1 Differential Equations for Solid Mechanics

Simple problems involving homogeneous stress states have been considered so far, wherein the stress is the same throughout the component under study. An exception to this was the varying stress field in the loaded beam, but there a simplified set of elasticity equations was used. Here the question of varying stress and strain fields in materials is considered. In order to solve such problems, a differential formulation is required. In this Chapter, a number of differential equations will be derived, relating the stresses and body forces (**equations of motion**), the strains and displacements (**strain-displacement relations**) and the strains with each other (**compatibility relations**). These equations are derived from physical principles and so apply to any type of material, although the latter two are derived under the assumption of small strain.

1.1 The Equations of Motion

In Book I, balance of forces and moments acting on any component was enforced in order to ensure that the component was in equilibrium. Here, allowance is made for stresses which vary continuously throughout a material, and force equilibrium of any portion of material is enforced.

One-Dimensional Equation

Consider a one-dimensional differential element of length Δx and cross sectional area *A*, Fig. 1.1.1. Let the *average* body force per unit volume acting on the element be *b* and the *average* acceleration and density of the element be *a* and ρ. Stresses σ act on the element.

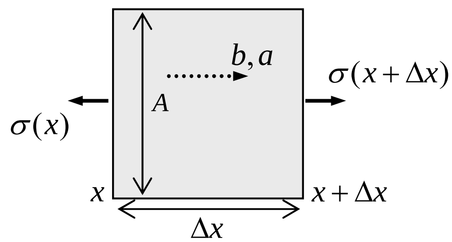


Figure 1.1.1: a differential element under the action of surface and body forces

The net surface force acting is $\sigma(x+\Delta x)A - \sigma(x)A$. If the element is small, then the body force and velocity can be assumed to vary linearly over the element and the average will act at the centre of the element. Then the body force acting on the element is $Ab\Delta x$ and the inertial force is $\rho A \Delta x a$. Applying Newton's second law leads to

$$\begin{aligned} &\sigma(x+\Delta x)A - \sigma(x)A + b\Delta x A = \rho a \Delta x A \\ &\rightarrow \frac{\sigma(x+\Delta x) - \sigma(x)}{\Delta x} + b = \rho a \end{aligned} \tag{1.1.1}$$

so that, by the definition of the derivative, in the limit as $\Delta x \to 0$,

$$\boxed{\frac{d\sigma}{dx} + b = \rho a} \quad \textbf{1-d Equation of Motion} \tag{1.1.2}$$

which is the one-dimensional **equation of motion**. Note that this equation was derived on the basis of a physical law and must therefore be satisfied for all materials, whatever they be composed of.

The derivative $d\sigma / dx$ is the **stress gradient** – physically, it is a measure of how rapidly the stresses are changing.

Example

Consider a bar of length *l* which hangs from a ceiling, as shown in Fig. 1.1.2.

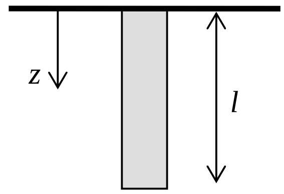


Figure 1.1.2: a hanging bar

The gravitational force is $F = mg$ downward and the body force per unit volume is thus $b = \rho g$. There are no accelerating material particles. Taking the z axis positive down, an integration of the equation of motion gives

$$\frac{d\sigma}{dz} + \rho g = 0 \quad \rightarrow \quad \sigma = -\rho g z + c \tag{1.1.3}$$

where c is an arbitrary constant. The lower end of the bar is free and so the stress there is zero, and so

$$\sigma = \rho g(l - z) \tag{1.1.4}$$

■

Two-Dimensional Equations

Consider now a two dimensional infinitesimal element of width and height Δx and Δy and unit depth (into the page).

Looking at the normal stress components acting in the x –direction, and allowing for variations in stress over the element surfaces, the stresses are as shown in Fig. 1.1.3.

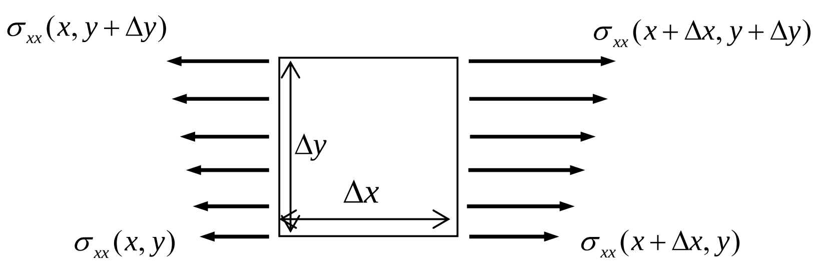


Figure 1.1.3: varying stresses acting on a differential element

Using a (two dimensional) Taylor series and dropping higher order terms then leads to the linearly varying stresses illustrated in Fig. 1.1.4. (where $\sigma_{xx} \equiv \sigma_{xx}(x, y)$ and the partial derivatives are evaluated at (x, y)), which is a reasonable approximation when the element is small.

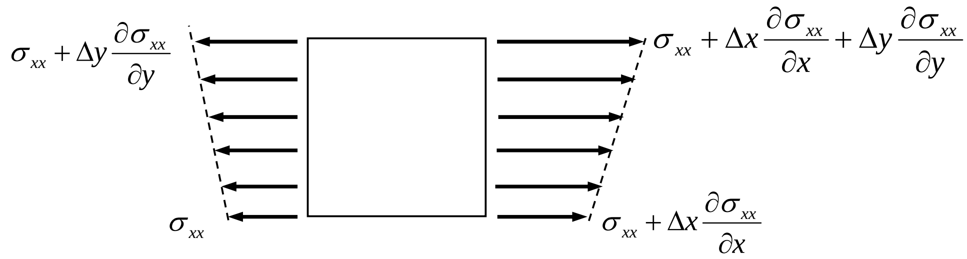


Figure 1.1.4: linearly varying stresses acting on a differential element

The effect (resultant force) of this linear variation of stress on the plane can be replicated by a *constant* stress acting over the whole plane, the size of which is the *average* stress. For the left and right sides, one has, respectively,

$$\sigma_{xx} + \frac{1}{2}\Delta y \frac{\partial \sigma_{xx}}{\partial y}, \quad \sigma_{xx} + \Delta x \frac{\partial \sigma_{xx}}{\partial x} + \frac{1}{2}\Delta y \frac{\partial \sigma_{xx}}{\partial y} \tag{1.1.5}$$

One can take away the stress $(1/2)\Delta y \partial \sigma_{xx} / \partial y$ from both sides without affecting the net force acting on the element so one finally has the representation shown in Fig. 1.1.5.

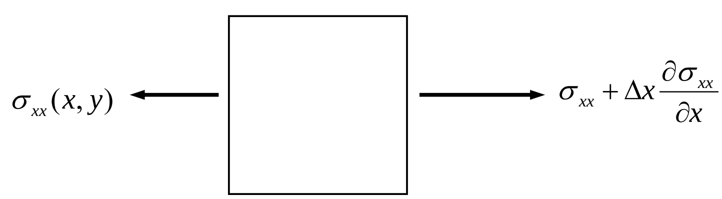


Figure 1.1.5: net stresses acting on a differential element

Carrying out the same procedure for the shear stresses contributing to a force in the x –direction leads to the stresses shown in Fig. 1.1.6.

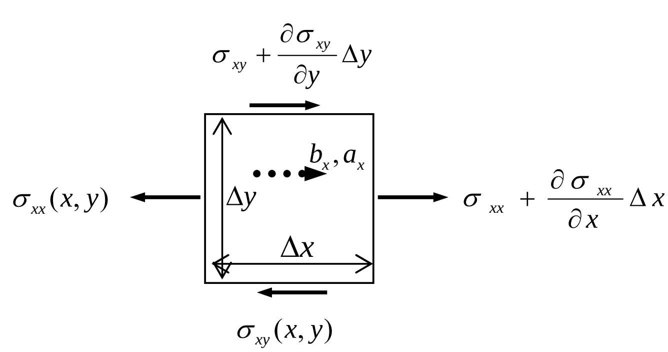


Figure 1.1.6: normal and shear stresses acting on a differential element

Take a_x, b_x to be the average acceleration and body force, and ρ to be the average density. Newton's law then yields

$$-\sigma_{xx}\Delta y+\left(\sigma_{xx}+\Delta x\frac{\partial\sigma_{xx}}{\partial x}\right)\Delta y-\sigma_{xy}\Delta x+\left(\sigma_{xy}+\Delta y\frac{\partial\sigma_{xy}}{\partial y}\right)\Delta x+b_x\Delta x\Delta y=\rho a_x\Delta x\Delta y \tag{1.1.6}$$

which, dividing through by $\Delta x\Delta y$ and taking the limit, gives

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+b_x=\rho a_x \tag{1.1.7}$$

A similar analysis for force components in the y –direction yields another equation and one then has the two-dimensional equations of motion:

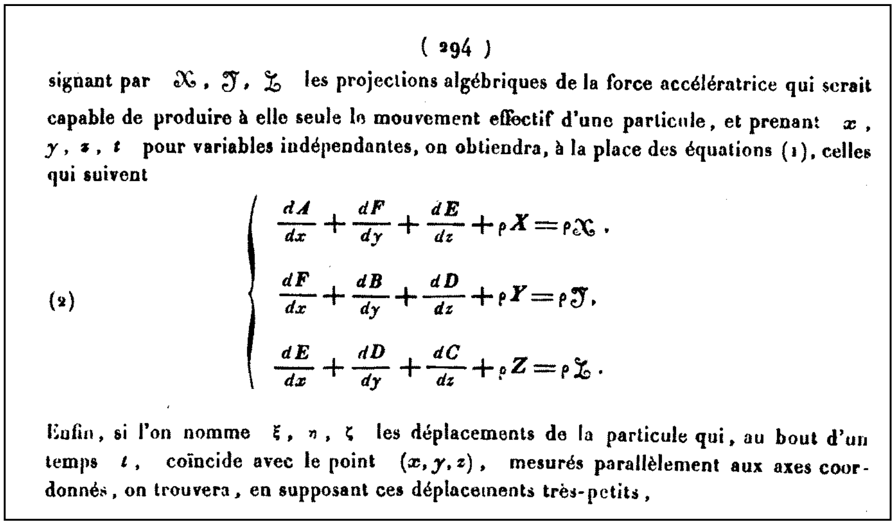
$$\boxed{\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+b_x&=\rho a_x\\ \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+b_y&=\rho a_y\end{aligned}} \quad \textbf{2-D Equations of Motion} \tag{1.1.8}$$

Three-Dimensional Equations

Similarly, one can consider a three-dimensional element, and one finds that

$$\boxed{\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+b_x&=\rho a_x\\ \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}+b_y&=\rho a_y\\ \frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+b_z&=\rho a_z\end{aligned}} \quad \textbf{3-D Equations of Motion} \tag{1.1.9}$$

These three equations express force-balance in, respectively, the x, y, z directions.



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signant par $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ les projections algébriques de la force accélératrice qui serait capable de produire à elle seule le mouvement effectif d'une particule, et prenant x, y, z, t pour variables indépendantes, on obtiendra, à la place des équations (1), celles qui suivent

$$
(2)\quad \begin{cases} \dfrac{dA}{dx}+\dfrac{dF}{dy}+\dfrac{dE}{dz}+\rho X=\rho\mathfrak{X}, \\ \dfrac{dF}{dx}+\dfrac{dB}{dy}+\dfrac{dD}{dz}+\rho Y=\rho\mathfrak{Y}, \\ \dfrac{dE}{dx}+\dfrac{dD}{dy}+\dfrac{dC}{dz}+\rho Z=\rho\mathfrak{Z}. \end{cases}
$$

Enfin, si l'on nomme ξ, η, ζ les déplacements de la particule qui, au bout d'un temps t, coïncide avec le point (x, y, z), mesurés parallèlement aux axes coordonnés, on trouvera, en supposant ces déplacements très-petits,

Figure 1.1.7: from Cauchy's Exercices de Mathematiques (1829)

The Equations of Equlibrium

If the material is not moving (or is moving at constant velocity) and is in static equilibrium, then the equations of motion reduce to the **equations of equilibrium**,

$$
\boxed{\begin{aligned} \frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+b_x&=0 \\ \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}+b_y&=0 \\ \frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+b_z&=0 \end{aligned}}
$$

3-D Equations of Equilibrium (1.1.10)

These equations express the force balance between surface forces and body forces in a material. The equations of equilibrium may also be used as a good approximation in the analysis of materials which have relatively small accelerations.

1.1.2 Problems

1. What does the one-dimensional equation of motion say about the stresses in a bar in the absence of any body force or acceleration?

2. Does equilibrium exist for the following two dimensional stress distribution in the absence of body forces?

$$
\begin{aligned} \sigma_{xx}&=3x^2+4xy-8y^2 \\ \sigma_{xy}=\sigma_{yx}&=x^2/2-6xy-2y^2 \\ \sigma_{yy}&=2x^2-xy+3y^2 \\ \sigma_{zz}=\sigma_{zx}=\sigma_{xz}&=\sigma_{zy}=\sigma_{yz}=0 \end{aligned}
$$

3. The elementary beam theory predicts that the stresses in a circular beam due to bending are

$$\sigma_{xx} = My/I, \quad \sigma_{xy} = \sigma_{yx} = V(R^2 - y^2)/3I \qquad (I = \pi R^4/4)$$

and all the other stress components are zero. Do these equations satisfy the equations of equilibrium?

4. With respect to axes $0xyz$ the stress state is given in terms of the coordinates by the matrix

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} xy & y^2 & 0 \\ y^2 & yz & z^2 \\ 0 & z^2 & xz \end{bmatrix}$$

Determine the body force acting on the material if it is at rest.

5. What is the acceleration of a material particle of density $\rho = 0.3\text{kgm}^{-3}$, subjected to the stress

$$[\sigma_{ij}] = \begin{bmatrix} 2x^2 - x^4 & 2xy & 2xz \\ 2xy & 2y^2 - y^4 & 2yz \\ 2xz & 2yz & 2z^2 - z^4 \end{bmatrix}$$

and gravity (the z axis is directed vertically upwards from the ground).

6. A fluid at rest is subjected to a hydrostatic pressure p and the force of gravity only.
   (a) Write out the equations of motion for this case.
   (b) A very basic formula of hydrostatics, to be found in any elementary book on fluid mechanics, is that giving the pressure variation in a static fluid,

   $$\Delta p = \rho g h$$

   where ρ is the density of the fluid, g is the acceleration due to gravity, and h is the vertical distance between the two points in the fluid (the relative depth). Show that this formula is but a special case of the equations of motion.

1.2 The Strain-Displacement Relations

The strain was introduced in Book I: §4. The concepts examined there are now extended to the case of strains which vary continuously throughout a material.

1.2.1 The Strain-Displacement Relations

Normal Strain

Consider a line element of length Δx emanating from position (x, y) and lying in the x-direction, denoted by AB in Fig. 1.2.1. After deformation the line element occupies $A'B'$, having undergone a translation, extension and rotation.

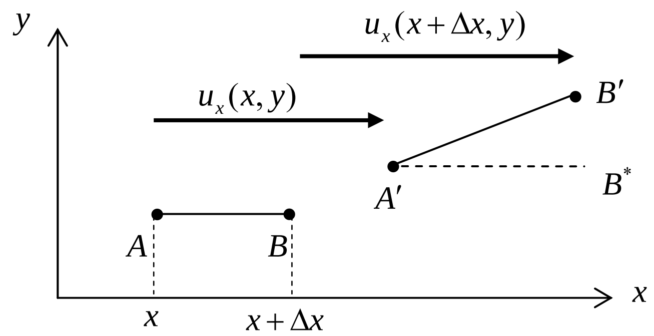

Figure 1.2.1: deformation of a line element

The particle that was originally at x has undergone a displacement $u_x(x, y)$ and the other end of the line element has undergone a displacement $u_x(x+\Delta x, y)$. By the definition of (small) normal strain,

$$\varepsilon_{xx} = \frac{A'B^* - AB}{AB} = \frac{u_x(x+\Delta x, y) - u_x(x, y)}{\Delta x} \tag{1.2.1}$$

In the limit $\Delta x \to 0$ one has

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \tag{1.2.2}$$

This partial derivative is a **displacement gradient**, a measure of how rapid the displacement changes through the material, and is the strain *at* (x, y). Physically, it represents the (approximate) unit change in length of a line element, as indicated in Fig. 1.2.2.

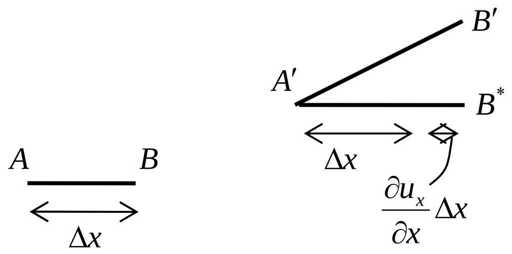

Figure 1.2.2: unit change in length of a line element

Similarly, by considering a line element initially lying in the y direction, the strain in the y direction can be expressed as

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \tag{1.2.3}$$

Shear Strain

The particles A and B in Fig. 1.2.1 also undergo displacements in the y direction and this is shown in Fig. 1.2.3. In this case, one has

$$B^* B' = \frac{\partial u_y}{\partial x}\Delta x \tag{1.2.4}$$

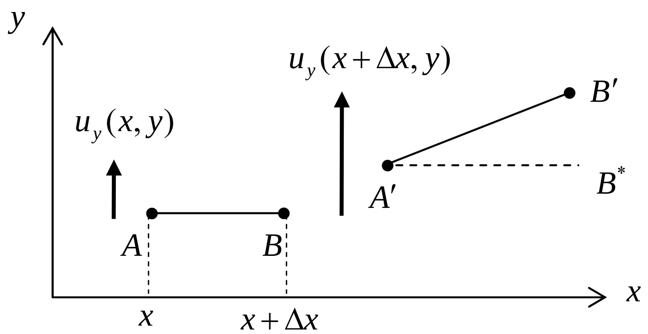

Figure 1.2.3: deformation of a line element

A similar relation can be derived by considering a line element initially lying in the y direction. A summary is given in Fig. 1.2.4. From the figure,

$$\theta \approx \tan\theta = \frac{\partial u_y / \partial x}{1 + \partial u_x / \partial x} \approx \frac{\partial u_y}{\partial x}$$

provided that (i) θ is small and (ii) the displacement gradient $\partial u_x / \partial x$ is small. A similar expression for the angle λ can be derived, and hence the shear strain can be written in terms of displacement gradients.

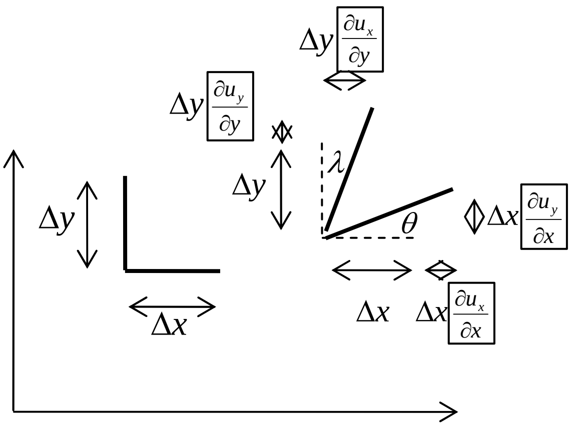

Figure 1.2.4: strains in terms of displacement gradients

The Small-Strain Stress-Strain Relations

In summary, one has

$$\boxed{\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \end{aligned}} \quad \textbf{2-D Strain-Displacement relations} \qquad (1.2.5)$$

1.2.2 Geometrical Interpretation of Small Strain

A geometric interpretation of the strain was given in Book I: §4.1.4. This interpretation is repeated here, only now in terms of displacement gradients.

Positive Normal Strain

Fig. 1.2.5a,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} > 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial y}{\partial x}\right) = 0 \qquad (1.2.6)$$

Negative Normal Strain

Fig 1.2.5b,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} < 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0 \qquad (1.2.7)$$

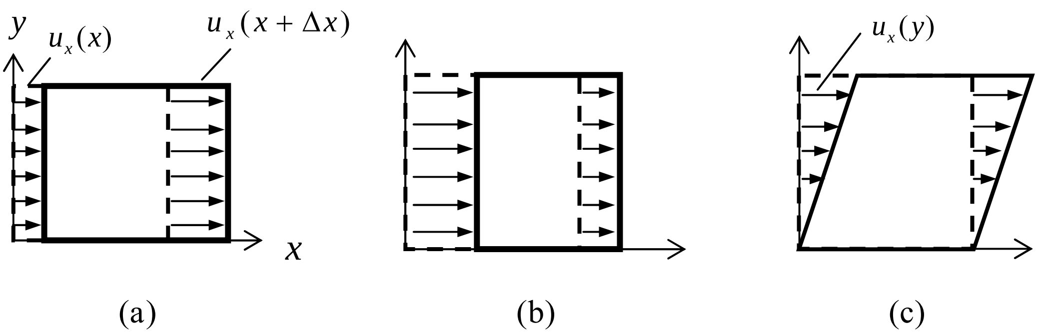

Figure 1.2.5: some simple deformations; (a) positive normal strain, (b) negative normal strain, (c) simple shear

Simple Shear

Fig. 1.2.5c,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\frac{\partial u_x}{\partial y} \qquad (1.2.8)$$

Pure Shear

Fig 1.2.6a,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x} \qquad (1.2.9)$$

1.2.3 The Rotation

Consider an arbitrary deformation (omitting normal strains for ease of description), as shown in Fig. 1.2.6. As usual, the angles θ and λ are small, equal to their tangents, and $\theta = \partial u_y / \partial x$, $\lambda = \partial u_x / \partial y$.

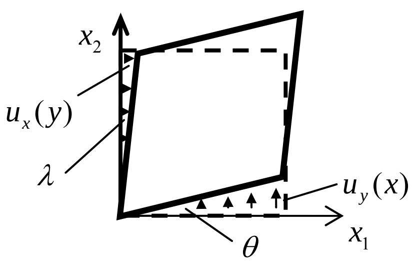

Figure 1.2.6: arbitrary deformation (shear and rotation)

Now this arbitrary deformation can be decomposed into a pure shear and a rigid rotation as depicted in Fig. 1.2.7. In the pure shear, $\theta = \lambda = \varepsilon_{xy} = \frac{1}{2}(\theta + \lambda)$. In the rotation, the angle of rotation is then $\frac{1}{2}(\theta + \lambda)$.

