v_j \mathbf{g}^j\right)}{\partial u^i} \\
&= \mathbf{g}^i \times \left(v_{j,i}\mathbf{g}^j + v_j \mathbf{g}^j_{,i}\right) \\
&= v_{j,i}(\mathbf{g}^i \times \mathbf{g}^j) + v_j\left(\mathbf{g}^i \times \mathbf{g}^j_{,i}\right)
\end{aligned} \tag{5.28}
$$

Using Eq. (2.189), the derivative of the contravariant basis $\mathbf{g}^j$ results in

$$
\mathbf{g}^j_{,i} = -\Gamma^j_{ik}\mathbf{g}^k \tag{5.29}
$$

Substituting Eq. (5.29) into Eq. (5.28), one obtains the curl of **v**.

$$
\begin{aligned}
\nabla \times \mathbf{v} &= v_{j,i}\left(\mathbf{g}^i \times \mathbf{g}^j\right) + v_j\left(\mathbf{g}^i \times \mathbf{g}^j_{,i}\right) \\
&= \hat{\varepsilon}^{ijk} v_{j,i}\mathbf{g}_k - v_j\Gamma^j_{ik}\left(\mathbf{g}^i \times \mathbf{g}^k\right) \\
&= \hat{\varepsilon}^{ijk} v_{j,i}\mathbf{g}_\mathrm{k} - \hat{\varepsilon}^{ikm} v_j\Gamma^j_{ik}\mathbf{g}_m
\end{aligned} \tag{5.30}
$$

where the contravariant permutation symbols can be defined as (cf. Appendix A).

$$
\hat{\varepsilon}^{ijk} = \begin{cases} +\dfrac{1}{\sqrt{g}} \text{ if } (i,\ j,\ k) \text{ is an even permutation} \\ -\dfrac{1}{\sqrt{g}} \text{ if } (i,\ j,\ k) \text{ is an odd permutation} \\ 0 \quad \text{if } i = j, \text{ or } i = k; \text{ or } j = k \end{cases} \tag{5.31}
$$

However, the second term in RHS of Eq. (5.30) vanishes due to the symmetric Christoffel symbols with respect to the indices of i and k, and the anticyclic permutation property with respect to i, k, and m.

$$
\begin{aligned}
\hat{\varepsilon}^{ikm} v_j\Gamma^j_{ik}\mathbf{g}_m &= \frac{v_j}{\sqrt{g}}\left(\Gamma^j_{ik} - \Gamma^j_{ki}\right)\mathbf{g}_m \\
&= \frac{v_j}{\sqrt{g}}\left(\Gamma^j_{ik} - \Gamma^j_{ik}\right)\mathbf{g}_m = 0
\end{aligned} \tag{5.32}
$$

Therefore, the curl of **v** in Eq. (5.30) becomes

$$
\nabla \times \mathbf{v} = \hat{\varepsilon}^{ijk} v_{j,i}\mathbf{g}_k = \hat{\varepsilon}^{ijk}\frac{\partial v_j}{\partial u^i}\mathbf{g}_k \tag{5.33}
$$

Obviously, curl of a vector is a vector (cf. Table 5.1).

5.3 Laplacian Operator

Laplacian operator is a linear map of an arbitrary tensor into an image tensor in N-dimensional curvilinear coordinates.

5.3.1 Laplacian of an Invariant

Laplacian of an invariant φ (function, zeroth-order tensor) is the divergence of grad φ. Using Eq. (5.9), this expression can be written in

$$Div\ (Grad\ \varphi) \equiv \nabla \cdot \nabla\varphi = \nabla^2\varphi \equiv \Delta\varphi \tag{5.34}$$

Substituting the gradient $\nabla\varphi$ of Eq. (5.9) into Eq. (5.34), one obtains the Laplacian $\Delta\varphi$.

$$\begin{aligned}\Delta\varphi &\equiv \nabla \cdot \nabla\varphi = \nabla \cdot \left(\varphi_{,k}\mathbf{g}^k\right) = \nabla \cdot \left(v_k\mathbf{g}^k\right) \\ &= \mathbf{g}^l \cdot \frac{\partial}{\partial u^l}\left(\varphi_{,k}\mathbf{g}^k\right) = \mathbf{g}^l \cdot \left(\varphi_{,kl}\mathbf{g}^k + \varphi_{,k}\mathbf{g}^k_{,l}\right)\end{aligned} \tag{5.35}$$

Using Eq. (2.189), the derivative of the contravariant basis $\mathbf{g}^j$ results in

$$\mathbf{g}^k_{,l} = -\Gamma^k_{lm}\mathbf{g}^m \tag{5.36}$$

Inserting Eq. (5.36) into Eq. (5.35) and using Eq. (5.19b), the Laplacian of φ can be computed as

$$\begin{aligned}\Delta\varphi &= \nabla^2\varphi \\ &= \mathbf{g}^l \cdot \left(\varphi_{,kl}\mathbf{g}^k + \varphi_{,k}\mathbf{g}^k_{,l}\right) = \left(\varphi_{,kl}\mathbf{g}^k - \varphi_{,k}\Gamma^k_{lm}\mathbf{g}^m\right) \cdot \mathbf{g}^l \\ &= \left(\varphi_{,kl}\mathbf{g}^k - \varphi_{,m}\Gamma^m_{lk}\mathbf{g}^k\right) \cdot \mathbf{g}^l \\ &= \left(\varphi_{,kl} - \varphi_{,m}\Gamma^m_{lk}\right)\mathbf{g}^k \cdot \mathbf{g}^l \\ &= \left(\varphi_{,kl} - \varphi_{,m}\Gamma^m_{kl}\right)g^{kl}\end{aligned} \tag{5.37}$$

The covariant vector components and their derivatives with respect to u^k and u^l are defined as

$$\begin{aligned}&\varphi_{,k} = \frac{\partial\varphi}{\partial u^k} = v_k; \quad \varphi_{,m} = \frac{\partial\varphi}{\partial u^m} = v_m; \quad \varphi_{,kl} = \frac{\partial^2\varphi}{\partial u^k \partial u^l} = v_{k,l} \\ &\Rightarrow \quad \Delta\varphi \ = \left(v_{k,l} - v_m\Gamma^m_{kl}\right)g^{kl} \equiv v_k|_l g^{kl}\end{aligned} \tag{5.38}$$

Obviously, Laplacian of a function is a scalar (cf. Table 5.1).

5.3.2 Laplacian of a Contravariant Vector

Laplacian of a contravariant vector (first-order tensor) is the divergence of grad **v** that can be computed as [3]

$$\begin{aligned} Div(Grad\,\mathbf{v}) &= \Delta\mathbf{v} \equiv \nabla\cdot\nabla\mathbf{v} = \nabla^2\mathbf{v} \\ &= \left(v^k|_{l,m} - v^k\big|_p \Gamma^p_{lm} + v^p|_l \Gamma^k_{pm}\right) g^{lm}\mathbf{g}_k \\ &\equiv v^k|_{lm} g^{lm}\mathbf{g}_k \end{aligned} \tag{5.39}$$

According to Eq. (5.39), Laplacian of a vector is a vector (cf. Table 5.1).

The second covariant derivative of the contravariant vector component v^k in Eq. (5.39) can be defined as

$$v^k|_{lm} \equiv v^k|_{l,m} - v^k\big|_p \Gamma^p_{lm} + v^p|_l \Gamma^k_{pm} \tag{5.40}$$

where

$$v^k|_{l,m} = \left(v^k|_l\right)_{,m} \equiv v^k_{,lm} + v^n_{,m}\Gamma^k_{nl} + v^n\Gamma^k_{nl,m} \tag{5.41}$$

$$v^k\big|_p \equiv v^k_{,p} + v^n\Gamma^k_{np} \tag{5.42}$$

$$v^p|_l \equiv v^p_{,l} + v^n\Gamma^p_{nl} \tag{5.43}$$

The vector triple product gives the relation of

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \tag{5.44}$$

Thus, Eq. (5.44) can be rewritten in the curl identity of the vector **v** is set into the position of the vector **c**, cf. Appendix C.

$$\begin{aligned} \nabla\times(\nabla\times\mathbf{v}) &= \nabla(\nabla\cdot\mathbf{v}) - (\nabla\cdot\nabla)\mathbf{v} \\ &= \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v} \end{aligned} \tag{5.45}$$

The Laplacian of a vector **v** results from Eq. (5.44) as

$$\begin{aligned} &\Delta\mathbf{v} \equiv \nabla^2\mathbf{v} = (\nabla\cdot\nabla)\mathbf{v} = \nabla\cdot(\nabla\mathbf{v}) \\ &\Leftrightarrow \nabla\cdot(\nabla\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v}) \\ &\Leftrightarrow Laplacian\mathbf{v} \equiv Div(Grad\mathbf{v}) = Grad(Div\mathbf{v}) - Curl(Curl\mathbf{v}) \end{aligned} \tag{5.46}$$

5.4 Applying Nabla Operators in Spherical Coordinates

Spherical coordinates (ρ,φ,θ) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary. Figure 5.1 shows a point P in the spherical coordinates (ρ,φ,θ) embedded in orthonormal Cartesian coordinates (x^1,x^2,x^3). However, the vector component changes as the spherical coordinates vary.

The vector ***OP*** can be written in Cartesian coordinates (x^1,x^2,x^3):

$$\begin{aligned}\mathbf{R} &= (\rho \sin\varphi \cos\theta)\,\mathbf{e}_1 + (\rho \sin\varphi \sin\theta)\,\mathbf{e}_2 + \rho\cos\varphi\,\mathbf{e}_3 \\ &\equiv x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3\end{aligned} \tag{5.47}$$

where

$\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the orthonormal bases of Cartesian coordinates;
φ is the equatorial angle;
θ is the polar angle.

To simplify the formulation with Einstein symbol, the coordinates of u^1, u^2, and u^3 can be used for ρ, φ, and θ, respectively. Therefore, the coordinates of the point P (u^1,u^2,u^3) can be written in Cartesian coordinates:

$$P\left(u^1,u^2,u^3\right) = \left\{\begin{array}{l} x^1 = \rho\sin\varphi\cos\theta \equiv u^1\sin u^2\cos u^3 \\ x^2 = \rho\sin\varphi\sin\theta \equiv u^1\sin u^2\cos u^3 \\ x^3 = \rho\cos\varphi \equiv u^1\cos u^2 \end{array}\right\} \tag{5.48}$$

The covariant bases result from Chap. 2.

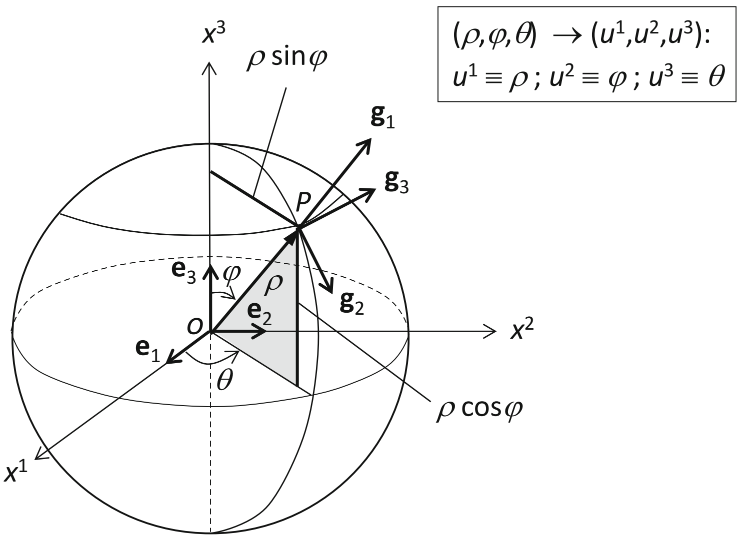


Fig. 5.1 Orthogonal spherical coordinates

$$
\begin{aligned}
\mathbf{g}_1 &= (\sin\varphi\cos\theta)\,\mathbf{e}_1 + (\sin\varphi\sin\theta)\,\mathbf{e}_2 + \cos\varphi\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_1| = |\mathbf{g}_\rho| = 1 \\
\mathbf{g}_2 &= (\rho\cos\varphi\cos\theta)\,\mathbf{e}_1 + (\rho\cos\varphi\sin\theta)\,\mathbf{e}_2 - (\rho\sin\varphi)\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_2| = |\mathbf{g}_\varphi| = \rho \\
\mathbf{g}_3 &= (-\rho\sin\varphi\sin\theta)\,\mathbf{e}_1 + (\rho\sin\varphi\cos\theta)\,\mathbf{e}_2 + 0.\mathbf{e}_3 \Rightarrow |\mathbf{g}_3| = |\mathbf{g}_\theta| = \rho\sin\varphi
\end{aligned}
\tag{5.49a}
$$

The covariant metric tensor **M** in the spherical coordinates can be computed from Eq. (5.49a).

$$
\mathbf{M} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & (\rho\sin\varphi)^2 \end{bmatrix} \tag{5.49b}
$$

Similarly, the contravariant bases result from Chap. 2.

$$
\begin{aligned}
\mathbf{g}^1 &= (\sin\varphi\cos\theta)\,\mathbf{e}_1 + (\sin\varphi\sin\theta)\,\mathbf{e}_2 + \cos\varphi\,\mathbf{e}_3 \Rightarrow |\mathbf{g}^1| = 1 \\
\mathbf{g}^2 &= \left(\frac{1}{\rho}\cos\varphi\cos\theta\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\cos\varphi\sin\theta\right)\mathbf{e}_2 - \left(\frac{1}{\rho}\sin\varphi\right)\mathbf{e}_3 \Rightarrow |\mathbf{g}^2| = \frac{1}{\rho} \\
\mathbf{g}^3 &= \left(-\frac{1}{\rho}\frac{\sin\theta}{\sin\varphi}\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\frac{\cos\theta}{\sin\varphi}\right)\mathbf{e}_2 + 0.\mathbf{e}_3 \Rightarrow |\mathbf{g}^3| = \frac{1}{\rho\sin\varphi}
\end{aligned}
\tag{5.50a}
$$

The contravariant metric coefficients in the contravariant metric tensor $\mathbf{M}^{-1}$ can be calculated from Eq. (5.50a).

$$
\mathbf{M}^{-1} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^{-2} & 0 \\ 0 & 0 & (\rho\sin\varphi)^{-2} \end{bmatrix} \tag{5.50b}
$$

5.4.1 Gradient of an Invariant

The gradient of an invariant $A \in \mathbf{R}$ can be written according to Eq. (5.9) in

$$
\nabla A = \mathbf{g}^i \frac{\partial A}{\partial u^i} \equiv A_{,i}\mathbf{g}^i \tag{5.51}
$$

Dividing the covariant basis by its vector length, the normalized covariant basis (covariant unitary basis) results in

$$\mathbf{g}_i^* = \frac{\mathbf{g}_i}{|\mathbf{g}_i|} = \frac{\mathbf{g}_i}{\sqrt{g_{(ii)}}} = \frac{\mathbf{g}_i}{h_i} \tag{5.52}$$

The covariant basis in Eq. (5.52) in given in

$$\mathbf{g}_i = h_i \mathbf{g}_i^* \tag{5.53}$$

where h_i are the vector lengths, as given in Eq. (5.49a).

$$\begin{aligned} h_1 &= \sqrt{g_{11}} = |\mathbf{g}_1| \equiv |\mathbf{g}_\rho| = 1 \\ h_2 &= \sqrt{g_{22}} = |\mathbf{g}_2| \equiv |\mathbf{g}_\varphi| = \rho \\ h_3 &= \sqrt{g_{33}} = |\mathbf{g}_3| \equiv |\mathbf{g}_\theta| = \rho \sin\varphi \end{aligned} \tag{5.54}$$

The contravariant bases can be transformed into the covariant bases in the orthogonal spherical contravariant basis, as given in Eq. (5.50b).

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j \Rightarrow \begin{cases} \mathbf{g}^1 = g^{11}\mathbf{g}_1 = \mathbf{g}_\rho \\ \mathbf{g}^2 = g^{22}\mathbf{g}_2 = \dfrac{1}{\rho^2}\mathbf{g}_\varphi \\ \mathbf{g}^3 = g^{33}\mathbf{g}_3 = \dfrac{1}{(\rho\sin\varphi)^2}\mathbf{g}_\theta \end{cases} \tag{5.55a}$$

Substituting Eqs. (5.53) and (5.54) into Eq. (5.55a), one obtains

$$\begin{cases} \mathbf{g}^1 = \mathbf{g}_\rho = \left(h_1\mathbf{g}_\rho^*\right) = \mathbf{g}_\rho^* \\ \mathbf{g}^2 = \dfrac{1}{\rho^2}\mathbf{g}_\varphi = \dfrac{1}{\rho^2}\left(h_2\mathbf{g}_\varphi^*\right) = \dfrac{1}{\rho}\mathbf{g}_\varphi^* \\ \mathbf{g}^3 = \dfrac{1}{(\rho\sin\varphi)^2}\mathbf{g}_\theta = \dfrac{1}{(\rho\sin\varphi)^2}\left(h_3\mathbf{g}_\theta^*\right) = \dfrac{1}{\rho\sin\varphi}\mathbf{g}_\theta^* \end{cases} \tag{5.55b}$$

Using Eqs. (5.51) and (5.55b), the gradient of A can be expressed in the physical vector components in the covariant unitary basis.

$$\begin{aligned} \nabla A &= \frac{\partial A}{\partial u^i}\mathbf{g}^i = \left(\frac{A_{,i}}{h_i}\right)\mathbf{g}_i^* \\ &= \frac{\partial A}{\partial \rho}\mathbf{g}_\rho^* + \frac{1}{\rho}\frac{\partial A}{\partial \varphi}\mathbf{g}_\varphi^* + \frac{1}{\rho\sin\varphi}\frac{\partial A}{\partial \theta}\mathbf{g}_\theta^* \end{aligned} \tag{5.56}$$

5.4.2 Divergence of a Vector

The divergence of $\mathbf{v}$ can be computed using the Christoffel symbols described in Eq. (5.15a).

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \mathbf{g}^i \frac{\partial \mathbf{v}}{\partial u^i} = \mathbf{g}^i \frac{\partial \left(v^j \mathbf{g}_j\right)}{\partial u^i} \\ &= \left(v^i_{,i} + v^j \Gamma^i_{ij}\right) \equiv v^i|_i\end{aligned} \tag{5.57}$$

At first, the covariant derivatives of the contravariant vector components in Eq. (5.57) have to be computed.

$$\begin{aligned}v^1|_1 &= v^1_{,1} + \Gamma^1_{11}v^1 + \Gamma^1_{12}v^2 + \Gamma^1_{13}v^3 \ \text{ for } i = 1;\ j = 1,2,3 \\ v^2|_2 &= v^2_{,2} + \Gamma^2_{21}v^1 + \Gamma^2_{22}v^2 + \Gamma^2_{23}v^3 \text{ for } i = 2;\ j = 1,2,3 \\ v^3|_3 &= v^3_{,3} + \Gamma^3_{31}v^1 + \Gamma^3_{32}v^2 + \Gamma^3_{33}v^3 \text{ for } i = 3;\ j = 1,2,3\end{aligned} \tag{5.58}$$

The second-kind Christoffel symbols in spherical coordinates can be calculated as [1].

$$\begin{aligned}&\Gamma^1_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\rho & 0 \\ 0 & 0 & -\rho \sin^2\varphi \end{pmatrix}; \quad \Gamma^2_{ij} = \begin{pmatrix} 0 & \frac{1}{\rho} & 0 \\ \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & -\sin\varphi\cos\varphi \end{pmatrix}; \\ &\Gamma^3_{ij} = \begin{pmatrix} 0 & 0 & \frac{1}{\rho} \\ 0 & 0 & \cot\varphi \\ \frac{1}{\rho} & \cot\varphi & 0 \end{pmatrix}\end{aligned} \tag{5.59}$$

The physical vector components v^{*i} result from the contravariant vector components in the covariant unitary basis $\mathbf{g}_i^*$ according to Eq. (B.11) in Appendix B.

$$v^i = \frac{v^{*i}}{h_i} \Rightarrow \begin{cases} v^1 = \dfrac{1}{h_1} v^{*1} \equiv v_\rho \\ v^2 = \dfrac{1}{h_2} v^{*2} \equiv \dfrac{1}{\rho} v_\varphi \\ v^3 = \dfrac{1}{h_3} v^{*3} \equiv \dfrac{1}{\rho \sin\varphi} v_\theta \end{cases} \tag{5.60}$$

Using Eqs. (5.59) and (5.60), the covariant derivatives of the contravariant vector components can be computed as

$$\begin{aligned} v^1|_1 &= v^1_{,1} = \frac{\partial v_\rho}{\partial \rho} \\ v^2|_2 &= v^2_{,2} + \Gamma^2_{21} v^1 = v^2_{,2} + \frac{1}{\rho} v^1 = \frac{1}{\rho}\frac{\partial v_\varphi}{\partial \varphi} + \frac{1}{\rho} v_\rho \\ v^3|_3 &= v^3_{,3} + \Gamma^3_{31} v^1 + \Gamma^3_{32} v^2 = \frac{1}{\rho \sin\varphi}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{\rho} v_\rho + \cot\varphi \frac{v_\varphi}{\rho} \end{aligned} \tag{5.61}$$

Thus, the divergence of $\mathbf{v}$ results from Eq. (5.61) in

$$\begin{aligned} \nabla \cdot \mathbf{v} &= v^i|_i \equiv v^1|_1 + v^2|_2 + v^3|_3 \\ &= \frac{\partial v_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial v_\varphi}{\partial \varphi} + \frac{1}{\rho \sin\varphi}\frac{\partial v_\theta}{\partial \theta} + \frac{2 v_\rho}{\rho} + \cot\varphi \frac{v_\varphi}{\rho} \end{aligned} \tag{5.62}$$

5.4.3 Curl of a Vector

The curl of a vector results from Eq. (5.33).

$$\begin{aligned} \nabla \times \mathbf{v} &= \hat{\varepsilon}^{ijk} v_{j,i} \mathbf{g}_k \\ &= \frac{1}{J}[(v_{3,2} - v_{2,3})\,\mathbf{g}_1 + (v_{1,3} - v_{3,1})\mathbf{g}_2 + (v_{2,1} - v_{1,2})\mathbf{g}_3] \end{aligned} \tag{5.63}$$

The Jacobian of the spherical coordinates were calculated in Eq. (2.37) as

$$J = \rho^2 \sin\varphi \tag{5.64}$$

Using Eqs. (B.19) and (5.49b), the covariant vector components can be computed in their physical vector components.

$$v_i = g_{ij}\left(\frac{v^{*j}}{h_j}\right) \Rightarrow \begin{cases} v_1 = g_{11}\left(\dfrac{v^{*1}}{h_1}\right) = v_\rho \\ v_2 = g_{22}\left(\dfrac{v^{*2}}{h_2}\right) = \rho\, v_\varphi \\ v_3 = g_{33}\left(\dfrac{v^{*3}}{h_3}\right) = \rho \sin\varphi\, v_\theta \end{cases} \tag{5.65}$$

According to Eq. (5.53), the covariant bases can be written in the covariant unitary basis.

$$\mathbf{g}_i = h_i \mathbf{g}_i^* \Rightarrow \begin{cases} \mathbf{g}_1 = h_1 \mathbf{g}_1^* = 1 \cdot \mathbf{g}_1^* \equiv \mathbf{g}_\rho^* \\ \mathbf{g}_2 = h_2 \mathbf{g}_2^* = \rho\, \mathbf{g}_2^* \equiv \rho\, \mathbf{g}_\varphi^* \\ \mathbf{g}_3 = h_3 \mathbf{g}_3^* = (\rho \sin \varphi)\, \mathbf{g}_3^* \equiv (\rho \sin \varphi)\, \mathbf{g}_\theta^* \end{cases} \tag{5.66}$$

Substituting Eqs. (5.64)–(5.66) into Eq. (5.63), the curl of $\mathbf{v}$ can be expressed in the unitary covariant basis $\mathbf{g}_i^*$.

$$\begin{aligned} \nabla \times \mathbf{v} &= \frac{1}{J}(v_{3,2} - v_{2,3})\, \mathbf{g}_1 + \frac{1}{J}(v_{1,3} - v_{3,1})\mathbf{g}_2 + \frac{1}{J}(v_{2,1} - v_{1,2})\mathbf{g}_3 \\ &= \left(\frac{\partial(\rho \sin\varphi \cdot v_\theta)}{\partial \varphi} - \frac{\partial\left(\rho \cdot v_\varphi\right)}{\partial \theta} \right) \frac{1}{\rho^2 \sin\varphi} \mathbf{g}_1^* \\ &\quad + \left(\frac{\partial v_\rho}{\partial \theta} - \frac{\partial(\rho \sin\varphi \cdot v_\theta)}{\partial \rho} \right) \frac{1}{\rho \sin\varphi} \mathbf{g}_2^* \\ &\quad + \left(\frac{\partial\left(\rho \cdot v_\varphi\right)}{\partial \rho} - \frac{\partial v_\rho}{\partial \varphi} \right) \frac{1}{\rho} \mathbf{g}_3^* \end{aligned} \tag{5.67}$$

Computing the partial derivatives in Eq. (5.67), one obtains the curl of $\mathbf{v}$ in the unitary spherical coordinate bases.

$$\begin{aligned} \nabla \times \mathbf{v} = &\left(\frac{1}{\rho} \frac{\partial v_\theta}{\partial \varphi} - \frac{1}{\rho \sin\varphi} \frac{\partial v_\varphi}{\partial \theta} + \cot\varphi \frac{v_\theta}{\rho} \right) \mathbf{g}_\rho^* \\ &+ \left(\frac{1}{\rho \sin\varphi} \frac{\partial v_\rho}{\partial \theta} - \frac{\partial v_\theta}{\partial \rho} - \frac{v_\theta}{\rho} \right) \mathbf{g}_\varphi^* \\ &+ \left(\frac{\partial v_\varphi}{\partial \rho} - \frac{1}{\rho} \frac{\partial v_\rho}{\partial \varphi} + \frac{v_\varphi}{\rho} \right) \mathbf{g}_\theta^* \end{aligned} \tag{5.68}$$

5.5 The Divergence Theorem

5.5.1 Gauss and Stokes Theorems

The divergence theorem, known as *Gauss theorem* deals with the relation between the flow of a vector or tensor field through the closed surface and the characteristics of the vector (tensor) in the volume closed by the surface. Gauss law states that the flux of a vector through any closed surface is proportional to the charge in the volume closed by the surface. This divergence theorem is a very useful tool that can be mostly applied to engineering and physics, such fluid dynamics and electrodynamics to derive the Navier-Stokes equations and Maxwell's equations, respectively.

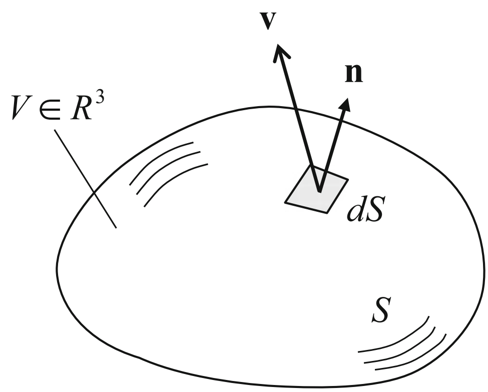


Fig. 5.2 Fluid flux through a closed surface S

The Gauss theorem can be generally written in a three-dimensional space (see Fig. 5.2).

$$\oint_S \mathbf{v} \cdot \mathbf{n} dS = \int_V \nabla \cdot \mathbf{v} \, dV \tag{5.69}$$

where $\mathbf{v}$ is the fluid vector through the surface S; $\mathbf{n}$ is the normal vector on the surface; and $\nabla \cdot \mathbf{v}$ is the divergence of the vector $\mathbf{v}$.

The outward fluid flux from the volume V causes the negative change rate of the volume mass with time.

$$\oint_S \rho \mathbf{v} \cdot \mathbf{n} dS = -\int_V \frac{\partial \rho}{\partial t} dV \tag{5.70}$$

Using Gauss divergence theorem, the balance of mass (also continuity equation) in the control volume V can be derived in

$$\oint_S \rho \mathbf{v} \cdot \mathbf{n} dS = -\int_V \frac{\partial \rho}{\partial t} dV = \int_V \nabla \cdot (\rho \mathbf{v}) dV \tag{5.71}$$

where ρ is the fluid density.

By rearranging the second and third terms in Eq. (5.71), the continuity equation can be written in the integral form for a control volume V:

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0 \Leftrightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{5.72}$$

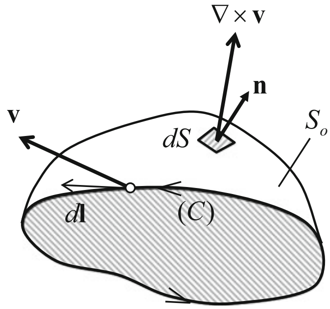


Fig. 5.3 Fluid flux through an open surface S_o

Stokes theorem can be used for an open surface S_o, as shown in Fig. 5.3. The Stokes theorem indicates that the flow velocity along the closed curve (C) is equal to the flux of curl $\mathbf{v}$ going through the open surface S_o.

$$\oint_{(C)} \mathbf{v} \cdot d\mathbf{l} = \int_{S_o} (\nabla \times \mathbf{v}) \cdot \mathbf{n} dS \tag{5.73}$$

where $d\mathbf{l}$ is the length differential on the curve closed (C) of the surface S_o; $\nabla \times \mathbf{v}$ is the curl of the vector $\mathbf{v}$.

5.5.2 Green's Identities

The Green's identities can be derived from the Gauss divergence theorem. Sometimes, they can be usefully applied to the boundary element method (BEM) using the Green's function [4–6]. Two Green's identities are discussed in the following section.

5.5.3 First Green's Identity

The vector $\mathbf{v}$ can be chosen as the product of two arbitrary scalars ψ and ϕ.

$$\mathbf{v} = \psi \nabla \phi \tag{5.74}$$

The divergence of $\mathbf{v}$ can be computed as follows:

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \nabla\cdot(\psi\,\nabla\phi)\\ &= \nabla\psi\cdot\nabla\phi+\psi\,\nabla\cdot\nabla\phi\\ &= \nabla\psi\cdot\nabla\phi+\psi\,\nabla^2\phi\end{aligned} \tag{5.75}$$

Applying the Gauss divergence law of Eq. (5.69) to Eq. (5.75), one obtains

$$\begin{aligned}&\int_V \nabla\cdot\mathbf{v}\,dV=\oint_S \mathbf{v}\cdot\mathbf{n}dS \Leftrightarrow\\ &\int_V \left(\nabla\psi\cdot\nabla\phi+\psi\,\nabla^2\phi\right)dV=\oint_S \psi(\nabla\phi\cdot\,\mathbf{n})dS=\oint_S \psi\frac{\partial\phi}{\partial n}dS\end{aligned} \tag{5.76}$$

5.5.4 Second Green's Identity

The vector $\mathbf{v}$ can be chosen as the function of two arbitrary scalar products of ψ and ϕ.

$$\mathbf{v}=\psi\,\nabla\phi-\phi\,\nabla\psi \tag{5.77}$$

Thus, the divergence of $\mathbf{v}$ can be calculated as follows:

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \nabla\cdot(\psi\,\nabla\phi-\phi\,\nabla\psi)\\ &= \nabla\psi\cdot\nabla\phi+\psi\,\nabla^2\phi-\nabla\phi\cdot\nabla\psi-\phi\,\nabla^2\psi\\ &= \psi\,\nabla^2\phi-\phi\,\nabla^2\psi\end{aligned} \tag{5.78}$$

Applying the Gauss divergence law of Eq. (5.69) to Eq. (5.78), one obtains

$$\begin{aligned}&\int_V \nabla\cdot\mathbf{v}\,dV=\oint_S \mathbf{v}\cdot\mathbf{n}dS \Leftrightarrow\\ &\int_V \left(\psi\,\nabla^2\phi-\phi\,\nabla^2\psi\right)dV=\oint_S (\psi\,\nabla\phi-\phi\,\nabla\psi)\cdot\mathbf{n}dS\\ &=\oint_S\left(\psi\frac{\partial\phi}{\partial n}-\phi\frac{\partial\psi}{\partial n}\right)dS\end{aligned} \tag{5.79}$$

where the gradients of the scalars in the normal direction can be defined as

$$\frac{\partial\phi}{\partial n}\equiv\nabla\phi\cdot\mathbf{n};\quad \frac{\partial\psi}{\partial n}\equiv\nabla\psi\cdot\mathbf{n} \tag{5.80}$$

5.5.5 Differentials of Area and Volume

In Gauss divergence theorem, the differentials of area dA and volume dV are changed from Cartesian coordinates to other general curvilinear coordinates by coordinate transformations.

5.5.6 Calculating the Differential of Area

Figure 5.4 shows the transformation of Cartesian coordinates $\{x^i\}$ into the curvilinear coordinate $\{u^i\}$. The differential $d\mathbf{r}$ can be written in the covariant basis.

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^i} du^i = \mathbf{g}_i du^i \tag{5.81}$$

Therefore,

$$d\mathbf{x}^{(i)} = \mathbf{e}_{(i)} dx^{(i)} = \mathbf{g}_{(i)} du^{(i)} \Leftrightarrow \begin{cases} d\mathbf{x}^1 = \mathbf{e}_1 dx^1 = \mathbf{g}_1 du^1 \\ d\mathbf{x}^2 = \mathbf{e}_2 dx^2 = \mathbf{g}_2 du^2 \end{cases} \tag{5.82}$$

The area differential can be calculated in the curvilinear coordinates [7].

$$\begin{aligned} dA &= \left| d\mathbf{x}^1 \times d\mathbf{x}^2 \right| = |(\mathbf{e}_1 \times \mathbf{e}_2)| dx^1 dx^2 = dx^1 dx^2 \\ &= |(\mathbf{g}_1 \times \mathbf{g}_2)| du^1 du^2 \end{aligned} \tag{5.83}$$

Using the Lagrange identity in Appendix E, one has the relation of

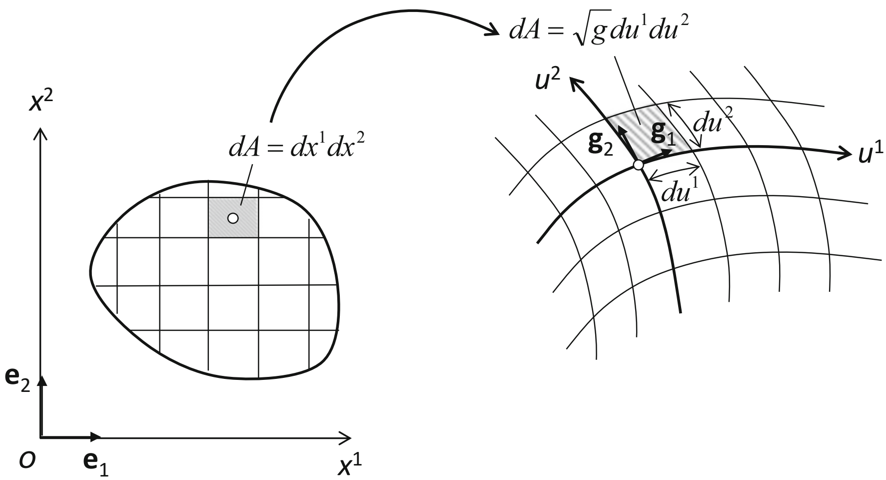


Fig. 5.4 Coordinate transformation in two-dimensional coordinates

$$
\begin{aligned}
|(\mathbf{g}_1 \times \mathbf{g}_2)| &= \sqrt{g_{11}g_{22} - (g_{12})^2} \\
&\equiv \sqrt{g}
\end{aligned}
\tag{5.84}
$$

where g_{ij} are the covariant metric coefficients, as defined in Eq. (2.43).

Substituting Eq. (5.84) into Eq. (5.83), the area differential becomes

$$
\begin{aligned}
dA &= dx^1 dx^2 \\
&= |(\mathbf{g}_1 \times \mathbf{g}_2)| du^1 du^2 \\
&= \sqrt{g_{11}g_{22} - (g_{12})^2} du^1 du^2 \\
&\equiv \sqrt{g} du^1 du^2
\end{aligned}
\tag{5.85}
$$

5.5.7 Calculating the Differential of Volume

The volume differential can be calculated in the curvilinear coordinates (Fig. 5.5).

$$
\begin{aligned}
dV &= \left|\left(d\mathbf{x}^1 \times d\mathbf{x}^2\right) \cdot d\mathbf{x}^3\right| = |(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3| dx^1 dx^2 dx^3 \\
&= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] dx^1 dx^2 dx^3 = dx^1 dx^2 dx^3 \\
&= |(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3| du^1 du^2 du^3 = [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] du^1 du^2 du^3 \\
&= J\, du^1 du^2 du^3
\end{aligned}
\tag{5.86}
$$

The scalar triple product of the covariant bases of the curvilinear coordinates can be defined as

$$
(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]
\tag{5.87}
$$

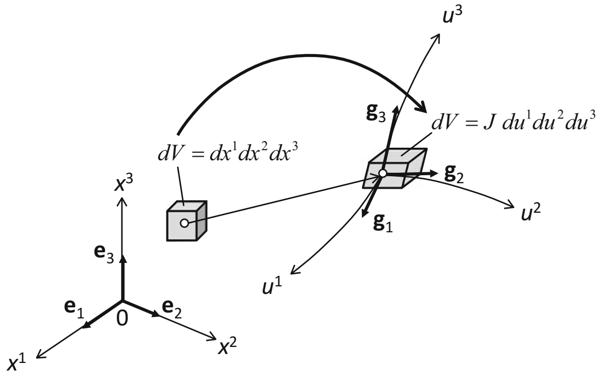


Fig. 5.5 Coordinate transformation in three-dimensional coordinates

The determinant of the covariant basis tensor equals the scalar triple product of the covariant bases, as given in Eq. (1.10).

$$[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{vmatrix} = J > 0 \tag{5.88}$$

where J (=Jacobian) is the determinant of the covariant basis tensor.

Using Eqs. (5.15a), (5.85) and (5.86), the Gauss divergence theorem in general curvilinear coordinates $\{u^i\} \in \mathbf{R}^3$ can be written in the tensor integral equation:

$$\begin{aligned} &\oint_S \mathbf{v} \cdot \mathbf{n} dS = \int_V \nabla \cdot \mathbf{v}\, dV \Leftrightarrow \oint_S v^i n_i \sqrt{g} du^1 du^2 = \\ &\int_V \left(v^i_{,i} + v^j \Gamma^i_{ij} \right) J du^1 du^2 du^3 \equiv \int_V v^i \big|_i J du^1 du^2 du^3 \end{aligned} \tag{5.89}$$

The physical vector components v^{*i} result from the contravariant vector components in the normalized covariant basis (unitary basis) $\mathbf{g}_i^*$ according to Eq. (B.11) in Appendix B.

$$v^i = \frac{v^{*i}}{h_i} \text{ for } i = 1, 2, 3 \tag{5.90}$$

The scale factor h_i can be defined as the covariant basis norm $|\mathbf{g}_i|$ of the curvilinear coordinates $\{u^i\}$.

$$h_i = |\mathbf{g}_i| = \sqrt{g_{(ii)}} \quad \text{(no summation over i)} \tag{5.91}$$

Therefore, the divergence theorem in Eq. (5.89) can be rewritten in the curvilinear coordinate $\{u^i\}$ with the physical vector components as follows:

$$\begin{aligned} \oint_S \frac{v^{*i}}{h_i} n_i \sqrt{g} du^1 du^2 &= \int_V \left(\frac{v^{*i}_{,i}}{h_i} + \frac{v^{*j}}{h_j} \Gamma^i_{ij} \right) J du^1 du^2 du^3 \\ &\equiv \int_V \frac{1}{h_i} v^{*i} \big|_i J du^1 du^2 du^3 \end{aligned} \tag{5.92}$$

In the following sections, some applications of tensor analysis and differential geometry are applied to computational fluid dynamics (CFD), continuum mechanics, classical electrodynamics, electrodynamics in relativity fields, and the Einstein field theory as well.

5.6 Governing Equations of Computational Fluid Dynamics

Navier-Stokes equations describe fluid flows in Computational Fluid Dynamics (CFD). In this book, Navier-Stokes equations are derived for compressible flows in a general rotating frame of the turbomachinery. The rotating frame rotates at an angular velocity $\boldsymbol{\omega}(t)$ with respect to the inertial coordinate system [8, 9]. At $\boldsymbol{\omega} = \mathbf{0}$, the Navier-Stokes equations can be used for a non-rotating frame (a special case). In this case, the velocity $\mathbf{w}$ in Eq. (5.94) is used in the equations instead of the absolute fluid velocity $\mathbf{v}$.

5.6.1 Continuity Equation

The continuity equation satisfies the mass balance of the fluid in the given control volume. According to Eq. (5.72), the continuity equation can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{5.93}$$

The absolute velocity $\mathbf{v}$ is the sum of the relative velocity $\mathbf{w}$ and the circumferential velocity $\mathbf{u}$.

$$\mathbf{v} = \mathbf{w} + \mathbf{u} = \mathbf{w} + (\boldsymbol{\omega} \times \mathbf{r}) \tag{5.94}$$

Substituting the absolute velocity $\mathbf{v}$ in Eq. (5.94) into Eq. (5.93), one obtains the continuity equation of compressible fluids in a rotating frame.

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot \rho(\mathbf{w} + (\boldsymbol{\omega} \times \mathbf{r})) = 0 \Leftrightarrow \\
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{w}) + \nabla \cdot \rho(\boldsymbol{\omega} \times \mathbf{r}) = 0 \Leftrightarrow \\
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{w}) + \rho \nabla \cdot (\boldsymbol{\omega} \times \mathbf{r}) + (\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla \rho = 0
\end{aligned} \tag{5.95}$$

Using the Gauss divergence theorem, the volumetric flow rate of the circumferential velocity $\mathbf{u}$ perpendicular to $\mathbf{n}$ over the control surface S equals zero (see Fig. 5.2).

$$\oint_S \mathbf{u} \cdot \mathbf{n} dA = \int_V \nabla \cdot \mathbf{u} dV = 0$$

Thus, the third term in Eq. (5.95) is equal to zero for any control volume $V \neq 0$.

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\boldsymbol{\omega} \times \mathbf{r}) = 0 \ \ (q.e.d.) \tag{5.96}$$

Using Eq. (5.96), the continuity equation (5.95) is written as

$$\left(\frac{\partial \rho}{\partial t} + (\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla \rho \right) + \nabla \cdot (\rho \mathbf{w}) \equiv \frac{\partial_R \rho}{\partial t} + \nabla \cdot (\rho \mathbf{w}) = 0 \tag{5.97}$$

The partial derivative with respect to the rotating frame can be defined by [9].

$$\frac{\partial_R \rho}{\partial t} \equiv \frac{\partial \rho}{\partial t} + (\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla \rho \tag{5.98}$$

The continuity equation (5.97) can be written in the tensor equation of

$$\begin{aligned} &\frac{\partial_R \rho}{\partial t} + \nabla \cdot (\rho \mathbf{w}) = 0 \\ \Leftrightarrow &\frac{\partial_R \rho}{\partial t} + \left(\rho w^i\right)\big|_i = 0 \\ \Leftrightarrow &\frac{\partial_R \rho}{\partial t} + \left(\rho w^i\right)_{,i} + \left(\rho w^j\right)\Gamma^i_{ij} = 0 \end{aligned} \tag{5.99a}$$

According to Eq. (5.18a), the continuity equation can be written in

$$\begin{aligned} &\frac{\partial_R \rho}{\partial t} + \left(\rho w^i\right)\big|_i = 0 \\ \Leftrightarrow &\frac{\partial_R \rho}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\rho w^i\right)}{\partial u^i} = 0 \\ \Leftrightarrow &\frac{\partial_R \rho}{\partial t} + \frac{1}{\sqrt{g}}\left(\sqrt{g}\rho w^i\right)_{,i} = 0 \end{aligned} \tag{5.99b}$$

The physical vector component in the normalized covariant basis $\mathbf{g}_i^*$ (unitary basis) can be obtained from Eq. (B.11) in Appendix B.

$$w^{*i} = h_i w^i \Rightarrow w^i = \frac{w^{*i}}{h_i} \tag{5.100}$$

where h_i is the norm of the covariant basis $\mathbf{g}_i$.

Therefore, the continuity tensor equation (5.99a) can be written in the physical velocity components of w^* using Eqs. (2.244) and (5.100).

$$\begin{aligned}&\frac{\partial_R\rho}{\partial t}+\left(\rho\frac{w^{*i}}{h_i}\right)_{,i}+\left(\rho\frac{w^{*j}}{h_j}\right)\Gamma^i_{ij}=0\\&\Leftrightarrow\frac{\partial_R\rho}{\partial t}+\left(\rho\frac{w^{*i}}{h_i}\right)_{,i}+\frac{1}{\sqrt{g}}\left(\rho\frac{w^{*j}}{h_j}\right)\frac{\partial\sqrt{g}}{\partial u^j}=0\end{aligned}\tag{5.101a}$$

and the continuity tensor equation (5.99b) becomes using Eq. (5.100)

$$\begin{aligned}&\frac{\partial_R\rho}{\partial t}+\frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\rho w^i\right)}{\partial u^i}=0\\&\Leftrightarrow\frac{\partial_R\rho}{\partial t}+\frac{1}{\sqrt{g}}\left(\sqrt{g}\rho\frac{w^{*i}}{h_i}\right)_{,i}=0\\&\Leftrightarrow\frac{\partial_R\rho}{\partial t}+\left(\rho\frac{w^{*i}}{h_i}\right)_{,i}+\frac{1}{\sqrt{g}}\left(\rho\frac{w^{*i}}{h_i}\right)\frac{\partial\sqrt{g}}{\partial u^i}=0\end{aligned}\tag{5.101b}$$

5.6.2 Navier-Stokes Equations

The Navier-Stokes equations describe the balance of forces in a given control volume V. According to Newton's second law, the balance of specific forces $\mathbf{f}_i$ (N/m^3) for an arbitrary unit volume is written as

$$\rho\frac{D\mathbf{v}}{Dt}=\sum_{i=1}^{n}\mathbf{f}_i\tag{5.102}$$

The substantial derivative in the inertial coordinate system is defined as

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\tag{5.103}$$

where the first term is the time derivative; $\mathbf{v}$ is the absolute fluid velocity.

Substituting the absolute velocity $\mathbf{v}$ in Eq. (5.94) into Eq. (5.103), one obtains the acceleration force acting upon a volumetric unit of the fluid in the inertial coordinate system.

$$\begin{aligned}\rho\frac{D\mathbf{v}}{Dt}&=\rho\frac{\partial\mathbf{v}}{\partial t}+\rho\mathbf{v}\cdot\nabla\mathbf{v}=\rho\frac{\partial(\mathbf{w}+\boldsymbol{\omega}\times\mathbf{r})}{\partial t}+\rho\mathbf{v}\cdot\nabla(\mathbf{w}+\boldsymbol{\omega}\times\mathbf{r})\\&=\rho\frac{\partial\mathbf{w}}{\partial t}+\rho\frac{\partial(\boldsymbol{\omega}\times\mathbf{r})}{\partial t}+\rho(\mathbf{w}+\boldsymbol{\omega}\times\mathbf{r})\cdot\nabla(\mathbf{w}+\boldsymbol{\omega}\times\mathbf{r})\end{aligned}\tag{5.104}$$

The second term on the RHS of Eq. (5.104) can be calculated as

$$\rho \frac{\partial(\boldsymbol{\omega} \times \mathbf{r})}{\partial t} = \rho\left(\boldsymbol{\omega} \times \frac{\partial \mathbf{r}}{\partial t}\right) + \rho\left(\frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{r}\right) \tag{5.105}$$

The first term on the RHS of Eq. (5.105) equals zero since the radius vector **r** does not vary with time.

Thus, Eq. (5.105) is simply written as

$$\rho \frac{\partial(\boldsymbol{\omega} \times \mathbf{r})}{\partial t} = \rho\left(\frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{r}\right) \tag{5.106}$$

The third term on the RHS of Eq. (5.104) can be written as

$$\begin{aligned} \rho \mathbf{v} \cdot \nabla \mathbf{v} &= \rho(\mathbf{w} + \boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla(\mathbf{w} + \boldsymbol{\omega} \times \mathbf{r}) \\ &= \rho \mathbf{w} \cdot \nabla \mathbf{w} + \rho \mathbf{w} \cdot \nabla(\boldsymbol{\omega} \times \mathbf{r}) + \rho(\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla \mathbf{w} \\ &\quad + \rho(\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla(\boldsymbol{\omega} \times \mathbf{r}) \end{aligned} \tag{5.107a}$$

where the last three terms on the RHS of Eq. (5.107a) result from [8, 9]:

$$\begin{aligned} &\rho \mathbf{w} \cdot \nabla(\boldsymbol{\omega} \times \mathbf{r}) = \rho \boldsymbol{\omega} \times \mathbf{w}; \\ &\rho(\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla \mathbf{w} = \boldsymbol{\omega} \times \rho \mathbf{w} = \rho \boldsymbol{\omega} \times \mathbf{w}; \\ &\rho(\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla(\boldsymbol{\omega} \times \mathbf{r}) = \rho \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \end{aligned} \tag{5.107b}$$

Substituting Eqs. (5.106), (5.107a) and (5.107b) into Eq. (5.104), the acceleration force acting upon a volumetric unit of the fluid results as

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \frac{\partial \mathbf{w}}{\partial t} + \rho \mathbf{w} \cdot \nabla \mathbf{w} + \rho\left(\frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{r}\right) + [\rho \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] + 2\rho(\boldsymbol{\omega} \times \mathbf{w}) \tag{5.108}$$

The acceleration force on the LHS of Eq. (5.108) consists of the following forces acting upon the volumetric unit of the fluid: the first term is the unsteady state term; the second term denotes the convective term; the third term is the circumferential force; the fourth term displays the centripetal force; and the last term is the Coriolis force.

