Ordinary Differential Equations of Non-Linear Elasticity I: Foundations of the Theories of Non-Linearly Elastic Rods and Shells

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Contents

1. Introduction

In this article, the first of a series, we obtain fully general theories of non-linearly elastic rods and shells from three-dimensional theories in a way that exposes their mathematical structure and physical significance while suppressing inessential detail. We specialize our results for shells to axisymmetric and cylindrical problems and show that the resulting one-dimensional equations have the same form as those for rods.

We thus obtain a simple mathematical characterization of the most general one-dimensional theories of non-linear elasticity in terms of the properties of a family of quasi-linear ordinary differential operators. The brevity of our derivations and the formal simplicity of our results are due to the compactness of the vectorial notation we employ.

Incorporated in our development are general one-dimensional consequences of the requirements that the deformation be locally invertible and orientation-preserving and that the constitutive functions satisfy the strong ellipticity condition. The resulting one-dimensional theories are accordingly endowed with a mathematical structure that preserves a number of the characteristic difficulties of the three-dimensional theory. This mathematical structure has a number of important physical consequences for rod and shell theories, which are discussed in Section 8. This structure supports not only the comprehensive existence, regularity, and multiplicity theories of Parts II and IV, but also the analysis of the qualitative behavior and solution branching for a variety of concrete problems (see Section 8). In marked contrast, general three-dimensional non-linear theories have so far proved to be mathematically intractable.

There are three ways of interpreting rod and shell theories (*cf.* ANTMAN (1972), NAGHDI (1972) and works cited therein):

i) They govern the behavior of intrinsically one and two-dimensional bodies (called *Cosserat continua*).

ii) They describe the behavior of constrained three-dimensional materials.

iii) They yield approximate solutions for three-dimensional problems.

We adopt interpretation (ii) as the most versatile. We obtain the governing equations by a process that generalizes the projection methods of Galerkin-Kantorovich type and that is reminiscent of techniques in the analytic dynamics of rigid bodies. Not only does this process yield rod and shell theories of more generality than those hitherto obtained either by intrinsically one- and two-dimensional formulations, but in a very natural way it also endows these theories with one- and two-dimensional analogs of the mathematical structure of the three-dimensional theory. To be sure, those aspects of this mathematical structure concerned with the ellipticity of the static part of the resulting differential operators can be obtained by alternative, purely intrinsic means. (*Cf.* ERICKSEN (1974), and SHAHINPOOR (1974).) On the other hand, our refined treatment of the invertibility condition relies completely on the three-dimensional origins of the theory. One of the most important and striking consequences of our development is that the mathematical structure of our rod and shell theories is independent of the class of holonomic constraints used to generate these theories. The mathematical structure characterizes every such theory. Thus there is a natural relation between interpretations (i) and (ii). (*Cf.* ANTMAN (1972, Sec. 15).)

Interpretation (iii) is in many ways the most important, because the practical justification for studying thin body theories is the supposition that they can accurately model the behavior of three-dimensional bodies. There is a voluminous literature devoted primarily to special linear problems that enforces, but does not rigorously justify, this view. Such justification demands effective error estimates. Toward this goal we obtain a preliminary convergence result in Section 7. We discuss technical aspects of developing error estimates in Section 8. (Our refined characterizations of local invertibility and orientation-preservation are essential to the construction of a viable approximation scheme.)

The work of this paper represents a substantial extension of that of ANTMAN (1972), whose format is closely followed, and of NAGHDI (1972). These articles in turn generalize many earlier investigations that they cite. We refer to these articles for full bibliographic and historical attributions. We do not formulate dynamical problems since we should add nothing to the treatment of ANTMAN (1972). Besides, as a consequence of our material formulation, the acceleration term effectively behaves like a second time derivative of displacement and provides nothing unexpected: It is the elliptic operator representing the constitutive relations that characterizes the non-linear structure of the hyperbolic problems of dynamic elasticity.

We remark that our treatment of three-dimensional non-linear elasticity is not merely a survey of standard material, but in fact contains a number of new developments, especially in the treatment of invertibility conditions, that play a fundamental role in the subsequent exposition.

The variational methods employed in Part II to treat conservative problems are so powerful that detailed descriptions of the topological properties of the class of admissible deformations are not needed. For the analysis of non-conservative problems, however, our simple development of Part I appears to be inadequate because it lacks such a description*. Consequently, in Part III we construct a much more refined characterization of one-dimensional non-linear elasticity under more restrictive assumptions. There we confront the one-dimensional consequences of the intricate relation among the requirements that the deformation be locally invertible and orientation-preserving, the strains be compatible, the fundamental geometric variables form a convex set of functions, the constitutive relations be frame-indifferent, and the constitutive functions satisfy a strong ellipticity condition and physically reasonable growth conditions. In Part IV, we study existence and regularity for non-conservative problems by using pseudo-monotone operator theory in conjunction with Brouwer degree theory.

2. Notation

We adopt the conventions that lower-case Greek indices have range 1, 2, that lower-case Latin indices have range 1, 2, 3, and that such diagonally repeated indices are summed over their range. Lower-case German indices are enumerative and are not summed.

Vectors in the Euclidean 3-space $\mathbb{E}^3$, triples in the real 3-space $\mathbb{R}^3$, and functions with values in either of these spaces are denoted by either lower-case or small upper-case bold-face letters. The components of the latter are denoted by full-size upper-case letters; thus $\mathbf{x}$ represents the triple (X^1, X^2, X^3). (Although $\mathbb{R}^3$ and other spaces are vector spaces, we restrict the unmodified term *vector* to elements of $\mathbb{E}^3$.) Second order tensors are elements of the nine-dimensional space $\mathscr{L}(\mathbb{E}^3, \mathbb{E}^3)$ of linear transformations of $\mathbb{E}^3$ into itself. Such tensors and functions with values in $\mathscr{L}(\mathbb{E}^3, \mathbb{E}^3)$ are denoted by full-size, upper-case, bold-face letters. We denote elements of $\mathbb{R}^N$ and functions with values in $\mathbb{R}^N$ by lower-case, bold-face, sans-serif letters.

We represent the usual operators of tensor algebra by a convenient dot notation. The inner product of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted $\boldsymbol{u} \cdot \boldsymbol{v}$. The norm $|\boldsymbol{u}|$ is defined to be $\sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$. The cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted $\boldsymbol{u} \times \boldsymbol{v}$. The value of the tensor $\boldsymbol{A}$ at the vector $\boldsymbol{u}$ is denoted $\boldsymbol{A} \cdot \boldsymbol{u}$. The product $\boldsymbol{A} \cdot \boldsymbol{B}$ of two tensors

* The need for topological restrictions for non-variational problems is evidenced by the simple problem of determining whether a two-dimensional continuous vector field can vanish on an annulus. If the vector field is the gradient of a continuously differentiable scalar function and if the gradient points outward from the annulus on the two circular boundaries, then the gradient must vanish at an interior point of the annulus where the scalar function is minimized. On the other hand, it is easy to construct examples of non-gradient vector fields that point outward on the boundaries but that fail to vanish on the annulus. The problem of determining where vector fields vanish is paradigmatic for the existence theories of Parts II and IV.

is the tensor defined as usual by $(\boldsymbol{A}\cdot\boldsymbol{B})\cdot\boldsymbol{u}=\boldsymbol{A}\cdot(\boldsymbol{B}\cdot\boldsymbol{u})$ for all $\boldsymbol{u}$. The transpose $\boldsymbol{A}^T$ of $\boldsymbol{A}$ is defined by $\boldsymbol{v}\cdot(\boldsymbol{A}\cdot\boldsymbol{u})=(\boldsymbol{A}^T\cdot\boldsymbol{v})\cdot\boldsymbol{u}$ for all $\boldsymbol{u}$, $\boldsymbol{v}$. Indeed we set $\boldsymbol{v}\cdot\boldsymbol{A}=\boldsymbol{A}^T\cdot\boldsymbol{v}$, so that parentheses are superfluous in the expression $\boldsymbol{v}\cdot(\boldsymbol{A}\cdot\boldsymbol{u})$. Thus a quadratic form based on $\boldsymbol{A}$ is denoted $\boldsymbol{u}\cdot\boldsymbol{A}\cdot\boldsymbol{u}$. If $\boldsymbol{A}=\boldsymbol{A}^T$, then $\boldsymbol{A}$ is called symmetric and we write $\boldsymbol{A}\in\mathscr{S}(\mathbb{E}^3,\mathbb{E}^3)$, where $\mathscr{S}(\mathbb{E}^3,\mathbb{E}^3)$ is the six-dimensional subspace of symmetric tensors. We set $\boldsymbol{A}:\boldsymbol{B}=\operatorname{tr}(\boldsymbol{A}\cdot\boldsymbol{B})$, where tr denotes the trace. We define the inner product of two tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ to be $\boldsymbol{A}:\boldsymbol{B}^T\equiv\boldsymbol{A}^T:\boldsymbol{B}$. We set $|\boldsymbol{A}|=\sqrt{\boldsymbol{A}:\boldsymbol{A}^T}$. We define the dyadic product $\boldsymbol{u}\boldsymbol{v}$ of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ to be the tensor satisfying $(\boldsymbol{u}\boldsymbol{v})\cdot\boldsymbol{w}=\boldsymbol{u}(\boldsymbol{v}\cdot\boldsymbol{w})$.

If $\{\boldsymbol{e}_i\}$ is a basis for $\mathbb{E}^3$, then its dual basis $\{\boldsymbol{e}^r\}$, is defined by $\boldsymbol{e}_i\cdot\boldsymbol{e}^r=\delta_i^r$, where δ_i^r is the Kronecker delta. We then have $\boldsymbol{v}=(\boldsymbol{v}\cdot\boldsymbol{e}^k)\,\boldsymbol{e}_k=(\boldsymbol{v}\cdot\boldsymbol{e}_k)\,\boldsymbol{e}^k$. We call $\{\boldsymbol{v}\cdot\boldsymbol{e}^k\}$ the *components of* $\boldsymbol{v}$ *with respect to the basis* $\{\boldsymbol{e}_k\}$. If $\{\boldsymbol{a}_k\}$ and $\{\boldsymbol{b}_k\}$ are bases for $\mathbb{E}^3$, then $\{\boldsymbol{a}_k\boldsymbol{b}_l\}$ is a basis for $\mathscr{L}(\mathbb{E}^3,\mathbb{E}^3)$. In particular, we can identify $\{\boldsymbol{a}_k\}$ with either $\{\boldsymbol{e}_k\}$ or $\{\boldsymbol{e}^k\}$ and $\{\boldsymbol{b}_k\}$ with either $\{\boldsymbol{e}_k\}$ or $\{\boldsymbol{e}^k\}$. A basis for $\mathscr{S}(\mathbb{E}^3,\mathbb{E}^3)$ is $\boldsymbol{e}_1\boldsymbol{e}_1,\ \frac{1}{\sqrt{2}}(\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_1),\ \ldots$.

If ϕ is a differentiable scalar function of the vector $\boldsymbol{u}$, if $\boldsymbol{f}$ is a differentiable vector function of the vector $\boldsymbol{u}$, if ψ is a differentiable scalar function of the tensor $\boldsymbol{A}$, and if θ is a differentiable scalar function of the symmetric tensor $\boldsymbol{D}$, then the vector function $\phi_{\boldsymbol{u}}\equiv\frac{\partial\phi}{\partial\boldsymbol{u}}$, the tensor functions $\boldsymbol{f}_{\boldsymbol{u}}\equiv\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{u}}$ and $\psi_{\boldsymbol{A}}\equiv\frac{\partial\psi}{\partial\boldsymbol{A}}$, and symmetric tensor function $\theta_{\boldsymbol{D}}\equiv\frac{\partial\theta}{\partial\boldsymbol{D}}$ are defined by

$$\frac{\partial\phi}{\partial\boldsymbol{u}}(\boldsymbol{v})\cdot\boldsymbol{w}=\lim_{t\to0}\frac{\phi(\boldsymbol{v}+t\boldsymbol{w})-\phi(\boldsymbol{v})}{t},\qquad\forall\,\boldsymbol{w}\in\mathbb{E}^3,\tag{2.1}$$

$$\frac{\partial\boldsymbol{f}}{\partial\boldsymbol{u}}(\boldsymbol{v})\cdot\boldsymbol{w}=\lim_{t\to0}\frac{\boldsymbol{f}(\boldsymbol{v}+t\boldsymbol{w})-\boldsymbol{f}(\boldsymbol{v})}{t},\qquad\forall\,\boldsymbol{w}\in\mathbb{E}^3,\tag{2.2}$$

$$\frac{\partial\psi}{\partial\boldsymbol{A}}(\boldsymbol{B}):\boldsymbol{C}^T=\lim_{t\to0}\frac{\psi(\boldsymbol{B}+t\boldsymbol{C})-\psi(\boldsymbol{B})}{t},\qquad\forall\,\boldsymbol{C}\in\mathscr{L}(\mathbb{E}^3,\mathbb{E}^3),\tag{2.3}$$

$$\frac{\partial\theta}{\partial\boldsymbol{D}}(\boldsymbol{E}):\boldsymbol{F}=\lim_{t\to0}\frac{\theta(\boldsymbol{E}+t\boldsymbol{F})-\theta(\boldsymbol{E})}{t},\qquad\forall\,\boldsymbol{F}\in\mathscr{S}(\mathbb{E}^3,\mathbb{E}^3).\tag{2.4}$$

If ψ is as above, if θ is the restriction of ψ to $\mathscr{S}(\mathbb{E}^3,\mathbb{E}^3)$, and if $\boldsymbol{E}\in\mathscr{S}(\mathbb{E}^3,\mathbb{E}^3)$, then $\theta(\boldsymbol{E})=\psi(\boldsymbol{E})$ and the preceding definitions imply that

$$\frac{\partial\theta}{\partial\boldsymbol{D}}(\boldsymbol{E})=\frac{1}{2}\left\{\frac{\partial\psi}{\partial\boldsymbol{A}}(\boldsymbol{E})+\left[\frac{\partial\psi}{\partial\boldsymbol{A}}(\boldsymbol{E})\right]^T\right\}.\tag{2.5}$$

If ϕ depends on vector $\boldsymbol{u}$ with components $\{u_k\}$, then we shall conventionally write $\phi(\boldsymbol{u})=\phi(u_k)$ and adopt the natural extensions of this policy.

Analogous definitions hold for $\mathbb{R}^N$: If $\boldsymbol{g}$ is a differentiable vector function of the N-tuple $\mathbf{u}=(u_1,\ldots,u_N)$, then $\frac{\partial\boldsymbol{g}}{\partial\mathbf{u}}(\mathbf{v})$, which belongs to the space $\mathscr{L}(\mathbb{R}^N,\mathbb{E}^3)$

of linear transformations of $\mathbb{R}^N$ to $\mathbb{E}^3$, is defined by

$$\frac{\partial g}{\partial \mathbf{u}}(\mathbf{v}) \cdot \mathbf{w} = \lim_{t \to 0} \frac{g(\mathbf{v}+t\mathbf{w}) - g(\mathbf{v})}{t}, \quad \forall \mathbf{w} \in \mathbb{R}^N, \tag{2.6}$$

etc. We have $\frac{\partial g}{\partial \mathbf{u}} \cdot \mathbf{w} = \sum_{p=1}^{N} \frac{\partial g}{\partial u_p} w_p$.

Let $\boldsymbol{h}$ be a vector function on $\mathbb{R}^N \times \mathbb{R}^3$ with values $\boldsymbol{h}(\mathbf{v}, \mathrm{Y})$ and let $\mathbf{u}$ be a function from $\mathbb{R}^3$ to $\mathbb{R}^N$ with values $\mathbf{u}(\mathrm{Y})$. We define

$$\boldsymbol{h}(\mathbf{u}(\mathrm{x}), \mathrm{x})_{,i} = \frac{\partial \boldsymbol{h}}{\partial \mathbf{v}}(\mathbf{u}(\mathrm{x}), \mathrm{x}) \cdot \frac{\partial \mathbf{u}}{\partial Y^i}(\mathrm{x}) + \frac{\partial \boldsymbol{h}}{\partial Y^i}(\mathbf{u}(\mathrm{x}), \mathrm{x}), \tag{2.7}$$

$$\frac{\partial \boldsymbol{h}}{\partial X^i}(\mathbf{u}(\mathrm{x}), \mathrm{x}) = \frac{\partial \boldsymbol{h}}{\partial Y^i}(\mathbf{u}(\mathrm{x}), \mathrm{x}). \tag{2.8}$$

With each such function $\boldsymbol{h}$, we associate the auxiliary functions

$$(\mathbf{u}, \mathbf{w}_i, \mathrm{x}) \mapsto \bar{\boldsymbol{h}}_i(\mathbf{u}, \mathbf{w}_i, \mathrm{x}) \equiv \frac{\partial \boldsymbol{h}}{\partial \mathbf{v}}(\mathbf{u}, \mathrm{x}) \cdot \mathbf{w}_i + \frac{\partial \boldsymbol{h}}{\partial Y^i}(\mathbf{u}, \mathrm{x}), \tag{2.9}$$

which have the property that

$$\boldsymbol{h}(\mathbf{u}(\mathrm{x}), \mathrm{x})_{,i} = \bar{\boldsymbol{h}}_i\left(u(\mathrm{x}), \frac{\partial \mathbf{u}}{\partial Y^i}(\mathrm{x}), \mathrm{x}\right). \tag{2.10}$$

We employ the obvious generalizations of this notational convention.

We denote sets by upper-case script letters. The closure of a set $\mathscr{S}$ is written $\mathrm{cl}\,\mathscr{S}$ and its boundary, $\partial\mathscr{S}$. $\mathscr{A} \setminus \mathscr{B}$ consists of all the elements of $\mathscr{A}$ not in $\mathscr{B}$. The range of a function f defined on $\mathscr{D}$ is denoted $f(\mathscr{D})$.

A function is called *smooth* if all of its derivatives that are exhibited in our exposition are continuous.

3. Classical and Weak Formulations of the Three-Dimensional Theory

We give a brief account of the three-dimensional theory of non-linear elasticity to establish notation, to list some fundamental assumptions, and primarily to emphasize those concepts that are central to the subsequent development. Several aspects of this description are novel.

Since the development of rod and shell theories relies upon certain distinguished material coordinate systems, we do not emphasize an entirely coordinate-free formulation of the three-dimensional theory.

a) Geometry of the Reference Configuration. The region occupied by a body in its reference configuration is assumed to admit a curvilinear material coordinate system $\mathrm{x} = (X^1, X^2, X^3)$. We assume that the body is such that x ranges over a domain $\mathscr{B}$ of $\mathbb{R}^3$ as the particles range over the body. Indeed, we simply identify the particles with their material coordinates x and term $\mathscr{B}$ the *body*. The position of x in the reference configuration is denoted $\mathrm{R}(\mathrm{x})$. R is assumed to be twice continuously differentiable and invertible on $\mathrm{cl}\,\mathscr{B}$. The metric tensor (identity tensor) **1** for these coordinates has covariant components

$$G_{ij} = \mathbf{R}_{,i} \cdot \mathbf{R}_{,j} \tag{3.1a}$$

(with respect to the basis dual to $\{\mathbf{R}_{,k}\,\mathbf{R}_{,l}\}$) and has contravariant components G^{ij} (with respect to the basis $\{\mathbf{R}_{,k}\,\mathbf{R}_{,l}\}$) defined by

$$G^{ij} G_{jk} = \delta^i_k. \tag{3.1b}$$

We set

$$G = \det(G_{ij}) = \left[\det\left(\frac{\partial \mathbf{R}}{\partial \mathrm{x}}\right)\right]^2. \tag{3.2}$$

We assume that G is bounded on $\mathrm{cl}\,\mathscr{B}$ and is positive almost everywhere on $\mathrm{cl}\,\mathscr{B}$. The differential volume of $\mathbf{R}(\mathscr{B})$ at x is

$$dV(\mathrm{x}) = \sqrt{G(\mathrm{x})}\, dX^1\, dX^2\, dX^3. \tag{3.3}$$

The differential surface area of $\partial \mathbf{R}(\mathscr{B})$ is denoted $dA(\mathrm{x})$. (Note that the boundary of the reference configuration of the body is $\partial \mathbf{R}(\mathscr{B})$ and not necessarily $\mathbf{R}(\partial B)$, which is a subset of $\partial \mathbf{R}(\mathscr{B})$.)

b) The Deformation. The position of x in an arbitrary deformed configuration is denoted $\boldsymbol{r}(\mathrm{x})$. Let $\boldsymbol{p}\colon \mathbf{R}(\mathscr{B}) \to \mathscr{E}^3$, be defined by $\boldsymbol{p}(\mathbf{z}) = \boldsymbol{r}(\mathbf{R}^{-1}(\mathbf{z}))$. $\boldsymbol{p}$ assigns position in the deformed configuration to the particle whose reference position is $\mathbf{z}$. The *deformation gradient* $\boldsymbol{F}$ is the tensor defined by $\boldsymbol{F} = \dfrac{\partial \boldsymbol{p}}{\partial \mathbf{z}}$. Note that

$$\frac{\partial \boldsymbol{r}}{\partial \mathrm{x}}(\mathrm{x}) = \boldsymbol{F}(\mathbf{R}(\mathrm{x})) \cdot \frac{\partial \mathbf{R}}{\partial \mathrm{x}}(\mathrm{x})$$

and that $\boldsymbol{F}(\mathbf{z}) \in \mathscr{L}(\mathbb{E}^3, \mathbb{E}^3)$, whereas $\dfrac{\partial \boldsymbol{r}}{\partial \mathrm{x}}(\mathrm{x}), \dfrac{\partial \mathbf{R}}{\partial \mathrm{x}}(\mathrm{x}) \in \mathscr{L}(\mathbb{R}^3, \mathbb{E}^3)$. We want $\boldsymbol{p}$ to be invertible so that two distinct parts of the body cannot occupy the same region of space in this deformed configuration. This global restriction on $\boldsymbol{p}$ or on $\boldsymbol{r}$, which demands the detailed prescription of boundary conditions that are to hold when distinct parts of the boundary come into contact, entails quite delicate questions of analysis (*cf.* FICHERA (1972b), DUVAUT & LIONS (1972), and works cited therein). We accordingly choose to ignore this requirement and thereby tolerate the possibility that two parts of a body may occupy the same region of space without interaction.