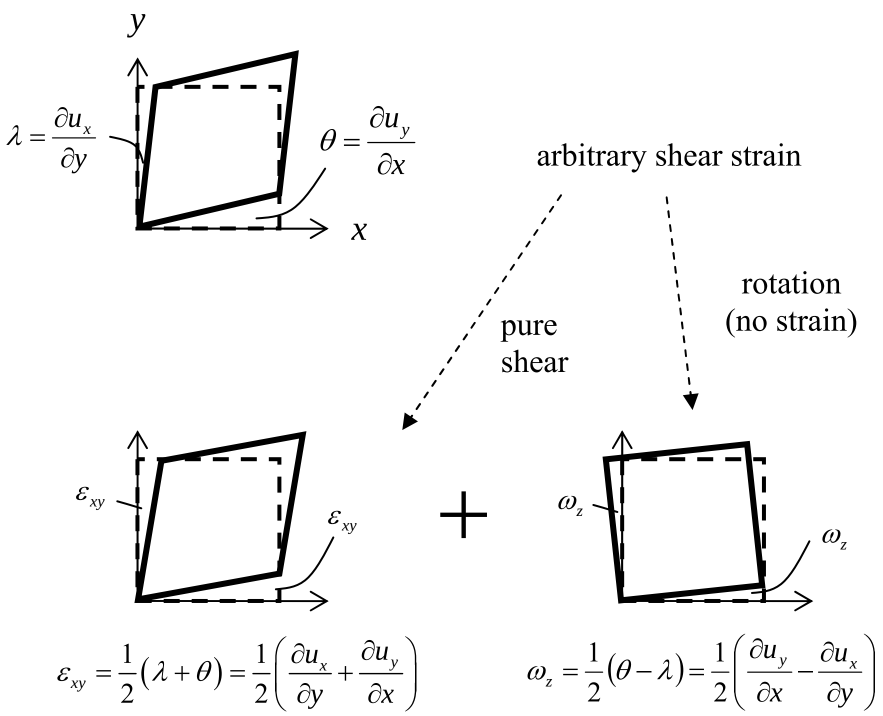

Figure 1.2.7: decomposition of a strain into a pure shear and a rotation

This leads one to define the **rotation** of a material particle, ω_z, the "z" signifying the axis about which the element is rotating:

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \tag{1.2.10}$$

The rotation will in general vary throughout a material. When the rotation is everywhere zero, the material is said to be **irrotational**.

For a pure rotation, note that

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0, \qquad \frac{\partial u_x}{\partial y} = -\frac{\partial u_y}{\partial x} \tag{1.2.11}$$

1.2.4 Fixing Displacements

The strains give information about the deformation of material particles but, since they do not encompass translations and rotations, they do not give information about the precise location in space of particles. To determine this, one must specify *three* displacement components (in two-dimensional problems). Mathematically, this is equivalent to saying that one cannot uniquely determine the displacements from the strain-displacement relations 1.2.5.

Example

Consider the strain field $\varepsilon_{xx} = 0.01, \varepsilon_{yy} = \varepsilon_{xy} = 0$. The displacements can be obtained by integrating the strain-displacement relations:

$$\begin{aligned} u_x &= \int \varepsilon_{xx} dx = 0.01x + f(y) \\ u_y &= \int \varepsilon_{yy} dy = g(x) \end{aligned} \tag{1.2.12}$$

where f and g are unknown functions of y and x respectively. Substituting the displacement expressions into the shear strain relation gives

$$f'(y) = -g'(x). \tag{1.2.13}$$

Any expression of the form $F(x) = G(y)$ which holds for all x and y implies that F and G are constant[1]. Since f', g' are constant, one can integrate to get $f(y) = A + Dy,\ \ g(x) = B + Cx$. From 1.2.13, $C = -D$, and

$$\begin{aligned} u_x &= 0.01x + A - Cy \\ u_y &= B + Cx \end{aligned} \tag{1.2.14}$$

There are three arbitrary constants of integration, which can be determined by specifying three displacement components. For example, suppose that it is known that

$$u_x(0,0) = 0, u_y(0,0) = 0, u_x(0,a) = b. \tag{1.2.15}$$

In that case, $A = 0, B = 0, C = -b/a$, and, finally,

$$\begin{aligned} u_x &= 0.01x + (b/a)y \\ u_y &= -(b/a)x \end{aligned} \tag{1.2.16}$$

which corresponds to Fig. 1.2.8, with (b/a) being the (tan of the small) angle by which the element has rotated.

[1] since, if this was not so, a change in x would change the left hand side of this expression but would not change the right hand side and so the equality cannot hold

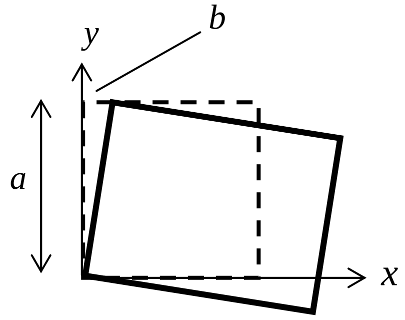

Figure 1.2.8: an element undergoing a normal strain and a rotation

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In general, the displacement field will be of the form

$$\begin{aligned} u_x &= \;\cdots\cdots\; + A - Cy \\ u_y &= \;\cdots\cdots\; + B + Cx \end{aligned} \qquad (1.2.17)$$

and indeed Eqn. 1.2.16 is of this form. Physically, *A*, *B* and *C* represent the possible rigid body motions *of the material as a whole*, since they are the same for all material particles. *A* corresponds to a translation in the x direction, *B* corresponds to a translation in the x direction, and *C* corresponds to a positive (counterclockwise) rotation.

1.2.5 Three Dimensional Strain

The three-dimensional stress-strain relations analogous to Eqns. 1.2.5 are

$$\boxed{\begin{aligned} &\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \\ &\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \end{aligned}}$$

3-D Stress-Strain relations (1.2.18)

The rotations are

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \quad \omega_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \qquad (1.2.19)$$

1.2.6 Problems

1. The displacement field in a material is given by

$$u_x = A(3x - y), \quad u_y = Axy^2$$

where *A* is a small constant.

(a) Evaluate the strains. What is the rotation ω_z? Sketch the deformation and any rigid body motions of a differential element at the point $(1,1)$
(b) Sketch the deformation and rigid body motions at the point $(0,2)$, by using a pure shear strain superimposed on the rotation.

2. The strains in a material are given by
$$\varepsilon_{xx} = \alpha x, \quad \varepsilon_{yy} = 0, \quad \varepsilon_{xy} = \alpha$$
Evaluate the displacements in terms of three arbitrary constants of integration, in the form of Eqn. 1.2.17,
$$\begin{aligned} u_x &= \quad \cdots\cdots \quad + A - Cy \\ u_y &= \quad \cdots\cdots \quad + B + Cx \end{aligned}$$
What is the rotation?

3. The strains in a material are given by
$$\varepsilon_{xx} = Axy, \quad \varepsilon_{yy} = Ay^2, \quad \varepsilon_{xy} = Ax$$
where A is a small constant. Evaluate the displacements in terms of three arbitrary constants of integration. What is the rotation?

4. Show that, in a state of plane strain ($\varepsilon_{zz} = 0$) with zero body force,
$$\frac{\partial e}{\partial x} - 2\frac{\partial \omega_z}{\partial y} = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}$$
where e is the volumetric strain (dilatation), the sum of the normal strains: $e = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$ (see Book I, §4.3).

1.3 Compatibility of Strain

As seen in the previous section, the displacements can be determined from the strains through integration, to within a rigid body motion. In the two-dimensional case, there are three strain-displacement relations but only two displacement components. This implies that the strains are not independent but are related in some way. The relations between the strains are called **compatibility conditions**.

1.3.1 The Compatibility Relations

Differentiating the first of 1.2.5 twice with respect to y, the second twice with respect to x and the third once each with respect to x and y yields

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2}, \quad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial x^2 \partial y}, \quad \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^3 u_x}{\partial x \partial y^2} + \frac{\partial^3 u_y}{\partial x^2 \partial y}\right).$$

It follows that

$$\boxed{\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}} \qquad \textbf{2-D Compatibility Equation} \quad (1.3.1)$$

This compatibility condition is an equation which must be satisfied by the strains at all material particles.

Physical Meaning of the Compatibility Condition

When all material particles in a component deform, translate and rotate, they need to meet up again very much like the pieces of a jigsaw puzzle must fit together. Fig. 1.3.1 illustrates possible deformations and rigid body motions for three line elements in a material. Compatibility ensures that they stay together after the deformation.

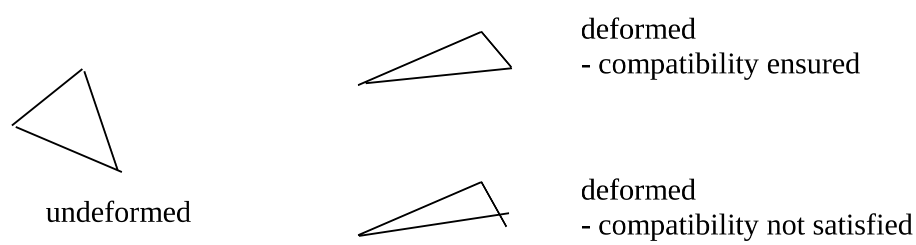


Figure 1.3.1: Deformation and Compatibility

The Three Dimensional Case

There are six compatibility relations to be satisfied in the three dimensional case:

$$\begin{aligned}
\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}, &\quad \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right) \\
\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x}, &\quad \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right). \\
\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, &\quad \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(+\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z} \right)
\end{aligned} \qquad (1.3.2)$$

By inspection, it will be seen that these are satisfied by Eqns. 1.2.19.

1.3.2 Problems

1. The displacement field in a material is given by

$$u_x = Axy, \quad u_y = Ay^2,$$

where A is a small constant. Determine
(a) the components of small strain
(b) the rotation
(c) the principal strains
(d) whether the compatibility condition is satisfied

2 One-Dimensional Elasticity

There are two types of one-dimensional problems, the **elastostatic** problem and the **elastodynamic** problem. The elastostatic problem gives rise to a second order differential equation in displacement which may be solved using elementary integration. The elastodynamic problem gives rise to the one-dimensional wave equation, whose solution predicts the propagation of stress waves and vibrations of material particles

2.1 One-dimensional Elastostatics

Consider a bar or rod made of linearly elastic material subjected to some load. Static problems will be considered here, by which is meant it is not necessary to know how the load was applied, or how the material particles moved to reach the stressed state; it is necessary only that the load was applied slowly enough so that the accelerations are zero, or that it was applied sufficiently long ago that any vibrations have died away and movement has ceased.

The equations governing the static response of the rod are:

$$\frac{d\sigma}{dx} + b = 0 \qquad \textbf{Equation of Equilibrium} \qquad (2.1.1a)$$

$$\varepsilon = \frac{du}{dx} \qquad \textbf{Strain-Displacement Relation} \qquad (2.1.1b)$$

$$\sigma = E\varepsilon \qquad \textbf{Constitutive Equation} \qquad (2.1.1c)$$

where E is the Young's modulus, ρ is the density and b is a body force (per unit volume). The unknowns of the problem are the stress σ, strain ε and displacement u.

These equations can be combined to give a second order differential equation in u, called **Navier's Equation**:

$$\boxed{\frac{d^2u}{dx^2} + \frac{b}{E} = 0} \qquad \textbf{1-D Navier's Equation} \qquad (2.1.2)$$

One requires two **boundary conditions** to obtain a solution. Let the length of the rod be L and the x axis be positioned as in Fig. 2.1.1. The possible boundary conditions are then

1. displacement specified at both ends ("fixed-fixed")
   $u(0) = \bar{u}_0, \quad u(L) = \bar{u}_L$
2. stress specified at both ends ("free-free")
   $\sigma(0) = \bar{\sigma}_0, \quad \sigma(L) = \bar{\sigma}_L$
3. displacement specified at left-end, stress specified at right-end ("fixed-free"):
   $u(0) = \bar{u}_0, \quad \sigma(L) = \bar{\sigma}_L$
4. stress specified at left-end, displacement specified at right-end ("free-fixed"):
   $\sigma(0) = \bar{\sigma}_0, \quad u(L) = \bar{u}_L$

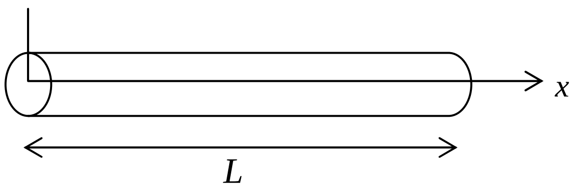


Figure 2.1.1: an elastic rod

Note that, from 2.1.1b-c, a stress boundary condition is a condition on the first derivative of *u*.

Example

Consider a rod in the absence of any body forces subjected to an applied stress σ_o, Fig. 2.1.2.

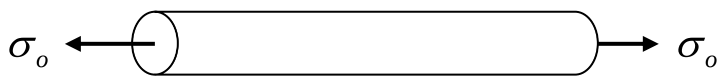


Figure 2.1.2: an elastic rod subjected to stress

The equation to solve is

$$\frac{d^2u}{dx^2} = 0 \tag{2.1.3}$$

subject to the boundary conditions

$$\left.\frac{\partial u}{dx}\right|_{x=0} = \frac{\sigma_0}{E}, \quad \left.\frac{\partial u}{dx}\right|_{x=L} = \frac{\sigma_0}{E} \tag{2.1.4}$$

Integrating twice and applying the conditions gives the solution

$$u = \frac{\sigma_0}{E} x + B \tag{2.1.5}$$

The stress is thus a constant σ_0 and the strain is σ_o / E. There is still an arbitrary constant *B* and this physically represents a possible rigid body translation of the rod. To remove this arbitrariness, one must specify the displacement at some point in the rod. For example, if $u(L/2) = 0$, the complete solution is

$$u = \frac{\sigma_o}{E}\left(x - \frac{L}{2}\right), \quad \varepsilon = \frac{\sigma_o}{E}, \quad \sigma = \sigma_o \tag{2.1.6}$$

2.1.1 Problems

1. What are the displacements of material particles in an elastic bar of length *L* and density ρ which hangs from a ceiling (see Fig. 1.1.2).

2. Consider a steel rod ($E = 210\ \text{GPa}$, $\rho = 7.85\ \text{g/cm}^3$) of length $30\ \text{cm}$, fixed at one end and subjected to a displacement $u = 1\text{mm}$ at the other. Solve for the stress, strain and displacement for the case of gravity acting along the rod. What is the solution in the absence of gravity. How significant is the effect of gravity on the stress?

2.2 One-dimensional Elastodynamics

In rigid body dynamics, it is assumed that when a force is applied to one point of an object, every other point in the object is set in motion simultaneously. On the other hand, in static elasticity, it is assumed that the object is at rest and is in equilibrium under the action of the applied forces; the material may well have undergone considerable changes in deformation when first struck, but one is only concerned with the final static equilibrium state of the object.

Elastostatics and rigid body dynamics are sufficiently accurate for many problems but when one is considering the effects of forces which are applied *rapidly*, or for very short periods of time, the effects must be considered in terms of the propagation of stress waves.

The analysis presented below is for one-dimensional deformations. Inherent are the assumptions that (1) material properties are uniform over a plane perpendicular to the longitudinal direction, (2) plane sections remain plane and perpendicular to the longitudinal direction and (3) there is no transverse displacement.

2.2.1 The Wave Equation

Consider now the dynamic problem. In this case $u = u(x,t)$ and one considers the governing equations:

$$\frac{\partial \sigma}{\partial x} + b = \rho a \qquad \textbf{Equation of Motion} \qquad (2.2.1a)$$

$$\varepsilon = \frac{\partial u}{\partial x} \qquad \textbf{Strain-Displacement Relation} \qquad (2.2.1b)$$

$$\sigma = E\varepsilon \qquad \textbf{Constitutive Equation} \qquad (2.2.1c)$$

where a is the acceleration. Expressing the acceleration in terms of the displacement, one then obtains the dynamic version of Navier's equation,

$$\boxed{E\frac{\partial^2 u}{\partial x^2} + b = \rho \frac{\partial^2 u}{\partial t^2}} \qquad \textbf{1-D Navier's Equation} \qquad (2.2.2)$$

In most situations, the body forces will be negligible, and so consider the partial differential equation

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}} \qquad \textbf{1-D Wave Equation} \qquad (2.2.3)$$

where

$$c = \sqrt{\frac{E}{\rho}} \qquad (2.2.4)$$

Equation 2.2.3 is the standard one-dimensional **wave equation** with wave speed *c*; note from 2.2.4 that *c* has dimensions of velocity.

The solution to 2.2.3 (see below) shows that a **stress wave** travels at speed *c* through the material from the point of disturbance, e.g. applied load. When the stress wave reaches a given material particle, the particle vibrates about an equilibrium position, Fig. 2.2.1. Since the material is elastic, no energy is lost, and the solution predicts that the particle will vibrate indefinitely, without damping or decay, unless that energy is transferred to a neighbouring particle.

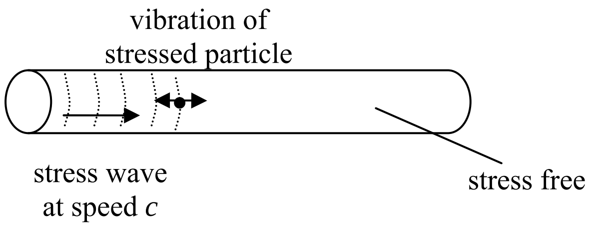


Figure 2.2.1: stress wave travelling at speed *c* through an elastic rod

This type of wave, where the disturbance (particle vibration) is in the same direction as the direction of wave propagation, is called a **longitudinal wave**.

The wave equation is solved subject to the **initial conditions** and **boundary conditions**. The initial conditions are that the displacement *u* and the particle velocity $\partial u / \partial t$ are specified at $t = 0$ (for all *x*). The boundary conditions are that the displacement *u* and the first derivative $\partial u / \partial x$ are specified (for all *t*). This latter derivative is the strain, which is proportional to the stress (see Eqn. 2.2.1b). In problems where there is no boundary (an infinite medium), no boundary conditions are explicitly applied. A semi-infinite medium will have one boundary. For a rod of finite length, there will be two boundaries and a boundary condition will be applied to each boundary.

2.2.2 Particle Velocities and Wave Speed

Before examining the wave equation 2.2.3 directly, first re-express it as

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial v}{\partial t} \tag{2.2.5}$$

where *v* is the velocity. Consider an element of material which has just been reached by the stress wave, Fig. 2.2.2. The length of material passed by the stress wave in a time interval Δt is $c\Delta t$. During this time interval, the stressed material at the left-hand side of the element moves at (average) velocity *v* and so moves an amount $v\Delta t$. The strain of the element is then the change in length divided by the original length:

$$\varepsilon = \frac{v}{c} \tag{2.2.6}$$

Under the small strain assumption, this implies that $v << c$ [1].

Let the stress acting on the element be $\Delta\sigma$; the stress on the free side of the element is zero. Then 2.2.5 leads to

$$\frac{\Delta\sigma}{c\Delta t} = \rho\frac{v}{\Delta t} \tag{2.2.7}$$

and so

$$\Delta\sigma = \rho c v \tag{2.2.8}$$

This is the discontinuity in stress across the wave front.

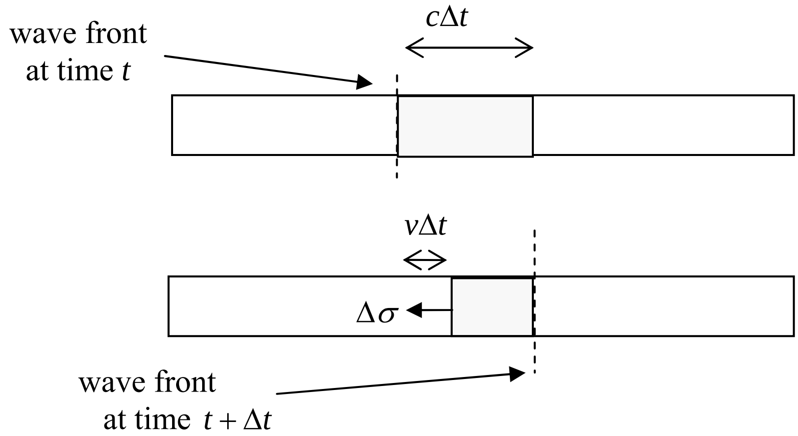


Figure 2.2.2: stress wave passing through a material element

Since $\Delta\sigma = E\varepsilon$, one has $c = \sqrt{E/\rho}$, as in 2.2.4. The wave speeds for some materials are given in Table 2.2.1. As can be seen, the wave speeds for typical engineering materials are of the order km/s and so particle velocities will be in the range $0 - 50\text{m/s}$.

Material	$\rho\ (\text{kg/m}^3)$	$E\ (\text{GPa})$	$c\ (\text{m/s})$
Aluminium Alloy	2700	70	5092
Brass	8300	95	3383
Copper	8500	114	3662
Lead	11300	17.5	1244
Steel	7800	210	5189
Glass	1870	55	5300
Granite	2700	26	3120
Limestone	2600	63	4920
Perspex	490	2.5	2260

Table 2.2.1: Elastic Wave Speeds for Several Materials

[1] note also that the density of the element will change as it is compressed, but again this change in density is small and can be neglected in the linear elastic theory

Consider steel: the velocity at which the material ceases to behave linearly elastic (taking the yield stress to be 400MPa) is $v = Y / \rho c \approx 10\text{m/s}$.

2.2.3 Waves

Before proceeding, it will be helpful to review and summarise the important facts and terminology regarding waves.

Suppose that there is a displacement *u* which is propagated along the *x* axis at velocity *c*. At time $t = 0$ say, the disturbance will have some **wave profile** $u = f(x)$. If the disturbance propagates *without change of shape*, then at some later time *t* the profile will look identical but it will have moved a distance *ct* in the positive direction. If we take a new origin at the point $x = ct$ and let the distance measured from this origin be $\bar{x}$, then the equation of the new wave profile referred to this new origin would be $u = f(\bar{x})$. Referred to the original fixed origin, then,

$$u = f\left(x - ct\right). \tag{2.2.9}$$

This is the most general expression for a wave travelling at constant velocity *c* and without change of shape, along the positive *x* axis. If the wave is travelling in the negative direction, then its form would be $u = f\left(x + ct\right)$.