The external forces acting upon the fluid control volume comprise the pressure, fluid viscous, and gravity forces. As a result, the unsteady-state Navier-Stokes equations for a compressible viscous fluid in a rotating frame with an angular velocity $\boldsymbol{\omega}(t)$ are written as

$$\begin{aligned} &\rho \frac{\partial \mathbf{w}}{\partial t} + \rho \mathbf{w} \cdot \nabla \mathbf{w} + \rho\left(\frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{r}\right) + [\rho \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] + 2\rho(\boldsymbol{\omega} \times \mathbf{w}) \\ &= -\nabla p + \nabla \cdot \boldsymbol{\Pi} - \nabla(\rho g z) \end{aligned} \tag{5.109}$$

where $\boldsymbol{\Pi}$ is the viscous stress tensor of fluid that is written using the Stokes relation [8].

$$\begin{aligned}\mathbf{\Pi} &= 2\mu\,\mathbf{E} - \frac{2}{3}\mu\,\nabla\cdot(\mathbf{wI}) \\ &= \pi^{ij}\mathbf{g}_i\mathbf{g}_j = \pi_{ij}\mathbf{g}^i\mathbf{g}^j\end{aligned} \tag{5.110a}$$

in which μ is the dynamic viscosity of fluid; $\mathbf{I}$ is the unit tensor; $\mathbf{E}$ is the strain tensor of fluid that is written in the strain components, cf. Eq. (5.176):

$$\mathbf{E} = \varepsilon^{ij}\mathbf{g}_i\mathbf{g}_j = \varepsilon_{ij}\mathbf{g}^i\mathbf{g}^j \tag{5.110b}$$

The first two terms on the LHS of Eq. (5.109) can be written in the tensor equation with the physical components as

$$\begin{aligned}&\rho\frac{\partial\mathbf{w}}{\partial t} + \rho\mathbf{w}\cdot\nabla\mathbf{w} = \rho\frac{\partial\left(w^k\mathbf{g}_k\right)}{\partial t} + \rho\left(w^j\mathbf{g}_j\right)\cdot\nabla\mathbf{w} \\ &= \rho\frac{\partial w^k}{\partial t}\mathbf{g}_k + \rho\left(w^j\mathbf{g}_j\right)\cdot\left(w^k\big|_i\mathbf{g}_k\mathbf{g}^i\right) = \rho\frac{\partial w^k}{\partial t}\mathbf{g}_k + \rho w^j\cdot w^k\Big|_i\delta^i_j\mathbf{g}_k \\ &= \rho\frac{\partial w^k}{\partial t}\mathbf{g}_k + \rho w^i\cdot w^k\big|_i\mathbf{g}_k = \rho\left(\frac{\partial w^k}{\partial t} + w^i\cdot w^k\big|_i\right)\mathbf{g}_k\end{aligned} \tag{5.111a}$$

Equation (5.111a) can be written in the unitary covariant basis with the physical velocity components of w^* using Eq. (5.100).

$$\begin{aligned}\rho\frac{\partial\mathbf{w}}{\partial t} + \rho\mathbf{w}\cdot\nabla\mathbf{w} &= \rho h_k\left(\frac{1}{h_k}\frac{\partial w^{*k}}{\partial t} + \frac{w^{*i}}{h_i h_k}\cdot w^{*k}\big|_i\right)\mathbf{g}^*_k \\ &= \rho\left(\frac{\partial w^{*k}}{\partial t} + \frac{1}{h_i}w^{*i}\cdot w^{*k}\big|_i\right)\mathbf{g}^*_k\end{aligned} \tag{5.111b}$$

where h_i and h_k are the norms of the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_k$, respectively.

Assumed that the fluid be a perfect gas. Using the state equation of gas, its density ρ is calculated from the pressure p and temperature T at any point P and time t as

$$\rho = \frac{p(P,t)}{R\cdot T(P,t)} \tag{5.112}$$

where R is the gas constant.

The third term on the LHS of Eq. (5.109) can be written in the tensor equation of the unitary covariant basis with the physical vector components and the contravariant permutation symbols (cf. Appendix A).

$$\begin{aligned}\rho\left(\frac{\partial \boldsymbol{\omega}}{\partial t}\times \mathbf{r}\right) &= \rho\frac{\partial \omega_i}{\partial t} r_j\left(\mathbf{g}^i \times \mathbf{g}^j\right) = \hat{\varepsilon}^{\,ijk}\rho\frac{\partial \omega_i}{\partial t} r_j \mathbf{g}_k \\ &= \hat{\varepsilon}^{\,ijk}\rho h_k \frac{\partial \omega_i}{\partial t} r_j \mathbf{g}_k^*\end{aligned} \tag{5.113}$$

The fourth term on the LHS of Eq. (5.109) can be written in the tensor equation and the covariant permutation symbols (cf. Appendix A).

$$\begin{aligned}\rho\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) &= \left(\rho\omega_l \mathbf{g}^l\right)\times\left(\omega^i r^j \hat{\varepsilon}_{\,ijk}\mathbf{g}^k\right) \\ &= \rho\omega_l\omega^i r^j \hat{\varepsilon}_{\,ijk}\left(\mathbf{g}^l\times\mathbf{g}^k\right) \\ &= \rho\omega_l\omega^i r^j \hat{\varepsilon}_{\,ijk}\hat{\varepsilon}^{\,lkm}\,\mathbf{g}_m \\ &= \rho\omega_l\omega^i r^j \delta_{ijk}^{lkm}\mathbf{g}_m \\ &= \rho\omega_l\omega^i r^j \delta_{ij}^{lm}\mathbf{g}_m\end{aligned} \tag{5.114}$$

It can be written in the tensor equation of the unitary covariant basis with the physical vector components.

$$\begin{aligned}\rho\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) &= h_m\rho\omega_l\omega^i r^j \delta_{ij}^{lm}\mathbf{g}_m^* \\ &= h_k\rho\omega_l\omega^i r^j \delta_{ij}^{lk}\mathbf{g}_k^*\end{aligned} \tag{5.115}$$

The Coriolis term on the LHS of Eq. (5.109) can be written in the tensor equation of the unitary covariant basis with the physical vector components and the contravariant permutation symbols (cf. Appendix A).

$$\begin{aligned}2\rho(\boldsymbol{\omega}\times\mathbf{w}) &= 2\rho\omega_i w^m\left(\mathbf{g}^i\times\mathbf{g}_m\right) \\ &= 2\rho\omega_i w^m\left(\mathbf{g}^i\times g_{mj}\mathbf{g}^j\right) \\ &= 2\hat{\varepsilon}^{\,ijk}\rho\omega_i g_{mj} w^m \mathbf{g}_k \\ &= 2\hat{\varepsilon}^{\,ijk}\rho\frac{h_k}{h_m}\omega_i g_{mj} w^{*m}\mathbf{g}_k^*\end{aligned} \tag{5.116}$$

Using Eqs. (B.2) and (B.3), the pressure tensor on the RHS of Eq. (5.109) can be written in the unitary covariant basis with the physical vector components.

$$\begin{aligned}-\nabla p = -p_{,}^{\,j}\mathbf{g}_j = -\left(p_{,}^{\,j}h_j\right)\mathbf{g}_j^* \equiv -p_{,}^{*j}\mathbf{g}_j^* = -p_{,}^{*k}\mathbf{g}_k^*; \\ -\nabla p = -p_{,i}\mathbf{g}^i = -\left(p_{,i}g^{ik}h_k\right)\mathbf{g}_k^* \equiv -p_{,}^{*k}\mathbf{g}_k^*\end{aligned} \tag{5.117}$$

Using Eq. (B.3), the physical covariant stress tensor components ${\pi_{ij}}^*$ in the unitary contravariant basis can be computed as

$$
\begin{aligned}
\mathbf{\Pi} &= \pi_{ij}\mathbf{g}^i\mathbf{g}^j = \pi_{ij}\left(g^{ik}\mathbf{g}_k\right)\left(g^{jl}\mathbf{g}_l\right) \\
&= \left(\pi_{ij}g^{ik}g^{jl}h_k h_l\right)\mathbf{g}_k^*\mathbf{g}_l^* \equiv \pi_{ij}^*\mathbf{g}_k^*\mathbf{g}_l^* \\
\Rightarrow \pi_{ij}^* &= \left(g^{ik}g^{jl}h_k h_l\right)\pi_{ij}
\end{aligned}
\tag{5.118}
$$

Using Eqs. (5.25), (5.118) and (B.2) and interchanging the index$l$with k, the friction covariant stress tensor on the RHS of Eq. (5.109) can be rewritten in the unitary contravariant basis with the physical tensor components.

$$
\begin{aligned}
\nabla\cdot\mathbf{\Pi} &= \pi_{ij}|_k g^{jk}\mathbf{g}^i \\
&= \pi_{ij}|_k g^{jk}\left(g^{li}\mathbf{g}_l\right) = \pi_{ij}|_k g^{jk}g^{li}\left(h_l\mathbf{g}_l^*\right) \\
&= \frac{g^{jk}g^{li}}{g^{ik}g^{jl}h_k}\ \pi_{ij}^*|_k\mathbf{g}_l^* = \frac{g^{jl}g^{ki}}{g^{il}g^{jk}h_l}\ \pi_{ij}^*|_l\mathbf{g}_k^*
\end{aligned}
\tag{5.119}
$$

The physical contravariant stress tensor components π^{*ij} in the unitary covariant basis can be computed using Eq. (B.2).

$$
\begin{aligned}
\mathbf{\Pi} &= \pi^{ij}\mathbf{g}_i\mathbf{g}_j = \pi^{ij}\left(h_i\mathbf{g}_i^*\right)\left(h_j\mathbf{g}_j^*\right) \\
&= \pi^{ij}h_i h_j\mathbf{g}_i^*\mathbf{g}_j^* \equiv \pi^{*ij}\mathbf{g}_i^*\mathbf{g}_j^* \\
\Rightarrow \pi^{*ij} &= h_i h_j\pi^{ij}
\end{aligned}
\tag{5.120}
$$

Using Eqs. (5.24a), (5.120) and (B.2), the friction contravariant stress tensor on the RHS of Eq. (5.109) can be written in the unitary covariant basis with the physical tensor components.

$$
\begin{aligned}
\nabla\cdot\mathbf{\Pi} &= \pi^{ij}|_i\mathbf{g}_j = \pi^{ij}|_i\ h_j\mathbf{g}_j^* \\
&= \frac{1}{h_i}\pi^{*ij}|_i\ \mathbf{g}_j^* = \frac{1}{h_i}\pi^{*ik}|_i\ \mathbf{g}_k^*
\end{aligned}
\tag{5.121}
$$

Using Eqs. (B.2) and (B.3), the gravity tensor on the RHS of Eq. (5.109) can be written in the unitary covariant basis with the physical vector components.

$$
\begin{aligned}
-\nabla(\rho g z) &= -\rho g z_,^{\,j}\mathbf{g}_j = -\rho g\left(z_,^{\,j}h_j\right)\mathbf{g}_j^* \equiv -\rho g z_,^{*k}\mathbf{g}_k^*; \\
-\nabla(\rho g z) &= -\rho g z_{,i}\mathbf{g}^i = -\rho g\left(z_{,i}g^{ik}h_k\right)\mathbf{g}_k^* \equiv -\rho g z_,^{*k}\mathbf{g}_k^*
\end{aligned}
\tag{5.122}
$$

where g (=9.81 m/s^2) is the earth gravity.

5.6.3 Energy (Rothalpy) Equation

The energy equation describes the balance of energies in a control volume. They are based on the first law of thermodynamics for an open system.

The specific rothalpy I for a unit volume is defined as

$$\begin{aligned} I &\equiv h + \frac{1}{2}v^2 - uv_u \\ &= c_pT + \frac{1}{2}v^2 - uv_u \end{aligned} \tag{5.123}$$

where h is the specific enthalpy of fluid.

Using the trigonometric calculation with the velocity triangle, the circumferential absolute velocity results as (Fig. 5.6)

$$v_u = u + w_u \tag{5.124}$$

Similarly, the absolute velocity can be written using Eq. (5.124).

$$\begin{aligned} v &= v_u + v_m \\ &= (u + w_u) + v_m \\ \Rightarrow v^2 &= (u + w_u)^2 + v_m^2 \\ &= u^2 + w_u^2 + 2uw_u + v_m^2 \end{aligned} \tag{5.125}$$

Substituting Eqs. (5.124) and (5.125) into Eq. (5.123), one obtains the specific rothalpy of the turbomachinery.

$$\begin{aligned} I &\equiv h + \frac{1}{2}v^2 - uv_u \\ &= h + \frac{1}{2}\left(u^2 + w_u^2 + 2uw_u + v_m^2\right) - u(u + w_u) \\ &= h + \frac{1}{2}\left(w_u^2 + v_m^2\right) - \frac{1}{2}u^2 \\ &= h + \frac{1}{2}w^2 - \frac{1}{2}u^2 \\ &= h + \frac{1}{2}\mathbf{w}\cdot\mathbf{w} - \frac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 \end{aligned} \tag{5.126}$$

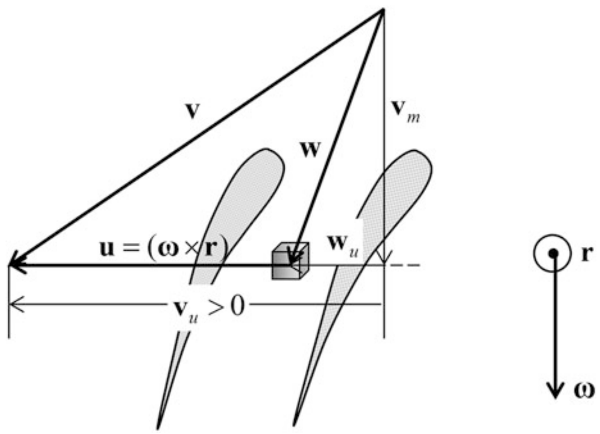


Fig. 5.6 Triangle of velocities in an axial turbomachinery

Therefore, the specific rothalpy I is written as

$$I = c_p T + \tfrac{1}{2}\mathbf{w}\cdot\mathbf{w} - \tfrac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 \tag{5.127}$$

Using the first law of thermodynamics for an open system in a rotating coordinate system, the energy equation can be written as [8].

$$\begin{aligned}\rho\left(\frac{DI}{Dt}\right)_{rot} &= \rho\frac{\partial I}{\partial t} + \rho\mathbf{w}\cdot\nabla I \\ &= \frac{\partial p}{\partial t} + \nabla\cdot\dot{\mathbf{q}} + [\mathbf{w}\cdot(\nabla\cdot\boldsymbol{\Pi}) + (\boldsymbol{\Pi}\cdot\mathbf{E})]\end{aligned} \tag{5.128}$$

in which the first term on the RHS of Eq. (5.128) denotes the specific pressure power, the second term is the specific heat transfer power, and both last terms are the induced specific viscous power of the stress and strain tensors **Π** and **E** of fluid acting upon the control volume.

In the following section, the relating terms in the energy equation (5.128) are calculated. At first, the substantial time derivative of the specific rothalpy I with respect to the rotating frame can be written in the tensor equation.

$$\begin{aligned}\rho\left(\frac{DI}{Dt}\right)_{rot} &\equiv \rho\frac{\partial I}{\partial t} + \rho\mathbf{w}\cdot\nabla I \\ &= \rho\frac{\partial I}{\partial t} + \rho w^j I_{,i}\mathbf{g}_j\cdot\mathbf{g}^i = \rho\frac{\partial I}{\partial t} + \rho w^j I_{,i}\delta_j^i \\ &= \rho\frac{\partial I}{\partial t} + \rho w^i I_{,i}\end{aligned} \tag{5.129a}$$

Using Eq. (5.100), the tensor equation (5.129a) can be written in the velocity physical component w^{*i}.

$$\rho\frac{\partial I}{\partial t} + \rho\mathbf{w}\cdot\nabla I = \rho\frac{\partial I}{\partial t} + \rho\frac{w^{*i}}{h_i}I_{,i} \tag{5.129b}$$

in which the gas density ρ is calculated from Eq. (5.112).

The time change rate of the rothalpy results from Eq. (5.127) as

$$\frac{\partial I}{\partial t} = c_p\frac{\partial T}{\partial t} + \mathbf{w}\cdot\frac{\partial\mathbf{w}}{\partial t} - (\boldsymbol{\omega}\times\mathbf{r})\cdot\frac{\partial(\boldsymbol{\omega}\times\mathbf{r})}{\partial t} \tag{5.130}$$

The heat transfer specific power due to heat conduction through the control volume can be written using the Laplacian of fluid temperature T, as given in Eq. (5.37).

$$\begin{aligned}\dot{\mathbf{q}} &= \lambda \nabla T \\ \Rightarrow \nabla \cdot \dot{\mathbf{q}} &= \nabla \cdot (\lambda \nabla T) = \lambda \nabla^2 T \\ &= \lambda g^{ij}\left(T_{,ij} - T_{,k}\Gamma_{ij}^{k}\right)\end{aligned} \tag{5.131}$$

where λ is the constant thermal conductivity.

The fluid viscous power on the control volume surface can be written in the tensor equation with the stress and strain tensor components of π^{ij} and ε_{ij}.

$$[\mathbf{w} \cdot (\nabla \cdot \mathbf{\Pi}) + (\mathbf{\Pi} \cdot \mathbf{E})] = \left(w_j \pi^{ij}|_i + \pi^{ij}\varepsilon_{ij}\right) = \left(g_{jk} w^k \pi^{ij}|_i + \pi^{ij}\varepsilon_{ij}\right) \tag{5.132a}$$

Using Eqs. (5.100) and (5.118), Eq. (5.132a) can be expressed in the physical vector components w^*.

$$[\mathbf{w} \cdot (\nabla \cdot \mathbf{\Pi}) + (\mathbf{\Pi} \cdot \mathbf{E})] = \frac{1}{h_i h_j h_k}\left(g_{jk} w^{*k} \pi^{*ij}\Big|_i + \frac{\pi^{*ij}\varepsilon^*_{ij}}{g^{ik} g^{jl} h_l}\right). \tag{5.132b}$$

5.7 Basic Equations of Continuum Mechanics

Continuum mechanics deals with the mechanical behavior of continuous materials on the macroscopic scale. The basic equations of continuum mechanics consist of two kinds of equations. First, the equations of conservation of mass, force, and energy can be applied to all materials in any coordinate system. Such equations are the *Cauchy's laws of motion*, which concern with the kinematics of a continuum medium (solids and fluids). Second, the *constitutive equations* describe the macroscopic responses resulting from the internal characteristics of an individual material. The additional constitutive equations help the Cauchy's equations of continuum mechanics in order to predict the responses of the individual material to the applied loads.

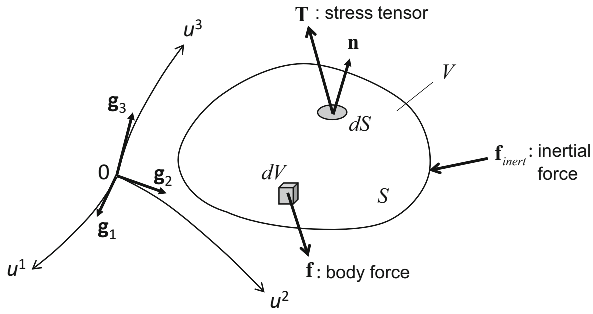
Physical laws must always be invariant and independent of the observers from different coordinate systems. Therefore, such equations describing physical laws are generally written in the tensor equations that are always valid for any coordinate system.

5.7.1 Cauchy's Law of Motion

The Cauchy's law of motion is based on the conservation of forces in a continuum medium. Figure 5.7 displays the balance of forces acting on the moving body in a general curvilinear coordinate system $\{u^1,u^2,u^3\}$.

The Cauchy's stress tensor **T** can be written in the contravariant second-order tensor as

Fig. 5.7 Acting forces on a moving body



$$\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j \tag{5.133}$$

The body force **f** per volume unit can be expressed in the contravariant first-order tensor (vector).

$$\mathbf{f} = f^j\mathbf{g}_j \tag{5.134}$$

The inertial force acting upon the body can be written in the contravariant first-order tensor (vector).

$$\mathbf{f}_{inert} = -\rho a^j\mathbf{g}_j = -\rho\ddot{u}^j\mathbf{g}_j \tag{5.135}$$

where ρ is the body density.

Using the D'Alembert principle, one obtains the Cauchy's law of motion in the integral form.

$$\sum \mathbf{F} = \oint_S \mathbf{T}\cdot\mathbf{n}dS + \int_V \mathbf{f}dV + \int_V \mathbf{f}_{inert}dV = \mathbf{0} \tag{5.136}$$

Applying the Gauss divergence theorem to Eq. (5.136), the first integral over surface S can be transformed into the integral over volume V.

$$\oint_S \mathbf{T}\cdot\mathbf{n}dS = \int_V \nabla\cdot\mathbf{T}dV = \int_V T^{ij}\big|_i\mathbf{g}_j dV \tag{5.137}$$

where the divergence $\nabla\cdot\mathbf{T}$ of a second-order tensor **T** results from Eq. (5.24a)

$$\begin{aligned}\nabla\cdot\mathbf{T} &= T^{ij}\big|_i\mathbf{g}_j \\ &= \left(T^{ij}_{,i} + \Gamma^i_{im}T^{mj} + \Gamma^j_{im}T^{im}\right)\mathbf{g}_j\end{aligned} \tag{5.138}$$

Applying the coordinate transformation in Eq. (5.89), Eq. (5.137) can be written in the curvilinear coordinates using the Jacobian J.

$$\int_V \nabla \cdot \mathbf{T} dV = \int_V T^{ij}|_i \mathbf{g}_j J du^1 du^2 du^3 \tag{5.139}$$

Thus, the conservation of forces can be rewritten in the curvilinear coordinates.

$$\int_V \left(T^{ij}|_i + f^j - \rho \ddot{u}^j \right) \mathbf{g}_j J du^1 du^2 du^3 = \mathbf{0} \tag{5.140}$$

The Cauchy's tensor equation for an arbitrary volume V becomes

$$T^{ij}|_i + f^j = \rho \ddot{u}^j \tag{5.141}$$

In case of equilibrium, the Cauchy's tensor equation becomes

$$T^{ij}|_i + f^j = 0 \tag{5.142}$$

The Cauchy stress tensor $\mathbf{T}$ contains nine tensor components:

$$T^{ij} = \tau^{ij} \text{ for } i, j = 1, 2, 3 \tag{5.143}$$

Using Eq. (5.143), one obtains the Cauchy's tensor equations for both cases:

$$\begin{aligned} \tau^{ij}|_i + f^j &= \rho \ddot{u}^j; \\ \tau^{ij}|_i + f^j &= 0 \end{aligned} \tag{5.144}$$

within using the Christoffel symbols of second kind, the covariant derivative of the stress tensor components with respect to u^i is defined as

$$\tau^{ij}|_i = \tau^{ij}_{,i} + \Gamma^i_{im} \tau^{mj} + \Gamma^j_{im} \tau^{im} \tag{5.145}$$

The *Cauchy's stress tensor* $\mathbf{T}$ can be written in a three-dimensional space as

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}^1 \\ \mathbf{T}^2 \\ \mathbf{T}^3 \end{bmatrix} = \begin{pmatrix} \sigma^{11} & \tau^{12} & \tau^{13} \\ \tau^{21} & \sigma^{22} & \tau^{23} \\ \tau^{31} & \tau^{32} & \sigma^{33} \end{pmatrix} \equiv \left(\tau^{ij} \right) \tag{5.146}$$

where σ is the normal stress; τ is the shear stress.

The stress tensor $\mathbf{T}^i$ on the surface S is defined as the acting force $\boldsymbol{F}$ per unit of surface area.

$$\mathbf{T}^i = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{F}}{\Delta S} = \frac{\partial \mathbf{F}}{\partial S} \tag{5.147}$$

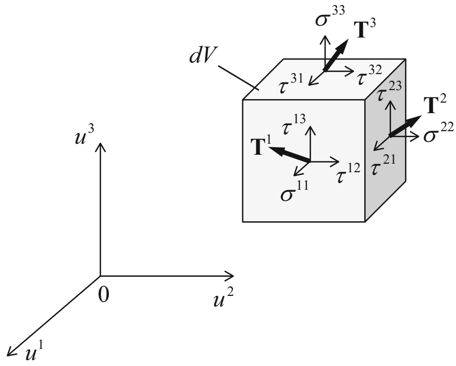


Fig. 5.8 Stress tensor components $\mathbf{T}^i$ acting upon a volume element

At each point in the surface S, there are a set of three stress tensor components, one normal stress component σ (pressure) perpendicular to the surface and two shear stress components τ parallel to the surface, as shown in Fig. 5.8. Note that the tensile normal stress denotes a positive normal stress; the compressive stress, a negative normal stress.

In the following section, it is to prove that the second-order Cauchy's stress tensor is symmetric in a free couple-stress body of non-polar materials. Note that *polar materials* contain couple stresses.

The conservation of angular momentum of the body in equilibrium (cf. Fig. 5.7) can be written as

$$\sum \mathbf{M} = \oint_S (\mathbf{r} \times \mathbf{T}) dS + \int_V (\mathbf{r} \times \mathbf{f}) dV = \mathbf{0} \tag{5.148}$$

The vector $\mathbf{r}$ can be written in the covariant basis of curvilinear coordinates:

$$\mathbf{r} = x^j \mathbf{g}_j \tag{5.149a}$$

The Cauchy's stress tensor $\mathbf{T}$ can be written in the covariant basis of curvilinear coordinates:

$$\mathbf{T} = T^{mk} \mathbf{g}_m \mathbf{g}_k \equiv \mathbf{T}^k \mathbf{g}_k \tag{5.149b}$$

Using Eqs. (5.149a) and (5.149b) and the Levi-Civita symbols in Eq. (A.6) in Appendix A, the angular momentum equation can be expressed as

$$
\begin{aligned}
&\oint_S \hat{\varepsilon}_{ijk} x^j \mathbf{T}^k \mathbf{g}^i dS + \int_V \hat{\varepsilon}_{ijk} x^j f^k \mathbf{g}^i dV = \mathbf{0} \Leftrightarrow \\
&\oint_S \hat{\varepsilon}_{ijk} x^j \left(T^{mk} \mathbf{g}_m\right) \mathbf{g}^i dS + \int_V \hat{\varepsilon}_{ijk} x^j f^k \mathbf{g}^i dV = \mathbf{0}
\end{aligned}
\tag{5.150}
$$

Applying the Gauss divergence theorem and using the transformation of curvilinear coordinates, Eq. (5.150) results in

$$
\begin{aligned}
&\int_V \hat{\varepsilon}_{ijk} \left(\left(x^j T^{mk}\right)\big|_m + x^j f^k \right) \mathbf{g}^i dV = \mathbf{0} \Rightarrow \\
&\int_V \hat{\varepsilon}_{ijk} \left(\left(x^j T^{mk}\right)\big|_m + x^j f^k \right) \mathbf{g}^i J du^1 du^2 du^3 = \mathbf{0}
\end{aligned}
\tag{5.151}
$$

Calculating the covariant derivative of the first term in Eq. (5.151), one obtains

$$
\int_V \hat{\varepsilon}_{ijk} \left(x^j\big|_m T^{mk} + x^j T^{mk}\Big|_m + x^j f^k \right) \mathbf{g}^i J du^1 du^2 du^3 = \mathbf{0} \tag{5.152}
$$

Rearranging the second and third terms in Eq. (5.152), one obtains

$$
\int_V \hat{\varepsilon}_{ijk} x^j\big|_m T^{mk} \mathbf{g}^i J du^1 du^2 du^3 + \int_V \hat{\varepsilon}_{ijk} x^j \left(T^{mk}\big|_m + f^k \right) \mathbf{g}^i J du^1 du^2 du^3 = \mathbf{0} \tag{5.153}
$$

Due to the balance of forces in Eq. (5.142), the second integral in Eq. (5.153) equals zero.

Therefore, the tensor equation of angular momentum Eq. (5.153) results in

$$
\begin{aligned}
&\int_V \hat{\varepsilon}_{ijk} x^j\big|_m T^{mk} \mathbf{g}^i J du^1 du^2 du^3 \\
&= \int_V \hat{\varepsilon}_{ijk} x^j\big|_m T^{mk} \mathbf{g}^i dV = \mathbf{0}
\end{aligned}
\tag{5.154}
$$

Using the relation of $x^j|_m = \delta^j_m$, one obtains for an arbitrary volume dV

$$
\begin{aligned}
&\hat{\varepsilon}_{ijk} x^j\big|_m T^{mk} = \hat{\varepsilon}_{ijk} \delta^j_m T^{mk} = \hat{\varepsilon}_{ijk} T^{jk} = 0 \\
&\Rightarrow \varepsilon_{ijk} \tau^{jk} = 0
\end{aligned}
\tag{5.155}
$$

Note that $\varepsilon_{ikj} = -\varepsilon_{ijk}$, one obtains from Eq. (5.155)

$$\begin{aligned} &\varepsilon_{ijk}\tau^{jk} = \varepsilon_{ikj}\tau^{kj} = 0 \\ &\Rightarrow \left(\varepsilon_{ijk}\tau^{jk} + \varepsilon_{ikj}\tau^{kj}\right) = \varepsilon_{ijk}\left(\tau^{jk} - \tau^{kj}\right) = 0 \\ &\Rightarrow \tau^{jk} = \tau^{kj} \text{ for } i \neq j \neq k; j,k = 1,2,3 \end{aligned} \tag{5.156}$$

This result proves that the Cauchy's stress tensor is *symmetric* in the free couple-stress body. In a three-dimensional space, there are six symmetric tensor components of shear stress τ^{ij} and three diagonal tensor components of normal stress σ^{ii}, as shown in Eq. (5.146). It is obvious, the Cauchy's stress tensor becomes *non-symmetric* if the couple stresses act on the body.

5.7.2 Principal Stresses of Cauchy's Stress Tensor

The Cauchy's stress tensor **T** in a three-dimensional space generally has three invariants that are independent of any chosen coordinate system. These invariants are the normal stresses in the principal directions perpendicular to the principal planes. In fact, the principal normal stresses are the eigenvalues of the stress tensor where only the normal stresses act on the principal planes in which the remaining shear stresses equal zero. The eigenvectors related to their eigenvalues have the same directions of the principal directions, as shown in Fig. 5.9.

The principal stress vector $\mathbf{T}^i$ of the stress tensor on the normal unit vector $\mathbf{n}_i$ (parallel to the principal direction) can be expressed as

$$\mathbf{T}^i = \lambda\, \mathbf{n}_i \tag{5.157}$$

Furthermore, the principal stress vector can be written in a linear form of the stress tensor components and their relating normal unit vectors.

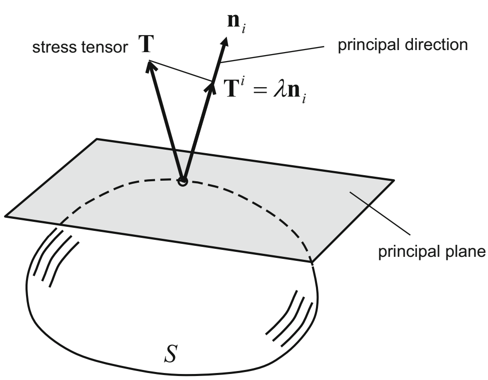


Fig. 5.9 Acting principal stress $\mathbf{T}^i$ on a principal plane

$$\mathbf{T}^i = \sigma^{ij}\mathbf{n}_j \text{ for } j = 1,2,3 \tag{5.158}$$

where using the Kronecker delta, the normal unit vector $\mathbf{n}_j$ is defined as

$$\mathbf{n}_i = \mathbf{n}_j\delta_i^j \text{ for } i,j = 1,2,3 \tag{5.159}$$

The relation between the unit vectors in Eq. (5.159) denotes that all shear stresses vanish in the principal planes perpendicular to the unit vectors $\mathbf{n}_j$ with all indices $j \neq i$. In case of $j \equiv i$, only the principal stresses act on the principal planes.

Substituting Eqs. (5.158) and (5.159) into Eq. (5.157), one obtains

$$\left(\sigma^{ij} - \lambda\delta_i^j\right)\mathbf{n}_j = \mathbf{0} \text{ for } i,j = 1,2,3 \tag{5.160}$$

For nontrivial solutions of Eq. (5.160), its determinant must equal zero.

Therefore,

$$det\left(\sigma^{ij} - \lambda\delta_i^j\right) = \begin{vmatrix} (\sigma^{11}-\lambda) & \sigma^{12} & \sigma^{13} \\ \sigma^{21} & (\sigma^{22}-\lambda) & \sigma^{23} \\ \sigma^{31} & \sigma^{32} & (\sigma^{33}-\lambda) \end{vmatrix} = 0 \tag{5.161}$$

This equation is called the *characteristic equation* of the eigenvalues of the second-order stress tensor **T**.

Calculating the determinant of Eq. (5.161), the third-order characteristic equation of λ results in

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0 \tag{5.162}$$

within

$$\begin{cases} I_1 = Tr(\sigma^{ii}); \\ I_2 = \dfrac{1}{2}\left(\sigma^{ii}\sigma^{jj} - \sigma^{ij}\sigma^{ji}\right); \\ I_3 = \det(\sigma^{ij}) \end{cases} \tag{5.163}$$

Generally, there are three real roots of Eq. (5.162) for the eigenvalues in the principal directions.

$$\begin{cases} \lambda_1 = \sigma^1 \equiv \sigma^{\max} \\ \lambda_2 = I_1 - \sigma^1 - \sigma^3 = \sigma^2 \\ \lambda_3 = \sigma^3 \equiv \sigma^{\min} \end{cases} \tag{5.164}$$

The eigenvalues of Eq. (5.164) are the roots of the *Cayley-Hamilton theorem*.

The stress tensor **T** can be written in the principal directions as follows:

$$\mathbf{T} \equiv \left(\sigma^{ij}\right) = \begin{pmatrix} \sigma^1 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^3 \end{pmatrix} \tag{5.165}$$

The maximum shear stress occurs at the angle of 45° between the smallest and largest principal stress planes with the value

$$\tau^{\max} = \frac{\left|\sigma^{\max} - \sigma^{\min}\right|}{2} = \frac{\left|\sigma^1 - \sigma^3\right|}{2} \quad \text{at} \quad \sigma^{middle} = \frac{\left|\sigma^1 + \sigma^3\right|}{2}. \tag{5.166}$$

5.7.3 Cauchy's Strain Tensor

The *Cauchy's strain tensor* describes the infinitesimal deformation of a solid body in which the displacement between two arbitrary points in the material is much less than any relevant dimension of the body. In this case, the Cauchy's strain tensor is based on the small deformation theory or linear deformation theory [18].

The vector $\mathbf{R}(u^1,u^2,u^3,t)$ of an arbitrary point P of the body at the time t after deformation results from the vector $\mathbf{r}(u^1,u^2,u^3)$ of the same point at the time t_0 before deformation and the small deformation vector $\mathbf{v}(u^1,u^2,u^3,t)$, as shown in Fig. 5.10.

$$\mathbf{R}\left(u^1, u^2, u^3, t\right) \;=\; \mathbf{r}\left(u^1, u^2, u^3\right) + \mathbf{v}\left(u^1, u^2, u^3, t\right) \tag{5.167}$$

The covariant basis vectors of both curvilinear coordinates at the times t_0 and t result from

$$\frac{\partial \mathbf{R}}{\partial u^i} = \frac{\partial \mathbf{r}}{\partial u^i} + \frac{\partial \mathbf{v}}{\partial u^i} \quad \Leftrightarrow \quad \mathbf{G}_i = \mathbf{g}_i + \mathbf{v}_{,i} \tag{5.168}$$

The difference of two segments can be calculated by

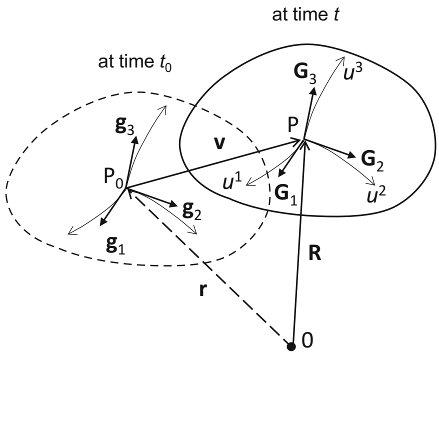


Fig. 5.10 Infinitesimal deformation of a solid body

$$\begin{aligned} d\mathbf{R}^2 - d\mathbf{r}^2 &= \mathbf{G}_i\mathbf{G}_j du^i du^j - \mathbf{g}_i\mathbf{g}_j du^i du^j \\ &= \left(G_{ij} - g_{ij}\right) du^i du^j \equiv 2\gamma_{ij} du^i du^j \end{aligned} \tag{5.169}$$

where γ_{ij} are the components of the second-order strain tensor.

Calculating the covariant metric coefficients, one obtains

$$\begin{aligned} &\mathbf{G}_i \cdot \mathbf{G}_j = (\mathbf{g}_i + \mathbf{v}_{,i}) \cdot \left(\mathbf{g}_j + \mathbf{v}_{,j}\right) = \mathbf{g}_i \cdot \mathbf{g}_j + \mathbf{g}_i \cdot \mathbf{v}_{,j} + \mathbf{g}_j \cdot \mathbf{v}_{,i} + \mathbf{v}_{,i} \cdot \mathbf{v}_{,j} \\ &\Rightarrow G_{ij} - g_{ij} = \mathbf{g}_i \cdot \mathbf{v}_{,j} + \mathbf{g}_j \cdot \mathbf{v}_{,i} + \mathbf{v}_{,i} \cdot \mathbf{v}_{,j} \end{aligned} \tag{5.170}$$

Thus, the stress tensor components can be calculated as

$$\begin{aligned} \gamma_{ij} = \frac{1}{2}\left(G_{ij} - g_{ij}\right) &= \frac{1}{2}\left(\mathbf{g}_i \cdot \mathbf{v}_{,j} + \mathbf{g}_j \cdot \mathbf{v}_{,i} + \mathbf{v}_{,i} \cdot \mathbf{v}_{,j}\right) \\ &= \frac{1}{2}\left(\mathbf{g}_i \cdot \frac{\partial \mathbf{v}}{\partial u^j} + \mathbf{g}_j \cdot \frac{\partial \mathbf{v}}{\partial u^i} + \frac{\partial \mathbf{v}}{\partial u^i} \cdot \frac{\partial \mathbf{v}}{\partial u^j}\right) \end{aligned} \tag{5.171}$$

The deformation vector $\mathbf{v}$ can be written in the contravariant and covariant bases.

$$\mathbf{v} = v_k \mathbf{g}^k = v^k \mathbf{g}_k \tag{5.172}$$

Using Eqs. (2.200) and (2.207), one obtains the covariant partial derivatives

$$\begin{aligned} \mathbf{v}_{,i} &= v_k\big|_i\, \mathbf{g}^k; \\ \mathbf{v}_{,j} &= v^k\big|_j\, \mathbf{g}_k \end{aligned} \tag{5.173}$$

within the covariant derivatives result from Eqs. (2.201) and (2.208) in

$$\begin{aligned} v_k\big|_i &= v_{k,i} - \Gamma_{ki}^j v_j; \\ v^k\big|_j &= v^k_{,j} + \Gamma_{ji}^k v^i \end{aligned} \tag{5.174}$$

Substituting Eq. (5.173) into Eq. (5.171), the strain tensor components of the Cauchy's strain tensor $\boldsymbol{\gamma}$ can be rewritten as

$$\begin{aligned} \gamma_{ij} &= \frac{1}{2}\left(\mathbf{g}_i \cdot \mathbf{v}_{,j} + \mathbf{g}_j \cdot \mathbf{v}_{,i} + \mathbf{v}_{,i} \cdot \mathbf{v}_{,j}\right) \\ &= \frac{1}{2}\left(v_k\big|_j\, \mathbf{g}^k \cdot \mathbf{g}_i + v_k\big|_i\, \mathbf{g}^k \cdot \mathbf{g}_j + v_l\big|_i\, v^k\big|_j\, \mathbf{g}_k \cdot \mathbf{g}^l\right) \\ &= \frac{1}{2}\left(v_k\big|_j\, \delta_i^k + v_k\big|_i\, \delta_j^k + v_l\big|_i\, v^k\big|_j\, \delta_k^l\right) \\ &= \frac{1}{2}\left(v_i\big|_j + v_j\big|_i + v_k\big|_i \cdot v^k\big|_j\right) \end{aligned} \tag{5.175}$$

The third term on the RHS of Eq. (5.175) is very small in the infinitesimal deformation. Therefore, the components of the Cauchy's strain tensor $\boldsymbol{\gamma}$ become

$$\gamma_{ij} \approx \frac{1}{2}\left(v_i\big|_j + v_j\big|_i \right) \tag{5.176}$$

Substituting Eq. (5.174) into Eq. (5.176) and using the symmetry of the Christoffel symbols, the Cauchy's tensor component can be written in the linear elasticity.

$$\begin{aligned} \gamma_{ij} &= \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) - \Gamma_{ij}^{k} v_k \\ &= \frac{1}{2}\left(v_{j,i} + v_{i,j}\right) - \Gamma_{ji}^{k} v_k \\ &= \gamma_{ji} \;\; (q.e.d.) \end{aligned} \tag{5.177}$$

This result proves that the Cauchy's strain tensor $\boldsymbol{\gamma}$ is also symmetric.

As an example, the *Cauchy's strain tensor* $\boldsymbol{\gamma}$ can be written in a three-dimensional space:

$$\boldsymbol{\gamma} = \begin{pmatrix} \varepsilon_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \varepsilon_{22} & \gamma_{21} \\ \gamma_{31} & \gamma_{32} & \varepsilon_{33} \end{pmatrix} \equiv \left(\gamma_{ij}\right) \tag{5.178}$$

where ε_{ii} are the normal strains; γ_{ij} the shear strains of the second-order strain tensor (matrix).

Note that the Christoffel symbols in Eq. (5.177) vanish in Cartesian coordinates (x,y,z) in a three-dimensional space $\mathbf{E}^3$. Therefore, the normal and shear strains of Eq. (5.178) can be computed as follows:

$$\left.\begin{aligned} &\varepsilon_{xx} = \frac{\partial u}{\partial x};\; \varepsilon_{yy} = \frac{\partial v}{\partial y};\; \varepsilon_{zz} = \frac{\partial w}{\partial z} \\ &\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \gamma_{yx} \\ &\gamma_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \gamma_{zy} \\ &\gamma_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \gamma_{zx} \end{aligned}\right\} \Rightarrow \boldsymbol{\gamma} = \begin{pmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{pmatrix} \tag{5.179}$$

where u, v, and w are the vector components of the deformation vector $\mathbf{v}$.

Similar to the principal stresses, the *principal strains* are invariants that are independent of any chosen coordinate system. In this case, only the principal strains (eigenvalues) exist in the principal directions (eigenvectors) of the principal planes in which the shear strains equal zero.

The characteristic equation of the eigenvalues λ of the strain tensor $\boldsymbol{\gamma}$ results from

$$\begin{aligned}\left(\gamma_{ij} - \lambda\delta_i^j\right)\mathbf{n}_j &= \mathbf{0} \text{ for } i,j = 1,2,3\\ \Rightarrow \det\left(\gamma_{ij} - \lambda\delta_i^j\right) &= 0\end{aligned} \tag{5.180}$$

There are three real roots of Eq. (5.180) for the principal strains (eigenvalues) in the principal directions (eigenvectors).

$$\lambda_1 = \varepsilon_1; \quad \lambda_2 = \varepsilon_2; \quad \lambda_3 = \varepsilon_3 \tag{5.181}$$

The Cauchy's strain tensor $\boldsymbol{\gamma}$ can be written in the principal directions:

$$\boldsymbol{\gamma} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \equiv \left(\varepsilon_{ij}\right) \tag{5.182}$$

5.7.4 *Constitutive Equations of Elasticity Laws*

The constitutive equations describe the macroscopic responses resulting from the internal characteristics of an individual material. The linear elasticity law of materials shows the relation between the stress tensor $\mathbf{T}$ and strain tensor $\boldsymbol{\gamma}$ in general curvilinear coordinates of an N-dimensional space.

In general, the Hooke's law is valid in the linear elasticity of material. As a result, the stress tensor component is proportional to the strain tensor component in the elasticity range of an individual material compared to the spring force that is proportional to its displacement by a spring constant.

The Hooke's law of the linear elasticity law is written as

$$\tau^{ij} = E^{ijkl}\gamma_{kl} \text{ for } i,j,k,l = 1,2,\ldots,N \tag{5.183}$$

where E^{ijkl} is the fourth-order *elasticity tensor* that only depends on the material characteristics. The elasticity tensor has 81 components (=3^4) in a three-dimensional space; N^4 components in an N-dimensional space.

The elasticity tensor is a function of the elasticity modulus E (Young's modulus) and the Poisson's ratio ν [10].

$$E^{ijkl} = \mu\left(g^{ik}g^{jl} + g^{il}g^{jk} + \frac{2\nu}{(1-2\nu)}g^{ij}g^{kl}\right) \tag{5.184}$$

where μ is the shear modulus (modulus of rigidity) that can be defined as

$$\mu = \frac{E}{2\,(1+\nu)} \tag{5.185}$$

Some moduli of steels are mostly used in engineering applications:

- $E \approx 212$ GPa (low-alloy steels); 230 GPa (high-alloy steels);
- $\mu \approx 0.385E \ldots 0.400E$;
- $\nu \approx 0.25\ldots0.30$ (most metals).

According to the Hooke's law, the elasticity equation results from Eqs. (5.183) to (5.185) for $i, j, k, l = 1,2,\ldots,N$.

$$\tau^{ij} = \mu \left(g^{ik}g^{jl} + g^{il}g^{jk} + \frac{2\nu}{(1-2\nu)} g^{ij}g^{kl} \right) \gamma_{kl} \tag{5.186}$$

The basic tensor equations of the linear elasticity theory comprise Eqs. (5.144), (5.177) and (5.186):

$$\begin{aligned} &\tau^{ij}_{,i} + \Gamma^{i}_{im}\tau^{mj} + \Gamma^{j}_{im}\tau^{im} = \rho\ddot{u}^{j} - f^{j} \\ &\gamma_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) - \Gamma^{k}_{ij}v_k \\ &\tau^{ij} = \mu \left(g^{ik}g^{jl} + g^{il}g^{jk} + \frac{2\nu}{(1-2\nu)} g^{ij}g^{kl} \right) \gamma_{kl}. \end{aligned} \tag{5.187}$$

5.8 Maxwell's Equations of Electrodynamics

Maxwell's equations are the fundamental equations for electrodynamics, telecommunication technologies, quantum electrodynamics, special and general relativity theories. They describe mutual interactions between the charges, currents, electric, and magnetic fields in a matter.

The Maxwell's equations of electromagnetism are a system of four inhomogeneous partial differential equations of the electric field strength **E**, magnetic field density **B**, electric displacement **D**, and magnetic field strength **H** in N-dimensional spaces and the four-dimensional spacetime (x,y,z,t) [11, 12].

5.8.1 Maxwell's Equations in Curvilinear Coordinate Systems

The Maxwell's equations in a matter can be written in integral, differential, and tensor equations with physical components using Eqs. (5.31) and (5.33) and SI units as

- **Gauss's law for electric fields:**

$$\oint_S \mathbf{D} \cdot \mathbf{n} dA = q_{enc} \quad \text{(Integral equation)} \tag{5.188a}$$

$$\nabla \cdot \mathbf{D} = \rho \quad \text{(Differential equation)} \tag{5.188b}$$

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left(\sqrt{g} D^i \right) = \rho \quad \text{(Tensor equation)} \tag{5.188c}$$

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left(\sqrt{g} \frac{D^{*i}}{h_i} \right) = \rho \quad \text{(* Physical components)} \tag{5.188d}$$

- **Gauss's law for magnetic fields**:

$$\oint_S \mathbf{B} \cdot \mathbf{n} dA = 0 \quad \text{(Integral equation)} \tag{5.189a}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(Differential equation)} \tag{5.189b}$$

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left(\sqrt{g} B^i \right) = 0 \quad \text{(Tensor equation)} \tag{5.189c}$$

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left(\sqrt{g} \frac{B^{*i}}{h_i} \right) = 0 \quad \text{(* Physical components)} \tag{5.189d}$$

- **Faraday's law**:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} dA \quad \text{(Integral equation)} \tag{5.190a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Differential equation)} \tag{5.190b}$$

$$\hat{\varepsilon}^{ijk} E_{k,j} = -\frac{\partial B^i}{\partial t} \quad \text{(Tensor equation)} \tag{5.190c}$$

$$\frac{1}{\sqrt{g}}\left(h_k E^*_{k,j} - h_j E^*_{j,k}\right) = -\frac{1}{h_i}\frac{\partial B^{*i}}{\partial t} \quad (\text{* Physical components}) \tag{5.190d}$$

– **Ampere-Maxwell law:**

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = I_{enc} + \frac{d}{dt}\int_S \mathbf{D}\cdot\mathbf{n}dA \quad (\text{Integral equation}) \tag{5.191a}$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \quad (\text{Differential equation}) \tag{5.191b}$$

$$\hat{\varepsilon}^{ijk}H_{k,j} = J^i + \frac{\partial D^i}{\partial t} \quad (\text{Tensor equation}) \tag{5.191c}$$

$$\frac{1}{\sqrt{g}}\left(h_k H^*_{k,j} - h_j H^*_{j,k}\right) = \frac{J^{*i}}{h_i} + \frac{1}{h_i}\frac{\partial D^{*i}}{\partial t} \quad (\text{* Physical components}) \tag{5.191d}$$

where

n is the normal unit vector to the surface S;
q_{enc} (**r**, t) is the four-dimensional spacetime enclosed electric charge;
ρ (**r**, t) is the four-dimensional spacetime electric charge volumetric density;
I_{enc} (**r**, t) is the four-dimensional spacetime enclosed electric current:
J (**r**, t) is the four-dimensional spacetime electric current volumetric density;
J^* is the physical component of **J**;
h_i is the norm of the covariant basis $\mathbf{g}_i$.

The electric displacement **D** is related to the electric field strength **E** by the matter permittivity ε.

$$\mathbf{D} = \varepsilon\mathbf{E} \tag{5.192}$$

in which $\varepsilon = 1/(\mu c^2)$ is the matter permittivity; $c \approx 3\times10^8$ m/s is the light speed in vacuum; μ is the matter permeability ($\mu_0 = 4\pi\times10^{-7}$ N/A^2 for vacuum).

Analogously, the relation between the magnetic field strength **H** and magnetic field density **B** can be written as

$$\mathbf{B} = \mu\mathbf{H} \tag{5.193}$$

where μ is the matter permeability.

Taking curl ($\nabla\times$) of the curl Eqs. (5.190b) and (5.191b) and using the curl identity, as given in Eq. (C.23), the homogenous wave equations for the electric and magnetic field strengths of **E** and **H** in vacuum (i.e. without any source exists, $\mathbf{J} = \rho = 0$) can be derived in

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2 \nabla^2 \mathbf{E} = \mathbf{0} \tag{5.194}$$

$$\frac{\partial^2 \mathbf{H}}{\partial t^2} - c^2 \nabla^2 \mathbf{H} = \mathbf{0} \tag{5.195}$$

where c is the propagating speed of the electromagnetic waves in vacuum, which equals the light speed.

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.998 \times 10^8 \ \mathrm{ms}^{-1} \tag{5.196}$$

with $\mu_0 = 4\pi \times 10^{-7}$ N/A^2 (H/m); $\varepsilon_0 = 8.85 \times 10^{-12}$ C^2/(Nm^2).

Therefore, Maxwell postulated that light is induced by the electromagnetic disturbance propagated through the field according to the electromagnetic laws. Furthermore, the law of conservation of the four-current density vector **J** is similar to the continuity equation of fluid dynamics, as given in Eq. (5.93).

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0 \tag{5.197}$$

Note that the Maxwell's equations are invariant at the Lorentz transformation [13]. However, they will be studied on the four-dimensional spacetime manifold by the Poincaré transformation.

