Thm 1.1

Let V be a vector space.

Let $x, y, z \in V$

If $x+z = y+z$

then $x = y$

PF

By VS 6, $\exists (-z) \in V$.

By VS 4, $x = x + \bar{0}_V = x + z + (-z) = y + z + (-z) = y$ ∎

Corollary1 of Thm 1.1

Let V be a vector space.

Then $\bar{0}_V$ is unique.

PF

Let $v \in V$

suppose that $\exists \bar{0}_V' \in V$ s.t. $v + \bar{0}_V' = v$

Then $v + \bar{0}_V = v = v + \bar{0}_V'$

By Thm 1.1 $\bar{0}_V = \bar{0}_V'$

There fore $\bar{0}_V$ is unique ∎

Corollary2 of Thm 1.1

Let V be a vector space.

Let $v \in V$

Then $(-v)$ is unique

PF

suppose that $\exists (-v)' \in V$ s.t. $v + (-v)' = \bar{0}_V$

Then $v + (-v) = \bar{0}_V = v + (-v)'$

By Thm 1.1 $(-v) = (-v)'$

There fore $(-v)$ is unique ∎

Thm 1.2

Let V be a vector space.

Let $v \in V$ and $a \in F$

(a) Then $\bar{0}_v = 0v$

(b) Then $(-av) = (-a)v = a(-v)$

PF(a)

By VS5, $0v = 0v + \bar{0}_v$

By VS10, $0v = (0+0)v = 0v + 0v$

By Thm 1.1 $\bar{0}_v = 0v$

PF (b)

By VS 6, $av + (-av) = \bar{0}_v$

By VS10, $av + (-a)v = (a-a)v = 0v$

By Thm 1.2 (a) $av + (-a)v = \bar{0}_v$

By uniqueness $(-av) = (-a)v$

By VS 8, $(-av) = (-a)v = a(-1)v = a(-v)$

Therefore $(-av) = (-a)v = a(-v)$ ∎

Thm 1.3

Let V be a vector space.

Let W be a subset of V

Let $w_1, w_2 \in W$ and $a \in F$

If $\bar{0}_v \in W$ and $aw_1 + w_2 \in W$

Then W is a subspace of V.

PF

By Assumption, VS 1, 2, 5 are satisfied

Let $w \in W$

By thm 1.2 $(-w) = (-1)w \in W$

Thus VS 6 also satisfied.

By condition of subspace W is a subspace of V.

Thm 1.5

Let V be a vector space.

Let W be a subset of V

Then span(W) is a subspace of V

More over, any subspace of V that contains W also contain the span(W)

pf)

if $W=\phi$, span$(\phi)=\{\bar{0}\}$,

$\{0\}$ is subspace of every vector space.

if $w \in W$, $0w=\bar{0} \in$ span(W)

if $x, y \in W$, $cx+y \in$ span(W)

by Condition of subspaces, W is a subspace of V

every subspace also vector space.

Let S be a subspace of V.

if $W \subseteq S$ then span$(W) \subseteq S$ ■

Thm 1.8

Let V be a vector space.

Let $v \in V$.

Then $\exists! \, a_i \in F$ s.t. $v=\sum_{i=1}^{n} a_i \beta_i^V$

PF

as β^V is generating set, $v \in$ span(β^V)

Thus $\exists a_i \in F$ s.t. $v=\sum_{i=1}^{n} a_i \beta_i^V$.

suppose that $v=\sum_{i=1}^{n} a_i \beta_i^V = \sum_{i=1}^{n} b_i \beta_i^V$

Then $\bar{0}_V = \sum_{i=1}^{n} (a_i-b_i)\beta_i^V$

Since β_i^V is Linearly independent, $a_i=b_i$

Thus a_i is unique ■

Thm 1.9

Let V be a vector space.

Let G be a finite generating set of V

Then maximum linearly independent subset of G is β^V

Hence V has a finite basis.