The simplest type of wave of this kind is the **harmonic wave**, in which the wave profile is a sine or cosine curve. If the wave profile at time $t = 0$ is $u = a\cos(kx)$, then at time *t* the profile is

$$u = a\cos\left[k\left(x - ct\right)\right]. \tag{2.2.10}$$

The maximum value of the disturbance, *a*, is called the **amplitude**. The wave profile repeats itself at regular distances $2\pi / k$, which is called the **wavelength** λ. The parameter *k* is called the **wave number**[2]; since there is one wave in λ units of distance, it is the number of waves in 2π units of distance:

$$k = \frac{2\pi}{\lambda}. \tag{2.2.11}$$

The distance travelled by the wave in time *t* is ct. The time taken for one complete wave to pass any point is called the **period** *T*, which is the time taken to travel one wavelength:

$$T = \frac{\lambda}{c}. \tag{2.2.12}$$

The **frequency** *f* is the number of waves passing a fixed point in unit time, so

[2] more specifically, this is the **angular wavenumber**, to distinguish it from the (**spectroscopic**) **wavenumber** $1/\lambda$

$$f = \frac{1}{T} = \frac{c}{\lambda} \tag{2.2.13}$$

The **angular frequency** is $\omega = 2\pi f = kc$.

As the wave travels along, the particle at any fixed point displaces back and forth about some equilibrium position; the particle is said to **vibrate**. The period and frequency were defined above in terms of the time taken for a wave to travel along the *x* axis. It can be seen that the period *T* is also equivalent to the time taken for a particle to displace away and then back to its original position, then off in the other direction and back again; the frequency *f* can also be seen to be equivalent to the number of times the particle vibrates about its equilibrium position in unit time.

The wave 2.2.10 can be expressed in the equivalent forms:

$$\begin{aligned}
u &= a\cos\left[k\left(x - ct\right)\right] \\
u &= a\cos\left[\frac{2\pi}{\lambda}\left(x - ct\right)\right] \\
u &= a\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \\
u &= a\cos\left(2\pi\frac{x}{\lambda} - \omega t\right) \\
u &= a\cos\left(kx - \omega t\right)
\end{aligned} \tag{2.2.14}$$

If one has two waves, $u_1 = a\cos\left(kx - \omega t\right)$ and $u_2 = a\cos\left(kx - \omega t + \phi\right)$, then the waves are the same except they are displaced relative to each other by an amount $\phi / k = \phi\lambda / 2\pi$; ϕ is called the **phase** of u_2 relative to u_1. If ϕ is a multiple of 2π, then the displaced distance is a multiple of the wavelength, and the waves are said to be **in phase**.

It can be verified by substitution that the wave 2.2.14 is a solution of the wave equation 2.2.3.

Example

Fig. 2.2.3 shows a wave travelling through steel and vibrating at frequency $f = 1\,\text{kHz}$. Using the data in Table 2.2.1, the wave number is $k = 2\pi f / c \approx 1.21$ and the wavelength is $\lambda = c / f \approx 5.2$. The period is $T = 1/1000$ sec. For unit amplitude, $a = 1$, the wave profiles are shown for $t = 0$ (blue) and $t = 1/1500$ sec $(= \frac{2}{3}T)$ (red). The dashed arrows show the movement of one particle as the wave passes.

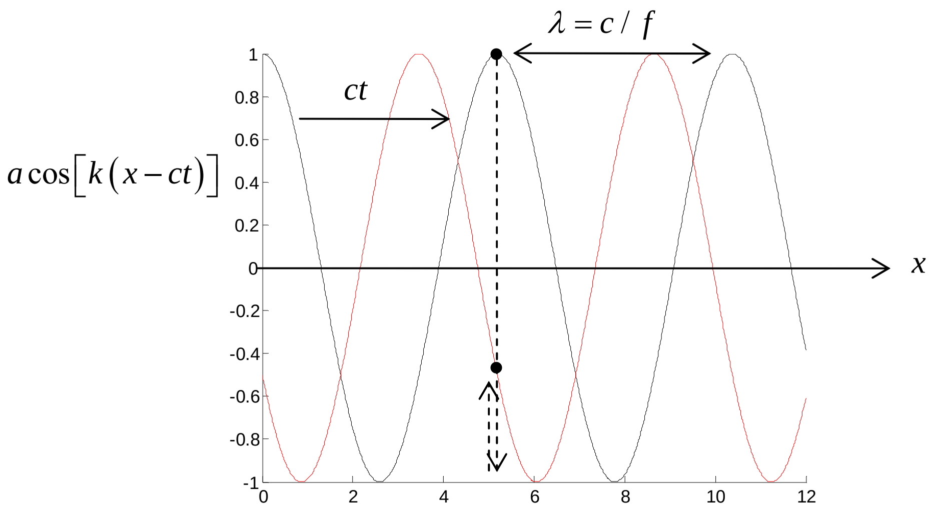


Figure 2.2.3: harmonic wave (Eqn. 2.2.10) travelling through steel at 1 kHz; $a = 1$ with $t = 0$ **(blue) and** $t = 1/1500$ **(red)**

Standing Waves

Because the wave equation is linear, any linear combination of waves is also a solution. In particular, consider two waves which are similar, only travelling in opposite directions; the superposition of these waves is the new wave

$$\begin{aligned} u &= a\cos(kx - \omega t) + a\cos(kx + \omega t) \\ &= 2a\cos(kx)\cos(\omega t) \end{aligned} \tag{2.2.15}$$

It will be seen that this wave profile does not move forward, and is therefore called a **standing wave** (to distinguish it from the **progressive waves** considered earlier). An example is shown in Fig. 2.2.4 (same parameters as for Fig. 2.2.3); at any fixed point, the wave moves up and down over time. The period is again $T = 1/1000$ sec. Shown is the wave at five instants, from $t = 0$ up to just short of the half-period.

Note that $u = 0$ for $x = \pm(2n+1)\pi / 2k$, $n = 0,1,2,\ldots$; these are called the **nodes** of the wave. The intermediate points, where the amplitude is greatest, are called **antinodes**. The distance between successive nodes (or antinodes) is half the wavelength.

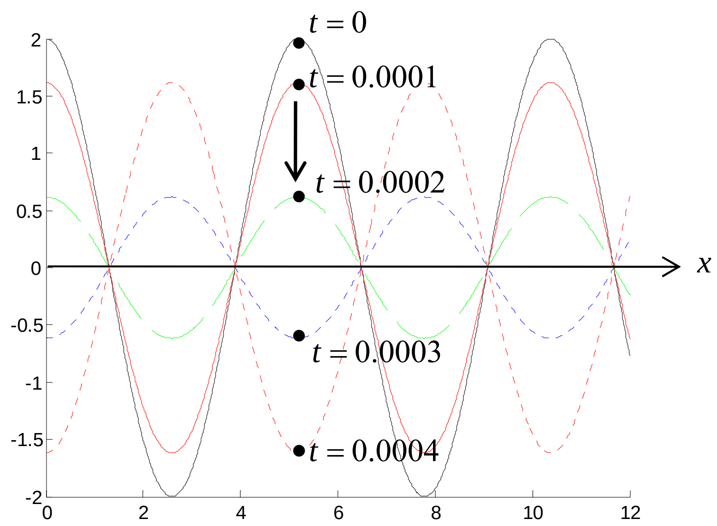


Figure 2.2.4: standing wave (Eqn. 2.2.15) in steel at 1 kHz; with $a = 1$ **at** $t = 0$ **(black),** $t = 0.0001$ **(red),** $t = 0.0002$ **(green dashed),** $t = 0.0003$ **(blue dotted) and** $t = 0.0004$ **(red dotted)**

If the wave is not harmonic, one can use a Fourier analysis (see below) to construct the wave out of a sum of individual harmonic waves; if the profile consists of a regularly repeating pattern, the definitions of wavelength, period, frequency and wave number, and the relations between them, Eqns. 2.2.11-13, still apply.

Complex Exponential Representation

When dealing with progressive waves of harmonic type, it is usually best to represent the wave using a complex exponential function. The reason for this is that exponentials are algebraically simpler than harmonic functions, and also the amplitude and phase are represented by one complex quantity rather than by two separate terms (as will be seen below).

The general wave of the form

$$u = a\cos\left(kx - \omega t + \phi\right) \tag{2.2.16}$$

is the real part of the complex exponential

$$ae^{i(kx-\omega t+\phi)} = a\left[\cos\left(kx - \omega t + \phi\right) + i\sin\left(kx - \omega t + \phi\right)\right] \tag{2.2.17}$$

The phase shift and amplitude can be absorbed into a new constant A:

$$u = Ae^{i(kx-\omega t)}, \quad A = ae^{i\phi} \tag{2.2.18}$$

It can be verified that this complex quantity is itself a solution of the wave equation, Eqn. 2.2.3 (and if a complex quantity is a solution, so are its real and imaginary parts). One can carry out analyses using the complex expression 2.2.18, keeping in mind that the "real" solution, Eqn. 2.2.16, is the real part of this expression. Since $\left|e^{i(kx-\omega t)}\right| = 1$, the true amplitude is $|A|$. The true phase shift ϕ is the argument of *A*, $\arg A$.

Eqn. 2.2.16 is a wave travelling to the "right". It has been seen how a wave travelling to the right is of the form $u = a\cos(kx + \omega t)$, suggesting a complex representation $u = Ae^{i(kx+\omega t)}$. However, this is not an ideal representation, because the difference between a wave travelling left or right, i.e. the difference between this expression and the one in Eqn. 2.2.17, is given by the sign of the frequency. This can make it difficult to solve problems involving reflecting waves[3] (see below), and therefore it is best to use the following representations when adding and subtracting waves:

$$\text{Travelling right: } Ae^{+i(kx-\omega t)} \tag{2.2.19a}$$

$$\text{Travelling left: } Ae^{-i(kx+\omega t)} \tag{2.2.19b}$$

(Note: another popular convention is to use $Ae^{-i(kx-\omega t)}$ for right and $Ae^{+i(kx+\omega t)}$ for left.)

2.2.4 Solution of the Wave Equation (D'Alembert's Solution)

The one-dimensional wave equation 2.2.3 has the very general solution (this is D'Alembert's solution – see the Appendix to this section for its derivation)

$$\boxed{u(x,t) = f(x - ct) + g(x + ct)} \tag{2.2.20}$$

where *f* and *g* are *any* functions[4]; for example, one solution is $f = e^{x-ct}$, $g = \sin(x + ct)$, which can be verified by substitution and carrying out the differentiation. The harmonic waves considered above are special cases of this solution, in which *f* and *g* are cosine functions. The actual forms of the functions *f* and *g* can be determined from the **initial conditions** of the problem, which are the initial displacement profile $u(x,0)$ and the initial velocity $v(x,0) = \partial u / \partial t\big|_{(x,0)}$. Consider the arbitrary initial conditions

$$\begin{aligned} u(x,0) &= U(x) \\ v(x,0) &= V(x) \end{aligned} \tag{2.2.21}$$

Then, as shown in the Appendix to this section, the solution is

$$u(x,t) = \frac{1}{2}\left[U(x+ct) + U(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(\alpha)\,d\alpha \tag{2.2.22}$$

---

[3] for example, when a wave hits a boundary and gets reflected, this representation would force the incident and reflected waves to have different frequencies, when in fact a solution in which the frequencies are the same is often sought
[4] provided they possess second derivatives

Example

Suppose for example that the initial displacement profile was triangular, with maximum displacement $u = \bar{u}$ at $x = 0$, extending to $x = \pm L$, Fig. 2.2.5.

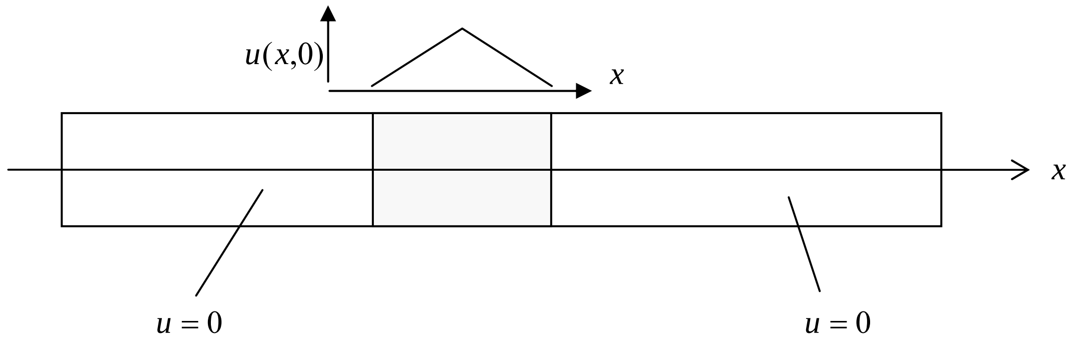


Figure 2.2.5: an initial triangular displacement

The initial conditions are

$$U(x) = u(x,0) = \begin{cases} 0, & |x| \geq L \\ \bar{u}(1 + x/L), & -L \leq x \leq 0 \\ \bar{u}(1 - x/L), & 0 \leq x \leq +L \end{cases}$$

and $V(x) = 0$. D'Alembert's solution is then

$$u(x,t) = \tfrac{1}{2}\left[U(x - ct) + U(x + ct)\right]$$

The solution predicts that at time $2L/c$ there are two triangular displacement profiles of half the magnitude of the original profile; one is to the left and the other is to the right of the original profile, Fig. 2.2.6.

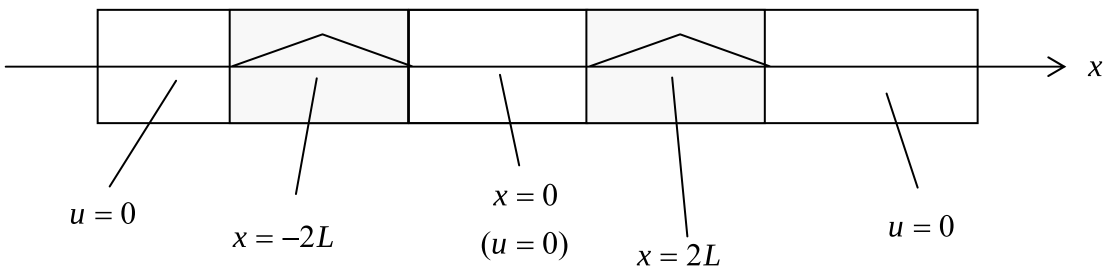


Figure 2.2.6: displacements at time 2*L*/*c*

As the wave passes, particles displace from their equilibrium point, up to the maximum position and then back again. It can be seen that the solution corresponds to a wave of disturbed material propagating through the material from the source, half in one direction and half in the other.

2.2.5 Reflection and Transmission

Let a train of harmonic waves travel from the negative *x* direction in a material with material properties E_1, ρ_1. The waves then meet a second material with different material properties E_2, ρ_2, at the origin $x = 0$. Let the displacements in the first material be u_1 and those in the second, u_2. As will be seen, the incident wave upon the second material will suffer partial reflection and partial transmission. Using the complex exponential representation, Eqn. 2.2.19, and superscripts "*i*" for incident, "*r*" for reflected and "*t*" for transmitted:

$$u_1 = u^{(i)} + u^{(r)}, \qquad u_2 = u^{(t)} \tag{2.2.23}$$

with

$$u^{(i)} = A_i e^{i(k_1 x - \omega t)}, \quad u^{(r)} = A_r e^{-i(k_1 x + \omega t)}, \quad u^{(t)} = A_t e^{i(k_2 x - \omega t)} \tag{2.2.24}$$

A_1 is real, but in general B_1, A_2 could be complex. The wave speeds *c* in each material will be different (if the material properties are different). The frequencies of all three waves are the same – since the material is connected to adjacent material, it must all be vibrating at the same frequency. It follows that the wavenumbers *k* differ also:

$$k_1 c_1 = k_2 c_2 \qquad \text{or} \qquad \frac{k_1}{k_2} = \sqrt{\frac{E_2 \rho_1}{E_1 \rho_2}} \tag{2.2.25}$$

The boundary conditions at the material interface are that

$$\begin{aligned} u_1(0,t) &= u_2(0,t) \\ E_1 \left.\frac{\partial u_1}{\partial x}\right|_{(0,t)} &= E_2 \left.\frac{\partial u_2}{\partial x}\right|_{(0,t)} \end{aligned} \tag{2.2.26}$$

The first of these says that the material remains continuous at the interface. The second says that the stress is also continuous there (see Eqns. 2.2.1b-c). Applying these to Eqn. 2.2.23 gives

$$\begin{aligned} A_i + A_r &= A_t \\ A_i E_1 k_1 - A_r E_1 k_1 &= A_t E_2 k_2 \end{aligned} \tag{2.2.27}$$

so that

$$A_r = A_i \frac{1-\varphi}{1+\varphi}, \qquad A_t = A_i \frac{2}{1+\varphi} \tag{2.2.28}$$

where

$$\varphi = \frac{E_2 k_2}{E_1 k_1} = \frac{c_2 \rho_2}{c_1 \rho_1} = \sqrt{\frac{E_2 \rho_2}{E_1 \rho_1}} \tag{2.2.29}$$

Note that, since A_i is real, so also are A_r, A_t.

The stresses are given by

$$\sigma^{(r)} = -\frac{A_r}{A_i}\sigma^{(i)} = \frac{\varphi - 1}{\varphi + 1}\sigma^{(i)}, \qquad \sigma^{(t)} = \frac{A_t}{A_i}\frac{E_2 k_2}{E_1 k_1}\sigma^{(i)} = \frac{2\varphi}{\varphi + 1}\sigma^{(i)} \tag{2.2.30}$$

The parameter φ determines the nature of the reflected and transmitted waves, and is the ratio of the quantities ρc of each material; this quantity ρc is often referred to as the **mechanical impedance** of the rod. Note that the stiffness E and density ρ are independent, so if $E_2 > E_1$, this does not imply that $\rho_2 > \rho_1$ or that $\varphi > 1$ (see Table 2.2.1).

When $\varphi > 1$, the reflected wave has opposite sign to that of the incident wave and has a smaller amplitude. The transmitted wave is of the same sign and is also smaller. In the limit as $\varphi \to \infty$, which would represent a perfectly rigid material 2 ($E_2 \to \infty$), there is no transmitted wave and the reflected wave has amplitude $A_r = -A_i$. The stress at the boundary is twice the stress due to the incident wave alone.

When $\varphi < 1$, the reflected wave has the same sign to that of the incident wave and has a smaller amplitude. The transmitted wave is of the same sign and is larger. In the limit as $\varphi \to 0$, which would represent "empty" material 2, the reflected wave is equal to the incident wave. The stress at the boundary is zero – this is called a "free boundary" (see below).

Examples of harmonic waves travelling through steel and granite are shown in Fig. 2.2.7. The frequency of vibration is taken to be $f = 1\text{kHz}$. Using the data in Table 2.2.1, the wave numbers are $k_s = 2\pi f / c_s \approx 1.21$ and $k_g = 2\pi f / c_g \approx 2.01$. The wavelengths of the waves are $\lambda_s = c_s / f \approx 5.2$ and $\lambda_g = c_g / f \approx 3.1$. The incident wave is taken to have unit amplitude. When the wave travels from steel into granite, $\varphi = 0.207$ and when it travels from granite into steel it is the inverse of this, $\varphi = 4.83$. The interference between the incident and reflected waves produce a new wave in material "1" (denoted by the green plots in Fig. 2.2.7):

$$u_1 = a\left\{\cos(kx - \omega t) + \frac{1-\varphi}{1+\varphi}\cos(kx + \omega t)\right\} \tag{2.2.31}$$

Note that $A_t = A_i$ at time $t = 0$ (full reflected and transmitted wave profiles are plotted at time zero, even though there is no actual wave present right through the material yet at this time).