5.8.2 Maxwell's Equations in the Four-Dimensional Spacetime

In the following section, the Maxwell's equations can be expressed on the spacetime manifold in which the Einstein special relativity theory has been usually formulated. The spacetime coordinates can be defined by Poincaré in the Minkowski space where three real space dimensions in Euclidean space of x, y, and z are combined with a single time dimension t to generate a four-dimensional spacetime manifold with a fourth imaginary dimension of $x^4 = jct$ in the Poincaré group [12, 13]. The coordinates of this four-dimensional flat spacetime are called the pseudo-Euclidean coordinates that can be written as (cf. Appendix G)

$$\begin{aligned} x^1 &= x; \\ x^2 &= y; \\ x^3 &= z; \\ x^4 &= \sqrt{-1}\, ct \equiv jct \end{aligned} \tag{5.198a}$$

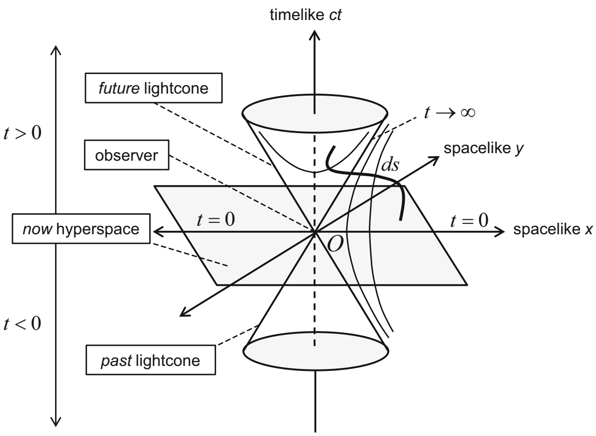


Fig. 5.11 Minkowski four-dimensional spacetime diagram

The spacetime interval ds in the Minkowski spacetime for the special relativity is written as

$$\begin{aligned} ds^2 = g = g_{\mu\nu} dx^\mu dx^\nu \\ = dx^2 + dy^2 + dz^2 + j^2 c^2 dt^2 \\ = dx^2 + dy^2 + dz^2 - c^2 dt^2 \end{aligned} \tag{5.198b}$$

The spacetime interval is the shortest path length in the four-dimensional spacetime. If the spacetime interval ds is zero, it is called the *lightcone* (*lightlike*); ds is negative, it is called the *timelike*; and ds is positive, it is called the *spacelike*, cf. Fig. 5.11.

Thus, the Minkowski metric with four spacetime coordinates (x, y, z, jct) results from Eq. (5.198b).

$$g = \left(g_{\mu\nu}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.198c}$$

Obviously, the Minkowski metric with four spacetime coordinates (x, y, z, ct) can be written as

$$g = \left(g_{\mu\nu}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5.198d}$$

The skew-symmetric tensor components F_{kl} and G_{kl} of the electromagnetic fields **B** and **H** are expressed in the pseudo-Euclidean coordinates for k, $l = 1,2,3,4$.

$$\begin{cases} F_{12} = B_z; \ \ F_{23} = B_x; \ \ F_{31} = B_y; \ \ F_{4k} = jE_k/c \\ G_{12} = H_z; \ \ G_{23} = H_x; \ \ G_{31} = H_y; \ \ G_{4k} = jcD_k \end{cases} \tag{5.199}$$

Similarly, the four-current density vector **J** can be defined as

$$\mathbf{J} \equiv \begin{cases} J_1 = J_x \\ J_2 = J_y \\ J_3 = J_z \\ J_4 = jc\rho = \sqrt{-1}\, c\rho \end{cases} \tag{5.200}$$

According to Eqs. (5.192), (5.193) and (5.196), the relation between G_{kl} and F_{kl} in Eq. (5.199) for vacuum results in

$$G_{kl} = \frac{F_{kl}}{\mu_0} \tag{5.201}$$

Therefore, the first and fourth inhomogeneous Maxwell's equations (5.188c) and (5.191c) can be expressed in the tensor equation of G_{kl}.

$$\begin{aligned} &\frac{\partial G_{kl}}{\partial x^l} \equiv G_{kl,l} = J_k \\ &\Rightarrow F_{kl,l} = \mu_0 J_k \text{ for } k, l = 1, 2, 3, 4 \end{aligned} \tag{5.202}$$

Thus, Eq. (5.202) can be written in the four-dimensional spacetime coordinates:

$$\begin{cases} \dfrac{\partial F_{12}}{\partial x^2} + \dfrac{\partial F_{13}}{\partial x^3} + \dfrac{\partial F_{14}}{\partial x^4} = \mu_0 J_1 \\ \dfrac{\partial F_{21}}{\partial x^1} + \dfrac{\partial F_{23}}{\partial x^3} + \dfrac{\partial F_{24}}{\partial x^4} = \mu_0 J_2 \\ \dfrac{\partial F_{31}}{\partial x^1} + \dfrac{\partial F_{32}}{\partial x^2} + \dfrac{\partial F_{34}}{\partial x^4} = \mu_0 J_3 \\ \dfrac{\partial F_{41}}{\partial x^1} + \dfrac{\partial F_{42}}{\partial x^2} + \dfrac{\partial F_{43}}{\partial x^3} = \mu_0 J_4 \end{cases} \tag{5.203}$$

Similarly, the second and third homogenous Maxwell's equations (5.189c) and (5.190c) can be written in the tensor equation of F_{kl}.

$$F_{kl,m} + F_{lm,k} + F_{mk,l} = 0 \text{ for } k, l, m = 1, 2, 3, 4 \tag{5.204}$$

where the covariant electromagnetic field tensors can be defined by [12].

$$(F_{ij}) \equiv \begin{pmatrix} 0 & B_z & -B_y & -jE_x/c \\ -B_z & 0 & B_x & -jE_y/c \\ B_y & -B_x & 0 & -jE_z/c \\ jE_x/c & jE_y/c & jE_z/c & 0 \end{pmatrix} \tag{5.205}$$

Equation (5.204) can be written in the four-dimensional spacetime coordinates:

$$\begin{cases} \dfrac{\partial F_{23}}{\partial x^1} + \dfrac{\partial F_{31}}{\partial x^2} + \dfrac{\partial F_{12}}{\partial x^3} = 0 \\ \dfrac{\partial F_{34}}{\partial x^2} + \dfrac{\partial F_{42}}{\partial x^3} + \dfrac{\partial F_{23}}{\partial x^4} = 0 \\ \dfrac{\partial F_{41}}{\partial x^3} + \dfrac{\partial F_{13}}{\partial x^4} + \dfrac{\partial F_{34}}{\partial x^1} = 0 \\ \dfrac{\partial F_{12}}{\partial x^4} + \dfrac{\partial F_{24}}{\partial x^1} + \dfrac{\partial F_{41}}{\partial x^2} = 0 \end{cases} \tag{5.206}$$

The Maxwell's equations written in the tensor components F_{ij} result from Eqs. (5.202) and (5.204) in the four-dimensional Minkowski spacetime (special relativity):

$$F_{kl,l} = \mu_0 J_k \text{ for } k, l = 1, 2, 3, 4$$
$$F_{kl,m} + F_{lm,k} + F_{mk,l} = 0 \text{ for } k, l, m = 1, 2, 3, 4$$

The potential vector **A (r,** ***t*****)** can be defined as a potential scalar φ.

$$\mathbf{A}(\mathbf{r}, t) = \begin{cases} A_1 = A_x \\ A_2 = A_y \\ A_3 = A_z \\ A_4 = \dfrac{j\varphi(\mathbf{r}, t)}{c} \end{cases} \tag{5.207}$$

The skew-symmetric electromagnetic field tensor component F_{kl} can be written in the potential vector components.

$$\begin{aligned} F_{kl} &= \frac{\partial A_l}{\partial x^k} - \frac{\partial A_k}{\partial x^l} \\ &= A_{l,k} - A_{k,l} \text{ for } k, l = 1, 2, 3, 4 \end{aligned} \tag{5.208}$$

Substituting Eqs. (5.202), (5.204) and (5.208), the Maxwell's equations can be expressed in the potential vector **A**.

$$\begin{aligned} F_{kl,l} &= \frac{\partial}{\partial x^l}(A_{l,k} - A_{k,l}) \\ &= \frac{\partial^2 A_l}{\partial x^k \partial x^l} - \frac{\partial^2 A_k}{\partial x^l \partial x^l} = \mu_0 J_k \end{aligned} \tag{5.209a}$$

$$\Leftrightarrow F_{kl,l} = A_{l,kl} - A_{k,ll} = \mu_0 J_k \text{ for } k,l = 1,2,3,4$$

According to Eqs. (5.204) and (5.209a), one obtains

$$\begin{aligned} &F_{kl,m} + F_{lm,k} + F_{mk,l} = \\ &\frac{\partial}{\partial x^m}(A_{l,k} - A_{k,l}) + \frac{\partial}{\partial x^k}(A_{m,l} - A_{l,m}) + \frac{\partial}{\partial x^l}(A_{k,m} - A_{m,k}) = 0 \end{aligned} \tag{5.209b}$$

$$\Leftrightarrow (A_{l,km} - A_{k,lm}) + (A_{m,lk} - A_{l,mk}) + (A_{k,ml} - A_{m,kl}) = 0 \text{ for } k,l,m = 1,2,3,4$$

The Maxwell's equations written in the potential vector components A_l result from Eqs. (5.209a) and (5.209b) in the four-dimensional Minkowski spacetime (special relativity):

$$A_{l,kl} - A_{k,ll} = \mu_0 J_k \text{ for } k,l = 1,2,3,4$$

$$(A_{l,km} - A_{k,lm}) + (A_{m,lk} - A_{l,mk}) + (A_{k,ml} - A_{m,kl}) = 0 \text{ for } k,l,m = 1,2,3,4$$

Using Eqs. (5.199) and (5.208), the magnetic field strength **H** can be rewritten in the potential vector **A**.

$$\mathbf{B} = \mu\,\mathbf{H} = \nabla \times \mathbf{A} \tag{5.210}$$

Using Faraday's law, the electric field strength **E** can be expressed in the potential vector **A** and the potential scalar φ.

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t} \tag{5.211}$$

In the Lorenz gauge, the relation between the potential scalar and vector can be written as [12].

$$\frac{1}{c^2}\frac{\partial \varphi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \tag{5.212}$$

Using the Maxwell's equations and product rules of vector calculus, the wave equations of the potential scalar and vector result in

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \tag{5.213}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \tag{5.214}$$

5.8.3 The Maxwell's Stress Tensor

Newton's third law (force and reaction force) is only valid for electrostatics and magnetostatics but not valid for electrodynamics. However, the momentum conservation law is still valid in electrodynamics.

In the following section, the specific electromagnetic force that consists of the electrostatic and magnetostatic forces is derived from the Maxwell's stress tensor.

First, the electrostatic force is written as

$$\mathbf{F}_e = \int \mathbf{E} dq = \oint_V \mathbf{E} \rho dV$$

where ρ is the electric charge volumetric density.

Second, the magnetostatic force is written as

$$\mathbf{F}_m = \int (\mathbf{v} \times \mathbf{B}) dq = \oint_V (\mathbf{v} \times \mathbf{B}) \rho dV$$

where $\mathbf{v}$ is the charge velocity of the electric charge q.

Thus, the electromagnetic force results from both electro- and magnetostatic forces as

$$\begin{aligned} \mathbf{F} = \mathbf{F}_e + \mathbf{F}_m &= \oint_V (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \rho dV \\ &= \oint_V (\rho \mathbf{E} + \rho \mathbf{v} \times \mathbf{B}) dV \\ &= \oint_V (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}) dV \end{aligned} \tag{5.215}$$

in which the current density vector $\mathbf{J}$ is defined as product of the electric charge density and its charge velocity.

$$\mathbf{J} \equiv \rho \mathbf{v}$$

The specific electromagnetic force **f** is defined as the electromagnetic force per unit volume.

$$\mathbf{f} = \rho\mathbf{E} + (\mathbf{J} \times \mathbf{B}) \tag{5.216}$$

Using the Gauss's law for electric fields, one obtains

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \rho = \varepsilon_0 \nabla \cdot \mathbf{E}$$

Thus, the first term on the RHS of Eq. (5.216) is written as

$$\rho\mathbf{E} = \varepsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} \tag{5.217}$$

The Ampere-Maxwell's law gives the electric current density

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \\ \Rightarrow \mathbf{J} &= \frac{1}{\mu_0}(\nabla \times \mathbf{B}) - \varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

Therefore, the second term on the RHS of Eq. (5.216) is written as

$$\begin{aligned} (\mathbf{J} \times \mathbf{B}) &= \left[\frac{1}{\mu_0}(\nabla \times \mathbf{B}) - \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right] \times \mathbf{B} \\ &= -\frac{\mathbf{B}}{\mu_0} \times (\nabla \times \mathbf{B}) - \varepsilon_0\left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\right) \end{aligned} \tag{5.218}$$

Furthermore, the last term on the RHS of Eq. (5.218) is calculated using the product rule of vector calculus as

$$\left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\right) = \frac{\partial(\mathbf{E} \times \mathbf{B})}{\partial t} - \left(\mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}\right)$$

Using the Faraday's law, one obtains

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Thus,

$$\left(\frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}\right) = \frac{\partial(\mathbf{E} \times \mathbf{B})}{\partial t} + \mathbf{E} \times (\nabla \times \mathbf{E}) \tag{5.219}$$

Substituting Eqs. (5.217)–(5.219) into Eq. (5.216), the specific electromagnetic force $\mathbf{f}$ is written as

$$\mathbf{f} = \varepsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})] - \frac{\mathbf{B}}{\mu_0} \times (\nabla \times \mathbf{B}) - \varepsilon_0 \frac{\partial(\mathbf{E} \times \mathbf{B})}{\partial t} \tag{5.220}$$

Using product rule of vector calculus, gradient of a vector product is calculated as (cf. Appendix C)

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

where $\mathbf{A}$ and $\mathbf{B}$ are two arbitrary vectors in $\mathbf{R}^N$.

In the case of $\mathbf{A} = \mathbf{B} \equiv \mathbf{E}$, one obtains

$$\nabla(\mathbf{E} \cdot \mathbf{E}) = \nabla \mathbf{E}^2 = 2(\mathbf{E} \cdot \nabla)\mathbf{E} + 2\mathbf{E} \times (\nabla \times \mathbf{E})$$

Thus, the relation of the electric field strength $\mathbf{E}$

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = \frac{1}{2}\nabla \mathbf{E}^2 - (\mathbf{E} \cdot \nabla)\mathbf{E} \tag{5.221}$$

Similarly, one obtains the relation of the magnetic field density $\mathbf{B}$

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = \frac{1}{2}\nabla \mathbf{B}^2 - (\mathbf{B} \cdot \nabla)\mathbf{B} \tag{5.222}$$

Using Gauss's law for magnetic fields ($\nabla \cdot \mathbf{B} = 0$), Eq. (5.222) can be written as

$$\begin{aligned} \mathbf{B} \times (\nabla \times \mathbf{B}) &= \frac{1}{2}\nabla \mathbf{B}^2 - (\mathbf{B} \cdot \nabla)\mathbf{B} \\ &= \frac{1}{2}\nabla \mathbf{B}^2 - (\mathbf{B} \cdot \nabla)\mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{B}) \end{aligned} \tag{5.223}$$

Substituting Eqs. (5.221) and (5.223) into Eq. (5.220), the specific electromagnetic force is expressed as

$$\begin{aligned} \mathbf{f} &= \varepsilon_0[(\nabla \cdot \mathbf{E})\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{E}] + \frac{1}{\mu_0}[(\nabla \cdot \mathbf{B})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}] \\ &\quad -\frac{1}{2}\nabla\left(\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2\right) - \varepsilon_0 \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) \end{aligned} \tag{5.224}$$

The Maxwell's stress tensor $\mathbf{M}$ is defined as a second-order tensor with the covariant components of

$$M_{ij} = \varepsilon_0\left(E_i E_j - \tfrac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \tfrac{1}{2}\delta_{ij}B^2\right) \text{ for } i,j = x,y,z$$

where

δ_{ij} is the Kronecker delta;

E_i is the i^{th} component of the electric field strength $\mathbf{E}$;

B_i is the i^{th} component of the magnetic field density $\mathbf{B}$;

E and B are the magnitudes of the electric field strength and the magnetic field density, respectively.

The magnitudes of $\mathbf{E}$ and $\mathbf{B}$ in a 3-D space $\mathbf{R}^3$ are calculated as

$$E \equiv |\mathbf{E}| = \sqrt{E_x^2 + E_y^2 + E_z^2};$$
$$B \equiv |\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}.$$

There are nine components of the Maxwell's stress tensor M_{ij} in a 3-D space $\mathbf{R}^3$ that are written as

$$\begin{aligned} M_{xx} &= \frac{1}{2}\varepsilon_0\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_x^2 - B_y^2 - B_z^2\right); \\ M_{xy} &= M_{yx} = \varepsilon_0 E_x E_y + \frac{1}{\mu_0} B_x B_y; \end{aligned} \tag{5.225a}$$

$$\begin{aligned} M_{yy} &= \frac{1}{2}\varepsilon_0\left(E_y^2 - E_x^2 - E_z^2\right) + \frac{1}{2\mu_0}\left(B_y^2 - B_x^2 - B_z^2\right); \\ M_{yz} &= M_{zy} = \varepsilon_0 E_y E_z + \frac{1}{\mu_0} B_y B_z; \end{aligned} \tag{5.225b}$$

$$\begin{aligned} M_{zz} &= \frac{1}{2}\varepsilon_0\left(E_z^2 - E_x^2 - E_y^2\right) + \frac{1}{2\mu_0}\left(B_z^2 - B_x^2 - B_y^2\right); \\ M_{zx} &= M_{xz} = \varepsilon_0 E_z E_x + \frac{1}{\mu_0} B_z B_x. \end{aligned} \tag{5.225c}$$

The second-order Maxwell's stress tensor $\mathbf{M}$ is written as in the contravariant bases

$$\begin{aligned} \mathbf{M} &= M_{ij}\mathbf{g}^i\mathbf{g}^j \text{ for } i,j = x,y,z \\ &= \begin{pmatrix} M_{xx} & M_{xy} & M_{xz} \\ M_{yx} & M_{yy} & M_{yz} \\ M_{zx} & M_{zy} & M_{zz} \end{pmatrix} \end{aligned}$$

Using the Christoffel symbols, the ith component of the divergence $\nabla \cdot \mathbf{M}$ is generally written in an N-dimensional space $\mathbf{R}^N$ as

$$\begin{aligned} &(\nabla \cdot \mathbf{M})_i = \left(M_{ij,k} - \Gamma_{ik}^m M_{mj} - \Gamma_{jk}^m M_{im}\right) g^{jk} \\ &\text{for } i,j,k,m = 1,2,\ldots,N \end{aligned}$$

For an orthonormal coordinate system in $\mathbf{R}^3$, the ith component of the divergence $\nabla\cdot\mathbf{M}$ results as [11]

$$\begin{aligned}(\nabla\cdot\mathbf{M})_i &= \varepsilon_0\Big((\nabla\cdot\mathbf{E})E_i+(\mathbf{E}\cdot\nabla)E_i-\tfrac{1}{2}(\nabla\cdot\mathbf{E}^2)_i\Big)\\ &+\frac{1}{\mu_0}\Big((\nabla\cdot\mathbf{B})B_i+(\mathbf{B}\cdot\nabla)B_i-\tfrac{1}{2}(\nabla\cdot\mathbf{B}^2)_i\Big)\ \text{ for } i=x,y,z\end{aligned} \tag{5.226}$$

The divergence of the Maxwell's stress tensor is calculated as

$$\begin{aligned}\nabla\cdot\mathbf{M} &= M_{ij}\big|_k g^{jk}\mathbf{g}^i\\ &= (\nabla\cdot\mathbf{M})_i\mathbf{g}^i \text{ for } i=x,y,z\end{aligned} \tag{5.227}$$

Substituting Eqs. (5.224), (5.226) and (5.227), one obtains the specific electromagnetic force in a simple form

$$\begin{aligned}\mathbf{f} &= \nabla\cdot\mathbf{M}-\varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})\\ &= \nabla\cdot\mathbf{M}-\varepsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}\\ &= \nabla\cdot\mathbf{M}-\frac{1}{c^2}\frac{\partial\mathbf{S}}{\partial t}\end{aligned} \tag{5.228}$$

The Poynting's vector $\mathbf{S}$ in Eq. (5.228) is defined as the energy flux density of the electromagnetic field.

$$\mathbf{S}=\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \tag{5.229}$$

Using Gauss's divergence theorem, the total electromagnetic force is calculated from Eq. (5.229) as

$$\begin{aligned}\mathbf{F} &= \oint_V \mathbf{f}\,dV=\oint_V\left(\nabla\cdot\mathbf{M}-\frac{1}{c^2}\frac{\partial\mathbf{S}}{\partial t}\right)dV\\ &= \oint_A \mathbf{M}\cdot\mathbf{n}\,dA-\frac{1}{c^2}\oint_V\frac{\partial\mathbf{S}}{\partial t}dV\end{aligned} \tag{5.230}$$

where $\mathbf{n}$ is the normal unit vector on the surface A.

The momentum density in the electromagnetic field is defined as

$$\mathbf{g}=\varepsilon_0\mu_0\mathbf{S}=\varepsilon_0(\mathbf{E}\times\mathbf{B})$$

5.8.4 The Poynting's Theorem

The Poynting's theorem is used to calculate the generated power of the electromagnetic force, as shown in Eqs. (5.215) and (5.216).

The generated work of the electromagnetic force is calculated as

$$\begin{aligned} dW_t &= \mathbf{F} \cdot d\mathbf{l} = \mathbf{F} \cdot \mathbf{v}dt \\ &= q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}dt = q\mathbf{E} \cdot \mathbf{v}dt + q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}dt \end{aligned}$$

Due to the orthogonality, the dot product of two vectors $(\mathbf{v} \times \mathbf{B})$ and $\mathbf{v}$ equals zero.

$$(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$$

As a result, the electromagnetic power P_t becomes

$$\begin{aligned} dW_t &= q\mathbf{E} \cdot \mathbf{v}dt \\ \Rightarrow P_t \equiv \frac{dW_t}{dt} &= q\mathbf{E} \cdot \mathbf{v} \ = \oint_V \rho\mathbf{E} \cdot \mathbf{v}dV \\ &= \oint_V (\mathbf{E} \cdot \mathbf{J})dV \end{aligned} \tag{5.231}$$

where q is the electric charge; $\mathbf{J}$ is the electric current density.

Using the Ampere and Maxwell's equation, the dot product between $\mathbf{E}$ and $\mathbf{J}$ is calculated as

$$\begin{aligned} \mathbf{E} \cdot \mathbf{J} &= \mathbf{E} \cdot \frac{1}{\mu_0}\left[(\nabla \times \mathbf{B}) - \varepsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t}\right] \\ &= \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \varepsilon_0\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{5.232}$$

Applying the product rule of vector calculus, one obtains using the Faraday's law

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{B}) &= (\nabla \times \mathbf{E}) \cdot \mathbf{B} - \mathbf{E} \cdot (\nabla \times \mathbf{B}) \\ &= -\frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{B} - \mathbf{E} \cdot (\nabla \times \mathbf{B}) \end{aligned}$$

Thus, the first term on the RHS of Eq. (5.232) results as

$$\begin{aligned} \mathbf{E} \cdot (\nabla \times \mathbf{B}) &= -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \\ &= -\frac{1}{2}\frac{\partial\left(\mathbf{B}^2\right)}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \end{aligned} \tag{5.233}$$

Substituting Eq. (5.233) into Eq. (5.232), one obtains

$$\begin{aligned}\mathbf{E}\cdot\mathbf{J} &= -\frac{1}{\mu_0}\left[\frac{1}{2}\frac{\partial\left(\mathbf{B}^2\right)}{\partial t}+\nabla\cdot(\mathbf{E}\times\mathbf{B})\right]-\varepsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\\ &= -\frac{1}{\mu_0}\left[\frac{1}{2}\frac{\partial\left(\mathbf{B}^2\right)}{\partial t}+\nabla\cdot(\mathbf{E}\times\mathbf{B})\right]-\frac{1}{2}\varepsilon_0\frac{\partial\left(\mathbf{E}^2\right)}{\partial t}\\ &= -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0E^2+\frac{B^2}{\mu_0}\right)-\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B})\end{aligned} \tag{5.234}$$

Inserting Eq. (5.234) into Eq. (5.231), one obtains the electromagnetic power

$$\begin{aligned}P_t &= -\frac{\partial}{\partial t}\oint_V\frac{1}{2}\left(\varepsilon_0E^2+\frac{B^2}{\mu_0}\right)dV-\oint_V\nabla\cdot\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})dV\\ &= -\left[\frac{\partial}{\partial t}\oint_V u\,dV+\oint_V\nabla\cdot\mathbf{S}dV\right]\\ &= -\oint_V\left(\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S}\right)dV\end{aligned} \tag{5.235}$$

In Eq. (5.235), the specific work u is defined as

$$u\equiv\frac{1}{2}\left(\varepsilon_0E^2+\frac{B^2}{\mu_0}\right) \tag{5.236}$$

The specific work results from the generated electro-magnetostatic power of the electro and magnetostatic forces.

The electrostatic work results from the electric field strength as

$$W_e=\frac{\varepsilon_0}{2}\oint_V E^2dV$$

The magnetostatic work results from the magnet field density as

$$W_m=\frac{1}{2\mu_0}\oint_V B^2dV$$

Therefore, the generated electro-magnetostatic power results as

$$W_{ems}=\oint_V\frac{1}{2}\left(\varepsilon_0E^2+\frac{B^2}{\mu_0}\right)dV\equiv\oint_V u\,dV$$

In an empty space, no electromagnetic power is generated. According to Eq. (5.235), the energy continuity equation is given as

$$P_t = -\oint_V \left(\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} \right) dV = 0$$
$$\Rightarrow \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = \frac{\partial u}{\partial t} - \frac{1}{\mu_0} \nabla \cdot (\mathbf{B} \times \mathbf{E}) = 0 \tag{5.237}$$

5.9 Einstein Field Equations

Einstein field equations (EFE) are the fundamental equations in the general relativity theory. They describe the interactions between the gravitational field, physical characteristics of matters, and energy momentum tensor [13]. According to Einstein, due to gravity the matter curves and distorts the spacetime. In return, the curved spacetime shows the matter how to move by means of energy momentum and its curvature. Obviously, the larger the mass is, the more the spacetime is wrapped. Hence, the mass is accelerated in the direction perpendicular to its moving direction due to centrifugal force. Gravitational waves that are caused by merging two rotating binary black holes could vibrate and wrap the universe spacetime. Recently, the wrapped spacetime with very tiny vibrations in different directions generating by gravitational waves has been confirmed on February 11th 2016 by LIGO Scientific Collaboration (Laser Interferometer Gravitational wave Observatory).

All tensors using in the general relativity theory have been mostly written in the abstract index notation defined by Penrose [14]. This index notation uses the indices to express the tensor types rather than their covariant components in the basis $\{\mathbf{g}^i\}$.

According to Eqs. (2.250a) and (2.251) and using the tensor contraction laws, the Einstein tensor can be written in the covariant tensor components.

$$\begin{aligned} G_{ij} &= g_{ik} G_j^k \\ &= g_{ik} \left(R_j^k - \tfrac{1}{2} \delta_j^k R \right) \\ &= R_{ij} - \tfrac{1}{2} g_{ij} R \end{aligned}$$

Therefore, the Einstein field equations can be expressed as

$$G_{ij} \equiv \left(R_{ij} - \tfrac{1}{2} g_{ij} R \right) = -\frac{8\pi G}{c^4} T_{ij} \tag{5.238}$$

where

G_{ij} is the covariant Einstein tensor in Eq. (2.251);
R_{ij} is the Ricci tensor in Eq. (2.242);
g_{ij} is the covariant metric tensor components;
R is the Ricci curvature tensor in Eq. (2.248);

G is the universal gravitational constant ($=6.673 \times 10^{-11}$ Nm^2/kg^2);
c is the light speed ($\approx 3 \times 10^8$ m/s);
T_{ij} is the kinetic energy-momentum tensor.

The kinetic energy-momentum tensor in vacuum can be calculated from the cosmological constant Λ and the covariant metric tensor components g_{ij}.

$$T_{ij} = \frac{-\Lambda c^4}{8\pi G} g_{ij} \tag{5.239}$$

in which the cosmological constant is equivalent to an energy density in a vacuum space. According to a recent measurement, the cosmological constant Λ is on the order of 10^{-52} m^{-2} and proportional to the dark-energy density ρ with a factor of $8\pi G$ used in the general relativity.

$$\Lambda = 8\pi G\rho \tag{5.240}$$

In case of a positive energy density of the vacuum space ($\Lambda > 0$), the related negative pressure will cause an accelerating expansion of the universe.

In relativity electromagnetism, the kinetic energy-momentum tensor T_{ij} can be calculated from the energy-momentum tensor of the electromagnetic field S_{ij} (electromagnetic stress-energy tensor) according to [12].

$$-T_{ij} = S_{ij} = \frac{1}{\mu_0}\left(F_{ik}F_{jk} - \tfrac{1}{4}\delta_{ij}F_{kl}F_{kl}\right) \tag{5.241}$$

in which δ_{ij} is the Kronecker delta.

Substituting Eq. (5.241) into Eq. (5.238), the Einstein-Maxwell equations can be generally written with the cosmological constant Λ.

$$\begin{aligned} G_{ij} - g_{ij}\Lambda &\equiv \left(R_{ij} - \tfrac{1}{2}g_{ij}R\right) - g_{ij}\Lambda = R_{ij} - \left(\tfrac{1}{2}R + \Lambda\right)g_{ij} \\ &= -\frac{8\pi G}{c^4}T_{ij} \\ &= \frac{8\pi G}{c^4\mu_0}\left(F_{ik}F_{jk} - \tfrac{1}{4}\delta_{ij}F_{kl}F_{kl}\right) \end{aligned} \tag{5.242}$$

where R is the Ricci curvature tensor can be obtained according to Eqs. (2.248) and (2.249).

$$\begin{aligned} R &= R_{ij}g^{ij} \\ &= \left(\frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^i \partial u^j} - \frac{1}{\sqrt{g}}\frac{\partial\left(J\Gamma_{ij}^m\right)}{\partial u^m} + \Gamma_{in}^m\Gamma_{jm}^n\right)g^{ij} \end{aligned} \tag{5.243}$$

The Ricci curvature tensor can be expressed in the kinetic energy-momentum tensor T [15]:

$$R = \frac{8\pi G}{c^4} T_i^i = \frac{8\pi G}{c^4} T \tag{5.244}$$

The covariant electromagnetic field tensors F_{ij} in Eq. (5.242) are given according to Eq. (5.205).

$$\left(F_{ij}\right) = \begin{pmatrix} 0 & B_z & -B_y & -\mathrm{j}E_x/c \\ -B_z & 0 & B_x & -\mathrm{j}E_y/c \\ B_y & -B_x & 0 & -\mathrm{j}E_z/c \\ \mathrm{j}E_x/c & \mathrm{j}E_y/c & \mathrm{j}E_z/c & 0 \end{pmatrix} \tag{5.245}$$

where E is the electric field strength; B is the magnetic field density; c is the light speed in vacuum.

5.10 Schwarzschild's Solution of the Einstein Field Equations

In case of ignoring the cosmological constant Λ and the negligibly small energy-momentum tensor ($T_{ij} = 0$) in an empty space of small scales ($R = 0$), the Einstein field equation in Eq. (5.242) can be written in a simple tensor equation as

$$R_{ij} = 0 \tag{5.246}$$

The Schwarzschild's solution of Eq. (5.246) can be derived for a spherically symmetrical empty space with four spacetime coordinates (jct, r, φ, θ) around an object with a mass M [15, 16].

The purely imaginary spacetime distance ds along a curve C in the spherical space is written as (cf. Appendix G)

$$\begin{aligned} ds^2 &= -c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \\ &= -g_{00} c^2 dt^2 + g_{rr} dr^2 + r^2 d\Omega^2 \\ &= -\left(1 - \frac{2MG}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 + r^2 d\varphi^2 + r^2 \sin^2\varphi \, d\theta^2 \end{aligned} \tag{5.247a}$$

According to Eq. (5.247a), the Schwarzschild metric describes the spacetime around the spherical black hole with four spacetime coordinates (jct, r, φ, θ)

$$g = \left(g_{\mu\nu}\right) = \begin{bmatrix} \left(1 - \frac{2MG}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2MG}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \varphi \end{bmatrix} \tag{5.247b}$$

The proper time in the Schwarzschild space results from Eq. (5.247a) in

$$d\tau^2 = \left(1 - \frac{2MG}{c^2 r}\right) dt^2 - \frac{1}{c^2} \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 - \frac{r^2}{c^2} \left(d\varphi^2 + \sin^2 \varphi \, d\theta^2\right) \tag{5.247c}$$

in which

$d\tau$ is the proper time (intrinsic time) unaffected by any gravitational field;

dt is the apparent time (laboratory time) in a rest frame of the clock affected by a gravitational field. Generally, $dt > d\tau$ due to gravitational time dilation;

$$d\Omega^2 \equiv d\varphi^2 + \sin^2 \varphi d\theta^2;$$

r is the radius of spherical coordinates (r,φ,θ), as shown in Fig. 5.1;

G is the gravitational constant ($G = 6.673 \times 10^{-11}$ m^3 kg^{-1}s^{-2});

M is the object mass;

c is the light speed.

Two singularities of the Schwarzschild's solution in Eq. (5.247a) exist as the space distance ds increases infinitely. Besides $r = 0$ (an unavoidable case), the space metric coefficient must be

$$g_{rr} = \frac{1}{1 - \frac{2MG}{c^2 r}} \to \infty \tag{5.248}$$

Thus,

$$1 - \frac{2MG}{c^2 r} = 0 \tag{5.249}$$

The Schwarzschild's radius can be defined at the singularity by

$$r_S \equiv r\big|_{g_{rr} \to \infty} = \frac{2MG}{c^2} \tag{5.250}$$

The proper time $d\tau$ without effect of any gravitational field results at $r \geq r_S$, i.e. the clock runs slower by the factor given in Eq. (5.251) when it comes near the gravitation field ($dt > d\tau$). This effect is called the gravitational time dilation [15].

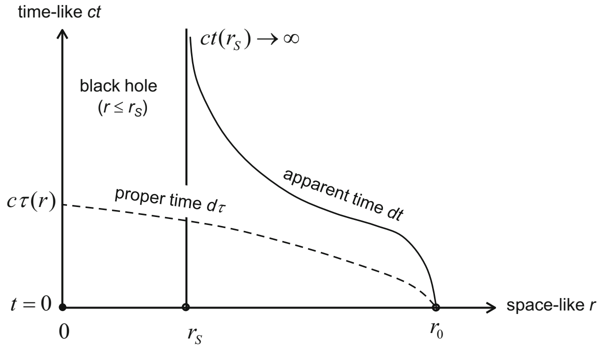


Fig. 5.12 Proper time vs. apparent time in the Schwarzschild spacetime

$$d\tau < \sqrt{g_{00}}dt = \sqrt{\left(1 - \frac{2MG}{c^2 r}\right)}\, dt < dt \tag{5.251}$$

According to Eq. (5.251), the laboratory time dt becomes infinite as r reaches the Schwarzschild's radius r_S. The clock would stop running near the black hole because dt goes to infinity (cf. Fig. 5.12).

$$\begin{aligned} dt &> \frac{1}{\sqrt{g_{00}}}d\tau = \frac{1}{\sqrt{\left(1 - \frac{2MG}{c^2 r}\right)}}d\tau \\ &= \sqrt{g_{rr}}d\tau \to \infty \quad \text{as} \quad r \to r_S. \end{aligned} \tag{5.252}$$

5.11 Schwarzschild Black Hole

In case of $r < r_S$, the uncharged spherically symmetric dwarf star (neutron star) with a mass M collapses within a cylinder of the radius r less than the Schwarzschild's radius r_S. This cylinder is called the Schwarzschild black hole, as shown in Fig. 5.13. According to Eq. (5.250), the Schwarzschild's radius r_S of the Sun is about 2.95×10^3 m compared to its radius R of 6.95×10^8 m (approximately 235,000 times larger than r_S). Therefore, our Sun with the mass $m \sim R^3$ would produce a huge energy $E = mc^2$ before it disappears into the black hole in the very far future.

Using the Hamiltonian H consisting of the generalized angular momentum p_i, active variable $\dot{q}_i$, and the Lagrangian L, the total conserved energy of the photon trajectory in the gravitational field is calculated as

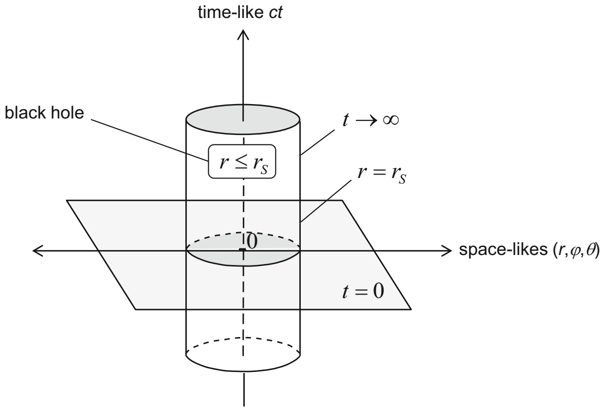


Fig. 5.13 Black hole in the Schwarzschild spacetime

$$
\begin{aligned}
H(q_i, p_i, t) &= \sum_i p_i \dot{q}_i - L = \sum_i \left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i - L(q_i, \dot{q}_i, t) \\
&= T + V = \frac{p^2}{2m} + V(q) \\
&\equiv E_P
\end{aligned}
$$

where the Lagrangian L is defined as

$$L \equiv T - V;$$

the generalized angular momentum and active variable can be written as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}; \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

The total conserved energy of the photon is approximately written according to [17] as

$$E_P \propto \frac{1}{r} \sqrt{1 - \frac{2MG}{c^2 r}} \tag{5.253}$$

In fact, the Hamiltonian is the total conserved energy of the kinetic and potential energy T and V of the system. Differentiating Eq. (5.253) with respect to r and calculating the second derivative at the extreme, the radius r_P results as the maximum energy of the photon trajectory occurs at the zero first derivative and negative second derivative.

$$\frac{\partial E_P}{\partial r} \propto \frac{\partial}{\partial r}\left(\frac{1}{r}\sqrt{1-\frac{2MG}{c^2 r}}\right) = \frac{\frac{3MG}{c^2} - r}{r^3\sqrt{1-\frac{2MG}{c^2 r}}} = 0$$

Thus,

$$r_P = \frac{3MG}{c^2} = \frac{3}{2} r_S \tag{5.254}$$

The second derivative of the photon energy is calculated as

$$\left.\frac{\partial^2 E_P}{\partial r^2}\right|_{r_P} \propto \frac{-1}{r_P^3\sqrt{1-\frac{2MG}{c^2 r_P}}} = \frac{-\sqrt{3}}{\left(\frac{3MG}{c^2}\right)^3} < 0$$

Thus, the maximum energy results at $r = r_P$, as shown in Fig. 5.14.

$$E_{P,\max} \propto \left.\frac{1}{r}\sqrt{1-\frac{2MG}{c^2 r}}\right|_{r=r_P} \propto \frac{c^2}{3\sqrt{3}MG} \tag{5.255}$$

The photon with the maximum energy at the radius r_P locates in a bi-stable state. Through a small disturbance at this bi-stable state, either the photon trajectory is rejected outwards the photon sphere ($r > r_P$) or attracted towards the black hole ($r \to r_S$) in order to reach the stable state of minimum energy, as shown in Fig. 5.14. The sphere with the radius of $r_P = (3/2)\ r_S$ is called the photon sphere of the

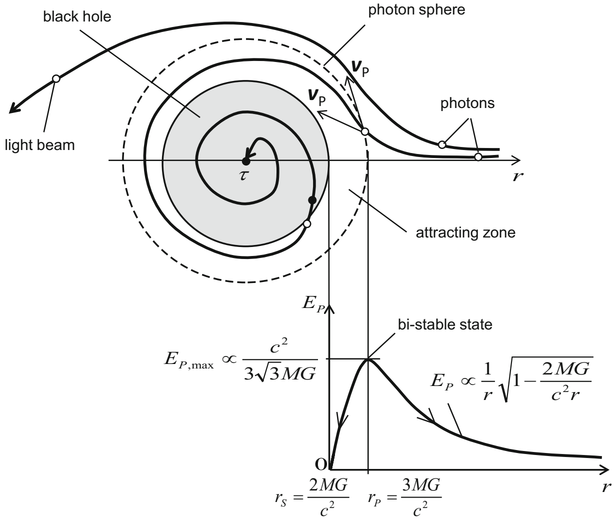


Fig. 5.14 Total energy of photons in the gravitational field of a curved space

gravitational field. Furthermore, the photon energy equals zero at the Schwarzschild's radius r_S according to Eqs. (5.250) and (5.253).

Considering a photon moving from the outside ($r \gg r_P$) near to the photon sphere, the photon energy reaches a maximum at $r = r_P$. Between the photon sphere and the black hole ($r_S < r < r_P$), the photon energy reduces from the maximum to zero at the Schwarzschild's radius r_S of the black hole. If the photon trajectory (light beam) moves outside the photon sphere, the photon trajectory is accelerated with a bending radius around the black hole, as shown in Fig. 5.14. In another case, the photon trajectory enters the photon sphere with a velocity $\mathbf{v}_P$.

There are two possibilities for the light beam depending on the moving direction of $\mathbf{v}_P$ [17]. First, if the velocity $\mathbf{v}_P$ is tangent to the photon sphere surface, the light beam moves on the photon sphere surface in a stable condition. Second, if the radial component of $\mathbf{v}_P$ moves towards the black hole center (called the time line τ), the light beam becomes unstable and collapses itself into the black hole after a number of cycles because the balance between the energy created by the strong force and gravitational force fails. The energy of the photon trajectory reduces to zero as it moves towards the black hole ($r \to r_S$) under the very strong gravitational field of the black hole. This is to blame for the attraction of the photon trajectory to the black hole. In this case, the light beam has never left the black hole back to the outside. The neutron star will be contracted in a infinitesimal point particle in the black hole under its huge gravitational field. Finally, the neutron star collapses in the black hole, crushing all its atoms into a highly dense ball of neutrons.

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Chapter 6
Tensors and Bra-Ket Notation in Quantum Mechanics

6.1 Introduction

Classical mechanics have been used to study Newtonian mechanics including conventional mechanics, electrodynamics, cosmology, and relativity physics of Einstein. Compared to quantum mechanics, classical mechanics is deterministic, in which the processes are well determined and foreseeable. On the contrary, processes in quantum mechanics are undeterministic, uncertain, nonlocal, and intrinsically random. Therefore, quantum mechanics deals with probability and uncertainty principles to study the atomic and subatomic world. Due to their extremely small sizes of the quantum particles, the capabilities of apparatuses are quite limited to measure the undeterministic and uncertain behaviors of the particles. Therefore, interactions between these very small particles in the quantum world, such as electrons and photons are simulated using mathematical abstractions that will be discussed in the following sections. Mathematics of quantum physics describes nonlocal correlations between the particles with the intrinsic randomness (e.g. Schrödinger's cat) in the combined system.

In quantum mechanics, Hilbert space is defined as any finite or infinite N-dimensional vector space in which abstract concepts of vectorial and functional analysis are carried out to study the interactions of the quantum particles. An element of the Hilbert space H can be expressed in coordinates of an orthonormal basis that is analogous with Cartesian coordinate system.

Quantum mechanics (also quantum physics) is a peculiar physics so that Richard Feynman had said, "nobody understands it after all". Contrary to classical physics, quantum mechanics has some special characteristics, such as entanglement, quantum teleportation, and nonlocality that lead to an intrinsic randomness at the measurement of a quantum state [1]. At first, such characteristics must be clearly understood before we try to use abstract mathematics to describe them.

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6.2 Quantum Entanglement and Nonlocality

Erwin Schrödinger had said that quantum entanglement is necessary for quantum mechanics that is also based on the Heisenberg's uncertainty principle. Firstly, entanglement of two entangled particles in a composite system could be understood that there is somehow an interaction between two particles in two different worlds at a long distance. Einstein called this action as the spooky action at a distance (in German: "spukhafte Fernwirkung") due to some hidden variables. Systems that contain no entanglement are defined as *separable systems* (non-entangled).

Secondly, quantum nonlocality describes the nonlocal correlations of two entangled subsystems in a composite system S_{AB} in the combined Hilbert space $H_A \times H_B$. The composite system consists of two or many subsystems. Each subsystem has objects with the matter and physical states. On the contrary, the *concept of locality* states that an object is only directly influenced by its surroundings. The action between two points is carried out by means of waves through a field that transports the information through the space between them. According to the special relativity theory, the transmitting velocity must be in any case less than the light speed.

In the Bell's game, two players A (Alice) in the subsystem of S_A and B (Bob) in the subsystem S_B have two similar game boxes. They play with the boxes at the same time in a very long distance between two planets without any physical connections (e.g., internet LAN, WLAN or any telecommunication techniques). Both game players did not know the strategy with that they intent to play in the game in advance. Note that the game is repeated in many times at a given time interval. Both results of the quantum state vectors $|\Psi_A\rangle$ and $|\Psi_B\rangle$ are instantaneously measured in the subsystems S_A and S_B, as shown in Fig. 6.1.

At the game end, the results show the average outcome of A is somehow correlated with the average outcome of B even if they had agreed in advance.

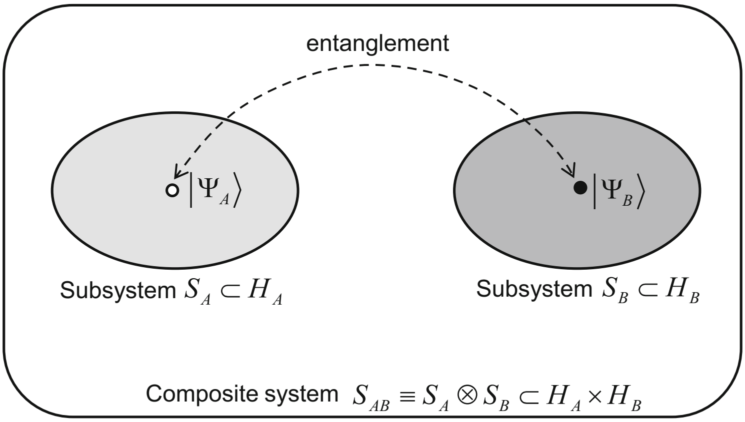

Fig. 6.1 Entanglement between two particles in a composite system

Both players A and B have a truly chance-like event in the Bell's game. This is called the intrinsic randomness that only exists in the nonlocal correlations (nonlocality in the composite system) without any physical communication between A and B. However, there is somehow any telepathic communication, which is simultaneously caused by quantum entanglement between the current states of the players [1–3]. Such correlations are nonlocal and intrinsically random in a composite system.

The results denote the general statement "*quantum entanglement is necessary but not sufficient for nonlocal correlations of the entangled particles*". To prove it, the first part of the statement is proved if the following proposition is true [4]. If the quantum entanglement does not occur in a composite system, then nonlocal correlations do not take place in it. This proposition is always true. Hence, the first part of the statement "*quantum entanglement is necessary for nonlocal correlations of the entangled particles*" has been proved. Next, a *quantum entanglement* could however occur with a *local correlation between the entangled particles* in a composite system of two very close subsystems. This proposition is true at least for this case. Therefore, the second part of the statement "*quantum entanglement is not sufficient for nonlocal correlations of the entangled particles*" is correct [4]. As a result, the general statement is proved.

Using the quantum entanglement of the states, quantum physics tries to predict the nonlocal correlations between the entangled particles of the different subsystems at a distance. Note that *quantum bit* (called *Qubit*) is a unit of quantum information that describes the quantum state of a system that is equivalent to the *von-Neumann entropy*. Therefore, the Qubit state is the current state of an object; and it is a linear superposition of the basis vectors $|0\rangle$ for *spin down* and $|1\rangle$ for *spin up* at the same time.

Note that particles in a composite system are entangled and correlated due to quantum entanglement in which time is its side effect. In general, the more uncertain the quantum system is, the more entangled the system becomes at increasing time [S. Lloyd]. In the composite system, quantum teleportation, teleporting entanglement, and intrinsic randomness are taken into account.

Quantum teleportation is defined as a direct teleportation of the Qubit state of the object (i.e. just only the state of quantum information, not the whole object of mass and energy) from one location to another location at a distance without passing through any medium. That means the state of the object must disappear from its original location; and its teleported state will appear in the new location independently of any distance [1, 5]. *Teleporting entanglement* is a quantum entanglement of the states between two entangled particles at measurements. The nonlocal correlations always occur in the composite system during the entanglement take place between two entangled particles. Therefore, the nonlocal correlations are a sufficient condition for the quantum entanglement of the two-particle state.

Such nonlocal correlations are well determined in the composite system. However, the quantum states of the subsystems are undetermined by the uncertainty principle. The *Heisenberg's uncertainty principle* states that the product of the uncertainties of the position and momentum of a particle must be always larger than a half of the reduced Planck constant (also Dirac constant) [6, 7]. As an example, the relative distance between two electrons is well determined in the composite system, but the positions of the electrons in the subsystems are not definite due to entanglement and the uncertainty principle.

Furthermore, the quantum entanglement between two entangled particles at an infinite distance leads to another issue at which the teleporting velocity of the quantum states between the particles is larger than the light speed. This new issue violates the Einstein special relativity theory, which is valid for quantum mechanics.

The above interpretations and discussions are quite confused; therefore, it is quite difficult to understand quantum mechanics. There are two contrary opinions about the quantum entanglement [1, 7]. Firstly, Einstein, Schrödinger, and De Broglie endorsed the hidden-variable theory with the "spooky action at a distance" and were against the interpretation of quantum entanglement. Secondly, Heisenberg and Bohr accepted the entanglement interpretation and rejected the hidden-variable theory.

6.3 Alternative Interpretation of Quantum Entanglement

Results of the recent measurements have shown that there are neither hidden variables nor the spooky action at a distance in quantum mechanics and the teleporting velocity between two entangled particles is less than the light speed. However, they do not explain how the quantum entanglement really works.

The first author of this book [Nguyen-Schäfer] attempts to interpret the quantum entanglement in the new way that is based on conservation laws of the quantum states.

Let A and B be two entangled particles in the subsystems S_A and S_B, respectively. The quantum state vectors $|\Psi_A\rangle$ and $|\Psi_B\rangle$ are measured using the apparatuses M_A and M_B in the subsystems. The composite system S_{AB} is a combined system of the subsystems S_A and S_B and belongs to the Hilbert space $H_A \times H_B$ (see Fig. 6.2).

Most of processes occurring in physics are almost completely symmetric. According to Noether's theorems [8], symmetric processes obey conservation laws; e.g., the conservation of angular momentum is also valid in a composite system that consists of two or many entangled subsystems. Note that conservation laws are the fundamental principles of physics. As a result, quantum mechanics strongly depends on the conservation of probability of the expectation values (average values) at the measurements.

Note that the quantum entanglement is broken when the entangled particles decohere by means of interaction with the environment, such as at measurement of an observable. In this case, the outcome of the measurement collapses onto the entangled state of the whole system. However, the conservation of the von-Neumann entropy is always held. The von-Neumann entropy that consists of the quantum information describes the disorder of the entangled quantum system.