PF

Let $W = \{g_1, \cdots g_k\}$ be maximum linearly independent subset of G

Since $W \subseteq V$, $span(W) \subseteq V$ by thm 1.5

Let $g_0 \in G - W$

Since $W \cup \{g_0\}$ is Linearly independent,

$a_0 g_0 + a_1 g_1 + \cdots + a_k g_k = \bar{0}_V \Rightarrow g_0 = a_0^{-1}(a_i g_i) \in span(W)$

Hence $G \subseteq span(W)$.

By thm 1.5 $span(G) \subseteq span(W) \Rightarrow V \subseteq span(W)$

Thus $span(W) = V$

Therefore W is β^V ∎

Thm 1.10

Let V be a vector space.

Let G be a generating set of V and $|G| = n$

Let L be a linearly independent subset of V and $|L| = m$

Then $n \geq m$

and $\exists H \subseteq G$ s.t. $L \cup H$ generates V and $|H| = n-m$

PF

if $m=0$ $L = \emptyset$ and $H = G$

if $m=k$ suppose that $\exists H$

if $m = k+1$ $L = \{l_1, \cdots, l_{k+1}\}$

Since $(L / \{l_{k+1}\}) \cup H$ generates V,

l_{k+1} is linear combination of $(L / \{l_{k+1}\}) \cup H$

Thus one of $h_j \in H$ is linear combination of $L \cup (H - \{h_j\})$

Hence $L \cup (H - \{h_j\})$ generates V

Therefore $\exists H$ when $m = k+1$ ∎

Corollary 1 of Thm 1.10

Let V be a finite dimension vector space.

Then every basis for V containing the same number of vectors

PF

Let $|\beta^{V_1}| = n$ and $|\beta^{V_2}| = m$

Since β^{V_1} is generating set for V

and β^{V_2} is linearly independent set for V

By Thm 1.10 $n \geq m$

At the same time, β^{V_1} is also linearly independent set for V

and β^{V_2} is also generating set for V

By Thm 1.10 $n \leq m$

Therefore $n = m$ ∎

Corollary 2 of Thm 1.10

Let V be a vector space with dimension n

(a) Let G be a generating set of V.

if G have n vectors

Then $G = \beta^V$

(b) Let W be an linearly independent subset of V

if W has n vectors

Then $W = \beta^V$

(c) Let W be a maximum linearly independent set of V

Then $W = \beta^V$

PF (a)

By Thm 1.9 Subset of G is a basis.

and By Corollary 1, every basis have same number of vector.

Thus G Should be a basis. ∎

PF (b)

By Thm 1.10 $\emptyset \cup W$ is generating set of V.

Thus W is basis for V ∎

PF (c)

By Thm 1.10 $|W| \leq |\beta^V|$.

Thus maximum $|W|$ is always $|\beta^V|$.

By (b), $W = \beta^V$ ∎

Thm 1.11

Let V be a finite dimensional vector space.

Let S be a subspace of V

Then S is finite dimensional & $\dim(S) \leq \dim(V)$

Moreover if $\dim(S) = \dim(V)$, then $V = S$

Pf)

Since β^S is a linearly independent subset of V

By thm 1.10 $|\beta^S| \leq |\beta^V| \Rightarrow \dim(S) \leq \dim(V)$

if $\dim(S) = \dim(V)$, by colloary 2 of Thm 1.10 $\beta^S = \beta^V$

Therefore $S = V$

Thm 2.1

Let V & W be vector spaces

Let $T \in \mathcal{L}(V, W)$

Then $N(T)$ & $R(T)$ are subspaces of V & W.

PF

$N(T) = \{ v \in V \mid T(v) = \bar{0}_W \}$

Let $T(\bar{0}_V) = w$

Then $w = T(\bar{0}_V) = T(\bar{0}_V + \bar{0}_V) = T(\bar{0}_V) + T(\bar{0}_V) = w + w$

Thus $w = \bar{0}_W$ and $\bar{0}_V \in N(T)$

Let $v_1, v_2 \in N(T)$ and $a \in F$.