Instead, we consider the much weaker requirement that the volume ratio be positive:

$$\det \boldsymbol{F}(\mathbf{R}(\mathrm{x})) = G^{-\frac{1}{2}}(\mathrm{x}) \det\left(\frac{\partial \boldsymbol{r}}{\partial \mathrm{x}}(\mathrm{x})\right) \equiv G^{-\frac{1}{2}}(\mathrm{x})[\boldsymbol{r}_{,1}(\mathrm{x}) \times \boldsymbol{r}_{,2}(\mathrm{x})] \cdot \boldsymbol{r}_{,3}(\mathrm{x}) > 0 \tag{3.4}$$

(at least almost everywhere). This ensures that $\boldsymbol{r}$ be locally invertible and orientation-preserving. We call this the *invertibility* condition. We can replace this pointwise condition with a related global condition that

$$\int_{\mathscr{B}} h\left(G^{-\frac{1}{2}}(\mathbf{x}) \det\left(\frac{\partial \boldsymbol{r}}{\partial \mathbf{x}}(\mathbf{x})\right)\right) dV(\mathbf{x}) < \infty \tag{3.5}$$

for some non-negative function h: $[0,\infty)\to[0,\infty)$ with $h(\delta)\to\infty$ as $\delta\to 0$. This ensures that (3.4) hold a.e. Note that these two conditions are strict inequalities. This poses some analytic difficulties. Below we shall assign a constitutive interpretation to the invertibility condition by relating a function like h to the growth rate of stress when the volume ratio is small. More generally, we may permit h to depend on other variables such as $|\boldsymbol{r}_{,3}(\mathbf{x})|$, whose vanishing causes $\det(\partial \boldsymbol{r}/\partial \mathbf{x})$ to vanish.

The set of functions $\boldsymbol{r}$ satisfying the invertibility condition (a.e.) on $\mathscr{B}$ form a cone in the most primitive sense that if $\boldsymbol{r}$ satisfies (3.4) (a.e.), then so does $\alpha\boldsymbol{r}$ for each positive number α. But this cone does not possess any convenient convexity properties. In fact, consider the family of mappings depending on the parameter θ:

$$\boldsymbol{r}(\mathbf{x};\theta)=\left(1+\frac{\theta}{\pi}\right)(X^1\cos\theta+X^2\sin\theta)\,\mathbf{E}_1$$
$$+\left(1-\frac{\theta}{2\pi}\right)(-X^1\sin\theta+X^2\cos\theta)\,\mathbf{E}_2+X^3\,\mathbf{E}_3.$$

Here $\{X^i\}$ are Cartesian coordinates and $\{\mathbf{E}_i\}$ is the corresponding orthonormal basis for $\mathbb{E}^3$. For $\theta\in[0,\pi]$, these mappings form a connected curve in the cone. On the other hand

$$\tfrac{1}{2}[\boldsymbol{r}(\mathbf{x};0)+\boldsymbol{r}(\mathbf{x};\pi)]=-\tfrac{1}{2}X^1\,\mathbf{E}_1+\tfrac{1}{4}X^2\,\mathbf{E}_2+X^3\,\mathbf{E}_3$$

so that $\frac{1}{2}[\boldsymbol{r}(\cdot\,;0)+\boldsymbol{r}(\cdot\,;\pi)]$ does not belong to the cone. Thus the cone has a connected component that is not convex. This suggests that finite dimensional sections of the cone may lack pleasant connectivity properties. In fact, for certain classes of problems we can assert that the cone has at least a countably infinite number of components. *E.g.*, consider the family of volume-preserving mappings (dislocations) $\{\boldsymbol{r}_{\mathfrak{k}}, \mathfrak{k}$ an integer$\}$ given by

$$\boldsymbol{r}_{\mathfrak{k}}(P,\Phi,\Psi)=[2+P\cos(\Psi+\mathfrak{k}\Phi)]\cos\Phi\,\mathbf{E}_1+[2+P\cos(\Psi+\mathfrak{k}\Phi)]$$
$$\cdot\sin\Phi\,\mathbf{E}_2+P\sin(\Psi+\mathfrak{k}\Phi)\,\mathbf{E}_3,$$

which take the solid torus $\boldsymbol{r}_0([0,1],[0,2\pi),[0,2\pi))$ homeomorphically into itself. Since each such mapping $\boldsymbol{r}_{\mathfrak{k}}$ belongs to a different homotopy class, our assertion about connectivity holds. These facts suggest that the incorporation of (3.4) into the formulation of a mechanics problem may present serious analytic difficulties. Below we suggest how these difficulties can be circumvented at some expense.

Remark. It is also instructive to consider other mappings that correspond to breaking the torus, tying it in a knot, and then rejoining the break so that particles originally contiguous are again contiguous. This collection of mappings provides us with no other homotopy class beyond those generated by $\{\boldsymbol{r}_{\mathfrak{k}}\}$ in the preceding example, since the local conditions (3.4) and the concept of homotopy permit us to pass one part of a body through another part. On the other hand, if we insist upon global invertibility, then such self-intersections are prohibited. This latter set of dislocations then generates a new collection of topologically distinct configurations according to a knot-theoretic classification of the image of the material circular line of centroids of the torus. (In this connection, *cf.* PIERCE (1973), who has examined the topological structure of the class of globally invertible mappings.)

An alternative characterization of local invertibility can be specified in terms of strain tensors. By the polar decomposition theorem, inequality (3.4) implies that $\partial \boldsymbol{r}/\partial \mathbf{x}$ has the unique decomposition

$$\frac{\partial \boldsymbol{r}}{\partial \mathbf{x}} = \boldsymbol{Q} \cdot \boldsymbol{U} \cdot \frac{\partial \mathbf{R}}{\partial \mathbf{x}}, \qquad (\boldsymbol{F} = \boldsymbol{Q} \cdot \boldsymbol{U}) \tag{3.6a}$$

where the *rotation tensor* $\boldsymbol{Q}$ is orthogonal and where the *right stretch tensor* $\boldsymbol{U}$ is the non-negative definite square root of the non-negative-definite *Green strain tensor* $\boldsymbol{C}$ that is defined by

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} \tag{3.6b}$$

and that has components

$$C_{ij} = \boldsymbol{r}_{,i} \cdot \boldsymbol{r}_{,j} \tag{3.6c}$$

relative to the basis dual to $\{\mathbf{R}_{,i}\, \mathbf{R}_{,j}\}$. Note that

$$\det \boldsymbol{C} = G^{-1} \left[\det \left(\frac{\partial \boldsymbol{r}}{\partial \mathbf{x}} \right) \right]^2. \tag{3.7}$$

When $\dfrac{\partial \boldsymbol{r}}{\partial \mathbf{x}}(\mathbf{x})$ is finite, the invertibility condition (3.4) ensures that $\boldsymbol{C}(\mathbf{x})$ is actually positive-definite and this in turn implies that the ratio of deformed to reference length of any fiber at $\mathbf{x}$ is positive. (Note that a deformation with a finite volume ratio can have some fibers stretched to infinite length and others compressed to zero length. The positive-definiteness of $\boldsymbol{C}$ prohibits this possibility.) On the other hand, the positive-definiteness of $\boldsymbol{C}(\mathbf{x})$ cannot tell us whether the material about $\mathbf{x}$ has suffered a change of orientation. This is not too serious because the stronger requirement that $\boldsymbol{C}$ be continuous and positive-definite everywhere on $\mathscr{B}$ ensures that the volume ratio cannot vanish on $\mathscr{B}$ and therefore the material about each $\mathbf{x}$ in $\mathscr{B}$ has the same orientation.

If we are willing to accept this mild ambiguity in the orientation of the deformed configuration, then we can adopt as an alternative invertibility condition the requirement that $\boldsymbol{U}$ (or $\boldsymbol{C}$) be positive-definite everywhere:

$$\boldsymbol{a} \cdot \boldsymbol{U}(\mathbf{x}) \cdot \boldsymbol{a} > 0 \qquad \forall\, \boldsymbol{a} \neq \mathbf{0} \text{ and } \forall\, \mathbf{x} \in \mathscr{B}. \tag{3.8}$$

By imposing suitable growth conditions on constitutive functions, we can preclude this ambiguity for classical solutions of problems of non-linear continuum mechanics. The actual treatment for non-linear elasticity is given below.

The collection of all mappings $\boldsymbol{r}$ such that $\boldsymbol{C}$ is positive-definite do not form a convex set since the average of the identity and a rotation of π about a fixed axis is a mapping for which $\det C = 0$. Indeed, the same geometric complexity of the class of mappings $\boldsymbol{r}$ satisfying (3.4) is observed for the class of mappings $\boldsymbol{r}$ satisfying (3.8).

On the other hand, the class of all positive-definite tensor-valued functions do form a convex set. It seems highly advantageous, if not critical, in non-conservative problems for the admissible class of dependent variables to form a convex set of functions. There seems to be no such urgency for conservative problems. (*Cf.* Part II.) In treating non-conservative problems we are accordingly motivated to choose $\boldsymbol{C}$ or $\boldsymbol{U}$, rather than $\boldsymbol{r}$ as our fundamental geometrical variable. (The pre-eminence of $\boldsymbol{U}$ arising from the consequences of frame-indifference also supports this choice.) In this case the 3-vector $\boldsymbol{r}$ must be found as a solution of the over-determined sixth order system (3.6) of partial differential equations. If $\boldsymbol{C}$ is smooth enough, then the compatibility conditions, which are necessary and sufficient conditions that (3.6) possess a classical solution $\boldsymbol{r}$ unique to within a rigid motion, include the requirement that the Riemann-Christoffel curvature tensor based on $\boldsymbol{C}$ vanish. Since this fourth order tensor consists of a non-linear combination of derivatives of $\boldsymbol{C}$ up to order 2, the use of this classical local version of compatibility portends serious regularity problems for any analysis. This difficulty might conceivably be mitigated by replacing the classical compatibility conditions by a corresponding set of integral relations. (*Cf.* TING (1974) for a treatment of this question for linearized compatibility equations.)

The one-dimensional analog of this approach will be adopted in our treatment of one-dimensional problems of non-linear elasticity. In this case, the difficulties caused by compatibility are easily resolved (in Part III). In Part IV we show that our choice of the strains as fundamental geometric variables is the cornerstone of a full existence and regularity theory.

We remark that it is unfortunately necessary to characterize the deformation alternately by $\boldsymbol{C}$, $\boldsymbol{U}$, and $\{\boldsymbol{r}_{,k}\}$. $\boldsymbol{C}$ is algebraically the most convenient measure of strain while $\boldsymbol{U}$ is most convenient geometrically. The vectors $\{\boldsymbol{r}_{,k}\}$ are more useful in our present work than the tensor $\boldsymbol{F}$ because they allow us to introduce a distinguished reference coordinate system in a simple way.

c) Stress and the Equilibrium Equations. Let $\{\boldsymbol{\tau}^i\}$ denote the Piola-Kirchhoff stress vectors: if $\mathrm{x} \in \mathscr{B}$, then $\boldsymbol{\tau}^k(\mathrm{x})$ is the force per unit reference area of $\{\mathrm{y} \in \mathscr{B} : Y^k = X^k\}$ exerted by $\{\mathrm{y} \in \mathscr{B} : Y^k > X^k\}$ on $\{\mathrm{y} \in \mathscr{B},\ Y^k \leqq X^k\}$ at x. The body force per unit reference volume at x is denoted $\boldsymbol{f}[\boldsymbol{r}; \mathrm{x}]$, where $\boldsymbol{f}$ is an assigned function from $\tilde{\mathscr{V}} \times \mathrm{cl}\,\mathscr{B}$ to $\mathbb{E}^3$ and $\tilde{\mathscr{V}}$ is a collection of admissible position fields $\boldsymbol{r}$. The classical form of the equilibrium equations is given by

$$(3.9)\qquad (\sqrt{G}\,\boldsymbol{\tau}^k)_{,k} + \sqrt{G}\,\boldsymbol{f} = \boldsymbol{0},$$

which is to hold wherever the left side of (3.9) is continuous. In addition, we require

$$(3.10)\qquad \boldsymbol{r}_{,k} \times \boldsymbol{\tau}^k = \boldsymbol{0},$$

wherever the left sides of (3.9) and (3.10) are continuous. This expression of local balance of moments will be adopted as a constitutive restriction.

To state classical boundary conditions easily, we assume that $\partial\mathscr{B}$ is smooth enough to possess a unit outer normal vector $\mathrm{N}(\mathrm{x})$ at each particle x. Let

$\partial\mathscr{B}=\bigcup_{c=0}^{3}\mathscr{S}_c$, with $\{\mathscr{S}_c,\ c=0,1,2,3\}$ mutually disjoint. We consider boundary conditions of the form

$$r=\hat{r}(\mathrm{x}) \quad \text{for } \mathrm{x}\in\mathscr{S}_0; \tag{3.11}$$

$$\gamma_a(r(\mathrm{x}),\mathrm{x})=0, \quad a=1,2, \quad \text{for } \mathrm{x}\in\mathscr{S}_1 \tag{3.12a}$$

with γ_1 and γ_2 independent,

$$[\tau^k(\mathrm{x})N_k(\mathrm{x})]\cdot e(r(\mathrm{x}),\mathrm{x})=\sigma[r|_{\partial\mathscr{B}};\mathrm{x}]\cdot e(r(\mathrm{x}),\mathrm{x}) \quad \text{for } \mathrm{x}\in\mathscr{S}_1 \tag{3.12b}$$

with

$$e(r,\mathrm{x})\cdot\frac{\partial\gamma_a}{\partial r}(r,\mathrm{x})=0, \quad a=1,2;$$

$$\gamma(r(\mathrm{x}),\mathrm{x})=0, \quad \text{for } \mathrm{x}\in\mathscr{S}_2, \tag{3.13a}$$

$$[\tau^k(\mathrm{x})N_k(\mathrm{x})]\cdot e_a(r(\mathrm{x}),\mathrm{x})=\sigma[r|_{\partial\mathscr{B}};\mathrm{x}]\cdot e_a(r(\mathrm{x}),\mathrm{x}), \quad a=1,2, \quad \text{for } \mathrm{x}\in\mathscr{S}_2 \tag{3.13b}$$

with e_1 and e_2 independent vectors satisfying

$$e_a(r,\mathrm{x})\cdot\frac{\partial\gamma}{\partial r}(r,\mathrm{x})=0;$$

$$\tau^k(\mathrm{x})N_k(\mathrm{x})=\sigma[r|_{\partial\mathscr{B}};\mathrm{x}] \quad \text{for } \mathrm{x}\in\mathscr{S}_3. \tag{3.14}$$

Here $\hat{r}, \gamma_1, \gamma_2, \gamma, \sigma, e, e_1, e_2$ are prescribed functions of their arguments, with $\hat{r}$ continuous and with $\gamma_1, \gamma_2, \gamma$ continuously differentiable. Condition (3.11) says that the position of the particle x is prescribed; (3.12), that the particle x is constrained to move on a curve and that the component of traction tangent to this curve is prescribed; (3.13), that the particle x is constrained to move on a fixed surface and the components of traction tangent to this surface are prescribed; and (3.14), that the position of the particle is free and the traction is completely prescribed. We can write (3.11)–(3.14) in a compact parametric form:

$$r(\mathrm{x})=\hat{r}(\mathrm{x},\mathrm{v}(\mathrm{x})), \tag{3.15a}$$

$$[\tau^k(\mathrm{x})N_k(\mathrm{x})]\cdot\frac{\partial r}{\partial\mathrm{v}}(\mathrm{x},\mathrm{v}(\mathrm{x}))=\sigma[r|_{\partial\mathscr{B}};\mathrm{x}]\cdot\frac{\partial\hat{r}}{\partial\mathrm{v}}(\mathrm{x},\mathrm{v}(\mathrm{x})) \tag{3.15b}$$

for $\mathrm{x}\in\partial\mathscr{B}$. Here $\hat{r}$ and σ are prescribed functions of their arguments with $\hat{r}$ differentiable in v. Where $\partial r/\partial\mathrm{v}$ has rank 0, 1, 2, 3, condition (3.15) reduces to (3.11), (3.12), (3.13), (3.14), respectively. When v actually appears, it is an unknown of the problem.

The classical formulation of equilibrium equations (3.8) and boundary conditions (3.11)–(3.14) or (3.15) is unduly restrictive. If $\{\tau^k\}$ and f are Lebesgue integrable on $\mathscr{B}$, if σ is Lebesgue integrable on $\partial\mathscr{B}$, and if $\partial\mathscr{B}$ is sufficiently well-behaved, then the weak form of (3.8) and (3.11)–(3.14) or (3.15) (which is just the principle of virtual work) is

$$\int_{\mathscr{B}}[\tau^k\cdot\eta_{,k}-f\cdot\eta]\,dV(\mathrm{x})-\int_{\partial\mathscr{B}}\sigma\cdot\eta\,dA(\mathrm{x})=0 \tag{3.16}$$

for all smooth enough vectors $\boldsymbol{\eta}$ satisfying

$$(3.17)\quad \boldsymbol{\eta}=\mathbf{0} \text{ on } \mathscr{S}_0, \quad \frac{\partial \gamma_\alpha}{\partial \boldsymbol{r}}\cdot\boldsymbol{\eta}=0, \quad \alpha=1,2 \text{ on } \mathscr{S}_1, \quad \frac{\partial \gamma}{\partial \boldsymbol{r}}\cdot\boldsymbol{\eta}=0 \text{ on } \mathscr{S}_2,$$

when the boundary conditions (3.11)–(3.14) are used, and satisfying

$$(3.18)\qquad \boldsymbol{\eta}(\mathrm{x})=\frac{\partial \hat{\boldsymbol{r}}}{\partial \mathrm{v}}(\mathrm{x},\mathrm{v}(\mathrm{x}))\cdot\mathrm{w}(\mathrm{x}), \qquad \mathrm{x}\in\partial\mathscr{B},$$

for arbitrary sufficiently smooth functions $\mathbf{w}$ when boundary condition (3.15) is used.

Equation (3.16) implies (3.8) whenever the left side of (3.8) is continuous. Moreover, (3.16) implies the balance of linear momentum:

$$(3.19)\qquad \int_{\partial\mathscr{P}} \boldsymbol{\tau}^k N_k\, dA(\mathrm{x}) + \int_{\mathscr{P}} \boldsymbol{f}\, dV(\mathrm{x}) = \mathbf{0}$$

for all compact $\mathscr{P}\subset\mathscr{B}$ with smooth enough boundary. This can be shown by choosing $\boldsymbol{\eta}$ to approximate $\chi_{\mathscr{P}}\mathrm{E}_k$ where $\chi_{\mathscr{P}}$ is the characteristic function for $\mathscr{P}$ and $\{\mathrm{E}_k\}$ is a set of fixed orthonormal vectors. Conversely, (3.19) can be shown to imply (3.16) by approximating $\boldsymbol{\eta}$ by finite linear combinations of $\chi_{\mathscr{P}}$, $\mathscr{P}\subset\mathscr{B}$.

d) Constitutive Relations. Let $\bar{\mathbb{E}}^3$ denote the extended Euclidean three-space consisting of vectors whose Cartesian components are extended real numbers. The material of $\mathscr{B}$ is *(Cauchy) elastic* if there exist functions

$$\boldsymbol{y}_j, \mathrm{x} \mapsto \boldsymbol{\tau}^k(\boldsymbol{y}_j, \mathrm{x}) \quad \text{and} \quad C_{ij}, \mathrm{x} \mapsto T^{kl}(C_{ij}, \mathrm{x})$$

such that

$$(3.20)\qquad \boldsymbol{\tau}^k(\mathrm{x})=\hat{\boldsymbol{\tau}}^k(\boldsymbol{r}_{,j}(\mathrm{x}),\mathrm{x}), \qquad \mathrm{x}\in \mathrm{cl}\,\mathscr{B},$$

with

$$(3.21)\qquad \hat{\boldsymbol{\tau}}^k:\{\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3:(\boldsymbol{y}_1\times\boldsymbol{y}_2)\cdot\boldsymbol{y}_3\geqq 0\}\times \mathrm{cl}\,\mathscr{B}\to\bar{\mathbb{E}}^3,$$

$$(3.22)\qquad \hat{\boldsymbol{\tau}}^k:\{\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3:(\boldsymbol{y}_1\times\boldsymbol{y}_2)\cdot\boldsymbol{y}_3> 0\}\times \mathscr{B}\to\mathbb{E}^3,$$

$$(3.23)\qquad \hat{\boldsymbol{\tau}}^k(\boldsymbol{y}_j,\mathrm{x})=T^{kl}(\boldsymbol{y}_i\cdot\boldsymbol{y}_j,\mathrm{x})\,\boldsymbol{y}_l,$$

$$(3.24)\qquad \boldsymbol{y}_k\times\boldsymbol{\tau}^k(\boldsymbol{y}_j,\mathrm{x})=\mathbf{0}, \quad \text{or equivalently,} \quad T^{kl}=T^{lk}.$$

Relation (3.23) ensures that the response is frame-indifferent (*cf.* TRUESDELL & NOLL (1965)) and (3.24) ensures that (3.9) is satisfied. $\{T^{kl}\}$ are the components of the second Piola-Kirchhoff stress tensor. For simplicity we assume that $\{\hat{\boldsymbol{\tau}}^k\}$ are continuously differentiable functions on the domain shown in (3.22). (BELL'S (1973) experimental work indicates that this differentiability assumption may not be valid for many materials. It is not difficult, however, to generalize the various special assumptions made below, such as strong ellipticity, to account for merely continuous constitutive functions.)