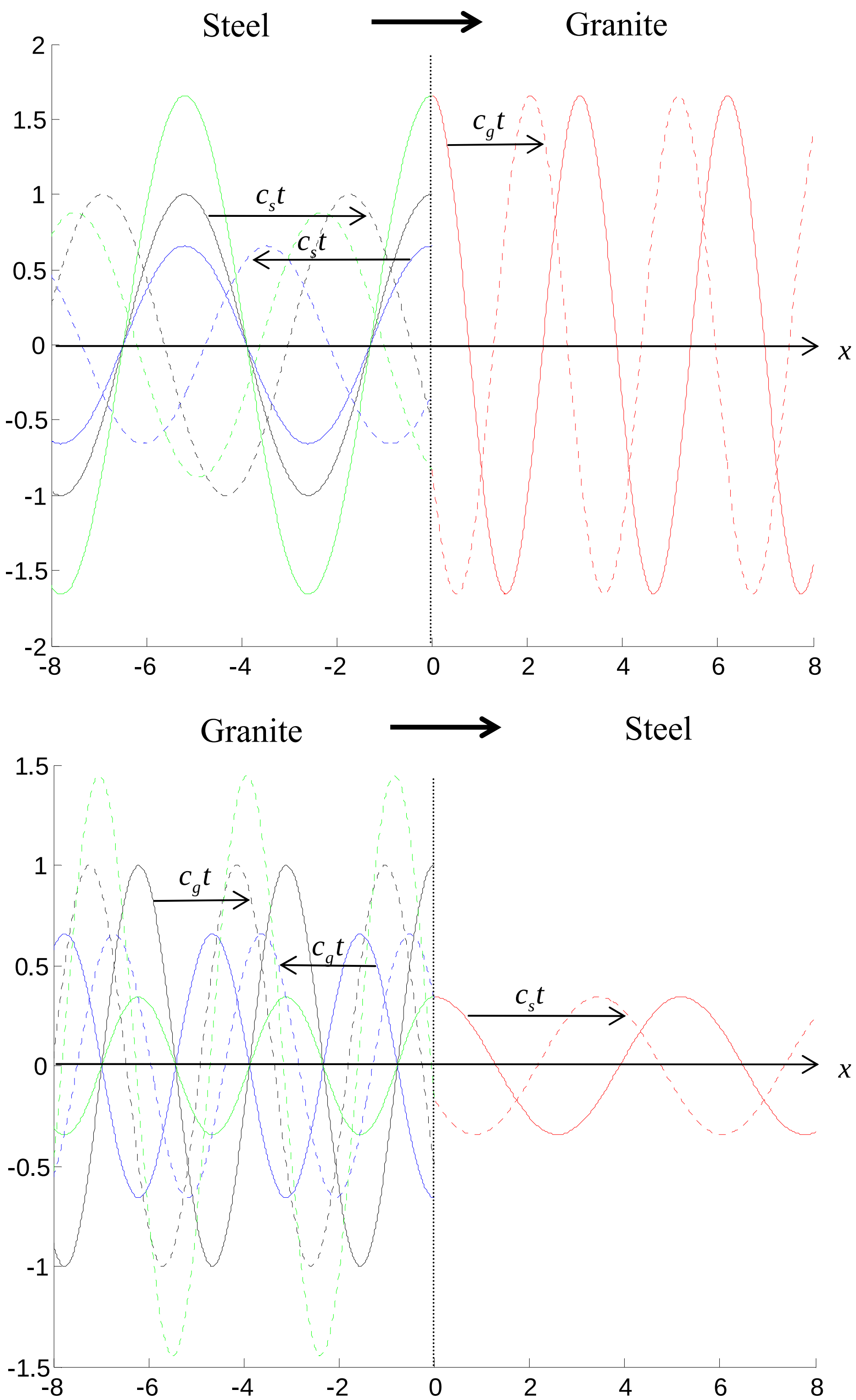


Figure 2.2.7: reflection and transmission of harmonic waves at the boundary between steel and granite; at time $t = 0$ (solid) and time $t = 1/1500$ (dashed); incident (black), reflected (blue), transmitted (red) and composite wave in material "1" (green)

2.2.6 Energy in Vibrating Bars

The kinetic energy in an element of length dx of the bar is $dK = \frac{1}{2} A\rho \left(\partial u / \partial t\right)^2 dx$, where A is the cross-sectional area. The total kinetic energy in a bar of length L is then

$$K = \frac{1}{2}\rho A \int_0^L \left(\partial u / \partial t\right)^2 dx, \tag{2.2.32}$$

The potential energy is the elastic strain energy; for a small element of length dx this is $dW = \frac{1}{2}\sigma\varepsilon A dx$, so

$$W = \frac{1}{2} AE \int_0^L \left(\partial u / \partial x\right)^2 dx, \tag{2.2.33}$$

2.2.7 Solution of the Wave Equation (Standing Waves)

D'Alembert's solution gives results for progressive waves travelling in an infinitely extended medium. Standing waves in an infinite medium can also be a solution. For example if one has the initial profile $U(x) = a\cos(kx)$ and zero initial velocity, $V(x) = 0$, one gets from Eqn. 2.2.22 the standing wave 2.2.15.

Standing waves can be generated more generally by using a **separation of variables** solution procedure for Eqn. 2.2.3. Using this method, detailed in the Appendix to this section, one has the general solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(\bar{A}_n \cos k_n x + \bar{B}_n \sin k_n x\right)\left(\bar{C}_n \cos ck_n t + \bar{D}_n \sin ck_n t\right) \tag{2.2.34}$$

The (infinite number of) constants $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ and **eigenvalues**[5] k can be obtained from the initial and boundary conditions (see later). What are termed "eigenvalues" in this context can be seen to be the wave number.

The terms $\cos k_n x$ and $\sin k_n x$ are called **modes** or **mode shapes**. At any given time *t*, the displacement is a linear combination of these modes. Example modes are shown in Fig. 2.2.8. Some modes will dominate over others, for example perhaps only the first few modes (terms in the series 2.2.34) are significant and need be considered.

[5] note that some authors use the term "eigenvalue" to mean the quantity (ck_n) in this expression

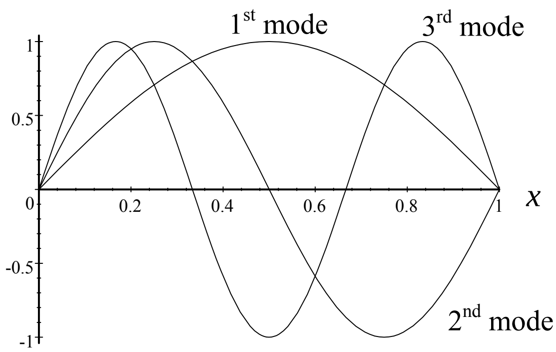


Figure 2.2.8: mode shapes for a vibrating elastic rod

Natural Frequencies

The eigenvalues (or, equivalently, the **natural frequencies** $\omega = ck$) depend on the boundary conditions. There are four possible cases for the one-dimensional rod. Taking the bar to have end-points $x = 0, L$, the boundary conditions are (these are the same as for the static elasticity problem):

$$
\begin{aligned}
&1.\ \textbf{fixed-fixed - } && u(0,t) = 0\,,\ u(L,t) = 0 \\
&2.\ \textbf{free-free - } && \partial u / \partial x\big|_{(0,t)} = 0\,,\ \partial u / \partial x\big|_{(L,t)} = 0 \\
&3.\ \textbf{fixed-free - } && u(0,t) = 0\,,\ \partial u / \partial x\big|_{(L,t)} = 0 \\
&4.\ \textbf{free-fixed - } && \partial u / \partial x\big|_{(0,t)} = 0\,,\ u(L,t) = 0
\end{aligned}
\qquad (2.2.35)
$$

The natural frequencies and modes for each of these boundary conditions are solved for and given in the Appendix to this section (in the boxes). For example, considering the "fixed-fixed" case, the solution is

$$
u(x,t) = \sum_{n=0}^{\infty} \left[A_n \cos(k_n ct) + B_n \sin(k_n ct) \right] \sin(k_n x) \qquad (2.2.36)
$$

with

$$
\begin{aligned}
&\text{Frequencies: } \omega_n = k_n c = \frac{n\pi c}{L}, \quad n = 0,1,\ldots \\
&\text{Modes: } \sin\left(k_n x\right), \quad n = 0,1,\ldots
\end{aligned}
\qquad (2.2.37)
$$

One can plot these sine functions over $[0, L]$ to see the displacement profile of each mode (the first three are those plotted in Fig. 2.2.8 – it can be seen that the higher the mode, the higher the frequency).

The complete solution and precise profile is then obtained by applying the initial conditions of the problem to determine the coefficients A_n, B_n in Eqn. 2.2.36. Some examples of this complete calculation are given in the Appendix.

Vibration Analysis

A **vibration analysis** is one in which the eigenvalues (natural frequencies) and modes are evaluated without regard to which of them might be important in an application. The boundary conditions *alone* determine the modes and natural frequencies. Thus a vibration analysis is carried out *without regard to how the vibration is initiated*. The exact combination of the modes for a particular problem is determined from the initial conditions; the initial conditions will determine the arbitrary constants in the above equations and hence the actual amplitude of vibration.

The vibration is termed **free** if the load is zero or constant; **forced vibration** occurs when the load itself oscillates.

Even though a vibration analysis does not completely solve the problem of a material model loaded in a certain way, for example solving for the propagation paths of stress waves, the amplitudes of vibration, and so on, the natural frequencies and modes are very useful information in themselves, for design and other purposes.

Dynamic response analysis or **transient response analysis** is the calculation of the *complete* response to any arbitrary boundary and initial conditions. This is more difficult than the vibration analysis, since it is a time-dependent problem.

Non-Homogeneous Boundary Conditions

The boundary conditions in 2.2.35 are all **homogeneous** (i.e. $u = 0$ or $\partial u / \partial x = 0$). In practice, the boundary conditions will not be homogeneous, but the natural frequencies do not depend on whether the boundary conditions are homogeneous or non-homogeneous. In other words, if one wants to determine the natural frequencies, *one needs only consider the case of homogeneous boundary conditions*, as will be seen now.

Consider the following **non-homogeneous** boundary conditions:

$$\text{BC's:} \quad u(0,t) = \hat{u}, \; u(L,t) = 0 \tag{2.2.38}$$

Since the wave equation is linear, the solution can be written as the superposition of two separate solutions,

$$u(x,t) = u_p(x,t) + u_h(x,t) \tag{2.2.39}$$

The u_h is the **homogeneous** solution, and is chosen to satisfy the wave equation with homogeneous boundary conditions; u_p is some **particular** solution and accounts for the non-homogeneous boundary condition:

$$\begin{aligned} \text{BC's:} \qquad & u_h(0,t) = 0, \quad u_h(L,t) = 0 \\ & u_p(0,t) = \hat{u}, \quad u_p(L,t) = 0 \end{aligned} \tag{2.2.40}$$

Substituting 2.2.39 into the wave equation 2.2.3 gives

$$\frac{\partial^2 u_h}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_h}{\partial t^2} = -\left(\frac{\partial^2 u_p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u_p}{\partial t^2}\right) \tag{2.2.41}$$

The left hand side is zero. The right hand side can be made zero by choosing u_p to be *any* particular solution of the wave equation. For a simple constant displacement boundary condition, one can choose the linear function

$$u_p(x) = \hat{u}\left(1 - \frac{x}{L}\right) \tag{2.2.42}$$

which can be seen to satisfy 2.2.40b. The complete solution u is illustrated in Fig. 2.2.9.

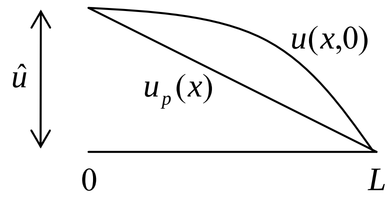


Figure 2.2.9: displacements as a superposition of two separate solutions

Suppose now that the initial conditions are

$$\text{IC's:} \quad \begin{aligned} u(x,0) &= \bar{u}(x) \\ v(x,0) &= \bar{v}(x) \end{aligned} \tag{2.2.43}$$

The initial conditions can be split between u_h and u_p according to

$$\text{IC's:} \quad \begin{aligned} u_h(x,0) &= \bar{u}(x) - u_p(x), \quad u_p(x,0) = u_p(x) \\ v_h(x,0) &= \bar{v}(x) - v_p(x), \quad v_p(x,0) = v_p(x) \end{aligned} \tag{2.2.44}$$

Thus, the complete solution is obtained by adding together:

(i) the function u_h which satisfies the wave equation with homogeneous boundary conditions on displacement, and initial conditions

$$\text{IC's:} \quad \begin{aligned} u_h(x,0) &= \bar{u}(x) - u_p(x) \\ v_h(x,0) &= \bar{v}(x) - v_p(x) \end{aligned}$$

(ii) the function

$$u_p(x) = \hat{u}\left(1 - \frac{x}{L}\right)$$

Thus, using the "fixed-fixed" homogeneous solution from the Appendix,

$$u(x,t) = \hat{u}\left(1 - \frac{x}{L}\right) + \sum_{n=0}^{\infty}\left[A_n \cos(k_n ct) + B_n \sin(k_n ct)\right]\sin(k_n x) \qquad (2.2.45)$$

and the natural frequencies are given by 2.2.37. The constants A_n, B_n can be obtained from the initial conditions, as outlined in the Appendix.

The important point to be made here is that the modes and natural frequencies are determined from (i), i.e. the problem involving the homogeneous boundary conditions, and so, as stated above, the non-homogeneous boundary condition does not affect the modes and natural frequencies.

Forced Vibration

Suppose now that the boundary conditions and initial conditions are given by

$$\text{BC's:} \quad \begin{array}{l} u(0,t) = \alpha\cos(\Omega t) \\ u(L,t) = 0 \end{array}, \qquad \text{IC's:} \quad \begin{array}{l} u(x,0) = \hat{u}\cos(x\pi/2L) \\ v(x,0) = 0 \end{array} \qquad (2.2.46)$$

Again, let $u(x,t) = u_p(x,t) + u_h(x,t)$ and substitute into the wave equation. In this case, the particular solution will be of the general form 2.2.34,

$$u_p = \left(A\cos kx + B\sin kx\right)\left(C\cos ckt + D\sin ckt\right) \qquad (2.2.47)$$

Applying the boundary conditions, one finds that {▲Problem 1}

$$u_p(x,t) = \alpha\left\{\cos\left(\frac{\Omega x}{c}\right) - \cot\left(\frac{\Omega L}{c}\right)\sin\left(\frac{\Omega x}{c}\right)\right\}\cos(\Omega t) \qquad (2.2.48)$$

As with the constant non-homogeneous boundary condition, the initial conditions can now be split appropriately between the homogeneous and particular solutions. Again, the complete solution is obtained by adding together:

(i) the function u_h which satisfies the wave equation with homogeneous boundary conditions on displacement, and initial conditions

$$\text{IC's:} \quad \begin{array}{l} u_h(x,0) = \hat{u}\cos\left(\dfrac{x\pi}{2L}\right) - \alpha\left\{\cos\left(\dfrac{\Omega x}{c}\right) - \cot\left(\dfrac{\Omega L}{c}\right)\sin\left(\dfrac{\Omega x}{c}\right)\right\} \\ v_h(x,0) = 0 \end{array}$$

(ii) the function 2.2.48

The complete solution is

$$u(x,t) = \alpha \left\{ \cos\left(\frac{\Omega x}{c}\right) - \cot\left(\frac{\Omega l}{c}\right) \sin\left(\frac{\Omega x}{c}\right) \right\} \cos(\Omega t) + \sum_{n=0}^{\infty} \left[A_n \cos(k_n ct) + B_n \sin(k_n ct) \right] \sin(k_n x) \tag{2.2.49}$$

Resonance occurs when the displacements become "infinite", which from 2.2.49 occurs when

$$\sin\frac{\Omega L}{c} = 0 \rightarrow \Omega = n\frac{\pi c}{L}.$$

These are precisely the natural frequencies of the system, i.e. the natural frequencies of (i). Thus the problem of resonance becomes more prominent when the forcing frequency Ω approaches any of the natural frequencies k_n.

2.2.8 Problems

1. Consider the case of forced vibration. Use the boundary conditions 2.2.46 to evaluate the constants in the particular solution 2.2.47 and hence derive the particular solution 2.2.48.

2. Consider a fixed-free problem, with the end $x = 0$ subjected to a forced displacement $u = \alpha \sin \Omega t$ and the end $x = L$ free.
   (a) Find the vibration of the material. What are the natural frequencies?
   (b) When does resonance occur?
   [note: the appropriate homogeneous solution and natural frequencies are given in the Appendix to this section]

3. Consider a vibrating bar with an oscillatory stress applied to one end, $\sigma(0) = \alpha \cos \Omega t$. The end $x = L$ is fixed, $u(L) = 0$.
   (a) Find the vibration of the material. What are the natural frequencies?
   (b) When does resonance occur?
   [note: the appropriate homogeneous solution and natural frequencies are given in the Appendix to this section]

2.2.9 Appendix to Section 2.2

1. D'Alembert's Solution of the Wave Equation

In the wave equation 2.2.3, change variables through

$$\xi = x - ct, \qquad \eta = x + ct \tag{2.2.50}$$

Then $u = u(\xi(x,t), \eta(x,t))$ and the chain rule gives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \tag{2.2.51}$$

and similarly for the variable *t*. Another differentiation gives

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \tag{2.2.52}$$

and similarly for the variable *t*. Substituting these expression into the wave equation 2.2.3 leads to

$$4\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \tag{2.2.53}$$

Integrating with respect to ξ gives $\partial u / \partial \eta = \gamma(\eta)$ where $\gamma(\eta)$ is some arbitrary function. A further integration then gives $u = \int \gamma(\eta) d\eta + f(\xi) = f(\xi) + g(\eta)$, which is D'Alembert's solution, Eqn. 2.2.20:

$$u(x,t) = f(x - ct) + g(x + ct) \tag{2.2.54}$$

Let the initial conditions be

$$\begin{aligned} u(x,0) &= U(x) \\ \left.\frac{\partial u}{\partial t}\right|_{(x,0)} &= V(x) \end{aligned} \tag{2.2.55}$$

Thus, from 2.2.54,

$$U(x) = f(x) + g(x). \tag{2.2.56}$$

Now

$$\frac{\partial u}{\partial t} = \frac{\partial f\left(\xi(x,t)\right)}{\partial t} + \frac{\partial g\left(\eta(x,t)\right)}{\partial t} = \frac{df}{d\xi}\frac{\partial \xi}{\partial t} + \frac{dg}{d\eta}\frac{\partial \eta}{\partial t} = -c\frac{df}{d\xi} + c\frac{dg}{d\eta} \tag{2.2.57}$$

At $t=0$, $f(\xi) = f(x)$ and $g(\eta) = g(x)$, so

$$V(x) = \left.\frac{\partial u}{\partial t}\right|_{(x,0)} = -c\frac{df(x)}{dx} + c\frac{dg(x)}{dx} \tag{2.2.58}$$

Integrating then gives

$$\frac{1}{c}\int_{x_0}^{x} V(\alpha)d\alpha = -\int_{x_0}^{x}\frac{df(\alpha)}{d\alpha}d\alpha + \int_{x_0}^{x}\frac{dg(\alpha)}{d\alpha}d\alpha = g(x) - f(x) + \vartheta(x_0), \quad \vartheta(x_0) = f(x_0) - g(x_0) \tag{2.2.59}$$

Subtracting this from Eqn. 2.2.56, and also adding it to Eqn. 2.2.56, gives

$$\begin{aligned} f(x) &= \frac{1}{2}U(x) - \frac{1}{2c}\int_{x_0}^{x} V(\alpha)d\alpha + \frac{1}{2}\vartheta(x_0) \\ g(x) &= \frac{1}{2}U(x) + \frac{1}{2c}\int_{x_0}^{x} V(\alpha)d\alpha - \frac{1}{2}\vartheta(x_0) \end{aligned} \tag{2.2.60}$$

If one now replaces *x* with $x - ct$ in the first of these, and with $x + ct$ in the latter, addition of the two expressions leads to Eqn. 2.2.22:

$$u(x,t) = \frac{1}{2}\left[U\left(x+ct\right) + U\left(x-ct\right)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(\alpha)d\alpha \tag{2.2.61}$$

2. Method of Separation of Variables Solution to the Wave Equation

Assuming a separable solution, write $u(x,t) = X(x)T(t)$ so that $\partial^2 u/\partial t^2 = X(x)\ddot{T}(t)$ and $\partial^2 u/\partial x^2 = X''(x)T(t)$. Inserting these into the wave equation gives

$$\begin{aligned} &X\frac{d^2T}{dt^2} = c^2\frac{d^2X}{dX}T \\ \rightarrow &\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dX} \end{aligned} \tag{2.2.62}$$

This relation states that a function of *t* equals a function of *x* and it must hold for all *t* and *x*. It follows that both sides of this expression must be equal to a constant, say *k* (if the left hand side were not constant it would change in value as *t* is changed, but then the equality would no longer hold because the right hand side does not change when *t* is changed – it is a function of *x* only). Thus there are two second order ordinary differential equations:

$$\frac{d^2 X}{dx^2} - kX = 0, \qquad \frac{d^2 T}{dt^2} - c^2 kT = 0 \tag{2.2.63}$$

which have solutions

$$X = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x} = 0, \qquad T = Ce^{c\sqrt{k}t} + De^{c\sqrt{k}t} \tag{2.2.64}$$

Modes and Natural Frequencies for Homogeneous Boundary Conditions

Suppose first that k is positive. Consider homogeneous boundary conditions, that is, $u = 0$ and/or $\partial u / \partial x = 0$ at the end points $x = 0, L$. Suppose first that $u(0,t) = 0$. Then $u(0,t) = X(0)T(t) = 0 \to X(0) = 0$ and so $A + B = 0$. If also $u(L,t) = 0$, then $Ae^{\sqrt{k}L} + Be^{-\sqrt{k}L} = 0$ which implies that $A = B = 0$, and $u(x,t) = 0$. Similarly, if one uses the conditions $\partial u / \partial x(0,t) = 0$ or $\partial u / \partial x(L,t) = 0$, or a combination of zero u and first derivative, one arrives at the same conclusion: a trivial zero solution. Therefore, to obtain a non-zero solution, one must have k negative, and

$$X(x) = \overline{A}\cos(\lambda x) + \overline{B}\sin(\lambda x), \qquad k = -\lambda^2 \tag{2.2.65}$$

The solution for $T(t)$ must then be

$$T(t) = \overline{C}\cos(\lambda ct) + \overline{D}\sin(\lambda ct) \tag{2.2.66}$$

and the full solution is

$$u(x,t) = \left(\overline{A}\cos(\lambda x) + \overline{B}\sin(\lambda x)\right)\left(\overline{C}\cos(\lambda ct) + \overline{D}\sin(\lambda ct)\right) \tag{2.2.67}$$

There are four possible combinations of boundary conditions.

1. Fixed-Fixed

Here, $u(0,t) = u(L,t) = 0$. Thus $X(0) = \overline{A} = 0$ and $X(L) = \overline{B}\sin(\lambda L) = 0$. For non-zero $\overline{B}$ one must have $\sin(\lambda L) = 0 \to \lambda = \pm n\pi / L,\ n = 0,1,\ldots$. Thus one has the infinite number of solutions $X_n(x) = \overline{B}_n \sin(\lambda_n x)$, and the complete general solution is ($A = \overline{B}\overline{C}$, $B = \overline{B}\overline{D}$)[6]

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct)\right]\sin(\lambda_n x) \tag{2.2.68}$$

with

[6] the solutions corresponding to negative values of n, i.e. $\lambda = -n\pi / L$, $n = 1, 2, \ldots$, can be subsumed into 2.2.68 through the constants A_n, B_n; the solution for $n = 0$ is zero

Frequencies: $\boxed{\omega_n = \lambda_n c = \frac{n\pi c}{L}, \quad n = 1,2,\ldots}$ Modes: $\boxed{\sin(\lambda_n x), \quad n = 1,2,\ldots}$

(2.2.69)

It can be proved that the series 2.2.68 converges and that it is indeed a solution of the wave equation, provided some fairly weak conditions are fulfilled (see a text on Advanced Calculus).

The first three modes are plotted in Fig. 2.2.10.