Then $T(av_1 + v_2) = aT(v_1) + T(v_2) = \bar{0}_W$

Thus $av_1 + v_2 \in N(T)$

Therefore $N(T)$ is subspace of V ■

$R(T) = \{ T(v) \mid v \in V \} \subseteq W$

Since $T(\bar{0}_V) = \bar{0}_W$, $\bar{0}_W \subseteq R(T)$

Let $w_1, w_2 \in R(T)$ and $a \in F$

Then $\exists v_1, v_2 \in V$ s.t. $T(v_1) = w_1$ and $T(v_2) = w_2$

$aw_1 + w_2 = aT(v_1) + T(v_2) = T(av_1 + v_2)$

Since $av_1 + v_2 \in V$, $aw_1 + w_2 \in R(T)$

Therefore $R(T)$ is subspace of W ■

Thm 2.2

Let V & W be Vector spaces.

Let $T \in \mathcal{L}(V, W)$

Then $R(T) = \text{span}(T(\beta^V))$

PF

Let $\dim(V) = n$

Since $T(\beta^V) \subseteq R(T)$ and $R(T)$ is subspace of W

By Thm 1.5 $\text{span}(T(\beta^V)) \subseteq R(T)$

Let $r \in R(T)$

Then $\exists v \in V$ s.t. $r = T(v)$

$r = T(v) = T(\sum_{i}^{n} a_i \beta_i^V) \in \text{span}(T(\beta^V))$

Thus $R(T) \subseteq \text{span}(T(\beta^V))$

Therefore $R(T) = \text{span}(T(\beta^V))$ ■

Thm 2.3

Let V & W be finite dimensional vector spaces.

Let $T \in \mathcal{L}(V, W)$

Then $\text{nullity}(T) + \text{rank}(T) = \dim(V)$

PF

Let $\beta^V = \{\beta_1^V, \cdots, \beta_n^V\}$ and $\beta^{N(T)} = \{\beta_1^V, \cdots, \beta_k^V\}$

Let $S = \beta^V / \beta^{N(T)}$

By Thm 2.2 $R(T) = \text{span}(T(\beta^V)) = \text{span}(T(S))$

Suppose that $\sum_{k+1}^{n} b_i T(\beta_i^V) = \bar{0}_V$ with some scalars $b_i \in F$

Since T is linear $T(\sum_{k+1}^{n} b_i \beta_i^V) = \bar{0}_V$, $\sum_{k+1}^{n} b_i \beta_i^V \in N(T)$

Thus $\sum_{k+1}^{n} b_i \beta_i^V = \sum_{1}^{k} c_i \beta_i^V \Rightarrow -\sum_{1}^{k} c_i \beta_i^V + \sum_{k+1}^{n} b_i \beta_i^V = 0$

As β^V is linearly independent, $b_i = 0$

Thus $T(S)$ is linearly independent

Consequently, $T(S) = \beta^{R(T)}$

because $|S| = |T(S)|$, $\dim(N(T)) + \dim(R(T)) = \dim(V)$ ∎

Thm 2.4

Let V & W be vector spaces.

Let $T \in \mathcal{L}(V, W)$

Then T is one-to-one if and only if $N(T) = \{\bar{0}_V\}$

Pf)

Suppose that T is one-to-one & $x \in N(T)$

Then $T(x) = \bar{0}_W = T(\bar{0}_V)$.

Since T is one to one, we have $x = \bar{0}_V$

Hence $N(T) = \{\bar{0}_V\}$

Assume that $N(T) = \{\bar{0}_V\}$ & $T(x) = T(y)$, $x, y \in V$

Then $T(x - y) = T(x) - T(y) = \bar{0}_V$

Therefore $x - y \in N(T) \Rightarrow x - y = \bar{0}_V \Rightarrow x = y$

This means T is one to one. ∎

Thm 2.5

Let V & W be vector spaces of equal dimension

Let $T \in \mathcal{L}(V, W)$

Then $\begin{pmatrix} T \text{ is one to one} \\ T \text{ is onto} \\ \text{rank}(T) = \dim(V) \end{pmatrix}$ is equivalent.