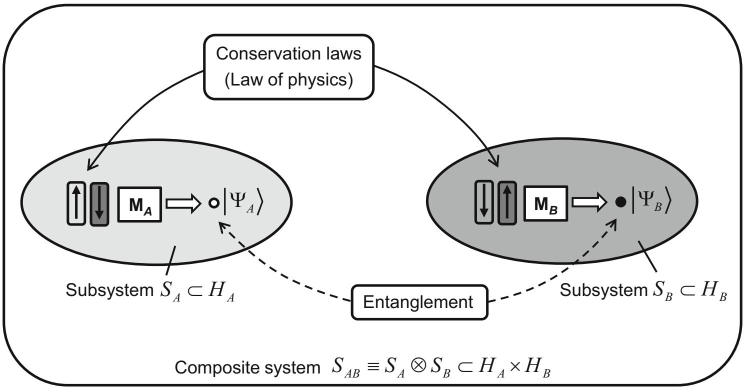

Fig. 6.2 Alternative interpretation of quantum entanglement

This entropy is used to measure the quantum entanglement. Generally, the larger the von-Neumann entropy is, the more entangled the composite system becomes.

The angular momentum of the quantum state (spin state) of a particle is conserved in a composite system but must not be necessarily invariant in its subsystems. According to the conservation law of the angular momentum, the current states of the subsystems are nonlocally correlated in the composite system instantaneously and independently of any distance between two entangled subsystems. At measurements in each subsystem, the instant state of A is correlated with the instant state of B in advance due to conservation of the quantum states in the composite system. In this case, the *conservation of the quantum states* (quantum information) is the *root cause* (upper hand) *for the quantum entanglement*. As a result, *quantum entanglement* is just *the logical consequence of the conservation law of quantum information*; and it is not caused by quantum teleportation or teleporting entanglement.

In this point of view, the conservation of the quantum states in the composite system is the root cause for the quantum entanglement.

An example is given to make this interpretation more plausible. It is assumed that the initial angular momentum of the quantum states of the composite system equals zero. The angular momentum of the composite system is always conserved and therefore invariant with respect to time. In the case if the quantum state of A (Alice) is changed into the spin up $|u\rangle$ in the subsystem S_A, then the quantum state of B (Bob) must be obviously changed into the spin down $|d\rangle$ in the subsystem S_B in order to satisfy the conservation of spin momentum in the composite system. At the game end, the average outcomes of A and B at measurements are correlated with each other.

In this interpretation, nonlocal correlations of the entangled particles occur in any composite system due to conservation laws of entropy. As a result, the statement "*quantum entanglement is necessary but not sufficient for nonlocal correlations of the entangled particles*" has been always valid. Therefore, the quantum entanglement is related to nonlocality of the entangled particles. This statement opposes the EPR (Einstein, Podolsky, and Rosen) paradox, which is based on the locality of the interacted particles in a composite system. Recent experiments have proved that Einstein and his colleagues were wrong on the EPR effect.

Furthermore, this interpretation of quantum entanglement could overcome the problems:

- Quantum entanglement is independent of any distance between the subsystems. Therefore, the teleporting velocity $v > c$ (light speed) is not involved in quantum entanglement;
- Quantum teleportation and teleporting entanglement are not involved; they are the consequences of the entanglement and not the causes;
- Hidden variables of the EPR paradox (Einstein, Podolsky, and Rosen) are irrelevant to quantum entanglement;
- The spooky action at a distance proposed by Einstein is not needed.

6.4 The Hilbert Space

Square integrable functions create the Hilbert space $H(\mathbf{R}^\infty)$ that has three properties:

- Being an infinite-dimensional complex vector space $\mathbf{C}^\infty$;
- Having Hermitian positive definite scalar product of two complex vectors;
- Containing the Hilbert bases of the Hermite functions $\phi_n(x)$.

A complex function $f(x)$ of a real variable $x \in \mathbf{R}$ is defined as a *square integrable function* if it satisfies the condition

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$$

Furthermore, each square integrable function $f(x)$ can be written in a linear expression of the orthonormal Hermite functions $\phi_n(x)$ as

$$f(x) = \sum_{n=0}^{\infty} C_n \phi_n(x) \in H(\mathbf{R})$$

where C_n is the component of $f(x)$ on the Hermite function $\phi_n(x)$ that is calculated as

$$C_n = \langle \phi_n | f \rangle = \int_0^{\infty} \phi_n^*(x) f(x) dx$$

The orthonormal Hermite functions $\phi_n(x)$ are used for the bases of the Hilbert space $H(\mathbf{R})$. They are called the *Hilbert bases*. As a result, each quantum state vector in the Hilbert space can be expressed using the orthonormal Hilbert bases as

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n |\phi_n\rangle \equiv \sum_{n=0}^{\infty} C_n |n\rangle$$

Obviously, C_n is the component of the square integrable function $\psi(x)$ on the Hilbert basis $|n\rangle$ of the Hilbert space.

$$C_n = \langle \phi_n | \psi \rangle = \langle n | \psi \rangle$$

Due to orthonormality, the Hermitian scalar product of two arbitrary Hilbert bases can be written in Kronecker delta

$$\langle m | n \rangle = \langle n | m \rangle = \delta_{nm}$$

6.5 State Vectors and Basis Kets

State vector denotes the quantum state of the object that is a set of numbers of Qubit. There are two possibilities of a coin: head (H) and tail (T). Similar to the coin, each object has two basis states $|0\rangle$ and $|1\rangle$, which are called Qubits. Thus, the possible number of a state vector results as

$$N_S = 2^n \tag{6.1}$$

where n is the number of objects.

As an example, there are four possible combinations ($N_S = 2^2$) for two objects a and b of the combined state vector (ket) that is superimposed by two basis states.

$$|a\ b\rangle = |0\ 0\rangle; |0\ 1\rangle; |1\ 0\rangle; |1\ 1\rangle.$$

In quantum mechanics, the spins up and down $|u\rangle$ and $|d\rangle$ are used for the basis states according to L. Susskind [7]. As a result, the possible combination of the combined state vector $|a\ b\rangle$ is written as

$$|a\ b\rangle = |u\ u\rangle; |u\ d\rangle; |d\ u\rangle; |d\ d\rangle \tag{6.2}$$

In Chap. 1, we dealt with the eigenvalues and eigenvectors of the Hermitian operator. At the special eigenvalues of +1 and −1, their corresponding eigenvectors are chosen as the basis kets (called basis spin or eigenbasis) in quantum mechanics. The normalized eigenvectors are orthonormal in Cartesian coordinate system

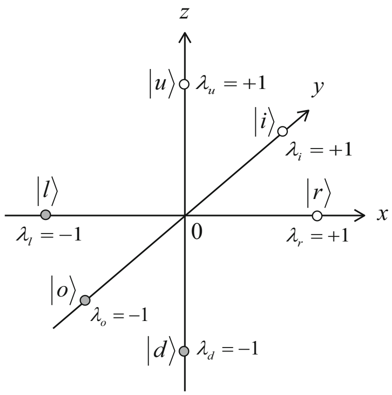

Fig. 6.3 Orthogonal basis kets (eigenkets) in quantum mechanics

(Euclidean space); therefore, the basis spins are also orthonormal in the Hilbert space. All possible basis spins (u: up, d: down, r: right; l: left; i: inner, and o: outer) are displayed in Fig. 6.3.

The state vector is defined as a linear superposition of the basis vectors. Using the bra-ket notation (cf. Chap. 1), the state ket of a single system (non-entangled system) can be written in the eigenkets of the spins up and down as

$$|\Psi\rangle = \sum_i \alpha_i |\lambda_i\rangle = \alpha_u |u\rangle + \alpha_d |d\rangle \tag{6.3}$$

where

α_i is the component of the state vector on the eigenvector;

$|\lambda_i\rangle$ is the orthonormal eigenvector (basis ket, eigenket) relating to its eigenvalue λ_i.

According to the Born rule, the probability of the measurement of the state vector in the eigenvector $|\lambda_i\rangle$ is the squared amplitude $|\alpha_i|^2$.

The component of the state vector (complex number) is defined as its projection on the basis ket according to Eq. (1.58).

$$\begin{aligned} &\alpha_i \equiv \langle\lambda_i|\Psi\rangle = \psi(\lambda_i) \in \mathbf{C} \\ &\Rightarrow |\alpha_i|^2 = |\psi(\lambda_i)|^2 \end{aligned} \tag{6.4}$$

At the measurement, the state vector could be collapsed onto a single term on any basis. In this case, the state vector becomes after collapsing on the basis ket $|\lambda_j\rangle$.

$$\begin{aligned} &|\Psi\rangle = \sum_i \alpha_i |\lambda_i\rangle \in H \\ &\Rightarrow \left|\Psi_{collapse}\right\rangle = \left|\Psi(\lambda_j)\right\rangle = \alpha_j \left|\lambda_j\right\rangle = \psi(\lambda_j)\left|\lambda_j\right\rangle \end{aligned} \tag{6.5}$$

Schrödinger's cat [6] in an imaginary box with a radioactive trigger is an example for the collapse of quantum state.

During the measurement at closing the box, the cat is neither alive nor dead but rather in a blend state of them (i.e. half alive and half dead in the sense of the statistical interpretation with a probability 50 % alive to 50 % dead). Therefore, his quantum state before opening the box is undeterministic in one half of the alive state $|\Psi_{alive}\rangle$ and one half of the dead state $|\Psi_{dead}\rangle$ since nobody knows whether the cat is alive or dead inside the box.

$$\left|\Psi_{before}\right\rangle = \frac{1}{\sqrt{2}}|\Psi_{alive}\rangle + \frac{1}{\sqrt{2}}|\Psi_{dead}\rangle.$$

At the measurement end, the cat has only a deterministic state either being alive or dead at the time of opening the box. As a result, the quantum state of the cat immediately collapses onto a single deterministic state either $|\Psi_{alive}\rangle$ or $|\Psi_{dead}\rangle$.

$$\left|\Psi_{collapse}\right\rangle = +1|\Psi_{alive}\rangle \;\; or + 1 \; |\Psi_{dead}\rangle.$$

According to the Born rule, the *relative probability* of the state vector in the basis ket $|\lambda_j\rangle$ is written as

$$\begin{aligned} P(\lambda_j) &= \left\|\langle\lambda_j|\Psi\rangle\right\|^2 = \langle\lambda_j|\Psi\rangle^*\langle\lambda_j|\Psi\rangle \\ &= \langle\Psi|\lambda_j\rangle\langle\lambda_j|\Psi\rangle = \psi^*(\lambda_j)\cdot\psi(\lambda_j) \\ &= \alpha_j^*\alpha_j = |\alpha_j|^2 = |\psi(\lambda_j)|^2 > 0 \end{aligned} \tag{6.6}$$

where the component $\psi^*(\lambda_j)$ is the **complex conjugate** of its component $\psi(\lambda_j)$.

The calculating rules of adjoints (also transpose conjugates) of entities are given in Table 6.1 (cf. Chap. 1). They are very useful to calculate the operators in quantum mechanics.

Let A be an *operator* (matrix or second-order tensor). Its *transpose* is written as

$$A = A_{ij} \Rightarrow A^T = A_{ji}$$

The *complex conjugate* of A is defined as

$$A^* = A_{ij}^*$$

The *transpose conjugate* of A is called the *adjoint* $A^\dagger$ that is calculated as

Table 6.1 Calculating rule of adjoints of entities

Calculating rules of adjoints (transpose conjugates)							
Entity	i	$\|\Psi\rangle$	A	AB	$A\|\Psi\rangle$	$\langle\Phi\|\Psi\rangle$	$\langle\Phi\|A\|\Psi\rangle$
Adjoint	$-i$	$\langle\Psi\|$	$A^\dagger$	$B^\dagger A^\dagger$	$\langle\Psi\|A^\dagger$	$\langle\Psi\|\Phi\rangle$	$\langle\Psi\|A^\dagger\|\Phi\rangle$

If the operator A is Hermitian, then $A^\dagger = A$

$$\begin{aligned}&\left(A^T\right)^* = \left(A_{ji}\right)^* = A_{ji}^* \equiv A^\dagger = A_{ij}^\dagger\\&\Leftrightarrow \left(A_{ji}^*\right)^* = A_{ji} = \left(A_{ij}^\dagger\right)^*\\&\Leftrightarrow A^T = \left(A^\dagger\right)^*\end{aligned}$$

If A is *hermitian*, then A is *self-adjoint*; i.e.,

$$A_{ij}^\dagger = A_{ij} \Leftrightarrow A^\dagger = A.$$

As an example, the probability of the spin up $|u\rangle$ in the state vector $|\Psi\rangle \equiv |r\rangle$, as shown in Eqs. (6.22a) and (6.22b), at the eigenvalue $\lambda_u = +1$ is calculated as (see Fig. 6.3)

$$\begin{aligned}P(\lambda_u) = \langle r|u\rangle \cdot \langle u|r\rangle &= \frac{1}{\sqrt{2}}(1 \quad 1)\begin{pmatrix}1\\0\end{pmatrix} \cdot (1 \quad 0)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\\&= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}\end{aligned}$$

The probability of the spin down $|d\rangle$ in the state vector $|\Psi\rangle \equiv |r\rangle$ at the eigenvalue $\lambda_d = -1$ is calculated as

$$\begin{aligned}P(\lambda_d) = \langle r|d\rangle \cdot \langle d|r\rangle &= \frac{1}{\sqrt{2}}(1 \quad 1)\begin{pmatrix}0\\1\end{pmatrix} \cdot (0 \quad 1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\\&= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}\end{aligned}$$

Thus,

$$\sum_i P(\lambda_i) = P(\lambda_u) + P(\lambda_d) = \frac{1}{2} + \frac{1}{2} = 1$$

Thus, the average measured value of the basis states $|u\rangle$ and $|d\rangle$ in the state vector $|\Psi\rangle \equiv |r\rangle$ results as (see Fig. 6.8)

$$\begin{aligned}\overline{P} \equiv \langle P\rangle = \sum_i P(\lambda_i)\lambda_i &= P(\lambda_u)\lambda_u + P(\lambda_d)\lambda_d\\&= \frac{1}{2}(1) + \frac{1}{2}(-1) = 0\end{aligned}$$

This result shows that the measured spins $|u\rangle$ and $|d\rangle$ have a chance of 50 % for each spin in the state vector $|r\rangle$ under consideration of quantum mechanics. However, the average value after many times of measurement is zero because the state vector $|r\rangle$ is perpendicular to both spins up and down.

The outer product (projection operator) of the basis ket $|\lambda_j\rangle$ is written as using Eq. (1.68).

$$\mathbf{P}_{\lambda_j} = |\lambda_j\rangle\langle\lambda_j| \Rightarrow \sum_j^N \mathbf{P}_{\lambda_j} = \mathbf{I} \quad \text{(identity operator)} \tag{6.7}$$

The *absolute probability* (*probability density*) of the state vector $|\Psi\rangle$ in the basis ket $|\lambda_j\rangle$ is defined as

$$\begin{aligned} p(\lambda_j) &= \frac{P(\lambda_j)}{\langle\Psi|\Psi\rangle} = \frac{\langle\Psi|\lambda_j\rangle\langle\lambda_j|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\langle\Psi|\mathbf{P}_{\lambda_j}|\Psi\rangle}{\langle\Psi|\Psi\rangle} \\ &\equiv \left\langle\Psi'|\mathbf{P}_{\lambda_j}|\Psi'\right\rangle \end{aligned} \tag{6.8}$$

The normalized state vector is defined as

$$\begin{aligned} \left|\Psi'\right\rangle &= \frac{|\Psi\rangle}{\sqrt{\langle\Psi|\Psi\rangle}} = \sum_i \left\langle\lambda_i|\Psi'\right\rangle|\lambda_i\rangle = \sum_i \psi'_i|\lambda_i\rangle \equiv \sum_i \alpha'_i|\lambda_i\rangle \\ &\Rightarrow \left\|\left|\Psi'\right\rangle\right\| = 1 \end{aligned} \tag{6.9}$$

Thus, the total absolute probability should be equal to 1 using Eq. (6.9).

$$\begin{aligned} \sum_j p(\lambda_j) &= \sum_j \left\langle\Psi'|\mathbf{P}_{\lambda_j}|\Psi'\right\rangle \\ &= \sum_j \left\langle\Psi'|\lambda_j\right\rangle\left\langle\lambda_j|\Psi'\right\rangle = \left\langle\Psi'\left|\sum_j |\lambda_j\rangle\langle\lambda_j|\right|\Psi'\right\rangle \\ &= \left\langle\Psi'|\mathbf{I}\Psi'\right\rangle = \left\langle\Psi'|\Psi'\right\rangle = \left\|\left|\Psi'\right\rangle\right\|^2 = 1 \quad (q.e.d.) \end{aligned} \tag{6.10a}$$

Proof

$$\mathbf{I}\left|\Psi'\right\rangle = \left|\Psi'\right\rangle \tag{6.10b}$$

The LHS of Eq. (6.10b) is written as

$$\begin{aligned} \mathbf{I}\left|\Psi'\right\rangle &= \sum_j |\lambda_j\rangle\langle\lambda_j|\left|\Psi'\right\rangle = \sum_j |\lambda_j\rangle\langle\lambda_j|\left(\sum_i \alpha'_i|\lambda_i\rangle\right) \\ &= \sum_{i,j} \alpha'_i\langle\lambda_j|\lambda_i\rangle|\lambda_j\rangle = \sum_{i,j} \alpha'_i\delta_{ij}|\lambda_j\rangle \\ &= \sum_i \alpha'_i|\lambda_i\rangle = \left|\Psi'\right\rangle \quad (q.e.d.) \end{aligned}$$

In continuum medium, the normalized state ket is written in the normalized wave function $\psi'(x)$ of a particle with the eigenket $|x\rangle$ as [6, 9]

$$\left|\Psi'\right\rangle = \int \left\langle x|\Psi'\right\rangle |x\rangle dx = \int \psi'(x)|x\rangle dx \tag{6.11}$$

where the normalized wave function of the particle is defined as

$$\psi'(x) = \left\langle x|\Psi'\right\rangle$$

Using Eq. (6.11), one obtains the same result of Eq. (6.10b) as

$$\begin{aligned}\mathbf{I}\left|\Psi'\right\rangle &= \left(\int |x\rangle\langle x|dx\right)\left|\Psi'\right\rangle = \int |x\rangle\left\langle x|\Psi'\right\rangle dx \\ &= \int \psi'(x)|x\rangle dx = \left|\Psi'\right\rangle\end{aligned}$$

According to the *Born rule* in quantum mechanics, the absolute probability (probability density) for finding the particle in position x at instant t is the squared amplitude of the particle normalized wave function.

$$p(x,t) = \|\psi'(x,t)\|^2$$

Generally, the absolute probability p_{ab} for finding the particle in a position between a and b at time t is written as (see Fig. 6.4)

$$p_{ab} = \int_a^b \|\psi'(x,t)\|^2 dx$$

According to the statistical interpretation, the integral over the entire space of the probability density function of the normalized wave function $\psi'(x)$ of the particle equals 1 at any time t.

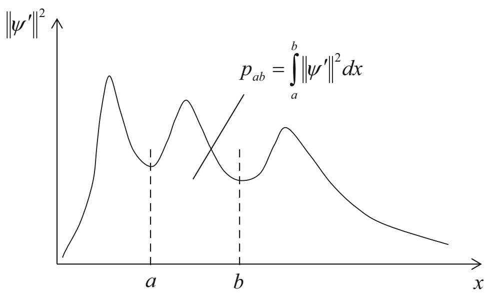

Fig. 6.4 Absolute probability of the particle occurring between a and b

$$\int_{-\infty}^{+\infty} p(x,t)dx = \int_{-\infty}^{+\infty} \|\psi'(x,t)\|^2 dx = 1.$$

The total absolute probability in a continuum medium is calculated as

$$\begin{aligned}\sum_j p\left(x_j,t\right) &\approx \int_{-\infty}^{+\infty} p(x,t)dx \\ &= \int_{-\infty}^{+\infty} \left\langle \Psi' | x \right\rangle \left\langle x | \Psi' \right\rangle dx = \int_{-\infty}^{+\infty} \left\langle \Psi' \middle| \mathbf{P}_x \middle| \Psi' \right\rangle dx \\ &= \left\langle \Psi' \middle| \int_{-\infty}^{+\infty} \mathbf{P}_x dx \middle| \Psi' \right\rangle = \left\langle \Psi' | \mathbf{I} \Psi' \right\rangle = \left\| \left| \Psi' \right\rangle \right\|^2 = 1 \quad (q.e.d.)\end{aligned}$$

In another way, one also obtains the total absolute probability

$$\begin{aligned}\int_{-\infty}^{+\infty} p(x,t)dx &= \int_{-\infty}^{+\infty} \psi'^*(x,t) \cdot \psi'(x,t)dx \\ &= \int_{-\infty}^{+\infty} \|\psi'(x,t)\|^2 dx = 1 \quad (q.e.d.)\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_j p\left(\lambda_j\right) = \int_{-\infty}^{+\infty} p(x,t)dx = 1 \\ &\Rightarrow \int_{-\infty}^{+\infty} p(\mathbf{r},t)d^3\mathbf{r} = 1\end{aligned} \tag{6.12}$$

The inner product of bra and ket in Eq. (1.60a) can be written as

$$\langle \Phi | \Psi \rangle = \int \langle \Phi | x \rangle \langle x | \Psi \rangle dx = \int \phi^*(x) \cdot \psi(x)dx \in \mathbf{R} \tag{6.13}$$

in which

$$\begin{aligned}\phi^*(x) &= \langle x|\Phi\rangle^* = \langle\Phi|x\rangle \\ &= \int \langle\Phi|\xi\rangle\langle\xi|x\rangle d\xi = \int \phi^*(\xi)\,\langle\xi|x\rangle d\xi; \\ \psi(x) &\equiv \langle x|\Psi\rangle \\ &= \int \langle x|\xi\rangle\langle\xi|\Psi\rangle d\xi = \int \langle x|\xi\rangle\,\psi(\xi) d\xi.\end{aligned}$$

Using Eq. (6.13), the norm of the state ket is calculated in a continuum medium as

$$\||\Psi\rangle\| = \sqrt{\langle\Psi|\Psi\rangle} = \sqrt{\int \psi^*(x)\cdot\psi(x)dx} \tag{6.14}$$

The *Dirac delta function* is defined as (see Fig. 6.5)

$$\delta(x-x_0) = \langle x|x_0\rangle = \begin{cases} 0 \;\; for\; x \neq x_0 \\ \infty \;\; for\; x = x_0 \end{cases} \tag{6.15}$$

The closure or completeness relation of the Dirac delta function is written as

$$\begin{aligned}\delta(x-x_0) = \langle x|x_0\rangle &= \sum_j \langle x|\Psi_j\rangle\langle\Psi_j|x_0\rangle \\ &= \sum_j \psi_j(x)\cdot\psi_j^*(x_0)\end{aligned}$$

The Dirac delta function can be written in the Gaussian function as

$$\delta(x-x_0) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon\sqrt{\pi}}\exp\left(\frac{-(x-x_0)^2}{\varepsilon^2}\right)$$

Using Eq. (6.15) and $\delta(x-x_0)=0$ for any $x \neq x_0$, the position wave function of the particle at the position x_0 is calculated as

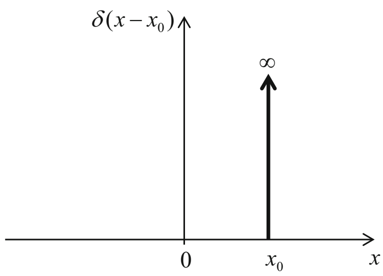

Fig. 6.5 Dirac delta function

$$\psi(x_0) = \langle x_0|\Psi\rangle = \langle x_0|x\rangle\langle x|\Psi\rangle = \int_{-\infty}^{+\infty} \langle x_0|x\rangle \cdot \psi(x)dx$$
$$= \int_{-\infty}^{+\infty} \delta^*(x - x_0)\psi(x)dx = \psi(x_0) \int_{-\infty}^{+\infty} \delta(x - x_0)dx$$

This result denotes that the integral of the Dirac delta function over dx must be equal to 1.

$$\int_{-\infty}^{+\infty} \delta(x - x_0)dx = 1 \tag{6.16}$$

Generally, the position wave function of a particle at any position x can be written as

$$\psi(x) = \langle x|\Psi\rangle = \langle x|\xi\rangle\langle\xi|\Psi\rangle = \int_{-\infty}^{+\infty} \langle x|\xi\rangle \cdot \psi(\xi)d\xi$$
$$= \int_{-\infty}^{+\infty} \delta(x - \xi)\psi(\xi)d\xi$$

Therefore, any function $f(x)$ can be written using the Dirac delta function as

$$f(x) = \int_{-\infty}^{+\infty} \langle x|\xi\rangle f(\xi)d\xi = \int_{-\infty}^{+\infty} \delta(x - \xi)f(\xi)d\xi$$

The Dirac and wave formalisms in quantum mechanics are shown in Table 6.2.

Table 6.2 Dirac and wave formalisms

Dirac Formalism $\Longleftrightarrow$	Wave formalism	Notation
$\|\psi\rangle$	$\psi(\mathbf{r})$	State vector ↔ Wave function
$\|\psi(t)\rangle$	$\psi(\boldsymbol{r}, t)$	Ket of state vector
$\langle\psi(t)\|$	$\psi^*(\mathbf{r}, t)$	Bra of state vector
$\langle\psi_1\|\psi_2\rangle$	$\int\psi_1^*(\mathbf{r})\psi_2(\mathbf{r})d^3\mathbf{r}$	Hermitian scalar product
$\|\| \psi\|\|^2 = \langle\psi\|\psi\rangle$	$\int\|\psi(\mathbf{r})\|^2 d^3\mathbf{r}$	Squared norm of ψ(r)
$\langle\psi_1\|\mathbf{L}\|\psi_2\rangle$	$\int\psi_1^*(\mathbf{r})\mathbf{L}\psi_2(r)d^3\mathbf{r}$	Linear operator **L**
$\overline{\mathbf{L}}=\langle\mathbf{L}\rangle = \langle\psi\|\mathbf{L}\|\psi\rangle$	$\int\psi^*(\mathbf{r})\mathbf{L}\psi(\mathbf{r})d^3\mathbf{r}$	Expectation value of **L**

6.6 The Pauli Matrices

The Pauli matrices are spin operators that are used to calculate the Hamiltonian, the possible energy of the system.

Let σ be the spin operator of the subsystem A; τ be the spin operator of the subsystem B; ω be the angular frequency. The Hamiltonian of the composite system of A and B is written as [7]

$$\mathbf{H} = \frac{\omega}{2}(\sigma \cdot \tau) = \frac{\omega}{2}\left(\sigma_x\tau_x + \sigma_y\tau_y + \sigma_z\tau_z\right) \tag{6.17}$$

In the following section, the Pauli matrices are derived in a Hilbert space.

The spin operator σ_z in the coordinate z is a (2×2) Pauli matrix with the eigenvectors of spins $|u\rangle$ and $|d\rangle$ at the eigenvalues $\lambda_u = +1$ and $\lambda_d = -1$, respectively (see Fig. 6.3).

The orthonormal eigenvectors in the direction z are defined as

$$\begin{aligned} &|u\rangle \equiv |z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |d\rangle \equiv |z-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\Rightarrow \langle u|d\rangle = \langle z + |z-\rangle = 0; \quad \langle d|u\rangle = \langle z - |z+\rangle = 0 \end{aligned}$$

The equation of the spin operator σ_z is written as

$$\begin{aligned} \sigma_z\,|u\rangle = +1\,|u\rangle \Leftrightarrow \begin{pmatrix} \sigma_{z,11} & \sigma_{z,12} \\ \sigma_{z,21} & \sigma_{z,22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \\ \sigma_z\,|d\rangle = -1\,|d\rangle \Leftrightarrow \begin{pmatrix} \sigma_{z,11} & \sigma_{z,12} \\ \sigma_{z,21} & \sigma_{z,22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{6.18}$$

Solving Eq. (6.18), one obtains the spin operator σ_z and τ_z.

$$\sigma_z \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \tau_z \equiv \tau_3 \tag{6.19a}$$

Analogously, the other spin operators result as

$$\begin{aligned} \sigma_x \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tau_x \equiv \tau_1; \\ \sigma_y \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \tau_y \equiv \tau_2 \end{aligned} \tag{6.19b}$$

The Pauli matrices have the following properties that are easily proved:

$$\begin{array}{l} \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbf{I}: \quad \text{unitary matrix} \\ \sigma_x\sigma_y + \sigma_y\sigma_x = 0; \quad \sigma_y\sigma_z + \sigma_z\sigma_y = 0; \quad \sigma_z\sigma_x + \sigma_x\sigma_z = 0 \\ \sigma_x\sigma_y = i\sigma_z; \quad \sigma_y\sigma_z = i\sigma_x; \quad \sigma_z\sigma_x = i\sigma_y \\ \left[\sigma_x, \sigma_y\right] = 2i\sigma_z; \quad \left[\sigma_y, \sigma_z\right] = 2i\sigma_x; \quad \left[\sigma_z, \sigma_x\right] = 2i\sigma_y \end{array}$$

Equations (6.19a) and (6.19b) delivers the terms that result from the spin operators act on the basis kets of $|u\rangle$ and $|d\rangle$.

$$\begin{array}{l} \sigma_z|u\rangle = |u\rangle; \quad \sigma_z|d\rangle = -|d\rangle \\ \sigma_x|u\rangle = |d\rangle; \quad \sigma_x|d\rangle = |u\rangle \\ \sigma_y|u\rangle = i|d\rangle; \quad \sigma_y|d\rangle = -i|u\rangle \end{array} \tag{6.20}$$

6.7 Combined State Vectors

To study quantum entanglement between two entangled subsystems, it is necessary to define a composite system (whole system) that involves in the subsystems. In the composite system, the combined state vectors of two state vectors of the particles are used. Similarly, the basis vectors of the composite system are the combined basis vectors of the subsystems. The following symbols are used by L. Susskind [7].

Let $|a\rangle$, $|b\rangle$ be the basis vectors of the subsystems S_A and S_B, respectively. Both subsystems S_A and S_B belong to the Hilbert spaces H_A and H_B. It is impossible to describe the entangled state vector of the composite system in a pure basis state either of the subsystem S_A or the subsystem S_B. In this case, the outcome of the measurement is random with each possibility having a probability of 50 % of both collapsed states of the composite system. Particles with integer spins of $\hbar$, $2\hbar$, $3\hbar$, and so on that are called bosons (e.g. photons) comply with the Bose-Einstein statistics, which involves counting particles, not counting wave frequencies. On the contrary, the Fermi-Dirac statistics, which describes the energy distribution of identical particles with half-integer spins of $(1/2)\hbar$, $(3/2)\hbar$, $(5/2)\hbar$, and so on in a system at thermodynamic equilibrium is generally applied to quantum mechanics. Such particles are called fermions, e.g. electrons, protons, and neutrons that obey the Fermi-Dirac statistics.

Using the Fermi-Dirac statistics, one of the entangled normalized states of the composite system can be expressed in the superimposed collapsed states using the basis vectors of each subsystem.

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|d\rangle_A \otimes |u\rangle_B - |u\rangle_A \otimes |d\rangle_B\right) \in H_A \times H_B$$

This entangled state vector of the composite system indicates that

– if the measured state of S_A is $|d\rangle_A$, the entangled state collapses onto $|d\rangle_A \otimes |u\rangle_B$;

– if the measured state of S_A is $|u\rangle_A$, the entangled state collapses onto $|u\rangle_A \otimes |d\rangle_B$.

The combined basis vector in the composite system is defined as the tensor product of two basis vectors of its subsystems.

$$|a\,b\rangle \equiv |a\rangle \otimes |b\rangle \in H_A \times H_B \tag{6.21}$$

Figure 6.3 shows the orthonormal bases in a Hilbert space H. They are used for the orthonormal basis kets (eigenkets) in quantum mechanics.

– Basis kets up and down:

$$\begin{aligned} |u\rangle &\equiv |z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \\ |d\rangle &\equiv |z-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{6.22a}$$

– Basis kets right and left:

$$\begin{aligned} |r\rangle &\equiv |x+\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}; \\ |l\rangle &\equiv |x-\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned} \tag{6.22b}$$

– Basis kets inner and outer:

$$\begin{aligned} |i\rangle &\equiv |y+\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}; \\ |o\rangle &\equiv |y-\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|d\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \end{aligned} \tag{6.22c}$$

The combined state vector of two single state kets of $|a\rangle$ and $|b\rangle$ is calculated as

$$\begin{aligned} |a\,b\rangle &= |a\rangle \otimes |b\rangle \\ &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}; \quad \forall a_i, b_j \in \mathbf{C} \end{aligned} \tag{6.23}$$

The combined state vector has four components ($N^2 = 2^2$) in a two-dimensional Hilbert space ($N = 2$) with two indices i and j, cf. Eq. (2.60).

Using Eqs. (6.22a) and (6.23), the combined basis kets of $|u\rangle$ and $|d\rangle$ in the Hilbert space H result as

$$|u\,u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix};\ |u\,d\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix};\ |d\,u\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix};\ |d\,d\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.24}$$

Generally, the entangled state vector can be expressed in the combined basis kets as

$$\begin{aligned} |a\,b\rangle &= \begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_2b_1 \\ a_2b_2 \end{pmatrix} = a_1b_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_1b_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a_2b_1\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + a_2b_2\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= a_1b_1|u\,u\rangle + a_1b_2|u\,d\rangle + a_2b_1|d\,u\rangle + a_2b_2|d\,d\rangle \\ &= \sum_{i=1}^{2}\sum_{j=1}^{2} a_ib_j|e_i\,e_j\rangle \end{aligned} \tag{6.25}$$

Furthermore, let A and B be two operators of second-order tensors in a two-dimensional Hilbert space. The tensor product of two tensors is calculated as (cf. Chap. 2).

$$\begin{aligned} A \otimes B &= A_{ij}\mathbf{g}^i\mathbf{g}^j \otimes B_{kl}\mathbf{g}^k\mathbf{g}^l \\ &= \left(A_{ij}B_{kl}\right)\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^l \end{aligned} \tag{6.26}$$

Using Eq. (6.26), the tensor product $A \otimes B$ results as a fourth-order tensor with 16 tensor elements ($N^4 = 2^4$) in a two-dimensional Hilbert space ($N = 2$) with four indices i, j, k, and l.

$$\begin{aligned} A \otimes B &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in H_A \times H_B \\ &= \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix} \end{aligned} \tag{6.27}$$

Furthermore, the tensor product of two operators that act on the state vectors is calculated using Eq. (6.27) as

$$A|a\rangle \otimes B|b\rangle = (A \otimes B)|a\,b\rangle \tag{6.28}$$

Proof Using Eq. (6.23), the RHS of Eq. (6.28) results as

$$(A \otimes B)|a\,b\rangle = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix} \begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_2b_1 \\ a_2b_2 \end{pmatrix}$$
$$= \begin{pmatrix} A_{11}B_{11}a_1b_1 + A_{11}B_{12}a_1b_2 + A_{12}B_{11}a_2b_1 + A_{12}B_{12}a_2b_2 \\ A_{11}B_{21}a_1b_1 + A_{11}B_{22}a_1b_2 + A_{12}B_{21}a_2b_1 + A_{12}B_{22}a_2b_2 \\ A_{21}B_{11}a_1b_1 + A_{21}B_{12}a_1b_2 + A_{22}B_{11}a_2b_1 + A_{22}B_{12}a_2b_2 \\ A_{21}B_{21}a_1b_1 + A_{21}B_{22}a_1b_2 + A_{22}B_{21}a_2b_1 + A_{22}B_{22}a_2b_2 \end{pmatrix}$$

The single terms on the LHS of Eq. (6.28) are calculated as

$$A|a\rangle = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} A_{11}a_1 + A_{12}a_2 \\ A_{21}a_1 + A_{22}a_2 \end{pmatrix};$$
$$B|b\rangle = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} B_{11}b_1 + B_{12}b_2 \\ B_{21}b_1 + B_{22}b_2 \end{pmatrix}$$

Using Eq. (6.23), the tensor product of two above terms on the LHS results as

$$A|a\rangle \otimes B|b\rangle = \begin{pmatrix} (A_{11}a_1 + A_{12}a_2) \cdot (B_{11}b_1 + B_{12}b_2) \\ (A_{11}a_1 + A_{12}a_2) \cdot (B_{21}b_1 + B_{22}b_2) \\ (A_{21}a_1 + A_{22}a_2) \cdot (B_{11}b_1 + B_{12}b_2) \\ (A_{21}a_1 + A_{22}a_2) \cdot (B_{21}b_1 + B_{22}b_2) \end{pmatrix}$$

Having compared the RHS to the LHS of Eq. (6.28), one obtains

$$A|a\rangle \otimes B|b\rangle = (A \otimes B)|a\,b\rangle \quad (q.e.d.)$$

Let σ_z be an observable (= measured quantity) operator in the subsystem S_A in which $|d\rangle$ is a state vector. In the subsystem S_B, $|u\rangle$ is a state vector. If the subsystem S_B does not affect the state vector $|d\rangle$ of the subsystem S_A, the combined entangled state vector of the composite system S_{AB} is derived in the following section, as shown in Fig. 6.6.

The observable operator σ_z acts on the state vector $|d\rangle$ in the subsystem S_A; therefore, the measured result of the entangled state vector $|d\rangle$ in the composite system S_{AB} is calculated using Eqs. (6.20) and (6.28) as

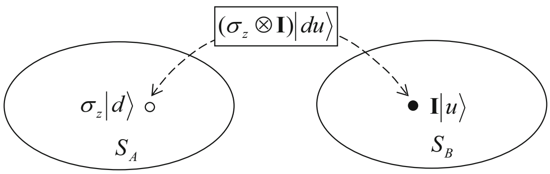

Fig. 6.6 Entangled state vector without effect of S_B in a composite system

$$\begin{aligned}(\sigma_z \otimes \mathbf{I})|d\ u\rangle &= \sigma_z|d\rangle \otimes \mathbf{I}|u\rangle \\ &= -|d\rangle \otimes |u\rangle = -|d\ u\rangle\end{aligned} \tag{6.29}$$

where $\mathbf{I}$ is the identity operator.

At first, the observable operator that acts on the combined basis state is calculated using Eqs. (6.19a) and (6.27) as

$$\sigma_z \otimes \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Using Eq. (6.24), one obtains

$$(\sigma_z \otimes \mathbf{I})|d\ u\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = -|d\ u\rangle$$

Similarly, one calculates the following terms:

$$\begin{aligned}(\sigma_x \otimes \mathbf{I})|u\ u\rangle = |d\ u\rangle; \quad (\sigma_x \otimes \mathbf{I})|u\ d\rangle = |d\ d\rangle; \\ (\sigma_x \otimes \mathbf{I})|d\ u\rangle = |u\ u\rangle; \quad (\sigma_x \otimes \mathbf{I})|d\ d\rangle = |u\ d\rangle\end{aligned} \tag{6.30a}$$

$$\begin{aligned}(\sigma_y \otimes \mathbf{I})|u\ u\rangle = i|d\ u\rangle; \quad (\sigma_y \otimes \mathbf{I})|u\ d\rangle = i|d\ d\rangle; \\ (\sigma_y \otimes \mathbf{I})|d\ u\rangle = -i|u\ u\rangle; \quad (\sigma_y \otimes \mathbf{I})|d\ d\rangle = -i|u\ d\rangle\end{aligned} \tag{6.30b}$$

$$\begin{aligned}(\sigma_z \otimes \mathbf{I})|u\ u\rangle = |u\ u\rangle; \quad (\sigma_z \otimes \mathbf{I})|u\ d\rangle = |u\ d\rangle; \\ (\sigma_z \otimes \mathbf{I})|d\ u\rangle = -|d\ u\rangle; \quad (\sigma_z \otimes \mathbf{I})|d\ d\rangle = -|d\ d\rangle\end{aligned} \tag{6.30c}$$

The observable operators σ_z and τ_x act on the state vectors $|d\rangle$ and $|u\rangle$ of the subsystems, respectively (see Fig. 6.7).

Using Eq. (6.28), the combined entangled state vector in the composite system S_{AB} is calculated as

$$\begin{aligned}\sigma_z|d\rangle \otimes \tau_x|u\rangle &= (\sigma_z \otimes \tau_x)(|d\rangle \otimes |u\rangle) \\ &= (\sigma_z \otimes \tau_x)|d\ u\rangle = -|d\ d\rangle\end{aligned} \tag{6.31}$$

Proof According Eq. (6.20), the LHS of Eq. (6.31) is written as

$$\sigma_z|d\rangle \otimes \tau_x|u\rangle = -|d\rangle \otimes |d\rangle = -|d\ d\rangle \quad (q.e.d.)$$

Using Eqs. (6.19a), (6.19b) and (6.27), the observable operator that acts on the combined basis state is calculated as

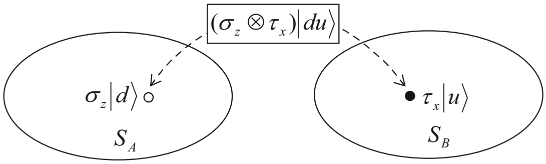

Fig. 6.7 Combined entangled state vector in a composite system

$$\sigma_z \otimes \tau_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Using Eq. (6.24), one obtains

$$(\sigma_z \otimes \tau_x)|d\,u\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -|d\,d\rangle$$

A general combined quantum state vector in the composite space is expressed as

$$\begin{aligned} |\Psi\rangle &= |\Psi_A\rangle \otimes |\Psi_B\rangle \in H_A \times H_B \\ &= \left(\psi_{A,u}|u\rangle + \psi_{A,d}|d\rangle\right) \otimes \left(\psi_{B,u}|u\rangle + \psi_{B,d}|d\rangle\right) \\ &= \left(\psi_{A,u}\psi_{B,u}\right)|u\rangle \otimes |u\rangle + \left(\psi_{A,u}\psi_{B,d}\right)|u\rangle \otimes |d\rangle \\ &\quad + \left(\psi_{A,d}\psi_{B,u}\right)|d\rangle \otimes |u\rangle + \left(\psi_{A,d}\psi_{B,d}\right)|d\rangle \otimes |d\rangle \\ &\equiv \psi_{uu}|u\,u\rangle + \psi_{ud}|u\,d\rangle + \psi_{du}|d\,u\rangle + \psi_{dd}|d\,d\rangle \end{aligned} \tag{6.32}$$

Due to normalized state vector, the relation between the state components is written as

$$\psi^*_{uu} \cdot \psi_{uu} + \psi^*_{ud} \cdot \psi_{ud} + \psi^*_{du} \cdot \psi_{du} + \psi^*_{dd} \cdot \psi_{dd} = 1 \tag{6.33}$$

6.8 Expectation Value of an Observable

Quantities, such as position, momentum, spin, polarization, energy, and entropy that can be measured by an apparatus, are called *observables*. The observables of entangled particles are correlated with each other in the whole system. An action on any particle has strong influences on the other particles and vice versa in the composite system. Quantum mechanics is intrinsically random; therefore, the expectation value is used for measurement of observables in the statistical quantum world.

Let **L** be an observable operator that has the eigenvalues λ_i for $i = 1,\ldots,N$ and their corresponding eigenkets $|\lambda_i\rangle$.

The expectation value (also average value) of the observable **L** in a normalized state vector $|\Psi\rangle$ at the measurement is defined as (see Fig. 6.8)

$$\begin{aligned}\langle \mathbf{L} \rangle \equiv \overline{\mathbf{L}} = \langle \Psi | \mathbf{L} | \Psi \rangle &= \int_{-\infty}^{+\infty} \psi^*(x,t) \cdot \mathbf{L} \cdot \psi(x,t) dx \\ &= \int_{-\infty}^{+\infty} \mathbf{L} \cdot \|\psi(x,t)\|^2 dx\end{aligned} \tag{6.34}$$

where the probability density of the normalized state vector $|\Psi\rangle$ satisfies

$$\int_{-\infty}^{+\infty} \|\psi(x,t)\|^2 dx = 1.$$

Note that the expectation value of an observable **L** is the average value of repeated measurements on a set of many identically prepared systems. In general, the more the repeated measurements are, the better the expectation value is achieved.

Some examples of expectation value of an observable are given here.

– The expectation value of the particle position x is written as

$$\langle x \rangle = \overline{x} = \int_{-\infty}^{+\infty} x \cdot \|\psi(x,t)\|^2 dx$$

– The expectation value of the particle momentum p is written as

$$\langle p \rangle = \overline{p} = \int_{-\infty}^{+\infty} p \cdot \|\psi(x,t)\|^2 dx$$

Trace of the observable operator **L** is defined as the sum of all diagonal elements of the operator.

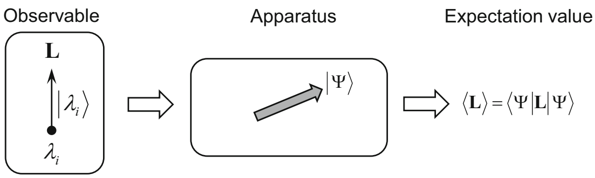

Fig. 6.8 Expectation value of an observable **L** in a state $|\Psi\rangle$

$$Tr\mathbf{L} = \sum_i \langle i|\mathbf{L}|i\rangle = \sum_i L_{ii} \tag{6.35}$$

According to Eq. (6.35), trace of the outer product of the state bra and ket and the operator $\mathbf{L}$ is written as

$$\begin{aligned} Tr[|\Psi\rangle\langle\Psi|\mathbf{L}] &= \sum_i \langle i||\Psi\rangle\langle\Psi|\mathbf{L}|i\rangle = \sum_i \langle\Psi|\mathbf{L}|i\rangle\langle i|\Psi\rangle \\ &= \sum_i \langle\Psi|\mathbf{L}I_i|\Psi\rangle = \langle\Psi|\mathbf{L}\mathbf{I}|\Psi\rangle \\ &= \langle\Psi|\mathbf{L}|\Psi\rangle = \langle\mathbf{L}\rangle = \overline{\mathbf{L}} \end{aligned} \tag{6.36}$$

Equation (6.36) describes the expectation value or average value of the outcome of the observable $\mathbf{L}$ in a state vector $|\Psi\rangle$ at the measurement.

Let $|\Psi\rangle$ be a state vector in a Hilbert space H. It can be written as

$$|\Psi\rangle = \sum_i \alpha_i|\lambda_i\rangle$$

Multiplying both sides by the observable $\mathbf{L}$, one obtains

$$\mathbf{L}|\Psi\rangle = \sum_i \alpha_i\mathbf{L}|\lambda_i\rangle$$

Using the characteristic equation of the observable $\mathbf{L}$ at the eigenvalue λ_i, one obtains

$$\mathbf{L}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$$

The expectation value of $\mathbf{L}$ of the state vector $|\Psi\rangle$ at the measurement results as

$$\begin{aligned} \langle\mathbf{L}\rangle = \langle\Psi|\mathbf{L}|\Psi\rangle &= \int_{-\infty}^{+\infty} \mathbf{L}\cdot\|\psi(x,t)\|^2 dx \\ &= \langle\Psi|\sum_i \alpha_i\mathbf{L}|\lambda_i\rangle = \sum_j \alpha_j^*\langle\lambda_j|\sum_i \alpha_i\mathbf{L}|\lambda_i\rangle \\ &= \sum_{i,j} \alpha_j^*\alpha_i\lambda_i\langle\lambda_j|\lambda_i\rangle = \sum_{i,j} \alpha_j^*\alpha_i\lambda_i\delta_{ji} \\ &= \sum_i \alpha_i^*\alpha_i\lambda_i = \sum_i \|\psi_i(x,t)\|^2\lambda_i \end{aligned} \tag{6.37}$$

where $\psi(x,t)$ is a wave function of which amplitude is calculated as

$$\|\psi(x,t)\| = \sqrt{\int \psi^*(x,t)\cdot\psi(x,t)dx}$$

in which the complex conjugate of $\psi(x,t)$ is defined as $\psi^*(x,t)$.

Substituting the relative probability $P(\lambda_i)$ in Eq. (6.6) into Eq. (6.37), one obtains

$$\langle \mathbf{L} \rangle = \sum_i \alpha_i^* \alpha_i \lambda_i = \sum_i P(\lambda_i)\lambda_i \tag{6.38}$$

Finally, the expectation value of $\mathbf{L}$ in a state vector $|\Psi\rangle$ is summarized as

$$\langle \mathbf{L} \rangle = \overline{\mathbf{L}} = \langle \Psi|\mathbf{L}|\Psi\rangle = Tr|\Psi\rangle\langle\Psi|\mathbf{L} = \sum_i P(\lambda_i)\lambda_i \tag{6.39}$$

6.9 Probability Density Operator

The probability density operator associates with an observable to generate the expectation value (average value) of the observable in a state vector.

6.9.1 Density Operator of a Pure Subsystem

Let $|\Psi\rangle$ be a state vector in a pure subsystem S_a in a Hilbert space H_A. The state vector is written in the basis ket $|a_n\rangle$ as

$$|\Psi\rangle = \sum_n \alpha_n |a_n\rangle = \int_{-\infty}^{+\infty} |x\rangle\langle x| \cdot |\Psi\rangle dx = \int_{-\infty}^{+\infty} |x\rangle\ \psi(x)dx \tag{6.40}$$

The density operator ρ is defined as the outer product of ket and bra of the state vector $|\Psi\rangle$.

$$\rho \equiv |\Psi\rangle\langle\Psi| \tag{6.41}$$

Substituting Eq. (6.40) into Eq. (6.41), the density operator is rewritten as

$$\begin{aligned} \rho \equiv |\Psi\rangle\langle\Psi| &= \sum_n \alpha_n^* \alpha_n |a_n\rangle\langle a_n| \\ &= \sum_n P\left(a_n\right)|a_n\rangle\langle a_n| \end{aligned} \tag{6.42}$$

in which $P(a_n)$ is the relative probability of the state vector in the basis ket is defined in Eq. (6.6).

$$P(a_n) = \alpha_n^* \alpha_n$$

Using Eqs. (6.39) and (6.41), the expectation value of an observable **L** in the state vector $|\Psi\rangle$ is calculated as

$$\langle \mathbf{L} \rangle = \overline{\mathbf{L}} = Tr|\Psi\rangle\langle\Psi|\mathbf{L} = Tr(\rho\mathbf{L}) \tag{6.43}$$

Therefore, the density operator is very important for calculating the expectation value of an observable. For this reason, the density operator is dealt with in the following section.

Due to orthogonality of the basis kets, the elements of the density operator are calculated using Eq. (1.78) as

$$\begin{aligned} \rho_{ij} &= \langle a_i|\rho|a_j\rangle = \langle a_i|\Psi\rangle\langle\Psi|a_j\rangle \\ &= \psi(a_i)\cdot\psi^*(a_j) = \alpha_i\alpha_j^* = \alpha_j^*\alpha_i \in \mathbf{R} \end{aligned} \tag{6.44}$$

An example is given here to show how to calculate the density operator.