The material of $\mathscr{B}$ is *hyperelastic* if there exist scalar functions ψ and ϕ defined on the domain of $\{\hat{\boldsymbol{\tau}}^k\}$ such that

$$(3.25)\qquad \hat{\boldsymbol{\tau}}^k(\boldsymbol{y}_i,\mathrm{x})=\frac{\partial\psi}{\partial \boldsymbol{y}^k}(\boldsymbol{y}_i,\mathrm{x})$$

with

$$\psi(\boldsymbol{y}_i, \mathrm{x}) = \phi(\boldsymbol{y}_i \cdot \boldsymbol{y}_j, \mathrm{x}). \tag{3.26}$$

We require that $\{\hat{\boldsymbol{\tau}}^k\}$ satisfy the strict form of the *strong ellipticity condition*

$$\mu_k \boldsymbol{\lambda} \cdot \frac{\partial \hat{\boldsymbol{\tau}}^k}{\partial \boldsymbol{y}_l} \cdot \boldsymbol{\lambda} \mu_l > 0 \quad \forall\, \boldsymbol{\lambda}, \boldsymbol{\mu} \neq \mathbf{0}. \tag{3.27a}$$

The componential version is far more complicated:

$$\left[2 \frac{\partial T^{kl}}{\partial C_{ij}} (\boldsymbol{\lambda} \cdot \boldsymbol{y}_l)(\boldsymbol{\lambda} \cdot \boldsymbol{y}_j) + T^{ki} \boldsymbol{\lambda} \cdot \boldsymbol{\lambda}\right] \mu_k \mu_i > 0 \quad \forall\, \boldsymbol{\lambda}, \boldsymbol{\mu} \neq \mathbf{0}. \tag{3.27b}$$

In the hyperelastic case, this condition reduces to the *strong Legendre-Hadamard condition*:

$$\mu_k \boldsymbol{\lambda} \cdot \frac{\partial^2 \psi}{\partial \boldsymbol{y}_k \partial \boldsymbol{y}_l} \cdot \boldsymbol{\lambda} \mu_l > 0 \quad \forall\, \boldsymbol{\lambda}, \boldsymbol{\mu} \neq \mathbf{0}. \tag{3.27c}$$

In our work, the restriction (3.27) on material response proves to be far more natural than other inequalities, such as the Coleman-Noll inequalities. The physical and mathematical implications of (3.27) and related inequalities have been discussed by COLEMAN & NOLL (1959), TRUESDELL & NOLL (1965), MORREY (1966), HAYES (1969), HILL (1970), KNOPS & PAYNE (1971), WANG & TRUESDELL (1973) and in works they cite. At the end of this section we shall contrast this condition with the unacceptably stringent *strict monotonicity condition**

$$\xi_k \cdot \frac{\partial \hat{\boldsymbol{\tau}}^k}{\partial \boldsymbol{y}_l} \cdot \xi_l > 0 \quad \forall\, \{\xi_k\} \text{ such that } G^{kl} \xi_k \cdot \xi_l \neq 0, \tag{3.28}$$

which in the hyperelastic case implies that ψ be strictly convex in $\{\boldsymbol{y}_i\}$. There we shall discuss how these two seemingly similar conditions hold significantly different implications for the boundary value problems of non-linear elasticity.

We now impose certain growth conditions, consistent with (3.27), on our constitutive functions $\{\hat{\boldsymbol{\tau}}^k\}$. These conditions effectively require that the Piola-Kirchhoff stress become large as the strain $\boldsymbol{U}$ becomes large and as the strain $\boldsymbol{U}$ loses its definiteness. This latter circumstance is indicated by the approach to zero of the principal subdeterminants of $\boldsymbol{U}$. We thereby incorporate (3.4) or (3.8) as a constitutive restriction. The violation of (3.4) or (3.8) is signified by the unboundedness of the stress, *i.e.*, by the loss of regularity of solutions of the governing equations. A somewhat stronger restriction is that an energy-like quantity get large as (3.4) or (3.8) is violated. We now give formal statements of these and related conditions.

---

* An operator g mapping a convex subset of a real linear space into the dual space is called *strictly monotone* if $\langle g(u) - g(v), u - v \rangle > 0$ for $u \neq v$. Here the bracket denotes the pairing of an element of the space with an element of its dual. In particular, if $\mathbf{h}$ is a differentiable mapping of a convex subset of $\mathbb{R}^N$ into $\mathbb{R}^N$, then $\mathbf{h}$ is strictly monotone if the quadratic form $\mathbf{a} \cdot \frac{\partial \mathbf{h}}{\partial \mathbf{u}} \cdot \mathbf{a}$ is positive definite.

We only examine hyperelastic materials here. Part III treats the general case. The frame-indifferent strain energy has four related functional forms:

$$\psi(\boldsymbol{r}_{,k}, \mathrm{x}) = \theta(\boldsymbol{F}, \mathrm{x}) = \theta(\boldsymbol{U}, \mathrm{x}) = \phi(\boldsymbol{C}, \mathrm{x}).$$

We shall use the form that seems most convenient in a given context. We regard position fields that differ only by a rigid displacement as equivalent. In our characterization of function spaces for position fields, *it is understood that the elements of the space are actually such equivalence classes.* We now state our hypotheses.

3.29. Hypothesis. $\left|\frac{\partial \theta}{\partial \boldsymbol{F}}(\boldsymbol{F}, \mathrm{x})\right| \to \infty$ *as* $|\boldsymbol{F}| = |\boldsymbol{U}| \to \infty$ *or as* $\det \boldsymbol{F} \to 0$.

A stronger restriction is

3.30. Hypothesis. $\theta(\boldsymbol{F}, \mathrm{x}) \to \infty$ *as* $|\boldsymbol{F}| \to \infty$ *or as* $\det \boldsymbol{F} \to 0$.

The next hypotheses are more technical.

Let $[\mathscr{W}_\alpha^1(\mathscr{B})]^3$, $\alpha \geqq 1$, denote the Sobolev space of functions $\boldsymbol{r}$ having locally integrable distributional derivatives for which

$$\int_{\mathscr{B}} |\boldsymbol{F}(\mathbf{R}(\mathrm{x}))|^\alpha \, dV(\mathrm{x}) < \infty. \tag{3.31}$$

We set

$$\mathscr{H} = \left\{ \boldsymbol{r} \in [\mathscr{W}_1^1(\mathscr{B})]^3 : \int_{\mathscr{B}} \theta(\boldsymbol{F}(\mathbf{R}(\mathrm{x})), \mathrm{x}) \, dV(\mathrm{x}) < \infty \right\}. \tag{3.32}$$

3.33. Hypothesis.

i) *There are numbers* α^-, α^+ *with* $1 < \alpha^- \leqq \alpha^+ < \infty$ *such that* $\mathscr{H}$ *belongs to a reflexive Banach space* $\mathscr{V}$ *with*

$$[\mathscr{W}_{\alpha^+}^1(\mathscr{B})]^3 \subset \mathscr{V} \subset [\mathscr{W}_{\alpha^-}^1(\mathscr{B})]^3. \tag{3.34}$$

ii) *If* $\boldsymbol{r} \in \mathscr{H}$ *and if there is a number* $\mu > 0$ *such that*

$$\det \boldsymbol{F}(\mathbf{R}(\mathrm{x})) > \mu \qquad \forall \mathrm{x} \in \mathscr{B}, \tag{3.35}$$

then

$$\begin{aligned} &\int_{\mathscr{B}} \hat{\boldsymbol{\tau}}^k(\boldsymbol{r}_{,p}(\mathrm{x}), \mathrm{x}) \cdot \boldsymbol{\eta}_{,k}(\mathrm{x}) \, dV(\mathrm{x}) \\ &\equiv \int_{\mathscr{B}} \frac{\partial \theta}{\partial \boldsymbol{F}}(\boldsymbol{F}(\mathbf{R}(\mathrm{x}), \mathrm{x})) : \left[\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{z}}(\mathbf{R}(\mathrm{x}))\right]^T dV(\mathrm{x}) \end{aligned} \tag{3.36}$$

is finite for all $\boldsymbol{\eta} \in \mathscr{V}$. (This means that when (3.35) holds, the formal gradient of the total strain energy functional belongs to the dual space $\mathscr{V}^*$ of $\mathscr{V}$.) Here $\boldsymbol{\eta}(\mathrm{x}) = \boldsymbol{\pi}(\mathbf{R}(\mathrm{x}))$.

As an alternative approach we could have restricted $\mathscr{H}$ and the spaces of (3.34) to consist of functions $\boldsymbol{\eta}$ satisfying (3.17) or (3.18). Moreover, if necessary we could have supplemented our boundary conditions (3.15a) with additional conditions that would remove the ambiguity of a rigid displacement from solutions without changing their essential properties. In this case each equivalence class of displacements would have but one element. This view is pursued in Part II. (When this approach is used, trace theorems must be used to ensure that the boundary conditions make sense.)

The admissible class of functions $\tilde{\mathscr{H}}$ consists of those elements of $\mathscr{H}$ that satisfy (3.15a).

3.37. Coercivity Hypothesis. *The functional* $\boldsymbol{r} \mapsto \int_{\mathscr{B}} \theta(\boldsymbol{F}(\mathbf{R}(\mathrm{x})), \mathrm{x})\, dV(\mathrm{x})$ *is bounded below on* $\tilde{\mathscr{H}}$ *and*

$$\int_{\mathscr{B}} \theta(\boldsymbol{F}(\mathbf{R}(\mathrm{x})), \mathrm{x})\, dV(\mathrm{x}) \to \infty \quad \text{as } \|\boldsymbol{r}\|_{\mathscr{V}} \to \infty \text{ on } \tilde{\mathscr{H}}. \tag{3.38}$$

We can illustrate these growth conditions by an example that shows that they are not inconsistent. We assume that there is a smooth function h^-: $(0,\infty) \times \mathrm{cl}\,\mathscr{B} \to [0,\infty)$ with $h^-(\delta, \mathrm{x}) \to \infty$ as $\delta \to 0$, that at each $\mathrm{x} \in \mathscr{B}$ there is a skew basis $\{\boldsymbol{e}_k(\mathrm{x})\}$ with a dual basis $\{\boldsymbol{e}^k(\mathrm{x})\}$ that depend continuously on x, that there are continuous functions α_k: $\mathrm{cl}\,\mathscr{B} \to [\alpha^-, \alpha^+]$ with $1 < \alpha^- \leqq \alpha^+ < \infty$, that there is an integrable function β on $\mathscr{B}$, and that there is a positive constant K^-, such that

$$\theta > K^- \theta^- \tag{3.39a}$$

where

$$\theta^-(\boldsymbol{F}, \mathrm{x}) = h^-(\det \boldsymbol{F}, \mathrm{x}) + \sum_{k=1}^{3} [\alpha_k(\mathrm{x})]^{-1} |\boldsymbol{F} \cdot \boldsymbol{e}_k(\mathrm{x})|^{\alpha_k(\mathrm{x})} + \beta(\mathrm{x}). \tag{3.39b}$$

(Note that $|\boldsymbol{F} \cdot \boldsymbol{e}_k| = \sqrt{(\boldsymbol{e}_k \cdot \boldsymbol{C} \cdot \boldsymbol{e}_k)}$.) This requirement ensures Hypotheses 3.29 and 3.30. We ensure Hypothesis 3.33i by taking $\theta < \mathrm{const}\, \theta^+$, with

$$\theta^+(\boldsymbol{F}, \mathrm{x}) = h^+(\det \boldsymbol{F}, \mathrm{x}) + \sum_{k=1}^{3} |\boldsymbol{F} \cdot \boldsymbol{e}_k(\mathrm{x})|^{\alpha_k(\mathrm{x})}, \tag{3.40}$$

where h^+, which satisfies the same requirements as h^-, is greater than h^-. (We think of $h^+(\delta, \mathrm{x})$ as behaving like $\left|\frac{\partial}{\partial \delta} h^-(\delta, \mathrm{x})\right|$ near $\delta = 0$.) Then we assume that there are functions β^k: $\mathscr{B} \to \mathbb{R}^1$ with

$$\int_{\mathscr{B}} |\beta^k(\mathrm{x})|^{\alpha_k^*(\mathrm{x})}\, dV(\mathrm{x}) < \infty$$

and that there is a constant $K^+ > 0$ such that

$$\begin{aligned} \left|\frac{\partial \theta}{\partial \boldsymbol{F}}(\boldsymbol{F}, \mathrm{x}) \cdot \boldsymbol{e}^k(\mathrm{x})\right| &< [h^+(\det \boldsymbol{F}, \mathrm{x})]^{1/\alpha_k^*(\mathrm{x})} \\ &\quad + K^+ \sum_{l=1}^{3} |\boldsymbol{F} \cdot \boldsymbol{e}_k(\mathrm{x})|^{\alpha_l(\mathrm{x})/\alpha_k^*(\mathrm{x})} + \beta^k(\mathrm{x}), \end{aligned} \tag{3.41}$$

where $\alpha_k^* = \alpha_k / (\alpha_k - 1)$. We can take

$$\mathscr{V} = \left\{ \boldsymbol{r} : \int_{\mathscr{B}} \sum_{k=1}^{3} [\alpha_k(\mathrm{x})]^{-1} |\boldsymbol{F}(\mathbf{R}(\mathrm{x}), \mathrm{x}) \cdot \boldsymbol{e}_k(\mathrm{x})|^{\alpha_k(\mathrm{x})}\, dV(\mathrm{x}) < \infty \right\}. \tag{3.42}$$

Observing that the integrand of (3.42) is a strictly convex function of $\boldsymbol{F}$, we can follow KRASNOSEL'SKIĬ & RUTITSKIĬ (1958) to show that $\mathscr{V}$ is a reflexive Banach space and therefore satisfies Hypothesis 3.33(ii). In particular, $\mathscr{V}$ has the dual space

$$\mathscr{V}^* = \left\{ r : \int_{\mathscr{B}} \sum_{k=1}^{3} [\alpha_k^*(\mathrm{x})]^{-1} \left| F(\mathrm{R}(\mathrm{x})) \cdot e^k(\mathrm{x}) \right|^{\alpha_k^*(\mathrm{x})} dV(\mathrm{x}) < \infty \right\}. \tag{3.43}$$

The integrand in (3.43) is just the convex function conjugate to that of the integrand of (3.42). That (3.40), (3.41) imply Hypothesis 3.33(ii) is easily shown by the use of the Young (or Fenchel) inequalities. (*Cf.* KRASNOSEL'SKIĬ & RUTITSKIĬ (1958) or ROCKAFELLAR (1970).) These inequalities justify the formal manipulation of the exponents $\{\alpha_k\}$ as if they were constants, in which case the Young inequality would reduce to the Hölder inequality.

Remarks. The need for separate functions h^- and h^+ can be motivated by the observation that h^- is used to give a lower bound for the strain energy in (3.39a), whereas h^+ is used to give an upper bound for the gradient of strain energy, *i.e.*, for the stress, in (3.41).) Wherever the strain energy approaches infinity for finite F, its gradient must grow at a faster rate. Our function space $\mathscr{V}$ of (3.42) is somewhat unusual because of the dependence of the (skew) basis and the exponents on x. The resulting versatility of this space is necessary to allow for aeolotropy and non-homogeneity in a non-linear setting. The appropriateness of expressions such as $F \cdot e_k$ for aeolotropy is discussed by ERICKSEN & RIVLIN (1954) (*cf.* TRUESDELL & TOUPIN (1960 § 62)). Thus $\mathscr{V}$ represents perhaps the simplest function space suitable for these phenomena. $\mathscr{V}$ is a space of Orlicz type; its study relies on convex analysis. On the other hand $\mathscr{V}$ has the same nice properties as the common Sobolev spaces. We finally note that the integrand of (3.42) is a strictly convex frame-indifferent function. The existence of such a function does not stand in contradiction to the result of TRUESDELL & TOUPIN (1963) (*cf.* TRUESDELL & NOLL (1965 p. 163)) that there exists no strictly convex, frame indifferent function of F that vanishes for a nonsingular F, since our convex function vanishes only at $F = \mathbf{0}$.

e) Boundary Value Problems. The *classical boundary value problem* is to find a function r, twice continuously differentiable on $\mathscr{B}$, that satisfies

$$[\sqrt{G(\mathrm{x})}\, \hat{\tau}^k(r_{,p}, \mathrm{x})]_{,k} + \sqrt{G(\mathrm{x})}\, f[r; \mathrm{x}] = \mathbf{0}, \quad \mathrm{x} \in \mathscr{B}, \tag{3.44}$$

that satisfies the boundary conditions (3.10)–(3.13) or (3.14), and that satisfies the invertibility condition (3.4) everywhere on cl$\mathscr{B}$.

It is well-known that the existence of a classical solution depends critically on the smoothness of the data and of $\partial\mathscr{B}$. It is therefore useful to formulate the problem in a weak form that has more likelihood of possessing a solution. Moreover, such weak forms prove eminently useful in the construction of rod and shell theories.

The weak form of the boundary value problem (principle of virtual work) is to find a function r in $\mathscr{H}$ satisfying (3.11), (3.12a), (3.13a) (in a generalized sense), such that

$$\begin{aligned} &\int_{\mathscr{B}} \{\hat{\tau}^k(r_{,j}(\mathrm{x}), \mathrm{x}) \cdot \eta_{,k}(\mathrm{x}) - f[r, \mathrm{x}] \cdot \eta(\mathrm{x})\}\, dV(\mathrm{x}) \\ &\quad - \int_{\partial\mathscr{B}} \sigma[r; \mathrm{x}] \cdot \eta(\mathrm{x})\, dA(\mathrm{x}) = 0 \end{aligned} \tag{3.45}$$

for all η in $\mathscr{V}$ satisfying (3.17). (In this connection see (3.36).) A variety of formally equivalent versions of (3.45) are available when the material is hyperelastic or when at least part of the external loading is conservative. One-dimensional analogs of these alternative formulations will be analyzed in Parts II and IV. They are very useful for treating problems in which the external loading is so ill-behaved that the existence theory for weak equations of the form (3.47) is difficult, if not impossible.

Since (3.4) and (3.8) are inequalities that we may permit to be violated on a set of measure zero, it might seem advantageous to replace the weak equation (3.45) by a variational inequality in which the left side of (3.45) is non-negative for all η in the tangent cone of admissible r's. (The sign of the left side of (3.45) is suggested by the naturalness of minimization problems for conservative problems. But by the methods of ANTMAN (1973, pp. 20–23) or by the variant of these methods used in Part II, Section 5, it is shown that our growth conditions force any solution of the variational inequality to be a solution of the weak equation.)

Unfortunately, there is no existence theory for even these weak problems under the hypothesis of strong ellipticity or any other such condition weak enough to permit multiplicity of solutions to problems with dead loadings.* (But see the comments of Section 7.) On the other hand, ther is a well-developed theory for monotone operators. (*Cf.* BROWDER (1970), LIONS (1969), *et al.*) Unfortunately, the monotonicity condition (3.27) ensures global uniqueness under dead loadings. (This means that a column subjected to a compressive end thrust would have a unique straight solution however great the thrust and however thin the column.) The existence theory of BEJU (1971) is essentially a variational equivalent to this monotone operator theory without the restrictions (3.4). One can consider a more general class of operators, called pseudo-monotone operators (introduced by BREZIS (1968)) which preserve existence but not uniqueness of solutions. These operators are effectively constructed by adding a lower order term to a monotone differential operator. In the context of the equations of non-linear elasticity under dead loadings, a monotone mapping generated by $\{\hat{\tau}^k\}$ satisfying the monotonicity condition (3.27) could only be weakened to a pseudo-monotone mapping by allowing $\{\hat{\tau}^k\}$ to depend upon r itself. But such a dependence upon r is prohibited by the principle of frame-indifference. (*Cf.* TRUESDELL & NOLL (1965).) Thus no trivial modifications of (3.27) suffice to introduce multiplicity. These comments on monotonicity conditions should be borne in mind when we discuss one-dimensional theories, because we shall show that a one-dimensional consequence of the strong ellipticity condition is that the differential operators of the one-dimensional theory are pseudo-monotone. It is this fact that makes one-dimensional theories tractable.