Also we say T is invertible if and only if $\text{rank}(T) = \dim(V)$

PF

Suppose that T is one to one.

By Thm 2.4, $\text{nullity}(T) = 0 \Rightarrow$ By Thm 2.3 $\text{rank}(T) = \dim(V)$

Since V and W have equal dimension, $\text{rank}(T) = \dim(W) \Rightarrow \dim(R(T)) = \dim(W)$

By Thm 1.11 $R(T) = W$

By definition of $R(T)$, $R(T) = T(V)$

Thus $T(V) = W \Rightarrow T$ is onto

Thm 2.6

Let V & W be vector spaces.

Let $\dim(V) = n$

Let $w_1, \cdots, w_n \in W$

then $\exists! \, T \in \mathcal{L}(V, W)$ s.t. $T(\beta_i^V) = w_i$ for $i = 1, \cdots, n$

PF

Let $x \in V$, then $x = \sum_1^n a_i \beta_i^V$

define $T: V \to W$ as $T(x) = \sum_1^n a_i w_i$

Since $T(\beta_i^V) = w_i \Rightarrow T(a_i \beta_i^V) = a_i w_i = a_i T(\beta_i^V)$

Thus, $T \in \mathcal{L}(V, W)$

now Suppose that $U \in \mathcal{L}(V, W)$ and $U(\beta_i^V) = w_i$

Then $U(x) = a_i U(\beta_i^V) = a_i w_i = T(x)$

Thus T is unique ∎

Corollary of Thm 2.6

Let $A \in M_{mn}$

Then $\exists! \, T \in \mathcal{L}(V, W)$ s.t. $[T]_{\beta^V}^{\beta^W} = A$

PF

Define T by $T(\beta_i^V) = \sum_{j=1}^m A_{ji} \beta_j^W$

By Thm 2.6 $\exists! \, T \in \mathcal{L}(V, W)$ and it satisfy $[T]_{\beta^V}^{\beta^W} = A$

Thm 2.7

Let V, W be vector spaces over a field F

Let $T, U \in \mathcal{L}(V, W)$

Then $aT + U : V \to W$ is linear

and $\mathcal{L}(V, W) = \{T \mid T : V \to W \text{ \& linear}\}$ is a vector space over F.

PF

Let $x, y \in V$ & $c \in F$

Then
$$
\begin{aligned}
(aT+U)(cx+y) &= aT(cx+y) + U(cx+y) \\
&= caT(x) + aT(y) + cU(x) + U(y) \\
&= c(aT(x) + U(x)) + aT(y) + U(y) \\
&= c(aT+U)(x) + (aT+U)(y)
\end{aligned}
$$

Thus $(aT+U)$ is Linear transformation

$T_0 \in \mathcal{L}(V, W)$ & $(T + T_0)(x) = T(x)$

Thus T_0 is $\overline{0}_{\mathcal{L}}$ for $\mathcal{L}(V, W)$

$(-T) \in \mathcal{L}(V, W)$ s.t. $(T + (-T))(x) = \overline{0}_W = T_0(x) = \overline{0}_{\mathcal{L}}$ ∎

Thm 2.8

Let V & W be finite dimensional vector spaces.

Let β^V & β^W be ordered basis

Let $T, U : V \to W$ be linear

Then $[T+U]_{\beta^V}^{\beta^W} = [T]_{\beta^V}^{\beta^W} + [U]_{\beta^V}^{\beta^W}$

and $[aT]_{\beta^V}^{\beta^W} = a[T]_{\beta^V}^{\beta^W}$

PF

Let $\dim(V) = n$ & $\dim(W) = m$

Then $\exists! A_{ji}$ & $\exists! B_{ji}$ s.t. $T(\beta_i^V) = \sum_{j=1}^{m} A_{ji} \beta_j^W$ and $U(\beta_i^V) = \sum_{j=1}^{m} B_{ji} \beta_j^W$

