1 Introduction

This brief chapter introduces the subject of Solid Mechanics and the contents of this book

1.1 What is Solid Mechanics?

Solid mechanics is the study of the deformation and motion of solid materials under the action of forces. It is one of the fundamental applied engineering sciences, in the sense that it is used to describe, explain and predict many of the physical phenomena around us.

Here are some of the wide-ranging questions which solid mechanics tries to answer:

When will this cliff collapse?

1

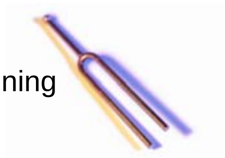
2

How does the heart contract and expand as it is pumped?

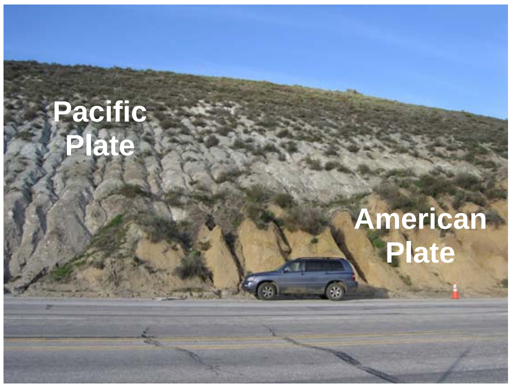
3

When will these gears wear out?

How long will a tuning fork vibrate for?



How will the San Andreas fault in California progress? How will the ground move during an earthquake?



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how do you build a bridge which will not collapse?

6

why does nature use the materials it does?

Solid mechanics is a vast subject. One reason for this is the wide range of materials which falls under its ambit: steel, wood, foam, plastic, foodstuffs, textiles, concrete, biological materials, and so on. Another reason is the wide range of applications in which these materials occur. For example, the hot metal being slowly forged during the manufacture of an aircraft component will behave very differently to the metal of an automobile which crashes into a wall at high speed on a cold day.

Here are some examples of Solid Mechanics of the cold, hot, slow and fast …

7 8

how did this Antarctic ice fracture?
what materials can withstand extreme heat?

how much will this glacier move in one year?
what damage will occur during a car crash?

Here are some examples of Solid Mechanics of the small, large, fragile and strong …

9

what affects the quality of paper? (shown are fibers 0.02mm thick)
how will a ship withstand wave slamming?

how strong is an eggshell and what prevents it from cracking?
how thick should a dam be to withstand the water pressure?

1.1.1 Aspects of Solid Mechanics

The theory of Solid Mechanics starts with the **rigid body**, an ideal material in which the distance between any two particles remains fixed, a good approximation in some applications. Rigid body mechanics is usually subdivided into

- **statics**, the mechanics of materials and structures at rest, for example of a cable-stayed bridge
- **dynamics**, the study of bodies which are changing speed, for example of an accelerating and decelerating elevator

Following on from statics and dynamics usually comes the topic of **Mechanics of Materials** (or **Strength of Materials**). This is the study of some elementary but very relevant deformable materials and structures, for example beams and pressure vessels. **Elasticity theory** is used, in which a material is assumed to undergo *small* deformations when loaded and, when unloaded, *returns to its original shape*. The theory well approximates the behaviour of most real solid materials at low loads, and the behaviour of the "engineering materials", for example steel and concrete, right up to fairly high loads.

More advanced theories of deformable solid materials include

- **plasticity theory**, which is used to model the behaviour of materials which undergo *permanent* deformations, which means pretty much anything loaded high enough
- **viscoelasticity theory**, which models well materials which exhibit many "fluid-like" properties, for example plastics, skin, wood and foam
- **viscoplasticity theory**, which is a combination of viscoelasticity and plasticity, and is good for materials like mud and gels.

Some other topics embraced by Solid Mechanics, are

- **rods, beams, shells and membranes**, the study of material components which can be approximated by various model geometries, such as "very thin"
- **vibrations of solids and structures**, where particles vibrate about some equilibrium position, giving rise to vibration and wave propagation
- **composite materials**, the study of components made up of more than one material, for example fibre-glass reinforced plastics
- **geomechanics**, the study of materials such as rock, soil and ice
- **contact mechanics**, the study of materials in contact, for example a set of gears
- **fracture and damage mechanics**, the mechanics of crack-growth and damage in materials
- **stability of structures**, the study of whether structures have the ability to return to some equilibrium position after being disturbed, or perhaps catastrophically fail
- **large deformation mechanics**, the study of materials such as rubber and muscle tissue, which stretch fairly easily
- **biomechanics**, the study of biological materials, such as bone and heart tissue
- **variational formulations and computational mechanics**, the study of the numerical (approximate) solution of the mathematical equations which arise in the various branches of solid mechanics, including the Finite Element Method
- **dynamical systems and chaos**, the study of mechanical systems which are highly sensitive to their initial position
- **experimental mechanics**, the design and analysis of experimental procedures for determining the behaviour of materials and structures
- **thermomechanics**, the analysis of materials using a formulation based on the principles of thermodynamics

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Images used:

1. http://www.allposters.com/-sp/Haute-Ville-on-Cliff-Edge-Bonifacio-South-Corsica-Corsica-France-Mediterranean-Europe-Posters_i8943529_.htm
2. http://www.natureworldnews.com/articles/1662/20130430/uk-begin-clinical-trials-gene-therapy-treat-heart-failure.htm
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13. http://edition.cnn.com/2008/US/07/06/nose.cone/index.html?iref=mpstoryview
14. http://rsna.kneadle.com/secure/Radiology/Modalities/CT/Clinical/MDCT-Peds-Skull-Frac.html

1.2 What is in this Book?

The aim of this book is to cover the essential concepts involved in solid mechanics, and the basic material models. It is primarily aimed at the Engineering or Science undergraduate student who has, perhaps, though not necessarily, completed some introductory courses on mechanics and strength of materials. Apart from giving a student a good grounding in the fundamentals, it should act as a stepping stone for further study in the field of Solid Mechanics, Continuum Mechanics, or any related field. The philosophy adopted is as follows:

- The mathematics is kept at a fairly low level; in particular, there are few differential equations, very little partial differentiation and there is no tensor mathematics
- The critical concepts – the ones which make what follows intelligible – are highlighted
- The physics involved, and not just the theory, is given attention
- A wide range of material models are considered, not just the standard Linear Elasticity

The outline of the book is as follows: Chapter 2 covers the essential material from a typical introductory course on mechanics; it serves as a brief review for those who have seen the material before, and serves as an introduction for those who are new to the subject. Chapters 3-8 cover much of the material typical of that included in a Strength of Materials or Mechanics of Materials course, and include the elementary beam theory and energy methods. The latter part of the book, Chapters 10-12 cover more advanced material models, namely viscoelasticity, plasticity and viscoplasticity.

2 Statics of Rigid Bodies

Statics is the study of materials at *rest*. The actions of all external forces acting on such materials are exactly counterbalanced and there is a zero net force effect on the material: such materials are said to be in a state of **static equilibrium**.

In much of this book (Chapters 6-8), **static elasticity** will be examined. This is the study of materials which, when loaded by external forces, deform by a small amount from some initial configuration, and which then take up the state of static equilibrium. An example might be that of floor boards deforming to take the weight of furniture. In this chapter, as an introduction to this subject, **rigid bodies** are considered. These are ideal materials which do not deform at all.

The chapter begins with the fundamental concepts and principles of mechanics – **Newton's laws of motion**. Then the mechanics of the **particle**, that is, of a very small amount of matter which is assumed to occupy a single point in space, is examined. Finally, an analysis is made of the mechanics of the rigid body.

The material in this chapter covers the essential material from a typical introductory course on statics. Although the concepts presented in this chapter serve mainly as an introduction for the later chapters, the ideas are very useful and important in themselves, for example in the design of machinery and in structural engineering.

2.1 The Fundamental Concepts and Principles of Mechanics

2.1.1 The Fundamental Concepts

The four fundamental concepts used in mechanics are **space**, **time**, **mass** and **force**[1]. It is not easy to define what these concepts are. Rather, one "knows" what they are, and they take on precise meaning when they appear in the principles and equations of mechanics discussed further below.

The concept of space is associated with the idea of the position of a point, which is described using coordinates (x, y, z) relative to an origin o as illustrated in Fig. 2.1.1.

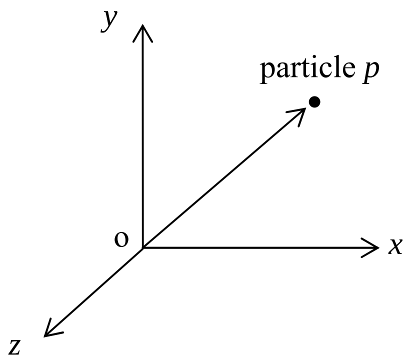

Figure 2.1.1: a particle in space

The time at which events occur must be recorded if a material is in motion. The concept of mass enters Newton's laws (see below) and in that way is used to characterize the relationship between the acceleration of a body and the forces acting on that body. Finally, a force is something that causes matter to accelerate; it represents the action of one body on another.

2.1.2 The Fundamental Principles

The fundamental laws of mechanics are Newton's three laws of motion. These are:

Newton's First Law:
if the resultant force acting on a particle is zero, the particle remains at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion)

By **resultant force**, one means the sum of the individual forces which act; the resultant is obtained by drawing the individual forces end-to-end, in what is known as the **vector**

[1] or at least the only ones needed outside more "advanced topics"

polygon law; this is illustrated in Fig. 2.1.2, in which three forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ act on a single particle, leading to a non-zero resultant force[2] $\mathbf{F}$.

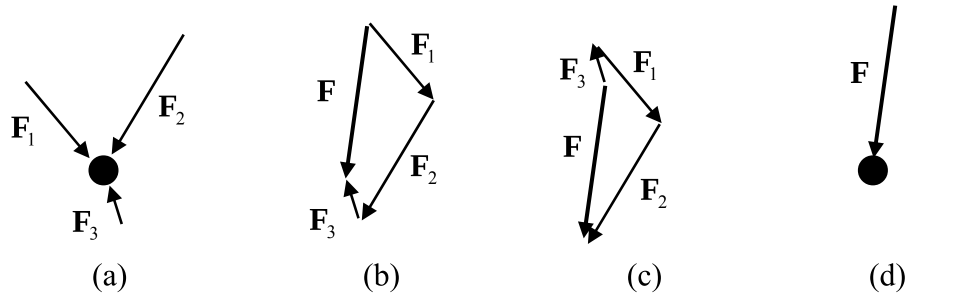

Figure 2.1.2: the resultant of a system of forces acting on a particle; (a) three forces acting on a particle, (b) construction of the resultant F, (c) an alternative construction, showing that the order in which the forces are drawn is immaterial, (d) the resultant force acting on the particle

Example (illustrating Newton's First Law)

In Fig. 2.1.3 is shown a floating boat. It can be assumed that there are two forces acting on the boat. The first is the boat's **weight** $\mathbf{F}_g$. There is also an upward buoyancy force $\mathbf{F}_b$ exerted *by* the water *on* the boat. If these two forces are equal and opposite, the resultant of these two forces will be zero, and therefore the boat will remain at rest (it will not move up or down).

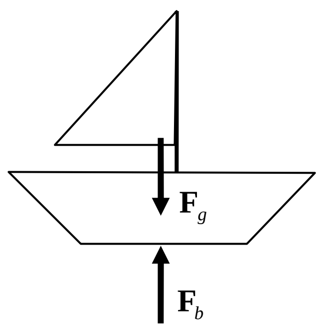

Figure 2.1.3: a zero resultant force acting on a boat

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The resultant force acting on the particle of Fig. 2.1.2 is non-zero, and in that case one applies Newton's second law:

[2] the construction of the resultant force can be regarded also as a principle of mechanics, in that it is not proved or derived, but is taken as "given" and is borne out by experiment

Newton's Second Law:
if the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant force and in the direction of this resultant force:

$$\mathbf{F} = m\mathbf{a} \tag{2.1.1}$$

where[3] $\mathbf{F}$ is the resultant force, $\mathbf{a}$ is the acceleration and m is the mass of the particle. The units of the force are the Newton (N), the units of acceleration are metres per second squared (m/s^2), and those of mass are the kilogram (kg); a force of 1 N gives a mass of 1 kg an acceleration of 1 m/s^2.

If the water were removed from beneath the boat of Fig. 2.1.3, a non-zero resultant force would act, and the boat would accelerate downward[4].

Newton's Third Law:
each force (of "action") has an equal and opposite force (of "reaction")

Again, considering the boat of Fig. 2.1.3, the water exerts an upward buoyancy force *on* the boat, and the boat exerts an equal and opposite force *on* the water. This is illustrated in Fig. 2.1.4.

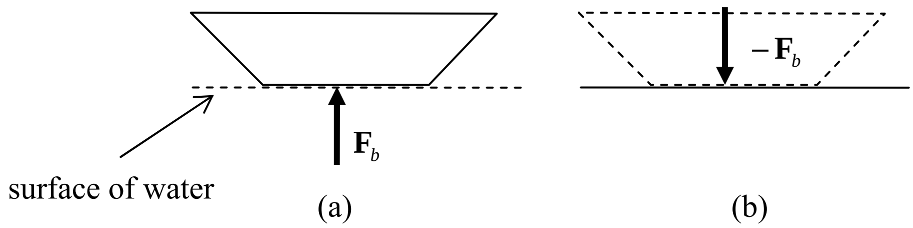

Figure 2.1.4: Newton's third law; (a) the water exerts a force on the boat, (b) the boat exerts an equal and opposite force on the water

Newton's laws are used in the analysis of the most basic problems and in the analysis of the most advanced, complex, problems. They appear in many guises and sometimes they appear hidden, but they are always there in a Mechanics problem.

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[3] **vector** quantities, that is, quantities which have both a magnitude and a direction associated with them, are represented by bold letters, like **F** here; scalars are represented by italics, like *m* here. The magnitude and direction of vectors are illustrated using arrows as in Fig. 2.1.2

[4] if we set **F** to be zero in Newton's Second Law, we get $\mathbf{a} = 0$, which seems to be saying the same thing as Newton's First Law, and in fact it appears to imply that Newton's First Law is redundant. For this reason, Newton's First Law is not actually used in analyzing problems (much); it is necessary only to deal with different frames of reference. For example, if you stand in an accelerating lift (your frame of reference) with glass walls, it appears to you that you are stationary and it is the "outside" (a different reference frame) which is accelerating, even though there is no "force" acting on the "outside", which appears to be a contradiction of Newton's Second Law. Newton's First Law discounts this option: it says that when the force is zero, the body remains at rest or at uniform velocity. Newton's First Law implies that Newton's Laws only apply to **Inertial Frames**, i.e. frames of reference in which a body remains at rest or uniform velocity unless acted upon by a force

2.2 The Statics of Particles

2.2.1 Equilibrium of a Particle

The statics of particles is the study of particles *at rest* under the action of forces. This situation is referred to as **equilibrium**, which is defined as follows:

> **Equilibrium of a Particle**
> A particle is in equilibrium when the resultant of all the forces acting on that particle is zero

In practical problems, one will want to introduce a coordinate system to describe the action of forces on a particle. It is important to note that a force exists independently of any coordinate system one might use to describe it. For example, consider the force **F** in Fig. 2.2.1. Using the vector polygon law, this force can be decomposed into combinations of any number of different individual forces; these individual forces are referred to as **components** of **F**. In particular, shown in Fig 2.2.1 are three cases in which **F** is decomposed into two rectangular (perpendicular) components, the components of **F** in "direction *x*" and in "direction *y*", $\mathbf{F}_x$ and $\mathbf{F}_y$.

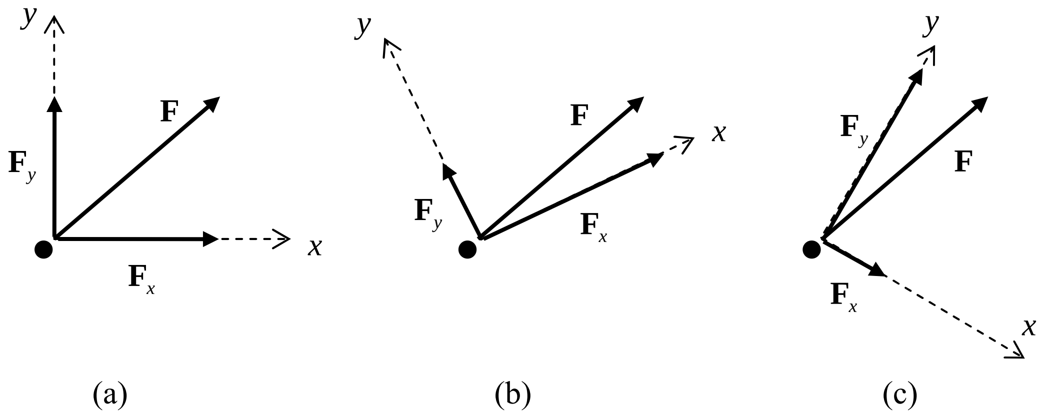

Figure 2.2.1: A force F decomposed into components $\mathbf{F}_x$ and $\mathbf{F}_y$ using three different coordinate systems

By resolving forces into rectangular components, one can obtain analytic solutions to problems, rather than relying on graphical solutions to problems, for example as done in Fig. 2.1.2. In order that the resultant force **F** on a body be zero, one must have that the resultant force in the *x* and *y* directions are zero individually[1], as illustrated in the following example.

[1] and in the *z* direction if one is considering a three dimensional problem

Example

Consider the particle in Fig. 2.2.2, subjected to forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$. The particle is in equilibrium and so by Newton's Laws the resultant force is zero, $\mathbf{F} = \mathbf{0}$. The forces are decomposed into horizontal and vertical components $\mathbf{F}_{1x}, \mathbf{F}_{2x}, \mathbf{F}_{3x}$ and $\mathbf{F}_{1y}, \mathbf{F}_{2y}, \mathbf{F}_{3y}$. The horizontal forces may be added together to get a single horizontal force $\mathbf{F}_x$, which must equal zero. This force $\mathbf{F}_x$ should be evaluated using the vector polygon law but, since the individual forces $\mathbf{F}_{1x}, \mathbf{F}_{2x}, \mathbf{F}_{3x}$ all lie along the same line, one need only add together the *magnitudes* of these vectors, which involves simply an addition of *scalars*: $F_{1x} + F_{2x} + F_{3x} = 0$. Similarly, one has $F_{1y} + F_{2y} + F_{3y} = 0$. These equations could be used to evaluate, for example, the force $\mathbf{F}_1$, if only $\mathbf{F}_2$ and $\mathbf{F}_3$ were known.

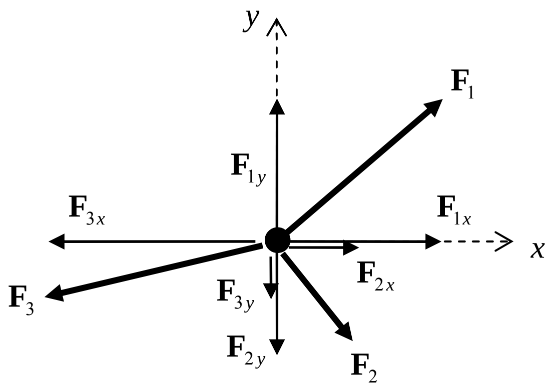

Figure 2.2.2: Calculating the resultant of three forces by decomposing them into horizontal and vertical components

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In general then, if a set of forces $\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_n$ act on a particle, the particle is in equilibrium if and only if

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0 \qquad \textbf{Equations of Equilibrium (particle)} \quad (2.2.1)$$

These are known as the **equations of equilibrium for a particle**. They are three equations and so can be used to solve problems involving three "unknowns", for example the three components of one of the forces. In two-dimensional problems (as in the next example), they are a set of two equations.

Example

Consider the system of two cables attached to a wall shown in Fig. 2.2.3a. The cables meet at C, and this point is subjected to the two forces shown. Assume now that there are forces arising in the cables AC and BC, indicated by the arrows in Fig. 2.2.3b[2]. One can now draw a **free body diagram** of the particle C. The free body diagram concept is incredibly important and it is used in the most simple and in the most complex of problems, and will be used again and again in what follows. A free body diagram isolates a body (in this case the particle C) from its surroundings, and one considers all the forces, and *only* those forces, acting *on* that body, as shown in Fig 2.2.3b.

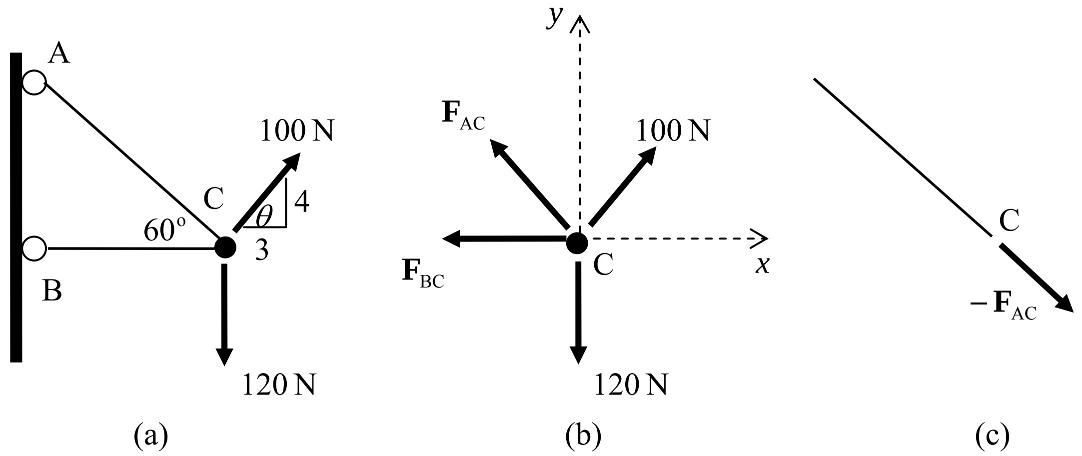


Figure 2.2.3: Calculating the tension in cables; (a) the cable system, (b) a free-body diagram of particle C, (c) cable AC in equilibrium

The equations of equilibrium for particle C are

$$\begin{aligned}\sum F_x &= -F_{\mathrm{BC}} - F_{\mathrm{AC}}\cos 60 + 100\cos\theta = 0, \\ \sum F_y &= F_{\mathrm{AC}}\cos 30 + 100\sin\theta - 120 = 0\end{aligned}$$

leading to $F_{\mathrm{AC}} = 46.2\,\mathrm{N}$, $F_{\mathrm{BC}} = 36.9\,\mathrm{N}$.

The results are positive numbers; if the answer was negative, the arrow we assumed to be going towards C would in fact have been going the other way, away from C. We guessed right.

The cable exerts a tension/pulling force on particle C and so, from Newton's third law, C must exert an equal and opposite force on the cable, as illustrated in Fig. 2.2.3c.

■

2.2.2 Rough and Smooth Surfaces

Fig 2.2.4a shows a particle in equilibrium, sitting on a rough surface and subjected to a force **F**. Such a surface is one where frictional forces are large enough to prevent tangential motion. The free body diagram of the particle is shown in Fig. 2.2.4b. The friction reaction force is $\mathbf{R}_f$ (preventing movement along the surface) and the normal

[2] it does not matter which way you draw the arrows (away from C or towards C); if you do the calculation correctly, you will still get the same, correct, answer

reaction force is **N** (preventing movement through the surface) and these lead to the resultant reaction force **R** which, by Newton's Laws, must balance **F**.

When a particle meets a smooth surface, there is no resistance to tangential movement. The particle is subjected to only a normal reaction force, and thus a particle in equilibrium can only sustain a purely normal force. This is illustrated in Fig. 2.2.4c.

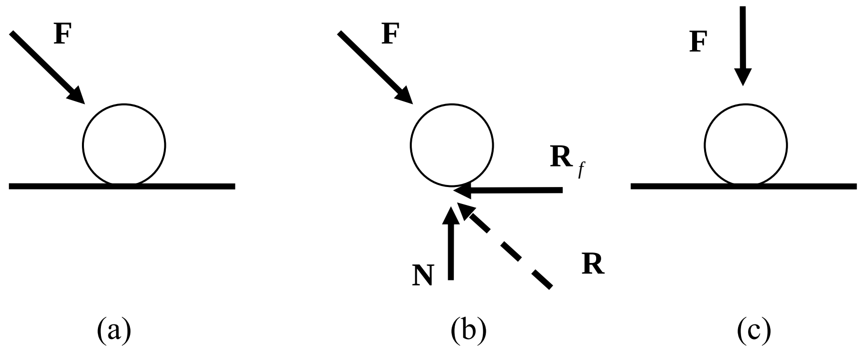
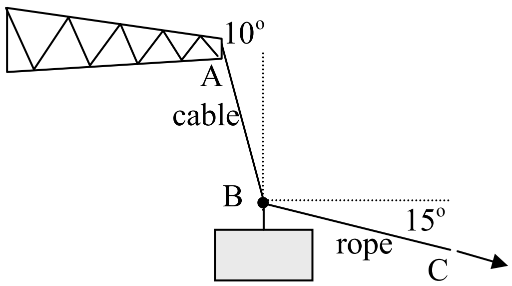


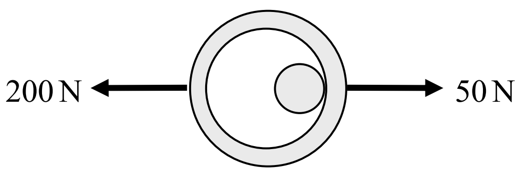
Figure 2.2.4: a particle sitting on a surface; (a) a rough surface, (b) a free-body diagram of the particle in (a), (c) a smooth surface

2.2.3 Problems

1. A 3000kg crate is being unloaded from a ship. A rope BC is pulled to position the crate correctly on the wharf. Use the Equations of Equilibrium to evaluate the tensions in the crane-cable AB and rope. [Hint: create a free body for particle B.]

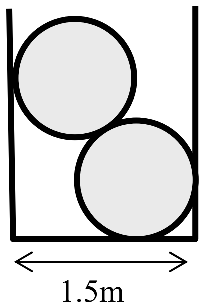


2. A metal ring sits over a stationary post, as shown in the plan view below. Two forces act on the ring, in opposite directions. Draw a free body diagram of the ring including the reaction force of the post *on* the ring. Evaluate this reaction force. Draw a free body diagram of the post and show also the forces acting on it.



3. Two cylindrical barrels of radius 500mm are placed inside a container, a cross section of which is shown below. The mass of each barrel is 10kg. All surfaces are

smooth. Draw free body diagrams of each barrel, including the reaction forces exerted by the container walls *on* the barrels, the weight of each barrel, which can be assumed to act through the barrel centres, and the reaction forces of barrel on barrel. Apply the Equations of Equilibrium to each barrel. Evaluate all forces. What forces act on the container walls?



2.3 The Statics of Rigid Bodies

A material body can be considered to consist of a very large number of particles. A rigid body is one which does not deform, in other words the distance between the individual particles making up the rigid body remains unchanged under the action of external forces.

A new aspect of mechanics to be considered here is that a rigid body under the action of a force has a tendency to *rotate* about some axis. Thus, in order that a body be at rest, one not only needs to ensure that the resultant force is zero, but one must now also ensure that the forces acting on a body do not tend to make it rotate. This issue is addressed in what follows.

2.3.1 Moments, Couples and Equivalent Forces

When you swing a door on its hinges, it will move more easily if (i) you push hard, i.e. if the force is large, and (ii) if you push furthest from the hinges, near the edge of the door. It makes sense therefore to measure the rotational effect of a force on an object as follows:

The tendency of a force to make a rigid body rotate is measured by the **moment** of that force about an axis. The moment of a force **F** about an axis through a point o is defined as the product of the magnitude of **F** times the perpendicular distance *d* from the **line of action** of **F** and the axis o. This is illustrated in Fig. 2.3.1.

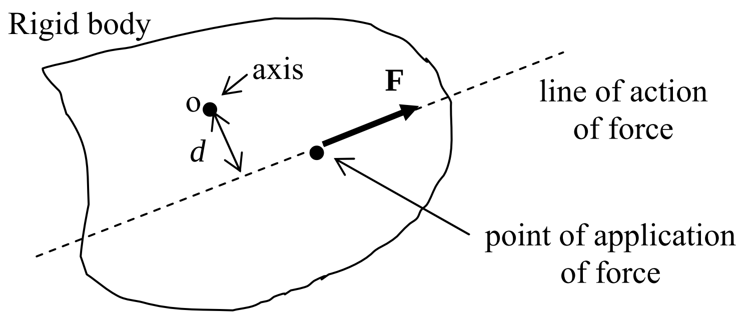


Figure 2.3.1: The moment of a force F about an axis o (the axis goes "into" the page)

The moment M_o of a force **F** can be written as

$$M_0 = Fd \tag{2.3.1}$$

Not only must the axis be specified (by the subscript *o*) when evaluating a moment, but the sense of that moment must be given; the convention that a tendency to rotate *counterclockwise* is taken to be a *positive* moment will be used here. Thus the moment in Fig. 2.3.1 is positive. The units of moment are the Newton metre (Nm).

Note that when the line of action of a force goes through the axis, the moment is zero.

It should be emphasized that there is not actually a physical axis, such as a rod, at the point *o* of Fig. 2.3.1; in this discussion, it is *imagined* that an axis is there.

Two forces of equal magnitude and acting along the same line of action have not only the same components F_x, F_y, but have equal moments about any axis. They are called **equivalent forces** since they have the same effect on a rigid body. This is illustrated in Fig. 2.3.2.

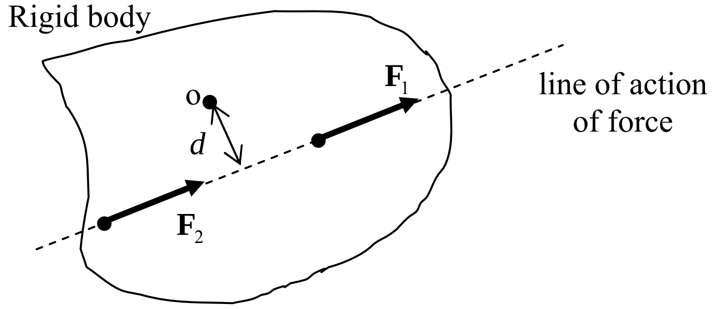


Figure 2.3.2: Two equivalent forces

Consider next the case of two forces of equal magnitude, parallel lines of action separated by distance *d*, and opposite sense. Any two such forces are said to form a **couple**. The only motion that a couple can impart is a rotation; unlike the forces of Fig. 2.3.2, the couple has no tendency to translate a rigid body. The moment of the couple of Fig. 2.3.3 about o is

$$M_o = Fd_2 - Fd_1 = Fd \tag{2.3.2}$$

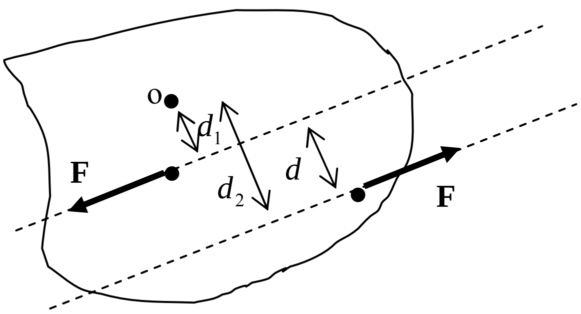


Figure 2.3.3: A couple

As with the moment, the sign convention which will be followed in what follows is that a couple is positive when it acts in a counterclockwise sense, as in Fig. 2.3.3.

It is straight forward to show the following three important properties of couples:

(a) the moment of Fig. 2.3.3 is also Fd about *any* axis in the rigid body, and so can be represented by *M*, without the subscript. In other words, this moment of the couple is independent of the choice of axis. {see ▲Problem 1}

(b) any two different couples having the same moment *M* are equivalent, in the sense that they tend to rotate the body in precisely the same way; it does not matter that the forces forming these couples might have different magnitudes, act in different directions and have different distances between them.

(c) any two couples may be replaced by a single couple of moment equal to the algebraic sum of the moments of the individual couples.

Example

Consider the two couples shown in Fig. 2.3.4a. These couples can conveniently be represented schematically by semi-circular arrows, as shown in Fig. 2.3.4b. They can also be denoted by the letter *M*, the magnitude of their moment, since the magnitude of the forces and their separation is unimportant, only their product. In this example, if the body is in static equilibrium, the couples must be equal and opposite, $M_2 = -M_1$, i.e. the sum of the moments is zero and the net effect is to impart zero rotation on the body.

Note that the curved arrow for M_2 has been drawn counterclockwise, even though it is negative. It could have been illustrated as in Fig. 2.3.4c, but the version of 2.3.4b is preferable as it is more consistent and reduces the likelihood of making errors when solving problems (see later). In other words, if your sign convention is counterclockwise positive, draw everything counterclockwise; if your sign convention is clockwise positive, draw everything clockwise.

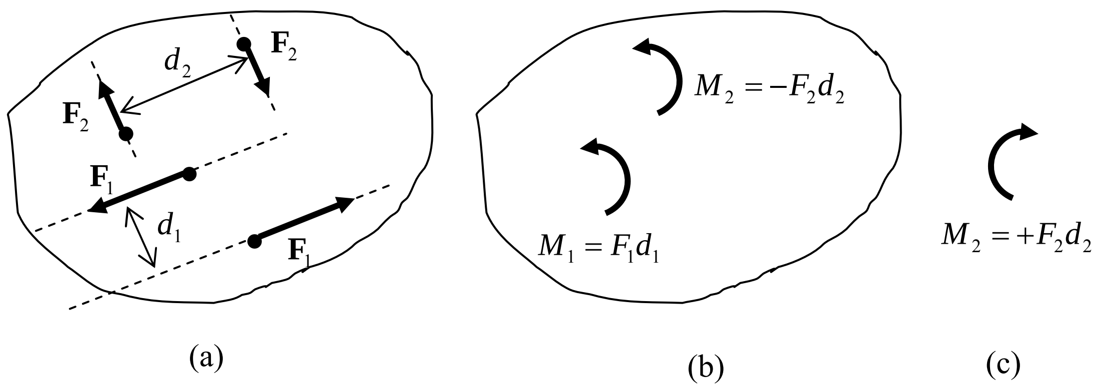


Figure 2.3.4: Two couples acting on a rigid body

■

A final point to be made regarding couples is the following: any force is equivalent to (i) a force acting at any (other) point and (ii) a couple. This is illustrated in Fig. 2.3.5.

Referring to Fig. 2.3.5, a force **F** acts at position A. This force tends to translate the rigid body along its line of action and also to rotate it about any chosen axis. The system of forces in Fig. 2.3.5b are equivalent to those in Fig. 2.3.5a: a set of equal and opposite forces have simply been added at position B. Now the force at A and one of the forces at B form a couple, of moment *M* say. As in the previous example, the couple can conveniently be represented by a curved arrow, and the letter *M*. For illustrative purposes, the curved arrow is usually grouped with the force **F** at B, as shown in Fig. 2.3.5c. However, note that the curved arrow representing the moment of a couple, which can be placed anywhere and have the same effect, *is not associated with any particular point in the rigid body*.

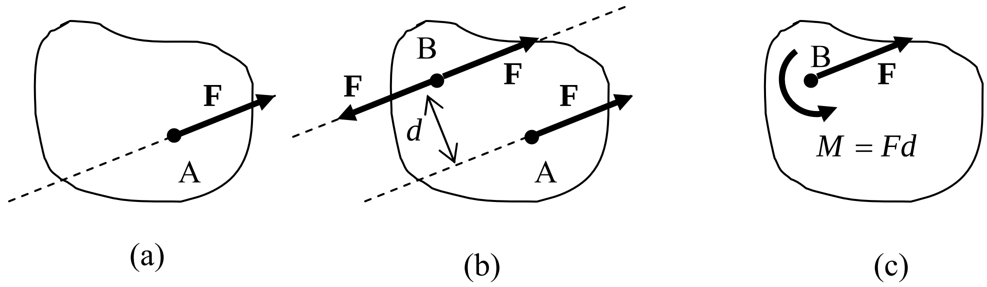


Figure 2.3.5: Equivalents force/moment systems; (a) a force F, (b) an equivalent system to (a), (c) an equivalent system involving a force and a couple *M*

Note that if the force at A was moved to a position other than B, the moment M of Fig. 2.3.5c would be different.

Example

Consider the spanner and bolt system shown in Fig. 2.3.6. A downward force of 200N is applied at the point shown. This force can be replaced by a force acting somewhere else, together with a moment. For the case of the force moved to the bolt-centre, the moment has the magnitude shown in Fig. 2.3.6b.

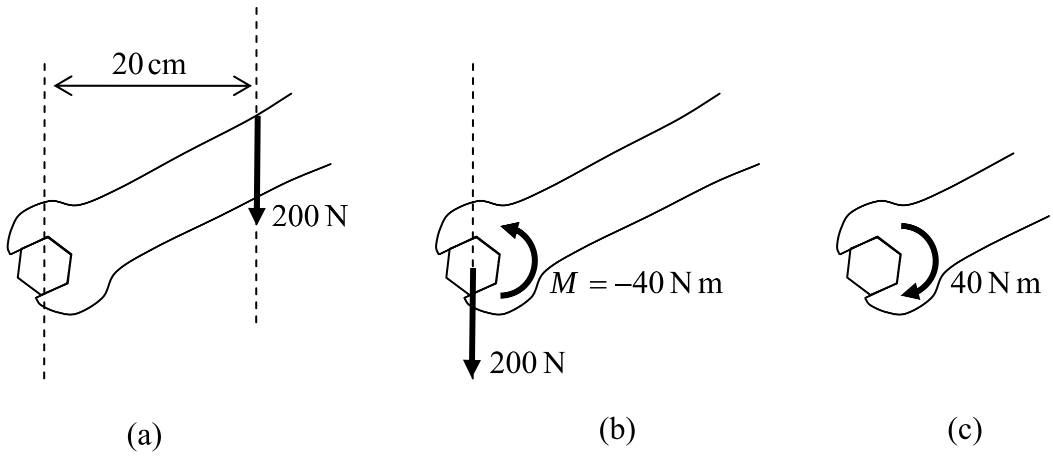


Figure 2.3.6: Equivalent force and force/moment acting on a spanner and bolt system

As mentioned, it is best to maintain consistency and draw the semi-circle representing the moment counterclockwise (positive) and given a value of -40 as in Fig. 2.3.6b; rather than as in Fig. 2.3.6c.

■

Example

Consider the plate subjected to the four external loads shown in Fig. 2.3.7a. An equivalent force-couple system **F**-M, with the force acting at the centre of the plate, can be calculated through

$$\sum F_x = 200\,\text{N}, \quad \sum F_y = 100\,\text{N}$$

$$\sum \text{M}_\text{o} = -(100)(100) - (50/\sqrt{2})(100) - (50/\sqrt{2})(100) + (200)(50) = -7071.07\,\text{Nmm}$$

and is shown in Fig. 2.3.7b. A **resultant force R** can also be derived, that is, an equivalent force positioned so that a couple is not necessary, as shown in Fig. 2.3.7.c.

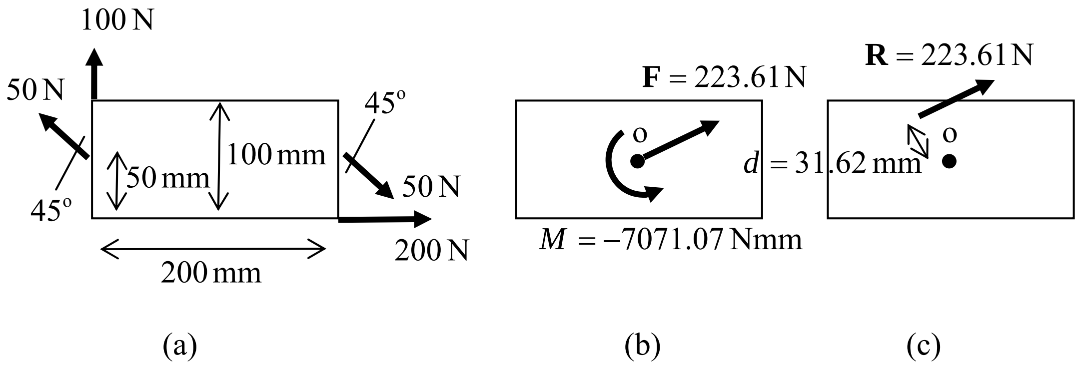


Figure 2.3.7: Forces acting on a plate; (a) individual forces, (b) an equivalent force-couple system at the plate-centre, (c) the resultant force

The force systems in the three figures are equivalent in the sense that they tend to impart (a) the same translation in the *x* direction, (b) the same translation in the *y* direction, and (c) the same rotation about *any* given point in the plate. For example, the moment about the upper left corner is

Fig 2.3.7a: $-(100)(0) - (50/\sqrt{2})(50) - (50/\sqrt{2})(150) + (200)(100)$
Fig 2.3.7b: $+(223.61)(89.44) - 7071$
Fig 2.3.7c: $+(223.61)(57.82)$

all leading to $M = 12928.93\,\text{Nmm}$ about that point.

■

2.3.2 Equilibrium of Rigid Bodies

The concept of equilibrium encountered earlier in the context of particles can now be generalized to the case of the rigid body:

Equilibrium of a Rigid Body
A rigid body is in equilibrium when the external forces acting on it form a system of forces equivalent to zero

The necessary and sufficient conditions that a (two dimensional) rigid body is in equilibrium are then

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_o = 0 \quad \textbf{Equilibrium Equations (2D Rigid Body)} \quad (2.3.3)$$

that is, there is no resultant force and no resultant moment. Note that the $x-y$ axes and the axis of rotation o can be chosen completely arbitrarily: if the resultant force is zero, and the resultant moment about *one* axis is zero, then the resultant moment about *any* other axis in the body will be zero also.

2.3.3 Joints and Connections

Components in machinery, buildings etc., connect with each other and are supported in a number of different ways. In order to solve for the forces acting in such assemblies, one must be able to analyse the forces acting at such connections/supports.

One of the most commonly occurring supports can be idealised as a **roller support**, Fig. 2.3.8a. Here, the contacting surfaces are smooth and the roller offers only a normal reaction force (see §2.2.2). This reaction force is labelled $\mathbf{R}_y$, according to the conventional $x-y$ coordinate system shown. This is shown in the free-body diagram of the component.

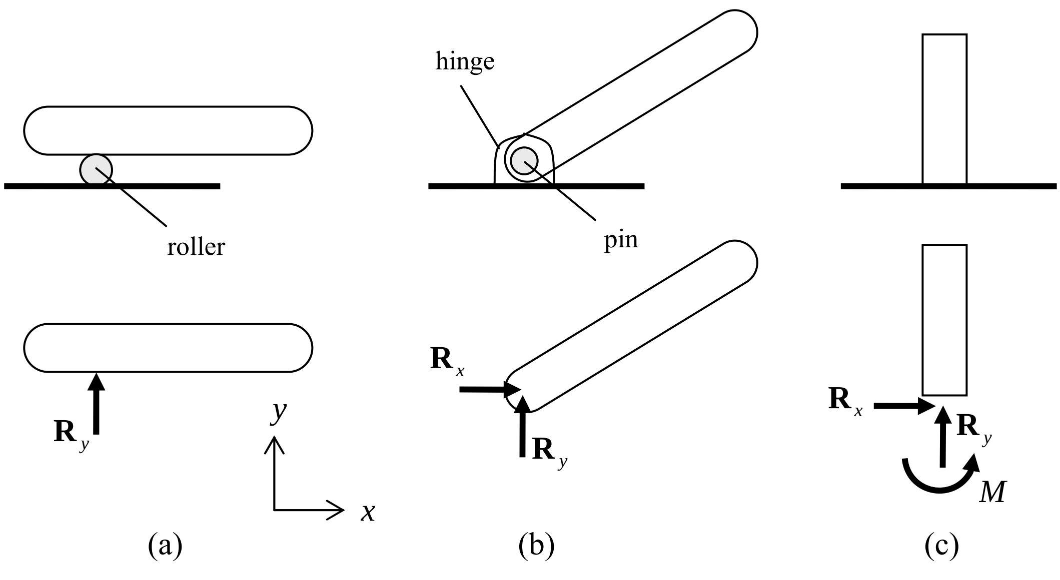


Figure 2.3.8: Supports and connections; (a) roller support, (b) pin joint, (c) clamped

Another commonly occurring connection is the **pin joint**, Fig. 2.3.8b. Here, the component is connected to a fixed hinge by a pin (going "into the page"). The component is thus constrained to move in one plane, and the joint does not provide resistance to this turning movement. The underlying support transmits a reaction force through the hinge pin to the component, which can have both normal ($\mathbf{R}_y$) and tangential ($\mathbf{R}_x$) components.

Finally, in Fig. 2.3.8c is shown a **fixed (clamped) joint**. Here the component is welded or glued and cannot move at the base. It is said to be **cantilevered**. The support in this case reacts with normal and tangential forces, but also with a couple of moment *M*, which resists any bending/turning at the base.

Example

For example, consider such a component loaded with a force **F** a distance *L* from the base, as shown in Fig. 2.3.9a. A free-body diagram of the component is shown in Fig. 2.3.9b. The known force **F** acts on the body and so do two unknown forces $\mathbf{R}_x$, $\mathbf{R}_y$, and a couple of moment *M*. The unknown forces and moment will be called **reactions** henceforth. If the component is static, the equilibrium equations 2.3.3 apply; one has, taking moments about the base of the component,

$$\sum F_x = F + R_x = 0, \quad \sum F_y = R_y = 0, \quad \sum M_o = -FL + M = 0$$

and so

$$R_x = -F, \quad R_y = 0, \quad M = FL$$

The moment is positive and so acts in the direction shown in the Figure.

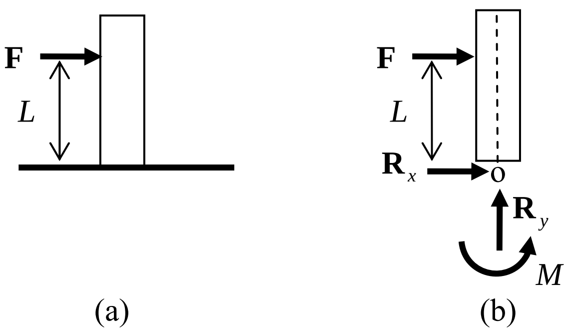


Figure 2.3.9: A loaded cantilevered component; (a) loaded component, (b) free body diagram of the component

The reaction moment of Fig. 2.3.9b can be experienced as follows: take a ruler and hold it firmly at one end, upright in your right hand. Simulate the applied force now by pushing against the ruler with a finger of your left hand. You will feel that, to maintain the ruler "vertical" at the base, you need to apply a twist with your right hand, in the direction of the moment shown in Fig. 2.3.9b.

Note that, when solving this problem, moments were taken about the base. As mentioned already, one can take the moment about *any* point in the column. For example, taking the moment about the point where the force **F** is applied, one has

$$\sum M_{\mathrm{F}} = R_x L + M = 0$$

This of course leads to the same result as before, but the final calculation of the forces is now slightly more complicated; in general, it is easier if the axis is chosen to coincide with the point where the reaction forces act – this is because the reaction forces do not then appear in the moment equation: $\sum M_{\mathrm{o}} = -FL + M = 0$.

■

For ease of discussion, from now on, “couples” such as that encountered in Fig. 2.3.9 will simply be called “moments”.

All the elements are now in place to tackle fairly complex static rigid body problems.

Example

Consider the plate subjected to the three external loads shown in Fig. 2.3.10a. The plate is supported by a roller at A and a pin-joint at B. The weight of the plate is assumed to be small relative to the applied loads and is neglected. A free body diagram of the plate is shown in Fig 2.3.10b. This shows all the forces acting *on* the plate. Reactions act at A and B: these forces represent the action of the base *on* the plate, preventing it from moving downward and horizontally. The equilibrium equations can be used to find the reactions:

$$\sum F_x = F_{xB} = 0 \quad \rightarrow \quad F_{xB} = 0$$
$$\sum F_y = +F_{yA} - 150 - 100 + 50 + F_{yB} = 0 \quad \rightarrow \quad F_{yA} + F_{yB} = 200\,\mathrm{N}$$
$$\sum M_{\mathrm{A}} = -(150)(50) - (100)(120) + (50)(200) + F_{yB}(200) = 0 \;\rightarrow\; F_{yB} = 47.5\,\mathrm{N},$$

$$\rightarrow \mathrm{F}_{yA} = 152.5\,\mathrm{N}$$

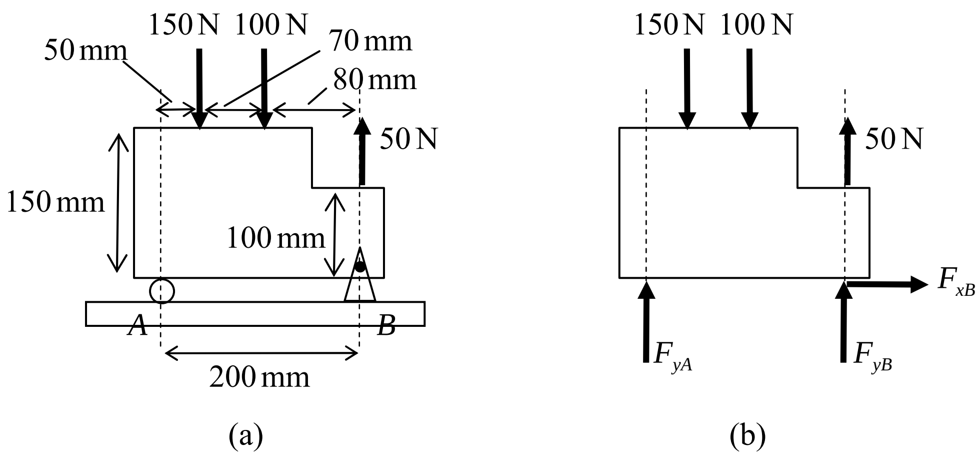


Figure 2.3.10: Equilibrium of a plate; (a) forces acting on the plate, (b) free-body diagram of the plate

The resultant moment was calculated by taking the moment about point A. As mentioned in relation to the previous example, one could have taken the moment about any other point in the plate. The "most convenient" point about which to take moments in this example would be point A or B, since in that case only one of the reaction forces will appear in the moment equilibrium equation.

■

In the above example there were three unknown reactions and three equilibrium equations with which to find them. If the roller was replaced with a pin, there would be four unknown reactions, and now there would not be enough equations with which to find the reactions. When this situation arises, the system is called **statically indeterminate**. To find the unknown reactions, one must relax the assumption of rigidity, and take into account the fact that all materials deform. By calculating deformations within the plate, the reactions can be evaluated. The deformation of materials is studied in the following chapters.

To end this Chapter, note the following:

(i) the equilibrium equations 2.3.3 result from Newton's laws, and are thus as valid for a body of water as they are for a body of hard steel; the external forces acting on a body of still water form a system of forces equivalent to zero.

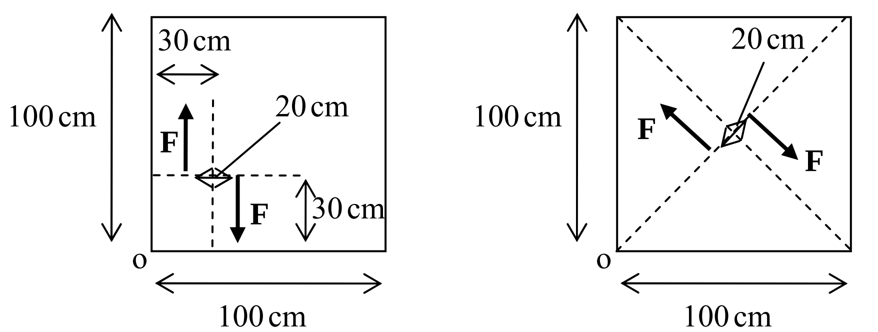
(ii) as mentioned already, Newton's laws apply not only to a complete body or structure, but to *any portion* of a body. The external forces acting *on* any free-body portion of static material form a system of forces equivalent to zero.

(iii) there is no such thing as a rigid body. Metals and other engineering materials can be considered to be "nearly rigid" as they do not deform by much under even fairly large loads. The analysis carried out in this Chapter is particularly relevant to these materials and in answering questions like: what forces act in the steel members of a suspension bridge under the load of self-weight and traffic? (which is just a more complicated version of the problem of Fig. 2.2.3 or Problem 3 below).

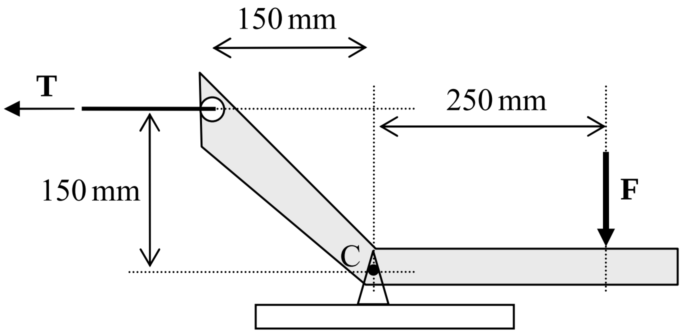
(iv) if the loads on the plate of Fig. 2.3.10a are too large, the plate will "break". The analysis carried out in this Chapter cannot answer where it will break or when it will break. The more sophisticated analysis carried out in the following Chapters is necessary to deal with this and many other questions of material response.

2.3.4 Problems

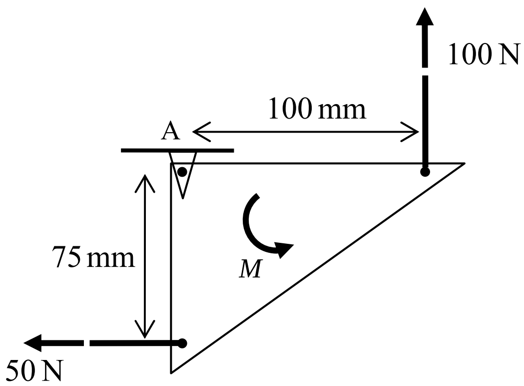
1. A plate is subjected to a couple Fd, with $d = 20\text{cm}$, as shown below left. Verify that the couple can be moved to the position shown below right, and the effect on the plate is the same, by showing that the moment about point o in both cases is $M = -20F$.



2. What force **F** must be applied to the following static component such that the tension in the cable, **T**, is 1kN? What are the reactions at the pin support C?



3. A machine part is hinged at A and subjected to two forces through cables as shown. What couple *M* needs to be applied to the machine part for equilibrium to be maintained? Where can this couple be applied?



3 Stress

Forces acting at the surfaces of components were considered in the previous chapter. The task now is to examine forces arising *inside* materials, **internal forces**. Internal forces are described using the idea of **stress**. There is a lot more to stress than the notion of "force over area", as will become clear in this chapter. First, the idea of surface (contact) stress distributions will be examined, together with their relationship to resultant forces and moments. Then internal stress and traction will be discussed. The means by which internal forces are described is through the **stress components**, for example σ_{zx}, σ_{yy}, and this "language" of sigmas and subscripts needs to be mastered in order to model sensibly the internal forces in real materials. **Stress analysis** involves representing the actual internal forces in a real physical component mathematically. Some of the limitations of this are discussed in §3.3.2.

Newton's laws are used to derive the **stress transformation equations**, and these are then used to derive expressions for the **principal stresses**, **stress invariants**, **principal directions** and **maximum shear stresses** acting at a material particle. The practical case of two dimensional **plane stress** is discussed.

3.1 Surface and Contact Stress

The concept of the force is fundamental to mechanics and many important problems can be cast in terms of forces only, for example the problems considered in Chapter 2. However, more sophisticated problems require that the action of forces be described in terms of *stress*, that is, force divided by area. For example, if one hangs an object from a rope, it is not the weight of the object which determines whether the rope will break, but the weight divided by the cross-sectional area of the rope, a fact noted by Galileo in 1638.

3.1.1 Stress Distributions

As an introduction to the idea of stress, consider the situation shown in Fig. 3.1.1a: a block of mass m and cross sectional area A sits on a bench. Following the methodology of Chapter 2, an analysis of a free-body of the block shows that a force equal to the weight mg acts upward on the block, Fig. 3.1.1b. Allowing for more detail now, this force will actually be distributed over the surface of the block, as indicated in Fig. 3.1.1c. Defining the stress to be force divided by area, the stress acting on the block is

$$\sigma = \frac{mg}{A} \tag{3.1.1}$$

The unit of stress is the Pascal (Pa): 1Pa is equivalent to a force of 1 Newton acting over an area of 1 metre squared. Typical units used in engineering applications are the kilopascal, kPa (10^3 Pa), the megapascal, MPa (10^6 Pa) and the gigapascal, GPa (10^9 Pa).

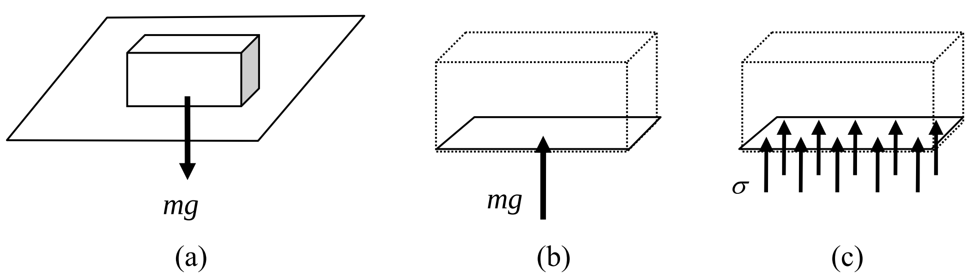


Figure 3.1.1: a block resting on a bench; (a) weight of the block, (b) reaction of the bench on the block, (c) stress distribution acting on the block

The stress distribution of Fig. 3.1.1c acts on the block. By Newton's third law, an equal and opposite stress distribution is exerted by the block on the bench; one says that the weight force of the block is *transmitted* to the underlying bench.

The stress distribution of Fig. 3.1.1 is **uniform**, i.e. constant everywhere over the surface. In more complex and interesting situations in which materials contact, one is more likely to obtain a *non-uniform* distribution of stress. For example, consider the case of a metal ball being pushed into a similarly stiff object by a force F, as illustrated in Fig. 3.1.2.[1] Again, an equal force F acts on the underside of the ball, Fig. 3.1.2b. As with the block, the force will actually be distributed over a **contact region**. It will be shown in Part II that the ball (and the large object) will deform and a circular contact region will arise where the ball and object meet[2], and that the stress is largest at the centre of the contact surface, dying away to zero at the edges of contact, Fig. 3.1.2c ($\sigma_1 > \sigma_2$ in Fig. 3.1.2c). In this case, we can consider a small area of the contact region dA, Fig. 3.1.2d; the force on this region is σdA. The total force is

$$F = \int_A dF = \int_A \sigma dA \qquad (3.1.2)$$

The stress varies from point to point over the surface but the sum (or integral) of the stresses (times areas) equals the total force applied to the ball.

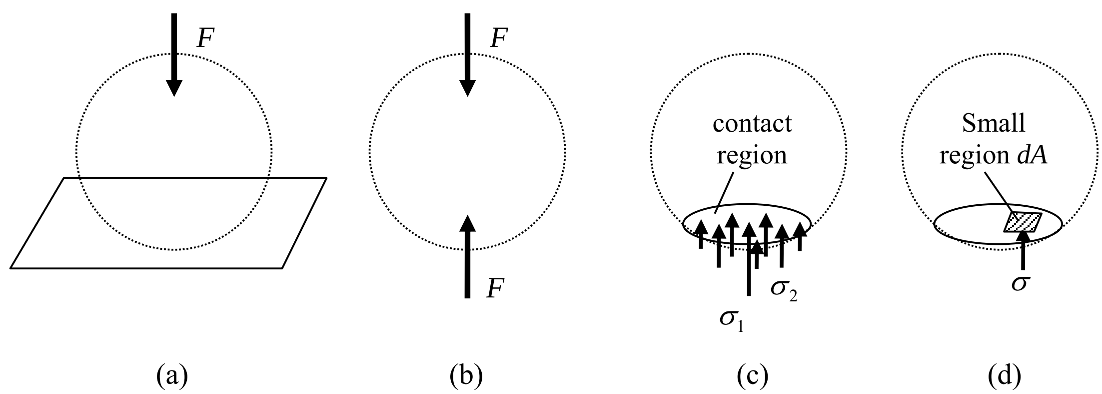


Figure 3.1.2: a ball being forced into a large object, (a) force applied to ball, (b) reaction of object on ball, (c) a non-uniform stress distribution over the contacting surface, (d) the stress acting on a small (infinitesimal) area

A given stress distribution gives rise to a resultant force, which is obtained by integration, Eqn. 3.1.2. It will also give rise to a resultant moment. This is examined in the following example.

Example

Consider the surface shown in Fig. 3.1.3, of length 2m and depth 2m (into the page). The stress over the surface is given by $\sigma = x$ kPa, with x measured in m from the left-hand side of the surface.

The force acting on an element of length dx at position x is (see Fig. 3.1.3b)

$$dF = \sigma\, dA = (x\,\text{kPa}) \times (dx\,\text{m} \times 2\,\text{m})$$

The resultant force is then, from Eqn. 3.1.2

[1] the weight of the ball is neglected here
[2] the radius of which depends on the force applied and the materials in contact

$$F = \int_A dF = 2\int_0^2 x dx \left(\text{kPa m}^2\right) = 4\,\text{kN}$$

The moment of the stress distribution is given by

$$M_0 = \int_A dM = \int_A \sigma \times l\, dA \tag{3.1.3}$$

where l is the length of the moment-arm from the chosen axis.

Taking the axis to be at $x = 0$, the moment-arm is $l = x$, Fig. 3.1.3b, and

$$M_{x=0} = \int_A dM = 2\int_0^2 x \times x\, dx \left(\text{kPa m}^3\right) = \frac{16}{3}\,\text{kN m}$$

Taking moments about the right-hand end, $x = 2$, one has

$$M_{x=2} = \int_A dM = -2\int_0^2 x \times (2 - x) dx \left(\text{kPa m}^3\right) = -\frac{8}{3}\,\text{kN m}$$

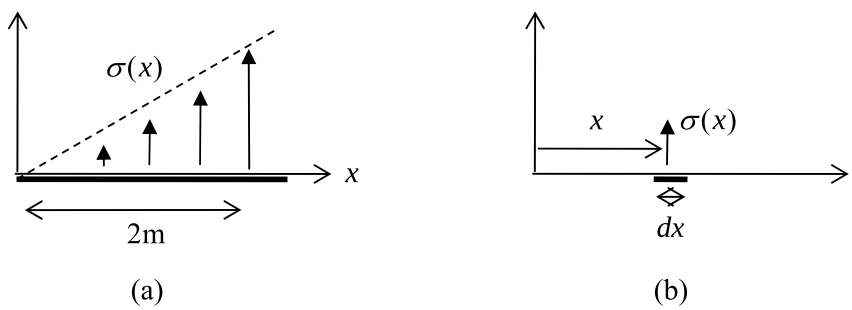


Figure 3.1.3: a non-uniform stress acting over a surface; (a) the stress distribution, (b) stress acting on an element of size d*x*

■

3.1.2 Equivalent Forces and Moments

Sometimes it is useful to replace a stress distribution σ with an **equivalent force** *F*, i.e. a force equal to the resultant force of the distribution and one which also gives the same moment about any axis as the distribution. Formulae for equivalent forces are derived in what follows for triangular and arbitrary linear stress distributions.

Triangular Stress Distribution

Consider the triangular stress distribution shown in Fig. 3.1.4. The stress at the end is σ_0, the length of the distribution is *L* and the thickness "into the page" is *t*. With $\sigma(x) = \sigma_0 x / L$, the equivalent force is, from Eqn. 3.1.2,

$$F = t\sigma_0 \int_0^L \frac{x}{L} dx = \frac{1}{2}\sigma_0 Lt \tag{3.1.4}$$

which is just the average stress times the "area of the triangle". The point of action of this force should be such that the moment of the force is equivalent to the moment of the stress distribution. Taking moments about the left hand end, for the distribution one has, from 3.1.3,

$$M_o = t\int_0^L x\sigma(x)dx = \frac{1}{3}\sigma_0 L^2 t$$

Placing the force at position $x = x_c$, Fig. 3.1.4, the moment of the force is $M_o = (\sigma_0 Lt/2)x_c$. Equating these expressions leads to the position at which the equivalent force acts, two-thirds the way along the triangle:

$$x_c = \frac{2}{3}L. \tag{3.1.5}$$

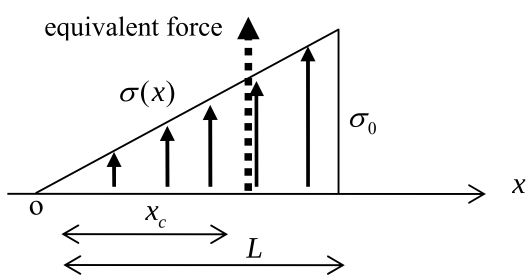


Figure 3.1.4: triangular stress distribution and equivalent force

Note that the moment about *any* axis is now the same for both the stress distribution and the equivalent force.

■

Arbitrary Linear Stress Distribution

Consider the linear stress distribution shown in Fig. 3.1.5. The stress at the ends are σ_1 and σ_2 and this time the equivalent force is

$$F = t\int_0^L [\sigma_1 + (\sigma_2 - \sigma_1)(x/L)]dx = Lt(\sigma_1 + \sigma_2)/2 \tag{3.1.6}$$

Taking moments about the left hand end, for the distribution one has

$$M_o = t\int_0^L x\sigma(x)dx = L^2 t(\sigma_1 + 2\sigma_2)/6 .$$

The moment of the force is $M_o = Lt(\sigma_1 + \sigma_2)x_c / 2$. Equating these expressions leads to

$$x_c = \frac{L(\sigma_1 + 2\sigma_2)}{3(\sigma_1 + \sigma_2)} \tag{3.1.7}$$

Eqn. 3.1.5 follows from 3.1.7 by setting $\sigma_1 = 0$.

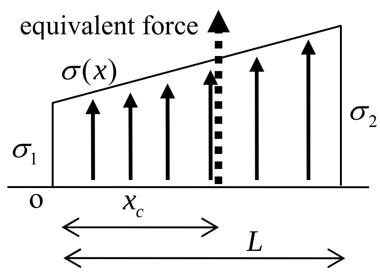


Figure 3.1.5: a non-uniform stress distribution and equivalent force

■

The Centroid

Generalising the above cases, the line of action of the equivalent force for any arbitrary stress distribution $\sigma(x)$ is

$$\boxed{x_c = \frac{t\int x\sigma(x)dx}{t\int \sigma(x)dx} = \frac{\int xdF}{F}} \qquad \textbf{Centroid} \tag{3.1.8}$$

This location is known as the **centroid** of the distribution.

Note that most of the discussion above is for two-dimensional cases, i.e. the stress is assumed constant "into the page". Three dimensional problems can be tackled in the same way, only now one must integrate two-dimensionally over a surface rather than one-dimensionally over a line.

Also, the forces considered thus far are *normal* forces, where the force acts perpendicular to a surface, and they give rise to **normal stresses**. Normal stresses are also called **pressures** when they are compressive as in Figs. 3.1.1-2.

3.1.3 Shear Stress

Consider now the case of **shear forces**, that is, forces which act tangentially to surfaces.

A normal force *F* acts on the block of Fig. 3.1.6a. The block does not move and, to maintain equilibrium, the force is resisted by a friction force $F = \mu mg$, where μ is the coefficient of friction. A free body diagram of the block is shown in Fig. 3.1.6b. Assuming a uniform distribution of stress, the stress and resultant force arising on the surfaces of the block and underlying object are as shown. The stresses are in this case called **shear stresses**.

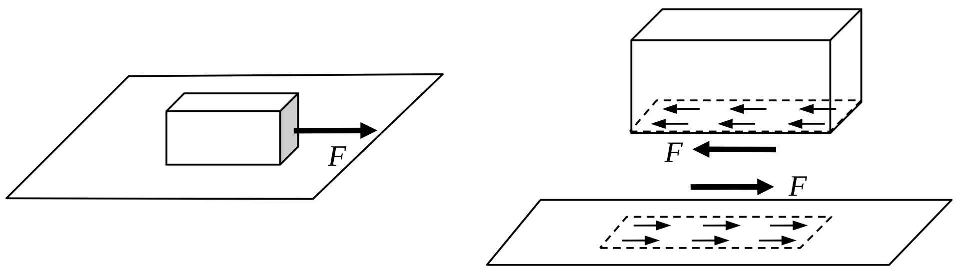


Figure 3.1.6: shear stress; (a) a force acting on a block, (b) shear stresses arising on the contacting surfaces

3.1.4 Combined Normal and Shear Stress

Forces acting inclined to a surface are most conveniently described by decomposing the force into components normal and tangential to the surface. Then one has both normal stress σ_N and shear stress σ_S, as in Fig. 3.1.7.

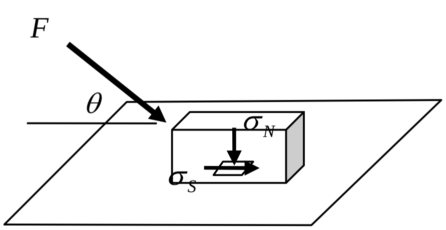
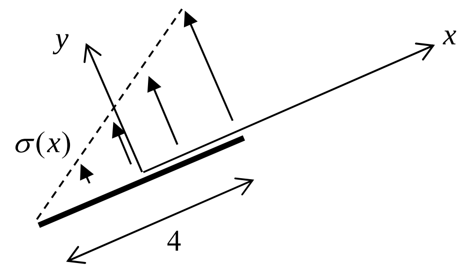


Figure 3.1.7: a force F giving rise to normal and shear stress over the contacting surfaces

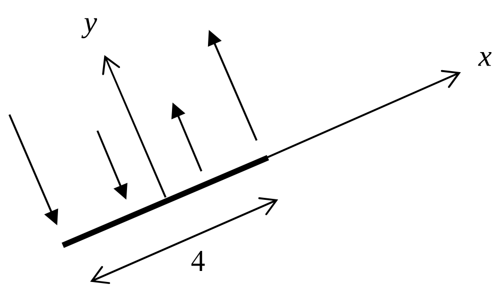
The stresses considered in this section are examples of **surface stresses** or **contact stresses**. They arise when materials meet at a common surface. Other examples would be sea-water pressurising a gas cylinder in deep water and the stress exerted by a train wheel on a train track.

3.1.5 Problems

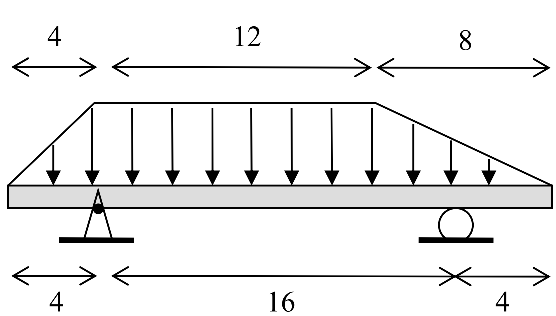
1. Consider the surface shown below, of length 4cm and unit depth (1cm into the page). The stress over the surface is given by $\sigma = 2 + x$ kPa, with x measured in cm from the surface *centre*.
   (a) Evaluate the resultant force acting on the surface (in Newtons).
   (b) What is the moment about an axis (into the page) through the left-hand end of the surface?
   (c) What is the moment about an axis (into the page) through the centre of the surface?



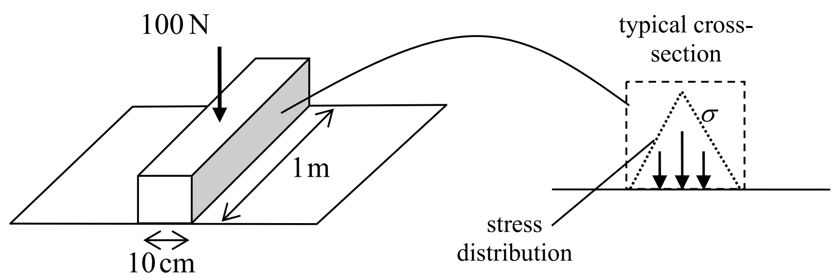
2. Consider the surface shown below, of length 4mm and unit depth (1mm into the page). The stress over the surface is given by $\sigma = x$ MPa, with x measured from the surface centre, i.e. x mm corresponds to x MPa. What is the total force acting on the surface, and the moment acting about the centre of the surface?



3. Find the reaction forces (per unit length) at the pin and roller for the following beam, which is subjected to a varying pressure distribution, the maximum pressure being $\sigma(x) = 20\,\text{kPa}$ (all lengths are in cm – give answer in N/m)
   [Hint: first replace the stress distribution with three equivalent forces]



4. A block of material of width 10cm and length 1m is pushed into an underlying substrate by a normal force of 100 N. It is found that a uniform triangular normal stress distribution arises at the contacting surfaces, that is, the stress is maximum at the centre and dies off linearly to zero at the block edges, as sketched below right. What is the maximum pressure acting on the surface?



3.2 Body Forces

Surface forces act on surfaces. As discussed in the previous section, these are the forces which arise when bodies are in contact and which give rise to stress distributions. Surface forces also arise *inside* materials, acting on internal surfaces, Fig. 3.2.1a, as will be discussed in the following section.

To complete the description of forces acting on real materials, one needs to deal with forces which arise even when bodies are not in contact; one can think of these forces as *acting at a distance*, for example the force of gravity. To describe these forces, one can define the **body force**, which acts on volume elements of material. Fig. 3.2.1b shows a sketch of a volume element subjected to a magnetic body force and a gravitational body force $\mathbf{F}_g$.

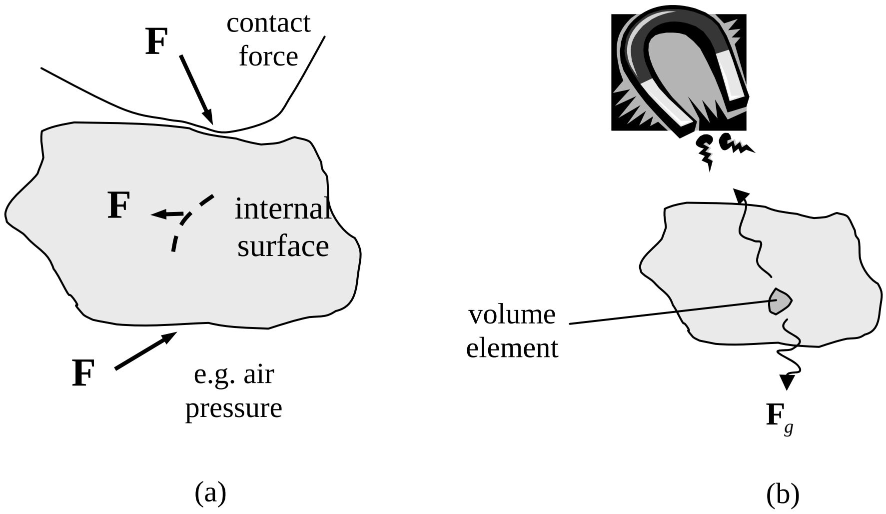


Figure 3.2.1: forces acting on a body; (a) surface forces acting on surfaces, (b) body forces acting on a material volume element

3.2.1 Weight

The most important body force is the force due to gravity, i.e. the weight force. In Chapter 2 there were examples involving the weight of components. In those cases it was simply stated that the weight could be taken to be a single force acting at the component centre (for example, Problem 3 in §2.2.3). This is true when the component is symmetrical, for example, in the shape of a circle or a square. However, it is not true in general for a component of arbitrary shape.

In what follows, the important case of a flat object of arbitrary shape will be examined.

The weight of a small volume element ΔV of material of density ρ is $dF = \rho g \Delta V$ and the total weight is

$$F = \int_V \rho g dV \tag{3.2.1}$$

Consider the general two-dimensional case, Fig. 3.2.2, where material elements of area ΔA_i (and constant thickness *t*) are subjected to forces $\Delta F_i = t\rho g \Delta A_i$.

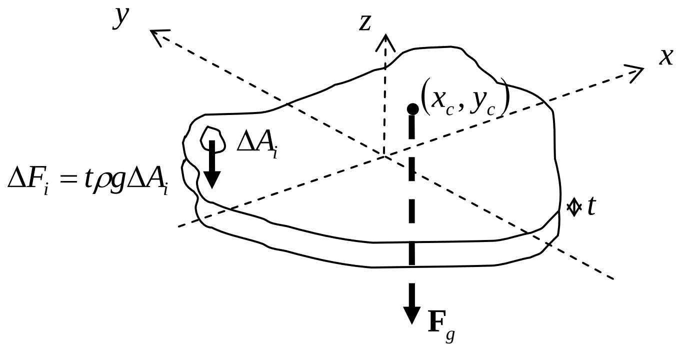


Figure 3.2.2: Resultant Weight on a body

The resultant, i.e. equivalent, weight force due to all elements, for a component with uniform density, is

$$F = \int dF = t\rho g \int dA = \rho g t A,$$

where *A* is the cross-sectional area.

The resultant moments about the *x* and *y* axes, which can be positioned anywhere in the body, are $M_x = t\rho g \int y dA$ and $M_y = t\rho g \int x dA$ respectively; the moment ΔM_x is shown in Fig. 3.2.3. The equivalent weight force is thus positioned at (x_c, y_c), Fig. 3.2.2, where

$$\boxed{x_c = \frac{\int x dA}{A}, \quad y_c = \frac{\int y dA}{A}} \qquad \textbf{Centroid of Area} \tag{3.2.2}$$

The position (x_c, y_c) is called the **centroid of the area**. The quantities $\int x dA$, $\int y dA$, are called the **first moments of area** about, respectively, the *y* and *x* axes.

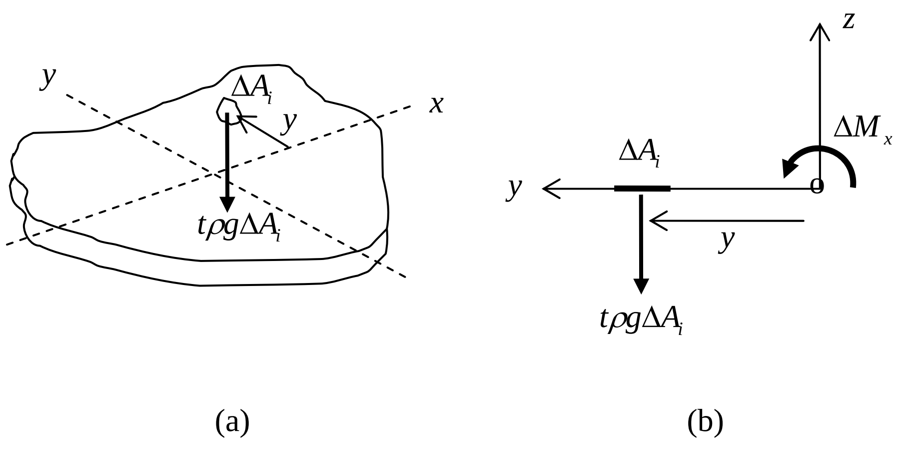
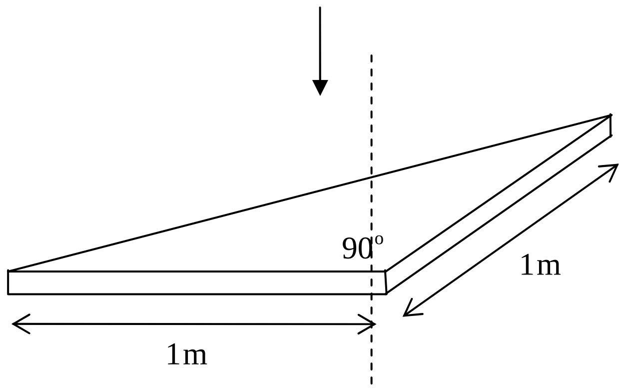


Figure 3.2.3: The moment M_x; (a) full view, (b) plane view

3.2.2 Problems

1. Where does the resultant force due to gravity act in the triangular component shown below? (Gravity acts downward in the direction of the arrow shown, perpendicular to the component's surface.)



3.3 Internal Stress

The idea of stress considered in §3.1 is not difficult to conceptualise since objects interacting with other objects are encountered all around us. A more difficult concept is the idea of forces and stresses acting *inside* a material, "within the interior where neither eye nor experiment can reach" as Euler put it. It took many great minds working for centuries on this question to arrive at the concept of stress we use today, an idea finally brought to us by Augustin Cauchy, who presented a paper on the subject to the Academy of Sciences in Paris, in 1822.

Augustin Cauchy

3.3.1 Cauchy's Concept of Stress

Uniform Internal Stress

Consider first a long slender block of material subject to equilibrating forces F at its ends, Fig. 3.3.1a. If the complete block is in equilibrium, then any sub-division of the block must be in equilibrium also. By imagining the block to be cut in two, and considering free-body diagrams of each half, as in Fig. 3.3.1b, one can see that forces F must be acting *within* the block so that each half is in equilibrium. Thus *external loads create internal forces*; internal forces represent the action of one part of a material on another part of the same material across an internal surface. We can take it that a uniform stress $\sigma = F / A$ acts over this interior surface, Fig. 3.3.1b.

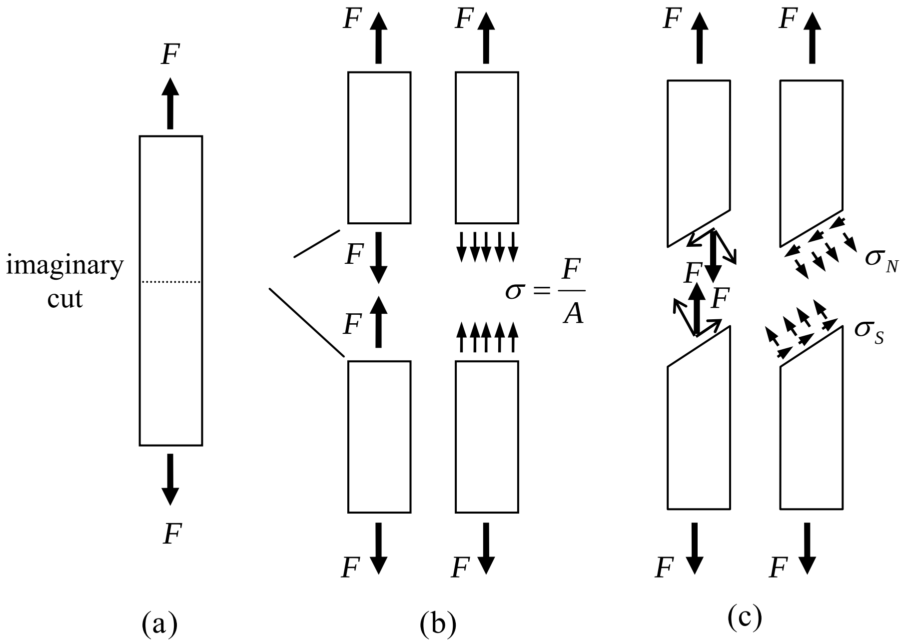


Figure 3.3.1: a slender block of material; (a) under the action of external forces F, (b) internal normal stress σ, (c) internal normal and shear stress

Note that, if the internal forces were not acting over the internal surfaces, the two half-blocks of Fig. 3.3.1b would fly apart; one can thus regard the internal forces as those required to maintain material in an un-cut state.

If the internal surface is at an incline, as in Fig. 3.3.1c, then the internal force required for equilibrium will not act normal to the surface. There will be components of the force normal and tangential to the surface, and thus both normal (σ_N) and shear (σ_S) stresses must arise. Thus, even though the material is subjected to a purely normal load, internal shear stresses develop.

From Fig. 3.3.2a, with the stress given by force divided by area, the normal and shear stresses arising on an interior surface inclined at angle θ to the horizontal are {▲Problem 1}

$$\sigma_N = \frac{F}{A}\cos^2\theta, \qquad \sigma_S = \frac{F}{A}\sin\theta\cos\theta \tag{3.3.1}$$

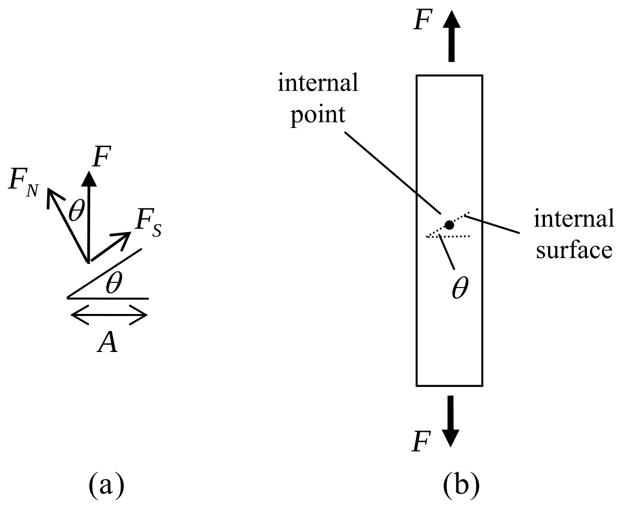


Figure 3.3.2: stress on inclined surface; (a) decomposing the force into normal and shear forces, (b) stress at an internal point

Although stress is associated with surfaces, one can speak of the stress "at a point". For example, consider some point interior to the block, Fig 3.3.2b. The stress there evidently depends on which surface through that point is under consideration. From Eqn. 3.3.1a, the normal stress at the point is a maximum F/A when $\theta = 0$ and a minimum of zero when $\theta = 90^\circ$. The maximum normal stress arising at a point within a material is of special significance, for example it is this stress value which often determines whether a material will fail ("break") there. It has a special name: the **maximum principal stress**. From Eqn. 3.3.1b, the **maximum shear stress** at the point is $\pm F/2A$ and arises on surfaces inclined at $\pm 45^\circ$.

Non-Uniform Internal Stress

Consider a more complex geometry under a more complex loading, as in Fig. 3.3.3. Again, using equilibrium arguments, there will be some stress distribution acting over any given internal surface. To evaluate these stresses is not a straightforward matter, suffice to say here that they will invariably be *non-uniform* over a surface, that is, the stress at some particle will differ from the stress at a neighbouring particle.

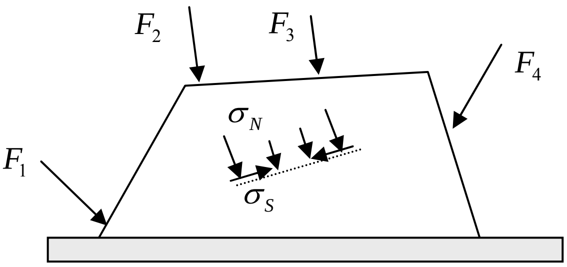


Figure 3.3.3: a component subjected to a complex loading, giving rise to a non-uniform stress distribution over an internal surface

Traction and the Physical Meaning of Internal Stress

All materials have a complex molecular microstructure and each molecule exerts a force on each of its neighbours. The complex interaction of countless molecular forces maintains a body in equilibrium in its unstressed state. When the body is disturbed and deformed into a new equilibrium position, net forces act, Fig. 3.3.4a. An imaginary plane can be drawn through the material, Fig. 3.3.4b. Unlike some of his predecessors, who attempted the extremely difficult task of accounting for all the molecular forces, Cauchy discounted the molecular structure of matter and simply replaced the imagined molecular forces acting on the plane by a single force **F**, Fig 3.3.4c. This is the force exerted by the molecules above the plane *on* the material below the plane and can be attractive or repulsive. Different planes can be taken through the *same* portion of material and, in general, a *different* force will act on the plane, Fig 3.3.4d.

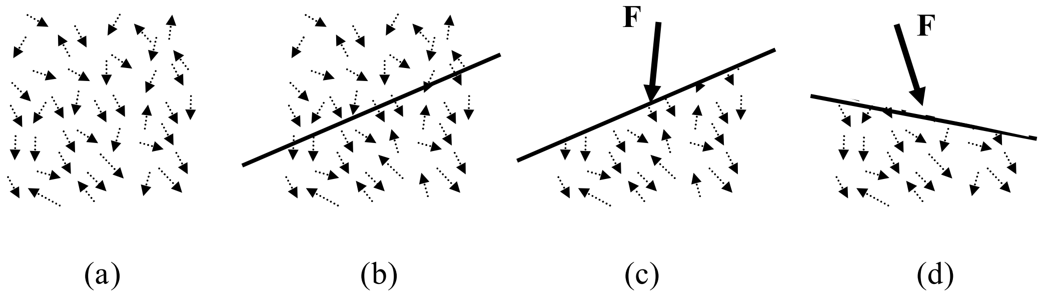


Figure 3.3.4: a multitude of molecular forces represented by a single force; (a) molecular forces, a plane drawn through the material, replacing the molecular forces with an equivalent force F, a different equivalent force F acts on a different plane through the same material

The definition of stress will now be made more precise. First, define the **traction** at some particular point in a material as follows: take a plane of surface area *S* through the point, on which acts a force *F*. Next shrink the plane – as it shrinks in size both *S* and *F* get smaller, and the direction in which the force acts may change, but eventually the ratio F/S will remain constant and the force will act in a particular direction, Fig. 3.3.5. The limiting value of this ratio of force over surface area is defined as the **traction vector** (or **stress vector**) **t**:[1]

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S} \tag{3.3.2}$$

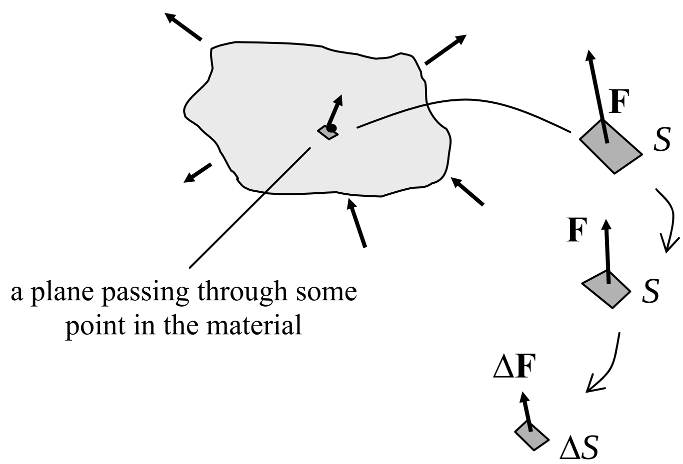


Figure 3.3.5: the traction vector - the limiting value of force over area, as the surface area of the element on which the force acts is shrunk

An infinite number of traction vectors act at any single point, since an infinite number of different planes pass through a point. Thus the notation $\lim_{\Delta S \to 0} \Delta F / \Delta S$ is ambiguous. For this reason the plane on which the traction vector acts must be specified; this can be done by specifying the normal **n** to the surface on which the traction acts, Fig 3.3.6. The traction is thus a special vector – associated with it is not only the direction in which it acts but also a second direction, the normal to the plane upon which it acts.

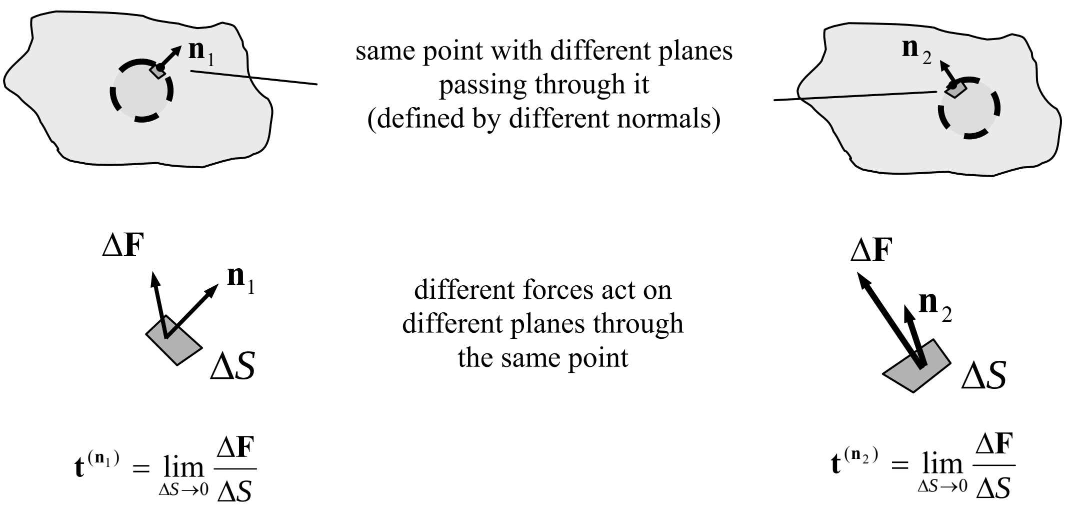


Figure 3.3.6: two different traction vectors acting at the same point

[1] this does not mean that the force is acting on a surface of zero area – the meaning of this limit is further examined in section 5.4, in the context of the continuum

Stress Components

The traction vector can be decomposed into components which act normal and parallel to the surface upon which it acts. These components are called the **stress components**, or simply **stresses**, and are denoted by the symbol σ; subscripts are added to signify the surface on which the stresses act and the directions in which the stresses act.

Consider a particular traction vector acting on a surface element. Introduce a Cartesian coordinate system with base vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ so that one of the base vectors is a normal to the surface, and the origin of the coordinate system is positioned at the point at which the traction acts. For example, in Fig. 3.3.7, the $\mathbf{k}$ direction is taken to be normal to the plane, and $\mathbf{t}^{(\mathbf{k})} = t_x \mathbf{i} + t_y \mathbf{j} + t_z \mathbf{k}$.

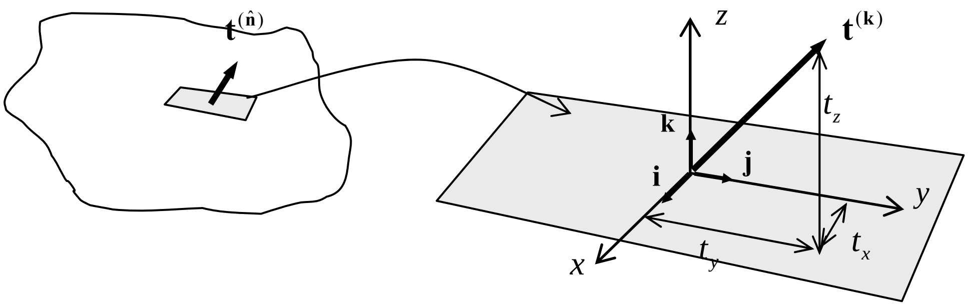


Figure 3.3.7: the components of the traction vector

Each of these components t_i is represented by σ_{ij} where the first subscript denotes the *direction of the normal* to the plane and the second denotes the *direction of the component*. Thus, re-drawing Fig. 3.3.7 as Fig. 3.3.8: $\mathbf{t}^{(\mathbf{k})} = \sigma_{zx} \mathbf{i} + \sigma_{zy} \mathbf{j} + \sigma_{zz} \mathbf{k}$. The first two stresses, the components acting tangential to the surface, are shear stresses, whereas σ_{zz}, acting normal to the plane, is a normal stress[2].

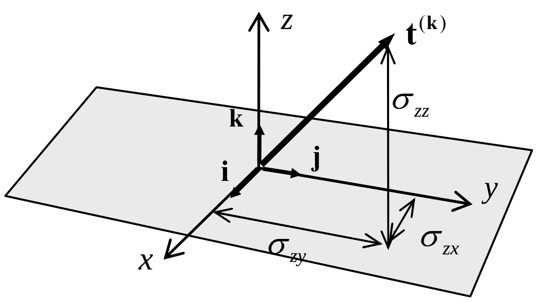


Figure 3.3.8: stress components – the components of the traction vector

[2] this convention for the subscripts is not universally followed. Many authors, particularly in the mathematical community, use the exact opposite convention, the first subscript to denote the direction and the second to denote the normal. It turns out that *both conventions are equivalent*, since, as will be shown later, $\sigma_{ij} = \sigma_{ji}$

The traction vector shown in Figs. 3.3.7, 3.3.8, represents the force (per unit area) exerted by the material above the surface *on* the material below the surface. By Newton's third law, an equal and opposite traction must be exerted by the material below the surface on the material above the surface, as shown in Fig. 3.3.9 (thick dotted line). If $\mathbf{t}^{(\mathbf{k})}$ has stress components $\sigma_{zx}, \sigma_{zy}, \sigma_{zz}$, then so should $\mathbf{t}^{(-\mathbf{k})}$: $\mathbf{t}^{(-\mathbf{k})} = \sigma_{zx}(-\mathbf{i}) + \sigma_{zy}(-\mathbf{j}) + \sigma_{zz}(-\mathbf{k}) = -\mathbf{t}^{(\mathbf{k})}$.

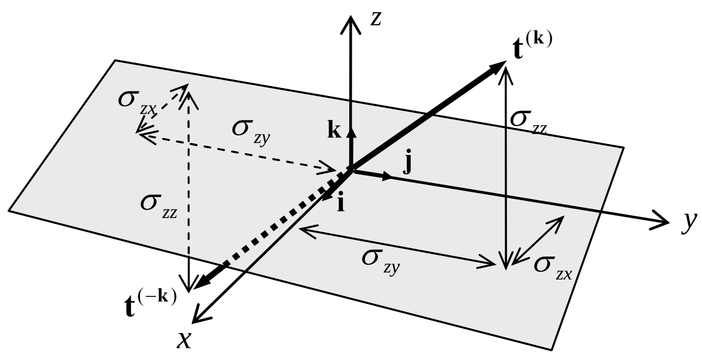


Figure 3.3.9: equal and opposite traction vectors – each with the same stress components

Sign Convention for Stress Components

The following convention is used:

> The stress is *positive* when the direction of the normal *and* the direction of the stress component are both positive *or* both negative
> The stress is *negative* when one of the directions is positive and the other is negative

According to this convention, the three stresses in Figs. 3.3.7-9 are all positive.