4. Structure of Rod Theories

Following ANTMAN (1972), we define a *theory of rods* as the characterization of the behavior of slender three-dimensional solid bodies by a finite set of equations

* In a dead loading f and σ depend only on x. Results recently announced by J.H. BALL may contradict this statement about existence.

having the parameter of a certain curve and the time as the only independent variables. In particular, we describe the equilibrium of elastic rods by a finite number of ordinary differential equations. These we obtain by constraining the displacement of the three-dimensional theory in a way that generalizes the projection methods of Galerkin-Kantorovich type.

a) Geometry of the Reference Configuration. Let $\mathscr{B}$ be connected and let X^1 and X^2 be bounded on $\mathrm{cl}\,\mathscr{B}$. We call $\mathscr{B}$ a *rod*. Set $S = X^3$ and denote d/dS by a prime. We define

$$S_1 = \inf\{S : \mathrm{x} \in \mathrm{cl}\,\mathscr{B}\}, \qquad S_2 = \sup\{S : \mathrm{x} \in \mathrm{cl}\,\mathscr{B}\}, \tag{4.1}$$

$$\mathscr{L} = \{\mathrm{x} \in \partial\mathscr{B} : S_1 < S < S_2\}, \tag{4.2}$$

$$\begin{aligned} &\mathscr{B}(S) = \{(X^1, X^2) : (X^1, X^2, S) \in \mathscr{B}\}, \\ &\mathscr{B}(S_\mathfrak{a}) = \{(X^1, X^2) : (X^1, X^2, S_\mathfrak{a}) \in \partial\mathscr{B}\}, \qquad \mathfrak{a} = 1, 2, \end{aligned} \tag{4.3}$$

$$\mathscr{A}(S) = \mathscr{B}(S) \times \{S\}, \qquad S \in [S_1, S_2]. \tag{4.4}$$

The set $\mathscr{L}$ is the *lateral surface*. The set $\mathrm{cl}\,\mathscr{A}(S)$ is the *section* at S, and the sets $\mathscr{A}(S_1)$ and $\mathscr{A}(S_2)$ are the *ends* of the rod. We assume that the area of $\mathbf{R}(\mathscr{A}(S))$ is positive for $S \in (S_1, S_2)$, but $\mathbf{R}(\mathscr{A}(S_1))$ and $\mathbf{R}(\mathscr{A}(S_2))$ are permitted to have zero areas. We further assume that G is positive on $\mathscr{B}$.

b) The Deformation. We generate a theory of rods by constraining the position field thus:

$$\boldsymbol{r}(\mathrm{x}) = \boldsymbol{b}(\mathbf{u}(S), \mathrm{x}), \tag{4.5}$$

where $\boldsymbol{b}$ is defined on $\mathbb{R}^N \times \mathrm{cl}\,\mathscr{B}$, is thrice continuously differentiable on $\mathbb{R}^N \times \mathscr{B}$, and satisfies the position boundary conditions (3.11), (3.12a), (3.13a) or (3.15a) (or approximate position boundary conditions) on $\mathscr{L}$ identically in $\mathbf{u}$ and v:

$$\boldsymbol{b}(\mathbf{u}, \mathrm{x}) = \hat{\boldsymbol{r}}(\mathrm{x}, \mathrm{v}(\mathrm{x})), \qquad \mathrm{x} \in \mathscr{L}. \tag{4.6}$$

In particular, this implies that

$$\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}} = \mathbf{0} \qquad \text{on } \mathscr{S}_0 \cap \mathscr{L}. \tag{4.7}$$

The construction of such functions $\boldsymbol{b}$ is described in detail by ANTMAN (1972). We further require $\boldsymbol{b}$ to satisfy the following *independency condition* for each S in (S_1, S_2):

$$\text{If } \frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}, \mathrm{x}) \cdot \mathbf{w} = \mathbf{0} \text{ for almost all } (X^1, X^2) \in \mathscr{B}(S), \text{ then } \mathbf{w} = \mathbf{0}. \tag{4.8}$$

(Alternative statements of independency in which the hypothesis of (4.8) is weakened to hold for all (X^1, X^2) in a much smaller set are possible, but (4.8) suffices for our needs.)

Most works on the foundation of rod theories take $\boldsymbol{b}$ linear in $\mathbf{u}$ and in fact consider just the special case in which $N = 3(K+1)$, $\mathbf{u}$ is the collection of vectors

$\{\boldsymbol{u}_a, a=1, \dots, K\}$, and

$$\boldsymbol{b}(\mathbf{u}, \mathrm{x}) = \sum_{a=0}^{K} \beta^a(\mathrm{x})\, \boldsymbol{u}_a(S). \tag{4.9}$$

(See ANTMAN (1972) for a discussion of the advantages of different representations and for references. The vectors $\boldsymbol{u}_a$ are called *directors*.) When this linear representation is used, the independency condition reduces to the requirement that $\{\beta^a(\cdot, \cdot, S)\}$ be independent functions.

For the purpose of demonstrating that the form of the system of ordinary differential equations governing the equilibrium of non-linearly elastic rods is independent of the choice of $\boldsymbol{b}$, we allow $\boldsymbol{b}$ to be an arbitrary non-linear function satisfying (4.8). For the purpose of casting non-variational boundary value problems in a form convenient for analysis, we shall employ this special linear representation (4.9) because its simplicity enables us to circumvent the many difficulties that would otherwise arise in the treatment of invertibility, compatibility, convexity of the class of admissible geometric variables, frame-indifference, strong ellipticity, and coercivity. We defer until Part III our fairly technical treatment of those questions that arise from the special linear representation. Here we restrict ourselves to a description of deformation that suffices for variational problems.

The representation (4.5) may be interpreted as requiring the position field for fixed S to have at most N degrees of freedom characterized by the N-tuple $\mathbf{u}$ of generalized coordinates. The independency condition (4.8) ensures that there can be no fewer than N degrees of freedom.

We define

$$\delta(\mathbf{u}, \mathbf{w}, X^\alpha, S) \equiv [\boldsymbol{b}_{,1}(\mathbf{u}, \mathrm{x}) \times \boldsymbol{b}_{,2}(\mathbf{u}, \mathrm{x})] \cdot [\bar{\boldsymbol{b}}_3(\mathbf{u}, \mathbf{w}, \mathrm{x})]. \tag{4.10}$$

In consonance with (4.5) we replace the invertibility condition (3.4) by its constrained form

$$\delta(\mathbf{u}(S), \mathbf{u}'(S), X^\alpha, S) > 0 \quad \text{for } \mathrm{x} \in \mathrm{cl}\,\mathscr{B}. \tag{4.11}$$

Since $\mathscr{B}$ is given, (4.10) is just a restriction on $\mathbf{u}$ and $\mathbf{u}'$. To convert (4.11) into a convenient form, we first observe that (4.11) is equivalent to the requirement that

$$\begin{aligned}(\mathbf{u}(S), \mathbf{u}'(S)) &\in \mathscr{G}(S) \\ &\equiv \bigcap_{X^\alpha \in \mathrm{cl}\,\mathscr{B}(S)} \{(\mathbf{u}, \mathbf{w}) \in \mathbb{R}^N \times \mathbb{R}^N : \delta(\mathbf{u}, \mathbf{w}, X^\alpha, S) > 0\} \quad \forall S \in [S_1, S_2].\end{aligned} \tag{4.12}$$

The *invertibility condition* is violated at S when

$$(\mathbf{u}(S), \mathbf{u}'(S)) \in \partial \mathscr{G}(S). \tag{4.13}$$

We set

$$\mathscr{G} \equiv \{(\mathbf{u}, \mathbf{w}, S) : (\mathbf{u}, \mathbf{w}) \in \mathscr{G}(S), S \in [S_1, S_2]\}. \tag{4.14}$$

Thus

$$\mathscr{G}(S) = \{(\mathbf{u}, \mathbf{w}) : (\mathbf{u}, \mathbf{w}, S) \in \mathscr{G}\}, \tag{4.15}$$

and (4.12) is equivalent to

$$(\mathbf{u}(S), \mathbf{u}'(S), S) \in \mathscr{G}. \tag{4.16}$$

To determine the nature of $\mathscr{G}$, we note that

$$\{\mathbf{w}: \delta(\mathbf{u}, \mathbf{w}, X^\alpha, S) > 0\} \tag{4.17}$$

is an open half-space in $\mathbb{R}^N$ when $\boldsymbol{b}_{,1}(\mathbf{u}, \mathrm{x}) \times \boldsymbol{b}_{,2}(\mathbf{u}, \mathrm{x}) \cdot \dfrac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}, \mathrm{x}) \neq \mathbf{0}$. Therefore

$$\begin{aligned} \mathscr{G}(\mathbf{u}, S) &\equiv \{\mathbf{w}: (\mathbf{u}, \mathbf{w}, S) \in \mathscr{G}\} \\ &\equiv \bigcap_{X^\alpha \in \mathrm{cl}\,\mathscr{B}} \{\mathbf{w}: \delta(\mathbf{u}, \mathbf{w}, X^\alpha, S) > 0\} \end{aligned} \tag{4.18}$$

is a convex subset of $\mathbb{R}^N$. (This result is critically important for questions of analysis.) We also note that $\mathscr{G}(S)$ is a subset of $\mathscr{I}(S) \times \mathbb{R}^n$, where

$$\begin{aligned} \mathscr{I}(S) \equiv \bigcap_{X^\alpha \in \mathrm{cl}\,\mathscr{B}(S)} \Big\{ \mathbf{u} \in \mathbb{R}^n : [\boldsymbol{b}_{,1}(\mathbf{u}, \mathrm{x}) \times \boldsymbol{b}_{,2}(\mathbf{u}, \mathrm{x})] \cdot \frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}, \mathrm{x}) \neq 0 \\ \text{or } [\boldsymbol{b}_{,1}(\mathbf{u}, \mathrm{x}) \times \boldsymbol{b}_{,2}(\mathbf{u}, \mathrm{x})] \cdot \frac{\partial \boldsymbol{b}}{\partial X^3}(\mathbf{u}, \mathrm{x}) > 0 \Big\}. \end{aligned} \tag{4.19}$$

In fact,

$$\partial \mathscr{G}(S) \cap \partial [\mathscr{I}(S) \times \mathbb{R}^n] \neq 0. \tag{4.20}$$

There are a number of other reasonable assumptions about the nature of $\mathscr{G}$ that hold for reasonable choices of $\boldsymbol{b}$, *e.g.*, for $\boldsymbol{b}$ of the form (4.9). Assumptions include requirements such as that $\mathscr{G}(\mathbf{u}, S)$ be unbounded. We postpone to Part II a statement of such requirements as are needed in the analysis there. We limit ourselves to a demonstration that $\partial\mathscr{G}$ may not be very well behaved. We consider the following special version of (4.9):

$$\boldsymbol{b}(\mathbf{u}, \mathrm{x}) = \boldsymbol{u}_0(S) + X^\alpha \boldsymbol{u}_\alpha(S).$$

(That this may not be a realistic assumption for the physics (*cf.* ANTMAN (1972, Sec. 11) does not concern us here; more complicated representations will manifest the same difficulties.) Let

$$\xi_{\mathrm{a}} = \boldsymbol{u}_1(S) \times \boldsymbol{u}_2(S) \cdot \boldsymbol{u}'_{\mathrm{a}}(S), \qquad \mathrm{a} = 0, 1, 2.$$

Then (4.12) is equivalent to

$$\{\boldsymbol{u}_{\mathrm{a}}(S), \boldsymbol{u}'_{\mathrm{a}}(S), \mathrm{a} = 0, 1, 2\} \in \mathscr{G}(S) \equiv \Big\{\{\boldsymbol{u}_{\mathrm{a}}, \boldsymbol{w}_{\mathrm{a}}, \mathrm{a} = 0, 1, 2\} : \min_{X^\alpha \in \mathrm{cl}\,\mathscr{B}(S)} \xi_0 + X^\beta \xi_\beta > 0\Big\}.$$

Suppose $\mathscr{B}(S)$ is convex. Then for fixed $\{\xi_{\mathrm{a}}\}$, the minimizer of the affine function $\xi_0 + X^\beta \xi_\beta$ can readily be found from the geometrical construction of Fig. 4.1 to be at the point $(\bar{X}^1, \bar{X}^2) \in \partial\mathscr{B}(S)$ where the ray with directions $-(\xi_1, \xi_2, 0)$ intersects $\partial\mathscr{B}(S)$. If we describe $\partial\mathscr{B}(S)$ by the polar coordinate representation

$$\sqrt{(X^1)^2 + (X^2)^2} = R(\theta), \qquad \tan\theta = (X^2/X^1),$$

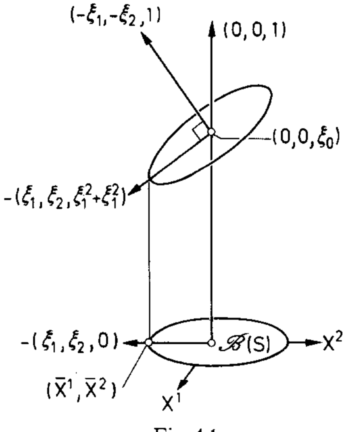

Fig. 4.1

then this construction implies that

$$\bar{X}^\beta = -R(\pi + \arctan(\xi_2/\xi_1))\,\xi_\beta(\xi_1^2+\xi_2^2)^{-\frac{1}{2}}$$

so that

$$\min_{X^\alpha \in \operatorname{cl}\mathscr{B}(S)} [\xi_0 + X^\beta \xi_\beta] = \xi_0 - R(\pi + \arctan(\xi_2/\xi_1))(\xi_1^2+\xi_2^2)^{\frac{1}{2}}.$$

Thus if $\boldsymbol{u}_1(S) \times \boldsymbol{u}_2(S) \neq \mathbf{0}$, $\mathscr{G}(S)$ consists of $\{\boldsymbol{u}_\alpha, \boldsymbol{w}_\alpha, \alpha = 0, 1, 2\}$ such that

$$\xi_0 > R(\pi + \arctan(\xi_2/\xi_1))(\xi_1^2+\xi_2^2)^{\frac{1}{2}}.$$

This is the equation of a solid cone in (ξ_0, ξ_1, ξ_2)-space (which has a cross-section that is an inversion of $\mathscr{B}(S)$). Thus we can expect that at best $\partial\mathscr{G}(S)$ is Lipschitz continuous whenever $\partial\mathscr{B}(S)$ is. (The only previous studies known to me of the role of invertibility conditions in boundary value problems are listed in Table 1.1 of Part II and these took as the invertibility conditions special cases of $\xi_0 > 0$. In such cases, the serious technical problems of Part II arising from the lack of smoothness of $\partial\mathscr{G}$ do not arise.)

We obtain a formal approximation scheme from (4.5) by considering not merely a single arbitrary $\boldsymbol{b}$, but a whole family of $\boldsymbol{b}$'s indexed by N. We demand that the family of such representations $\boldsymbol{b}$ be dense in $\mathscr{H}$, *i.e.*, for arbitrary admissible $\boldsymbol{r}$ and for arbitrary positive ε, there be a $\boldsymbol{b}$ in this family and a function $\mathbf{u}\colon [S_1, S_2] \to \mathbb{R}^N$ such that $\|\boldsymbol{b}(\mathbf{u}(\cdot), \cdot) - \boldsymbol{r}\|_{\mathscr{V}} \leqq \varepsilon$. (The questions of approximation and convergence are touched on briefly in Sections 7 and 8.)

The constraint (4.5) does not allow arbitrary position boundary conditions on $\mathbf{u}$ to be prescribed at the ends. We therefore either restrict position boundary conditions at the ends or else approximate arbitrary position boundary conditions on the ends by those consistent with (4.5). *E.g.*, if $\mathscr{A}(S_1) \subset \mathscr{S}_0$ (see (3.11), (4.4)), then we choose $\mathbf{u}(S_1)$ so that $\boldsymbol{b}(\mathbf{u}(S_1), \cdot, \cdot, S_1)$ approximates $\boldsymbol{r}(\cdot, \cdot, S_1)$. (In Section 8 we describe a strategy for treating general end conditions.) More generally, we may assume that representation (4.5) generates position boundary conditions in the parametric form (analogous to (3.15a))

$$\mathbf{u}(S_\alpha) = \mathbf{u}_\alpha(\mathbf{v}_\alpha), \qquad \alpha = 1, 2. \tag{4.21}$$

Here, the parameters $\mathbf{v}_1, \mathbf{v}_2$, which are in $\mathbb{R}^N$, are unknowns and $\dfrac{\partial \mathbf{u}_a}{\partial \mathbf{v}_a}$ has rank $\leq N$. Alternatively, if the rod is toroidal, *i.e.*, if the ends consist of the same particles, then we impose periodicity conditions

$$\mathbf{u}(S+S_2-S_1)=\mathbf{u}(S). \tag{4.22}$$

We could even contemplate some combination of (4.21) and (4.22) for a ring with an incomplete transverse slit. We treat traction conditions below (equation (4.34)) because they arise naturally from the weak form of the governing equations for rods.

c) Stress Resultants and the Equilibrium Equations. To obtain the weak form of the equilibrium equations for rods, we first choose $\boldsymbol{\eta}$ of (3.16) to be

$$\boldsymbol{\eta}(\mathbf{x})=\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}, \mathbf{x})\cdot \mathbf{y}(S), \tag{4.23}$$

with $\mathbf{y}$ smooth enough and satisfying the end conditions

$$\mathbf{y}(S_a)=\frac{\partial \mathbf{u}_a}{\partial \mathbf{v}_a}(\mathbf{v}_a)\cdot \mathbf{w}_a, \qquad \mathbf{w}_a\in\mathbb{R}^N,\ a=1,2 \tag{4.24}$$

when (4.10) holds and with $\mathbf{y}$ satisfying

$$\mathbf{y}(S+S_2-S_1)=\mathbf{y}(S) \tag{4.25}$$

when (4.22) holds. (The $\boldsymbol{\eta}$ of (4.23) is just the virtual displacement corresponding to the holonomic constraint (4.5).) The substitution of (4.23) into (3.16) yields the *weak form of the equilibrium equations for rods*

$$\int_{S_1}^{S_2}[\mathbf{m}\cdot\mathbf{y}'+\mathbf{n}\cdot\mathbf{y}-\mathbf{f}\cdot\mathbf{y}]\,dS-[\mathbf{p}\cdot\mathbf{y}]_{S_1}^{S_2}=0, \tag{4.26}$$

for all sufficiently smooth $\mathbf{y}$ satisfying (4.24) or (4.25), where

$$\mathbf{m}(S)\equiv\int_{\mathscr{B}(S)}\boldsymbol{\tau}^3(\mathbf{x})\cdot\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}(S),\mathbf{x})\sqrt{G(\mathbf{x})}\,dX^1\,dX^2, \tag{4.27}$$

$$\mathbf{n}(S)=\int_{\mathscr{B}(S)}\boldsymbol{\tau}^k(\mathbf{x})\cdot\left[\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}(S),\mathbf{x})\right]_{,k}\sqrt{G(\mathbf{x})}\,dX^1\,dX^2, \tag{4.28}$$

$$\begin{aligned}\mathbf{f}[\mathbf{u};S]=&\int_{\mathscr{B}(S)}\boldsymbol{f}[\boldsymbol{b}(\mathbf{u},\cdot);\mathbf{x}]\cdot\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}(S),\mathbf{x})\sqrt{G(\mathbf{x})}\,dX^1\,dX^2\\&+\int_{\partial\mathscr{B}(S)}\boldsymbol{\sigma}[\boldsymbol{b}(\mathbf{u},\cdot)|_{\partial\mathscr{B}};\mathbf{x}]\cdot\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}(S),\mathbf{x})\frac{N^1\,dX^2-N^2\,dX^1}{1-N^3N_3},\end{aligned} \tag{4.29}$$

$$\begin{aligned}\mathbf{p}_a[\mathbf{u}]=&\int_{\mathscr{B}(S_a)}\boldsymbol{\sigma}[\boldsymbol{b}(\mathbf{u},\cdot)|_{\partial\mathscr{B}};X^1,X^2,S_a]\cdot\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}(S_a);X^1,X^2,S_a)\\&\cdot\sqrt{G(X^1,X^2,S_a)}\,dX^1\,dX^2.\end{aligned} \tag{4.30}$$

$\mathbf{m}$ and $\mathbf{n}$ are *stress resultants, i.e.*, weighted averages of the stress across a section. $\mathbf{f}$ is the *external load* and contains contributions from the body force and tractions on the lateral surface. The requirement (4.6) ensures that the integral over $\partial\mathscr{B}(S)$ in (4.29) is well-defined, *i.e.*, it includes only contributions from those components of $\boldsymbol{\sigma}$ that are prescribed. To see this we note that (3.18) and (4.6) imply that

$$\frac{\partial \hat{\boldsymbol{r}}}{\partial \mathbf{v}} \cdot \mathbf{w} = \frac{\partial \boldsymbol{b}}{\partial \mathbf{u}} \cdot \mathbf{y} \quad \text{on } \mathscr{L} \tag{4.31}$$

and that (3.15b) holds. $\mathbf{p}_1$ and $\mathbf{p}_2$ are resultants at the ends. Only

$$\mathbf{p}_a[\mathbf{u}] \cdot \frac{\partial \mathbf{u}_a}{\partial \mathbf{v}_a}(\mathbf{v}_a), \qquad a = 1, 2 \tag{4.32}$$

are prescribed, the other components of $\mathbf{p}_a$, $a = 1, 2$ being left free to accomodate the end conditions (4.21) and (4.22). (Note the form of $\mathbf{y}(S_a)$ given by (4.24).)

Wherever $\mathbf{m}'$, $\mathbf{n}$, $\mathbf{f}$ are continuous functions of S, (4.26) yields the *classical equilibrium equations for rods*

$$\mathbf{m}' - \mathbf{n} + \mathbf{f} = \mathbf{0} \tag{4.33}$$

and boundary conditions consisting of either (4.21) and

$$\mathbf{m}(S_a) \cdot \frac{\partial \mathbf{u}_a}{\partial \mathbf{v}_a} = \mathbf{p}_a[\mathbf{u}] \cdot \frac{\partial \mathbf{u}_a}{\partial \mathbf{v}_a} \tag{4.34}$$

or (4.22) and the requirement that $\mathbf{m}$ and $\mathbf{n}$ have period $S_2 - S_1$ in S.

d) Constitutive Relations. The constitutive equations for elastic elastic rods are simply obtained by substituting (4.5) into (3.20) and then substituting the resulting expression into (4.27) and (4.28). In accordance with (2.7)–(2.10), they are

$$\mathbf{m}(S) = \hat{\mathbf{m}}(\mathbf{u}(S), \mathbf{u}'(S), S), \tag{4.35}$$

$$\mathbf{n}(S) = \hat{\mathbf{n}}(\mathbf{u}(S), \mathbf{u}'(S), S), \tag{4.36}$$

where

$$\hat{\mathbf{m}}(\mathbf{u}, \mathbf{w}, S) \equiv \int_{\mathscr{B}(S)} \hat{\boldsymbol{\tau}}^3(\bar{\boldsymbol{b}}_j(\mathbf{u}, \mathbf{w}, \mathrm{x})) \cdot \frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}, \mathrm{x}) \sqrt{G(\mathrm{x})}\, dX^1\, dX^2. \tag{4.37}$$

$$\hat{\mathbf{n}}(\mathbf{u}, \mathbf{w}, S) \equiv \int_{\mathscr{B}(S)} \hat{\boldsymbol{\tau}}^k(\bar{\boldsymbol{b}}_j(\mathbf{u}, \mathbf{w}, \mathrm{x})) \cdot \overline{\left(\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\right)_k} (\mathbf{u}, \mathbf{w}, \mathrm{x}) \sqrt{G(\mathrm{x})}\, dX^1\, dX^2. \tag{4.38}$$

Expression (3.22) and our preceding remarks imply that the domain of $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ is $\mathscr{G}$. We may also define $\hat{\mathbf{m}}$, $\hat{\mathbf{n}}$ in the extended sense on the closure of $\mathscr{G}$ in the spirit of (3.21).