Then $(T+U)(\beta_i^V) = T(\beta_i^V) + U(\beta_i^V) = \sum_{j=1}^{m} (A_{ji} + B_{ji}) \beta_j^W$

$$\Rightarrow \left([T+U]_{\beta^V}^{\beta^W}\right)_{ji} = A_{ji} + B_{ji} = \left([T]_{\beta^V}^{\beta^W}\right)_{ji} + \left([U]_{\beta^V}^{\beta^W}\right)_{ji}$$

$$\Rightarrow [T+U]_{\beta^V}^{\beta^W} = [T]_{\beta^V}^{\beta^W} + [U]_{\beta^V}^{\beta^W}$$

and $(aT)(\beta_i^V) = aT(\beta_i^V) = a\sum_{j=1}^{m} A_{ji} \beta_j^W$

$$\Rightarrow \left([aT]_{\beta^V}^{\beta^W}\right)_{ji} = aA_{ji} = a\left([T]_{\beta^V}^{\beta^W}\right)_{ji}$$

$$\Rightarrow [aT]_{\beta^V}^{\beta^W} = a[T]_{\beta^V}^{\beta^W}$$ ∎

Thm 2.14

Let V & W be finite-dimensional vector spaces

Let β^V & β^W be ordered bases for V & W

Let $T \in \mathcal{L}(V, W)$

Then for each $u \in V$, $[T(u)]_{\beta^W} = [T]_{\beta^V}^{\beta^W} [u]_{\beta^V}$

Pf1)

Let $u = \sum_1^n a_i \beta_i^V$

Let $w = T(u) = \sum_1^m b_j \beta_j^W$

Let $T(\beta_i^V) = \sum_1^m A_{ji} \beta_j^W$

Then $T(u) = T(\sum_1^n a_i \beta_i^V) = \sum_1^n a_i T(\beta_i^V)$

$$= \sum_{i=1}^n a_i \left(\sum_{j=1}^m A_{ji} \beta_j^W \right) = \sum_{j=1}^m \left(\sum_{i=1}^n a_i A_{ji} \right) \beta_j^W = \sum_1^m b_j \beta_j^W$$

$$\therefore b_j = \sum_{i=1}^n A_{ji} a_i$$

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \Rightarrow [T(u)]_{\beta^W} = [T]_{\beta^V}^{\beta^W} [u]_{\beta^V}$$

Pf2)

Let $u \in V$

Let $f \in \mathcal{L}(F, V)$ by $f(a) = au$ and $g \in \mathcal{L}(F, W)$ by $g(a) = aT(u)$

Let $\beta^F = \{1\}$ be the standard ordered basis for F

Then $g = Tf$

$$[T(u)]_{\beta^W} = [g(1)]_{\beta^W} = [g]_{\beta^F}^{\beta^W} = [Tf]_{\beta^F}^{\beta^W} = [T]_{\beta^V}^{\beta^W} [f]_{\beta^F}^{\beta^V} = [T]_{\beta^V}^{\beta^W} [f(1)]_{\beta^V} = [T]_{\beta^V}^{\beta^W} [u]_{\beta^V}$$

Thm 2.15

Let $A \& B \in M_{m\times n}(F)$

Let β^n & β^m is standard ordered bases for F^n & F^m

Then $L_A : F^n \to F^m$ is linear

(a) $[L_A]_{\beta^n}^{\beta^m} = A$

(b) $L_A = L_B$ if and only if $A=B$

(c) $L_{cA+B} = c \cdot L_A + L_B$

(d) If $T: F^n \to F^m$ is linear, $\exists! \, C \in M_{m\times n}$ s.t. $T = L_C$. In fact $C = [T]_{\beta^n}^{\beta^m}$

(e) If $E \in M_{n\times p}(F)$ then $L_{AE} = L_A L_E$

(f) If $m=n$, then $L_I = I_{F^n}$

PF

Let $x, y \in F^n$ & $a \in F$

Then $L_A(ax+y) = A(ax+y) = aAx + Ay = aL_A(x) + L_A(y)$

Thus L_A is linear ■

PF(a)