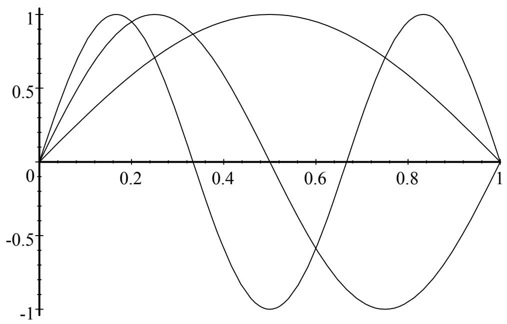


Figure 2.2.10: first three mode shapes for fixed-fixed

Case 2. Free-Free

Here, $\partial u / \partial x(0,t) = \partial u / \partial x(L,t) = 0$. Thus $X'(0) = \lambda \overline{B} = 0$ and $X'(L) = -\lambda \overline{A} \sin(\lambda L) = 0$. Thus the general solution is ($A = \overline{A}\overline{C}$, $B = \overline{A}\overline{D}$)

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct) \right] \cos(\lambda_n x) \tag{2.2.70}$$

with the λ_n as for fixed-fixed.

Frequencies: $\boxed{\omega_n = \lambda_n c = \frac{n\pi c}{L}, \quad n = 1,2,\ldots}$ Modes: $\boxed{\cos(\lambda_n x), \quad n = 1,2,\ldots}$

(2.2.71)

The displacement profiles of the first three modes are shown in Fig. 2.2.11.

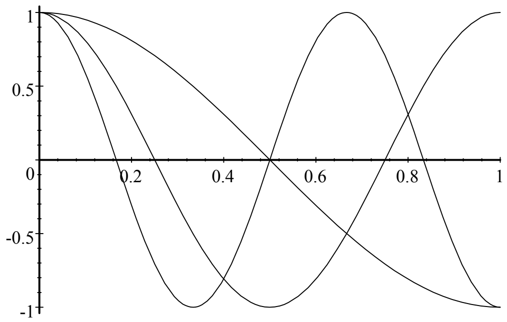


Figure 2.2.11: first three mode shapes for free-free

Case 3. Fixed-Free

Here, $u(0,t) = \partial u / \partial x(L,t) = 0$. Thus $X(0) = \overline{A} = 0$ and $X(L) = \lambda\overline{B}\cos(\lambda L) = 0$. For non-zero $\overline{B}$ one must have $\cos(\lambda L) = 0 \rightarrow \lambda = (2n-1)\pi / 2L,\ n = \ldots -2, -1, 0, 1, 2, \ldots$. The solution is again given by 2.2.68, which is repeated here,

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct) \right] \sin(\lambda_n x) \tag{2.2.72}$$

only now

$$\text{Frequencies: } \boxed{\omega_n = \lambda_n c = \frac{(2n-1)\pi c}{2L}, \quad n = 1, 2, \ldots} \qquad \text{Modes: } \boxed{\sin(\lambda_n x), \quad n = 1, 2, \ldots} \tag{2.2.73}$$

The displacement profiles of the first three modes are shown in Fig. 2.2.12.

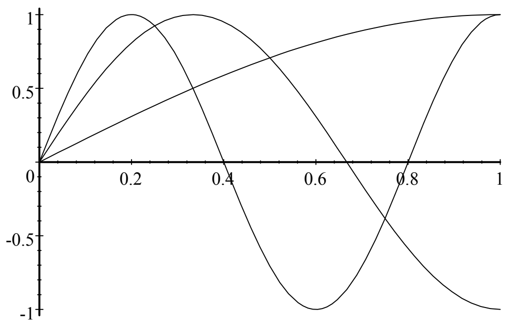


Figure 2.2.12: first three mode shapes for fixed-free

Case 4. Free-Fixed

Here, $\partial u / \partial x(0,t) = u(L,t) = 0$. Thus $X'(0) = \lambda \overline{B} = 0$ and $X(L) = \overline{A}\cos(\lambda L) = 0$. For non-zero $\overline{A}$ one must have $\cos(\lambda L) = 0$ so the general solution is as for free-free, Eqn. 2.2.70, but with $A_0 = 0$:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(\lambda_n ct) + B_n \sin(\lambda_n ct) \right] \cos(\lambda_n x) \qquad (2.2.74)$$

with the λ_n as for fixed-free.

$$\text{Frequencies: } \boxed{\omega_n = \lambda_n c = \frac{(2n-1)\pi c}{2L}, \quad n = 1,2,\dots} \qquad \text{Modes: } \boxed{\cos(\lambda_n x), \quad n = 1,2,\dots} \qquad (2.2.75)$$

The displacement profiles of the first three modes are shown in Fig. 2.2.13.

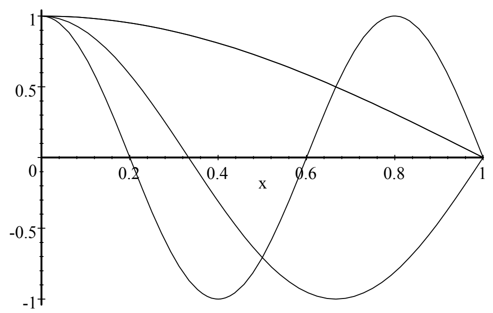


Figure 2.2.13: first three mode shapes for free-fixed

Full Solution (incorporating Initial Conditions)

(a) Initial Condition on Displacement

The initial condition on displacement is

$$u(x,0) = u_0(x) \qquad (2.2.76)$$

which give, from 2.2.68, 2.2.70, 2.2.72, 2.2.74,

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x) = u_0(x) \qquad \text{fixed-fixed/fixed-free}$$

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(\lambda_n x) = u_0(x) \qquad \text{free-free} \qquad (2.2.77)$$

$$u(x,0)=\sum_{n=1}^{\infty}A_n\cos(\lambda_n x)=u_0(x) \qquad \text{free-fixed}$$

These can be solved by using the **orthogonality condition** of the trigonometric functions:

$$\int_0^L \sin(\lambda_n x)\sin(\lambda_m x)dx=\int_0^L \cos(\lambda_n x)\cos(\lambda_m x)dx=\begin{cases}0, & m\neq n\\ L/2, & m=n\end{cases} \qquad (2.2.78)$$

for either of $\lambda_n = n\pi/L,\ (2n-1)\pi/2L$. Thus multiplying both sides of 2.2.77a by $\sin(\lambda_m x)$ and 2.2.77b-c by $\cos(\lambda_m x)$ and integrating over $[0,L]$ gives

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L u_0(x)\sin(\lambda_n x)dx && \text{fixed-fixed/fixed-free}\\
A_0 &= \frac{1}{L}\int_0^L u_0(x)dx,\quad A_n=\frac{2}{L}\int_0^L u_0(x)\cos(\lambda_n x)dx,\ n=1,2,\dots && \text{free-free} \qquad (2.2.79)\\
A_n &= \frac{2}{L}\int_0^L u_0(x)\cos(\lambda_n x)dx && \text{free-fixed}
\end{aligned}$$

(b) Initial Condition on Velocity

The initial condition on velocity, $\dot{u}(x,0)=v_0(x)$, gives

$$\begin{aligned}
\dot{u}(x,0) &= \sum_{n=1}^{\infty}\lambda_n cB_n\sin(\lambda_n x)=v_0(x) && \text{fixed-fixed/fixed-free}\\
\dot{u}(x,0) &= \sum_{n=1}^{\infty}\lambda_n cB_n\cos(\lambda_n x)=v_0(x) && \text{free-fixed/free-free} \qquad (2.2.80)
\end{aligned}$$

Using the orthogonality conditions again gives

$$\begin{aligned}
B_n &= \frac{2}{Lc\lambda_n}\int_0^L v_0(x)\sin(\lambda_n x)dx && \text{fixed-fixed/fixed-free}\\
B_n &= \frac{2}{Lc\lambda_n}\int_0^L v_0(x)\cos(\lambda_n x)dx && \text{free-fixed/free-free} \qquad (2.2.81)
\end{aligned}$$

Example

Consider the fixed-free case with initial conditions $u_0(x)=0,\ v_0(x)=2x/L$. Thus $A_n=0$ and

$$B_n = \frac{8}{(2n-1)\pi Lc}\int_0^L x\sin\left(\frac{(2n-1)\pi}{2L}x\right)dx = \frac{8}{(2n-1)\pi Lc}\frac{4L^2(-1)^{n+1}}{\pi^2(2n-1)^2}$$
$$= \frac{32(-1)^{n+1}}{(2n-1)^3\pi^3}\frac{L}{c}$$

so that

$$u(x,t) = \frac{32L}{\pi^3 c}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\sin(\lambda_n x)\sin(\lambda_n ct), \qquad \omega_n = \lambda_n c = \frac{(2n-1)\pi c}{2L}, \qquad n = 1,2,\ldots$$

The period for the first (dominant) mode is $T_1 = 2\pi/\lambda_1 c = 4L/c$. The solution is plotted in Fig. 2.2.14 for $c = 5000\text{m/s}$, $L = 0.1\text{m}$, for the five times $iT_1/16,\ i = 0\ldots4$ (up to the quarter-period). Thereafter, the solution decreases back to zero, down through negative displacements, back to zero and then repeats.

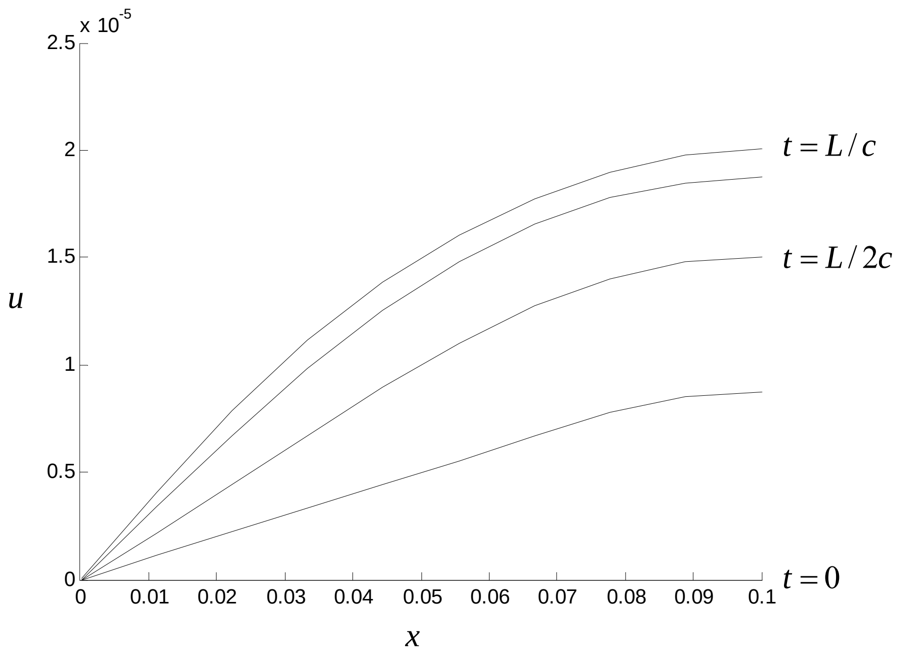


Figure 2.2.14: displacements for fixed-free example

Example

Consider the free-free case with initial conditions $u_0(x) = \bar{u}x,\ v_0(x) = 0$. Thus $B_n = 0$ and

$$A_0 = \frac{\bar{u}}{L}\int_0^L x dx = \frac{\bar{u}L}{2}$$

$$A_n = \frac{2\bar{u}}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2\bar{u}L}{\pi^2 n^2}\left((-1)^n - 1\right), \quad n = 1, 2, \ldots$$

so that

$$u(x,t) = \bar{u}L\left[\frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\left((-1)^n - 1\right)}{n^2}\cos(\lambda_n ct)\cos(\lambda_n x)\right], \quad \lambda_n = \frac{n\pi}{L} \tag{2.2.34}$$

The period for the first (dominant) mode is $T_1 = 2\pi/\lambda_1 c = 2L/c$. The solution is plotted in Fig. 2.2.15 again for $c = 5000\text{m/s}$, $L = 0.1\text{m}$, for the nine times $iT_1/16,\ i = 0\ldots 8$ (up to the half-period). Thereafter, the solution returns back to the initial position and then repeats.

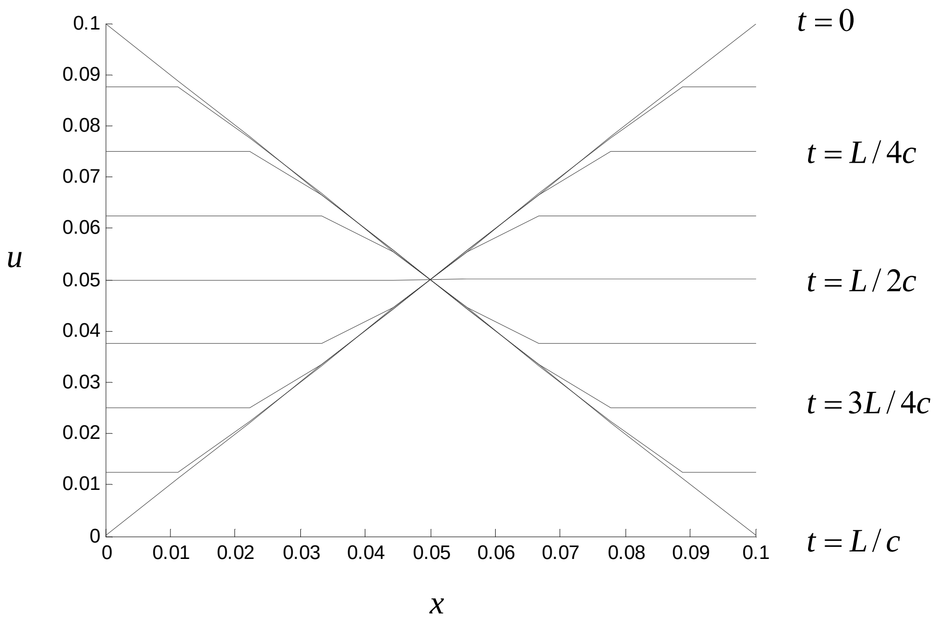


Figure 2.2.15: displacements for free-free example

3 2D Elastostatic Problems in Cartesian Coordinates

Two dimensional elastostatic problems are discussed in this Chapter, that is, static problems of either plane stress or plane strain. Cartesian coordinates are used, which are appropriate for geometries which are have straight boundaries. The two-dimensional Navier equations are derived and the Airy stress function technique is used to solved exactly some important problems.

3.1 Plane Problems

What follows is to be applicable to any two dimensional problem, so it is taken that $\sigma_{yz} = \sigma_{xz} = 0$, which is true of both plane stress and plane strain.

3.1.1 Governing Equations for Plane Problems

To recall, the equations governing the elastostatic problem are the *elastic stress-strain law* (Part I, Eqns. 4.2.11-14), the *strain-displacement relations* (Eqns. 1.2.5) and the *equations of equilibrium* (1.1.10)

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right], \quad \varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right], \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$
$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right) \qquad \textbf{Plane Stress} \qquad (3.1.1a)$$

$$\varepsilon_{xx} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right], \quad \varepsilon_{yy} = \frac{1+\nu}{E}\left[-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}\right], \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$
$$\sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right)$$
$$\textbf{Plane Strain} \qquad (3.1.1b)$$

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \end{aligned} \qquad \textbf{Strain-displacement relations} \qquad (3.1.2)$$

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x &= 0 \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y &= 0 \\ \frac{\partial \sigma_{zz}}{\partial z} + b_z &= 0 \end{aligned} \qquad \textbf{Equations of Equilibrium} \qquad (3.1.3)$$

One way of solving these equations is to re-write the stresses in 3.1.3 in terms of strains by using 3.1.1, and then using 3.1.2 to re-write the resulting equations in terms of displacements only. For example in the case of plane strain one arrives at

$$\frac{E}{2(1+\nu)(1-2\nu)}\left\{2(1-\nu)\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial x \partial y}+(1-2\nu)\frac{\partial^2 u_x}{\partial y^2}\right\}+b_x=0$$
$$\frac{E}{2(1+\nu)(1-2\nu)}\left\{2(1-\nu)\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_x}{\partial x \partial y}+(1-2\nu)\frac{\partial^2 u_y}{\partial x^2}\right\}+b_y=0 \tag{3.1.4}$$

These are the 2D Navier's equations, analogous to the 1D version, Eqn. 2.1.2. This set of partial differential equations can be solved subject to boundary conditions on the displacement. Obviously, in the absence of body forces, any linear displacement field satisfies 3.1.4, for example the field

$$u_x=\frac{\sigma_o}{E}x+A-Cy, \qquad u_y=-\frac{\nu\sigma_o}{E}y+B+Cy \tag{3.1.5}$$

with A, B, C representing the possible rigid body motions; this corresponds to a simple tension $\sigma_{xx}=\sigma_o$.

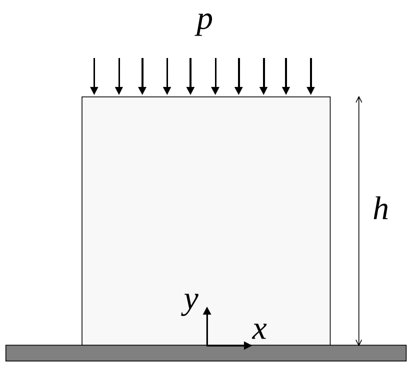
Solving Eqns. 3.1.4 directly for more complex cases is not an easy task. An alternative solution strategy for the plane elastostatic problem is the Airy stress function method described in the next section.

3.1.2 Problems

1. Derive the plane stress Navier equations analogous to 3.1.4.

2. Show that the displacement field $u_x = Ax^2$, $u_y=-4Axy/(1+\nu)$, in the absence of body forces, satisfies the plane stress governing equations derived in Problem 1 (this solution does not satisfy 3.1.4). Determine the corresponding stress field and verify that it satisfies the equilibrium equations.

3. Consider the *thin* plate shown below subjected to a uniform pressure *p* on the top and its own body weight. The plate is *perfectly bonded* to the base plate.
   (a) Does the stress distribution
   $$\sigma_{yy}(x,y)=-p+\rho g(y-h), \quad \sigma_{xx}=\sigma_{xy}=0$$
   satisfy the equations of equilibrium?
   (b) Does it satisfy the boundary conditions at the upper surface, and at the two free surfaces?
   (c) Suppose now that the plate was made out of elastic material. Show that, in that case, the stresses given above are actually not a correct solution to the problem.


p
h
y
x

3.2 The Stress Function Method

An effective way of dealing with many two dimensional problems is to introduce a new "unknown", the **Airy stress function** ϕ, an idea brought to us by George Airy in 1862. The stresses are written in terms of this new function and a new differential equation is obtained, one which can be solved more easily than Navier's equations.

3.2.1 The Airy Stress Function

The stress components are written in the form

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$$
$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \qquad (3.2.1)$$
$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

Note that, unlike stress and displacement, the Airy stress function has no obvious physical meaning.

The reason for writing the stresses in the form 3.2.1 is that, *provided the body forces are zero*, the equilibrium equations are automatically satisfied, which can be seen by substituting Eqns. 3.2.1 into Eqns. 2.2.3 {▲Problem 1}. On this point, the body forces, for example gravitational forces, are generally very small compared to the effect of typical surface forces in elastic materials and may be safely ignored (see Problem 2 of §2.1). When body forces are significant, Eqns. 3.2.1 can be amended and a solution obtained using the Airy stress function, but this approach will not be followed here. A number of examples including non-zero body forces are examined later on, using a different solution method.

3.2.2 The Biharmonic Equation

The Compatability Condition and Stress-Strain Law

In the previous section, it was shown how one needs to solve the equilibrium equations, the stress-strain constitutive law, and the strain-displacement relations, resulting in the differential equation for displacements, Eqn. 3.1.4. An alternative approach is to ignore the displacements and attempt to solve for the *stresses and strains only*. In other words, the strain-displacement equations 3.1.2 are ignored. However, if one is solving for the strains but not the displacements, one must ensure that the compatibility equation 1.3.1 is satisfied.

Eqns. 3.2.1 already ensures that the equilibrium equations are satisfied, so combine now the two dimensional compatibility relation and the stress-strain relations 3.1.1 to get {▲Problem 2}

$$\text{plane stress}: \quad \left(\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4}\right) = 0$$
$$\text{plane strain}: \quad \left(\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4}\right)(1-\nu) = 0 \qquad (3.2.2)$$

Thus one has what is known as the biharmonic equation:

$$\boxed{\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0} \quad \textbf{The biharmonic equation} \qquad (3.2.3)$$

The biharmonic equation is often written using the short-hand notation $\nabla^4\phi = 0$.

By using the Airy stress function representation, the problem of determining the stresses in an elastic body is reduced to that of finding a solution to the biharmonic partial differential equation 3.2.3 whose derivatives satisfy certain boundary conditions.

Note that the biharmonic equation is independent of elastic constants, Young's modulus E and Poisson's ratio ν. Thus for bodies in a state of plane stress or plane strain, the stress field is independent of the material properties, provided the boundary conditions are expressed in terms of tractions (stress) [1]; boundary conditions on displacement will bring the elastic constants in through the stress-strain law. Further, the plane stress and plane strain stress fields are identical.

3.2.3 Some Simple Solutions

Clearly, any polynomial of degree 3 or less will satisfy the biharmonic equation. Here follow some elementary examples.

(i) $\phi = Ay^2$

one has $\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2} = 2A, \quad \sigma_{yy} = \sigma_{xy} = 0$, a state if uniaxial tension

(ii) $\phi = Bxy$

here, $\sigma_{xx} = \sigma_{yy} = 0, \quad \sigma_{xy} = -B$, a state of pure shear

(iii) $\phi = Ay^2 + Bxy$

here, $\sigma_{xx} = 2A, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -B$, a superposition of (i) and (ii)

[1] technically speaking, this is true only in simply connected bodies, i.e. ones without any "holes", since problems involving bodies with holes have an implied displacement condition (see, for example, Barber (1992), §2.2).

3.2.4 Pure Bending of a Beam

Consider the bending of a rectangular beam by a moment M_0, as shown in Fig. 3.2.1. The elementary beam theory predicts that the stress σ_{xx} varies linearly with y, Fig. 3.2.1, with the $y = 0$ axis along the beam-centre, so a good place to start would be to choose, or guess, as a stress function $\phi = Cy^3$, where C is some constant to be determined. Then $\sigma_{xx} = 6Cy,\ \sigma_{yy} = 0,\ \sigma_{xy} = 0$, and the boundary conditions along the top and bottom of the beam are clearly satisfied.