Let $|\Psi\rangle$ be a normalized state vector that is written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \||\Psi\rangle\| = 1$$

The orthonormal basis kets in the direction z result from Eq. (6.22a) as

$$|u\rangle \equiv |z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |d\rangle \equiv |z-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Using Eq. (6.44), the elements of the density operator are calculated as

$$\begin{aligned} \rho_{uu} &= \langle u|\rho|u\rangle = \langle u|\Psi\rangle\langle\Psi|u\rangle \\ &= (1 \quad 0)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cdot\frac{1}{\sqrt{2}}(1 \quad -1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}(1+0)\cdot\frac{1}{\sqrt{2}}(1+0) = \frac{1}{2} \end{aligned}$$

Similarly, one obtains

$$\rho_{ud} = \langle u|\rho|d\rangle = \langle u|\Psi\rangle\langle\Psi|d\rangle = -\frac{1}{2};$$
$$\rho_{du} = \langle d|\rho|u\rangle = \langle d|\Psi\rangle\langle\Psi|u\rangle = -\frac{1}{2};$$
$$\rho_{dd} = \langle d|\rho|d\rangle = \langle d|\Psi\rangle\langle\Psi|d\rangle = \frac{1}{2}$$

The density operator ρ in the pure subsystem results as

$$\rho = \begin{pmatrix} \rho_{uu} & \rho_{ud} \\ \rho_{du} & \rho_{dd} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}.$$

6.9.2 Density Operator of an Entangled Composite System

Quantum entanglement occurs in a composite system S_{AB} that consists of the entangled subsystems S_A and S_B. The calculation of the density operator of an entangled state vector of a composite system is quite complicated. However, we have it done step by step.

The entangled normalized state vector of a composite system S_{AB} is defined as

$$\begin{aligned} |\Psi_{ab}\rangle &= \sum_{ab} \psi(ab)|a\rangle \otimes |b\rangle \\ &= \sum_{ab} \psi(ab)|a\,b\rangle \in S_A \otimes S_B \end{aligned} \tag{6.45}$$

where

$|a\,b\rangle$ is the basis kets in the composite system;

$\psi(ab)$ is the wave function of the state vector in the composite system.

The expectation value of an observable $\mathbf{L}$ in the entangled state vector $|\Psi_{ab}\rangle$ in the composite system is calculated using Eq. (6.39) as

$$\begin{aligned} \langle\mathbf{L}\rangle_{S_{AB}} = \overline{\mathbf{L}}_{S_{AB}} &= \left\langle\Psi_{a'b'}\middle|\mathbf{L}\middle|\Psi_{ab}\right\rangle \\ &= \sum_{a'b',ab} \psi^*\left(a'b'\right)\left\langle a'b'\middle|\mathbf{L}\middle|ab\right\rangle\psi(ab) \\ &\equiv \sum_{a'b',ab} \psi^*\left(a'b'\right)\cdot L_{a'b',ab}\cdot\psi(ab) \end{aligned} \tag{6.46}$$

where $\langle a'b'|$ is the basis bra in the composite system that is defined as

$$\left\langle a'\,b'\right| \equiv \left\langle a'\right| \otimes \left\langle b'\right|$$

In the following section, the observable $\mathbf{L}$ only acts on the subsystem S_A; and the subsystem S_B does not affect the observable $\mathbf{L}$. In this case, the basis bra $\langle b'|$ is identical to $\langle b|$. Therefore, the expectation value of $\mathbf{L}$ in the entangled normalized state vector $|\Psi_{ab}\rangle$ becomes

$$\begin{aligned}\langle \mathbf{L}\rangle_{S_A} &= \overline{\mathbf{L}}_{S_A} = \langle \Psi_{a'b}|\mathbf{L}|\Psi_{ab}\rangle \\ &= \sum_{a',ab} \psi^*\left(a'b\right)\cdot L_{a',a}\cdot\psi(ab)\end{aligned} \tag{6.47}$$

The probability of the entangled state vector $|\Psi_{ab}\rangle$ in the composite system S_{AB} is calculate as

$$\begin{aligned}&P(ab) = \psi^*(ab)\cdot\psi(ab) \\ &\Rightarrow \sum_{ab} P(ab) = \sum_{ab}\psi^*(ab)\cdot\psi(ab) = 1\end{aligned} \tag{6.48}$$

The density operator ρ in the composite system S_{AB} is defined as

$$\rho_{S_{AB}} \equiv |\Psi_{ab}\rangle\langle\Psi_{a'b'}| \tag{6.49}$$

Similar to Eq. (6.44), the elements of the density operator of the composite system are calculated using Eq. (6.49) as [7]

$$\begin{aligned}\rho_{a'b',ab} &= \sum_{a'b',ab}\langle ab|\rho_{S_{AB}}|a'b'\rangle = \sum_{a'b',ab}\langle ab||\Psi_{ab}\rangle\langle\Psi_{a'b'}||a'b'\rangle \\ &= \sum_{a'b',ab}\langle ab|\Psi_{ab}\rangle\langle\Psi_{a'b'}|a'b'\rangle = \sum_{a'b',ab}\psi^*\left(a'b'\right)\cdot\psi(ab)\end{aligned} \tag{6.50}$$

If only the observable $\mathbf{L}$ acts on the subsystem S_A, the elements of the *reduced density operator* of the one-half mixed state of S_A results from Eq. (6.50) as

$$\begin{aligned}&\rho_{S_A} \equiv |\Psi_{ak}\rangle\langle\Psi_{a'k}| \Rightarrow \\ &\rho_{a'a} = \sum_k \langle ak|\Psi_{ak}\rangle\langle\Psi_{a'k}|a'k\rangle = \sum_k \psi^*\left(a'k\right)\cdot\psi(ak)\end{aligned} \tag{6.51}$$

in which k is the integrating variable that is defined as

$$\begin{aligned}&\langle k| = \langle u|, \langle d|; \quad |k\rangle = |u\rangle, |d\rangle \\ &\langle a'| = \langle u|, \langle d|; \quad |a\rangle = |u\rangle, |d\rangle \\ &\Rightarrow \langle a'\,k| = \langle u\,u|; \langle u\,d|; \langle d\,u|; \langle d\,d| \\ &\Rightarrow |a\,k\rangle = |u\,u\rangle; |u\,d\rangle; |d\,u\rangle; |d\,d\rangle\end{aligned}$$

An example is given here to show how to calculate the density operator in the composite system.

Let $|\Psi_{ak}\rangle$ be a normalized entangled state vector that is written as

$$|\Psi_{ak}\rangle = \frac{1}{\sqrt{2}}(|u\,d\rangle - |d\,u\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \langle\Psi_{a'k}| = |\Psi_{ak}\rangle^{*} = \frac{1}{\sqrt{2}}(0 \quad 1 \quad -1 \quad 0)$$

According to Eq. (6.24), one calculates the combined basis kets in S_{AB} as

$$|u\,u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \langle u\,u| = (1 \quad 0 \quad 0 \quad 0);$$

$$|u\,d\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \langle u\,d| = (0 \quad 1 \quad 0 \quad 0);$$

$$|d\,u\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \langle d\,u| = (0 \quad 0 \quad 1 \quad 0);$$

$$|d\,d\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \langle d\,d| = (0 \quad 0 \quad 0 \quad 1).$$

The wave functions are calculated.

$$\psi(uu) = (1 \quad 0 \quad 0 \quad 0)\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\cdot 0 = 0;$$

$$\psi^{*}(uu) = \frac{1}{\sqrt{2}}(0 \quad 1 \quad -1 \quad 0)\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\cdot 0 = 0.$$

Analogously, one obtains the other terms

$$\begin{aligned}
&\psi(ud) = \frac{1}{\sqrt{2}}; \quad \psi^*(ud) = \frac{1}{\sqrt{2}}; \\
&\psi(du) = -\frac{1}{\sqrt{2}}; \quad \psi^*(du) = -\frac{1}{\sqrt{2}}; \\
&\psi(dd) = 0; \quad \psi^*(dd) = 0.
\end{aligned}$$

Using Eq. (6.51), the elements of the reduced density operator are calculated as

$$\begin{aligned}
\rho_{uu} &= \sum_k \psi^*(uk)\,\psi(uk) \\
&= \psi^*(uu)\,\psi(uu) + \psi^*(ud)\,\psi(ud) = \frac{1}{2}; \\
\rho_{ud} &= \sum_k \psi^*(uk)\,\psi(dk) \\
&= \psi^*(uu)\,\psi(du) + \psi^*(ud)\,\psi(dd) = 0; \\
\rho_{du} &= \sum_k \psi^*(dk)\,\psi(uk) \\
&= \psi^*(du)\,\psi(uu) + \psi^*(dd)\,\psi(ud) = 0; \\
\rho_{dd} &= \sum_k \psi^*(dk)\,\psi(dk) \\
&= \psi^*(du)\,\psi(du) + \psi^*(dd)\,\psi(dd) = \frac{1}{2}.
\end{aligned}$$

Finally, the probability reduced density operator of the subsystem S_A in the combined system S_{AB} results as

$$\rho_{S_A} = \begin{pmatrix} \rho_{uu} & \rho_{ud} \\ \rho_{du} & \rho_{dd} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, the element of the squared density operator results as

$$\rho_{S_A}^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \neq \rho_{a'a} \Rightarrow Tr\left(\rho_{S_A}^2\right) = \frac{1}{2} < 1$$

This result of the elements of the reduced density operators shows that the entangled state vector is an one-half mixed state in the subsystem S_A [7].

Using Eqs. (6.43) and (6.51), the expectation value (average value) of an observable $\mathbf{L}$ in the entangled state vector $|\Psi_{ab}\rangle$ in the subsystem S_A results as

$$\langle \mathbf{L} \rangle = \overline{\mathbf{L}} = Tr\left(\rho_{S_A}\mathbf{L}\right) = Tr\left(\sum_k |\Psi_{ak}\rangle\langle\Psi_{a'k}|\,\mathbf{L}\right) \tag{6.52}$$

6.10 Heisenberg's Uncertainty Principle

Heisenberg's uncertainty principle (HUP) shows the relationship between the position of a particle and its momentum state. The more precisely measured the particle position is, the less precisely is its momentum state at the measurements.

The momentum of a particle in quantum mechanics results from De Broglie formula as

$$p = \frac{h}{\lambda} \equiv \frac{2\pi\hbar}{\lambda} \tag{6.53}$$

where λ is the wavelength of the particle; $\hbar$ (read h bar) is the reduced Planck's constant or Dirac constant, which is defined as

$$\hbar \equiv \frac{h}{2\pi} = 1.054 \times 10^{-34}\text{Js}$$

The uncertainty principle is based on the standard variation that is recapitulated at first. The individual deviation of the particle position from its average value is written as (see Fig. 6.9).

$$\Delta x = x - \langle x \rangle = x - \overline{x} \tag{6.54}$$

The expectation value (average value) of the particle position x is calculated as

$$\begin{aligned}\overline{x} = \langle x \rangle \equiv \langle \Psi | \mathbf{x} | \Psi \rangle &= \sum_i x_i P(x_i, t) \\ &= \int_{-\infty}^{+\infty} x \cdot \psi^*(x,t)\psi(x,t)dx = \int_{-\infty}^{+\infty} x \cdot \|\Psi(x,t)\|^2 dx\end{aligned} \tag{6.55}$$

where

x is the operator of the particle position, as defined in Eq. (6.69);

$P(x_i,t)$ is the relative probability of the particle position at x_i at time t; and it is written as

$$P(x_i, t) = \|\Psi(x_i, t)\|^2$$

Using the product rule of differentiation, one obtains

$$\frac{\partial \|\psi\|^2}{\partial t} = \frac{\partial(\psi^*\psi)}{\partial t} = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}$$

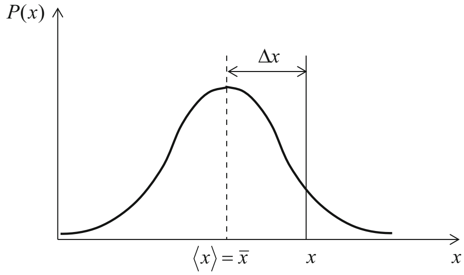

Fig. 6.9 Probability function of the particle position x

The Schrödinger equation (cf. Sect. 6.12) gives the following relations

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar}V\psi$$
$$\Rightarrow \frac{\partial \psi^*}{\partial t} = \frac{-i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar}V\psi^*$$

Using the product rule of integration, the expectation value of the particle velocity is calculated [6] as

$$\begin{aligned}\langle v \rangle &= \frac{d\langle x \rangle}{dt} = \int_{-\infty}^{+\infty} x \frac{\partial \|\psi\|^2}{\partial t} dx \\ &= \frac{i\hbar}{2m}\int_{-\infty}^{+\infty} x \frac{\partial}{\partial x}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right) dx \\ &= \frac{-i\hbar}{2m}\int_{-\infty}^{+\infty}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right) dx\end{aligned} \tag{6.56a}$$

The product rule of integration of the function $d(\psi^*\psi)/dx$ gives

$$\int_{-\infty}^{+\infty} \psi^* \frac{\partial \psi}{\partial x} dx = -\int_{-\infty}^{+\infty} \psi \frac{\partial \psi^*}{\partial x} dx \tag{6.56b}$$

Substituting Eq. (6.56b) into Eq. (6.56a), one obtains

$$\langle v \rangle = \frac{d\langle x \rangle}{dt} = \frac{-i\hbar}{m}\int_{-\infty}^{+\infty} \psi^*(x,t)\frac{\partial \psi(x,t)}{\partial x} dx$$

Therefore, the expectation value (average value) of the particle momentum $\langle p \rangle$ results as

$$\overline{p} = \langle p \rangle = m\langle v \rangle = -i\hbar \int_{-\infty}^{+\infty} \psi^*(x,t) \frac{\partial \psi(x,t)}{\partial x} dx \tag{6.56c}$$

The squared standard deviation of the particle position x is defined as the expectation value of squared Δx [6, 7], cf. Appendix H.

$$\begin{aligned} \sigma_x^2 &\equiv \left\langle (\Delta x)^2 \right\rangle = \sum_i (\Delta x_i)^2 P(x_i) = \sum_i (x_i - \overline{x})^2 P(x_i) \\ &= \sum_i x_i^2 P(x_i) - 2\overline{x} \sum_i x_i P(x_i) + \overline{x}^2 \sum_i P(x_i) \\ &= \left\langle x^2 \right\rangle - 2\overline{x}^2 + \overline{x}^2 = \left\langle x^2 \right\rangle - (\overline{x})^2 \end{aligned} \tag{6.57a}$$

Therefore, the standard deviation of the particle position x is written as

$$\sigma_x \equiv \sqrt{\left\langle (\Delta x)^2 \right\rangle} = \sqrt{\langle x^2 \rangle - (\overline{x})^2} \tag{6.57b}$$

Analogously, the standard deviation of the particle momentum p results as

$$\sigma_p^2 \equiv \left\langle (\Delta p)^2 \right\rangle = \left\langle p^2 \right\rangle - (\overline{p})^2; \tag{6.58a}$$

$$\sigma_p \equiv \sqrt{\left\langle (\Delta p)^2 \right\rangle} = \sqrt{\langle p^2 \rangle - (\overline{p})^2} \tag{6.58b}$$

In general, the standard deviation of an observable Hermitian operator A is written using Eq. (6.57a) as

$$\begin{aligned} \sigma_A^2 &\equiv \left\langle (\Delta A)^2 \right\rangle = \left\langle \left(A - \overline{A}\right)^2 \right\rangle \\ &= \langle \Psi | \left(A - \overline{A}\right)^2 | \Psi \rangle = \left\langle \left(A - \overline{A}\right)\Psi | \left(A - \overline{A}\right)\Psi \right\rangle \\ &\equiv \langle f_A | f_A \rangle \end{aligned} \tag{6.59a}$$

in which the function f_A is defined as

$$f_A \equiv \left(A - \overline{A}\right)\Psi(x) \tag{6.59b}$$

Similarly, the standard deviation of a Hermitian operator B is written as

$$\sigma_B^2 = \left\langle \left(B - \overline{B}\right)\Psi | \left(B - \overline{B}\right)\Psi \right\rangle \equiv \langle f_B | f_B \rangle \tag{6.60a}$$

with the function f_B is defined as

$$f_B \equiv \left(B - \overline{B}\right)\Psi(x) \tag{6.60b}$$

Using the Schwarz inequality, the product of the standard deviations in written as

$$\sigma_A^2 \sigma_B^2 = \langle f_A | f_A \rangle \langle f_B | f_B \rangle \geq \| \langle f_A | f_B \rangle \|^2 \tag{6.61}$$

At first, the RHS of Eq. (6.61) is calculated as

$$\begin{aligned}
\langle f_A | f_B \rangle &= \big\langle \big(A - \overline{A}\big)\Psi \big| \big(B - \overline{B}\big)\Psi \big\rangle = \langle \Psi | \big(A - \overline{A}\big)\big(B - \overline{B}\big) | \Psi \rangle \\
&= \langle \Psi | \big(AB - A\overline{B} - \overline{A}B + \overline{A}\,\overline{B}\big) | \Psi \rangle \\
&= \langle \Psi | AB | \Psi \rangle - \overline{B} \langle \Psi | A | \Psi \rangle - \overline{A} \langle \Psi | B | \Psi \rangle + \langle \Psi | \overline{AB} | \Psi \rangle \\
&= \langle AB \rangle - \overline{B}\,\overline{A} - \overline{A}\,\overline{B} + \overline{A}\,\overline{B} = \langle AB \rangle - \overline{B}\,\overline{A} \\
&= \langle AB \rangle - \langle B \rangle \langle A \rangle
\end{aligned} \tag{6.62}$$

Calculating the adjoint of Eq. (6.62), it results

$$\begin{aligned}
\langle f_B | f_A \rangle &= \langle f_A | f_B \rangle^* = \langle AB \rangle^* - \big(\overline{B}\,\overline{A}\big)^* \\
&= \langle BA \rangle - \overline{A}\,\overline{B} \\
&= \langle BA \rangle - \langle A \rangle \langle B \rangle
\end{aligned} \tag{6.63}$$

Subtracting Eq. (6.62) from Eq. (6.63), one obtains

$$\begin{aligned}
\langle f_A | f_B \rangle - \langle f_B | f_A \rangle &= \langle AB \rangle - \langle BA \rangle = \langle AB - BA \rangle \\
&\equiv \langle [A, B] \rangle
\end{aligned} \tag{6.64}$$

The term $[A, B]$ is called the *commutator* of two Hermitian operators A and B that results from Eq. (6.64) as

$$[A, B] = AB - BA \neq 0 \tag{6.65}$$

The commutator is sometimes called the *Lie bracket* (cf. Sect. 3.12.2).

If $[A, B] \neq 0$, a complete set of mutual eigenkets of A and B does not exist. In this case, both operators are not commutative with each other. Conversely, if $[A, B] = 0$, a complete set of mutual eigenkets of A and B exists, in which the operators A and B are commutative.

The commutator of the Hermitian operators has the following properties:

$$\begin{aligned}
&[A, B] = -[B, A] \quad : \textit{anticommutativity} \\
&[A, A] = 0 \\
&[A + B,\ C] = [A, C] + [B, C] \\
&[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \quad : \textit{Jacobi identity} \\
&[A, BC] = [A, B]C + B[A, C] \\
&[AB, C] = A[B, C] + [A, C]B
\end{aligned} \tag{6.66}$$

The relation between the commutator (*Lie bracket*) applied to quantum mechanics and *Poisson bracket* used in classical mechanics was recognized by Dirac.

$$[A, B] \quad \leftrightarrow \quad i\hbar\{A, B\} \tag{6.67}$$

Next, the norm (absolute value) of the inner product of Eq. (6.62) is calculated to derive Heisenberg's uncertainty principle (HUP).

The inner product is a complex number that can be written as

$$z \equiv \langle f_A | f_B \rangle = \alpha + i\beta \in \mathbf{C}$$

Its transpose conjugate is calculated as

$$z^* \equiv \langle f_A | f_B \rangle^* = \langle f_B | f_A \rangle = \alpha - i\beta \in \mathbf{C}$$

Using Eq. (6.64), one obtains

$$\begin{aligned} & z - z^* \equiv \langle f_A | f_B \rangle - \langle f_B | f_A \rangle = \langle [A, B] \rangle = 2i\beta \\ & \Rightarrow \beta = \frac{\langle [A, B] \rangle}{2i} \end{aligned}$$

Using Eqs. (6.61) and (6.64), one obtains the inequality equation

$$\begin{aligned} \sigma_A^2 \sigma_B^2 \geq \| z \| \equiv \| \langle f_A | f_B \rangle \| = \alpha^2 + \beta^2 \\ \geq \beta^2 = \left(\tfrac{\langle [A,B] \rangle}{2i} \right)^2 \end{aligned}$$

Therefore,

$$\sigma_A \sigma_B \geq \frac{\langle [A, B] \rangle}{2i} \geq 0 \tag{6.68}$$

Finally, let A be the position operator $\mathbf{x}$ of particle and B be its momentum operator $\mathbf{p}_x$. Note that both operators $\mathbf{x}$ and $\mathbf{p}_x$ are Hermitian operators.

The position operator $\mathbf{x}$ of particle at a position x is defined as

$$\mathbf{x} \equiv \int_{-\infty}^{+\infty} x | x \rangle \langle x | dx \tag{6.69}$$

The momentum operator $\mathbf{p}_x$ of particle at a position x is defined using the imaginary unit i as

$$\mathbf{p}_x \equiv -i\hbar \frac{\partial}{\partial x} \tag{6.70}$$

In the following section, the transformation of position and momentum coordinates x and p is carried out.

The position wave function of the particle is written using Eq. (6.15) as

$$\psi_x(x) = \langle \mathbf{x}|\Psi\rangle = \langle \mathbf{x}|\mathbf{p}\rangle\langle \mathbf{p}|\Psi\rangle = \int_{-\infty}^{+\infty} \langle \mathbf{x}|\mathbf{p}\rangle\langle \mathbf{p}|\Psi\rangle dp$$
$$= \int_{-\infty}^{+\infty} \psi_p(p)\langle \mathbf{x}|\mathbf{p}\rangle dp = \int_{-\infty}^{+\infty} \psi_p(p)\,\delta(x-p)dp$$

Analogously, the momentum wave function of the particle results from Eq. (6.15) as

$$\psi_p(p) = \langle \mathbf{p}|\Psi\rangle = \langle \mathbf{p}|\mathbf{x}\rangle\langle \mathbf{x}|\Psi\rangle = \int_{-\infty}^{+\infty} \langle \mathbf{p}|\mathbf{x}\rangle\langle \mathbf{x}|\Psi\rangle dx$$
$$= \int_{-\infty}^{+\infty} \psi_x(x)\langle \mathbf{p}|\mathbf{x}\rangle dx = \int_{-\infty}^{+\infty} \psi_x(x)\,\delta(p-x)dx$$

The momentum operator $\mathbf{p}_x$ acting on the state vector $|\Psi(x)\rangle$ is expressed as

$$\langle x|\mathbf{p}_x|\Psi\rangle = \mathbf{p}_x\psi(x) = \left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)$$

Hence,

$$\langle x|\mathbf{p}_x|\Psi\rangle = -i\hbar\frac{\partial\psi(x)}{\partial x}$$

Generally, this above equation is valid for any state vectors $|\Phi\rangle$ and $|\Psi\rangle$. Using partially integrating the RHS term of the below equation, one obtains

$$\begin{aligned}\langle\Phi|\mathbf{p}_x|\Psi\rangle &= -i\hbar\int\phi^*(x)\frac{\partial\psi(x)}{\partial x}dx\\ &= -i\hbar\left[\phi^*\psi\right]_{-\infty}^{+\infty} + i\hbar\int\psi(x)\frac{\partial\phi^*(x)}{\partial x}dx\\ &= i\hbar\int\psi(x)\frac{\partial\phi^*(x)}{\partial x}dx\end{aligned}$$

The above-calculated term is equal to its transpose conjugate (cf. Chap. 1).

$$\begin{aligned}\langle \Phi|\mathbf{p}_x|\Psi\rangle &= i\hbar\int \psi(x)\frac{\partial \phi^*(x)}{\partial x}dx\\ &= \langle \Psi|\mathbf{p}_x|\Phi\rangle^\dagger = \langle \Phi|\mathbf{p}_x^\dagger|\Psi\rangle\\ &\Rightarrow \mathbf{p}_x^\dagger = \mathbf{p}_x \rightarrow \mathit{Hermitian}\end{aligned}$$

This result indicates that momentum operator $\mathbf{p}_x$ is a Hermitian operator.

Using Eq. (6.65), the commutator of $\mathbf{x}$ and $\mathbf{p}_x$ results as

$$\begin{aligned}[\mathbf{x},\mathbf{p}_x]\,\psi(x) &= \mathbf{x}\left(-i\hbar\frac{\partial\psi}{\partial x}\right) - \left(-i\hbar\frac{\partial(\psi\mathbf{x})}{\partial x}\right)\\ &= -\mathbf{x}\left(i\hbar\frac{\partial\psi}{\partial x}\right) + \mathbf{x}\left(i\hbar\frac{\partial\psi}{\partial x}\right) + i\hbar\frac{\partial\mathbf{x}}{\partial x}\psi(x)\\ &= i\hbar\psi(x)\end{aligned} \tag{6.71}$$

The result of Eq. (6.71) gives the very useful commutator for HUP

$$[\mathbf{x},\mathbf{p}_x] \equiv (\mathbf{x}\mathbf{p}_x - \mathbf{p}_x\mathbf{x}) = i\hbar \tag{6.72}$$

This is the fundamental equation of matrix mechanics in quantum physics (Heisenberg, Born, and Jordan). However, this equation is very peculiar in the everyday experience.

Having squared both sides of Eq. (6.72), one obtains using Eq. (6.65) the squared value of the commutator

$$[\mathbf{x},\mathbf{p}_x]^2 = (\mathbf{x}\mathbf{p}_x - \mathbf{p}_x\mathbf{x})^2 = -\hbar^2 < 0$$

This results shows that the squared value of a quantum quantity is *negative*. That is the quantum mathematical language that opens a new view of the quantum world, which differs from the mathematical world of classical physics.

In this case, both Hermitian operators $\mathbf{x}$ and $\mathbf{p}_x$ are called the canonical conjugate entities that have a canonical commutation relation.

Generally, the commutator (Lie bracket) of differential Hermitian operators $\mathbf{r}_k$ and $\mathbf{p}_l$ of position and momentum of a particle in an N-dimensional space is written using Kronecker delta as

$$[\mathbf{r}_k,\mathbf{p}_l] = i\hbar\delta_{kl};\quad k,l = 1,2,\ldots,N.$$

Substituting Eq. (6.72) into Eq. (6.68), the well-known Heisenberg's uncertainty principle is derived as

$$\sigma_x\sigma_p \geq \frac{\langle[\mathbf{x},\mathbf{p}_x]\rangle}{2i} = \frac{\langle i\hbar\rangle}{2i} = \frac{\hbar}{2} \qquad \Leftrightarrow \Delta x\cdot\Delta p \geq \frac{\hbar}{2} \tag{6.73}$$

where Δx and Δp are the uncertainties of the position and momentum of the particle, respectively.

In 1927, W. Heisenberg stated that *the more precisely measured the position of a particle is, the less precisely is its quantum momentum state at the measurements*. The Heisenberg's uncertainty principle in quantum mechanics is statistically based on the repeated measurements that are carried out in a set of many identically prepared systems

6.11 The Wave-Particle Duality

A quantum object (e.g. photon, light quanta) has both characteristics of wave and particle as well. That is called the wave-particle duality of a quantum object. Sometimes the object behavior is like a wave; and another time, like a particle, but the object itself is the same.

In fact, the quantum object with a wave-like or particle-like characteristic depends on how it is considered like in topology and differential geometry (cf. Sect. 4.1). If we consider the object at a distance as a whole without considering it in detail like topologists do, the object responses as wave-like. On the contrary, if we consider it much closer and concern it much more in detail with all smallest scales of it like geometers do, the object reacts as particle-like.

In quantum mechanics, Born and Heisenberg preferred the particle-like characteristic for a quantum object; and Einstein, Schrödinger, and De Broglie preferred rather the wave-like characteristic for the object [3, 5]. On the one hand, the wave-like properties are found in the phenomena of interference behind two slots and diffraction of light. On the other hand, the particle-like properties are used to explain the photoelectric and Compton effects. Furthermore, Fourier analysis shows that the wave and particle duality is an intrinsic mathematical equivalence that always occurs in all quantum entities [3].

In fact, a quantum object can be considered as either a particle-like or a wave-like behavior. This point of view depends on how to interpret the possibility of the quantum-mechanical formalism. The existence of light quanta denotes that the light wave is quantized into many quanta portions that move together in a waveform. Thus, this quantized wave theory goes well with the quantized particle theory. Both interpretations of the wave-particle duality do not contradict to each other. Furthermore, electrons cannot be considered as single particles because their positions could not be exactly determined. In this case, they are regarded as rather a cloud of many electrons than single particles in quantum mechanics.

The wave-particle duality is based on wave mechanics (Schrödinger) and matrix mechanics (Heisenberg), respectively. They are only a special case of quantum algebra that was developed by Dirac. The quantum algebra involves additions and multiplications of quantum variables, such as canonical coordinates q_r and momenta p_s of the particles in three-dimensional Hilbert spaces.

The *Bohr's complementarity* is a fundamental part of nature, in which a quantum object has both particle and wave properties that depend on how one observes it at measurements. As a result, the process of measurement triggers the current wavefunction from the superposition of eigenstates into one of the component eigenstates. This process is called the *wavefunction collapse* at the measurement. The unity of the Bohr's complementarity, Heisenberg's uncertainty principle, and wavefunction collapse triggered by measurements has been known as the *"Copenhagen interpretation of quantum mechanics"*.

The fundamental quantum conditions of the quantum variables satisfy the Dirac quantum-algebra equations using Kronecker delta δ_{rs} [10]:

$$\begin{aligned} &q_r q_s - q_s q_r = 0; \\ &p_r p_s - p_s p_r = 0; \\ &q_r p_s - p_s q_r = i\hbar\{q_r, p_s\} = i\hbar\delta_{rs}. \end{aligned}$$

The Poisson bracket (PB) of two dynamical variables u and v is defined as, cf. Eq. (6.67)

$$\{u, v\} = \frac{-i}{\hbar}(uv - vu) \equiv \sum_{r=1}^{3P}\left(\frac{\partial u}{\partial q_r}\frac{\partial v}{\partial p_r} - \frac{\partial u}{\partial p_r}\frac{\partial v}{\partial q_r}\right)$$

where r is the index from 1 to three times of the number of particles P.

6.11.1 De Broglie Wavelength Formula

De Broglie wavelength formula gives the relation between wave and particle characteristics of a free particle by means of its momentum and wavelength.

Let a particle be moving with a velocity $v < c$ (light speed). According to the special relativity theory, its relativistic mass m results as

$$m = m_0\gamma \equiv \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{6.74}$$

where

m_0 is the mass at rest ($v = 0$);

γ is the Lorentz factor, which is defined as

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \geq 1 \tag{6.75}$$

Equation (6.74) denotes that the higher the velocity of the moving mass is, the heavier its relativistic mass (i.e. moving mass) becomes. At rest, its mass equals the mass at rest m_0; its relativistic mass goes to infinity as the velocity reaches the light speed.

Multiplying both sides of Eq. (6.74) by c^2 and squaring them, one obtains the energy-momentum relation

$$\begin{aligned} & m^2c^4\left(1-\frac{v^2}{c^2}\right)=m_0^2c^4 \\ & \Leftrightarrow m^2c^4=m_0^2c^4+(m^2v^2)c^2 \\ & \Leftrightarrow E^2=E_0^2+p^2c^2 \end{aligned} \tag{6.76}$$

Let the particle be a photon that has no mass; i.e. $m_0 = 0$. Hence, $E_0 = 0$.

Equation (6.76) is written for a photon as

$$E = pc \tag{6.77}$$

The momentum of the particle results as from Eq. (6.77)

$$p = \frac{E}{c} \tag{6.78}$$

The energy of the light quanta only depends on the light frequency.

$$E = hv \equiv \hbar(2\pi v) = \hbar\omega \tag{6.79}$$

in which

h is the Planck constant (=6.63 × 10^{-34} J s);

$\hbar$ is the reduced Planck constant or Dirac constant, which is defined as

$$\hbar \equiv \frac{h}{2\pi}$$

v is the particle frequency;

ω is the particle angular frequency ($\omega = 2\pi v$).

Substituting Eq. (6.79) into Eq. (6.78), the particle momentum results as

$$p = \frac{hv}{c} \tag{6.80}$$

The wavelength λ of the particle is written as

$$\lambda = \frac{c}{v} \tag{6.81}$$

Substituting Eqs. (6.80) and (6.81), the particle wavelength is calculated as

$$\lambda = \frac{h}{p} \Rightarrow p = \frac{h}{\lambda} \tag{6.82}$$

This is the *De Broglie wavelength formula* of a free particle.

The momentum of a particle with a mass $m \equiv m_0$ is defined in a non-relativistic system ($v << c$) as

$$p = m_0 v \equiv mv \tag{6.83}$$

Using Eq. (6.83), the kinetic energy of the particle with a mass $m \equiv m_0$ is calculated in a non-relativistic system as

$$E_{ke} = \frac{1}{2} m_0 v^2 = \frac{1}{2} m_0 \left(\frac{p}{m_0} \right)^2 = \frac{p^2}{2m_0} \equiv \frac{p^2}{2m} \tag{6.84}$$

6.11.2 The Compton Effect

The Compton Effect is used to demonstrate that the quantum object has a particle-like behavior rather than a wave-like behavior. The experiment is carried out with a single photon (i.e. light quanta with $m = 0$) at a frequency of v that collides with a rest electron. Due to the collision, the incident photon is scattered with a scattering angle θ from the incident direction at a frequency of v'. The electron with a mass at rest m_e is deflected from the incident direction of the photon with a velocity v (see Fig. 6.10). Note that all considered particles are *relativistic* in this experiment.

Due to energy conservation, the balance of the relativistic particle energy of the system before and after collision is written as

$$\begin{gathered} E_{0,e^-} + E = E_{e^-} + E' \\ \Leftrightarrow m_e c^2 + hv = (m_e \gamma) c^2 + hv' \end{gathered} \tag{6.85a}$$

where m_e is the mass at rest of the electron; c is the light speed¸ and γ is the Lorentz factor in Eq. (6.75). Note that $E_{0,e-}$ is the quantum energy of the electron even at rest; and E_{e-} is the quantum energy of the electron at moving according to Einstein's special relativity theory.

Equation (6.85a) can be written as

$$\begin{aligned} m_e^2 \gamma^2 &= \left(m_e + \tfrac{hv}{c^2} - \tfrac{hv'}{c^2} \right)^2 \equiv \left(m_e + \tfrac{h\Delta v}{c^2} \right)^2 \\ &= m_e^2 + \frac{2 m_e h \Delta v}{c^2} + \frac{h^2 (\Delta v)^2}{c^4} \approx m_e^2 + \frac{2 m_e h \Delta v}{c^2} \end{aligned}$$

Multiplying both sides of the equation by c^2, one obtains

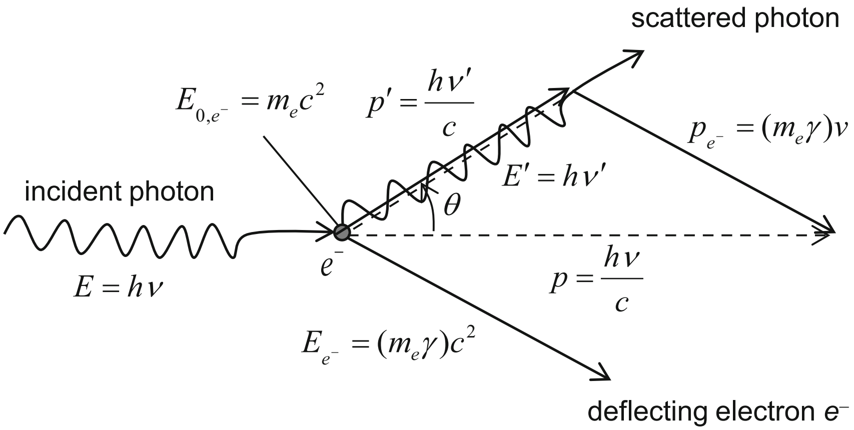

Fig. 6.10 The Compton Effect of relativistic particles

$$m_e^2\gamma^2c^2 \approx m_e^2c^2 + 2m_e h\Delta\nu \tag{6.85b}$$

in which

$$\Delta\nu \equiv \nu - \nu'$$

The conservation of momentum before and after collision is expressed as

$$\vec{p} = \vec{p}' + \vec{p}_{e^-} \tag{6.86a}$$

Using the law of cosines for a triangle, one obtains the relation of the momenta.

$$(m_e\gamma v)^2 = \left(\frac{h\nu}{c}\right)^2 + \left(\frac{h\nu'}{c}\right)^2 - 2\left(\frac{h\nu}{c}\right)\cdot\left(\frac{h\nu'}{c}\right)\cdot\cos\theta \tag{6.86b}$$

The RHS of Eq. (6.86b) is simplified as

$$\begin{aligned}\text{RHS} &= \frac{h^2}{c^2}\left(\nu - \nu'\right)^2 + 2\left(\frac{h\nu}{c}\right)\cdot\left(\frac{h\nu'}{c}\right)\cdot(1 - \cos\theta)\\ &\approx \frac{2h^2\nu\nu'}{c^2}(1 - \cos\theta)\end{aligned} \tag{6.86c}$$

Using the relation

$$c^2\left(\gamma^2 - 1\right) = c^2\left(\frac{1}{1 - \frac{v^2}{c^2}} - 1\right) = \frac{v^2}{1 - \frac{v^2}{c^2}} = \gamma^2 v^2$$

and Eq. (6.85b), the LHS of Eq. (6.86b) is written as

$$\begin{aligned} m_e^2\gamma^2 v^2 &= m_e^2 c^2(\gamma^2 - 1) = m_e^2 c^2\gamma^2 - m_e^2 c^2 \\ &\approx \left(m_e^2 c^2 + 2m_e h\Delta v\right) - m_e^2 c^2 \\ &\approx 2m_e h\Delta v \end{aligned} \tag{6.87}$$

Substituting Eqs. (6.87) and (6.86c) into Eq. (6.86b), one obtains the scattered frequency of the photon.

$$2m_e h\Delta v \approx \frac{2h^2 v v'}{c^2}(1 - \cos\theta)$$

Using the trigonometric formula, one obtains the scattered photon frequency.

$$\begin{aligned} &\Delta v \equiv v - v' \approx \frac{hvv'}{m_e c^2}(1 - \cos\theta) = \frac{2hvv'}{m_e c^2}\sin^2\frac{\theta}{2} \\ &\Rightarrow v' = \frac{v}{1 + \frac{2hv}{m_e c^2}\sin^2\frac{\theta}{2}} = \frac{v}{1 + \frac{4\pi\hbar v}{m_e c^2}\sin^2\frac{\theta}{2}} \le v \end{aligned} \tag{6.88}$$

Substituting Eq. (6.81) into Eq. (6.88), it gives the scattered photon wavelength.

$$\begin{aligned} v - v' &= c \cdot \left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) \approx \frac{2h\frac{c\,c}{\lambda\lambda'}}{m_e c^2}\sin^2\frac{\theta}{2} \\ &= \frac{4\pi\hbar}{m_e\lambda\lambda'}\sin^2\frac{\theta}{2} \end{aligned}$$

Thus,

$$\begin{aligned} &\Delta\lambda \equiv \lambda' - \lambda = \frac{4\pi\hbar}{m_e c}\sin^2\frac{\theta}{2} = \frac{h}{m_e c}(1 - \cos\theta) \\ &\Rightarrow \lambda' = \lambda + \frac{4\pi\hbar}{m_e c}\sin^2\frac{\theta}{2} = \lambda + \frac{h}{m_e c}(1 - \cos\theta) \ge \lambda \end{aligned} \tag{6.89}$$

The quantity of $h/(m_e c)$ in Eq. (6.89) is called the *Compton wavelength* of the electron. This wavelength is equal to about 2.43×10^{-12} m; the quantity of $\hbar/(m_e c)$, the *reduced Compton wavelength* of the electron equals ca 3.9×10^{-13} m.

In *non-relativistic* systems, the kinetic energy of any moving mass $m \equiv m_0$ at a velocity $v \ll c$ is calculated as the difference between the quantum energies of the moving object and the rest object.

$$
\begin{aligned}
E_{ke} &= E - E_0 = (m\gamma)c^2 - mc^2 \\
&= mc^2(\gamma - 1) = m\left(\frac{c^2}{\sqrt{1-\frac{v^2}{c^2}}} - c^2\right) \\
&\approx m\left(\frac{c^2 - c^2\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)}{1 - \frac{1}{2}\frac{v^2}{c^2}}\right) \approx \frac{1}{2}mv^2
\end{aligned}
$$

The Compton Effect indicates that the frequency of scattered photon is less than its incident frequency due to the necessary energy that needs to deflect the electron from the rest position. Obviously, the lower the wave frequency is, the longer the wavelength of the particle is, as shown in Eqs. (6.88) and (6.89).

This effect could be easily explained by Einstein's light quantum theory (photoelectric effect) based on the particle-like theory. In this case, the wave-like theory fails to explain this Compton Effect. However, the particle-like theory cannot explain such phenomena of the light interference and diffraction that could be explained by means of the wave-like theory [3].

According to the wave-like approach, the energy of the quantum object is on the one hand written in its frequency as

$$E = hv$$

On the other hand, the energy of the quantum object with the mass at rest m_0 can be expressed in the particle-like approach in a relativistic system as, cf. Eq. (6.76)

$$E = \sqrt{p^2c^2 + m_0^2c^4} = E_{ke} + m_0c^2$$

The momentum p of the quantum object with the mass at rest m_0 is defined in a relativistic system as

$$p = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}$$

Using the Taylor series, the kinetic energy E_{ke} of the quantum object with the mass at rest m_0 is calculated in a relativistic system as

$$\begin{aligned} E_{ke} &= \sqrt{p^2c^2 + m_0^2c^4} - m_0c^2 = m_0c^2\left[\sqrt{1 + \left(\frac{p}{m_0c}\right)^2} - 1\right] \\ &\approx m_0c^2\left[1 + \frac{1}{2}\left(\frac{p}{m_0c}\right)^2 - \frac{1}{8}\left(\frac{p}{m_0c}\right)^4 \pm \cdots - 1\right] \\ &\approx \frac{p^2}{2m_0} - \frac{p^4}{8m_0^3c^2} \pm \cdots \end{aligned}$$

The second term on the RHS of the above equation is the relativistic correction term of the kinetic energy that is considered in the relativistic system.

6.11.3 Double-Slit Experiments with Electrons

The double-slit experiments (Young's experiment) are carried out in which electrons are generated by an electron generator, as shown in Fig. 6.11.

The electrons go through two slits A and B. At first, only the slit A is opened and the slit B is closed, the probability function $P_A(z)$ of the electrons from the slit A is counted on the screen after a long observing time. The same experiment is repeated at closing the slit A and opening the slit B, the probability function $P_B\ (z)$ of the electrons from the slit B is displayed on the screen. Note that the maximum of the probability distribution on the screen locates in the middle of the slit axis for each experiment. In this case, no interference pattern that is caused by the interference of the waves of the electrons from A and B is found on the screen (see Fig. 6.11a).

The probability function $P_i(z)$ of the electrons for finding the electron at position z in a long observing time is the squared amplitude of the wave function $\psi_i(z)$, cf. Eq. (6.6).

$$P_i(z) \equiv \|\psi_i(z,t)\|^2 \text{ for } i = A,\ B$$

The experiment with opening both slits A and B is repeated. After a long observing time, the probability function $P_{AB}(z)$ of the electrons is very different from the earlier experiments (see Fig. 6.11b). The interference pattern (interference stripes) of the diffracting electrons from both slits occurs on the screen, in which no electron hits in some certain positions.

The results show that the electrons behave like waves behind the slits where the waves of the electrons coming from A and B interfere with each other, add the amplitudes of waves together at the crests (bright spots), and cancel them at the troughs (dark spots) of the interference fringes (Young's experiment). The amplitudes of the electrons are superimposed on the screen in form of the interference pattern.

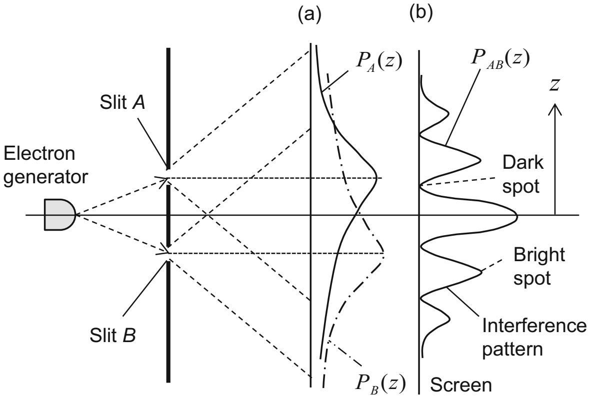

Fig. 6.11 Double-slit experiments with an electron generator

In case of the interference pattern, the probability function of the electrons for finding the electron at position z in a long observing time is the squared amplitude of the sum of the wave functions $\psi_A(z)$ and $\psi_B(z)$.

The probability function $P_{AB}(z)$ at position z is calculated as, cf. Appendix F.

$$\begin{aligned} P_{AB}(z) &\equiv \|\psi_{AB}(z,t)\|^2 = \|\psi_A(z,t) + \psi_B(z,t)\|^2 \\ &= \|\psi_A\|^2 + \|\psi_B\|^2 + 2\mathrm{Re}\left(\psi_A \psi_B^*\right) \\ &= P_A(z) + P_B(z) + 2\mathrm{Re}\left(\psi_A \psi_B^*\right) \\ &= P_A(z) + P_B(z) + 2\mathrm{Re}\left(\psi_A^* \psi_B\right) \end{aligned}$$

The third term on the RHS of the above equation generates the quantum-interference pattern, as displayed in Fig. 6.11b. The additional term can be positive or negative. Note that the interference term is mathematically created; and it only exists in quantum mechanics [9]. This quantum-interference term is not taken into account in the probability function $P_{AB}(z)$ in the classical probability theory.

So far so good, such experimental results can be explained using the wave theory of classical mechanics. However, some points of view about the particle-like behavior should be discussed in the following section. How about is the light beam (quanta photon) in quantum mechanics? Note that the light beam consists of many quanta photons; each of them has a certain portion of energy (quanta entity) according to Planck and Einstein. How about are the photoelectric and Compton effects, in which the particle-like behavior is preferred?

To make sure that the electron beam is made of single particles (quanta), the intensity of the electron generator is reduced so that a single electron is generated one after another. Each electron of them travels through the slit A or B at time. The experiment is repeated at opening both slits A and B. At first, no interference pattern occurs on the screen. However, the interference pattern takes places again on the screen after a long observing time [11]. The question emerges in the experiment why a single electron can interfere with itself to cause the interference pattern.

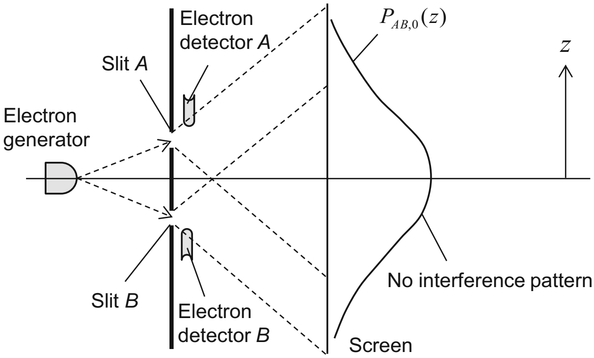

Fig. 6.12 Double-slit experiments with electron detectors

Would the single electron split into two halves? Each of them goes through both slits at the same time. Then, these two halves of the electron interfere with each other to generate the interference pattern.

To answer this question, two detectors are installed at both opening slits A and B. Every electron travels through the slit is detected by the electron detectors using the light beam at a frequency ν_d acting upon the moving electron. As a result, no split of the electrons occurs; and the interference pattern does not happen on the screen at a long observing time. The probability function $P_{AB,0}(z)$ of the electrons is shown in Fig. 6.12.

In case without the interference pattern, the probability function $P_{AB,0}(z)$ of the electrons for finding the electron at position z in a long observing time is the sum of the squared amplitudes of the wave functions $\psi_A(z)$ and $\psi_B(z)$.

$$P_{AB,0}(z) \equiv \|\psi_A(z,t)\|^2 + \|\psi_B(z,t)\|^2$$

This result shows that such detecting the electrons that travel through the slits may destroy the interference pattern. Possibly, the detection of the exact positions of the electrons prevents their interference ability. The experiment is repeated once again with reducing the intensities of the detector light beams so that both detectors have not affected the electrons travelling through the slits any longer. To lower the intensity of the detecting light in W/m^2 or J/(m^{2}s), either the number of light quanta or the light frequency ν_d acting upon the electron must be reduced since the energy of the light beam in J equals n times $h\nu_d$ in which n is the number of the light quanta.

Even reducing one photon for each electron passing through the slit, the interference pattern does not occur on the screen. In another way, the detecting light frequency ν_d must be reduced at least its wavelength λ_d is larger than the distance between two slits. In this case, we do not know which electron travels through which slit; i.e., the positions of the electrons are not determined at the measurement. That means the detecting light does not affect the electrons travelling through the slits. After an observing time, the interference pattern returns on the screen. The same result is obtained if both electron detectors at the slits are removed at the same time. Note that if the electrons are not observed or detected, they could be able to interfere with each other to generate the interference fringes that lead to the interference pattern on the screen.

Using the Heisenberg's uncertainty principle (HUP), the phenomena occurring in the double-slit experiments with the electron detectors could be explained. In the case of detecting the travelling electrons through each slit, the positions of the electrons are quite determined, which electron travels through the slit A or the slit B. Therefore, their momenta in direction z cannot be exactly determined according to the HUP, cf. Eq. (6.73) as

$$\Delta p_z \geq \frac{\hbar}{2\Delta z}$$

The momentum uncertainty of the electron travelling through the slit is usually equal to the momentum uncertainty of the photon beaming on the electron that depends on the De-Broglie wavelength λ_d of the detecting light according to Eq. (6.82). Thus, the momentum uncertainty of the electron is written as

$$\Delta p_z \approx \Delta p_d = \frac{h}{\lambda_d}$$

Therefore, the velocity uncertainty of the electron on the screen is calculated as

$$\Delta v_z = \frac{\Delta p_z}{m_{e^-}} \approx \frac{h}{m_{e^-}\lambda_d}$$

At detecting the electrons, the position uncertainty Δz of the electrons in direction z is in the order of the wavelength of the detecting light λ_d ($\Delta z \sim \lambda_d$). Therefore, the momentum uncertainty of the electrons in direction z satisfies the HUP.

$$\begin{aligned}\Delta p_z &\approx \frac{h}{\lambda_d} = \frac{2\pi\hbar}{\lambda_d} \\ &\geq \frac{\hbar}{2\lambda_d} \approx \frac{\hbar}{2\Delta z}\,(q.e.d.)\end{aligned}$$

At a measurement using a wavelength of the detecting violet light $\lambda_d = 400$ nm, the electron mass $m_{e-} = 9.11 \times 10^{-31}$ kg, and the Planck's constant $h = 6.63 \times 10^{-34}$ J s, one obtains the velocity uncertainty of the electrons

$$\Delta v_z \approx \frac{h}{m_{e^-}\lambda_d} = \frac{6.63 \times 10^{-34}}{\left(9.11 \times 10^{-31}\right) \cdot \left(4 \times 10^{-7}\right)} \approx 1.82 \times 10^3 \frac{m}{s}$$

The HUP is a statistical average over a set of many identical measurements. In this case, the velocity uncertainty of the electrons is too high so that we do not know where these electrons locate after 1 s, in the large uncertain range between 0 and 1.8 km from the measurement. As a result, the interference phenomenon does not take place at the measurement. On the contrary, if the wavelength of the detecting light λ_d is long enough that must be much larger than the distance l_s between two slits ($l_s \approx 0.7$ mm), the velocity uncertainty of the electrons becomes much smaller. In this case, the light detectors do not affect the electrons moving through the slits. As a result, the electrons locate in the measurement and interfere with each other. Hence, the interference pattern returns on the screen.