From (4.37) and (4.38) we readily obtain

$$\frac{\partial \hat{\mathbf{m}}}{\partial \mathbf{w}} = \int_{\mathscr{B}(S)} \left(\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\right)^T \cdot \frac{\partial \hat{\boldsymbol{\tau}}^3}{\partial \boldsymbol{y}_3} \cdot \frac{\partial \boldsymbol{b}}{\partial \mathbf{u}} \sqrt{G}\, dX^1\, dX^2, \tag{4.39}$$

$$(4.40)\qquad \frac{\partial \hat{\mathbf{m}}}{\partial \mathbf{u}}=\int_{\mathscr{B}(S)}\left[\left(\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\right)^T\cdot\frac{\partial \hat{\boldsymbol{\tau}}^3}{\partial y_k}\cdot\overline{\left(\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\right)_k}+\hat{\boldsymbol{\tau}}^3\cdot\frac{\partial^2 \boldsymbol{b}}{\partial \mathbf{u}\,\partial \mathbf{u}}\right]\sqrt{G}\,dX^1\,dX^2,$$

$$(4.41)\qquad \frac{\partial \hat{\mathbf{n}}}{\partial \mathbf{w}}=\int_{\mathscr{B}(S)}\left[\overline{\left(\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\right)_k^T}\cdot\frac{\partial \hat{\boldsymbol{\tau}}^k}{\partial y_3}\cdot\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}+\hat{\boldsymbol{\tau}}^3\cdot\frac{\partial^2 \boldsymbol{b}}{\partial \mathbf{u}\,\partial \mathbf{u}}\right]\sqrt{G}\,dX^1\,dX^2,$$

$$(4.42)\qquad \frac{\partial \hat{\mathbf{n}}}{\partial \mathbf{u}}=\int_{\mathscr{B}(S)}\left[\overline{\left(\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\right)_k^T}\cdot\frac{\partial \hat{\boldsymbol{\tau}}^k}{\partial y_j}\cdot\overline{\left(\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\right)_j}+\hat{\boldsymbol{\tau}}^k\cdot\overline{\left(\frac{\partial^2 \boldsymbol{b}}{\partial \mathbf{u}\,\partial \mathbf{u}}\right)_k}\right]\sqrt{G}\,dX^1\,dX^2.$$

Here we suppress the arguments appearing in (4.37), (4.38).

Let us examine the quadratic form

$$(4.43)\qquad \begin{aligned} &\mathbf{a}\cdot\frac{\partial \hat{\mathbf{m}}}{\partial \mathbf{w}}\cdot\mathbf{a}+\mathbf{a}\cdot\left[\frac{\partial \hat{\mathbf{m}}}{\partial \mathbf{u}}-\int_{\mathscr{B}(S)}\hat{\boldsymbol{\tau}}^3\cdot\frac{\partial^2 \boldsymbol{b}}{\partial \mathbf{u}\,\partial \mathbf{u}}\sqrt{G}\,dX^1\,dX^2\right]\cdot\mathbf{c}\\ &+\mathbf{c}\cdot\left[\frac{\partial \hat{\mathbf{n}}}{\partial \mathbf{w}}-\int_{\mathscr{B}(S)}\hat{\boldsymbol{\tau}}^3\cdot\frac{\partial^2 \boldsymbol{b}}{\partial \mathbf{u}\,\partial \mathbf{u}}\sqrt{G}\,dX^1\,dX^2\right]\cdot\mathbf{a}\\ &+\mathbf{c}\cdot\left[\frac{\partial \hat{\mathbf{n}}}{\partial \mathbf{u}}-\int_{\mathscr{B}(S)}\hat{\boldsymbol{\tau}}^k\cdot\overline{\left(\frac{\partial^2 \boldsymbol{b}}{\partial \mathbf{u}\,\partial \mathbf{u}}\right)_k}\sqrt{G}\,dX^1\,dX^2\right]\cdot\mathbf{c}, \end{aligned}$$

which can be written as

$$(4.44)\qquad \int_{\mathscr{B}(S)}\xi_k\cdot\frac{\partial \hat{\boldsymbol{\tau}}^k}{\partial y_l}\cdot\xi_l\sqrt{G}\,dX^1\,dX^2,$$

where

$$(4.45)\qquad \xi_k=\overline{\left(\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\right)_k}\cdot\mathbf{c}+\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\cdot\mathbf{a}\,\delta_k^3.$$

If the monotonicity condition (3.28) were to hold, then the integrand of (4.44) would be positive definite. Were we also to require that $\left(\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\right)_{,\alpha}=\frac{\partial^2 \boldsymbol{b}}{\partial \mathbf{u}\,\partial X^\alpha}$ satisfy the same independency condition (4.8) as $\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}$, then the vanishing of ξ_1, ξ_2, ξ_3 would imply that of $\mathbf{a}$ and $\mathbf{c}$. Since the measure of $\mathscr{B}(S)$ is positive for $S\in(S_1, S_2)$, (4.43) would be positive definite for $S\in(S_1, S_2)$. If, finally, $\boldsymbol{b}$ were linear in $\mathbf{u}$, then the integrals in (4.43) would vanish. Under these conditions, the differential operator

$$(4.46)\qquad \frac{d}{dS}\hat{\mathbf{m}}(\mathbf{u}, \mathbf{u}', S)-\hat{\mathbf{n}}(\mathbf{u}, \mathbf{u}', S)$$

would be monotone and the boundary value problems under dead loading would have unacceptable unique solutions just as in the three-dimensional case.

On the other hand, if we set $\mathbf{c}=\mathbf{0}$ in (4.43) and (4.45) and if we choose

$$(4.47)\qquad \boldsymbol{\lambda}=\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\cdot\mathbf{a},\qquad \mu_k=\delta_k^3,$$

then

$$(4.48)\qquad \mathbf{a}\cdot\frac{\partial \hat{\mathbf{m}}}{\partial \mathbf{w}}\cdot\mathbf{a}$$

is positive definite for $S \in (S_1, S_2)$ by virtue of the strong ellipticity condition (3.27) and the independency condition (4.8). This means that the principal part $\frac{\partial \hat{\mathbf{m}}}{\partial \mathbf{u}'} \cdot \mathbf{u}''$ of (4.46) is monotone, which in turn implies that (4.46) belongs to the class of *strictly pseudo-monotone operators*. In Parts II and IV, we shall show that the positive definiteness of (4.48) is strong enough to ensure existence (on certain manifolds) but is not so restrictive as to prohibit non-uniqueness. Thus it is the lower order terms in (4.46) that give rise to multiplicity of solutions. Recall that such lower order terms could not appear in the three-dimensional theory because of frame-indifference.

In certain special cases when *a priori* information about solutions is available, one may be able to obtain further restrictions on the constitutive equations from (4.43)–(4.45) that are consistent with the strong ellipticity condition because ξ_k would reduce to the permissible form $\lambda \mu_k$. This occurs, *e.g.*, in the study of necking of straight rods under tension (*cf.* ANTMAN (1973a), (1974a).)

If the material is hyperelastic, then we can define a one-dimensional strain energy function

$$\Psi(\mathbf{u}, \mathbf{w}, S) = \int_{\mathscr{B}(S)} \psi(\bar{\boldsymbol{b}}_j(\mathbf{u}, \mathbf{w}, \mathrm{x}), \mathrm{x}) \sqrt{G}\, dX^1\, dX^2 \tag{4.49}$$

on $\mathscr{G}$ from which we readily obtain

$$\hat{\mathbf{m}} = \frac{\partial \Psi}{\partial \mathbf{w}}, \qquad \hat{\mathbf{n}} = \frac{\partial \Psi}{\partial \mathbf{u}} \tag{4.50}$$

in place of (4.35) and (4.36). In this case (4.48) implies that

$$\mathbf{a} \cdot \frac{\partial^2 \Psi}{\partial \mathbf{w}\, \partial \mathbf{w}} \cdot \mathbf{a} \tag{4.51}$$

is positive definite, *i.e.*, Ψ is convex in $\mathbf{w}$.

Remark. Since the Piola-Kirchhoff stress tensor is a symmetric and frame-indifferent function of the strain, our one-dimensional stress resultants inherit this structure by their definition (4.35)–(4.38). It is only for special linear representations of $\boldsymbol{b}$, however, that we can obtain explicit one-dimensional consequences of these results. (*Cf.* ANTMAN (1972). Some of these matters become critical for non-variational problems. See Part III.) Since this frame-indifference is merely implicit in our present treatment, the critically important positive-definiteness of (4.48) and (4.51) must be treated with some circumspection. These inequalities are given in a coordinate-free form. It is possible to introduce componential versions that are inconsistent with frame-indifference. Naturally such versions give physically inconsistent results. (*Cf.* ANTMAN (1971, §5) for an example of this.) Thus we interpret this positive-definiteness of (4.48) or (4.51) as meaning that there is a basis for $\mathbf{u}$ for which a componential form of the positive-definiteness of (4.48) or (4.51) that is consistent with frame-indifference is valid.

We now develop some one-dimensional consequences of the growth hypotheses for hyperelastic rods laid down in Section 3d. With slight modification, a number of our results carry over to non-hyperelastic rods. Other hypotheses are treated in Part III.

Relations (4.49) and (4.50) imply that

$$(4.52)\qquad |\hat{\mathbf{m}}(\mathbf{u},\mathbf{w},S)|+|\hat{\mathbf{n}}(\mathbf{u},\mathbf{w},S)|\equiv \left|\frac{\partial\Psi}{\partial\mathbf{w}}(\mathbf{u},\mathbf{w},S)\right|+\left|\frac{\partial\Psi}{\partial\mathbf{u}}(\mathbf{u},\mathbf{w},S)\right|=\infty$$

if

$$(4.53)\qquad \sum_{k=1}^{3}|\tau^k(\overline{\boldsymbol{b}_p}(\mathbf{u},\mathbf{w},X^\alpha,S),X^\alpha,S)|=\infty$$

for X^α in a subset of $\mathscr{B}(S)$ of positive measure, and that

$$(4.54)\qquad \Psi(\mathbf{u},\mathbf{w},S)=\infty$$

if

$$(4.55)\qquad \psi(\overline{\boldsymbol{b}_p}(\mathbf{u},\mathbf{w},X^\alpha,S),X^\alpha,S)=\infty$$

for X^α in a subset of $\mathscr{B}(S)$ of positive measure. If

$$(4.56)\qquad |\boldsymbol{F}(\mathbf{R}(X^\alpha,S))|=\infty\quad\text{or}\quad\det\boldsymbol{F}(\mathbf{R}(X^\alpha,S))=0$$

for X^α in a subset of $\mathscr{B}(S)$ of positive measure, then Hypothesis 3.29 implies (4.53) which implies (4.52) and Hypothesis 3.30 implies (4.55) which implies (4.54). Thus (4.52) and (4.54) signify extreme strain. Sharper results are given below.

To determine results corresponding to Hypothesis 3.33, we first note that

$$(4.57)\qquad |\boldsymbol{F}|^\alpha=|\boldsymbol{U}|^\alpha=(C_k^k)^{\frac{\alpha}{2}}=(G^{kl}\boldsymbol{r}_{,k}\cdot\boldsymbol{r}_{,l})^{\frac{\alpha}{2}}.$$

Now $G^{kl}\overline{\boldsymbol{b}_k}\cdot\overline{\boldsymbol{b}_l}$ is a non-homogeneous quadratic form in $\mathbf{w}$ with coefficients depending on $\mathbf{u}$ and x. This form is positive definite for almost all x in $\mathscr{B}$. If $\boldsymbol{b}$ is linear in $\mathbf{u}$, then $G^{kl}\overline{\boldsymbol{b}_k}\cdot\overline{\boldsymbol{b}_l}$ is a homogeneous quadratic form in $\mathbf{u}$ and $\mathbf{w}$ with coefficients depending on x. In this case the boundedness of $\mathscr{B}(S)$ implies that there is a weight function $\rho^-:(S_1,S_2)\to(0,\infty)$ and a positive constant K^+ such that

$$(4.58)\qquad \begin{aligned}\int_{S_1}^{S_2}\rho^-(S)[|\mathbf{u}|^\alpha+|\mathbf{w}|^\alpha]\,dS&\leqq\int_{\mathscr{B}}|\boldsymbol{F}|^\alpha\,dV\\ &\leqq K^+\int_{S_1}^{S_2}[|\mathbf{u}|^\alpha+|\mathbf{w}|^\alpha]\,dS.\end{aligned}$$

We identify functions $\mathbf{u}$ for which $\mathrm{x}\to\boldsymbol{b}(\mathbf{u}(S),\mathrm{x})$ differ at most by a rigid displacement. Then (4.58) and Hypothesis 3.33 imply that the set of functions $\mathbf{u}$ for which

$$(4.59)\qquad \int_{S_1}^{S_2}\psi(\mathbf{u}(S),\mathbf{u}'(S),S)\,dS$$

is finite belong to a reflexive Banach space $\mathscr{W}$ with

$$(4.60)\qquad [\mathscr{W}^1_{\alpha^+,1}]^N\subset\mathscr{W}\subset[\mathscr{W}^1_{\alpha^-,\rho^-}]^N$$

and satisfy the requirements that if there is a number $\mu>0$ such that

$$(4.61)\qquad \operatorname{dist}((\mathbf{u}(S),\mathbf{u}'(S)),\partial\mathscr{G}(S))>\mu\qquad\forall S\in[S_1,S_2]$$

then

$$\int_{S_1}^{S_2} [\hat{\mathbf{m}}(\mathbf{u},\mathbf{u}',S)\cdot\mathbf{y}'+\hat{\mathbf{n}}(\mathbf{u},\mathbf{u}',S)\cdot\mathbf{y}]\,dS \tag{4.62}$$

is finite for all $\mathbf{y}$ in $\mathscr{W}$. Here $[\mathscr{W}^1_{\alpha,\rho}]^N$ is the weighted Sobolev space of functions $\mathbf{u}$ having distributional derivatives $\mathbf{u}'$ for which

$$\int_{S_1}^{S_2} \rho(S)[|\mathbf{u}|^\alpha+|\mathbf{u}'|^\alpha]\,dS<\infty. \tag{4.63}$$

It is clear that this result holds for a much larger class of $\boldsymbol{b}$'s. We accordingly adopt it as a constitutive hypothesis.

We now examine the consequences of the violation of (4.11) at a particle of $\mathscr{B}$. Suppose that

$$\delta(\mathbf{u},\mathbf{w},Y^\alpha,S)=0 \tag{4.64}$$

for some fixed $\mathbf{u},\mathbf{w},Y^\alpha,S$ with $S\in(S_1,S_2)$ and $Y^\alpha\in\mathscr{B}(S)$. Since $\boldsymbol{b}$ is a twice continuously differentiable function of its arguments it follows from (4.10) that $G^{-\frac{1}{2}}\delta$ is a twice continuously differentiable function of its arguments. Since $G^{-\frac{1}{2}}\delta$ is non-negative there are positive numbers $Q(\mathbf{u},\mathbf{w},Y^\alpha,S)$ and ε such that

$$G^{-\frac{1}{2}}(\mathrm{x})\,\delta(\mathbf{u},\mathbf{w},X^\alpha,S)\leqq Q\rho^2 \quad \text{for } \rho^2\equiv(X^1-Y^1)^2+(X^2-Y^2)^2<\varepsilon^2. \tag{4.65}$$

Suppose that there is a smooth function

$$h^-\colon (0,\infty)\times \mathrm{cl}\,\mathscr{B}\to[0,\infty),\qquad \frac{\partial h^-}{\partial\delta}(\delta,\mathrm{x})>0 \tag{4.66}$$

such that

$$\psi(y_i,\mathrm{x})>h^-(\det \boldsymbol{F},\mathrm{x}). \tag{4.67}$$

(This assumption is completely in the spirit of the coercivity conditions of Section 3.d.) Then

$$\begin{aligned}\Psi(\mathbf{u},\mathbf{w},S)&\geqq \int_{\mathscr{B}(S)} h^-(G^{-\frac{1}{2}}\delta,X)\sqrt{G}\,dX^1\,dX^2\\ &\geqq \text{const.}\int_0^\varepsilon h^-(Q\rho^2)\,\rho\,d\rho.\end{aligned} \tag{4.68}$$

If h^- has the property that

$$\lim_{a\to 0^+}\int_a^\varepsilon h^-(\rho^2)\,\rho\,d\rho=\infty, \tag{4.69}$$

then $\Psi(\mathbf{u},\mathbf{w},S)=\infty$ if δ vanishes at any point of $\mathscr{B}(S)$. Relation (4.69) holds if $h^-(\delta)>\text{const.}/\delta$.) Thus when (4.69) holds, the violation of the strict invertibility condition (4.11) at a single interior point of a section $\mathscr{B}(S)$ is signaled by the unboundedness of $\Psi(\mathbf{u},\mathbf{w},S)$. Note the interdependence of the convexity of the set $\mathscr{G}(\mathbf{u},S)$, the convexity of the function $\Psi(\mathbf{u},\cdot,S)$ and this growth condition: It is impossible for a convex function to be infinite everywhere on the boundary of a non-convex set.

Next, let $(Y^1, Y^2) \in \partial\mathscr{B}(S)$. If δ is continuously differentiable in X^1, X^2 on $\mathrm{cl}\,\mathscr{B}(S)$ and if $\partial\mathscr{B}(S)$ is smooth at (Y^1, Y^2), then the vanishing of δ at (Y^1, Y^2) implies that there are numbers $\varepsilon > 0$ and $\{Q_\alpha(\mathbf{u}, \mathbf{w}, Y^\beta, S)\}$ with

(4.70) $$[X^\alpha - Y^\alpha]\, Q_\alpha > 0 \quad \text{for } (X^1, X^2) \text{ in } \mathscr{B}(S),$$

such that

(4.71) $$G^{-\frac{1}{2}}\, \delta(\mathbf{u}, \mathbf{w}, X^\alpha, S) \leqq Q_\alpha (X^\alpha - Y^\alpha) \quad \text{for } \rho \leqq \varepsilon.$$

We readily find

(4.72) $$\Psi(\mathbf{u}, \mathbf{w}, S) \geqq \text{const.} \int_0^\varepsilon h^-(\text{const.}\, \rho)\, \rho\, d\rho .$$

Thus if h^- has the property that

(4.73) $$\lim_{a \to 0^+} \int_a^\varepsilon h^-(\rho)\, \rho\, d\rho = \infty,$$

then $\Psi(\mathbf{u}, \mathbf{w}, S) = \infty$ if δ vanishes at any point of $\partial\mathscr{B}(S)$. (Condition (4.73) is ensured by the requirement that $h^-(\delta) > \text{const.}/\delta^2$.)