Let $[L_A]_{\beta^n}^{\beta^m} = B$

By thm 2.14, $L_A(e_j) = [L_A]_{\beta^n}^{\beta^m} [e_j]_{\beta^n} = Be_j = Ae_j$

Therefore $[L_A]_{\beta^n}^{\beta^m} = A$

PF (b)

as $L_A = L_B$, $[L_A]_{\beta^n}^{\beta^m} = [L_B]_{\beta^n}^{\beta^m}$

By (a), $A=B$ ■

PF (d)

Let $C = [T]_{\beta^n}^{\beta^m}$ & $x \in F^n$

By thm 2.14 $[T(x)]_{\beta^m} = [T]_{\beta^n}^{\beta^m} [x]_{\beta^n} \Rightarrow T(x) = Cx = L_C(x)$

So $T = L_C$, $C = [T]_{\beta^n}^{\beta^m}$

general $T(x) = w \Rightarrow [w]_{\beta^W} = [T]_{\beta^V}^{\beta^W} [x]_{\beta^V}$

$F^n \to F^m$ $T(x) = w \Rightarrow w = [T]_{\beta^V}^{\beta^W} x$

PF(e)

$L_{AE}(e_j) = AEe_j = L_A(Ee_j) = L_A(L_E(e_j))$

By Thm 2.6 $L_{AE} = L_A L_E$

Thm 2.17

Let V & W be vector spaces.

Let T be an isomorphism from V to W.

Then $T^{-1}: W \to V$ is linear.

PF

Let $w_1, w_2 \in W$ & $c \in F$

Since T is onto and one to one, $\exists!\ v_1, v_2 \in V$ s.t. $T(v_1) = w_1$ & $T(v_2) = w_2$

$T^{-1}(cw_1 + w_2) = T^{-1}(cT(v_1) + T(v_2)) = T^{-1}(T(cv_1 + v_2)) = cv_1 + v_2$

$\Rightarrow T^{-1}(cw_1 + w_2) = cT^{-1}(w_1) + T^{-1}(w_2)$ ∎

Lemma of Thm 2.18

Let V & W be vector spaces.

Let T be an isomorphism

Then V is finite dimensional if and only if W is finite dimensional.

In this case, $\dim(V) = \dim(W)$

Pf)

Suppose that V is finite dimensional.

Since T is onto, $W = T(V) = R(T)$ $\Rightarrow \dim(W) = \dim(R(T))$

by Thm 2.2 $R(T) = \text{span}(T(\beta^V))$

Hence W is finite dimensional by Thm 1.9

Since T is one-to-one, $\text{nullity}(T) = 0$ by Thm 2.4

By Thm 2.3 $\text{range}(T) = \dim(R(T)) = \dim(V)$

Therefore $\dim(V) = \dim(W)$ ∎

Thm 2.18.

Let V & W be vector spaces.

Let β^V & β^W be ordered bases.

Let $T \in \mathcal{L}(V, W)$

Then T is invertible if and only if $[T]_{\beta^V}^{\beta^W}$ is invertible

Furthermore, $[T^{-1}]_{\beta^W}^{\beta^V} = ([T]_{\beta^V}^{\beta^W})^{-1}$

Pf

Suppose that T is invertible.

By the Lemma $\dim(V) = \dim(W) = n \Rightarrow [T]_{\beta^V}^{\beta^W} \in M_{n\times n}$

$I = [I_V]_{\beta^V} = [T^{-1}T]_{\beta^V} = [T^{-1}]_{\beta^W}^{\beta^V}[T]_{\beta^V}^{\beta^W}$, where I is identity matrix.

Thus $[T]_{\beta^V}^{\beta^W}$ is invertible $[T^{-1}]_{\beta^W}^{\beta^V} = \left([T]_{\beta^V}^{\beta^W}\right)^{-1}$

now suppose that $A = [T]_{\beta^V}^{\beta^W}$ is invertible.