This result indicates that the measured results are strongly influenced by the apparatus or the measurement method in the quantum world. Therefore, the electrons behave sometimes like waves and in another time like particles depending on the measurement and observing method [11]. If the experiment is set up to study the wave-like behavior of the electrons, the wave phenomena of the interference and diffraction that are interacted between the electrons and the observing method are found. In another case, the particle-like behaviors, such as the photoelectric and Compton effects are given if the experiment is set up to detect the particle characteristic of the electrons. It would be better off letting the nature laws organize themselves, especially in the quantum world. In general, physics teaches us how to understand the nature laws, not how to manipulate them.

In the double-slit experiment with detectors at each slit, each observer sees the electron go through just one slit. In accordance with *the Copenhagen interpretation* of many worlds, this experiment of the real world splits in two worlds that are separate and non-interacting with each other. Both worlds have the same probability of electron going through the slit (i.e. 50 % to 50 %). In fact, they are not parallel but perpendicular to each other in the many-worlds interpretation (MWI). As a result, no interference between the separate and non-interacting worlds takes place on the screen.

In fact, the quantum objects with a wave-like or particle-like characteristic depend on how they are considered like in topology and differential geometry (cf. Sect. 4.1). If we consider the objects at a distance as a whole without considering them in detail (i.e. without detecting the electron positions) like topologists do, the measured objects response as wave-like behavior with the interference pattern (see Fig. 6.11b). On the contrary, if we consider them much closer and concern them much more in detail with all smallest scales of them (i.e. detecting the electron positions) like geometers do, the measured objects react as particle-like behavior (see Fig. 6.12).

6.12 The Schrödinger Equation

Schrödinger equation describes the behavior of non-relativistic quantum objects that are based on the wave-like behavior of wave mechanics.

6.12.1 Time Evolution in Quantum Mechanics

Time evolution is derived from two main principles of the state vector of the quantum object [7].

The first principle states that the time-dependent state vector is consists of a function of time and its state vector at time $t = 0$.

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle \tag{6.90a}$$

Thus,

$$U(0) = 1 \;\; at \;\; t = 0.$$

The formalism in Eq. (6.90a) is the time-dependent state vector in quantum mechanics.

The formulation of the complex number is used here to describe the time dependence of the state vector. If the Hamiltonian **H** is independent of time, the time-evolution operator is defined as

$$U(t) \equiv e^{\frac{-i\mathbf{H}t}{\hbar}} = \sum_{k=0}^{\infty} \frac{\left(\frac{-i\mathbf{H}t}{\hbar}\right)^k}{k!} \approx 1 - \frac{i\mathbf{H}t}{\hbar} + \frac{1}{2!}\left(\frac{-i\mathbf{H}t}{\hbar}\right)^2 + \cdots \tag{6.90b}$$

In the other case, if the Hamiltonian **H** is dependent on time, the time-evolution operator is defined as

$$U(t) \equiv \exp\left(\frac{-i}{\hbar}\int_0^t \mathbf{H}(\tau)d\tau\right) \tag{6.90c}$$

The second principle states that the below relation is always valid for any time.

$$\langle\Phi(t)|\Psi(t)\rangle = \langle\Phi(0)|\Psi(0)\rangle \, for \; \forall t \geq 0 \tag{6.91}$$

The bra $\langle\Phi(t)|$ is calculated from Eq. (6.90a) as

$$\begin{aligned} &|\Phi(t)\rangle = U(t)|\Phi(0)\rangle \\ &\Rightarrow \langle\Phi(t)| = U^{\dagger}(t)\langle\Phi(0)| \end{aligned}$$

Using Eqs. (6.90a) and (6.91), the inner product of bra and ket is written as

$$\begin{gathered} \langle\Phi(t)|\Psi(t)\rangle = \langle\Phi(0)|U^{\dagger}(t)\cdot U(t)|\Psi(0)\rangle = \langle\Phi(0)|\Psi(0)\rangle \\ \Rightarrow U^{\dagger}(t)\cdot U(t) = 1 \end{gathered} \tag{6.92}$$

This result is called the uniquetarity on time.

For time $t = \varepsilon \ll 1$, the time function $U(t)$ can be written using Eq. (6.90b) as

$$\begin{aligned} U(\varepsilon) &= 1 - \frac{i\varepsilon}{\hbar}\mathbf{H} \\ \Rightarrow U^{\dagger}(\varepsilon) &= 1 + \frac{i\varepsilon}{\hbar}\mathbf{H}^{\dagger} \end{aligned} \tag{6.93}$$

Using Eqs. (6.92) and (6.93), one calculates

$$\begin{aligned} U^{\dagger}(\varepsilon)\cdot U(\varepsilon) &= \left(1 + \frac{i\varepsilon}{\hbar}\mathbf{H}^{\dagger}\right)\cdot\left(1 - \frac{i\varepsilon}{\hbar}\mathbf{H}\right) \\ &= 1 - \frac{i\varepsilon}{\hbar}\mathbf{H} + \frac{i\varepsilon}{\hbar}\mathbf{H}^{\dagger} + \frac{\varepsilon^2}{\hbar^2}\mathbf{H}^{\dagger}\mathbf{H} \\ &\approx 1 + \frac{i\varepsilon}{\hbar}\left(\mathbf{H}^{\dagger} - \mathbf{H}\right) = 1 \end{aligned}$$

Thus, $\mathbf{H}^{\dagger} = \mathbf{H}$. This result denotes that the Hamiltonian $\mathbf{H}$ is a Hermitian operator (cf. Chap. 1).

The time derivative of the state vector is defined using Eq. (6.90a) as

$$\begin{aligned} \frac{d|\Psi(\varepsilon)\rangle}{d\varepsilon} &= \lim_{\varepsilon\to 0}\frac{U(\varepsilon)|\Psi(0)\rangle - |\Psi(0)\rangle}{(\varepsilon - 0)} = \lim_{\varepsilon\to 0}\frac{|\Psi(0)\rangle(U(\varepsilon) - 1)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{-|\Psi(0)\rangle\frac{i\varepsilon}{\hbar}\mathbf{H}}{\varepsilon} = -\frac{i\mathbf{H}}{\hbar}|\Psi(0)\rangle;\ \ \forall\varepsilon << 1 \end{aligned}$$

Multiplying both sides of the above equation by $U(t)$, one obtains

$$\begin{aligned} \frac{\partial|\Psi(t)\rangle}{\partial t} &= -\frac{i\mathbf{H}}{\hbar}|\Psi(t)\rangle \\ \Rightarrow i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} &= \mathbf{H}|\Psi(t)\rangle \end{aligned}$$

This above equation is called the fundamental Schrödinger equation for non-relativistic particles. Its general solution of the state vector results from Eqs. (6.90a) and (6.90b) as

$$\begin{aligned} &|\Psi(t)\rangle = U(t)|\Psi(0)\rangle = e^{\frac{-i\mathbf{H}t}{\hbar}}|\Psi(0)\rangle \\ &\Rightarrow \langle\Psi(t)| = e^{\frac{i\mathbf{H}t}{\hbar}}\langle\Psi(0)| \end{aligned}$$

6.12.2 The Schrödinger and Heisenberg Pictures

In the following section, the Schrödinger and Heisenberg pictures are introduced for the expectation value of the time-dependent operator **A**.

The *Schrödinger picture* is defined as the expectation value of a time-dependent observable **A** that is a Hermitian linear operator.

$$\begin{aligned}\langle \mathbf{A} \rangle_t &= \langle \Psi_S(t)|\mathbf{A}_S|\Psi_S(t)\rangle; \\ |\Psi_S(t)\rangle &= e^{\frac{-i\mathbf{H}t}{\hbar}}|\Psi(0)\rangle\end{aligned}$$

where $\mathbf{A}_S$ is an operator in the Schrödinger picture (Schrödinger representation).

The *Heisenberg picture* is defined using the time-dependent state vector as

$$\begin{aligned}\langle \mathbf{A} \rangle_t &= \langle \Psi_S(t)|\mathbf{A}_S|\Psi_S(t)\rangle = \langle \Psi(0)|e^{\frac{i\mathbf{H}t}{\hbar}}\mathbf{A}_S e^{\frac{-i\mathbf{H}t}{\hbar}}|\Psi(0)\rangle \\ &\equiv \langle \Psi(0)|\mathbf{A}(t)|\Psi(0)\rangle\end{aligned}$$

Thus, the time-dependent operator **A** in the Heisenberg picture results as

$$\mathbf{A}(t) = e^{\frac{i\mathbf{H}t}{\hbar}}\mathbf{A}_S e^{\frac{-i\mathbf{H}t}{\hbar}}$$

Replacing $\mathbf{A}_S$ by $\partial \mathbf{A}_S/\partial t$ considered as an observable operator in the Heisenberg picture [12], one obtains the corresponding Heisenberg operator

$$\frac{\partial \mathbf{A}(t)}{\partial t} = e^{\frac{i\mathbf{H}t}{\hbar}}\frac{\partial \mathbf{A}_S}{\partial t}e^{\frac{-i\mathbf{H}t}{\hbar}}$$

Time differentiating the time-dependent operator $\mathbf{A}(t)$, one obtains the Heisenberg equation of motion.

$$\begin{aligned}\frac{d\mathbf{A}}{dt} &= \frac{i\mathbf{H}}{\hbar}\cdot\left(e^{\frac{i\mathbf{H}t}{\hbar}}\mathbf{A}_S e^{\frac{-i\mathbf{H}t}{\hbar}}\right) + e^{\frac{i\mathbf{H}t}{\hbar}}\frac{\partial \mathbf{A}_S}{\partial t}e^{\frac{-i\mathbf{H}t}{\hbar}} \\ &\quad + \left(e^{\frac{i\mathbf{H}t}{\hbar}}\mathbf{A}_S e^{\frac{-i\mathbf{H}t}{\hbar}}\right)\cdot\left(-\frac{i\mathbf{H}}{\hbar}\right) \\ &= \frac{i}{\hbar}(\mathbf{HA}-\mathbf{AH}) + e^{\frac{i\mathbf{H}t}{\hbar}}\frac{\partial \mathbf{A}_S}{\partial t}e^{\frac{-i\mathbf{H}t}{\hbar}} = \frac{i}{\hbar}[\mathbf{H},\mathbf{A}] + \frac{\partial \mathbf{A}}{\partial t}\end{aligned}$$

where $[\mathbf{H}, \mathbf{A}]$ is the commutator of the operators **H** and **A**, cf. Eq. (6.65). Note that both operators usually do not commute.

However, there are some properties in the Heisenberg picture of a non-relativistic particle:

- The state vector in the Schrödinger picture is *time-dependent*.

$$|\Psi_S(t)\rangle = e^{\frac{-i\mathbf{H}t}{\hbar}}|\Psi(0)\rangle$$

- The operator $\mathbf{A}_S$ in the Schrödinger picture is *time-independent*.

$$\frac{\partial \mathbf{A}_S}{\partial t} = \frac{\partial \mathbf{A}}{\partial t} = 0$$

- The state vector in the Heisenberg picture is *time-independent*.

$$|\Psi(t)\rangle = e^{\frac{i\mathbf{H}t}{\hbar}}|\Psi_S(t)\rangle = e^{\frac{i\mathbf{H}t}{\hbar}}e^{\frac{-i\mathbf{H}t}{\hbar}}|\Psi(0)\rangle = |\Psi(0)\rangle$$

Therefore, *the Heisenberg equation* of motion can be written as

$$i\hbar\frac{d\mathbf{A}}{dt} = [\mathbf{A}, \mathbf{H}]$$

Having replaced the operator **A** by the Hamiltonian **H** in the Heisenberg equation, it gives using Eq. (6.66)

$$i\hbar\frac{d\mathbf{H}}{dt} = [\mathbf{H}, \mathbf{H}] = 0 \Rightarrow \mathbf{H}(t) = \mathbf{H}(0) \equiv \mathbf{H} = const.$$

This result indicates that the Hamiltonian **H** (Hermitian operator) does not depend on time explicitly, cf. Eq. (6.90b).

In summary, we obtain

- Solutions of the Schrödinger equation:

$$i\hbar\frac{d|\Psi(t)\rangle}{dt} = \mathbf{H}|\Psi\rangle \Rightarrow |\Psi_S(t)\rangle = e^{\frac{-i\mathbf{H}t}{\hbar}}|\Psi(0)\rangle$$

The state vector in the Schrödinger picture is determined by the initial condition $|\psi(0)\rangle$.

- Solutions of the Heisenberg equation of motion:

$$i\hbar\frac{d\mathbf{A}}{dt} = [\mathbf{A}, \mathbf{H}] \Rightarrow \mathbf{A}(t) = e^{\frac{i\mathbf{H}t}{\hbar}}\mathbf{A}_S e^{\frac{-i\mathbf{H}t}{\hbar}}$$

The Heisenberg picture $\mathbf{A}(t)$ is the solution of the motion equation.

Using either the Schrödinger picture or the Heisenberg picture, the expectation value of the time-dependent operator **A** results as the same value.

$$\langle \mathbf{A} \rangle_t = \langle \Psi_S(t) | \mathbf{A}_S | \Psi_S(t) \rangle = \langle \Psi(0) | \mathbf{A}(t) | \Psi(0) \rangle$$

6.12.3 Time-Dependent Schrödinger Equation (TDSE)

For simplicity, the Schrödinger equation is written in a one-dimensional space (1-D) using the wave function instead of the state vector as

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \mathbf{H}\Psi(x,t) \tag{6.94}$$

The Hamiltonian $\mathbf{H}$ is defined in classical mechanics as the sum of the kinetic energy T and potential energy V using Eq. (6.84) as

$$\mathbf{H} = T + V = \frac{\mathbf{p}^2}{2m} + V(x)$$

Using the momentum operator in Eq. (6.70), one obtains

$$\mathbf{p} = -i\hbar \frac{\partial}{\partial x} \Rightarrow \mathbf{p}^2 = -\hbar^2 \frac{\partial^2}{\partial x^2} \tag{6.95}$$

Thus, the RHS of Eq. (6.94) is expressed as

$$\begin{aligned} \mathbf{H}\Psi(x,t) &= \left(\frac{\mathbf{p}^2}{2m} + V(x) \right) \Psi(x,t) \\ &= \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \Psi(x,t) \end{aligned} \tag{6.96}$$

Substituting Eq. (6.96) into Eq. (6.94), one obtains *the time-dependent Schrödinger equation* (TDSE) in a one-dimensional space for a single non-relativistic particle.

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t) \tag{6.97}$$

In the case that the Hamiltonian $\mathbf{H}$ is not an explicit function of time t; i.e., the potential energy V only depends on x, the solution of the general wave function of Eq. (6.97) can be written using the separation method of variables as

$$\Psi(x,t) \equiv \psi(x) \cdot \varphi(t) \tag{6.98}$$

Thus,

$$\frac{\partial \Psi(x,t)}{\partial t} \equiv \psi(x)\frac{d\varphi(t)}{dt};$$
$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} \equiv \varphi(t)\frac{d^2\psi(x)}{dx^2}.$$

Substituting them into Eq. (6.97), one obtains

$$i\hbar\psi(x)\frac{d\varphi(t)}{dt} = \frac{-\hbar^2}{2m}\varphi(t)\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x)\cdot\varphi(t)$$

Dividing both sides of the above equation by $\psi(x)\varphi(t)$, it gives

$$\frac{i\hbar}{\varphi(t)}\frac{d\varphi(t)}{dt} = \frac{-\hbar^2}{2m\psi(x)}\frac{d^2\psi(x)}{dx^2} + V(x)$$

Both terms of the above equation are independent of each other; therefore, each term is independent of each other and must be therefore an invariant.

$$\begin{aligned} &\frac{i\hbar}{\varphi(t)}\frac{d\varphi(t)}{dt} = \frac{-\hbar^2}{2m\psi(x)}\frac{d^2\psi(x)}{dx^2} + V(x) \equiv E \\ &\Rightarrow \frac{i\hbar}{\varphi(t)}\frac{d\varphi(t)}{dt} = E \end{aligned} \tag{6.99}$$

The solution of the above equation results from integrating over dt as

$$\int\frac{d\varphi}{\varphi} = \int\frac{E}{i\hbar}dt = \int -\frac{iE}{\hbar}dt \Rightarrow \varphi(t) = e^{\frac{-iEt}{\hbar}} \tag{6.100}$$

Multiplying both sides of Eq. (6.99) by $\psi(x)$, one obtains

$$\begin{aligned} &\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) \equiv E\psi(x) \\ &\Leftrightarrow \mathbf{H}\psi(x) = E\psi(x) \end{aligned} \tag{6.101}$$

Equation (6.101) is called *the time independent Schrödinger equation* (TISE). It is obvious that E is the energy eigenvalue at the energy eigenstate wave function $\psi(x)$ of the Hamiltonian operator **H**.

The energy eigenstate solution is called the *stationary state solution* (also *standing wave solution*) at the energy eigenvalue E. The stationary state solution is the basis solution for the general solution of the wave function of the TDSE.

The energy eigenvalue E results from the expectation value (average value) of the Hamiltonian operator **H** using Eq. (6.101) as

$$\begin{aligned}\langle \mathbf{H} \rangle = \overline{\mathbf{H}} &= \int_{-\infty}^{+\infty} \psi^*(x)\mathbf{H}\psi(x)dx \\ &= \int_{-\infty}^{+\infty} \psi^*(x)E\psi(x)dx = E\int_{-\infty}^{+\infty} \psi^*(x)\psi(x)dx \\ &= E\end{aligned}$$

Inserting Eq. (6.100) into Eq. (6.98), the general solution of the wave function of the TDSE results as

$$\Psi(x,t) = \psi(x)e^{\frac{-iEt}{\hbar}} = \psi(x)e^{\frac{-i\overline{\mathbf{H}}t}{\hbar}} \tag{6.102}$$

In general, the time-dependent Schrödinger equation (3-D TDSE) is written in a three-dimensional space (3-D) in Cartesian coordinates as

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \mathbf{H}\Psi(\mathbf{r},t)$$

The momentum operator **p** of the particle in a Hilbert space is defined as

$$\begin{aligned}&\mathbf{p} \equiv -i\hbar\nabla \\ &\Rightarrow \mathbf{p}^2 = (-i\hbar\nabla)^2 = -\hbar^2\nabla^2 = -\hbar^2\Delta\end{aligned}$$

Thus, the Hamiltonian operator results from the kinetic and potential energies as

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\Delta + V(x)$$

Thus, the 3-D time-dependent Schrödinger equation becomes

$$\begin{aligned}i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} &= \frac{-\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t) + V(\mathbf{r})\Psi(\mathbf{r},t) \\ &= \frac{-\hbar^2}{2m}\Delta\Psi(\mathbf{r},t) + V(\mathbf{r})\Psi(\mathbf{r},t)\end{aligned} \tag{6.103}$$

where $\mathbf{r}$ is the position vector of the particle, $\mathbf{r}(x, y, z) \in \mathbf{R}^3$.

The general solution of Eq. (6.103) is written using Fourier series as

$$\Psi(\mathbf{r},t) = \sum_n c_n\psi_n(\mathbf{r})e^{\frac{-iE_nt}{\hbar}} \tag{6.104}$$

The coefficients c_n are used to satisfy the initial conditions at $t = 0$.

$$f(\mathbf{r}) \equiv \Psi(\mathbf{r}, 0) = \sum_{n=1}^{\infty} c_n \psi_n(\mathbf{r})$$

Applying Dirichlet's theorem to function $f(\mathbf{r})$, one obtains the coefficient c_m using Kronecker delta due to orthonormality of $\psi_n(\mathbf{r})$

$$\int_{-\infty}^{+\infty} \psi_m^*(\mathbf{r}) f(\mathbf{r}) d\mathbf{r} = \sum_{n=1}^{\infty} c_n \int_{-\infty}^{+\infty} \psi_m^*(\mathbf{r})\, \psi_n(\mathbf{r}) d\mathbf{r}$$
$$= \sum_{n=1}^{\infty} c_n \delta_{mn} = c_m$$

Thus, the coefficients c_n in Eq. (6.104) are calculated as

$$c_n = \int_{-\infty}^{+\infty} \psi_n^*(\mathbf{r}) f(\mathbf{r}) d\mathbf{r} \tag{6.105}$$

Analogously, the 3-D time independent Schrödinger equation (3-D TISE) of Eq. (6.101) results as

$$\frac{-\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}) + V(\mathbf{r}) \psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{6.106}$$

For N non-interacting particles in 3-D, the Hamiltonian operator $\mathbf{H}$ is written as

$$\mathbf{H} = \sum_n \left[\frac{\mathbf{p}_n \mathbf{p}_n}{2m_n} + V_n(x_n, y_n, z_n) \right]$$

The momentum operator is defined as

$$\mathbf{p}_n \equiv -i\hbar \nabla_n$$
$$\Rightarrow \mathbf{p}_n \mathbf{p}_n = \mathbf{p}_n^2 = -\hbar^2 \nabla_n^2 = -\hbar^2 \left(\frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_n^2} + \frac{\partial^2}{\partial z_n^2} \right)$$

Therefore, the 3-D time independent Schrödinger equation (3-D TISE) for N non-interacting particles results as

$$\left(-\sum_{n=1}^{N} \frac{\hbar^2}{2m_n} \nabla_n^2 + V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) \right) \psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = E\psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) \tag{6.107}$$

The general solution of the wave function of N non-interacting particles results as

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t) = e^{\frac{-iEt}{\hbar}} \prod_{n=1}^{N} \psi(\mathbf{r}_n) \tag{6.108}$$

6.12.4 Discussions of the Schrödinger Wave Functions

The quantum wave functions in Eqs. (6.104) and (6.108) of the Schrödinger equations have the amplitudes in complex numbers that always contain the imaginary number *i*. As a result, one can never measure their amplitudes of complex numbers in reality. Note that only the amplitudes of real numbers can be measured in experiments.

As having known, quantum mechanics is inherently a probabilistic theory. Thus, the squared amplitude of the Schrödinger wave function is interpreted as the probability of finding the particle at any point in space and at any particular time according to Born's interpretation as

$$P(\mathbf{r}, t) = \|\Psi(\mathbf{r}, t)\|^2 = \langle\Psi|\Psi\rangle = \Psi^*(\mathbf{r}, t) \cdot \Psi(\mathbf{r}, t) \in \mathbf{R}$$

In fact, the Schrödinger wave functions can never be measured by experiments in quantum mechanics. However, the squared amplitudes of the wave functions are measurable probabilistic quantities that describe the finding probabilities of the particle at any point in space and at any given time. Moreover, these particle probabilities strongly obey the Heisenberg uncertainty principle according to the Born's interpretation. Nowadays, the probabilistic theory of the Schrödinger wave function has been accepted as the nature of physical reality and therefore plays a fundamental component of modern physics [13].

6.13 The Klein-Gordon Equation

The Schrödinger wave equation is valid for non-relativistic particles in quantum mechanics. In fact, the Klein-Gordon equation is a *relativistic variance* of the Schrödinger equation.

The relativistic energy equation for a free particle is written as [2, 3, 13]

$$\mathbf{E}^2 = \mathbf{p}^2 c^2 + m^2 c^4 \tag{6.109}$$

The energy operator is defined as

$$\begin{aligned} \mathbf{E} &\equiv i\hbar\frac{\partial}{\partial t} \\ \Rightarrow \mathbf{E}^2 &= -\hbar^2\frac{\partial^2}{\partial t^2} \end{aligned} \tag{6.110}$$

The momentum operator is defined as

$$\begin{aligned} \mathbf{p} &\equiv -i\hbar\nabla \\ \Rightarrow \mathbf{p}^2 &= -\hbar^2\nabla^2 \end{aligned} \tag{6.111}$$

Substituting Eqs. (6.110) and (6.111) into Eq. (6.109), one obtains the Klein-Gordon equation

$$\begin{aligned} &\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \nabla^2\Psi + \frac{m^2c^2}{\hbar^2}\Psi \equiv \\ &\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \nabla^2\Psi + \mu^2\Psi = 0 \end{aligned} \tag{6.112}$$

where the factor μ is defined as

$$\mu \equiv \frac{mc}{\hbar}$$

In the spacetime, the Minkowski metric with four spacetime coordinates (t, x, y, z) results as (cf. Chap. 5)

$$g = \left(g_{\mu\nu}\right) = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In this case, the Klein-Gordon equation is written in the contravariant notation as

$$-\left(g_{\mu\nu}\partial^\mu\partial^\nu\right)\Psi + \mu^2\Psi = 0 \tag{6.113}$$

The general solution of Eq. (6.112) is the wave function of

$$\Psi(\mathbf{r}, t) = Ae^{[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]} \tag{6.114}$$

where

r is the position vector of the particle;
k is the wave vector of the particle;
ω is the angular frequency of the particle.
The position vector **r** is written as

$$\mathbf{r} = (x,\ y,\ z) \in \mathbf{R}^3$$

The wave vector **k** is written as

$$\mathbf{k} = \left(k_x,\ k_y,\ k_z\right) \in \mathbf{R}^3$$

in which the wavenumber k is defined as

$$k = \frac{2\pi}{\lambda} = \frac{p}{\hbar} = \frac{E}{\hbar c}$$

The angular frequency ω is defined as

$$\omega = 2\pi\nu = \frac{2\pi c}{\lambda} = \frac{cp}{\hbar} = ck$$
$$\Rightarrow k = \frac{\omega}{c}$$

The 4-wavevector in special relativity is written as

$$\mathbf{k} = \left(\frac{\omega}{c},\ k_x,\ k_y,\ k_z\right) \in \mathbf{R}^4$$

Partially differentiating with respect to t and **r** the wave function in Eq. (6.114), one obtains

$$\begin{aligned} \frac{\partial^2\Psi(\mathbf{r},t)}{\partial t^2} &= -\omega^2\Psi(\mathbf{r},t); \\ \frac{\partial^2\Psi(\mathbf{r},t)}{\partial \mathbf{r}^2} &= -\mathbf{k}^2\Psi(\mathbf{r},t). \end{aligned} \tag{6.115}$$

Substituting Eq. (6.115) into Eq. (6.112), it gives the dispersion relation

$$\left(\frac{\omega}{c}\right)^2 - \|\mathbf{k}\|^2 = \left(\frac{\omega}{c}\right)^2 - \left(k_x^2 + k_y^2 + k_z^2\right) = \frac{m^2c^2}{\hbar^2} \tag{6.116}$$

6.14 The Dirac Equation

The Schrödinger wave equation does not include spin vectors or rotations that happen in the everyday sense. Furthermore, they are very important in the Pauli matrices, cf. Sect. 6.6.

Let $\Psi(\mathbf{r},t)$ be the state vector (state ket) of the wave function. The wave state ket is written in the non-relativistic case as

$$|\Psi(\mathbf{r},t)\rangle = \begin{pmatrix} \psi_1(\mathbf{r},t) \\ \psi_2(\mathbf{r},t) \\ \cdots \\ \psi_N(\mathbf{r},t) \end{pmatrix} \tag{6.117}$$

The sth component of the state vector on the spin vector s is defined as

$$\psi_s(\mathbf{r},t) \equiv \langle \mathbf{r}s|\psi(t)\rangle \tag{6.118}$$

where $\mathbf{r}$ is the orbital variable vector, and s are the spin variables ($s = 1, 2,\ldots,N$) of the state vector.

The probability function is the squared magnitude of the wave function, cf. Sect. 6.11.3.

$$P(\mathbf{r},t) = \|\Psi(\mathbf{r},t)\|^2 = \sum_{s=1}^{N} \left\|\psi_s^2\right\| \tag{6.119}$$

Thus, the wave function is expressed as

$$i\hbar \frac{\partial \Psi(\mathbf{r},t)}{\partial t} = \mathbf{H}_D \Psi(\mathbf{r},t) \tag{6.120}$$

where $\mathbf{H}_D$ is the Dirac Hamilton operator that is also a Hermitian operator of the state-vector space.

The Dirac Hamilton operator $\mathbf{H}_D$ is defined as

$$\mathbf{H}_D \equiv \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m \tag{6.121}$$

where

$\boldsymbol{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ is the spin operator with three spin variables;
$\mathbf{p}$ is the momentum operator;
β is the spin variable;
m is the rest mass of the particle.

The energy operator is written using Eq. (6.110) as

$$\mathbf{E} \equiv i\hbar \frac{\partial}{\partial t}$$

The momentum operator is written using Eq. (6.111) as

$$\mathbf{p} \equiv -i\hbar \nabla$$

Substituting Eq. (6.121) and both operators into Eq. (6.120), the wave function becomes

$$\left(i\hbar\frac{\partial}{\partial t} - \mathbf{H}_D\right)\Psi(\mathbf{r},t) = [\mathbf{E} - \boldsymbol{\alpha}\cdot\mathbf{p} - \beta m]\Psi(\mathbf{r},t) = 0 \tag{6.122}$$

Equation (6.122) is called the Dirac equation.

Multiplying the LHS of Eq. (6.122) by the term $[\mathbf{E} + \alpha \cdot \mathbf{p} + \beta m]$, one obtains [12]

$$\begin{pmatrix} \mathbf{E}^2 - \sum_k \alpha_k^2 p_k^2 - m^2\beta^2 - \sum_{k<l} (\alpha_k\alpha_l + \alpha_l\alpha_k) p_k p_l \\ -\sum_k (\alpha_k\beta + \beta\alpha_k) m p_k \end{pmatrix} \Psi(\mathbf{r},t) = 0 \tag{6.123}$$

The Klein-Gordon equation (6.112) can be written in forms of the energy and momentum operators as

$$\begin{aligned} &\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \nabla^2\Psi + \frac{m^2c^2}{\hbar^2}\Psi = 0 \\ &\Leftrightarrow \left[\mathbf{E}^2 - \mathbf{p}^2c^2 - m^2c^4\right]\Psi(\mathbf{r},t) = 0 \end{aligned} \tag{6.124}$$

Equation (6.123) is identical to the Klein-Gordon equation if the spin variables in Eq. (6.123) are anticommute and their amplitudes satisfy the following conditions

$$\begin{aligned} &(\alpha_k\alpha_l + \alpha_l\alpha_k) = 0 \ \text{ for } k \neq l; \\ &(\alpha_k\beta + \beta\alpha_k) = 0; \\ &\alpha_k^2 = \beta \equiv c^2 \end{aligned} \tag{6.125}$$

The spin variables result from the Pauli's matrices in case of the light speed is chosen $c \equiv 1$.

$$\begin{aligned} &\sigma_x = -i\alpha_y\alpha_z; \ \ \sigma_y = -i\alpha_z\alpha_x; \ \ \sigma_z = -i\alpha_x\alpha_y \\ &\tau_1 = \sigma_z\alpha_z = -i\alpha_x\alpha_y\alpha_z; \\ &\tau_2 = -i\tau_1\tau_3 = -\beta\alpha_x\alpha_y\alpha_z; \\ &\tau_3 = \beta \end{aligned} \tag{6.126}$$

The Pauli's matrices in Eq. (6.126) are defined as, cf. Sect. 6.6

$$\begin{aligned} \sigma_x \equiv \tau_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \\ \sigma_y \equiv \tau_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \\ \sigma_z \equiv \tau_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{6.127}$$

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Appendix A: Relations Between Covariant and Contravariant Bases

The contravariant basis vector $\mathbf{g}^k$ of the curvilinear coordinate of u^k at the point P is perpendicular to the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$, as shown in Fig. A.1. This contravariant basis $\mathbf{g}^k$ can be defined as

$$\alpha\, \mathbf{g}^k \equiv \mathbf{g}_i \times \mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial u^i} \times \frac{\partial \mathbf{r}}{\partial u^j} \tag{A.1}$$

where α is the scalar factor; $\mathbf{g}^k$ is the contravariant basis of the curvilinear coordinate of u^k.

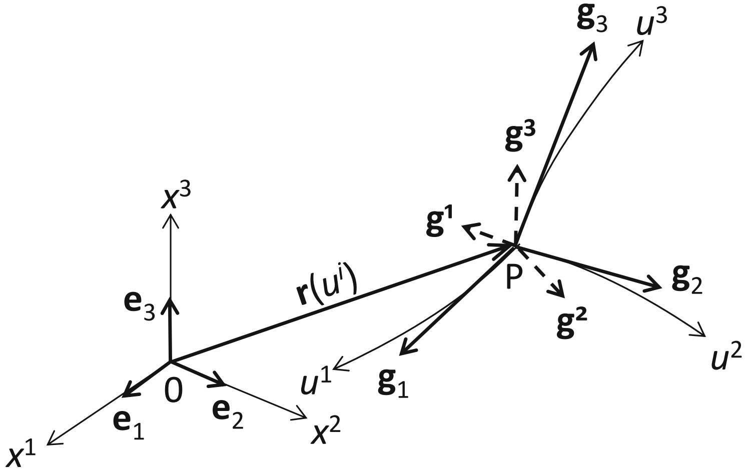

Fig. A.1 Covariant and contravariant bases of curvilinear coordinates

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Multiplying Eq. (A.1) by the covariant basis $\mathbf{g}_k$, the scalar factor α results in

$$\begin{aligned} &\left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot \mathbf{g}_k = \alpha\left(\mathbf{g}^k \cdot \mathbf{g}_k\right) = \alpha\, \delta_k^k = \alpha \\ &\Rightarrow \alpha = \left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot \mathbf{g}_k \equiv \left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right] \end{aligned} \tag{A.2}$$

The scalar triple product of the covariant bases can be written as

$$\alpha = [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 = \sqrt{g} = J \tag{A.3}$$

where Jacobian J is the determinant of the covariant basis tensor $\mathbf{G}$.

The direction of the cross-product vector in Eq. (A.1) is opposite if the dummy indices are interchanged with each other in Einstein summation convention. Therefore, the Levi-Civita permutation symbols (pseudo-tensor components) can be used in expression of the contravariant basis.

$$\begin{aligned} &\sqrt{g}\, \mathbf{g}^k = J\, \mathbf{g}^k = \left(\mathbf{g}_i \times \mathbf{g}_j\right) = -\left(\mathbf{g}_j \times \mathbf{g}_i\right) \\ &\Rightarrow \mathbf{g}^k = \frac{\varepsilon_{ijk}\left(\mathbf{g}_i \times \mathbf{g}_j\right)}{\sqrt{g}} = \frac{\varepsilon_{ijk}\left(\mathbf{g}_i \times \mathbf{g}_j\right)}{J} \end{aligned} \tag{A.4}$$

where the Levi-Civita permutation symbols are defined by

$$\begin{aligned} &\varepsilon_{ijk} = \begin{cases} +1 \text{ if } (i,\ j,\ k) \text{ is an even permutation;} \\ -1 \text{ if } (i,\ j,\ k) \text{ is an odd permutation;} \\ 0 \quad \text{if } i = j, \text{ or } i = k; \text{ or } j = k \end{cases} \\ &\Leftrightarrow \varepsilon_{ijk} = \frac{1}{2}(i-j) \cdot (j-k) \cdot (k-i) \text{ for } i,j,k = 1,2,3 \end{aligned} \tag{A.5}$$

Thus, the cross product of the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$ results from Eq. (A.4):

$$\begin{aligned} &\left(\mathbf{g}_i \times \mathbf{g}_j\right) = \varepsilon_{ijk}\sqrt{g}\, \mathbf{g}^k = \varepsilon_{ijk} J \mathbf{g}^k \equiv \hat{\varepsilon}_{ijk} \mathbf{g}^k \\ &\Rightarrow \mathbf{g}^k = \frac{\varepsilon_{ijk}\left(\mathbf{g}_i \times \mathbf{g}_j\right)}{\sqrt{g}} = \hat{\varepsilon}^{\,ijk}\left(\mathbf{g}_i \times \mathbf{g}_j\right) \\ &\Rightarrow \hat{\varepsilon}_{ijk} = \left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot \mathbf{g}_k \end{aligned} \tag{A.6}$$

The covariant permutation symbols in Eq. (A.6) can be defined as

$$\hat{\varepsilon}_{ijk} = \begin{cases} +\sqrt{g} \text{ if } (i,\ j,\ k) \text{ is an even permutation;} \\ -\sqrt{g} \text{ if } (i,\ j,\ k) \text{ is an odd permutation;} \\ \quad 0 \quad \text{if } i = j, \text{ or } i = k; \text{ or } j = k \end{cases} \tag{A.7}$$

The contravariant permutation symbols in Eq. (A.6) can be defined as

$$\hat{\varepsilon}^{ijk} = \begin{cases} +\dfrac{1}{\sqrt{g}} \text{ if } (i,\ j,\ k) \text{ is an even permutation;} \\ -\dfrac{1}{\sqrt{g}} \text{ if } (i,\ j,\ k) \text{ is an odd permutation;} \\ \quad 0 \quad \text{if } i = j, \text{ or } i = k; \text{ or } j = k \end{cases} \tag{A.8}$$

The covariant basis vector $\mathbf{g}_k$ of the curvilinear coordinate of u^k at the point P is perpendicular to the contravariant bases $\mathbf{g}^i$ and $\mathbf{g}^j$, as shown in Fig. A.1. Therefore, the cross product of the contravariant bases $\mathbf{g}^i$ and $\mathbf{g}^j$ can be written as

$$\begin{aligned} &(\mathbf{g}^i \times \mathbf{g}^j) = \frac{\varepsilon_{ijk}}{\sqrt{g}}\mathbf{g}_k = \frac{\varepsilon_{ijk}}{J}\mathbf{g}_k \equiv \hat{\varepsilon}^{ijk}\mathbf{g}_k \\ &\Rightarrow \hat{\varepsilon}^{ijk} = (\mathbf{g}^i \times \mathbf{g}^j) \cdot \mathbf{g}^k \end{aligned} \tag{A.9}$$

Thus, the covariant basis results from Eq. (A.9):

$$\begin{aligned} \mathbf{g}_k &= \varepsilon_{ijk}\sqrt{g}\left(\mathbf{g}^i \times \mathbf{g}^j\right) = \varepsilon_{ijk}J\left(\mathbf{g}^i \times \mathbf{g}^j\right) \\ &= \hat{\varepsilon}_{ijk}\left(\mathbf{g}^i \times \mathbf{g}^j\right) \end{aligned} \tag{A.10}$$

Obviously, there are some relations between the covariant and contravariant permutation symbols:

$$\begin{aligned} &\hat{\varepsilon}^{ijk}\hat{\varepsilon}_{ijk} = 1 \quad \text{(no summation)} \\ &\hat{\varepsilon}_{ijk} = \hat{\varepsilon}^{ijk}J^2 \text{(no summation)} \end{aligned} \tag{A.11}$$

The tensor product of the covariant and contravariant permutation pseudo-tensors is a sixth-order tensor.

$$\hat{\varepsilon}^{ijk}\hat{\varepsilon}_{pqr} = \delta^{ijk}_{pqr} = \begin{cases} +1; \quad (i,\ j,\ k) \text{ and } (l, m, n) \text{ even permutation} \\ -1; \quad (i,\ j,\ k) \text{ and } (l, m, n) \text{ odd permutation} \\ \quad 0; \quad \qquad \qquad \qquad \text{otherwise} \end{cases} \tag{A.12}$$

The sixth-order Kronecker tensor can be written in the determinant form:

$$\hat{\varepsilon}^{ijk}\hat{\varepsilon}_{pqr} = \delta^{ijk}_{pqr} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^k_p & \delta^k_q & \delta^k_r \end{vmatrix} \tag{A.13}$$

Using the tensor contraction rules with $k = r$, one obtains

$$\begin{aligned} \delta^{ij}_{pq} &= \delta^{ijr}_{pqr} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^r_p & \delta^r_q & \delta^r_r \end{vmatrix} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ 0 & 0 & 1 \end{vmatrix} \\ \Rightarrow \hat{\varepsilon}^{ij}\hat{\varepsilon}_{pq} &= \delta^{ij}_{pq} = 1.\begin{vmatrix} \delta^i_p & \delta^i_q \\ \delta^j_p & \delta^j_q \end{vmatrix} = \delta^i_p\delta^j_q - \delta^i_q\delta^j_p \end{aligned} \tag{A.14}$$

Further contraction of Eq. (A.14) with $j = q$ gives

$$\begin{aligned} \hat{\varepsilon}^{iq}\hat{\varepsilon}_{pq} &= \delta^i_p\delta^q_q - \delta^i_q\delta^q_p \\ &= \delta^i_p\delta^q_q - \delta^i_p = 2\delta^i_p - \delta^i_p = \delta^i_p \ \text{ for } i,p \ = \ 1,2 \end{aligned} \tag{A.15}$$

From Eq. (A.15), the next contraction with $i = p$ gives

$$\begin{aligned} \hat{\varepsilon}^{pq}\hat{\varepsilon}_{pq} &= \delta^p_p \quad (\text{summation over p}) \\ &= \delta^1_1 + \delta^2_2 = 2 \ \text{ for } \ p \ = \ 1,2 \end{aligned} \tag{A.16}$$

Similarly, contracting Eq. (A.13) with $k = r$; $j = q$, one has for a three-dimensional space.

$$\begin{aligned} \hat{\varepsilon}^{iq}\hat{\varepsilon}_{pq} &= \delta^i_p\delta^q_q - \delta^q_p\delta^i_q \\ &= \delta^i_p\delta^q_q - \delta^i_p = 3\delta^i_p - \delta^i_p = 2\delta^i_p \ \text{ for } i,p \ = \ 1,2,3 \end{aligned} \tag{A.17}$$

Contracting Eq. (A.17) with $i = p$, one obtains

$$\begin{aligned} \hat{\varepsilon}^{pq}\hat{\varepsilon}_{pq} &= 2\delta^p_p \quad (\text{summation over p}) \\ &= 2\left(\delta^1_1 + \delta^2_2 + \delta^3_3\right) \\ &= 2(1+1+1) = 6 \ \text{ for } p \ = \ 1,2,3 \end{aligned} \tag{A.18}$$

The covariant metric tensor **M** can be written as

$$\mathbf{M} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{A.19}$$

where the covariant metric coefficients are defined by $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$.

The contravariant metric coefficients in the contravariant metric tensor $\mathbf{M}^{-1}$ result from inverting the covariant metric tensor $\mathbf{M}$.

$$\mathbf{M}^{-1} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} \tag{A.20}$$

where the contravariant metric coefficients are defined by $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$.

Thus, the relation between the covariant and contravariant metric coefficients can be written as

$$g^{ik} g_{kj} = g_{kj} g^{ik} = \delta^i_j \Leftrightarrow \mathbf{M}^{-1}\mathbf{M} = \mathbf{M}\mathbf{M}^{-1} = \mathbf{I} \tag{A.21}$$

In case of $i \neq j$, all terms of $g^{ik} g_{kj}$ equal zero. Thus, only nine terms of $g^{ik} g_{ki}$ for $i = j$ remain in a three-dimensional space $\mathbf{R}^3$:

$$\begin{aligned} g^{ik} g_{ki} &= g^{1k} g_{k1} + g^{2k} g_{k2} + g^{3k} g_{k3} \text{ for } i, k = 1,\ 2,\ 3 \\ &= \delta^1_1 + \delta^2_2 + \delta^3_3 = \delta^i_i \text{ for } i\ =\ 1, 2, 3 \\ &= 1 + 1 + 1 = 3 \end{aligned} \tag{A.22}$$

The relation between the covariant and contravariant bases in the general curvilinear coordinates results in

$$\begin{aligned} \mathbf{g}^i \cdot \mathbf{g}_j &= g^{ik} g_{kj} = \delta^i_j; \text{ for } i\ \equiv\ j\ \ \Rightarrow \\ \mathbf{g}^i \cdot \mathbf{g}_i &= \mathbf{g}^1 \cdot \mathbf{g}_1 + \mathbf{g}^2 \cdot \mathbf{g}_2 + \mathbf{g}^3 \cdot \mathbf{g}_3 \text{ for } i\ = 1, 2, 3 \\ &= g^{ik} g_{ki} = \delta^i_i \text{ for } i, k\ = 1, 2, 3 \end{aligned} \tag{A.23}$$

According to Eq. (A.23), nine terms of $g^{ik} g_{ki}$ for $k = 1,2,3$ result in

$$\begin{cases} \mathbf{g}^1 \cdot \mathbf{g}_1 = g^{1k} g_{k1} = g^{11} g_{11} + g^{12} g_{21} + g^{13} g_{31}\ = \delta^1_1 \text{ for } i\ =\ 1; \\ \mathbf{g}^2 \cdot \mathbf{g}_2 = g^{2k} g_{k2} = g^{21} g_{12} + g^{22} g_{22} + g^{23} g_{32}\ = \delta^2_2 \text{ for } i\ = 2; \\ \mathbf{g}^3 \cdot \mathbf{g}_3 = g^{3k} g_{k3} = g^{31} g_{13} + g^{32} g_{23} + g^{33} g_{33} = \delta^3_3 \text{ for } i =\ 3. \end{cases} \tag{A.24}$$

The scalar product of the covariant and contravariant bases gives

$$\begin{aligned} \mathbf{g}^{(i)} \cdot \mathbf{g}_{(i)} &= \left|\mathbf{g}^{(i)}\right| \cdot \left|\mathbf{g}_{(i)}\right| \cos\left(\mathbf{g}^{(i)}, \mathbf{g}_{(i)}\right) \\ &= \sqrt{g^{(ii)}} \cdot \sqrt{g_{(ii)}} \cos\left(\mathbf{g}^{(i)}, \mathbf{g}_{(i)}\right) = 1 \end{aligned} \tag{A.25}$$

where the index (i) means no summation is carried out over i.

Equation (A.25) indicates that the product of the covariant and contravariant basis norms generally does not equal 1 in the curvilinear coordinates.

$$\sqrt{g^{(ii)}} \cdot \sqrt{g_{(ii)}} = \frac{1}{\cos\left(\mathbf{g}^{(i)}, \mathbf{g}_{(i)}\right)} \geq 1 \tag{A.26}$$

In <u>orthogonal</u> coordinate systems, $\mathbf{g}^{(i)}$ is parallel to $\mathbf{g}_{(i)}$. Therefore, Eq. (A.26) becomes

$$\sqrt{g^{(ii)}} \cdot \sqrt{g_{(ii)}} = 1 \Rightarrow \sqrt{g^{(ii)}} = \frac{1}{\sqrt{g_{(ii)}}} = \frac{1}{h_i} \tag{A.27}$$

Appendix B: Physical Components of Tensors

The physical component of a tensor can be defined as the tensor component on its unitary covariant basis. Therefore, the covariant basis of the general curvilinear coordinates has to be normalized.

Dividing the covariant basis by its vector length, the unitary covariant basis (covariant normalized basis) results in

$$\mathbf{g}_i^* = \frac{\mathbf{g}_i}{|\mathbf{g}_i|} = \frac{\mathbf{g}_i}{\sqrt{g_{(ii)}}} \Rightarrow \left|\mathbf{g}_i^*\right| = 1 \tag{B.1a}$$

The covariant basis norm $|\mathbf{g}_i|$ can be considered as a scale factor h_i without summation over (i).

$$h_i = |\mathbf{g}_i| = \sqrt{g_{(ii)}} \tag{B.1b}$$

Thus, the covariant basis can be related to its unitary covariant basis by the relation

$$\mathbf{g}_i = \sqrt{g_{(ii)}}\mathbf{g}_i^* = h_i \mathbf{g}_i^* \tag{B.2}$$

The contravariant basis can be related to its unitary covariant basis using Eqs. (2.47) and (B.2).

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j = g^{ij} h_j \mathbf{g}_j^* \tag{B.3}$$

The contravariant second-order tensor can be written in the unitary covariant bases using Eq. (B.2).

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$$\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j = \left(T^{ij}h_ih_j\right)\mathbf{g}_i^*\mathbf{g}_j^* \equiv T^{*ij}\mathbf{g}_i^*\mathbf{g}_j^* \tag{B.4}$$

Thus, the physical contravariant tensor components denoted by *star* result in

$$T^{*ij} \equiv h_ih_jT^{ij} \tag{B.5}$$

The covariant second-order tensor can be written in the unitary contravariant bases using Eq. (B.3).

$$\mathbf{T} = T_{ij}\mathbf{g}^i\mathbf{g}^j = \left(T_{ij}g^{ik}g^{jl}h_kh_l\right)\mathbf{g}_k^*\mathbf{g}_l^* \equiv T_{ij}^*\mathbf{g}_k^*\mathbf{g}_l^* \tag{B.6}$$

Similarly, the physical covariant tensor components denoted by *star* result in

$$T_{ij}^* \equiv g^{ik}g^{jl}h_kh_lT_{ij} \tag{B.7}$$

The mixed tensors can be written in the unitary covariant bases using Eqs. (B.2) and (B.3)

$$\begin{aligned}\mathbf{T} = T_j^i\mathbf{g}_i\mathbf{g}^j &= T_j^i\mathbf{g}_i\left(g^{jk}\mathbf{g}_k\right)\\ &= T_j^i\left(h_i\mathbf{g}_i^*\right)\left(g^{jk}h_k\mathbf{g}_k^*\right)\\ &= \left(T_j^ig^{jk}h_ih_k\right)\mathbf{g}_i^*\mathbf{g}_k^*\\ &\equiv \left(T_j^i\right)^*\mathbf{g}_i^*\mathbf{g}_k^*\end{aligned} \tag{B.8}$$

Thus, the physical mixed tensor components denoted by *star* result in

$$\left(T_j^i\right)^* \equiv g^{jk}h_ih_kT_j^i \tag{B.9}$$

Analogously, the contravariant vector can be written using Eq. (B.2).

$$\begin{aligned}\mathbf{v} = v^i\mathbf{g}_i &= (v^ih_i)\mathbf{g}_i^*\\ &\equiv v^{*i}\mathbf{g}_i^* = \frac{v^{*i}}{h_i}\mathbf{g}_i\end{aligned} \tag{B.10}$$

Thus, the physical component of the contravariant vector $\mathbf{v}$ on the unitary basis $\mathbf{g}_i^*$ is defined as

$$v^{*i} \equiv h_iv^i = \sqrt{g_{(ii)}}v^i \tag{B.11}$$

The contravariant basis can be normalized dividing by its vector length without summation over (i).

$$\mathbf{g}^{*i} = \frac{\mathbf{g}^i}{|\mathbf{g}^i|} = \frac{\mathbf{g}^i}{\sqrt{g^{(ii)}}} \tag{B.12}$$

where $g^{(ii)}$ is the contravariant metric coefficient that results from Eq. (A.20).