We finally examine the interrelation among the equations $\Psi = \infty$, $|\Psi_{\mathbf{u}}| = \infty$, $|\Psi_{\mathbf{w}}| = \infty$ at $\partial\mathscr{G}$. For $(\mathbf{u}, \mathbf{w})$ near $\partial\mathscr{G}(S)$ suppose that $\Psi(\mathbf{u}, \mathbf{w}, S)$ behaves like

(4.74) $$\Omega(\mathbf{u}, \mathbf{w}, S) \equiv \int_{\mathscr{B}(S)} [G^{-\frac{1}{2}}(\mathrm{x})\, \delta(\mathbf{u}, \mathbf{w}, X^\alpha, S)]^{-p} \sqrt{G(\mathrm{x})}\, dX^1\, dX^2$$

that $\Psi_{\mathbf{u}}$ behaves like $\Omega_{\mathbf{u}}$, and that $\Psi_{\mathbf{w}}$ behaves like $\Omega_{\mathbf{w}}$. Here p is a real number. We have just shown that if (4.64) holds for $Y^\alpha \in \mathscr{B}(S)$ and if $p \geqq 1$, or if (4.64) holds for $Y^\alpha \in \partial\mathscr{B}(S)$ and if $p \geqq 2$, then $\Omega(\mathbf{u}, \mathbf{w}, S) = \infty$. Now the integrand of $\Omega_{\mathbf{w}}$ is

(4.75) $$-p\,\delta_{\mathbf{w}}\, \delta^{-(p+1)}\, G^{(p+1)/2} = -p\,\delta_{\mathbf{w}} \left[\delta_{\mathbf{w}} \cdot \mathbf{w} + \frac{\partial \boldsymbol{b}}{\partial X^3}\right]^{-(p+1)} G^{(p+1)/2}.$$

Let (4.64) hold. If $\dfrac{\partial \boldsymbol{b}}{\partial X^3}(\mathbf{u}, Y^\alpha, S) \neq \mathbf{0}$, then $\delta_{\mathbf{w}}(\mathbf{u}, \mathbf{w}, Y^\alpha, S) \neq \mathbf{0}$ and there are constants ε, $k > 0$ such that the Euclidean norm of (4.75) exceeds $k\delta^{-(p+1)}$ for $p < \varepsilon$. If $\dfrac{\partial \boldsymbol{b}}{\partial X^3}(\mathbf{u}, Y^\alpha, S) = \mathbf{0}$, then $\mathbf{w} \cdot \delta_{\mathbf{w}} / |\mathbf{w}|\, \delta^{(p+1)} = 1/|\mathbf{w}|\, \delta^p$. We may perform a similar analysis on $\Omega_{\mathbf{u}}$. By following the lines of the arguments leading to (4.69) and (4.73) we may conclude the following results: If (4.64) holds for $Y^\alpha \in \mathscr{B}(S)$ and if $p \geqq 1$, then $\Omega(\mathbf{u}, \mathbf{w}, S) = \infty$ and $|\Omega_{\mathbf{w}}(\mathbf{u}, \mathbf{w}, S)| = \infty$. If (4.64) holds for $Y^\alpha \in \partial\mathscr{B}(S)$ and if $p \geqq 2$, then $\Omega(\mathbf{u}, \mathbf{w}, S) = \infty$ and $|\Omega_{\mathbf{w}}(\mathbf{u}, \mathbf{w}, S)| = \infty$. If $p \geqq 1$ and if $\Omega(\mathbf{u}, \mathbf{w}, S) = \infty$ or if $|\Omega_{\mathbf{u}}(\mathbf{u}, \mathbf{w}, S)| = \infty$, then $|\Omega_{\mathbf{w}}(\mathbf{u}, \mathbf{w}, S)| = \infty$. Since Ψ behaves like Ω, we can replace Ω in these statements by Ψ. Thus if $p \geqq 1$ and if $\Psi_{\mathbf{w}}(\mathbf{u}(S), \mathbf{u}'(S), S)$ is defined and finite, then the invertibility condition is satisfied for all $X^\alpha \in \mathscr{B}(S)$; if $p \geqq 2$ and if $\Psi_{\mathbf{w}}(\mathbf{u}(S), \mathbf{u}'(S), S)$ is defined and finite, then the invertibility condition is satisfied for all $X^\alpha \in \mathrm{cl}\,\mathscr{B}(S)$. In Part II, we shall need more refined growth conditions.

e) Boundary Value Problems. The *classical boundary value problem* for non-linearly elastic rods is to find a twice continuously differentiable function $\mathbf{u}$ on

(S_1, S_2) that satisfies

$$\frac{d}{dS}\hat{\mathbf{m}}(\mathbf{u}, \mathbf{u}', S) - \hat{\mathbf{n}}(\mathbf{u}, \mathbf{u}', S) + \mathbf{f}[\mathbf{u}; S] = \mathbf{0} \quad \text{on } (S_1, S_2), \tag{4.76}$$

that satisfies the boundary conditions (4.21) or (4.22) and (4.34) or periodicity conditions on $\mathbf{m}$ and $\mathbf{n}$, and that satisfies the invertibility condition (4.12) everywhere on $[S_1, S_2]$.

The weak form of the boundary value problem for non-linearly elastic rods is to find a function $\mathbf{u}$ rendering (4.59) finite and satisfying boundary conditions (4.21) or (4.22) such that

$$\begin{aligned}\int_{S_1}^{S_2} [\hat{\mathbf{m}}(\mathbf{u}(S), \mathbf{u}'(S), S) \cdot \mathbf{y}' + \hat{\mathbf{n}}(\mathbf{u}(S), \mathbf{u}'(S), S) \cdot \mathbf{y}(S) \\ - \mathbf{f}[\mathbf{u}; S] \cdot \mathbf{y}(S)]\, dS - \mathbf{p}[\mathbf{u}] \cdot \mathbf{y}|_{S_1}^{S_2} = 0\end{aligned} \tag{4.77}$$

for all $\mathbf{y}$ in $\mathscr{W}$ satisfying (4.24) or (4.25).

The mathematical structure of theories of non-linearly elastic rods is embodied not merely in the form of the classical and weak boundary value problems, but also in the pseudo-monotonicity condition (4.48), in the growth conditions (4.60)–(4.62), in the characterization of invertibility by conditions such as (4.59) and (4.63). There are more refined aspects of this mathematical structure, concerned with matters such as the convexity of spaces of admissible strains, that will be discussed in Part III. It is this totality of mathematical structure that makes rods theories both physically rich and mathematically tractable.

Our development of rod theories from weak formulations of the three-dimensional theories is somewhat more attractive from the mathematical and philosophical viewpoint than the development of ANTMAN (1972) from classical formulations. The two approaches are formally equivalent. Our results on the mathematical structure described in the preceeding paragraph are new.

5. Structure of Shell Theories

We define a *theory of shells* as the characterization of the behavior of thin three-dimensional solid bodies by a finite set of equations having the parameters of a certain surface and the time as the only independent variables. The equilibrium of elastic shells is described by a finite number of partial differential equations in two independent variables. There is a duality in the derivation of rod and shell theories that is effected by interchanging the roles of (X^1, X^2) with X^3. It therefore suffices for us merely to outline the main steps in constructing a shell theory.

a) Geometry of the Reference Configuration. Let $\mathscr{B}$ be connected and let X^3 be bounded on $\mathrm{cl}\,\mathscr{B}$. Let $\mathrm{s} = (X^1, X^2)$. We call $\mathscr{B}$ a *shell.* Set

$$\mathscr{M} = \{\mathrm{s} : \mathrm{x} \in \mathscr{B}\}. \tag{5.1}$$

The *edge* of the shell is $\{x \in \partial\mathscr{B} : s \in \partial\mathscr{M}\}$. For simplicity, we assume that $\partial\mathscr{B}$ is the union of the edge and of the two surfaces

$$\mathscr{L}_1 = \{x : X^3 = l_1(s)\}, \qquad \mathscr{L}_2 = \{x : X^3 = l_2(s)\}, \tag{5.2}$$

with $l_1(s) < l_2(s)$ for $s \in \mathscr{M}$. $\mathscr{L}_1$ and $\mathscr{L}_2$ are called the *faces* of $\mathscr{B}$.

b) The Deformation. We generate a theory of shells by constraining the position field thus:

$$\boldsymbol{r}(x) = \boldsymbol{b}(\mathbf{u}(s), x), \tag{5.3}$$

where $\boldsymbol{b}$ is defined on $\mathbb{R}^N \times \mathrm{cl}\,\mathscr{B}$, is thrice continuously differentiable on $\mathbb{R}^N \times \mathscr{B}$, satisfies the position boundary conditions (3.11), (3.12a), (3.13a) or (3.15a) (or approximate position boundary conditions) on $\mathscr{L}_1 \cup \mathscr{L}_2$ identically in $\mathbf{u}$ and $\mathbf{v}$:

$$\boldsymbol{b}(\mathbf{u}, x) = \hat{\boldsymbol{r}}(x, v(x)), \qquad x \in \mathscr{L}_1 \cup \mathscr{L}_2, \tag{5.4}$$

and satisfies the following *independency condition* for each s in $\mathscr{M}$:

$$\text{If } \frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}, x) \cdot \mathbf{w} = \mathbf{0} \text{ for almost all } X^3 \text{ in } (l_1(s), l_2(s)), \text{ then } \mathbf{w} = \mathbf{0}. \tag{5.5}$$

We define

$$\delta(\mathbf{u}, \mathbf{w}_\alpha, x) \equiv [\overline{\boldsymbol{b}_1}(\mathbf{u}, \mathbf{w}_\alpha, x) \times \overline{\boldsymbol{b}_2}(\mathbf{u}, \mathbf{w}_\alpha, x)] \cdot \boldsymbol{b}_{,3}(\mathbf{u}, x) \tag{5.6a}$$

and replace the invertibility condition (3.4) by

$$\delta(\mathbf{u}(s), \mathbf{u}(s)_{,\alpha}, s, X^3) > 0 \quad \text{for } x \in \mathrm{cl}\,\mathscr{B} \tag{5.6b}$$

or by the analog of (4.12). We assume that the representation (5.4) generates position boundary conditions in the parametric form

$$\mathbf{u}(s) = \hat{\mathbf{u}}(s, \mathbf{v}(s)), \qquad s \in \partial\mathscr{M} \tag{5.7}$$

or periodicity conditions of the form

$$\mathbf{u}(X^1 + T^1, X^2) = \mathbf{u}(X^1, X^2), \tag{5.8a}$$

$$\mathbf{u}(X^1, X^2 + T^2) = \mathbf{u}(X^1, X^2), \tag{5.8b}$$

or combinations of components of (5.7), (5.8).

c) Stress Resultants and the Equilibrium Equations. By the same procedure used in Section 4, we obtain the *weak form of the equilibrium equations for shells*:

$$\int_{\mathscr{M}} [\mathbf{m}^\alpha \cdot \mathbf{y}_{,\alpha} + \mathbf{n} \cdot \mathbf{y} - \mathbf{f} \cdot \mathbf{y}]\, dX^1\, dX^2 - \int_{\partial\mathscr{M}} \mathbf{p} \cdot \mathbf{y}\, d\Sigma(x) = 0 \tag{5.9}$$

for all sufficiently smooth $\mathbf{y}$ satisfying

$$\mathbf{y}(s) = \frac{\partial \mathbf{u}}{\partial \mathbf{v}}(s, \mathbf{v}(s)) \cdot \mathbf{w}(s), \qquad s \in \partial\mathscr{M} \tag{5.10}$$

when (5.7) holds and satisfying appropriate periodicity conditions when (5.8) holds. Here $d\Sigma$ is the differential arc length on $\partial\mathcal{M}$ and

$$\mathbf{m}^\alpha(\mathrm{s})=\int_{l_1(\mathrm{s})}^{l_2(\mathrm{s})}\boldsymbol{\tau}^\alpha(\mathrm{x})\cdot\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}(\mathrm{s}),\mathrm{x})\sqrt{G}(\mathrm{x})\,dX^3, \tag{5.11}$$

$$\mathbf{n}(\mathrm{s})=\int_{l_1(\mathrm{s})}^{l_2(\mathrm{s})}\boldsymbol{\tau}^k(\mathrm{x})\cdot\left[\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}(\mathrm{s}),\mathrm{x})\right]_{,k}\sqrt{G(\mathrm{x})}\,dX^3, \tag{5.12}$$

$$\begin{aligned}\mathbf{f}[\mathbf{u};\mathrm{s}]=&\int_{l_1(\mathrm{s})}^{l_2(\mathrm{s})}\boldsymbol{f}[\boldsymbol{b}(\mathbf{u},\cdot);\mathrm{x}]\cdot\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}(\mathrm{s}),\mathrm{x})\sqrt{G(\mathrm{x})}\,dX^3\\&+\left\{\boldsymbol{\sigma}[\boldsymbol{b}(\mathbf{u},\cdot)|_{\partial\mathcal{B}};\mathrm{x}]\cdot\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}(\mathrm{s}),\mathrm{x})\sqrt{G}[N^3-N^\alpha l_{,\alpha}]\right\}\Bigg|_{l_1(\mathrm{s})}^{l_2(\mathrm{s})},\end{aligned} \tag{5.13}$$

$$\begin{aligned}\mathbf{p}[\mathbf{u};\mathrm{s}]=&\int_{l_1(\mathrm{s})}^{l_2(\mathrm{s})}\boldsymbol{\sigma}[\boldsymbol{b}(\mathbf{u},\cdot)|_{\partial\mathcal{B}};\mathrm{x}]\cdot\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}(\mathbf{u}(\mathrm{s}),\mathrm{s},X^3)\\&\cdot\sqrt{G(\mathrm{s},X^3)}\,dX^3\qquad\text{for }\mathrm{s}\in\partial\mathcal{M}.\end{aligned} \tag{5.14}$$

Only

$$\mathbf{p}[\mathbf{u};\mathrm{s}]\cdot\frac{\partial\hat{\mathbf{u}}}{\partial \mathbf{v}}(\mathrm{s},\mathbf{v}(\mathrm{s})),\qquad \mathrm{s}\in\partial\mathcal{M} \tag{5.15}$$

is prescribed.

Wherever $\mathbf{m}^\alpha{}_{,\alpha}$, $\mathbf{n}$, $\mathbf{f}$ are continuous functions of s, equation (5.9) yields the *classical equilibrium equations for shells*

$$\mathbf{m}^\alpha{}_{,\alpha}-\mathbf{n}+\mathbf{f}=\mathbf{0}. \tag{5.16}$$

A special but useful variant of these equations is obtained by selecting a material reference surface $\mathbf{R}(\mathcal{M}^*)$, where

$$\mathcal{M}^*=\{\mathrm{x}:X^3=l(\mathrm{s}),\ \mathrm{s}\in\mathcal{M}\}, \tag{5.17}$$

and by setting

$$\mathbf{m}^\alpha=\Gamma\mathbf{m}^\alpha_*,\qquad \mathbf{n}=\Gamma\mathbf{n}_*,\qquad \mathbf{f}=\Gamma\mathbf{f}_*,\qquad \mathbf{p}=\Gamma\mathbf{p}_* \tag{5.18}$$

where

$$\Gamma(\mathrm{s})=\sqrt{G(\mathrm{s},l(\mathrm{s}))}\,[N^3_*(\mathrm{s})-N^\alpha_*\,l_{,\alpha}(\mathrm{s})], \tag{5.19}$$

$$\mathbf{N}_*(\mathrm{s})=\frac{\mathbf{G}^3(\mathrm{s},l(\mathrm{s}))-\mathbf{G}^\alpha(\mathrm{s},l(\mathrm{s}))\,l_{,\alpha}(\mathrm{s})}{|\mathbf{G}^3(\mathrm{s},l(\mathrm{s}))-\mathbf{G}^\alpha(\mathrm{s},l(\mathrm{s}))\,l_{,\alpha}(\mathrm{s})|}. \tag{5.20}$$

Then (5.9) becomes

$$\begin{aligned}&\int_{\mathcal{M}^*}[\mathbf{m}^\alpha_*\cdot\mathbf{y}_{,\alpha}+\mathbf{n}_*\cdot\mathbf{y}-\mathbf{f}_*\cdot\mathbf{y}]\,dA(\mathrm{x})\\&\qquad-\int_{\partial\mathcal{M}^*}\mathbf{p}_*\cdot\mathbf{y}\,ds(\mathrm{x})=0,\end{aligned} \tag{5.21}$$

where ds is the differential arc length on $[\mathbf{R}(\partial\mathcal{M}^*)]$, and (5.16) becomes

$$(\Gamma\mathbf{m}^\alpha_*)_{,\alpha}-\Gamma\mathbf{n}_*-\Gamma\mathbf{f}_*=\mathbf{0}. \tag{5.22}$$

In the special case when $\mathcal{M}^*$ is given by $X^3=\text{const.}$, we find that $\Gamma=\sqrt{G}$. The advantage of this formulation is that integration is with respect to the area of the reference surface $\mathbf{R}(\mathcal{M}^*)$ and that the role of $\sqrt{G}$ is explicit. This is important because $\sqrt{G}$ may vanish on $\partial\mathcal{M}$ in certain natural coordinate systems and this formulation does not obscure this singularity. (This is the case for spherical shells.)

d) Constitutive Relations. The constitutive equations for elastic shells are

$$\mathbf{m}^\alpha(\mathrm{s})=\hat{\mathbf{m}}^\alpha(\mathbf{u}(\mathrm{s}),\mathbf{u}_{,\beta}(\mathrm{s}),\mathrm{s}), \tag{5.23}$$

$$\mathbf{n}(\mathrm{s})=\hat{\mathbf{n}}(\mathbf{u}(\mathrm{s}),\mathbf{u}_{,\beta}(\mathrm{s}),\mathrm{s}), \tag{5.24}$$

where

$$\hat{\mathbf{m}}^\alpha(\mathbf{u},\mathbf{w}_\beta,\mathrm{s})=\int_{l_1(\mathrm{s})}^{l_2(\mathrm{s})}\hat{\tau}^\alpha(\bar{b}_j(\mathbf{u},\mathbf{w}_\beta,\mathrm{x}),\mathrm{x})\cdot\frac{\partial \boldsymbol{b}}{\partial\mathbf{u}}(\mathbf{u},\mathrm{x})\sqrt{G(\mathrm{x})}\,dX^3, \tag{5.25}$$

$$\hat{\mathbf{n}}(\mathbf{u},\mathbf{w}_\beta,\mathrm{s})=\int_{l_1(\mathrm{s})}^{l_2(\mathrm{s})}\hat{\tau}^k(\bar{b}_j(\mathbf{u},\mathbf{w}_\beta,\mathrm{x}),\mathrm{x})\cdot\overline{\left(\frac{\partial \boldsymbol{b}}{\partial\mathbf{u}}\right)_k}(\mathbf{u},\mathbf{w}_\beta,\mathrm{x})\sqrt{G(\mathrm{x})}\,dX^3. \tag{5.26}$$

From (5.25) and (3.23) we conclude that $\{\hat{\mathbf{m}}^\alpha\}$ satisfy the two-dimensional *strong ellipticity condition*

$$\mu_\alpha\mathbf{a}\cdot\frac{\partial\mathbf{m}^\alpha}{\partial\mathbf{w}_\beta}\cdot\mathbf{a}\mu_\beta>0,\qquad\forall\mathbf{a}\neq 0,\ (\mu_1,\mu_2)\neq(0,0), \tag{5.27}$$

for $\mathrm{s}\in\mathcal{M}$.

If the material is hyperelastic, then we set

$$\Psi(\mathbf{u},\mathbf{w}_\beta,\mathrm{s})=\int_{l_1(\mathrm{s})}^{l_2(\mathrm{s})}\psi(\bar{b}_j(\mathbf{u},\mathbf{w}_\beta,\mathrm{x}),\mathrm{x})\sqrt{G}\,dX^3 \tag{5.28a}$$

and find that

$$\hat{\mathbf{m}}^\alpha=\frac{\partial\Psi}{\partial\mathbf{w}_\alpha},\qquad\hat{\mathbf{n}}=\frac{\partial\Psi}{\partial\mathbf{u}}. \tag{5.28b}$$

Remark. At the end of Section 3, we observed that the requirement that the strongly elliptic operator of the three-dimensional theory be strictly pseudo-monotone and frame-indifferent implies that this operator actually be strictly monotone. Strictly monotone operators are unacceptable in non-linear elasticity because they yield uniqueness of solutions under dead loads. On the other hand, the operator for rod theories was shown to be a strictly pseudo-monotone operator in (4.48). As such it has an existence theory that allows multiplicity. (See Parts II and IV.) At first glance, it would therefore seem attractive to replace (5.27) by

$$\mathbf{a}_\alpha\cdot\frac{\partial\hat{\mathbf{m}}^\alpha}{\partial\mathbf{w}_\beta}\cdot\mathbf{a}_\beta>0\quad\forall\{\mathbf{a}_\alpha\}\text{ such that }|\mathbf{a}_1|+|\mathbf{a}_2|\neq 0. \tag{5.29a}$$

This assumption would make the operator of shell theory (formally) strictly pseudo-monotone and likely to give rise to an existence theory without uniqueness. If we examine the restrictions on the strong ellipticity condition (3.27) needed to make the operator of shell theory (formally) pseudo-monotone, we find that they prohibit buckling in certain planes however large the load and however thin the body. Thus this requirement is physically unacceptable. Moreover, its mathematical advantages are illusory: The domain

$$\bigcap_{X^3 \in l_1(\mathrm{s}), l_2(\mathrm{s})} \{\mathbf{w}_\beta : \delta(\mathbf{u}, \mathbf{w}_\beta, \mathrm{s}, X^3) > 0\} \tag{5.29b}$$

of the monotone mapping $\hat{\mathbf{m}}^1(\mathbf{u}, \cdot, S)$, $\hat{\mathbf{m}}^2(\mathbf{u}, \cdot, S)$ is not convex. If (5.28) holds, then it would be impossible for Ψ to be infinite on the boundary of (5.29b) and this is an eminently natural way to characterize invertibility.

We require $\hat{\mathbf{m}}^\alpha$ and $\hat{\mathbf{n}}$ to satisfy growth conditions completely analogous to those for rods. In particular, the set of functions u satisfying

$$\int_{\mathscr{M}} \Psi(\mathbf{u}(\mathrm{s}), \mathbf{u}_{,\alpha}(\mathrm{s}), \mathrm{s})\, dX^1\, dX^2 < \infty \tag{5.30}$$

satisfies

$$\delta(\mathbf{u}(\mathrm{s}), \mathbf{u}_{,\alpha}(\mathrm{s}), \mathrm{s}, X^3) > 0 \quad \text{a.e. on } \mathrm{cl}\,\mathscr{B}. \tag{5.31}$$

We assume that there is a smooth function h^- satisfying (4.66) and (4.67).

We now examine the consequences of the violation of (5.6b) at a point of $\mathscr{B}$. Suppose that

$$\delta(\mathbf{u}, \mathbf{w}_\alpha, \mathrm{s}, Y^3) = 0 \tag{5.32}$$

for some fixed $\mathbf{u}$, $\mathbf{w}_\alpha$, s, Y^3 with $\mathrm{s} \in \mathscr{M}$ and $l_1(\mathrm{s}) < Y^3 < l_2(\mathrm{s})$. The smoothness and positivity of δ ensure that there are positive numbers $Q(\mathbf{u}, \mathbf{w}_\alpha, \mathrm{s}, Y^3)$ and ε such that

$$G^{-\frac{1}{2}} \delta(\mathbf{u}, \mathbf{w}_\alpha, \mathrm{s}, X^3) < Q\, |X^3 - Y^3|^2 \quad \text{for } |X^3 - Y^3| < \varepsilon.$$

If h^- has the property that

$$\lim_{a \to 0^+} \int_a^\varepsilon h^-(\rho^2)\, d\rho = \infty, \tag{5.33}$$

then $\Psi(\mathbf{u}, \mathbf{w}_\alpha, S) = \infty$ if δ vanishes at any point of $(l_1(\mathrm{s}), l_2(\mathrm{s}))$. (This is ensured if $h^-(\delta) > \mathrm{const.}\, \delta^{-\frac{1}{2}}$.)