Looking at the two-dimensional case for ease of visualisation, the (positive and negative) normal stresses and shear stresses on either side of a surface are as shown in Fig. 3.3.10. To clarify this, consider the σ_{yy} stress in Fig. 3.310a: "above" the plane, the normal to the plane is in the positive *y* direction (up) and the component σ_{yy} acts in the positive direction (up), so this stress is positive; "below" the plane, the normal to the plane is in the negative *y* direction (down) and the component σ_{yy} acts in the negative direction (down), so this stress is positive. The simple consequence of this sign convention is that normal stresses which "pull" (tension) are positive and normal stresses which "push" (compression) are negative. Note that the shear stresses always go in opposite directions.

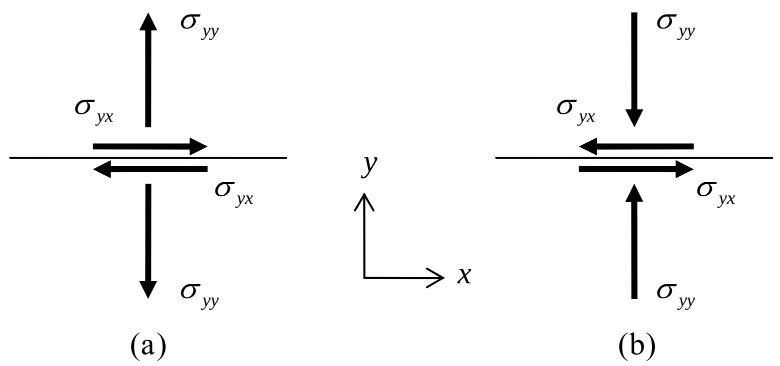


Figure 3.3.10: stresses acting on either side of a material surface: (a) positive stresses, (b) negative stresses

Examples of negative stresses are shown in Fig. 3.3.11 {▲Problem 4}.

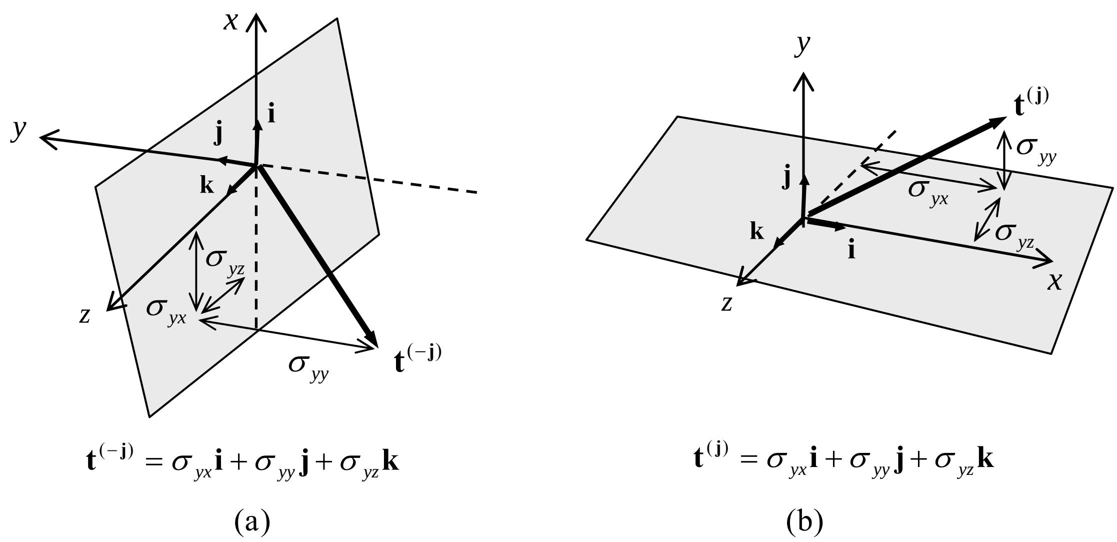


Figure 3.3.11: examples of negative stress components

3.3.2 Real Problems and Saint-Venant's Principle

Some examples have been given earlier of external forces acting on materials. In reality, an external force will be applied to a real material component in a complex way. For example, suppose that a block of material, welded to a large object at one end, is pulled at its other end by a rope attached to a metal hoop, which is itself attached to the block by a number of bolts, Fig. 3.3.12a. The block can be idealised as in Fig 3.3.12b; here, the precise details of the region in which the external force is applied are neglected.

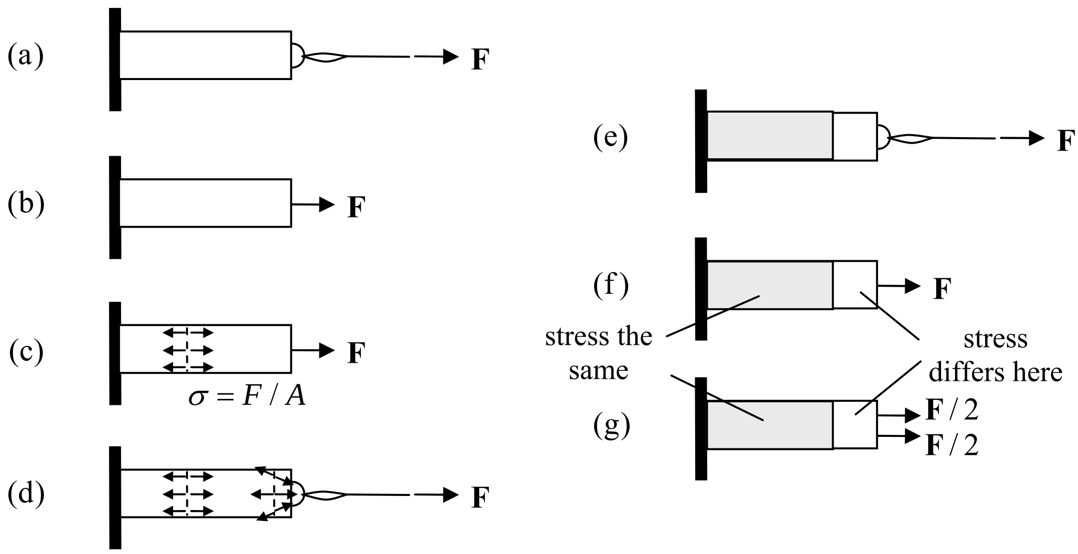


Figure 3.3.12: a block subjected to an external force: (a) real case, (b) ideal model, (c) stress in ideal model, (d) stress in actual material, (e) the stress in the real material, away from the right hand end, is modelled well by either (f) or (g)

According to the earlier discussion, the stress in the ideal model is as in Fig. 3.3.12c. One will find that, in the *real* material, the stress is indeed (approximately) as predicted, but only at an appreciable distance from the right hand end. Near where the rope is attached, the force will differ considerably, as sketched in Fig.3.3.12d.

Thus the ideal models of the type discussed in this section, and in much of this book, are useful only in predicting the stress field in real components in regions away from points of application of loads. This does not present too much of a problem, since the stresses internal to a structure in such regions are often of most interest. If one wants to know what happens near the bolted connection, then one will have to create a complex model incorporating all the details and the problem will be more difficult to solve.

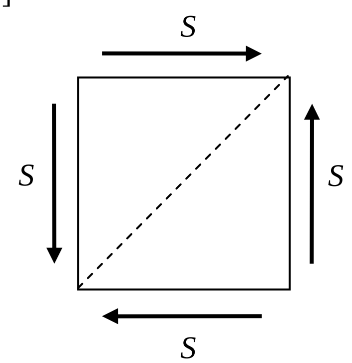
That said, it is an *experimental* fact that if two different force systems are applied to a material, but they are *equivalent force systems*, as in Fig. 3.3.12(f,g), then the stress fields in regions away from where the loads are applied will be the same. This is known as **Saint-Venant's Principle**. Typically, one needs to move a distance away from where the loads are applied roughly equal to the distance over which the loads are applied.

Saint-Venant's principle is extremely important in practical applications: we can replace a complicated problem by a simple model problem; the solution to this latter problem will often give us the information we require.

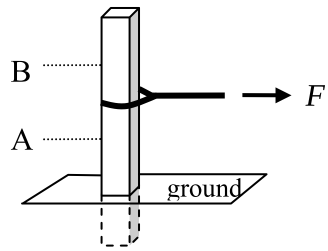
3.3.3 Problems

1. Derive Eqns. 3.3.1.

2. The four sides of a square block are subjected to equal *forces S*, as illustrated. The length of each side is *l* and the block has unit depth (into the page). What normal and shear *stresses* act along the (dotted) diagonal? [Hint: draw a free body diagram of the upper left hand triangle.]

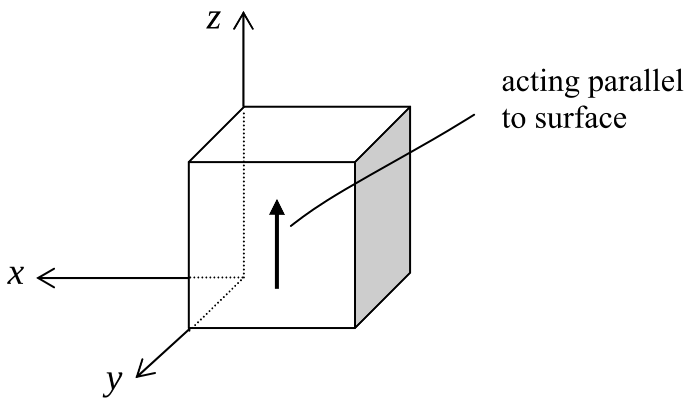


3. A shaft is concreted firmly into the ground. A thick steel rope is looped around the shaft and a force is applied normal to the shaft, as shown. The shaft is in static equilibrium. Draw a free body diagram of the shaft (from the top down to ground level) showing the forces/moments acting on the shaft (including the reaction forces at the ground-level; ignore the weight of the shaft). Draw a free body diagram of the section of shaft from the top down to the cross section at A. Draw a free body diagram of the section of shaft from the top to the cross section at B. Roughly sketch the stresses acting over the (horizontal) internal surfaces of the shaft at A and B.

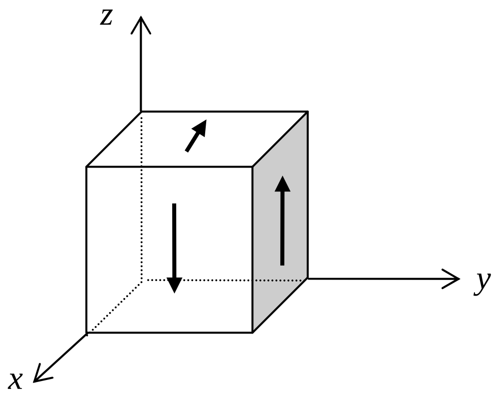


4. In Fig. 3.3.11, which of the stress components is/are negative?

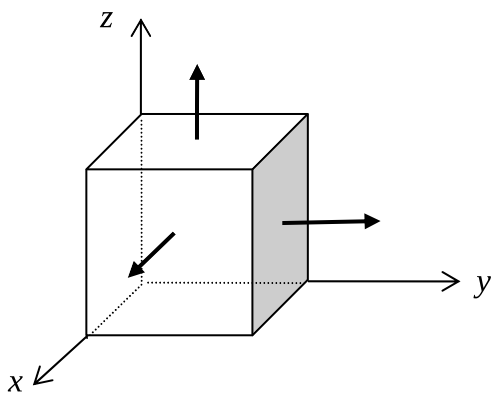
5. Label the following stress component acting on an internal material surface. Is it a positive or negative stress?



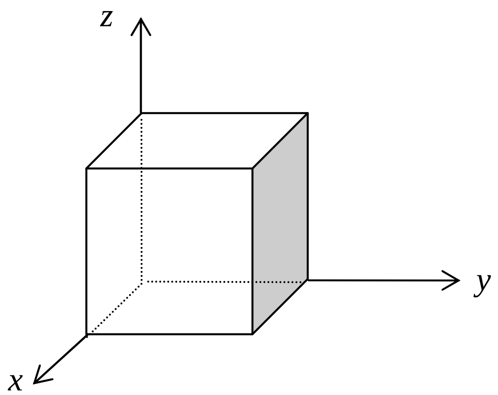
6. Label the following shear stresses. Are they positive or negative?



7. Label the following normal stresses. Are they positive or negative?



8. By the definition of the traction vector **t** which acts on the $x-z$ plane, $\mathbf{t}^{(\mathrm{j})} = \sigma_{yx}\mathbf{i} + \sigma_{yy}\mathbf{j} + \sigma_{yz}\mathbf{k}$. Sketch these three stress components on the figure below.



3.4 Equilibrium of Stress

Consider two perpendicular planes passing through a point *p*. The stress components acting on these planes are as shown in Fig. 3.4.1a. These stresses are usually shown together acting on a small material element of finite size, Fig. 3.4.1b. It has been seen that the stress may vary from point to point in a material but, if the element is very small, the stresses on one side can be taken to be (more or less) equal to the stresses acting on the other side. By convention, in analyses of the type which will follow, all stress components shown are *positive*.

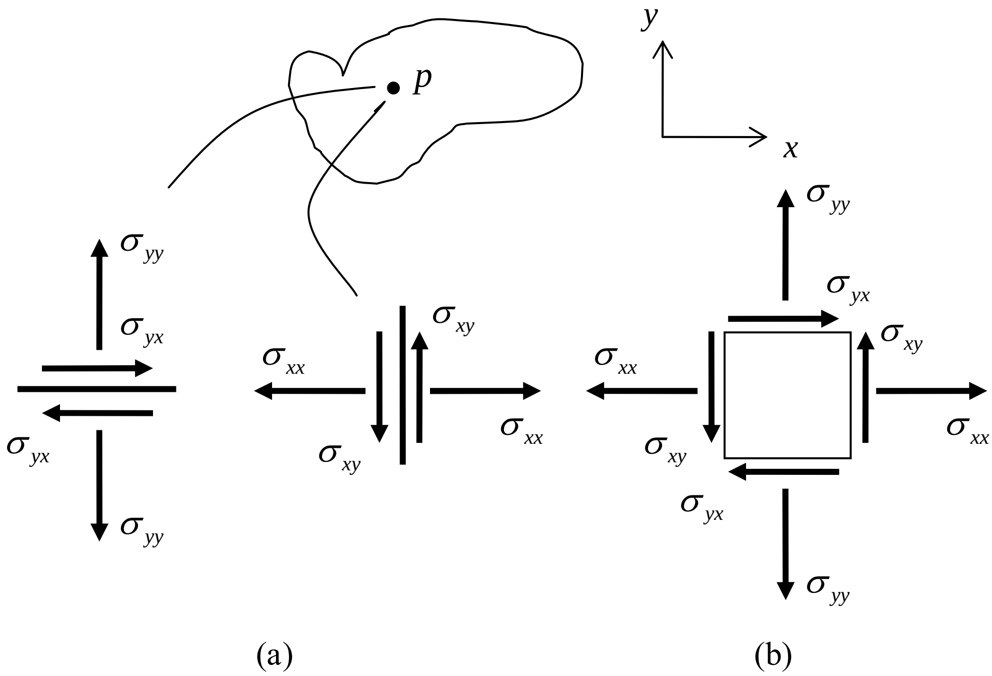


Figure 3.4.1: stress components acting on two perpendicular planes through a point; (a) two perpendicular surfaces at a point, (b) small material element at the point

The four stresses can conveniently be written in the form of a **stress matrix**:

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \tag{3.4.1}$$

It will be shown below that the stress components acting on *any* other plane through *p* can be evaluated from a knowledge of only these stress components.

3.4.1 Symmetry of the Shear Stress

Consider the material element shown in Fig. 3.4.1b, reproduced in Fig. 3.4.2a below. The element has dimensions is $\Delta x \times \Delta y$ and is subjected to uniform stresses over its sides. The resultant forces of the stresses acting on each side of the element act through the side-centres, and are shown in Fig. 3.4.2b. The stresses shown are positive, but note how *positive stresses can lead to negative forces*, depending on the definition of the $x-y$ axes used. The resultant force on the complete element is seen to be zero.

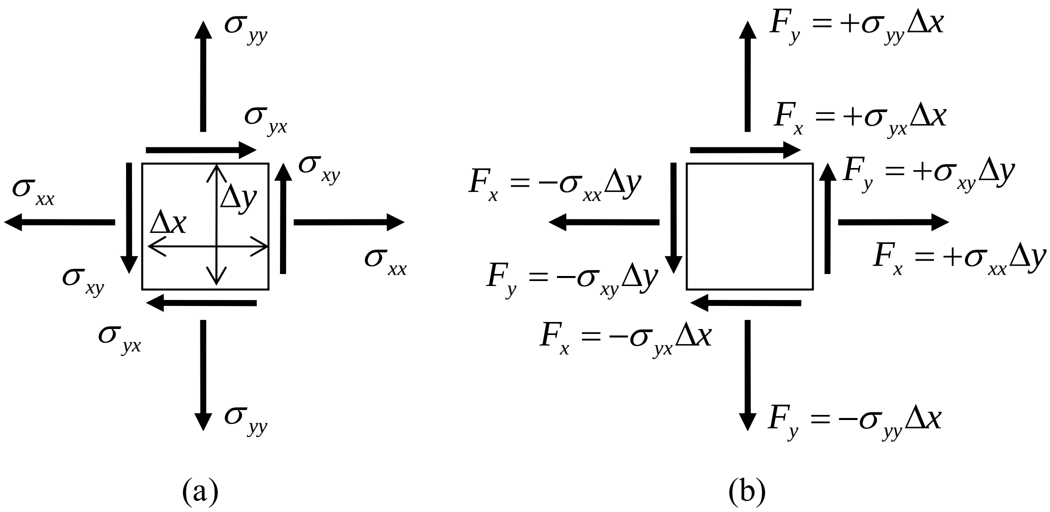


Figure 3.4.2: stress components acting on a material element; (a) stresses, (b) resultant forces on each side

By taking moments about any point in the block, one finds that {▲Problem 1}

$$\sigma_{xy} = \sigma_{yx} \tag{3.4.2}$$

Thus the shear stresses acting on the element are all equal, and for this reason the σ_{yx} stresses are usually labelled σ_{xy}, Fig. 3.4.3a, or simply labelled τ, Fig. 3.4.3b.

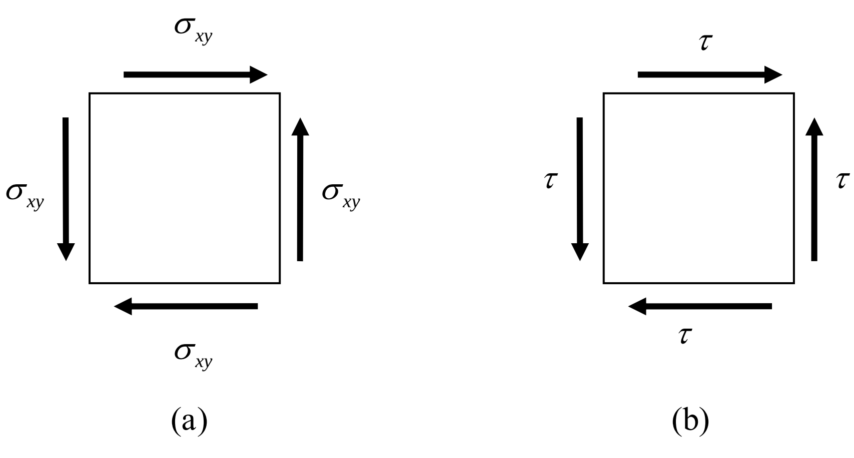


Figure 3.4.3: shear stress acting on a material element

In fact, in two-dimensional problems, the double-subscript notation is often dispensed with for simplicity, and the stress matrix can be expressed as

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau \\ \tau & \sigma_y \end{bmatrix}, \tag{3.4.3}$$

to go along with the representation shown in Fig. 3.4.4.

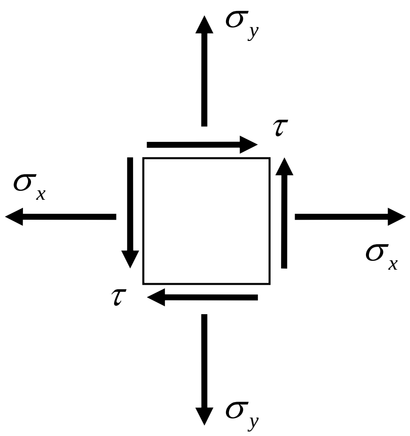


Figure 3.4.4: a simpler notation for 2D stress components (without the double subscripts)

3.4.2 Three Dimensional Stress

The three-dimensional counterpart to the two-dimensional element of Fig. 3.4.2 is shown in Fig. 3.4.5. Again, all stresses shown are positive.

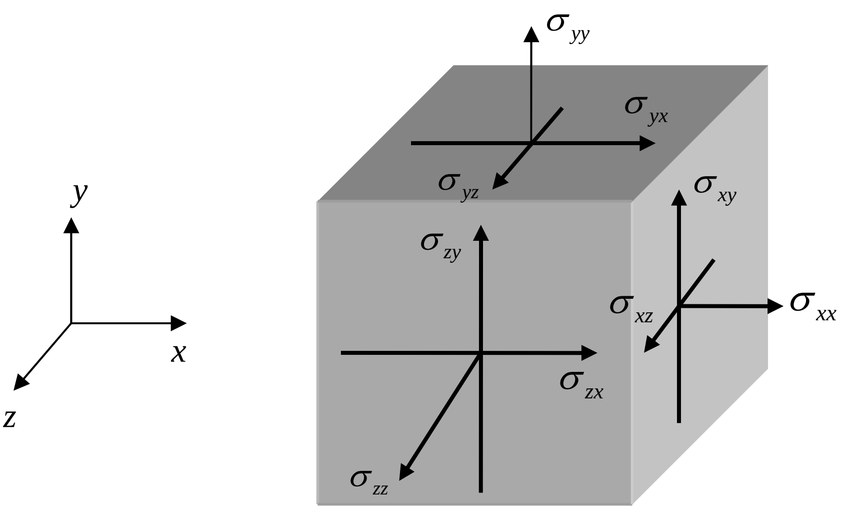


Figure 3.4.5: a three dimensional material element

Moment equilibrium in this case requires that

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{xz} = \sigma_{zx}, \quad \sigma_{yz} = \sigma_{zy} \tag{3.4.4}$$

The nine stress components, six of which are independent, can be written in the matrix form

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \qquad (3.4.5)$$

A vector **F** has one direction associated with it and is characterised by *three* components (F_x, F_y, F_z). The stress is a quantity which has two directions associated with it (the direction of a force and the normal to the plane on which the force acts) and is characterised by the *nine* components of Eqn. 3.4.5. Such a mathematical object is called a **tensor**. Just as the three components of a vector change with a change of coordinate axes (for example, as in Fig. 2.2.1), so the nine components of the **stress tensor** change with a change of axes. This is discussed in the next section for the two-dimensional case.

3.4.3 Stress Transformation Equations

Consider the case where the nine stress components acting on three perpendicular planes through a material particle are known. These components are σ_{xx}, σ_{xy}, etc. when using x, y, z axes, and can be represented by the cube shown in Fig. 3.4.6a. Rotate now the planes about the three axes – these new planes can be represented by the rotated cube shown in Fig. 3.4.6b; the axes normal to the planes are now labelled x', y', z' and the corresponding stress components with respect to these new axes are $\sigma'_{xx}, \sigma'_{xy}$, etc.

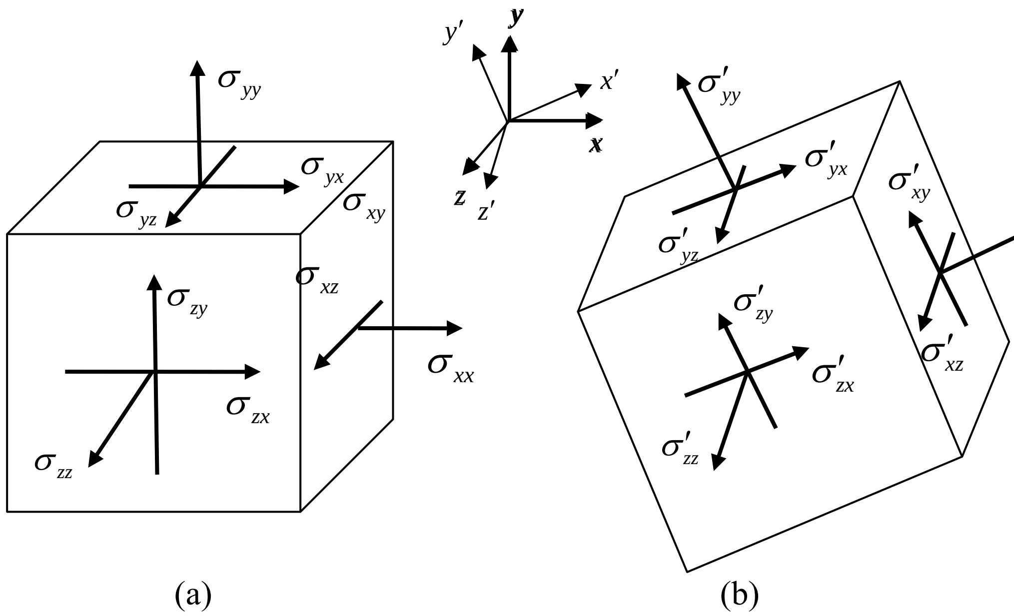


Figure 3.4.6: a three dimensional material element; (a) original element, (b) rotated element

There is a relationship between the stress components σ_{xx}, σ_{xy}, etc. and the stress components $\sigma'_{xx}, \sigma'_{xy}$, etc. The relationship can be derived using Newton's Laws. The equations describing the relationship in the fully three-dimensional case are very lengthy. Here, the relationship for the two-dimensional case will be derived – this 2D relationship will prove very useful in analysing many practical situations.

Two-dimensional Stress Transformation Equations

Assume that the stress components of Fig. 3.4.7a are known. It is required to find the stresses arising on other planes through *p*. Consider the perpendicular planes shown in Fig. 3.4.7b, obtained by rotating the original element through a positive (counterclockwise) angle θ. The new surfaces are defined by the axes $x' - y'$.

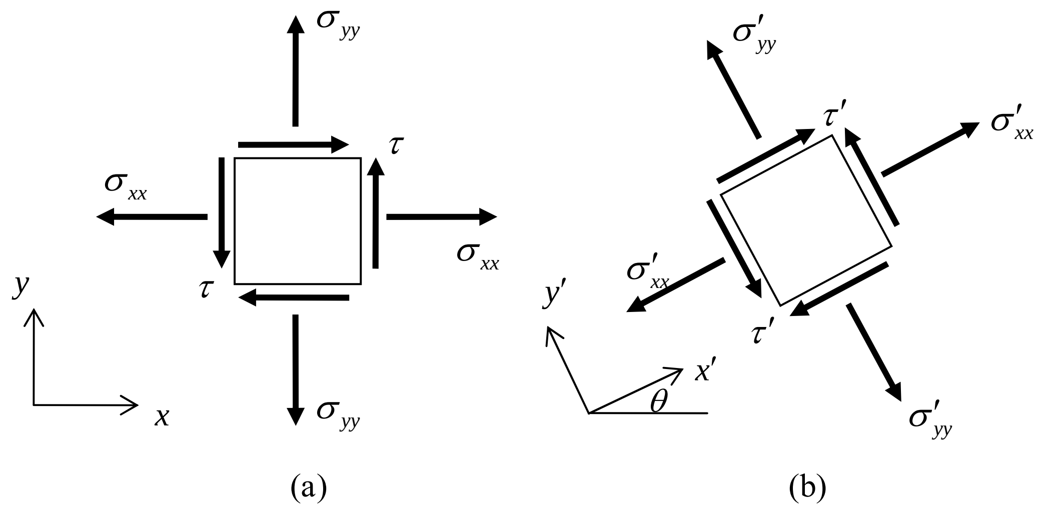


Figure 3.4.7: stress components acting on two different sets of perpendicular surfaces, i.e. in two different coordinate systems; (a) original system, (b) rotated system

To evaluate these new stress components, consider a triangular element of material at the point, Fig. 3.4.8. Carrying out force equilibrium in the direction x', one has (with unit depth into the page)

$$\sum F_{x'}: \quad \sigma'_{xx}|AB| - \sigma_{xx}|OB|\cos\theta - \sigma_{yy}|OA|\sin\theta - \tau|OB|\sin\theta - \tau|OA|\cos\theta = 0 \qquad (3.4.6)$$

Since $|OB| = |AB|\cos\theta,\ |OA| = |AB|\sin\theta$, and dividing through by $|AB|$,

$$\sigma'_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \tau\sin 2\theta \qquad (3.4.7)$$

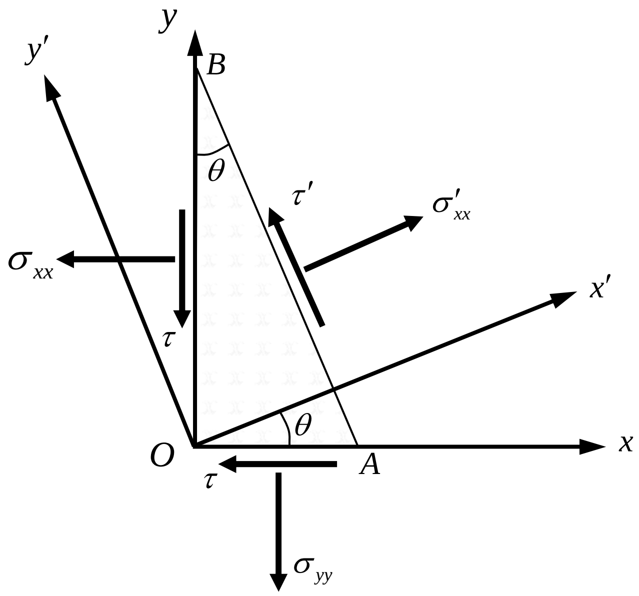


Figure 3.4.8: a free body diagram of a triangular element of material

The forces can also be resolved in the y' direction and one obtains the relation

$$\tau' = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \tau\cos 2\theta \qquad (3.4.8)$$

Finally, consideration of the element in Fig. 3.4.9 yields two further relations, one of which is the same as Eqn. 3.4.8.

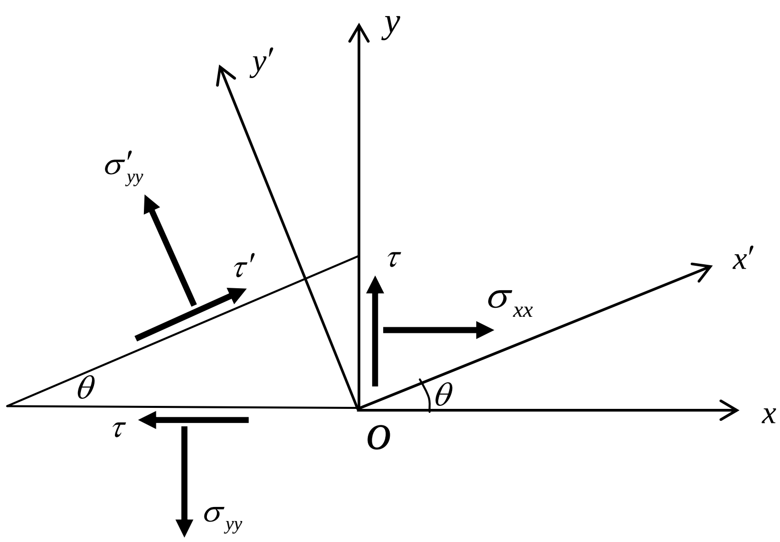


Figure 3.4.9: a free body diagram of a triangular element of material

In summary, one obtains the **stress transformation equations**:

$$\boxed{\begin{aligned} \sigma'_{xx} &= \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} + \sin 2\theta\sigma_{xy} \\ \sigma'_{yy} &= \sin^2\theta\sigma_{xx} + \cos^2\theta\sigma_{yy} - \sin 2\theta\sigma_{xy} \\ \sigma'_{xy} &= \sin\theta\cos\theta(\sigma_{yy} - \sigma_{xx}) + \cos 2\theta\sigma_{xy} \end{aligned}}$$

2D Stress Transformation Equations (3.4.9)

These equations have many uses, as will be seen in the next section.

In matrix form,

$$\begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{yx} & \sigma'_{yy} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{3.4.10}$$

These relations hold also in the case when there are body forces, when the material is accelerating and when there are non-uniform stress fields. (This is discussed in the next section.)

3.4.4 Problems

1. Derive Eqns. 3.4.2 by taking moments about the lower left corner of the block in Fig. 3.4.2.

2. Suppose that the stresses acting on two perpendicular planes through a point are

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

   Use the stress transformation formulae to evaluate the stresses acting on two new perpendicular planes through the point, obtained from the first set by a positive rotation of 30°. Use the conventional notation $x' - y'$ to represent the coordinate axes parallel to these new planes.

3.4b Stress Transformation: Further Aspects

Here, it will be shown that the Stress Transformation Equations are valid also when (i) there are body forces, (ii) the body is accelerating and (iii) the stress and other quantities are not uniform. We will also examine the fully three-dimensional stress subject to the transformation.

Suppose that a body force $\mathbf{F}_b = (\mathbf{F}_b)_x \mathbf{i} + (\mathbf{F}_b)_y \mathbf{j}$ acts on the material and that the material is accelerating with an acceleration $\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j}$. The components of body force and acceleration are shown in Fig. 3.4.10 (a reproduction of Fig. 3.4.8).

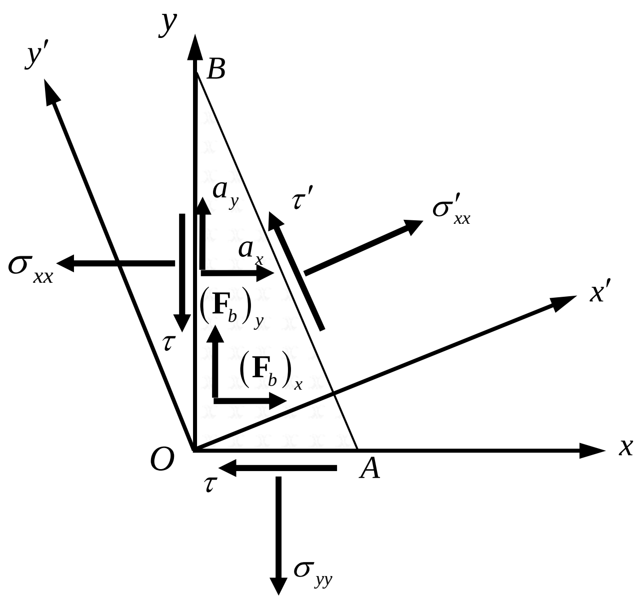


Figure 3.4.10: a free body diagram of a triangular element of material, including a body force and acceleration

The body force will vary depending on the size of the material under consideration, e.g. the force of gravity $\mathbf{F}_b = m\mathbf{g}$ will be larger for larger materials; therefore consider a quantity which is independent of the amount of material: the body force per unit mass, $\mathbf{F}_b / m$. Then, Eqn 3.4.6 now reads

$$\begin{aligned}\sum F_{x'}: \quad & \sigma'_{xx}|AB| - \sigma_{xx}|OB|\cos\theta - \sigma_{yy}|OA|\sin\theta - \tau|OB|\sin\theta - \tau|OA|\cos\theta \\ & + (\mathbf{F}_b / m)_x m\cos\theta + (\mathbf{F}_b / m) m\sin\theta + ma_x\cos\theta + ma_y\sin\theta = 0\end{aligned} \tag{3.4.11}$$

where m is the mass of the triangular portion of material. The volume of the triangle is $\frac{1}{2}|OA||OB| = |AB|^2 / \sin 2\theta$ so that, this time, when 3.4.11 is divided through by $|AB|$, one is left with

$$\begin{aligned}\sigma'_{xx} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \tau\sin 2\theta \\ &\quad - |AB|\rho\left\{(\mathbf{F}_b / m)_x / 2\sin\theta + (\mathbf{F}_b / m) / 2\cos\theta + a_x / 2\sin\theta + a_y / 2\cos\theta\right\}\end{aligned} \tag{3.4.12}$$

where ρ is the density. Now, as the element is shrunk in size down to the vertex *O*, $|AB| \to 0$, and Eqn. 3.4.6 is recovered. Thus the Stress Transformation Equations are valid provided the element under consideration is very small; in the limit, they are valid "at the point" *O*.

Finally, consider the case where the stress is not uniform over the faces of the triangular portion of material. Intuitively, it can be seen that, if one again shrinks the portion of material down in size to the vertex *O*, the Stress Transformation Equations will again be valid, with the quantities $\sigma'_{xx}, \sigma_{xx}, \sigma_{yy}$ etc. being the values "at" the vertex. To be more precise, consider the σ_{xx} stress acting over the face $|OB|$ in Fig. 3.4.11. No matter how the stress varies in the material, if the distance $|OB|$ is small, the stress can be approximated by a linear stress distribution, Fig. 3.4.11b. This linear distribution can itself be decomposed into two components, a uniform stress of magnitude σ^o_{xx} (the value of σ_{xx} at the vertex) and a triangular distribution with maximum value $\Delta\sigma_{xx}$. The resultant force on the face is then $|OB|\left(\sigma^o_{xx} + \Delta\sigma_{xx} / 2\right)$. This time, as the element is shrunk in size, $\Delta\sigma_{xx} \to 0$ and Eqn. 3.4.6 is again recovered. The same argument can be used to show that the Stress Transformation Equations are valid for any varying stress, body force or acceleration.

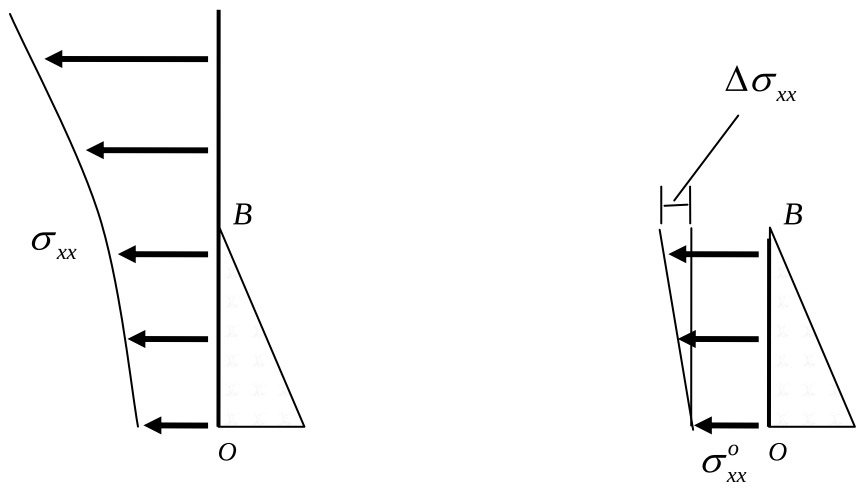


Figure 3.4.11: stress varying over a face; (a) stress is linear over OB if OB is small, (b) linear distribution of stress as a uniform stress and a triangular stress

Three Dimensions Re-visited

As the planes were rotated in the two-dimensional analysis, no consideration was given to the stresses acting in the "third dimension". Considering again a three dimensional block, Fig. 3.4.12, there is only one traction vector acting *on* the $x-y$ plane at the material particle, **t**. This traction vector can be described in terms of the x, y, z axes as $\mathbf{t} = \sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} + \sigma_{zz}\mathbf{k}$, Fig 3.4.12a. Alternatively, it can be described in terms of the x', y', z' axes as $\mathbf{t} = \sigma'_{zx}\mathbf{i}' + \sigma'_{zy}\mathbf{j}' + \sigma'_{zz}\mathbf{k}'$, Fig 3.4.12b.

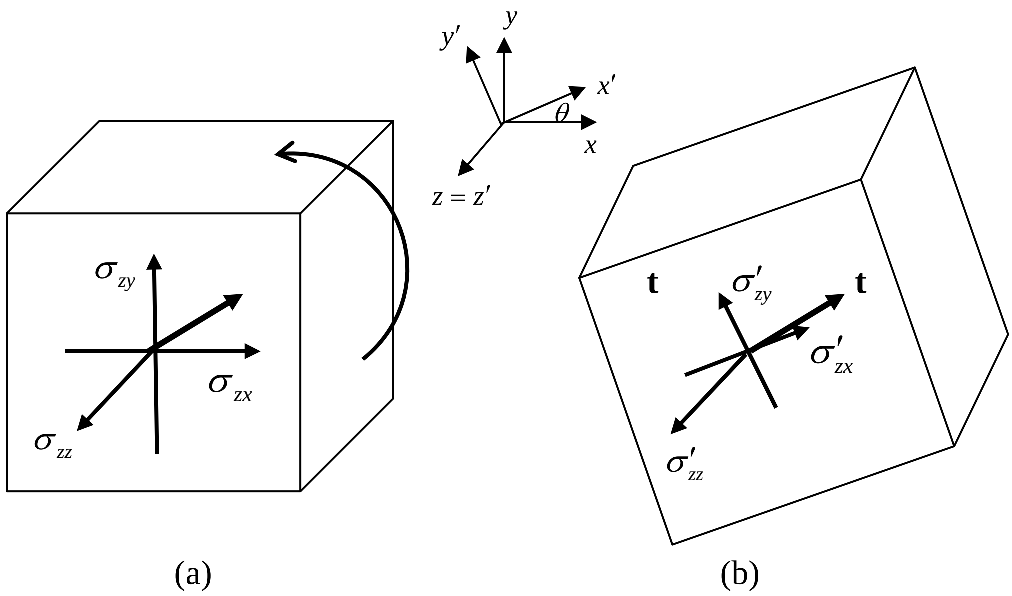


Figure 3.4.12: a three dimensional material element; (a) original element, (b) rotated element (rotation about the *z* axis)

With the rotation only happening *in* the $x-y$ plane, about the *z* axis, one has $\sigma_{zz} = \sigma'_{zz}$, $\mathbf{k} = \mathbf{k}'$. One can thus examine the two dimensional $x-y$ plane shown in Fig. 3.4.13, with

$$\sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} = \sigma'_{zx}\mathbf{i}' + \sigma'_{zy}\mathbf{j}' . \tag{3.4.13}$$

Using some trigonometry, one can see that

$$\begin{aligned} \sigma'_{zx} &= +\sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta \\ \sigma'_{zy} &= -\sigma_{zx}\sin\theta + \sigma_{zy}\cos\theta \end{aligned} . \tag{3.4.14}$$

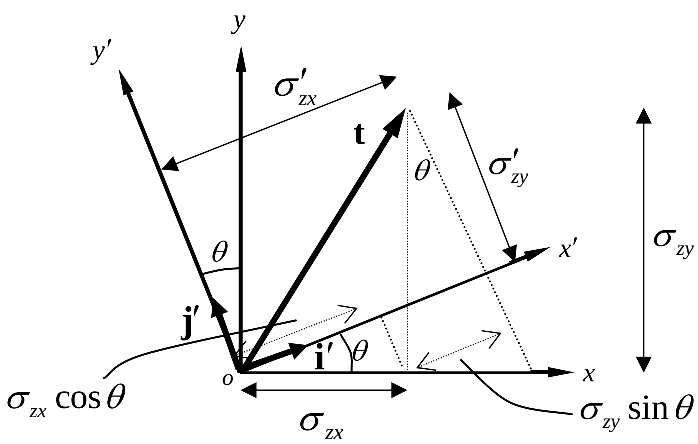


Figure 3.4.12: the traction vector represented using two different coordinate systems

3.5 Plane Stress

This section is concerned with a special two-dimensional state of stress called **plane stress**. It is important for two reasons: (1) it arises in real components (particularly in thin components loaded in certain ways), and (2) it is a two dimensional state of stress, and thus serves as an excellent introduction to more complicated three dimensional stress states.

3.5.1 Plane Stress

The state of plane stress is defined as follows:

Plane Stress:
If the stress state at a material particle is such that the only non-zero stress components act in one plane only, the particle is said to be in plane stress.

The axes are usually chosen such that the $x-y$ plane is the plane in which the stresses act, Fig. 3.5.1.

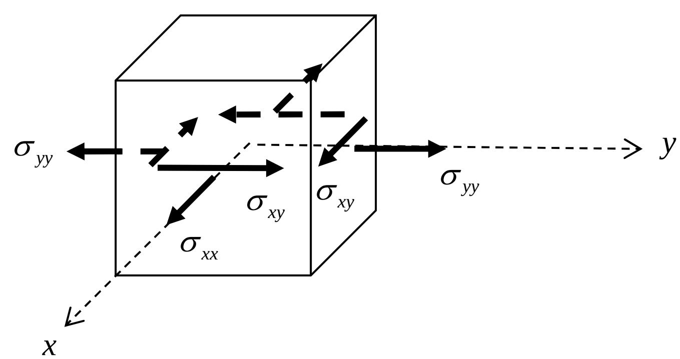


Figure 3.5.1: non-zero stress components acting in the $x-y$ plane

The stress can be expressed in the matrix form 3.4.1.

Example

The thick block of uniform material shown in Fig. 3.5.2, loaded by a constant stress σ_o in the x direction, will have $\sigma_{xx} = \sigma_0$ and all other components zero everywhere. It is therefore in a state of plane stress.

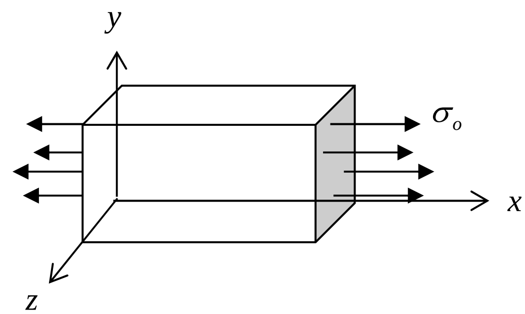


Figure 3.5.2: a thick block of material in plane stress

3.5.2 Analysis of Plane Stress

Next are discussed the **stress invariants**, **principal stresses** and **maximum shear stresses** for the two-dimensional plane state of stress, and tools for evaluating them. These quantities are useful because they tell us the complete state of stress at a point in simple terms. Further, these quantities are directly related to the strength and response of materials. For example, the way in which a material plastically (permanently) deforms is often related to the maximum shear stress, the directions in which flaws/cracks grow in materials are often related to the principal stresses, and the energy stored in materials is often a function of the stress invariants.

Stress Invariants

A stress invariant is some function of the stress components which is independent of the coordinate system being used; in other words, they have the same value no matter where the $x-y$ axes are drawn through a point. In a two dimensional space there are two stress invariants, labelled I_1 and I_2. These are

$$\boxed{\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} \\ I_2 &= \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 \end{aligned}} \qquad \textbf{Stress Invariants} \tag{3.5.1}$$

These quantities can be proved to be invariant directly from the stress transformation equations, Eqns. 3.4.9 {▲Problem 1}. Physically, invariance of I_1 and I_2 means that they are the same for any chosen perpendicular planes through a material particle.

Combinations of the stress invariants are also invariant, for example the important quantity

$$\tfrac{1}{2}I_1 \pm \sqrt{\tfrac{1}{4}I_1^2 - I_2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\tfrac{1}{4}\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \sigma_{xy}^2} \tag{3.5.2}$$

Principal Stresses

Consider a material particle for which the stress, with respect to some $x-y$ coordinate system, is

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.5.3}$$

The stress acting on different planes through the point can be evaluated using the Stress Transformation Equations, Eqns. 3.4.9, and the results are plotted in Fig. 3.5.3. The original planes are re-visited after rotating 180°.

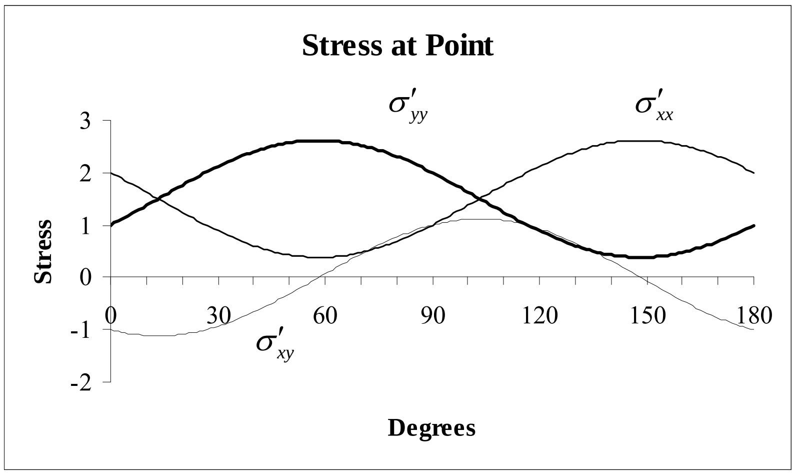


Figure 3.5.3: stresses on different planes through a point

It can be seen that there are two perpendicular planes for which the shear stress is zero, for θ ≈ 58° and $\theta \approx (58+90)^{\circ}$. In fact it can be proved that for every point in a material there are two (and only two) perpendicular planes on which the shear stress is zero (see below). These planes are called the **principal planes**. It will also be noted from the figure that the normal stresses acting on the planes of zero shear stress are either a maximum or minimum. Again, this can be proved (see below). These normal stresses are called principal stresses. The principal stresses are labelled σ_1 and σ_2, Fig. 3.5.4.

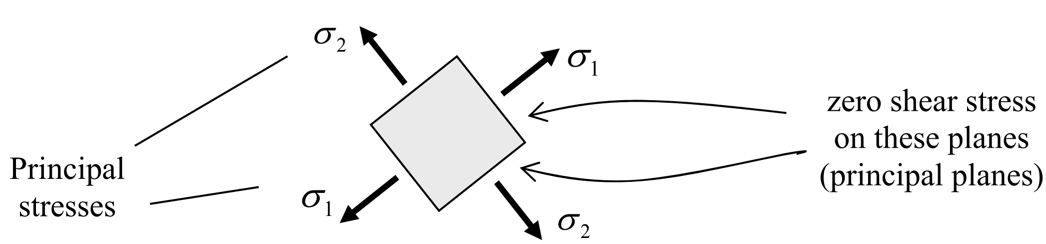


Figure 3.5.4: principal stresses

The principal stresses can be obtained by setting $\sigma'_{xy} = 0$ in the Stress Transformation Equations, Eqns. 3.4.9, which leads to the value of θ for which the planes have zero shear stress:

$$\boxed{\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}} \qquad \textbf{Location of Principal Planes} \qquad (3.5.4)$$

For the example stress state, Eqn. 3.5.3, this leads to

$$\theta = \frac{1}{2}\arctan(-2)$$

and so the perpendicular planes are at $\theta = -31.72^\circ \ (148.28^\circ)$ and $\theta = 58.3^\circ$.

Explicit expressions for the principal stresses can be obtained by substituting the value of θ from Eqn. 3.5.4 into the Stress Transformation Equations, leading to (see the Appendix to this section, §3.5.7)

$$\boxed{\begin{aligned} \sigma_1 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \\ \sigma_2 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} \end{aligned}} \qquad \textbf{Principal Stresses} \qquad (3.5.5)$$

For the example stress state Eqn.3.5.3, one has

$$\sigma_1 = \frac{3+\sqrt{5}}{2} \approx 2.62, \qquad \sigma_2 = \frac{3-\sqrt{5}}{2} \approx 0.38$$

Note here that one uses the symbol σ_1 to represent the maximum principal stress and σ_2 to represent the minimum principal stress. By maximum, it is meant the algebraically largest stress so that, for example, $+1 > -3$.

From Eqns. 3.5.2, 3.5.5, the principal stresses are invariant; they are intrinsic features of the stress state at a point and do not depend on the coordinate system used to describe the stress state.

The question now arises: why are the principal stresses so important? One part of the answer is that the maximum principal stress is the largest normal stress acting on any plane through a material particle. This can be proved by differentiating the stress transformation formulae with respect to θ,

$$
\begin{aligned}
\frac{d\sigma'_{xx}}{d\theta} &= -\sin 2\theta(\sigma_{xx}-\sigma_{yy}) + 2\cos 2\theta\sigma_{xy} \\
\frac{d\sigma'_{yy}}{d\theta} &= +\sin 2\theta(\sigma_{xx}-\sigma_{yy}) - 2\cos 2\theta\sigma_{xy} \\
\frac{d\sigma'_{xy}}{d\theta} &= -\cos 2\theta(\sigma_{xx}-\sigma_{yy}) - 2\sin 2\theta\sigma_{xy}
\end{aligned}
\tag{3.5.6}
$$

The maximum/minimum values can now be obtained by setting these expressions to zero. One finds that the normal stresses are a maximum/minimum at the very value of θ in Eqn. 3.5.4 – the value of θ for which the shear stresses are zero – the principal planes.

Very often the only thing one knows about the stress state at a point are the principal stresses. In that case one can derive a very useful formula as follows: align the coordinate axes in the principal directions, so

$$
\sigma_{xx} = \sigma_1, \quad \sigma_{yy} = \sigma_2, \quad \sigma_{xy} = 0 \tag{3.5.7}
$$

Using the transformation formulae with the relations $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ then leads to

$$
\boxed{
\begin{aligned}
\sigma'_{xx} &= \frac{1}{2}(\sigma_1+\sigma_2) + \frac{1}{2}(\sigma_1-\sigma_2)\cos 2\theta \\
\sigma'_{yy} &= \frac{1}{2}(\sigma_1+\sigma_2) - \frac{1}{2}(\sigma_1-\sigma_2)\cos 2\theta \\
\sigma'_{xy} &= -\frac{1}{2}(\sigma_1-\sigma_2)\sin 2\theta
\end{aligned}
}
\tag{3.5.8}
$$

Here, θ is measured *from* the principal directions, as illustrated in Fig. 3.5.5.

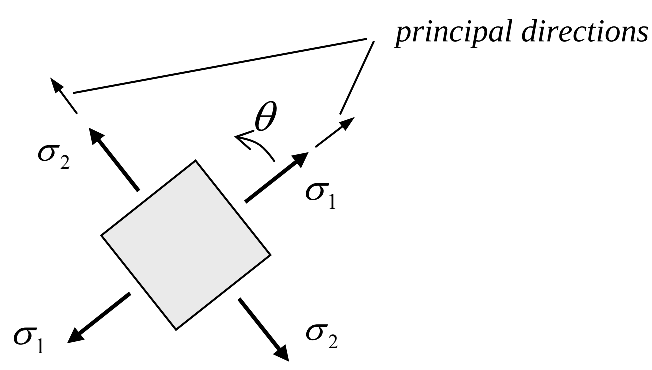


Figure 3.5.5: principal stresses and principal directions

The Third Principal Stress

Although plane stress is essentially a two-dimensional stress-state, it is important to keep in mind that any real particle is three-dimensional. The stresses acting *on* the $x-y$ plane are the normal stress σ_{zz} and the shear stresses σ_{zx} and σ_{zy}, Fig. 3.5.6. These are all zero (in plane stress). It was discussed above how the principal stresses occur on planes of zero shear stress. Thus the σ_{zz} stress is also a principal stress. Technically speaking, there are always three principal stresses in three dimensions, and (at least) one of these will be zero in plane stress. This fact will be used below in the context of maximum shear stress.

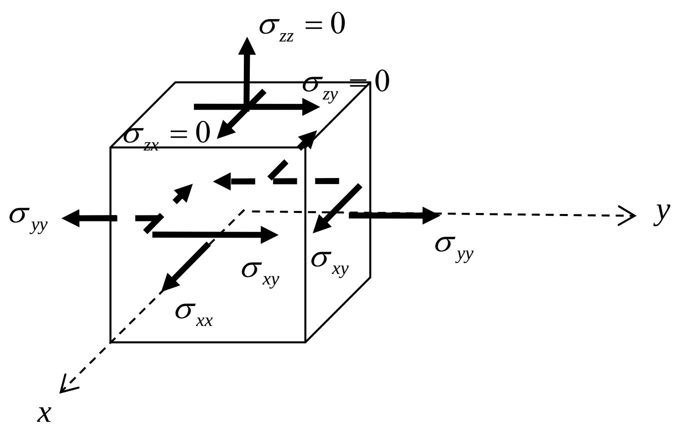


Figure 3.5.6: stresses acting on the *x – y* plane

Maximum Shear Stress

Eqns. 3.5.8 can be used to derive an expression for the maximum shear stress. Differentiating the expression for shear stress with respect to θ, setting to zero and solving, shows that the maximum/minimum occurs at $\theta = \pm 45$, in which case

$$\sigma_{xy}\Big|_{\theta=+45} = -\frac{1}{2}(\sigma_1 - \sigma_2), \qquad \sigma_{xy}\Big|_{\theta=-45} = +\frac{1}{2}(\sigma_1 - \sigma_2)$$

or

$$\boxed{\max(\sigma_{xy}) = \frac{1}{2}\left|\sigma_1 - \sigma_2\right|} \qquad \textbf{Maximum Shear Stress} \qquad (3.5.9)$$

Thus the shear stress reaches a maximum on planes which are oriented at $\pm 45^\circ$ to the principal planes, and the value of the shear stress acting on these planes is as given above. Note that the formula Eqn. 3.5.9 does not let one know in which *direction* the shear stresses are acting but this is not usually an important issue. Many materials respond in certain ways when the maximum shear stress reaches a critical value, and the actual direction of shear stress is unimportant. The direction of the maximum principal stress is, on the other hand, important – a material will in general respond differently according to whether the normal stress is compressive or tensile.

The normal stress acting on the planes of maximum shear stress can be obtained by substituting $\theta = \pm 45$ back into the formulae for normal stress in Eqn. 3.5.8, and one sees that

$$\sigma'_{xx} = \sigma'_{yy} = (\sigma_1 + \sigma_2)/2 \tag{3.5.10}$$

The results of this section are summarised in Fig. 3.5.7.

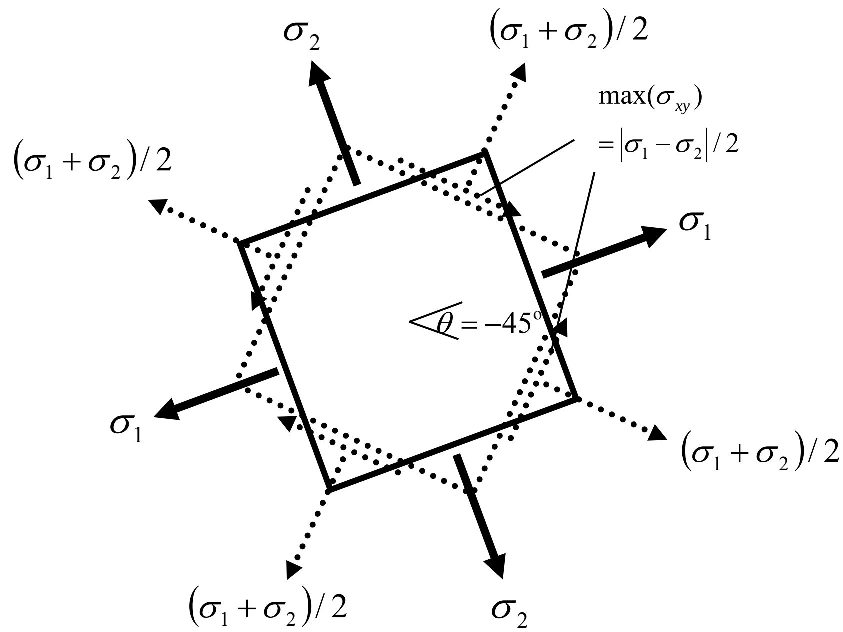


Figure 3.5.7: principal stresses and maximum shear stresses acting in the $x - y$ plane

The maximum shear stress in the $x - y$ plane was calculated above, Eqn. 3.5.9. This is not necessarily the maximum shear stress acting at the material particle. In general, it can be shown that the maximum shear stress is the maximum of the following three terms (see Part III, §3.4.3):

$$\frac{1}{2}|\sigma_1 - \sigma_2|, \quad \frac{1}{2}|\sigma_1 - \sigma_3|, \quad \frac{1}{2}|\sigma_2 - \sigma_3|$$

The first term is the maximum shear stress in the $1-2$ plane, i.e. the plane containing the σ_1 and σ_2 stresses (and given by Eqn. 3.5.9). The second term is the maximum shear stress in the $1-3$ plane and the third term is the maximum shear stress in the $2-3$ plane. These are sketched in Fig. 3.5.8 below.

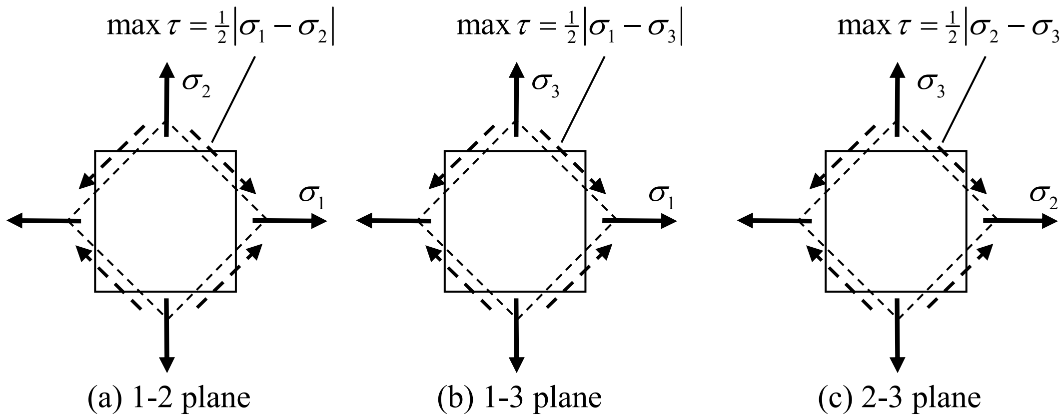


Figure 3.5.8: principal stresses and maximum shear stresses

In the case of plane stress, $\sigma_3 = \sigma_{zz} = 0$, and the maximum shear stress will be (see the Appendix to this section, §3.5.7)

$$\max\left\{\frac{1}{2}|\sigma_1 - \sigma_2|, \quad \frac{1}{2}|\sigma_1|, \quad \frac{1}{2}|\sigma_2|\right\} \tag{3.5.11}$$

3.5.3 Stress Boundary Conditions

When solving problems, information is usually available on what is happening at the boundaries of materials. This information is called the **boundary conditions**. Information is usually not available on what is happening in the interior of the material – information there is obtained by solving the equations of mechanics.

A number of different conditions can be known at a boundary, for example it might be known that a certain part of the boundary is fixed so that the displacements there are zero. This is known as a **displacement boundary condition**. On the other hand the stresses over a certain part of the material boundary might be known. These are known as **stress boundary conditions** – this case will be examined here.

General Stress Boundary Conditions

It has been seen already that, when one material contacts a second material, a force, or distribution of stress arises. This force F will have arbitrary direction, Fig. 3.5.9a, and can be decomposed into the sum of a normal stress distribution σ_N and a shear distribution σ_S, Fig. 3.5.9b. One can introduce a coordinate system to describe the applied stresses, for example the $x-y$ axes shown in Fig. 3.5.9c (the axes are most conveniently defined to be normal and tangential to the boundary).

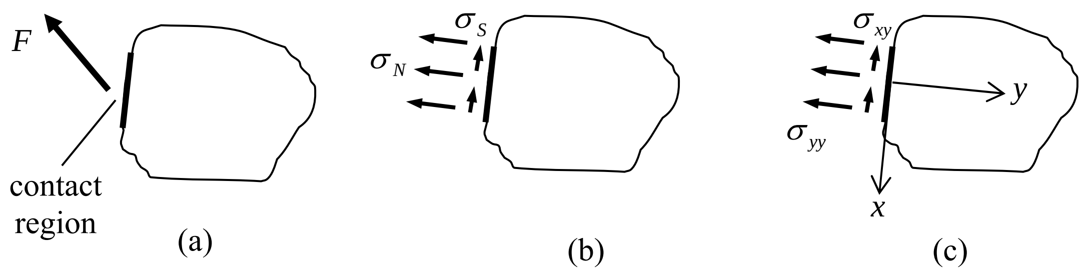


Figure 3.5.9: Stress boundary conditions; (a) force acting on material due to contact with a second material, (b) the resulting normal and shear stress distributions, (c) applied stresses as stress components in a given coordinate system

Figure 3.5.10 shows the same component as Fig. 3.5.9. Shown in detail is a small material element at the boundary. From equilibrium of the element, stresses σ_{xy}, σ_{yy}, equal to the applied stresses, must be acting inside the material, Fig. 3.5.10a. Note that the **tangential stresses**, which are the σ_{xx} stresses in this example, can take on any value and the element will still be in equilibrium with the applied stresses, Fig. 3.5.10b.

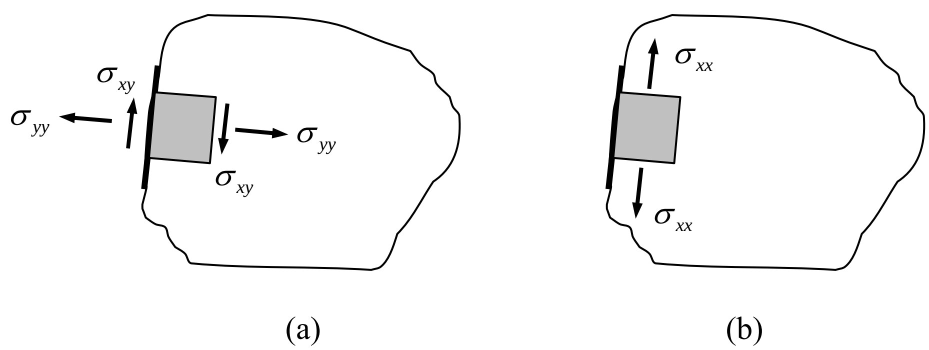


Figure 3.5.10: Stresses acting on a material element at the boundary, (a) normal and shear stresses, (b) tangential stresses

Thus, if the applied stresses are *known*, then so also are the normal and shear stresses acting at the boundary of the material.

Stress Boundary Conditions at a Free Surface

A free surface is a surface that has "nothing" on one side and so there is nothing to provide reaction forces. Thus there must also be no normal or shear stress on the other side (the inside).

This leads to the following, Fig. 3.5.11:

Stress boundary conditions at a free surface:
the normal and shear stress at a free surface are zero

This simple fact is used again and again to solve practical problems.

Again, the stresses acting normal to any other plane at the surface do not have to be zero – they can be balanced as, for example, the tangential stresses σ_T and the stress $\bar{\sigma}$ in Fig. 3.5.11.

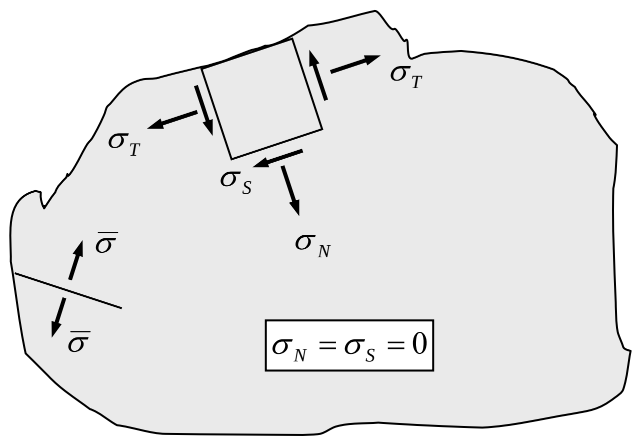


Figure 3.5.11: A free surface - the normal and shear stresses there are zero

Atmospheric Pressure

There *is* something acting on the outside "free" surfaces of materials – the atmospheric pressure. This is a type of stress which is **hydrostatic**, that is, it acts normal at all points, as shown in Fig. 3.5.12. Also, it does not vary much. This pressure is present when one characterises a material, that is, when its material properties are determined from tests and so on, for example, its Young's Modulus (see Chapter 5). The atmospheric pressure is therefore a datum – stresses are really measured relative to this value, and so the atmospheric pressure is ignored.

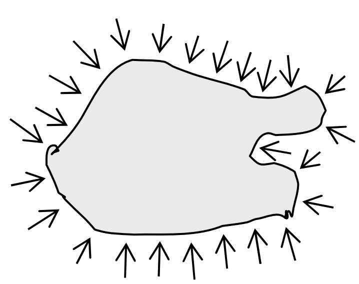


Figure 3.5.12: a material subjected to atmospheric pressure

3.5.4 Thin Components

Consider a thin component as shown in Fig. 3.5.13. With the coordinate axes aligned as shown, and with the large face free of loading, one has $\sigma_{zx} = \sigma_{zy} = \sigma_{zz} = 0$. Strictly speaking, these stresses are zero *only* at the free surfaces of the material but, because it is thin, these stresses should not vary much from zero within. Taking the "z" stresses to be identically zero throughout the material, the component is in a state of plane stress[1]. On the other hand, were the sheet not so thin, the stress components that were zero at the free-surfaces might well deviate significantly from zero deep within the material and one could not safely argue that the component was in a state of plane stress.

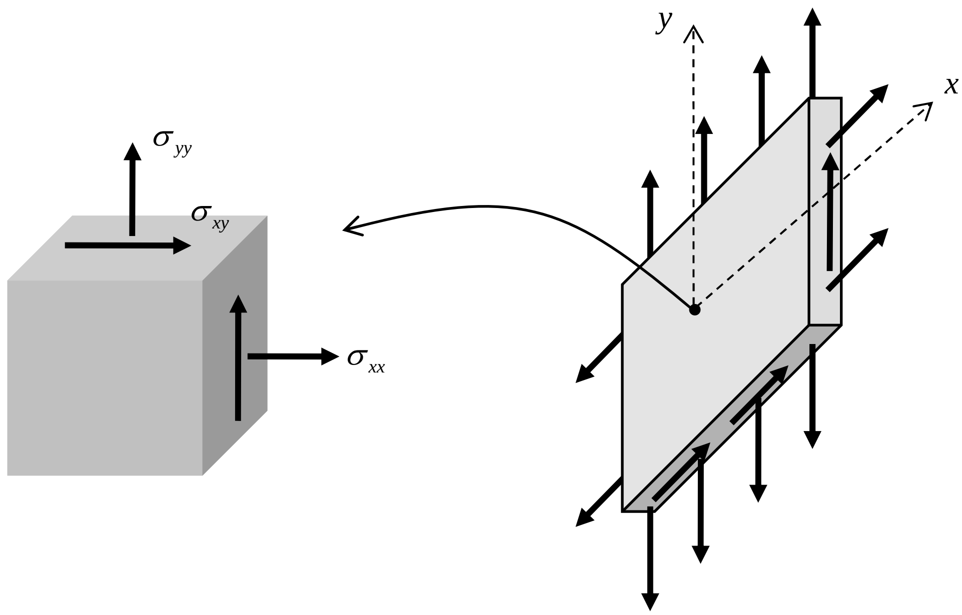
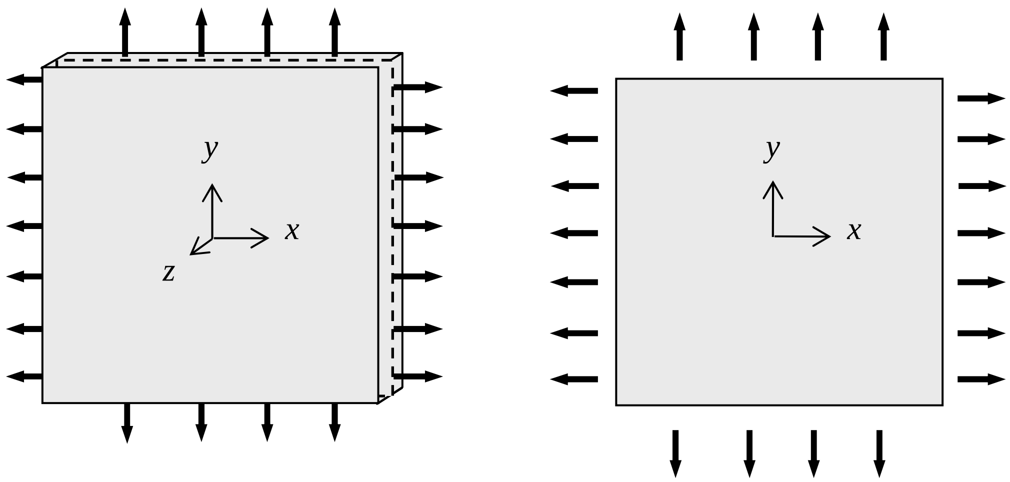


Figure 3.5.13: a thin material loaded in-plane, leading to a state of plane stress

When analysing plane stress states, only one cross section of the material need be considered. This is illustrated in Fig. 3.5.14.

[1] it can be shown that, when the applied stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ vary only *linearly* over the thickness of the component, the stresses $\sigma_{zz}, \sigma_{zx}, \sigma_{zy}$ are exactly zero throughout the component, otherwise they are only approximately zero



3 – dimensional material

2 – dimensional cross-section of material

Figure 3.5.14: one two-dimensional cross-section of material

Note that, although the stress normal to the plane, σ_{zz}, is zero, the three dimensional sheet of material *is* deforming in this direction – it will obviously be getting thinner under the tensile loading shown in Fig. 3.5.14.

Note that plane stress arises in *all* thin materials (loaded in –plane), no matter what they are made of.

3.5.5 Mohr's Circle

Otto Mohr devised a way of describing the state of stress at a point using a single diagram, called the **Mohr's circle**.

To construct the Mohr circle, first introduce the **stress coordinates** (σ, τ), Fig. 3.5.15; the abscissae (horizontal) are the normal stresses σ and the ordinates (vertical) are the shear stresses τ. On the horizontal axis, locate the principal stresses σ_1, σ_2, with $\sigma_1 > \sigma_2$. Next, draw a circle, centred at the average principal stress $(\sigma, \tau) = ((\sigma_1 + \sigma_2)/2, 0)$, having radius $(\sigma_1 - \sigma_2)/2$.

The normal and shear stresses acting on a single plane are represented by a single point on the Mohr circle. The normal and shear stresses acting on two perpendicular planes are represented by two points, one at *each end of a diameter* on the Mohr circle. Two such diameters are shown in the figure. The first is horizontal. Here, the stresses acting on two perpendicular planes are $(\sigma, \tau) = (\sigma_1, 0)$ and $(\sigma, \tau) = (\sigma_2, 0)$ and so this diameter represents the principal planes/stresses.

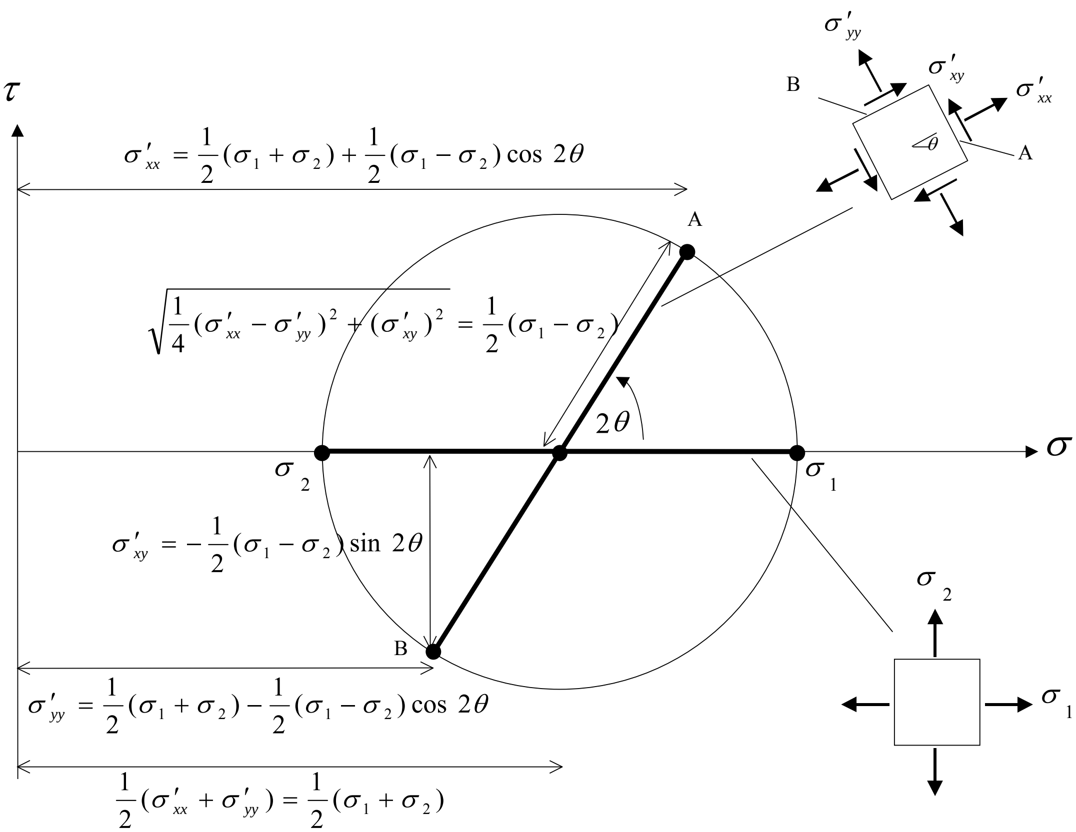


Figure 3.5.15: Mohr's Circle

The stresses on planes rotated by an amount θ from the principal planes are given by Eqn. 3.5.8. Using elementary trigonometry, these stresses are represented by the points A and B in Fig. 3.5.15. Note that a rotation of θ in the physical plane corresponds to a rotation of 2θ in the Mohr diagram.

Note also that the conventional labeling of shear stress has to be altered when using the Mohr diagram. On the Mohr circle, a shear stress is positive if it yields a clockwise moment about the centre of the element, and is "negative" when it yields a negative moment. For example, at point A the shear stress is "positive" ($\tau > 0$), which means the direction of shear on face A of the element is actually opposite to that shown. This agrees with the formula $\sigma'_{xy} = -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta$, which is less than zero for $\sigma_1 > \sigma_2$ and $\theta \le 90^{\circ}$. At point B the shear stress is "negative" ($\tau < 0$), which again agrees with formula.

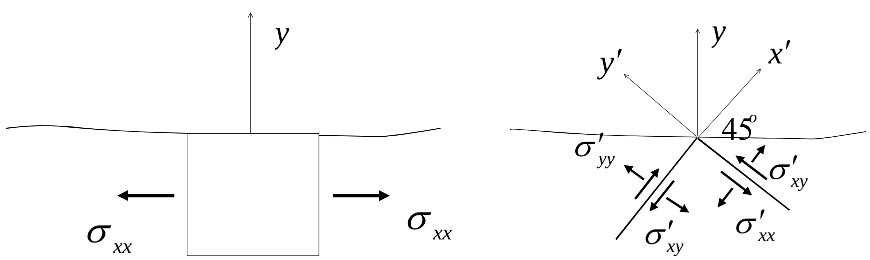
3.5.6 Problems

1. Prove that the function $\sigma_x + \sigma_y$, i.e. the sum of the normal stresses acting at a point, is a stress invariant. [Hint: add together the first two of Eqns. 3.4.9.]

2. Consider a material in plane stress conditions. An element at a free surface of this material is shown below left. Taking the coordinate axes to be orthogonal to the surface as shown (so that the tangential stress is σ_{xx}), one has

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & 0 \\ 0 & 0 \end{bmatrix}$$

   (a) what are the two in-plane principal stresses at the point? Which is the maximum and which is the minimum?
   (b) examine planes inclined at 45° to the free surface, as shown below right. What are the stresses acting on these planes and what have they got to do with maximum shear stress?



3. The stresses at a point in a state of plane stress are given by

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 3 & 2 \end{bmatrix}$$

   (a) Draw a little box to represent the point and draw some arrows to indicate the magnitude and direction of the stresses acting at the point.
   (b) What relationship exists between Oxy and a second coordinate set $Ox'y'$, such that the shear stresses are zero in $Ox'y'$?
   (c) Find the two in-plane principal stresses.
   (d) Draw another box whose sides are aligned to the principal directions and draw some arrows to indicate the magnitude and direction of the principal stresses acting at the point.
   (e) Check that the sum of the normal stresses at the point is an invariant.

4. A material particle is subjected to a state of stress given by

$$[\sigma_{ij}] = \begin{bmatrix} \alpha & \alpha & 0 \\ \alpha & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   Find the principal stresses (all three), maximum shear stresses (see Eqn. 3.5.11), and the direction of the planes on which these stresses act.

5. Consider the following state of stress (with respect to an x, y, z coordinate system):

$$\begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   (a) Use the stress transformation equations to derive the stresses acting on planes obtained from the original planes by a counterclockwise rotation of 45° about z axis.
   (b) What is the maximum normal stress acting at the point?
   (c) What is the maximum shear stress? On what plane(s) does it act? (See Eqn. 3.5.11.)

6. Consider the two dimensional stress state

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

   Show that this is an **isotropic state of stress**, that is, the stress components are the same on *all* planes through a material particle.

7. (a) Is a trampoline (the material you jump on) in a state of plane stress? When someone is actually jumping on it?
   (b) Is a picture hanging on a wall in a state of plane stress?
   (c) Is a glass window in a state of plane stress? On a very windy day?
   (d) A piece of rabbit skin is stretched in a testing machine – is it in a state of plane stress?

3.5.7 Appendix to §3.5

A Note on the Formulae for Principal Stresses

To derive Eqns. 3.5.5, first rewrite the transformation equations in terms of 2θ using $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ to get

$$\begin{aligned} \sigma'_{xx} &= \tfrac{1}{2}(1+\cos 2\theta)\sigma_{xx} + \tfrac{1}{2}(1-\cos 2\theta)\sigma_{yy} + \sin 2\theta\sigma_{xy} \\ \sigma'_{yy} &= \tfrac{1}{2}(1-\cos 2\theta)\sigma_{xx} + \tfrac{1}{2}(1+\cos 2\theta)\sigma_{yy} - \sin 2\theta\sigma_{xy} \\ \sigma'_{xy} &= \tfrac{1}{2}\sin 2\theta(\sigma_{yy} - \sigma_{xx}) + \cos 2\theta\sigma_{xy} \end{aligned}$$

Next, from Eqn. 3.5.4,

$$\sin 2\theta = \frac{2\sigma_{xy}}{\sqrt{(\sigma_{xx}-\sigma_{yy})^2 + 4\sigma_{xy}^2}}, \qquad \cos 2\theta = \frac{\sigma_{xx}-\sigma_{yy}}{\sqrt{(\sigma_{xx}-\sigma_{yy})^2 + 4\sigma_{xy}^2}}$$

Substituting into the rewritten transformation formulae then leads to

$$\begin{aligned}
\sigma'_{xx} &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy})+\sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2} \\
\sigma'_{yy} &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy})-\sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2} \\
\sigma'_{xy} &= 0
\end{aligned}$$

Here $\sigma'_{xx} > \sigma'_{yy}$ so that the maximum principal stress is $\sigma_1 = \sigma'_{xx}$ and the minimum principal stress is $\sigma'_2 = \sigma'_{yy}$. Here it is implicitly assumed that $\tan 2\theta > 0$, i.e. that $0 < 2\theta < 90$ or $180 < 2\theta < 270$. On the other hand one could assume that $\tan 2\theta < 0$, i.e. that $90 < 2\theta < 180$ or $270 < 2\theta < 360$, in which case one arrives at the formulae

$$\begin{aligned}
\sigma'_{xx} &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy})-\sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2} \\
\sigma'_{yy} &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy})+\sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2}
\end{aligned}$$

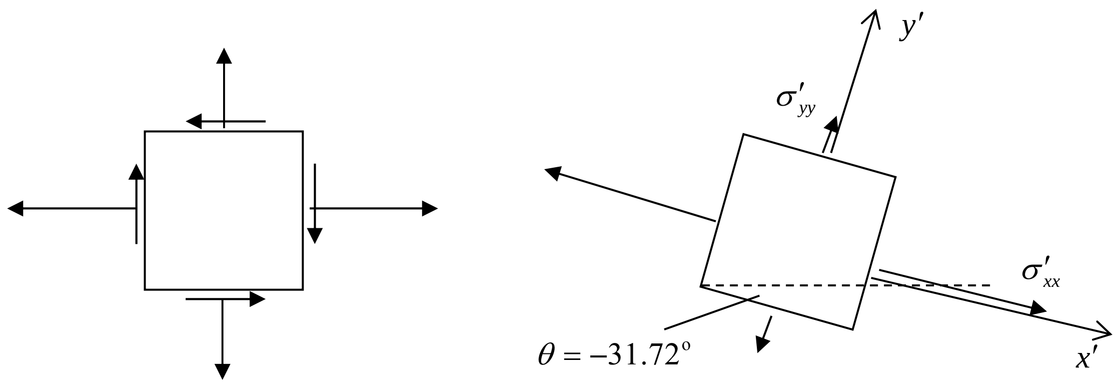
The results can be summarised as Eqn. 3.5.5,

$$\begin{aligned}
\sigma_1 &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy})+\sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2} \\
\sigma_2 &= \tfrac{1}{2}(\sigma_{xx}+\sigma_{yy})-\sqrt{\tfrac{1}{4}(\sigma_{xx}-\sigma_{yy})^2+\sigma_{xy}^2}
\end{aligned}$$

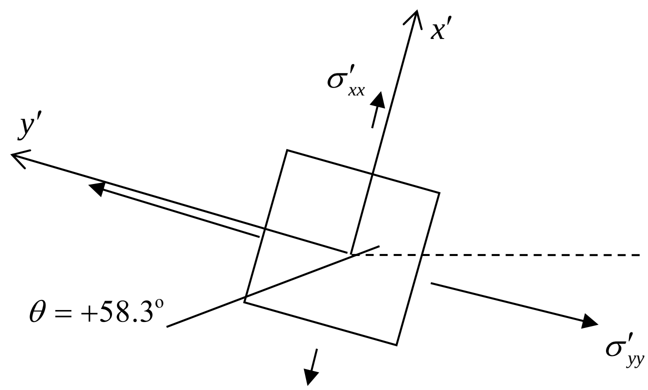
These formulae do not tell one on which of the two principal planes the maximum principal stress acts. This might not be an important issue, but if this information is required one needs to go directly to the stress transformation equations. In the example stress state, Eqn. 3.5.3, one has

$$\begin{aligned}
\sigma'_{xx} &= \cos^2\theta(2)+\sin^2\theta(1)+\sin 2\theta(-1) \\
\sigma'_{yy} &= \sin^2\theta(2)+\cos^2\theta(1)-\sin 2\theta(-1)
\end{aligned}$$

For $\theta = -31.72^\circ\ (148.28^\circ)$, $\sigma'_{xx} = 2.62$ and $\sigma'_{yy} = 0.38$. So one has the situation shown below.



If one takes the other angle, $\theta = 58.3°$, one has $\sigma'_{xx} = 0.38$ and $\sigma'_{yy} = 2.62$, and the situation below



A Note on the Maximum Shear Stress

Shown below left is a box element with sides perpendicular to the $1,2,z$ axes, i.e. aligned with the principal directions. The stresses in the new x', y' axis system shown are given by Eqns. 3.5.8, with θ measured from the principal directions:

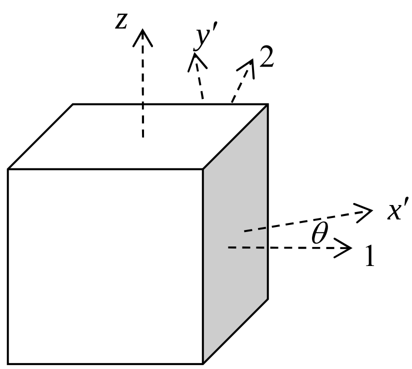
$$\begin{aligned}
\sigma'_{xx} &= \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\
\sigma'_{yy} &= \frac{1}{2}(\sigma_1 + \sigma_2) - \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\
\sigma'_{xy} &= -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta
\end{aligned}$$

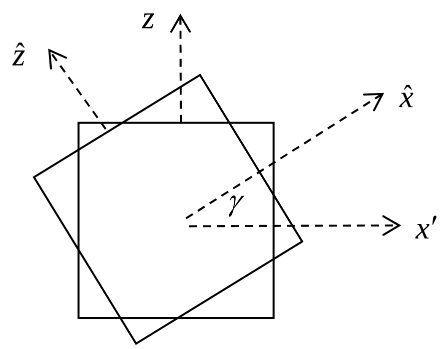
Now as well as rotating around in the $1-2$ plane through an angle θ, rotate also in the x', z plane through an angle γ (see below right). This rotation leads to the new stresses

$$\begin{aligned}
\hat{\sigma}_{xx} &= \cos^2\gamma\sigma'_{xx} + \sin^2\gamma\sigma_{zz} + \sin 2\gamma\sigma_{x'z} \\
\hat{\sigma}_{zz} &= \sin^2\gamma\sigma'_{xx} + \cos^2\gamma\sigma_{zz} - \sin 2\gamma\sigma_{x'z} \\
\hat{\sigma}_{xz} &= \sin\gamma\cos\gamma(\sigma_{zz} - \sigma'_{xx}) + \cos 2\gamma\sigma_{x'z}
\end{aligned}$$

In plane stress, $\sigma_{zz} = \sigma_{x'z} = 0$, so one has the stresses

$$\hat{\sigma}_{xx} = \cos^2\gamma\sigma'_{xx}, \quad \hat{\sigma}_{yy} = \sin^2\gamma\sigma'_{xx}, \quad \hat{\sigma}_{xy} = -\tfrac{1}{2}\sin 2\gamma\sigma'_{xx}$$





The shear stress can be written out in full:

$$\hat{\sigma}_{xy}(\gamma,\theta) = -\tfrac{1}{2}\sin 2\gamma\left[\tfrac{1}{2}(\sigma_1 + \sigma_2) + \tfrac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta\right].$$

This is a function of two variables; its minimum value can be found by setting the partial derivatives with respect to these variables to zero. Differentiating,

$$\begin{aligned}
\partial\hat{\sigma}_{xy} / \partial\gamma &= -\cos 2\gamma\left[\tfrac{1}{2}(\sigma_1 + \sigma_2) + \tfrac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta\right] \\
\partial\hat{\sigma}_{xy} / \partial\theta &= -\tfrac{1}{2}\sin 2\gamma\left[-(\sigma_1 - \sigma_2)\sin 2\theta\right]
\end{aligned}$$

Setting to zero gives the solutions $\sin 2\theta = 0$, $\cos 2\gamma = 0$, i.e. $\theta = 0$, $\gamma = 45^{\circ}$. Thus the maximum shear stress occurs at 45° to the $1-2$ plane, and in the $1-z$, i.e. $1-3$ plane (as in Fig. 3.5.8b). The value of the maximum shear stress here is then $\left|\hat{\sigma}_{xy}\right| = \left|\tfrac{1}{2}\sigma_1\right|$, which is the expression in Eqn. 3.5.11.

3.5b Stress Boundary Conditions: Continued

Consider now in more detail a surface between two different materials, Fig. 3.5.16. One says that the normal and shear stresses are **continuous** across the surface, as illustrated.

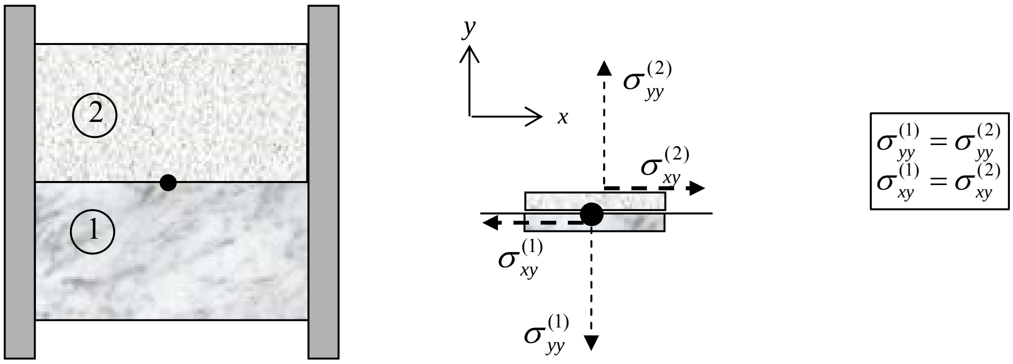


Figure 3.5.16: normal and shear stress continuous across an interface between two different materials, material '1' and material '2'

Note also that, since the shear stress σ_{xy} is the same on both sides of the surface, the shear stresses acting on both sides of a perpendicular plane passing *through* the interface between the materials, by the symmetry of stress, must also be the same, Fig. 3.5.17a.

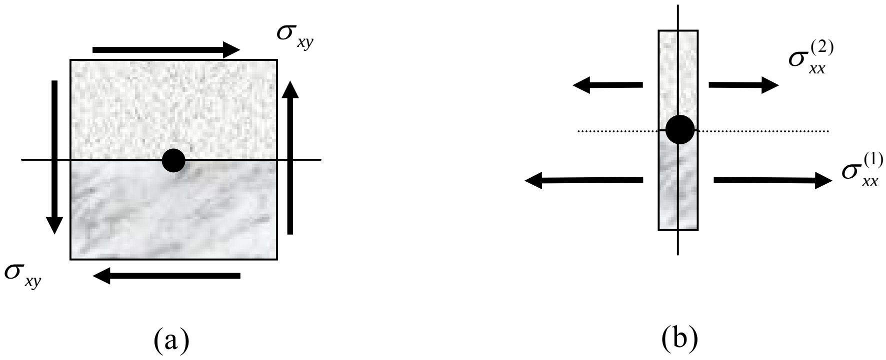


Figure 3.5.17: stresses at an interface; (a) shear stresses continuous across the interface, (b) tangential stresses not necessarily continuous

However, again, the tangential stresses, those acting parallel to the interface, do *not* have to be equal. For example, shown in Fig. 3.5.17b are the tangential stresses acting in the upper material, $\sigma_{xx}^{(2)}$ - they balance no matter what the magnitude of the stresses $\sigma_{xx}^{(1)}$.

Description of Boundary Conditions

The following example brings together the notions of stress boundary conditions, stress components, equilibrium and equivalent forces.

Example

Consider the plate shown in Fig. 3.5.18. It is of width $2a$, height *b* and depth *t*. It is subjected to a tensile stress *r*, pressure *p* and shear stresses *s*. The applied stresses are uniform through the thickness of the plate. It is welded to a rigid base.

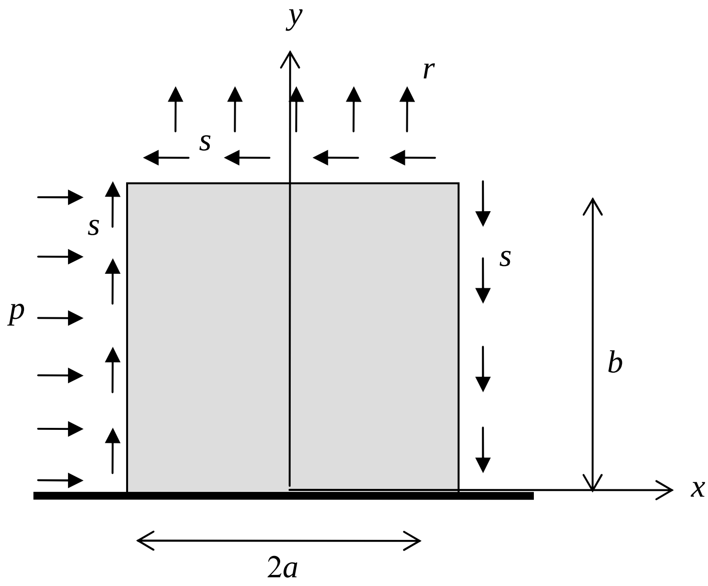


Figure 3.5.18: a plate subjected to stress distributions

Using the $x-y$ axes shown, the stress boundary conditions can be expressed as:

Left-hand surface: $\begin{cases} \sigma_{xx}(-a,y) = -p \\ \sigma_{xy}(-a,y) = -s \end{cases}, \quad 0 < y < b$

Top surface: $\begin{cases} \sigma_{yy}(x,b) = +r \\ \sigma_{xy}(x,b) = -s \end{cases}, \quad -a < x < +a$

Right-hand surface: $\begin{cases} \sigma_{xx}(+a,y) = 0 \\ \sigma_{xy}(+a,y) = -s \end{cases}, \quad 0 < y < b$

Note carefully the description of the normal and shear stresses over each side and the signs of the stress components.

The stresses at the lower edge are unknown (there is a displacement boundary condition there: zero displacement). They will in general not be uniform. Using the given $x-y$ axes, these unknown reaction stresses, exerted by the base on the plate, are (see Fig 3.5.19)

Lower surface: $\begin{cases} \sigma_{yy}(x,0) \\ \sigma_{xy}(x,0) \end{cases}, \qquad -a < x < +a$

Note the directions of the arrows in Fig. 3.5.19, they have been drawn in the direction of positive $\sigma_{yy}(x,0)$, $\sigma_{xy}(x,0)$.

