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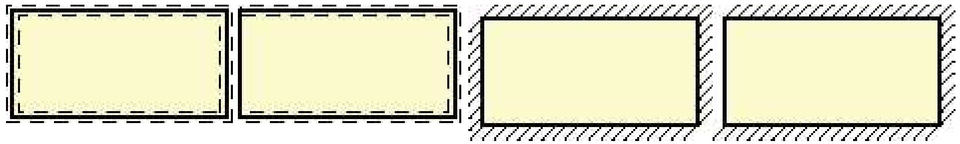
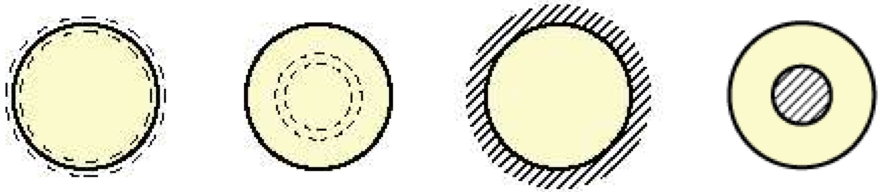


Simply Supported (SSSS)	Free-SSS (FSSS)	Clamped (Fixed) (CCCC)	Free-CCC (FCCC)



Simply Supported (cpS)	Free-Simply Supported (cpFS)	Clamped (Fixed) (cpC)	Free-Clamped (apFC)

Classical Plate Equation

The *small* transverse (out-of-plane) displacement w of a *thin* plate is governed by the Classical Plate Equation,

$$\nabla^2 D \nabla^2 w = p$$

where p is the distributed load (force per unit area) acting in the same direction as z (and w), and D is the in which E is the Young's modulus, ν is the Poisson's ratio of the plate material, and t is the thickness of the plate.

Furthermore, the differential operator ∇^2 is called the Laplacian differential operator Δ,

$$\Delta \equiv \nabla^2 = \begin{cases} \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \varphi^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} & \text{cylindrical coordinate (circular plates)} \\ \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} & \text{Cartesian coordinate (rectangular plates)} \end{cases}$$

If the bending rigidity D is constant throughout the plate, the plate equation can be simplified to,

$$\nabla^4 w = \frac{p}{D}$$

where $\nabla^4 = \nabla^2\nabla^2 = \Delta\Delta$ is called the biharmonic differential operator.

- This small deflection theory assumes that w is small in comparison to the thickness of the plate t, and the strains and the midplane slopes are much smaller than 1.

- A plate is called thin when its thickness t is at least one order of magnitude smaller than the span or diameter of the plate.

Origin of the Plate Equation

The classical plate equation arises from a combination of four distinct subsets of plate theory: the kinematic, constitutive, force resultant, and equilibrium equations.

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = z\begin{bmatrix}\kappa_x\\ \kappa_y\\ 2\kappa_{xy}\end{bmatrix} = -z\begin{bmatrix}\dfrac{\partial^2 w_0}{\partial x^2}\\ \dfrac{\partial^2 w_0}{\partial y^2}\\ 2\dfrac{\partial^2 w_0}{\partial x \partial y}\end{bmatrix}$$

where w_0 is the displacement of the middle plane in z direction.

Constitutive:

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix}$$

Resultants:

$$\begin{bmatrix}M_x\\ M_y\\ M_{xy}\end{bmatrix} = \int_{-t/2}^{t/2} z\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix}dz \qquad \begin{bmatrix}Q_{xz}\\ Q_{yz}\end{bmatrix} = \int_{-t/2}^{t/2}\begin{bmatrix}\sigma_{xz}\\ \sigma_{yz}\end{bmatrix}dz$$

Equilibrium:

$$Q_{yz} = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}$$

$$Q_{xz} = \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y}$$

$$\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} = -p_z$$

The plate is assumed to be constructed by isotropic material and subjected to transverse loading. Also, the Cartesian coordinate system is used.

We'll demonstrate this hierarchy by working backwards. We first combine the 3 equilibrium equations to eliminate Q_{xz} and Q_{yz},

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p_z$$

Next, replace the moment resultants with its definition in terms of the direct stress,

$$\int_{-t/2}^{t/2} z\left(\frac{\partial^2 \sigma_x}{\partial x^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) dz = -p_z$$

Note that uniform thickness is assumed.

Use the constitutive relation to eliminate stress in favor of the strain,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

and then use kinematics to replace strain in favor of the normal displacement w_0,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = -\frac{Ez}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ \dfrac{\partial^2 w_0}{\partial x \partial y} \end{bmatrix}$$

The equation of equilibrium can then be expressed in terms of the normal displacement w_0

$$\int_{-t/2}^{t/2} \frac{Ez^2}{1-\nu^2} \left(\left(\frac{\partial^4 w_0}{\partial x^4} + \nu \frac{\partial^4 w_0}{\partial x^2 \partial y^2} \right) + 2(1-\nu) \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \left(\frac{\partial^4 w_0}{\partial y^4} + \nu \frac{\partial^4 w_0}{\partial x^2 \partial y^2} \right) \right) dz = p_z$$

which yields

$$\int_{-t/2}^{t/2} \frac{Ez^2}{1-\nu^2} dz \left(\frac{\partial^4 w_0}{\partial x^4} + 2 \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \frac{\partial^4 w_0}{\partial y^4} \right) = p_z$$

Note that homogeneous material across the plate (*x* and *y* directions) is assumed.

As a final step, assuming homogeneous material along the thickness of the plate, the bending stiffness of the plate can be written as

$$D = \int_{-t/2}^{t/2} \frac{E}{\left(1-\nu^2\right)} z^2 dz = \frac{Et^3}{12\left(1-\nu^2\right)}$$

or a slimmer form

$$\nabla^4 w = \frac{p}{D}$$

where w_0 is replaced by w and p_z replaced by p to be consistent with the notations in most published literatures.