Using Eq. (B.12), the covariant vector $\mathbf{v}$ can be written as

$$\mathbf{v} = v_i \mathbf{g}^i = v_i \sqrt{g^{(ii)}} \mathbf{g}^{*i} \equiv v_i^* \mathbf{g}^{*i} \tag{B.13}$$

Thus, the physical component of the covariant vector $\mathbf{v}$ results in

$$v_i^* = v_i \sqrt{g^{(ii)}} \tag{B.14}$$

According to Eq. (A.27), Eq. (B.14) can be rewritten in <u>orthogonal</u> coordinate systems:

$$v_i^* = \sqrt{g^{(ii)}} v_i = \frac{1}{\sqrt{g_{(ii)}}} v_i = \frac{1}{h_i} v_i \tag{B.15}$$

Using Eq. (B.3), the covariant vector can be written in

$$\begin{aligned} \mathbf{v} &= v_i \mathbf{g}^i = v_i g^{ij} \mathbf{g}_j \\ &= \left(v_i g^{ij} h_j\right) \mathbf{g}_j^* \equiv v_j^* \mathbf{g}_j^* \\ &= \frac{v_j^*}{g^{ij} h_j} \mathbf{g}^i \end{aligned} \tag{B.16}$$

Thus, the physical contravariant vector component of $\mathbf{v}$ on the unitary basis $\mathbf{g}_j^*$ can be defined as

$$v_j^* \equiv g^{ij} h_j v_i \tag{B.17}$$

Furthermore, the vector $\mathbf{v}$ can be written in both covariant and contravariant bases.

$$\begin{aligned} &\mathbf{v} = v_j \mathbf{g}^j = v^i \mathbf{g}_i \\ &\Rightarrow \left(v_j \mathbf{g}^j\right) \cdot \mathbf{g}_k = \left(v^i \mathbf{g}_i\right) \cdot \mathbf{g}_k \\ &\Rightarrow v_j \delta_k^j = v^i g_{ik} \\ &\Rightarrow v_k = v^i g_{ik} \end{aligned} \tag{B.18}$$

Interchanging i with j and k with i, one obtains

$$v_i = v^j g_{ij} \tag{B.19}$$

Only in <u>orthogonal</u> coordinate systems, we have

$$g_{ij} = 0 \text{ for } i \neq j; \quad g_{(ii)} = h_i^2 \tag{B.20}$$

Thus, one obtains from Eq. (B.19)

$$\begin{aligned} v_i &= v^j g_{ij} = v^1 g_{i1} + v^2 g_{i2} + \ldots + v^N g_{iN} \\ &= v^i g_{(ii)} = v^i h_i^2 \end{aligned} \tag{B.21}$$

Substituting Eq. (B.11) into Eq. (B.21), one obtains Eq. (B.22) that is equivalent to Eq. (B.15).

$$v_i = v^i h_i^2 = \left(\frac{v^{*i}}{h_i} \right) h_i^2 = h_i v^{*i} \tag{B.22}$$

Appendix C: Nabla Operators

Some useful Nabla operators are listed in Cartesian and general curvilinear coordinates:

1. **Gradient of an invariant *f***
   - Cartesian coordinate $\{x^i\}$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y + \frac{\partial f}{\partial z}\mathbf{e}_z \tag{C.1}$$

   - General curvilinear coordinate $\{u^i\}$

$$\nabla f = f_{,i}\mathbf{g}^i = \frac{\partial f}{\partial u^i}\mathbf{g}^i = \frac{\partial f}{\partial u^i} g^{ij}\mathbf{g}_j{}^j \tag{C.2}$$

2. **Gradient of a vector v**
   - General curvilinear coordinate $\{u^i\}$

$$\begin{aligned}\nabla \mathbf{v} &= \left(v^k_{,i} + v^j \Gamma^k_{ij}\right)\mathbf{g}_k\mathbf{g}^i = v^k\big|_i \mathbf{g}_k\mathbf{g}^i \\ &= \left(v_{k,i} - v_j \Gamma^j_{ik}\right)\mathbf{g}^k\mathbf{g}^i = v_k|_i \mathbf{g}^k\mathbf{g}^i\end{aligned} \tag{C.3}$$

3. **Divergence of a vector v**
   - Cartesian coordinate $\{x^i\}$

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$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{C.4}$$

- General curvilinear coordinate $\{u^i\}$

$$\begin{aligned}\nabla \cdot \mathbf{v} &= v^i|_i \equiv \left(v^i_{,i} + v^j \Gamma^i_{ij}\right) \\ &= \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g} v^i\right)}{\partial u^i} = \frac{1}{\sqrt{g}}\left(\sqrt{g} v^i\right)_{,i} \\ \nabla \cdot \mathbf{v} &= v_k|_i g^{ki} = \left(v_{k,i} - v_j \Gamma^j_{ik}\right) g^{ki}\end{aligned} \tag{C.5}$$

4. **Gradient of a second-order tensor T**

- General curvilinear coordinate $\{u^i\}$ for a covariant second-order tensor

$$\begin{aligned}\nabla \mathbf{T} &= \left(T_{ij,k} - \Gamma^m_{ik} T_{mj} - \Gamma^m_{jk} T_{im}\right) \mathbf{g}^i \mathbf{g}^j \mathbf{g}^k \\ &= T_{ij}|_k \mathbf{g}^i \mathbf{g}^j \mathbf{g}^k\end{aligned} \tag{C.6}$$

- General curvilinear coordinate $\{u^i\}$ for a contravariant second-order tensor

$$\begin{aligned}\nabla \mathbf{T} &= \left(T^{ij}_{,k} + \Gamma^i_{km} T^{mj} + \Gamma^j_{km} T^{im}\right) \mathbf{g}_i \mathbf{g}_j \mathbf{g}^k \\ &= T^{ij}|_k \mathbf{g}_i \mathbf{g}_j \mathbf{g}^k\end{aligned} \tag{C.7}$$

- General curvilinear coordinate $\{u^i\}$ for a mixed second-order tensor

$$\begin{aligned}\nabla \mathbf{T} &= \left(T^i_{j,k} + \Gamma^i_{km} T^m_j - \Gamma^m_{jk} T^i_m\right) \mathbf{g}_i \mathbf{g}^j \mathbf{g}^k \\ &= T^i_j|_k \mathbf{g}_i \mathbf{g}^j \mathbf{g}^k\end{aligned} \tag{C.8}$$

5. **Divergence of a second-order tensor T**

- General curvilinear coordinate $\{u^i\}$ for a covariant second-order tensor

$$\begin{aligned}\nabla \cdot \mathbf{T} &= \left(T_{ij,k} - \Gamma^m_{ik} T_{mj} - \Gamma^m_{jk} T_{im}\right) \mathbf{g}^i \left(\mathbf{g}^j \cdot \mathbf{g}^k\right) \\ &\equiv T_{ij}|_k g^{jk} \mathbf{g}^i\end{aligned} \tag{C.9}$$

- General curvilinear coordinate $\{u^i\}$ for a contravariant second-order tensor

$$\begin{aligned}\nabla \cdot \mathbf{T} &= \left(T^{ij}_{,k} + \Gamma^i_{km} T^{mj} + \Gamma^j_{km} T^{im}\right) \delta^k_i \mathbf{g}_j \\ &= \left(T^{ij}_{,i} + \Gamma^i_{im} T^{mj} + \Gamma^j_{im} T^{im}\right) \mathbf{g}_j \\ &\equiv T^{ij}|_i \mathbf{g}_j\end{aligned} \tag{C.10}$$

- General curvilinear coordinate $\{u^i\}$ for a mixed second-order tensor

$$\begin{aligned}\nabla \cdot \mathbf{T} &= \left(T^i_{j,k} + \Gamma^i_{km}T^m_j - \Gamma^m_{jk}T^i_m\right)\delta^k_i \mathbf{g}^j \\ &= \left(T^i_{j,i} + \Gamma^i_{im}T^m_j - \Gamma^m_{ji}T^i_m\right)\mathbf{g}^j \\ &\equiv T^i_j|_i \mathbf{g}^j = T^i_j|_i g^{jk}\mathbf{g}_k\end{aligned} \tag{C.11}$$

6. **Curl of a vector v**

- Cartesian coordinate $\{x^i\}$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \tag{C.12}$$

The curl of **v** results from calculating the determinant of Eq. (C.12).

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{e}_z \tag{C.13}$$

- General curvilinear coordinate $\{u^i\}$

$$\nabla \times \mathbf{v} = \hat{\varepsilon}^{\,ijk} v_{j,i}\mathbf{g}_k \tag{C.14}$$

The contravariant permutation symbol is defined by

$$\hat{\varepsilon}^{\,ijk} = \begin{cases} +\dfrac{1}{\sqrt{g}} \text{ if } (i,\ j,\ k) \textit{ is an even permutation}; \\ -\dfrac{1}{\sqrt{g}} \text{ if } (i,\ j,\ k) \textit{ is an odd permutation}; \\ \ 0 \quad \text{if } i = j, \textit{ or } i = k; \textit{ or } j = k \end{cases} \tag{C.15}$$

where J is the Jacobian.

7. **Laplacian of an invariant *f***

- Cartesian coordinate $\{x^i\}$

$$\nabla^2 f \equiv \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{C.16}$$

- General curvilinear coordinate $\{u^i\}$

$$\begin{aligned}\nabla^2 f \equiv \Delta f &= \left(f_{,ij} - f_{,k}\Gamma^k_{ij}\right)g^{ij} \\ &= \left(v_{i,j} - v_k\Gamma^k_{ij}\right)g^{ij} \equiv v_i\big|_j g^{ij}\end{aligned} \tag{C.17}$$

where the covariant vector component and its derivative with respect to u^k are defined by

$$v_i = f_{,i} = \frac{\partial f}{\partial u^i};\;\; v_k = f_{,k} = \frac{\partial f}{\partial u^k};\; v_{i,j} = f_{,ij} = \frac{\partial^2 f}{\partial u^i \partial u^j} \tag{C.18}$$

8. **Calculation rules of the Nabla operators**

$$Div\, Grad\, f = \nabla \cdot (\nabla f) = \nabla^2 f = \Delta f \quad \text{(Laplacian)} \tag{C.19}$$

$$Curl\, Grad\, f = \nabla \times (\nabla f) = 0 \tag{C.20}$$

$$Div\, Curl\, \mathbf{v} = \nabla \cdot (\nabla \times \mathbf{v}) = 0 \tag{C.21}$$

$$\Delta(fg) = f\Delta g + 2\nabla f \cdot \nabla g + g\Delta f \tag{C.22}$$

$$Curl\, Curl\, \mathbf{v} = \nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \Delta \mathbf{v} \quad \text{(Curl identity)} \tag{C.23}$$

The Laplacian of $\mathbf{v}$ in Eq. (C.23) is computed in the tensor formulation for general curvilinear coordinates.

$$\begin{aligned} Div\, Grad\, \mathbf{v} &= Laplacian\, \mathbf{v} = \Delta \mathbf{v} \\ &= \nabla \cdot (\nabla \mathbf{v}) = \nabla^2 \mathbf{v} \\ &= \left(v^i\big|_{j,k} - v^i\big|_p \Gamma^p_{jk} + v^p\big|_j \Gamma^i_{pk} \right) g^{jk} \mathbf{g}_i \\ &\equiv v^i\big|_{jk} g^{jk} \mathbf{g}_i \end{aligned} \tag{C.24}$$

9. **Essential Vector and Nabla Identities**
Let f and g be arbitrary scalars in $\mathbf{R}$; $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ arbitrary vectors in $\mathbf{R}^N$.

Products of vectors

$$\begin{aligned} &\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \\ &\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) : \text{ even permutation} \\ &\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -\mathbf{C} \cdot (\mathbf{B} \times \mathbf{A}) = -\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}) : \text{ odd permutation} \\ &\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \end{aligned}$$

Gradient $\nabla(\,)$

$$\begin{aligned} &\nabla(f + g) = \nabla f + \nabla g \in \mathbf{R}^N \\ &\nabla(fg) = f\nabla g + g\nabla f \in \mathbf{R}^N \\ &\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \in \mathbf{R}^N \\ &\nabla f \equiv \nabla \otimes f \in \mathbf{R}^N : 1^{\text{st}} \text{ order tensor, vector} \\ &\nabla \mathbf{A} \equiv \nabla \otimes \mathbf{A} \in \mathbf{R}^N \times \mathbf{R}^N : 2^{\text{nd}} \text{ order tensor} \end{aligned}$$

Divergence $\nabla\cdot(\,)$

$$\nabla\cdot(\mathbf{A}+\mathbf{B})=\nabla\mathbf{A}+\nabla\mathbf{B}\in\ \mathbf{R}$$
$$\nabla\cdot(f\mathbf{A})=f\nabla\cdot\mathrm{A}+\mathbf{A}\cdot\nabla f\in\ \mathbf{R}$$
$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})\in\ \mathbf{R}$$
$$(\mathbf{A}\cdot\nabla)\mathbf{A}=\frac{1}{2}\nabla\mathbf{A}^2-\mathbf{A}\times(\nabla\times\mathbf{A})\in\ \mathbf{R}^N$$

Curl $\nabla\times(\,)$

$$\nabla\times(\mathbf{A}+\mathbf{B})=(\nabla\times\mathbf{A})+(\nabla\times\mathbf{B})\in\ \mathbf{R}^N$$
$$\nabla\times(f\mathbf{A})=f(\nabla\times\mathrm{A})-\mathbf{A}\times\nabla f\in\ \mathbf{R}^N$$
$$\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\nabla\cdot\mathbf{B})-(\mathbf{A}\cdot\nabla)\mathbf{B}-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}\in\ \mathbf{R}^N$$
$$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}\in\ \mathbf{R}^N$$
$$(\mathbf{A}\cdot\nabla)\mathbf{A}=\frac{1}{2}\nabla\mathbf{A}^2-\mathbf{A}\times(\nabla\times\mathbf{A})\in\ \mathbf{R}^N$$

Laplacian $\boldsymbol{\Delta}(\,)$, $\nabla^2(\,)$

$$\Delta f\equiv\nabla^2 f=(\nabla\cdot\nabla)f=\nabla\cdot(\nabla f)\ \ :\text{Laplacian}\,f\ \in\ \mathbf{R}$$
$$\Delta(fg)\equiv\nabla^2(fg)=f\nabla^2 g+2\nabla f\cdot\nabla g+g\nabla^2 f\in\ \mathbf{R}$$
$$\Delta\mathbf{A}=\nabla^2\mathbf{A}\equiv\nabla(\nabla\cdot\mathbf{A})-\nabla\times(\nabla\times\mathbf{A})\quad:\text{Laplacian }\mathbf{A}\ \in\ \mathbf{R}^N$$
$$\Delta(f\mathbf{A})\equiv\nabla^2(f\mathbf{A})=\mathbf{A}\nabla^2 f+2(\nabla f\cdot\nabla)\mathbf{A}+f\nabla^2\mathbf{A}\in\ \mathbf{R}^N$$
$$\begin{aligned}\Delta(\mathbf{A}\cdot\mathbf{B})\equiv\nabla^2(\mathbf{A}\cdot\mathbf{B})&=\mathbf{A}\cdot\nabla^2\mathbf{B}-\mathbf{B}\cdot\nabla^2\mathbf{A}\\&\quad+2\nabla\cdot[(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{B}\times(\nabla\times\mathbf{A})]\in\ \mathbf{R}\end{aligned}$$

Differentiations

$$\nabla\times(\nabla f)\equiv\nabla\times(\nabla\otimes f)=\mathbf{0}$$
$$\nabla\cdot(\nabla\times\mathbf{A})=0$$
$$\nabla\cdot(f\nabla g)=f\nabla^2 g+\nabla f\cdot\nabla g\in\ \mathbf{R}$$
$$\nabla\cdot(f\nabla g-g\nabla f)=f\nabla^2 g-g\nabla^2 f\in\ \mathbf{R}$$

10. **Summary of essential Nabla operators**

	Operator			
Operand ( )	Grad $\nabla(\,)$	Div $\nabla\cdot(\,)$	Curl $\nabla\times(\,)$	Laplacian $\Delta(\,)$
Function $f\in\mathbf{R}$ (0th order tensor)	Vector $\in\mathbf{R}^N$	–	–	Scalar $\in\mathbf{R}$
Vector $\mathbf{v}\in\mathbf{R}^N$ (1st order tensor)	2nd order tensor $\in\mathbf{R}^N\times\mathbf{R}^N$	Scalar $\in\mathbf{R}$	Vector $\in\mathbf{R}^N$	Vector $\in\mathbf{R}^N$
Tensor $\mathbf{T}\in\mathbf{R}^N\times\mathbf{R}^N$ (2nd order tensor)	3rd order tensor $\in\mathbf{R}^N\times\mathbf{R}^N\times\mathbf{R}^N$	Vector $\in\mathbf{R}^N$	–	–
Order of results	One order higher	One order lower	Order unchanged	Order unchanged

Appendix D: Essential Tensors

Derivative of the covariant basis

$$\mathbf{g}_{i,j} = \Gamma_{ij}^{k} \mathbf{g}_k \tag{D.1}$$

Derivative of the contravariant basis

$$\mathbf{g}_{,j}^{i} = \frac{\partial \mathbf{g}^i}{\partial u^j} \equiv \hat{\Gamma}^{\,i}_{jk} \mathbf{g}^k = -\Gamma_{jk}^{i} \mathbf{g}^k \tag{D.2}$$

Derivative of the covariant metric coefficient

$$g_{ij,k} = \Gamma_{ik}^{p} g_{pj} + \Gamma_{jk}^{p} g_{p\,i} \tag{D.3}$$

First-kind Christoffel symbol

$$\Gamma_{ijk} = \frac{1}{2}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) = g_{lk}\Gamma_{ij}^{l} \qquad \Rightarrow \Gamma_{ij}^{l} = g^{lk}\Gamma_{ijk} \tag{D.4}$$

Second-kind Christoffel symbol based on the covariant basis

$$\Gamma_{ij}^{k} = \mathbf{g}_{i,j} \cdot \mathbf{g}^k = g^{kl}\Gamma_{ijl} \tag{D.5}$$

$$\Gamma_{ij}^{k} = \frac{\partial u^k}{\partial x^p} \cdot \frac{\partial^2 x^p}{\partial u^i \partial u^j} = \Gamma_{ji}^{k} \tag{D.6}$$

$$\Gamma_{ij}^{k} = g^{kp}\frac{1}{2}\left(g_{ip,j} + g_{jp,i} - g_{ij,p}\right) \tag{D.7}$$

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$$\begin{aligned}\Gamma_{ij}^{i} &= \frac{1}{J}\frac{\partial J}{\partial u^j} = \frac{\partial(\ln J)}{\partial u^j} \\ &= \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial u^j} = \frac{\partial\left(\ln\sqrt{g}\right)}{\partial u^j}\end{aligned} \tag{D.8}$$

Second-kind Christoffel symbol based on the contravariant basis

$$\hat{\Gamma}\,_{kj}^{\;i} = -\Gamma_{kj}^{i} = \hat{\Gamma}\,_{jk}^{\;i} \tag{D.9}$$

Covariant derivative of covariant first-order tensors

$$T_i\big|_j = T_{i,j} - \Gamma_{ij}^{k}T_k = \mathbf{T}_{,j}\cdot\mathbf{g}_i \tag{D.10}$$

Covariant derivative of contravariant first-order tensors

$$T^i\big|_j = T^i_{,j} + \Gamma_{jk}^{i}T^k = \mathbf{T}_{,j}\cdot\mathbf{g}^i \tag{D.11}$$

Covariant derivative of covariant and contravariant second-order tensors

$$\begin{aligned}T_{ij}|_k &= T_{ij,k} - \Gamma_{ik}^{m}T_{mj} - \Gamma_{jk}^{m}T_{im} \\ T^{ij}|_k &= T^{ij}_{,k} + \Gamma_{km}^{i}T^{mj} + \Gamma_{km}^{j}T^{im}\end{aligned} \tag{D.12}$$

Covariant derivative of mixed second-order tensors

$$\begin{aligned}T_j^i|_k &= T_{j,k}^i + \Gamma_{km}^{i}T_j^m - \Gamma_{jk}^{m}T_m^i \\ T_i^j|_k &= T_{i,k}^j + \Gamma_{km}^{j}T_i^m - \Gamma_{ik}^{m}T_m^j\end{aligned} \tag{D.13}$$

Second covariant derivative of covariant first-order tensors

$$\begin{aligned}T_i\big|_{kj} = T_{i,jk} &- \Gamma_{ik,j}^{m}T_m - \Gamma_{ik}^{m}T_{m,j} \\ &-\Gamma_{ij}^{m}T_{m,k} + \Gamma_{ij}^{m}\Gamma_{mk}^{n}T_n \\ &-\Gamma_{jk}^{m}T_{i,m} + \Gamma_{jk}^{m}\Gamma_{im}^{n}T_n\end{aligned} \tag{D.14}$$

Second covariant derivative of the contravariant vector

$$v^k|_{lm} \equiv v^k|_{l,m} - v^k\big|_p\Gamma_{lm}^{p} + v^p|_l\Gamma_{pm}^{k} \tag{D.15}$$

where

$$v^k|_{l,m} = \left(v^k|_l\right)_{,m} \equiv v^k_{,lm} + v^n_{,m}\Gamma_{nl}^{k} + v^n\Gamma_{nl,m}^{k} \tag{D.16}$$

$$v^k\big|_p \equiv v^k_{,p} + v^n \Gamma^k_{np} \tag{D.17}$$

$$v^p|_l \equiv v^p_{,l} + v^n \Gamma^p_{nl} \tag{D.18}$$

Riemann-Christoffel tensor

$$R^n_{ijk} \equiv \Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik}\Gamma^n_{mj} - \Gamma^m_{ij}\Gamma^n_{mk} \tag{D.19}$$

Riemann curvature tensor

$$R_{lijk} \equiv g_{nl} R^n_{ijk} \tag{D.20}$$

First-kind Ricci tensor

$$\begin{aligned} R_{ij} &= \frac{\partial \Gamma^k_{ik}}{\partial u^j} - \frac{\partial \Gamma^k_{ij}}{\partial u^k} - \Gamma^r_{ij}\Gamma^k_{rk} + \Gamma^r_{ik}\Gamma^k_{rj} \\ &= \frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^i \partial u^j} - \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\Gamma^k_{ij}\right)}{\partial u^k} + \Gamma^r_{ik}\Gamma^k_{rj} \end{aligned} \tag{D.21}$$

Second-kind Ricci tensor

$$\begin{aligned} R^i_j &\equiv g^{ik} R_{kj} \\ &= g^{ik}\left(\frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^k \partial u^j} - \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\Gamma^m_{kj}\right)}{\partial u^m} + \Gamma^m_{kn}\Gamma^n_{mj}\right) \end{aligned} \tag{D.22}$$

Ricci curvature

$$R = g^{ij}\left(\frac{\partial^2\left(\ln\sqrt{g}\right)}{\partial u^i \partial u^j} - \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\Gamma^k_{ij}\right)}{\partial u^k} + \Gamma^k_{ir}\Gamma^r_{kj}\right) \tag{D.23}$$

Einstein tensor

$$G^i_j \equiv R^i_j - \tfrac{1}{2}\delta^i_j R = g^{ik} G_{kj}$$

$$G_{ij} = g_{ik} G^k_j = R_{ij} - \tfrac{1}{2} g_{ij} R \tag{D.24}$$

$$G^i_j\Big|_i = 0 \tag{D.25}$$

First fundamental form

$$\begin{aligned}&I = Edu^2 + 2Fdudv + Gdv^2;\\ &\mathbf{M} = \left(g_{ij}\right) \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \mathbf{r}_u \cdot \mathbf{r}_u & \mathbf{r}_u \cdot \mathbf{r}_v \\ \mathbf{r}_v \cdot \mathbf{r}_u & \mathbf{r}_v \cdot \mathbf{r}_v \end{bmatrix}\end{aligned} \tag{D.26}$$

Second fundamental form

$$\begin{aligned}&II = Ldu^2 + 2Mdudv + Ndv^2;\\ &\mathbf{H} = \left(h_{ij}\right) \equiv \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{uu} \cdot \mathbf{n} & \mathbf{r}_{uv} \cdot \mathbf{n} \\ \mathbf{r}_{uv} \cdot \mathbf{n} & \mathbf{r}_{vv} \cdot \mathbf{n} \end{bmatrix}\end{aligned} \tag{D.27}$$

Gaussian curvature of a curvilinear surface

$$K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2} = \frac{\det\left(h_{ij}\right)}{\det\left(g_{ij}\right)} \tag{D.28}$$

Mean curvature of a curvilinear surface

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{EN - 2MF + LG}{2\left(EG - F^2\right)} \tag{D.29}$$

Unit normal vector of a curvilinear surface

$$\mathbf{n} = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{|\mathbf{g}_1 \times \mathbf{g}_2|} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{\det\left(g_{ij}\right)}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{EG - F^2}} \tag{D.30}$$

Differential of a surface area

$$\begin{aligned} dA &= |\mathbf{g}_1 \times \mathbf{g}_2| dudv = \sqrt{g_{11} g_{22} - (g_{12})^2}\, dudv \\ &= \sqrt{\det\left(g_{ij}\right)}\, dudv = \sqrt{EG - F^2}\, dudv \end{aligned} \tag{D.31}$$

Gauss derivative equations

$$\begin{aligned}&\mathbf{g}_{i,j} = \Gamma_{ij}^k \mathbf{g}_k + h_{ij} \mathbf{g}_3 = \Gamma_{ij}^k \mathbf{g}_k + h_{ij} \mathbf{n} \\ &\Leftrightarrow \quad \mathbf{g}_i\big|_j \equiv \mathbf{g}_{i,j} - \Gamma_{ij}^k \mathbf{g}_k = h_{ij} \mathbf{n}\end{aligned} \tag{D.32}$$

Weingarten's equations

$$\mathbf{n}_i = -h_i^j \mathbf{g}_j = -\left(h_{ik} g^{jk}\right) \mathbf{g}_j \tag{D.33}$$

Codazzi's equations

$$K_{ij,k} = K_{ik,j} \Rightarrow \quad (K_{11,2} = K_{12,1}\ ; \quad K_{21,2} = K_{22,1}) \tag{D.34}$$

Gauss equations

$$\begin{aligned} K &= \det\left(K_i^j\right) = \left(K_1^1 K_2^2 - K_2^1 K_1^2\right) \\ &= \frac{K_{11}K_{22} - K_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{R_{1212}}{g} \end{aligned} \tag{D.35}$$

Appendix E: Euclidean and Riemannian Manifolds

In the following appendix, we summarize fundamental notations and basic results from vector analysis in Euclidean and Riemannian manifolds. This section can be written informally and is intended to remind the reader of some fundamentals of vector analysis in general curvilinear coordinates. For the sake of simplicity, we abstain from being mathematically rigorous. Therefore, we recommend some literature given in References for the mathematically interested reader.

E.1 N-Dimensional Euclidean Manifold

N-dimensional Euclidean manifold $\mathbf{E}^N$ can be represented by two kinds of coordinate systems, Cartesian (orthonormal) und curvilinear (non-orthogonal) coordinate systems with N dimensions. Lines, curves, and surfaces can be considered as subsets of Euclidean manifold. Two lines or two curves can generate a flat (planes) and curvilinear surface (cylindrical and spherical surfaces), respectively. Both kinds of surfaces can be embedded in Euclidean space.

E.1.1 Vector in Cartesian Coordinates

Cartesian coordinates are an orthonormal coordinate system in which the bases (**i**,**j**, **k**) are mutually perpendicular (orthogonal) and unitary (normalized vector length). The orthonormal bases (**i**,**j**,**k**) are fixed in Cartesian coordinates. Any vector could be described by its components and the relating bases in Cartesian coordinates.

The vector **r** can be written in Euclidean space $\mathbf{E}^3$ (three-dimensional space) in Cartesian coordinates (cf. Fig. E.1).

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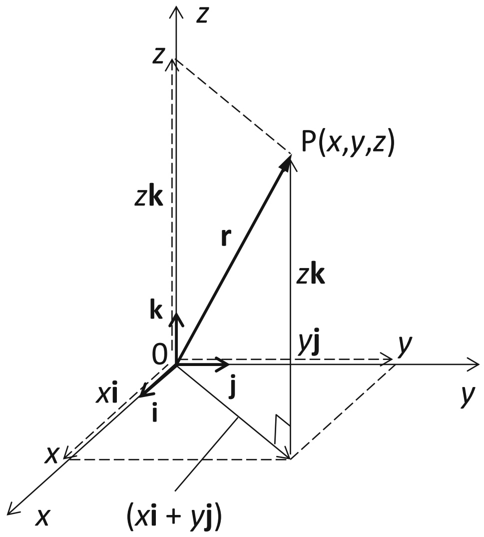

Fig. E.1 Vector **r** in Cartesian coordinates

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{E.1}$$

where

x, y, z are the vector components in the coordinate system (x,y,z);

i, **j**, **k** are the orthonormal bases of the corresponding coordinates.

The vector length of **r** can be computed using the Pythagorean theorem as

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \geq 0 \tag{E.2}$$

E.1.2 Vector in Curvilinear Coordinates

We consider a curvilinear coordinate system (u^1,u^2,u^3) of Euclidean space $\mathbf{E}^3$, i.e. a coordinate system which is generally non-orthogonal and non-unitary (non-orthonormal basis). By abuse of notation, we denote the basis vector simply basis.

In other words, the bases are not mutually perpendicular and their vector lengths are not equal to one [1, 2]. In the curvilinear coordinate system (u^1,u^2,u^3), there are three covariant bases $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ and three contravariant bases $\mathbf{g}^1$, $\mathbf{g}^2$, and $\mathbf{g}^3$ at the origin $0'$, as shown in Fig. E.2. Generally, the origin $0'$ of the curvilinear coordinates could move everywhere in Euclidean space; therefore, the bases of the curvilinear coordinates only depend on each considered origin $0'$. For this reason, the bases are not fixed in the whole curvilinear coordinates such as in Cartesian coordinates, as displayed in Fig. E.1.

The vector $\mathbf{r}$ of the point $P(u^1,u^2,u^3)$ can be written in the covariant and contravariant bases:

$$\begin{aligned}\mathbf{r} &= u^1\mathbf{g}_1 + u^2\mathbf{g}_2 + u^3\mathbf{g}_3 \\ &= u_1\mathbf{g}^1 + u_2\mathbf{g}^2 + u_3\mathbf{g}^3\end{aligned} \tag{E.3}$$

where

u^1, u^2, u^3 are the contravariant vector components of the coordinates (u^1,u^2,u^3);
$\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ are the covariant bases of the coordinate system (u^1,u^2,u^3);
u_1, u_2, u_3 are the covariant vector components of the coordinates (u^1,u^2,u^3);
$\mathbf{g}^1$, $\mathbf{g}^2$, $\mathbf{g}^3$ are the contravariant bases of the coordinate system (u^1,u^2,u^3).

The covariant basis $\mathbf{g}_i$ can be defined as the tangential vector to the corresponding curvilinear coordinate u^i for $i = 1, 2, 3$. Both bases $\mathbf{g_1}$ and $\mathbf{g_2}$ generate a tangential surface to the curvilinear surface (u^1u^2) at the considered origin $0'$, as shown in Fig. E.2. Note that the basis $\mathbf{g_1}$ is not perpendicular to the bases $\mathbf{g_2}$ and $\mathbf{g_3}$. However, the contravariant basis $\mathbf{g^3}$ is perpendicular to the tangential surface $(\mathbf{g_1g_2})$ at the origin $0'$. Generally, the contravariant basis $\mathbf{g}^k$ results from the cross product of the other covariant bases $(\mathbf{g}_i \times \mathbf{g}_j)$.

$$\alpha\, \mathbf{g}^k = \mathbf{g}_i \times \mathbf{g}_j \;\text{ for } i,j,k = 1,\ 2,\ 3 \tag{E.4a}$$

where α is a scalar factor (scalar triple product) given in Eq. (1.6).

$$\begin{aligned}\alpha &= \left(\mathbf{g}_i \times \mathbf{g}_j\right) \cdot \mathbf{g}_k \\ &\equiv \left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right]\end{aligned} \tag{E.4b}$$

Thus,

$$\mathbf{g}^1 = \frac{\mathbf{g}_2 \times \mathbf{g}_3}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]};\ \mathbf{g}^2 = \frac{\mathbf{g}_3 \times \mathbf{g}_1}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]};\ \mathbf{g}^3 = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \tag{E.4c}$$

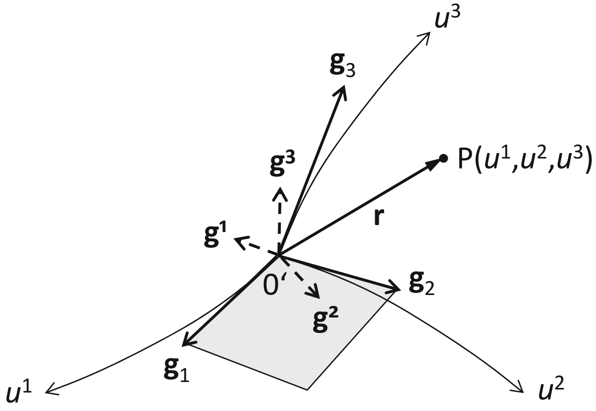

Fig. E.2 Bases of the curvilinear coordinates

E.1.3 Orthogonal and Orthonormal Coordinates

The coordinate system is called orthogonal if its bases are mutually perpendicular, as displayed in Fig. E.1. The dot product of two orthonormal bases is defined as

$$\begin{aligned}\mathbf{i} \cdot \mathbf{j} &= |\mathbf{i}| \cdot |\mathbf{j}| \cdot \cos(\mathbf{i}, \mathbf{j}) \\ &= (1) \cdot (1) \cdot \cos\left(\frac{\pi}{2}\right) \\ &= 0\end{aligned} \tag{E.5}$$

Thus,

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0 \tag{E.6}$$

If the length of each basis equals 1, the bases are unitary vectors.

$$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1 \tag{E.7}$$

If the coordinate system satisfies both conditions (E.6) and (E.7), it is called the orthonormal coordinate system, which exists in Cartesian coordinates.

Therefore, the vector length in the orthonormal coordinate system results from

$$\begin{aligned}|\mathbf{r}|^2 &= \mathbf{r}\cdot\mathbf{r}\\ &= (x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}).(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})\\ &= x^2(\mathbf{i}\cdot\mathbf{i})+xy(\mathbf{i}\cdot\mathbf{j})+xz(\mathbf{i}\cdot\mathbf{k})\\ &\quad +yx(\mathbf{j}\cdot\mathbf{i})+y^2(\mathbf{j}\cdot\mathbf{j})+yz(\mathbf{j}\cdot\mathbf{k})\\ &\quad +zx(\mathbf{k}\cdot\mathbf{i})+zy(\mathbf{k}\cdot\mathbf{j})+z^2(\mathbf{k}\cdot\mathbf{k})\end{aligned} \tag{E.8}$$

Due to Eqs. (E.6) and (E.7) the vector length in Eq. (E.8) becomes

$$|\mathbf{r}|^2 = x^2+y^2+z^2 \Rightarrow |\mathbf{r}| = \sqrt{x^2+y^2+z^2} \tag{E.9}$$

The cross product (called vector product) of a pair of bases of the orthonormal coordinate system is (informally) given by means of *right-handed rule*; i.e., if the right-hand fingers move in the rotating direction from the basis **j** to the basis **k**, the thumb will point in the direction of the basis $\mathbf{i} = \mathbf{j}\times\mathbf{k}$. The bases (**i**, **j**, **k**) form a right-handed triple.

$$\begin{cases}\mathbf{i} = \mathbf{j}\times\mathbf{k} = -\mathbf{k}\times\mathbf{j}\\ \mathbf{j} = \mathbf{k}\times\mathbf{i} = -\mathbf{i}\times\mathbf{k}\\ \mathbf{k} = \mathbf{i}\times\mathbf{j} = -\mathbf{j}\times\mathbf{i}\end{cases} \tag{E.10}$$

The magnitude of the cross product of two orthonormal bases is calculated as

$$\begin{aligned}&\mathbf{i}\times\mathbf{j} = |\mathbf{i}|\cdot|\mathbf{j}|\cdot\sin(\mathbf{i},\mathbf{j})\,\mathbf{k}\\ &\Rightarrow |\mathbf{i}\times\mathbf{j}| = |\mathbf{i}|\cdot|\mathbf{j}|\cdot\sin(\mathbf{i},\mathbf{j})\,|\mathbf{k}| = 1\cdot 1\cdot\sin\left(\frac{\pi}{2}\right)|\mathbf{k}| = |\mathbf{k}|\end{aligned} \tag{E.11}$$

E.1.4 Arc Length Between Two Points in a Euclidean Manifold

We consider two points $P(x^1,x^2,x^3)$ and $Q(x^1,x^2,x^3)$ in Euclidean space $\mathbf{E}^3$ in Cartesian and curvilinear coordinate systems, as shown in Figs. E.3 and E.4. Both points P and Q have three components x^1, x^2, and x^3 in Cartesian coordinates ($\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$). To simplify some mathematically written expressions, the coordinates x, y, and z in Cartesian coordinates can be transformed into x^1, x^2, and x^3; the bases (**i**, **j**, **k**) turn to ($\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$).

We now turn to the notation of the differential $d\mathbf{r}$ of a vector **r**. The differential $d\mathbf{r}$ can be expressed using the Einstein summation convention [1, 3]:

$$\begin{aligned}d\mathbf{r} &\equiv \mathbf{e}_i dx^i \text{for } i = 1,\ 2,\ 3\\ &= \sum_{i=1}^{3}\mathbf{e}_i dx^i\end{aligned} \tag{E.12}$$

The Einstein summation convention used in Eq. (E.12) indicates that $d\mathbf{r}$ equals the sum of $\mathbf{e}_i\, dx^i$ by running the dummy index i from 1 to 3.

The *arc length ds* between the points P and Q (cf. Fig. E.3) can be calculated by the dot product of two differentials.

$$\begin{aligned}(ds)^2 &= d\mathbf{r}\cdot\, d\mathbf{r}\\ &= (\mathbf{e}_i dx^i)\cdot\left(\mathbf{e}_j dx^j\right)\\ &= \left(\mathbf{e}_i\cdot\, \mathbf{e}_j\right) dx^i dx^j\\ &= dx^i dx^i \text{ for } i = 1,\ 2,\ 3.\end{aligned} \tag{E.13}$$

Thus, the arc length in the orthonormal coordinate system results in

$$\begin{aligned}ds &= \sqrt{dx^{\mathrm{i}}dx^{\mathrm{i}}} \text{ for } \ i = 1,\ 2,\ 3\\ &= \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}\end{aligned} \tag{E.14}$$

The points P and Q have three components u^1, u^2, and u^3 in the curvilinear coordinate system with the basis($\mathbf{g_1}$, $\mathbf{g_2}$, $\mathbf{g_3}$) in Euclidean 3-space, as displayed in Fig. E.4. The location vector $\mathbf{r}(u^1,u^2,u^3)$ of the point P is a function of u^i. Therefore, the differential $d\mathbf{r}$ of the vector $\mathbf{r}$ can be rewritten in a linear formulation of du^i.

$$\begin{aligned}d\mathbf{r} &= \frac{\partial\mathbf{r}}{\partial u^i} du^i\\ &\equiv \mathbf{g}_i du^i\end{aligned} \tag{E.15}$$

where $\mathbf{g}_i$ is the covariant basis of the curvilinear coordinate u^i.

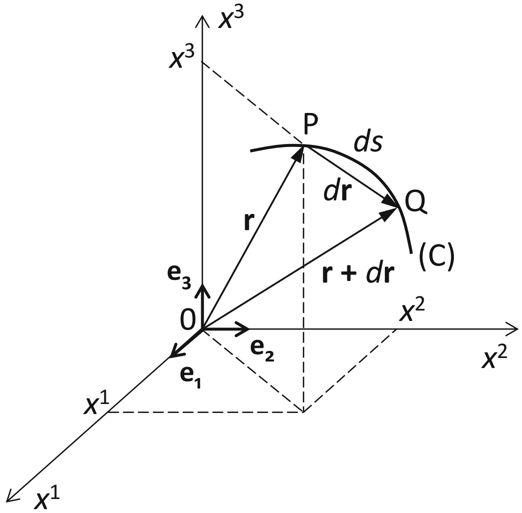

Fig. E.3 Arc length ds of P and Q in Cartesian coordinates

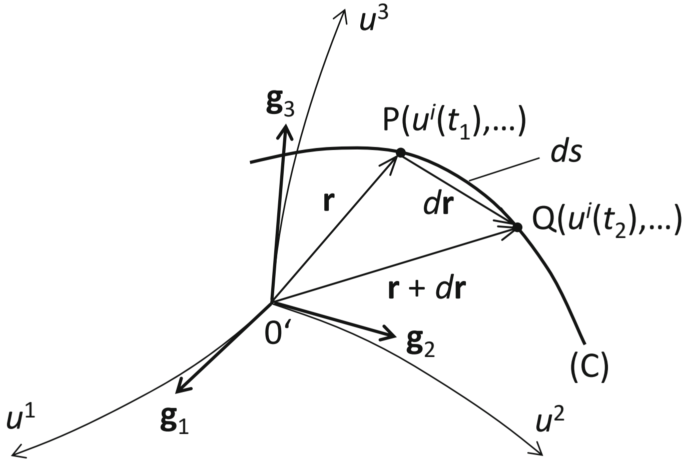

Fig. E.4 Arc length *d*s of P and Q in the curvilinear coordinates

Analogously, the arc length ds between two points of P and Q in the curvilinear coordinate system can be calculated by

$$\begin{aligned}(ds)^2 &= d\mathbf{r} \cdot d\mathbf{r} \\ &= \left(\mathbf{g}_i du^i\right) \cdot \left(\mathbf{g}_j du^j\right) \\ &= \left(\mathbf{g}_i \cdot \mathbf{g}_j\right) du^i du^j \\ &= g_{ij}\, du^i du^j \text{ for } i, j = 1, 2, 3\end{aligned} \tag{E.16}$$

Therefore,

$$\begin{aligned} ds &= \sqrt{\left|g_{ij}\, du^i du^j\right|} \\ \Rightarrow s &= \int_{t_1}^{t_2} \sqrt{\left|g_{ij}\, \frac{du^i}{dt}\frac{du^j}{dt}\right|}\, dt \end{aligned} \tag{E.17}$$

where

t is the parameter in the curve C with the coordinate $u^i(t)$;

g_{ij} is defined as the metric coefficient of two non-orthonormal bases:

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \mathbf{g}_j \cdot \mathbf{g}_i = g_{ji} \neq \delta_i^j \tag{E.18}$$

It is obvious that the symmetric metric coefficients g_{ij} vanishes for any $i \neq j$ in the orthogonal bases because $\mathbf{g}_i$ is perpendicular to $\mathbf{g}_j$; therefore, the metric tensor can be rewritten as

$$g_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ g_{ii} \text{ if } i = j \end{cases} \tag{E.19}$$

In the orthonormal bases, the metric coefficients g_{ij} in Eq. (E.19) becomes

$$g_{ij} \equiv \delta_i^j = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \tag{E.20}$$

where δ_i^j is called the Kronecker delta.

E.1.5 Bases of the Coordinates

The vector $\mathbf{r}$ can be rewritten in Cartesian coordinates of Euclidean space $\mathbf{E}^3$.

$$\mathbf{r} = x^i \mathbf{e}_i \tag{E.21}$$

The differential $d\mathbf{r}$ results from Eq. (E.21) in

$$\begin{aligned} d\mathbf{r} &= \mathbf{e}_i dx^i \\ &= \frac{\partial \mathbf{r}}{\partial x^i} dx^i \end{aligned} \tag{E.22}$$

Thus, the orthonormal bases $\mathbf{e}_i$ of the coordinate x^i can be defined as

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i} \quad \text{for } i = 1, 2, 3 \tag{E.23}$$

Analogously, the basis of the curvilinear coordinate u^i can be calculated in the curvilinear coordinate system of $\mathbf{E}^3$.

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial u^i} \quad \text{for} \quad i = 1, 2, 3 \tag{E.24}$$

Substituting Eq. (E.24) into Eq. (E.18) we obtain the metric coefficients g_{ij} that are generally symmetric in Euclidean space; i.e., $g_{ij} = g_{ji}$.

$$\begin{aligned} g_{ij} &= \mathbf{g}_i \cdot \mathbf{g}_j = g_{ji} \neq \delta_i^j \\ &= \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial \mathbf{r}}{\partial u^j} = \left(\frac{\partial \mathbf{r}}{\partial x^m} \frac{\partial x^m}{\partial u^i} \right) \cdot \left(\frac{\partial \mathbf{r}}{\partial x^n} \frac{\partial x^n}{\partial u^j} \right) \\ &= \frac{\partial x^m}{\partial u^i} \frac{\partial x^n}{\partial u^j} (\mathbf{e}_m \cdot \mathbf{e}_n) = \frac{\partial x^m}{\partial u^i} \frac{\partial x^n}{\partial u^j} \delta_m^n \\ &= \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial u^j} \text{ for } k = 1,\ 2,\ 3 \end{aligned} \tag{E.25}$$

According to Eqs. (E.4a) and (E.4b), the contravariant basis $\mathbf{g}^k$ is perpendicular to both covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$. Additionally, the contravariant basis $\mathbf{g}^k$ is chosen such that the vector length of the contravariant basis equals the inversed vector length of its relating covariant basis; thus, $\mathbf{g}^k.\ \mathbf{g}_k = 1$. As a result, the scalar products of the covariant and contravariant bases can be written in general curvilinear coordinates $(u^1, \ldots, u^N)$.

$$\begin{cases} \mathbf{g}_i \cdot \mathbf{g}^k = \mathbf{g}^k \cdot \mathbf{g}_i = \delta_i^k \text{ for } i, k = 1,\ 2,\ \ldots, N \\ \mathbf{g}_i \cdot \mathbf{g}_k = g_{ik} = g_{ki} \neq \delta_i^k \text{ for } i, k = 1,\ 2,\ \ldots, N \end{cases} \tag{E.26}$$

E.1.6 Orthonormalizing a Non-orthonormal Basis

The basis $\{\mathbf{g}_i\}$ is non-orthonormal in the curvilinear coordinates. Using the Gram-Schmidt scheme [4], an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ can be created from the basis $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$. The orthonormalization procedure of the basis $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ will be derived in this section; and the orthonormalizing scheme is demonstrated in Fig. E.5.

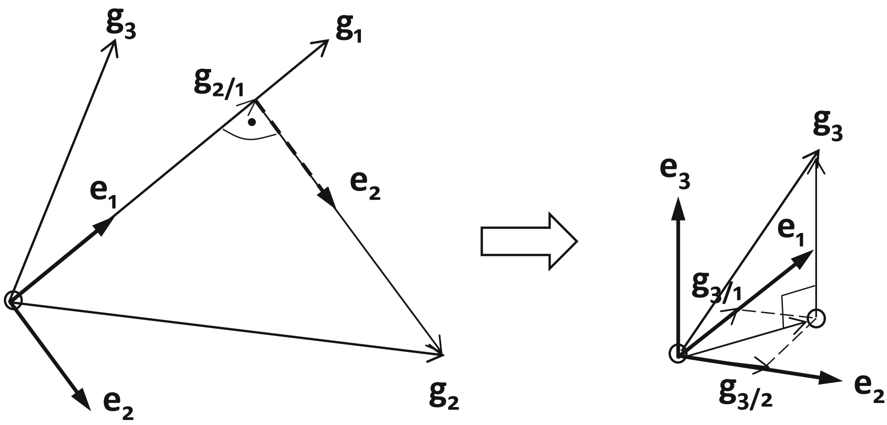

Fig. E.5 Schematic visualization of the Gram-Schmidt procedure

The Gram-Schmidt scheme for $N = 3$ has three orthonormalization steps:

1. Normalize the first basis vector $\mathbf{g}_1$ by dividing it by its length to get the *normalized* basis $\mathbf{e}_1$.

$$\mathbf{e}_1 = \frac{\mathbf{g}_1}{|\mathbf{g}_1|}$$

2. Project the basis $\mathbf{g}_2$ onto the basis $\mathbf{g}_1$ to get the projection vector $\mathbf{g}_{2/1}$ on the basis $\mathbf{g}_1$. The normalized basis $\mathbf{e}_2$ results from subtracting the projection vector $\mathbf{g}_{2/1}$ from the basis $\mathbf{g}_2$. Then, iteratively, normalize this vector by dividing it by its length to generate the basis $\mathbf{e}_2$.

$$\mathbf{e}_2 = \frac{\mathbf{g}_2 - \mathbf{g}_{2/1}}{\left|\mathbf{g}_2 - \mathbf{g}_{2/1}\right|} = \frac{\mathbf{g}_2 - (\mathbf{g}_2.\,\mathbf{e}_1)\mathbf{e}_1}{|\mathbf{g}_2 - (\mathbf{g}_2.\,\mathbf{e}_1)\mathbf{e}_1|}$$

3. Subtract projections along the bases of $\mathbf{e}_1$ and $\mathbf{e}_2$ from the basis $\mathbf{g}_3$ and normalize it to obtain the normalized basis $\mathbf{e}_3$.

$$\mathbf{e}_3 = \frac{\mathbf{g}_3 - \mathbf{g}_{3/1} - \mathbf{g}_{3/2}}{\left|\mathbf{g}_3 - \mathbf{g}_{3/1} - \mathbf{g}_{3/2}\right|} = \frac{\mathbf{g}_3 - (\mathbf{g}_3.\,\mathbf{e}_1)\mathbf{e}_1 - (\mathbf{g}_3.\,\mathbf{e}_2)\mathbf{e}_2}{|\mathbf{g}_3 - (\mathbf{g}_3.\,\mathbf{e}_1)\mathbf{e}_1 - (\mathbf{g}_3.\,\mathbf{e}_2)\mathbf{e}_2|}$$

Using the Gram-Schmidt scheme, the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ results from the non-orthonormal bases $\{\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3\}$.