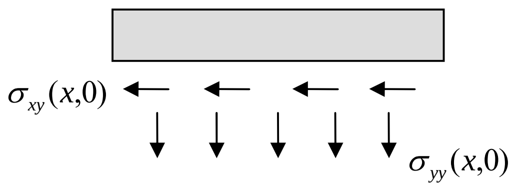


Figure 3.5.19: unknown reaction stresses acting on the lower edge

For force equilibrium of the complete plate, consider the free-body diagram 3.5.20; shown are the resultant forces of the stress distributions. Force equilibrium requires that

$$\sum F_x = bpt - 2ast - t\int_{-a}^{+a} \sigma_{xy}(x,0)dx = 0$$

$$\sum F_y = 2art - t\int_{-a}^{+a} \sigma_{yy}(x,0)dx = 0$$

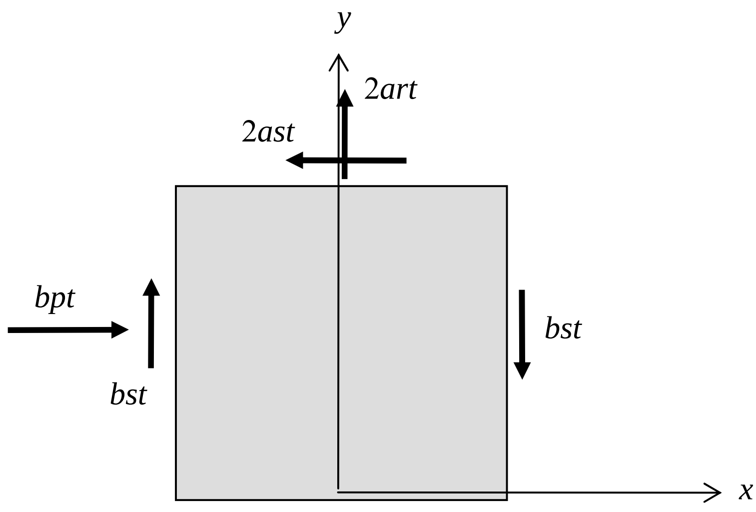


Figure 3.5.20: a free-body diagram of the plate in Fig. 3.5.18 showing the known resultant forces (forces on the lower boundary are not shown)

For moment equilibrium, consider the moments about, for example, the lower left-hand corner. One has

$$\sum M_0 = -bpt(b/2) + 2ast(b) + 2art(a) - bst(2a) - t\int_{-a}^{+a} \sigma_{yy}(x,0) \times (a+x) dx = 0$$

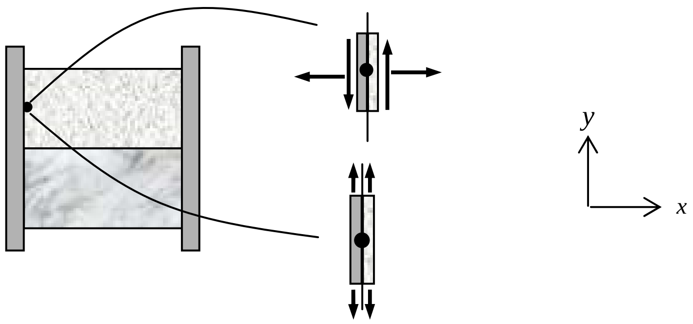
If one had taken moments about the top-left corner, the equation would read

$$\sum M_0 = +bpt(b/2) + 2art(a) - bst(2a)$$
$$- t\int_{-a}^{+a} \sigma_{xy}(x,0) \times b\, dx - t\int_{-a}^{+a} \sigma_{yy}(x,0) \times (a+x) dx = 0$$

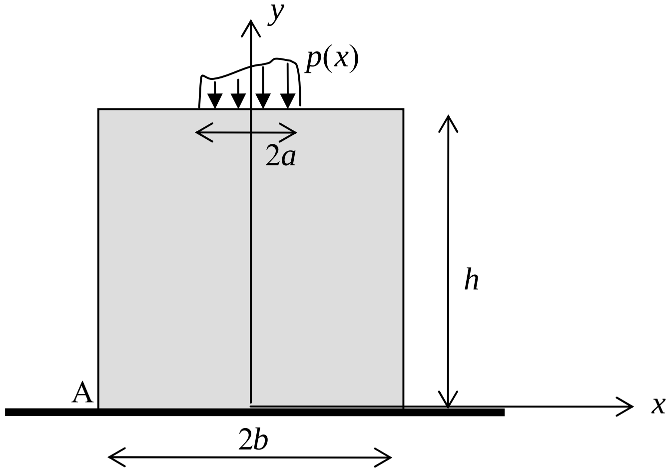
■

Problems

8. Consider the point shown below, at the boundary between a wall and a dissimilar material. Label the stress components displayed using the coordinate system shown. Which stress components are continuous across the wall/material boundary? (Add a superscript 'w' for the stresses in the wall.)



9. A thin metal plate of width $2b$, height h and depth t is loaded by a pressure distribution $p(x)$ along $-a < x < +a$ and welded at its base to the ground, as shown in the figure below. Write down expressions for the stress boundary conditions (two on each of the three edges). Write down expressions for the force equilibrium of the plate and moment equilibrium of the plate about the corner A.



4 Strain

The concept of strain is introduced in this Chapter. The approximation to the **true strain** of the **engineering strain** is discussed. The practical case of two dimensional **plane strain** is discussed, along with the **strain transformation formulae**, **principal strains**, **principal strain directions** and the **maximum shear strain**.

4.1 Strain

If an object is placed on a table and then the table is moved, each material particle moves in space. The particles undergo a **displacement**. The particles have moved in space as a **rigid body**. The material remains unstressed. On the other hand, when a material is acted upon by a set of forces, it *changes size and/or shape*, it **deforms**. This deformation is described using the concept of **strain**. The study of this movement and deformation, without reference to the forces or anything else which might "cause" it, is called **kinematics**.

4.1.1 One Dimensional Strain

The Engineering Strain

Consider a slender rod, fixed at one end and stretched, as illustrated in Fig. 4.1.1; the original position of the rod is shown dotted.

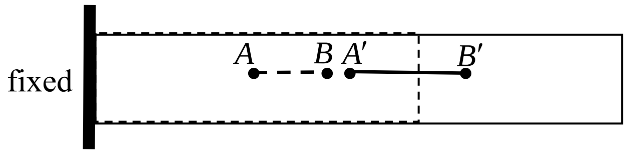


Figure 4.1.1: the strain at a point *A* in a stretched slender rod; *AB* is a line element in the unstretched rod, $A'B'$ is the same line element in the stretched rod

There are a number of different ways in which this stretching/deformation can be described (see later). Here, what is perhaps the simplest measure, the **engineering strain**, will be used. To determine the strain at point *A*, Fig. 4.1.1, consider a small line element *AB* emanating from *A* in the unstretched rod. The points *A* and *B* move to A' and B' when the rod has been stretched. The (engineering) strain ε at *A* is then defined as[1]

$$\varepsilon^{(A)} = \frac{|A'B'| - |AB|}{|AB|} \tag{4.1.1}$$

The strain at other points in the rod can be evaluated in the same way.

If a line element is stretched to twice its original length, the strain is 1. If it is unstretched, the strain is 0. If it is shortened to half its original length, the strain is -0.5. The strain is often expressed as a percentage; a 100% strain is a strain of 1, a 200% strain is a strain of 2, etc. Most engineering materials, such as metals and concrete, undergo extremely small strains in practical applications, in the range 10^{-6} to 10^{-2}; rubbery materials can easily undergo large strains of 100%.

[1] this is the strain at point *A*. The strain at *B* is evidently the same – one can consider the line element *AB* to emanate fom point *B* (it does not matter whether the line element emanates out from the point to the "left" or to the "right")

Consider now two adjacent line elements AE and EB (not necessarily of equal length), which move to $A'E'$ and $E'B'$, Fig. 4.1.2. If the rod is stretching **uniformly**, that is, if all line elements are stretching in the same proportion along the length of the rod, then $|A'E'|/|AE| = |E'B'|/|EB|$, and $\varepsilon^{(A)} = \varepsilon^{(E)}$; the strain is the same at all points along the rod.

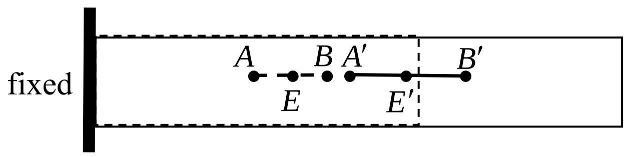


Figure 4.1.2: the strain at a point *A* and the strain at point *E* in a stretched rod

In this case, one could equally choose the line element AB or the element AE in the calculation of the strain at A, since

$$\varepsilon^{(A)} = \frac{|A'B'| - |AB|}{|AB|} = \frac{|A'E'| - |AE|}{|AE|}$$

In other words it does not matter what the length of the line element chosen for the calculation of the strain at A is. In fact, if the length of the rod before stretching is L_0 and after stretching it is L, Fig. 4.1.3, the strain everywhere is (this is equivalent to choosing a "line element" extending the full length of the rod)

$$\varepsilon = \frac{L - L_0}{L_0} \tag{4.1.2}$$

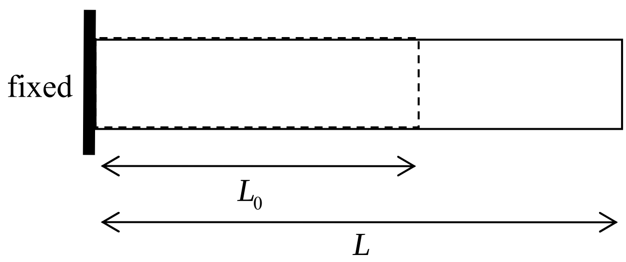


Figure 4.1.3: a stretched slender rod

On the other hand, when the strain is *not* uniform, for example $|A'E'|/|AE| \neq |E'B'|/|EB|$, then the length of the line element does matter. In this case, to be precise, the line element AB in the definition of strain in Eqn. 4.1.1 should be "infinitely small"; the smaller the line element, the more accurate will be the evaluation of the strain. The strains considered in this book will be mainly uniform.

Displacement, Strain and Rigid Body Motions

To highlight the difference between displacement and strain, and their relationship, consider again the stretched rod of Fig 4.1.1. Fig 4.1.4 shows the same rod: the two points *A* and *B* undergo displacements $u^{(A)} = |AA'|, u^{(B)} = |BB'|$. The strain at *A*, Eqn 4.1.1, can be re-expressed in terms of these displacements:

$$\varepsilon^{(A)} = \frac{u^{(B)} - u^{(A)}}{|AB|} \tag{4.1.3}$$

In words, the strain is a measure of the *change* in displacement as one moves along the rod.

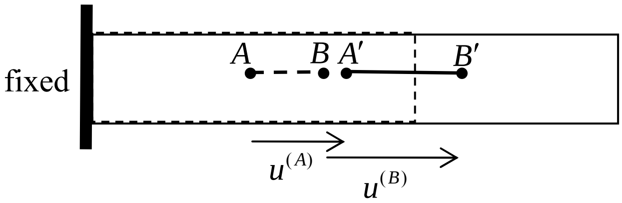


Figure 4.1.4: displacements in a stretched rod

Consider a line element emanating from the left-hand fixed end of the rod. The displacement at the fixed end is zero. However, the strain at the fixed end is *not* zero, since the line element there will change in length. This is a case where the displacement is zero but the strain is not zero.

Consider next the case where the rod is not fixed and simply moves/translates in space, without any stretching, Fig. 4.1.5. This is a case where the displacements are all non-zero (and in this case everywhere the same) but the strain is everywhere zero. This is in fact a feature of a good measure of strain: it should be zero for any rigid body motion; the strain should only measure the deformation.

Figure 4.1.5: a rigid body translation of a rod

Note that if one knows the strain at all points in the rod, one cannot be sure of the rod's exact position in space – again, this is because strain does not include information about possible rigid body motion. To know the precise position of the rod, one must also have some information about the displacements.

The True Strain

As mentioned, there are many ways in which deformation can be measured. Many different strains measures are in use apart from the engineering strain, for example the Green-Lagrange strain and the Euler-Almansi strain: referring again to Fig. 4.1.1, these are

$$\text{Green-Lagrange } \varepsilon^{(A)} = \frac{|A'B'|^2 - |AB|^2}{2|AB|^2}, \qquad \text{Euler-Alamnsi } \varepsilon^{(A)} = \frac{|A'B'|^2 - |AB|^2}{2|A'B'|^2} \tag{4.1.4}$$

Many of these strain measures are used in more advanced theories of material behaviour, particularly when the deformations are very large. Apart from the engineering strain, just one other measure will be discussed in any detail here: the **true strain** (or **logarithmic strain**), since it is often used in describing material testing (see Chapter 5).

The true strain may be defined as follows: define a small increment in strain to be the change in length divided by the *current* length: $d\varepsilon_t = dL / L$. As the rod of Fig. 4.1.1 stretches (uniformly), this current length continually changes, and the total strain thus defined is the accumulation of these increments:

$$\varepsilon_t = \int_{L_0}^{L} \frac{dL}{L} = \ln\left(\frac{L}{L_0}\right). \tag{4.1.5}$$

If a line element is stretched to twice its original length, the (true) strain is 0.69. If it is unstretched, the strain is 0. If it is shortened to half its original length, the strain is -0.69. The fact that a stretching and a contraction of the material by the same factor results in strains which differ only in sign is one of the reasons for the usefulness of the true strain measure.

Another reason for its usefulness is the fact that the true strain is additive. For example, if a line element stretches in two steps from lengths L_1 to L_2 to L_3, the total true strain is

$$\varepsilon_t = \ln\left(\frac{L_3}{L_2}\right) + \ln\left(\frac{L_2}{L_1}\right) = \ln\left(\frac{L_3}{L_1}\right),$$

which is the same as if the stretching had occurred in one step. This is not true of the engineering strain.

The true strain and engineering strain are related through (see Eqn. 4.1.2, 4.1.5)

$$\varepsilon_t = \ln(1+\varepsilon) \tag{4.1.6}$$

One important consequence of this relationship is that the smaller the deformation, the less the difference between the two strains. This can be seen in Table 4.1 below, which shows the values of the engineering and true strains for a line element of initial length 1mm, at different stretched lengths. (In fact, using a Taylor series expansion, $\varepsilon_t = \ln(1+\varepsilon) \approx \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \cdots$, for small ε.) Almost all strain measures in use are similar in this way: they are defined such that they are more or less equal when the deformation is small. Put another way, when the deformations are small, it does not really matter which strain measure is used, since they are all essentially the same – in that case it is often sensible to use the simplest measure.

L_0 (mm)	L (mm)	ε	ε_t
1	2	1	0.693
1	1.5	0.5	0.405
1	1.4	0.4	0.336
1	1.3	0.3	0.262
1	1.2	0.2	0.182
1	1.1	0.1	0.095
1	1.01	0.01	0.00995
1	1.001	0.001	0.000995

Table 4.1: true strain and engineering strain at different stretches

If one defines strain to be "change in length over length", then the true strain would be more "correct" than the engineering strain. On the other hand, if the strain is considered to have a variety of (related) definitions, such as Eqns 4.1.2, 4.1.4-5, then no strain measure is really more "correct" than any other; the usefulness of a strain measure will depend on the application and the problem at hand.

4.1.2 Two Dimensional Strain

The two dimensional case is similar to the one dimensional case, in that material deformation can be described by imagining the material to be a collection of small line elements. As the material is deformed, the line elements stretch, or get shorter, only now they can also rotate in space relative to each other. This movement of line elements is encompassed in the idea of strain: the "strain at a point" is all the stretching, contracting and rotating of *all* line elements emanating from that point, with all the line elements together making up the continuous material, as illustrated in Fig. 4.1.6.

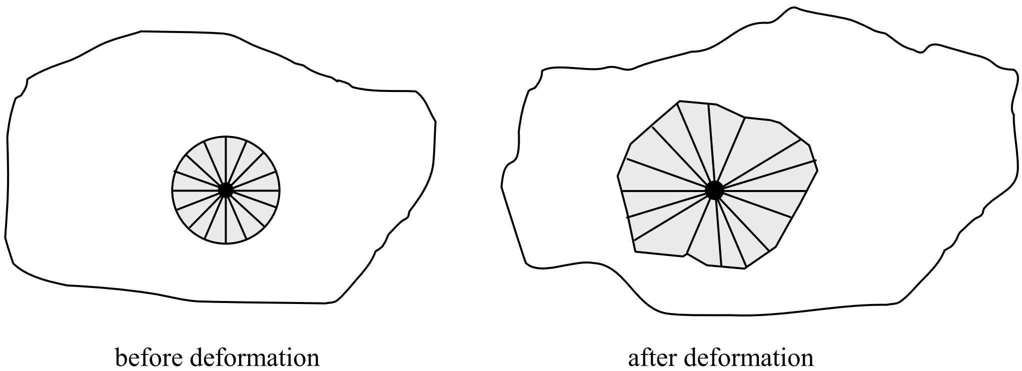


Figure 4.1.6: a deforming material element; original state of line elements and their final position after straining

It turns out that the strain at a point is completely characterised by the movement of *any two mutually perpendicular line-segments*. If it is known how these perpendicular line-segments are stretching, contracting and rotating, it will be possible to determine how any other line element at the point is behaving, by using a **strain transformation rule** (see later). This is analogous to the way the stress at a point is characterised by the stress acting on perpendicular planes through a point, and the stress components on other planes can be obtained using the stress transformation formulae.

So, for the two-dimensional case, consider two perpendicular line-elements emanating from a point. When the material that contains the point is deformed, two things (can) happen:

(1) the line segments will *change length* and
(2) the *angle* between the line-segments *changes*.

The change in length of line-elements is called **normal strain** and the change in angle between initially perpendicular line-segments is called **shear strain**.

As mentioned earlier, a number of different definitions of strain are in use; here, the following, most commonly used, definition will be employed, which will be called the **exact strain**:

Normal strain in direction x: (denoted by ε_{xx})
change in length (per unit length) of a line element originally lying in the x–direction

Normal strain in direction y: (denoted by ε_{yy})
change in length (per unit length) of a line element originally lying in the y–direction

Shear strain: (denoted by ε_{xy})
(half) the change in the original right angle between the two perpendicular line elements

Referring to Fig. 4.1.7, the (exact) strains are

$$\varepsilon_{xx} = \frac{A'B' - AB}{AB}, \quad \varepsilon_{yy} = \frac{A'C' - AC}{AC}, \quad \varepsilon_{xy} = \frac{1}{2}(\theta + \lambda). \tag{4.1.7}$$

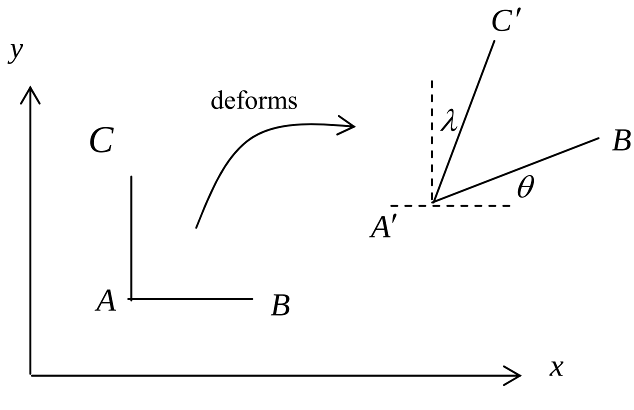


Figure 4.1.7: strain at a point *A*

These 2D strains can be represented in the matrix form

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \tag{4.1.8}$$

As with the stress, the strain matrix is symmetric, with, by definition, $\varepsilon_{xy} = \varepsilon_{yx}$.
Note that the point *A* in Fig. 4.1.7 has also undergone a displacement **u**(*A*). This displacement has two components, u_x and u_y, as shown in Fig. 4.1.8 (and similarly for the points *B* and *C*).

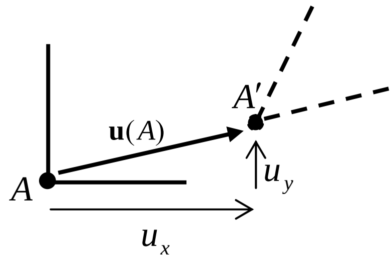


Figure 4.1.8: displacement of a point *A*

The line elements not only change length and the angle between them changes – they can also move in space as rigid-bodies. Thus, for example, the normal and shear strain in the three examples shown in Fig. 4.1.9 are the same, even though the displacements occurring in each case are different – *strain is independent of rigid body motions*.

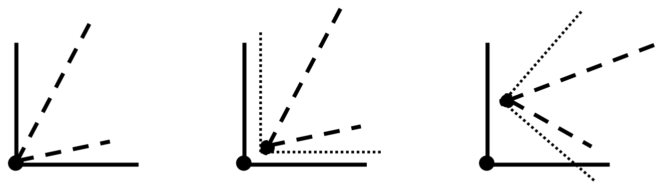


Figure 4.1.9: rigid body motions

The Engineering Strain

Suppose now that the deformation is very small, so that, in Fig. 4.1.10, $A'B' \approx A'B^*$; here $A'B^*$ is the projection of $A'B'$ in the $x-$direction. In that case,

$$\varepsilon_{xx} \approx \frac{A'B^* - AB}{AB}. \tag{4.1.9}$$

Similarly, one can make the approximations

$$\varepsilon_{yy} \approx \frac{A'C^* - AC}{AC}, \quad \varepsilon_{xy} \approx \frac{1}{2}\left(\frac{B^*B'}{AB} + \frac{C^*C'}{AC}\right), \tag{4.1.10}$$

the expression for shear strain following from the fact that, for a *small angle*, the angle (measured in radians) is approximately equal to the tan of the angle.

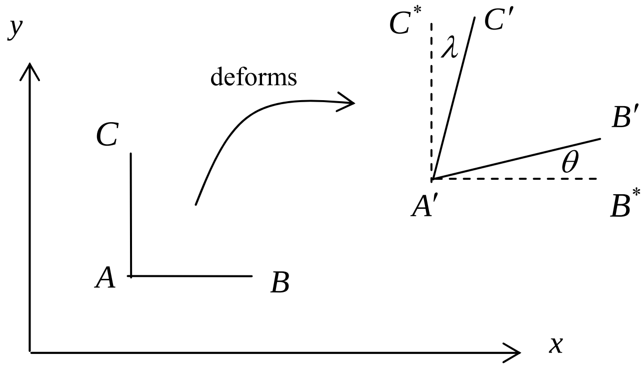


Figure 4.1.10: small deformation

This approximation for the normal strains is called the **engineering strain** or **small strain** or **infinitesimal strain** and is valid when the *deformations are small*. The advantage of the small strain approximation is that the mathematics is simplified greatly.

Example

Two perpendicular lines are etched onto the fuselage of an aircraft. During testing in a wind tunnel, the perpendicular lines deform as in Fig. 4.1.10. The coordinates of the line end-points (referring to Fig. 4.1.10) are:

$$
\begin{array}{ll}
C:(0.0000, 1.0000) & C':(0.0025, 1.0030) \\
A:(0.0000, 0.0000) & A':(0.0000, 0.0000) \\
B:(1.0000, 0.0000) & B':(1.0045, 0.0020)
\end{array}
$$

The exact strains are, from Eqn. 4.1.9, (to 8 decimal places)

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\sqrt{|A'B^*|^2 + |B^*B'|^2}}{|AB|} - 1 = 0.00450199 \\
\varepsilon_{yy} &= \frac{\sqrt{|A'C^*|^2 + |C^*C'|^2}}{|AC|} - 1 = 0.00300312 \\
\varepsilon_{xy} &= \frac{1}{2}\left(\arctan\left(\frac{|B^*B'|}{|A'B^*|} \right) + \arctan\left(\frac{|C^*C'|}{|A'C^*|} \right) \right) = 0.00224178
\end{aligned}
$$

The engineering strains are, from Eqns. 4.1.10-11,

$$
\varepsilon_{xx} = \frac{|A'B^*|}{|AB|} - 1 = 0.0045, \quad \varepsilon_{yy} = \frac{|A'C^*|}{|AC|} - 1 = 0.003, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{|B^*B'|}{|AB|} + \frac{|C^*C'|}{|AC|} \right) = 0.00225
$$

As can be seen, for the small deformations which occurred, the errors in making the small-strain approximation are extremely small, less than 0.11% for all three strains.

■

Small strain is useful in characterising the small deformations that take place in, for example, (1) engineering materials such as concrete, metals, stiff plastics and so on, (2) linear viscoelastic materials such as many polymeric materials (see Chapter 10), (3) some porous media such as soils and clays at moderate loads, (4) almost any material if the loading is not too high.

Small strain is inadequate for describing large deformations that occur, for example, in many rubbery materials, soft tissues, engineering materials at large loads, etc. In these cases the more precise definition 4.1.7 (or a variant of it) is required. That said, the engineering strain and the concepts associated with it are an excellent introduction to the more involved large deformation strain measures.

In one dimension, there is no distinction between the exact strain and the engineering strain – they are the same. Differences arise between the two in the two-dimensional case when the material shears (as in the example above), or rotates as a rigid body (as will be discussed further below).

Engineering Shear Strain and Tensorial Shear Strain

The definition of shear strain introduced above is the **tensorial shear strain** ε_{xy}. The **engineering shear strain**[2] γ_{xy} is defined as twice this angle, i.e. as $\theta + \lambda$, and is often used in Strength of Materials and elementary Solid Mechanics analyses.

4.1.3 Sign Convention for Strain

A positive normal strain means that the line element is lengthening. A negative normal strain means the line element is shortening.

For shear strain, one has the following convention: when the two perpendicular line elements are both directed in the positive directions (say x and y), or both directed in the negative directions, then a positive shear strain corresponds to a *decrease* in right angle. Conversely, if one line segment is directed in a positive direction whilst the other is directed in a negative direction, then a positive shear strain corresponds to an *increase* in angle. The four possible cases of shear strain are shown in Fig. 4.1.11a (all four shear strains are positive). A box undergoing a positive shear and a negative shear are also shown, in Figs. 4.1.11b,c.

[2] not to be confused with the term *engineering strain*, i.e. *small strain*, used throughout this Chapter

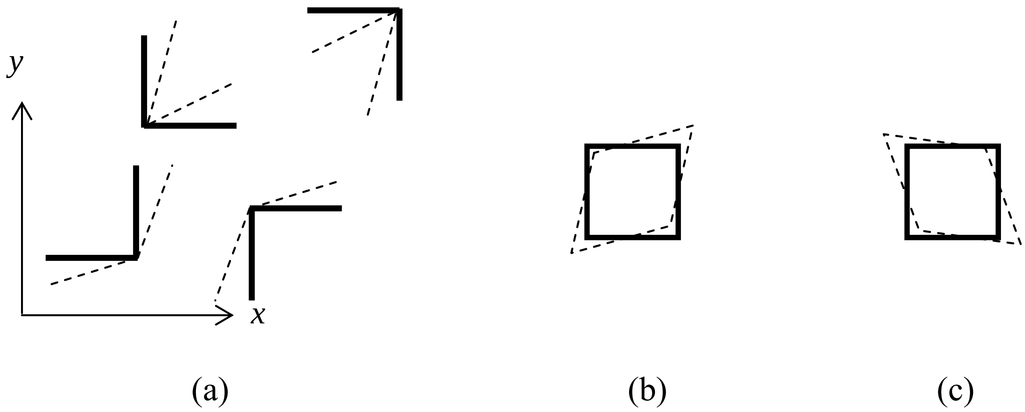


Figure 4.1.11: sign convention for shear strain; (a) line elements undergoing positive shear, (b) a box undergoing positive shear, (c) a box undergoing negative shear

4.1.4 Geometrical Interpretation of the Engineering Strain

Consider a small "box" element and suppose it to be so small that the strain is constant/uniform throughout - one says that the strain is **homogeneous**. This implies that straight lines remain straight after straining and parallel lines remain parallel. A few simple deformations are examined below and these are related to the strains.

A positive normal strain $\varepsilon_{xx} > 0$ is shown in Fig. 4.1.12a. Here the undeformed box element (dashed) has elongated. As mentioned already, knowledge of the strain alone is not enough to determine the position of the strained element, since it is free to move in space as a rigid body. The displacement over some part of the box is usually specified, for example the left hand end has been fixed in Fig. 4.1.12b. A negative normal strain acts in Fig. 4.1.12c.

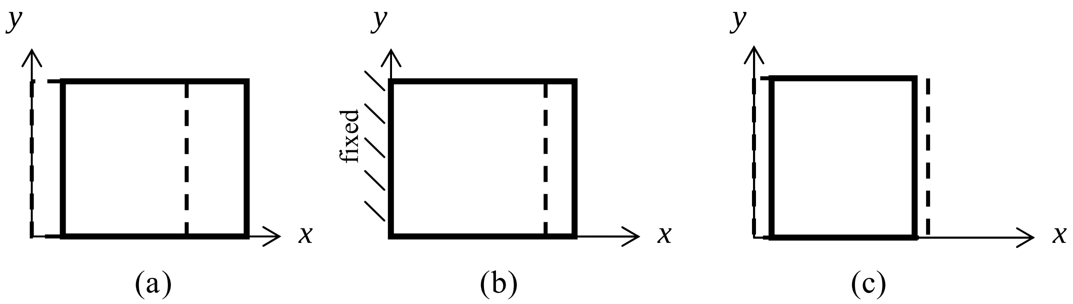


Figure 4.1.12: normal strain; (a) positive normal strain, (b) positive normal strain with the left-hand end fixed in space, (c) negative normal strain

A case known as **simple shear** is shown in Fig. 4.1.13a, and that of **pure shear** is shown in Fig. 4.1.13b. In both illustrations, $\varepsilon_{xy} > 0$. A pure (rigid body) **rotation** is shown in Fig. 4.1.13c (zero strain).

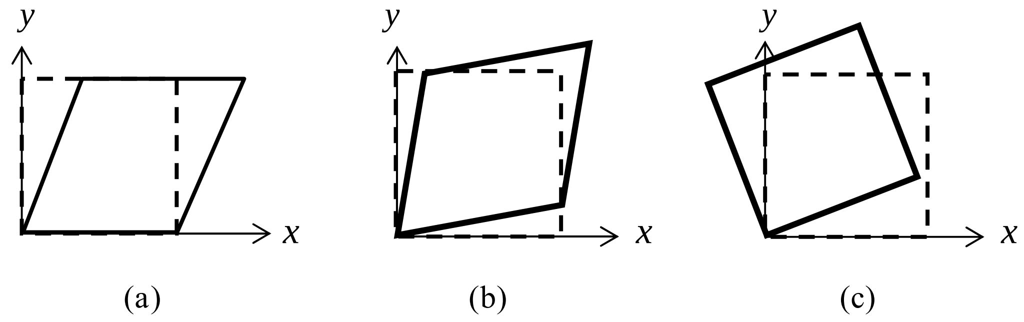


Figure 4.1.13: (a) simple shear, (b) pure shear, (c) pure rotation

Any shear strain can be decomposed into a pure shear and a pure rotation, as illustrated in Fig. 4.1.14.

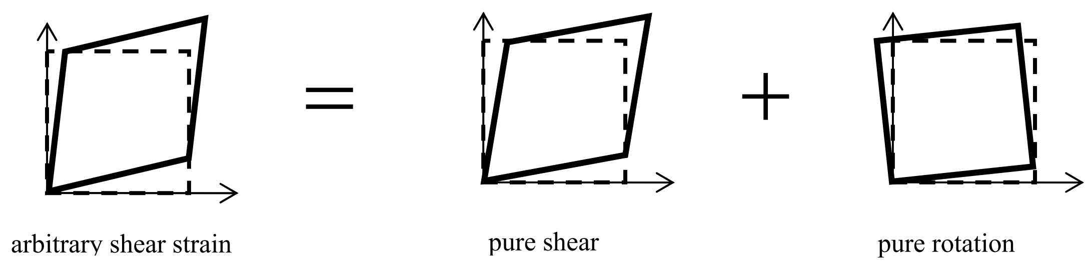


Figure 4.1.14: shear strain decomposed into a pure shear and a pure rotation

4.1.5 Large Rotations and the Small Strain

The example in section 4.1.2 above illustrated that the small strain approximation is good, provided the deformations are small. However, this is provided also that any *rigid body rotations are small*. To illustrate this, consider a square material element (with sides of unit length) which undergoes a pure rigid body rotation of θ, Fig. 4.1.15. The exact strains 4.1.7 remain zero. The small shear strain remains zero also. However, the small normal strains are seen to be $\varepsilon_{xx} = \varepsilon_{yy} = \cos\theta - 1$. Using a Taylor series expansion, this is equal to $\varepsilon_{xx} = \varepsilon_{yy} \approx -\theta^2 / 2 + \theta^4 / 24 - \cdots$. Thus, when θ is small, the rotation-induced strains are of the magnitude/order θ^2. If θ is of the same order as the strains themselves, i.e. in the range $10^{-6} - 10^{-2}$, then θ^2 will be very much smaller than θ and the rotation-induced strains will not introduce any inaccuracy; the small strains will be a good approximation to the actual strains. If, however, the rotation is large, then the engineering normal strains will be wildly inaccurate. For example, when $\theta = 45^{\circ}$, the rotation-induced normal strains are ≈ -0.3, and will likely be larger than the actual strains occurring in the material.

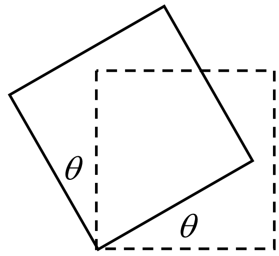


Figure 4.1.15: an element undergoing a rigid body rotation

As an example, consider a cantilevered beam which undergoes large bending, Fig. 4.1.16. The shaded element shown might well undergo small normal and shear strains. However, because of the large rotation of the element, additional spurious engineering normal strains are induced. Use of the precise definition, Eqn. 4.1.7, is required in cases such as this.

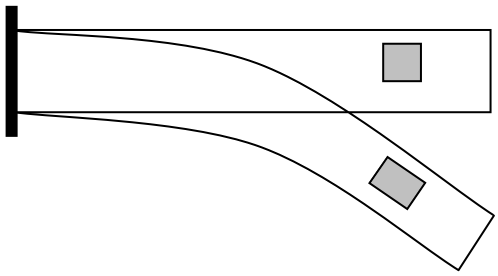


Figure 4.1.16: Large rotations of an element in a bent beam

4.1.6 Three Dimensional Strain

The above can be generalized to three dimensions. In the general case, there are three normal strains, $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$, and three shear strains, $\varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}$. The ε_{zz} strain corresponds to a change in length of a line element initially lying along the *z* axis. The ε_{yz} strain corresponds to half the change in the originally right angle of two perpendicular line elements aligned with the *y* and *z* axes, and similarly for the ε_{zx} strain. Straining in the $y-z$ plane ($\varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{yz}$) is illustrated in Fig. 4.1.17 below.

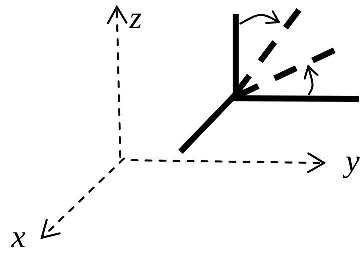


Figure 4.1.17: strains occurring in the $y-z$ plane

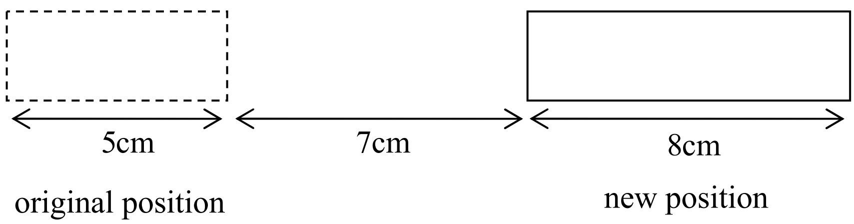
The 3D strains can be represented in the (symmetric) matrix form

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{4.1.11}$$

As with the stress (see Eqn. 3.4.5), there are nine components in 3D, with 6 of them being independent.

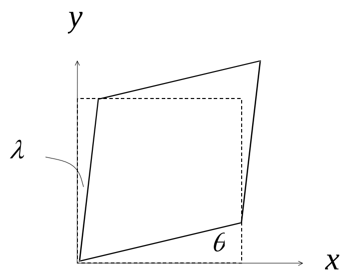
4.1.7 Problems

1. Consider a rod which moves and deforms (uniformly) as shown below.
   (a) What is the displacement of the left-hand end of the rod?
   (b) What is the engineering strain at the left-hand end of the rod



2. A slender rod of initial length 2cm is extended (uniformly) to a length 4cm. It is then compressed to a length of 3cm.
   (a) Calculate the engineering strain and the true strain for the extension
   (b) Calculate the engineering strain and the true strain for the compression
   (c) Calculate the engineering strain and the true strain for one step, i.e. an extension from 2cm to 3cm.
   (d) From your calculations in (a,b,c), which of the strain measures is additive?

3. An element undergoes a homogeneous strain, as shown. There is no normal strain in the element. The angles are given by $\lambda = 0.001$ and $\theta = 0.002$ radians. What is the (tensorial) shear strain in the element?

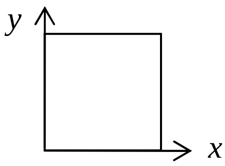


4. In a fixed $x - y$ reference system established for the test of a large component, three points *A*, *B* and *C* on the component have the following coordinates before and after loading (see Figure 4.1.10):

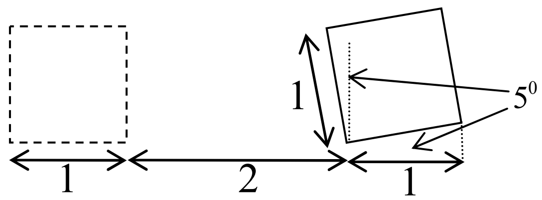
$$C:(0.0000, 1.5000) \qquad C':(-0.0025, 1.5030)$$
$$A:(0.0000, 0.0000) \qquad A':(0.0000, 0.0000)$$
$$B:(2.0000, 0.0000) \qquad B':(2.0045, 0.0000)$$

Determine the actual strains and the small strains (at/near point *A*). What is the error in the small strain compared to the actual strains?

5. Sketch the deformed shape for the material shown below under the following strains (*A*, *B* constant):
   (i) $\varepsilon_{xx} = A > 0$ (taking $\varepsilon_{yy} = \varepsilon_{xy} = 0$) – assume that the right-hand edge is fixed
   (ii) $\varepsilon_{yy} = B < 0$ (with $\varepsilon_{xx} = \varepsilon_{xy} = 0$) – assume that the lower edge is fixed
   (iii) $\varepsilon_{xy} = B < 0$ (with $\varepsilon_{xx} = \varepsilon_{yy} = 0$) – assume that the left-hand edge is fixed



6. The element shown below undergoes the change in position and dimensions shown (dashed square = undeformed). What are the three engineering strains $\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}$?



4.2 Plane Strain

A state of plane strain is defined as follows:

Plane Strain: If the strain state at a material particle is such that the only non-zero strain components act in one plane only, the particle is said to be in plane strain.

The axes are usually chosen such that the $x-y$ plane is the plane in which the strains are non-zero, Fig. 4.2.1.

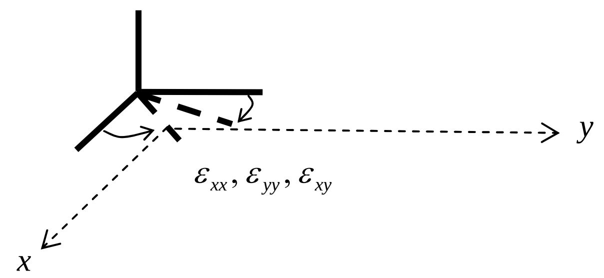


Figure 4.2.1: non-zero strain components acting in the $x-y$ plane

Then $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$. The fully three dimensional strain matrix reduces to a two dimensional one:

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad \to \quad \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \tag{4.2.1}$$

4.2.1 Analysis of Plane Strain

Stress transformation formulae, principal stresses, stress invariants and formulae for maximum shear stress were presented in §4.4-§4.5. The strain is very similar to the stress. They are both mathematical objects called tensors, having nine components, and all the formulae for stress hold also for the strain. All the equations in section 3.5.2 are valid again in the case of plane strain, with σ replaced with ε. This will be seen in what follows.

Strain Transformation Formula

Consider two perpendicular line-elements lying in the coordinate directions x and y, and suppose that it is known that the strains are $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$, Fig. 4.2.2. Consider now a second coordinate system, with axes x', y', oriented at angle θ to the first system, and consider line-elements lying along these axes. Using some trigonometry, it can be shown that the line-elements in the second system undergo strains according to the following (two dimensional) **strain transformation equations** (see the Appendix to this section, §4.2.5, for their derivation):

$$\boxed{\begin{aligned}\varepsilon'_{xx} &= \cos^2\theta\varepsilon_{xx} + \sin^2\theta\varepsilon_{yy} + \sin 2\theta\varepsilon_{xy}\\ \varepsilon'_{yy} &= \sin^2\theta\varepsilon_{xx} + \cos^2\theta\varepsilon_{yy} - \sin 2\theta\varepsilon_{xy}\\ \varepsilon'_{xy} &= \sin\theta\cos\theta(\varepsilon_{yy} - \varepsilon_{xx}) + \cos 2\theta\varepsilon_{xy}\end{aligned}}$$

Strain Transformation Formulae (4.2.2)

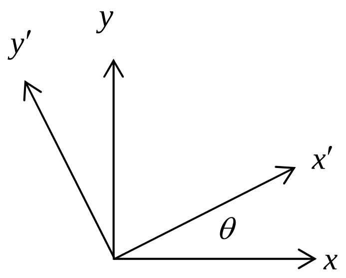


Figure 4.2.2: A rotated coordinate system

Note the similarity between these equations and the stress transformation formulae, Eqns. 3.4.9. Although they have the same structure, the stress transformation equations were derived using Newton's laws, whereas no physical law is used to derive the strain transformation equations 4.2.2, just geometry.

Eqns. 4.2.2 are valid only when the strains are small (as can be seen from their derivation in the Appendix to this section), and the engineering/small strains are assumed in all of which follows. The exact strains, Eqns. 4.1.7, do not satisfy Eqn. 4.2.2 and for this reason they are rarely used – when the strains are large, other strain measures, such as those in Eqns. 4.1.4, are used.

Principal Strains

Using exactly the same arguments as used to derive the expressions for principal stress, there is always at least one set of perpendicular line elements which stretch and/or contract, but which do not undergo angle changes. The strains in this special coordinate system are called **principal strains**, and are given by (compare with Eqns. 3.5.5)

$$\boxed{\begin{aligned}\varepsilon_1 &= \tfrac{1}{2}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + \sqrt{\tfrac{1}{4}(\varepsilon_{xx} - \varepsilon_{yy})^2 + \varepsilon_{xy}^2}\\ \varepsilon_2 &= \tfrac{1}{2}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) - \sqrt{\tfrac{1}{4}(\varepsilon_{xx} - \varepsilon_{yy})^2 + \varepsilon_{xy}^2}\end{aligned}}$$

Principal Strains (4.2.3)

Further, it can be shown that ε_1 is the maximum normal strain occurring at the point, and that ε_2 is the minimum normal strain occurring at the point.

The **principal directions**, that is, the directions of the line elements which undergo the principal strains, can be obtained from (compare with Eqns. 3.5.4)

$$\tan 2\theta = \frac{2\varepsilon_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \tag{4.2.4}$$

Here, θ is the angle at which the principal directions are oriented with respect to the x axis, Fig. 4.2.2.

Maximum Shear Strain

Analogous to Eqn. 3.5.9, the maximum shear strain occurring at a point is

$$\varepsilon_{xy}\Big|_{\max} = \frac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right) \tag{4.2.5}$$

and the perpendicular line elements undergoing this maximum angle change are oriented at 45° to the principal directions.

Example (of Strain Transformation)

Consider the block of material in Fig. 4.2.3a. Two sets of perpendicular lines are etched on its surface. The block is then stretched, Fig. 4.2.3b.

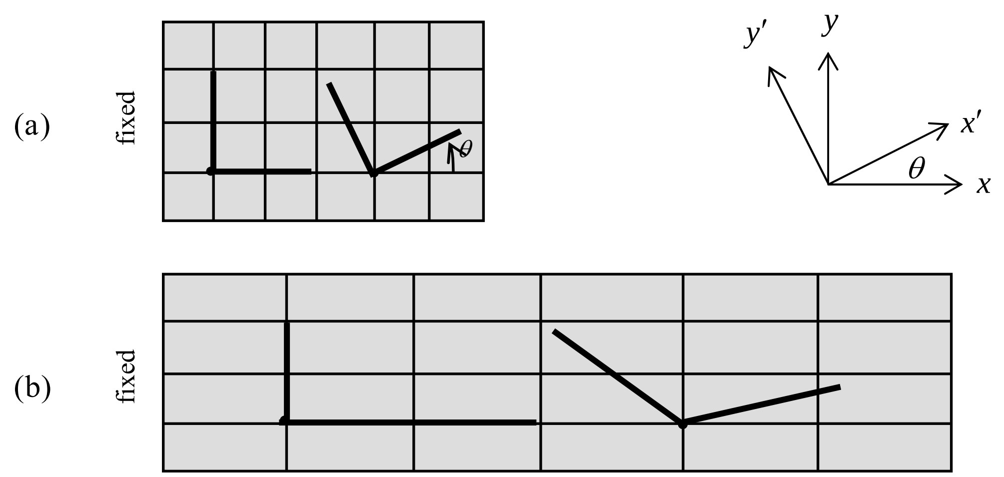


Figure 4.2.3: A block with strain measured in two different coordinate systems

This is a homogeneous deformation, that is, *the strain is the same at all points*. However, in the $x-y$ description, $\varepsilon_{xx} > 0$ and $\varepsilon_{yy} = \varepsilon_{xy} = 0$, but in the $x'-y'$ description, none of the strains is zero. The two sets of strains are related through the strain transformation equations.

■

Example (of Strain Transformation)

As another example, consider a square material element which undergoes a pure shear, as illustrated in Fig. 4.2.4, with

$$\varepsilon_{xx} = \varepsilon_{yy} = 0, \quad \varepsilon_{xy} = 0.01$$

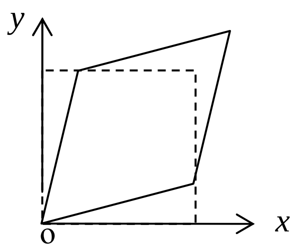


Figure 4.2.4: A block under pure shear

From Eqn. 4.2.3, the principal strains are $\varepsilon_1 = +0.01, \varepsilon_2 = -0.01$ and the principal directions are obtained from Eqn. 4.2.4 as $\theta = \pm 45^\circ$. To find the direction in which the maximum normal strain occurs, put $\theta = +45^\circ$ in the strain transformation formulae to find that $\varepsilon_1 = \varepsilon'_{xx} = +0.01$, so the deformation occurring in a piece of material whose sides are aligned in these principal directions is as shown in Fig. 4.2.5.

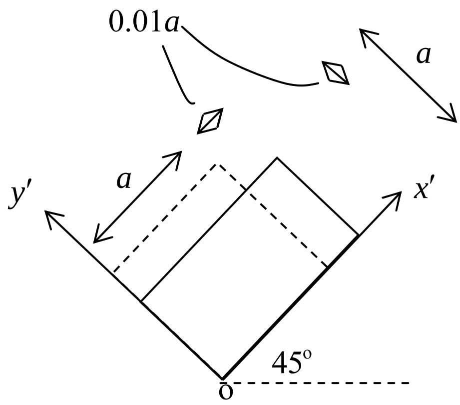


Figure 4.2.5: Principal strains for the block in pure shear

The strain as viewed along the principal directions, and also using the $x - y$ system, are as shown in Fig. 4.2.6.

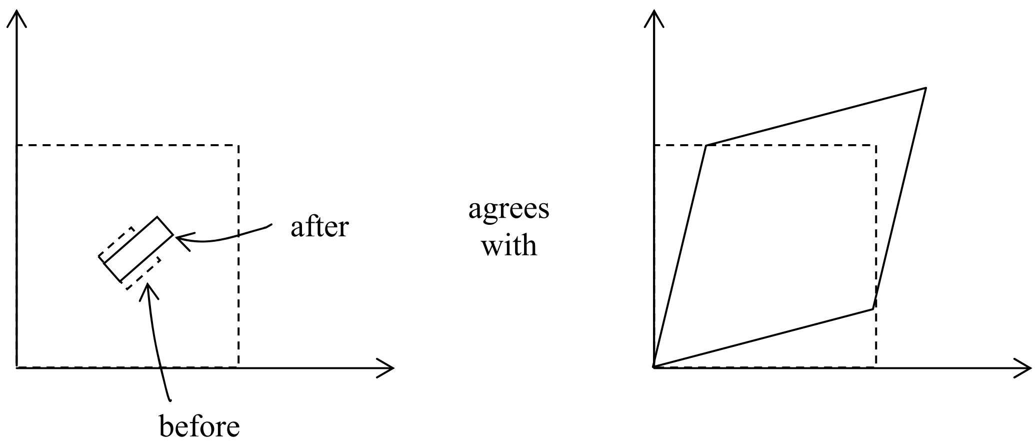


Figure 4.2.6: Strain viewed from two different coordinate systems

The two deformations, square into diamond and square into rectangle, look very different, but they are actually the same thing. For example, the square which deforms into the diamond can be considered to be made up of an infinite number of small rotated squares, Fig. 4.2.7. These then deform into rectangles, which then form the diamond.

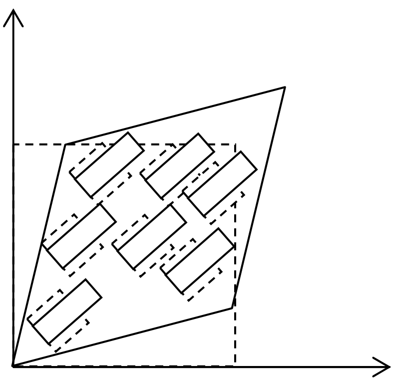


Figure 4.2.7: Alternative viewpoint of the strains in Fig. 4.2.6

Note also that, since the original $x-y$ axes were oriented at $\pm 45°$ to the principal directions, these axes are those of maximum shear strain – the original $\varepsilon_{xy} = 0.01$ is the maximum shear strain occurring at the material particle.

■

4.2.2 Thick Components

It turns out that, just as the state of plane stress often arises in thin components, a state of plane strain often arises in very thick components.

Consider the three dimensional block of material in Fig. 4.2.7. The material is constrained from undergoing normal strain in the z direction, for example by preventing movement with rigid immovable walls – and so $\varepsilon_{zz} = 0$.

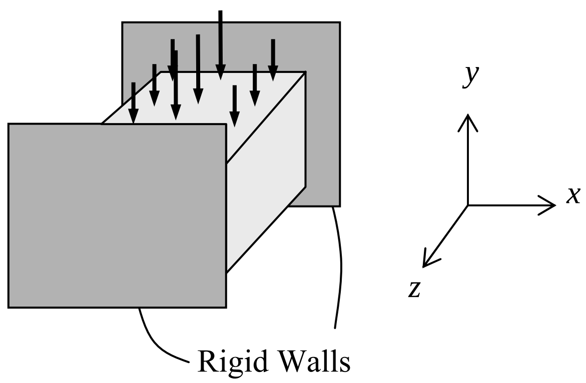


Figure 4.2.7: A block of material constrained by rigid walls

If, in addition, the loading is as shown in Fig. 4.2.7, i.e. it is the same on all cross sections parallel to the $y - z$ plane (or $x - z$ plane) – then the line elements shown in Fig. 4.2.8 will remain perpendicular (although they might move out of plane).

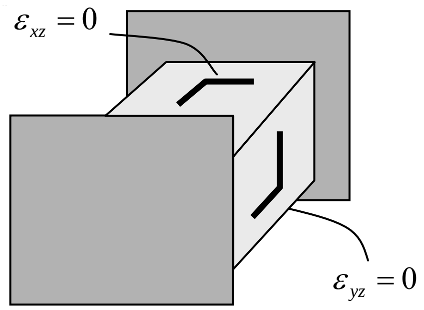


Figure 4.2.8: Line elements etched in a block of material – they remain perpendicular in a state of plane strain

Then $\varepsilon_{xz} = \varepsilon_{yz} = 0$. Thus a state of plane strain will arise.

The problem can now be analysed using the three independent strains, which simplifies matters considerable. Once a solution is found for the deformation of one plane, the solution has been found for the deformation of the whole body, Fig. 4.2.9.

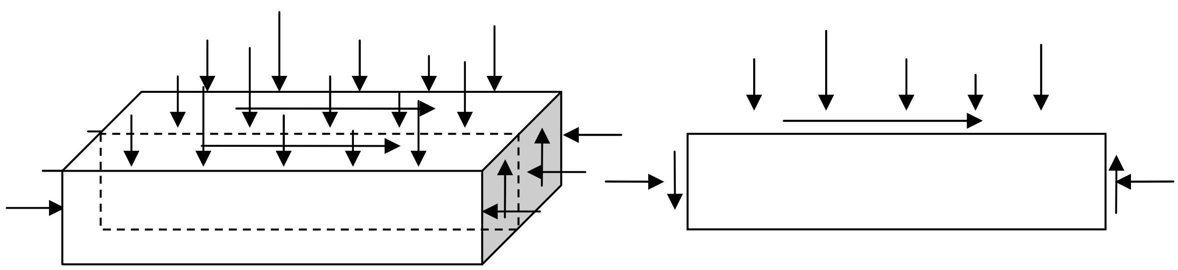


Figure 4.2.9: three dimensional problem reduces to a two dimensional one for the case of plane strain

Note that reaction stresses σ_{zz} act over the ends of the large mass of material, to prevent any movement in the z direction, i.e. ε_{zz} strains, Fig. 4.2.10.

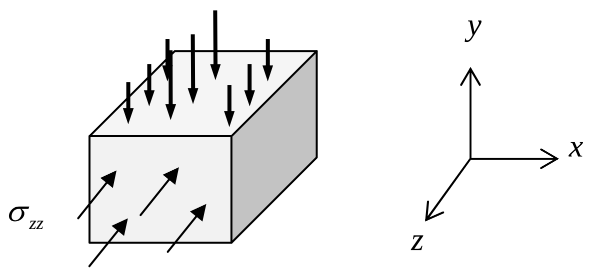


Figure 4.2.10: end-stresses required to prevent material moving in the *z* direction

A state of plane strain will also exist in thick structures without end walls. Material towards the centre is constrained by the mass of material on either side and will be (approximately) in a state of plane strain, Fig. 4.2.10.

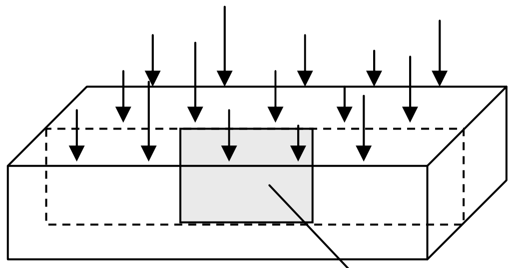


Figure 4.2.10: material in an approximate state of plane strain

Plane Strain is useful when solving many types of problem involving thick components, even when the ends of the mass of material are allowed to move (as in Fig. 4.2.10), using a concept known as **generalised plane strain** (see more advanced mechanics material).

4.2.3 Mohr's Circle for Strain

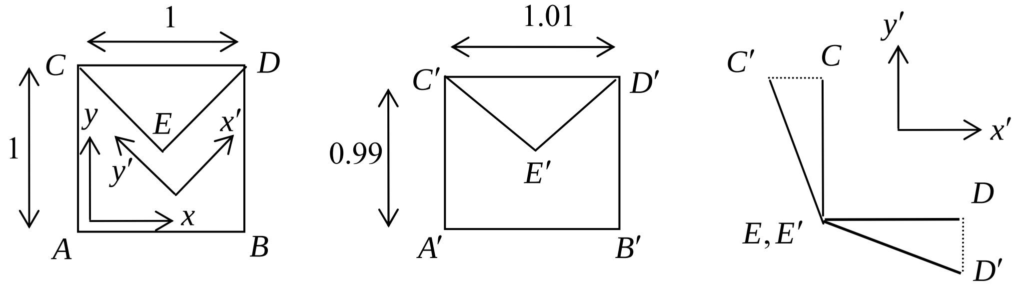
Because of the similarity between the stress transformation equations 3.4.9 and the strain transformation equations 4.2.2, Mohr's Circle for strain is identical to Mohr's Cirlce for stress, section 3.5.5, with σ replaced by ε (and τ replaced by ε_{xy}).

4.2.4 Problems

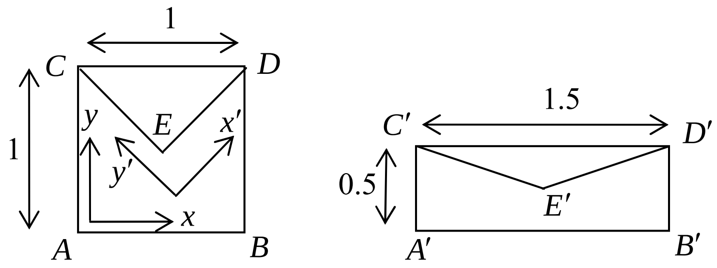
1. In Fig. 4.2.3, take $\theta = 30^{\circ}$ and $\varepsilon_{xx} = 0.02$.
   (a) Calculate the strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$.
   (b) What are the principal strains?
   (c) What is the maximum shear strain?
   (d) Of all the line elements which could be etched in the block, at what angle θ to the *x* axis are the perpendicular line elements which undergo the largest angle change from the initial right angle?

2. Consider the undeformed rectangular element below left which undergoes a uniform strain as shown centre.
   (a) Calculate the engineering strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$
   (b) Calculate the engineering strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$. Hint: use the two half-diagonals *EC* and *ED* sketched; by superimposing points E, E' (to remove the rigid body motion of *E*), it will be seen that point *D* moves straight down and *C* moves left, when viewed along the x', y' axes, as shown below right.
   (c) Use the strain transformation formulae 4.2.2 and your results from (a) to check your results from (b). Are they the same?

(d) What is the actual unit change in length of the half-diagonals? Does this agree with your result from (b)?



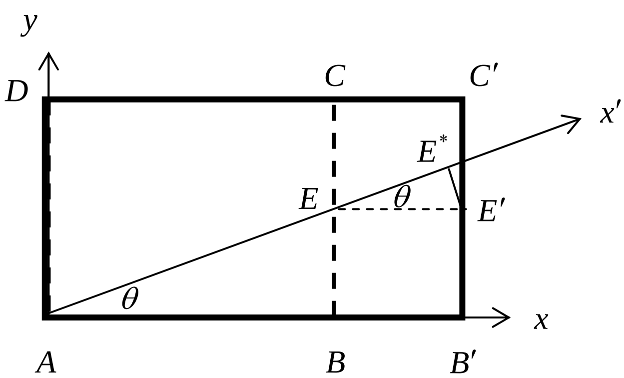
3. Repeat problem 2 only now consider the larger deformation shown below:
   (a) Calculate the engineering strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$
   (b) Calculate the engineering strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$
   (c) Use the strain transformation formulae 2.4.2 and your results from (a) to check your results from (b). Are they accurate?
   (d) What is the actual unit change in length of the half-diagonals? Does this agree with your result from (b)?



4.2.5 Appendix to §4.2

Derivation of the Strain Transformation Formulae

Consider an element $ABCD$ undergoing a strain ε_{xx} with $\varepsilon_{yy} = \varepsilon_{xy} = 0$ to $AB'C'D$ as shown in the figure below.



In the $x - y$ coordinate system, by definition, $\varepsilon_{xx} = BB' / AB$. In the $x' - y'$ system, AE moves to AE', and one has

$$\varepsilon'_{xx} = \frac{EE^*}{AE} = \frac{\cos\theta \, EE'}{AB / \cos\theta} = \cos^2\theta \frac{BB'}{AB}$$

which is the first term of Eqn. 4.2.2a. Also,

$$\varepsilon'_{xy} = -\frac{E'E^*}{AE} = \frac{\sin\theta \, EE'}{AB / \cos\theta} = -\sin\theta\cos\theta \frac{BB'}{AB}$$

which is in Eqn. 4.2.2c. The remainder of the transformation formulae can be derived in a similar manner.

4.3 Volumetric Strain

The volumetric strain is defined as follows:

Volumetric Strain:
The volumetric strain is the unit change in volume, i.e. the change in volume divided by the original volume.

4.3.1 Two-Dimensional Volumetric Strain

Analogous to Eqn 3.5.1, the strain invariants are

$$\boxed{\begin{aligned} I_1 &= \varepsilon_{xx} + \varepsilon_{yy} \\ I_2 &= \varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}^2 \end{aligned}} \qquad \textbf{Strain Invariants} \qquad (4.3.1)$$

Using the strain transformation formulae, Eqns. 4.2.2, it will be verified that these quantities remain unchanged under any rotation of axes.

The first of these has a very significant physical interpretation. Consider the deformation of the material element shown in Fig. 4.3.1a. The volumetric strain is

$$\begin{aligned} \frac{\Delta V}{V} &= \frac{(a+\Delta a)(b+\Delta b) - ab}{ab} \\ &= (1+\varepsilon_{xx})(1+\varepsilon_{yy}) - 1 \\ &= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{yy} \end{aligned} \qquad (4.3.2)$$

If the strains are small, the term $\varepsilon_{xx}\varepsilon_{yy}$ will be very much smaller than the other two terms, and the volumetric strain in that case is given by

$$\boxed{\frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy}} \qquad \textbf{Volumetric Strain} \qquad (4.3.3)$$

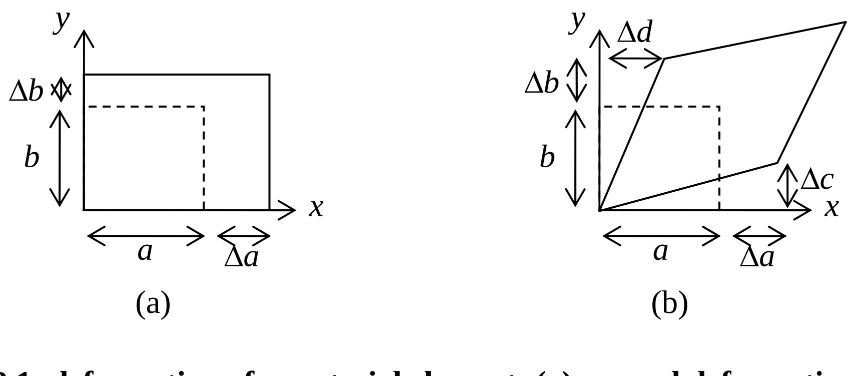


Figure 4.3.1: deformation of a material element; (a) normal deformation, (b) with shearing

Since by Eqn. 4.3.1 the volume change is an invariant, the normal strains in any coordinate system may be used in its evaluation. This makes sense: the volume change cannot depend on the particular axes we choose to measure it. In particular, the principal strains may be used:

$$\frac{\Delta V}{V} = \varepsilon_1 + \varepsilon_2 \tag{4.3.4}$$

The above calculation was carried out for stretching in the *x* and *y* directions, but the result is valid for any arbitrary deformation. For example, for the general deformation shown in Fig. 4.3.1b, some geometry shows that the volumetric strain is $\Delta V / V = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}^2$, which again reduces to Eqns 4.3.3, 4.3.4, for small strains.

An important consequence of Eqn. 4.3.3 is that *normal strains induce volume changes*, whereas *shear strains induce a change of shape but no volume change*.

4.3.2 Three Dimensional Volumetric Strain

A slightly different approach will be taken here in the three dimensional case, so as not to simply repeat what was said above, and to offer some new insight into the concepts.

Consider the element undergoing strains $\varepsilon_{xx}, \varepsilon_{xy}$, etc., Fig. 4.3.2a. The same deformation is viewed along the principal directions in Fig. 4.3.2b, for which only normal strains arise.

The volumetric strain is:

$$\begin{aligned}\frac{\Delta V}{V} &= \frac{(a+\Delta a)(b+\Delta b)(c+\Delta c) - abc}{abc} \\ &= (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1 \\ &\approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3\end{aligned} \tag{4.3.5}$$

and the squared and cubed terms can be neglected because of the small-strain assumption.

Since any elemental volume such as that in Fig. 4.3.2a can be constructed out of an infinite number of the elemental cubes shown in Fig. 4.3.3b (as in Fig. 4.2.7), this result holds for any elemental volume irrespective of shape.

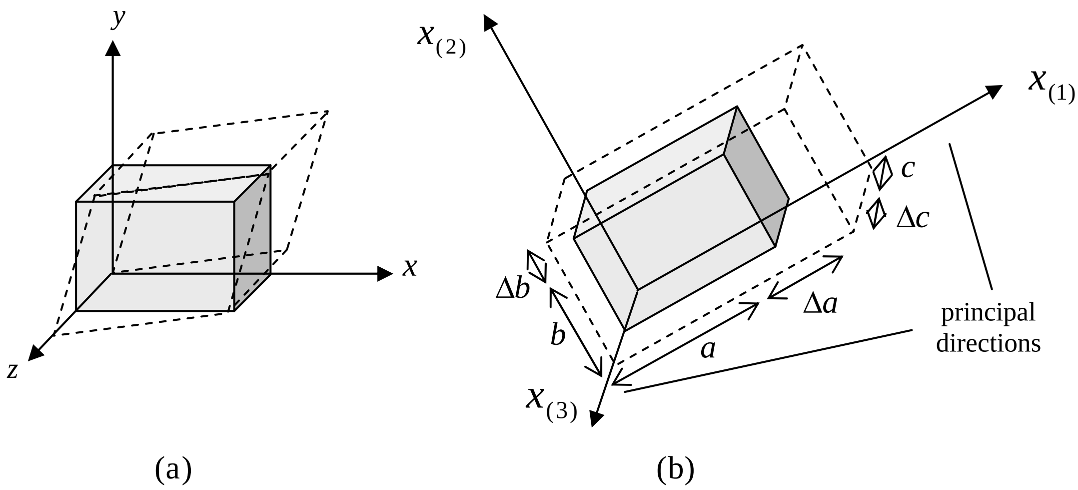


Figure 4.3.2: A block of deforming material; (a) subjected to an arbitrary strain; (a) principal strains

5 Material Behaviour and Mechanics Modelling

In this Chapter, the real physical response of various types of material to different types of loading conditions is examined. The means by which a mathematical model can be developed which can predict such real responses is also considered.

5.1 Mechanics Modelling

5.1.1 The Mechanics Problem

Typical questions which mechanics attempts to answer were given in Section 1.1. In the examples given, one invariably knows (some of) the forces (or stresses) acting on the material under study, be it due to the wind, water pressure, the weight of the human body, a moving train, and so on. One also often knows something about the displacements along some portion of the material, for example it might be fixed to the ground and so the displacements there are zero. A schematic of such a generic material is shown in Fig. 5.1.1 below.

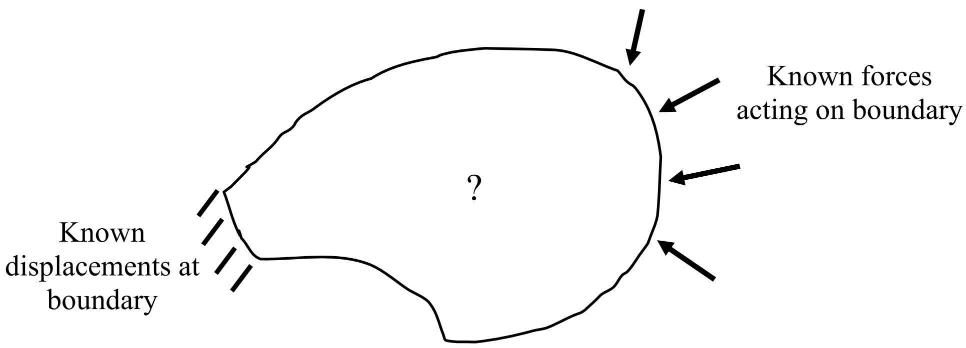


Figure 5.1.1: a material component; force and displacement are known along some portion of the boundary

The basic problem of mechanics is to determine what is happening *inside* the material. This means: what are the stresses and strains inside the material? With this information, one can answer further questions: Where are the stresses high? Where will the material first fail? What can we change to make the material function better? Where will the component move to? What is going on inside the material, at the microscopic level? Generally speaking, what is happening and what will happen?

One can relate the loads on the component to the stresses inside the body using equilibrium equations and one can relate the displacement to internal strains using kinematics relations. For example, consider again the simple rod subjected to tension forces examined in Section 3.3.1, shown again in Fig. 5.1.2. The internal normal stress σ_N on any plane oriented at an angle θ to the rod cross-section is related to the external force F through the equilibrium equation 3.3.1: $\sigma_N = F\cos^2\theta / A$, where A is the cross-sectional area. Similarly, if the ends undergo a separation/displacement of $\Delta = l - l_0$, Fig. 5.1.2b, the strain of any internal line element, at orientation θ, is $\varepsilon_N = \Delta\cos^2\theta / l_0$. However, there is no relationship between this internal stress and internal strain: for any given force, there is no way to determine the internal strain (and hence displacement of the rod); for any given displacement of the rod, there is no way to determine the internal stress (and hence force applied to the rod). The required relationship between stress and strain is discussed next.

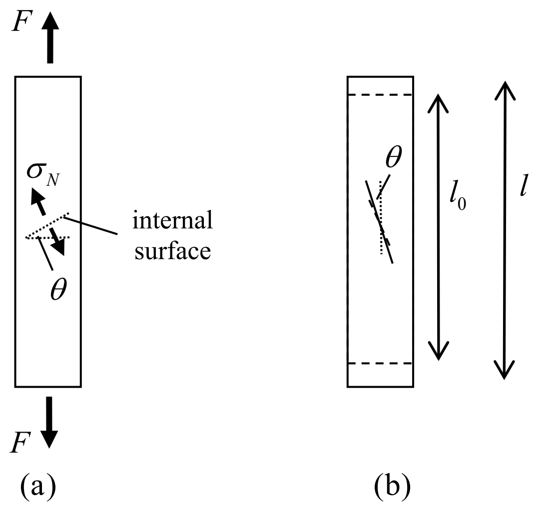


Figure 5.1.2: a slender rod; (a) internal stress due to external force, (b) internal strain due to gross displacement of rod (dotted = before straining)

5.1.2 Constitutive Equation

Stress was discussed in Chapter 3 and strain in Chapter 4. In all that discussion, no mention was made of the particular material under study, be it metallic, polymeric, biological or foodstuff (apart from the necessity that the strain be small when using the engineering strain). The concept of stress and the resulting theory of stress transformation, principal stresses and so on, are based on physical principles (Newton's Laws), which apply to *all* materials. The concept of strain is based, essentially, on geometry and trigonometry; again, it applies to all materials. However, it is the relationship *between* stress and strain which differs from material to material.

The relationship between the stress and strain for any particular material will depend on the microstructure of that material – what constitutes that material. For this reason, the stress-strain relationship is called the **constitutive relation**, or **constitutive law**. For example, metals consist of a closely packed lattice of atoms, whereas a rubber consists of a tangled mass of long-chain polymer molecules; for this reason, the strain in a metal will be different to that in rubber, when they are subjected to the same stress.

The constitutive equation allows the mechanics problem to be solved – this is shown schematically in Fig. 5.1.3.

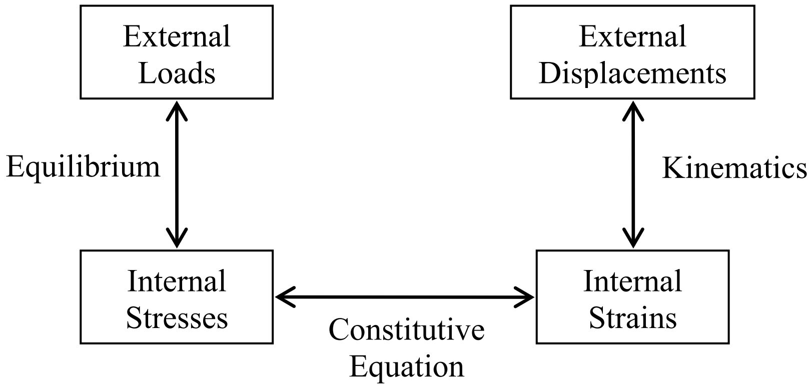


Figure 5.1.3: the role of the constitutive equation in the equations of mechanics

Example Constitutive Equations

A constitutive equation will be of the general form

$$\sigma = f(\varepsilon). \tag{5.1.1}$$

The simplest constitutive equation is a **linear elastic** relation, in which the stress is proportional to the strain:

$$\sigma \propto \varepsilon. \tag{5.1.2}$$

Although no real material satisfies precisely Eqn. 5.1.2, many do so approximately – this type of relation will be discussed in Chapters 6-8. More complex relations can involve the *rate* at which a material is strained or stressed; these types of relation will be discussed in Chapter 10.

More on constitutive equations will follow in Section 5.3.

5.1.3 Mathematical Model

Some of the questions asked earlier can be answered using experimentation. For example, one could use a car-crash test to determine the weakest points in a car. However, one cannot carry out multiple tests for each and every possible scenario – different car speeds, different obstacles into which it crashes, and so on; it would be too time-consuming and too expensive. The only practical way in which these questions can be answered is to develop a **mathematical model**. This model consists of the various equations of equilibrium and the kinematics, the constitutive relation, equations describing the shape of the material, etc. (see Fig. 5.1.3). The mathematical model will have many approximations to reality associated with it. For example, it might be assumed that the material is in the shape of a perfect sphere, when in fact it only resembles a sphere. It may be assumed that a load is applied at a “point” when in fact it is applied over a region of the material’s surface. Another approximation in the mathematical model is the constitutive equation itself; the relation between stress and strain in any material can be extremely complex, and the constitutive equation can only be an approximation of the reality.

Once the mathematical model has been developed, the various equations can be solved and the model can then be used to *make a prediction*. The prediction of the model can now be tested against reality: a set of well-defined experiments can be carried out – does the material really move to where the model says it will move?

Simple models (simple constitutive relations) should be used as a first step. If the predictions of the model are wildly incorrect, the model can be adjusted (made more complicated), and the output tested again.

The equations associated with simple models can often be solved analytically, i.e. using a pen and paper. More complex models result in complex sets of equations which can only be solved approximately (though, hopefully, accurately) using a computer.

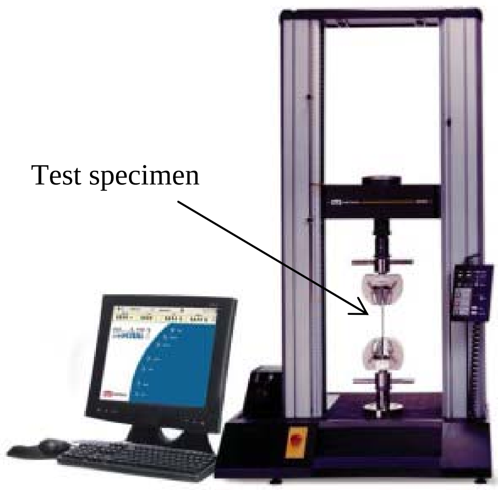
5.2 The Response of Real Materials

The constitutive equation was introduced in the previous section. The means by which the constitutive equation is determined is by carrying out experimental tests on the material in question. This topic is discussed in what follows.

5.2.1 The Tension Test

Consider the following key experiment, the **tensile test**, in which a small, usually cylindrical, specimen is gripped and stretched, usually at some given rate of stretching. A typical specimen would have diameter about 1cm and length 5cm, and larger ends so that it can be easily gripped, Fig. 5.2.1a. Specialised machines are used, for example the Instron testing machine shown in Fig. 5.2.1b.

1



2

Figure 5.2.1: the tension test; (a) test specimen, (b) testing machine

As the specimen is stretched, the force required to hold the specimen at a given displacement/stretch is recorded[1].

The Engineering Materials

For many of the (hard) engineering materials, the force/displacement curve will look something like that shown in Fig. 5.2.2. It will be found that the force is initially proportional to displacement as with the linear portion OA in Fig. 5.2.2. The following observations will also be made:

(1) if the load has not reached point *A*, and the material is then unloaded, the force/displacement curve will trace back along the line OA down to zero force and zero displacement; further loading and unloading will again be up and down OA.

(2) the loading curve remains linear up to a certain force level, the **elastic limit** of the material (point *A*). Beyond this point, **permanent deformations** are induced[2]; on

[1] the very precise details of how the test should be carried out are contained in the special standards for materials testing developed by the American Society for Testing and Materials (ASTM)

unloading to zero force (from point *B* to *C*), the specimen will have a permanent elongation. An example of this response (although not a tension test) can be seen with a paper clip – gently bend the clip and it will "spring back" (this is the *OA* behaviour); bend the clip too much (*AB*) and it will stay bent after you let go (*C*).

(3) above the elastic limit (from *A* to *B*), the material **hardens**, that is, the force required to maintain further stretching, unsurprisingly, keeps increasing. (However, some materials can **soften**, for example granular materials such as soils).

(4) the rate (speed) at which the specimen is stretched makes little difference to the results observed (at least if the speed and/or temperature is not too high).

(5) the strains up to the elastic limit are small, less than 1% (see below for more on strains).

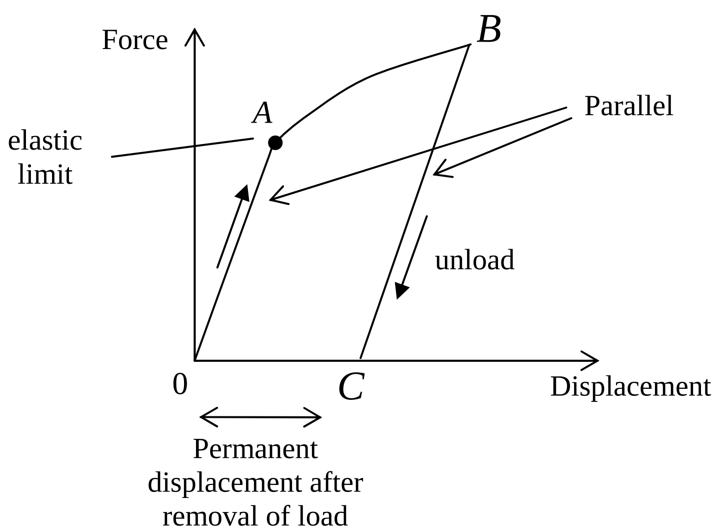


Figure 5.2.2: force/displacement curve for the tension test; typical response for engineering materials

Stress-Strain Curve

There are two definitions of stress used to describe the tension test. First, there is the force divided by the *original* cross sectional area of the specimen A_0; this is the **nominal stress** or **engineering stress**,

$$\sigma_n = \frac{F}{A_0} \tag{5.2.1}$$

Alternatively, one can evaluate the force divided by the (smaller) *current* cross-sectional area *A*, leading to the **true stress**

[2] if the tension tests are carried out extremely carefully, one might be able to distinguish between a point where the stress-strain curve ceases to be linear (the **proportional limit**) and the elastic limit (which will occur at a slightly higher stress)

$$\sigma = \frac{F}{A} \tag{5.2.2}$$

in which *F* and *A* are both changing with time. For small elongations, within the linear range OA, the cross-sectional area of the material undergoes negligible change and both definitions of stress are more or less equivalent.

Similarly, one can describe the deformation in two alternative ways. As discussed in Section 4.1.1, one can use the engineering strain

$$\varepsilon = \frac{l - l_0}{l_0} \tag{5.2.3}$$

or the true strain

$$\varepsilon_t = \ln\left(\frac{l}{l_0}\right) \tag{5.2.4}$$

Here, l_0 is the original specimen length and *l* is the current length. Again, at small deformations, the difference between these two strain measures is negligible (see Tabel 4.1).

The stress-strain diagram for a tension test can now be described using the true stress/strain or nominal stress/strain definitions, as in Fig. 5.2.3. The shape of the nominal stress/strain diagram, Fig. 5.2.3a, is of course the same as the graph of force versus displacement. *C* here denotes the point at which the maximum force the specimen can withstand has been reached. The nominal stress at *C* is called the **Ultimate Tensile Strength** (UTS) of the material.

After the UTS is reached, the specimen "necks", that is, the specimen begins to deform locally – with a very rapid reduction in cross-sectional area somewhere about the centre of the specimen until the specimen breaks, as indicated by the asterisk in Fig. 5.2.3. The appearance of a test specimen at each of these stages of the stress-strain curve is shown top of Fig. 5.2.3a.

For many materials, it will be observed that there is very little volume change during the permanent deformation phase, so $A_0 l_0 \approx Al$ and $\sigma = \sigma_N (1+\varepsilon)$. This nominal stress to true stress conversion formula will only be valid up to the point of necking.

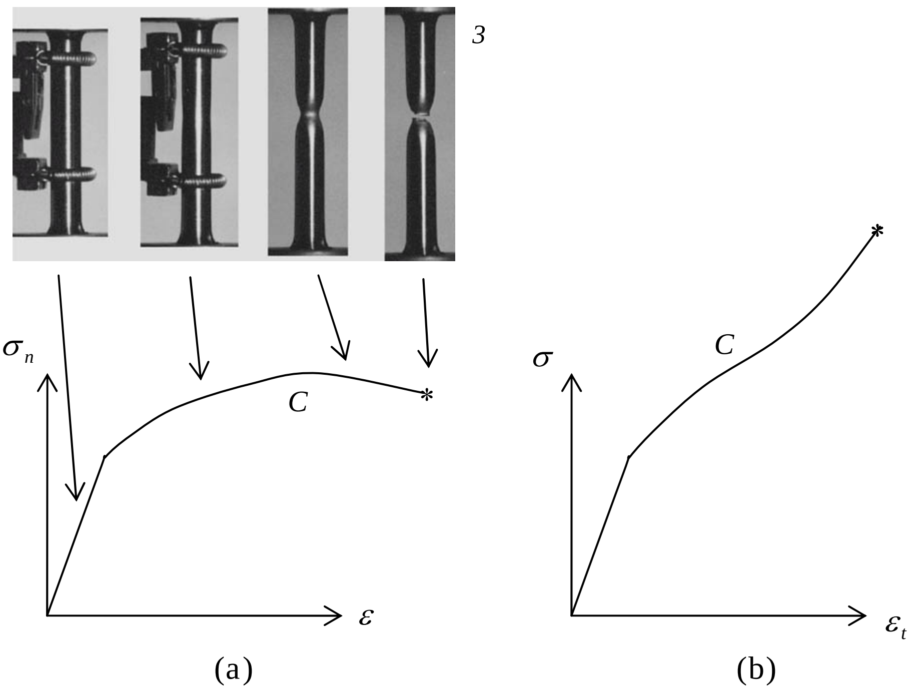


Figure 5.2.3: typical stress-strain curve for an engineering material; (a) engineering stress and strain, (b) true stress and strain

The stress-strain curves for mild steel and aluminium are shown in Fig. 5.2.4. For mild steel, the stress at first increases after reaching the elastic limit, but then decreases. The curve contains a distinct **yield point**; this is where a large increase in strain begins to occur with little increase in required stress[3], i.e. little hardening. There is no distinct yield point for aluminium (or, in fact, for most materials), Fig. 5.2.4b. In this case, it is useful to define a **yield strength** (or **offset yield point**). This is the maximum stress that can be applied without exceeding a specified value of permanent strain. This offset strain is usually taken to be 0.1 or 0.2% and the yield strength is found by following a line parallel to the linear portion until it intersects the stress-strain curve.

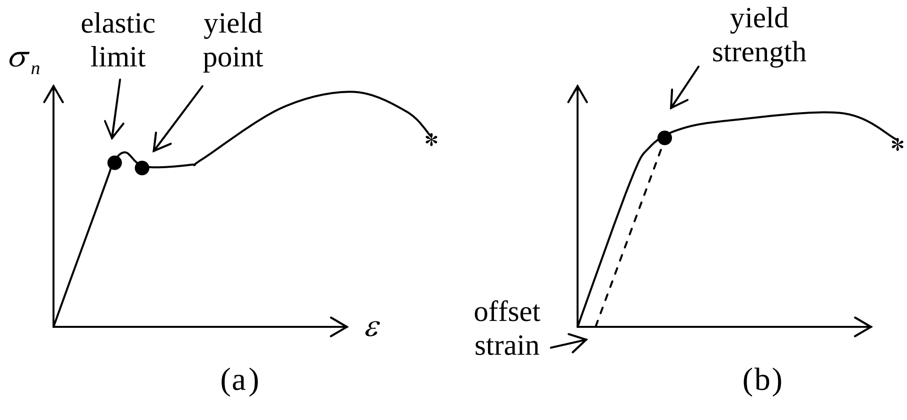


Figure 5.2.4: typical stress-strain curves for (a) mild steel, (b) aluminium

[3] this is also called the **lower yield point**; the **upper yield point** is then the higher stress value just above the elastic limit

The Young's Modulus

The slope of the stress-strain curve over the linear region, before the elastic limit is reached, is the **Young's Modulus** E:

$$E = \frac{\sigma}{\varepsilon} \tag{5.2.5}$$

The Young's Modulus has units of stress and is a measure of how "stiff" a material is.

Eqn. 5.2.5 is a constitutive relation (it is of the general form of Eqn. 5.1.1-2); it is the **one-dimensional linear elastic** constitutive relation.

Use of the Tension Test Data

What is the data from the tension test used for? First of all, it is of direct use in many structural applications. Many structures, such as bridges, buildings and the human skeleton, are composed in part of relatively long and slender components. In service, these components undergo tension and/or compression, very much like the test specimen in the tension test. The tension test data (the Young's Modulus, the Yield Strength and the UTS) then gives direct information on the amount of stress that these components can safely handle, before undergoing dangerous straining or all-out failure.

More importantly, the tension test data (and similar test data – see below) can be used to predict what will happen when a component of complex three-dimensional shape is loaded in a complex way, nothing like as in the simple tension test. This can be put another way: one must be able to predict the world around us without having to resort to complex, expensive, time-consuming materials testing – one should be able to use the test data from the tension test (and similar simple tests) to achieve this. How this is actually done is a major theme of mechanics modelling.

Test data for a number of metals are listed in Table 5.2.1 below. Note that although some materials can have similar stiffnesses, for example Nickel and Steel, their relative strengths can be very different.

	Young's Modulus *E* (GPa)	0.2% Yield Strength (MPa)	Ultimate Tensile Strength (MPa)
Ni	200	70	400
Mild steel	203	220	430
Steel (AISI 1144)	210	540	840
Cu	120	60	400
Al	70	40	200
Al Alloy (2014-T651)	73	415	485

Table 5.2.1: Tensile test data for some metals (at room temperature)

Data as listed above should be treated with caution – it should be used only as a rough guide to the actual material under study; the data can vary wildly depending on the purity and precise nature of the material. For example, the tensile strength of glass as found in a typical glass window is about 50MPa. For fine glass fibres as used in fibre-reinforced plastics and composite materials, the tensile strength can be 4000MPa. In fact, glass is a good reminder as to why the tensile values differ from material to material – it is due to the difference in microstructure. The glass window has many very fine flaws and cracks in it, invisible to the naked eye, and so this glass is not very strong; very fine slivers of glass have no such flaws and are extremely strong – hence their use in engineering applications.

The Poisson's Ratio

Another useful material parameter is the **Poisson's ratio** ν.[4] As the material stretches in the tension test, it gets thinner; the Poisson's ratio is a measure of the ease with which it thins:

$$\nu = -\frac{\Delta w / w_0}{\Delta l / l_0} = -\frac{\varepsilon_w}{\varepsilon} \tag{5.2.6}$$

Here, $\Delta w = w - w_0$, w_0 are the change in thickness and original thickness of the specimen, Fig. 5.2.5; $\Delta l = l - l_0$, l_0 are the change in length and original length of the specimen; $\varepsilon_w = (w - w_0)/w_0$ is the strain in the thickness direction. A negative sign is included because Δw is negative, making the Poisson's ratio a positive number. (It is implicitly assumed here that the material is getting thinner by the same amount in all directions; see below in the context of anisotropy for when this is not the case.)

Most engineering materials have a Poisson's Ratio of about 0.3. Values for a range of materials are listed in Table 5.2.2 further below.

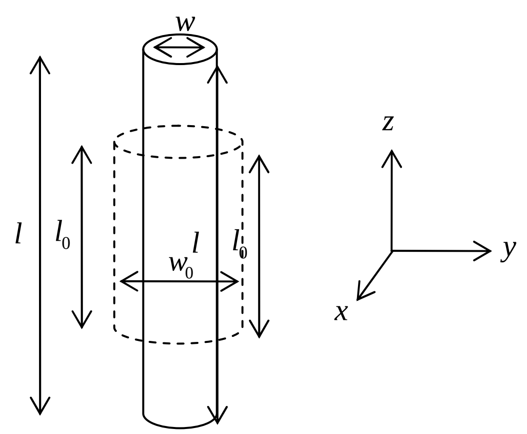


Figure 5.2.5: Change in dimensions of a test specimen

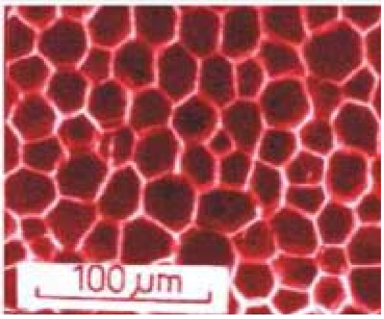
Recall from Section 4.3 that the volumetric strain is given by the sum of the normal strains. There is no harm in re-calculating this for the tensile test specimen of Fig. 5.2.5. One has $\Delta V / V = w^2 l / w_0^2 l_0 - 1$, so that, assuming the strains are small so that the terms $\varepsilon\varepsilon_w$, ε_w^2 and $\varepsilon\varepsilon_w^2$ can be neglected, $\Delta V / V = \varepsilon + 2\varepsilon_w$ (this is the sum of the normal

[4] this is the Greek letter *nu*, not the letter "v"

strains, $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$, Fig. 5.2.5). Using the definition of the Poisson's ratio, Eqn. 5.2.6, one has

$$\frac{\Delta V}{V} = \varepsilon(1 - 2\nu) \tag{5.2.7}$$

A material which undergoes little volume change thus has a Poisson's Ratio close to 0.5; rubber and other soft tissues, for example biological materials, have Poisson's Ratios very close to 0.5. A material which undergoes zero volume change ($\nu = 0.5$) is called **incompressible** (see more on incompressibility in Section 5.2.4 below). At the other extreme, materials such as cork can have Poisson's Ratios close to zero. The reason for this can be seen from the microstructure of cork shown in Fig. 5.2.6; when tested in compression, the hexagonal honeycomb structure simply folds down, with no necessary lateral expansion.



[4]

Figure 5.2.6: Microstructure of Cork

Auxetic materials are materials which have a negative Poisson's Ratio; when they are stretched, they get thicker. Examples can be found amongst polymers, foams, rocks and biological materials. These materials obviously have a very peculiar microstructure. A typical example is the network microstructure shown in Fig. 5.2.7.

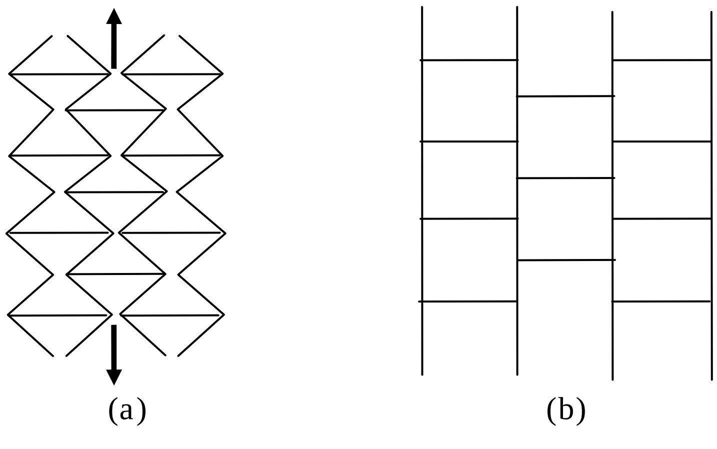


Figure 5.2.7: Auxetic material (a) before loading, (b) after loading

Ductile and Brittle Materials

The engineering materials can be grouped into two broad classes: the ductile materials and the brittle materials. The ductile materials undergo large permanent deformations, stretching and necking before failing[5]. The term ductile **rupture** is usually reserved for materials which fail in this way. The separate pieces of the specimen pull away from each other gradually, leaving rough surfaces. A simple measure of ductility is the engineering strain at failure. The brittle materials are generally more stiff and strong, but fail without undergoing much permanent deformation – the tension specimen undergoes a sudden clean break – a **fracture**. The UTS in the case of a brittle material is the same as the failure/fracture stress. Ceramics and glasses are extremely brittle – they fractures suddenly without undergoing any permanent deformation. The difference is illustrated schematically in Fig. 5.2.8 below.

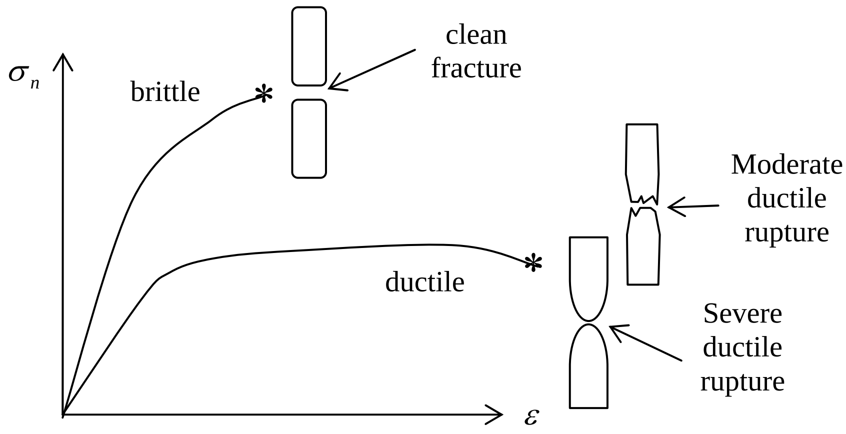


Figure 5.2.8: the difference between ductile and brittle materials

Ductility will depend on temperature – a very cold metal will tend to shatter suddenly, whereas it will stretch more easily when hot.

Soft Materials

Tension test data for (the traditionally) non-engineering materials can be very different to that given above. For example, the typical response of a "soft" material, such as rubber, is shown in Fig. 5.2.9. For many soft materials, the elastic limit (or yield strength) can be very high on the stress-strain curve, close to failure. Most of the curve is elastic, meaning that when one unloads the material, the unloading curve traces over the loading curve back down to zero stress and zero strain: the material does not undergo any permanent deformation[6]. Note that the stress-strain curve is non-linear (curved), unlike the straight line elastic portion for a typical metal, Fig. 5.2.2-4, so these materials do not have a single Young's Modulus through which their response can be described.

[5] the term ductile is used for a specimen in tension; the analogous term for compression is **malleability** – a malleable material is easily "squashed"

[6] here, as elsewhere, these statements should not be taken literally; a real rubber will undergo *some* permanent deformation, only it will often be so small that it can be discounted, and an unload curve will never "exactly" trace over a loading curve

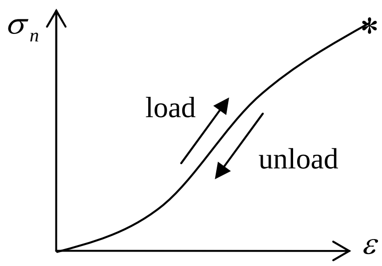
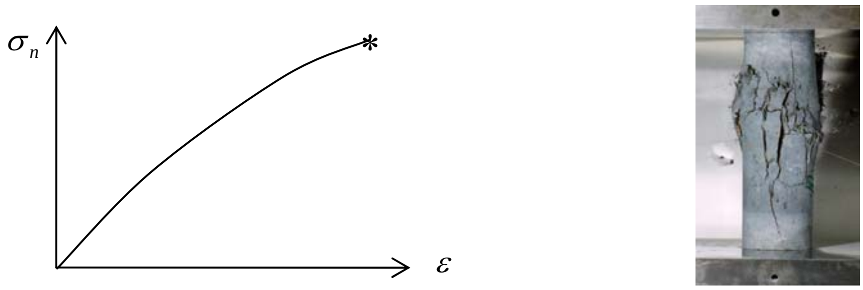


Figure 5.2.9: typical load/unload curves for rubber

5.2.2 Compression Tests

Many materials are used, or designed for use, in compression only, for example soils and concrete. These materials are tested in compression. A common testing method for concrete is to place a cylindrical specimen between two parallel plates and bring the plates together. The typical response of concrete is shown in Fig. 5.2.10a; at failure, the concrete crushes catastrophically, as in the specimen shown in Fig. 5.2.10b. Nominal stresses in the region 20-70MPa are typical and a good concrete would strain to much less than 1% at failure.



[5]

(a) (b)

Figure 5.2.10: typical compressive response of concrete; (a) stress-strain curve, (b) specimen at failure

For many materials, e.g. metals, a compression test will lead to similar results as the tensile stress. The yield strength in compression will be approximately the same as (the negative of) the yield strength in tension. If one plots the true stress versus true strain curve for both tension and compression (absolute values for the compression), and the two curves more or less coincide, this would indicate that the behaviour of the material under compression is broadly similar to that under tension. However, if one were to use the nominal stress and strain, then the two curves would not coincide even if the real tensile/compressive behaviour was similar (although they would of course in the small-strain linear region); this is due to the definition of the engineering strain/stress.

5.2.3 Shear Tests

In the **shear test**, the material is subjected to a shear strain $\gamma \equiv 2\varepsilon_{xy}$ by applying a shear stress[7] $\tau \equiv \sigma_{xy}$, Fig. 5.2.11a. The resulting shear stress-strain curve will be similar to the tensile stress-strain curve, Fig. 5.2.11b. The shear stress at failure, the **shear strength**, can be greater or smaller than the UTS. The shear yield strength, on the other hand, is usually in the region of 0.5-0.75 times the tensile yield strength. In the linear small-strain region, the shear stress will be proportional to the shear strain; the constant of proportionality is the **shear modulus** G:

$$G = \frac{\tau}{\gamma} \tag{5.2.8}$$

For many of the engineering materials, $G \approx 0.4E$.