Then $\exists B \in M_{n\times n}$ s.t. $AB = BA = I$

By Thm 2.6 $\exists!\ U \in \mathcal{L}(W, V)$ s.t. $[U]_{\beta^V}^{\beta^W} = B$

Then $[UT]_{\beta^V} = [U]_{\beta^W}^{\beta^V}[T]_{\beta^V}^{\beta^W} = BA = I = [I_V]_{\beta^V}$

Thus $UT = I_V$ and similarly $TU = I_W$ ∎

Corollary of Thm 2.18

Let V be finite dimensional vector space

Let $T \in \mathcal{L}(V)$

Then T is invertible if and only if $[T]_{\beta^V}$ is invertible.

Corollary2 of thm 2.18

Let $A \in M_{n\times n}$

Then A is invertible if and only if L_A is invertible.

Thm 2.19

Let V, W be finite dimensional vector spaces.

Then V and W are isomorphic if and only if $\dim(V) = \dim(W)$

PF

Suppose that V and W are isomorphic

Let T be an isomorphism from V to W.

By Lemma of Thm 2.18, $\dim(V) = \dim(W)$

now suppose that $\dim(V) = \dim(W)$

By Thm 2.6, $\exists! \, T \in \mathcal{L}(V, W)$ s.t $T(\beta_i^V) = \beta_i^W$

using Thm 2.2, $R(T) = \text{span}(T(\beta^V)) = \text{span}(\beta^W) = W \Rightarrow T$ is onto

From Thm 2.5, T is also one-to-one.

Thus T is an isomorphism from V to W

Therefore V and W are isomorphic ∎

Thm 2.20

Let V, W be finite dimensional vector spaces and $\dim(V) = n$ $\dim(W) = m$

Then a function $\Phi : \mathcal{L}(V, W) \to M_{mn}(F)$, defined by $\Phi(T) = [T]_{\beta^V}^{\beta^W}$ is an isomorphism.

PF

Let $T, U \in \mathcal{L}(V, W)$ and $a \in F$

Then $\Phi(aT + U) = [aT+U]_{\beta^V}^{\beta^W} = a[T]_{\beta^V}^{\beta^W} + [U]_{\beta^V}^{\beta^W} = a\Phi(T) + \Phi(U)$ By Thm 2.8.

Thus Φ is Linear.

Let $A \in M_{mn}$

By Corollary of Thm 2.6, $\exists! \, T \in \mathcal{L}(V, W)$ where $[T]_{\beta^V}^{\beta^W} = A$

and every Linear transformation have matrix representation.

It is shown that Φ is both one to one and onto.

Thus Φ is invertible.

Therefore Φ is an isomorphism. ∎

Corollary of Thm 2.20

Let V & W be finite dimensional vector spaces and $\dim(V) = n$ $\dim(W) = m$

Then $\mathcal{L}(V, W)$ is finite dimensional & $\dim(\mathcal{L}(V, W)) = mn$

PF

By Thm 2.20 $\mathcal{L}(V, W)$ and $M_{mn}(F)$ are isomorphic

By Thm 2.19 $\dim(\mathcal{L}(V, W)) = \dim(M_{mn}(F))$

Therefore $\dim(\mathcal{L}(V, W)) = mn$ ∎

Thm 2.21

Let V be a finite dimensional vector space and $\dim(V)=n$

Then ϕ_{β^V} is an isomorphism.

PF

Let $v_1, v_2 \in V$ & $a \in F$

Then $v_1 = \sum_{i=1}^{n} b_i \beta_i^V$, $v_2 = \sum_{i=1}^{n} C_i \beta_i^V$ $\Rightarrow$ $av_1 + v_2 = \sum_{i=1}^{n} (ab_i + C_i) \beta_i^V$

$[av_1+v_2]_{\beta^V} = a[v_1]_{\beta^V} + [v_2]_{\beta^V}$ $\Rightarrow$ $\phi_{\beta^V}(av_1+v_2) = a\phi_{\beta^V}(v_1) + \phi_{\beta^V}(v_2)$

Thus ϕ_{β^V} is Linear.

By thm 1.8 ϕ_{β^V} is both one to one and onto.

Thus ϕ_{β^V} is invertible.