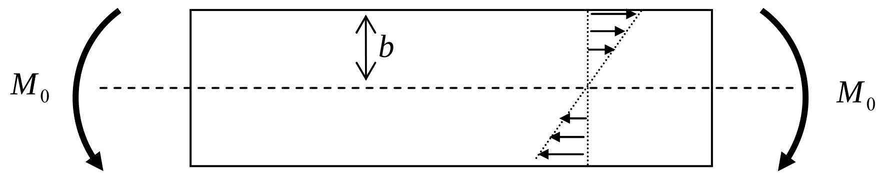


Figure 3.2.1: a beam in pure bending

The moment and stress distribution are related through

$$M_0 = \int_{-b}^{+b} \sigma_{xx} y dy = 6C \int_{-b}^{+b} y^2 dy = 4Cb^3 \tag{3.2.4}$$

and so $C = M_0 / 4b^3$ and $\sigma_{xx} = 3M_0 y / 2b^3$. The fact that this last expression agrees with the elementary beam theory ($\sigma = -My/I$ with $I = 2b^3 h/3$, where h is the depth "into the page") shows that that the beam theory is *exact* in this simple loading case.

Assume now plane strain conditions. In that case, there is another non-zero stress component, acting "perpendicular to the page", $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = 3M\nu y / 2b^3$. Using Eqns. 3.1.1b,

$$\begin{aligned} \varepsilon_{xx} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] = \left(\frac{1-\nu^2}{E}\frac{3M}{2b^3}\right) y = \alpha y, \text{ say} \\ \varepsilon_{yy} &= \frac{1+\nu}{E}\left[-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}\right] = \left(-\frac{\nu(1+\nu)}{E}\frac{3M}{2b^3}\right) y = \beta y, \text{ say} \end{aligned} \tag{3.2.5}$$

and the other four strains are zero.

As in §1.2.4, once the strains have been found, the displacements can be found by integrating the strain-displacement relations. Thus

$$
\begin{aligned}
&\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \alpha y \\
&\quad \to u_x = \alpha xy + f(y) \\
&\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = \beta y \\
&\quad \to u_y = \tfrac{1}{2}\beta y^2 + g(x) \\
&\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\big(\alpha x + f'(y) + g'(x)\big) \equiv 0 \\
&\quad \to g'(x) + \alpha x = -f'(y)
\end{aligned}
\tag{3.2.6}
$$

Therefore $f'(y)$ must be some constant, $-C$ say, so $f(y) = -Cy + A$, and $g(x) = Cx - \frac{1}{2}\alpha x^2 + B$. Finally,

$$
\begin{aligned}
u_x &= \alpha xy - Cy + A \\
u_y &= -\tfrac{1}{2}\alpha x^2 + \tfrac{1}{2}\beta y^2 + Cx + B
\end{aligned}
\tag{3.2.7}
$$

which are of the form 1.2.17. For the case when the mid-point of the beam is fixed, so has no translation, $u_x(0,0) = u_y(0,0) = 0$, and if it has no rotation there, $\omega_z(0,0) = 0$, then the three arbitrary constants are zero, and

$$
\begin{aligned}
u_x &= \alpha xy \\
u_y &= -\tfrac{1}{2}\alpha x^2 + \tfrac{1}{2}\beta y^2
\end{aligned}
\tag{3.2.8}
$$

3.2.5 A Cantilevered Beam

Consider now the cantilevered beam shown in Fig. 3.2.2. The beam is subjected to a uniform shear stress $\sigma_{xy} = \tau$ over its free end, Fig. 3.2.2a. The boundary conditions are

$$
\sigma_{xx}(0,y) = 0, \quad \sigma_{xy}(0,y) = \tau, \quad \sigma_{yy}(x,\pm b) = \sigma_{xy}(x,\pm b) = 0 \tag{3.2.9}
$$

It is difficult, if not impossible, to obtain concise expressions for stress and strain for problems even as simple as this[2]. However, a concise solution can be obtained by relaxing one of the above conditions. To this end, consider the similar problem of Fig. 3.2.2b – this beam is subjected to a shear force *F*, the resultant of the shear stresses. The applied force of Fig. 3.2.2b is equivalent to that in Fig. 3.2.2a if

$$
\int_{-b}^{+b} \sigma_{xy}(0,y)\,dy = F \tag{3.2.10}
$$

[2] an exact solution will usually require an infinite series of terms for the stress and strain

This is known as a **weak boundary condition**, since the stress is not specified in a point-wise sense along the boundary – only the resultant is. However, from Saint-Venant's principle (Part I, §3.3.2), the stress field in both beams will be the same except for in a region close to the applied load.

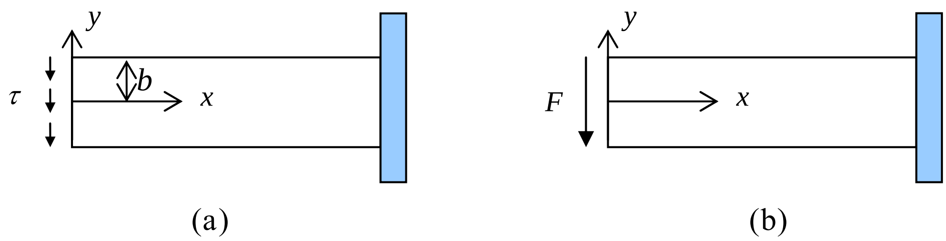


Figure 3.2.2: A cantilevered beam subjected to; (a) a uniform distribution of shear stresses along its free end, (b) a shear force along its free end

The elementary beam theory predicts a stress $\sigma_{xx} = -My/I = Fxy/I$. Thus a good place to start is to choose the stress function $\phi = \alpha xy^3$, where α is a constant to be determined. The stresses are then

$$\sigma_{xx} = 6\alpha xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = -3\alpha y^2 \tag{3.2.11}$$

However, it can be seen that $\sigma_{xy}(x, \pm b) = -3\alpha b^2 \neq 0$. To offset this, one can superimpose a constant shear stress $3\alpha b^2$, in other words amend the stress function to

$$\phi = \alpha xy^3 - 3\alpha b^2 xy \tag{3.2.12}$$

The boundary conditions are now satisfied and, from Eqn. 3.2.10,

$$\alpha = \frac{F}{4b^3} \tag{3.2.13}$$

and so

$$\sigma_{xx} = \frac{3F}{2b^3} xy, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = \frac{3F}{4b^3}\left(b^2 - y^2\right) \tag{3.2.14}$$

3.2.6 Problems

1. Verify that the relations 3.2.1 satisfy the equilibrium equations 2.2.3.

2. Derive Eqn. 3.2.2.

3. A large thin plate is subjected to certain boundary conditions on its thin edges (with its large faces free of stress), leading to the stress function

$$\phi = Ax^3 y^2 - Bx^5$$

(i) use the biharmonic equation to express A in terms of B
(ii) calculate all stress components
(iii) calculate all strain components (in terms of B, E, ν)
(iv) derive an expression for the volumetric strain, in terms of B, E, ν, x and y.
(v) check that the compatibility equation is satisfied
(vi) check that the equilibrium equations are satisfied

4. A very thick component has the same boundary conditions on any given cross-section, leading to the following stress function:

$$\phi = x^4 y + 4x^2 y^3 - y^5$$

(i) is this a valid stress function, i.e. does it satisfy the biharmonic equation?
(ii) calculate all stress components (with $\nu = 1/4$)
(iii) calculate all strain components
(iv) find the displacements
(v) specify any three displacement components which will render the arbitrary constant displacements of (iv) zero

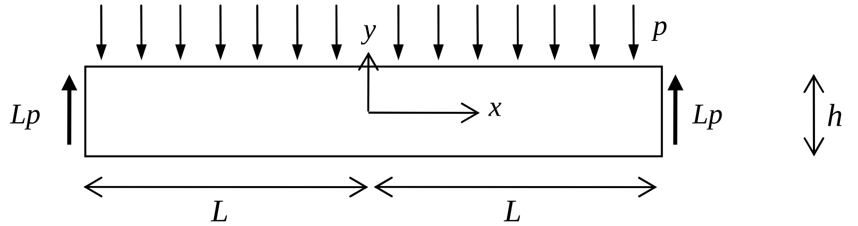
5. For the cantilevered beam discussed in §3.2.5, evaluate the resultant shear force and moment on an arbitrary cross-section $x = \bar{x}$. Are they as you expect? (You will find that the beam is in equilibrium, as expected, since the equilibrium equations have been satisfied.)

6. For the cantilevered beam discussed in §3.2.5, evaluate the strains and displacements, assuming plane stress conditions.
Note: to evaluate the three arbitrary constants of integration, one would be tempted to apply the obvious $u_x = u_y = 0$ all along the built-in end. However, since only weak boundary conditions were imposed, one cannot enforce these strong conditions (try it). Instead, apply the following weaker conditions: (i) the displacement at the built-in end at $y = 0$ is zero ($u_x = u_y = 0$), (ii) the slope there, $\partial u_y / \partial x$, is zero.

7. Show that the stress function

$$\phi = \frac{p}{20h^3}\left[-20y^3\left(L^2 - x^2\right) - 4y^5 - 15h^2x^2y + 2h^2y^3 - 5h^3x^2\right]$$

satisfies the boundary conditions for the simply supported beam subjected to a uniform pressure p shown below. Check the boundary conditions in the *weak* (Saint-Venant) sense on the shorter left and right hand sides (for both normal and shear stress). Since the normal stress σ_{xx} is not zero at the ends, but only its resultant, check also that the moment is zero at each end.



Note that the elementary beam theory predicts an approximate flexural stress but an exact shear stress:

$$\sigma_{xx} = -\frac{6p}{h^3} y\left(L^2 - x^2\right), \quad \sigma_{xy} = \frac{6p}{h^3} x\left(\frac{h^2}{4} - y^2\right)$$

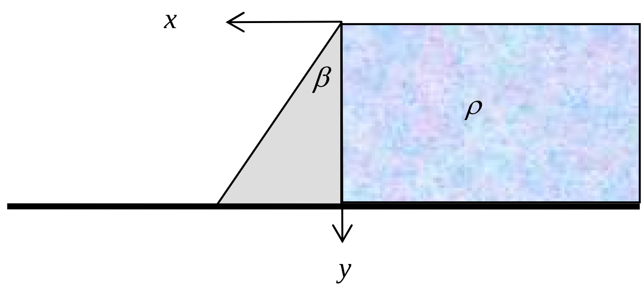
8. Consider the dam shown in the figure below. Assume first a general cubic stress function

$$\phi = \frac{1}{6}C_1 x^3 + \frac{1}{2}C_2 x^2 y + \frac{1}{2}C_3 x y^2 + \frac{1}{6}C_4 y^3$$

Apply the boundary conditions to determine the constants and hence the stresses in the dam, in terms of ρ, the density of water. (Use the stress transformation equations for the sloped boundary and ignore the weight of the dam.)
[Just consider the effect of the water; to these must be added the stresses resulting from the weight of the dam itself, which are given by

$$\sigma_{xx} = 0, \quad \sigma_{yy} = \rho_s g\left[\frac{1}{\tan\beta} x - y\right], \quad \sigma_{xy} = 0$$

where ρ_s is the density of the dam material.]



4 2D Elastostatic Problems in Polar Coordinates

Many problems are most conveniently cast in terms of polar coordinates. To this end, first the governing differential equations discussed in Chapter 1 are expressed in terms of polar coordinates. Then a number of important problems involving polar coordinates are solved.

4.1 Cylindrical and Polar Coordinates

4.1.1 Geometrical Axisymmetry

A large number of practical engineering problems involve geometrical features which have a natural **axis of symmetry**, such as the solid cylinder, shown in Fig. 4.1.1. The axis of symmetry is an **axis of revolution**; the feature which possesses **axisymmetry** (axial symmetry) can be generated by revolving a surface (or line) about this axis.

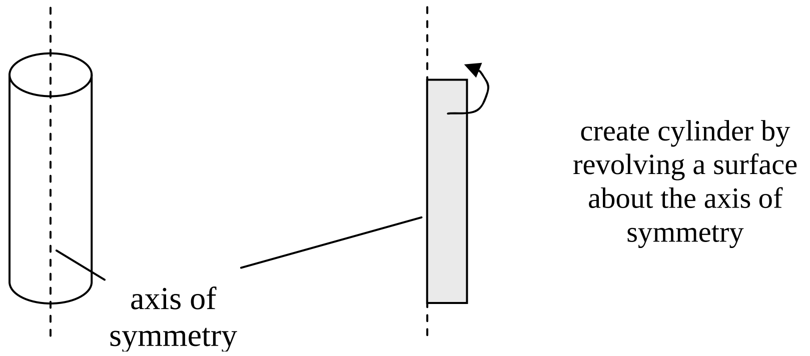


Figure 4.1.1: a cylinder

Some other axisymmetric geometries are illustrated Fig. 4.1.2; a frustum, a disk on a shaft and a sphere.

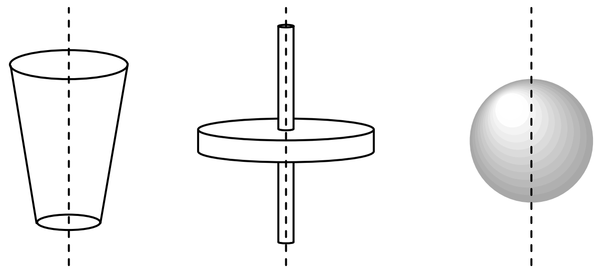


Figure 4.1.2: axisymmetric geometries

Some features are not only axisymmetric – they can be represented by a plane, which is similar to other planes right through the axis of symmetry. The hollow cylinder shown in Fig. 4.1.3 is an example of this **plane axisymmetry**.

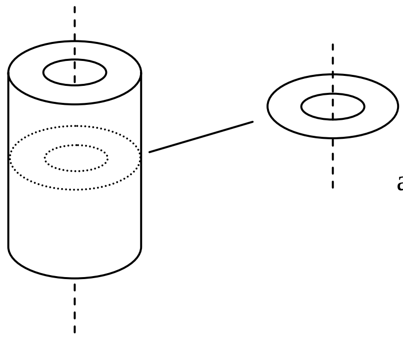


Figure 4.1.3: a plane axisymmetric geometries

Axially Non-Symmetric Geometries

Axially non-symmetric geometries are ones which have a natural axis associated with them, but which are not completely symmetric. Some examples of this type of feature, the curved beam and the half-space, are shown in Fig. 4.1.4; the half-space extends to "infinity" in the axial direction and in the radial direction "below" the surface – it can be thought of as a solid half-cylinder of infinite radius. One can also have plane axially non-symmetric features; in fact, both of these are examples of such features; a slice through the objects perpendicular to the axis of symmetry will be representative of the whole object.

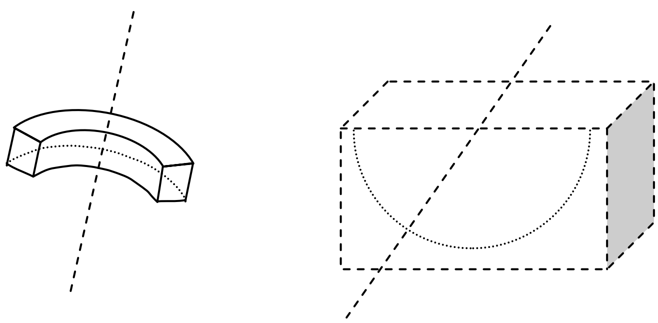


Figure 4.1.4: a plane axisymmetric geometries

4.1.2 Cylindrical and Polar Coordinates

The above features are best described using **cylindrical coordinates**, and the plane versions can be described using **polar coordinates**. These coordinates systems are described next.

Stresses and Strains in Cylindrical Coordinates

Using cylindrical coordinates, any point on a feature will have specific (r, θ, z) coordinates, Fig. 4.1.5:

- r – the **radial** direction ("out" from the axis)
- θ – the **circumferential** or **tangential** direction ("around" the axis – counterclockwise when viewed from the positive z side of the $z = 0$ plane)
- z – the **axial** direction ("along" the axis)

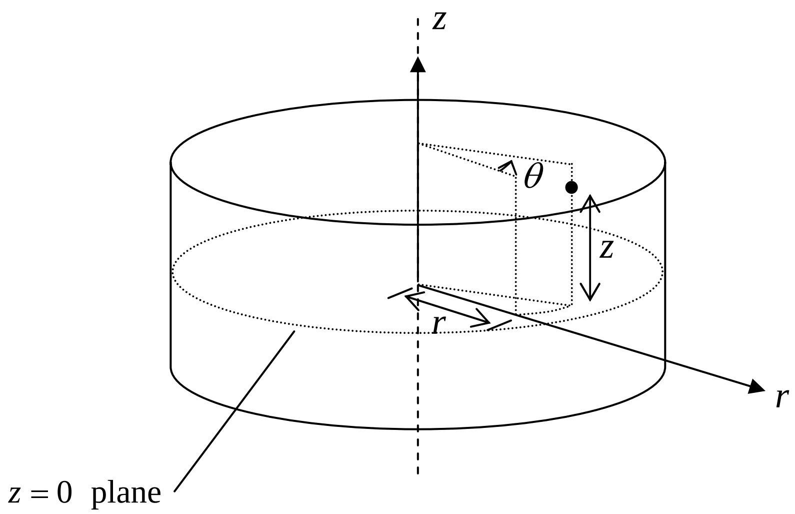


Figure 4.1.5: cylindrical coordinates

The displacement of a material point can be described by the three components in the radial, tangential and axial directions. These are often denoted by

$$u \equiv u_r,\ v \equiv u_\theta \ \text{and}\ w \equiv u_z$$

respectively; they are shown in Fig. 4.1.6. Note that the displacement v is positive in the positive θ direction, i.e. the direction of increasing θ.

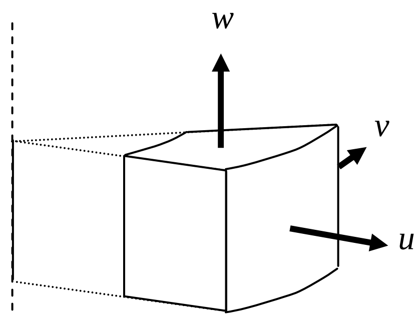


Figure 4.1.6: displacements in cylindrical coordinates

The stresses acting on a small element of material in the cylindrical coordinate system are as shown in Fig. 4.1.7 (the normal stresses on the left, the shear stresses on the right).

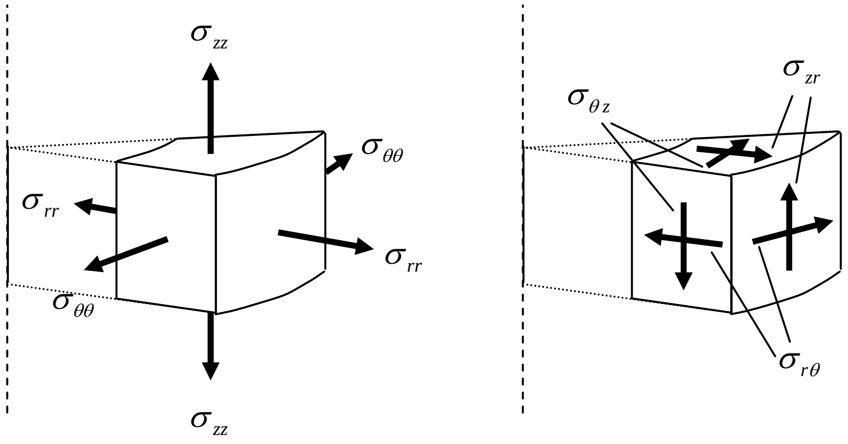
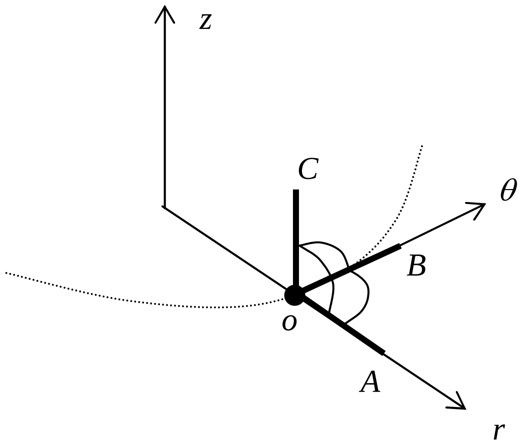


Figure 4.1.7: stresses in cylindrical coordinates

The normal strains ε_{rr}, $\varepsilon_{\theta\theta}$ and ε_{zz} are a measure of the elongation/shortening of material, per unit length, in the radial, tangential and axial directions respectively; the shear strains $\varepsilon_{r\theta}$, $\varepsilon_{\theta z}$ and ε_{zr} represent (half) the change in the right angles between line elements along the coordinate directions. The physical meaning of these strains is illustrated in Fig. 4.1.8.



strain at point *o*

ε_{rr} = unit elongation of *oA*
$\varepsilon_{\theta\theta}$ = unit elongation of *oB*
ε_{zz} = unit elongation of *oC*
$\varepsilon_{r\theta}$ = ½ change in angle $\angle AoB$
$\varepsilon_{\theta z}$ = ½ change in angle $\angle BoC$
ε_{zr} = ½ change in angle $\angle AoC$

Figure 4.1.8: strains in cylindrical coordinates

Plane Problems and Polar Coordinates

The stresses in any particular plane of an axisymmetric body can be described using the two-dimensional polar coordinates (r, θ) shown in Fig. 4.1.9.

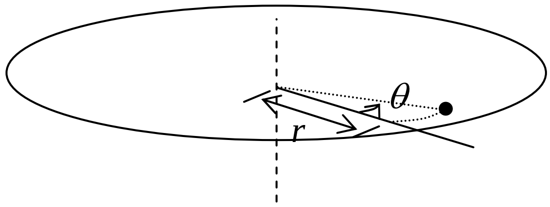


Figure 4.1.9: polar coordinates

There are *three* stress components acting *in* the plane $z = 0$: the radial stress σ_{rr}, the circumferential (tangential) stress $\sigma_{\theta\theta}$ and the shear stress $\sigma_{r\theta}$, as shown in Fig. 4.1.10. Note the direction of the (positive) shear stress – it is conventional to take the *z* axis out of the page and so the θ direction is counterclockwise. The three stress components which do not act in this plane, but which act *on* this plane (σ_{zz}, $\sigma_{\theta z}$ and σ_{zr}), may or may not be zero, depending on the particular problem (see later).