Now let Y^3 equal either $l_1(\mathrm{s})$ or $l_2(\mathrm{s})$. The vanishing of δ at Y^3 implies there are positive numbers $Q(\mathbf{u}, \mathbf{w}_\alpha, \mathrm{s}, Y^3)$ and ε with

$$Q(X^3 - Y^3) > 0 \quad \text{for } X^3 \in (l_1(\mathrm{s}), l_2(\mathrm{s}))$$

such that

$$\delta(\mathbf{u}, \mathbf{w}_\alpha, \mathrm{s}, X^3) \leqq Q(X^3 - Y^3) \quad \text{for } |X^3 - Y^3| < \varepsilon.$$

If h^- has the property that

$$\lim_{a \to 0^+} \int_a^\varepsilon h^-(\rho)\, d\rho = \infty, \tag{5.34}$$

then $\Psi(\mathbf{u}, \mathbf{w}_\alpha, \mathrm{s}) = \infty$ if δ vanishes at $l_1(\mathrm{s})$ or $l_2(\mathrm{s})$. (This is ensured by the requirement that $h^-(\delta) > \mathrm{const.}/\delta$.) Thus, under mild growth conditions for the constitutive assumption, the violation of the invertibility condition is again signaled by the infinity of Ψ. By following the development at the end of Section 4, we may even show that this violation is signaled by the unboundedness of $|\Psi_{\mathbf{w}_\beta}|$.

e) Boundary Value Problems. The classical boundary value problem is obtained by substituting the constitutive equations (5.23), (5.24) into the equilibrium equations (5.16) and by appending an appropriate set of complementary boundary conditions. The weak form of the boundary value problem for shells is obtained as in Section 4. Because of the strong ellipticity condition (5.27), such a problem presents many of the same formidable difficulties for analysis as the three-dimensional theory. We accordingly examine cases in which the governing equations reduce to ordinary differential equations.

f) Axisymmetric Deformations of Axisymmetric Shells. Let $\mathscr{B}$ be axisymmetric, *i.e.*, let there be a cylindrical coordinate system (R, Θ, Z) with a corresponding orthonormal basis $(\boldsymbol{e}_R(\Theta), \boldsymbol{e}_\Theta(\Theta), \boldsymbol{k})$ such that $R\boldsymbol{e}_R(\Theta)+Z\boldsymbol{k}\in\mathbf{R}(\mathscr{B})$ if and only if $R\boldsymbol{e}_R(0)+Z\boldsymbol{k}\in\mathbf{R}(\mathscr{B})$. We choose $X^1=\Theta$ and take X^2 to be a coordinate orthogonal to Θ. (X^2 must satisfy restrictions implicit in the discussion in Section 5a.) We denote X^2 by S. Then there are numbers $S_1<S_2$ such that

(5.35a) $$\mathscr{M}=\{\Theta, S: 0\leqq\Theta<2\pi, S_1<S<S_2\},$$

(5.35b) $$\partial\mathscr{M}=\{\Theta, S: 0\leqq\Theta<2\pi, S=S_1, S_2\}.$$

Let $\boldsymbol{g}$ be a vector-valued function on $\mathscr{M}$. $\boldsymbol{g}$ is called *axisymmetric* if its components with respect to $\boldsymbol{e}_R$, $\boldsymbol{e}_\Theta$, k are independent of Θ. This implies that $\boldsymbol{g}$ is differentiable with respect to Θ and that

(5.36) $$\frac{\partial\boldsymbol{g}}{\partial\Theta}(\Theta, S)=\boldsymbol{k}\times\boldsymbol{g}(\Theta, S).$$

To introduce the notion of axisymmetry for shells in a natural way, we assume that $\mathbf{u}$ is a $(K+1)$-tuple of Euclidean 3-vectors:

(5.37) $$\mathbf{u}=(\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_K), \qquad N=3(K+1).$$

This implies that $\mathbf{m}^1$, $\mathbf{m}^2$, $\mathbf{n}$, $\mathbf{f}$, $\mathbf{p}$ are likewise $(K+1)$-tuples of Euclidean 3-vectors. We define

(5.38) $$\boldsymbol{k}\times\mathbf{u}=(\boldsymbol{k}\times\boldsymbol{u}_0, \boldsymbol{k}\times\boldsymbol{u}_1, \ldots, \boldsymbol{k}\times\boldsymbol{u}_K).$$

Then $\mathbf{u}$ is *axisymmetric* if

(5.39) $$\frac{\partial\mathbf{u}}{\partial\Theta}=\boldsymbol{k}\times\mathbf{u}.$$

A shell problem is *axisymmetric** if $\mathscr{B}$ is axisymmetric, if the position boundary conditions are axisymmetric, and if the functions $\hat{\mathbf{m}}^1$, $\hat{\mathbf{m}}^2$, $\hat{\mathbf{n}}$, $\mathbf{f}$, $\mathbf{p}$ are such that axisymmetric $\mathbf{u}$ gives rise to axisymmetric $\mathbf{m}^1$, $\mathbf{m}^2$, $\mathbf{n}$, $\mathbf{f}$, $\mathbf{p}$. By (5.35a), (5.39), and the chain rule, this means that (5.7) has the form

(5.40) $$\mathbf{u}(\Theta, S_a)=\mathbf{u}_a(\Theta, \mathbf{v}_a(\Theta)), \qquad a=1,2,$$

* Cf. ERICKSEN (1972), WANG (1972, 1973), CARROLL & NAGHDI (1972), and works cited therein for treatments of homogeneity and isotropy in shells. Simple aspects of these questions are implicit in our notion of axisymmetry.

with

$$\frac{\partial \mathbf{u}_a}{\partial \Theta}+\frac{\partial \mathbf{u}_a}{\partial \mathbf{v}_a}\cdot(\boldsymbol{k}\times\mathbf{v}_a)=\boldsymbol{k}\times\mathbf{u}_a, \tag{5.41}$$

and that

$$\frac{\partial \hat{\mathbf{m}}^\alpha}{\partial \mathbf{u}}\cdot(\boldsymbol{k}\times\mathbf{u})+\frac{\partial \hat{\mathbf{m}}^\alpha}{\partial \mathbf{w}_1}\cdot[\boldsymbol{k}\times(\boldsymbol{k}\times\mathbf{u})]+\frac{\partial \hat{\mathbf{m}}^\alpha}{\partial \mathbf{w}_2}\cdot(\boldsymbol{k}\times\mathbf{w}_2)+\frac{\partial \hat{\mathbf{m}}^\alpha}{\partial \Theta}=\boldsymbol{k}\times\hat{\mathbf{m}}^\alpha, \quad \text{etc.} \tag{5.42}$$

Thus in an axisymmetric shell problem (*i.e.*, in the axisymmetric deformation of an axisymmetric shell), the classical equilibrium equations assume the form

$$\mathbf{m}_0'-\mathbf{n}_0+\mathbf{f}=\mathbf{0}, \tag{5.43}$$

where the prime denotes differentiation with respect to S, and where

$$\mathbf{m}_0=\mathbf{m}^2, \qquad \mathbf{n}_0=-\boldsymbol{k}\times\mathbf{m}^1+\mathbf{n}. \tag{5.44}$$

By setting $\mu_1=0$, $\mu_2=1$ in (5.27), we obtain

$$\mathbf{a}\cdot\frac{\partial \hat{\mathbf{m}}_0'}{\partial \mathbf{w}}\cdot\mathbf{a}>0 \qquad \forall\,\mathbf{a}\neq\mathbf{0}, \tag{5.45}$$

which states that under axisymmetry $\hat{\mathbf{m}}_0$ $(\equiv\hat{\mathbf{m}}^2)$ is a monotone function of $\mathbf{w}$ $(\equiv\mathbf{w}_2)$.

For hyperelastic materials, we set

$$\Psi_0(\mathbf{u},\mathbf{w},s)=\Psi(\mathbf{u},\boldsymbol{k}\times\mathbf{u},\mathbf{w},S),$$

so by the chain rule

$$\hat{\mathbf{m}}_0=\frac{\partial \Psi_0}{\partial \mathbf{w}}, \qquad \hat{\mathbf{n}}_0=\frac{\partial \Psi_0}{\partial \mathbf{u}}. \tag{5.46}$$

It is easily seen that the coercivity requirements for $\hat{\mathbf{m}}_0$ and $\hat{\mathbf{n}}_0$ are the same as those for $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ described in Section 4.

Note that the vectors of this theory of axisymmetric shells depend on Θ only through the base vectors. We can eliminate this insignificant parametric dependence, if we wish, by merely regarding $\mathbf{u}$, $\mathbf{m}$, $\mathbf{n}$, $\mathbf{f}$, $\mathbf{p}$, not as $(K+1)$-tuples of Euclidean 3-vectors, but as N-tuples of their cylindrical components. Thus axisymmetric shell problems give rise to boundary value problems of exactly the same form as those for rods.

g) Cylindrical Deformations of Cylindrical Shells. If $\mathrm{R}(\mathscr{B})$ is bounded by cylindrical surfaces with generators parallel to a given straight line and by two planes perpendicular to this line, then the shell is said to be *cylindrical.* If the equations are independent of the coordinate along the generators, then they are said to describe the *cylindrical deformation of a cylindrical shell.* (This is just a plane strain problem.) It is readily shown that the corresponding boundary value problem has the same form as that for rods.

6. Special Force Systems

We examine the form of $\mathbf{f}$ for some commonly encountered force systems in one-dimensional problems. Our purpose is merely to produce some examples showing the variety of ways that $\mathbf{f}$ can depend on $\mathbf{u}$. Our attention is especially directed to the asymptotic rate of growth of $\mathbf{f}$ in $\mathbf{u}$ for large $\mathbf{u}$, since the balance between this growth rate and the coercivity conditions plays a critical role both in the existence theory and in the interpretation of the physical significance of the analytic results obtained in Parts II and IV.

We study dead loadings, a class of pressure loadings, and effective loadings due to steady rotation. These are conservative force systems. The techniques we employ to construct $\mathbf{f}$ are applicable to any force system. For general analyses of conservative vis-a-vis non-conservative loadings, see SEWELL (1967), BATRA (1972), and works cited therein. The distinction is critical in the study of stability.

a) Dead Loading. The body force $\boldsymbol{f}$ of the three-dimensional theory is *dead* if it depends only on x. In this case, it is derivable from the potential ω:

$$f(\mathrm{x}) = -\frac{\partial \omega}{\partial \boldsymbol{r}}(\boldsymbol{r}, \mathrm{x}), \qquad \omega(\boldsymbol{r}, \mathrm{x}) = -\boldsymbol{f}(\mathrm{x}) \cdot \boldsymbol{r}. \tag{6.1}$$

Then the corresponding body force for rods is given by (4.29):

$$\mathbf{f}(\mathbf{u}, S) = \int_{\mathscr{B}(S)} \boldsymbol{f}(\mathrm{x}) \cdot \frac{\partial \boldsymbol{b}}{\partial \mathbf{u}} \sqrt{G}\, dX^1\, dX^2 = -\frac{\partial \Omega}{\partial \mathbf{u}}, \tag{6.2}$$

where

$$\begin{aligned} \Omega[\mathbf{u}, S] &= -\int_{\mathscr{B}(S)} \boldsymbol{f}(\mathrm{x}) \cdot \boldsymbol{b}(\mathbf{u}, \mathrm{x}) \sqrt{G(\mathrm{x})}\, dX^1\, dX^2 \\ &= \int_{\mathscr{B}(S)} \omega(\boldsymbol{b}(\mathbf{u}, \mathrm{x}), \mathrm{x}) \sqrt{G(\mathrm{x})}\, dX^1\, dX^2. \end{aligned} \tag{6.3}$$

Thus the rate of growth of $\mathbf{f}$ is determined by that of $\dfrac{\partial \boldsymbol{b}}{\partial \mathbf{u}}$. In particular, if $\boldsymbol{b}$ is linear in $\mathbf{u}$, then $\mathbf{f}$ is independent of $\mathbf{u}$. The results for shells are completely analogous.

b) Pressure Loading. The equilibrium pressure exerted by a gas on a surface is a conservative loading with a potential that is a function of the volume of the gas and possibly of the points of the surface. If the gas is confined by a surface $\mathscr{S}$, with surface coordinates Ξ^1, Ξ^2, then by Green's theorem, the volume enclosed by $\mathscr{S}$ is

$$v = \tfrac{1}{3} \int_{\mathscr{S}} \boldsymbol{\rho} \cdot \mathrm{N}\, dA = \tfrac{1}{3} \int_{\boldsymbol{\rho}^{-1}(\mathscr{S})} \boldsymbol{\rho} \cdot \left(\frac{\partial \boldsymbol{\rho}}{\partial \Xi^1} \times \frac{\partial \boldsymbol{\rho}}{\partial \Xi^2} \right) d\Xi^1\, d\Xi^2, \tag{6.4}$$

where $\boldsymbol{\rho}(\Xi^1, \Xi^2)$ is the position vector to the point with coordinates Ξ^1, Ξ^2 and N is the outer normal to $\mathscr{S}$. We are particularly concerned with the case in which $\mathscr{S}$ consists of the face $\mathscr{L}_1$ of a shell and part of a fixed surface $\mathscr{T}$ on which $\partial \mathscr{L}_1$ may slide (Fig. 6.1). $\mathscr{L}_1$ is parametrized by X^1, X^2, so on $\mathscr{L}_1$ we identify Ξ^α with X^α and set

$$\boldsymbol{\rho}(X^1, X^2) = \boldsymbol{r}(X^1, X^2, l_1(X^1, X^2)). \tag{6.5}$$

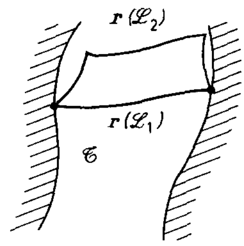

Fig. 6.1

We parametrize the part of $\mathscr{T}$ bounded by $\mathbf{R}(\partial\mathscr{L}_1)$ and $r(\partial\mathscr{L}_1)$ by V^1, V^2 and denote this region by $\mathscr{T}_0$. Then the volume enclosed by $\mathscr{S}$ is

$$v[r]=\tfrac{1}{3}\int_{\mathscr{M}} r\cdot\left[\left(\frac{\partial r}{\partial X^1}+\frac{\partial r}{\partial X^3}l_{1,1}\right)\times\left(\frac{\partial r}{\partial X^2}+\frac{\partial r}{\partial X^3}l_{1,2}\right)\right]dX^1\,dX^2 \tag{6.6}$$
$$+\tfrac{1}{3}\int_{\rho^{-1}(\mathscr{T}_0)}\rho\cdot\left(\frac{\partial\rho}{\partial V^1}\times\frac{\partial\rho}{\partial V^2}\right)dV^1\,dV^2+\text{const.}$$

The arguments of r and its derivatives in the first integral of (6.6) are X^1, X^2, $l_1(X^1,X^2)$. The constant is just the volume in the reference configuration. Note that $r(\partial\mathscr{L}_1)$, which determines $\mathscr{T}_0$, consists of points of the form

$$\hat{r}(X^\alpha, l_1(X^\alpha), V^\beta(X^\alpha, l_1(X^\alpha))),\qquad (X^1,X^2)\in\partial\mathscr{M}. \tag{6.7}$$

(*Cf.* (3.15a).) The replacement of r by its constrained version (5.3) yields a corresponding volume functional $v_1[\mathbf{u}]$. The potential for the pressure loading has the form

$$\Omega[\mathbf{u}]=\mu(v_1[\mathbf{u}],\mathrm{s}), \tag{6.8}$$

where μ: $\mathbb{R}\times\mathscr{M}\to\mathbb{R}$.

We now specialize this representation for v_1 to axisymmetric shell problems. Then $X^1=\Theta$ and $X^2=S$ are orthogonal and

$$\partial\mathscr{M}=\{(\Theta,S): S=S_1,S_2\}$$

so that $r(\partial\mathscr{M})$ consists of two circles (possibly degenerate). (*Cf.* (6.7).) Since $\mathscr{T}$ must also be axisymmetric, we also take $V^1=\Theta$. If we take V^2 to be orthogonal to Θ on $\mathscr{T}$, then axisymmetry ensures that the location of the circles forming $r(\partial\mathscr{M})$ is determined by constant values of $V^2=\Lambda_1,\Lambda_2$. In particular, if ρ_1 and ρ_2 represent position vectors to the part of $\mathscr{T}$ on which the edges S_1 and S_2 slide, then v of (6.6) reduces to

$$v[r]=\frac{2\pi}{3}\int_{S_1}^{S_2} r\cdot[(k\times r)\times r_{,2}]\,dS \tag{6.9}$$
$$+\frac{2\pi}{3}\int_0^{\Lambda_1}\rho_1(T)\cdot[(k\times\rho_1(T))\times\rho_1'(T)]\,dT$$
$$-\frac{2\pi}{3}\int_0^{\Lambda_2}\rho_2(T)\cdot[(k\times\rho_2(T))\times\rho_2'(T)]\,dT,$$

where the arguments of $\boldsymbol{r}$ in the first integral of (6.9) are Θ, S, $l_1(S)$, where $\boldsymbol{r}_{,2}=\frac{\partial \boldsymbol{r}}{\partial X^2}+\frac{\partial \boldsymbol{r}}{\partial X^3}\,l_1'(S)$, and where the argument Θ of $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ is suppressed in consonance with the axisymmetry. The numbers Λ_1, Λ_2 satisfy the identity

$$\hat{\boldsymbol{r}}(\Theta, S_\alpha, l_1(S_\alpha), \Theta, \Lambda_\alpha)=\boldsymbol{\rho}_\alpha(\Lambda_\alpha). \tag{6.10}$$

The introduction of the constraint (5.3) now gives

$$\begin{aligned}\frac{3}{2\pi}\,v_1[\mathbf{u}]=&\int_{S_1}^{S_2}\boldsymbol{b}\cdot\left[(\boldsymbol{k}\times\boldsymbol{b})\times\left(\frac{\partial \boldsymbol{b}}{\partial \mathbf{u}}\cdot\mathbf{u}'+\frac{\partial \boldsymbol{b}}{\partial X^3}\,l_1'\right)\right]dS\\ &+\int_0^{\Lambda_1}\boldsymbol{\rho}_1(T)\cdot\{[\boldsymbol{k}\times\boldsymbol{\rho}_1(T)]\times\boldsymbol{\rho}_1'(T)\}\,dT\\ &-\int_0^{\Lambda_2}\boldsymbol{\rho}_2(T)\cdot\{[\boldsymbol{k}\times\boldsymbol{\rho}_2(T)]\times\boldsymbol{\rho}_2'(T)\}\,dT.\end{aligned} \tag{6.11}$$

The arguments of $\boldsymbol{b}$ in the first integral are

$$\mathbf{u}(0,S), 0, S, l_1(S) \quad \text{with} \quad \frac{\partial \mathbf{u}}{\partial \Theta}=\boldsymbol{k}\times\boldsymbol{u}.$$

A constrained version of (6.10) now obtains. (*Cf.* (5.7).)

An identical analysis can be performed on $\mathscr{L}_2$. The total potential is the sum of separate potentials due to the pressures on the two surfaces.

It is important to note that if $\boldsymbol{b}$ is linear in $\mathbf{u}$, then v_1 is cubic in $\mathbf{u}$. Thus the rate of growth of Ω depending on the composition of μ and v_1 is unlikely to be linear as for dead loadings. Indeed, if μ is linear, then Ω itself is cubic. This is a reasonable model for the simplest kinds of hydrostatic pressures. For specific details about Ω for a simple model of a Cosserat shell, see ANTMAN (1971).