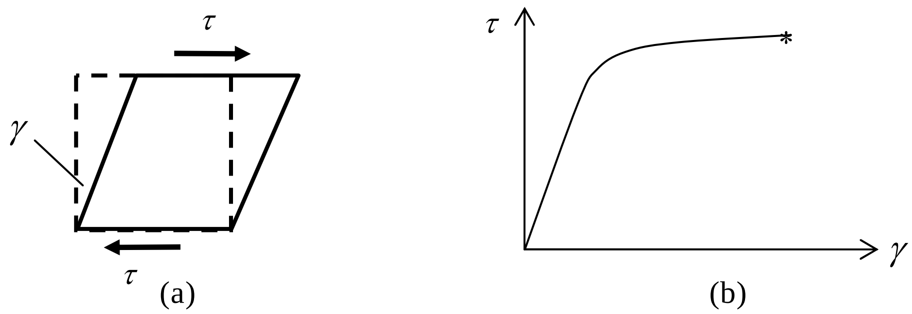


Figure 5.2.11: the shear test; (a) specimen subjected to shear stress and shear strain (dotted = undeformed), (b) shear stress-strain curve

5.2.4 Compressibility

In the **confined compression test**, a sample is placed in a container and a piston is used to compress it at some pressure p, Fig. 5.2.12a. This test can be used to determine how compressible a material is. When a material is compressed by equal pressures on all sides, the ratio of applied pressure p to (unit) volume change, i.e. volumetric strain $\Delta V / V$, is called the **Bulk Modulus** K, Fig. 5.2.12b (this is not quite the situation in Fig. 5.2.12a – the reaction pressures on the side walls will only be about half the applied surface pressure p; see Section 6.2):

$$K = -\frac{p}{\Delta V / V} \tag{5.2.9}$$

The negative sign is included since a positive pressure implies a negative volumetric strain, so that the Bulk Modulus is a positive value.

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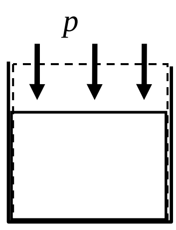
[7] there are many ways that this can be done, for example by pushing blocks of the material over each other, or using more sophisticated methods such as twisting thin tubes of the material (see Section 7.2)



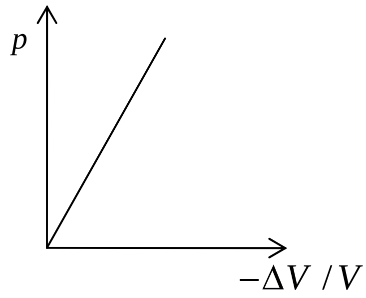


Figure 5.2.12: the confined compression test; (a) specimen subjected to confined compression, (b) pressure plotted against volume change

A material which can be easily compressed has a low Bulk Modulus. As mentioned earlier, a material which cannot be compressed at all is called incompressible ($K \to \infty$).

No real material is incompressible, but some can be regarded as incompressible so as to make the mechanics modelling easier. For example, the Shear Modulus of rubber is very much smaller than its Bulk Modulus, Table 5.2.2. Essentially, this means that the shape of rubber can be easily changed as compared to its volume. Thus, in applications where a rubber component is being deformed or subjected to arbitrary stressing, it is perfectly reasonable to simply assume that rubber is incompressible. The same applies, only more so, to water; the Shear Modulus is effectively zero and there is no resistance to change in shape (which will be observed on pouring a glass of water on to the ground); it is thus regarded almost always as completely incompressible. On the other hand, even though the Bulk Modulus of the metals and other engineering materials is very much *larger* than that of water or rubber, they are still regarded as compressible in applications – the extremely small changes in volume are significant.

	Young's Modulus E (GPa)	Shear Modulus G (GPa)	Bulk Modulus K (GPa)	Poissons Ratio
Ni	200	76	180	0.31
Mild steel	203	78	138	0.30
Steel (AISI 1144)	210	80	140	0.31
Cu	120	46	142	0.34
Al	70	26	76	0.35
Rubber	14.9×10^{-4}	5×10^{-4}	1	0.49
Water	$\approx 10^{-14}$	$\approx 10^{-14}$	2.2	

Table 5.2.2: Moduli and Poisson's Ratios for a number of materials

5.2.5 Cyclic Tests

Many materials are subjected to complex loading regimes when in service, not simply a one-off stretching, shearing or compression. A classic example are the wings of an aircraft which are continually loaded in tension, then compression, then tension and so on, as in Fig. 5.2.13. Another example would be the stresses experienced by cardiac tissue in a pumping heart. Anything moving back and forward is likely to be subjected to this tension/compression-type cyclic loading.

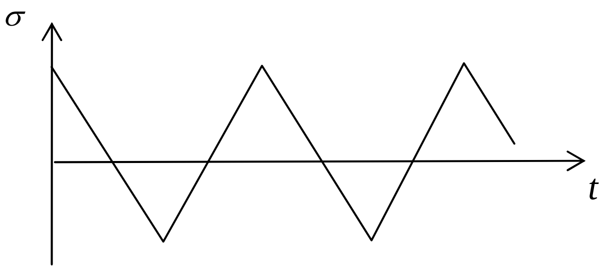
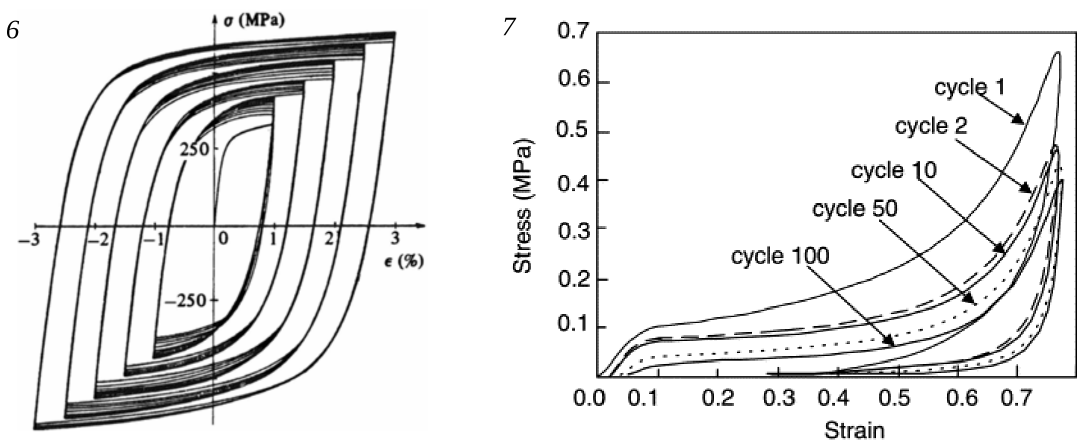


Figure 5.2.13: cyclic loading; alternating between tension (positive stress) and compression (negative stress) over time *t*

Cyclic tests can be carried out to determine the response of materials to such loading cycles. An example is shown in Fig. 5.2.14a, the stress-strain response of a Stainless Steel. The Steel is first cycled between two strain values (one positive, one negative, differing only in sign) a number of times. The stress is seen to increase on each successive cycle. The strain is then increased for a number of further cycles, and so on.

One does not have to move from tension to compression; many materials cycle in only tension or compression. For example, the response to cyclic (compressive) loading of polyurethane foam is shown in Fig. 5.2.14b (note how the loading curve is similar to that in 5.2.9).



(a) (b)

Figure 5.2.14: cyclic loading; (a) cyclic straining of a Stainless Steel, (b) cyclic loading (in compression) of a polyurethane foam

5.2.6 Other Tests

There are other important tests, for example the Vickers and Brinell **hardness tests**, and the **three-point bending test** (the bending test is discussed in section 7.4.9, in the context of beam theory). Another two very important tests, the **creep test** and the **stress relaxation test**, will be discussed in Chapter 10.

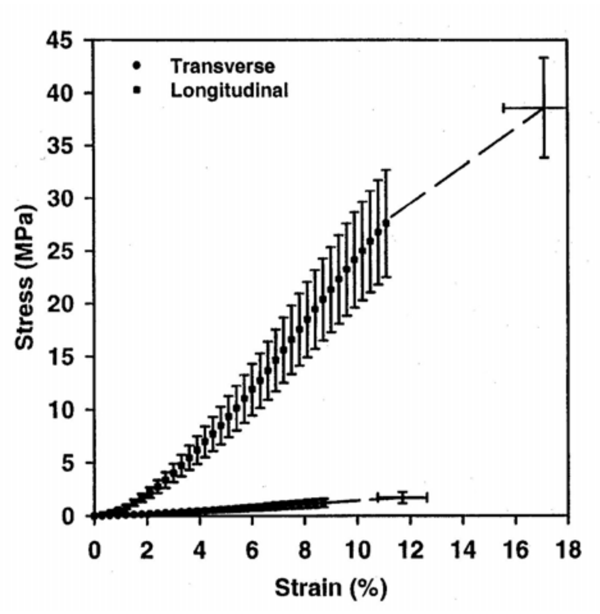
5.2.7 Isotropy and Anisotropy

Many materials have a strong direction-dependence. The classic example is wood, which has a clear structure –along the grain, along which fine lines can be seen, and against the grain, Fig. 5.2.15. The wood is stiffer and stronger along the grain than against the grain. A material which has this direction-dependence of mechanical (and physical) properties is called **anisotropic**.

8

Figure 5.2.15: Wood

Fig. 5.2.16 shows stress-strain curves for human ligament tissue; in one test, the ligament is stretched along its length (the **longitudinal** direction), in the second, across the width of the ligament (the **transverse** direction). It can be seen that the stiffness is much higher in the longitudinal direction. Another example is bone – it is much stiffer along the length of the bone than across the width of the bone. In fact, many biological materials are strongly anisotropic.



9

Figure 5.2.16: Anisotropic response of human ligament

A material whose properties are the same in all directions is called **isotropic**. In particular, the relationship between stress and strain *at any single location* in a material is the same in all directions. This implies that if a specimen is cut from an isotropic material and subjected to a load, it would not matter in which orientation the specimen is cut, the resulting deformation would be the same – as illustrated in Fig. 5.2.17. Most metals and ceramics can be considered to be isotropic (see Section 5.4).

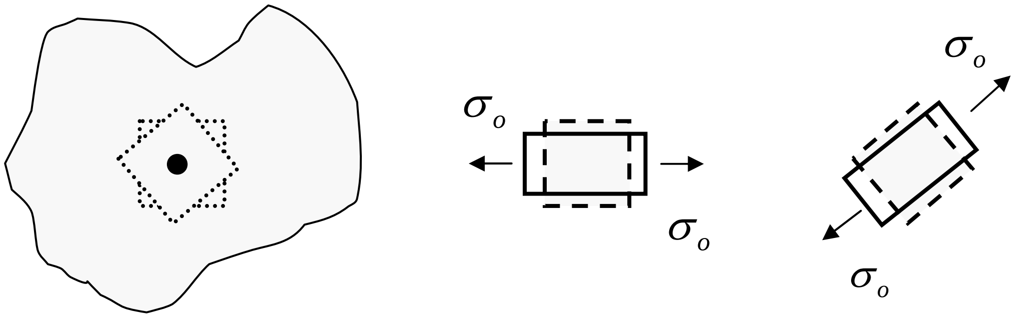


Figure 5.2.17: Illustration of Isotropy; the relationship between stress and strain is the same no matter in what "direction" the test specimen is cut from the material

Anisotropy will be examined in more detail in §6.3. It will be shown there, for example, that an anisotropic material can have a Poisson's ratio greater than 0.5.[8]

5.2.8 Homogeneous Materials

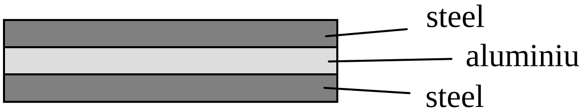
The term homogeneous means that the mechanical properties are the same at each point throughout the material. In other words, the relationship between stress and strain is the same for all material particles. Most materials can be assumed to be homogeneous.

In engineering applications, it is sometimes beneficial to design materials/components which are specifically not homogeneous, i.e. **inhomogeneous**. Such materials whose properties vary gradually throughout are called **Functionally Graded Materials**, and have been gaining popularity since the 1980s-90s in advanced technologies.

Note that a material can be homogeneous and not isotropic, and *vice versa* – homogeneous refers to different locations whereas isotropy refers to the same location.

5.2.9 Problems

1. Steel and aluminium can be considered to be isotropic and homogeneous materials. Is the composite sandwich-structure shown here isotropic and/or homogeneous? Everywhere in the sandwich?



[8] cork was mentioned earlier and it was pointed out that it has a near-zero Poisson's Ratio; actually, cork is quite anisotropic and the Poisson's Ratio in other "directions" will be different (close to 1.0)

Images used:

1. http://site.metacos.com/main/3108/index.asp?pageid=84386&t=&AlbumID=0&page=2
2. http://travisarp.wordpress.com/2012/01/24/how-do-they-do-that-tenderness/
3. http://www.ara.com/Projects/SVO/popups/weld_geometry.html
4. http://10minus9.wordpress.com/2010/03/23/10minus9-interview-philip-moriarty-part-2/
5. http://info.admet.com/blog/?Tag=Compression%20Test
6. Chaboche JL, On some modifications of kinematic hardening to improve the description of ratcheting effects, Int. J. Plasticity 7(7), 661-678, 1991.
7. Shen Y, Golnaraghi F, Plumtree A, Modelling compressive cyclic stress-strain behaviour of structural foam, Int J Fatigue, 23(6), 491-497, 2001.
8. http://www.sufaparket.com/solid-parquet/product-profile-solid/sungkai
9. Quapp KM, Weiss JA, Material characterization of human medial collateral ligament, J Biomech Engng, 120(6), 757-763, 1998.

5.3 Material Models

The response of real materials to various loading conditions was discussed in the previous section. Now comes the task of creating mathematical models which can predict this response. To this end, it is helpful to categorise the material responses into ideal models. There are four broad **material models** which are used for this purpose: (1) the **elastic model**, (2) the **viscoelastic model**, (3) the **plastic model**, and (4) the **viscoplastic model**. These models will be discussed briefly in what follows, and in more depth throughout the rest of this book.

5.3.1 The Elastic Model

An ideal elastic material has the following characteristics:

(i) the unloading stress-strain path is the same as the loading path
(ii) there is no dependence on the rate of loading or straining
(iii) it does not undergo permanent deformation; it returns to its precise original shape when the loads are removed

Typical stress-strain curves for an ideal elastic model subjected to a tension (or (compression) test are shown in Fig. 5.3.1. The response of a **linear elastic material**, where the stress is *proportional* to the strain, is shown in Fig. 5.3.1a and that for a **non-linear elastic material** is shown in Fig. 5.3.1b.

From the discussion in the previous section, the linear elastic model will well represent the engineering materials up to their elastic limit (see, for example, Figs. 5.2.2-4). It will also represent the complete stress-strain response up to the point of fracture of many very brittle materials. The model can also be used to represent the response of almost any material, provided the stresses are sufficiently small.

The non-linear elastic model is useful for predicting the response of soft materials like rubber and biological soft tissue (see, for example Fig. 5.2.9).

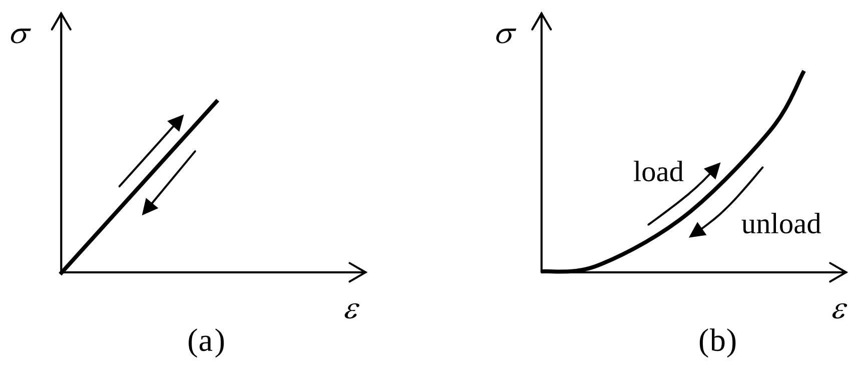


Figure 5.3.1: The Elastic Model; (a) linear elastic, (b) non-linear elastic

It goes without saying that there is no such thing as a purely elastic material. All materials will undergo at least some permanent deformations, even at low loads; no material's response will be exactly the same when stretched at different speeds, and so on.

However, if these occurrences and differences are small enough to be neglected, the ideal elastic model will be useful.

Note also that a prediction of a material's response may be made with accuracy using the elastic model in some circumstances, but not in others. An example would be metal; the elastic model might be able to predict the response right up to high stress levels when the metal is cold, but not so well when the temperature is high, when inelastic effects may not be so easily disregarded (see below).

5.3.2 Viscoelasticity

When solid materials have some "fluid-like" characteristics, they are said to be viscoelastic. A fluid is something which flows easily when subjected to loading – it cannot keep to any particular shape. If a fluid is one (the "viscous") extreme and the elastic solid is at the other extreme, then the viscoelastic material is somewhere in between.

The typical response of a viscoelastic material is sketched in Fig. 5.3.2. The following will be noted:

(i) the loading and unloading curves do not coincide, Fig. 5.3.2a, but form a **hysteresis loop**
(ii) there is a dependence on the rate of straining $d\varepsilon / dt$, Fig. 5.3.2b; the faster the stretching, the larger the stress required
(iii) there may or may not be some permanent deformation upon complete unloading, Fig. 5.3.2a

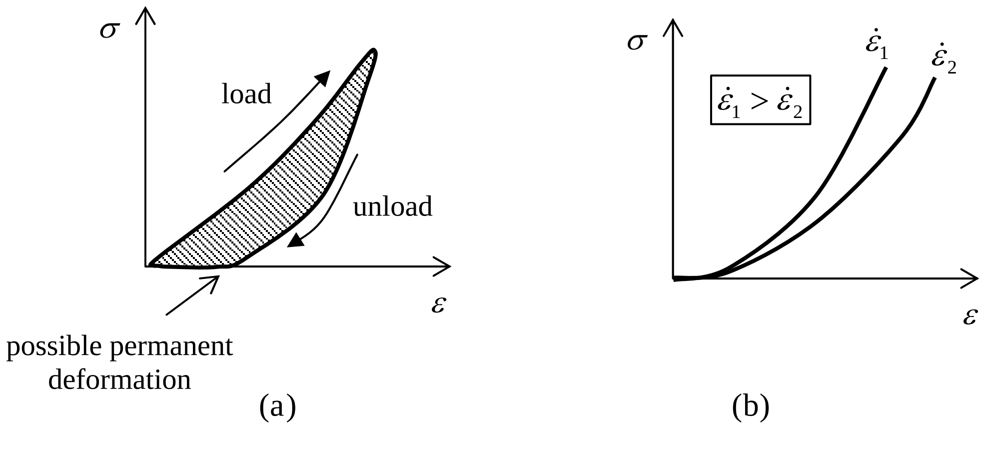


Figure 5.3.2: Response of a Viscoelastic material in the Tension test; (a) loading and unloading with possible permanent deformation (non-zero strain at zero stress), (b) different rates of stretching

The effect of *rate* of stretching shows that the viscoelastic material *depends on time*. This contrasts with the elastic material; it makes no difference whether an elastic material is loaded to some given stress level for one second or one day, or quickly or slowly, the resulting strain will be the same. This rate effect can be seen when you push your hand through water – it is easier to do so when you push slowly than when you push fast.

Depending on how “fluid-like” or “solid-like” a material is, it can be considered to be a **viscoelastic fluid**, for example blood or toothpaste, or a **viscoelastic solid**, for example Silly Putty™ or foam. That said, the model for both and the theory behind each will be similar.

Viscoelastic materials will be discussed in detail in Chapter 10.

5.3.3 Plasticity

Plasticity has the following characteristics:

(i) The loading is elastic up to some threshold limit, beyond which permanent deformations occur
(ii) The permanent deformation, i.e. the **plasticity**, is time independent

This plasticity can be seen in Figs. 5.2.2-4. The threshold limit – the elastic limit – can be quite high but it can also be extremely small, so small that significant permanent deformations occur at almost any level of loading. The plasticity model is particularly useful in describing the permanent deformations which occur in metals, soils and other engineering materials. It will be discussed in further detail in Chapter 11.

5.3.4 Viscoplasticity

Finally, the viscoplastic model is a combination of the viscoelastic and plastic models. In this model, the plasticity is rate-dependent. One of the main applications of the model is in the study of metals at high temperatures, but it is used also in the modeling of a huge range of materials and other applications, for example asphalt, concrete, clay, paper pulp, biological cells growth, etc. This model will be discussed in Chapter 12.

5.4 Continuum Models and Micromechanics

The models mentioned in the previous section are **continuum models**. What this means is explained in what follows.

5.4.1 Stress and Scale

In the definition of the traction vector, §3.3.1, it was assumed that the ratio of force over area would reach some definite limit as the area ΔS of the surface upon which the force ΔF acts was shrunk to zero. This issue can be explored further by considering Fig. 5.4.1. Assume first that the plane upon which the force acts is fairly large; it is then shrunk and the ratio F/S tracked. A schematic of this ratio is shown in Fig. 5.4.2. At first (to the right of Fig. 5.4.2) the ratio F/S undergoes change, assuming the stress to vary within the material, as it invariably will if the material is loaded in some complex way. Eventually the plane will be so small that the ratio changes very little, perhaps with some small variability ε. If the plane is allowed to get too small, however, down below some distance h^* say and down towards the atomic level, where one might encounter "intermolecular space", there will be large changes in the ratio and the whole concept of a force acing on a single surface breaks down.

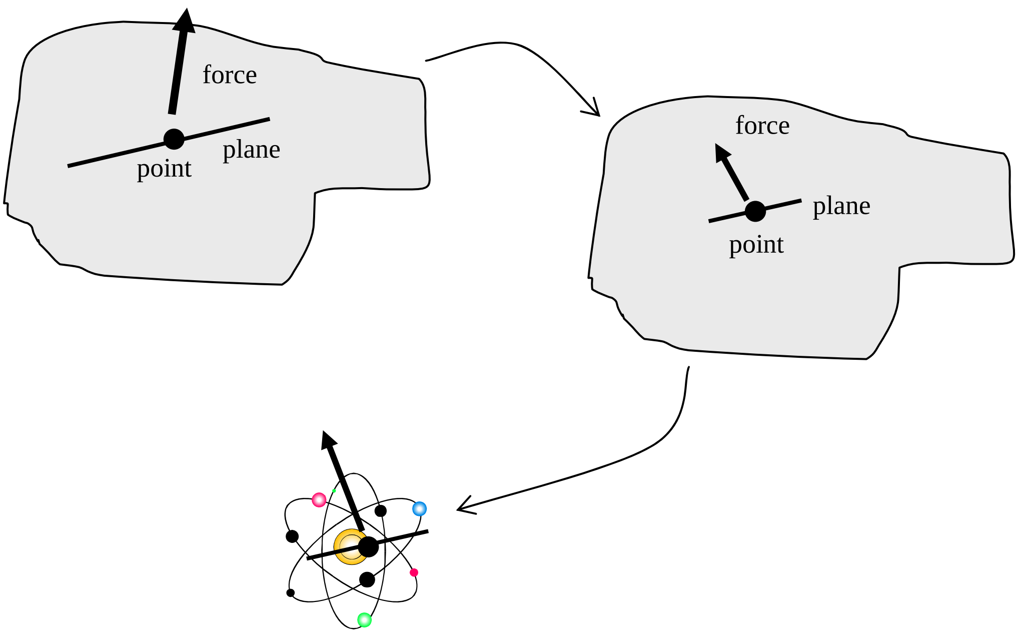


Figure 5.4.1: A force acting on an internal surface; allowing the plane on which the force acts to get progressively smaller

In a continuum model, it is assumed that the ratio F/S follows the dotted path shown in Fig. 5.4.2; a definite limit is reached as the plane shrinks to *zero size*. It should be kept in mind that the traction in a *real* material should be evaluated through

$$\mathbf{t} = \lim_{\Delta S \to (h^*)^2} \frac{\Delta F}{\Delta S} \qquad (5.4.1)$$

where $h*$ is some minimum dimension below which there is no acceptable limit. On the other hand, it is necessary to take the limit to zero in the *mathematical* modelling of materials since that is the basis of calculus[1].

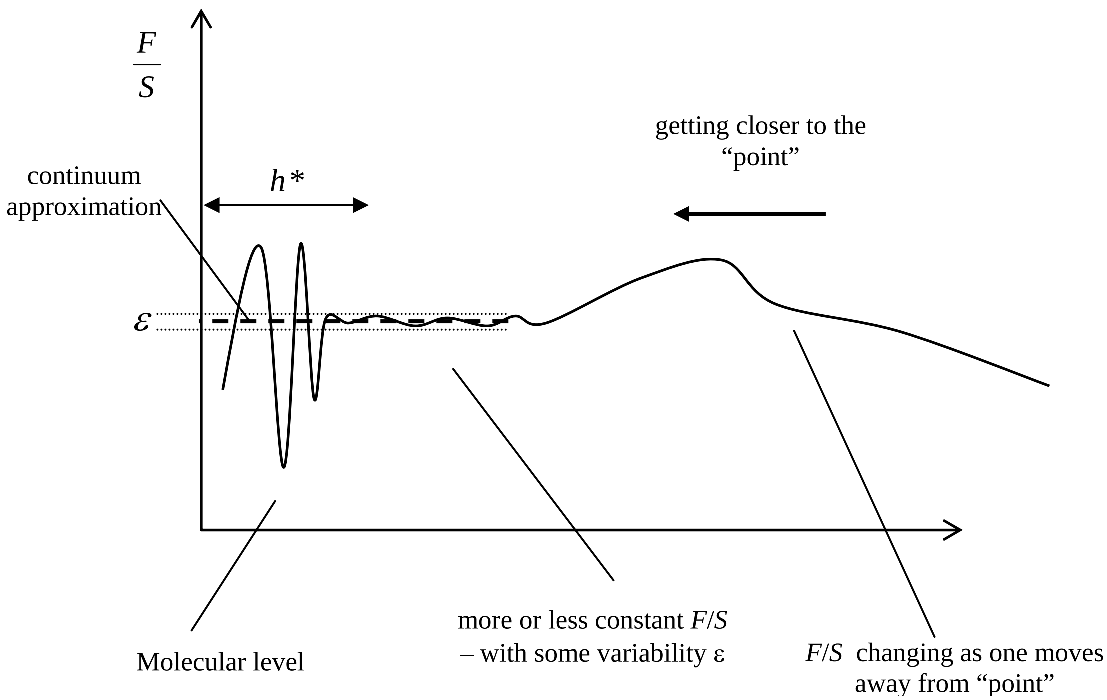


Figure 5.4.2: the change in traction as the plane upon which a force acts is reduced in size

In a continuum model, then, there is a minimum sized element one can consider, say of size $\Delta V = (h*)^3$. When one talks about the stress on this element, the mass of this element, the density, velocity and acceleration of this element, one means the average of these quantities throughout or over the surface of the element – the discrete atomic structure within the element is ignored and is averaged out, or "smeared" out, into a **continuum element**.

The continuum element is also called a **representative volume element** (RVE), an element of material large enough for the heterogeneities to be replaced by homogenised mean values of their properties. The order of the dimensions of RVE's for some common engineering materials would be approximately (see the metal example which follows)

Metal:	0.1mm
Polymers/composites:	1mm
Wood:	10mm
Concrete:	100mm

One does not have any information about what is happening inside the continuum element – it is like a "black box". The scale of the element (and higher) is called the

[1] calculus is not used too much in this book – it is absolutely necessary and ubiquitous in more refined and advanced mechanics theories

macroscale – continuum mechanics is mechanics on the macroscale. The scale of entities within the element is termed the **microscale** – continuum models cannot give any information about what happens on the microscale.

5.4.2 Example: Metal

Metal, from a distance, appears fairly uniform. With the help of a microscope, however, it will be seen to consist of many individual grains of metal. For example, the metal shown in Fig. 5.4.3 has grains roughly 0.05mm across, and each one has very individual properties (the crystals in each grain are aligned in different directions).

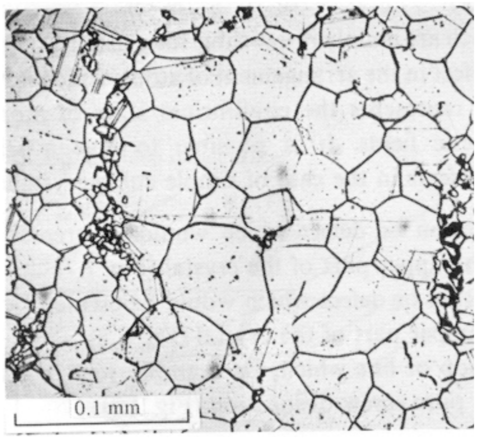


Figure 5.4.3: metal grains

If one is interested in the gross deformation of a moderately sized component of this metal, it would be sufficient to consider deformations that are averaged over volumes which are large compared to individual grains, but small compared to the whole component. A minimum dimension of, say, $h^* = 0.5\text{mm}$ for the metal of Fig. 5.4.3 would seem to suffice, and this would be the macro/micro-scale boundary, with a minimum surface area of dimension $(h^*)^2$ for the definition of stress.

When one measures physical properties of the metal "at a point", for example the density, one need only measure an average quantity over an element of the order, say, $(0.5\text{mm})^3$ or higher. It is not necessary to consider the individual grains of metal – these are inside the "black box". The model will return valuable information about the deformation of the gross material, but it will not be able to furnish any information about movement of individual grains.

It was shown how to evaluate the Young's Modulus and other properties of a metal in Section 5.2.1. The test specimens used for such tests are vastly larger than the continuum elements discussed above. Thus the test data is perfectly adequate to describe the response of the metal, on the macroscale.

What if the response of individual grains to applied loads is required? In that case a model would have to be constructed which accounted for the different mechanical properties of each grain. The metal could no longer be considered to be a uniform material, but a complex one with many individual grains, each with different properties and orientation. The macro/micro boundary could be set at about $h^* = 0.1\mu\text{m}$. There are now two problems which need to be dealt with: (1) experiments such as the tensile test would have to be conducted on specimens much smaller than the grain size in order to provide data for any mathematical model, and (2) the mathematical model will be more complex and difficult to solve.

5.4.3 Micromechanical Models

Consider the schematic of a continuum model shown in Fig. 5.4.4 below. One can determine the material's properties, such as the Young's modulus *E*, through experimentation, and the resulting mathematical continuum model can be used to make predictions about the material's response. With the improved power of computers, especially since the 1990s, it has now become possible to complement continuum models with **micromechanical models**. These models take into account more fine detail of the material's structure (for example of the individual grains of the metal discussed earlier). Usually, one will have a micromechanical model of a small (typical) RVE of material. This then provides information regarding the properties of the RVE to be included in a continuum model (rather than having a micromechanical model of the *complete* material, which is in most cases still not practical). The means by which the properties at the micro scale are averaged (for example into a "smeared out" single *E* value) and passed "up" to the continuum model is through **homogenisation theory**. Such micromechanical models can provide further insight into material behaviour than the simpler continuum model.

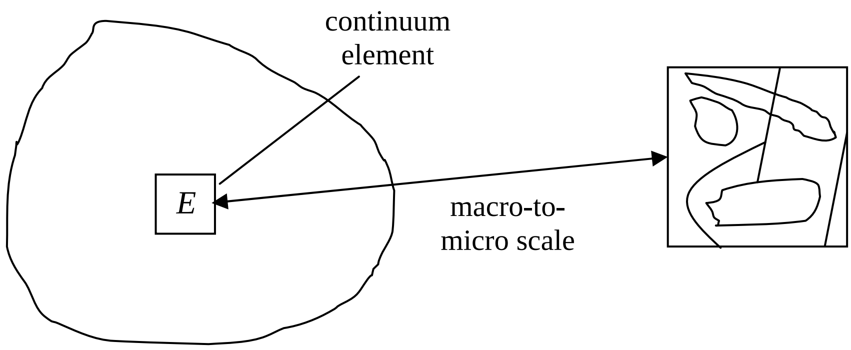


Figure 5.4.4: continuum model and micromechanical model

5.4.4 Problems

1. You want to evaluate the stiffness *E* of a metal for inclusion in a mechanics model. What *minimum* size specimen would you use in your test - $10\mu\text{m}$, 0.1mm, 5mm or 5cm?

2. Individual rice grains are separate solid particles. However, rice flowing down a chute at a food processing plant can be considered to be a fluid, and the flow of rice can be solved using the equations of mechanics. What minimum dimension h^*

should be employed for measurements in this case to ensure the validity of a continuum model of flowing rice?

6 Linear Elasticity

The simplest constitutive law for solid materials is the linear elastic law, which assumes a linear relationship between stress and engineering strain. This assumption turns out to be an excellent predictor of the response of components which undergo small deformations, for example steel and concrete structures under large loads, and also works well for practically any material at a sufficiently small load.

The linear elastic model is discussed in this chapter and some elementary problems involving elastic materials are solved. Anisotropic elasticity is discussed in Section 6.3.

6.1 The Linear Elastic Model

6.1.1 The Linear Elastic Model

Repeating some of what was said in Section 5.3: the Linear Elastic model is used to describe materials which respond as follows:

(i) the strains in the material are small[1] (**linear**)

(ii) the stress is proportional to the strain, $\sigma \propto \varepsilon$ (**linear**)

(iii) the material returns to its original shape when the loads are removed, and the unloading path is the same as the loading path (**elastic**)

(iv) there is no dependence on the rate of loading or straining (**elastic**)

From the discussion in the previous chapter, this model well represents the engineering materials up to their elastic limit. It also models well almost any material provided the stresses are sufficiently small.

The stress-strain (loading and unloading) curve for the Linear Elastic solid is shown in Fig. 6.1.1a. Other possible responses are shown in Figs. 6.1.1b,c. Fig. 6.1.1b shows the typical response of a rubbery-type material and many biological tissues; these are **non-linear elastic** materials. Fig. 6.1.1c shows the typical response of **viscoelastic** materials (see Chapter 10) and that of many plastically and viscoplastically deforming materials (see Chapters 11 and 12).

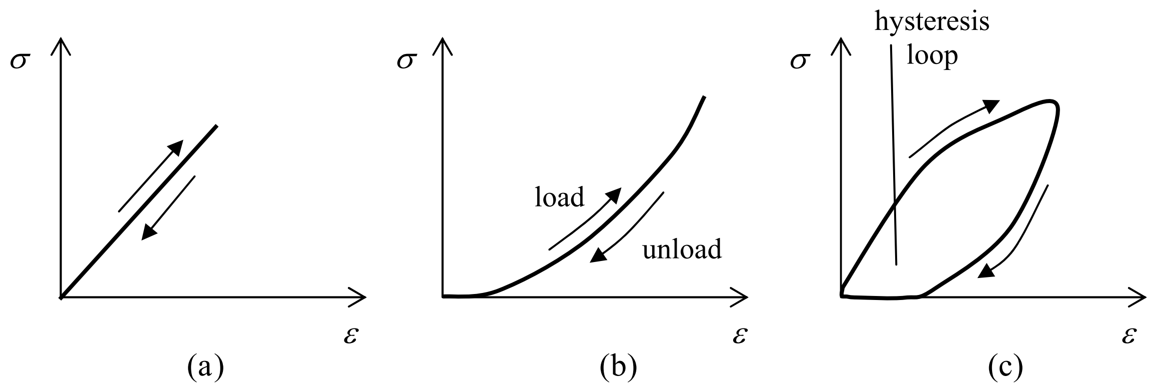


Figure 6.1.1: Different stress-strain relationships; (a) linear elastic, (b) non-linear elastic, (c) viscoelastic/plastic/viscoplastic

It will be assumed at first that the material is isotropic and homogeneous. The case of an anisotropic elastic material is discussed in Section 6.3.

[1] if the small-strain approximation is not made, the stress-strain relationship will be inherently non-linear; the actual strain, Eqn. 4.1.7, involves (non-linear) squares and square-roots of lengths

6.1.2 Stress-Strain Law

Consider a cube of material subjected to a uniaxial tensile stress σ_{xx}, Fig. 6.1.2a. One would expect it to respond by extending in the *x* direction, $\varepsilon_{xx} > 0$, and to contract laterally, so $\varepsilon_{yy} = \varepsilon_{zz} < 0$, these last two being equal because of the isotropy of the material. With stress proportional to strain, one can write

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx}, \quad \varepsilon_{yy} = \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} \tag{6.1.1}$$

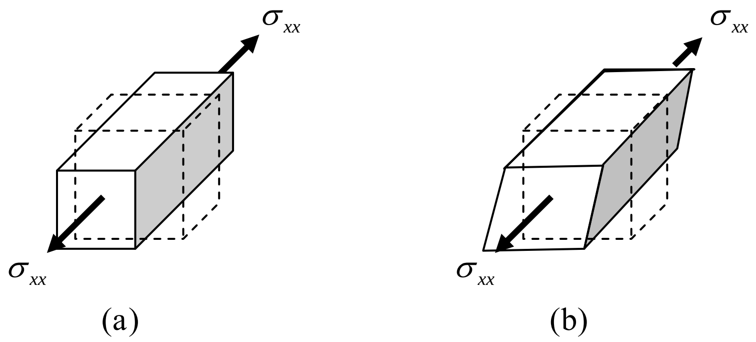


Figure 6.1.2: an element of material subjected to a uniaxial stress; (a) normal strain, (b) shear strain

The constant of proportionality between the normal stress and strain is the Young's Modulus, Eqn. 5.2.5, the measure of the stiffness of the material. The material parameter ν is the Poisson's ratio, Eqn. 5.2.6. Since $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}$, it is a measure of the contraction relative to the normal extension.

Because of the isotropy/symmetry of the material, the shear strains are zero, and so the deformation of Fig. 6.1.2b, which shows a non-zero ε_{xy}, is not possible – shear strain can arise if the material is not isotropic.

One can write down similar expressions for the strains which result from a uniaxial tensile σ_{yy} stress and a uniaxial σ_{zz} stress:

$$\begin{aligned} \varepsilon_{yy} &= \frac{1}{E}\sigma_{yy}, \quad \varepsilon_{xx} = \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{yy} \\ \varepsilon_{zz} &= \frac{1}{E}\sigma_{zz}, \quad \varepsilon_{xx} = \varepsilon_{yy} = -\frac{\nu}{E}\sigma_{zz} \end{aligned} \tag{6.1.2}$$

Similar arguments can be used to write down the shear strains which result from the application of a shear stress:

$$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}, \quad \varepsilon_{yz} = \frac{1}{2\mu}\sigma_{yz}, \quad \varepsilon_{xz} = \frac{1}{2\mu}\sigma_{xz} \tag{6.1.3}$$

The constant of proportionality here is the Shear Modulus μ, Eqn. 5.2.8, the measure of the resistance to shear deformation (the letter *G* was used in Eqn. 5.2.8 – both *G* and μ are used to denote the Shear Modulus, the latter in more "mathematical" and "advanced" discussions) .

The strain which results from a combination of all six stresses is simply the sum of the strains which result from each[2]:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right], \quad \varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}, \quad \varepsilon_{xz} = \frac{1}{2\mu}\sigma_{xz}, \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right], \quad \varepsilon_{yz} = \frac{1}{2\mu}\sigma_{yz} \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]
\end{aligned} \tag{6.1.4}$$

These equations involve three material parameters. It will be proved in §6.3 that an isotropic linear elastic material can have only *two* independent material parameters and that, in fact,

$$\mu = \frac{E}{2(1+\nu)}. \tag{6.1.5}$$

This relation will be verified in the following example.

Example: Verification of Eqn. 6.1.5

Consider the simple shear deformation shown in Fig. 6.1.3, with $\varepsilon_{xy} > 0$ and all other strains zero. With the material linear elastic, the only non-zero stress is $\sigma_{xy} = 2\mu\varepsilon_{xy}$.

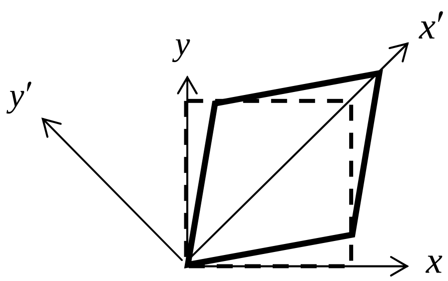


Figure 6.1.3: a simple shear deformation

[2] this is called the **principle of linear superposition**: the "effect" of a sum of "causes" is equal to the sum of the individual "effects" of each "cause". For a linear relation, e.g. $\sigma = E\varepsilon$, the effects of two causes σ_1, σ_2 are $E\varepsilon_1$ and $E\varepsilon_2$, and the effect of the sum of the causes $\sigma_1 + \sigma_2$ is indeed equal to the sum of the individual effects: $E(\varepsilon_1 + \varepsilon_2) = E\varepsilon_1 + E\varepsilon_2$. This is not true of a non-linear relation, e.g. $\sigma = E\varepsilon^2$, since $E(\varepsilon_1 + \varepsilon_2)^2 \neq E\varepsilon_1^2 + E\varepsilon_2^2$

Using the strain transformation equations, Eqns. 4.2.2, the only non-zero strains in a second coordinate system $x' - y'$, with x' at $\theta = 45^{\circ}$ from the x axis (see Fig. 6.1.3), are $\varepsilon'_{xx} = +\varepsilon_{xy}$ and $\varepsilon'_{yy} = -\varepsilon_{xy}$. Because the material is isotropic, Eqns 6.1.4 hold also in this second coordinate system and so the stresses in the new coordinate system can be determined by solving the equations

$$\begin{aligned}
\varepsilon'_{xx} &= +\varepsilon_{xy} = \frac{1}{E}\left[\sigma'_{xx} - \nu\left(\sigma'_{yy} + \sigma'_{zz}\right)\right], \quad \varepsilon'_{xy} = 0 = \frac{1}{2\mu}\sigma'_{xy}, \quad \varepsilon'_{xz} = 0 = \frac{1}{2\mu}\sigma'_{xz}, \\
\varepsilon'_{yy} &= -\varepsilon_{xy} = \frac{1}{E}\left[\sigma'_{yy} - \nu\left(\sigma'_{xx} + \sigma'_{zz}\right)\right], \quad \varepsilon'_{yz} = 0 = \frac{1}{2\mu}\sigma'_{yz} \\
\varepsilon'_{zz} &= 0 \quad = \frac{1}{E}\left[\sigma'_{zz} - \nu\left(\sigma'_{xx} + \sigma'_{yy}\right)\right]
\end{aligned} \qquad (6.1.6)$$

which results in

$$\sigma'_{xx} = +\frac{E}{1+\nu}\varepsilon_{xy}, \quad \sigma'_{yy} = -\frac{E}{1+\nu}\varepsilon_{xy} \qquad (6.1.7)$$

But the stress transformation equations, Eqns. 3.4.8, with $\sigma_{xy} = 2\mu\varepsilon_{xy}$, give $\sigma'_{xx} = +2\mu\varepsilon_{xy}$ and $\sigma'_{yy} = -2\mu\varepsilon_{xy}$ and so Eqn. 6.1.5 is verified.

■

Relation 6.1.5 allows the Linear Elastic Solid stress-strain law, Eqn. 6.1.4, to be written as

$$\boxed{\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\varepsilon_{xz} &= \frac{1+\nu}{E}\sigma_{xz} \\
\varepsilon_{yz} &= \frac{1+\nu}{E}\sigma_{yz}
\end{aligned}} \qquad \textbf{Stress-Strain Relations} \qquad (6.1.8)$$

This is known as **Hooke's Law**. These equations can be solved for the stresses to get

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\left(\varepsilon_{xx} + \varepsilon_{zz}\right)\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} \\
\sigma_{xz} &= \frac{E}{1+\nu}\varepsilon_{xz} \\
\sigma_{yz} &= \frac{E}{1+\nu}\varepsilon_{yz}
\end{aligned}$$

Stress-Strain Relations (6.1.9)

Values of E and ν for a number of materials are given in Table 6.1.1 below (see also Table 5.2.2).

Material	*E* (GPa)	ν
Grey Cast Iron	100	0.29
A316 Stainless Steel	196	0.3
A5 Aluminium	68	0.33
Bronze	130	0.34
Plexiglass	2.9	0.4
Rubber	0.001-2	0.4-0.49
Concrete	23-30	0.2
Granite	53-60	0.27
Wood (pinewood) fibre direction transverse direction	 17 1	 0.45 0.79

Table 6.1.1: Young's Modulus *E* and Poisson's Ratio ν for a selection of materials at 20°C

Volume Change

Recall that the volume change in a material undergoing small strains is given by the sum of the normal strains (see Section 4.3). From Hooke's law, normal stresses cause normal strain and shear stresses cause shear strain. It follows that *normal stresses produce volume changes* and *shear stresses produce distortion* (change in shape), but no volume change.

6.1.3 Two Dimensional Elasticity

The above three-dimensional stress-strain relations reduce in the case of a two-dimensional stress state or a two-dimensional strain state.

Plane Stress

In plane stress (see Section 3.5), $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$, Fig. 6.1.5, so the stress-strain relations reduce to

$$\boxed{\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\\
\sigma_{xx} &= \frac{E}{1-\nu^2}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \\
\sigma_{yy} &= \frac{E}{1-\nu^2}\left[\nu\varepsilon_{xx} + \varepsilon_{yy}\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy}
\end{aligned}}$$

Stress-Strain Relations (Plane Stress) (6.1.10)

with

$$\begin{aligned}
&\varepsilon_{zz} = -\frac{\nu}{E}\left[\sigma_{xx} + \sigma_{yy}\right], \quad \varepsilon_{xz} = \varepsilon_{yz} = 0 \\
&\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0
\end{aligned} \qquad (6.1.11)$$

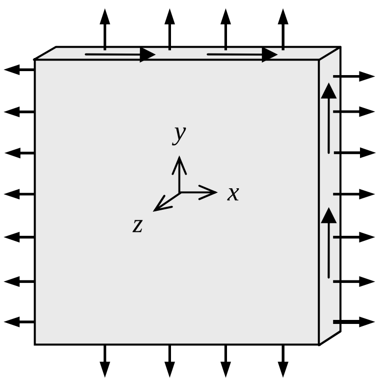


Figure 6.1.5: Plane stress

Note that the ε_{zz} strain is *not* zero. Physically, ε_{zz} corresponds to a change in thickness of the material perpendicular to the direction of loading.

Plane Strain

In plane strain (see Section 4.2), $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$, Fig. 6.1.6, and the stress-strain relations reduce to

$$\boxed{\begin{aligned}
\varepsilon_{xx} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] \\
\varepsilon_{yy} &= \frac{1+\nu}{E}\left[-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\\
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{xx}\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy}
\end{aligned}}$$

Stress-Strain Relations (Plane Strain) (6.1.12)

with

$$\begin{aligned}
&\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \\
&\sigma_{zz} = \nu\left[\sigma_{xx} + \sigma_{yy}\right], \quad \sigma_{xz} = \sigma_{yz} = 0
\end{aligned} \qquad (6.1.13)$$

Again, note here that the stress component σ_{zz} is *not* zero. Physically, this stress corresponds to the forces preventing movement in the *z* direction.

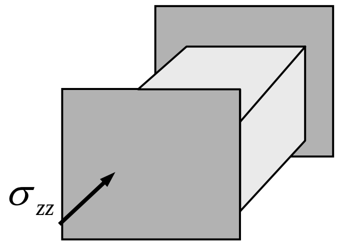


Figure 6.1.6 Plane strain - a thick component constrained in one direction

Similar Solutions

The expressions for plane stress and plane strain are very similar. For example, the plane strain constitutive law 6.1.12 can be derived from the corresponding plane stress expressions 6.1.10 by making the substitutions

$$E = \frac{E'}{1-\nu'^2}, \quad \nu = \frac{\nu'}{1-\nu'} \qquad (6.1.14)$$

in 6.1.10 and then dropping the primes. The plane stress expressions can be derived from the plane strain expressions by making the substitutions

$$E = E'\frac{1+2\nu'}{(1+\nu')^2}, \quad \nu = \frac{\nu'}{1+\nu'} \qquad (6.1.15)$$

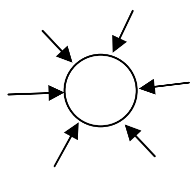
in 6.1.12 and then dropping the primes. Thus, if one solves a plane stress problem, one has automatically solved the corresponding plane strain problem, and *vice versa*.

6.1.4 Problems

1. A strain gauge at a certain point on the surface of a thin A5 Aluminium component (loaded in-plane) records strains of $\varepsilon_{xx} = 60\,\mu\text{m}$, $\varepsilon_{yy} = 30\,\mu\text{m}$, $\varepsilon_{xy} = 15\,\mu\text{m}$. Determine the principal stresses. (See Table 6.1.1 for the material properties.)

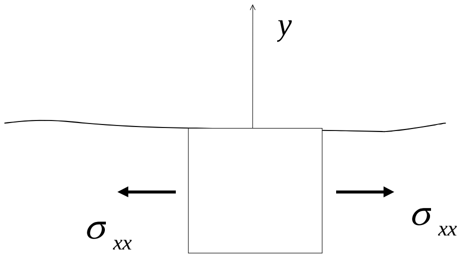
2. Use the stress-strain relations to prove that, for a linear elastic solid,
$$\frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}} = \frac{2\varepsilon_{xy}}{\varepsilon_{xx}-\varepsilon_{yy}}$$
and, indeed,
$$\frac{2\sigma_{xz}}{\sigma_{xx}-\sigma_{zz}} = \frac{2\varepsilon_{xz}}{\varepsilon_{xx}-\varepsilon_{zz}}, \quad \frac{2\sigma_{yz}}{\sigma_{yy}-\sigma_{zz}} = \frac{2\varepsilon_{yz}}{\varepsilon_{yy}-\varepsilon_{zz}}$$
Note: from Eqns. 3.5.4 and 4.2.4, these show that the principal axes of stress and strain coincide for an *isotropic* elastic material

3. Consider the case of hydrostatic pressure in a linearly elastic solid:
$$[\sigma_{ij}] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$
as might occur, for example, when a spherical component is surrounded by a fluid under high pressure, as illustrated in the figure below. Show that the volumetric strain (Eqn. 4.3.5) is equal to
$$-p\frac{3(1-2\nu)}{E},$$
so that the Bulk Modulus, Eqn. 5.2.9, is
$$K = \frac{E}{3(1-2\nu)}$$



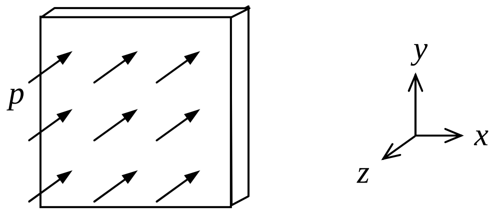
4. Consider again Problem 2 from §3.5.7.
   (a) Assuming the material to be linearly elastic, what are the strains? Draw a second material element (superimposed on the one shown below) to show the deformed shape of the square element – assume the displacement of the box-centre to be zero and that there is no rotation. Note how the free surface moves, even though there is no stress acting on it.

(b) What are the principal strains ε_1 and ε_2? You will see that the principal directions of stress and strain coincide (see Problem 2) – the largest normal stress and strain occur in the same direction.



5. Consider a very thin sheet of material subjected to a normal pressure *p* on one of its large surfaces. It is fixed along its edges. This is an example of a **plate** problem, an important branch of elasticity with applications to boat hulls, aircraft fuselage, etc.
   (a) write out the complete three dimensional stress-strain relations (both cases, strain in terms of stress, Eqns. 6.1.8, stress in terms of strain, Eqns. 6.1.9). Following the discussion on thin plates in section 3.5.4, the shear stresses σ_{xz}, σ_{yz}, can be taken to be zero throughout the plate. Simplify the relations using this fact, the pressure boundary condition on the large face and the coordinate system shown.
   (b) assuming that the through thickness change in the sheet can be neglected, show that

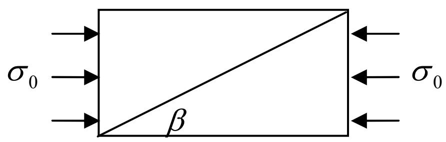
$$p = -\nu(\sigma_{xx} + \sigma_{yy})$$



6. A thin linear elastic rectangular plate with width *a* and height *b* is subjected to a uniform compressive stress σ_0 as shown below. Show that the slope of the plate diagonal shown after deformation is given by

$$\tan(\beta + \delta\beta) = \frac{b}{a}\left(\frac{1 + \nu\sigma_0 / E}{1 - \sigma_0 / E}\right)$$

What is the magnitude of $\delta\beta$ for a steel plate ($E = 210\text{GPa}$, $\nu = 0.3$) of dimensions $20 \times 20\ \text{cm}^2$ with $\sigma_0 = 1\text{MPa}$?



6.2 Homogeneous Problems in Linear Elasticity

A **homogeneous** stress (strain) field is one where the stress (strain) is the same at all points in the material. Homogeneous conditions will arise when the geometry is simple and the loading is simple.

6.2.1 Elastic Rectangular Cuboids

Hooke's Law, Eqns. 6.1.8 or 6.1.9, can be used to solve problems involving homogeneous stress and deformation. Hoooke's law is 6 equations in 12 unknowns (6 stresses and 6 strains). If some of these unknowns are given, the rest can be found from the relations.

Example

Consider the block of linear elastic material shown in Fig. 6.2.1. It is subjected to an equi-biaxial stress of $\sigma_{xx} = \sigma_{yy} = \bar{\sigma} > 0$.

Since this is an isotropic elastic material, the shears stresses and strains will be all zero for such a loading. One thus need only consider the three normal stresses and strains.

There are now 3 equations (the first 3 of Eqns. 6.1.8 or 6.1.9) in 6 unknowns. One thus needs to know *three* of the normal stresses and/or strains to find a solution. From the loading, one knows that $\sigma_{xx} = \bar{\sigma}$ and $\sigma_{yy} = \bar{\sigma}$. The third piece of information comes from noting that the surfaces parallel to the $x-y$ plane are free surfaces (no forces acting on them) and so $\sigma_{zz} = 0$.

From Eqn. 6.1.8 then, the strains are

$$\varepsilon_{xx} = \varepsilon_{yy} = (1-\nu)\frac{\bar{\sigma}}{E}, \quad \varepsilon_{zz} = -2\nu\frac{\bar{\sigma}}{E}, \quad \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

As expected, $\varepsilon_{xx} = \varepsilon_{yy}$ and $\varepsilon_{zz} < 0$.

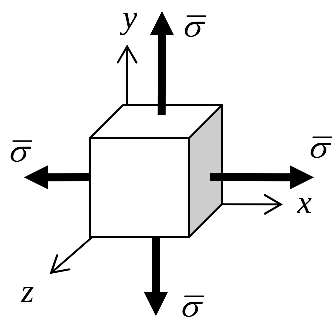


Figure 6.2.1: A block of linear elastic material subjected to an equi-biaxial stress

■

6.2.2 Problems

1. A block of isotropic linear elastic material is subjected to a compressive normal stress σ_o over two opposing faces. The material is constrained (prevented from moving) in one of the direction normal to these faces. The other faces are free.
   (a) What are the stresses and strains in the block, in terms of σ_o, ν, E ?
   (b) Calculate three maximum shear stresses, one for each plane (parallel to the faces of the block). Which of these is the overall maximum shear stress acting in the block?

2. Repeat problem 1a, only with the free faces now fixed also.

6.3 Anisotropic Elasticity

There are many materials which, although well modelled using the linear elastic model, are not nearly isotropic. Examples are wood, composite materials and many biological materials. The mechanical properties of these materials differ in different directions. Materials with this direction dependence are called anisotropic (see Section 5.2.7).

6.3.1 Material Constants

The most general form of Hooke's law, the **generalised Hooke's Law**, for a linear elastic material is

$$\begin{bmatrix} \sigma_1 = \sigma_{xx} \\ \sigma_2 = \sigma_{yy} \\ \sigma_3 = \sigma_{zz} \\ \sigma_4 = \sigma_{yz} \\ \sigma_5 = \sigma_{xz} \\ \sigma_6 = \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 = \varepsilon_{xx} \\ \varepsilon_2 = \varepsilon_{yy} \\ \varepsilon_3 = \varepsilon_{zz} \\ \varepsilon_4 = \varepsilon_{yz} \\ \varepsilon_5 = \varepsilon_{xz} \\ \varepsilon_6 = \varepsilon_{xy} \end{bmatrix} \tag{6.3.1}$$

where each stress component depends (linearly) on all strain components. This new notation, with only one subscript for the stress and strain, numbered from 1…6, is helpful as it allows the equations of anisotropic elasticity to be written in matrix form. The 36 C_{ij}'s are material constants called the **stiffnesses**, and in principle are to be obtained from experiment. The matrix of stiffnesses is called the **stiffness matrix**. Note that these equations imply that a normal stress σ_{xx} will induce a material element to not only stretch in the *x* direction and contract laterally, but to undergo shear strain too, as illustrated schematically in Fig. 6.3.1.

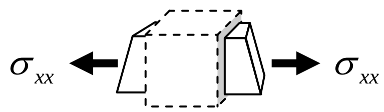


Figure 6.3.1: an element undergoing shear strain when subjected to a normal stress only

In section 8.4.3, when discussing the strain energy in an elastic material, it will be shown that it is necessary for the stiffness matrix to be *symmetric* and so there are only 21 independent elastic constants in the most general case of anisotropic elasticity.

Eqns. 6.3.1 can be inverted so that the strains are given explicitly in terms of the stresses:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{6.3.2}$$

The S_{ij}'s here are called **compliances**, and the matrix of compliances is called the **compliance matrix**. The bottom half of the compliance matrix has been omitted since it too is symmetric.

It is difficult to model fully anisotropic materials due to the great number of elastic constants. Fortunately many materials which are not fully isotropic still have certain **material symmetries** which simplify the above equations. These material types are considered next.

6.3.2 Orthotropic Linear Elasticity

An **orthotropic** material is one which has three orthogonal planes of microstructural symmetry. An example is shown in Fig. 6.3.2a, which shows a glass-fibre composite material. The material consists of thousands of very slender, long, glass fibres bound together in bundles with oval cross-sections. These bundles are then surrounded by a plastic binder material. The continuum model of this composite material is shown in Fig. 6.3.2b wherein the fine microstructural details of the bundles and surrounding matrix are “smeared out” and averaged. Three mutually perpendicular planes of symmetry can be passed through each point in the continuum model. The x, y, z axes forming these planes are called the **material directions**.

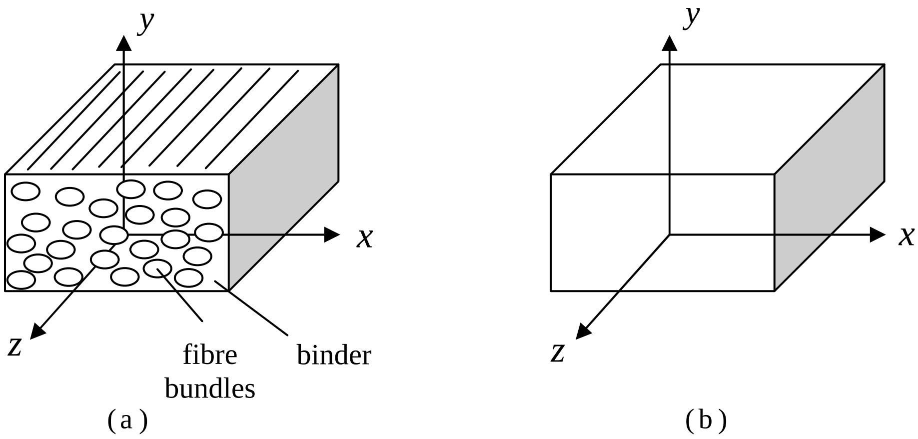


Figure 6.3.2: an orthotropic material; (a) microstructural detail, (b) continuum model

The material symmetry inherent in the orthotropic material reduces the number of independent elastic constants. To see this, consider an element of orthotropic material subjected to a shear strain $\varepsilon_6 \left(= \varepsilon_{xy}\right)$ and also a strain $-\varepsilon_6 \left(= -\varepsilon_{xy}\right)$, as in Fig. 6.3.3.

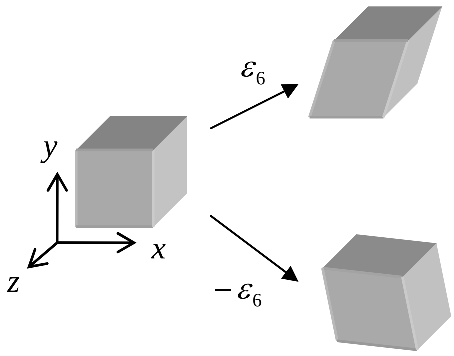


Figure 6.3.3: an element of orthotropic material undergoing shear strain

From Eqns. 6.3.1, the stresses induced by a strain ε_6 only are

$$\begin{aligned} \sigma_1 &= C_{16}\varepsilon_6, \quad \sigma_2 = C_{26}\varepsilon_6, \quad \sigma_3 = C_{36}\varepsilon_6 \\ \sigma_4 &= C_{46}\varepsilon_6, \quad \sigma_5 = C_{56}\varepsilon_6, \quad \sigma_6 = C_{66}\varepsilon_6 \end{aligned} \tag{6.3.3}$$

The stresses induced by a strain $-\varepsilon_6$ only are (the prime is added to distinguish these stresses from those of Eqn. 6.3.3)

$$\begin{aligned} \sigma_1' &= -C_{16}\varepsilon_6, \quad \sigma_2' = -C_{26}\varepsilon_6, \quad \sigma_3' = -C_{36}\varepsilon_6 \\ \sigma_4' &= -C_{46}\varepsilon_6, \quad \sigma_5' = -C_{56}\varepsilon_6, \quad \sigma_6' = -C_{66}\varepsilon_6 \end{aligned} \tag{6.3.4}$$

These stresses, together with the strain, are shown in Fig. 6.3.4 (the microstructure is also indicated)

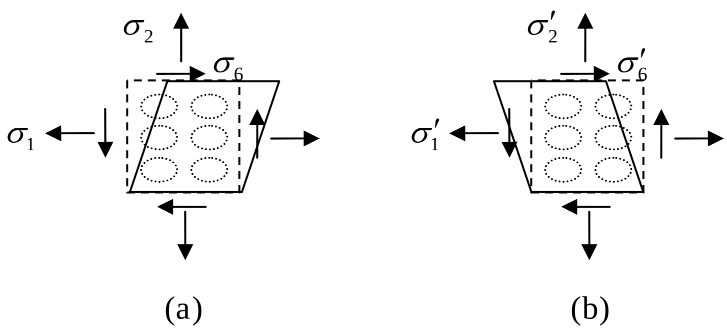


Figure 6.3.4: an element of orthotropic material undergoing shear strain; (a) positive strain, (b) negative strain

Because of the symmetry of the material (print this page out, turn it over, and Fig. 6.3.4a viewed from the "other side" of the page is the same as Fig. 6.3.4b on "this side" of the page), one would expect the normal stresses in Fig. 6.3.4 to be the same, $\sigma_1 = \sigma_1'$,

$\sigma_2 = \sigma_2'$, but the shear stresses to be of opposite sign, $\sigma_6 = -\sigma_6'$. Eqns. 6.3.3-4 then imply that

$$C_{16} = C_{26} = C_{36} = C_{46} = C_{56} = 0 \tag{6.3.5}$$

Similar conclusions follow from considering shear strains in the other two planes:

$$\begin{aligned} \varepsilon_5 &: C_{15} = C_{25} = C_{35} = C_{45} = 0 \\ \varepsilon_4 &: C_{14} = C_{24} = C_{34} = 0 \end{aligned} \tag{6.3.6}$$

The stiffness matrix is thus reduced, and there are only *nine* independent elastic constants:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{6.3.7}$$

These equations can be inverted to get, introducing elastic constants E, ν and G in place of the S_{ij}'s:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{E_1} & -\dfrac{\nu_{21}}{E_2} & -\dfrac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\dfrac{\nu_{12}}{E_1} & \dfrac{1}{E_2} & -\dfrac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\dfrac{\nu_{13}}{E_1} & -\dfrac{\nu_{23}}{E_2} & \dfrac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{2G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{2G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{2G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{6.3.8}$$

The nine independent constants here have the following meanings:

E_i is the Young's modulus (stiffness) of the material in direction $i = 1, 2, 3$; for example, $\sigma_1 = E_1 \varepsilon_1$ for uniaxial tension in the direction 1.

ν_{ij} is the Poisson's ratio representing the ratio of a transverse strain to the applied strain in *uniaxial tension*; for example, $\nu_{12} = -\varepsilon_2 / \varepsilon_1$ for uniaxial tension in the direction 1.

G_{ij} are the shear moduli representing the shear stiffness in the corresponding plane; for example, G_{12} is the shear stiffness for shearing in the 1-2 plane.

If the 1-axis has long fibres along that direction, it is usual to call G_{12} and G_{13} the **axial shear moduli** and G_{23} the **transverse** (**out-of-plane**) **shear modulus**.

Note that, from symmetry of the stiffness matrix,

$$\nu_{23}E_3 = \nu_{32}E_2, \quad \nu_{13}E_3 = \nu_{31}E_1, \quad \nu_{12}E_2 = \nu_{21}E_1 \tag{6.3.9}$$

An important feature of the orthotropic material is that there is no **shear coupling** with respect to the material axes. In other words, normal stresses result in normal strains only and shear stresses result in shear strains only.

Note that there will in general be shear coupling when the reference axes used, x, y, z, are not aligned with the material directions $1, 2, 3$. For example, suppose that the $x-y$ axes were oriented to the material axes as shown in Fig. 6.3.5. Assuming that the material constants were known, the stresses and strains in the constitutive equations 6.3.8 can be transformed into $\varepsilon_{xx}, \varepsilon_{xy}$, etc. and σ_{xx}, σ_{xy}, etc. using the strain and stress transformation equations. The resulting matrix equations relating the strains $\varepsilon_{xx}, \varepsilon_{xy}$ to the stresses σ_{xx}, σ_{xy} will then not contain zero entries in the stiffness matrix, and normal stresses, e.g. σ_{xx}, will induce shear strain, e.g. ε_{xy}, and shear stress will induce normal strain.

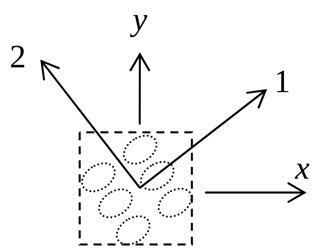


Figure 6.3.5: reference axes not aligned with the material directions

6.3.3 Transversely Isotropic Linear Elasticity

A **transversely isotropic** material is one which has a single material direction and whose response in the plane orthogonal to this direction is isotropic. An example is shown in Fig. 6.3.6, which again shows a glass-fibre composite material with aligned fibres, only now the cross-sectional shapes of the fibres are circular. The characteristic material direction is z and the material is isotropic in any plane parallel to the $x-y$ plane. The material properties are the same in all directions transverse to the fibre direction.

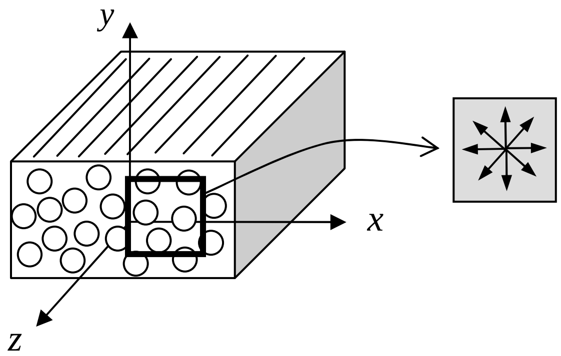


Figure 6.3.6: a transversely isotropic material

This extra symmetry over that inherent in the orthotropic material reduces the number of independent elastic constants further. To see this, consider an element of transversely isotropic material subjected to a normal strain $\varepsilon_1 \left(=\varepsilon_{xx}\right)$ only of magnitude ε, Fig. 6.3.7a, and also a normal strain $\varepsilon_2 \left(=\varepsilon_{yy}\right)$ of the same magnitude, ε, Fig. 6.3.7b. The $x-y$ plane is the plane of isotropy.

Figure 6.3.7: elements of a transversely isotropic material undergoing normal strain in the plane of isotropy

From Eqns. 6.3.7, the stresses induced by a strain $\varepsilon_1 = \varepsilon$ only are

$$\begin{gathered}\sigma_1 = C_{11}\varepsilon, \quad \sigma_2 = C_{21}\varepsilon, \quad \sigma_3 = C_{31}\varepsilon \\ \sigma_4 = 0, \quad \sigma_5 = 0, \quad \sigma_6 = 0\end{gathered} \tag{6.3.10}$$

The stresses induced by the strain $\varepsilon_2 = \varepsilon$ only are (the prime is added to distinguish these stresses from those of Eqn. 6.3.10)

$$\begin{gathered}\sigma_1' = C_{12}\varepsilon, \quad \sigma_2' = C_{22}\varepsilon, \quad \sigma_3' = C_{32}\varepsilon \\ \sigma_4' = 0, \quad \sigma_5' = 0, \quad \sigma_6' = 0\end{gathered} \tag{6.3.11}$$

Because of the isotropy, the $\sigma_1(=\sigma_{xx})$ due to the ε_1 should be the same as the $\sigma_2 = (\sigma_{yy})$ due to the ε_2, and it follows that $C_{11} = C_{22}$. Further, the $\sigma_3(=\sigma_{zz})$ should be the same for both, and so $C_{31} = C_{32}$.

Further simplifications arise from consideration of shear deformations, and rotations about the material axis, and one finds that $C_{44} = C_{55}$ and $C_{66} = C_{11} - C_{12}$.

The stiffness matrix is thus reduced, and there are only *five* independent elastic constants:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{11} - C_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \qquad (6.3.12)$$

with '3' being the material direction. These equations can be inverted to get, introducing elastic constants E, ν and G in place of the S_{ij}'s. One again gets Eqn. 6.3.8, but now

$$E_1 = E_2, \quad \nu_{12} = \nu_{21}, \quad \nu_{13} = \nu_{23}, \quad \nu_{31} = \nu_{32}, \quad G_{13} = G_{23} \qquad (6.3.13)$$

so

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{13}}{E_1} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \qquad (6.3.14)$$

with, due to symmetry,

$$\nu_{13} / E_1 = \nu_{31} / E_3 \qquad (6.3.15)$$

Eqns. 6.3.13-15 seem to imply that there are 6 independent constants; however, the transverse modulus G_{12} is related to the transverse Poisson ratio and the transverse stiffness through (see Eqn. 6.1.5, and 6.3.20 below, for the isotropic version of this relation)

$$G_{12} = \frac{E_1}{2(1+\nu_{12})} \qquad (6.3.16)$$

These equations are often expressed in terms of "*a*" for fibre (or "*a*" for axial) and "*t*" for transverse:

$$
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} =
\begin{bmatrix}
\frac{1}{E_t} & -\frac{\nu_t}{E_t} & -\frac{\nu_f}{E_f} & 0 & 0 & 0 \\
-\frac{\nu_t}{E_t} & \frac{1}{E_t} & -\frac{\nu_f}{E_f} & 0 & 0 & 0 \\
-\frac{\nu_f}{E_f} & -\frac{\nu_f}{E_f} & \frac{1}{E_f} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2G_f} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2G_f} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2G_t}
\end{bmatrix}
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}
\tag{6.3.17}
$$

6.3.4 Isotropic Linear Elasticity

An isotropic material is one for which the material response is independent of orientation. The symmetry here further reduces the number of elastic constants to *two*, and the stiffness matrix reads

$$
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
 & C_{11} & C_{12} & 0 & 0 & 0 \\
 & & C_{11} & 0 & 0 & 0 \\
 & & & C_{11}-C_{12} & 0 & 0 \\
 & & & & C_{11}-C_{12} & 0 \\
 & & & & & C_{11}-C_{12}
\end{bmatrix}
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}
\tag{6.3.18}
$$

These equations can be inverted to get, introducing elastic constants E, ν and G,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{6.3.19}$$

with

$$\frac{1}{2G} = \frac{1+\nu}{E} \tag{6.3.20}$$

which are Eqns. 6.1.8 and 6.1.5.

Eqns. 6.3.18 can also be written concisely in terms of the engineering constants E, ν and G with the help of the **Lamé constants**, λ and μ:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & 2\mu & 0 & 0 \\ & & & & 2\mu & 0 \\ & & & & & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{6.3.21}$$

with

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \quad (=G) \tag{6.3.22}$$

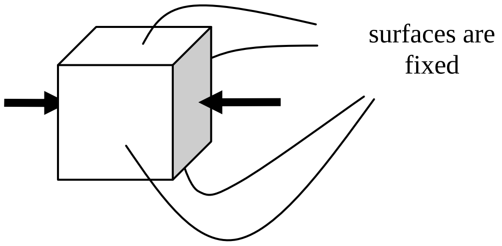
6.3.5 Problems

1. A piece of orthotropic material is loaded by a uniaxial stress σ_1 (aligned with the material direction '1'). What are the strains in the material, in terms of the engineering constants?

2. A specimen of bone in the shape of a cube is fixed and loaded by a compressive stress $\sigma = 1\text{MPa}$ as shown below. The bone can be considered to be orthotropic, with material properties

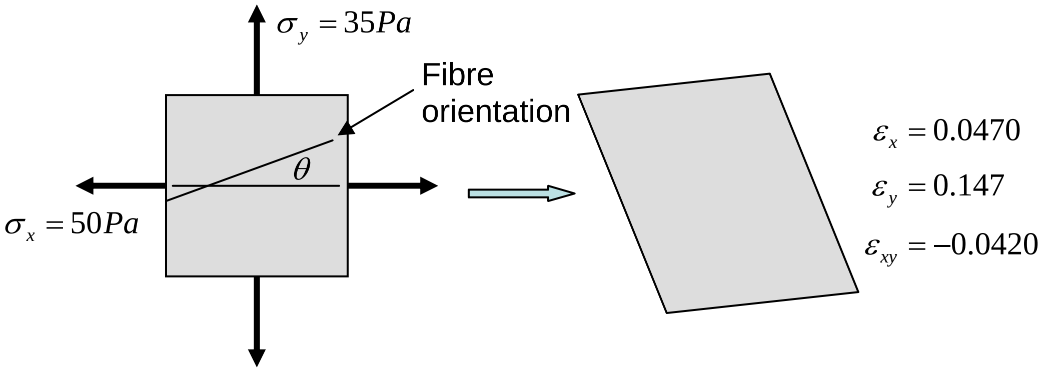
$$E_1 = 6.91\text{GPa}, \quad E_2 = 8.51\text{GPa}, \quad E_3 = 18.4\text{GPa}$$
$$G_{12} = 2.41\text{GPa}, \quad G_{13} = 3.56\text{GPa}, \quad G_{23} = 4.91\text{GPa}$$
$$\nu_{21} = 0.62, \quad \nu_{31} = 0.32, \quad \nu_{32} = 0.31$$

What are the stresses and strains which arise from the test according to this model (the bone is compressed along the ‘1’ direction)?



3. Consider a block of transversely isotropic material subjected to a compressive stress $\sigma_1 = -p$ (perpendicular to the material direction) and constrained from moving in the other two perpendicular directions (as in Problem 2). Evaluate the stresses σ_2 and σ_3 in terms of the engineering constants E_t, E_f and ν_t, ν_f.

4. A strip of skin is tested in biaxial tension as shown below. The measured stresses and strains are as given in the figure. The orientation of the fibres in the material is later measured to be $\theta = 20^o$.



(a) Calculate the normal stresses along and transverse to the fibres, and the corresponding shear stress. (Hint: use the stress transformation equations.)
(b) Calculate the normal strains along and transverse to the fibres, and the corresponding shear strain. (Hint: use the strain transformation equations.)
(c) Assuming the material to be orthotropic, determine the elastic constants of the material (assume the stiffness in the fibre direction to be five times greater than the stiffness in the transverse direction). Note: because the material is thin, one can take $\sigma_3 = \sigma_4 = \sigma_5 = 0$.
(d) Calculate the magnitude and orientations of the principal normal stresses and strains. (Hint: the principal directions of stress are where there is zero shear stress.)
(e) Do the principal directions of stress and strain coincide?

5. A biaxial test is performed on a roughly planar section of skin (thickness 1mm) from the back of a test-animal. The test axes (*x* and *y*) are aligned such that deformation is induced in the skin along the spinal direction and transverse to this direction, under the assumption that the fibres are oriented principally in these directions. However, it is found during the experiment that shear stresses are necessary to maintain a biaxial deformation state. Measured stresses are

$$\sigma_{xx} = 5\text{kPa}, \quad \sigma_{yy} = 2\text{kPa}, \quad \sigma_{xy} = 1\text{kPa}$$

Determine the in-plane orientation of the fibres given the data $E_1 = 1000\,\text{kPa}$, $E_2 = 500\,\text{kPa}$, $G_6 = 500\,\text{kPa}$, $\nu_{21} = 0.2$.

[Hint: derive an expression for ε_{xy} involving θ only, where θ is the inclination of the material axes from the $x-y$ axes]

7 Applications of Elasticity

The linear elastic model was introduced in the previous chapter and some elementary problems involving elastic materials were solved there (in particular in section 6.2). In this Chapter, five important, practical, theories are presented concerning elastic materials; they all have specific geometries and are subjected to particular types of load. In §7.1, the geometry is that of a long slender bar and the load is one which acts along the length of the bar; in §7.2, the geometry is that of a long slender circular bar and the load is one which twists the bar; in §7.3 the geometry is that of a thin-walled cylindrical or spherical component, and the load is normal to these walls; in §7.4 the geometry is that of a long and slender beam, and the load is transverse to the beam length. Finally, in §7.5, the geometry is a column, fixed at one end and loaded at the other so that it deflects. These five particular situations allow for simplifications (or approximations) to be made to the full three-dimensional linear elastic stress-strain relations; this allows one to write down simple expressions for the stress and strain and so solve some important practical problems analytically.

7.1 One Dimensional Axial Deformations

In this section, a specific simple geometry is considered, that of a long and thin straight component loaded in such a way that it deforms in the axial direction only. The x-axis is taken as the longitudinal axis, with the cross-section lying in the $x-y$ plane, Fig. 7.1.1.

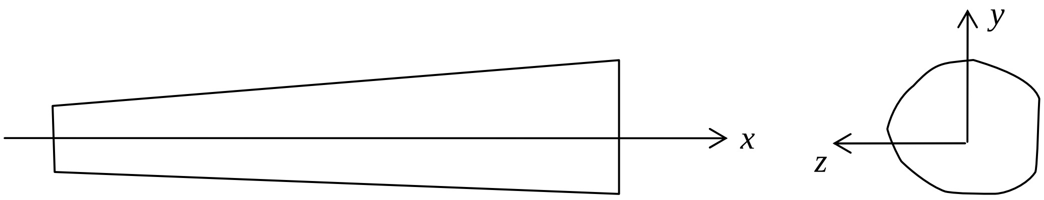


Figure 7.1.1: A slender straight component; (a) longitudinal axis, (b) cross-section

7.1.1 Basic relations for Axial Deformations

Any static analysis of a structural component involves the following three considerations:
(1) constitutive response
(2) kinematics
(3) equilibrium

In this Chapter, it is taken for (1) that the material responds as an isotropic linear elastic solid. It is assumed that the only significant stresses and strains occur in the axial direction, and so the stress-strain relations 6.1.8-9 reduce to the one-dimensional equation $\sigma_{xx} = E\varepsilon_{xx}$ or, dropping the subscripts,

$$\sigma = E\varepsilon \tag{7.1.1}$$

Kinematics (2), the study of deformation, was the subject of Chapter 4. In the theory developed here, known as **axial deformation**, it is assumed that the axis of the component remains straight and that cross-sections that are initially perpendicular to the axis remain perpendicular after deformation. This implies that, although the strain might vary along the axis, it remains *constant over any cross section*. The axial strain occurring over any section is defined by Eqn. 4.1.2,

$$\varepsilon = \frac{L - L_0}{L_0} \tag{7.1.2}$$

This is illustrated in Fig. 7.1.2, which shows a (shaded) region undergoing a compressive (negative) strain.

Recall that individual particles/points undergo displacements whereas regions/line-elements undergo strain. In Fig. 7.1.2, the particle originally at A has undergone a displacement $u(A)$ whereas the particle originally at B has undergone a displacement $u(B)$. From Fig. 7.1.2, another way of expressing the strain in the shaded region is (see Eqn. 4.1.3)

$$\varepsilon = \frac{u(B) - u(A)}{L_0} \tag{7.1.3}$$

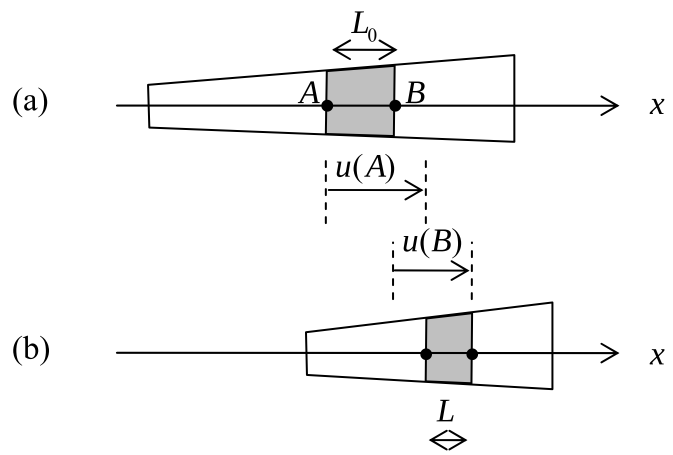


Figure 7.1.2: axial strain; (a) before deformation, (b) after deformation

Both displacements $u(A)$ and $u(B)$ of Fig. 7.1.2 are *positive*, since the particles displace in the positive *x* direction – if they moved to the left, for consistency, one would say they underwent *negative* displacements. Further, positive stresses are as shown in Fig. 7.1.3a and negative stresses are as shown in Fig. 7.1.3b. From Eqn. 7.1.1, a positive stress implies a positive strain (lengthening) and a compressive stress implies a negative strain (contracting)

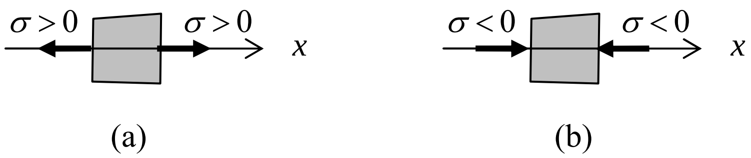


Figure 7.1.3: Stresses arising in the slender component; (a) positive (tensile) stress, (b) negative (compressive) stress

Equilibrium, (3), will be considered in the individual examples below.

Note that, in the previous Chapter, problems were solved using only the stress-strain law (1). Kinematics (2) and equilibrium (3) were not considered, the reason being the problems were so simple, with uniform (homogeneous) stress and strain (as indeed also in the first example which follows). Whenever more complex problems are encountered, with non-uniform stress and strains, (3) and perhaps (2) need to be considered to solve for the stress and strain.

7.1.2 Structures with Uniform Members

A uniform axial member is one with cross-section *A* and modulus *E* constant along its length, and loaded with axial forces at its ends only.

Example

Consider the bar of initial length L shown in Fig. 7.1.4, subjected to equal and opposite end-forces F. The free-body (equilibrium) diagram of a section of the bar shown in Fig. 7.1.4b shows that the internal force is also F everywhere along the bar. The stress is thus everywhere $\sigma = F / A$ and the strain is everywhere

$$\varepsilon = \frac{F}{EA} \tag{7.1.4}$$

and, from Eq. 7.1.2, the bar extends in length by an amount

$$\Delta = \frac{FL}{EA} \tag{7.1.5}$$

Note that, although the force acting on the left-hand end is negative (acting in the $-x$ direction), the stress there is positive (see Fig. 7.1.3).

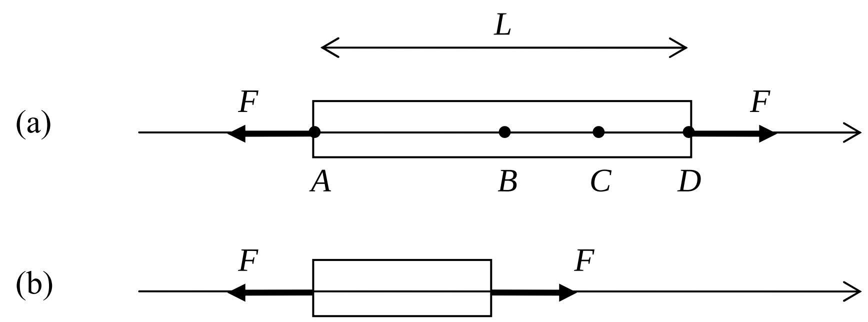


Figure 7.1.4: A uniform axial member; (a) subjected to axial forces *F*, (b) free-body diagram

Displacements need to be calculated relative to some datum displacement[1]. For example, suppose that the displacement at the centre of the bar is zero, $u(B) = 0$, Fig. 7.1.4. Then, from Eqn. 7.1.3,

$$\begin{aligned} u(C) &= u(B) + \varepsilon(C - B) = \frac{F}{EA}\frac{L}{4} \\ u(D) &= u(B) + \varepsilon(D - B) = \frac{F}{EA}\frac{L}{2} \\ u(A) &= u(B) + \varepsilon(A - B) = -\frac{F}{EA}\frac{L}{2} \end{aligned} \tag{7.1.6}$$

■

[1] which is another way of saying that one can translate the bar left or right as a rigid body without affecting the stress or strain – but it does affect the displacements

Example

Consider the two-element structure shown in Fig. 7.1.5. The first element is built-in to a wall at end A, is of length L_1, cross-sectional area A_1 and Young's modulus E_1. The second element is attached at B and has properties L_2, A_2, E_2. External loads F and P are applied at B and C as shown. An unknown reaction force R acts at the wall, at A. This can be determined from the force equilibrium equation for the complete structure:

$$R - F + P = 0 \tag{7.1.7}$$

Note that, as is usual, the reaction is assumed to act in the positive (x) direction. With R known, the stress $\sigma^{(1)}$ in the first element can be evaluated using the free-body diagram 7.1.5b, and $\sigma^{(2)}$ using Fig. 7.1.5c:

$$\sigma^{(1)} = \frac{P-F}{A_1}, \qquad \sigma^{(2)} = \frac{P}{A_2} \tag{7.1.8}$$

and so the strain is

$$\varepsilon^{(1)} = \frac{P-F}{E_1 A_1}, \qquad \varepsilon^{(2)} = \frac{P}{E_2 A_2} \tag{7.1.9}$$

Note that the stress and strain are *discontinuous* at B [2].

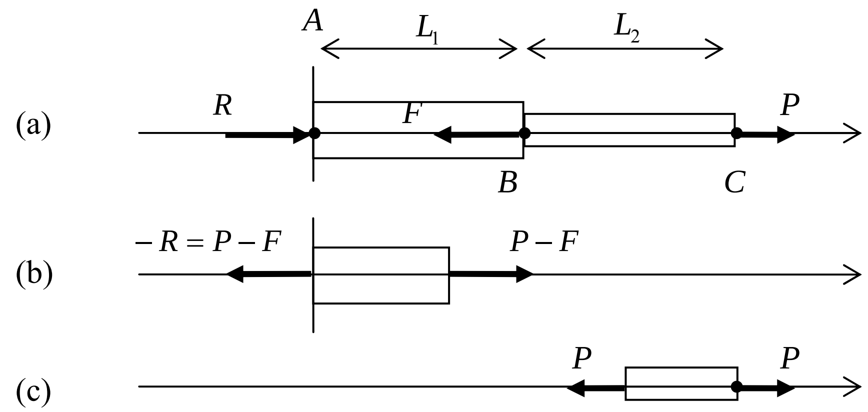


Figure 7.1.5: A two-element structure (a) subjected to axial forces *F* and *P*, (b,c) free-body diagrams

For each element, the total elongations Δ_i are

---

[2] this result, which can be viewed as a violation of equilibrium at B, is a result of the one-dimensional approximation of what is really a two-dimensional problem

$$\Delta_1 = u(B) - u(A) = \frac{(P-F)L_1}{E_1 A_1}$$
$$\Delta_2 = u(C) - u(B) = \frac{PL_2}{E_2 A_2} \tag{7.1.10}$$

If $P > F$, then $\Delta_1 > 0$ as expected, with $R < 0$ and $\sigma > 0$.

Thus far, the stress and strain (and elongations) have been obtained. If one wants to evaluate the displacements, then one needs to ensure that the strains in each of the two elements are **compatible**, that is, that the elements fit together after deformation just like they did before deformation. In this example, the displacements at *B* and *C* are

$$u(B) = u(A) + \Delta_1, \quad u(C) = u(B) + \Delta_2 \tag{7.1.11}$$

A **compatibility condition**, bringing together the separate relations in 7.1.11, is then

$$u(C) = u(A) + \frac{(P-F)L_1}{E_1 A_1} + \frac{PL_2}{E_2 A_2} \tag{7.1.12}$$

ensuring that $u(B)$ is unique. As in the previous example, the displacements can now be calculated if the displacement at any one (datum) point is known. Indeed, it is known that $u(A) = 0$.

■

Example

Consider next the similar situation shown in Fig. 7.1.6. Here, both ends of the two-element structure are built-in and there is only one applied force, *F*, at *B*. There are now two reaction forces, at ends *A* and *C*, but there is only one equilibrium equation to determine them:

$$R_A + F + R_C = 0 \tag{7.1.13}$$

Any structure for which there are more unknowns than equations of equilibrium, so that the stresses cannot be determined without considering the deformation of the structure, is called a **statically indeterminate** structure[3].

[3] See the end of §2.3.3

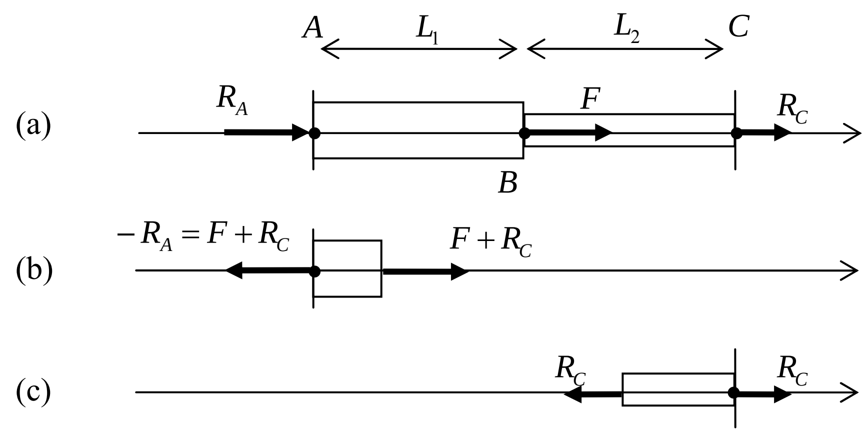


Figure 7.1.6: A two-element structure built-in and both ends; (a) subjected to an axial force *F*, (b,c) free-body diagrams

In terms of the unknown reactions, the strains are

$$\varepsilon^{(1)} = \frac{\sigma^{(1)}}{E_1} = -\frac{R_A}{E_1 A_1} = \frac{F + R_C}{E_1 A_1}, \qquad \varepsilon^{(2)} = \frac{\sigma^{(2)}}{E_2} = \frac{R_C}{E_2 A_2} \tag{7.1.14}$$

and, for each element, the total elongations are

$$\Delta_1 = \frac{R_A L_1}{E_1 A_1}, \qquad \Delta_2 = \frac{R_C L_2}{E_2 A_2} \tag{7.1.15}$$

Finally, compatibility of both elements implies that the total elongation $\Delta_1 + \Delta_2 = 0$. Using this relation with Eqn. 7.1.13-14 then gives

$$R_A = +F \frac{L_2 E_1 A_1}{L_1 E_2 A_2 + L_2 E_1 A_1}, \qquad R_C = -F \frac{L_1 E_2 A_2}{L_1 E_2 A_2 + L_2 E_1 A_1} \tag{7.1.16}$$

The displacements can now be evaluated, for example,

$$u(B) = +F \frac{1}{E_1 A_1 / L_1 + E_2 A_2 / L_2} \tag{7.1.17}$$

so that a positive *F* displaces *B* to the right and a negative *F* displaces *B* to the left.

■

Note the general solution procedure in this last example, known as the **basic force method**:

Equilibrium + Compatibility of Strain in terms of unknown Forces
→ Solve equations for unknown Forces

The Stiffness Method

The **stiffness method** (also known as the **displacement method**) is a slight modification of the above solution procedure, where the final equations to be solved involve known forces and unknown displacements only:

Equilibrium in terms of Displacement
→ Solve equations for unknown Displacements

If one deals in displacements, one does not need to ensure compatibility (it will automatically be satisfied); compatibility only needs to be considered when dealing in strains (as in the previous example)[4].

Example (The Stiffness Method)

Consider a series of three bars of cross-sectional areas A_1, A_2, A_3, Young's moduli E_1, E_2, E_3 and lengths L_1, L_2, L_3, Fig. 7.1.7. The first and third bars are built-in at points *A* and *D*, bars one and two meet at *B* and bars two and three meet at *C*. Forces P_B and P_C act at *B* and *C* respectively.

The force is constant in each bar, and for each bar there is a relation between the force F_i, and elongation, Δ_i, Eqn. 7.1.5:

$$F_i = k_i \Delta_i \quad \text{where} \quad k_i = \frac{A_i E_i}{L_i} \tag{7.1.18}$$

Here, k_i is the effective **stiffness** of each bar. The elongations are related to the displacements, $\Delta_1 = u_B - u_A$ etc., so that, with $u_A = u_D = 0$,

$$F_1 = k_1 u_B, \qquad F_2 = k_2 (u_C - u_B), \qquad F_3 = -k_3 u_C \tag{7.1.19}$$

There are two **degrees of freedom** in this problem, that is, two nodes are free to move. One therefore needs two equilibrium equations. One could use any two of

$$-F_1 + P_B + P_C + F_3 = 0, \qquad -F_1 + P_B + F_2 = 0, \qquad -F_2 + P_C + F_3 = 0 \tag{7.1.20}$$

In the stiffness method, one uses the second and third of these; the second is the "node *B*" equation and the third is the "node *C*" equation. Substituting Eqns. 7.1.19 into 7.1.20 leads to the system of two equations

$$\begin{aligned} -(k_1 + k_2)u_B + k_2 u_C &= -P_B \\ +k_2 u_B - (k_2 + k_3)u_C &= -P_C \end{aligned} \tag{7.1.21}$$

[4] the reason is: if you know the displacements, you know where every particle is and you know the strains and everything else; if you *only know the strains*, you know the *change in displacement*, but you do not know the actual displacements. You need some extra information to know the displacements – this is the compatibility equation

which can be solved for the two unknown nodal displacements.

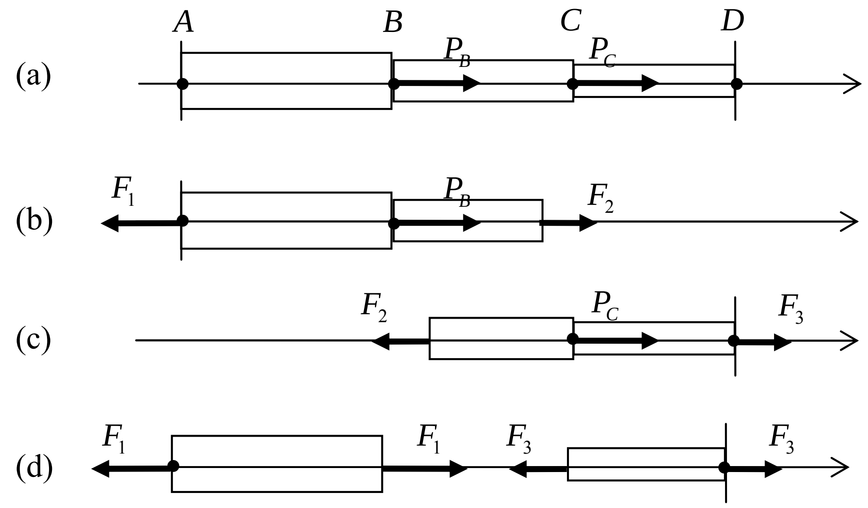


Figure 7.1.7: three bars in series; (a) subjected to external loads, (b,c,d) free-body diagrams

Equations 7.1.21 can also be written in the matrix form

$$\begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix} \begin{bmatrix} u_B \\ u_C \end{bmatrix} = \begin{bmatrix} -P_B \\ -P_C \end{bmatrix} \tag{7.1.22}$$

Note that it was not necessary to evaluate the reactions to obtain a solution. Once the forces have been found, the reactions can be found using the free-body diagram of Fig. 7.1.7d.

The stiffness method is a very systematic procedure. It can be used to solve for structures with many elements, with the two equations 7.1.21, 7.1.22, replaced by a large system of equations which can be solved numerically using a computer.

7.1.3 Structures with Non-uniform Members

Consider the structure shown in Fig. 7.1.8, an axial bar consisting of two separate components bonded together. The components have Young's moduli E_1, E_2 and cross-sectional areas A_1, A_2. The bar is subjected to equal and opposite forces F as shown, in such a way that axial deformations occur, that is, the cross-sections remain perpendicular to the x axis throughout the deformation.