Generally, the orthogonal bases $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_N\}$ for the N-dimensional space can be generated from the non-orthonormal bases $\{\mathbf{g}_1,\mathbf{g}_2,\ldots,\mathbf{g}_N\}$ according to the Gram-Schmidt scheme as follows:

$$\mathbf{e}_j = \frac{\mathbf{g}_j - \sum_{i=1}^{j-1} \left(\mathbf{g}_j \cdot \mathbf{e}_i\right)\mathbf{e}_i}{\left|\mathbf{g}_j - \sum_{i=1}^{j-1} \left(\mathbf{g}_j \cdot \mathbf{e}_i\right)\mathbf{e}_i\right|} \text{ for } j = 1, 2, \ldots, N$$

E.1.7 Angle Between Two Vectors and Projected Vector Component

The angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ can be defined by means of the scalar product (Fig. E.6)

$$\begin{aligned} \cos\theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{\mathbf{g}_i \cdot \mathbf{g}_j \, a^i b^j}{|\mathbf{a}| \cdot |\mathbf{b}|} \\ &= \frac{g_{ij} a^i b^j}{\sqrt{g_{ij} a^i a^j} \cdot \sqrt{g_{kl} b^k b^l}} = \frac{g_{ij} a^i b^j}{\sqrt{a^i a_i} \cdot \sqrt{b^j b_j}} \end{aligned} \tag{E.27}$$

where

$$\begin{aligned} |\mathbf{a}|^2 &= \mathbf{a} \cdot \mathbf{a} \\ &= g_{ij} a^i a^j = g^{ij} a_i a_j = a^i a_i \text{ for } i, j = 1, 2, \ldots, N \end{aligned} \tag{E.28}$$

in which

a^i, b^j are the contravariant vector components;

a_i, b_j are the covariant vector components;

g_{ij}, g^{ij} are the covariant and contravariant metric coefficients of the bases.

The projected component of the vector $\mathbf{a}$ on vector $\mathbf{b}$ results from its vector length and Eq. (E.27).

$$\begin{aligned} a_b &= |\mathbf{a}| \cdot \cos\theta \\ &= |\mathbf{a}| \cdot \frac{g_{ij} a^i b^j}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{g_{ij} a^i b^j}{|\mathbf{b}|} \\ &= \frac{g_{ij} a^i b^j}{\sqrt{g_{kl} b^k b^l}} \text{ for } i, j, k, l = 1, 2, \ldots, N \end{aligned} \tag{E.29}$$

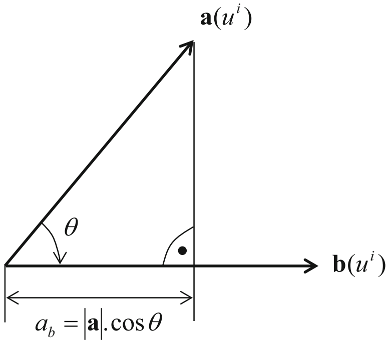

Fig. E.6 Angle between two vectors and projected vector component

Examples
Given two vectors **a** and **b**:

$$\mathbf{a} = 1 \cdot \mathbf{e}_1 + \sqrt{3} \cdot \mathbf{e}_2 = a^i \mathbf{g}_i;$$
$$\mathbf{b} = 1 \cdot \mathbf{e}_1 + 0 \cdot \mathbf{e}_2 = b^j \mathbf{g}_j$$

Thus, the relating vector components are

$$\begin{aligned} &\mathbf{g}_1 = \mathbf{e}_1; \quad \mathbf{g}_2 = \mathbf{e}_2 \\ &a^1 = 1; \quad a^2 = \sqrt{3} \\ &b^1 = 1; \quad b^2 = 0 \end{aligned}$$

The covariant metric coefficients g_{ij} in the orthonormal basis ($\mathbf{e}_1$,$\mathbf{e}_2$) can be calculated according to Eq. (E.18).

$$\left(g_{ij}\right) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The angle θ between two vectors results from Eq. (E.27).

$$\begin{aligned} \cos\theta &= \frac{g_{ij}a^i b^j}{\sqrt{g_{ij}a^i a^j}.\sqrt{g_{kl}b^k b^l}} for\ i,j,k,l = 1,2 \\ &= \frac{g_{11}a^1 b^1 + g_{12}a^1 b^2 + g_{21}a^2 b^1 + g_{22}a^2 b^2}{\sqrt{g_{ij}a^i a^j}.\sqrt{g_{kl}b^k b^l}} \\ &= \frac{(1 \cdot 1 \cdot 1) + (0 \cdot 1 \cdot 0) + \left(0 \cdot \sqrt{3} \cdot 1\right) + \left(1 \cdot \sqrt{3} \cdot 0\right)}{\sqrt{1+0+0+3}\ .\ \sqrt{1+0+0+0}} = \frac{1}{2} \end{aligned}$$

Therefore,

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

The projected vector component can be calculated according to Eq. (E.29).

$$\begin{aligned} a_b &= \frac{g_{ij}a^i b^j}{\sqrt{g_{kl}b^k b^l}} \text{ for } i,j,k,l = 1,2 \\ &= \frac{(1\cdot 1\cdot 1) + (0\cdot 1\cdot 0) + \left(0\cdot\sqrt{3}\cdot 1\right) + \left(1\cdot\sqrt{3}\cdot 0\right)}{\sqrt{1+0+0+0}} = 1 \end{aligned}$$

E.2 General N-Dimensional Riemannian Manifold

The concept of the Riemannian geometry is a very important fundamental brick in the modern physics of relativity and quantum field theories, theoretical elementary particles physics, and string theory. In contrast to the homogenous Euclidean manifold, the nonhomogenous Riemannian manifold only contains a tuple of fiber bundles of N arbitrary curvilinear coordinates of $u^1,\ldots,u^N$. Each of the fiber bundle is related to a point and belongs to the N-dimensional differentiable Riemannian manifold. In case of the infinitesimally small fiber lengths in all dimensions, the fiber bundle now becomes a single point. Therefore, the tuple of fiber bundles becomes a tuple of points on the manifold. In fact, Riemannian manifold only contains a point tuple [10].

In turn, each point of the point tuple can move along a fiber bundle in N arbitrary directions (dimensions) in the N-dimensional Riemannian manifold. Generally, a hypersurface of the fiber bundle of curvilinear coordinates $\{u^i\}$ for $i = 1, 2,\ldots,N$ at a certain point can be defined as a differentiable $(N-1)$-dimensional subspace with a codimension of one. This definition can be understood that the $(N-1)$-dimensional subbundle of fibers moves along the one-dimensional remaining fiber.

E.2.1 Point Tuple in Riemannian Manifold

We now consider an N-dimensional differentiable Riemannian manifold $\mathbf{R}^N$ that contains a tuple of points. In general, each point on the manifold locally has N curvilinear coordinates of $u^1,\ldots,u^N$ embedded at this point. Therefore, the considered point P_i can be expressed in the curvilinear coordinates as $P_i(u^1,\ldots,u^N)$. The notation of Riemannian manifold allows the local embedding of an N-dimensional affine tangential manifold (called affine tangential vector space) into the point P_i, as

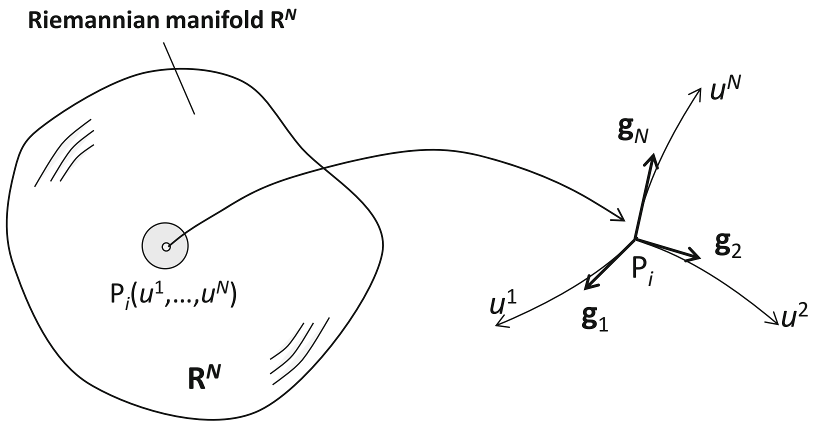

Fig. E.7 Bundle of N coordinates at P_i in Riemannian manifold

displayed in Fig. E.7. The arc length between any two points of N tuples of coordinates on the manifold does not physically change in any chosen basis. However, its components are changed in the coordinate bases that vary on the manifold. Therefore, these components must be taken into account in the transformation between different curvilinear coordinate systems in Riemannian manifold. To do that, each point in Riemannian manifold can be embedded with the individual metric coefficients g_{ij} for the relating point. Note that the metric coefficients g_{ij} of the coordinates $(u^1,\ldots,u^N)$ at any point are symmetric; and they totally have N^2 components in an N-dimensional manifold. That means one can embed an affine tangential manifold $\mathbf{E}^N$ at any point in Riemannian manifold $\mathbf{R}^N$ in which the metric coefficients g_{ij} could be only applied to this point and change from one point to another point. However, the dot product (inner product) is not valid any longer in the affine tangential manifold [1, 10].

E.2.2 Flat and Curved Surfaces

By abuse of notation and by completely abstaining from mathematical rigorousness, we introduce the notation of flat and curved surfaces. A surface in Euclidean space is called *flat* if the sum of angles in any triangle ABC is equal to 180° or, alternatively, if the arc length between any two points fulfills the condition in Eq. (E.13). Therefore, the flat surface is a plane in Euclidean space. On the contrary, an arbitrary surface in a Riemannian manifold, is called *curved* if the angular sum in an arbitrary triangle ABC is not equal to 180°, as displayed in Fig. E.8.

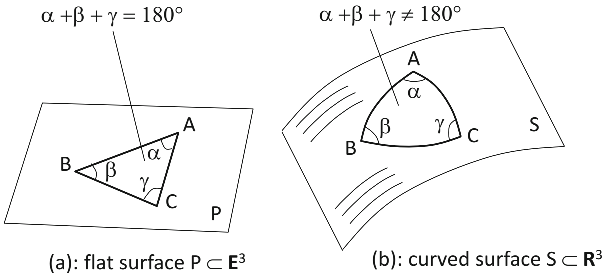

Fig. E.8 (**a**) Flat surfaces $\mathrm{P} \subset \mathbf{E}^3$ and (**b**) curved surfaces $\mathrm{S} \subset \mathbf{R}^3$

Conditions for the flat and curved surfaces [5]:

$$\begin{cases} \alpha + \beta + \gamma = 180^\circ \text{ for a flat surface} \\ \alpha + \beta + \gamma \neq 180^\circ \text{ for a curved surface} \end{cases} \tag{E.30}$$

Furthermore, the surface curvature in Riemannian manifold can be used to determine the surface characteristics. Additionally, the line curvature is also applied to studying the curve and surface characteristics.

E.2.3 Arc Length Between Two Points in Riemannian Manifold

We now consider a differentiable Riemannian manifold and calculate the arc length between two points $\mathrm{P}(u^1,\ldots,u^N)$ and $\mathrm{Q}(u^1,\ldots,u^N)$ in the curve C (see Fig. E.9). The arc length is an important notation in Riemannian manifold theory. The coordinates $(u^1,\ldots,u^N)$ can be considered as a function of the parameter t that varies from $\mathrm{P}(t_1)$ to $\mathrm{Q}(t_2)$.

The *arc length ds* between the points P and Q thus results from

$$\left(\frac{ds}{dt}\right)^2 = \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} \tag{E.31}$$

where the derivative of the vector $\mathbf{r}(u^1,\ldots,u^N)$ can be calculated as

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \frac{d(\mathbf{g}_i u^i)}{dt} \\ &\equiv \mathbf{g}_i \dot{u}^i(t) \text{ for } i = 1,\ 2,\ \ldots, N \end{aligned} \tag{E.32}$$

Substituting Eq. (E.32) into Eq. (E.31), one obtains the arc length

$$
\begin{aligned}
ds &= \sqrt{\varepsilon(\mathbf{g}_i \dot{u}^i) \cdot \left(\mathbf{g}_j \dot{u}^j\right)}\, dt \\
&= \sqrt{\varepsilon\, g_{ij} \dot{u}^i(t) \dot{u}^j(t)}\, dt \;\text{ for } i,j = 1,2,\ldots,N
\end{aligned}
\tag{E.33}
$$

where ε $(= \pm 1)$ is the functional indicator that ensures the square root always exists.

Therefore, the arc length of PQ is given by integrating Eq. (E.33) from the parameter t_1 to the parameter t_2.

$$
s = \int_{t_1}^{t_2} \sqrt{\varepsilon\, g_{ij} \dot{u}^i(t) \dot{u}^j(t)}\, dt \;\text{ for } i,j = 1,2,\ldots,N \tag{E.34}
$$

where the covariant metric coefficients g_{ij} are defined by

$$
\begin{aligned}
g_{ij} &= \mathbf{g}_i \cdot \mathbf{g}_j \neq \delta_i^j \\
&= \frac{\partial x^k}{\partial u^i} \cdot \frac{\partial x^k}{\partial u^j} \;\text{ for } k = 1,\ 2,\ \ldots, N
\end{aligned}
\tag{E.35}
$$

We now assume that the points $P(u^1,u^2)$ and $Q(u^1,u^2)$ lie on the Riemannian surface S, which is embedded in Euclidean space $\mathbf{E}^3$. Each point on the surface only depends on two parameterized curvilinear coordinates of u^1 and u^2 that are called the Gaussian surface parameters, as shown in Fig. E.9.

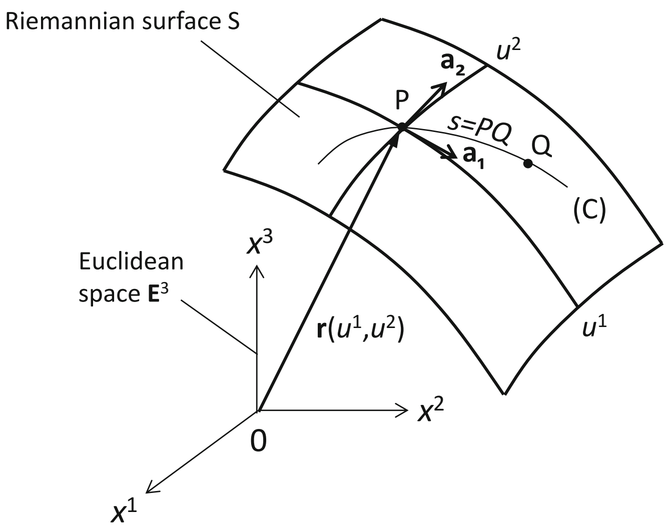

Fig. E.9 Arc length between two points in a Riemannian surface

The differential $d\mathbf{r}$ of the vector $\mathbf{r}$ can be rewritten in the coordinates (u^1,u^2):

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial u^i} du^i \\ &\equiv \mathbf{r}_{,\,i} du^i \\ &\equiv \mathbf{a}_i du^i \text{ for } i = 1,\ 2 \end{aligned} \tag{E.36}$$

where $\mathbf{a}_i$ is the tangential vector of the coordinate u^i on the Riemannian surface.

Therefore, the arc length ds on the differentiable Riemannian parameterized surface can be computed as

$$\begin{aligned} (ds)^2 &= d\mathbf{r} \cdot d\mathbf{r} \\ &= \mathbf{a}_i \cdot \mathbf{a}_j du^i du^j \\ &\equiv a_{ij} du^i du^j \text{ for } \ i,\ j = 1,\ 2 \end{aligned} \tag{E.37}$$

whereas a_{ij} are the surface metric coefficients only at the point P in the coordinates (u^1, u^2) on the Riemannian curved surface S. The formulation of $(ds)^2$ in Eq. (E.37) is called the first fundamental form for the intrinsic geometry of Riemannian manifold [6–9].

The surface metric coefficients of the covariant and contravariant components have the similar characteristics such as the metric coefficients:

$$\begin{aligned} a_{ij} = a_{ji} &= \mathbf{a}_i \cdot\ \mathbf{a}_j \neq \delta_i^j \\ &= \frac{\partial x^k}{\partial u^i} \cdot \frac{\partial x^k}{\partial u^j} \text{ for } k = 1,\ 2,\ \ldots, N; \end{aligned} \tag{E.38a}$$

$$a_i^j = \mathbf{a}_i \cdot\ \mathbf{a}^j = \delta_i^j \tag{E.38b}$$

Instead of the metric coefficients g_{ij} in the curvilinear Euclidean space, the surface metric coefficients a_{ij} are used in the general curvilinear Riemannian manifold.

E.2.4 Tangent and Normal Vectors on the Riemannian Surface

We consider a point $P(u^1,u^2)$ on a differentiable Riemannian surface that is parameterized by u^1 and u^2. Furthermore, the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are the covariant bases of the curvilinear coordinates (u^1,u^2), respectively. In general, a hypersurface in an N-dimensional manifold with coordinates $\{u^i\}$ for $i = 1, 2, \ldots, N$ can be defined as a differentiable $(N-1)$-dimensional subspace with a codimension of 1.

The basis $\mathbf{a}_i$ of the coordinate u^i can be rewritten as

$$\begin{aligned} \mathbf{a}_i &= \frac{\partial \mathbf{r}}{\partial u^i} \\ &\equiv \mathbf{r}_{,\,i} \text{ for } i = 1,\ 2 \end{aligned} \tag{E.39}$$

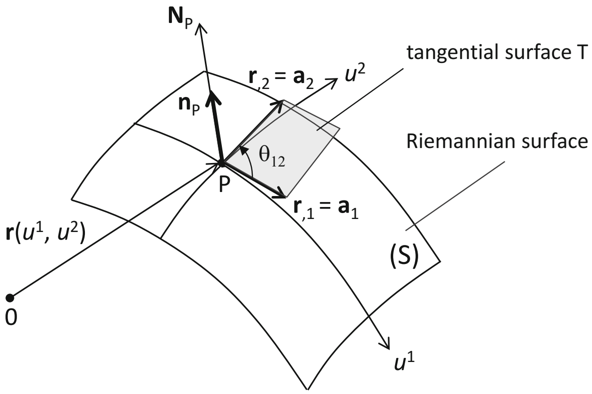

Fig. E.10 Tangent vectors to the curvilinear coordinates (u^1,u^2)

The covariant basis $\mathbf{a}_i$ is tangent to the coordinate u^i at the point P. Both bases $\mathbf{a}_1$ and $\mathbf{a}_2$ generate the tangential surface T tangent to the Riemannian surface S at the point P, which is defined by the curvilinear coordinates of u^1 and u^2, as shown in Fig. E.10.

The angle of two intersecting Gaussian parameterized curves u^i and u^j results from the dot product of the bases at the point $\text{P}(u^i,u^j)$.

$$\begin{aligned} &\mathbf{a}_i \cdot \mathbf{a}_j = |\mathbf{a}_i| \cdot |\mathbf{a}_j| \cos\left(\mathbf{a}_i, \mathbf{a}_j\right) \Rightarrow \\ &\cos\,\theta_{ij} \equiv \cos\left(\mathbf{a}_i, \mathbf{a}_j\right) = \frac{\mathbf{a}_i \cdot \mathbf{a}_j}{|\mathbf{a}_i| \cdot |\mathbf{a}_j|} \\ &\qquad = \frac{a_{ij}}{\sqrt{a_{(ii)}} \cdot \sqrt{a_{(jj)}}} \le 1 \text{ for } \theta_{ij} \in \left[0, \frac{\pi}{2}\right] \end{aligned} \tag{E.40}$$

Note that

$|\mathbf{a}_i| = \sqrt{\mathbf{a}_i \cdot \mathbf{a}_i} = \sqrt{a_{(ii)}}$, no summation over (ii)

where a_{ii} and a_{jj} are the vector lengths of $\mathbf{a}_i$ and $\mathbf{a}_j$; a_{ij}, the surface metric coefficients.

The surface metric coefficients can be defined by

$$\begin{aligned} a_{ij} &= \mathbf{a}_i \cdot \mathbf{a}_j \neq \delta_i^j \\ &= \frac{\partial x^k}{\partial u^i} \cdot \frac{\partial x^k}{\partial u^j} \text{ for } k = 1, 2, \ldots, N \end{aligned} \tag{E.41}$$

E.2.5 Angle Between Two Curvilinear Coordinates

We now give a concrete example of the computation of the angle between two curvilinear coordinates. Given two arbitrary basis vectors at the point P(u^1,u^2), we can write them with the covariant basis $\{\mathbf{e}_i\}$:

$$\begin{aligned}\mathbf{a}_1 &= 1\cdot\mathbf{e}_1 + 0\cdot\ \mathbf{e}_2;\\ \mathbf{a}_2 &= 0\cdot\ \mathbf{e}_1 + 1\cdot\ \mathbf{e}_2\end{aligned}$$

The covariant metric coefficients a_{ij} can be calculated:

$$(\mathbf{a}_{ij}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The angle between two base vectors results from Eq. (E.40):

$$\begin{aligned}\cos\ \theta_{ij} &= \frac{a_{ij}}{\sqrt{a_{(ii)}}\ \cdot\ \sqrt{a_{(jj)}}}\\ \Rightarrow\ \cos\ \theta_{12} &= \frac{a_{12}}{\sqrt{a_{11}}\cdot\sqrt{a_{22}}} = \frac{0}{\sqrt{1}\cdot\sqrt{1}} = 0\end{aligned}$$

Thus,

$$\theta_{12} = \cos^{-1}\left(\frac{a_{12}}{\sqrt{a_{11}}\cdot\sqrt{a_{22}}}\right) = \cos^{-1}(0) = \frac{\pi}{2}$$

In this case, the curvilinear coordinates of u^1 and u^2 are orthogonal at the point P on the Riemannian surface S, as shown in Fig. E.10.

The tangent vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ generate the tangential surface T tangent to the Riemannian surface S at the point P. The normal vector $\mathbf{N}_P$ to the tangential surface T at the point P is given by

$$\begin{aligned}\mathbf{N}_{\mathrm{P}} &= \frac{\partial\mathbf{r}}{\partial u^i}\times\frac{\partial\mathbf{r}}{\partial u^j} = \mathbf{r}_{,\,i}\times\mathbf{r}_{,\,j}\\ &\equiv \mathbf{a}_i\times\mathbf{a}_j \text{ for } i,j = 1,\ 2\\ &= \alpha\,\mathbf{a}^k\end{aligned} \tag{E.42}$$

where

α is the scalar factor;

$\mathbf{a}^k$ is the contravariant basis of the curvilinear coordinate of u^k.

Multiplying Eq. (E.42) by the covariant basis $\mathbf{a}_k$, the scalar factor α results in

$$\begin{aligned} &\alpha\left(\mathbf{a}^k \cdot \mathbf{a}_k\right) = \alpha\,\delta_k^k = \alpha = \left(\mathbf{a}_i \times \mathbf{a}_j\right) \cdot \mathbf{a}_k \\ &\Rightarrow \alpha = \left(\mathbf{a}_i \times \mathbf{a}_j\right) \cdot \mathbf{a}_k \equiv \left[\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k\right] \end{aligned} \tag{E.43}$$

The scalar factor α equals the scalar triple product that is given in [2]:

$$\begin{aligned} \alpha &\equiv [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] = \left(\mathbf{a}_i \times \mathbf{a}_j\right) \cdot \mathbf{a}_k = (\mathbf{a}_k \times \mathbf{a}_i) \cdot \mathbf{a}_j = \left(\mathbf{a}_j \times \mathbf{a}_k\right) \cdot \mathbf{a}_i \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{\frac{1}{2}} = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}^{\frac{1}{2}} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}^{\frac{1}{2}} \\ &= \sqrt{\det\left(\mathbf{a}_{ij}\right)} \equiv J \end{aligned} \tag{E.44}$$

where Jacobian J is the determinant of the covariant basis tensor.

The unit normal vector $\mathbf{n}_P$ in Eq. (E.42) becomes using the Lagrange identity.

$$\mathbf{n}_\mathrm{P} = \frac{\mathbf{a}_i \times \mathbf{a}_j}{\left|\mathbf{a}_i \times \mathbf{a}_j\right|} = \frac{\mathbf{a}_i \times \mathbf{a}_j}{\sqrt{a_{(ii)} \cdot a_{(jj)} - \left(a_{ij}\right)^2}} \tag{E.45}$$

Note that

$|\mathbf{a}_i|^2 = \mathbf{a}_i \cdot \mathbf{a}_i = a_{(ii)}$, no summation over (ii)

The Lagrange identity results from the cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}| \cdot |\mathbf{b}| \sin(\mathbf{a}, \mathbf{b}) \Rightarrow \\ |\mathbf{a} \times \mathbf{b}|^2 &= |\mathbf{a}|^2 \cdot |\mathbf{b}|^2 \sin^2(\mathbf{a}, \mathbf{b}) \\ &= |\mathbf{a}|^2 \cdot |\mathbf{b}|^2 \cdot \left(1 - \cos^2(\mathbf{a}, \mathbf{b})\right) \\ &= (|\mathbf{a}| \cdot |\mathbf{b}|)^2 - (|\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos(\mathbf{a}, \mathbf{b}))^2 \\ &= (|\mathbf{a}| \cdot |\mathbf{b}|)^2 - (\mathbf{a} \cdot \mathbf{b})^2 \end{aligned}$$

Thus,

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{|\mathbf{a}|^2 \cdot |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2} \tag{E.46}$$

Equation (E.46) is called the *Lagrange identity*.

E.2.6 Surface Area in Curvilinear Coordinates

The surface area S in the differentiable Riemannian curvilinear surface, as displayed in Fig. E.11, can be calculated using the Lagrange identity [2].

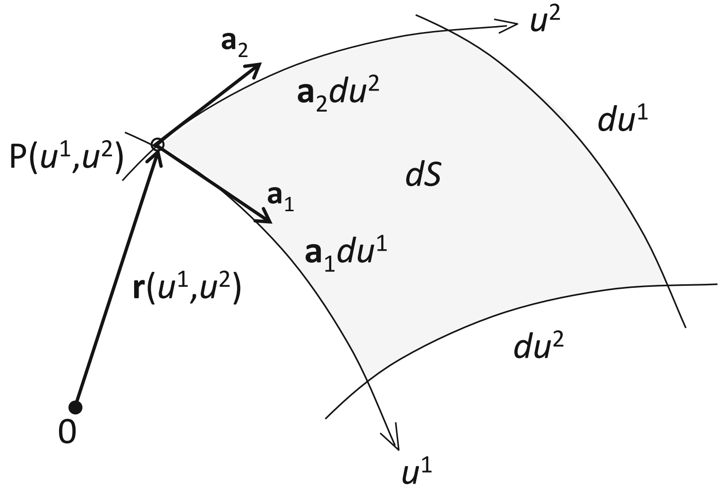

Fig. E.11 Surface area in the curvilinear coordinates

$$\begin{aligned} S &= \iint \left| \frac{\partial \mathbf{r}}{\partial u^i} \times \frac{\partial \mathbf{r}}{\partial u^j} \right| du^i du^j \\ &= \iint \left| \mathbf{a}_i \times \mathbf{a}_j \right| du^i du^j \\ &= \iint \sqrt{\left|\mathbf{a}_i\right|^2 \cdot \ \left|\mathbf{a}_j\right|^2 - \left(\mathbf{a}_i \cdot \mathbf{a}_j\right)^2}\, du^i du^j \\ &= \iint \sqrt{a_{(ii)} \cdot \ a_{(jj)} - \left(a_{ij}\right)^2}\, du^i du^j \end{aligned} \tag{E.47}$$

Therefore,

$$\begin{aligned} S &= \iint \left|\mathbf{a}_1 \times \mathbf{a}_2\right| du^1 du^2 \ \text{ for } \ i = 1;\ j = 2 \\ &= \iint \sqrt{a_{11} \cdot \ a_{22} - \left(a_{12}\right)^2}\, du^1 du^2 \end{aligned} \tag{E.48}$$

In Eq. (E.47), the vector length squared can be calculated as
$|\mathbf{a}_i|^2 = \mathbf{a}_i \cdot \ \mathbf{a}_i = a_{(ii)}$, no summation over (ii)

E.3 Kronecker Delta

The Kronecker delta is very useful in tensor analysis and is defined as

$$\delta_i^j \equiv \frac{\partial u^j}{\partial u^i} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \tag{E.49}$$

where u^i and u^j are in the same coordinate system and independent of each other.

Some properties of the Kronecker delta (Kronecker tensor) are considered [2, 5]. We summarize a few properties of the Kronecker delta:

Property 1 The chain rule of differentiation of the Kronecker delta using the contraction rule (cf. Appendix A)

$$\delta_i^j = \frac{\partial u^j}{\partial u^i} = \frac{\partial u^j}{\partial u^k}\frac{\partial u^k}{\partial u^i} = \delta_k^j \delta_i^k \tag{E.50}$$

Property 2 Kronecker delta in Einstein summation convention

$$\begin{aligned} \delta_j^i a^{jk} &= \delta_1^i a^{1k} + \cdots + \delta_i^i a^{ik} + \cdots + \delta_N^i a^{Nk} \\ &= 0 + \cdots + 1 \cdot a^{ik} + \cdots + 0 \\ &= a^{ik} \end{aligned} \tag{E.51}$$

Property 3 Product of Kronecker deltas

$$\begin{aligned} \delta_j^i \delta_k^j &= \delta_1^i \delta_k^1 + \cdots + \delta_i^i \delta_k^i + \cdots + \delta_N^i \delta_k^N \\ &= 0 + \cdots + 1 \cdot \delta_k^i + \cdots + 0 \\ &= \delta_k^i \end{aligned} \tag{E.52}$$

Note that
$\delta_{(i)}^{(i)} \equiv \delta_1^1 = \delta_2^2 = \cdots = \delta_N^N = 1$ (no summation over the free index i);
However,
$\delta_i^i \equiv \delta_1^1 + \delta_2^2 + \cdots + \delta_N^N = N$ (summation over the dummy index i).

E.4 Levi-Civita Permutation Symbols

Levi-Civita permutation symbols in a three-dimensional space are third-order pseudo-tensors. They are a useful tool to simplify the mathematical expressions and computations [1–3].

The Levi-Civita permutation symbols can simply be defined as

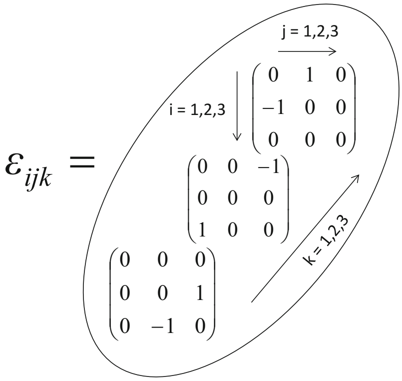

Fig. E.12 27 Levi-Civita permutation symbols

$$\begin{aligned}\varepsilon_{ijk} &= \begin{cases} +1 \text{ if } (i,\ j,\ k) \text{ is an even permutation;} \\ -1 \text{ if } (i,\ j,\ k) \text{ is an odd permutation;} \\ 0 \quad \text{if } i = j, \ or \ i = k; \ or \ j = k \end{cases} \\ \Leftrightarrow \varepsilon_{ijk} &= \frac{1}{2}(i-j)\cdot(j-k)\cdot(k-i) \text{ for } i,j,k = 1,2,3\end{aligned} \tag{E.53}$$

Here, we abstain from giving an exact definition of even and odd permutation because this would go beyond the scope of this book. The reader is referred to the literature [8, 9].

According to Eq. (E.53), the Levi-Civita permutation symbols can be expressed as

$$\varepsilon_{ijk} = \begin{cases} \varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} \text{ (even permutation);} \\ -\varepsilon_{ikj} = -\varepsilon_{kji} = -\varepsilon_{jik} \text{(odd permutation)} \end{cases} \tag{E.54}$$

The 27 Levi-Civita permutation symbols for a three-dimension coordinate system are graphically displayed in Fig. E.12.

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Appendix F: Probability Function for the Quantum Interference

In case of the interference pattern, the probability function of the electrons for finding the electron at position z in a long observing time is the squared amplitude of the sum of the wave functions $\psi_A(z)$ and $\psi_B(z)$.

The complex wave functions $\psi_A(z)$ and $\psi_B(z)$ are displayed in a complex plane, as shown in Fig. F.1.

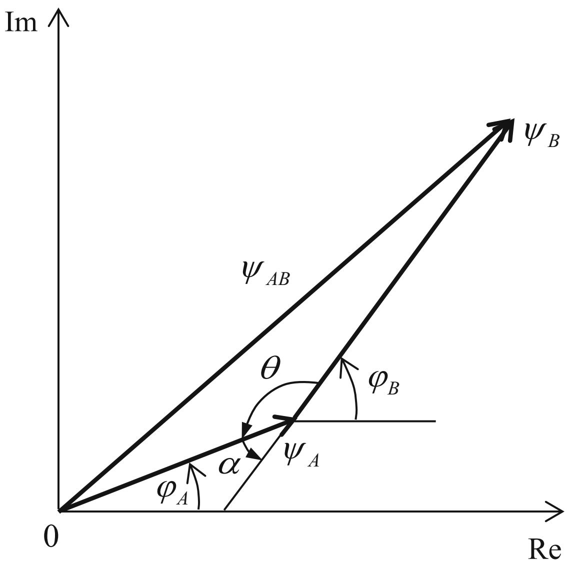

Fig. F.1 The sum function ψ_{AB} of two complex wave functions

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Using the law of cosines for a triangle, the probability function $P_{AB}(z)$ at position z is calculated as

$$\begin{aligned} P_{AB}(z) &\equiv \|\psi_{AB}\|^2 = \|\psi_A + \psi_B\|^2 \\ &= \|\psi_A\|^2 + \|\psi_B\|^2 - 2\|\psi_A\| \cdot \|\psi_B\| \cos\theta \\ &= \|\psi_A\|^2 + \|\psi_B\|^2 + 2\|\psi_A\| \cdot \|\psi_B\| \cos\alpha \\ &= P_A(z) + P_B(z) + 2\|\psi_A\| \cdot \|\psi_B\| \cos\alpha \end{aligned} \tag{F.1}$$

where $P_A(z)$ and $P_B(z)$ are the probability functions at position z of the wave functions $\psi_A(z)$ and $\psi_B(z)$.

The wave functions are expressed in the complex form as

$$\begin{aligned} \psi_A &= \|\psi_A\| \cdot (\cos\varphi_A + i\sin\varphi_A) = \|\psi_A\| e^{i\varphi_A}; \\ \psi_B &= \|\psi_B\| \cdot (\cos\varphi_B + i\sin\varphi_B) = \|\psi_B\| e^{i\varphi_B} \end{aligned} \tag{F.2}$$

The complex conjugate of the wave function of ψ_B results as

$$\begin{aligned} \psi_B^* &= \|\psi_B\| \cdot (\cos\varphi_B - i\sin\varphi_B) \\ &= \|\psi_B\| \cdot [\cos(-\varphi_B) + i\sin(-\varphi_B)] = \|\psi_B\| e^{-i\varphi_B} \end{aligned} \tag{F.3}$$

The product of two complex wave functions is calculated as

$$\begin{aligned} \psi_A \psi_B^* &= \|\psi_A\| \cdot \|\psi_B\| e^{i(\varphi_A - \varphi_B)} \\ &= \|\psi_A\| \cdot \|\psi_B\| \cdot [\cos(\varphi_A - \varphi_B) + i\sin(\varphi_A - \varphi_B)] \end{aligned} \tag{F.4}$$

The difference angle between the phase angles of the wave functions results as

$$\varphi_B - \varphi_A = \alpha \Rightarrow \varphi_A - \varphi_B = -\alpha \tag{F.5}$$

Substituting Eqs. (F.4) and (F.5), one calculates the real part of Eq. (F.4)

$$\begin{aligned} \mathrm{Re}(\psi_A \psi_B^*) &= \|\psi_A\| \cdot \|\psi_B\| \cos(\varphi_A - \varphi_B) \\ &= \|\psi_A\| \cdot \|\psi_B\| \cos(-\alpha) \\ &= \|\psi_A\| \cdot \|\psi_B\| \cos\alpha \\ &= \mathrm{Re}(\psi_A^* \psi_B) \end{aligned} \tag{F.6}$$

Substituting Eq. (F.6) into Eq. (F.1), one obtains the probability function $P_{AB}(z)$ at position z

$$\begin{aligned} P_{AB}(z) &= \|\psi_A\|^2 + \|\psi_B\|^2 + 2\mathrm{Re}(\psi_A \psi_B^*) \\ &= P_A(z) + P_B(z) + 2\mathrm{Re}(\psi_A \psi_B^*) \\ &= P_A(z) + P_B(z) + 2\mathrm{Re}(\psi_A^* \psi_B) \end{aligned} \tag{F.7}$$

The third term on the RHS of Eq. (F.7) represents the quantum interference in the probability function.

Appendix G: Lorentz and Minkowski Transformations in Spacetime

The Lorentz and Minkowski transformations deal with the special relativity theory (SRT) in a four-dimensional spacetime.

Let S be an inertial coordinate system; S' be a moving coordinate system with a velocity v relative to S (s. Fig. G.1).

In the case that the velocity v is parallel to the coordinate x, the Lorentz transformation between two coordinate systems is written as

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma \cdot (x - vt) \\ y' &= y \\ z' &= z \\ t' &= \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma \cdot \left(t - \frac{vx}{c^2}\right) \end{aligned} \tag{G.1}$$

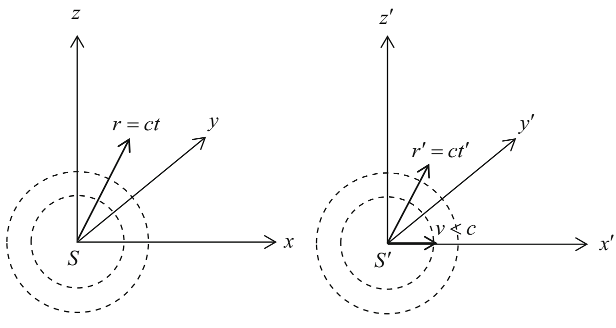

Fig. G.1 Inertial and moving coordinate systems

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The transformed coordinates of the moving system S' are written in the transformation matrix $\mathbf{L}$ as

$$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} = \mathbf{L} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c^2} & 0 & 0 & \gamma \end{pmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \tag{G.2}$$

where c is the constant light speed, t and t' are the time coordinates in the systems S and S', respectively; γ is called the Lorentz factor, which is defined as

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \geq 1$$

The Lorentz backtransformation results from Eq. (G.2) as

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \mathbf{L}^{-1} \begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\gamma v}{c^2} & 0 & 0 & \gamma \end{pmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} \tag{G.3}$$

Thus, the coordinates of the inertial system S are written as

$$\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \cdot \left(x' + vt'\right) \\ y &= y' \\ z &= z' \\ t &= \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma \cdot \left(t' + \frac{vx'}{c^2}\right) \end{aligned} \tag{G.4}$$

Note that the larger the moving speed v, the shorter the length of an object in the moving coordinate system S' becomes. This effect is called the *length contraction*.

$$l' = \frac{l}{\gamma} = l \cdot \sqrt{1 - \frac{v^2}{c^2}} \leq l \tag{G.5}$$

On the contrary, the larger the moving speed v, the longer the time interval between two events encounters compared to the standing coordinate system S. This effect is defined as the *time dilation*. This result indicates that the clock runs more slowly in the moving coordinate system. If the moving speed v reaches the light speed c, the clock would stop running since the time interval between two events will go to infinity ($t' \to \infty$) in the moving coordinate system.

$$t^{'} = \gamma t = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}} \geq t \tag{G.6}$$

However, if the velocity v is in an arbitrary direction in the coordinate system S, the propagating distances of a light signal transmitting with a constant light speed c at the times t and t' result in the coordinate systems S and S' as

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} = ct; \\ r^{'} &= \sqrt{x^{'2} + y^{'2} + z^{'2}} = ct^{'} \end{aligned} \tag{G.7}$$

Having squared both sides of Eq. (G.7), one obtains

$$\begin{aligned} & x^2 + y^2 + z^2 - c^2 t^2 = 0; \\ & x^{'2} + y^{'2} + z^{'2} - c^2 t^{'2} = 0. \end{aligned} \tag{G.8}$$

The generalized Lorentz transformation between two arbitrary coordinate systems results from Eq. (G.8) for any proportional parameter σ as

$$x^{'2} + y^{'2} + z^{'2} - c^2 t^{'2} = \sigma \cdot \left(x^2 + y^2 + z^2 - c^2 t^2 \right) = 0 \tag{G.9}$$

Equation (G.9) is also valid for $\sigma = 1$; therefore, the generalized Lorentz transformation becomes

$$x^{'2} + y^{'2} + z^{'2} - c^2 t^{'2} = x^2 + y^2 + z^2 - c^2 t^2 \tag{G.10}$$

The Minkowski transformation in the field-free spacetime with four independent dimensions of x_1, x_2, x_3, and x_4 is written as

$$\begin{pmatrix} x_1 = x \\ x_2 = y \\ x_3 = z \\ x_4 = \sqrt{-1}ct = jct \end{pmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & jc \end{pmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \tag{G.11}$$

Therefore,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x^2 + y^2 + z^2 - c^2 t^2 \tag{G.12}$$

Thus, the Minkowski transformation satisfies the generalized Lorentz transformation of Eq. (G.10) using in the special relativity theory, cf. Fig. 5.11. Both transformations are used for spacetimes without gravitational field.

The general relativity theory (GRT) is the field theory of gravitation, in which the spacetime is curved with nonzero Riemannian curvature at any point on its surface. On the contrary, the SRT is only valid in a flat spacetime in which gravity does not exist. Hence, the GRT is a geometric theory of gravity that determines the structure of spacetime by means of curvatures at each point on the spacetime surface.

The *principle of equivalence* indicates that the effect of an accelerating coordinate system has the same effect of gravity, in which the spacetime is affected by a gravitational field. As a result, the geometric structure of an accelerated spacetime must be curved.

If a matter is under a gravitational or electromagnetic field in a spacetime, it is moved in the spacetime. In turn, the matter curves the spacetime by means of its gravitational or electromagnetic force acting upon the spacetime. Therefore, the four-dimensional Minkowski transformation is a pseudo-Euclidean flat spacetime is only obtained in the absence of gravity.

In the GRT, a metric tensor $g_{\mu\nu}$ is necessary for the *curved spacetime* under a gravitational field, in which the geodesic distance ds (i.e. the shortest distance between two arbitrary points) is written in a four-dimensional spacetime as

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu ; \forall\mu, \forall\nu = 1,2,3,4 \tag{G.13}$$

Using the Schwarzschild metric tensor for a gravitational field, the geodesic in a curved spacetime is expressed in the rectangular coordinates as

$$\begin{aligned} ds^2 &= g_{11} dx_1^2 + g_{22} dx_2^2 + g_{33} dx_3^2 + g_{44} dx_4^2 \\ &= g_{11} dx^2 + g_{22} dy^2 + g_{33} dz^2 + g_{44} dt^2 \end{aligned} \tag{G.14}$$

The Schwarzschild metric tensor in Eq. (G.14) is written as

$$g_{\mu\nu} = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & g_{33} & 0 \\ 0 & 0 & 0 & g_{44} \end{pmatrix} \tag{G.15}$$

The principle components of the Schwarzschild metric tensor are defined as

$$\begin{aligned} g_{11} = g_{22} = g_{33} &\equiv \frac{1}{1 - \dfrac{2MG}{c^2 r}}; \\ g_{44} &\equiv -c^2 \left(1 - \frac{2MG}{c^2 r}\right) \end{aligned} \tag{G.16}$$

where

G is the gravitational constant ($G = 6.673 \times 10^{-11}$ m^3 kg^{-1} s^{-2});
M is the mass of the object;
r is the radial coordinate of the spherical spacetime.

Appendix H: The Law of Large Numbers in Statistical Mechanics

Let $X_1, X_2, \ldots, X_N$ be N independent variables that have the probability densities $\rho_1(X_1), \rho_2(X_2), \ldots, \rho_N(X_N)$, respectively.

The probability densities satisfy the condition

$$\int_{-\infty}^{+\infty} \rho_i(X_i)dX_i = 1 \ \ for\ i = 1,\ 2,\ \ldots, N \tag{H.1}$$

The joint probability density of N independent variables gives the relation

$$\rho(X_1, \ldots, X_N)dX_1 \ldots dX_N = \rho_1(X_1) \ldots \rho_N(X_N)dX_1 \ldots . dX_N \tag{H.2}$$

Integrating the joint probability density over the entire space, one obtains using Eq. (H.1)

$$\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \rho(X_1, \ldots, X_N)dX_1 \ldots dX_N = \int_{-\infty}^{+\infty} \rho_1(X_1)dX_1 \ldots \int_{-\infty}^{+\infty} \rho_N(X_N)dX_N \\ = 1 \tag{H.3}$$

Let $\Delta X_1, \ldots, \Delta X_N$ be the differences of the independent variables $X_1, \ldots, X_N$ to their average value, respectively. They can be expressed as

$$\Delta X_i = X_i - \langle X_i \rangle \ \ for\ i = 1,\ 2,\ \ldots, N \tag{H.4}$$

where the expectation (average value) of the variable X_i is defined as

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$$\langle X_i \rangle \equiv \int_{-\infty}^{+\infty} X_i \rho_i(X_i) dX_i \tag{H.5}$$

Similarly, the variance of variable X_i is written as

$$\begin{aligned} Var(X_i) &\equiv \left\langle (\Delta X_i)^2 \right\rangle = \left\langle (X_i - \langle X_i \rangle)^2 \right\rangle \\ &= \int_{-\infty}^{+\infty} (X_i - \langle X_i \rangle)^2 \rho_i(X_i) dX_i \end{aligned} \tag{H.6}$$

The mean value X_m of the N independent variables is defined as

$$X_m = \frac{1}{N} \sum_{j=1}^{N} X_j \tag{H.7}$$

Using Eqs. (H.5) and (H.7), the expectation of the mean value X_m is calculated as

$$\begin{aligned} \langle X_m \rangle &\equiv \int_{-\infty}^{+\infty} X_m \rho(X_1, \ldots, X_N) dX_1 \ldots dX_N \\ &= \int_{-\infty}^{+\infty} X_m \rho_1(X_1) \ldots \rho_N(X_N) dX_1 \ldots dX_N \\ &= \frac{1}{N} \sum_{i=1}^{N} \int_{-\infty}^{+\infty} X_i \rho_i(X_i) dX_i \equiv \frac{1}{N} \sum_{i=1}^{N} \langle X_i \rangle \end{aligned} \tag{H.8}$$

The variance of the mean value X_m is written as

$$\begin{aligned} Var(X_m) &\equiv \left\langle (\Delta X_m)^2 \right\rangle = \left\langle (X_m - \langle X_m \rangle)^2 \right\rangle \\ &= \int_{-\infty}^{+\infty} (X_m - \langle X_m \rangle)^2 \rho(X_1, \ldots, X_N) dX_1 \ldots dX_N \end{aligned} \tag{H.9}$$

Substituting Eqs. (H.7) and (H.8) into Eq. (H.9), one obtains the variance of X_m

$$\begin{aligned}\left\langle (\Delta X_m)^2 \right\rangle &= \frac{1}{N^2}\sum_{i=1}^{N} \int_{-\infty}^{+\infty} (X_i - \langle X_i \rangle)^2 \rho_i(X_i) dX_i \\ &= \frac{1}{N^2}\sum_{i=1}^{N} \left\langle (\Delta X_i)^2 \right\rangle \equiv \sigma_m^2\end{aligned} \tag{H.10}$$

Thus, the standard deviation of the mean value X_m results as

$$\sigma_m = \sqrt{\left\langle (\Delta X_m)^2 \right\rangle} = \frac{1}{N}\sqrt{\sum_{i=1}^{N} \left\langle (\Delta X_i)^2 \right\rangle} \tag{H.11}$$

In the case of equal probabilities for all N independent variables, the standard deviation of the mean value X_m results as

$$\begin{aligned}\sigma_m^2 &= \left\langle (\Delta X_m)^2 \right\rangle = \frac{N\left\langle (\Delta X_i)^2 \right\rangle}{N^2} \equiv \frac{Var(X_i)}{N} \\ \Rightarrow \sigma_m &= \frac{\sqrt{Var(X_i)}}{\sqrt{N}} \equiv \frac{\sigma}{\sqrt{N}}\end{aligned} \tag{H.12}$$

where σ is the standard deviation of variable X_i.

Equation (H.12) is *the law of large numbers* in statistical mechanics. It denotes that the mean value X_m of N independent variables X_i for $i = 1, 2, \ldots, N$ has a standard deviation that is only $1/N^{1/2}$ of the standard deviation of variable X_i in case of equal probabilities for N independent variables.

In statistics, there are two laws of large numbers, the weak and strong laws.

The *weak law* states that the mean value of N independent variables converges *in probability* towards the expected value when the number of variables is very large.

$$\lim_{N\to\infty} X_m \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} X_i \overset{P}{=} \langle X_m \rangle \tag{H.13}$$

In this case, the probability density function for the weak law is written for any positive real number ε as

$$\lim_{N\to\infty} \rho(|X_m - \langle X_m \rangle| > \varepsilon) = 0 \tag{H.14}$$

The *strong law* denotes that the mean value of N independent variables *always* converges to the expected value when the number of variables is very large.

$$\lim_{N\to\infty} X_m \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} X_i = \langle X_m \rangle \tag{H.15}$$

In this case, the probability density function for the strong law results as

$$\rho\left(\lim_{N\to\infty} X_m = \langle X_m \rangle\right) = 1. \tag{H.16}$$

Mathematical Symbols in This Book

- First partial derivative of a second-order tensor with respect to u^k

$$T^{ij}_{,k} \equiv \frac{\partial T^{ij}}{\partial u^k} \tag{1}$$

Do not confuse Eq. (1) with the symbol used in some books:

$$T^{ij}_{,k} \equiv \frac{\partial T^{ij}}{\partial u^k} + \Gamma^i_{km} T^{mj} + \Gamma^j_{km} T^{im}$$

This symbol is equivalent to Eq. (2) used in this book.

- Covariant derivative of a second-order tensor with respect to u^k

$$T^{ij}|_k \equiv T^{ij}_{,k} + \Gamma^i_{km} T^{mj} + \Gamma^j_{km} T^{im} \tag{2}$$

- Second partial derivative of a first-order tensor with respect to u^j and u^k

$$T_{i,jk} \equiv \frac{\partial^2 T_i}{\partial u^j \partial u^k} \tag{3}$$

Do not confuse Eq. (3) with the symbol used in some books:

$$\begin{aligned} T_{i,jk} \equiv \; & \frac{\partial^2 T_i}{\partial u^j \partial u^k} - \Gamma^m_{ik,j} T_m - \Gamma^m_{ik} \frac{\partial T_m}{\partial u^j} - \Gamma^m_{ij} \frac{\partial T_m}{\partial u^k} \\ & + \Gamma^m_{ij} \Gamma^n_{mk} T_n - \Gamma^m_{jk} \frac{\partial T_i}{\partial u^m} + \Gamma^m_{jk} \Gamma^n_{im} T_n \end{aligned}$$

This symbol is equivalent to Eq. (4) used in this book.

- Second covariant derivative of a first-order tensor with respect to u^j and u^k

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$$T_i\big|_{kj} \equiv T_{i,jk} - \Gamma^m_{ik,j} T_m - \Gamma^m_{ik} T_{m,j} - \Gamma^m_{ij} T_{m,k} + \Gamma^m_{ij} \Gamma^n_{mk} T_n - \Gamma^m_{jk} T_{i,m} + \Gamma^m_{jk} \Gamma^n_{im} T_n \tag{4}$$

- Christoffel symbols of first kind
  Γ_{ijk} instead of $[i\ j,\quad k]$ used in some books
- Christoffel symbols of second kind
  Γ^k_{ij} instead of $\left\{ \begin{matrix} \mathrm{k} \\ \mathrm{i} \quad \mathrm{j} \end{matrix} \right\}$ used in some books.

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