If $\mathscr{S}$ is a cylindrical surface with a closed cross-section and if $\mathscr{S}$ has coordinates X^1 along the generators and $X^2=S$ orthogonal to the generators, then the pressure exerted on $\mathscr{S}$ by a gas gives rise to a potential per unit of X^1 that depends upon the area a of a cross-section $\mathscr{C}$ of $\mathscr{S}$. If $\boldsymbol{k}$ is the direction of the generators and if $\boldsymbol{\rho}(S)$ is the position to a point of $\mathscr{C}$, then a is given by

$$a[\boldsymbol{r}]=\frac{\boldsymbol{k}}{2}\int_{S_1}^{S_2}[\boldsymbol{\rho}(S)\times\boldsymbol{\rho}'(S)]\,dS, \qquad [S_1, S_2]=\boldsymbol{\rho}^{-1}(\mathscr{C}). \tag{6.12}$$

A modification of the previous argument leads to a full description of the potential due to a pressure loading on the face $\mathscr{L}_1$ with $\partial\mathscr{L}_1$ free to slide on a fixed cylindrical surface $\mathscr{T}$. This depends on the area $a_1[\mathbf{u}]$ enclosed by $\mathscr{L}_1$ and $\mathscr{T}$ when the constraint (5.3) holds. If $\boldsymbol{b}$ is linear in $\mathbf{u}$, then a_1 is quadratic in $\mathbf{u}$. The different rates of growth of a_1 and v_1 may lead to major differences in the types of solutions that can hold for cylindrical and axisymmetric shell problems. This point is treated in Part II. Also see ANTMAN (1971), (1972, § 23). For details on the potential for a simple model of a cylindrical Cosserat shell, *cf.* ANTMAN (1972, Chapter III).

c) Loading Due to Centrifugal Forces. If a body is rotating with a constant angular velocity w about an axis with direction $\boldsymbol{k}$ through the origin $\mathbf{O}$ of $\mathbb{E}^3$, then the acceleration term is equivalent to a centrifugal body force

$$f(\boldsymbol{r}, \mathrm{x}) = \rho(\mathrm{x})\, w^2 [\boldsymbol{r} - (\boldsymbol{r} \cdot \boldsymbol{k})\boldsymbol{k}] = \frac{\partial}{\partial \boldsymbol{r}} \omega(\boldsymbol{r}, \mathrm{x}) \tag{6.13}$$

with

$$\omega(\boldsymbol{r}, \mathrm{x}) = \tfrac{1}{2}\rho(\mathrm{x})\, w^2\, |\boldsymbol{r} - (\boldsymbol{r} \cdot \boldsymbol{k})\boldsymbol{k}|^2, \tag{6.14}$$

where $\rho(\mathrm{x})$ is the mass density at x. In particular, if the body is a rod and if (4.5) holds, then the corresponding external force is given by (4.29):

$$\begin{aligned} \mathbf{f}(\mathbf{u}, S) &= w^2 \int_{\mathscr{B}(S)} \rho(\mathrm{x})[\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{k})\boldsymbol{k}] \frac{\partial \boldsymbol{b}}{\partial \mathbf{u}} \sqrt{G}\, dX^1\, dX^2 \\ &= \frac{\partial \Omega}{\partial \mathbf{u}}(\mathbf{u}, S), \end{aligned} \tag{6.15}$$

where

$$\Omega(\mathbf{u}, S) = \int_{\mathscr{B}(S)} \omega(\boldsymbol{b}(\mathbf{u}, S), \mathrm{x}) \sqrt{G}\, dX^1\, dX^2. \tag{6.16}$$

If $\boldsymbol{b}$ is linear in $\mathbf{u}$, then $\mathbf{f}$ is linear in $\mathbf{u}$ and Ω is quadratic in $\mathbf{u}$. Similar results hold for shells.

d) Functionals Fixing Rigid Displacements. When position boundary conditions allow at least a one-parameter family of configurations differing by a rigid displacement, we can remove this ambiguity by demanding $\boldsymbol{r}$ and its constrained version $\boldsymbol{b}(\mathbf{u}, \mathrm{x})$ to satisfy suitable side conditions. We may fix translation in the direction $\boldsymbol{e}$ by requiring the first moment of mass of $\mathscr{B}$ in this direction to vanish:

$$\boldsymbol{e} \cdot \int_{\mathscr{B}} \rho(\mathrm{x})\, \boldsymbol{r}(\mathrm{x})\; dV(\mathrm{x}) = 0. \tag{6.17}$$

Here $\rho(\mathrm{x})$ is the mass density at x. If we set $\rho = 1$ in (6.17), then (6.17) would imply that the first moment of volume of $\mathscr{B}$ vanishes; this has the same effect. The constrained version of (6.17) for rods is

$$\int_{S_1}^{S_2} \Omega(\mathbf{u}(S), S)\, dS = 0, \tag{6.18a}$$

where

$$\Omega(\mathbf{u}, S) = \int_{\mathscr{B}(S)} \rho(\mathrm{x})\, \boldsymbol{e} \cdot \boldsymbol{b}(\mathbf{u}, \mathrm{x}) \sqrt{G(\mathrm{x})}\, dX^1\, dX^2. \tag{6.18b}$$

Note that (6.18 b) has the same form as (6.3). Analogous results hold for shells.

To fix the orientation, we first note that the material curl of the displacement is a measure of mean rotation at a particle. (*Cf.* TRUESDELL & TOUPIN (1960, Section 36).) Thus we may require

$$\boldsymbol{e} \cdot \int_{\mathscr{B}} \mathbf{G}^k(\mathrm{x}) \frac{\partial}{\partial X^k} \times [\boldsymbol{r}(\mathrm{x}) - \mathbf{R}(\mathrm{x})]\, dV(\mathrm{x}) = 0. \tag{6.19}$$

The constrained version of (6.19) for rods is

$$\int_{S_1}^{S_2} \boldsymbol{e} \cdot \left\{ \int_{\mathscr{B}(S)} \mathbf{G}^k(\mathbf{x}) \frac{\partial}{\partial X^k} \times \boldsymbol{b}(\mathbf{u}(S), \mathbf{x}) \sqrt{G(\mathbf{x})}\, dX^1\, dX^2 \right\} dS$$
$$= \text{const} = \boldsymbol{e} \cdot \int_{\mathscr{B}} \mathbf{G}^k(\mathbf{x}) \frac{\partial}{\partial X^k} \times \mathbf{R}(\mathbf{x})\, dV(\mathbf{x}). \tag{6.20}$$

7. The Question of Convergence and Error Estimates

We only treat rod problems here; the treatment of shell problems is identical. Consider a sequence $\{\boldsymbol{b}^{(N)}\}$ of functions form $\mathbb{R}^N \times \mathrm{cl}\mathscr{B}$ to $\mathbb{R}^3$ with the property:

7.1. *For arbitrary $\mu > 0$, for arbitrary $\boldsymbol{r} \in \mathscr{F}_\mu \equiv \{\boldsymbol{r} \in \mathscr{H} : \det \boldsymbol{F} \geqq \mu\}$, and for arbitrary $\varepsilon > 0$ there exists a positive integer N and a function $\mathbf{v}^{(N)}: [S_1, S_2] \to \mathbb{R}^N$ satisfying*

$$\boldsymbol{b}^{(N)}(\mathbf{v}^{(N)}(S), \mathbf{x}) \in \mathscr{F}_\mu, \tag{7.2}$$

$$\|\boldsymbol{r} - \boldsymbol{b}^{(N)}(\mathbf{v}^{(N)}(\cdot), \cdot)\|_{\mathscr{V}} < \varepsilon. \tag{7.3}$$

The construction of such sequences $\{\boldsymbol{b}^{(N)}\}$ is standard.

A fundamental problem of thin body theories is to determine conditions under which a corresponding sequence of solutions $\{\mathbf{u}^{(N)}\}$ to one-dimensional boundary value problems converges to a solution of the three-dimensional theory. (*Cf.* ANTMAN (1972, Section 11) for a general discussion of this and related questions.) We examine certain aspects of this question by adapting standard methods of variational calculus to our situation.

Let $\boldsymbol{r} \to \Gamma[\boldsymbol{r}]$ be the potential energy functional for a conservative static problem in three-dimensional hyper-elasticity: The Euler-Lagrange equations for Γ are the classical equilibrium equations of the theory. For each $\mu > 0$, we suppose that Γ is continuous on $\mathscr{F}_\mu$ in the norm of $\mathscr{V}$. (This is true of the strain energy functional $\int_{\mathscr{B}} \psi(\boldsymbol{r}_{,k}, \mathbf{x})\, dV$, which is not continuous on $\mathscr{H}$, however.) Set

$$U^{(N)}[\mathbf{v}^{(N)}] = \Gamma[\boldsymbol{b}^{(N)}(\mathbf{v}^{(N)}(\cdot), \cdot)]. \tag{7.4}$$

7.5. Theorem. *Let $\mathbf{u}^{(N)}$ minimize $U^{(N)}$ over its domain and let there exist a number $\mu_N > 0$ such that $\boldsymbol{b}^{(N)}(\mathbf{u}^{(N)}(\cdot), \cdot) \in \mathscr{F}_{\mu_N}$.* Then $\{\boldsymbol{b}^{(N)}(\mathbf{u}^{(N)}(\cdot), \cdot)\}$ is an infimizing sequence for Γ,* i.e.,

$$\Gamma[\boldsymbol{b}^{(N)}(\mathbf{u}^{(N)}(\cdot), \cdot)] \to \inf_{\mathscr{H}} \Gamma[\boldsymbol{r}]. \tag{7.6}$$

(In Part II, we supply conditions ensuring the existence of such minimizers $\mathbf{u}^{(N)}$.)

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* The results of Part II show that this requirement is usually a consequence of the minimizing properties of $\mathbf{u}^{(N)}$.

Proof. Let $\{\boldsymbol{r}^{(K)}\}$ be an infimizing sequence for Γ with $\{\boldsymbol{r}^{(K)}\} \in \mathscr{F}_{\mu_K}$, $\mu_K > 0$:

$$\Gamma[\boldsymbol{r}^{(K)}] \to \inf_{\tilde{\mathscr{H}}} \Gamma[\boldsymbol{r}]. \tag{7.7}$$

(If $\Gamma[\boldsymbol{r}^{(K)}] < \infty$, then $\boldsymbol{r}^{(K)}$ can violate (3.4) only on a set of zero measure. Thus we may assume that $\boldsymbol{r}^{(K)} \in \mathscr{F}_{\mu_K}$ for some $\mu_K > 0$, for if not, we could suitably modify $\boldsymbol{r}^{(K)}{}_{,k}$ whenever the left side of (3.4) is small without affecting (7.7).) By (7.3), for arbitrary $\varepsilon > 0$ there exists a number $N = N(K, \varepsilon)$ and a function $\mathbf{v}^{(N)}$ satisfying (7.2) and either (4.10) or (4.11) such that

$$\|\boldsymbol{r}^{(K)} - \boldsymbol{b}^{(N)}(\mathbf{v}^{(N)}(\cdot), \cdot)\|_{\mathscr{V}} < \varepsilon. \tag{7.8}$$

By the continuity of Γ, we can take ε so small that

$$\Gamma[\boldsymbol{b}^{(N)}(\mathbf{v}^{(N)}(\cdot), \cdot)] - \Gamma[\boldsymbol{r}^{(K)}] \leqq \frac{1}{K}. \tag{7.9}$$

Let $K \to \infty$. Since $\mathbf{u}^{(N)}$ minimizes $U^{(N)}$, we find

$$\begin{aligned} \inf_{\tilde{\mathscr{H}}} \Gamma[\boldsymbol{r}] &\leqq \liminf \Gamma[\boldsymbol{b}^{(N)}(\mathbf{u}^{(N)}(\cdot), \cdot)] \leqq \limsup \Gamma[\boldsymbol{b}^{(N)}(\mathbf{u}^{(N)}(\cdot), \cdot)] \\ &\equiv \limsup U^{(N)}[\mathbf{u}^{(N)}] \leqq \limsup U^{(N)}[\mathbf{v}^{(N)}] \\ &\equiv \limsup \Gamma[\boldsymbol{b}^{(N)}(\mathbf{v}^{(N)}(\cdot), \cdot) = \lim \Gamma[\boldsymbol{r}^{(K)}] = \inf_{\tilde{\mathscr{H}}} \Gamma[\boldsymbol{r}]. \end{aligned} \tag{7.10}$$

This implies (7.6). □

This result does not imply that Γ is minimized on $\tilde{\mathscr{H}}$. Neither does it imply that if Γ is minimized on $\tilde{\mathscr{H}}$ then $\boldsymbol{b}^N(\mathbf{u}^{(N)}(\cdot), \cdot)$ converges to the minimizer. Assuming the hypotheses of Section 3, we do however have

7.11. Theorem. *If Γ is weakly lower semi-continuous on $\mathscr{H}_\mu \equiv \{\boldsymbol{r} \in \tilde{\mathscr{H}} : \Gamma[\boldsymbol{r}] \leqq \mu\}$ for all $\mu > 0$, then Γ is minimized on $\mathscr{H}_\mu$ and a subsequence of $\{\boldsymbol{b}^{(N)}(\mathbf{u}^{(N)}(\cdot), \cdot)\}$ converges weakly to the minimizer.*

Sketch of Proof. Since $\tilde{\mathscr{H}}$ belongs to a reflexive Banach space, the coercivity hypotheses of Section 3 immediately imply that the set $\mathscr{H}_\mu$ of functions is weakly compact. Thus the infimizing sequence $\{\boldsymbol{b}^{(N)}(\mathbf{u}^{(N)}(\cdot), \cdot)\}$, whose members lie in this set for large enough N, has a subsequence, denoted the same way, that converges weakly to some limit $\boldsymbol{r}_0$ in $\mathscr{H}_\mu$. ($\mathscr{H}_\mu$ is weakly closed.) By the weak lower semi-continuity of Γ we have

$$\inf_{\tilde{\mathscr{H}}} \Gamma[\boldsymbol{r}] \leqq \Gamma[\boldsymbol{r}_0] \leqq \lim\, \inf \Gamma[\boldsymbol{b}^{(N)}(\mathbf{u}^{(N)}(\cdot), \cdot)] = \inf_{\tilde{\mathscr{H}}} \Gamma[\boldsymbol{r}]. \tag{7.12}$$

Thus $\Gamma[\boldsymbol{r}_0] = \inf_{\tilde{\mathscr{H}}} \Gamma[\boldsymbol{r}]$. □

In Part II we furnish sufficient conditions ensuring that the minimizers of $\{U^{(N)}\}$ are weak solutions of the Euler equations, which are just the equilibrium equations for rods (or shells). (Under slightly stronger hypotheses, we prove that these solutions are classical.) If Γ is Fréchet differentiable at its minimizer, then its derivative must vanish there, *i.e.*, the minimizer is a weak solution of the three-dimensional equations. Thus we have

7.13. Theorem. *If Γ is weakly lower semi-continuous on $\{\mathscr{H}_{\mu>0}\}$ and if Γ is Fréchet differentiable at its minimizer, then a subsequence of $\{\boldsymbol{b}^{(N)}(\mathbf{u}^{(N)}(\cdot),\cdot)\}$ converges weakly to the weak solution of the three-dimensional problem.*

At present the range of applicability of Theorems 7.11 and 7.13 is not known: I know of no proof that frame-indifferent functionals satisfying the strong Legendre-Hadamard condition are weakly lower semi-continuous. If, however, the strain energy function is quadratic in $\partial(\boldsymbol{r}-\mathbf{R})/\partial\mathrm{x}$ and if it satisfies the strong Legendre-Hadamard condition, then its integral is weakly lower semi-continuous (*cf.* MORREY (1966, Section 4.4). Also see FICHERA (1972a).) Thus Theorems 7.11 and 7.13 apply to linear theories of rods and shells. Moreover, J.M. BALL (1974) has shown me evidence indicating that certain conditions somewhat more stringent than the strong Legendre-Hadamard condition are physically realistic, allow non-uniqueness, and ensure that the strain energy functional is weakly lower semi-continuous.

Stronger convergence results obtain when the strain energy function is convex in $\partial\boldsymbol{r}/\partial\mathrm{x}$ or more generally when the Piola-Kirchhoff stress $\{\boldsymbol{\tau}^k\}$ is a monotone function of $\{\boldsymbol{r}_{,k}\}$. *Cf.* BROWDER (1967). Such results are physically meaningful only for linear and "nearly" linear problems.

An even more difficult problem than convergence is that of estimating the error between the approximations generated by rod and shell theories and the exact solution of the three-dimensional problem. When the strain energy function is convex, complementary energy principles may yield effective error estimates for specific choices of $\boldsymbol{b}$ linear in $\mathbf{u}$. For general aspects of this question, *cf.* EKELAND & TEMAN (1974), LAVERY (1976) and LEVENTHAL (1975). By using alternative geometric methods, JOHN has estimated errors for geometric quantities in terms of an *a priori* bound on the total strain. For a description of these techniques, *cf.* the works cited by JOHN (1975).

8. Conclusion and Commentary

a) Structure of One-Dimensional Problems. We have shown that equilibrium problems for the spatial deformation of elastic rods, for the axisymmetric deformation of axisymmetric elastic shells, and for the cylindrical deformation of cylindrical elastic shells have a rich and definitive mathematical structure that is independent of the method of constraints used to generate these one-dimensional models. This mathematical structure consists of the following facets:

i) The classical versions of the equilibrium problems are boundary value problems for quasi-linear systems of second-order ordinary differential equations (4.76).

ii) The weak versions of the equilibrium problem have the form (4.77). Despite the unilaterality of the invertibility conditions, these are not variational inequalities. The reason for this is discussed in Part II, Section 6.

iii) The form of these problems is independent of the particular type of constraint used to generate these one-dimensional models and is independent of the complexity of the model (characterized by the number N of "degrees of freedom"). *Cf.* (4.76), (4.77).

iv) The differential operators (or, equivalently, the constitutive functions) of the one-dimensional theory satisfy the monotonicity condition (4.48) (or the convexity condition (4.51)). The domain of $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ is $\mathscr{G}$ given by (4.14). $\mathscr{G}(\mathbf{u}, S)$, defined in (4.18), is convex.

v) The differential operators satisfy growth conditions of the form (4.52)–(4.55), (4.60)–(4.62). These conditions and the concomitant choice of $\mathscr{W}$ permit the constitutive functions to have different principal directions of asymptotic growth and different rates of asymptotic growth at each particle. (See the discussion following (3.43), especially the example treated in (3.40)–(3.45).) This means that the material can exhibit considerable latitude in the aeolotropy and non-homogeneity of its response.

vi) The class of admissible position functions $\mathbf{u}$ satisfy

$$\int_{S_1}^{S_2} \Psi(\mathbf{u}(S), \mathbf{u}'(S), S)\, dS < \infty. \tag{8.1}$$

Hypotheses requiring that $\Psi(\mathbf{u}, \mathbf{w}, S) \to \infty$ as $(\mathbf{u}, \mathbf{w}) \to \partial\mathscr{G}(S)$ imply that the invertibility conditions (4.12) holds a.e. Thus (8.1) furnishes a weakened statement of invertibility.

vii) If δ were to vanish at some particle in the section at S, then under reasonable circumstances, $\Psi(\mathbf{u}(S), \mathbf{u}'(S), S)$ would equal ∞ at S. (See Section 4d.) In this case the finiteness of Ψ serves as a classical statement of the invertibility conditions. Whether or not these reasonable circumstances obtain, the finiteness of Ψ is a suitable criterion of regularity.

Results (i) and (iii) were developed by ANTMAN (1972) by a different approach. Result (ii) was foreshadowed by ANTMAN (1973b). Results (iv)–(vii) are new.

b) Applications and Physical Significance of Monotonicity and Invertibility Conditions. The monotonicity and invertibility conditions play key roles in the analysis of one-dimensional problems of non-linear elasticity. For problems of existence and regularity (Parts II and IV) these conditions are in conflict in that the invertibility condition presents a host of serious obstacles that interfere with the straightforward use of monotonicity to reduce the existence and regularity questions to simple applications of the now well-established theories of pseudo-monotone operators and theories of regularity for weak solutions of ordinary differential equations. On the other hand, these conditions work in harmony in the study of the qualitative behavior of solutions. The invertibility condition with its concomitant coercivity conditions excludes from consideration physically unrealistic solutions whose properties would be far more difficult to determine than those of physically admissible solutions. The monotonicity condition permits various applications of global implicit function theorems. These play a crucial role in reducing problems to forms readily accessible to qualitative study. The papers of ANTMAN (1970; 1973a, b; 1974a, b) and ANTMAN & JORDAN (1975) exploit these properties in the qualitative study of buckling, necking, and shearing of rods under terminal loads and of rings under hydrostatic pressure. The constitutive equations for these bodies are special cases of those developed here. Bifurcation analyses (ANTMAN (1970, 1973a), OLMSTEAD & MESCHELOFF (1974), ANTMAN & CARBONE (1976)) also rely on the monotonicity condition.

The monotonicity condition gives a precise mathematical sense to the notion of order-preservation in the stress-strain laws. The order-preservation embodied in the monotonicity condition turns out to be very natural for the physics. *E.g.*, it may seem physically reasonable to expect an increase in shear deformation to accompany an increase in shear force. But this statement is imprecise because we do not specify exactly what component of force is the shear component and exactly what component of deformation is to measure shear. (Indeed, in the literature distinct entities have been labeled shear; although their values may agree for small deformations, they can disagree significantly for large deformations.) It can be shown that a concept of order preservation for the wrong components may lead not only to serious analytic difficulties (see ANTMAN (1971)), but also to the prohibition of kinds of solutions that have been observed empirically. The monotonicity condition prevents this difficulty; as an invariant condition it produces consistent componential versions of order preservation. Various aspects of this question in relation to shear are examined in more detail by ANTMAN (1972, Section 18; 1973a, b, c; 1974).

c) Global Invertibility. If one wishes to prescribe boundary conditions to hold when two distinct parts of a body come into contact, then the resulting conditions, by themselves, naturally give rise to a variational inequality. (*Cf.* DUVAUT & LIONS (1972), FICHERA (1972b).) The analytic questions arising from such unilateral constraints turn out to be of an entirely different nature from those resulting from the invertibility condition. (See Parts II and IV.)

d) End and Edge Conditions. There are a variety of strategies by which general end conditions of the form (3.15a) can be reduced to those of the form (4.21). *E.g.*, let N be an integral multiple of 3. If any N particles are chosen in cl $\mathscr{A}(S_a)$, then the independency condition (4.8) ensures that $\mathbf{u}$ can be chosen such that $\boldsymbol{b}$ exactly satisfies position boundary conditions (3.15a) at these particles.

e) The Role of Thinness. Thickness ratios do not appear in our development. They can be introduced, however, by an appropriate scaling. Their importance would be manifested in error estimates, which presumably would show that the thinner the body the smaller the N needed to provide a satisfactory approximation for a given class of data.

One may contemplate the construction of one-dimensional theories by means of an asymptotic expansion in a thickness parameter. This entails several difficulties: (α) There may be many thickness parameters available since a given problem may involve many typical lengths. (β) The resulting theory may not be truly one-dimensional. (γ) Much of the elegance of structure of non-linear problems may be lost. For linear static problems with sufficiently simple geometries, the difficulties (α) and (γ) do not arise, and there are satisfactory formal asymptotic theories. (Rigorous asymptotic results for linear theories have been developed by RIGOLOT (1972).) The same is not true for linear dynamical problems, however. A comparison of our projection theories with satisfactory asymptotic theories would be worthwhile since it would illuminate the connection between weak solutions of the three dimensional equations that are approximated by our one-dimensional theories and the boundary layer effects that are treated naturally by asymptotic methods. (Note that stress resultants appear naturally in asymptotic developments. (*Cf.* HAY (1942) or REISS & LOCKE (1961), e.g.) This is essentially the sort of question that arises in the study of the St. Venant Principle.

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