Problem 2.4.17

Let V & W be finite dimensional vector spaces.

Let T be an isomorphism from V to W.

Let V_0 be a subspace of V

(a) Then $T(V_0)$ is a subspace of W

(b) Then $\dim(V_0) = \dim(T(V_0))$

PF(b)

Let $T' \in \mathcal{L}(V_0, T(V_0))$ defined by $T'(x) = T(x)$

T' is obviously onto.

T' is also one-to-one since T is isomorphism.

Thus T' is also isomorphism

By Lemma of Thm 2.18 $\dim(V_0) = \dim(T(V_0))$

PF(a)

$T(V_0) \subseteq W$.

Since $\bar{0}_V \in V_0$, $\bar{0}_W \in T(V_0)$ by the fact that $T(\bar{0}_V) = \bar{0}_W$

If $w_1, w_2 \in T(V_0)$ & $a \in F$

Then $\exists! v_{01}, v_{02} \in V_0$ s.t. $T(v_{01}) = w_1$ $T(v_{02}) = w_2$

$aw_1 + w_2 = aT(v_{01}) + T(v_{02}) = T(av_{01} + v_{02})$

Since $av_{01} + v_{02} \in V_0$, $aw_1 + w_2 \in T(V_0)$

Therefore $T(V_0)$ is subspace of W

Prb 2.4.20

Let $T \in \mathcal{L}(V,W)$

Let $\dim(V)=n$ and $\dim(W)=m$

(a) Then $\text{rank}(T) = \text{rank}(L_A)$

(b) Then $\text{nullity}(T) = \text{nullity}(L_A)$

where $A=[T]_{\beta^V}^{\beta^W}$

PF(a)

$R(L_A) = L_A(F^n) = L_A(\phi_{\beta^V}(V)) = \phi_{\beta^W}(T(V)) = \phi_{\beta^W}(R(T))$ by Thm 2.14

Since $R(T)$ is a subspace of W, $\phi_{\beta^W}(R(T))$ is a subspace of F^m and

$\dim(R(T)) = \dim(\phi_{\beta^W}(R(T))) = \dim(R(L_A))$ by Prb 2.4.17

Thus $\text{rank}(T) = \text{rank}(L_A)$

Let $v \in N(T)$

Then $L_A(\phi_{\beta^V}(v)) = \phi_{\beta^W}(T(v)) = \phi_{\beta^W}(\overline{0}_W) = \overline{0}_m \Rightarrow \phi_{\beta^V}(N(T)) \subseteq N(L_A)$

Let $n \in N(L_A)$ Then $\exists!\ v \in V$ s.t. $\phi_{\beta^V}(v) = n$

Hence $\phi_{\beta^W}(T(v)) = L_A(\phi_{\beta^V}(v)) = \overline{0}_m \Rightarrow T(v) = \overline{0}_W \Rightarrow \phi_{\beta^V}(N(T)) \supseteq N(L_A)$

Thus $\phi_{\beta^V}(N(T)) = N(L_A) \Rightarrow \dim(N(T)) = \dim(\phi_{\beta^V}(N(T))) = \dim(N(L_A))$

by Prb 2.4.17

Thus $\text{nullity}(N(T)) = \text{nullity}(N(L_A))$