Since there are only axial deformations, the strain is constant over a cross-section. However, the stress is not uniform, with $\sigma_1 = E_1\varepsilon$ and $\sigma_2 = E_2\varepsilon$; on any cross-section, the stress is higher in the stiffer component. The resultant force acting on each component is $F_1 = E_1A_1\varepsilon$ and $F_2 = E_2A_2\varepsilon$. Since $F_1 + F_2 = F$, the total elongation is

$$\Delta = \frac{FL}{E_1 A_1 + E_2 A_2} \tag{7.1.23}$$

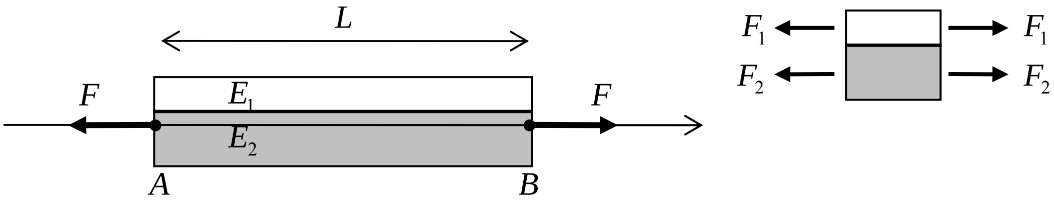


Figure 7.1.8: A bar consisting of two separate materials bonded together

7.1.4 Resultant Force and Moment

Consider the force and moments acting over any cross-section, Fig. 7.1.9. The resultant force is the integral of the stress times elemental area over the cross section, Eqn. 3.1.2,

$$F = \int_A \sigma dA \tag{7.1.24}$$

There are two moments; the moment M_y about the *y* axis and M_z about the *z* axis,

$$M_y = \int_A z\sigma dA, \quad M_z = -\int_A y\sigma dA \tag{7.1.25}$$

Positive moments are defined through the **right hand rule**, i.e. with the thumb of the right hand pointing in the positive *y* direction, the closing of the fingers indicates the positive M_y; the negative sign in Eqn. 7.1.25b is due to the fact that a positive stress with $y > 0$ would lead to a negative moment M_z.

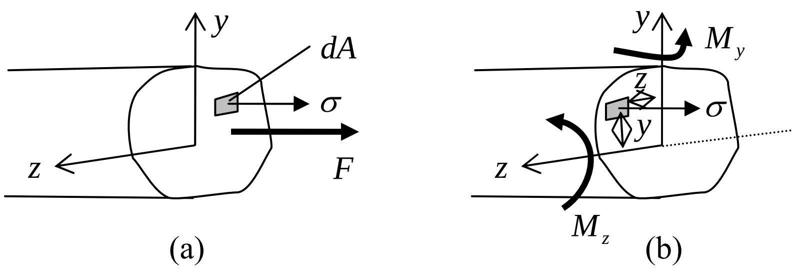


Figure 7.1.9: Resultants on a cross-section; (a) resultant force, (b) resultant moments

Consider now the case where *the stress is constant over a cross-section*. (Since it is assumed that the strain is constant over the cross-section, from Eqn. 7.1.1 this will occur when the Young's modulus is constant.) In that case, Eqns. 7.1.24-25 can be re-written as

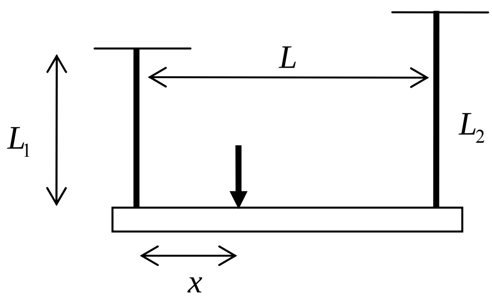
$$F = \sigma A, \quad M_y = \sigma \int_A z dA, \quad M_z = -\sigma \int_A y dA \tag{7.1.25}$$

The quantities $\int_A z dA$ and $\int_A y dA$ are the **first moments of area** about, respectively, the *y* and *z* axes. These are equal to $\bar{z}A$ and $\bar{y}A$, where $(\bar{y}, \bar{z})$ are the coordinates of the centroid of the section (see Eqn. 3.2.2). Taking the *x* axis to run through the centroid, $\bar{y} = \bar{z} = 0$, which results in $M_y = M_z = 0$. Thus, a resultant axial force which acts through the centroid of the cross-section ensures that there is no moment/rotation of that cross-section, the main assumption of this section.

For the non-uniform member of Fig. 7.1.8, since the resultant of a constant stress over an area is a force acting through the centroid of that area, the forces F_1, F_2 act through the centroids of the respective areas A_1, A_2. The precise location of the total resultant force *F* can be determined by taking the moments of the forces F_1, F_2 about the *y* and *z* axes, and equating this to the moment of the force *F* about these axes.

7.1.5 Problems

1. Consider the *rigid* beam supported by two deformable bars shown below. The bars have properties L_1, A_1 and L_2, A_2 and have the same Young's modulus *E*. They are separated by a distance *L*. The beam supports an arbitrary load at position *x*, as shown. What is *x* if the beam is to remain horizontal after deformation.



7.2 Torsion

In this section, the geometry to be considered is that of a long slender circular bar and the load is one which twists the bar. Such problems are important in the analysis of twisting components, for example lug wrenches and transmission shafts.

7.2.1 Basic relations for Torsion of Circular Members

The theory of torsion presented here concerns **torques**[1] which twist the members but which *do not induce any warping*, that is, cross sections which are perpendicular to the axis of the member remain so after twisting. Further, radial lines remain straight and radial as the cross-section rotates – they merely rotate with the section.

For example, consider the member shown in Fig. 7.2.1, built-in at one end and subject to a torque T at the other. The x axis is drawn along its axis. The torque shown is positive, following the right-hand rule (see §7.1.4). The member twists under the action of the torque and the radial plane $ABCD$ moves to $ABC'D$.

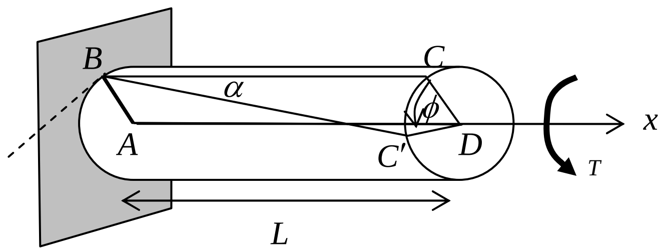


Figure 7.2.1: A cylindrical member under the action of a torque

Whereas in the last section the measure of deformation was elongation of the axial members, here an appropriate measure is the amount by which the member twists, the rotation angle ϕ. The rotation angle will vary along the member – the sign convention is that ϕ is positive in the same direction as positive T as indicated by the arrow in Fig. 7.2.1. Further, whereas the measure of strain used in the previous section was the normal strain ε_{xx}, here it will be the engineering shear strain γ_{xy} (twice the tensorial shear strain ε_{xy}). A relationship between γ (dropping the subscripts) and ϕ will next be established.

As the line BC deforms into BC', Fig. 7.2.1, it undergoes an angle change α. As defined in §4.1.2, the shear strain γ is the change in the original right angle formed by BC and a tangent at B (indicated by the dotted line – this is the y axis to be used in γ_{xy}). If α is small, then

$$\gamma \approx \alpha \approx \tan\alpha \approx \frac{CC'}{BC} \approx \frac{R\varphi(L)}{L} \tag{7.2.1}$$

[1] the term torque is usually used instead of moment in the context of twisting shafts such as those considered in this section

where L is the length, R the radius of the member and $\phi(L)$ means the magnitude of ϕ at L. Note that the strain is constant along the length of the member although ϕ is not. Considering a general cross-section within the member, as in Fig. 7.2.2, one has

$$\gamma \approx \alpha \approx \frac{R\varphi(x)}{x} \tag{7.2.2}$$

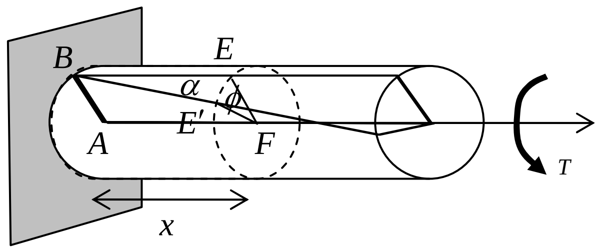


Figure 7.2.2: A section of a twisting cylindrical member

The shear strain at an arbitrary radial location r, $0 < r < R$, is

$$\gamma(r) = \frac{r\phi(x)}{x} \tag{7.2.3}$$

showing that the shear strain varies from zero at the centre of the shaft to a maximum $R\phi(L)/L$ $\left(= R\phi(x)/x\right)$ on the outer surface of the shaft.

The only strain is this shear strain and so the only stress which will arise is a shear stress τ. From Hooke's Law

$$\tau = G\gamma \tag{7.2.4}$$

where G is the shear modulus (the μ of Eqn. 6.1.5). Following the shear strain, the shear stress is zero at the centre of the shaft and a maximum on the outer surface.

Considering a free-body diagram of any portion of the shaft of Fig. 7.2.1, a torque T acts on all cross-sections. This torque must equal the resultant of the shear stresses acting over the section, as schematically illustrated in Fig. 7.2.3a.

The elemental force acting over an element of area dA is τdA and so the resultant moment about $r = 0$ is

$$T = \int_{dA} r\tau(r)dr \tag{7.2.5}$$

But γ / r is a constant and so therefore also is τ / r (provided G is) and Eqn. 7.2.5 can be re-written as

$$T = \frac{\tau(r)}{r}\left[\int_A r^2 dA\right] = \frac{\tau(r)J}{r} \tag{7.2.6}$$

The quantity in square brackets is called the **polar moment of inertia** of the cross-section (also called the **polar second moment of area**) and is denoted by J:

$$\boxed{J = \int_A r^2 dA} \qquad \textbf{Polar Moment of Area} \tag{7.2.7}$$

where dA is an element of area and the integration is over the complete cross-section.

For the circular cross-section under consideration, the area element has sides dr and $rd\theta$, Fig. 7.2.3c, so

$$J = \int_0^{2\pi}\int_0^R r^3 dr d\theta = 2\pi\int_0^R r^3 dr = \frac{\pi R^4}{2} = \frac{\pi D^4}{32} \tag{7.2.8}$$

where D is the diameter.

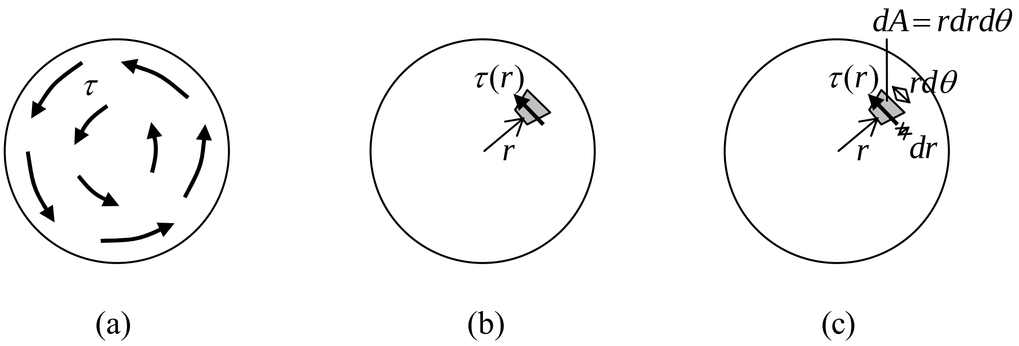


Figure 7.2.3: Shear stresses acting over a cross-section; (a) shear stress, (b,c) moment for an elemental area

From Eqn. 7.2.6, the shear stress at any radial location is given by

$$\boxed{\tau(r) = \frac{rT}{J}} \tag{7.2.9}$$

From Eqn. 7.2.1, 7.2.4, 7.26 and 7.2.9, the angle of twist at the end of the member – or the twist at one end relative to that at the other end – is

$$\boxed{\phi = \frac{TL}{GJ}} \tag{7.2.10}$$

Example

Consider the problem shown in Fig.7.2.4, two torsion members of lengths L_1, L_2, diameters d_1, d_2 and shear moduli G_1, G_2, built-in at A and subjected to torques T_B and T_C. Equilibrium of moments can be used to determine the unknown torques acting in each member:

$$-T_1 + T_B + T_C = 0, \qquad -T_2 + T_C = 0 \tag{7.2.11}$$

so that $T_1 = T_B + T_C$ and $T_2 = T_C$.

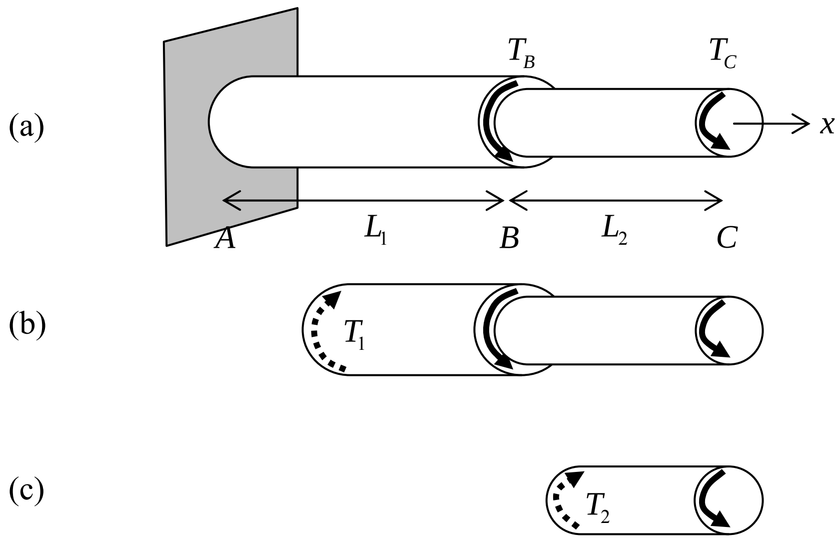


Figure 7.2.4: A structure consisting of two torsion members; (a) subjected to torques T_B and T_C, (b,c) free-body diagrams

The shear stresses in each member are therefore

$$\tau_1 = \frac{r(T_B + T_C)}{J_1}, \qquad \tau_2 = \frac{rT_C}{J_2} \tag{7.2.12}$$

where $J_1 = \pi d_1^4 / 32$ and $J_2 = \pi d_2^4 / 32$.

From Eqn. 7.2.10, the angle of twist at B is given by $\phi_B = T_1 L_1 / G_1 J_1$. The angle of twist at C is then

$$\phi_C = \frac{T_2 L_2}{G_2 J_2} - \phi_B \tag{7.2.13}$$

■

Statically indeterminate problems can be solved using methods analogous to those used in the section 7.1 for uniaxial members.

Example

Consider the structure in Fig. 7.2.5, similar to that in Fig. 7.2.4 only now both ends are built-in and there is only a single applied torque, T_B.

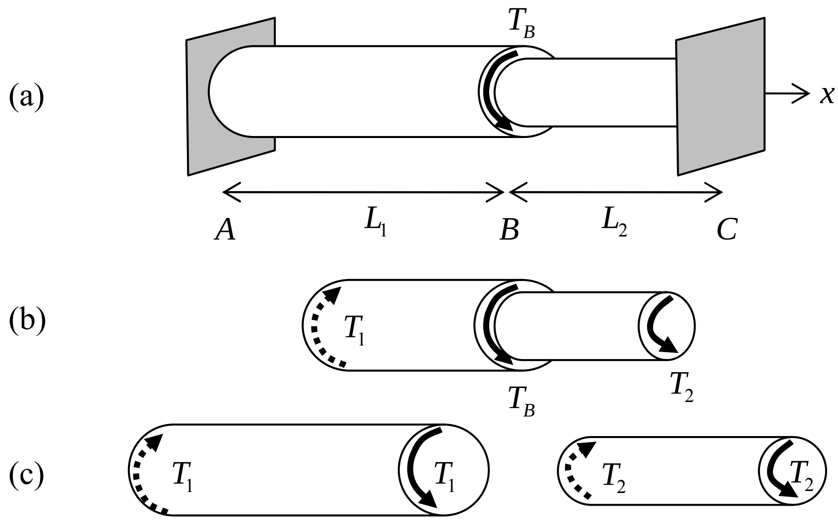


Figure 7.2.5: A structure consisting of two torsion members; (a) subjected to a Torque T_B, (b) free-body diagram, (c) separate elements

Referring to the free-body diagram of Fig. 7.2.5b, there is only one equation of equilibrium with which to determine the two unknown member torques:

$$-T_1 + T_B + T_2 = 0 \tag{7.2.14}$$

and so the deformation of the structure needs to be considered. A systematic way of dealing with this situation is to consider each element separately, as in Fig. 7.2.5c. The twist in each element is

$$\phi_1 = \frac{T_1 L_1}{G_1 J_1}, \quad \phi_2 = \frac{T_2 L_2}{G_2 J_2} \tag{7.2.15}$$

The total twist is zero and so $\phi_1 + \phi_2 = 0$ which, with Eqn. 7.2.14, can be solved to obtain

$$T_1 = +\frac{L_2 G_1 J_1}{L_1 G_2 J_2 + L_2 G_1 J_1} T_B, \quad T_2 = -\frac{L_1 G_2 J_2}{L_1 G_2 J_2 + L_2 G_1 J_1} T_B \tag{7.2.16}$$

The rotation at B can now be determined, $\phi_B = \phi_1 = -\phi_2$.

■

7.2.2 Stress Distribution in Torsion Members

The shear stress in Eqn. 7.2.9 is acting over a cross-section of a torsion member. From the symmetry of the stress, it follows that shear stresses act also along the length of the member, as illustrated to the left of Fig. 7.2.6. Shear stresses do not act *on* the surface of the element shown, as it is a free surface.

Any element of material not aligned with the axis of the cylinder will undergo a complex stress state, as shown to the right of Fig. 7.2.6. The stresses acting on an element are given by the stress transformation equations, Eqns. 3.4.9:

$$\sigma'_{xx} = +\sin 2\theta\tau, \qquad \sigma'_{yy} = -\sin 2\theta\tau, \qquad \sigma'_{xy} = +\cos 2\theta\tau \tag{7.2.17}$$

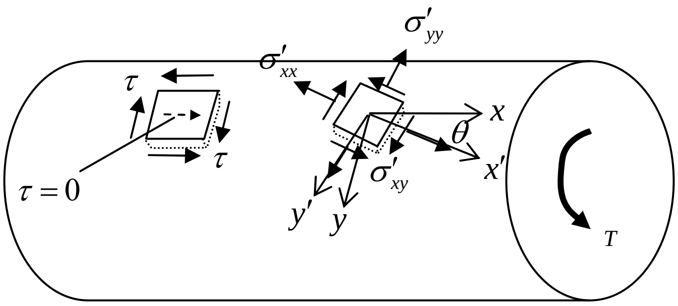
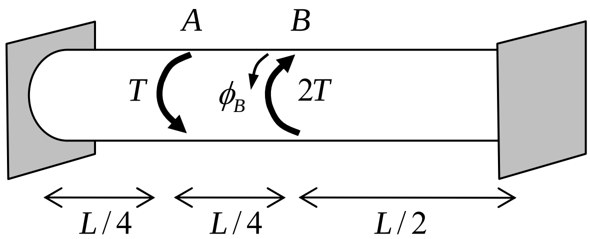


Figure 7.2.6: Stress distribution in a torsion member

From Eqns. 3.5.4-5, the maximum normal (principal) stresses arise on planes at $\theta = \pm 45^o$ and are $\sigma_1 = +\tau$ and $\sigma_2 = -\tau$. Thus the maximum tensile stress in the member occurs at 45^o to the axis and arises at the surface. The maximum shear stress is simply τ, with $\theta = 0$.

7.2.3 Problems

1. A shaft of length *L* and built-in at both ends is subjected to two external torques, T at *A* and $2T$ at *B*, as shown below. The shaft is of diameter *d* and shear modulus *G*. Determine the maximum (absolute value of) shear stress in the shaft and determine the angle of twist at *B*.



7.3 The Thin-walled Pressure Vessel Theory

An important practical problem is that of a cylindrical or spherical object which is subjected to an internal pressure p. Such a component is called a **pressure vessel**, Fig. 7.3.1. Applications arise in many areas, for example, the study of cellular organisms, arteries, aerosol cans, scuba-diving tanks and right up to large-scale industrial containers of liquids and gases.

In many applications it is valid to assume that

(i) the material is isotropic
(ii) the strains resulting from the pressures are small
(iii) the wall thickness t of the pressure vessel is much smaller than some characteristic radius: $t = r_o - r_i << r_o, r_i$

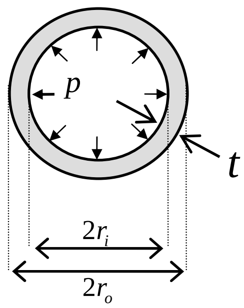


Figure 7.3.1: A pressure vessel (cross-sectional view)

Because of (i,ii), the isotropic linear elastic model is used. Because of (iii), it will be assumed that there is negligible variation in the stress field across the thickness of the vessel, Fig. 7.3.2.

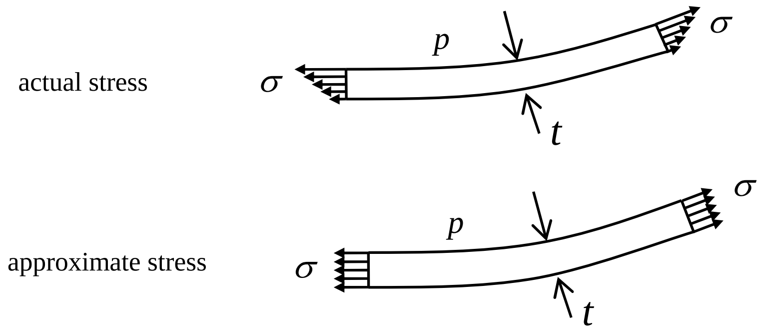


Figure 7.3.2: Approximation to the stress arising in a pressure vessel

As a rule of thumb, if the thickness is less than a tenth of the vessel radius, then the actual stress will vary by less than about 5% through the thickness, and in these cases the constant stress assumption is valid.

Note that a pressure $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p_i$ means that the stress on *any* plane drawn inside the vessel is subjected to a normal stress $-p_i$ and zero shear stress (see problem 6 in section 3.5.7).

7.3.1 Thin Walled Spheres

A thin-walled spherical shell is shown in Fig. 7.3.3. Because of the symmetry of the sphere and of the pressure loading, the **circumferential** (or **tangential** or **hoop) stress** σ_t at any location and in any tangential orientation must be the same (and there will be zero shear stresses).

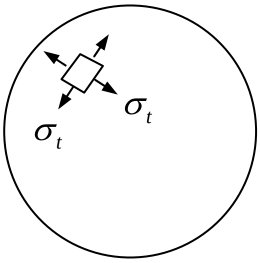


Figure 7.3.3: a thin-walled spherical pressure vessel

Considering a free-body diagram of one half of the sphere, Fig. 7.3.4, force equilibrium requires that

$$\pi\left(r_o^2 - r_i^2\right)\sigma_t - \pi r_i^2 p = 0 \qquad (7.3.1)$$

and so, with $r_0 = r_i + t$,

$$\sigma_t = \frac{r_i^2 p}{2r_i t + t^2} \qquad (7.3.2)$$

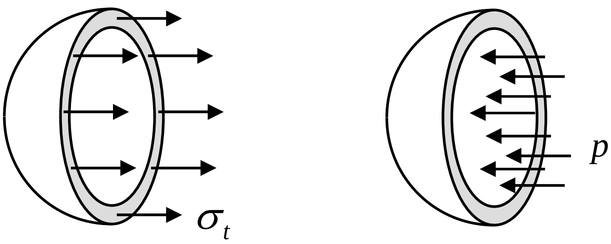


Figure 7.3.4: a free body diagram of one half of the spherical pressure vessel

One can now take as a characteristic radius the dimension *r*. This could be the inner radius, the outer radius, or the average of the two – results for all three should be close. Setting $r = r_i$ and neglecting the small terms $t^2 \ll 2r_i t$,

$$\boxed{\sigma_t = \frac{pr}{2t}} \quad \textbf{Tangential stress in a thin-walled spherical pressure vessel} \tag{7.3.3}$$

This tangential stress accounts for the stress in the plane of the surface of the sphere. The stress normal to the walls of the sphere is called the **radial stress**, σ_r. The radial stress is zero on the outer wall since that is a free surface. On the inner wall, the normal stress is $\sigma_r = -p$, Fig. 7.3.5. From Eqn. 7.3.3, since $t/r << 1$, $p << \sigma_t$, and it is reasonable to take $\sigma_r = 0$ not only on the outer wall, but on the inner wall also. The stress state in the spherical wall is then one of plane stress.

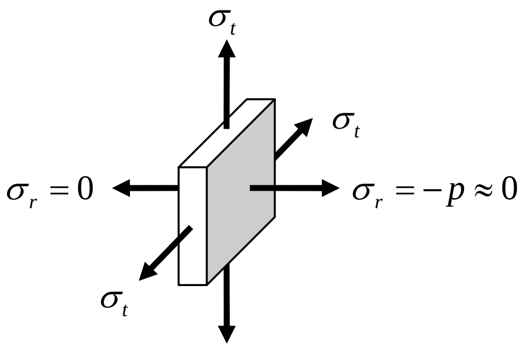


Figure 7.3.5: An element at the surface of a spherical pressure vessel

There are no in-plane shear stresses in the spherical pressure vessel and so the tangential and radial stresses are the principal stresses: $\sigma_1 = \sigma_2 = \sigma_t$, and the minimum principal stress is $\sigma_3 = \sigma_r = 0$. Thus the radial direction is one principal direction, and any two perpendicular directions in the plane of the sphere's wall can be taken as the other two principal directions.

Strain in the Thin-walled Sphere

The thin-walled pressure vessel expands when it is internally pressurised. This results in three principal strains, the **circumferential strain** ε_c (or **tangential strain** ε_t) in two perpendicular in-plane directions, and the **radial strain** ε_r. Referring to Fig. 7.3.6, these strains are

$$\varepsilon_c = \frac{A'C' - AC}{AC} = \frac{C'D' - CD}{CD}, \quad \varepsilon_r = \frac{A'B' - AB}{AB} \tag{7.3.4}$$

From Hooke's law (Eqns. 6.1.8 with *z* the radial direction, with $\sigma_r = 0$),

$$\begin{bmatrix} \varepsilon_c \\ \varepsilon_c \\ \varepsilon_r \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E \\ -\nu/E & 1/E & -\nu/E \\ -\nu/E & -\nu/E & 1/E \end{bmatrix} \begin{bmatrix} \sigma_t \\ \sigma_t \\ \sigma_r \end{bmatrix} = \frac{1}{E}\frac{pr}{2t}\begin{bmatrix} 1-\nu \\ 1-\nu \\ -2\nu \end{bmatrix} \tag{7.3.5}$$

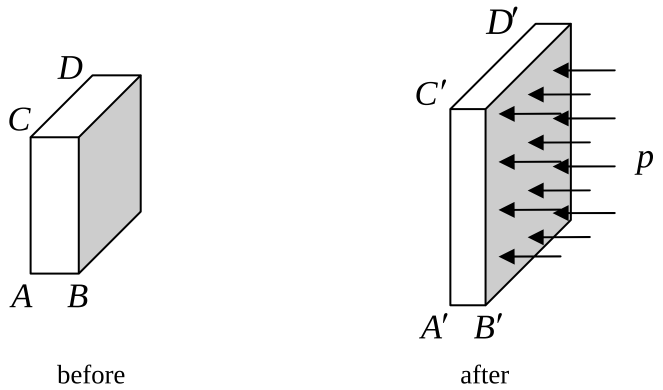


Figure 7.3.6: Strain of an element at the surface of a spherical pressure vessel

To determine the amount by which the vessel expands, consider a circumference at average radius *r* which moves out with a displacement δ_r, Fig. 7.3.7. From the definition of normal strain

$$\varepsilon_c = \frac{(r+\delta_r)\Delta\theta - r\Delta\theta}{r\Delta\theta} = \frac{\delta_r}{r} \tag{7.3.6}$$

This is the circumferential strain for points on the mid-radius. The strain at other points in the vessel can be approximated by this value.

The expansion of the sphere is thus

$$\delta_r = r\varepsilon_c = \frac{1-\nu}{E}\frac{pr^2}{2t} \tag{7.3.7}$$

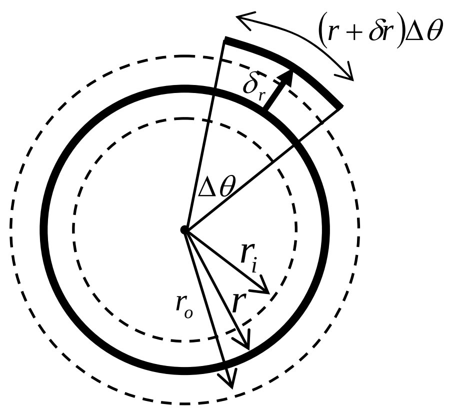


Figure 7.3.7: Deformation in the thin-walled sphere as it expands

To determine the amount by which the circumference increases in size, consider Fig. 7.3.8, which shows the original circumference at radius *r* of length *c* increase in size by an amount δ_c. One has

$$\delta_c = c\varepsilon_c = 2\pi r\varepsilon_c = 2\pi \frac{1-\nu}{E}\frac{pr^2}{2t} \tag{7.3.8}$$

It follows from Eqn. 7.3.7-8 that the circumference and radius increases are related through

$$\delta_c = 2\pi\delta_r \tag{7.3.9}$$

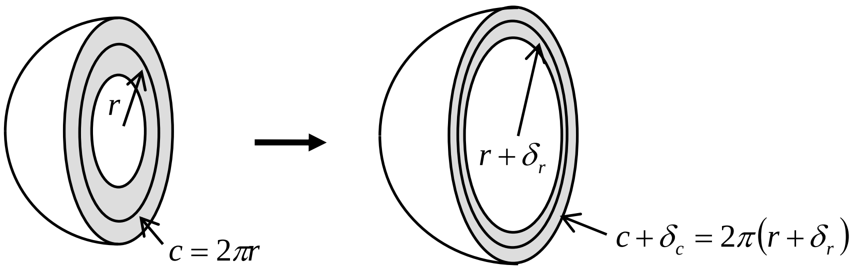


Figure 7.3.8: Increase in circumference length as the vessel expands

Note that the circumferential strain is *positive*, since the circumference is increasing in size, but the radial strain is *negative* since, as the vessel expands, the thickness decreases.

7.3.2 Thin Walled Cylinders

The analysis of a thin-walled internally-pressurised cylindrical vessel is similar to that of the spherical vessel. The main difference is that the cylinder has three different principal stress values, the circumferential stress, the radial stress, and the **longitudinal stress** σ_l, which acts in the direction of the cylinder axis, Fig. 7.3.9.

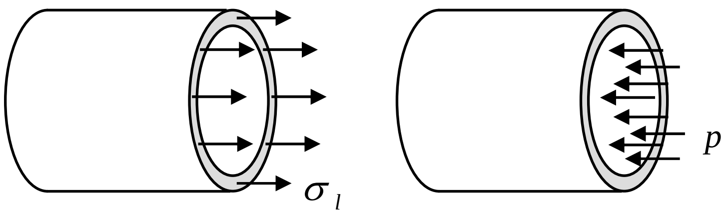


Figure 7.3.9: free body diagram of a cylindrical pressure vessel

Again taking a free-body diagram of the cylinder and carrying out an equilibrium analysis, one finds that, as for the spherical vessel,

$$\boxed{\sigma_l = \frac{pr}{2t}} \quad \textbf{Longitudinal stress in a thin-walled cylindrical pressure vessel} \tag{7.3.10}$$

Note that this analysis is only valid at positions sufficiently far away from the cylinder ends, where it might be closed in by caps – a more complex stress field would arise there.

The circumferential stress can be evaluated from an equilibrium analysis of the free body diagram in Fig. 7.3.10:

$$-\sigma_c 2tL + 2r_i Lp = 0 \tag{7.3.11}$$

and so

$$\boxed{\sigma_c = \frac{pr}{t}} \quad \textbf{Circumferential stress in a thin-walled cylindrical pressure vessel} \tag{7.3.12}$$

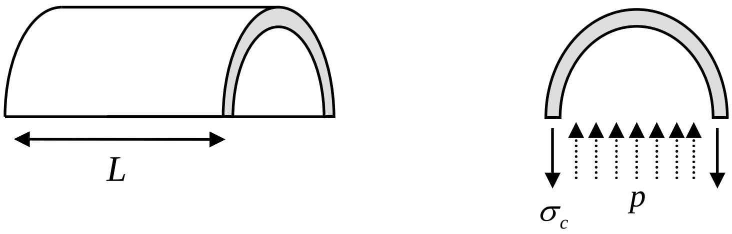


Figure 7.3.10: free body diagram of a cylindrical pressure vessel

As with the sphere, the radial stress varies from $-p$ at the inner surface to zero at the outer surface, but again is small compared with the other two stresses, and so is taken to be $\sigma_r = 0$.

Strain in the Thin-walled cylinder

The analysis of strain in the cylindrical pressure vessel is very similar to that of the spherical vessel. Eqns. 7.3.6 and 7.3.9 hold also here. Eqn. 7.3.5 would need to be amended to account for the three different principal stresses in the cylinder.

7.3.3 External Pressure

The analysis given above can be extended to the case where there is also an external pressure acting on the vessel. The internal pressure is now denoted by p_i and the external pressure is denoted by p_o, Fig. 7.3.11.

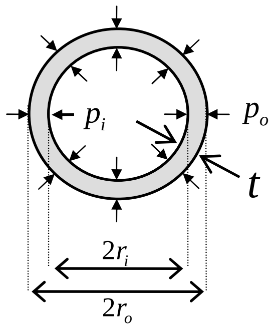


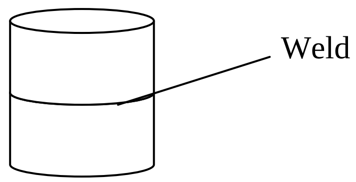
Figure 7.3.11: A pressure vessel subjected to internal and external pressure

In this case, the pressure p in formulae derived above can simply be replaced by $(p_i - p_o)$, which is known as the **gage pressure** (see the Appendix to this section, §7.3.5, for justification).

7.3.4 Problems

1. A 20m diameter spherical tank is to be used to store gas. The shell plating is 10 mm thick and the working stress of the material, that is, the maximum stress to which the material should be subjected, is 125 MPa. What is the maximum permissible gas pressure?

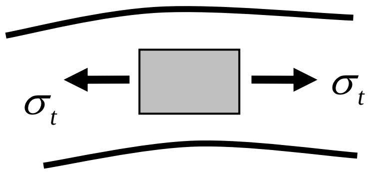
2. A steel propane tank for a BBQ grill has a 25cm diameter[1] and a wall thickness of 5mm (see figure). The tank is pressurised to 1.2 MPa.
   (a) determine the longitudinal and circumferential stresses in the cylindrical body of the tank
   (b) determine the absolute maximum shear stress in the cylindrical portion of the tank
   (c) determine the tensile force per cm length being supported by a weld joining the upper and lower sections of the tank.



3. What are the strains in the BBQ tank of question 2? What is the radial displacement? [take the steel to be isotropic with $E = 200\text{GPa}, \nu = 0.3$]

4. What are the strains in the cylindrical pressure vessel, in terms of E, ν, p, t and r?

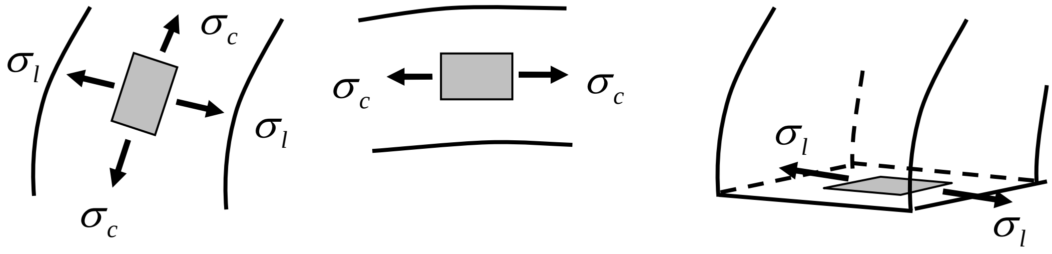
5. There are no shear stresses in the tangential plane of the spherical pressure vessel. However, there are shear stresses acting on planes through the thickness of the wall. A cross-section through the thickness is shown below. Take it that the radial stresses are zero. What are the maximum shear stresses occurring on this cross section?



6. The three perpendicular planes in the cylindrical pressure vessel are the in-plane, through the thickness and longitudinal sections, as shown below. The non-zero (principal) stresses acting on these planes are also shown. Evaluate the maximum

[1] this is an average diameter – the inside is 250-5mm and the outside is 250+5mm

shear stresses on each of these three planes. Which of these three maxima is the overall maximum shear stress acting in the vessel?

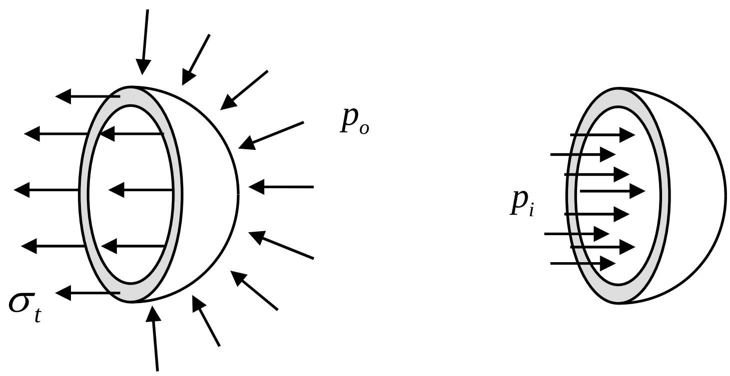


in-plane   through the thickness   longitudinal section

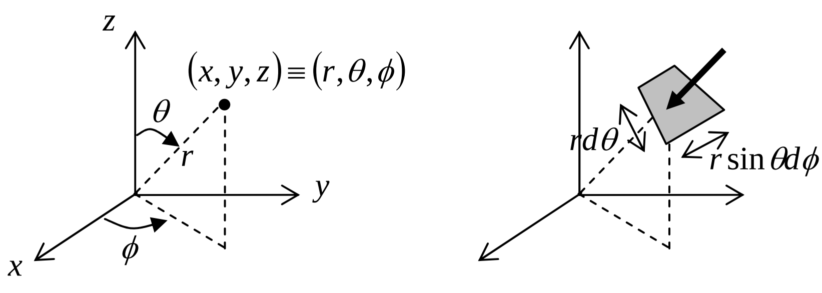
7.3.5 Appendix to §7.3

Equilibrium of a Pressure Vessel with both internal and external pressure

Consider the spherical pressure vessel. An external pressure p_o is distributed around its outer surface. Consider a free-body diagram of one half of the vessel, as shown below.



The force due to the external pressure acting in the horizontal direction can be evaluated using the spherical coordinates shown below.



An element of surface area upon which the pressure acts, swept out when the angles change by $d\theta$ and $d\phi$, has sides $r d\theta$ and $r \sin\theta d\phi$. The force acting on this area is then $p_o r^2 \sin\theta d\theta d\phi$. Force equilibrium in the horizontal (y) direction then leads to

$$-r_o^2 p_o \int_0^\pi \sin^2 \theta d\theta \int_0^\pi \sin \phi d\phi - \pi\left(r_o^2 - r_i^2\right)\sigma_t + \pi r_i^2 p_i = 0$$

and so,

$$\sigma_t = \frac{r_i^2 p_i - r_o^2 p_o}{\left(r_0 + r_i\right)t}$$

or $\sigma_t \approx (p_i - p_o)r/2t$ – see Eqn. 7.3.3.

7.4 The Elementary Beam Theory

In this section, problems involving long and slender beams are addressed. As with pressure vessels, the geometry of the beam, and the specific type of loading which will be considered, allows for approximations to be made to the full three-dimensional linear elastic stress-strain relations.

The beam theory is used in the design and analysis of a wide range of structures, from buildings to bridges to the load-bearing bones of the human body.

7.4.1 The Beam

The term **beam** has a very specific meaning in engineering mechanics: it is a component that is designed to support **transverse loads**, that is, loads that act perpendicular to the longitudinal axis of the beam, Fig. 7.4.1. The beam supports the load by *bending only*. Other mechanisms, for example twisting of the beam, are not allowed for in this theory.

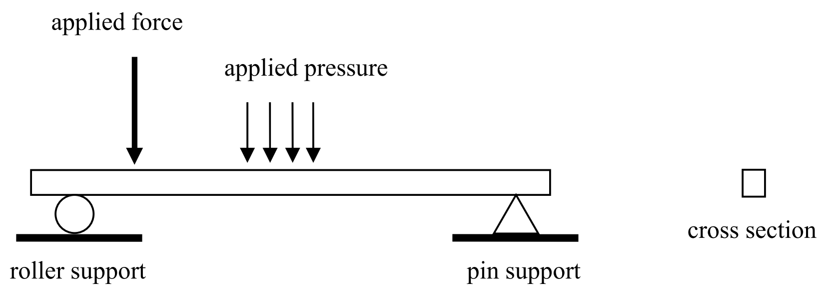


Figure 7.4.1: A supported beam loaded by a force and a distribution of pressure

It is convenient to show a two-dimensional cross-section of the three-dimensional beam together with the beam cross section, as in Fig. 7.4.1. The beam can be supported in various ways, for example by roller supports or pin supports (see section 2.3.3). The cross section of this beam happens to be rectangular but it can be any of many possible shapes.

It will assumed that the beam has a **longitudinal plane of symmetry**, with the cross section symmetric about this plane, as shown in Fig. 7.4.2. Further, it will be assumed that the loading and supports are also symmetric about this plane. With these conditions, the beam has no tendency to twist and will undergo bending only[1].

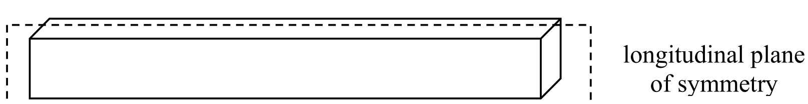


Figure 7.4.2: The longitudinal plane of symmetry of a beam

[1] certain very special cases, where there is *not* a plane of symmetry for geometry and/or loading, can lead also to bending with no twist, but these are not considered here

Imagine now that the beam consists of many fibres aligned longitudinally, as in Fig. 7.4.3. When the beam is bent by the action of downward transverse loads, the fibres near the top of the beam contract in length whereas the fibres near the bottom of the beam extend. Somewhere in between, there will be a plane where the fibres do not change length. This is called the **neutral surface**. The intersection of the longitudinal plane of symmetry and the neutral surface is called the **axis of the beam**, and the deformed axis is called the **deflection curve**.

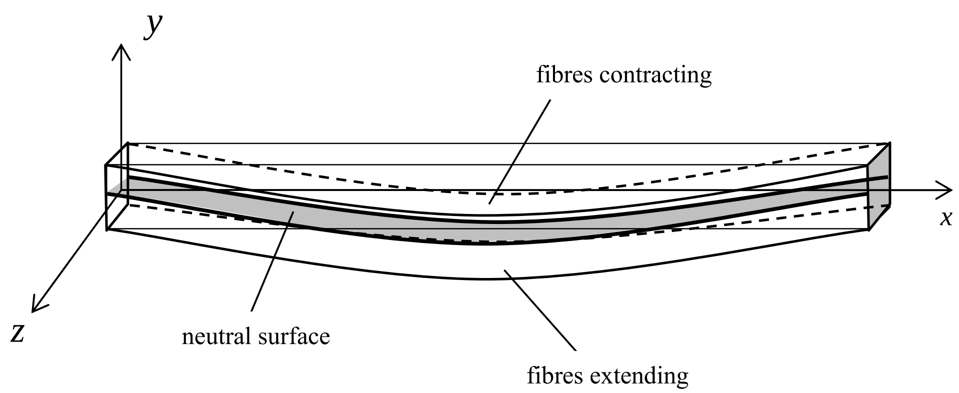


Figure 7.4.3: the neutral surface of a beam

A conventional coordinate system is attached to the beam in Fig. 7.4.3. The x axis coincides with the (longitudinal) axis of the beam, the y axis is in the transverse direction and the longitudinal plane of symmetry is in the $x-y$ plane, also called the **plane of bending**.

7.4.2 Moments and Forces in a Beam

Normal and shear stresses act over any cross section of a beam, as shown in Fig. 7.4.4. The normal and shear stresses acting on each side of the cross section are equal and opposite for equilibrium, Fig. 7.4.4b. The normal stresses σ will vary over a section during bending. Referring again to Fig. 7.4.3, over one part of the section the stress will be tensile, leading to extension of material fibres, whereas over the other part the stresses will be compressive, leading to contraction of material fibres. This distribution of normal stress results in a moment M acting on the section, as illustrated in Fig. 7.4.4c. Similarly, shear stresses τ act over a section and these result in a shear force V.

The beams of Fig. 7.4.3 and Fig. 7.4.4 show the normal stress and deflection one would expect when a beam bends downward. There are situations when parts of a beam bend upwards, and in these cases the signs of the normal stresses will be opposite to those shown in Fig. 7.4.4. However, the moments (and shear forces) shown in Fig. 7.4.4 will be regarded as *positive*. This sign convention to be used is shown in Fig. 7.4.5.

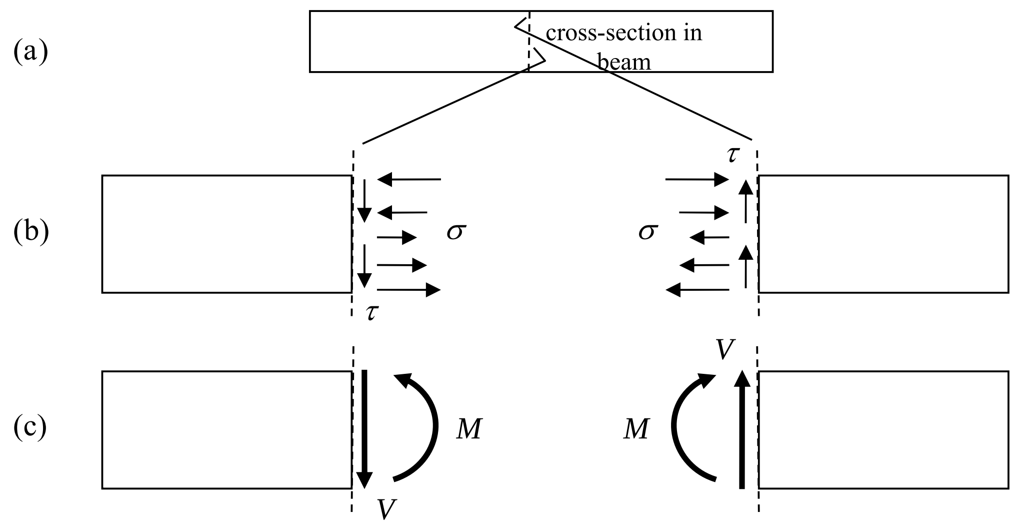


Figure 7.4.4: stresses and moments acting over a cross-section of a beam; (a) a cross-section, (b) normal and shear stresses acting over the cross-section, (c) the moment and shear force resultant of the normal and shear stresses

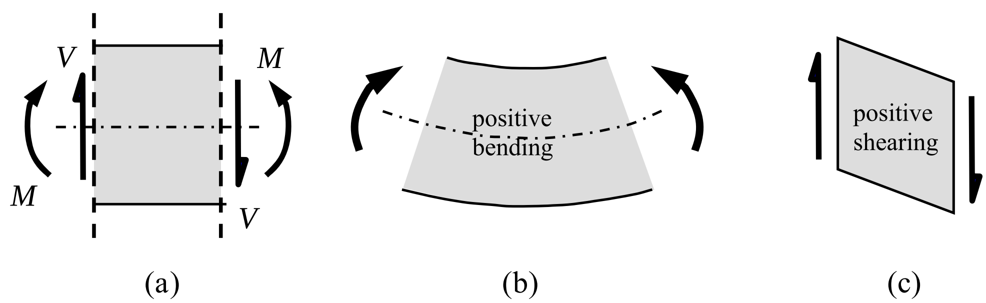


Figure 7.4.5: sign convention for moments and shear forces

Note that the sign convention for the shear stress conventionally used the beam theory conflicts with the sign convention for shear stress used in the rest of mechanics, introduced in Chapter 3. This is shown in Fig. 7.4.6.

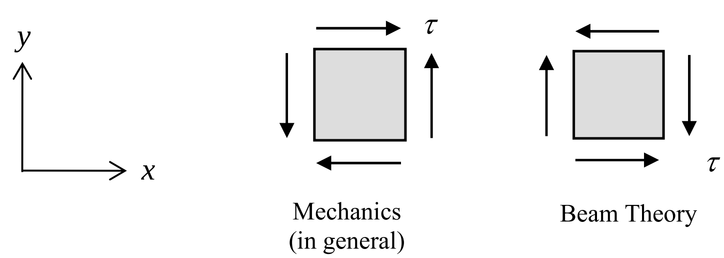


Figure 7.4.6: sign convention for shear stress in beam theory

The moments and forces acting within a beam can in many simple problems be evaluated from equilibrium considerations alone. Some examples are given next.

Example 1

Consider the **simply supported** beam in Fig. 7.4.7. From the loading, one would expect the beam to deflect something like as indicated by the deflection curve drawn. The reaction at the roller support, end *A*, and the vertical reaction at the pin support[2], end *B*, can be evaluated from the equations of equilibrium, Eqns. 2.3.3:

$$R_{Ay} = P/3, \quad R_{By} = 2P/3 \tag{7.4.1}$$

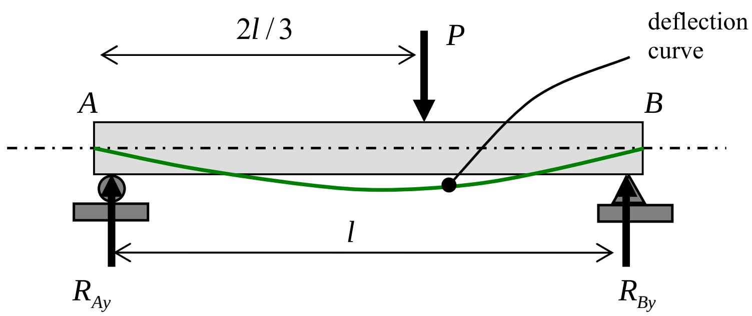


Figure 7.4.7: a simply supported beam

The moments and forces acting *within* the beam can be evaluated by taking free-body diagrams of sections of the beam. There are clearly two distinct regions in this beam, to the left and right of the load. Fig. 7.4.8a shows an arbitrary portion of beam representing the left-hand side. A coordinate system has been introduced, with *x* measured from *A*.[3] An unknown moment *M* and shear force *V* act at the end. A *positive* moment and force have been drawn in Fig. 7.4.8a. From the equilibrium equations, one finds that the shear force is constant but that the moment varies linearly along the beam:

$$V = \frac{P}{3}, \qquad M = \frac{P}{3}x \qquad \left(0 < x < \frac{2l}{3}\right) \tag{7.4.2}$$

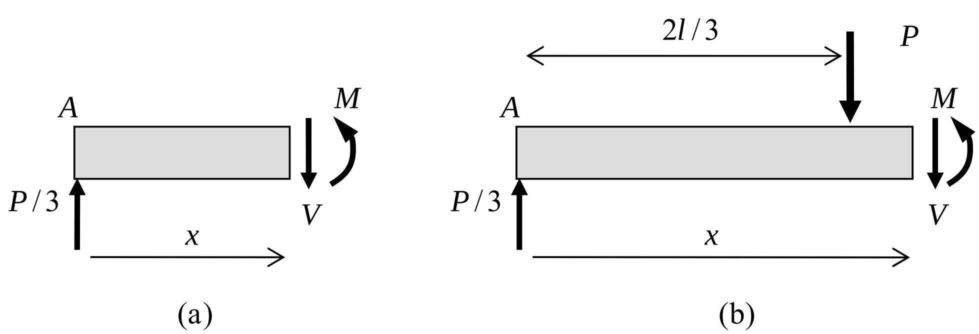


Figure 7.4.8: free body diagrams of sections of a beam

[2] the horizontal reaction at the pin is zero since there are no applied forces in this direction; the beam theory does not consider such types of (axial) load; further, one does not have a pin at each support, since this would prevent movement in the horizontal direction which in turn would give rise to forces in the horizontal direction – hence the pin at one end and the roller support at the other end
[3] the coordinate *x* can be measured from any point in the beam; in this example it is convenient to measure it from point *A*

Cutting the beam to the right of the load, Fig. 7.4.8b, leads to

$$V = -\frac{2P}{3}, \qquad M = \frac{2P}{3}(l - x) \qquad \left(\frac{2l}{3} < x < l\right) \tag{7.4.3}$$

The shear force is negative, so acts in the direction opposite to that initially assumed in Fig. 7.4.8b.

The results of the analysis can be displayed in what are known as a **shear force diagram** and a **bending moment diagram**, Fig. 7.4.9. Note that there is a "jump" in the shear force at $x = 2l/3$ equal to the applied force, and in this example the bending moment is everywhere positive.

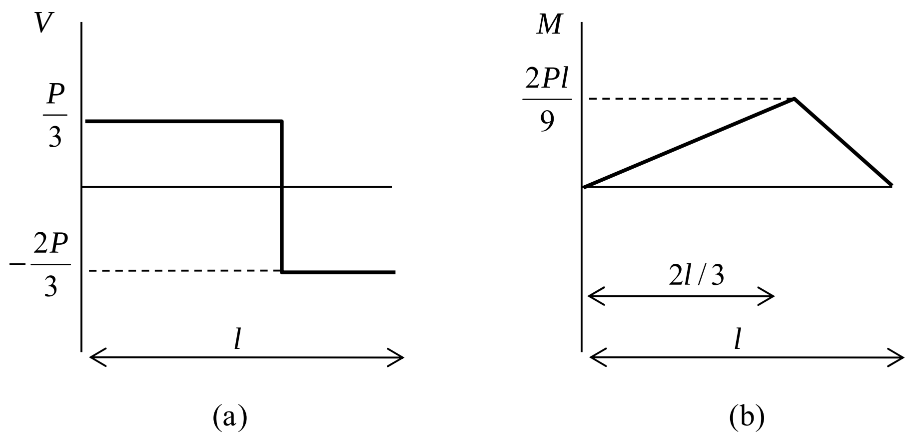


Figure 7.4.9: results of analysis; (a) shear force diagram, (b) bending moment diagram

■

Example 2

Fig. 7.4.10 shows a **cantilever**, that is, a beam supported by clamping one end (refer to Fig. 2.3.8). The cantilever is loaded by a force at its mid-point and a (negative) moment at its end.

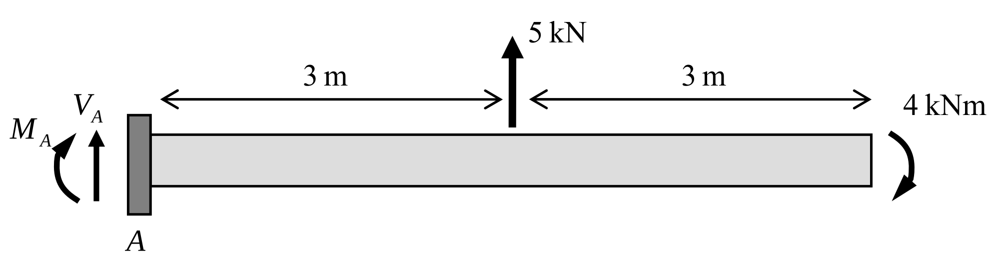


Figure 7.4.10: a cantilevered beam loaded by a force and moment

Again, positive unknown reactions M_A and V_A are considered at the support *A*. From the equilibrium equations, one finds that

$$M_A = 11\ \text{kNm}, \quad V_A = -5\ \text{kN} \tag{7.4.4}$$

As in the previous example, there are two distinct regions along the beam, to the left and to the right of the applied concentrated force. Again, a coordinate x is introduced and the beam is sectioned as in Fig. 7.4.11. The unknown moment M and shear force V can then be evaluated from the equilibrium equations:

$$\begin{array}{lll} V = -5\,\text{kN}, & M = 11 - 5x\ \text{kNm} & (0 < x < 3) \\ V = 0, & M = -4\ \text{kNm} & (3 < x < 6) \end{array} \tag{7.4.5}$$

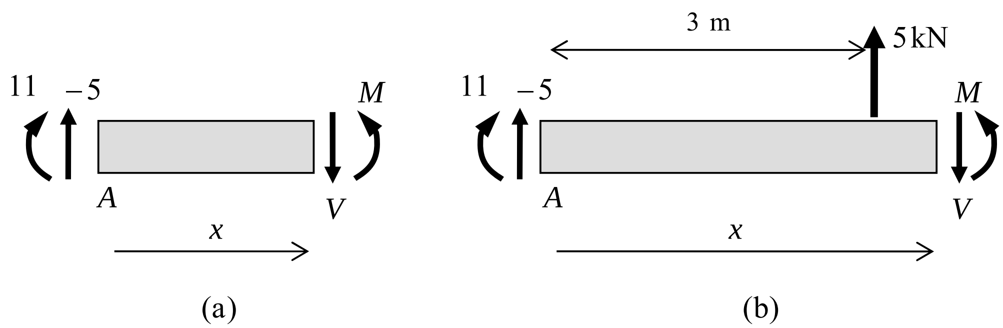


Figure 7.4.11: free body diagrams of sections of a beam

The results are summarized in the shear force and bending moment diagrams of Fig. 7.4.12.

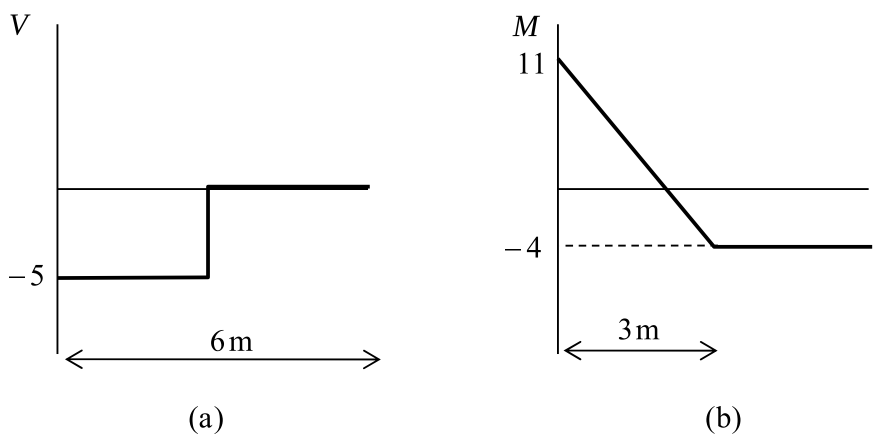


Figure 7.4.12: results of analysis; (a) shear force diagram, (b) bending moment diagram

In this example the beam experiences negative bending moment over most of its length.

■

Example 3

Fig. 7.4.13 shows a simply supported beam subjected to a distributed load (force per unit length). The load is uniformly distributed over half the length of the beam, with a triangular distribution over the remainder.

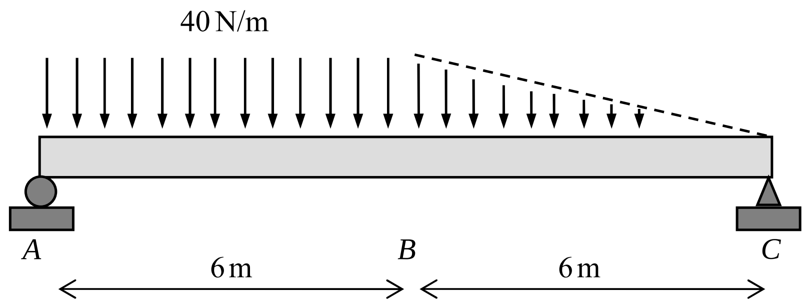


Figure 7.4.13: a beam subjected to a distributed load

The unknown reactions can be determined by replacing the distributed load with statically equivalent forces as in Fig. 7.4.14 (see §3.1.2). The equilibrium equations then give

$$R_A = 220 \text{ N}, \quad R_C = 140 \text{ N} \tag{7.4.6}$$

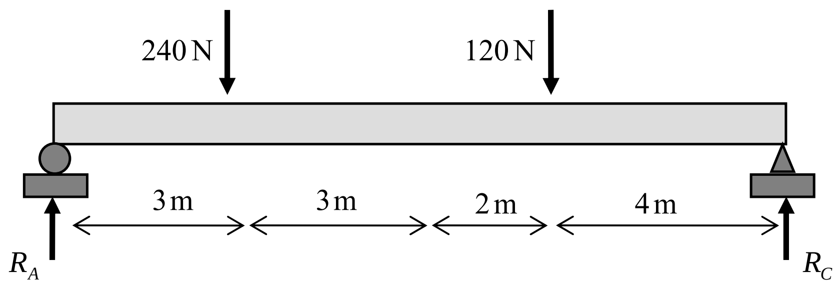


Figure 7.4.14: equivalent forces acting on the beam of Fig. 7.4.13

Referring again to Fig. 7.4.13, there are two distinct regions in the beam, that under the uniform load and that under the triangular distribution of load. The first case is considered in Fig. 7.4.15.

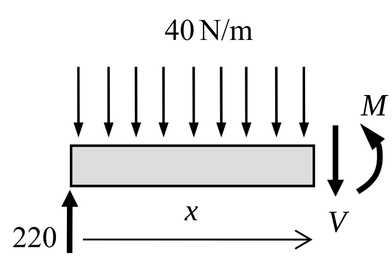


Figure 7.4.15: free body diagram of a section of a beam

The equilibrium equations give

$$V = 220 - 40x, \qquad M = 220x - 20x^2 \qquad (0 < x < 6) \tag{7.4.7}$$

The region beneath the triangular distribution is shown in Fig. 7.4.16. Two possible approaches are illustrated: in Fig. 7.4.16a, the free body diagram consists of the complete length of beam to the left of the cross-section under consideration; in Fig. 7.4.16b, only the portion to the right is considered, with distance measured from the right hand end, as $12-x$. The problem is easier to solve using the second option; from Fig. 7.4.16b then, with the equilibrium equations, one finds that

$$V=-140+10(12-x)^2/3, \quad M=140(12-x)-10(12-x)^3/9 \quad (6<x<12) \tag{7.4.8}$$

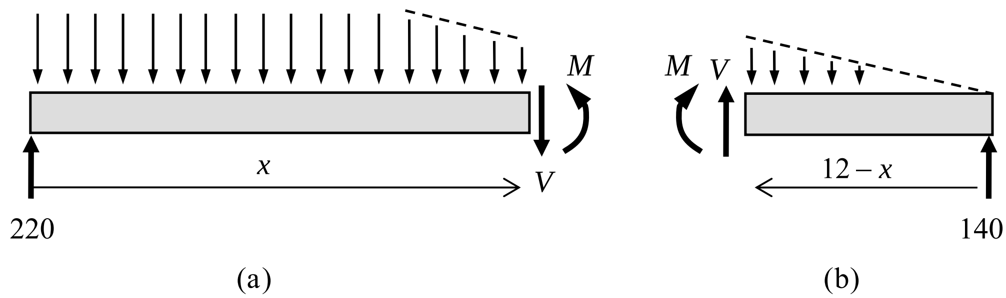


Figure 7.4.16: free body diagrams of sections of a beam

The results are summarized in the shear force and bending moment diagrams of Fig. 7.4.17.

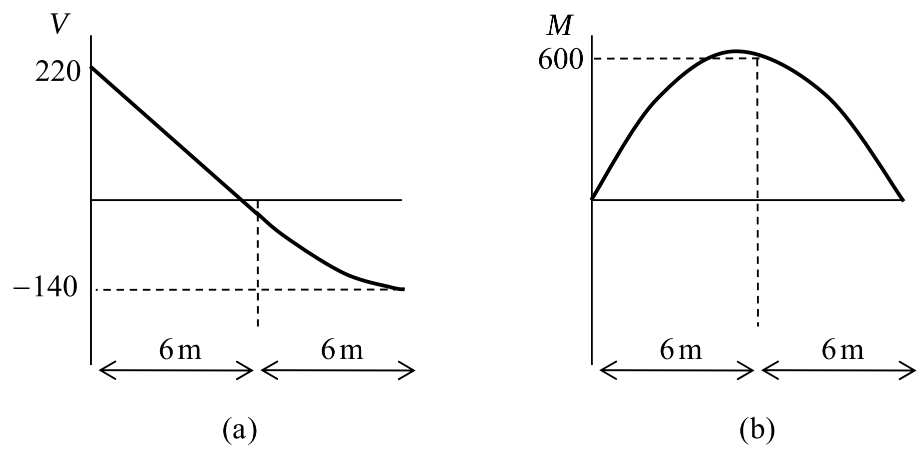


Figure 7.4.17: results of analysis; (a) shear force diagram, (b) bending moment diagram

■

7.4.3 The Relationship between Loads, Shear Forces and Bending Moments

Relationships between the applied loads and the internal shear force and bending moment in a beam can be established by considering a small beam element, of width Δx, and subjected to a distributed load $p(x)$ which varies along the section of beam, and which is *positive upward*, Fig. 7.4.18.

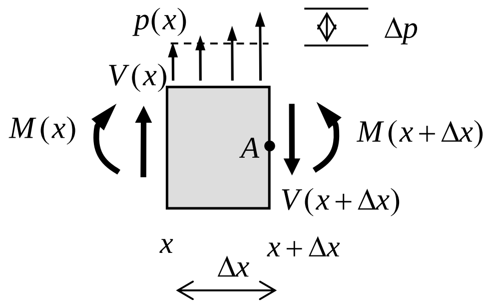


Figure 7.4.18: forces and moments acting on a small element of beam

At the left-hand end of the free body, at position *x*, the shear force, moment and distributed load have values $V(x)$, $M(x)$ and $p(x)$ respectively. On the right-hand end, at position $x+\Delta x$, their values are slightly different: $V(x+\Delta x)$, $M(x+\Delta x)$ and $p(x+\Delta x)$. Since the element is very small, the distributed load, even if it is varying, can be approximated by a *linear* variation over the element. The distributed load can therefore be considered to be a uniform distribution of intensity $p(x)$ over the length Δx together with a triangular distribution, 0 at x and Δp say, a *small* value, at $x+\Delta x$. Equilibrium of vertical forces then gives

$$\begin{aligned} & V(x)+p(x)\Delta x+\frac{1}{2}\Delta p\Delta x-V(x+\Delta x)=0 \\ \rightarrow\ & \frac{V(x+\Delta x)-V(x)}{\Delta x}=p(x)+\frac{1}{2}\Delta p \end{aligned} \tag{7.4.9}$$

Now let the size of the element decrease towards zero. The left-hand side of Eqn. 7.4.9 is then the definition of the derivative, and the second term on the right-hand side tends to zero, so

$$\boxed{\frac{dV}{dx}=p(x)} \tag{7.4.10}$$

This relation can be seen to hold in Eqn. 7.4.7 and Fig. 7.4.17a, where the shear force over $0<x<6$ has a slope of -40 and the pressure distribution is uniform, of intensity $-40\,\text{N/m}$. Similarly, over $6<x<12$, the pressure decreases linearly and so does the slope in the shear force diagram, reaching zero slope at the end of the beam.

It also follows from 7.4.10 that the change in shear along a beam is equal to the area under the distributed load curve:

$$V(x_2)-V(x_1)=\int_{x_1}^{x_2} p(x)dx \tag{7.4.11}$$

Consider now moment equilibrium, by taking moments about the point A in Fig. 7.4.18:

$$\begin{aligned}
&-M(x) - V(x)\Delta x + M(x+\Delta x) - p(x)\Delta x\frac{\Delta x}{2} - \frac{1}{2}\Delta p\Delta x\frac{\Delta x}{3} = 0 \\
&\rightarrow \frac{M(x+\Delta x) - M(x)}{\Delta x} = V(x) + p(x)\frac{\Delta x}{2} + \Delta p\frac{\Delta x}{6}
\end{aligned} \tag{7.4.12}$$

Again, as the size of the element decreases towards zero, the left-hand side becomes a derivative and the second and third terms on the right-hand side tend to zero, so that

$$\boxed{\frac{dM}{dx} = V(x)} \tag{7.4.13}$$

This relation can be seen to hold in Eqns. 7.4.2-3, 7.4.5 and 7.4.7-8. It also follows from Eqn. 7.4.13 that the change in moment along a beam is equal to the area under the shear force curve:

$$M(x_2) - M(x_1) = \int_{x_1}^{x_2} V(x)dx \tag{7.4.14}$$

7.4.4 Deformation and Flexural Stresses in Beams

The moment at any given cross-section of a beam is due to a distribution of normal stress, or **flexural stress** (or **bending stress**) across the section (see Fig. 7.4.4). As mentioned, the stresses to one side of the neutral axis are tensile whereas on the other side of the neutral axis they are compressive. To determine the distribution of normal stress over the section, one must determine the precise location of the neutral axis, and to do this one must consider the *deformation* of the beam.

Apart from the assumption of there being a longitudinal plane of symmetry and a neutral axis along which material fibres do not extend, the following two assumptions will be made concerning the deformation of a beam:

1. Cross-sections which are plane and are perpendicular to the axis of the undeformed beam remain plane and remain perpendicular to the deflection curve of the deformed beam. In short: "plane sections remain plane". This is illustrated in Fig. 7.4.19. It will be seen later that this assumption is a valid one provided the beam is sufficiently long and slender.

2. Deformation in the vertical direction, i.e. the transverse strain ε_{yy}, may be neglected in deriving an expression for the longitudinal strain ε_{xx}. This assumption is summarised in the deformation shown in Fig. 7.4.20, which shows an element of length l and height h undergoing transverse and longitudinal strain.

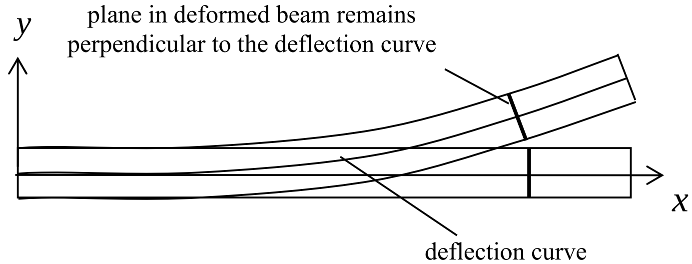


Figure 7.4.19: plane sections remain plane in the elementary beam theory

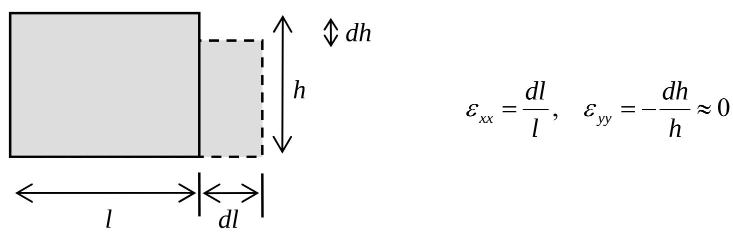


Figure 7.4.20: transverse strain is neglected in the elementary beam theory

With these assumptions, consider now the element of beam shown in Fig. 7.4.21. Here, two material fibres ab and pq, of length Δx in the undeformed beam, deform to $a'b'$ and $p'q'$. The deflection curve has a radius of curvature *R*. The above two assumptions imply that, referring to the figure:

$$\begin{aligned} &\angle p'a'b' = \angle a'b'q' = \pi/2 && \text{(assumption 1)} \\ &|ap| = |a'p'|, \quad |bq| = |b'q'| && \text{(assumption 2)} \end{aligned} \qquad (7.4.15)$$

Since the fibre ab is on the neutral axis, by definition $|a'b'| = |ab|$. However the fibre pq, a distance *y* from the neutral axis, extends in length from Δx to length $\Delta x'$. The longitudinal strain for this fibre is

$$\varepsilon_{xx} = \frac{\Delta x' - \Delta x}{\Delta x} = \frac{(R-y)\Delta\theta - R\Delta\theta}{R\Delta\theta} = -\frac{y}{R} \qquad (7.4.16)$$

As one would expect, this relation implies that a small *R* (large curvature) is related to a large strain and a large *R* (small curvature) is related to a small strain. Further, for $y > 0$ (above the neutral axis), the strain is negative, whereas if $y < 0$ (below the neutral axis), the strain is positive[4], and the variation across the cross-section is linear.

[4] this is under the assumption that *R* is positive, which means that the beam is concave up; a negative *R* implies that the centre of curvature is below the beam

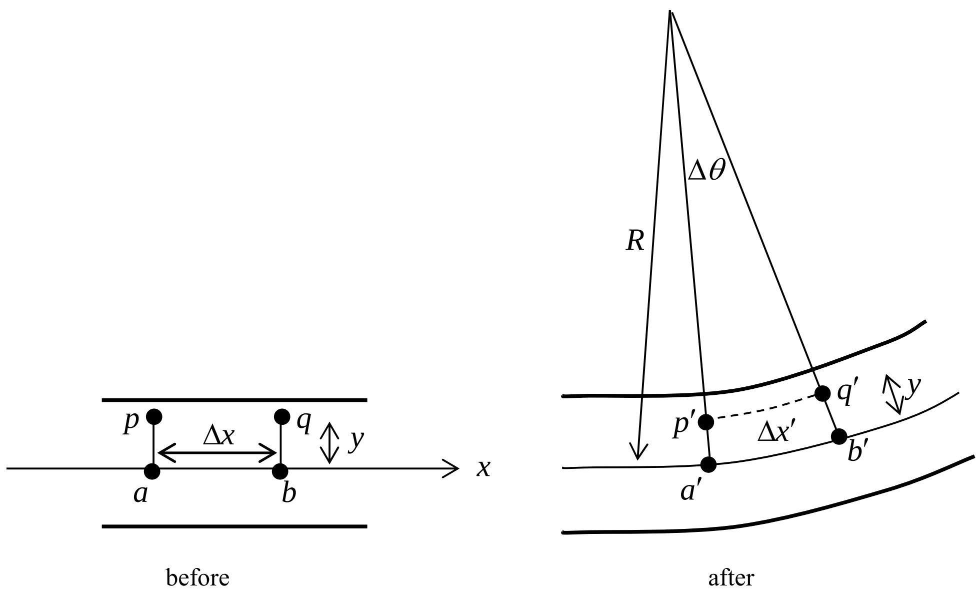


Figure 7.4.21: deformation of material fibres in an element of beam

To relate this deformation to the stresses arising in the beam, it is necessary to postulate the stress-strain law for the material out of which the beam is made. Here, it is assumed that the beam is isotropic linear elastic[5].

The beam is a three-dimensional object, and so will in general experience a fairly complex three-dimensional stress state. We will show in what follows that a simple one-dimensional approximation, $\sigma_{xx} = E\varepsilon_{xx}$, whilst disregarding all other stresses and strains, will be sufficiently accurate for our purposes.

Since there are no forces acting in the *z* direction, the beam is in a state of plane stress, and the stress-strain equations are (see Eqns. 6.1.10)

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] \\
\varepsilon_{zz} &= -\frac{\nu}{E}\left[\sigma_{xx} + \sigma_{yy}\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy}, \qquad \varepsilon_{xz} = \varepsilon_{yz} = 0
\end{aligned} \tag{7.4.17}$$

Yet another assumption is now made, that the transverse normal stresses, σ_{yy}, may be neglected in comparison with the flexural stresses σ_{xx}. This is similar to the above assumption #2 concerning the deformation, where the transverse normal strain was neglected in comparison with the longitudinal strain. It might seem strange at first that the transverse stress is neglected, since all loads are in the transverse direction. However,

[5] the beam theory can be extended to incorporate more complex material models (constitutive equations)

just as the tangential stresses are much larger than the radial stresses in the pressure vessel, it is found that the longitudinal stresses in a beam are very much greater than the transverse stresses. With this assumption, the first of Eqn. 7.4.17 reduces to a one-dimensional equation:

$$\varepsilon_{xx} = \sigma_{xx} / E \tag{7.4.18}$$

and, from Eqn. 7.4.16, dropping the subscripts on σ,

$$\sigma = -\frac{E}{R} y \tag{7.4.19}$$

Finally, the resultant force of the normal stress distribution over the cross-section must be zero, and the resultant moment of the distribution is *M*, leading to the conditions

$$\begin{aligned} 0 &= \int_A \sigma dA = -\frac{E}{R}\int_A y dA \\ M &= -\int_A \sigma y dA = \frac{E}{R}\int_A y^2 dA = -\frac{\sigma}{y}\int_A y^2 dA \end{aligned} \tag{7.4.20}$$

and the integration is over the complete cross-sectional area *A*. The minus sign in the second of these equations arises because a positive moment and a positive *y* imply a compressive (negative) stress (see Fig. 7.4.4).

The quantity $\int_A y dA$ is the first moment of area about the neutral axis, and is equal to $\bar{y}A$, where $\bar{y}$ is the centroid of the section (see, for example, §3.2.1). Note that the horizontal component ("in-out of the page") of the centroid will always be at the centre of the beam due to the symmetry of the beam about the plane of bending. Since the first moment of area is zero, it follows that $\bar{y} = 0$: *the neutral axis passes through the centroid of the cross-section*.

The quantity $\int_A y^2 dA$ is called the **second moment of area** or the **moment of inertia** about the neutral axis, and is denoted by the symbol *I*. It follows that the flexural stress is related to the moment through

$$\boxed{\sigma = -\frac{My}{I}} \qquad \textbf{Flexural stress in a beam} \tag{7.4.21}$$

This is one of the most famous and useful formulas in mechanics.

The Moment of Inertia

The moment of inertia depends on the shape of a beam's cross-section. Consider the important case of a rectangular cross section. Before determining the moment of inertia one must locate the centroid (neutral axis). Due to symmetry, the neutral axis runs through the centre of the cross-section. To evaluate I for a rectangle of height h and width b, consider a small strip of height dy at location y, Fig. 7.4.22. Then

$$I = \int_A y^2 dA = b \int_{-h/2}^{+h/2} y^2 dy = \frac{bh^3}{12} \tag{7.4.22}$$

This relation shows that the "taller" the cross-section, the larger the moment of inertia, something which holds generally for I. Further, the larger is I, the smaller is the flexural stress, which is always desirable.

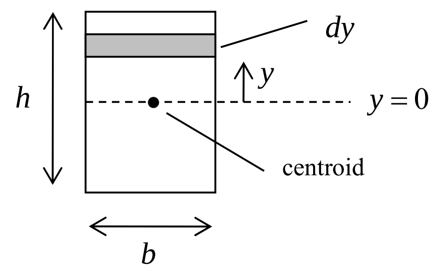


Figure 7.4.22: Evaluation of the moment of inertia for a rectangular cross-section

For a circular cross-section with radius R, consider Fig. 7.4.23. The moment of inertia is then

$$I = \int_A y^2 dA = \int_0^{2\pi} \int_0^R r^3 \sin^2 \theta dr d\theta = \frac{\pi R^4}{4} \tag{7.4.23}$$

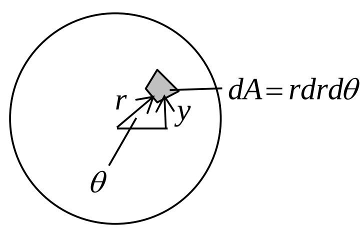


Figure 7.2.23: Moment of inertia for a circular cross-section

Example

Consider the beam shown in Fig. 7.4.24. It is loaded symmetrically by two concentrated forces each of magnitude 100N and has a circular cross-section of radius 100mm. The reactions at the two supports are found to be 100N. Sectioning the beam to the left of the forces, and then to the right of the first force, one finds that

$$\begin{aligned} V &= 100, \qquad M = 100x \qquad (0 < x < 250) \\ V &= 0, \qquad M = 25000 \qquad (250 < x < l/2) \end{aligned} \tag{7.4.24}$$

where l is the length of the beam.

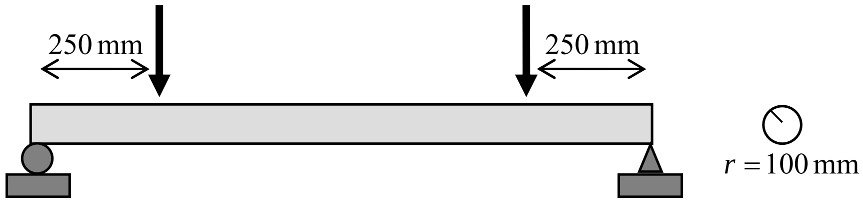


Figure 7.4.24: a loaded beam with circular cross-section

The maximum tensile stress is then

$$\sigma_{\max} = -\frac{M_{\max}(-y_{\max})}{I} = \frac{25000r}{\pi r^4/4} = 31.8\,\text{MPa} \tag{7.4.25}$$

and occurs at all sections between the two loads (at the base of the beam).

■

7.4.5 Shear Stresses in Beams

In the derivation of the flexural stress formula, Eqn. 7.4.21, it was assumed that plane sections remain plane. This implies that there is no shear strain and, for an isotropic elastic material, no shear stress, as indicated in Fig. 7.4.25.

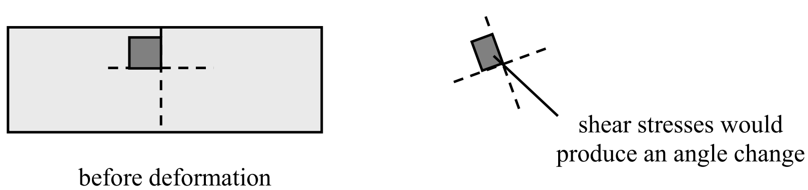


Figure 7.4.25: a section of beam before and after deformation

This fact will now be ignored, and an expression for the shear stress τ within a beam will be developed. It is implicitly assumed that this shear stress has little effect on the (calculation of the) flexural stress.

As in Fig. 7.4.18, consider the equilibrium of a thin section of beam, as shown in Fig. 7.4.26. The beam has *rectangular* cross-section (although the theory developed here is strictly for rectangular cross sections only, it can be used to give approximate shear stress values in any beam with a plane of symmetry). Consider the equilibrium of a section *of* this section, at the upper surface of the beam, shown hatched in Fig. 7.4.26. The stresses acting on this section are as shown. Again, the normal stress is compressive at the surface, consistent with the sign convention for a positive moment. Note that there are no shear stresses acting at the surface – there may be distributed normal loads or forces acting at the surface but, for clarity, these are not shown, and they are not necessary for the following calculation.

From equilibrium of forces in the horizontal direction of the surface section:

$$\left[-\int_A \sigma dA\right]_x + \left[\int_A \sigma dA\right]_{x+\Delta x} + \tau b \Delta x = 0 \qquad (7.4.26)$$

The third term on the left here assumes that the shear stress is uniform over the section – this is similar to the calculations of §7.4.3 – for a very small section, the variation in stress is a small term and may be neglected. Using the bending stress formula, Eqn. 7.4.21,

$$-\int_A \frac{M(x+\Delta x)-M(x)}{\Delta x}\frac{y}{I}dA + \tau b = 0 \qquad (7.4.27)$$

and, with Eqn. 7.4.13, as $\Delta x \to 0$,

$$\boxed{\tau = \frac{VQ}{Ib}} \quad \textbf{Shear stress in a beam} \qquad (7.4.28)$$

where Q is the first moment of area $\int_A ydA$ of the *surface section* of the cross-section.

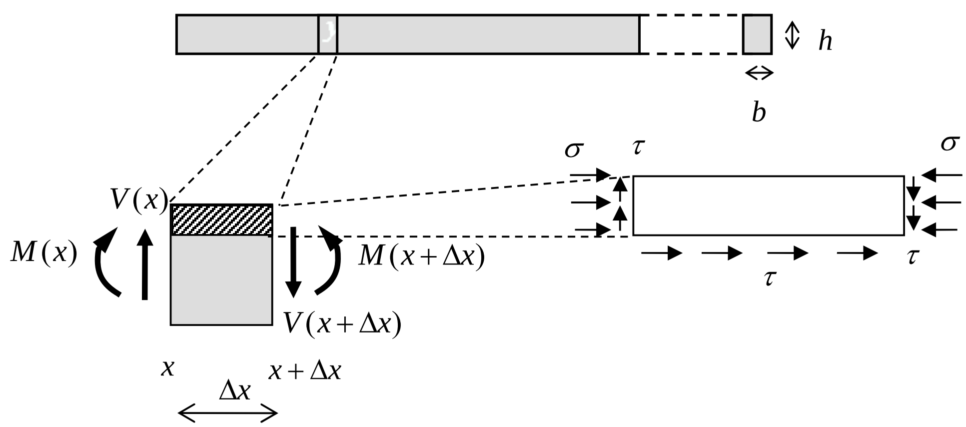


Figure 7.4.26: stresses and forces acting on a small section of material at the surface of a beam

As mentioned, this formula 7.4.28 can be used as an approximation of the shear stress in a beam of arbitrary cross-section, in which case b can be regarded as the depth of the beam at that section. For the rectangular beam, one has

$$Q = b\int_y^{h/2} ydy = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right) \qquad (7.4.29)$$

so that

$$\tau = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right) \tag{7.4.30}$$

The maximum shear stress in the cross-section arises at the neutral surface:

$$\tau_{\max} = \frac{3V}{2bh} = \frac{3V}{2A} \tag{7.4.31}$$

and the shear stress dies away towards the upper and lower surfaces. Note that the average shear stress over the cross-section is V / A and the maximum shear stress is 150% of this value.

Finally, since the shear stress on a vertical cross-section has been evaluated, the shear stress on a longitudinal section has been evaluated, since the shear stresses on all four sides of an element are the same, as in Fig.7.4.6.

Example

Consider the simply supported beam loaded by a concentrated force shown in Fig. 7.4.27. The cross-section is rectangular with height $100\,\text{mm}$ and width $50\,\text{mm}$. The reactions at the supports are $5\,\text{kN}$ and $15\,\text{kN}$. To the left of the load, one has $V = 5\,\text{kN}$ and $M = 5x\,\text{kNm}$. To the right of the load, one has $V = -15\,\text{kN}$ and $M = 30 - 15x\,\text{kNm}$.

The maximum shear stress will occur along the neutral axis and will clearly occur where V is largest, so anywhere to the right of the load:

$$\tau_{\max} = \frac{3V_{\max}}{2A} = 4.5\,\text{MPa} \tag{7.4.32}$$

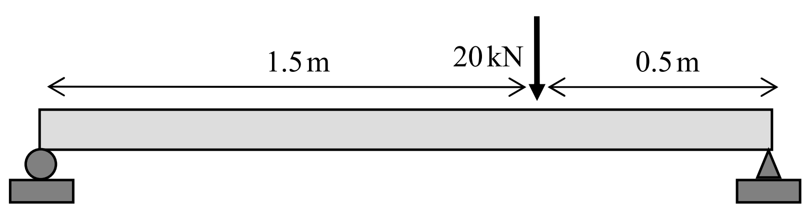


Figure 7.4.27: a simply supported beam

As an example of general shear stress evaluation, the shear stress at a point 25 mm below the top surface and 1 m in from the left-hand end is, from Eqn 7.4.30, $\tau = +1.125\,\text{MPa}$. The shear stresses acting on an element at this location are shown in Fig. 7.4.28.

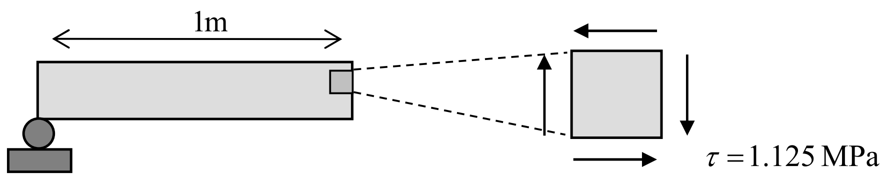


Figure 7.4.28: shear stresses acting at a point in the beam

■

7.4.6 Approximate nature of the beam theory

The beam theory is only an approximate theory, with a number of simplifications made to the full equations of elasticity. A more advanced (and exact) mechanics treatment of the beam problem would not make any assumptions regarding plane sections remaining plane, etc. The accuracy of the beam theory can be explored by comparing the beam theory results with the results of the more exact theory.

When a beam is in **pure bending**, that is when the shear force is everywhere zero, the full elasticity solution shows that plane sections *do* actually remain plane and the beam theory is exact. For more complex loadings, plane sections *do* actually deform. For example, it can be shown that the initially plane sections of a cantilever subjected to an end force, Fig. 7.4.29, do not remain plane. Nevertheless, the beam theory prediction for normal and shear stress is exact in this simple case.

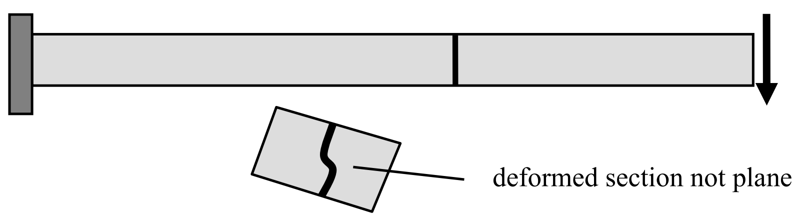


Figure 7.4.29: a cantilevered beam loaded by a force and moment

Consider next a cantilevered beam of length l and rectangular cross section, height h and width b, subjected to a uniformly distributed load p. With x measured from the cantilevered end, the shear force and moment are given by $V = p(l-x)$ and $M = (pl^2/2)\left(-1+2x/l-(x/l)^2\right)$. The shear stress is

$$\tau = \frac{6p}{bh^3}\left(\frac{h^2}{4}-y^2\right)(l-x) \tag{7.4.33}$$

and the flexural stresses at the cantilevered end, at the upper surface, are

$$\frac{\sigma}{p} = \frac{3}{b}\left(\frac{l}{h}\right)^2 \tag{7.4.34}$$

The solution for shear stress, Eqn 7.4.33, turns out to be exact; however, the exact solution corresponding to Eqn 7.4.34 is[6]

$$\frac{\sigma}{p} = \frac{1}{b}\left[3\left(\frac{l}{h}\right)^2 - \frac{1}{5}\right] \tag{7.4.35}$$

It can be seen that the beam theory is a good approximation for the case when l/h is large, in which case the term 1/5 is negligible.

Following this type of analysis, a general rule of thumb is this: for most configurations, the elementary beam theory formulae for flexural stress and transverse shear stress are accurate to within about 3% for beams whose length-to-height ratio is greater than about 4.

7.4.7 Beam Deflection

Consider the deflection curve of a beam. The displacement of the neutral axis is denoted by v, positive upwards, as in Fig. 7.4.30. The slope at any point is then given by the first derivative, dv/dx.

For any type of material, provided the slope of the deflection curve is small, it can be shown that the radius of curvature R is related to the second derivative d^2v/dx^2 through (see the Appendix to this section, §7.4.10)

$$\frac{1}{R} = \frac{d^2v}{dx^2} \tag{7.4.36}$$

and for this reason d^2v/dx^2 is called the **curvature** of the beam. Using Eqn. 7.4.19, $\sigma = -Ey/R$, and the flexural stress expression, Eqn. 7.4.21, $\sigma = -My/I$, one has the **moment-curvature equation**

$$\boxed{M(x) = EI\frac{d^2v}{dx^2}} \qquad \textbf{moment-curvature equation} \tag{7.4.37}$$

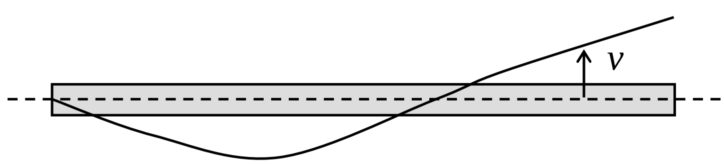


Figure 7.4.30: the deflection of a beam

With the moment known, this differential equation can be integrated twice to obtain the deflection. Boundary conditions must be supplied to obtain constants of integration.

[6] this can be derived using the Stress Function method discussed in Book 2, section 3.2

Example

Consider the cantilevered beam of length L shown in Fig. 7.4.31, subjected to an end-force F and end-moment M_0. The moment is found to be $M(x) = F(L - x) + M_0$, with x measured from the clamped end. The moment-curvature equation is then

$$
\begin{aligned}
& EI\frac{d^2v}{dx^2} = (FL + M_0) - Fx \\
\to \quad & EI\frac{dv}{dx} = (FL + M_0)x - \frac{1}{2}Fx^2 + C_1 \\
\to \quad & EIv = \frac{1}{2}(FL + M_0)x^2 - \frac{1}{6}Fx^3 + C_1 x + C_2
\end{aligned}
\tag{7.4.38}
$$

The boundary conditions are that the displacement and slope are both zero at the clamped end, from which the two constant of integration can be obtained:

$$
\begin{aligned}
v(0) = 0 \quad & \to \quad C_2 = 0 \\
v'(0) = 0 \quad & \to \quad C_1 = 0
\end{aligned}
\tag{7.4.39}
$$

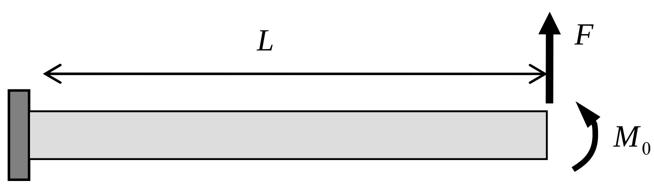


Figure 7.4.31: a cantilevered beam loaded by an end-force and moment

The slope and deflection are therefore

$$
v = \frac{1}{EI}\left[\frac{1}{2}(FL + M_0)x^2 - \frac{1}{6}Fx^3\right], \qquad \frac{dv}{dx} = \frac{1}{EI}\left[(FL + M_0)x - \frac{1}{2}Fx^2\right] \tag{7.4.40}
$$

The maximum deflection occurs at the end, where

$$
v(L) = \frac{1}{EI}\left[\frac{1}{2}M_0 L^2 + \frac{1}{3}FL^3\right] \tag{7.4.41}
$$

■

The term EI in Eqns. 7.4.40-41 is called the **flexural rigidity**, since it is a measure of the resistance of the beam to deflection.

Example

Consider the simply supported beam of length L shown in Fig. 7.4.32, subjected to a uniformly distributed load p over half its length. In this case, the moment is given by

$$M(x) = \begin{cases} \frac{3}{8} pLx - \frac{1}{2} px^2 & 0 < x < \frac{L}{2} \\ \frac{1}{8} pL(L - x) & \frac{L}{2} < x < L \end{cases} \tag{7.4.42}$$

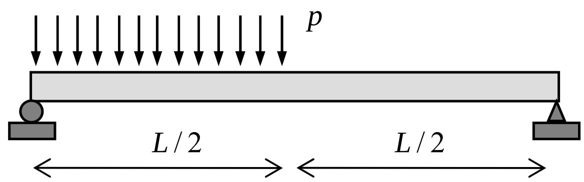


Figure 7.4.32: a simply supported beam subjected to a uniformly distributed load over half its length

It is necessary to apply the moment-curvature equation to each of the two regions $0 < x < L/2$ and $L/2 < x < L$ separately, since the expressions for the moment in these regions differ. Thus there will be four constants of integration:

$$\begin{aligned} & EI \frac{d^2 v}{dx^2} = \frac{3}{8} pLx - \frac{1}{2} px^2 && EI \frac{d^2 v}{dx^2} = \frac{1}{8} pL^2 - \frac{1}{8} pLx \\ & \to EI \frac{dv}{dx} = \frac{3}{16} pLx^2 - \frac{1}{6} px^3 + C_1 && \to EI \frac{dv}{dx} = \frac{1}{8} pL^2 x - \frac{1}{16} pLx^2 + D_1 \\ & \to EIv = \frac{3}{48} pLx^3 - \frac{1}{24} px^4 + C_1 x + C_2 && \to EIv = \frac{1}{16} pL^2 x^2 - \frac{1}{48} pLx^3 + D_1 x + D_2 \end{aligned} \tag{7.4.43}$$

The boundary conditions are: (i) no deflection at roller support, $v(0) = 0$, from which one finds that $C_2 = 0$, and (ii) no deflection at pin support, $v(L) = 0$, from which one finds that $D_2 = -pL^4 / 24 - D_1 L$. The other two necessary conditions are the **continuity conditions** where the two solutions meet. These are that (i) the deflection of both solutions agree at $x = L/2$ and (ii) the slope of both solutions agree at $x = L/2$. Using these conditions, one finds that

$$C_1 = -\frac{9 pL^3}{384}, \quad C_2 = -\frac{17 pL^3}{384} \tag{7.4.44}$$

so that

$$v = \begin{cases} \frac{wL^4}{384EI} \left[-9\left(\frac{x}{L}\right) + 24\left(\frac{x}{L}\right)^3 - 16\left(\frac{x}{L}\right)^4 \right] & 0 < x < \frac{L}{2} \\ \frac{wL^4}{384EI} \left[1 - 17\left(\frac{x}{L}\right) + 24\left(\frac{x}{L}\right)^2 - 8\left(\frac{x}{L}\right)^3 \right] & \frac{L}{2} < x < L \end{cases} \tag{7.4.45}$$

The deflection is shown in Fig. 7.4.33. Note that the maximum deflection occurs in $0 < x < L/2$; it can be located by setting $dv/dx = 0$ there and solving.