Therefore ϕ_{β^V} is an isomorphism. ∎

Thm 2.22

Let V be a vector space.

Let β^{V_1} & β^{V_2} be two ordered bases for V

Let $Q = [I_V]_{\beta^{V_1}}^{\beta^{V_2}}$

Let $v \in V$

Then Q is invertible and

$[v]_{\beta^{V_2}} = Q[v]_{\beta^{V_1}}$

PF

Since I_V is invertible, By Thm 2.18 Q is invertible.

By Thm 2.14 $[v]_{\beta^{V_2}} = [I_V(v)]_{\beta^{V_2}} = [I_V]_{\beta^{V_1}}^{\beta^{V_2}} [v]_{\beta^{V_1}} = Q[v]_{\beta^{V_1}}$

Thm 2.23

Let V be a Vector space.

Let β^{V_1}, β^{V_2} be two ordered bases.

Let $T \in \mathcal{L}(V)$

Let $Q = [I_V]_{\beta^{V_1}}^{\beta^{V_2}}$

Then $[T]_{\beta^{V_1}} = Q^{-1}[T]_{\beta^{V_2}}Q$ and $[T]_{\beta^{V_1}}$ & $[T]_{\beta^{V_2}}$ are similar.

PF

$$Q[T]_{\beta^{V_1}} = [I_V]_{\beta^{V_1}}^{\beta^{V_2}}[T]_{\beta^{V_1}}^{\beta^{V_1}} = [I_V T]_{\beta^{V_1}}^{\beta^{V_2}} = [T I_V]_{\beta^{V_1}}^{\beta^{V_2}} = [T]_{\beta^{V_2}}^{\beta^{V_2}}[I_V]_{\beta^{V_1}}^{\beta^{V_2}} = [T]_{\beta^{V_2}}Q$$

$\therefore [T]_{\beta^{V_1}} = Q^{-1}[T]_{\beta^{V_2}}Q$ ∎

Corollary for Thm 2.23

Let $A \in M_{n \times n}(F)$

Let β^{F_1} be an standard ordered basis for F_n

Let β^{F_2} be an arbitrary ordered basis for F_n

Then $[L_A]_{\beta^{F_2}} = Q^{-1}AQ$

PF

$A = [L_A]_{\beta^{F_1}}$ & $Q = [I_V]_{\beta^{F_2}}^{\beta^{F_1}}$

Thm 3.3

Let V, W be finite dimensional vector spaces.

Let β^V, β^W be ordered bases

Let $T \in \mathcal{L}(V, W)$

Then $\operatorname{rank}(T) = \operatorname{rank}([T]_{\beta^V}^{\beta^W})$

PF)

Let $A = [T]_{\beta^V}^{\beta^W}$

Then $\operatorname{rank}([T]_{\beta^V}^{\beta^W}) = \operatorname{rank}(L_A) = \operatorname{rank}(T)$ by Prb 2.4.20

Thm 3.4

Let $A \in M_{mn}$

Let $P \in M_{mm}$, $Q \in M_{nn}$ and both are Invertible.

Then (a) $\text{rank}(AQ) = \text{rank}(A)$

(b) $\text{rank}(PA) = \text{rank}(A)$

(c) $\text{rank}(PAQ) = \text{rank}(A)$

PF (a)

$R(L_{AQ}) = R(L_A L_Q) = L_A L_Q(F^n) = L_A(F^n) = R(L_A)$ since L_Q is onto.

Thus $\text{rank}(AQ) = \text{rank}(A)$ ■

PF (b)

Since L_P is isomorphism,

$\dim(R(L_{PA})) = \dim(L_P L_A(F^n)) = \dim(L_A(F^n)) = \dim(R(L_A))$

by Prb 2.4.17

Thus $\text{rank}(PA) = \text{rank}(A)$ ■

PF (c)

$\dim(R(L_{PAQ})) = \dim(L_P L_A L_Q(F^n)) - \dim(L_A(F^n))$

by Thm 3.4 (a), (b)

Thus $\text{rank}(PAQ) = \text{rank}(A)$ ■