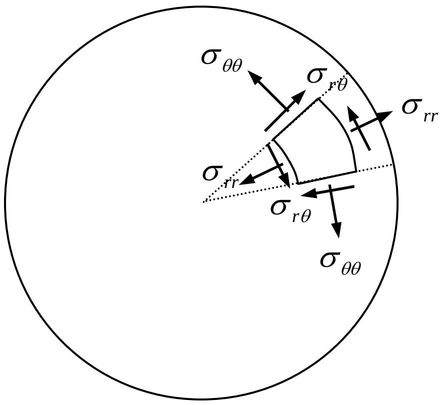


Figure 4.1.10: stresses in polar coordinates

4.2 Differential Equations in Polar Coordinates

Here, the two-dimensional Cartesian relations of Chapter 1 are re-cast in polar coordinates.

4.2.1 Equilibrium equations in Polar Coordinates

One way of expressing the equations of equilibrium in polar coordinates is to apply a change of coordinates directly to the 2D Cartesian version, Eqns. 1.1.8, as outlined in the Appendix to this section, §4.2.6. Alternatively, the equations can be derived from first principles by considering an element of material subjected to stresses σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$, as shown in Fig. 4.2.1. The dimensions of the element are Δr in the radial direction, and $r\Delta\theta$ (inner surface) and $(r+\Delta r)\Delta\theta$ (outer surface) in the tangential direction.

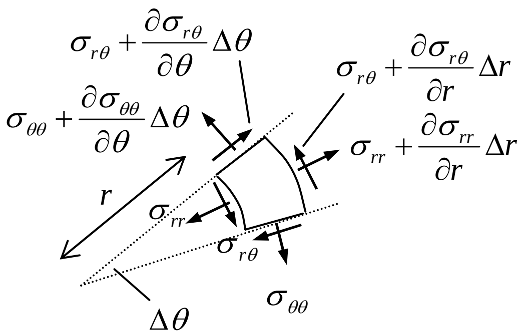


Figure 4.2.1: an element of material

Summing the forces in the radial direction leads to

$$\begin{aligned}\sum F_r = &\left(\sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r}\Delta r\right)(r+\Delta r)\Delta\theta - \sigma_{rr} r\Delta\theta \\ &- \sin\frac{\Delta\theta}{2}\left(\sigma_{\theta\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\Delta\theta\right)\Delta r - \sin\frac{\Delta\theta}{2}(\sigma_{\theta\theta})\Delta r \\ &+ \cos\frac{\Delta\theta}{2}\left(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta}\Delta\theta\right)\Delta r - \cos\frac{\Delta\theta}{2}(\sigma_{r\theta})\Delta r \equiv 0\end{aligned} \tag{4.2.1}$$

For a small element, $\sin\theta \approx \theta, \cos\theta \approx 1$ and so, dividing through by $\Delta r\Delta\theta$,

$$\frac{\partial \sigma_{rr}}{\partial r}(r+\Delta r) + \sigma_{rr} - \sigma_{\theta\theta} - \frac{\Delta\theta}{2}\left(\frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right) + \frac{\partial \sigma_{r\theta}}{\partial \theta} \equiv 0 \tag{4.2.2}$$

A similar calculation can be carried out for forces in the tangential direction {▲Problem 1}. In the limit as $\Delta r, \Delta\theta \to 0$, one then has the two-dimensional equilibrium equations in polar coordinates:

$$\boxed{\begin{aligned}&\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{1}{r}\left(\sigma_{rr}-\sigma_{\theta\theta}\right)=0\\&\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{2\sigma_{r\theta}}{r}=0\end{aligned}}$$

Equilibrium Equations (4.2.3)

4.2.2 Strain Displacement Relations and Hooke's Law

The two-dimensional strain-displacement relations can be derived from first principles by considering line elements initially lying in the r and θ directions. Alternatively, as detailed in the Appendix to this section, §4.2.6, they can be derived directly from the Cartesian version, Eqns. 1.2.5,

$$\boxed{\begin{aligned}\varepsilon_{rr}&=\frac{\partial u_r}{\partial r}\\\varepsilon_{\theta\theta}&=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\\\varepsilon_{r\theta}&=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)\end{aligned}}$$

2-D Strain-Displacement Expressions (4.2.4)

The stress-strain relations in polar coordinates are completely analogous to those in Cartesian coordinates – the axes through a small material element are simply labelled with different letters. Thus Hooke's law is now

$$\varepsilon_{rr}=\frac{1}{E}\left[\sigma_{rr}-\nu\sigma_{\theta\theta}\right],\quad\varepsilon_{\theta\theta}=\frac{1}{E}\left[\sigma_{\theta\theta}-\nu\sigma_{rr}\right],\quad\varepsilon_{r\theta}=\frac{1+\nu}{E}\sigma_{r\theta}$$
$$\varepsilon_{zz}=-\frac{\nu}{E}\left(\sigma_{rr}+\sigma_{\theta\theta}\right)$$

Hooke's Law (Plane Stress) (4.2.5a)

$$\varepsilon_{rr}=\frac{1+\nu}{E}\left[(1-\nu)\sigma_{rr}-\nu\sigma_{\theta\theta}\right],\quad\varepsilon_{\theta\theta}=\frac{1+\nu}{E}\left[-\nu\sigma_{rr}+(1-\nu)\sigma_{\theta\theta}\right],\quad\varepsilon_{r\theta}=\frac{1+\nu}{E}\sigma_{r\theta}$$

Hooke's Law (Plane Strain) (4.2.5b)

4.2.3 Stress Function Relations

In order to solve problems in polar coordinates using the stress function method, Eqns. 3.2.1 relating the stress components to the Airy stress function can be transformed using the relations in the Appendix to this section, §4.2.6:

$$\sigma_{rr}=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2},\quad\sigma_{\theta\theta}=\frac{\partial^2\phi}{\partial r^2},\quad\sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)=\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\qquad(4.2.6)$$

It can be verified that these equations automatically satisfy the equilibrium equations 4.2.3 {▲Problem 2}.

The biharmonic equation 3.2.3 becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)^2 \phi = 0 \tag{4.2.7}$$

4.2.4 The Compatibility Relation

The compatibility relation expressed in polar coordinates is (see the Appendix to this section, §4.2.6)

$$\frac{1}{r^2}\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} + \frac{\partial^2 \varepsilon_{\theta\theta}}{\partial r^2} - \frac{2}{r}\frac{\partial^2 \varepsilon_{r\theta}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{2}{r}\frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{2}{r^2}\frac{\partial \varepsilon_{r\theta}}{\partial \theta} = 0 \tag{4.2.8}$$

4.2.5 Problems

1. Derive the equilibrium equation 4.2.3b

2. Verify that the stress function relations 4.2.6 satisfy the equilibrium equations 4.2.3.

3. Verify that the strains as given by 4.2.4 satisfy the compatibility relations 4.2.8.

4.2.6 Appendix to §4.2

From Cartesian Coordinates to Polar Coordinates

To transform equations from Cartesian to polar coordinates, first note the relations

$$\begin{gathered} x = r\cos\theta, \quad y = r\sin\theta \\ r = \sqrt{x^2 + y^2}, \quad \theta = \arctan(y/x) \end{gathered} \tag{4.2.9}$$

Then the Cartesian partial derivatives become

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned} \tag{4.2.10}$$

The second partial derivatives are then

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\
&= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial}{\partial r}\right) - \cos\theta\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial r}\right) + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} + \sin^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) + \sin 2\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)
\end{aligned}
\tag{4.2.11}
$$

Similarly,

$$
\begin{aligned}
\frac{\partial^2}{\partial y^2} &= \sin^2\theta\frac{\partial^2}{\partial r^2} + \cos^2\theta\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) - \sin 2\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right) \\
\frac{\partial^2}{\partial x\partial y} &= -\sin\theta\cos\theta\left(-\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) - \cos 2\theta\left(\frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}\right)
\end{aligned}
\tag{4.2.12}
$$

Equilibrium Equations

The Cartesian stress components can be expressed in terms of polar components using the stress transformation formulae, Part I, Eqns. 3.4.7. Using a *negative* rotation (see Fig. 4.2.2), one has

$$
\begin{aligned}
\sigma_{xx} &= \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - \sigma_{r\theta}\sin 2\theta \\
\sigma_{yy} &= \sigma_{rr}\sin^2\theta + \sigma_{\theta\theta}\cos^2\theta + \sigma_{r\theta}\sin 2\theta \\
\sigma_{xy} &= \sin\theta\cos\theta(\sigma_{rr} - \sigma_{\theta\theta}) + \sigma_{r\theta}\cos 2\theta
\end{aligned}
\tag{4.2.13}
$$

Applying these and 4.2.10 to the 2D Cartesian equilibrium equations 3.1.3a-b lead to

$$
\begin{aligned}
&\cos\theta\left[\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta})\right] - \sin\theta\left[\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2\sigma_{r\theta}}{r}\right] = 0 \\
&\sin\theta\left[\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta})\right] + \cos\theta\left[\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2\sigma_{r\theta}}{r}\right] = 0
\end{aligned}
\tag{4.2.14}
$$

which then give Eqns. 4.2.3.

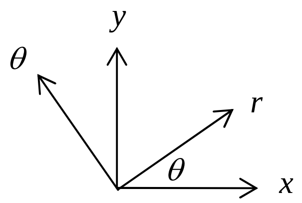


Figure 4.2.2: rotation of axes

The Strain-Displacement Relations

Noting that

$$\begin{aligned} u_x &= u_r \cos\theta - u_\theta \sin\theta \\ u_y &= u_r \sin\theta + u_\theta \cos\theta \end{aligned}, \tag{4.2.15}$$

the strains in polar coordinates can be obtained directly from Eqns. 1.2.5:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ &= \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \right) \left(u_r \cos\theta - u_\theta \sin\theta \right) \\ &= \cos^2\theta \frac{\partial u_r}{\partial r} + \sin^2\theta \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) - \sin 2\theta \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \end{aligned} \tag{4.2.16}$$

One obtains similar expressions for the strains ε_{yy} and ε_{xy}. Substituting the results into the strain transformation equations Part I, Eqns. 3.8.1,

$$\begin{aligned} \varepsilon_{rr} &= \varepsilon_{xx} \cos^2\theta + \varepsilon_{yy} \sin^2\theta + \varepsilon_{xy} \sin 2\theta \\ \varepsilon_{\theta\theta} &= \varepsilon_{xx} \sin^2\theta + \varepsilon_{yy} \cos^2\theta - \varepsilon_{xy} \sin 2\theta \\ \varepsilon_{r\theta} &= \sin\theta\cos\theta \left(\varepsilon_{yy} - \varepsilon_{xx} \right) + \varepsilon_{xy} \cos 2\theta \end{aligned} \tag{4.2.17}$$

then leads to the equations given above, Eqns. 4.2.4.

The Stress – Stress Function Relations

The stresses in polar coordinates are related to the stresses in Cartesian coordinates through the stress transformation equations (this time a positive rotation; compare with Eqns. 4.2.13 and Fig. 4.2.2)

$$\begin{aligned} \sigma_{rr} &= \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + \sigma_{xy} \sin 2\theta \\ \sigma_{\theta\theta} &= \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - \sigma_{xy} \sin 2\theta \\ \sigma_{r\theta} &= \sin\theta\cos\theta \left(\sigma_{yy} - \sigma_{xx} \right) + \sigma_{xy} \cos 2\theta \end{aligned} \tag{4.2.18}$$

Using the Cartesian stress – stress function relations 3.2.1, one has

$$\sigma_{rr} = \frac{\partial^2 \phi}{\partial y^2} \cos^2\theta + \frac{\partial^2 \phi}{\partial x^2} \sin^2\theta - \frac{\partial^2 \phi}{\partial x \partial y} \sin 2\theta \tag{4.2.19}$$

and similarly for $\sigma_{\theta\theta}, \sigma_{r\theta}$. Using 4.2.11-12 then leads to 4.2.6.

The Compatibility Relation

Beginning with the Cartesian relation 1.3.1, each term can be transformed using 4.2.11-12 and the strain transformation relations, for example

$$\frac{\partial^2 \varepsilon_{xx}}{\partial x^2} = \left(\cos^2\theta \frac{\partial^2}{\partial r^2} + \sin^2\theta \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) + \sin 2\theta \left(\frac{1}{r^2}\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} \right) \right) \times \left(\varepsilon_{rr}\cos^2\theta + \varepsilon_{\theta\theta}\sin^2\theta - \varepsilon_{r\theta}\sin 2\theta \right) \qquad (4.2.20)$$

After some lengthy calculations, one arrives at 4.2.8.

4.3 Plane Axisymmetric Problems

In this section are considered **plane axisymmetric problems**. These are problems in which both the geometry *and* loading are axisymmetric.

4.3.1 Plane Axisymmetric Problems

Some three dimensional (not necessarily plane) examples of axisymmetric problems would be the thick-walled (hollow) cylinder under internal pressure, a disk rotating about its axis[1], and the two examples shown in Fig. 4.3.1; the first is a complex component loaded in a complex way, but exhibits axisymmetry in both geometry and loading; the second is a sphere loaded by concentrated forces along a diameter.

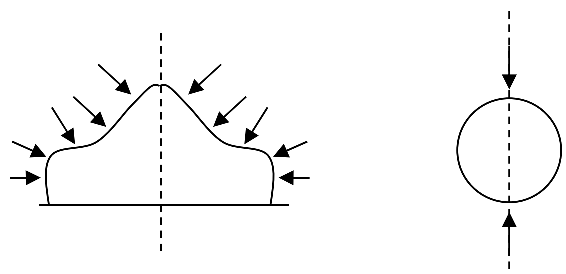


Figure 4.3.1: axisymmetric problems

A two-dimensional (plane) example would be one plane of the thick-walled cylinder under internal pressure, illustrated in Fig. 4.3.2[2].

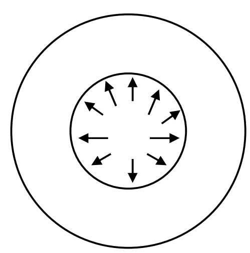


Figure 4.3.2: a cross section of an internally pressurised cylinder

It should be noted that many problems involve axisymmetric geometries but non-axisymmetric loadings, and *vice versa*. These problems are *not* axisymmetric. An example is shown in Fig. 4.3.3 (the problem involves a plane axisymmetric geometry).

[1] the rotation induces a stress in the disk
[2] the rest of the cylinder is coming out of, and into, the page

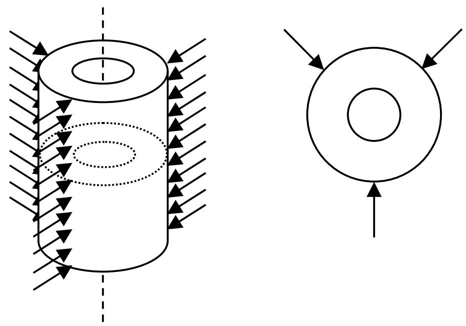


Figure 4.3.3: An axially symmetric geometry but with a non-axisymmetric loading

The important characteristic of these axisymmetric problems is that all quantities, be they stress, displacement, strain, or anything else associated with the problem, *must be independent of the circumferential variable* θ. As a consequence, any term in the differential equations of §4.2 involving the derivatives $\partial / \partial\theta, \partial^2 / \partial\theta^2$, etc. can be immediately set to zero.

4.3.2 Governing Equations for Plane Axisymmetric Problems

The two-dimensional strain-displacement relations are given by Eqns. 4.2.4 and these simplify in the axisymmetric case to

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\ \varepsilon_{\theta\theta} &= \frac{u_r}{r} \\ \varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \end{aligned} \tag{4.3.1}$$

Here, it will be assumed that the displacement $u_\theta = 0$. Cases where $u_\theta \neq 0$ but where the stresses and strains are still independent of θ are termed **quasi-axisymmetric problems**; these will be examined in a later section. Then 4.3.1 reduces to

$$\boxed{\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{r\theta} = 0} \tag{4.3.2}$$

It follows from Hooke's law that $\sigma_{r\theta} = 0$. The non-zero stresses are illustrated in Fig. 4.3.4.

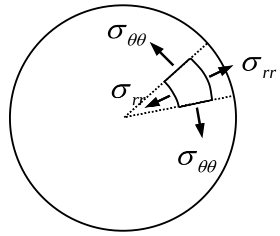


Figure 4.3.4: stress components in plane axisymmetric problems

4.3.3 Plane Stress and Plane Strain

Two cases arise with plane axisymmetric problems: in the plane stress problem, the feature is very thin and unloaded on its larger free-surfaces, for example a thin disk under external pressure, as shown in Fig. 4.3.5. Only two stress components remain, and Hooke's law 4.2.5a reads

$$\begin{aligned} \varepsilon_{rr} &= \frac{1}{E}\left[\sigma_{rr} - \nu\sigma_{\theta\theta}\right] \\ \varepsilon_{\theta\theta} &= \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\sigma_{rr}\right] \end{aligned} \qquad \text{or} \qquad \begin{aligned} \sigma_{rr} &= \frac{E}{1-\nu^2}\left[\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right] \\ \sigma_{\theta\theta} &= \frac{E}{1-\nu^2}\left[\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\right] \end{aligned} \tag{4.3.3}$$

with $\varepsilon_{zz} = \frac{-\nu}{E}\left(\sigma_{rr} + \sigma_{\theta\theta}\right)$, $\varepsilon_{zr} = \varepsilon_{z\theta} = 0$ and $\sigma_{zz} = 0$.

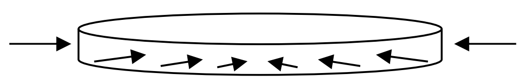


Figure 4.3.5: plane stress axisymmetric problem

In the plane strain case, the strains ε_{zz}, $\varepsilon_{z\theta}$ and ε_{zr} are zero. This will occur, for example, in a hollow cylinder under internal pressure, with the ends fixed between immovable platens, Fig. 4.3.6.

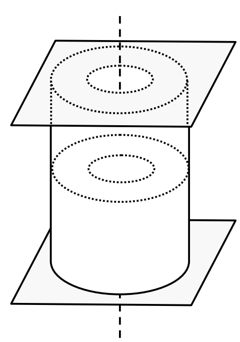


Figure 4.3.6: plane strain axisymmetric problem

Hooke's law 4.2.5b reads

$$\begin{aligned}\varepsilon_{rr} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}\right] \\ \varepsilon_{\theta\theta} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right]\end{aligned} \quad \text{or} \quad \begin{aligned}\sigma_{rr} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{\theta\theta} + (1-\nu)\varepsilon_{rr}\right] \\ \sigma_{\theta\theta} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{rr} + (1-\nu)\varepsilon_{\theta\theta}\right]\end{aligned} \tag{4.3.4}$$

with $\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$.

Shown in Fig. 4.3.7 are the stresses acting in the axisymmetric plane body (with σ_{zz} zero in the plane stress case).

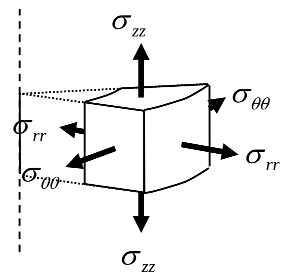


Figure 4.3.7: stress components in plane axisymmetric problems

4.3.4 Solution of Plane Axisymmetric Problems

The equations governing the plane axisymmetric problem are the equations of equilibrium 4.2.3 which reduce to the single equation

$$\boxed{\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0}, \tag{4.3.5}$$

the strain-displacement relations 4.3.2 and the stress-strain law 4.3.3-4.

Taking the plane stress case, substituting 4.3.2 into the second of 4.3.3 and then substituting the result into 4.3.5 leads to (with a similar result for plane strain)

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2}u = 0 \tag{4.3.6}$$

This is Navier's equation for plane axisymmetry. It is an "Euler-type" ordinary differential equation which can be solved exactly to get (see Appendix to this section, §4.3.8)

$$u = C_1 r + C_2 \frac{1}{r} \tag{4.3.7}$$

With the displacement known, the stresses and strains can be evaluated, and the full solution is

$$
\begin{gathered}
u = C_1 r + C_2 \frac{1}{r} \\
\varepsilon_{rr} = C_1 - C_2 \frac{1}{r^2}, \quad \varepsilon_{\theta\theta} = C_1 + C_2 \frac{1}{r^2} \\
\sigma_{rr} = \frac{E}{1-\nu} C_1 - \frac{E}{1+\nu} C_2 \frac{1}{r^2}, \quad \sigma_{\theta\theta} = \frac{E}{1-\nu} C_1 + \frac{E}{1+\nu} C_2 \frac{1}{r^2}
\end{gathered} \tag{4.3.8}
$$

For problems involving stress boundary conditions, it is best to have simpler expressions for the stress so, introducing new constants $A = -EC_2/(1+\nu)$ and $C = EC_1/2(1-\nu)$, the solution can be re-written as

$$
\boxed{
\begin{gathered}
\sigma_{rr} = +A\frac{1}{r^2} + 2C, \quad \sigma_{\theta\theta} = -A\frac{1}{r^2} + 2C \\
\varepsilon_{rr} = +\frac{(1+\nu)A}{E}\frac{1}{r^2} + \frac{2(1-\nu)C}{E}, \quad \varepsilon_{\theta\theta} = -\frac{(1+\nu)A}{E}\frac{1}{r^2} + \frac{2(1-\nu)C}{E}, \quad \varepsilon_{zz} = -\frac{4\nu C}{E} \\
u = -\frac{(1+\nu)A}{E}\frac{1}{r} + \frac{2(1-\nu)C}{E} r
\end{gathered}
} \tag{4.3.9}
$$

Plane stress axisymmetric solution

Similarly, the plane strain solution turns out to be again 4.3.8a-b only the stresses are now {▲Problem 1}

$$
\sigma_{rr} = \frac{E}{(1+\nu)(1-2\nu)}\left[-(1-2\nu)C_2\frac{1}{r^2} + C_1\right], \quad \sigma_{\theta\theta} = \frac{E}{(1+\nu)(1-2\nu)}\left[+(1-2\nu)C_2\frac{1}{r^2} + C_1\right] \tag{4.3.10}
$$

Then, with $A = -EC_2/(1+\nu)$ and $C = EC_1/2(1+\nu)(1-2\nu)$, the solution can be written as

$$
\boxed{
\begin{gathered}
\sigma_{rr} = +A\frac{1}{r^2} + 2C, \quad \sigma_{\theta\theta} = -A\frac{1}{r^2} + 2C, \quad \sigma_{zz} = 4\nu C \\
\varepsilon_{rr} = \frac{1+\nu}{E}\left[+A\frac{1}{r^2} + 2(1-2\nu)C\right], \quad \varepsilon_{\theta\theta} = \frac{1+\nu}{E}\left[-A\frac{1}{r^2} + 2(1-2\nu)C\right] \\
u = \frac{1+\nu}{E}\left[-A\frac{1}{r} + 2(1-2\nu)Cr\right]
\end{gathered}
} \tag{4.3.11}
$$

Plane strain axisymmetric solution

The solutions 4.3.9, 4.3.11 involve two constants. When there is a solid body with one boundary, A must be zero in order to ensure finite-valued stresses and strains; C can be determined from the boundary condition. When there are two boundaries, both A and C are determined from the boundary conditions.