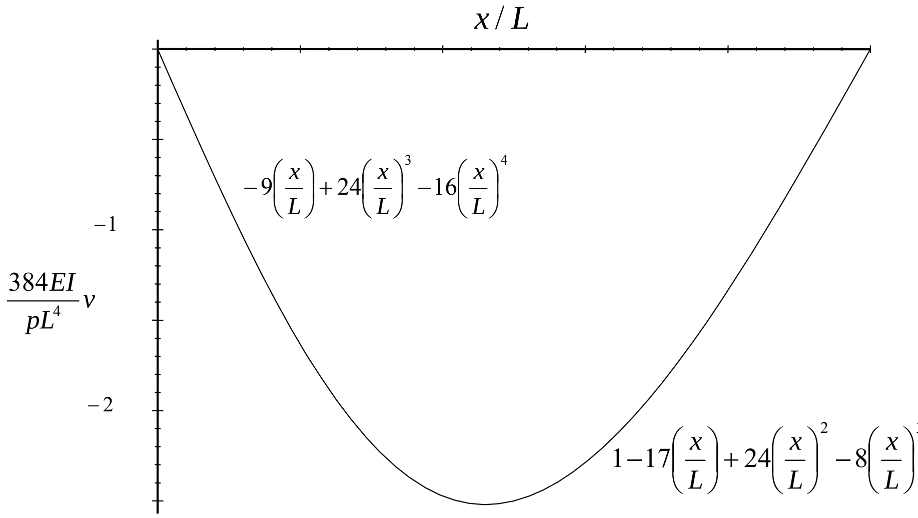


Figure 7.4.33: deflection of a beam

■

7.4.8 Statically Indeterminate Beams

Consider the beam shown in Fig. 7.4.34. It is cantilevered at one end and supported by a roller at its other end. A moment is applied at its centre. There are three unknown reactions in this problem, the reaction force at the roller and the reaction force and moment at the built-in end. There are only two equilibrium equations with which to determine these three unknowns and so it is not possible to solve the problem from equilibrium considerations alone. The beam is therefore statically indeterminate (see the end of section 2.3.3).

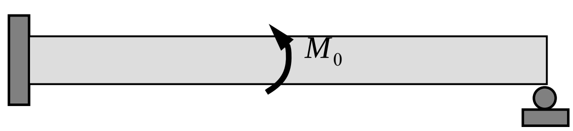


Figure 7.4.34: a cantilevered beam supported also by a roller

More examples of statically indeterminate beam problems are shown in Fig. 7.4.35. To solve such problems, one must consider the deformation of the beam. The following example illustrates how this can be achieved.

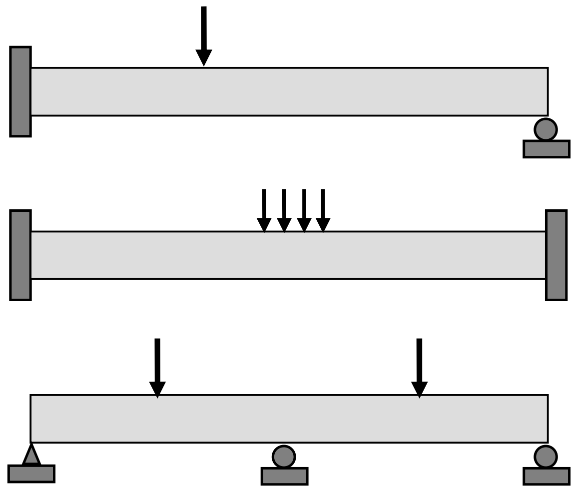


Figure 7.4.35: examples of statically indeterminate beams

Example

Consider the beam of length *L* shown in Fig. 7.4.36, cantilevered at end *A* and supported by a roller at end *B*. A moment M_0 is applied at *B*.

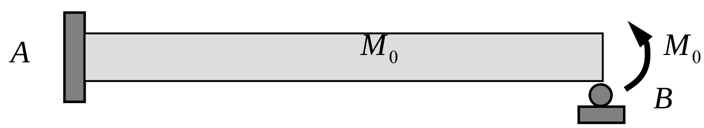


Figure 7.4.36: a statically indeterminate beam

The moment along the beam can be expressed in terms of the *unknown* reaction force at end *B*: $M(x) = R_B(L-x) + M_0$. As before, one can integrate the moment-curvature equation:

$$
\begin{aligned}
&EI\frac{d^2 v}{dx^2} = R_B(L-x) + M_0 \\
&\rightarrow EI\frac{dv}{dx} = \left(R_B L + M_0\right)x - \frac{1}{2}R_B x^2 + C_1 \\
&\rightarrow EIv = \frac{1}{2}\left(R_B L + M_0\right)x^2 - \frac{1}{6}R_B x^3 + C_1 x + C_2
\end{aligned}
\tag{7.4.46}
$$

There are three boundary conditions, two to determine the constants of integration and one can be used to determine the unknown reaction R_B. The boundary conditions are (i) $v(0) = 0 \rightarrow C_2 = 0$, (ii) $dv/dx(0) = 0 \rightarrow C_1 = 0$ and (iii) $v(L) = 0$ from which one finds that $R_B = -3M_0/2L$. The slope and deflection are therefore

$$v = \frac{M_B L^2}{4EI}\left[\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^2\right]$$
$$\frac{dv}{dx} = \frac{M_B L}{4EI}\left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)\right] \qquad (7.4.47)$$

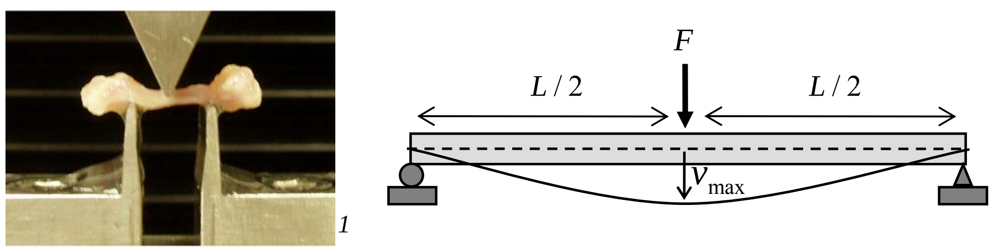
One can now return to the equilibrium equations to find the remaining reactions acting on the beam, which are $R_A = -R_B$ and $M_A = M_0 + LR_B$

■

7.4.9 The Three-point Bending Test

The 3-point bending test is a very useful experimental procedure. It is used to gather data on materials which are subjected to bending in service. It can also be used to get the Young's Modulus of a material for which it might be more difficult to get *via* a tension or other test.

A mouse bone is shown in the standard 3-point bend test apparatus in Fig. 7.4.37a. The idealised beam theory model of this test is shown in Fig. 7.4.37b. The central load is *F*, so the reactions at the supports are $F/2$. The moment is zero at the supports, varying linearly to a maximum $FL/4$ at the centre.



[1]

(a) (b)

Figure 7.4.37: the three-point bend test; (a) a mouse bone specimen, (b) idealised model

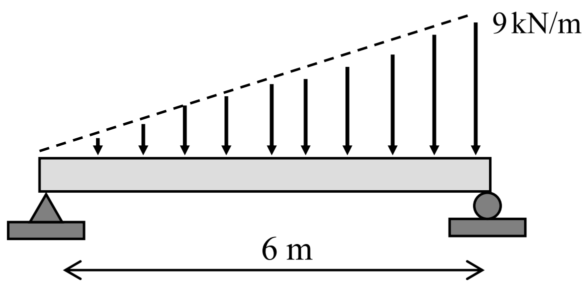
The maximum flexural stress then occurs at the outer fibres at the centre of the beam: for a circular cross-section, $\sigma_{\max} = FL/\pi R^3$. Integrating the moment-curvature equation, and using the fact that the deflection is zero at the supports and, from symmetry, the slope is zero at the centre, the maximum deflection is seen to be $v_{\max} = FL^3/12\pi R^4 E$. If one plots the load *F* against the deflection $v_{\max}$, one will see a straight line (initially, before the elastic limit is reached); let the slope of this line be $\hat{E}$. The Young's modulus can then be evaluated through

$$E = \frac{L^3}{12\pi R^4}\hat{E} \qquad (7.4.48)$$

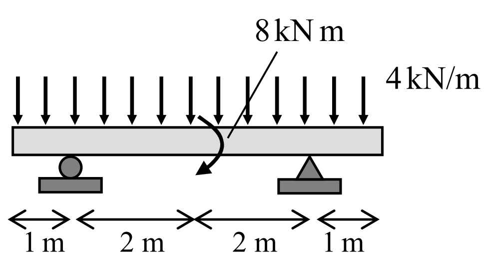
With $\sigma = E\varepsilon$, the maximum strain is $\varepsilon_{\max} = FL/\pi ER^3 = 12Rv_{\max}/L^2$. By carrying the test on beyond the elastic limit, the strength of the material at failure can be determined.

7.4.10 Problems

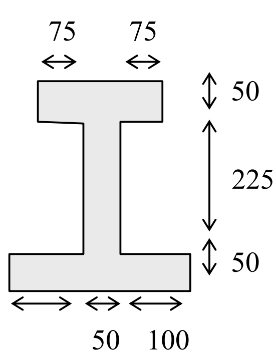
1. The simply supported beam shown below carries a vertical load that increases uniformly from zero at the left end to a maximum value of 9 kN/m at the right end. Draw the shearing force and bending moment diagrams



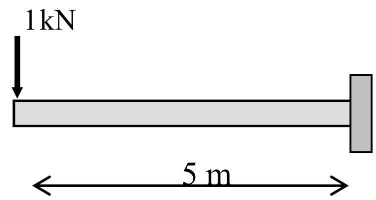
2. The beam shown below is simply supported at two points and overhangs the supports at each end. It is subjected to a uniformly distributed load of 4 kN/m as well as a couple of magnitude 8 kN m applied to the centre. Draw the shearing force and bending moment diagrams



3. Evaluate the centroid of the beam cross-section shown below (all measurements in mm)

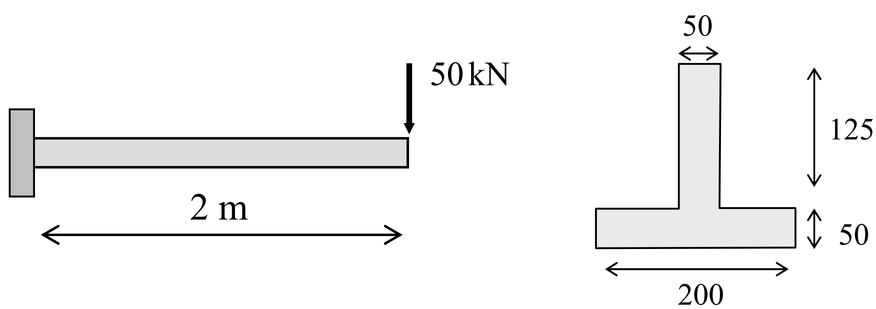


4. Determine the maximum tensile and compressive stresses in the following beam (it has a rectangular cross-section with height 75 mm and depth 50 mm)

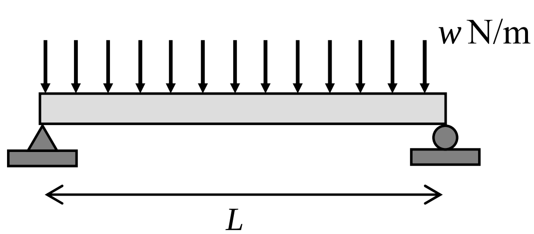


5. Consider the cantilever beam shown below. Determine the maximum shearing stress in the beam and determine the shearing stress 25 mm from the top surface of the beam at a section adjacent to the supporting wall. The cross-section is the "T" shape shown, for which $I = 40 \times 10^6 \text{ mm}^4$.

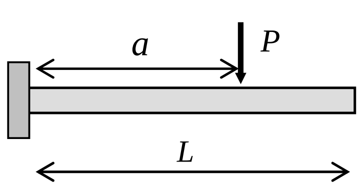
   [note: use the shear stress formula derived for rectangular cross-sections – as mentioned above, in this formula, *b* is the thickness of the beam *at the point where the shear stress is being evaluated*]



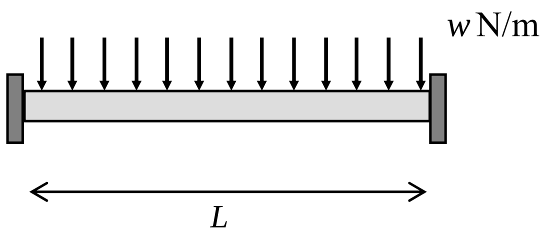
6. Obtain an expression for the maximum deflection of the simply supported beam shown here, subject to a uniformly distributed load of $w\,\text{N/m}$.



7. Determine the equation of the deflection curve for the cantilever beam loaded by a concentrated force P as shown below.



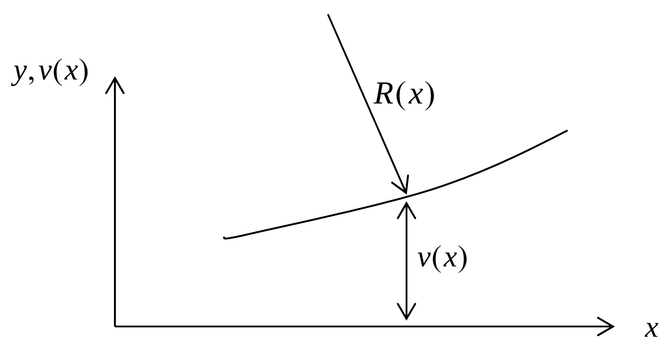
8. Determine the reactions for the following uniformly loaded beam clamped at both ends.



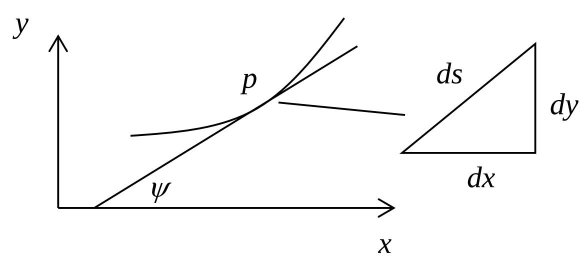
7.4.11 Appendix to §7.4

Curvature of the deflection curve

Consider a deflection curve with deflection $v(x)$ and radius of curvature $R(x)$, as shown in the figure below. Here, *deflection* is the transverse displacement (in the *y* direction) of the points that lie along the axis of the beam. A relationship between $v(x)$ and $R(x)$ is derived in what follows.



First, consider a curve (arc) *s*. The tangent to some point *p* makes an angle ψ with the *x* – axis, as shown below. As one move along the arc, ψ changes.



Define the **curvature** κ of the curve to be the rate at which ψ increases relative to *s*,

$$\kappa = \frac{d\psi}{ds}$$

Thus if the curve is very "curved", ψ is changing rapidly as one moves along the curve (as one increase *s*) and the curvature will be large.

From the above figure,

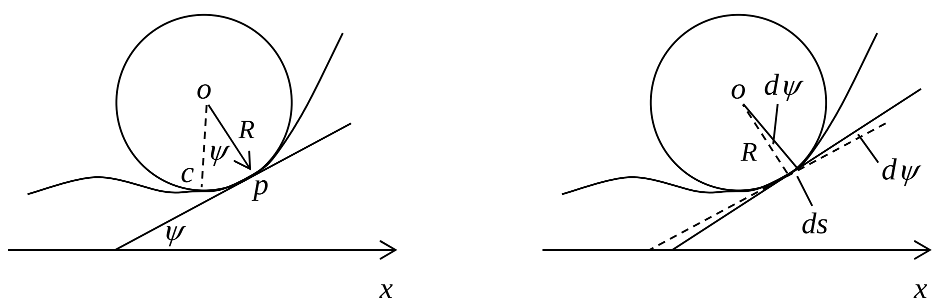
$$\tan\psi = \frac{dy}{dx}, \quad \frac{ds}{dx} = \frac{\sqrt{(dx)^2 + (dy)^2}}{dx} = \sqrt{1 + (dy/dx)^2} \,,$$

so that

$$\kappa = \frac{d\psi}{ds} = \frac{d\psi}{dx}\frac{dx}{ds} = \frac{d\left(\arctan(dy/dx)\right)}{dx}\frac{dx}{ds} = \frac{1}{1 + (dy/dx)^2}\frac{d^2y}{dx^2}\frac{dx}{ds}$$
$$= \frac{d^2y/dx^2}{\left[1 + (dy/dx)^2\right]^{3/2}}$$

Finally, it will be shown that the curvature is simply the reciprocal of the radius of curvature. Draw a circle to the point *p* with radius R. Arbitrarily measure the arc length *s* from the point *c*, which is a point on the circle such that $\angle cop = \psi$. Then arc length $s = R\psi$, so that

$$\kappa = \frac{d\psi}{ds} = \frac{1}{R}$$



Thus

$$\frac{1}{R} = \frac{\dfrac{d^2v}{dx^2}}{\left[1 + \left(\dfrac{dv}{dx}\right)^2\right]^{\frac{3}{2}}}$$

If one assumes now that the slopes of the deflection curve are small, then $dv/dx << 1$ and

$$\frac{1}{R} \approx \frac{d^2v}{dx^2}$$

Images used:
1. http://www.mc.vanderbilt.edu/root/vumc.php?site=CenterForBoneBiology&doc=20412

7.5 Elastic Buckling

The initial theory of the buckling of columns was worked out by Euler in 1757, a nice example of a theory preceding the application, the application mainly being for the later "invented" metal and concrete columns in modern structures.

7.5.1 Columns and Buckling

A **column** is a long slender bar under axial compression, Fig. 7.5.1. A column can be horizontal, vertical or inclined; in the latter cases it is termed a **strut**.

The column under axial compression responds elastically in exactly the same way as the axial bar of §7.1. For example, it decreases in length under a compressive force P by an amount given by Eqn. 7.1.5, $\Delta = PL/EA$. However, when the compressive force is large enough, the column will **buckle** with lateral deflection. This possibility is the subject of this section.

Euler's Theory of Buckling

Consider an elastic column of length L, pin-ended so free to rotate at its ends, subjected to an axial load P, Fig. 7.5.1. Assume that it undergoes a lateral deflection denoted by v. Moment equilibrium of a section of the deflected column cut at a typical point x, and using the moment-curvature Eqn. 7.4.37, results in

$$-Pv(x) = M(x) = EI\frac{d^2v}{dx^2} \tag{7.5.1}$$

Hence the deflection v satisfies the differential equation

$$\frac{d^2v}{dx^2} + k^2 v(x) = 0 \tag{7.5.2}$$

where

$$k^2 = \frac{P}{EI} \tag{7.5.3}$$

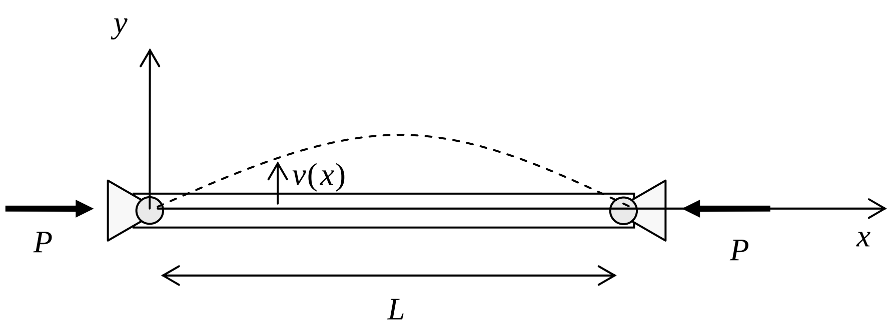


Fig. 7.5.1: a column with deflection *v*

The ordinary differential equation 7.5.2 is linear, homogeneous and with constant coefficients. Its solution can be found in any standard text on differential equations and is given by (for $k^2 > 0$)

$$v(x) = A\cos(kx) + B\sin(kx) \tag{7.5.4}$$

where *A* and *B* are as yet unknown constants. The boundary conditions for pinned-ends are

$$v(0) = 0, \qquad v(L) = 0 \tag{7.5.5}$$

The first condition requires *A* to be zero and the second leads to

$$B\sin(kL) = 0 \tag{7.5.6}$$

It follows that either:
(a) $B = 0$, in which case $v(x) = 0$ for all *x* and the column is not deflected

or

(b) $\sin(kL) = 0$, which holds when kL is an integer number of π's, i.e.

$$k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots, \tag{7.5.7}$$

As mentioned, the solution (a) is governed by the axial deformation theory discussed in §7.1. Concentrating on (b), the corresponding solution for the deflection is

$$v_n(x) = B\sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots \tag{7.5.8}$$

The parameter *k* is defined by Eqn. 7.5.3, so that, using 7.5.7,

$$P_n = EI\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots, \tag{7.5.9}$$

It has hence been shown that buckling, i.e. $v \neq 0$, can only occur at a discreet set of applied loads - the **buckling loads** - given by 7.5.9. In practice the most important buckling load is the first, corresponding to $n = 1$, since this will be the first of the loads reached as the applied load *P* is increased from zero; this is called the **critical buckling load**:

$$\boxed{P_c = EI\left(\frac{\pi}{L}\right)^2} \tag{7.5.10}$$

with associated deflection

$$v_1(x) = B \sin\left(\frac{\pi x}{L}\right) \tag{7.5.11}$$

The column hence deforms into a single sine wave, which is termed the **mode** or **mode shape** of the deflected column. Note that B, the amplitude of the deflection, can not be determined by this model. This is a consequence of assuming the deflection is small; of **linearising** the problem (which is inherent in the derivation of the moment-deflection curve, Eqn. 7.5.1). A more exact finite deformation theory has been worked out and is called the **theory of the elastica**, but this is not pursued here.

This mathematical structure, where one finds one can only get non-zero solutions of an equation for certain values of a parameter is very common in engineering and theoretical physics. The critical values of the parameter, in this case *k*, are termed the **eigenvalues** of the problem, and the corresponding non-zero solutions, $v(x)$, are the **eigenfunctions**.

The second moment of area *I* has dimensions of $(\text{length})^4$, and for columns is often written in the form $I = Ar^2$ where *A* is the cross-sectional area of the column and the length *r* is called the **radius of gyration**. For example in the case of a circular shaft of radius *a*, $I = \pi a^4 / 4$ (see Eqn. 7.4.23) so $r = a/2$.

Failure of the Column

The expression 7.5.10 for the critical buckling load can be written in terms of the radius of gyration:

$$P_{cr} = EAr^2\left(\frac{\pi}{L}\right)^2 \qquad \text{or} \qquad \frac{\sigma_{cr}}{E} = \frac{\pi^2}{(L/r)^2} \tag{7.5.12}$$

where σ_{cr} is the mean compressive stress on the loaded end of the column.

The second equation in 7.5.12 is the most convenient non-dimensional form of presenting theoretical and experimental results for buckling problems. The ratio L/r is called the **slenderness ratio**.

Failure of the column will occur in purely axial compression if the stress in the column reaches the yield stress of the material (see §5.2). On the other hand, if the critical buckling stress σ_{cr} is less than the yield stress, then the column will fail by buckling before the yield stress is reached.

Eqn. 7.5.12 is plotted in Fig. 7.5.2. The yield stress of the material is denoted by *Y*. A critical slenderness ratio is denoted by $(L/r)_{cr}$. For slenderness ratios less than the critical value, that is, for relatively squat columns, the stress in the column will reach the yield stress before buckling occurs.

For example, consider a steel column for which $E = 210\ \text{GPa}$ and $Y = 210\ \text{MPa}$. The critical value of the slenderness ratio is then $L/r = 99.35$, which is a length to diameter ratio of about 25 for a circular column. Buckling will then occur in such columns which have $L/r > 99.35$, for sufficiently high applied axial compressive force.

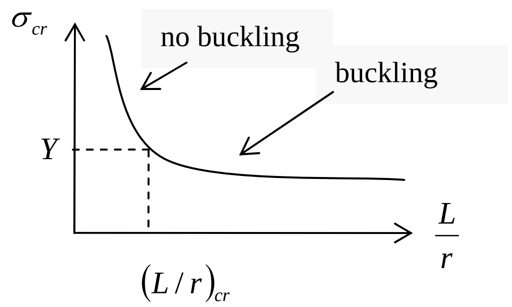


Fig. 7.5.2: critical values of the slenderness ratio

7.5.2 A General Approach to Buckling

The model developed above only applies to columns simply supported at each end. To discuss the more general case one can return to the formulation of the bending of a beam discussed in §7.4.3, but include also axial forces. Fig. 7.4.18 is reproduced as Fig. 7.5.3 but now with compressive axial forces, the forces offset by a small increment in deflection Δv.

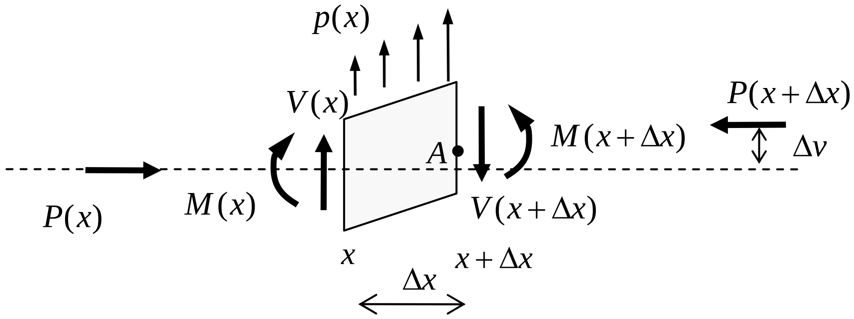


Figure 7.5.3: forces and moments acting on a column

Resolving vertically, one again arrives at Eqn. 7.4.10:

$$\frac{dV}{dx} = p(x) \tag{7.5.13}$$

Resolving horizontally, one simply gets $P(x) = P(x + \Delta x)$, so that P is constant. Taking moments, one has, instead of 7.4.13,

$$\frac{dM}{dx} + P\frac{dv}{dx} = V \tag{7.5.14}$$

Note the extra term involving *P*, which is not present in pure bending theory. Eliminating *M* between 7.5.14 and the moment-curvature equation 7.4.37 then leads to an expression for the shear force:

$$\frac{V}{EI} = \frac{d^3 v}{dx^3} + \frac{P}{EI}\frac{dv}{dx} \tag{7.5.15}$$

Note that, in the beam theory, where $P = 0$, the third derivative of the deflection is zero whenever the shear force is zero, in particular at a free, i.e. unsupported, end. Here, however, it is no longer true that the third derivative is zero.

The final differential equation is now obtained by differentiating 7.5.15 and using 7.5.13:

$$\frac{d^4 v}{dx^4} + \frac{P}{EI}\frac{d^2 v}{dx^2} = \frac{p}{EI} \tag{7.5.16}$$

Concentrating on the buckling behaviour and so neglecting the transverse load $p(x)$ [1], one arrives at the differential equation

$$\frac{d^4 v}{dx^4} + k^2 \frac{d^2 v}{dx^2} = 0 \tag{7.5.17}$$

where again $k^2 = P / EI$ (Eqn. 7.5.3). Eqn. 7.5.17 is a homogeneous fourth-order differential equation and its solution is

$$v(x) = A\cos(kx) + B\sin(kx) + Cx + D \tag{7.5.18}$$

The four constants are determined by the end conditions on $v(x)$, two conditions at each end. There are three cases:

(1) Pinned end:
boundary conditions are $v = 0$ and $M = 0$; from the moment-curvature equation, $M = 0$ can be replaced with $d^2 v / dx^2 = 0$

(2) Fixed end:
boundary conditions are $v = 0$, $dv / dx = 0$

(3) Free end:
Boundary conditions are $M = 0$ and $V = 0$; again, this implies that $d^2 v / dx^2 = 0$ and, from Eqn. 7.5.15, $V = 0$ can be replaced with $d^3 v / dx^3 + k^2 (dv / dx) = 0$

The case of pinned-pinned results again in the Euler solution given above. Consider now the case where one end is clamped and the other, loaded, end, is unrestrained ("fixed-free"), Fig. 7.5.4.

[1] bars subjected to both axial compressive loads and transverse loads are called **beam-columns**

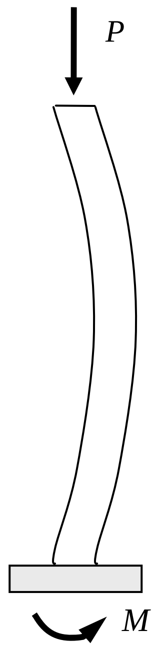


Fig. 7.5.4: a fixed-free column

At the clamped end, $v(0) = v'(0) = 0$, giving

$$A + D = 0, \qquad C + kB = 0 \tag{7.5.19}$$

At the free end, $v''(L) = 0$ and $v'''(L) + k^2 v'(L) = 0$, leading to

$$A\cos(kx) + B\sin(kx) = 0, \qquad C = 0 \tag{7.5.20}$$

Thus, from 7.5.19, B too is zero and A satisfies

$$A\cos(kL) = 0 \tag{7.5.21}$$

Buckling hence can only occur when $\cos(kL) = 0$, i.e. when

$$kL = \pi\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots \tag{7.5.22}$$

Using the definition of the parameter k the buckling loads are given by

$$P = EI\left[\frac{\pi\left(n + \frac{1}{2}\right)}{L}\right]^2, \quad n = 0, 1, 2, \ldots \tag{7.5.23}$$

with the critical buckling load now

$$P_{cr} = EI\left(\frac{\pi}{2L}\right)^2 \tag{7.5.24}$$

which is one quarter of the value for a pinned strut, Eqn. 7.5.10. The buckling modes are given by 7.5.18:

$$v(x) = D\left\{1 - \cos\left[\left(n + \tfrac{1}{2}\right)\pi \frac{x}{L}\right]\right\}, \quad n = 0, 1, 2, \ldots \tag{7.5.25}$$

The first three modes are sketched in Fig. 7.5.5; again, the amplitude is unknown, only the shape.

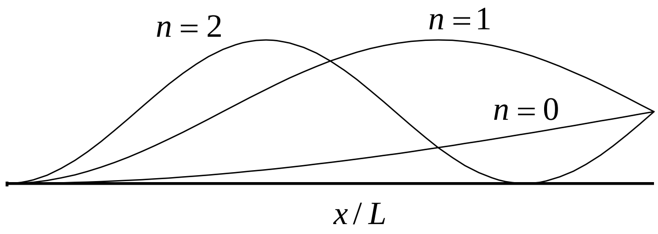


Figure 7.5.5: mode shapes for the fixed-free column

Other cases of end-support can be treated in the same way. Results for the critical buckling stress for various cases are sketched in Fig. 7.5.6.

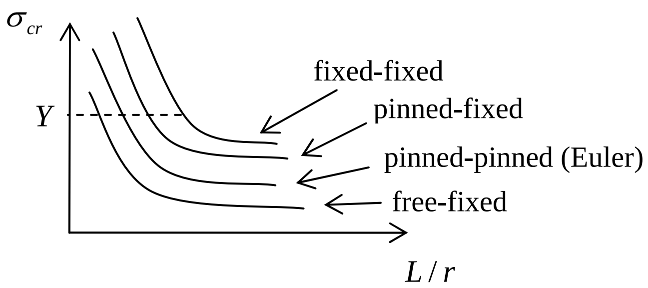


Fig. 7.5.6: critical values of the slenderness ratio for different end-cases

8 Energy and Virtual Work

Thus far in this book, problems have been solved by using a combination of force-equilibrium and kinematics. Here, another approach is explored, in which expressions for work and energy are derived and utilised.

Two important topics are discussed in this Chapter. The first is Energy Methods, which are techniques for solving problems involving elastic materials. Some of these methods, for example Castigliano's second theorem, apply only to linear elastic materials, but most apply to generally non-linear elastic materials.

The second topic is that of Virtual Work. The virtual work approach leads to powerful methods which can be used to solve static or dynamic problems involving any material model.

8.1 Energy in Deforming Materials

There are many different types of **energy**: mechanical, chemical, nuclear, electrical, magnetic, etc. Energies can be grouped into **kinetic energies** (which are due to movement) and **potential energies** (which are stored energies – energy that a piece of matter has because of its position or because of the arrangement of its parts).

A rubber ball held at some height above the ground has (gravitational) potential energy. When dropped, this energy is progressively converted into kinetic energy as the ball's speed increases until it reaches the ground where all its energy is kinetic. When the ball hits the ground it begins to deform elastically and, in so doing, the kinetic energy is progressively converted into **elastic strain energy**, which is stored *inside* the ball. This elastic energy is due to the re-arrangement of molecules in the ball – one can imagine this to be very like numerous springs being compressed inside the ball. The ball reaches maximum deformation when the kinetic energy has been completely converted into strain energy. The strain energy is then converted back into kinetic energy, "pushing" the ball back up for the rebound.

Elastic strain energy is a potential energy – elastically deforming a material is in many ways similar to raising a weight off the ground; in both cases the potential energy is increased.

Similarly, **work** is done in stretching a rubber band. This work is converted into elastic strain energy within the rubber. If the applied stretching force is then slowly reduced, the rubber band will use this energy to "pull" back. If the rubber band is stretched and then released suddenly, the band will retract quickly; the strain energy in this case is converted into kinetic energy – and sound energy (the "snap").

When a small weight is placed on a large metal slab, the slab will undergo minute strains, too small to be noticed visually. Nevertheless, the metal behaves like the rubber ball and when the weight is removed the slab uses the internally stored strain energy to return to its initial state. On the other hand, a metal bar which is bent considerably, and then laid upon the ground, will not nearly recover its original un-bent shape. It has undergone *permanent* deformation. Most of the energy supplied has been lost; it has been converted into heat energy, which results in a very slight temperature rise in the bar. Permanent deformations of this type are accounted for by **plasticity theory**, which is treated in Chapter 11.

In any real material undergoing deformation, at least some of the supplied energy will be converted into heat. However, with the ideal elastic material under study in this chapter, it is assumed that *all* the energy supplied is converted into strain energy. When the loads are removed, the material returns to its precise initial shape and there is no energy loss; for example, a purely elastic ball dropped onto a purely elastic surface would bounce back up to the precise height from which it was released.

As a prelude to a discussion of the energy of elastic materials, some important concepts from elementary particle mechanics are reviewed in the following sections. It is shown that Newton's second law, the **principle of work and kinetic energy** and the **principle of conservation of mechanical energy** are equivalent statements; each can be derived from the other. These concepts are then used to study the energetics of elastic materials.

8.1.1 Work and Energy in Particle Mechanics

Work

Consider a force F which acts on a particle, causing it to move through a displacement *s*, the directions in which they act being represented by the arrows in Fig. 8.1.1a. The work *W* done by *F* is defined to be $Fs\cos\theta$ where θ is the angle formed by positioning the start of the *F* and *s* arrows at the same location with $0 \le \theta \le 180$. Work can be positive or negative: when the force and displacement are in the same direction, then $0 \le \theta \le 90$ and the work done is positive; when the force and displacement are in opposite directions, then $90 \le \theta \le 180$ and the work done is negative.

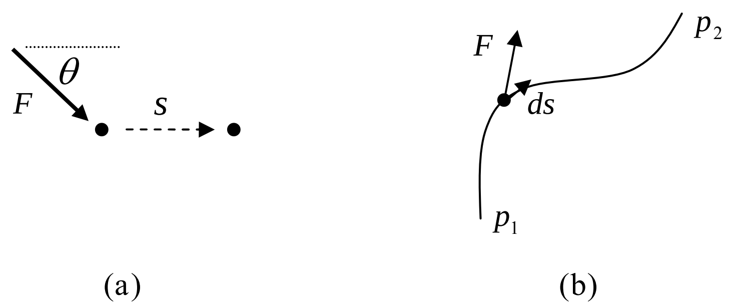


Figure 8.1.1: (a) force acting on a particle, which moves through a displacement *s*; (b) a varying force moving a particle along a path

Consider next a particle moving along a certain path between the points p_1, p_2 by the action of some force *F*, Fig. 8.1.1b. The work done is

$$W = \int_{p_1}^{p_2} F\cos\theta\, ds \tag{8.1.1}$$

where s is the displacement. For motion along a straight line, so that $\theta = 0$, the work is $W = \int_{p_1}^{p_2} F ds$; if *F* here is *constant* then the work is simply *F* times the distance between p_1 to p_2 but, in most applications, *the force will vary* and an integral needs to be evaluated.

Conservative Forces

From Eqn. 8.1.1, the work done by a force in moving a particle through a displacement will in general *depend on the path* taken. There are many important practical cases, however, when the work is *independent* of the path taken, and simply depends on the initial and final positions, for example the work done in deforming elastic materials (see later) – these lead to the notion of a **conservative** (or **potential**) **force**. Looking at the one-dimensional case, a conservative force F_{con} is one which can always be written as the derivative of a function *U* (the minus sign will become clearer in what follows),

$$F_{\text{con}} = -\frac{dU}{dx}, \tag{8.1.2}$$

since, in that case,

$$W = \int_{p_1}^{p_2} F_{\text{con}}\, dx = -\int_{p_1}^{p_2} \frac{dU}{dx} dx = -\int_{p_1}^{p_2} dU = -\left(U(p_2) - U(p_1)\right) = -\Delta U \tag{8.1.3}$$

In this context, the function U is called the **potential energy** and ΔU is the change in potential energy of the particle as it moves from p_1 to p_2. If the particle is moved from p_1 to p_2 and then back to p_1, the net work done is zero and the potential energy U of the particle is that with which it started.

Potential Energy

The potential energy of a particle/system can be defined as follows:

Potential Energy: the work done in moving a system from some standard configuration to the current configuration

Potential energy has the following characteristics:
(1) The existence of a force field
(2) To move something in the force field, work must be done
(3) The force field is conservative
(4) There is some reference configuration
(5) The force field itself does negative work when another force is moving something against it
(6) It is recoverable energy

These six features are evident in the following example: a body attached to the coil of a spring is extended slowly by a force F, overcoming the spring (restoring) force F_{spr} (so that there are no accelerations and $F = -F_{\text{spr}}$ at all times), Fig. 8.1.2.

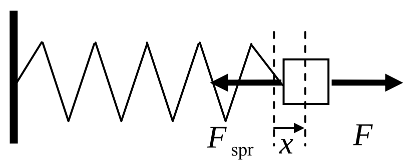


Figure 8.1.2: a force extending an elastic spring

Let the initial position of the block be x_0 (relative to the reference configuration, $x = 0$). Assuming the force to be proportional to deflection, $F = kx$, the work done by F in extending the spring to a distance x is

$$W = \int_{x_0}^{x} F dx = \int_{x_0}^{x} kx dx = \tfrac{1}{2}kx^2 - \tfrac{1}{2}kx_0^2 \equiv U(x) - U(x_0) = \Delta U \tag{8.1.4}$$

This is the work done to move something in the elastic spring "force field" and by definition is the potential energy (change in the body). The energy supplied in moving the body is said to be **recoverable** because the spring is ready to pull back and do the same amount of work.

The corresponding work done by the conservative spring force F_{spr} is

$$W_{\text{spr}} = -\int_{x_0}^{x} F dx = -\left(\tfrac{1}{2}kx^2 - \tfrac{1}{2}kx_0^2\right) \equiv -\Delta U \tag{8.1.5}$$

This work can be seen from the area of the triangles in Fig. 8.1.3: the spring force is zero at the equilibrium/reference position ($x = 0$) and increases linearly as x increases.

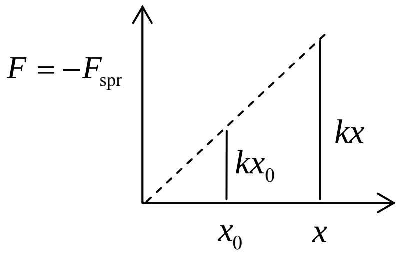


Figure 8.1.3: force-extension curve for a spring

■

The forces in this example depend on the amount by which the spring is stretched. This is similar to the potential energy stored in materials – the potential force will depend in some way on the separation between material particles (see below).

Also, from the example, it can be seen that an alternative definition for the potential energy U of a system is *the negative of the work done by a conservative force in moving the system from some standard configuration to the current configuration*.

In general then, the work done by a conservative force is related to the potential energy through

$$W_{con} = -\Delta U \tag{8.1.6}$$

Dissipative (Non-Conservative) Forces

When the forces are not conservative, that is, they are **dissipative**, one cannot find a universal function U such that the work done is the difference between the values of U at the beginning and end points – *one has to consider the path* taken by the particle and the work done will be different in each case. A general feature of non-conservative forces is that if one moves a particle and then returns it to its original position the net work done will not be zero. For example, consider a block being dragged across a rough surface,

Fig. 8.1.4. In this case, if the block slides over and back a number of times, the work done by the pulling force F keeps increasing, and the work done is not simply determined by the final position of the block, but by its complete path history. The energy used up in moving the block is dissipated as heat (the energy is **irrecoverable**).

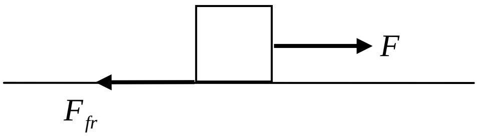


Figure 8.1.4: Dragging a block over a frictional surface

8.1.2 The Principle of Work and Kinetic Energy

In general, a mechanics problem can be solved using either Newton's second law or the principle of work and energy (which is discussed here). These are two different equations which basically say the same thing, but one might be preferable to the other depending on the problem under consideration. Whereas Newton's second law deals with *forces*, the work-energy principle casts problems in terms of *energy*.

The kinetic energy of a particle of mass m and velocity v is defined to be $K = \frac{1}{2}mv^2$. The rate of change of kinetic energy is, using Newton's second law $F = ma$,

$$\dot{K} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = mv\frac{dv}{dt} = (ma)v = Fv \tag{8.1.7}$$

The change in kinetic energy over a time interval (t_0, t_1) is then

$$\Delta K = K_1 - K_0 = \int_{t_0}^{t_1} \frac{dK}{dt}dt = \int_{t_0}^{t_1} Fv\,dt \tag{8.1.8}$$

where K_0 and K_1 are the initial and final kinetic energies. The work done over this time interval is

$$W = \int_{W(t_0)}^{W(t_1)} dW = \int_{x(t_0)}^{x(t_1)} F\,dx = \int_{t_0}^{t_1} Fv\,dt \tag{8.1.9}$$

and it follows that

$$\boxed{W = \Delta K} \qquad \textbf{Work – Energy Principle} \tag{8.1.10}$$

One has the following:

The principle of work and kinetic energy:
the total work done by the external forces acting on a particle equals the change in kinetic energy of the particle

It is not a new principle of mechanics, rather a rearrangement of Newton's second law of motion (or one could have started with this principle, and derived Newton's second law).

The following example shows how the principle holds for conservative, dissipative and applied forces.

Example

A block of mass m is attached to a spring and dragged along a rough surface. It is dragged from left to right, Fig. 8.1.5. Three forces act on the block, the applied force F_{apl} (taken to be constant), the spring force F_{spr} and the friction force F_{fri} (assumed constant).

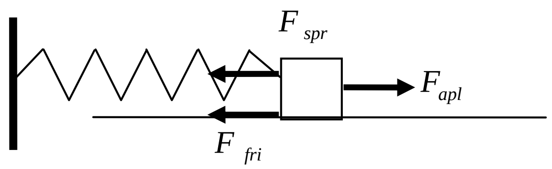


Figure 8.1.5: a block attached to a spring and dragged along a rough surface

Newton's second law, with $F_{spr} = kx$, leads to a standard non-homogeneous second order linear ordinary differential equation with constant coefficients:

$$m\frac{d^2x}{dt^2} = F_{apl} - F_{fri} - F_{spr} \tag{8.1.11}$$

Taking the initial position of the block to be x_0 and the initial velocity to be $\dot{x}_0$, the solution can be found to be

$$x(t) = \frac{F_{apl} - F_{fri}}{k} + \left(x_0 - \frac{F_{apl} - F_{fri}}{k} \right)\cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t \tag{8.1.12}$$

where $\omega = \sqrt{k/m}$. The total work done W is the sum of the work done by the applied force W_{apl}, the work done by the spring force W_{spr} and that done by the friction force W_{fri}:

$$W = W_{apl} + W_{spr} + W_{fri} = F_{apl}(x - x_0) - \frac{1}{2}k\left(x^2 - x_0^2\right) - F_{fri}(x - x_0) \tag{8.1.13}$$

The change in kinetic energy of the block is

$$\Delta K = \frac{1}{2} m\left(\dot{x}^2 - \dot{x}_0^2\right) \tag{8.1.14}$$

Substituting Eqn. 8.1.12 into 8.1.13-14 and carrying out the algebra, one indeed finds that $W = \Delta K$:

$$W = W_{apl} + W_{spr} + W_{fri} = \Delta K \tag{8.1.15}$$

Now the work done by the spring force is equivalent to the negative of the potential energy change, so the work-energy equation (8.1.15) can be written in the alternative form[1]

$$W_{apl} + W_{fri} = \Delta U_{spr} + \Delta K \tag{8.1.16}$$

The friction force is dissipative – it leads to energy loss. In fact, the work done by the friction force is converted into heat which manifests itself as a temperature change in the block. Denoting this energy loss by (see Eqn. 8.1.13) $H_{fri} = F_{fri}(x - x_0)$, one has

$$W_{apl} - H_{fri} = \Delta U_{spr} + \Delta K \tag{8.1.17}$$

■

8.1.3 The Principle of Conservation of Mechanical Energy

In what follows, it is assumed that *there is no energy loss*, so that no dissipative forces act. Define the **total mechanical energy** of a body to be the sum of the kinetic and potential energies of the body. The work-energy principle can then be expressed in two different ways, for this special case:

1. The total work done by the external forces acting on a body equals the change in kinetic energy of the body:

$$W = W_{con} + W_{apl} = \Delta K \tag{8.1.18}$$

2. The total work done by the external forces acting on a body, exclusive of the conservative forces, equals the change in the total mechanical energy of the body

$$W_{\text{apl}} = \Delta U + \Delta K \tag{8.1.19}$$

The special case where there are no external forces, or where all the external forces are conservative/potential, leads to $0 = \Delta U + \Delta K$, so that the mechanical energy is *constant*. This situation occurs, for example, for a body in free-fall {▲Problem 3} and for a freely oscillating spring {▲Problem 4}. Both forms of the work-energy principle can also be seen to apply for a spring subjected to an external force {▲Problem 5}.

[1] it is conventional to keep work terms on the left and energy terms on the right

The Principle of Conservation of Mechanical Energy

The **principle of conservation of energy** states that the total energy of a system remains constant – energy cannot be created or destroyed, it can only be changed from one form of energy to another.

The principle of conservation of energy in the case where there is no energy dissipation is called the **principle of conservation of mechanical energy** and states that, *if a system is subject only to conservative forces, its mechanical energy remains constant*; any system in which non-conservative forces act will inevitable involve non-mechanical energy (heat transfer).

So, when there are only conservative forces acting, one has

$$0 = \Delta U + \Delta K \tag{8.1.20}$$

or, equivalently,

$$K_f + U_f = K_i + U_i \tag{8.1.21}$$

where K_i, K_f are the initial and final kinetic energies and U_i, U_f are the initial and final potential energies.

Note that the principle of mechanical energy conservation is not a new separate law of mechanics, it is merely a re-expression of the work-energy principle (or of Newton's second law).

8.1.4 Deforming Materials

The discussion above which concerned particle mechanics is now generalized to that of a deforming material.

Any material consists of many molecules and particles, all interacting in some complex way. There will be a complex system of **internal forces** acting between the molecules, even when the material is in a natural (undeformed) equilibrium state. If **external forces** are applied, the material will deform and the molecules will move, and hence not only will work be done by the external forces, but *work will be done by the internal forces*. The work-energy principle in this case states that the total work done by the external and internal forces equals the change in kinetic energy,

$$W_{\text{ext}} + W_{\text{int}} = \Delta K \tag{8.1.22}$$

In the special case where no external forces act on the system, one has

$$W_{\text{int}} = \Delta K \tag{8.1.23}$$

which is a situation known as **free vibration**. The case where the kinetic energy is unchanging is

$$W_{\mathrm{ext}} + W_{\mathrm{int}} = 0 \tag{8.1.24}$$

and this situation is known as **quasi-static** (the quantities here can still depend on time).

The force interaction between the molecules can be grouped into:
(1) conservative internal force systems
(2) non-conservative internal force systems (or at least partly non-conservative)

Conservative Internal Forces

First, assuming a conservative internal force system, one can imagine that the molecules interact with each other in the manner of elastic springs. Suppose one could apply an external force to pull two of these molecules apart, as shown in Fig. 8.1.6.

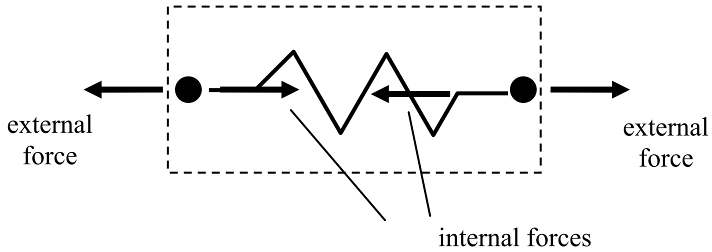


Figure 8.1.6: external force pulling two molecules/particles apart

In this ideal situation one can say that the work done by the external forces equals the change in potential energy plus the change in kinetic energy,

$$W_{\mathrm{ext}} = \Delta U + \Delta K \tag{8.1.25}$$

The energy U in this case of deforming materials is called the **elastic strain energy**, the energy due to the molecular arrangement relative to some equilibrium position.

The free vibration case is now $0 = \Delta U + \Delta K$ and the quasi-static situation is $W_{\mathrm{ext}} = \Delta U$.

Non-Conservative Internal Forces

Consider now another example of internal forces acting within materials, that of a polymer with long-chain molecules. If one could somehow apply an external force to a pair of these molecules, as shown in Fig. 8.1.7, the molecules would slide over each other. Frictional forces would act between the molecules, very much like the frictional force between the block and rough surface of Fig. 8.1.4. This is called **internal friction**. Assuming that the internal forces are dissipative, the external work cannot be written in terms of a potential energy, $W_{\mathrm{ext}} \neq \Delta U + \Delta K$, since *the work done depends on the path taken*. One would have to calculate the work done by evaluating an integral.

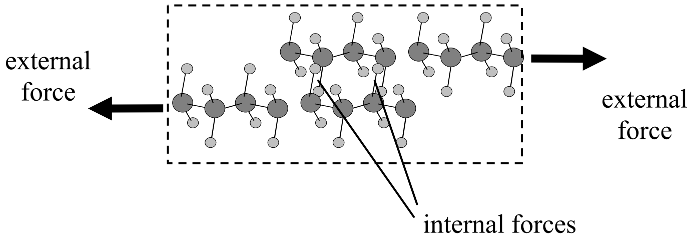


Figure 8.1.7: external force pulling two molecules/particles apart

Similar to Eqn. 8.1.17, however, the energy balance can be written as

$$W_{\text{ext}} - H = \Delta K + \Delta U \tag{8.1.26}$$

where H is the energy dissipated during the deformation *and will depend on the precise deformation process*. This energy is dissipated through heat transfer and is conducted away through the material.

8.1.5 Energy Methods

The work-energy principle provides a method for obtaining solutions to conservative static problems and will be pursued in the next section. The principle is one of a number of tools which can be grouped under the heading **Energy Methods**, such as Castigliano's theorems and the Crotti-Engesser theorem (see later). These methods can be used to solve a wide range of problems involving elastic (linear or non-linear) materials.

Virtual work methods are closely related to energy methods and provide powerful means for solving problems whether they involve elastic materials or not; for the case of elastic materials, they lead naturally to the principle of minimum potential energy discussed in a later section. These virtual work methods will be discussed in sections 8.5-8.6.

8.1.6 Problems

1. Consider the conservative force field
$$F = \frac{1}{x^4} - \frac{1}{x^2}$$
What is the potential energy of a particle at some position $x = x_1$ (define the point at infinity to be the reference point)? What work is done by F as the particle moves from the reference point to $x = x_1$? What is the work done by the applied force which moves the particle from the reference point to $x = x_1$?

2. Consider the gravitational force field mg. Consider a body acted upon by its weight $w = mg$ and by an equal and opposite upward force F (arising, for instance, in a string). Suppose the weight to be moved at slow speed from one position to another one (so that there is no acceleration and $F = -w$). Calculate the work done by F and

show that it is independent of the path taken. What is the potential energy of the body? What is the work done by the gravitational force?

3. Show that both forms of the work-energy principle, Eqns. 8.1.18, 8.1.19, hold for a body in free-fall and that the total mechanical energy is constant. (Use Newton's second law with *x* positive up, initial height *h* and zero initial velocity.)

4. Consider a mass *m* attached to a freely oscillating spring, at initial position x_0 and with initial velocity $\dot{x}_0$. Use Newton's second law to show that
$$x = x_0 \cos\omega t + (\dot{x}_0 / \omega)\sin\omega t$$
$$\dot{x} = \omega\left(-x_0 \sin\omega t + (\dot{x}_0 / \omega)\cos\omega t\right)$$
where $\omega = \sqrt{k/m}$. Show that both forms of the work-energy principle, Eqns. 8.1.18, 8.1.19, hold for the mass and that the total mechanical energy is constant.

5. Consider the case of an oscillating mass *m* attached to a spring with a *constant* force *F* applied to the mass. From Newton's second law, one has $m\ddot{x} = -kx + F$ which can be solved to obtain
$$x = \left(x_0 - \frac{F}{k}\right)\cos\omega t + (\dot{x}_0 / \omega)\sin\omega t + \frac{F}{k}$$
$$\dot{x} = \omega\left\{-\left(x_0 - \frac{F}{k}\right)\sin\omega t + (\dot{x}_0 / \omega)\cos\omega t\right\}$$
Evaluate the change in kinetic energy and the total work done by the applied force to show that $W = \Delta K$. Show also that the total work done by the applied force, exclusive of the conservative spring force, is equivalent to $\Delta U + \Delta K$.

6. Consider a body dragged a distance *s* along a rough horizontal surface by a force *F*, Fig. 8.1.4. By Newton's second law, $F - F_{fr} = m\ddot{x}$. By directly integrating this equation twice and letting the initial position and velocity of the body be x_0 and $\dot{x}_0$ respectively, show that the work done and the change in kinetic energy of the block are both given by
$$(F - F_{fr})\left\{\frac{1}{2}\left(\frac{F - F_{fr}}{m}\right)t^2 + \dot{x}_0 t\right\}$$
so that the principle of work and kinetic energy holds. How much energy is dissipated?

8.2 Elastic Strain Energy

The strain energy stored in an elastic material upon deformation is calculated below for a number of different geometries and loading conditions. These expressions for stored energy will then be used to solve some elasticity problems using the energy methods mentioned in the previous section.

8.2.1 Strain energy in deformed Components

Bar under axial load

Consider a bar of elastic material fixed at one end and subjected to a steadily increasing force *P*, Fig. 8.2.1. The force is applied slowly so that kinetic energies are negligible. The initial length of the bar is L. The work dW done in extending the bar a small amount $d\Delta$ is[1]

$$dW = Pd\Delta \tag{8.2.1}$$

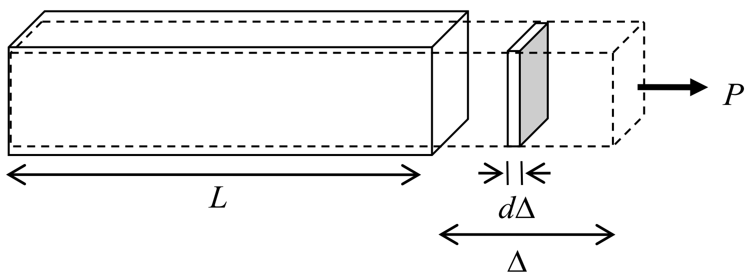


Figure 8.2.1: a bar loaded by a force

It was shown in §7.1.2 that the force and extension Δ are linearly related through $\Delta = PL/EA$, Eqn. 7.1.5, where *E* is the Young's modulus and *A* is the cross sectional area. This linear relationship is plotted in Fig. 8.2.2. The work expressed by Eqn. 8.2.1 is the white region under the force-extension curve (line). The total work done during the complete extension up to a *final* force *P* and *final* extension Δ is the total area beneath the curve.

The work done is stored as elastic strain energy *U* and so

$$U = \frac{1}{2}P\Delta = \frac{P^2L}{2EA} \tag{8.2.2}$$

If the axial force (and/or the cross-sectional area and Young's modulus) varies along the bar, then the above calculation can be done for a small element of length dx. The energy stored in this element would be $P^2dx/2EA$ and the total strain energy stored in the bar would be

[1] the small change in force d*P* which occurs during this small extension may be neglected, since it will result in a smaller-order term of the form d*P*dΔ

$$U = \int_0^L \frac{P^2}{2EA} dx \tag{8.2.3}$$

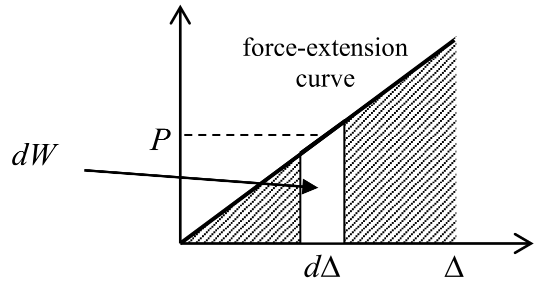


Figure 8.2.2: force-displacement curve for uniaxial load

The strain energy is always positive, due to the square on the force *P*, regardless of whether the bar is being compressed or elongated.

Note the factor of one half in Eqn. 8.2.2. The energy stored is not simply force times displacement because *the force is changing* during the deformation.

Circular Bar in Torsion

Consider a circular bar subjected to a torque *T*. The torque is equivalent to a couple: two forces of magnitude *F* acting in opposite directions and separated by a distance $2r$ as in Fig. 8.2.3; $T = 2Fr$. As the bar twists through a small angle $\Delta\phi$, the forces each move through a distance $\Delta s = r\Delta\phi$. The work done is therefore $\Delta W = 2\left(F\Delta s\right) = T\Delta\phi$.

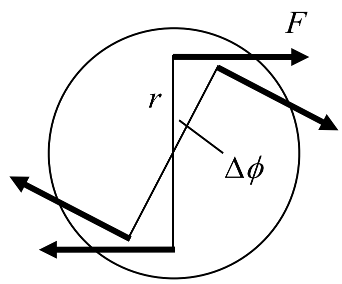


Figure 8.2.3: torque acting on a circular bar

It was shown in §7.2 that the torque and angle of twist are linearly related through Eqn. 7.2.10, $\phi = TL/GJ$, where *L* is the length of the bar, *G* is the shear modulus and *J* is the polar moment of inertia. The angle of twist can be plotted against the torque as in Fig. 8.2.4.

The total strain energy stored in the cylinder during the straining up to a final angle of twist ϕ is the work done, equal to the shaded area in Fig. 8.2.4, leading to

$$U = \frac{1}{2}\phi T = \frac{T^2 L}{2GJ} \tag{8.2.4}$$

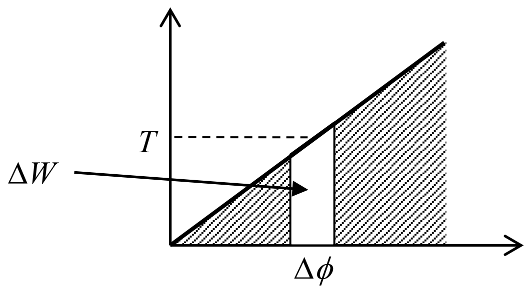


Figure 8.2.4: torque – angle of twist plot for torsion

Again, if the various quantities are varying along the length of the bar, then the total strain energy can be expressed as

$$U = \int_0^L \frac{T^2}{2GJ} dx \tag{8.2.5}$$

Beam subjected to a Pure Moment

As with the bar under torsion, the work done by a moment M as it moves through an angle $d\theta$ is $Md\theta$. The moment is related to the radius of curvature R through Eqns. 7.4.36-37, $M = EI/R$, where E is the Young's modulus and I is the moment of inertia. The length L of a beam and the angle subtended θ are related to R through $L = R\theta$, Fig. 8.2.5, and so moment and angle θ are linearly related through $\theta = ML/EI$.

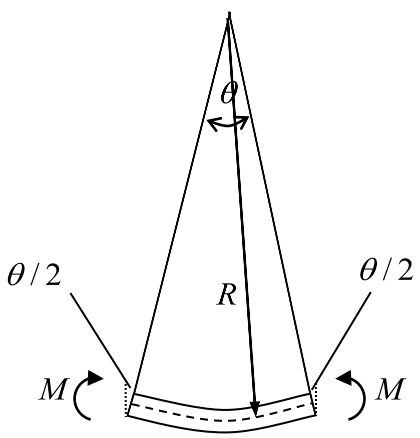


Figure 8.2.5: beam of length L under pure bending

The total strain energy stored in a bending beam is then

$$U = \frac{1}{2}\theta M = \frac{M^2 L}{2EI} \tag{8.2.6}$$

and if the moment and other quantities vary along the beam,

$$U = \int_0^L \frac{M^2}{2EI} dx \tag{8.2.7}$$

This expression is due to the flexural stress σ. A beam can also store energy due to shear stress τ; this latter energy is usually much less than that due to the flexural stresses provided the beam is slender – this is discussed further below.

Example

Consider the bar with varying circular cross-section shown in Fig. 8.2.6. The Young's modulus is 200GPa.

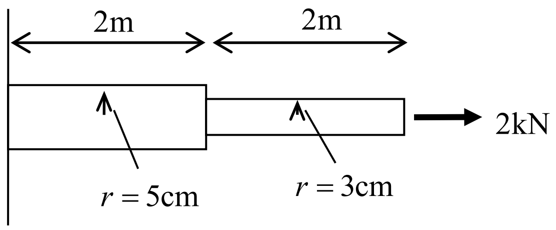


Figure 8.2.6: a loaded bar

The strain energy stored in the bar when a force of 2kN is applied at the free end is

$$U = \int_0^L \frac{P^2}{2EA} dx = \frac{\left(2\times 10^3\right)^2 (2)}{2\left(2\times 10^{11}\right)\pi}\left(\frac{1}{\left(5\times 10^{-2}\right)^2} + \frac{1}{\left(3\times 10^{-2}\right)^2}\right) = 9.62\times 10^{-3}\ \text{Nm} \tag{8.2.8}$$

■

8.2.2 The Work-Energy Principle

The work-energy principle for elastic materials, that is, the fact that the work done by external forces is stored as elastic energy, can be used directly to solve some simple problems. To be precise, it can be used to solve problems involving a single force and for solving for the displacement in the direction of that force. By force and displacement here it is meant **generalised force** and **generalised displacement**, that is, a force/displacement pair, a torque/angle of twist pair or a moment/bending angle pair.

More complex problems need to be solved using more sophisticated energy methods, such as Castigliano's method discussed further below.

Example

Consider the beam of length *L* shown in Fig. 8.2.7, pinned at one end (A) and simply supported at the other (C). A moment M_0 acts at B, a distance L_1 from the left-hand end. The cross-section is rectangular with depth *b* and height *h*. The work-energy principle can be used to calculate the angle θ_B through which the moment at B rotates.

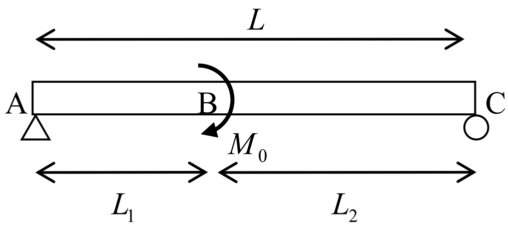


Figure 8.2.7: a beam subjected to a moment at B

The moment along the beam can be calculated from force and moment equilibrium,

$$M = \begin{cases} -M_0 x/L, & 0 < x < L_1 \\ M_0(1 - x/L), & L_1 < x < L \end{cases} \tag{8.2.9}$$

The strain energy stored in the bar (due to the flexural stresses only) is

$$U = \int_0^L \frac{M^2}{2EI} dx = \frac{6M_0^2}{Ebh^3} \left\{ \int_0^{L_1} \left(\frac{x}{L} \right)^2 dx + \int_{L_1}^{L} \left(1 - \frac{x}{L} \right)^2 dx \right\} = \frac{6M_0^2 L}{Ebh^3} \left[\frac{1}{3} - \left(\frac{L_1}{L} \right) - \left(\frac{L_1}{L} \right)^2 \right] \tag{8.2.10}$$

The work done by the applied moment is $M_0 \theta_B / 2$ and so

$$\theta_B = \frac{12 M_0 L}{Ebh^3} \left[\frac{1}{3} - \left(\frac{L_1}{L} \right) - \left(\frac{L_1}{L} \right)^2 \right] \tag{8.2.11}$$

■

8.2.3 Strain Energy Density

The strain energy will in general vary throughout a body and for this reason it is useful to introduce the concept of **strain energy density**, which is a measure of how much energy is stored in small volume elements throughout a material.

Consider again a bar subjected to a uniaxial force *P*. A small volume element with edges aligned with the x, y, z axes as shown in Fig. 8.2.8 will then be subjected to a stress σ_{xx} only. The volume of the element is $dV = dxdydz$.

From Eqn. 8.2.2, the strain energy in the element is

$$U = \frac{(\sigma_{xx} dydz)^2 dx}{2Edydz} \tag{8.2.12}$$

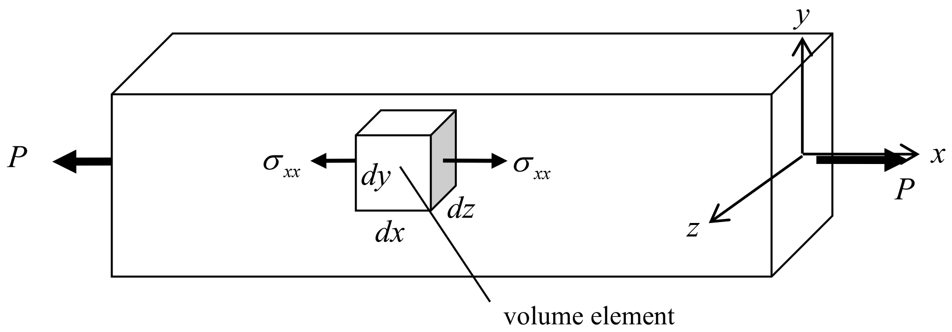


Figure 8.2.8: a volume element under stress

The strain energy density u is defined as the strain energy *per unit volume*:

$$u = \frac{\sigma_{xx}^2}{2E} \tag{8.2.13}$$

The total strain energy in the bar may now be expressed as this quantity integrated over the whole volume,

$$U = \int_V u dV , \tag{8.2.14}$$

which, for a constant cross-section A and length L reads $U = A\int_0^L u dx$. From Hooke's law, the strain energy density of Eqn. 8.2.13 can also be expressed as

$$u = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} \tag{8.2.15}$$

As can be seen from Fig. 8.2.9, this is the area under the uniaxial stress-strain curve.

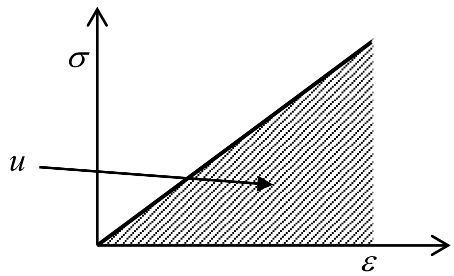


Figure 8.2.9: stress-strain curve for elastic material

Note that the element *does* deform in the y and z directions but no work is associated with those displacements since there is no force acting in those directions.

The strain energy density for an element subjected to a σ_{yy} stress only is, by the same arguments, $\sigma_{yy}\varepsilon_{yy}/2$, and that due to a σ_{zz} stress is $\sigma_{zz}\varepsilon_{zz}/2$. Consider next a shear stress σ_{xy} acting on the volume element to produce a shear strain ε_{xy} as illustrated in Fig. 8.2.10. The element deforms with small angles θ and λ as illustrated. Only the stresses on the upper and right-hand surfaces are shown, since the stresses on the other two surfaces do no work. The force acting on the upper surface is $\sigma_{xy}dxdz$ and moves through a displacement λdy. The force acting on the right-hand surface is $\sigma_{xy}dydz$ and moves through a displacement θdx. The work done when the element moves through angles $d\theta$ and $d\lambda$ is then, using the definition of shear strain,

$$dW = (\sigma_{xy}dxdz)(d\lambda dy) + (\sigma_{xy}dydz)(d\theta dx) = (dxdydz)\sigma_{xy}(2d\varepsilon_{xy}) \qquad (8.2.16)$$

and, with shear stress proportional to shear strain, the strain energy density is

$$u = 2\int \sigma_{xy}d\varepsilon_{xy} = \sigma_{xy}\varepsilon_{xy} \qquad (8.2.17)$$

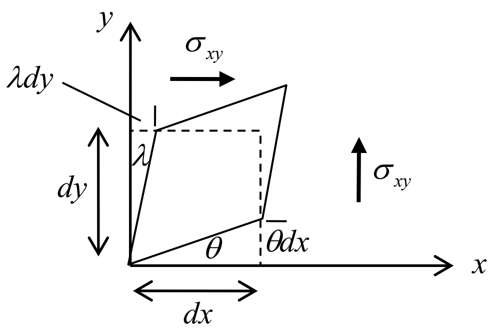


Figure 8.2.10: a volume element under shear stress

The strain energy can be similarly calculated for the other shear stresses and, in summary, the strain energy density for a volume element subjected to arbitrary stresses is

$$u = \frac{1}{2}(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz}) + (\sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx}) \qquad (8.2.18)$$

Using Hooke's law, Eqns. 6.1.9, and Eqn. 6.1.5, the strain energy density can also be written in the alternative and useful forms {▲Problem 4}

$$\begin{aligned} u &= \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\nu}{E}(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \frac{1}{2\mu}(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2) \\ &= \frac{\mu}{1-2\nu}\left[(1-\nu)(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + 2\nu(\varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{yy}\varepsilon_{zz} + \varepsilon_{zz}\varepsilon_{xx})\right] + 2\mu(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2) \\ &= \frac{\nu\mu}{1-2\nu}(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz})^2 + \mu(\varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{zz}^2) + 2\mu(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2) \end{aligned} \qquad (8.2.19)$$

Strain Energy in a Beam due to Shear Stress

The shear stresses arising in a beam at location y from the neutral axis are given by Eqn. 7.4.28, $\tau(y) = Q(y)V / Ib(y)$, where Q is the first moment of area of the section of beam from y to the outer surface, V is the shear force, I is the moment of inertia of the complete cross-section and b is the thickness of the beam at y. From Eqns. 8.2.19a and 8.2.14 then, the total strain energy in a beam of length L due to shear stress is

$$U = \int_V \frac{\tau^2}{2\mu} dV = \frac{1}{2}\int_0^L \frac{V^2}{\mu I^2}\left[\int_A \frac{Q^2}{b^2} dA\right] dx \tag{8.2.20}$$

Here V, μ and I are taken to be constant for any given cross-section but may vary along the beam; Q varies and b may vary over any given cross-section. Expression 8.2.20 can be simplified by introducing the **form factor for shear** f_s, defined as

$$f_s(x) = \frac{A}{I^2}\int_A \frac{Q^2}{b^2} dA \tag{8.2.21}$$

so that

$$U = \frac{1}{2}\int_0^L \frac{f_s V^2}{\mu A} dx \tag{8.2.22}$$

The form factor depends only on the shape of the cross-section. For example, for a rectangular cross-section, using Eqn. 7.4.29,

$$f_s(x) = \frac{bh}{\left(bh^3/12\right)^2}\int_{-h/2}^{+h/2} \frac{1}{b^2}\left[\frac{b}{2}\left(\frac{h^2}{4} - y^2\right)\right]^2 dy \int_{-b/2}^{+b/2} dz = \frac{6}{5} \tag{8.2.23}$$

In a similar manner, the form factor for a circular cross-section is found to be $10/9$ and that of a very thin tube is 2.

8.2.4 Castigliano's Second Theorem

The work-energy method is the simplest of energy methods. A more powerful method is that based on **Castigliano's second theorem**[2], which can be used to solve problems involving *linear* elastic materials. As an introduction to Castigliano's second theorem, consider the case of uniaxial tension, where $U = P^2 L / 2EA$. The displacement through which the force moves can be obtained by a differentiation of this expression with respect to that force,

$$\frac{dU}{dP} = \frac{PL}{EA} = \Delta \tag{8.2.24}$$

[2] Casigliano's first theorem will be discussed in a later section

Similarly, for torsion of a circular bar, $U = T^2 L / 2GJ$, and a differentiation gives $dU / dT = TL / GJ = \phi$. Further, for bending of a beam it is also seen that $dU / dM = \theta$.

These are examples of Castigliano's theorem, which states that, provided the body is in equilibrium, *the derivative of the strain energy with respect to the force gives the displacement corresponding to that force, in the direction of that force*. When there is more than one force applied, then one takes the partial derivative. For example, if *n* independent forces $P_1, P_2, \ldots, P_n$ act on a body, the displacement corresponding to the *i*th force is

$$\Delta_i = \frac{\partial U}{\partial P_i} \tag{8.2.25}$$

Before proving this theorem, here follow some examples.

Example

The beam shown in Fig. 8.2.11 is pinned at A, simply supported half-way along the beam at B and loaded at the end C by a force *P* and a moment M_0.

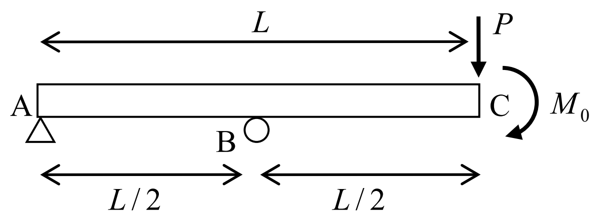


Figure 8.2.11: a beam subjected to a force and moment at C

The moment along the beam can be calculated from force and moment equilibrium,

$$M = \begin{cases} -Px - 2M_0 x / L, & 0 < x < L/2 \\ -M_0 - P(L - x), & L/2 < x < L \end{cases} \tag{8.2.26}$$

The strain energy stored in the bar (due to the flexural stresses only) is

$$\begin{aligned} U &= \frac{1}{2EI}\left\{ \left(P + \frac{2M_0}{L}\right)^2 \int_0^{L/2} x^2 dx + \int_{L/2}^{L} \left(M_0 + P(L - x)\right)^2 dx \right\} \\ &= \frac{P^2 L^3}{24EI} + \frac{5PM_0 L^2}{24EI} + \frac{M_0^2 L}{3EI} \end{aligned} \tag{8.2.27}$$

In order to apply Castigliano's theorem, the strain energy is considered to be a function of the two external loads, $U = U(P, M_0)$. The displacement associated with the force *P* is then

$$\Delta_C = \frac{\partial U}{\partial P} = \frac{PL^3}{12EI} + \frac{5M_0L^2}{24EI} \tag{8.2.28}$$

The rotation associated with the moment is

$$\theta_C = \frac{\partial U}{\partial M_0} = \frac{5PL^2}{24EI} + \frac{2M_0L}{3EI} \tag{8.2.29}$$

■

Example

Consider next the beam of length *L* shown in Fig. 8.2.12, built in at both ends and loaded centrally by a force *P*. This is a statically indeterminate problem. In this case, the strain energy can be written as a function of the applied load and one of the unknown reactions.

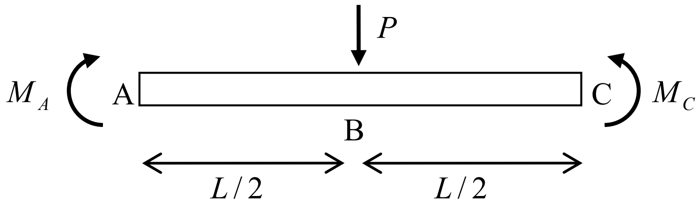


Figure 8.2.12: a statically indeterminate beam

First, the moment in the beam is found from equilibrium considerations to be

$$M = M_A + \frac{P}{2}x, \qquad 0 < x < L/2 \tag{8.2.30}$$

where M_A is the unknown reaction at the left-hand end. Then the strain energy in the left-hand half of the beam is

$$U = \frac{1}{2EI}\int_0^{L/2}\left(M_A + \frac{P}{2}x\right)^2 dx = \frac{P^2L^3}{192EI} + \frac{PM_AL^2}{16EI} + \frac{M_A^2L}{4EI} \tag{8.2.31}$$

The strain energy in the complete beam is double this:

$$U = \frac{P^2L^3}{96EI} + \frac{PM_AL^2}{8EI} + \frac{M_A^2L}{2EI} \tag{8.2.32}$$

Writing the strain energy as $U = U(P, M_A)$, the rotation at A is

$$\theta_A = \frac{\partial U}{\partial M_A} = \frac{PL^2}{8EI} + \frac{M_AL}{EI} \tag{8.2.33}$$

But $\theta_A = 0$ and so Eqn. 8.2.33 can be solved to get $M_A = -PL/8$. Then the displacement at the centre of the beam is

$$\Delta_B = \frac{\partial U}{\partial P} = \frac{PL^3}{48EI} + \frac{M_A L^2}{8EI} = \frac{PL^3}{192EI} \qquad (8.2.34)$$

This is positive in the direction in which the associated force is acting, and so is downward.

■

Proof of Castigliano's Theorem

A proof of Castiligliano's theorem will be given here for a structure subjected to a single load. The load P produces a displacement Δ and the strain energy is $U = P\Delta/2$, Fig. 8.2.13. If an additional force dP is applied giving an additional deformation $d\Delta$, the additional strain energy is

$$dU = Pd\Delta + \frac{1}{2}dPd\Delta \qquad (8.2.35)$$

If the load $P + dP$ is applied from zero in one step, the work done is $(P + dP)(\Delta + d\Delta)/2$. Equating this to the strain energy $U + dU$ given by Eqn. 8.2.35 then gives $Pd\Delta = \Delta dP$. Substituting into Eqn. 8.2.35 leads to

$$dU = \Delta dP + \frac{1}{2}dPd\Delta \qquad (8.2.36)$$

Dividing through by dP and taking the limit as $d\Delta \to 0$ results in Castigliano's second theorem, $dU/dP = \Delta$.

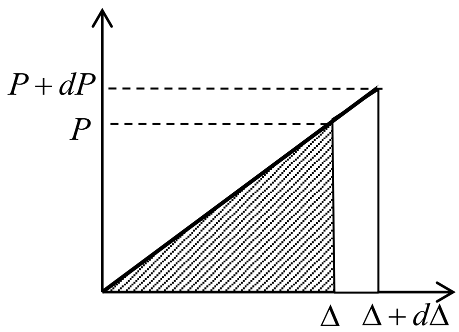


Figure 8.2.13: force-displacement curve

In fact, dividing Eqn. 8.2.35 through by $d\Delta$ and taking the limit as $d\Delta \to 0$ results in **Castigliano's first theorem**, $dU/d\Delta = P$. It will be shown later that this first theorem, unlike the second, in fact holds also for the case when the elastic material is *non-linear*.

8.2.5 Dynamic Elasticiy

Impact and Dynamic Loading

Consider the case of a weight *P* dropped instantaneously onto the end of an elastic bar. If the weight *P* had been applied gradually from zero, the strain energy stored at the final force *P* and final displacement Δ_0 would be $\frac{1}{2}P\Delta_0$. However, the instantaneously applied load is constant throughout the deformation and work done up to a displacement Δ_0 is $P\Delta_0$, Fig. 8.2.14. The difference between the two implies that the bar acquires a kinetic energy (see Eqn. 8.1.19); the material particles accelerate from their equilibrium positions during the compression.

As deformation proceeds beyond Δ_0, it is clear from Fig. 8.2.14 that the strain energy is increasing faster than the work being done by the weight and so there must be a drop in kinetic energy; the particles begin to decelerate. Eventually, at $\Delta_{max} = 2\Delta_0$, the work done by the weight exactly equals the strain energy stored and the material is at rest. However, the material is not in equilibrium – the equilibrium position for a load *P* is Δ_0 – and so the material begins to accelerate back to Δ_0.

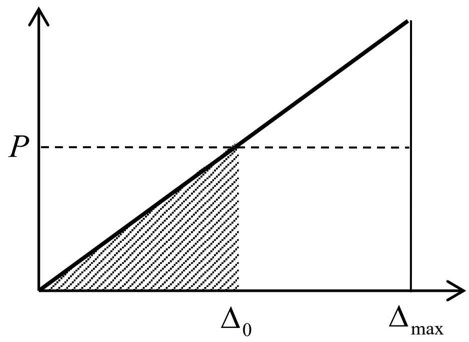


Figure 8.2.14: non-equilibrium loading

The bar and weight will continue to oscillate between 0 and Δ_{max} indefinitely. In a real (inelastic) material, internal friction will cause the vibration to decay.

Thus the maximum compression of a bar under impact loading is twice that of a bar subjected to the same load gradually.

Example

Consider a weight *w* dropped from a height *h*. If one is interested in the final, maximum, displacement of the bar, Δ_{max}, one does not need to know about the detailed and complex transfer of energies during the impact; the energy lost by the weight equals the strain energy stored in the bar:

$$w(h + \Delta_{max}) = \frac{1}{2}P\Delta_{max} \tag{8.2.37}$$

where P is the force acting on the bar at its maximum compression. For an elastic bar, $P = \Delta_{max} EA / L$, or, introducing the **stiffness** k so that $P = k\Delta_{max}$,

$$w(h + \Delta_{max}) = \frac{1}{2} k\Delta_{max}^2, \qquad k = \frac{EA}{L} \tag{8.2.38}$$

which is a quadratic equation in Δ_{max} and can be solved to get

$$\Delta_{max} = \frac{w}{k}\left\{1 + \sqrt{1 + \frac{2hk}{w}}\right\} \tag{8.2.39}$$

If the force w had been applied gradually, then the displacement would have been $\Delta_{st} = w / k$, the "st" standing for "static", and Eqn. 8.2.39 can be re-written as

$$\Delta_{max} = \Delta_{st}\left\{1 + \sqrt{1 + \frac{2h}{\Delta_{st}}}\right\} \tag{8.2.40}$$

If $h = 0$, so that the weight is just touching the bar when released, then $\Delta_{max} = 2\Delta_{st}$.

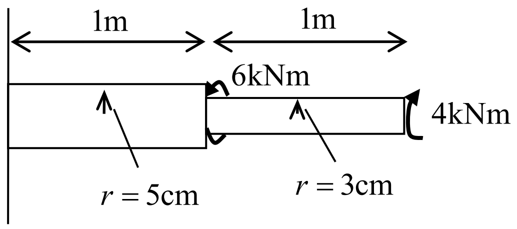
■

8.2.6 Problems

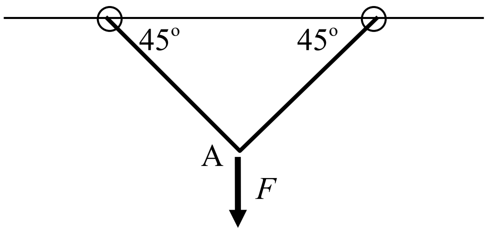
1. Show that the strain energy in a bar of length L and cross sectional area A hanging from a ceiling and subjected to its own weight is given by (at any section, the force acting is the weight of the material below that section)

$$U = \frac{A\rho^2 g^2 L^3}{6E}$$

2. Consider the circular bar shown below subject to torques at the free end and where the cross-sectional area changes. The shear modulus is $G = 80\text{GPa}$. Calculate the strain energy in the bar(s).



3. Two bars of equal length L and cross-sectional area A are pin-supported and loaded by a force F as shown below. Derive an expression for the vertical displacement at point A using the direct work-energy method, in terms of L, F, A and the Young's modulus E.

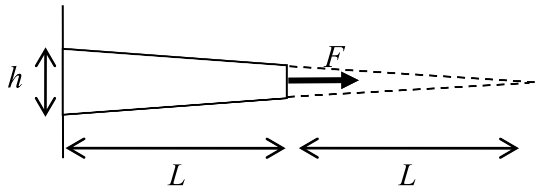


4. Derive the strain energy density equations 8.2.19.

5. For the beam shown in Fig. 8.2.7, use the expression 8.2.22 to calculate the strain energy due to the shear stresses. Take the shear modulus to be $G = 80\text{GPa}$. Compare this with the strain energy due to flexural stress given by Eqn. 8.2.10.

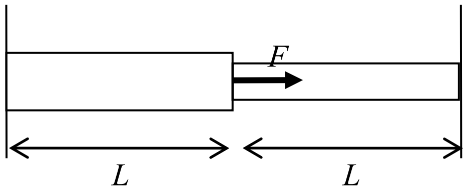
6. Consider a simply supported beam of length L subjected to a uniform load w N/m. Calculate the strain energy due to both flexural stress and shear stress for (a) a rectangular cross-section of depth times height $b \times h$, (b) a circular cross-section with radius r. What is the ratio of the shear-to-flexural strain energies in each case?

7. Consider the tapered bar of length L and square cross-section shown below, built-in at one end and subjected to a uniaxial force F at its free end. The thickness is h at the built-in end. Evaluate the displacement in terms of the (constant) Young's modulus E at the free end using (i) the work-energy theorem, (ii) Castigliano's theorem



8. Consider a cantilevered beam of length L and constant cross-section subjected to a uniform load w N/m. The beam is built-in at $x = 0$ and has a Young's modulus E. Use Castigliano's theorem to calculate the deflection at $x = L$. Consider only the flexural strain energy. [Hint: place a fictitious "dummy" load F at $x = L$ and set to zero once Castigliano's theorem has been applied]

9. Consider the statically indeterminate uniaxial problem shown below, two bars joined at $x = L$, built in at $x = 0$ and $x = 2L$, and subjected to a force F at the join. The cross-sectional area of the bar on the left is $2A$ and that on the right is A. Use (i) the work-energy theorem and (ii) Castigliano's theorem to evaluate the displacement at $x = L$.



8.3 Complementary Energy

The linear elastic solid was considered in the previous section, with the characteristic straight force-deflection curve for axial deformations, Fig.8.2.2. Here, consider the more general case of a bar of *non-linear* elastic material, of length *L*, fixed at one end and subjected to a steadily increasing force *P*. The work dW done in extending the bar a small amount $d\Delta$ is

$$dW = Pd\Delta \,. \tag{8.3.1}$$

Force is now no longer proportional to extension Δ, Fig. 8.3.1. However, the total work done during the complete extension up to a final force *P* and final extension Δ is once again the total area beneath the force-extension curve. The work done is equal to the stored elastic strain energy which must now be expressed as an integral,

$$U = \int_0^\Delta Pd\Delta \tag{8.3.2}$$

The strain energy can be calculated if the precise force-deflection relationship is known.

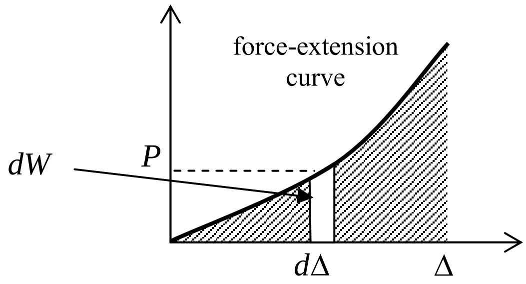


Figure 8.3.1: force-displacement curve for a non-linear material

8.3.1 Complementary Energy

The force-deflection curve is naturally divided into two regions, beneath the curve and above the curve, Fig. 8.3.2. The area of the region under the curve is the strain energy. It is helpful to introduce a new concept, the **complementary energy** *C*, which is the area above the curve; this can be seen to be given by

$$C = \int_0^P \Delta dP \,. \tag{8.3.3}$$

For a linear elastic material, $C = U$. Although *C* has units of energy, it has no real physical meaning.

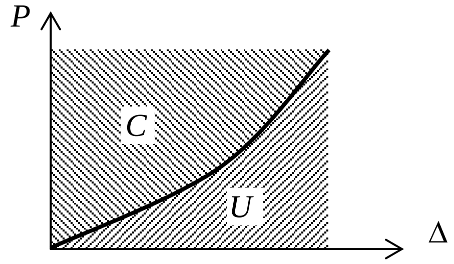


Figure 8.3.2: strain energy and complementary energy for an elastic material

8.3.2 The Crotti-Engesser Theorem

Suppose an elastic body is loaded by *n* independent loads $P_1, P_2, \ldots, P_n$. The strain energy is then the work done by these loads,

$$U = \int_0^{\Delta_1} P_1 d\Delta_1 + \int_0^{\Delta_2} P_2 d\Delta_2 + \cdots + \int_0^{\Delta_n} P_n d\Delta_n \tag{8.3.4}$$

It follows that

$$\frac{\partial U}{\partial \Delta_j} = P_j \tag{8.3.5}$$

which is known as Castigliano's first theorem.

Similarly, the total complementary energy is

$$C = \int_0^{P_1} \Delta_1 dP_1 + \int_0^{P_2} \Delta_2 dP_2 + \cdots + \int_0^{P_n} \Delta_n dP_n \tag{8.3.6}$$

and it follows that

$$\frac{\partial C}{\partial P_j} = \Delta_j \tag{8.3.7}$$

which is known as the **Crotti-Engesser theorem**. For a *linear* elastic material, $C = U$, and the Crotti-Engesser theorem reduces to Castigliano's second theorem, $\Delta_j = \partial U / \partial P_j$, Eqn. 8.2.25.

8.3.3 Problems

1. The force-deflection equation for a non-linear elastic material is given by $P = \alpha \Delta^3$. Find expressions for the strain energy and the complementary energy in terms of (i) P only, (ii) Δ only. Check that $U + C = P\Delta$. What is the ratio C/U?

8.4 Strain Energy Potentials

8.4.1 The Linear Elastic Strain Energy Potential

The strain energy u was introduced in §8.2[1]. From Eqn 8.2.19, the strain energy can be regarded as a function of the strains:

$$\begin{aligned} u &= u(\varepsilon_{ij}) \\ &= \frac{\mu}{1-2\nu}\left[(1-\nu)\left(\varepsilon_{xx}^2+\varepsilon_{yy}^2+\varepsilon_{zz}^2\right)+2\nu\left(\varepsilon_{xx}\varepsilon_{yy}+\varepsilon_{yy}\varepsilon_{zz}+\varepsilon_{zz}\varepsilon_{xx}\right)\right]+2\mu\left(\varepsilon_{xy}^2+\varepsilon_{yz}^2+\varepsilon_{zx}^2\right) \end{aligned} \tag{8.4.1}$$

Differentiating with respect to ε_{xx} (holding the other strains constant),

$$\frac{\partial u}{\partial \varepsilon_{xx}} = \frac{2\mu}{1-2\nu}\left[(1-\nu)(\varepsilon_{xx})+\nu\left(\varepsilon_{yy}+\varepsilon_{zz}\right)\right] \tag{8.4.2}$$

From Hooke's law, Eqn 6.1.9, with Eqn 6.1.5, $\mu = E/[2(1+\nu)]$, the expression on the right is simply σ_{xx}. The strain energy can also be differentiated with respect to the other normal strain components and one has

$$\frac{\partial u}{\partial \varepsilon_{xx}} = \sigma_{xx}, \quad \frac{\partial u}{\partial \varepsilon_{yy}} = \sigma_{yy}, \quad \frac{\partial u}{\partial \varepsilon_{zz}} = \sigma_{zz} \tag{8.4.3}$$

The strain energy is a **potential**, meaning that it provides information through a differentiation. Note the similarity between these equations and the equation relating a conservative force and the potential energy seen in §8.1: $dU/dx = F$.

Differentiating Eqn. 8.4.1 with respect to the shear stresses results in

$$\frac{\partial u}{\partial \varepsilon_{xy}} = 2\sigma_{xy}, \quad \frac{\partial u}{\partial \varepsilon_{yz}} = 2\sigma_{yz}, \quad \frac{\partial u}{\partial \varepsilon_{zx}} = 2\sigma_{zx} \tag{8.4.4}$$

The fact that Eqns. 8.4.4 has the factor of 2 on the right hands side but Eqns. 8.4.3 do not is not ideal. There are two common ways of viewing the strain energy potential to overcome this lack of symmetry. First, the strain energy can be taken to be a function of the *six* independent strains, $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$, the latter three being the engineering shear strains, $\gamma_{xy} = 2\varepsilon_{xy}$, etc. Re-writing Eqn. 8.4.1 in terms of the engineering shear strains then leads to the set of equations {▲Problem 1}

$$\frac{\partial u}{\partial \varepsilon_{xx}} = \sigma_{xx}, \quad \frac{\partial u}{\partial \varepsilon_{yy}} = \sigma_{yy}, \quad \frac{\partial u}{\partial \varepsilon_{zz}} = \sigma_{zz}, \quad \frac{\partial u}{\partial \gamma_{xy}} = \sigma_{xy}, \quad \frac{\partial u}{\partial \gamma_{yz}} = \sigma_{yz}, \quad \frac{\partial u}{\partial \gamma_{zx}} = \sigma_{zx} \tag{8.4.5}$$

[1] strictly speaking, this is the strain energy *density*, but it should be clear from the context whether it is energy per unit volume or not; the word density will often be omitted henceforth for brevity

The second method is to treat the strain energy as a function of *nine* independent strains, the three normal strains and $\varepsilon_{xy}, \varepsilon_{yx}, \varepsilon_{yz}, \varepsilon_{zy}, \varepsilon_{zx}, \varepsilon_{xz}$. In other words the fact that the strains ε_{xy} and ε_{yx} are the same is ignored and the strain energy is differentiated with respect to these as though they were independent. In order to implement this approach, the strain energy needs to be derived anew treating σ_{xy} and σ_{yx} as independent quantities. This simply means that Fig. 8.2.10 is re-drawn as Fig. 8.4.1 below, and Eqn. 8.2.16 is re-expressed using $\varepsilon_{xy} = (\varepsilon_{xy} + \varepsilon_{yx})/2$ as

$$dW = (\sigma_{yx} dxdz)(d\lambda dy) + (\sigma_{xy} dydz)(d\theta dx) = (dxdydz)\left[\sigma_{xy} d\varepsilon_{xy} + \sigma_{yx} d\varepsilon_{yx}\right] \qquad (8.4.6)$$

so that

$$u = \frac{1}{2}\sigma_{xy}\varepsilon_{xy} + \frac{1}{2}\sigma_{yx}\varepsilon_{yx} \qquad (8.4.7)$$

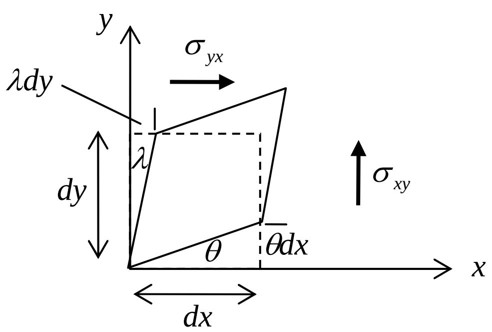


Figure 8.4.1: a volume element under shear stress

Re-writing Eqn. 8.4.1 and differentiation then leads to {▲Problem 2}

$$\frac{\partial u}{\partial \varepsilon_{xx}} = \sigma_{xx}, \quad \frac{\partial u}{\partial \varepsilon_{yy}} = \sigma_{yy}, \quad \frac{\partial u}{\partial \varepsilon_{zz}} = \sigma_{zz},$$
$$\frac{\partial u}{\partial \varepsilon_{xy}} = \sigma_{xy}, \quad \frac{\partial u}{\partial \varepsilon_{yz}} = \sigma_{yz}, \quad \frac{\partial u}{\partial \varepsilon_{zx}} = \sigma_{zx}, \quad \frac{\partial u}{\partial \varepsilon_{yx}} = \sigma_{yx}, \quad \frac{\partial u}{\partial \varepsilon_{zy}} = \sigma_{zy}, \quad \frac{\partial u}{\partial \varepsilon_{xz}} = \sigma_{xz} \qquad (8.4.8)$$

These equations can be expressed in the succinct form

$$\boxed{\frac{\partial u}{\partial \varepsilon_{ij}} = \sigma_{ij}} \qquad (8.4.9)$$

8.4.2 The Elastic Strain Energy Potential

Eqns. 8.4.9 was derived for an isotropic linear elastic material. In fact these equations are valid very generally, for non-linear and not-necessarily isotropic materials. Generalising the above discussion, recall that the strain energy is the area beneath the stress-strain curve, Fig. 8.4.2, and

$$dW = du = \sigma d\varepsilon \tag{8.4.10}$$

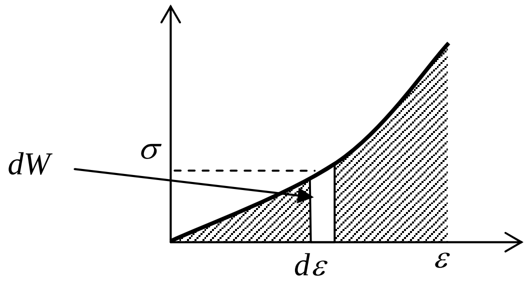


Figure 8.4.2: stress-strain curve for a non-linear material

When the material undergoes increments in strain $d\varepsilon_{xx}$, $d\varepsilon_{xy}$, etc., the increment in strain energy is

$$du = \sigma_{xx} d\varepsilon_{xx} + \sigma_{yy} d\varepsilon_{yy} + \sigma_{xy} d\varepsilon_{xy} + \cdots \tag{8.4.11}$$

If the strain energy is a function of the nine strains ε_{ij}, $u = u(\varepsilon_{ij})$, its increment can also be expressed as

$$du = \frac{\partial u}{\partial \varepsilon_{xx}} d\varepsilon_{xx} + \frac{\partial u}{\partial \varepsilon_{yy}} d\varepsilon_{yy} + \frac{\partial u}{\partial \varepsilon_{xy}} d\varepsilon_{xy} + \cdots \tag{8.4.12}$$

Subtracting Eqns 8.4.12 from 8.4.11 then gives

$$0 = \left(\sigma_{xx} - \frac{\partial u}{\partial \varepsilon_{xx}} \right) d\varepsilon_{xx} + \left(\sigma_{yy} - \frac{\partial u}{\partial \varepsilon_{yy}} \right) d\varepsilon_{yy} + \left(\sigma_{xy} - \frac{\partial u}{\partial \varepsilon_{xy}} \right) d\varepsilon_{xy} + \cdots \tag{8.4.13}$$

Because the strains are independent, that is, any one of them can be adjusted without changing the others, one again arrives at Eqns. 8.4.8-9, only now it has been shown that this result holds very generally.

Note that in the case of an incompressible material, $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0$, so that the strains are *not* independent, and Eqns. 8.4.8-9 must be amended.

8.4.3 Symmetry of the Elastic Stiffness Matrix

Consider again the generalised Hooke's Law, Eqn. 6.3.1. Using Eqn. 8.4.9:

$$\frac{\partial}{\partial \varepsilon_{xy}} \left(\frac{\partial u}{\partial \varepsilon_{xx}} \right) = \frac{\partial}{\partial \varepsilon_{xy}} (\sigma_{xx}) = C_{16}, \qquad \frac{\partial}{\partial \varepsilon_{xx}} \left(\frac{\partial u}{\partial \varepsilon_{xy}} \right) = \frac{\partial}{\partial \varepsilon_{xx}} (\sigma_{xy}) = C_{61} \tag{8.4.14}$$

Since the order of partial differentiation for these second partial derivatives should be immaterial, it follows that $C_{16} = C_{61}$. Following the same procedure for the rest of the stresses and strains, it can be seen that the stiffness matrix in Eqn. 6.3.1 is *symmetric* and so there are only 21 independent elastic constants in the most general case of anisotropic elasticity.

8.4.4 The Complementary Energy Potential

Analogous to Eqns. 8.4.10-13, an increment in complementary energy density can be expressed as

$$dc = \varepsilon d\sigma \tag{8.4.15}$$

with

$$dc = \varepsilon_{xx} d\sigma_{xx} + \varepsilon_{yy} d\sigma_{yy} + \varepsilon_{xy} d\sigma_{xy} + \cdots \tag{8.4.16}$$

and

$$dc = \frac{\partial c}{\partial \sigma_{xx}} d\sigma_{xx} + \frac{\partial c}{\partial \sigma_{yy}} d\sigma_{yy} + \frac{\partial c}{\partial \sigma_{xy}} d\sigma_{xy} + \cdots \tag{8.4.17}$$

so that

$$\left(\varepsilon_{xx} - \frac{\partial u}{\partial \sigma_{xx}} \right) d\sigma_{xx} + \left(\varepsilon_{yy} - \frac{\partial u}{\partial \sigma_{yy}} \right) d\sigma_{yy} + \left(\varepsilon_{xy} - \frac{\partial u}{\partial \sigma_{xy}} \right) d\sigma_{xy} + \cdots \tag{8.4.18}$$

With the stresses independent, one has an expression analogous to 8.4.9,

$$\boxed{\frac{\partial c}{\partial \sigma_{ij}} = \varepsilon_{ij}} \tag{8.4.19}$$

8.4.5 Problems

1. Derive equations 8.4.5

2. Derive equations 8.4.8

8.5 Virtual Work

Consider a mass attached to a spring and pulled by an applied force F_{apl}, Fig. 8.5.1a. When the mass is in equilibrium, $F_{spr} + F_{apl} = 0$, where $F_{spr} = -kx$ is the spring force with x the distance from the spring reference position.

(a)

(b)

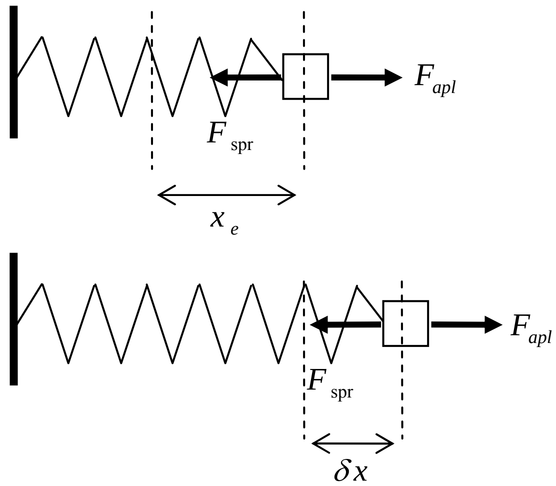


Figure 8.5.1: a force extending an elastic spring; (a) block in equilibrium, (b) block not at its equilibrium position

In order to develop a number of powerful techniques based on a concept known as **virtual work**, imagine that the mass is not in fact at its equilibrium position but at an (incorrect) non-equilibrium position $x + \delta x$, Fig. 8.5.1b. The imaginary displacement δx is called a **virtual displacement**. Define the *virtual work* δW done by a force to be the equilibrium force times this small imaginary displacement δx. It should be emphasized that virtual work is not real work – no work has been performed since δx is not a real displacement which has taken place; this is more like a "thought experiment". The virtual work of the spring force is then $\delta W_{spr} = F_{spr}\delta x = -kx\delta x$. The virtual work of the applied force is $\delta W_{apl} = F_{apl}\delta x$. The total virtual work is

$$\delta W = \delta W_{spr} + \delta W_{apl} = \left(-kx + F_{apl}\right)\delta x \tag{8.5.1}$$

There are two ways of viewing this expression. First, if the system is in equilibrium ($-kx + F_{apl} = 0$) then the virtual work is zero, $\delta W = 0$. Alternatively, if the virtual work is zero then, since δx is arbitrary, the system must be in equilibrium. Thus the virtual work idea gives one an alternative means of determining whether a system is in equilibrium.

The symbol δ is called a **variation** so that, for example, δx is a *variation in the displacement* (from equilibrium).

Virtual work is explored further in the following section.

8.5.1 Principle of Virtual Work: a single particle

A particle of mass m is acted upon by a number of forces, $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_N$, Fig. 8.5.2. Suppose the particle undergoes a virtual displacement $\delta\mathbf{u}$; to reiterate, these impressed forces $\mathbf{f}_i$ do not *cause* the particle to move, one imagines it to be incorrectly positioned a little away from the true equilibrium position.

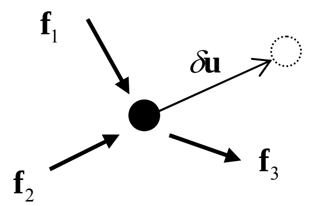


Figure 8.5.2: a particle in equilibrium under the action of a number of forces

If the particle is moving with an acceleration **a**, the quantity $-m\mathbf{a}$ is treated as an inertial force. The total virtual work is then (each term here is the dot product of two vectors)

$$\delta W = \left(\sum_{i=1}^{N} \mathbf{f}_i - m\mathbf{a} \right) \cdot \delta\mathbf{u} \tag{8.5.2}$$

Now *if* the particle is in equilibrium by the action of the effective (impressed plus inertial) force, then

$$\delta W = 0 \tag{8.5.3}$$

This can be expressed as follows:

The principle of virtual work (or **principle of virtual displacements**) **I**: if a particle is in equilibrium under the action of a number of forces (including the inertial force) the total work done by the forces for a virtual displacement is zero

Alternatively, one can define the external virtual work $\delta W_{\text{ext}} = \sum \mathbf{f}_i \cdot \delta\mathbf{u}$ and the virtual kinetic energy $\delta K = m\mathbf{a} \cdot \delta\mathbf{u}$ in which case the principle takes the form $\delta W_{\text{ext}} = \delta K$ (compare with the work-energy principle, Eqn. 8.1.10).

In the above, the principle of virtual work was derived using Newton's second law. One could just as well regard the principle of virtual work as the fundamental principle and from it derive the conditions for equilibrium. In this case one can say that[1]

[1] note the word *any* here: this must hold for *all* possible virtual displacements, for it will always be possible to find one virtual displacement which is perpendicular to the resultant of the forces, so that $\left(\sum \mathbf{f}\right) \cdot \delta\mathbf{u} = 0$ even though $\sum \mathbf{f}$ is not necessarily zero

The principle of virtual work (or **principle of virtual displacements**) **II**:
a particle is in equilibrium under the action of a system of forces (including the inertial force) if the total work done by the forces is zero for any virtual displacement of the particle

Constraints

In many practical problems, the particle will usually be constrained to move in only certain directions. For example consider a ball rolling over a table, Fig. 8.5.3. If the ball is in equilibrium then all the forces sum to zero, $\mathbf{R} + \sum \mathbf{f} - m\mathbf{a} = \mathbf{0}$, where one distinguishes between the non-reaction forces $\mathbf{f}_i$ and the reaction force **R**. If the virtual displacement $\delta\mathbf{u}$ is such that the constraint is not violated, that is the ball is not allowed to go "through" the table, then $\delta\mathbf{u}$ and **R** are perpendicular, the virtual work done by the reaction force is zero and $\delta W = \left(\sum \mathbf{f} - m\mathbf{a}\right) \cdot \delta\mathbf{u} = 0$. This is one of the benefits of the principle of virtual work; one does not need to calculate the forces of constraint **R** in order to determine the forces $\mathbf{f}_i$ which maintain the particle in equilibrium.

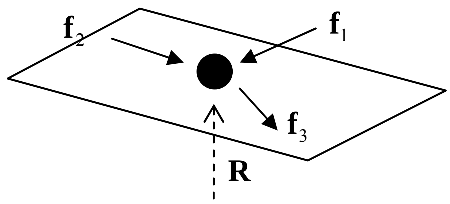


Figure 8.5.3: a particle constrained to move over a surface

The term **kinematically admissible displacement** is used to mean one that does not violate the constraints, and hence one arrives at the version of the principle which is often used in practice:

The principle of virtual work (or **principle of virtual displacements**) **III**:
a particle is in equilibrium under the action of a system of forces (including the inertial force) if the total work done by the forces (excluding reaction forces) is zero for any kinematically admissible virtual displacement of the particle

Whether one uses a kinematically admissible virtual displacement and so disregard reaction forces, or permit a virtual displacement that violates the constraint conditions will usually depend on the problem at hand. In this next example, use is made of a kinematically inadmissible virtual displacement.

Example

Consider a rigid bar of length L supported at its ends and loaded by a force F a distance a from the left hand end, Fig. 8.5.3a. Reaction forces R_A, R_C act at the ends. Let point C undergo a virtual displacement δu. From similar triangles, the displacement at B is $(a/L)\delta u$. End A does not move and so no virtual work is performed there. The total virtual work is

$$\delta W = R_C \delta u - F \frac{a}{L} \delta u \tag{8.5.4}$$

Note the minus sign here – the displacement at B is in a direction opposite to that of the action of the load and hence the work is negative. The beam is in equilibrium when $\delta W = 0$ and hence $R_C = aF / L$.

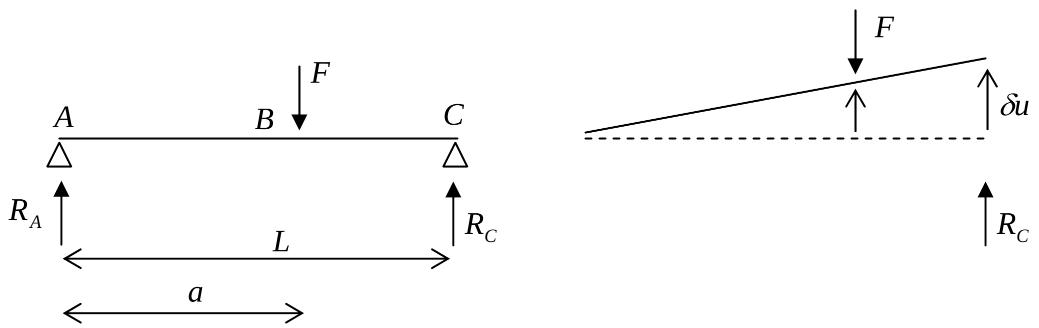


Figure 8.5.3: a loaded rigid bar; (a) bar geometry, (b) a virtual displacement at end C

■

8.5.2 Principle of Virtual Work: deformable bodies

A deformable body can be imagined to undergo virtual displacements (not necessarily the same throughout the body). Virtual work is done by the externally applied forces – **external virtual work** – and by the internal forces – **internal virtual work**. Looking again at the spring problem of Fig. 8.5.1, the external virtual work is $\delta W_{ext} = F_{apl} \delta x$ and, considering the spring force to be an "internal" force, the internal virtual work is $\delta W_{\text{int}} = -kx\delta x$. This latter virtual work can be re-written as $\delta W_{\text{int}} = -\delta U$ where δU is the virtual potential energy change which occurs when the spring is moved a distance δx (keeping the spring force constant).

In the same way, the internal virtual work of an elastic body is the (negative of the) virtual strain energy and the principle of virtual work can be expressed as

$$\boxed{\delta W_{ext} = \delta U} \quad \textbf{Principle of Virtual Work for an Elastic Body} \tag{8.5.4}$$

The principle can be extended to accommodate dissipation (energy loss), but only elastic materials will be examined here.

The virtual strain energy for a uniaxial rod is derived next.

8.5.3 Virtual Strain Energy for a Uniaxially Loaded Bar

In what follows, to distinguish between the strain energy and the displacement, the former will now be denoted by *w* and the latter by *u*.

Consider a uniaxial bar which undergoes strains ε. The strain is the unit change in length and, considering an element of length dx, Fig. 8.5.4a, the strain is

$$\varepsilon = \frac{[\Delta x + u(x+\Delta x) - u(x)] - \Delta x}{\Delta x} = \frac{du}{dx} \tag{8.5.5}$$

in the limit as $\Delta x \to 0$. With $dw = \sigma d\varepsilon$, the strain energy density is

$$w = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2 = \frac{1}{2}E\left(\frac{du}{dx}\right)^2 \tag{8.5.6}$$

and the strain energy is

$$U = \int_v \frac{1}{2}E\left(\frac{du}{dx}\right)^2 dV = \int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx \tag{8.5.7}$$

This is the actual strain energy change when the bar undergoes actual strains ε. For the simple case of constant *A* and *L* and constant strain $du/dx = \Delta / L$ where Δ is the elongation of the bar, Eqn. 8.5.7 reduces to $U = AE\Delta^2 / 2L$ (equivalent to Eqn. 8.2.2).

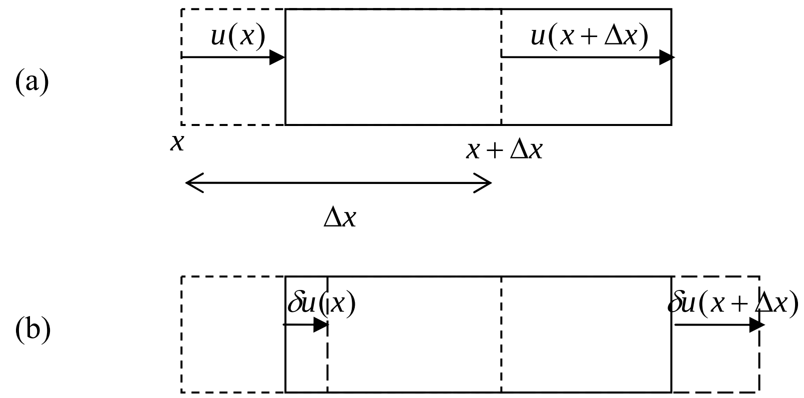


Figure 8.5.4: element undergoing actual and virtual displacements; (a) actual displacements, (b) virtual displacements

It will now be shown that the internal virtual work done as material particles undergo virtual displacements δu is given by δU, with *U* given by Eqn. 8.5.7.

Consider an element to "undergo" virtual displacements δu, Fig. 8.5.4b, which are, by definition, *measured from the actual displacements*. The virtual displacements give rise to **virtual strains**:

$$\delta\varepsilon = \frac{\delta u(x+\Delta x) - \delta u(x)}{\Delta x} = \frac{d(\delta u)}{dx} \tag{8.5.8}$$

again in the limit as $\Delta x \to 0$. Since $\delta\varepsilon = \delta(du/dx)$, it follows that

$$\delta\left(\frac{du}{dx}\right)=\frac{d(\delta u)}{dx} \tag{8.5.9}$$

In other words, the variation of the derivative is equal to the derivative of the variation[2].

One other result is needed before calculating the internal virtual work. Consider a function of the displacement, $f(u)$. The variation of f when u undergoes a virtual displacement is, by definition,

$$\delta f \equiv f(u+\delta u)-f(u)=\frac{f(u+\delta u)-f(u)}{\delta u}\delta u=\frac{df}{du}\delta u \tag{8.5.10}$$

now in the limit as the virtual displacement $\delta u \to 0$. From this one can write

$$\delta\left[\left(\frac{du}{dx}\right)^2\right]=2\left(\frac{du}{dx}\right)\delta\left(\frac{du}{dx}\right) \tag{8.5.11}$$

The stress σ applied to the surface of the element under consideration is an "external force". The internal force is the equal and opposite stress on the other side of the surface inside the element. The internal virtual work (per unit volume) is then $\delta W=-\sigma\delta\varepsilon$. Since σ is the *actual* stress, unaffected by the virtual straining,

$$\delta W=-E\varepsilon\delta\varepsilon=-E\left(\frac{du}{dx}\right)\delta\left(\frac{du}{dx}\right)=-\frac{1}{2}E\delta\left(\frac{du}{dx}\right)^2=-\delta\left[\frac{1}{2}E\left(\frac{du}{dx}\right)^2\right] \tag{8.5.12}$$

since the Young's modulus is unaffected by any virtual displacement. The total work done is then

$$\delta W_{\text{int}}=-\delta\int_V \frac{1}{2}E\left(\frac{du}{dx}\right)^2 dV \tag{8.5.13}$$

which, comparing with Eqn. 8.5.7, is the desired result, $\delta W_{\text{int}}=-\delta U$.

Example

Two rods with cross sectional areas A_1, A_2, lengths L_1, L_2 and Young's moduli E_1, E_2 and joined together with the other ends fixed, as shown in Fig. 8.5.5. The rods are subjected to a force P where they meet. As the rods elongate/contract, the strain is simply $\varepsilon=u_B/L$, where u_B is the displacement of the point at which the force is applied. The total elastic strain energy is, from Eqn. 8.5.7,

[2] this holds in general for any function; manipulations with variations form a part of a branch of mathematics known as the **Calculus of Variations**, which is concerned in the main with minima/maxima problems

$$U = \frac{E_1 A_1}{2L_1} u_B^2 + \frac{E_2 A_2}{2L_2} u_B^2 \tag{8.5.14}$$

Introduce now a virtual displacement δu_B at *B*. The external virtual work is $\delta W_{\text{ext}} = P\delta u_B$. The principle of virtual work, Eqn. 8.5.4, states that

$$P\delta u_B = \delta\left\{\left(\frac{E_1 A_1}{2L_1} + \frac{E_2 A_2}{2L_2}\right) u_B^2\right\} \tag{8.5.15}$$

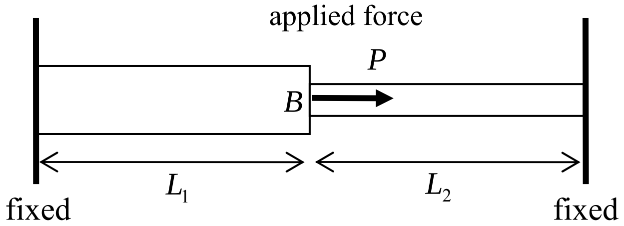


Figure 8.8.5: two rods subjected to a force *P*

From relation 8.5.10,

$$P\delta u_B = \left(\frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2}\right) u_B \delta u_B \tag{8.5.16}$$

The virtual displacement δu_B is arbitrary and so can be cancelled out, giving the result

$$u_B = P\left(\frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2}\right)^{-1} \tag{8.5.17}$$

from which the strains and hence stresses can be evaluated. Note that the reaction forces were not involved in this solution method.

■

8.5.4 Virtual Strain Energy for a Beam

The strain energy in a beam is given by Eqn. 8.2.7, *viz*.

$$U = \int_0^L \frac{M^2}{2EI} dx \tag{8.5.18}$$

Using the moment-curvature relation 7.4.37, $M = EI\left(d^2 v / dx^2\right)$, where *v* is the deflection of the beam,

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2 v}{dx^2}\right)^2 dx \qquad (8.5.19)$$

and the virtual strain energy is

$$\delta U = \delta \int_0^L \frac{EI}{2}\left(\frac{d^2 v}{dx^2}\right)^2 dx \qquad (8.5.20)$$

It is not easy to analyse problems using this expression and the principle of virtual work directly, but this expression will be used in the next section in conjunction with the related principle of minimum potential energy.

8.5.5 Problems

1. Consider a uniaxial bar of length L with constant cross section A and Young's modulus E, fixed at one end and subjected to a force P at the other. Use the principle of virtual work to show that the displacement at the loaded end is $u = PL / EA$.

2. Consider a uniaxial bar of length L, cross sectional area A and Young's modulus E. What factor of EAL is the strain energy when the displacements in the bar are $u = 10^{-3} x$, with x measured from one end of the bar? What is the internal virtual work for a virtual displacement $\delta u = 10^{-5} x$? For a constant virtual displacement along the bar?

3. A rigid bar rests upon three columns, a central column with Young's modulus 100GPa and two equidistant outer columns with Young's moduli 200GPa. The columns are of equal length 1m and cross-sectional area 1cm^2. The rigid bar is subjected to a downward force of 10kN. Use the principle of virtual work to evaluate the vertical displacement downward of the rigid bar.

4. Re-solve problem 3 from §8.2.6 using the principle of virtual displacements.

8.6 The Principle of Minimum Potential Energy

The **principle of minimum potential energy** follows directly from the principle of virtual work (for elastic materials).

8.6.1 The Principle of Minimum Potential Energy

Consider again the example given in the last section; in particular re-write Eqn. 8.5.15 as

$$\delta\left\{Pu_B - \left(\frac{E_1A_1}{2L_1} + \frac{E_2A_2}{2L_2}\right)u_B^2\right\} = 0 \tag{8.6.1}$$

The quantity inside the curly brackets is defined to be the **total potential energy** of the system, Π, and the equation states that the variation of Π is zero – that this quantity does not vary when a virtual displacement is imposed:

$$\delta\Pi = 0 \tag{8.6.2}$$

The total potential energy as a function of displacement *u* is sketched in Fig. 8.6.1. With reference to the figure, Eqn. 8.6.2 can be interpreted as follows: the total potential energy attains a stationary value (maximum or minimum) at the *actual* displacement (u_1); for example, $\delta\Pi \neq 0$ for an incorrect displacement u_2. Thus the solution for displacement can be obtained by finding a stationary value of the total potential energy. Indeed, it can be seen that the quantity inside the curly brackets in Fig. 8.6.1 attains a minimum for the solution already derived, Eqn. 8.5.17.

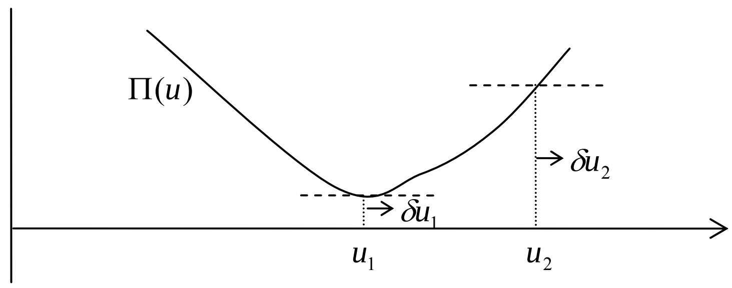


Figure 8.6.1: the total potential energy of a system

To generalise, define the "potential energy" of the applied loads to be $\delta V = -\delta W_{ext}$ so that

$$\delta\Pi = \delta U + \delta V \tag{8.6.3}$$

The external loads must be conservative, precluding for example any sliding frictional loading. Taking the total potential energy to be a function of displacement *u*, one has

$$\delta\Pi = \frac{d\Pi(u)}{du}\delta u = 0 \tag{8.6.4}$$

Thus of all possible displacements u satisfying the loading and boundary conditions, the actual displacement is that which gives rise to a stationary point $d\Pi / du = 0$ and the problem reduces to finding a stationary value of the total potential energy $\Pi = U + V$.

Stability

To be precise, Eqn. 8.6.2 only demands that the total potential energy has a stationary point, and in that sense it is called the **principle of stationary potential energy**. One can have a number of stationary points as sketched in Fig. 8.6.2. The true displacement is one of the stationary values u_1, u_2, u_3.

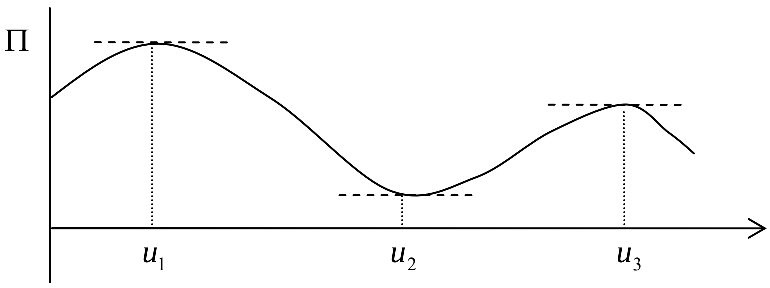


Figure 8.6.2: the total potential energy of a system

Consider the system with displacement u_2. If an external force acts to give the particles of the system some small initial velocity and hence kinetic energy, one has $0 = \Delta\Pi + \Delta K$. The particles will now move and so the displacement u_2 changes. Since Π is a minimum there it must increase and so the kinetic energy must decrease, and so the particles remain close to the equilibrium position. For this reason u_2 is defined as a **stable** equilibrium point of the system. If on the other hand the particles of the body were given small initial velocities from an initial displacement u_1 or u_3, the kinetic energy would increase dramatically; these points are called **unstable** equilibrium points. Only the state of stable equilibrium is of interest here and the principle of stationary potential energy in this case becomes the principle of minimum potential energy.

8.6.2 The Rayleigh-Ritz Method

In applications, the principle of minimum potential energy is used to obtain *approximate* solutions to problems which are otherwise difficult or, more usually, impossible to solve exactly. It forms one basis of the **Finite Element Method** (FEM), a general technique for solving systems of equations which arise in complex mechanics problems.

Example

Consider a uniaxial bar of length L, young's modulus E and varying cross-section $A = A_0(1 + x/L)$, fixed at one end and subjected to a force F at the other. The true solution for displacement to this problem can be shown to be $u = (FL/EA_0)\ln(1 + x/L)$. To see how this might be approximated using the principle, one writes

$$\Pi = U + V = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - Fu\Big|_{x=L} \tag{8.6.5}$$

First, substituting in the exact solution leads to

$$\Pi = \frac{EA_0}{2}\int_0^L (1 + x/L)\left(\frac{F}{EA_0}\frac{1}{1 + x/L}\right)^2 dx - F\frac{FL}{EA_0}\ln 2 = -\frac{\ln 2}{2}\frac{F^2 L}{EA_0} \tag{8.6.6}$$

According to the principle, any other displacement solution (which satisfies the displacement boundary condition $u(0) = 0$) will lead to a greater potential energy Π.

Suppose now that the solution was unknown. In that case an estimate of the solution can be made in terms of some unknown parameter(s), substituted into Eqn. 8.6.5, and then minimised to find the parameters. This procedure is known as the **Rayleigh Ritz method**. For example, let the guess, or **trial function**, be the linear function $u = \alpha + \beta x$. The boundary condition leads to $\alpha = 0$. Substituting $u = \beta x$ into Eqn. 8.6.5 leads to

$$\Pi = \frac{1}{2}EA_0\beta^2\int_0^L (1 + x/L)dx - F\beta L = \frac{3}{4}EA_0 L\beta^2 - F\beta L \tag{8.6.7}$$

The principle states that $\delta\Pi = (d\Pi/d\beta)\delta\beta = 0$, so that

$$\frac{d\Pi}{d\beta} = \frac{3}{2}EA_0 L\beta - FL = 0 \quad \rightarrow \quad \beta = \frac{2F}{3EA_0} \quad \rightarrow \quad u = \frac{2Fx}{3EA_0} \tag{8.6.8}$$

The exact and approximate Ritz solution are plotted in Fig. 8.6.3.

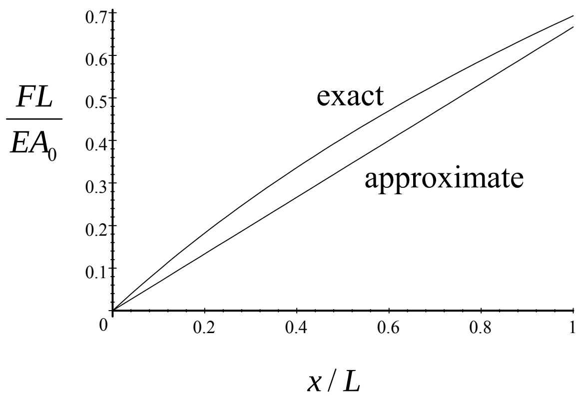


Figure 8.6.3: exact and (Ritz) approximate solution for axial problem

The total potential energy due to this approximate solution $2Fx/3EA_0$ is, from Eqn. 8.6.5,

$$\Pi = -\frac{1}{3}\frac{F^2 L}{EA_0} \tag{8.6.9}$$

which is indeed greater than the minimum value Eqn. 8.6.6 ($\approx -0.347F^2 L/EA_0$).

■

The accuracy of the solution 8.6.9 can be improved by using as the trial function a quadratic instead of a linear one, say $u = \alpha + \beta x + \gamma x^2$. Again the boundary condition leads to $\alpha = 0$. Then $u = \beta x + \gamma x^2$ and there are now two unknowns to determine. Since Π is a function of two variables,

$$\delta\Pi(\beta,\gamma) = \frac{\partial\Pi}{\partial\beta}\delta\beta + \frac{\partial\Pi}{\partial\gamma}\delta\gamma = 0 \tag{8.6.10}$$

and the two unknowns can be obtained from the two conditions

$$\frac{\partial\Pi}{\partial\beta} = 0, \qquad \frac{\partial\Pi}{\partial\gamma} = 0 \tag{8.6.11}$$

Example

A beam of length L and constant Young's modulus E and moment of inertia I is supported at its ends and subjected to a uniform distributed force per length f. Let the beam undergo deflection $v(x)$. The potential energy of the applied loads is

$$V = -\int_0^L fv(x)dx \tag{8.6.12}$$

and, with Eqn. 8.5.19, the total potential energy is

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{d^2 v}{dx^2}\right)^2 dx - f\int_0^L v dx \tag{8.6.13}$$

Choose a quadratic trial function $v = \alpha + \beta x + \gamma x^2$. The boundary conditions lead to $v = \gamma x(x - L)$. Substituting into 8.6.13 leads to

$$\Pi = 2\gamma^2 EIL - f\gamma L^3/6 \tag{8.6.14}$$

With $\delta\Pi = (d\Pi/d\gamma)\delta\gamma = 0$, one finds that

$$\gamma = \frac{fL^2}{24EI} \quad \to \quad v(x) = -\frac{fL^3}{24EI}x + \frac{fL^2}{24EI}x^2 \tag{8.6.15}$$

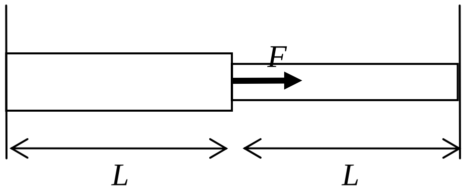
which compares with the exact solution

$$v(x) = -\frac{fL^3}{24EI}x + \frac{fL}{12EI}x^3 - \frac{f}{24EI}x^4 \tag{8.6.16}$$

■

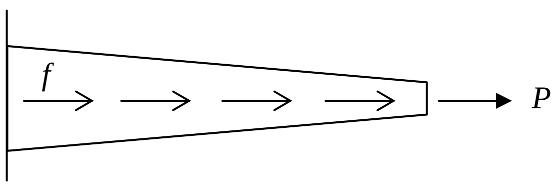
8.6.3 Problems

1. Consider the statically indeterminate uniaxial problem shown below, two bars joined at $x = L$, built in at $x = 0$ and $x = 2L$, and subjected to a force F at the join. The cross-sectional area of the bar on the left is $2A$ and that on the right is A. Use the principle of minimum potential energy in conjunction with the Rayleigh-Ritz method with a trial displacement function of the form $u = \alpha + \beta x + \gamma x^2$ to approximate the exact displacement and in particular the displacement at $x = L$.



2. A beam of length L and constant Young's modulus E and moment of inertia I is supported at its ends and subjected to a uniform distributed force per length f *and* a concentrated force P at its centre. Use the principle of minimum potential energy in conjunction with the Rayleigh-Ritz method with a trial deflection $v = \alpha \sin(\pi x / L)$, to approximate the exact deflection.

3. Use the principle of minimum potential energy in conjunction with the Rayleigh-Ritz method with a trial solution $u = \alpha x$ to approximately solve the problem of axial deformation of an elastic rod of varying cross section, built in at one end and loaded by a uniform distributed force/length f, and a force P at the free end, as shown below. The cross sectional are is $A(x) = A_0(2 - x / L)$ and the length of the rod is L.



9 Failure

In this short Chapter, some methods which can be used to predict the failure of a component are discussed.

9.1 Failure of Elastic Materials

In terms of material behavior, **failure** means *a change in the normal constitutive behavior* of a material, usually in response to excessive loads or deformations that cause irreparable changes to the microstructure. For example, compressed rock will respond elastically up to a certain point but, if the load is high enough, the rock will crush with permanent deformations (see Fig. 5.2.10b). A model of crushing rock will involve a non-elastic constitutive law and is hence beyond the scope of elasticity theory. However, at issue here is the attempt to predict when the material first ceases to respond elastically, not what happens after it does so. The failure of a specimen of rock under uniaxial tension[1] can be predicted if a tension test has been carried out on a similar rock – it will fail when the applied tension reaches the yield strength (see §5.2.1). However, the question to be addressed here is how to predict the failure of a component which is loaded in a complex way, with a consequent complex three-dimensional state of stress state at any material particle.

The theory of **stress modulated failure** assumes that failure occurs once some function of the stresses reaches some critical value. This function of the stress, or **stress metric**, might be the maximum principal stress, the maximum shear stress or some more complicated function of the stress components. Once the stress metric exceeds the critical value, the material no longer behaves elastically.

This section examines the case of failure being a transition from elastic to inelastic behaviour. Another type of failure, which occurs before this transition point is reached, is the case of buckling of a column, considered in section 7.5.

9.1.1 Failure Theories

Three theories of material failure will be discussed in what follows. They are used principally in predicting the failure of the (hard) engineering materials, but can be used or modified for many other types of material.

1. Maximum Principal Stress Theory

Consider a very brittle material, such as a ceramic or glass, or cold metal. Such a material will fracture with a "clean break". If there is no permanent deformation, the stress-strain curve in a tension test up to the point of failure will look something like that in Fig. 9.1.1. The clean break can be hypothesised to be brought about by the stresses acting normal to the fracture surface, as sketched in Fig. 9.1.1a; when these stresses reach the failure stress σ_f, the material "breaks".

Using this argument for a complex three-dimensional component, Fig. 9.1.1b, one can hypothesise that the material will fail at any location where the normal stress of largest magnitude, i.e. the maximum principal stress σ_1, reaches the appropriate failure stress σ_f of the material:

[1] i.e. acting along one axis, so one-dimensional

$$\sigma_1 = \sigma_f \tag{9.1.1}$$

Assuming the material fails similarly in tension as in compression, one can write $|\sigma_1| = \sigma_f$.

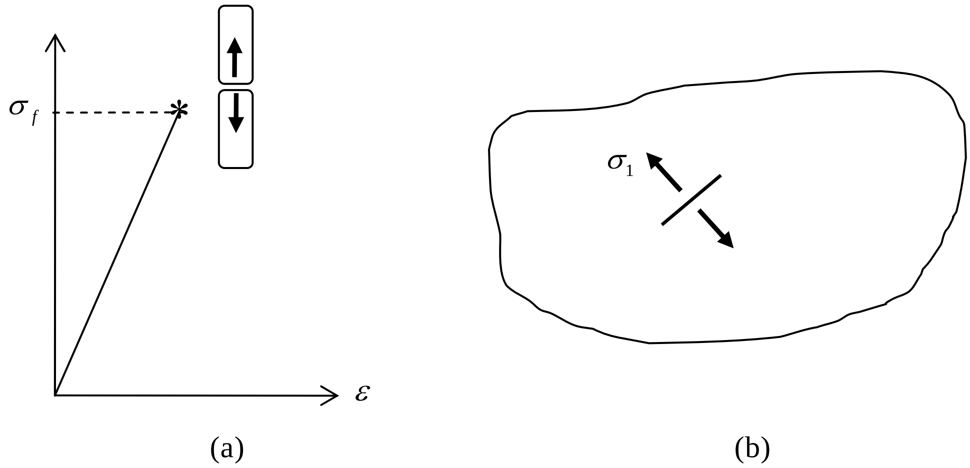


Figure 9.1.1: Brittle failure; (a) stress-strain curve in a tension test, (b) fracture occurring in a three-dimensional component

2. The Maximum Shear Stress (Tresca) Theory

Consider now a different type of failure. Instead of the clean fracture described above, the material deforms in a more complex way, and develops visible deformation bands. The bands are called **Lüders** (**slip**) **bands** (in the case of ductile steels and other metals) and **shear bands** (in the case of more brittle materials). These bands appear at roughly 45 degrees to the direction of loading. Lüders bands in a steel and shear bands in an amorphous Zirconium alloy[2] after tensile testing are shown in Fig. 9.1.2.

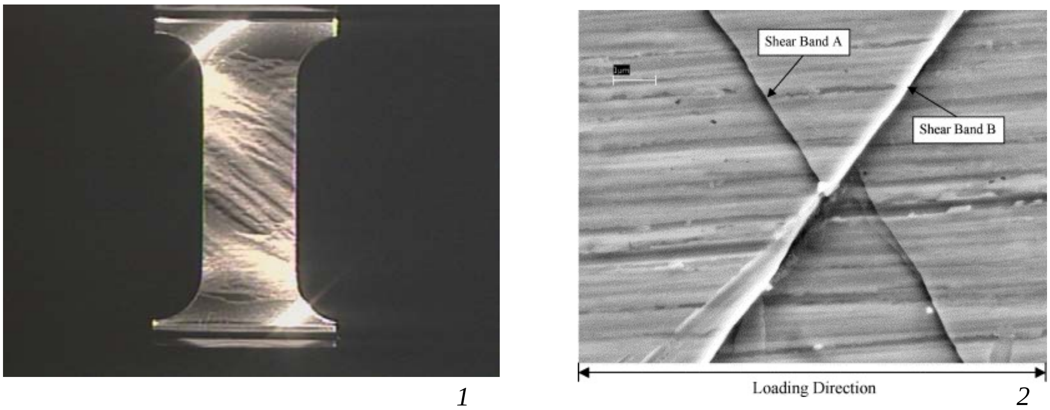
This evidence points to a shearing-type failure, with the metal sheared along these bands/planes to failure. The hypothesis here, then, is that the material fails when the shear stress acting on these planes is large enough to shear the material along these planes. In the tension test, let the applied tension be Y at the point when the elastic limit (or the yield stress) is reached, Fig. 9.1.3a. The shear stress in the specimen is given by Eqn. 3.3.1 and the maximum shear stress $\tau_{\max} = Y/2$ occurs at 45 degrees to the direction of loading.

Using this argument for a complex three-dimensional component, Fig. 9.1.3b, one can hypothesise that the material will fail at any location where the maximum shear stress reaches one half the tensile yield stress Y of the material (see Eqns. 3.5.9. *et seq.*):

$$\max\left(|\sigma_1 - \sigma_2|, |\sigma_1 - \sigma_3|, |\sigma_2 - \sigma_3|\right) = Y \tag{9.1.2}$$

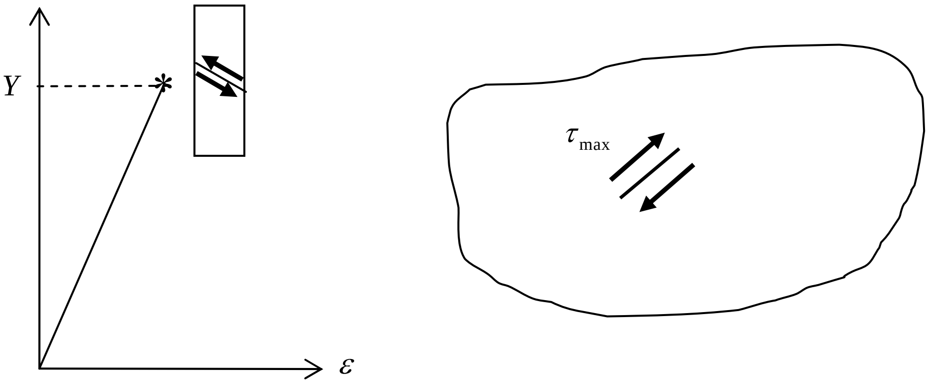
---

[2] these metals have a non-crystalline disordered molecular structure, like a glass, and behave in a more brittle fashion than the standard crystalline metals



(a) (b)

Figure 9.1.2: Tension test; (a) Lüders bands in a steel, (b) shear bands in an amorphous metal



(a) (b)

Figure 9.1.3: Failure through shear; (a) stress-strain curve in a tension test, (b) shear failure in a three-dimensional component

3. The Von Mises Theory

The **Von Mises theory** predicts that failure of a material subjected to any state of stress occurs when the following expression, involving the sum of the squares of the differences between the principal stresses, is satisfied

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}=Y \tag{9.1.3}$$

where Y is the yield stress in a tension test. Although it appears quite different, this criterion is very similar to the Tresca criterion – they both give similar predictions, the Tresca criterion being slightly more conservative, i.e. the Tresca criterion will predict that a material will fail at a lower stress than the Von Mises.

The Von Mises criterion is usually used in preference to the Tresca criterion; one reason is that one does not have to deal with the cumbersome absolute signs in Eqn. 9.1.2, and also the Von Mises criterion is usually more accurate, particularly for ductile metals.

The Von Mises criterion was really developed from theoretical grounds and "works". Well after the criterion was proposed, the following physical interpretations were made: the first is that the quantity on the left hand side of Eqn. 9.1.3 is proportional to the **octahedral shear stress** τ_{oct} (Eqn. 9.1.3 can be expressed as $\frac{3}{\sqrt{2}}\tau_{\text{oct}} = Y$). This is the shear stress acting on planes which make equal angles to the principal stress axes, as shown in Fig. 9.1.4.

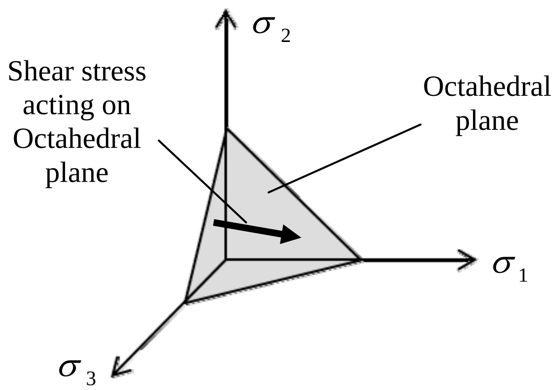


Figure 9.1.4: Interpretation of the Von Mises criterion as the shear stress acting on the Octahedral Plane

A second interpretation is that, when a linear elastic material deforms, its deformation can be decomposed into the addition of a pure volume change (as in a uniform pressure) and a distortion (change in shape). The strain energy in a deforming material subject to an arbitrary stress state can be found from Eqn. 8.2.19. Expressing these equations in terms of principal stresses, one can show that the left hand side of Eqn. 9.1.3 is proportional to the distortional component of strain energy u_d (Eqn. 9.1.3 can be expressed as $\sqrt{3Eu_d / (1+\nu)} = Y$).

Graphical Interpretation of the Failure Theories

The three failure theories can conveniently be displayed on a single graph. Assuming plane stress conditions, with $\sigma_3 = 0$, the two principal stresses σ_1 and σ_2 can be plotted against each other in what is known as **stress space**, Fig. 9.1.5. Each closed curve is called the **failure locus** of the associated failure theory. When the stress state is such that the point (σ_1, σ_2) lies inside the locus, then the material remains elastic. If the stress state is such that (σ_1, σ_2) reaches the locus, then the failure criterion is satisfied and failure occurs.

As mentioned, what happens once failure occurs is beyond elasticity theory. "Beyond failure" of ductile metals and other materials which undergo plasticity is examined in Chapter 11.

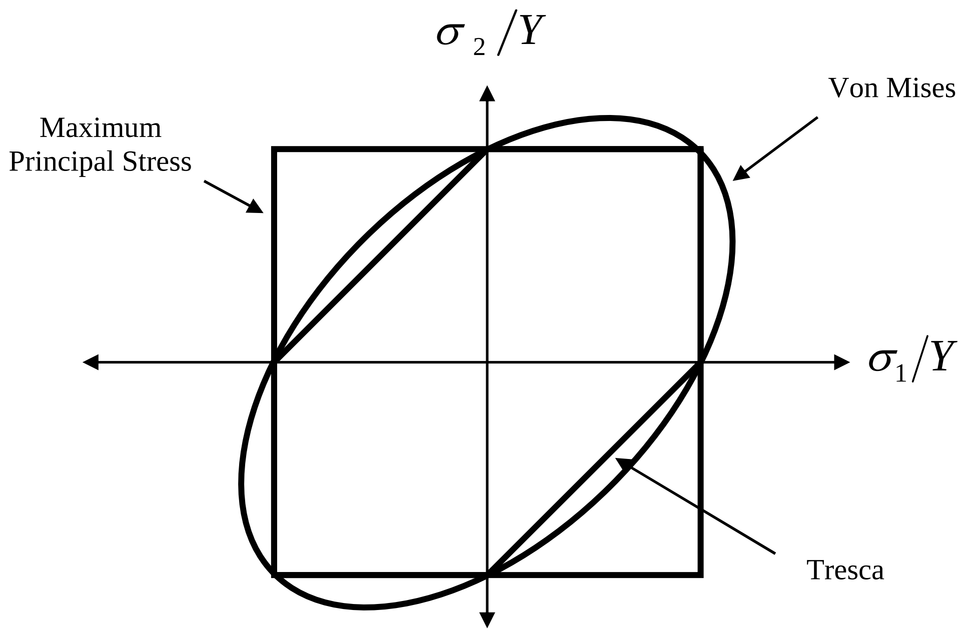


Figure 9.1.5: Failure theories in stress space

Images used:

1. http://vimeo.com/4586024
2. Yang B *et al*, Temperature evolution during fatigue damage, Intermetallics, 13(3-4), 419-428, 2005.

10 Viscoelasticity

The Linear Elastic Solid has been the main material model analysed in this book thus far. It has a long history and is still the most widely used model in applications today.

Viscoelasticity is the study of materials which have a **time-dependence**. Vicat, a French engineer from the Department of Road Construction, noticed in the 1830's that bridge-cables continued to elongate over time even though under constant load, a viscoelastic phenomenon known as **creep**. Many other investigators, such as Weber and Boltzmann, studied viscoelasticity throughout the nineteenth century, but the real driving force for its study came later – the increased demand for power and the associated demand for materials which would stand up to temperatures and pressures that went beyond previous experience. By then it had been recognised that significant creep occurred in metals at high temperatures. The theory developed further with the emergence of synthetic polymer plastics, which exhibit strong viscoelastic properties. The study of viscoelasticity is also important in Biomechanics, since many biomaterials respond viscoelastically, for example, heart tissue, muscle tissue and cartilage.

Viscoelastic materials are defined in section 10.1 and some everyday viscoelastic materials and phenomena are discussed in section 10.2. The basic mechanical models of viscoelasticity, the Maxwell and Kelvin models, are introduced in section 10.3, as is the general differential equation form of the linear viscoelastic law. The hereditary integral form of the constitutive equation is discussed in section 10.4 and it is shown how the Laplace transform can be used to solve linear viscoelastic problems in section 10.5. In section 10.6, dynamic loading, impact and vibrations of viscoelastic materials are considered. Finally, in the last section, temperature effects are briefly discussed, including the important concept of thermorheologically simple materials.

10.1 The Response of Viscoelastic Materials

10.1.1 Viscoelastic Materials

The basic response of the viscoelastic material was discussed in section 5.3.2. Repeating what was said there, the typical response of a viscoelastic material is as sketched in Fig. 10.1.1. The following will be noted:

(i) the loading and unloading curves do not coincide, Fig. 10.1.1a, but form a hysteresis loop
(ii) there is a dependence on the rate of straining $d\varepsilon / dt$, Fig. 10.1.1b; the faster the stretching, the larger the stress required
(iii) there may or may not be some permanent deformation upon complete unloading, Fig. 10.1.1a

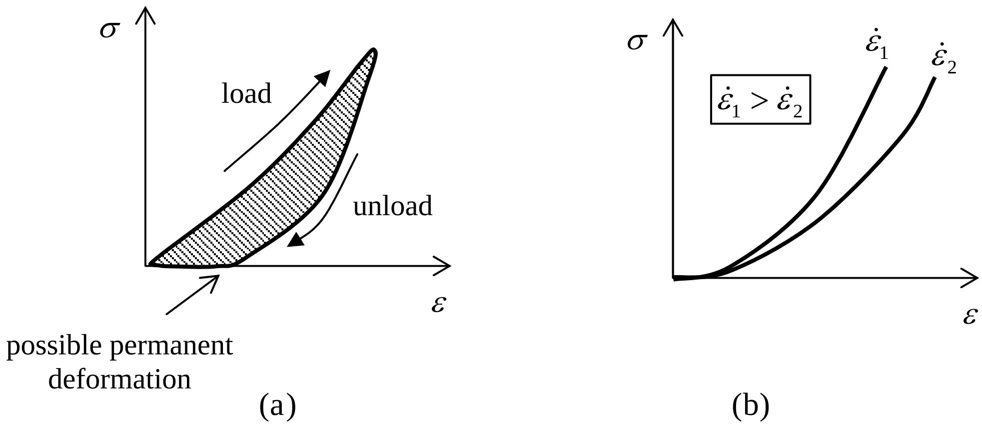


Figure 10.1.1: Response of a Viscoelastic material in the Tension test; (a) loading and unloading with possible permanent deformation (non-zero strain at zero stress), (b) different rates of stretching

The effect of *rate* of stretching shows that the viscoelastic material *depends on time*. This contrasts with the elastic material, whose constitutive equation is independent of time, for example it makes no difference whether an elastic material is loaded to some given stress level for one second or one day, or loaded slowly or quickly; the resulting strain will be the same.

It was shown in Chapter 5 that the area beneath the stress-strain curve is the energy per unit volume; during loading, it is the energy stored in a material, during unloading it is the energy recovered. There is a difference between the two for the viscoelastic material, indicated by the shaded region in Fig. 10.1.1a. This shaded region is a measure of the energy lost through heat transfer mechanisms during the deformation.

Most engineering materials undergo plasticity, meaning permanent deformations occur once the stress goes above the elastic limit. The stress-strain curve for these materials can look very similar to that of Fig. 10.1.1a, but, in contrast to viscoelasticity, plasticity is rate independent. Plasticity will be discussed in chapter 11.

Linear Viscoelasticity

Linear viscoelastic materials are those for which there is a linear relationship between stress and strain (at any given time), $\sigma \propto \varepsilon$. As mentioned before, this requires also that the strains are small, so that the engineering strain measure can be used (since the exact strain is inherently non-linear).

Strain-time curves for a linear viscoelastic material subjected to various constant stresses are shown in Fig. 10.1.2. At any given time, say t_1, the strain is proportional to stress, so that the strain there due to $3\sigma_o$ is three times the strain due to σ_o.

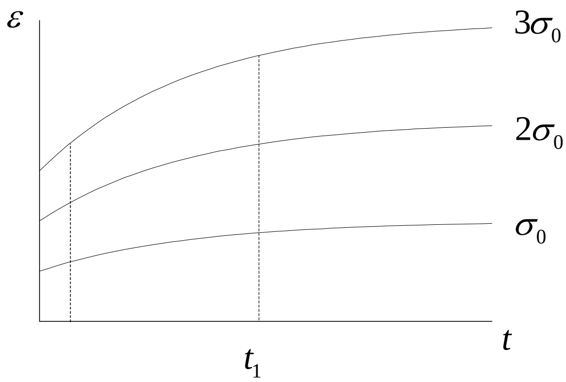


Figure 10.1.2: Strain as a function of time at different loads

Linear viscoelasticity is a reasonable approximation to the time-dependent behaviour of metals and ceramics at relatively low temperatures and under relatively low stress. However, its most widespread application is in the modelling of polymers.

10.1.2 Testing of Viscoelastic Materials

The tension test described in section 5.2 is the standard materials test. A number of other tests which are especially useful for the characterisation of viscoelastic materials have been developed, and these are discussed next.

The Creep and Recovery Test

The **creep-recovery test** involves loading a material at constant stress, holding that stress for some length of time and then removing the load. The response of a typical viscoelastic material to this test is show in Fig. 10.1.3.

First there is an instantaneous elastic straining, followed by an ever-increasing strain over time known as **creep strain**. The creep strain usually increases with an ever decreasing strain rate so that eventually a more-or-less constant-strain steady state is reached, but many materials often do not reach such a noticeable steady-state, even after a very long time.

When unloaded, the elastic strain is recovered immediately. There is then **anelastic** recovery – strain recovered over time; this anelastic strain is usually very small for metals, but may be significant in polymeric materials. A permanent strain may then be left in the material[1].

A test which focuses on the loading phase only is simply called the **creep test**.

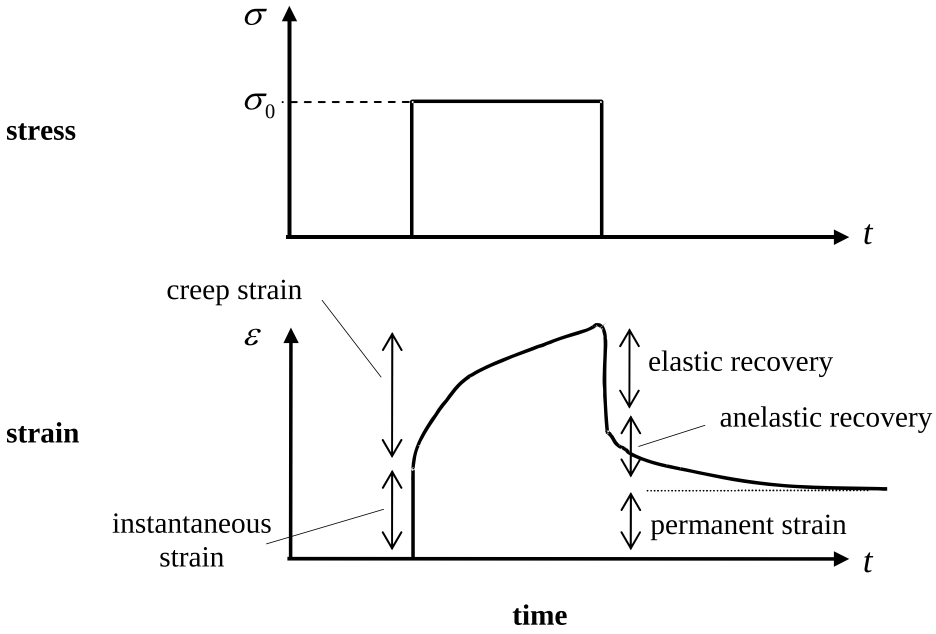


Figure 10.1.3: Strain response to the creep-recovery test

Stress Relaxation Test

The stress relaxation test involves straining a material at constant strain and then holding that strain, Fig. 10.1.4. The stress required to hold the viscoelastic material at the constant strain will be found to decrease over time. This phenomenon is called **stress relaxation**; it is due to a re-arrangement of the material on the molecular or micro-scale.

[1] if the load is above the yield stress, then some of the permanent deformation will be instantaneous plastic (rate-independent) strain; the subject of this chapter is confined to materials which are loaded up to a stress below any definable yield stress; rate–dependent materials with a yield stress above which permanent deformation take place, the viscoplastic materials, are discussed in Chapter 12

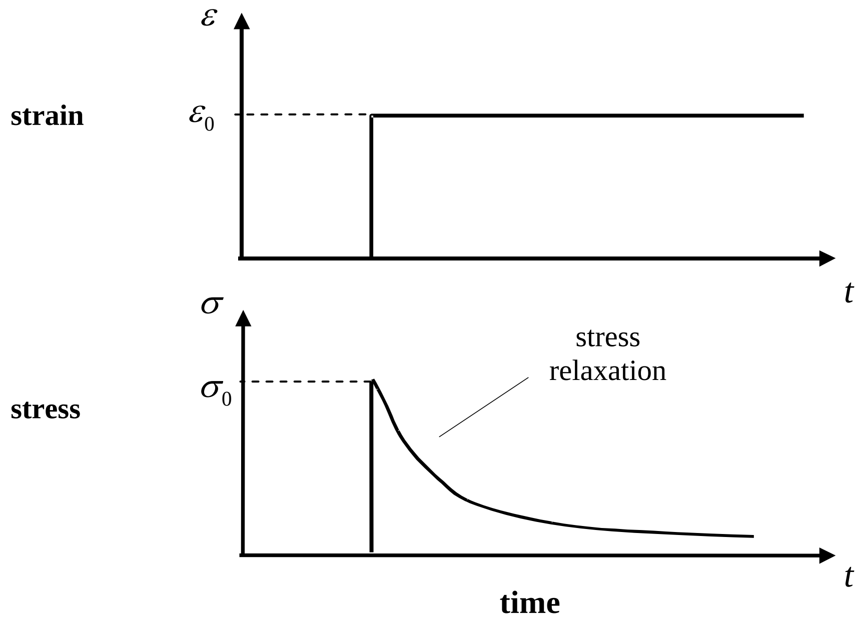


Figure 10.1.4: Stress response to the stress-relaxation test

The Cyclic Test

The cyclic test involves a repeating pattern of loading-unloading, Fig. 10.1.5 (see section 5.2.5). It can be strain-controlled (with the resulting stress observed), as in Fig. 10.1.5, or stress-controlled (with the resulting strain observed). The results of a cyclic test can be quite complex, due to the creep, stress-relaxation and permanent deformations.

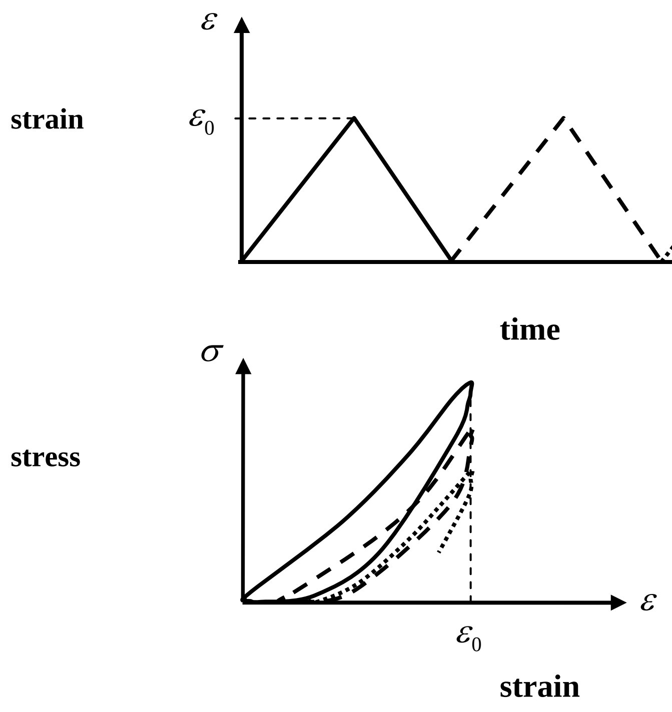


Figure 10.1.5: Typical stress response to the cyclic test

10.2 Examples and Applications of Viscoelastic Materials

Some of the properties of viscoelastic materials are their ability to creep, recover, undergo stress relaxation and absorb energy. Some examples of these phenomena are discussed in this section[1].

10.2.1 Creep and Recovery

The disks in the human spine are viscoelastic. Under normal body weight, the disks creep, that is they get shorter with time. Lying down allows the spinal disks to recover and this means that most people are taller in the morning than in the evening. Astronauts have gained up to 5cm in height under near-zero gravity conditions.

Skin tissue is viscoelastic. This can be seen by pinching the skin at the back of the hand; it takes time to recover back to its original flat position. The longer the skin is held in the pinched position, the longer it takes to recover. The more rapidly it is pinched, the less time it takes to recover – it behaves "more elastically". Skin is an **ageing material**, that is, its physical properties change over time. Younger skin recovers more rapidly than older skin.

Wood is viscoelastic. The beams of old wooden houses can often be seen to sag, but this creeping under the weight of the roof and gravity can take many decades or centuries to be noticeable. Concrete and soils are other materials which creep, as is ice, which has consequences for glacial movements.

Materials which behave elastically at room temperature often attain significant viscoelastic properties when heated. Such is the case with metal turbine blades in jet engines, which reach very high temperatures and need to withstand very high tensile stresses. Conventional metals can creep significantly at high temperatures and this has led to the development of creep-resistant alloys; turbine blades are now often made of so-called superalloys which contain some or all of nickel, cobalt, chromium, aluminium, titanium, tungsten and molybdenum.

Creep is also one of the principal causes of failure in the electric light bulb. The filaments in light bulbs are made of tungsten, a metal with a very high melting point (>3300^{o}C); this is essential because the filament needs to be electrically heated to a temperature high enough for light emission ($\approx 2000^{o}$C). If the filament creeps too much it sags and its coils touch each other, leading to a localised short circuit. Light bulbs last longer if the temperature is reduced, as in dimmed lights. Creep can also be reduced by adding potassium bubbles to the tungsten.

Polymer foams used in seat cushions creep, allowing progressive conformation of the cushion to the body shape. These cushions help reduce the pressures on the body and are very helpful for people confined to wheelchairs or hospital beds for lengthy periods.

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[1] quite a few of the applications and examples here are taken from Viscoelastic Solids, by R. S. Lakes, CRC Press, 1999

They often have to be replaced after about 6 months because creep causes them to become more dense and stiff.

A newly born baby's head is viscoelastic and its ability to creep and recover helps in the birthing process. Also, if a baby lies in one specific position for long, for example the same way of sleeping all the time, its head can become misshapen due to creep deformation. A baby's skull becomes more solid after about a year.

Viscoelasticity is also involved in the movement and behaviour of the tectonic plates, the plates which float on and travel independently over the mantle of the earth, and which are responsible for earthquakes, volcanoes, etc.

10.2.2 Stress Relaxation

Guitar strings are viscoelastic. When tightened they take up a tensile stress. However, when fixed at constant length (strain), stress relaxation occurs. The speed of sound in a string is $c = \sqrt{\sigma / \rho}$, where σ is the stress and ρ the density. The frequency is $f = c / \lambda$, where λ is the wavelength. The length of the string L is equal to half the wavelength: $f = \sqrt{\sigma / \rho} /(2L)$. The reduction in stress thus implies a reduction in frequency and a lowering of pitch – the guitar goes out of tune. The strings of a Classical Guitar are made of Nylon, a **synthetic polymer**. The great classical guitarists of the 19th Century did not have Nylon, invented in 1938, but used Catgut strings, usually made from the intestines of sheep; Catgut is a **natural polymer**. Metal guitar strings do not go out of tune so easily since metals are less viscoelastic than polymers.

10.2.3 Energy Absorption

Tall buildings vibrate when dynamically loaded by wind or earthquakes. Viscoelastic materials have the property of absorbing such vibrational energy – **damping** the vibrations. Viscoelastic dampers are used in some tall buildings, for example in the Columbia Center in Seattle, in which the dampers consist of steel plates coated with a viscoelastic polymer compound - the dampers are fixed to some of the diagonal bracing members.

Sometimes it is necessary to control vibrations but the use of a polymer is inappropriate - in this case it is necessary to use some other material with good vibration-control properties. A good example is the use of copper-manganese alloy to reduce vibration and noise from naval ship propellers. This alloy has also been used in pneumatic rock crushers. Zinc is also relatively viscoelastic for a metal and zinc-aluminium alloys are used in pneumatic drills - the alloy damps the vibrations and makes it a little less uncomfortable for anyone holding a pneumatic drill. Viscoelastic materials are also used to line the gloves worn by people working with pneumatic drills and jackhammers.

Helicopters make a lot of noise, which comes mainly from the turbine (rotary engine) and gears, but it is usually exacerbated by resonance of the fuselage skin. Acoustic blankets consisting of a layer of fibreglass sandwiched between layers of vinyl cloth, placed inside the fuselage, can reduce the noise. Sikorsky, in their HH-53C rescue helicopter, coated a small portion of the fuselage skin with damping treatments, which helped reduce the high-frequency noise in the cabin by 10 dB.

In quartz watches, vibrations are set up in quartz crystal at ultrasonic frequency (32.768 kHz). The vibrations are then used to generate periodic signals, which may be divided into intervals of time, like the second. Quartz (SiO_2) is a very **low loss** material, meaning that it is very un-viscoelastic. This ensures that the vibrations are not dampened and the watch keeps good time.

Tuning forks are often made of aluminium as it is also a low-loss material. An aluminium tuning fork will continue vibrating for quite a long time after being struck – the vibrations eventually die down because of sound-energy loss, but also because of the small energy loss due to viscoelasticity within the aluminium fork.

Viscoelastic materials are excellent impact absorbers. A peak impact force can be reduced by a factor of two if an impact buffer is made of viscoelastic, rather than elastic, material. **Elastomers** are highly viscoelastic and make good impact absorbers; these are any of various substances resembling rubber - they have trade names like Sorbothane, Implus and Noene.

Viscoelastic materials are used in automobile bumpers, on computer drives to protect from mechanical shock, in helmets (the foam padding inside), in wrestling mats, etc. Viscoelastic materials are also used in shoe insoles to reduce impact transmitted to a person's skeleton.

The cartilage at the ends of the femur and tibia, in the knee joint, is a natural shock absorber. In an osteoarthritic knee, the cartilage has degraded – sometimes the bones grind against each other causing great pain. Synthetic viscoelastic materials can be injected directly into an osteoarthritic knee, enveloping cartilage-deficient joints and acting as a lubricant and shock absorber.

10.3 Rheological Models

In this section, a number of one-dimensional linear viscoelastic models are discussed.

10.3.1 Mechanical (rheological) models

The word viscoelastic is derived from the words "viscous" + "elastic"; a viscoelastic material exhibits both viscous and elastic behaviour – a bit like a fluid and a bit like a solid. One can build up a model of linear viscoelasticity by considering combinations of the linear elastic spring and the linear viscous dash-pot. These are known as **rheological models** or **mechanical models**.

The Linear Elastic Spring

The constitutive equation for a material which responds as a linear elastic spring of stiffness E is (see Fig. 10.3.1)

$$\varepsilon = \frac{1}{E}\sigma \tag{10.3.1}$$

The response of this material to a creep-recovery test is to undergo an instantaneous elastic strain upon loading, to maintain that strain so long as the load is applied, and then to undergo an instantaneous de-straining upon removal of the load.

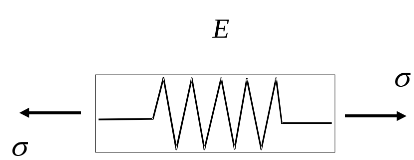


Figure 10.3.1: the linear elastic spring

The Linear Viscous Dash-pot

Imagine next a material which responds like a viscous dash-pot; the dash-pot is a piston-cylinder arrangement, filled with a viscous fluid, Fig. 10.3.2 – a strain is achieved by dragging the piston through the fluid. By definition, the dash-pot responds with a strain-rate proportional to stress:

$$\dot{\varepsilon} = \frac{1}{\eta}\sigma \tag{10.3.2}$$

where η is the **viscosity** of the material. This is the typical response of many **fluids**; the larger the stress, the faster the straining (as can be seen by pushing your hand through water at different speeds).

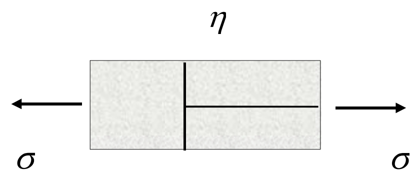


Figure 10.3.2: the linear dash-pot

The strain due to a suddenly applied load σ_o may be obtained by integrating the constitutive equation 10.3.2. Assuming zero initial strain, one has

$$\varepsilon = \frac{\sigma_o}{\eta} t \tag{10.3.3}$$

The strain is seen to increase linearly and without bound so long as the stress is applied, Fig. 10.3.3. Note that there is no movement of the dash-pot at the onset of load; it takes time for the strain to build up. When the load is removed, there is no stress to move the piston back through the fluid, so that any strain built up is permanent. The slope of the creep-line is σ_o / η.

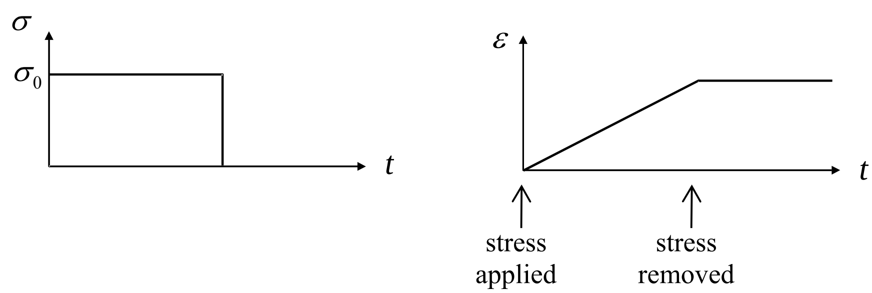


Figure 10.3.3: Creep-Recovery Response of the Dash-pot

The relationship between the stress and strain during the creep-test may be expressed in the form

$$\varepsilon(t) = \sigma_o J(t), \quad J(t) = \frac{t}{\eta} \tag{10.3.4}$$

J here is called the **creep (compliance) function** ($J = 1/E$ for the elastic spring).

10.3.2 The Maxwell Model

Consider next a spring and dash-pot in series, Fig. 10.3.4. This is the **Maxwell model**. One can divide the total strain into one for the spring (ε_1) and one for the dash-pot (ε_2).

Equilibrium requires that the stress be the same in both elements. One thus has the following three equations in four unknowns[1]:

$$\varepsilon_1 = \frac{1}{E}\sigma, \qquad \dot{\varepsilon}_2 = \frac{1}{\eta}\sigma, \qquad \varepsilon = \varepsilon_1 + \varepsilon_2 \tag{10.3.5}$$

To eliminate ε_1 and ε_2, differentiate the first and third equations, and put the first and second into the third:

$$\boxed{\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon}} \qquad \textbf{Maxwell Model} \tag{10.3.6}$$

This constitutive equation has been put in what is known as **standard form** – stress on left, strain on right, increasing order of derivatives from left to right, and coefficient of σ is 1.

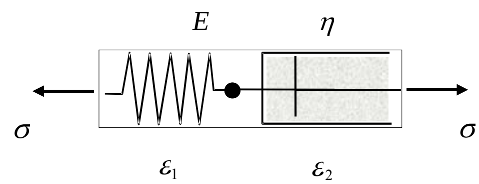


Figure 10.3.4: the Maxwell Model

Creep-Recovery Response

Consider now a creep test. Physically, when the Maxwell model is subjected to a stress σ_0, the spring will stretch immediately and the dash-pot will take time to react. Thus the initial strain is $\varepsilon(0) = \sigma_o / E$. Using this as the initial condition, an integration of 10.3.6 (with a zero stress-rate[2]) leads to

---

[1] If one considers an actual spring of length L_1 and a dashpot of length L_2 as in Fig. 10.3.4, and corresponding elongations d_1 and d_2 due to strains ε_1 and ε_2, the total elongation would be $\varepsilon(L_s + L_d) = \varepsilon_s L_s + \varepsilon_d L_d$, which is not quite the same as Eqn. 10.3.5c. It is best to think of the Maxwell model as: the total strain at a material particle can be decomposed additively according to Eqn. 10.3.5c, with the separate strains being linear elastic and viscous; Fig. 10.3.4 is simply an attempt to visualise this concept.

[2] there is a jump in stress from zero to σ_o when the load is applied, implying an infinite stress-rate $\dot{\sigma}$. One is not really interested in this jump here because the corresponding jump in strain can be predicted from the physical response of the spring. One is more interested in what happens just "after" the load is applied. In that sense, when one speaks of initial strains and stress-rates, one means their values at 0^+, just after $t = 0$; the stress-rate is zero from 0^+ on. To be more precise, one can deal with the sudden jump in stress by integrating the constitutive equation across the point $t = 0$ as follows:

$$(E/\eta)\int_{-\Delta\tau}^{+\Delta\tau}\sigma(t)dt + \int_{-\Delta\tau}^{+\Delta\tau}\dot{\sigma}(t)dt = E\int_{-\Delta\tau}^{+\Delta\tau}\dot{\varepsilon}(t)dt$$

$$\to (E/\eta)\int_{-\Delta\tau}^{+\Delta\tau}\sigma(t)dt + \left[\sigma(+\Delta\tau) - \sigma(-\Delta\tau)\right] = E\left[\varepsilon(+\Delta\tau) - \varepsilon(-\Delta\tau)\right]$$

In the limit as $\Delta\tau \to 0$, the integral tends to zero (σ is finite), the values of stress and strain at 0^-, i.e. in the limit as $\Delta\tau \to 0$ from the left, are zero. All that remains are the values to the right, giving

$$\dot{\varepsilon} = \frac{\sigma_o}{\eta} \quad \rightarrow \quad \varepsilon(t) = \frac{\sigma_o}{\eta} t + C$$
$$\rightarrow \quad \varepsilon(t) = \sigma_o \left(\frac{1}{\eta} t + \frac{1}{E} \right) \tag{10.3.7}$$

The creep-response can again be expressed in terms of a creep compliance function:

$$\varepsilon(t) = \sigma_o J(t) \quad \text{where} \quad J(t) = \frac{t}{\eta} + \frac{1}{E} \tag{10.3.8}$$

When the load is removed, the spring again reacts immediately, but the dash-pot has no tendency to recover. Hence there is an immediate elastic recovery σ_o / E, with the creep strain due to the dash-pot remaining. The full creep and recovery response is shown in Fig. 10.3.5.

The Maxwell model predicts creep, but not of the ever-decreasing strain-rate type. There is no anelastic recovery, but there is the elastic response and a permanent strain.

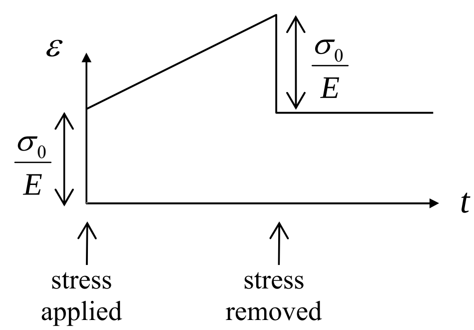


Figure 10.3.5: Creep-Recovery Response of the Maxwell Model

Stress Relaxation

In the stress relaxation test, the material is subjected to a constant strain ε_0 at $t = 0$. The Maxwell model then leads to {▲Problem 1}

$$\sigma(t) = \varepsilon_o E(t) \quad \text{where} \quad E(t) = E e^{-t/t_R}, \quad t_R = \frac{\eta}{E} \tag{10.3.9}$$

Analogous to the creep function *J* for the creep test, $E(t)$ is called the **relaxation modulus** function.

$\sigma(0^+) = E\varepsilon(0^+)$, as expected. One can deal with this sudden behaviour more easily using integral formulations or with the Laplace Transform (see §10.4, §10.5)

The parameter t_R is called the **relaxation time** of the material and is a measure of the time taken for the stress to relax; the shorter the relaxation time, the more rapid the stress relaxation.

10.3.3 The Kelvin (Voigt) Model

Consider next the other two-element model, the **Kelvin** (or **Voigt**) **model**, which consists of a spring and dash-pot in parallel, Fig. 10.3.6. It is assumed there is no bending in this type of parallel arrangement, so that the strain experienced by the spring is the same as that experienced by the dash-pot. This time,

$$\varepsilon = \frac{1}{E}\sigma_1, \quad \dot{\varepsilon} = \frac{1}{\eta}\sigma_2, \quad \sigma = \sigma_1 + \sigma_2 \qquad (10.3.10)$$

where σ_1 is the stress in the spring and σ_2 is the dash-pot stress. Eliminating σ_1, σ_2 leaves the constitutive law

$$\boxed{\sigma = E\varepsilon + \eta\dot{\varepsilon}} \quad \textbf{Kelvin (Voigt) Model} \qquad (10.3.11)$$

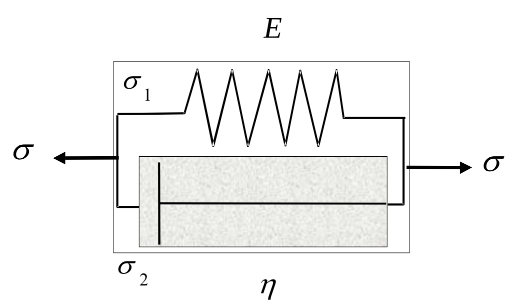


Figure 10.3.6: the Kelvin (Voigt) Model

Creep-Recovery Response

If a load σ_0 is applied suddenly to the Kelvin model, the spring will want to stretch, but is held back by the dash-pot, which cannot react immediately. Since the spring does not change length, the stress is initially taken up by the dash-pot. The creep curve thus starts with an initial slope σ_o / η.

Some strain then occurs and so some of the stress is transferred from the dash-pot to the spring. The slope of the creep curve is now σ_2 / η, where σ_2 is the stress in the dash-pot, with σ_2 ever-decreasing. In the limit when $\sigma_2 = 0$, the spring takes all the stress and thus the maximum strain is σ_o / E.

Solving the first order non-homogeneous differential equation 10.3.11 with the initial condition $\varepsilon(0) = 0$ gives

$$\varepsilon(t) = \frac{\sigma_o}{E}\left(1 - e^{-(E/\eta)t}\right) \tag{10.3.12}$$

which agrees with the above physical reasoning; the creep compliance function is now

$$J(t) = \frac{1}{E}\left(1 - e^{-t/t_R}\right), \quad t_R = \frac{\eta}{E} \tag{10.3.13}$$

The parameter t_R, in contrast to the relaxation time of the Maxwell model, is here called the **retardation time** of the material and is a measure of the time taken for the creep strain to accumulate; the shorter the retardation time, the more rapid the creep straining.

When the Kelvin model is unloaded, the spring will want to contract but again the dash pot will hold it back. The spring will however eventually pull the dash-pot back to its original zero position given time and full recovery occurs.

Suppose the material is unloaded at time $t = \tau$. The constitutive law, with zero stress, reduces to $0 = E\varepsilon + \eta\dot{\varepsilon}$. Solving leads to

$$\varepsilon(t) = Ce^{-(E/\eta)t} \tag{10.3.14}$$

where C is a constant of integration. The t here is measured from the point where "zero load" begins. If one wants to measure time from the onset of load, t must be replaced with $t - \tau$. From Eqn. 10.3.12, the strain at $t = \tau$ is $\varepsilon(\tau) = (\sigma_o / E)\left(1 - e^{-(E/\eta)\tau}\right)$. Using this as the initial condition, one finds that

$$\varepsilon(t) = \frac{\sigma_o}{E} e^{-(E/\eta)t}\left(e^{(E/\eta)\tau} - 1\right), \quad t > \tau \tag{10.3.15}$$

The creep and recovery response is shown in Fig. 10.3.7. There is a transient-type creep and anelastic recovery, but no instantaneous or permanent strain.

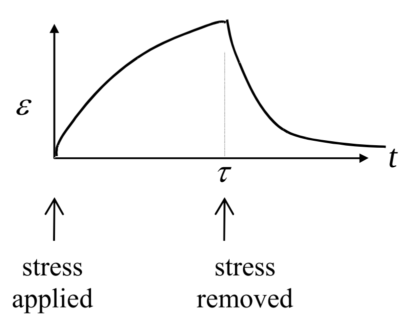


Figure 10.3.7: Creep-Recovery Response of the Kelvin (Voigt) Model

Stress Relaxation

Consider next a stress-relaxation test. Setting the strain to be a constant ε_0, the constitutive law 10.3.11 reduces to $\sigma = E\varepsilon_0$. Thus the stress is taken up by the spring and is constant, so there is in fact no stress relaxation over time. Actually, in order that the Kelvin model undergoes an instantaneous strain of ε_0, an infinite stress needs to be applied, since the dash-pot will not respond instantaneously to a finite stress[3].

10.3.4 Three – Element Models

The Maxwell and Kelvin models are the simplest viscoelastic models. More realistic material responses can be modelled using more elements. The four possible three-element models are shown in Fig. 10.3.8 below. The models of Fig. 10.3.8a-b are referred to as "solids" since they react instantaneously as elastic materials and recover completely upon unloading. The models of Figs. 10.3.8c-d are referred to as "fluids" since they involve dashpots at the initial loading phase and do not recover upon unloading.

The differential constitutive relations for the Maxwell and Kelvin models were not to difficult to derive. However, even with three elements, deriving them can be a difficult task. This is because one needs to eliminate variables from a set of equations, one or more of which is a differential equation (for example, see 10.3.5). The task is more easily accomplished using integral formulations and the Laplace transform, which are discussed in §10.4-§10.5.

Only results are given here: the constitutive relations for the four models shown in Fig. 10.3.8 are

$$\boxed{\begin{array}{ll} \text{(a)} & \sigma + \dfrac{\eta}{E_1+E_2}\dot{\sigma} = \dfrac{E_1E_2}{E_1+E_2}\varepsilon + \dfrac{\eta E_1}{E_1+E_2}\dot{\varepsilon} \\ \text{(b)} & \sigma + \dfrac{\eta}{E_2}\dot{\sigma} = E_1\varepsilon + \dfrac{\eta(E_1+E_2)}{E_2}\dot{\varepsilon} \\ \text{(c)} & \sigma + \dfrac{\eta_2}{E}\dot{\sigma} = (\eta_1+\eta_2)\dot{\varepsilon} + \dfrac{\eta_1\eta_2}{E}\ddot{\varepsilon} \\ \text{(d)} & \sigma + \dfrac{\eta_1+\eta_2}{E}\dot{\sigma} = \eta_1\dot{\varepsilon} + \dfrac{\eta_1\eta_2}{E}\ddot{\varepsilon} \end{array}} \tag{10.3.16}$$

The response of these models can be determined by specifying stress (strain) and solving the differential equations 10.3.16 for strain (stress).

[3] the stress required is $\sigma(0) = \eta\varepsilon_0\delta(t)$, where $\delta(t)$ is the Dirac delta function (this can be determined using the integral representations of §10.4)

(a)

(b)

(c)

(d)

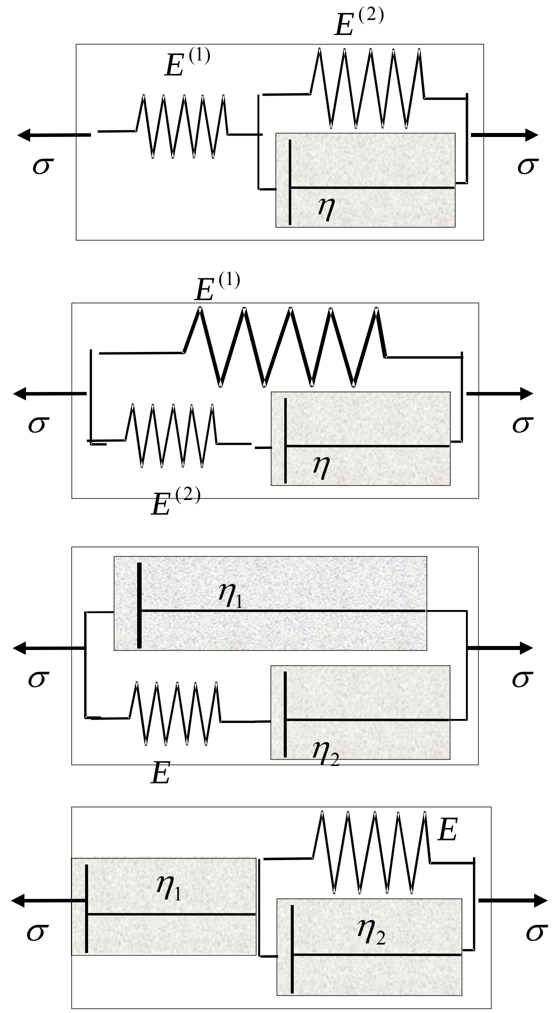


Figure 10.3.8: Three-element Models: (a) Standard Solid I, (b) Standard Solid II, (c) Standard Fluid I, (d) Standard Fluid II

10.3.5 The Creep Compliance and the Relaxation Modulus

The creep compliance function and the relaxation modulus have been mentioned in the context of the two-element models discussed above. More generally, they are defined as follows: the creep compliance is the strain due to unit stress:

$$\boxed{\varepsilon(t) = \sigma_o J(t), \qquad \varepsilon(t) = J(t) \text{ when } \sigma_o = 1} \qquad \textbf{Creep Compliance} \qquad (10.3.17)$$

The relaxation modulus is the stress due to unit strain:

$$\boxed{\sigma(t) = \varepsilon_o E(t), \qquad \sigma(t) = E(t) \text{ when } \varepsilon_o = 1} \qquad \textbf{Relaxation Modulus} \qquad (10.3.18)$$

Whereas the creep function describes the response of a material to a creep test, the relaxation modulus describes the response to a stress-relaxation test.

10.3.6 Generalized Models

More complex models can be constructed by using more and more elements. A complex viscoelastic rheological model will usually be of the form of the **generalized Maxwell model** or the **generalized Kelvin chain**, shown in Fig. 10.3.9. The generalized Maxwell model consists of N different Maxwell units in parallel, each unit with different parameter values. The absence of the isolated spring would ensure fluid-type behaviour, whereas the absence of the isolated dash-pot would ensure an instantaneous response. The generalised Kelvin chain consists of a chain of Kelvin units and again the isolated spring may be omitted if a fluid-type response is required.

In general, the more elements one has, the more accurate a model will be in describing the response of real materials. That said, the more complex the model, the more material parameters there are which need to be evaluated by experiment – the determination of a large number of material parameters might be a difficult, if not an impossible, task.

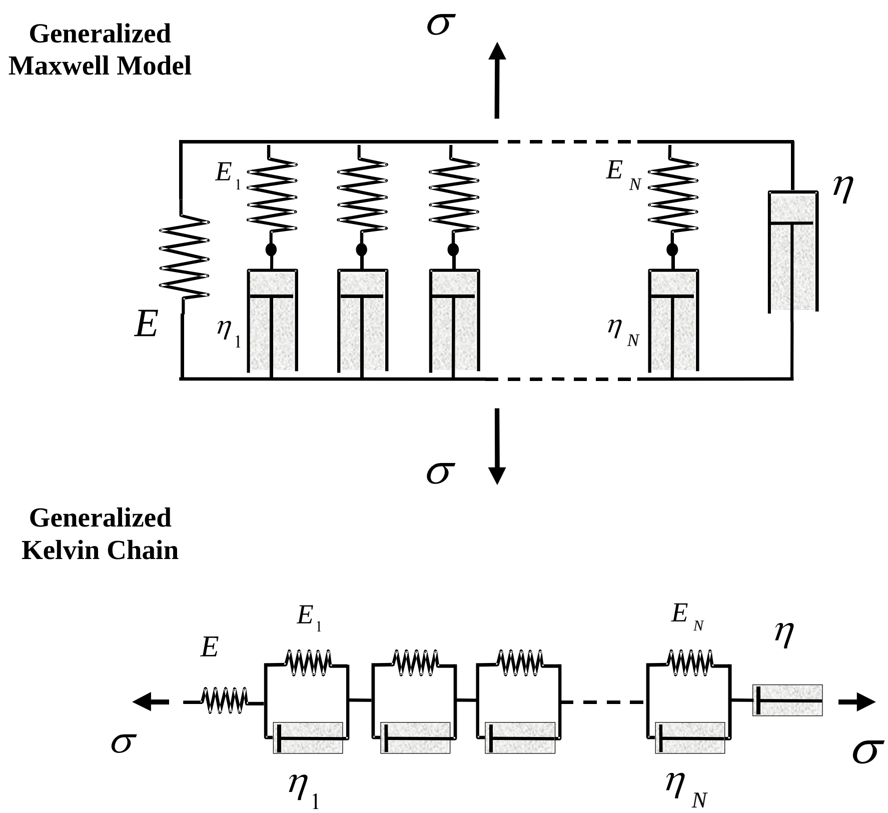


Figure 10.3.9: Generalised Viscoelastic Models

In general, a linear viscoelastic constitutive equation will be of the general form

$$p_o\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} + p_3\dddot{\sigma} + p_4\sigma^{(IV)} + \cdots = q_o\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon} + q_3\dddot{\varepsilon} + q_4\varepsilon^{(IV)} + \cdots \tag{10.3.19}$$

The more elements (springs/dashpots) one uses, the higher the order of the differential equation.

Eqn. 10.3.19 is sometimes written in the short-hand notation

$$\mathbf{P}\sigma = \mathbf{Q}\varepsilon \tag{10.3.20}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are the linear differential operators

$$\mathbf{P} = \sum_{i=0}^{n} p_i \frac{\partial^i}{\partial t^i}, \qquad \mathbf{Q} = \sum_{i=0}^{n} q_i \frac{\partial^i}{\partial t^i} \tag{10.3.21}$$

A viscoelastic model can be created by simply entering values for the coefficients p_i, q_i, in 10.3.19, without referring to any particular rheological spring – dashpot arrangement. In that sense, springs and dashpots are not necessary for a model, all one needs is a differential equation of the form 10.3.19. However, the use of springs and dashpots is helpful as it gives one a physical feel for the way a material might respond, rather than simply using an abstract mathematical expression such as 10.3.19.

10.3.7 Non-Linear Models

More realistic material responses can be achieved by using non-linear models. For example, the springs of the previous section can be replaced with more general non-linear stress-strain relations of the form:

$$\sigma = E\varepsilon^n \tag{10.3.22}$$

Various non-linear expressions for dash-pots can also be used, for example,

$$\sigma = A\varepsilon^n \dot{\varepsilon}, \qquad \sigma = Ae^{-\varepsilon/\varepsilon_0}\dot{\varepsilon} \tag{10.3.23}$$

Material data will certainly be better matched by such non-linear expressions; however, of course, they will result in non-linear differential equations which will be more difficult to solve than their linear counterparts.

10.3.8 Problems

1. Derive the Relaxation Modulus $E(t)$ for the Maxwell material.

2. What are the values of the coefficients p_i, q_i in the general differential equation 10.3.19 for
   (a) the Maxwell model and the Kelvin model?
   (b) The three-element models

10.3b Retardation and Relaxation Spectra

Generalised models can contain many parameters and will exhibit a whole array of relaxation and retardation times. For example, consider two Kelvin units in series, as in the generalised Kelvin chain; the first unit has properties E_1, η_1 and the second unit has properties E_2, η_2. Using the methods discussed in §10.4-§10.5, it can be shown that he constitutive equation is

$$\sigma + \frac{\eta_1 + \eta_2}{E_1 + E_2}\dot{\sigma} = \frac{E_1 E_2}{E_1 + E_2}\varepsilon + \frac{E_1\eta_2 + E_2\eta_1}{E_1 + E_2}\dot{\varepsilon} + \frac{\eta_1\eta_2}{E_1 + E_2}\ddot{\varepsilon} \qquad (10.3.24)$$

Consider the case of specified stress, so that this is a second order differential equation in $\varepsilon(t)$. The homogeneous solution is {▲Problem 3}

$$\varepsilon_h(t) = Ae^{-t/t_R^1} + Be^{-t/t_R^2} \qquad (10.3.25)$$

where $t_R^1 = \eta_1 / E_1, t_R^2 = \eta_2 / E_2$ are the eigenvalues of 10.3.24. For a constant load σ_0, the full solution is {▲Problem 3}

$$\varepsilon(t) = \sigma_0\left[\frac{1}{E_1}\left(1 - e^{-t/t_R^1}\right) + \frac{1}{E_2}\left(1 - e^{-t/t_R^2}\right)\right] \qquad (10.3.26)$$

Thus, whereas the single Kelvin unit has a single retardation time, Eqn. 10.3.13, this model has two retardation times, which are the eigenvalues of the differential constitutive equation. The term inside the square brackets is evidently the creep compliance of the model.

Note that, for constant strain, the model predicts a static response with no stress relaxation (as in the single Kelvin model).

In a similar way, for *N* units, it can be shown that the response of the generalised Kelvin chain to a constant load σ_0 is, neglecting the effect of the free spring/dashpot, of the form

$$\varepsilon(t) = \sigma_0 \sum_{i=1}^{N} \frac{1}{E_i}\left(1 - e^{-t/t_R^i}\right), \quad t_R^i = \frac{\eta_i}{E_i} \qquad (10.3.27)$$

where E_i, η_i are the spring stiffness and dashpot viscosity of Kelvin element i, $i = 1 \ldots N$, Fig. 10.3.9. The response of real materials can be modelled by allowing for a number of different retardation times of different orders of magnitude, e.g. $t_R^i = \{\ldots, 10^{-1}, 1, 10^1, 10^2, \ldots\}$.

If one considers many elements, Eqn. 10.3.27 can be expressed as

$$\varepsilon(t) = \sigma_0 \sum_{i=1}^{N} \Delta\varphi\left(t_R^i\right)\left(1 - e^{-t/t_R^i}\right), \quad \Delta\varphi\left(t_R^i\right) = \frac{1}{\eta_i} t_R^i \qquad (10.3.28)$$

If one is to obtain the same order of magnitude of strain for applied stress, these $\Delta\phi$'s will have to get smaller and smaller for increasing number of Kelvin units. In the limit as $N \to \infty$, letting $d\varphi = (d\varphi / dt_R) dt_R$, one has, changing the dummy variable of integration from dt_R to λ, and letting $\phi(t_R) = d\varphi / dt_R$,

$$\varepsilon(t) = \sigma_0 \int_0^\infty \phi(\lambda)\left(1 - e^{-t/\lambda}\right) d\lambda \tag{10.3.29}$$

The representation 10.3.29 allows for a continuous retardation time, in contrast to the discrete times of the model 10.3.27. The function $\phi(\lambda)$ is called the **retardation spectrum** of the model. Different responses can be modelled by simply choosing different forms for the retardation spectrum.

An alternative form of Eqn. 10.3.29 is often used, using the fact that $d\lambda / d(\ln\lambda) = \lambda$:

$$\varepsilon(t) = \sigma_0 \int_0^\infty \overline{\phi}(\lambda)\left(1 - e^{-t/\lambda}\right) d(\ln\lambda) \tag{10.3.30}$$

where $\overline{\phi} = \lambda\phi$.

A similar analysis can be carried out for the Generalised Maxwell model. For two Maxwell elements in parallel, the constitutive equation can be shown to be

$$\sigma + \frac{E_1\eta_2 + E_2\eta_1}{E_1E_2}\dot{\sigma} + \frac{\eta_1\eta_2}{E_1E_2}\ddot{\sigma} = (\eta_1 + \eta_2)\dot{\varepsilon} + \frac{E_1 + E_2}{E_1E_2}\eta_1\eta_2\ddot{\varepsilon} \tag{10.3.31}$$

Consider the case of specified strain, so that this is a second order differential equation in $\sigma(t)$. The homogeneous solution is, analogous to 10.3.25, {▲Problem 4}

$$\sigma_h(t) = Ae^{-t/t_R^1} + Be^{-t/t_R^2} \tag{10.3.32}$$

where again $t_R^1 = \eta_1 / E_1, t_R^2 = \eta_2 / E_2$, and are the eigenvalues of 10.3.31. For a constant strain ε_0, the full solution is {▲Problem 4}

$$\sigma(t) = \varepsilon_0\left[E_1 e^{-t/t_R^1} + E_2 e^{-t/t_R^2}\right] \tag{10.3.33}$$

Thus, whereas the single Maxwell unit has a single relaxation time, Eqn. 10.3.9, this model has two relaxation times, which are the eigenvalues of the differential constitutive equation. The term inside the square brackets is evidently the relaxation modulus of the model.

By considering a model with an indefinite number of Maxwell units in parallel, each with vanishingly small elastic moduli ΔE_i, one has the expression analogous to 10.3.29,

$$\sigma(t) = \varepsilon_0 \int_0^\infty \vartheta(t_R) e^{-t/t_R} dt_R \tag{10.3.34}$$

and $\vartheta(t_R)$ is called the **relaxation spectrum** of the model.

To complete this section, note that, for the two Maxwell units in parallel, a constant stress σ_0 leads to the creep strain {▲Problem 5}

$$\begin{gathered}\varepsilon(t) = \sigma_0 \left[\frac{1}{E_1 + E_2} e^{-t/t_R} + \left(\frac{\eta_1 / E_1 + \eta_2 / E_2}{\eta_1 + \eta_2} - \frac{t_R}{\eta_1 + \eta_2} \right)\left(1 - e^{-t/t_R}\right) + \frac{t}{\eta_1 + \eta_2} \right], \\ t_R = \frac{\eta_1 \eta_2}{\eta_1 + \eta_2} \frac{E_1 + E_2}{E_1 E_2}\end{gathered} \tag{10.3.35}$$

Problems

3. Consider two Kelvin units in series, as in the generalised Kelvin chain; the first unit has properties E_1, η_1 and the second unit has properties E_2, η_2. The constitutive equation is given by Eqn. 10.3.24.
   (a) The homogeneous equation is of the form $A\ddot{\varepsilon} + B\dot{\varepsilon} + C\varepsilon = 0$. By considering the characteristic equation $A\lambda^2 + B\lambda + C = 0$, show that the eigenvalues are $\lambda = -E_1 / \eta_1, -E_2 / \eta_2$ and hence that the homogeneous solution is 10.3.25.
   (b) Consider now a constant load σ_0. Show that the particular solution is $\varepsilon(t) = \sigma_0 (E_1 + E_2)/ E_1 E_2$.
   (c) One initial condition of the problem is that $\varepsilon(0) = 0$. The second condition results from the fact that only the dashpots react at time $t = 0$ (equivalently, one can integrate the constitutive equation across $t = 0$ as in the footnote in §10.3.2). Show that this condition leads to $\dot{\varepsilon}(0) = \sigma_0 (\eta_1 + \eta_2)/ \eta_1 \eta_2$.
   (d) Use the initial conditions to show that the constants in 10.3.24 are given by $A = -\sigma_0 / E_1, B = -\sigma_0 / E_2$ and hence that the complete is given by 10.3.26.
   (e) Consider again the constitutive equation 10.3.24. What values do the constants E_2, η_2 take so that it reduces to the single Kelvin model, Eqn. 10.3.11.

4. Consider two Maxwell units in parallel, as in the generalised Maxwell model; the first unit has properties E_1, η_1 and the second unit has properties E_2, η_2. The constitutive equation is given by Eqn. 10.3.31.
   (a) Suppose we have a prescribed strain history and we want to determine the stress. The homogeneous equation is of the form $A\ddot{\sigma} + B\dot{\sigma} + C\sigma = 0$. By considering the characteristic equation $A\lambda^2 + B\lambda + C = 0$, show that the eigenvalues are $t_R^1 = \eta_1 / E_1, t_R^2 = \eta_2 / E_2$ and hence that the homogeneous solution is 10.3.32.
   (b) Consider now a constant load ε_0. Show that the particular solution is zero.
   (c) One initial condition results from the fact that only the springs react at time $t = 0$, which leads to the condition $\sigma(0) = \varepsilon_0 (E_1 + E_2)$. A second condition can be

obtained by integrating the constitutive equation across $t = 0$ as in the footnote in §10.3.2. Show that this leads to the condition $\dot{\sigma}(0^+) = -\left(E_1^2/\eta_1 + E_2^2/\eta_2\right)\varepsilon_0$.

(d) Use the initial conditions to show that the constants in 10.3.32 are given by $A = E_1\varepsilon_0$, $B = E_2\varepsilon_0$ and hence that the complete solution is given by 10.3.33.

(e) Consider again the constitutive equation 10.3.31. What values do the constants E_2, η_2 take so that it reduces to the single Maxwell model, Eqn. 10.3.6.

5. Consider again the two Maxwell units in parallel, as in Problem 4. This time consider a stress-driven problem.

(a) From the constitutive equation 10.3.31, the differential equation to be solved is of the form $A\ddot{\varepsilon} + B\dot{\varepsilon} = \ldots$. By considering the characteristic equation $A\lambda^2 + B\lambda = 0$, show that the eigenvalues are

$$\lambda_1 = 0,\ \lambda_2 = -\frac{(\eta_1 + \eta_2)E_1E_2}{\eta_1\eta_2(E_1 + E_2)}$$

and hence that the homogeneous solution is $\varepsilon(t) = C_1 + C_2 e^{-t/t_R}$ where $t_R = -1/\lambda_2$.

(b) Consider now a constant stress σ_0. By using the condition that only the springs react at time $t = 0$, show that the particular solution is $\sigma_0 t/(\eta_1 + \eta_2)$.

(c) One initial condition results from the fact that only the springs react at time $t = 0$, which leads to the condition $\varepsilon(0) = \sigma_0/(E_1 + E_2)$. A second condition can be obtained by integrating the constitutive equation across $t = 0$ as in the footnote in §10.3.2. Be careful to consider all terms in 10.3.31. Show that this leads to the condition

$$\dot{\varepsilon}(0^+) = (\sigma_0/t_R)\left[\left(\eta_1/E_1 + \eta_2/E_2\right)/\left(\eta_1 + \eta_2\right) - 1/\left(E_1 + E_2\right)\right].$$

(d) Use the initial conditions to show that the complete solution is given by 10.3.35.

10.4 The Hereditary Integral

In the previous section, it was shown that the constitutive relation for a linear viscoelastic material can be expressed in the form of a linear differential equation, Eqn. 10.3.19. Here it is shown that the stress-strain relation can also be expressed in the form of an integral, called the **hereditary integral**. Quite a few different forms of this integral are commonly used; to begin this section, the different forms are first derived for the Maxwell model, before looking at the more general case(s).

10.4.1 An Example: the Maxwell Model

Consider the differential equation for the Maxwell model, Eqn. 10.3.6,

$$\frac{d\sigma}{dt} + \frac{E}{\eta}\sigma = E\frac{d\varepsilon}{dt} \tag{10.4.1}$$

The first order differential equation can be solved using the standard **integrating factor** method. This converts 10.4.1 into an integral equation. Three similar integral equations will be derived in what follows[1].

Hereditary Integral over $[-\infty, \mathbf{t}]$

It is sometimes convenient to regard 10.4.1 as a differential equation over the time interval $[-\infty, t]$, even though the time interval of interest is really $[0, t]$. This can make it easier to deal with sudden "jumps" in stress or strain at time $t = 0$. The initial condition on 10.4.1 is then

$$\sigma(-\infty) = 0\,. \tag{10.4.2}$$

Using the integrating factor $e^{Et/\eta}$, re-write 10.4.1 in the form

$$\frac{d}{dt}\left(e^{Et/\eta}\sigma(t)\right) = Ee^{Et/\eta}\frac{d\varepsilon(t)}{dt} \tag{10.4.3}$$

Integrating both sides over $[-\infty, \hat{t}]$ gives

$$\left(e^{Et/\eta}\sigma\right)_{\hat{t}} - \left(e^{Et/\eta}\sigma\right)_{-\infty} = \int_{-\infty}^{\hat{t}} Ee^{Et/\eta}\frac{d\varepsilon(t)}{dt}dt \tag{10.4.4}$$

or

[1] note that Eqn. 10.4.1 predicts that sudden changes in the strain-rate, $\dot{\varepsilon}$, will lead to sudden changes in the stress-rate, $\dot{\sigma}$, but the stress σ will remain continuous. The strain ε does not appear explicitly in 10.4.1; sudden changes in strain can be dealt with by (i) integrating across the point where the jump occurs, or (ii) using step functions and the integral formulation (see later)

$$\sigma(\hat{t}) = \int_{-\infty}^{\hat{t}} E e^{-E(\hat{t}-t)/\eta} \frac{d\varepsilon(t)}{dt} dt \tag{10.4.5}$$

Changing the notation,

$$\boxed{\sigma(t) = \int_{-\infty}^{t} E(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau} \tag{10.4.6}$$

where $E(t)$, the relaxation modulus for the Maxwell model, is

$$E(t) = E e^{-Et/\eta} \tag{10.4.7}$$

This is known as a hereditary integral; given the **strain history** over $[-\infty, t]$, one can evaluate the stress at the current time. It is the *same* constitutive equation as Eqn. 10.4.1, only in a different form.

Hereditary Integral over $[\mathbf{0}, \mathbf{t}]$

The hereditary integral can also be expressed in terms of an integral over $[0, t]$. Let there be a sudden non-zero strain $\varepsilon(0)$ at $t = 0$, with the strain possibly varying, but continuously, thereafter. The strain, which in Eqn. 10.4.6 is to be regarded as a single function over $[-\infty, t]$ with a jump at $t = 0$, is sketched in Fig. 10.4.1.

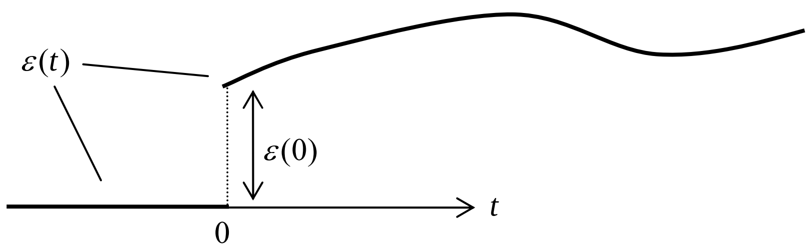


Figure 10.4.1: Strain with a sudden jump to a non-zero strain at $t = 0$

There are two ways to proceed. First, write the integral over three separate intervals:

$$\sigma(t) = \lim_{\vartheta \to 0} \left\{ \int_{-\infty}^{-\vartheta} E(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau + \int_{-\vartheta}^{+\vartheta} E(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau + \int_{+\vartheta}^{t} E(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau \right\} \tag{10.4.8}$$

With $\varepsilon(t) = 0$ over $[-\infty, -\vartheta]$, the first integral is zero. With a jump in strain only at $t = 0$, the integrand in the third integral remains finite. The second integral can be evaluated by considering the function $f(t)$ illustrated in Fig. 10.4.2, a straight line with slope $\varepsilon(0)/2\vartheta$. As $\vartheta \to 0$, it approaches the actual strain function $\varepsilon(t)$, which jumps to $\varepsilon(0)$ at $t = 0$. Then

$$\lim_{\vartheta \to 0} \int_{-\vartheta}^{+\vartheta} E(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau = \lim_{\vartheta \to 0} \frac{\varepsilon(0)}{2\vartheta} \eta e^{-Et/\eta} \left(e^{+E\vartheta/\eta} - e^{-E\vartheta/\eta} \right) \tag{10.4.9}$$

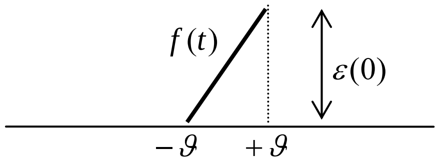


Figure 10.4.2: A function used to approximate the strain for a sudden jump

Using the approximation $e^x \approx 1 + x$ for small *x*, the value of this integral is $\varepsilon(0) E e^{-Et/\eta}$. Thus Eqn. 10.4.6 can be expressed as

$$\boxed{\sigma(t) = E(t)\varepsilon(0) + \int_0^t E(t-\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau} \tag{10.4.10}$$

By "0" here in the lower limit of the integral, one means 0^+, just after any possible non-zero initial strain. In that sense, the strain $\varepsilon(t)$ in Eqn. 10.4.10 is to be regarded as a continuous function, i.e. with no jumps over $[0, t]$. Jumps in strain after $t = 0$ can be dealt with in a similar manner.

A second and more elegant way to arrive at Eqn. 10.4.10 is to re-express the above analysis in terms of the Heaviside step function $H(t)$ and the Dirac delta function $\delta(t)$ (see the Appendix to this section for a discussion of these functions).

The function sketched in Fig. 10.4.1 can be expressed as $H(t)\varepsilon(t)$ where now $\varepsilon(t)$ is to be regarded as a continuous function over $[-\infty, t]$ – the jump is now contained within the step function $H(t)$. Eqn. 10.4.6 now becomes

$$\sigma(t) = \int_{-\infty}^t E(t-\tau) H(\tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau + \int_{-\infty}^t E(t-\tau) \varepsilon(\tau) \frac{dH(\tau)}{d\tau} d\tau \tag{10.4.11}$$

The first integral becomes the integral in 10.4.10. From the brief discussion in the Appendix to this section, the second integral becomes

$$\int_{-\infty}^t E(t-\tau) \varepsilon(\tau) \frac{dH(\tau)}{d\tau} d\tau = \int_{-\infty}^t E(t-\tau) \varepsilon(\tau) \delta(\tau) d\tau = E(t)\varepsilon(0) \tag{10.4.12}$$

A Third Hereditary Integral

Finally, the integral can also be expressed as a function of $\varepsilon(t)$, rather than its derivative. To achieve this, one can integrate 10.4.10 by parts:

$$\sigma(t) = E(0)\varepsilon(t) + \int_0^t \frac{dE(t-\tau)}{d(t-\tau)} \varepsilon(\tau) d\tau \qquad (10.4.13)$$

This can be expressed as

$$\boxed{\sigma(t) = E(0)\varepsilon(t) - \int_0^t R(t-\tau)\varepsilon(\tau) d\tau} \qquad (10.4.14)$$

where $R(t) = -dE(t)/dt$.

Note that integration by parts is only possible when there are no "jumps" in the functions under the integral sign and this is assumed for the integrand in 10.4.10. If there are jumps, the integral can either be split into separate integrals as in 10.4.8, or the functions can be represented in terms of step functions, which automatically account for jumps.

The formulae 10.4.6, 10.4.10 and 10.4.14 give the stress as functions of the strain. Similar formulae can be derived for the strain in terms of the stress (see the Appendix to this Section).

Relaxation Test

To illustrate the use of the hereditary integral formulae, consider a relaxation test, where the strain history is given by

$$\varepsilon(t) = \begin{cases} 0, & t < 0 \\ \varepsilon_0, & \text{otherwise} \end{cases} \qquad (10.4.15)$$

Expressing the strain history as $\varepsilon(t) = \varepsilon_0 H(t)$, Eqn. 10.4.6 gives

$$\sigma(t) = \varepsilon_0 \int_{-\infty}^t E(t-\tau)\delta(\tau) d\tau = \varepsilon_0 E(t) \qquad (10.4.16)$$

From 10.4.10, with the derivative in the integrand zero, one has $\sigma(t) = E(t)\varepsilon(0) = \varepsilon_0 E(t)$. Finally, from 10.4.14, with $R(t) = +(E^2/\eta)e^{-Et/\eta}$, one again has

$$\sigma(t) = E\varepsilon_0 - \varepsilon_0 \int_0^t \frac{E^2}{\eta} e^{-E(t-\tau)/\eta} d\tau = \varepsilon_0 E e^{-Et/\eta} = \varepsilon_0 E(t) \qquad (10.4.17)$$

10.4.2 Hereditary Integrals: General Formulation

Although derived for the Maxwell mode, these formulae Eqns. 10.4.6, 10.4.10, 10.4.14, are in fact quite general, for example they can be derived from the differential equation for the Kelvin model (see Appendix to this section).

The hereditary integrals were derived directly from the Maxwell model differential equation so as to emphasis that they are one and the same constitutive equation. Here they are derived more generally from first principles.

The strain due to a constant step load $\sigma(0)$ applied at time $t = 0$ is by definition $\varepsilon(t) = \sigma(0)J(t)$, where $J(t)$ is the creep compliance function. The strain due to a second load, $\Delta\sigma$ say, applied at some later time τ, is $\varepsilon(t) = \Delta\sigma J(t-\tau)$. The total strain due to both loads is[2], Fig. 10.4.3,

$$\varepsilon(t) = \sigma(0)J(t) + \Delta\sigma J(t-\tau) \qquad (10.4.18)$$

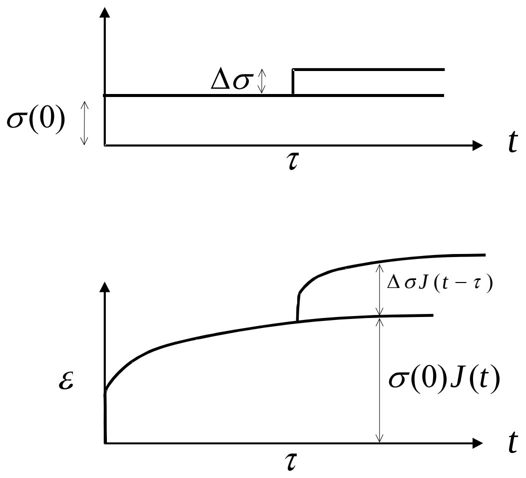


Figure 10.4.3: Superposition of loads

Generalising to an indefinite number of applied loads of infinitesimal magnitude, $d\sigma_i$, one has

$$\varepsilon(t) = \sigma(0)J(t) + \sum_{i=1}^{\infty} d\sigma_i J(t-\tau_i) \qquad (10.4.19)$$

In the limit, the summation becomes the integral $\int J(t-\tau)d\sigma$, or[3] (see Fig. 10.4.4)

$$\boxed{\varepsilon(t) = \sigma(0)J(t) + \int_0^t J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau} \quad \textbf{Hereditary Integral (for Strain)} \quad (10.4.20)$$

[2] this is again an application of the linear superposition principle, mentioned in §6.1.2; because the material is linear (and only because it is linear), the "effect" of a sum of "causes" is equal to the sum of the individual "effects" of each "cause"

[3] this integral equation allows for a sudden non-zero stress at $t = 0$. Other jumps in stress at later times can be allowed for in a similar manner – one would split the integral into separate integrals at the point where the jump occurs

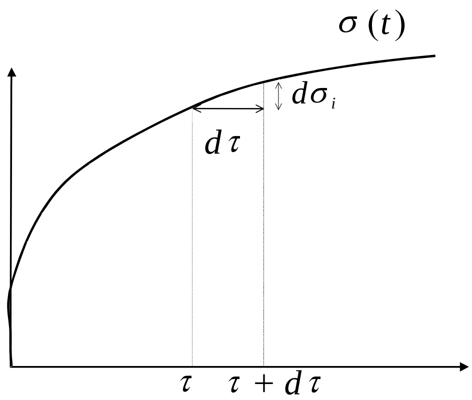


Figure 10.4.4: Formation of the hereditary integral

One can also derive a corresponding hereditary integral in terms of the relaxation modulus {▲Problem 1}:

$$\boxed{\sigma(t) = \varepsilon(0)E(t) + \int_0^t E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau} \quad \textbf{Hereditary Integral (for Stress)} \quad (10.4.21)$$

This is Eqn. 10.4.10, which was derived specifically from the Maxwell model.

The hereditary integrals only require a knowledge of the creep function (or relaxation function). One does not need to construct a rheological model (with springs/dashpots) to determine a creep function. For example, the creep function for a material may be determined from test-data from a creep test. The hereditary integral formulation is thus not restricted to particular combinations of springs and dash-pots.

Example

Consider the Maxwell model and the two load histories shown in Fig. 10.4.5. The maximum stress is the same in both, $\hat{\sigma}$, but load (1) is applied more gradually.

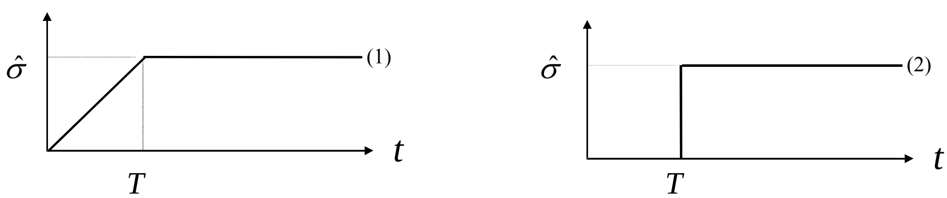


Figure 10.4.5: two stress histories

Examine load (1) first. The stress history is

$$\sigma(t) = \begin{cases} \dfrac{\hat{\sigma}}{T} t, & t < T \\ \\ \hat{\sigma}, & t > T \end{cases}$$

In the hereditary integral 10.4.20, the creep compliance function $J(t)$ is given by 10.3.8, $J(t) = t/\eta + 1/E$, and the stress is zero at time zero, so $\sigma(0) = 0$. The strain is then

$0 < t < T:\ d\sigma/dt = \hat{\sigma}/T$

$$\varepsilon(t) = \sigma(0)J(t) + \int_0^t J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau = \frac{\hat{\sigma}}{T}\int_0^t \left[\frac{t-\tau}{\eta} + \frac{1}{E}\right]d\tau = \frac{\hat{\sigma}}{T}\left[\frac{t^2}{2\eta} + \frac{t}{E}\right]$$

$T < t:\ d\sigma/dt = 0$

$$\begin{aligned} \varepsilon(t) &= \sigma(0)J(t) + \int_0^T J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau + \int_T^t J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau \\ &= \int_0^T J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau = \frac{\hat{\sigma}}{T}\int_0^T \left[\frac{t-\tau}{\eta} + \frac{1}{E}\right]d\tau = \hat{\sigma}\left[\frac{t-T/2}{\eta} + \frac{1}{E}\right] \end{aligned}$$

For load history (2),

$$\sigma(t) = \begin{cases} 0, & t < T \\ \\ \hat{\sigma}, & t > T \end{cases}$$

The strain is then $\varepsilon(t) = 0$ for $t < T$. The hereditary integral 10.4.20 allows for a jump at $t = 0$. For a jump from zero stress to a non-zero stress at $t = T$ it can be modified to

$$\varepsilon(t) = \sigma(T)J(t-T) = \hat{\sigma}\left[\frac{t-T}{\eta} + \frac{1}{E}\right]$$

which is less than the strain due to load (1). (Alternatively, one could use the Heaviside step function and let $\sigma(t) = \hat{\sigma}H(t-T)$ in 10.4.20, leading to the same result,

$$\varepsilon(t) = \sigma(0)J(t) + \hat{\sigma}\int_0^t J(t-\tau)\delta(\tau - T)d\tau = \hat{\sigma}J(t-T)\text{ .)}$$

This example illustrates two points:
(1) the material has a "memory". It remembers the previous loading history, responding differently to different loading histories
(2) the rate of loading is important in viscoelastic materials. This result agrees with an observed phenomenon: the strain in viscoelastic materials is larger for stresses which grow gradually to their final value, rather than when applied more quickly[4].

[4] for the Maxwell model, if one applied the second load at time $t = T/2$, so that the total stress applied in (1) and (2) was the same, one would have obtained the same response after time *T*, but this is not the case in general

10.4.3 Non-linear Hereditary Integrals

The linear viscoelastic models can be extended into the non-linear range in a number of ways. For example, generalising expressions of the form 10.4.14,

$$\sigma(t) = f_1(\varepsilon(t)) + \int_0^t R(t-\tau) f_2(\varepsilon(t)) d\tau \tag{10.4.22}$$

where f_1, f_2 are non-linear functions of the strain history. The relaxation function can also be assumed to be a function of strain as well as time:

$$\sigma(t) = f_1(\varepsilon(t)) + \int_0^t K(t-\tau, \varepsilon) f_2(\varepsilon(t)) d\tau \tag{10.4.23}$$

10.4.4 Problems

1. Derive the hereditary integral 10.4.21,

$$\sigma(t) = \varepsilon(0)E(t) + \int_0^t E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau$$

2. Use the hereditary integral form of the constitutive equation for a linear viscoelastic material, Eqn. 10.4.20, to evaluate the response of a material with creep compliance function

$$J(t) = \ln(t+1)$$

to a load $\sigma(t) = \sigma_o(1 + Bt)$. Sketch $J(t)$, which of course gives the strain response due to a unit load $\sigma(t) = 1$. Sketch also the load $\sigma(t)$ and the calculated strain $\varepsilon(t)$.
[note: $\int_0^t \ln[(b-x)+1]dx = -(b-t+1)\ln(b-t+1) + (b+1)\ln(b+1) - t$]

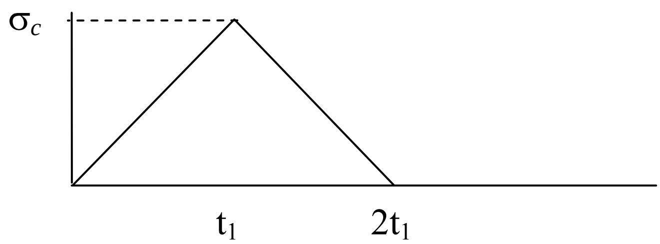
3. A creep test was carried out on a certain linear viscoelastic material and the data was fitted approximately by the function

$$\varepsilon(t) = \hat{\sigma}\left(1 - e^{-2t}\right),$$

where $\hat{\sigma}$ was the constant applied load.
(i) Sketch this strain response over $0 \le t \le 3$ (very roughly, with $\hat{\sigma} = 1$).
(ii) Which of the following three rheological models could be used to model the material:
(a) the full generalized Kelvin chain of Fig. 10.3.9
(b) the Kelvin chain minus the free spring
(c) the generalized Maxwell model minus the free spring and free dash-pot
Give reasons for your choice (and reasons for discounting the other two).
(iii) For the rheological model you chose in part (ii), roughly sketch the response to a standard creep-recovery test (the response during the loading phase has already been done in part (i)).
(iv) Find the material's response to a load $\sigma(t) = t^2 + 1$.

4. Determine the strain response of the Kelvin model to a stress history which is triangular in time:

$$\sigma(t)=\begin{cases}\sigma(t)=0, & t<0\\ \sigma(t)=(\sigma_c/t_1)t, & 0<t<t_1\\ \sigma(t)=2\sigma_c-(\sigma_c/t_1)t, & t_1<t<2t_1\\ \sigma(t)=0, & 2t_1<t<\infty\end{cases}$$



10.4.5 Appendix to §10.4

1. The Heaviside Step Function and the Dirac Delta Functions

The **Heaviside step function** $H(t)$ is defined through

$$H(t-a)=\begin{cases}0, & t<a\\ 1/2, & t=a\\ 1, & t>a\end{cases} \tag{10.4.24}$$

and is illustrated in Fig. 10.4.6a. The derivative of the Heaviside step function, dH/dt, can be evaluated by considering $H(t-a)$ to be the limit of the function $f(t)$ shown in Fig. 10.4.6b as $\vartheta\to 0$. This derivative df/dt is shown in Fig. 10.4.6c and in the limit is

$$\frac{dH(t-a)}{dt}=\lim_{\vartheta\to 0}\frac{df}{dt}=\delta(t-a) \tag{10.4.25}$$

where δ is the **Dirac delta function** defined through (the integral here states that the "area" is unity, as illustrated in Fig. 10.4.6c)

$$\delta(t-a)=\begin{cases}\infty, & t=a\\ 0 & \text{otherwise}\end{cases}, \qquad \int_{-\infty}^{+\infty}\delta(t-a)dt=1 \tag{10.4.26}$$

Integrals involving delta functions are evaluated as follows: consider the integral

$$\int_{-\infty}^{+\infty} g(t)\delta(t-b)dt \tag{10.4.27}$$

The delta function here is zero, and hence the integrand is zero, everywhere except at $t = b$. Thus the integral is

$$\int_{-\infty}^{+\infty} g(t)\delta(t-b)dt = \int_{-\infty}^{+\infty} g(b)\delta(t-b)dt = g(b)\int_{-\infty}^{+\infty} \delta(t-b)dt = g(b) \tag{10.4.28}$$

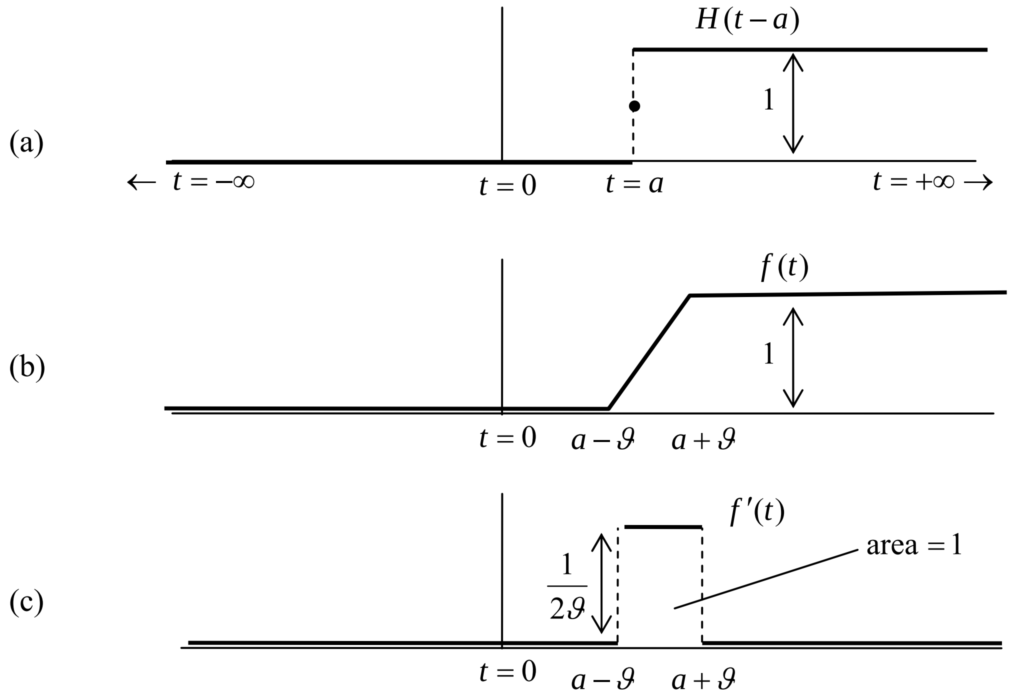


Figure 10.4.6: The Heaviside Step Function and evaluation of its derivative

2. The Maxwell Model: Functions of the Stress

In §10.4.1, the hereditary integrals for the Maxwell model were derived for the stress in terms of integrals of the strain. Here, they are derived for the strain in terms of integrals of the stress.

Consider again the differential equation for the Maxwell model, Eqn. 7.3.6,

$$\frac{d\varepsilon}{dt} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{1}{\eta}\sigma \tag{10.4.29}$$

Direct integration gives

$$\varepsilon(t) = \frac{1}{E}\sigma(t) + \int_{-\infty}^{t} \frac{1}{\eta}\sigma(\tau)d\tau \tag{10.4.30}$$

Integrating by parts leads to

$$\varepsilon(t) = \left(\frac{1}{E} + \frac{t}{\eta}\right)\sigma(t) - \int_{-\infty}^{t} \frac{\tau}{\eta} \frac{d\sigma(\tau)}{d\tau} d\tau \tag{10.4.31}$$

Bringing the first term inside the integral,

$$\varepsilon(t) = \int_{-\infty}^{t} \left(\frac{1}{E} + \frac{t-\tau}{\eta}\right) \frac{d\sigma(\tau)}{d\tau} d\tau \tag{10.4.32}$$

or

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau) \frac{d\sigma(\tau)}{d\tau} d\tau \tag{10.4.33}$$

where the creep function is $J(t) = 1/E + t/\eta$.

If there is a jump in stress at $t = 0$, 10.4.33 can be expressed as an integral over $[0,t]$ by evaluating the contribution of the jump to the integral in 10.4.33:

$$\begin{aligned} \lim_{\vartheta\to 0} \int_{-\vartheta}^{+\vartheta} \left[\frac{1}{E} + \frac{t-\tau}{\eta}\right] \frac{\sigma(0)}{2\vartheta} d\tau &= \lim_{\vartheta\to 0} \frac{\sigma(0)}{2\vartheta} \left[\frac{1}{E}\tau + \frac{t\tau - \tau^2/2}{\eta}\right]_{-\vartheta}^{+\vartheta} = \lim_{\vartheta\to 0} \frac{\sigma(0)}{2\vartheta} 2\vartheta \left(\frac{1}{E} + \frac{t}{\eta}\right) \\ &= \left(\frac{1}{E} + \frac{t}{\eta}\right)\sigma(0) \end{aligned} \tag{10.4.34}$$

leading to

$$\varepsilon(t) = J(t)\sigma(0) + \int_{0}^{t} J(t-\tau) \frac{d\sigma(\tau)}{d\tau} d\tau \tag{10.4.35}$$

Alternatively, one could also have simply let $\sigma(t) \to H(t)\sigma(t)$ in 10.4.33, again leading to the term $\int_{-\infty}^{t} J(t-\tau)\sigma(\tau)\delta(\tau) d\tau = J(t)\sigma(0)$.

Finally, integrating by parts, one also has

$$\varepsilon(t) = J(0)\sigma(t) - \int_{0}^{t} S(t-\tau)\sigma(\tau) d\tau \tag{10.4.36}$$

where $S(t) = -dJ(t)/dt$.

3. The Kelvin Model: Functions of the Stress

Consider the differential equation for the Kelvin model, Eqn. 10.3.11,

$$\frac{d\varepsilon}{dt} + \frac{E}{\eta}\varepsilon = \frac{1}{\eta}\sigma \tag{10.4.37}$$

Using the integrating factor $e^{Et/\eta}$, one has

$$\frac{d}{dt}\left(e^{Et/\eta}\varepsilon(t)\right) = \frac{1}{\eta}e^{Et/\eta}\sigma(t) \tag{10.4.38}$$

Integrating both sides over $\left[-\infty, \hat{t}\right]$ gives

$$\left(e^{Et/\eta}\varepsilon\right)_{\hat{t}} - \left(e^{Et/\eta}\varepsilon\right)_{-\infty} = \int_{-\infty}^{\hat{t}} \frac{1}{\eta}e^{Et/\eta}\sigma(t)dt \tag{10.4.39}$$

or

$$\varepsilon(\hat{t}) = \int_{-\infty}^{\hat{t}} \frac{1}{\eta}e^{-E\left(\hat{t}-t\right)/\eta}\sigma(t)dt \tag{10.4.40}$$

Changing the notation,

$$\varepsilon(t) = \int_{-\infty}^{t} \frac{1}{\eta}e^{-E(t-\tau)/\eta}\sigma(\tau)d\tau \tag{10.4.41}$$

An integration by parts leads to

$$\varepsilon(t) = \frac{1}{E}\sigma(t) - \int_{-\infty}^{t} \frac{1}{E}e^{-E(t-\tau)/\eta}\frac{d\sigma(\tau)}{d\tau}d\tau \tag{10.4.42}$$

Finally, taking the free term inside the integral:

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau \tag{10.4.43}$$

where $J(t) = \left(1 - e^{-Et/\eta}\right)/E$ is the creep compliance function for the Kelvin model.

The other versions of the hereditary integral,. Eqn. 10.4.10, 10.4.14 can be derived from this as before.

10.5 Linear Viscoelasticity and the Laplace Transform

The Laplace transform is very useful in constructing and analysing linear viscoelastic models.

10.5.1 The Laplace Transform

The formula for the Laplace transform of the derivative of a function is[1]:

$$\begin{aligned} L(\dot{f}) &= s\bar{f} - f(0) \\ L(\ddot{f}) &= s^2 \bar{f} - sf(0) - \dot{f}(0), \qquad \text{etc.} \end{aligned} \tag{10.5.1}$$

where s is the transform variable, the overbar denotes the Laplace transform of the function, and $f(0)$ is the value of the function at time $t = 0$. The Laplace transform is defined in such a way that $f(0)$ refers to $t = 0^-$, that is, just before time zero. Some other important Laplace transforms are summarised in Table 10.5.1, in which α is a constant.

$f(t)$	$\bar{f}(s)$
α	α / s
$H(t)$	$1/s$
$\delta(t-\tau)$	$e^{-\tau s}$
$\dot{\delta}(t)$	s
$e^{-\alpha t}$	$1/(\alpha + s)$
$\left(1-e^{-\alpha t}\right)/\alpha$	$1/s(\alpha + s)$
$t/\alpha - \left(1-e^{-\alpha t}\right)/\alpha^2$	$1/s^2(\alpha + s)$
t^n	$n!/s^{1+n}, \quad n = 0, 1, \cdots$

Table 10.5.1: Laplace Transforms

Another useful formula is the time-shifting formula:

$$L\left[f(t-\tau)H(t-\tau)\right] = e^{-\tau s}\bar{f}(s) \tag{10.5.2}$$

10.5.2 Mechanical models revisited

The Maxwell Model

The Maxwell model is governed by the set of three equations 10.3.5:

[1] this rule actually only works for functions whose derivatives are continuous, although the derivative of the function being transformed may be piecewise continuous. Discontinuities in the function or its derivatives introduce additional terms

$$\varepsilon_1 = \frac{1}{E}\sigma, \quad \dot{\varepsilon}_2 = \frac{1}{\eta}\sigma, \quad \varepsilon = \varepsilon_1 + \varepsilon_2 \tag{10.5.3}$$

Taking Laplace transforms gives

$$\bar{\varepsilon}_1 = \frac{1}{E}\bar{\sigma}, \quad s\bar{\varepsilon}_2 = \frac{1}{\eta}\bar{\sigma}, \quad \bar{\varepsilon} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2 \tag{10.5.4}$$

and it has been assumed that the strain ε_2 is zero at $t = 0^-$. The three differential equations have been reduced to a set of three algebraic equations, which may now be solved to get

$$\bar{\sigma} + \frac{\eta}{E}s\bar{\sigma} = \eta s\bar{\varepsilon} \tag{10.5.5}$$

Transforming back then gives Eqn. 10.3.6:

$$\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon} \tag{10.5.6}$$

Now examine the response to a sudden load. When using the Laplace transform, the load is written as $\sigma(t) = \sigma_o H(t)$, where $H(t)$ is the Heaviside step function (see the Appendix to the previous section). Then 10.5.6 reads

$$\sigma_o H(t) + \frac{\eta}{E}\sigma_o \delta(t) = \eta\dot{\varepsilon} \tag{10.5.7}$$

Using the Laplace transform gives

$$\frac{\sigma_o}{s} + \frac{\eta}{E}\sigma_o = \eta s\bar{\varepsilon} \quad \rightarrow \quad \bar{\varepsilon} = \frac{\sigma_o}{E}\frac{1}{s} + \frac{\sigma_o}{\eta}\frac{1}{s^2} \quad \rightarrow \quad \varepsilon(t) = \frac{\sigma_o}{E}H(t) + \frac{\sigma_o}{\eta}t \tag{10.5.8}$$

which is the same result as before, Eqn. 10.3.7-8. Subsequent unloading, at time $t = \tau$ say, can be dealt with most conveniently by superimposing another load $\sigma(t) = -\sigma_o H(t-\tau)$ onto the first. Putting this into the constitutive equation and using the Laplace transform gives

$$\bar{\varepsilon} = -\frac{\sigma_o}{\eta}\frac{1}{s^2}e^{-\tau s} - \frac{\sigma_o}{E}\frac{1}{s}e^{-\tau s} \tag{10.5.9}$$

Transforming back, again using the time-shifting rule, gives

$$\varepsilon(t) = -\frac{\sigma_o}{\eta}(t-\tau)H(t-\tau) - \frac{\sigma_o}{E}H(t-\tau) \tag{10.5.10}$$

Adding this to the strain due to the first load then gives the expected result

$$\varepsilon(t) = \begin{cases} \dfrac{\sigma_o}{E} + \dfrac{\sigma_o}{\eta} t, & 0 < t < \tau \\ \dfrac{\sigma_o}{\eta} \tau, & \tau < t \end{cases} \tag{10.5.11}$$

The Kelvin Model

Taking Laplace transforms of the three equations for the Kelvin model, Eqns. 10.3.10, gives $\overline{\sigma} = E\overline{\varepsilon} + \eta s \overline{\varepsilon}$, which yields 10.3.11, $\sigma = E\varepsilon + \eta\dot{\varepsilon}$. The response to a load $\sigma(t) = \sigma_o H(t)$ is

$$\sigma_o H(t) = E\varepsilon + \eta\dot{\varepsilon} \quad \rightarrow \quad \overline{\varepsilon} = \frac{\sigma_o}{\eta} \frac{1}{s(E/\eta + s)} \quad \rightarrow \quad \varepsilon(t) = \frac{\sigma_o}{E}\left(1 - e^{-(E/\eta)t}\right) \tag{10.5.12}$$

The response to another load of magnitude $\sigma(t) = -\sigma_o H(t-\tau)$ is

$$\begin{aligned} -\sigma_o H(t-\tau) = E\varepsilon + \eta\dot{\varepsilon} \quad &\rightarrow \quad \overline{\varepsilon} = -\frac{\sigma_o}{\eta} \frac{e^{-\tau s}}{s(E/\eta + s)} \\ &\rightarrow \quad \varepsilon(t) = -\frac{\sigma_o}{E}\left(1 - e^{-(E/\eta)(t-\tau)}\right) H(t-\tau) \end{aligned} \tag{10.5.13}$$

The response to both loads now gives the complete creep and recovery response:

$$\varepsilon(t) = \begin{cases} \dfrac{\sigma_o}{E}\left(1 - e^{-(E/\eta)t}\right), & 0 < t < \tau \\ \dfrac{\sigma_o}{E} e^{-(E/\eta)t}\left(e^{(E/\eta)\tau} - 1\right), & t > \tau \end{cases} \tag{10.5.14}$$

To analyse the response to a suddenly applied strain, substitute $\varepsilon(t) = \varepsilon_o H(t)$ into the constitutive equation $\sigma = E\varepsilon + \eta\dot{\varepsilon}$ to get $\sigma = E\varepsilon_0 H(t) + \eta\varepsilon_0 \delta(t)$, which shows that the relaxation modulus of the Kelvin model is

$$E(t) = E + \eta\delta(t) \tag{10.5.15}$$

The Standard Linear Model

Consider next the standard linear model, which consists of a spring in series with a Kelvin unit, Fig. 10.5.1 (see Fig. 10.3.8a). Upon loading one expects the left-hand spring to stretch immediately. The dash pot then takes up the stress, transferring the load to the second spring as it slowly opens over time. Upon unloading one expects the left-hand spring to contract immediately and for the right-hand spring to slowly contract, being held back by the dash-pot.

The equations for this model are, from the figure,

$$\begin{aligned}
\sigma &= \sigma_1 + \sigma_2 \\
\varepsilon &= \varepsilon_1 + \varepsilon_2 \\
\sigma &= E_1 \varepsilon_1 \\
\sigma_1 &= E_2 \varepsilon_2 \\
\sigma_2 &= \eta \dot{\varepsilon}_2
\end{aligned} \tag{10.5.16}$$

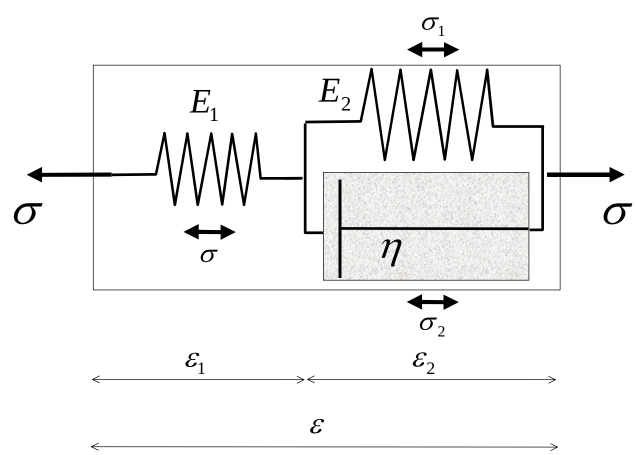


Figure 10.5.1: the standard linear model

One can eliminate the four unknowns from these five equations using the Laplace transform, giving

$$(E_1 + E_2)\bar{\sigma} + \eta s \bar{\sigma} = E_1 E_2 \bar{\varepsilon} + E_1 \eta s \bar{\varepsilon} \tag{10.5.17}$$

which transforms back to (in standard form)

$$\boxed{\sigma + \frac{\eta}{E_1 + E_2}\dot{\sigma} = \frac{E_1 E_2}{E_1 + E_2}\varepsilon + \frac{E_1 \eta}{E_1 + E_2}\dot{\varepsilon}} \tag{10.5.18}$$

which is Eqn. 10.3.16a.

The response to a load $\sigma(t) = \sigma_o H(t)$ is

$$\begin{aligned}
&(E_1 + E_2)\sigma_o H(t) + \eta \sigma_o \delta(t) = E_1 E_2 \varepsilon + E_1 \eta \dot{\varepsilon} \\
&\quad \to \quad \bar{\varepsilon} = \frac{\sigma_o}{E_1} \frac{1}{((E_2/\eta) + s)} + \frac{\sigma_o (E_1 + E_2)}{E_1 \eta} \frac{1}{s((E_2/\eta) + s)} \\
&\quad \to \quad \varepsilon(t) = \sigma_o J(t)
\end{aligned} \tag{10.5.19}$$

and the creep compliance is

$$J(t) = \frac{1}{E_1} e^{-(E_2/\eta)t} + \frac{E_1 + E_2}{E_1 E_2}\left(1 - e^{-(E_2/\eta)t}\right)$$
$$= \frac{1}{E_1} + \frac{1}{E_2}\left(1 - e^{-(E_2/\eta)t}\right) \tag{10.5.20}$$

Note that $\varepsilon(0) = \sigma_o / E_1$ as expected.

For recovery one can superimpose an opposite load onto the first, at time τ say:

$$-\left(E_1 + E_2\right)\sigma_o H(t-\tau) - \eta\sigma_o \delta(t-\tau) = E_1 E_2 \varepsilon + E_1 \eta \dot{\varepsilon}$$
$$\rightarrow \quad \bar{\varepsilon} = -\frac{\sigma_o}{E_1}\frac{1}{(E_2/\eta + s)} e^{-\tau s} - \sigma_o \left(\frac{E_1 + E_2}{E_1 \eta}\right)\frac{1}{s(E_2/\eta + s)} e^{-\tau s} \tag{10.5.21}$$
$$\rightarrow \quad \varepsilon(t) = -\sigma_o H(t-\tau)\left\{\frac{1}{E_1} e^{-(E_2/\eta)(t-\tau)} + \left(\frac{E_1 + E_2}{E_1 E_2}\right)\left(1 - e^{-(E_2/\eta)(t-\tau)}\right)\right\}$$

The response after time τ is then

$$\varepsilon(t) = \frac{\sigma_o}{E_2} e^{-(E_2/\eta)t}\left(e^{(E_2/\eta)\tau} - 1\right) \tag{10.5.22}$$

This is, as expected, simply the recovery response of the Kelvin unit. The full response is as shown in Fig. 10.5.2. This seems to be fairly close now to the response of a real material as discussed in §10.1, although it does not allow for a permanent strain.

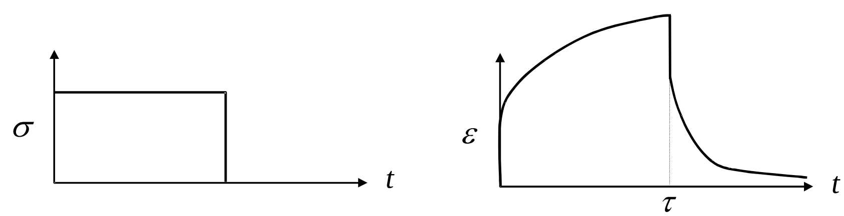


Figure 10.5.2: Creep-recovery response of the standard linear model

Non-constant Loading

The response to a complex loading history can be evaluated by solving the differential constitutive equation (or the corresponding hereditary integral). The differential equation can be most easily solved using Laplace transforms.

Example

Consider the example treated earlier using hereditary integrals, at the end of §10.4.2. Load (1) of Fig. 10.4.5 can be thought of as consisting of the two loads (1a) $\sigma = (\hat{\sigma}/T)t$ and (1b) $\sigma = -(\hat{\sigma}/T)(t-T)H(t-T)$ applied at time $t = T$. Load (2) consists of a constant load applied at time $t = T$.

For load (1a),

$$\frac{\hat{\sigma}}{T}t + \frac{\eta}{E}\frac{\hat{\sigma}}{T} = \eta\dot{\varepsilon} \quad \rightarrow \quad \bar{\varepsilon} = \frac{\hat{\sigma}}{\eta T}\frac{1}{s^3} + \frac{1}{E}\frac{\hat{\sigma}}{T}\frac{1}{s^2} \quad \rightarrow \quad \frac{\varepsilon(t)}{\hat{\sigma}} = \frac{1}{ET}t + \frac{1}{2\eta T}t^2$$

which gives the response for $t < T$.

For load (1b) one has [note: $L\left((t-\tau)\delta(t-\tau)\right) = 0$]

$$\begin{aligned}
\eta\dot{\varepsilon} &= -\frac{\hat{\sigma}}{T}(t-T)H(t-T) - \frac{\eta}{E}\frac{\hat{\sigma}}{\tau}\left[(t-T)\delta(t-T) + H(t-T)\right] \\
\rightarrow \quad s\bar{\varepsilon} &= -\frac{\hat{\sigma}}{\eta T}e^{-Ts}\frac{1}{s^2} - \frac{1}{E}\frac{\hat{\sigma}}{T}e^{-Ts}\frac{1}{s} \\
\rightarrow \quad \bar{\varepsilon} &= -\frac{\hat{\sigma}}{\eta\tau}e^{-Ts}\frac{1}{s^3} - \frac{1}{E}\frac{\hat{\sigma}}{\tau}e^{-Ts}\frac{1}{s^2} \\
\rightarrow \quad \frac{\varepsilon(t)}{\hat{\sigma}} &= H(t-T)\left\{-\frac{1}{2\eta T}(t-T)^2 - \frac{1}{E\tau}(t-T)\right\}
\end{aligned}$$

The response after time T is then given by adding the two results:

$$\frac{\varepsilon(t)}{\hat{\sigma}} = \frac{1}{E} + \frac{1}{\eta}\left(t - \frac{T}{2}\right)$$

10.5.3 Relationship between Creep and Relaxation

Taking the Laplace transform of the general constitutive equation 10.3.19, $\mathbf{P}\sigma = \mathbf{Q}\varepsilon$, leads to

$$\left(p_o + p_1 s + p_2 s^2 + p_3 s^3 + p_4 s^4 + \cdots\right)\bar{\sigma} = \left(q_o + q_1 s + q_2 s^2 + q_3 s^3 + q_4 s^4 + \cdots\right)\bar{\varepsilon} \tag{10.5.23}$$

which can also be written in the contracted form

$$P(s)\bar{\sigma} = Q(s)\bar{\varepsilon} \tag{10.5.24}$$

where P and Q are the polynomials

$$P(s) = \sum_{i=0}^{n} p_i s^i , \qquad Q(s) = \sum_{i=0}^{n} q_i s^i \tag{10.5.25}$$

The Laplace transforms of the creep compliance $\left(J(t) \rightarrow \bar{J}(s)\right)$ and relaxation modulus $\left(E(t) \rightarrow \bar{E}(s)\right)$ can be written in terms of these polynomials as follows. First, the strain due to a unit load $\sigma = H(t)$ is $J(t)$. Since $\bar{\sigma} = 1/s$, substitution into the above equation gives

$$\bar{J}(s) = \frac{P(s)}{sQ(s)} \tag{10.5.26}$$

Similarly, the stress due to a unit strain $\varepsilon = H(t)$ is $E(t)$ and so

$$\bar{E}(s) = \frac{Q(s)}{sP(s)} \tag{10.5.27}$$

It follows that

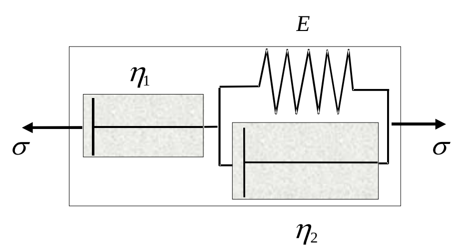
$$\boxed{\bar{J}(s)\bar{E}(s) = \frac{1}{s^2}} \tag{10.5.28}$$

Thus, for a linear viscoelastic material, there is a unique and simple relationship between the creep and relaxation behaviour.

10.5.4 Problems

1. Check that the relation 10.5.28, $\bar{J}(s)\bar{E}(s) = 1/s^2$, holds for the Kelvin model

2. (a) Derive the constitutive relation (in standard form) for the three-element model shown below using the Laplace transform (this is the Standard Fluid II of Fig. 10.3.8d and the constitutive relation is given by Eqn. 10.3.16d)
   (b) Derive the creep compliance $J(t)$ by considering a suddenly applied load.



10.6 Oscillatory Stress, Dynamic Loading and Vibrations

Creep and relaxation experiments do not provide complete information concerning the mechanical behaviour of viscoelastic materials. These experiments usually provide test data in the time-range from 10 seconds to 10 years. It is often of interest to know the response of materials to loads of very short duration. For example, duration of the impact of a steel ball on a viscoelastic block may be of the order of 10^{-5} sec [1]. In order to be able to determine the response for such conditions, it is necessary to know the behaviour of a material at high rates of loading (or short duration loading).

The techniques and apparatus for investigating the response of a material to very short term loading are different to those involved in longer-term testing. For very short time loading it is more convenient to use oscillatory than static loading, and in order to predict the behaviour of a viscoelastic material subjected to an oscillatory load, one needs to formulate the theory based on oscillatory stresses and strains.

10.6.1 Oscillatory Stress

Consider a dynamic load of the form

$$\sigma(t) = \sigma_o \cos(\omega t) \tag{10.6.1}$$

where σ_o is the stress amplitude and ω is the angular frequency[2]. Assume that the resulting strain is of the form[3]

$$\varepsilon(t) = \varepsilon_o \cos(\omega t - \delta) \tag{10.6.2}$$

so that the strain is an oscillation at the same frequency as the stress but lags behind by a phase angle δ, Fig. 10.6.1. This angle is referred to as the **loss angle** of the material, for reasons which will become clear later.

Expanding the strain trigonometric terms,

$$\varepsilon(t) = \varepsilon_o \cos\delta \cos\omega t + \varepsilon_o \sin\delta \sin\omega t \tag{10.6.3}$$

The first term here is completely in phase with the input; the second term is completely out of phase with the input. If the phase angle δ is zero, then the stress and strain are in

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[1] dynamic experiments usually provide data from about 10^{-8} sec. to about 10^{3} sec. so there is a somewhat overlapping region where data can be obtained from both types of experiment

[2] when an oscillatory force is first applied, transient vibrations result at the natural frequency of the material – these soon die out leaving the vibrations at the source frequency

[3] if one substitutes 10.6.1 into the general constitutive equation 10.3.17, one sees that the strain and its derivatives contain sine and cosine terms, so that the strain must be of the general form $A\cos(\omega t) + B\sin(\omega t)$, where *A* and *B* are constants. For convenience, this can be written as $C\cos(\omega t - D)$ where *C* and *D* are new constants

phase (as happens with an ideal elastic material), whereas if $\delta = \pi / 2$, the stress and strain are completely out of phase.

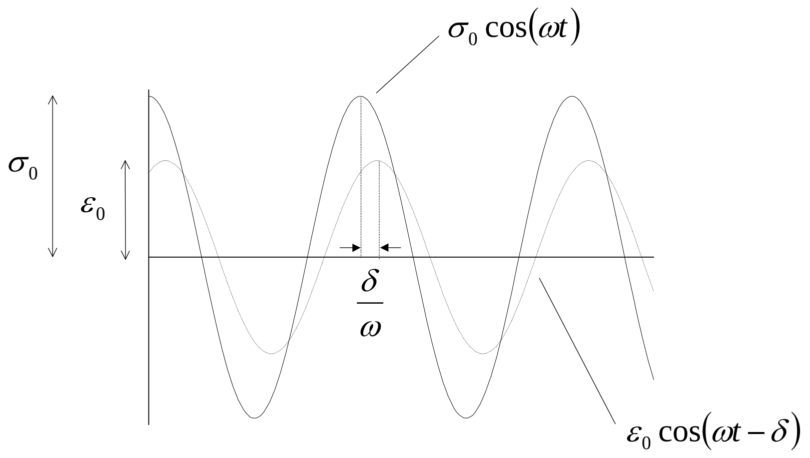

Figure 10.6.1: Oscillatory stress and strain

The Complex Compliance

Define

$$J_1 = \frac{\varepsilon_o}{\sigma_o}\cos\delta, \qquad J_2 = \frac{\varepsilon_o}{\sigma_o}\sin\delta \tag{10.6.4}$$

so that

$$\varepsilon(t) = \sigma_o\left(J_1 \cos\omega t + J_2 \sin\omega t\right) \tag{10.6.5}$$

The quantities J_1 and J_2 are a measure of how in, or out of, phase the stress is with the strain. The former, J_1, is called the **storage compliance** and the latter, J_2, is called the **loss compliance**. They are usually written as the components of a **complex compliance**, J^*:

$$J^* = J_1 - iJ_2 \tag{10.6.6}$$

If one has a stress input in the form of a sine function, then

$$\begin{aligned}
\sigma(t) &= \sigma_o \sin(\omega t) \\
\varepsilon(t) &= \varepsilon_o \sin(\omega t - \delta) \\
&= \varepsilon_o \cos\delta \sin\omega t - \varepsilon_o \sin\delta\cos\omega t \\
&= \sigma_o\left(J_1 \sin\omega t - J_2 \cos\omega t\right)
\end{aligned} \tag{10.6.7}$$

and again the storage compliance is a measure of the amount "in phase" and the loss compliance is a measure of the amount "out of phase".

The Complex Modulus

One can also regard of the strain as the input and the stress as the output. In that case one can write (δ is again the phase angle by which the strain lags behind the stress)

$$\begin{aligned}\varepsilon(t) &= \varepsilon_o \cos(\omega t) \\ \sigma(t) &= \sigma_o \cos(\omega t + \delta) \\ &= \sigma_o \cos\delta \cos\omega t - \sigma_o \sin\delta \sin\omega t\end{aligned} \tag{10.6.8}$$

This is in effect the same stress-strain relationship as that used above, only the stress/strain are shifted along the *t*-axis.

Define next the two new quantities

$$E_1 = \frac{\sigma_o}{\varepsilon_o}\cos\delta, \qquad E_2 = \frac{\sigma_o}{\varepsilon_o}\sin\delta \tag{10.6.9}$$

so that

$$\sigma(t) = \varepsilon_o\left(E_1 \cos\omega t - E_2 \sin\omega t\right) \tag{10.6.10}$$

Again, these quantities are a measure of how much the response is in phase with the input. The former, E_1, is called the **storage modulus** and the latter, E_2, is called the **loss modulus**. As with the compliances, they are usually written as the components of a **complex modulus**[4], E^*:

$$E^* = E_1 + iE_2 \tag{10.6.11}$$

Again, if one has a sinusoidal strain as input, one can write

$$\begin{aligned}\varepsilon(t) &= \varepsilon_o \sin(\omega t) \\ \sigma(t) &= \sigma_o \sin(\omega t + \delta) \\ &= \sigma_o \cos\delta \sin\omega t + \sigma_o \sin\delta \cos\omega t \\ &= \varepsilon_o\left(E_1 \sin\omega t + E_2 \cos\omega t\right)\end{aligned} \tag{10.6.12}$$

It is apparent from the above that

$$J^* E^* = 1 \tag{10.6.13}$$

which is a much simpler relationship than that between the creep compliance function and the relaxation modulus (which involved Laplace transforms, Eqn. 10.5.28).

[4] typical values for the storage and loss moduli for a polymer would be around $E_1 = 10$ MPa, $E_2 = 0.1$ MPa. The ratio of the amplitudes is called the **dynamic modulus**, $\left|E^*\right| = \sigma_o / \varepsilon_o$.

Complex Formulation

The above equations can be succinctly written using a complex formulation, using Euler's formula

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta \tag{10.6.14}$$

For a stress input,

$$\begin{aligned}\sigma(t) &= \sigma_o e^{i\omega t}\\ \varepsilon(t) &= \varepsilon_o e^{i(\omega t-\delta)}\\ &= \varepsilon_o\left(\cos\delta - i\sin\delta\right)e^{i\omega t}\\ &= \sigma_o\left[J_1 - iJ_2\right]e^{i\omega t}\\ &= \sigma_o J^* e^{i\omega t}\end{aligned} \tag{10.6.15}$$

The creep compliance function $J(t)$ is the strain response to a unit load. In the same way, from 10.6.15, the complex compliance J^* can be interpreted as the strain amplitude response to a sinusoidal stress input of unit magnitude.

Similarly, for a strain input, one has

$$\begin{aligned}\varepsilon(t) &= \varepsilon_o e^{i\omega t}\\ \sigma(t) &= \sigma_o e^{i(\omega t+\delta)}\\ &= \sigma_o\left(\cos\delta + i\sin\delta\right)e^{i\omega t}\\ &= \varepsilon_o E^* e^{i\omega t}\end{aligned} \tag{10.6.16}$$

and the term in brackets is, by definition, the complex modulus E^*.

The relationship between the complex compliance/modulus and the differential constitutive equation

Putting $\sigma(t) = \sigma_o e^{i\omega t}$ and the resulting strain $\varepsilon(t) = \varepsilon_o e^{(i\omega-\delta)t}$ into the general differential operator form of the constitutive equation 10.3.19, one has

$$\begin{aligned}&\left[p_o + p_1(i\omega) + p_2(i\omega)^2 + p_3(i\omega)^3 + \cdots\right]\sigma_o e^{i\omega t}\\ &\quad = \left[q_o + q_1(i\omega) + q_2(i\omega)^2 + q_3(i\omega)^3 + \cdots\right]\sigma_o J^* e^{i\omega t}\end{aligned} \tag{10.6.17}$$

This equation thus gives the relationship between the complex compliance and the constants p_i, q_i. A similar relationship can be found for the complex modulus:

$$E^* = \frac{q_o + q_1(i\omega) + q_2(i\omega)^2 + q_3(i\omega)^3 + \cdots}{p_o + p_1(i\omega) + p_2(i\omega)^2 + p_3(i\omega)^3 + \cdots} \tag{10.6.18}$$

Again one sees that $J^* E^* = 1$.

From 10.6.17-18, the complex compliance and complex modulus are functions of the frequency ω, and thus, from the definitions 10.6.4, 10.6.6, 10.6.9, 10.6.11, so is the phase angle δ. Thus ω is the primary variable influencing the viscoelastic properties (whereas time *t* was used for this purpose in the analysis of static loading).

The relationship between the complex compliance/modulus and the creep compliance/ relaxation modulus

It can be shown[5] that the complex compliance $J^*(\omega)$ and the complex modulus $E^*(\omega)$ are related to the creep compliance $J(t)$ and relaxation modulus $E(t)$ through

$$\begin{aligned} J^*(\omega) &= (i\omega)L[J(t)]_{s=i\omega} \\ E^*(\omega) &= (i\omega)L[E(t)]_{s=i\omega} \end{aligned}, \qquad (10.6.19)$$

Here, the Laplace transform is first taken and then evaluated at $s = i\omega$ [6].

A Note on Frequency

Frequencies below 0.1 Hz are associated with seismic waves. Vibrations of structures and solid objects occur from about 0.1 Hz to 10 kHz depending on the size of the structure. Stress waves from 20 Hz to 20 kHz are perceived as sound - above 20 kHz is the ultrasonic range. Frequencies above 10^{12} Hz correspond to molecular vibration and represent an upper limit for stress waves in real solids.

10.6.2 Example: The Maxwell Model

The constitutive equation for the Maxwell model is given by Eqn. 10.3.6,

$$\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon} \qquad (10.6.20)$$

Consider an oscillatory stress $\sigma = \sigma_o \cos(\omega t)$. We thus have[7]

$$\begin{aligned} \sigma_o \cos(\omega t) - \frac{\eta}{E}\omega\sigma_o \sin(\omega t) = \eta\dot{\varepsilon} &\to \int d\varepsilon = \sigma_o \left\{ -\frac{\omega}{E}\int \sin(\omega t)dt + \frac{1}{\eta}\int \cos(\omega t)dt \right\} \\ &\to \varepsilon(t) = \sigma_o \left\{ \frac{1}{E}\cos(\omega t) + \frac{1}{\omega\eta}\sin(\omega t) \right\} \end{aligned} \qquad (10.6.21)$$

[5] using Fourier transform theory for example

[6] J_1 and J_2 are also related to each other (as are E_1 and E_2) by an even more complicated rule known as the **Kramers-Kronig relation**

[7] the constant of integration is zero (assuming that the initial strain is that in the spring, σ_o / E).

Thus the complex compliance is

$$J^* = J_1 - iJ_2 = \frac{1}{E} - i\frac{1}{\omega\eta} \qquad (10.6.22)$$

This result can be obtained more easily using the relationship between the complex compliance and the constitutive equation: the constitutive equation can be rewritten as

$$p_o\sigma + p_1\dot{\sigma} = q_0\varepsilon + q_1\dot{\varepsilon}, \quad \text{where} \quad p_o = 1,\ p_1 = \frac{\eta}{E},\ q_0 = 0,\ q_1 = \eta \qquad (10.6.23)$$

From Eqn. 10.6.17,

$$J^* = \frac{p_o + p_1(i\omega) + p_2(i\omega)^2 + \cdots}{q_o + q_1(i\omega) + q_2(i\omega)^2 + \cdots} = \frac{1 + (\eta / E)(i\omega)}{\eta(i\omega)} = \frac{1}{E} - i\frac{1}{\omega\eta} \qquad (10.6.24)$$

Also, the complex modulus is related to the complex compliance through 10.6.13, $E^* = 1/J^*$, so that

$$E^* = \frac{(\omega\eta)^2 E}{(\omega\eta)^2 + E^2} + i\frac{\omega\eta E^2}{(\omega\eta)^2 + E^2} \qquad (10.6.25)$$

For very low frequencies, $\omega \to 0$, $\sin(\omega t)/\omega \to t$, and the response, as expected, reduces to that for a static load, $\varepsilon(t) = \sigma_o\left(1/E + t/\eta\right)$.

For very high frequencies, $1/\omega \to 0$, and the response is $\varepsilon(t) = \left(\sigma_o / E\right)\cos(\omega t)$. Thus the strain is completely in-phase with the load, but the dash-pot is not moving – it has no time to respond at such high frequencies - the spring/dash-pot model is reacting like an isolated spring, that is, like a solid, with no fluid behaviour.

10.6.3 Energy Dissipation

Because the equations 10.6.12

$$\varepsilon(t) = \varepsilon_o \sin(\omega t), \quad \sigma(t) = \sigma_o \sin(\omega t + \delta) \qquad (10.6.26)$$

are the parametric equations for an ellipse, that is, they trace out an ellipse for values of *t*, the stress-strain curve for an oscillatory stress is an elliptic hysteresis loop, Fig. 10.6.2.

The work done in stressing a material (per unit volume) is given by

$$W = \int \sigma d\varepsilon \qquad (10.6.27)$$

The energy lost ΔW through internal friction and heat is given by the area of the ellipse. Thus

$$\Delta W = \int_{t_1}^{t_1+T} \sigma d\varepsilon = \int_{t_1}^{t_1+T} \sigma \frac{d\varepsilon}{dt} dt \tag{10.6.28}$$

where t_1 is some starting time and T is the period of oscillation, $T = 2\pi / \omega$. Substituting in Eqns. 10.6.26 for strain and stress then gives

$$\begin{aligned} \Delta W &= \omega \sigma_o \varepsilon_o \int_{t_1}^{t_1+T} \sin(\omega t + \delta)\cos(\omega t) dt \\ &= \frac{1}{2}\omega \sigma_o \varepsilon_o \int_{t_1}^{t_1+T} [\sin(2\omega t + \delta) + \sin\delta] dt \\ &= \frac{1}{2}\omega \sigma_o \varepsilon_o \left[-\frac{\cos(2\omega t + \delta)}{2\omega} + t\sin\delta \right]_{t_1}^{t_1+T} \end{aligned} \tag{10.6.29}$$

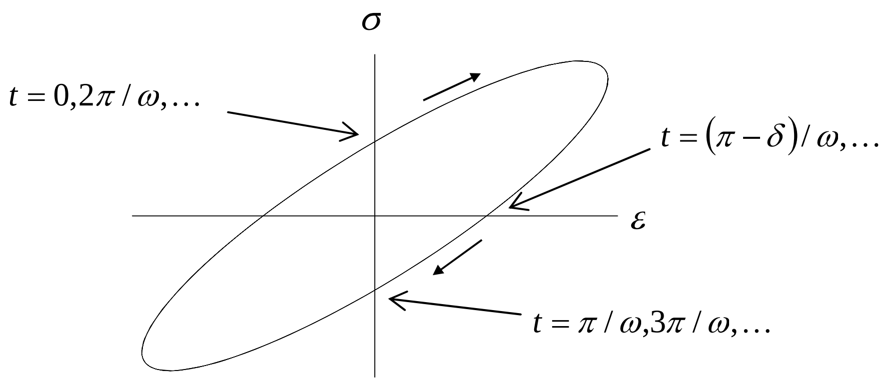

Figure10.6.2: Elliptic Stress-Strain Hysteresis Loop

Taking $t_1 = 0$ then gives[8]

$$\boxed{\Delta W = \pi \sigma_o \varepsilon_o \sin\delta} \qquad \textbf{Energy Loss} \tag{10.6.30}$$

When $\delta = 0$, the energy dissipated is zero, as in an elastic material. It can also be seen that

$$\Delta W = \pi \varepsilon_0^2 E_2 = \pi \sigma_0^2 J_2 \tag{10.6.31}$$

and hence the names *loss* modulus and *loss* compliance.

[8] the same result is obtained for $\sigma = \sigma_o \sin(\omega t),\ \varepsilon = \varepsilon_o \sin(\omega t - \delta)$ or when the stress and strain are cosine functions

Damping Energy

The energy stored after one complete cycle is zero since the material has returned to its original configuration. The maximum energy stored during any one cycle can be computed by integrating the increment of work $\sigma d\varepsilon$ from zero up to a maximum stress, that is over one quarter the period T of one cycle. Thus, integrating from $t_1 = -\delta/\omega$ (where $\sigma = 0$) to $t_2 = t_1 + \pi/2\omega$, Fig. 10.6.3[9]

$$W = \sigma_o \varepsilon_o \left[\frac{\cos\delta}{2} + \frac{\pi}{4}\sin\delta \right] \tag{10.6.32}$$

The second term is $\pi\sigma_o\varepsilon_o \sin\delta/4$, which is one quarter of the energy dissipated per cycle, and so can be considered to represent the dissipated energy. The remaining, first, term represents the area of the shaded triangle in Fig. 10.6.3 and can be considered to be the energy stored, $W_s = \sigma_o\varepsilon_o \cos\delta/2$ (it reduces to the elastic solution $W = \sigma_o\varepsilon_o/2$ when $\delta = 0$).

The **damping energy** of a viscoelastic material is defined as $\Delta W / W_S$, where W_S is the maximum energy the system can store in a given stress/strain amplitude. Thus (dividing ΔW by 4 so it is consistent with the integration over a quarter-cycle to obtain the stored energy)

$$\boxed{\frac{\Delta W}{W_s} = \frac{\pi}{2}\tan\delta} \qquad \textbf{Damping Energy} \tag{10.6.33}$$

Thus the damping ability of a linearly viscoelastic material is only dependent on the phase/loss angle δ.

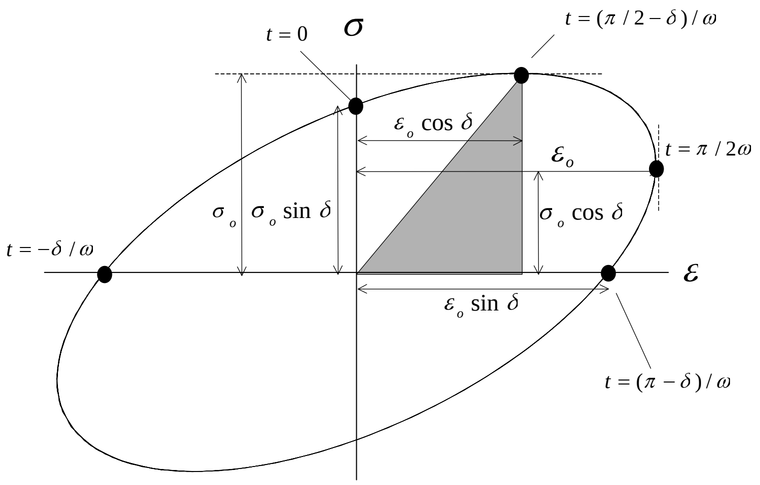

Figure10.6.3: Elliptic Stress-Strain Hysteresis Loop

[9] or one could integrate from zero to maximum strain, over $[0, \pi/2\omega]$, giving the same result

The quantity $\tan\delta$ is known as the **mechanical loss**, or the **loss tangent**. It can be considered to be the fundamental measure of damping in a linear material (other measures, for example δ, $2\pi\tan\delta$, etc., are often used)[10]. Typical values for a range of materials at various temperatures and frequencies are shown in Table 10.6.1.

Material	Temperature	Frequency (ν)	Loss Tangent ($\tan\delta$)
Sapphire	4.2 K	30 kHz	2.5×10^{-10}
Sapphire	rt	30 kHz	5×10^{-9}
Silicon	rt	20 kHz	3×10^{-8}
Quartz	rt	1 MHz	$\approx 10^{-7}$
Aluminium	rt	20 kHz	$< 10^{-5}$
Cu-31%Zn	rt	6 kHz	9×10^{-5}
Steel	rt	1 Hz	0.0005
Aluminium	rt	1 Hz	0.001
Fe-0.6%V	33°C	0.95 Hz	0.0016
Basalt	rt	0.001-0.5 Hz	0.0017
Granite	rt	0.001-0.5 Hz	0.0031
Glass	rt	1 Hz	0.0043
Wood	rt	≈ 1 Hz	0.02
Bone	37°C	1-100 Hz	0.01
Lead	rt	1-15 Hz	0.029
PMMA	rt	1 Hz	0.1

Table 10.6.1: Loss Tangents of Common Materials[11]

10.6.4 Impact

Consider the impact of a viscoelastic ball dropped from a height h_d onto a rigid floor. During the impact, a proportion of the initial potential energy mgh_d, which is now kinetic energy $\frac{1}{2}mv^2$, where v is the velocity at impact, is lost and only some is stored. The stored energy is converted back to kinetic energy which drives the ball up on the rebound, reaching a height $h_r < h_d$, with final potential energy mgh_r. The ratio of the two heights is[12]

$$f \equiv \frac{h_r}{h_d} = \frac{mgh_r}{mgh_d} = \frac{W_s}{W_s + W_d} \tag{10.6.34}$$

where W_s is the energy stored and W_d is the energy dissipated during the impact.

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[10] some investigators recommend that one uses the maximum storable energy when $\delta = 0$, in which case the stored energy is $\sigma_o\varepsilon_o/2$ and the damping measure would be $\Delta W/W_S = \pi\sin\delta/2$

[11] from Table 7.1 of Viscoelastic Solids, by R. S. Lakes, CRC Press, 1999

[12] the **coefficient of restitution** e is defined as the ratio of the velocities before and after impact, $e = v_r/v_d$, so $f = e^2$.

The impact event can be approximated by a half-cycle of the oscillatory stress-strain curve, Fig. 10.6.4. Integrating over $[0, \pi/\omega]$ or $[-\delta/\omega, (\pi-\delta)/\omega]$, one has[13]

$$W = \frac{1}{2}\sigma_o \varepsilon_o [\cos\delta + \pi \sin\delta] \tag{10.6.35}$$

and so the "height lost" is given by

$$f = 1 - \frac{W_d}{W_s + W_d} \approx 1 - \frac{W_d}{W_s} = 1 - \pi \tan\delta \tag{10.6.36}$$

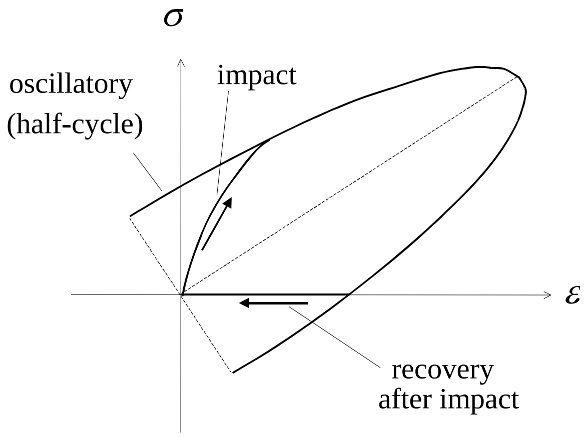

Figure10.6.4: Impact approximated as a half-cycle of oscillatory stress and strain

Note some other approximations made:
(i) energy losses due to air resistance, friction and radiation of sound energy during impact have been neglected
(ii) in a real impact, the stress and strain are both initially zero. In the current analysis, when one of these quantities is zero, the other is finite, and this will inevitably introduce some error[14].

10.6.5 Damping of Vibrations

The inertial force in many applications can be neglected. However, when dealing with vibrations, the product of acceleration times mass can be appreciable when compared to the other forces present.

Vibrational damping can be examined by looking at a simple oscillator with one degree of freedom, Fig. 10.6.5. A mass m is connected to a wall by a viscoelastic bar of length L and cross sectional area A. The motion of the system is described by the equations

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[13] although it might be more accurate to integrate over $[0, (\pi-\delta)/\omega]$

[14] as mentioned, there is a transient term involved in the oscillation which has been ignored, and which dies out over time, leaving the strain to lag behind the stress at a constant phase angle

Dynamic equation: $m\ddot{x} + F = 0$
Kinematic relation: $\varepsilon = x / L$
Constitutive relation: (depends on model)

Assuming an oscillatory motion, $x = x_o e^{i\omega t}$, and using the first two of these,

$$-\omega^2 m x_o e^{i\omega t} + A\sigma = 0 \qquad \to \quad \sigma = \frac{x_o}{L}\frac{L\omega^2 m}{A} e^{i\omega t} = \varepsilon_o \left[\frac{L\omega^2 m}{A}\right] e^{i\omega t} \tag{10.6.37}$$

The quantity in brackets is the complex modulus E^* (see Eqn. 10.6.16).

As an example, for the Maxwell model (see Eqn. 10.6.24)

$$E^* = \left[\frac{1}{E} - \frac{i}{\eta\omega}\right]^{-1} \tag{10.6.38}$$

and so

$$\frac{1}{E}\omega^2 - \frac{i}{\eta}\omega = \frac{A}{Lm}, \tag{10.6.39}$$

which can be solved to get

$$\omega = \left\{\frac{E}{2\eta} i \pm \sqrt{\frac{AE}{Lm} - \frac{E^2}{4\eta^2}}\right\} \tag{10.6.40}$$

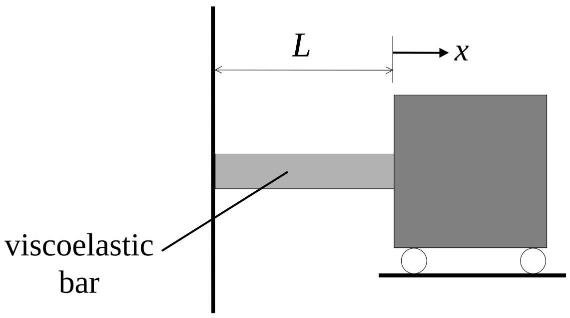

Figure10.6.5: Vibration

If m is small or E is large (and E / η is not too large) the root has a real part, v say, so that

$$\omega = i(E / 2\eta) \pm v \tag{10.6.41}$$

and one has the damped vibration

$$\begin{aligned} x &= x_o\left(c_1 e^{i\omega_1 t} + c_2 e^{i\omega_2 t}\right) \\ &= x_o e^{-(E/2\eta)t}\left(c_1 e^{ivt} + c_2 e^{-ivt}\right) \\ &= x_o e^{-(E/2\eta)t}\left(A\cos(vt) + B\sin(vt)\right) \end{aligned} \tag{10.6.42}$$

If, on the other hand, the mass is large or the spring compliant, one gets a pure imaginary root, $\omega = i(E/2\eta) \pm iv$, so that $i\omega$ is real (and less than zero) and one has the aperiodic damping

$$x = x_o\left(c_1 e^{(E/2\eta+v)t} + c_2 e^{(E/2\eta-v)t}\right) \tag{10.6.43}$$

10.6.6 Problems

1. Use the differential form of the constitutive equation for a linearly viscoelastic material to derive the *complex compliance*, the *complex modulus*, and the *loss tangent* for a Kelvin material. (put the first two in the form $\alpha + i\beta$). Use your expression for the complex compliance to derive the strain response to a stress $\sigma_o \cos(\omega t)$, in terms of $\sigma_o, \omega, t, E, \eta$, in the form
$$\varepsilon(t) = \sigma_o\left(A\cos\omega t + B\sin\omega t\right)$$
What happens at very low frequencies?

10.7 Temperature-dependent Viscoelastic Materials

Many materials, for example polymeric materials, have a response which is strongly temperature-dependent. Temperature effects can be incorporated into the theory discussed thus far in a simple way by allowing for the coefficients of the differential constitutive equations to be functions of temperature. Thus, Eqn. 10.3.19 can be expressed more generally as

$$p_o(\theta)\sigma + p_1(\theta)\dot{\sigma} + p_2(\theta)\ddot{\sigma} + \cdots = q_o(\theta)\varepsilon + q_1(\theta)\dot{\varepsilon} + q_2(\theta)\ddot{\varepsilon} + \cdots \qquad (10.7.1)$$

where θ denotes temperature. Equivalently, one can allow for the creep and relaxation functions to be functions of temperature in the hereditary integral formulation. Thus Eqns. 10.4.20-21 read

$$\begin{aligned} \varepsilon(t,\theta) &= \sigma(0)J(t,\theta) + \int_0^t J(t-\tau,\theta)\frac{d\sigma(\tau)}{d\tau}d\tau \\ \sigma(t,\theta) &= \varepsilon(0)E(t,\theta) + \int_0^t E(t-\tau,\theta)\frac{d\varepsilon(\tau)}{d\tau}d\tau \end{aligned} \qquad (10.7.2)$$

10.7.1 Example: The Maxwell Model

Consider a Maxwell material whose dash-pot viscosity η is a function of temperature θ. The differential constitutive equation is then

$$\sigma + \frac{\eta(\theta)}{\overline{E}}\frac{d\sigma}{dt} = \eta(\theta)\frac{d\varepsilon}{dt} \qquad (10.7.3)$$

where $\overline{E}$ is the temperature-independent spring stiffness. This equation is a function of both temperature and time. With temperature a function of time, $\theta = \theta(t)$, it is a linear differential equation with non-constant coefficients. For constant temperature, it has constant coefficients.

Consider first the case of constant temperature. The relaxation modulus and creep compliance functions can be evaluated by applying unit strain and unit stress. From the previous work, one has

$$\begin{aligned} &E(t,\theta) = \overline{E}e^{-t/t_R(\theta)}, \quad t_R(\theta) = \frac{\eta(\theta)}{\overline{E}} \\ &J(t,\theta) = \frac{1}{\overline{E}} + \frac{t}{\eta(\theta)} \end{aligned} \qquad (10.7.4)$$

Thus any given material has temperature-dependent relaxation and creep functions.

Consider now the change of variable

$$\xi = A\frac{t}{\eta(\theta)} \tag{10.7.5}$$

where A is any constant (which can be chosen arbitrarily for convenience – see later). This transforms Eqn. 10.7.3 into

$$\sigma(\xi) + \frac{A}{\overline{\overline{E}}}\frac{d\sigma}{d\xi} = A\frac{d\varepsilon}{d\xi} \tag{10.7.6}$$

This is now an equation with dependence on only one variable, ξ. From this equation, one obtains relaxation and creep functions

$$\begin{aligned} E(\xi) &= \overline{E}e^{-\xi/t_R}, \quad t_R = \frac{A}{\overline{\overline{E}}} \\ J(\xi) &= \frac{1}{\overline{\overline{E}}} + \frac{\xi}{A} \end{aligned} \tag{10.7.7}$$

These equations generate **master curves** from which the different temperature-dependent curves 10.7.4 can be obtained.

Example Data

For example, consider a viscosity which varies linearly over the range $-100^{\circ}\mathrm{C} < \theta < 100^{\circ}\mathrm{C}$ according to the relation

$$\eta(\theta) = \eta_0\left[1 - A_\eta\left(\frac{\theta}{\theta_0} - 1\right)\right] \tag{10.7.8}$$

where η_0 is a constant viscosity, $A_\eta = 0.2$ and $\theta_0 = 20^{\circ}\mathrm{C}$ (a reference temperature at which $\eta(\theta) = \eta_0$. This function is plotted in Fig. 10.7.1 below.

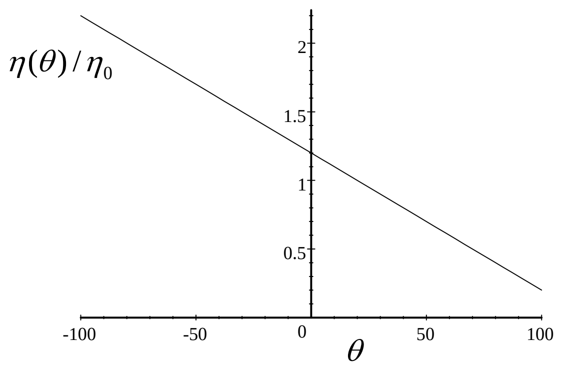

Figure 10.7.1: linear dependence of viscosity on temperature

Also, let $\eta_0 / \overline{E} = m$. The resulting relaxation and creep functions of Eqn. 10.7.4 are plotted in Fig. 10.7.2 below (for $m = 5$).

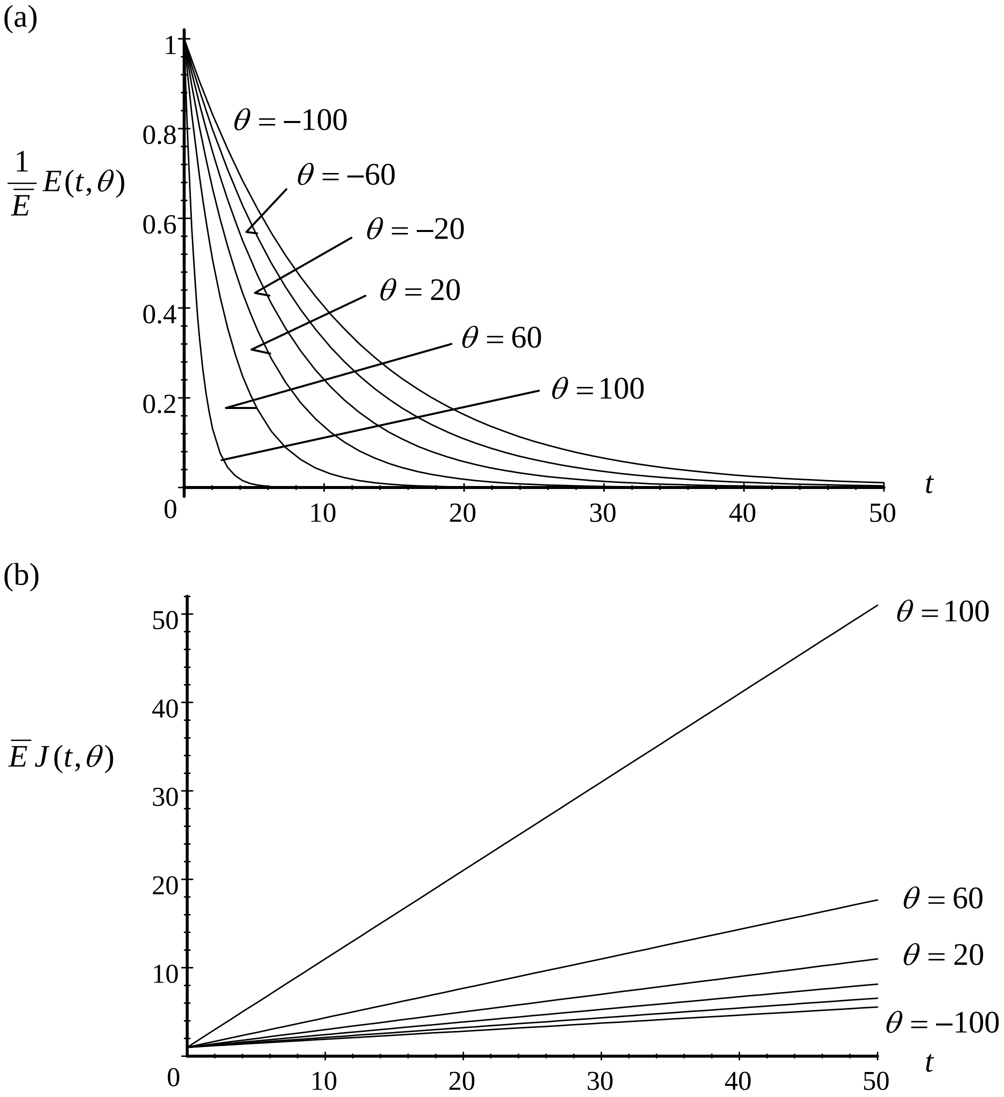

Figure 10.7.2: temperature-dependent functions; (a) relaxation modulus, (b) creep compliance

Note the following, referring to Fig. 10.7.2:

(i) for temperatures greater than the reference temperature $\theta = \theta_0 = 20^{\mathrm{o}}$ (see Eqn. 10.7.8), the viscosity is $\eta(\theta) < \eta_0$. This implies that, for $\theta > \theta_0$, the relaxation times are shorter than for $\theta = \theta_0$ (see Eqn. 10.7.4a), Fig. 10.7.2a, and the slope of the creep curves is greater than for $\theta = \theta_0$ (see Eqn. 10.7.4b)., Fig. 10.7.2b.

(ii) for temperatures smaller than the reference temperature, $\eta(\theta) > \eta_0$. Thus, for $\theta < \theta_0$, the relaxation times are longer than for $\theta = \theta_0$ and the slope of the creep curves is smaller than for $\theta = \theta_0$.

Now choose the constant A in Eqn. 10.7.5 to be equal to η_0. This ensures that $\xi = t$ at the reference temperature θ_0 (see 10.7.8). In other words, the master curves of Eqn. 10.7.7 and the functions 10.7.4 corresponding to θ_0 coincide (with the t axis and ξ axis coincident).

The master relaxation and creep curves of Eqn. 10.7.7 are now $E(\xi)/\bar{E} = e^{-\xi/m}$ and $\bar{E}J(\xi) = 1 + \xi/m$. These are plotted in Fig. 10.7.3 below (for $m = 5$).

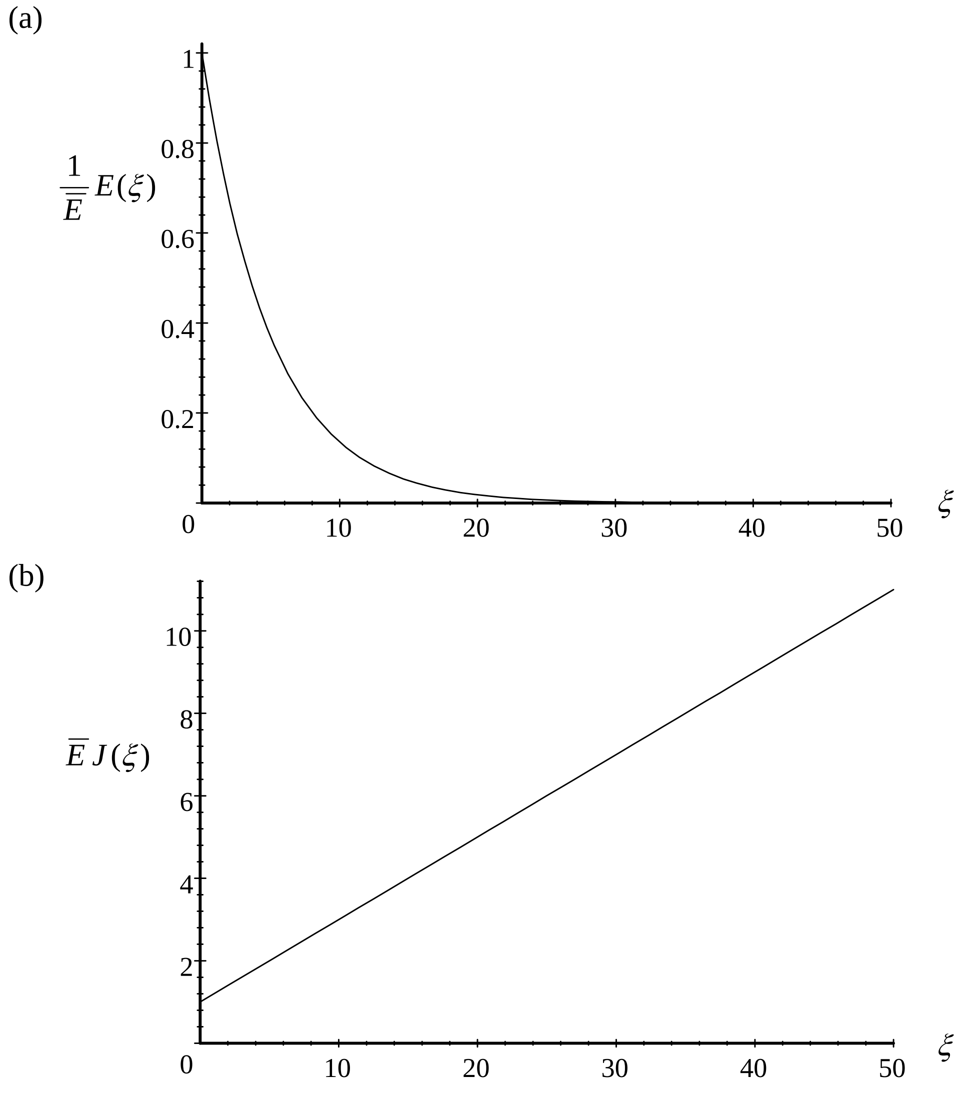

Figure 10.7.3: master curves; (a) relaxation modulus, (b) creep compliance

All the curves of Fig. 10.7.2 collapse onto the master curve of Fig. 10.7.3 as follows:

(i) the curves corresponding to the reference temperature, $\theta = \theta_0 = 20^\circ$, in Figs. 10.7.2 lie on the master curves (with the t axis and ξ axis coincident)

(ii) for a curve with $\theta > \theta_0$, if the time axis of Fig. 10.7.2a,b is "stretched" (according to 10.7.5), the curve will come to lie along the $\theta = \theta_0$ curve (and hence on the master curve); for a curve with $\theta < \theta_0$, if the time axis of Fig. 10.7.2a is "shrunk" (according to 10.7.5), the curve will come to lie along the $\theta = \theta_0$ curve (and hence on the master curve)

10.7.2 Thermorheologically Simple Materials

The fact that the relaxation and creep curves of Fig. 10.7.2 collapsed onto the master curves of Fig. 10.7.3 relied on the change of variable, Eqn. 10.7.5, reducing the time and temperature dependent constitutive relation 10.7.3 to an equation in one variable, ξ, only, Eqn. 10.7.6. This in turn depended critically on the form of the differential equation 10.7.3. For example, if the spring stiffness $\overline{E}$ in the Maxwell model is temperature-dependent, the collapsing of curves is not possible.

Temperature-dependent viscoelastic materials for which this collapsing of curves is possible are called **thermorheologically simple** materials. In this context, the parameter ξ is called the **reduced time**. More generally, the transformation 10.7.5 is expressed in the form

$$\xi = \frac{t}{a_\theta(\theta)} \tag{10.7.9}$$

and the function $a_\theta(\theta)$ is called the **shift factor** function. The shift factor is chosen so that the relaxation and creep curves corresponding to the chosen reference temperature θ_0 coincide (as in the Maxwell model example above), i.e. so that $a_\theta(\theta_0) = 1$.

The relaxation and creep functions now transform as

$$E(t,\theta) \;\; \to \;\; E(\xi,\theta_0), \qquad\qquad J(t,\theta) \;\; \to \;\; J(\xi,\theta_0) \tag{10.7.10}$$

For temperatures below the reference temperature, $\theta < \theta_0$, $a_\theta(\theta_0)$ will be greater than 1, and the corresponding relaxation/creep curves collapse onto the master curve by "shrinking" the time axis *t*, which looks like a "shifting" of the curve "to the left" onto the $\theta = \theta_0$ curve. On the other hand, for $\theta > \theta_0$, $a_\theta(\theta_0) < 1$, and the corresponding curves collapse by a "stretching" of the time axis, which looks like a "shifting" of the curves "to the right" onto the master curve. This is summarised in Fig. 10.7.4 below.

The result of this is that materials at high temperatures and high strain rates behave similarly to materials at low temperatures and low strain rates.

The method discussed can also be used when the temperature is time-dependent, for then the transformation can be expressed as

$$\xi(t) = \int_0^t \frac{d\tau}{a_\theta(\theta(\tau))} \tag{10.7.11}$$

so that

$$\frac{d\xi}{dt} = \frac{1}{a_\theta(\theta(t))} \tag{10.7.12}$$

leading to the same reduced differential equation.

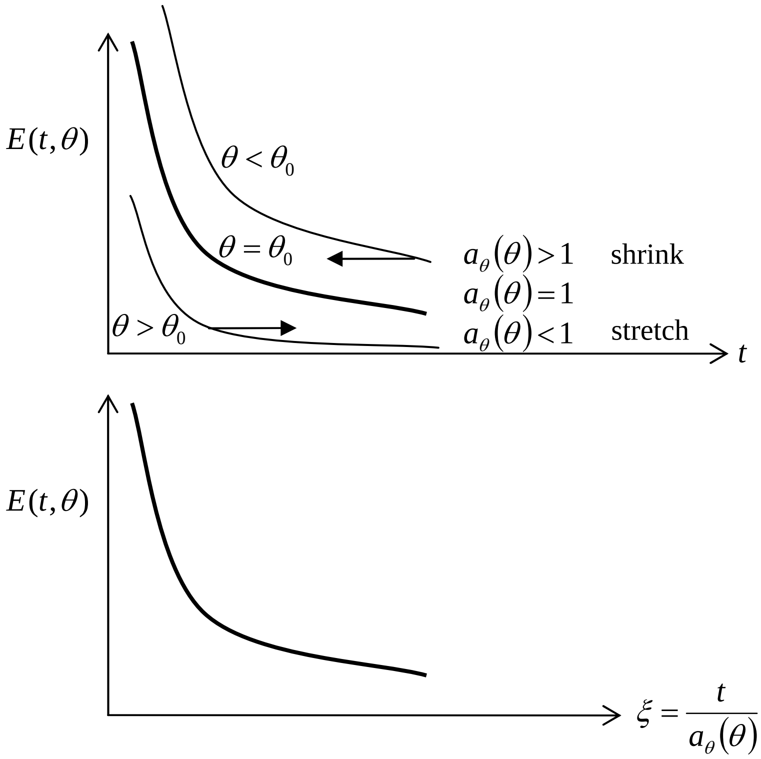

Figure 10.7.4: Relaxation modulus, as a function of (a) time, (b) reduced time

The above discussion has related to the differential constitutive equation 10.7.1. The analysis can also be expressed in terms of hereditary integrals of the form 10.7.2. For example, the equivalent hereditary integral in terms of reduced time, corresponding to the reduced differential equation (see Eqn. 10.7.6 for the Maxwell model equation) is

$$\sigma(\xi) = \int_{-\infty}^{\xi} E(\xi - \tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau \tag{10.7.13}$$

where $E(\xi)$ is as before (see Eqn. 10.7.7 for the Maxwell model expression).