4.3.5 Example: Expansion of a thick circular cylinder under internal pressure

Consider the problem of Fig. 4.3.8. The two unknown constants A and C are obtained from the boundary conditions

$$\begin{aligned}\sigma_{rr}(a) &= -p \\ \sigma_{rr}(b) &= 0\end{aligned} \tag{4.3.12}$$

which lead to

$$\sigma_{rr}(a) = \frac{A}{a^2} + 2C = -p, \quad \sigma_{rr}(b) = \frac{A}{b^2} + 2C = 0 \tag{4.3.13}$$

so that

$$\boxed{\sigma_{rr} = -p\frac{b^2/r^2 - 1}{b^2/a^2 - 1}, \quad \sigma_{\theta\theta} = +p\frac{b^2/r^2 + 1}{b^2/a^2 - 1}, \quad \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})} \tag{4.3.14}$$

Cylinder under Internal Pressure

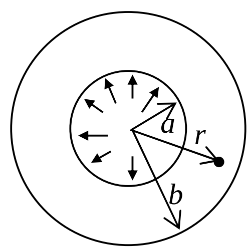


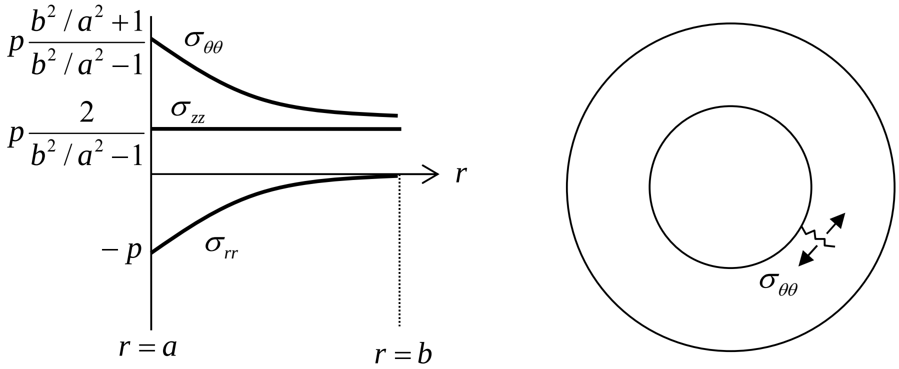
Figure 4.3.8: an internally pressurised cylinder

The stresses through the thickness of the cylinder walls are shown in Fig. 4.3.9a. The maximum principal stress is the $\sigma_{\theta\theta}$ stress and this attains a maximum at the inner face. For this reason, internally pressurized vessels often fail there first, with microcracks perpendicular to the inner edge been driven by the tangential stress, as illustrated in Fig. 4.3.9b.

Note that by setting $b = a + t$ and taking the wall thickness to be very small, $t, t^2 << a$, and letting $a = r$, the solution 4.3.14 reduces to:

$$\sigma_{rr} = -p, \quad \sigma_{\theta\theta} = +p\frac{r}{t}, \quad \sigma_{zz} = \nu p\frac{r}{t} \tag{4.3.15}$$

which is equivalent to the thin-walled pressure-vessel solution, Part I, §4.5.2 (if $\nu = 1/2$, i.e. incompressible).



(a) (b)

Figure 4.3.9: (a) stresses in the thick-walled cylinder, (b) microcracks driven by tangential stress

Generalised Plane Strain and Other Solutions

The solution for a pressurized cylinder in plane strain was given above, i.e. where ε_{zz} was enforced to be zero. There are two other useful situations:

(1) The cylinder is free to expand in the axial direction. In this case, ε_{zz} is not forced to zero, but allowed to be a constant along the length of the cylinder, say $\bar{\varepsilon}_{zz}$. The σ_{zz} stress is zero, as in plane stress. This situation is called **generalized plane strain**.

(2) The cylinder is closed at its ends. Here, the axial stresses σ_{zz} inside the walls of the tube are counteracted by the internal pressure *p* acting on the closed ends. The force acting on the closed ends due to the pressure is $p\pi a^2$ and the balancing axial force is $\sigma_{zz}\pi\left(b^2-a^2\right)$, assuming σ_{zz} to be constant through the thickness. For equilibrium

$$\sigma_{zz}=\frac{p}{b^2/a^2-1} \tag{4.3.16}$$

Returning to the full three-dimensional stress-strain equations (Part I, Eqns. 6.1.9), set $\varepsilon_{zz}=\bar{\varepsilon}_{zz}$, a constant, and $\varepsilon_{xz}=\varepsilon_{yz}=0$. Re-labelling x,y,z with r,θ,z, and again with $\sigma_{r\theta}=0$, one has

$$\begin{aligned}
\sigma_{rr}&=\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{rr}+\nu\left(\varepsilon_{\theta\theta}+\bar{\varepsilon}_{zz}\right)\right]\\
\sigma_{\theta\theta}&=\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{\theta\theta}+\nu\left(\varepsilon_{rr}+\bar{\varepsilon}_{zz}\right)\right]\\
\sigma_{zz}&=\frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\bar{\varepsilon}_{zz}+\nu\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}\right)\right]
\end{aligned} \tag{4.3.17}$$

Substituting the strain-displacement relations 4.3.2 into 4.3.16a-b, and, as before, using the axisymmetric equilibrium equation 4.3.5, again leads to the differential equation 4.3.6 and the solution $u = C_1 r + C_2 / r$, with $\varepsilon_{rr} = C_1 - C_2 / r^2$, $\varepsilon_{\theta\theta} = C_1 + C_2 / r^2$, but now the stresses are

$$\begin{aligned}
\sigma_{rr} &= \frac{E}{(1+\nu)(1-2\nu)}\left[C_1 - (1-2\nu)C_2\frac{1}{r^2} + \nu\bar{\varepsilon}_{zz}\right] \\
\sigma_{\theta\theta} &= \frac{E}{(1+\nu)(1-2\nu)}\left[C_1 + (1-2\nu)C_2\frac{1}{r^2} + \nu\bar{\varepsilon}_{zz}\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[2\nu C_1 + (1-\nu)\nu\bar{\varepsilon}_{zz}\right]
\end{aligned} \tag{4.3.18}$$

As before, to make the solution more amenable to stress boundary conditions, we let $A = -EC_2/(1+\nu)$ and $C = EC_1/2(1+\nu)(1-2\nu)$, so that the solution is

$$\boxed{\begin{gathered}
\sigma_{rr} = +A\frac{1}{r^2} + 2C + \frac{\nu E}{(1+\nu)(1-2\nu)}\bar{\varepsilon}_{zz}, \quad \sigma_{\theta\theta} = -A\frac{1}{r^2} + 2C + \frac{\nu E}{(1+\nu)(1-2\nu)}\bar{\varepsilon}_{zz} \\
\sigma_{zz} = 4\nu C + \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\bar{\varepsilon}_{zz} \\
\varepsilon_{rr} = \frac{1+\nu}{E}\left[+A\frac{1}{r^2} + 2(1-2\nu)C\right], \quad \varepsilon_{\theta\theta} = \frac{1+\nu}{E}\left[-A\frac{1}{r^2} + 2(1-2\nu)C\right] \\
u = \frac{1+\nu}{E}\left[-A\frac{1}{r} + 2(1-2\nu)Cr\right]
\end{gathered}} \tag{4.3.19}$$

Generalised axisymmetric solution

For internal pressure *p*, the solution to 4.3.19 gives the same solution for radial and tangential stresses as before, Eqn. 4.3.14. The axial displacement is $u_z = z\bar{\varepsilon}_{zz}$ (to within a constant).

In the case of the cylinder with open ends (generalized plane strain), $\sigma_{zz} = 0$, and one finds from Eqn. 4.3.19 that $\bar{\varepsilon}_{zz} = -2\nu p / E\left(b^2/a^2 - 1\right) < 0$. In the case of the cylinder with closed ends, one finds that $\bar{\varepsilon}_{zz} = (1-2\nu)\, p / E\left(b^2/a^2 - 1\right) > 0$.

A Transversely isotropic Cylinder

Consider now a transversely isotropic cylinder. The strain-displacement relations 4.3.2 and the equilibrium equation 4.3.5 are applicable to any type of material. The stress-strain law can be expressed as (see Part I, Eqn. 6.2.14)

$$\begin{aligned}
\sigma_{rr} &= C_{11}\varepsilon_{rr} + C_{12}\varepsilon_{\theta\theta} + C_{13}\varepsilon_{zz} \\
\sigma_{\theta\theta} &= C_{12}\varepsilon_{rr} + C_{11}\varepsilon_{\theta\theta} + C_{13}\varepsilon_{zz} \\
\sigma_{zz} &= C_{13}\varepsilon_{rr} + C_{13}\varepsilon_{\theta\theta} + C_{33}\varepsilon_{zz}
\end{aligned} \tag{4.3.20}$$

Here, take $\varepsilon_{zz} = \bar{\varepsilon}_{zz}$, a constant. Then, using the strain-displacement relations and the equilibrium equation, one again arrives at the differential equation 4.3.6 so the solution for displacement and strain is again 4.3.8a-b. With $A = C_2 / (C_{12} - C_{11})$ and $C = C_1 / 2(C_{11} + C_{12})$, the stresses can be expressed as

$$
\begin{aligned}
\sigma_{rr} &= +A\frac{1}{r^2} + 2C + C_{13}\bar{\varepsilon}_{zz} \\
\sigma_{\theta\theta} &= -A\frac{1}{r^2} + 2C + C_{13}\bar{\varepsilon}_{zz} \\
\sigma_{zz} &= 4C\frac{C_{13}}{C_{11} + C_{12}} + C_{33}\bar{\varepsilon}_{zz}
\end{aligned}
\tag{4.3.21}
$$

The plane strain solution then follows from $\bar{\varepsilon}_{zz} = 0$ and the generalized plane strain solution from $\sigma_{zz} = 0$. These solutions reduce to 4.3.11, 4.3.19 in the isotropic case.

4.3.6 Stress Function Solution

An alternative solution procedure for axisymmetric problems is the stress function approach. To this end, first specialise equations 4.2.6 to the axisymmetric case:

$$
\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}, \quad \sigma_{r\theta} = 0 \tag{4.3.22}
$$

One can check that these equations satisfy the axisymmetric equilibrium equation 4.3.4.

The biharmonic equation in polar coordinates is given by Eqn. 4.2.7. Specialising this to the axisymmetric case, that is, setting $\partial / \partial\theta = 0$, leads to

$$
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right) = 0 \tag{4.3.23}
$$

or

$$
\frac{d^4\phi}{dr^4} + \frac{2}{r}\frac{d^3\phi}{dr^3} - \frac{1}{r^2}\frac{d^2\phi}{dr^2} + \frac{1}{r^3}\frac{d\phi}{dr} = 0 \tag{4.3.24}
$$

Alternatively, one could have started with the compatibility relation 4.2.8, specialised that to the axisymmetric case:

$$
\frac{\partial^2\varepsilon_{\theta\theta}}{\partial r^2} - \frac{1}{r}\frac{\partial\varepsilon_{rr}}{\partial r} + \frac{2}{r}\frac{\partial\varepsilon_{\theta\theta}}{\partial r} = 0 \tag{4.3.25}
$$

and then combine with Hooke's law 4.3.3 or 4.3.4, and 4.3.22, to again get 4.3.24.

Eqn. 4.3.24 is an Euler-type ODE and has solution (see Appendix to this section, §4.3.8)

$$\phi = A\ln r + Br^2 \ln r + Cr^2 + D \tag{4.3.26}$$

The stresses then follow from 4.3.22:

$$\begin{aligned} \sigma_{rr} &= +\frac{A}{r^2} + B(1 + 2\ln r) + 2C \\ \sigma_{\theta\theta} &= -\frac{A}{r^2} + B(3 + 2\ln r) + 2C \end{aligned} \tag{4.3.27}$$

The strains are obtained from the stress-strain relations. For plane strain, one has, from 4.3.4,

$$\begin{aligned} \varepsilon_{rr} &= \frac{1+\nu}{E}\left\{+\frac{A}{r^2} + B[1 - 4\nu + 2(1-2\nu)\ln r] + 2C(1-2\nu)\right\} \\ \varepsilon_{\theta\theta} &= \frac{1+\nu}{E}\left\{-\frac{A}{r^2} + B[3 - 4\nu + 2(1-2\nu)\ln r] + 2C(1-2\nu)\right\} \end{aligned} \tag{4.3.28}$$

Comparing these with the strain-displacement relations 4.3.2, and integrating ε_{rr}, one has

$$\begin{aligned} u_r &= \int \varepsilon_{rr}\,dr = \frac{1+\nu}{E}\left\{-\frac{A}{r} + Br[-1 + 2(1-2\nu)\ln r] + 2C(1-2\nu)r\right\} + F \\ u_r &= r\varepsilon_{\theta\theta} \quad = \frac{1+\nu}{E}\left\{-\frac{A}{r} + Br[+1 + 2(1-2\nu)\ln r] + 2C(1-2\nu)r\right\} \end{aligned} \tag{4.3.27}$$

To ensure that one has a unique displacement u_r, one must have $B = 0$ and the constant of integration $F = 0$, and so one again has the solution 4.3.11[3].

4.3.7 Problems

1. Derive the solution equations 4.3.11 for axisymmetric plane strain.

2. A cylindrical rock specimen is subjected to a pressure *p* over its cylindrical face and is constrained in the axial direction. What are the stresses, including the axial stress, in the specimen? What are the displacements?

[3] the biharmonic equation was derived using the expression for compatibility of strains (4.3.23 being the axisymmetric version). In simply connected domains, i.e. bodies without holes, compatibility is assured (and indeed *A* and *B* must be zero in 4.3.26 to ensure finite strains). In multiply connected domains, however, for example the hollow cylinder, the compatibility condition is necessary but *not sufficient* to ensure compatible strains (see, for example, Shames and Cozzarelli (1997)), and this is why compatibility of strains must be explicitly enforced as in 4.3.25

3. A long hollow tube is subjected to internal pressure p_i and external pressures p_o and constrained in the axial direction. What is the stress state in the walls of the tube? What if $p_i = p_o = p$?

4. A long mine tunnel of radius *a* is cut in deep rock. Before the mine is constructed the rock is under a uniform pressure *p*. Considering the rock to be an infinite, homogeneous elastic medium with elastic constants *E* and ν, determine the radial displacement at the surface of the tunnel due to the excavation. What radial stress $\sigma_{rr}(a) = -P$ should be applied to the wall of the tunnel to prevent any such displacement?

5. A long hollow elastic tube is fitted to an inner rigid (immovable) shaft. The tube is perfectly bonded to the shaft. An external pressure *p* is applied to the tube. What are the stresses and strains in the tube?

6. Repeat Problem 3 for the case when the tube is free to expand in the axial direction. How much does the tube expand in the axial direction (take $u_z = 0$ at $z = 0$)?

4.3.8 Appendix

Solution to Eqn. 4.3.6

The differential equation 4.3.6 can be solved by a change of variable $r = e^t$, so that

$$r = e^t, \quad \log r = t, \quad \frac{1}{r} = \frac{dt}{dr} \tag{4.3.28}$$

and, using the chain rule,

$$\begin{aligned} \frac{du}{dr} &= \frac{du}{dt}\frac{dt}{dr} = \frac{1}{r}\frac{du}{dt} \\ \frac{d^2u}{dr^2} &= \frac{d}{dr}\left(\frac{du}{dt}\frac{dt}{dr}\right) = \frac{d^2u}{dt^2}\frac{dt}{dr}\frac{dt}{dr} + \frac{du}{dt}\frac{d^2t}{dr^2} = \frac{1}{r^2}\frac{d^2u}{dt^2} - \frac{1}{r^2}\frac{du}{dt} \end{aligned} \tag{4.3.29}$$

The differential equation becomes

$$\frac{d^2u}{dt^2} - u = 0 \tag{4.3.30}$$

which is an ordinary differential equation with constant coefficients. With $u = e^{\lambda t}$, one has the characteristic equation $\lambda^2 - 1 = 0$ and hence the solution

$$\begin{aligned} u &= C_1 e^{+t} + C_2 e^{-t} \\ &= C_1 r + C_2 \frac{1}{r} \end{aligned} \tag{4.3.31}$$

Solution to Eqn. 4.3.24

The solution procedure for 4.3.24 is similar to that given above for 4.3.6. Using the substitution $r = e^t$ leads to the differential equation with constant coefficients

$$\frac{d^4\phi}{dt^4} - 4\frac{d^3\phi}{dt^3} + 4\frac{d^2\phi}{dt^2} = 0 \tag{4.3.32}$$

which, with $\phi = e^{\lambda t}$, has the characteristic equation $\lambda^2(\lambda - 2)^2 = 0$. This gives the repeated roots solution

$$\phi = At + Bte^{2t} + Ce^{2t} + D \tag{4.3.33}$$

and hence 4.3.24.

4.4 Rotating Discs

4.4.1 The Rotating Disc

Consider a thin disc rotating with constant angular velocity ω, Fig. 4.4.1. Material particles are subjected to a centripetal acceleration $a_r = -r\omega^2$. The subscript r indicates an acceleration in the radial direction and the minus sign indicates that the particles are accelerating towards the centre of the disc.

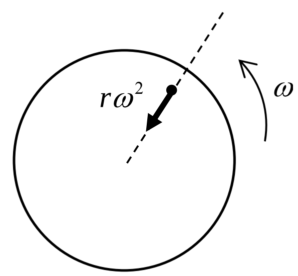


Figure 4.4.1: the rotating disc

The accelerations lead to an inertial force (per unit volume) $F_a = -\rho r\omega^2$ which in turn leads to stresses in the disc. The inertial force is an axisymmetric "loading" and so this is an axisymmetric problem. The axisymmetric equation of equilibrium is given by 4.3.5. Adding in the acceleration term gives the corresponding equation of motion:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = -\rho r\omega^2, \qquad (4.4.1)$$

This equation can be expressed as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) + b_r = 0, \qquad (4.4.2)$$

where $b_r = \rho r\omega^2$. Thus the dynamic rotating disc problem has been converted into an equivalent static problem of a disc subjected to a known body force. Note that, in a general dynamic problem, and unlike here, one does not know what the accelerations are – they have to be found as part of the solution procedure.

Using the strain-displacement relations 4.3.2 and the plane stress Hooke's law 4.3.3 then leads to the differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{1}{r^2}u = -\frac{1-\nu^2}{E}\rho r\omega^2 \qquad (4.4.3)$$

This is Eqn. 4.3.6 with a non-homogeneous term. The solution is derived in the Appendix to this section, §4.4.3:

$$u = C_1 r + C_2 \frac{1}{r} - \frac{1}{8}\frac{1-\nu^2}{E}\rho r^3 \omega^2 \qquad (4.4.4)$$

As in §4.3.4, let $A = -EC_2/(1+\nu)$ and $C = EC_1/2(1-\nu)$, and the full general solution is, using 4.3.2 and 4.3.3, {▲Problem 1}

$$\begin{aligned}
\sigma_{rr} &= +A\frac{1}{r^2} + 2C - \frac{1}{8}(3+\nu)\rho\omega^2 r^2 \\
\sigma_{\theta\theta} &= -A\frac{1}{r^2} + 2C - \frac{1}{8}(1+3\nu)\rho\omega^2 r^2 \\
\varepsilon_{rr} &= \frac{1}{E}\left[+(1+\nu)\frac{A}{r^2} + 2(1-\nu)C - \frac{3}{8}(1-\nu^2)\rho\omega^2 r^2\right] \\
\varepsilon_{\theta\theta} &= \frac{1}{E}\left[-(1+\nu)\frac{A}{r^2} + 2(1-\nu)C - \frac{1}{8}(1-\nu^2)\rho\omega^2 r^2\right] \\
u &= \frac{1}{E}\left[-(1+\nu)\frac{A}{r} + 2(1-\nu)Cr - \frac{1}{8}(1-\nu^2)\rho\omega^2 r^3\right]
\end{aligned} \qquad (4.4.5)$$

which reduce to 4.3.9 when $\omega = 0$.

A Solid Disc

For a solid disc, A in 4.4.5 must be zero to ensure finite stresses and strains at $r = 0$. C is then obtained from the boundary condition $\sigma_{rr}(b) = 0$, where b is the disc radius:

$$A = 0, \qquad C = \frac{1}{16}(3+\nu)\rho\omega^2 b^2 \qquad (4.4.6)$$

The stresses and displacements are

$$\begin{aligned}
\sigma_{rr}(r) &= \frac{3+\nu}{8}\rho\omega^2\left[b^2 - r^2\right] \\
\sigma_{\theta\theta}(r) &= \frac{3+\nu}{8}\rho\omega^2\left[b^2 - \frac{1+3\nu}{3+\nu}r^2\right] \\
u(r) &= \frac{3+\nu}{8}\rho\omega^2\frac{1-\nu}{E}r\left[b^2 - \frac{1+\nu}{3+\nu}r^2\right]
\end{aligned} \qquad (4.4.7)$$

Note that the displacement is zero at the disc centre, as it must be, but the strains (and hence stresses) do not have to be, and are not, zero there.

Dimensionless stress and displacement are plotted in Fig. 4.4.2 for the case of $\nu = 0.3$. The maximum stress occurs at $r = 0$, where

$$\sigma_{rr}(0) = \sigma_{\theta\theta}(0) = \frac{3+\nu}{8}\rho\omega^2 b^2 \qquad (4.4.8)$$

The disc expands by an amount

$$u(b) = \frac{1-\nu}{4E}\rho\omega^2 b^3 \tag{4.4.9}$$

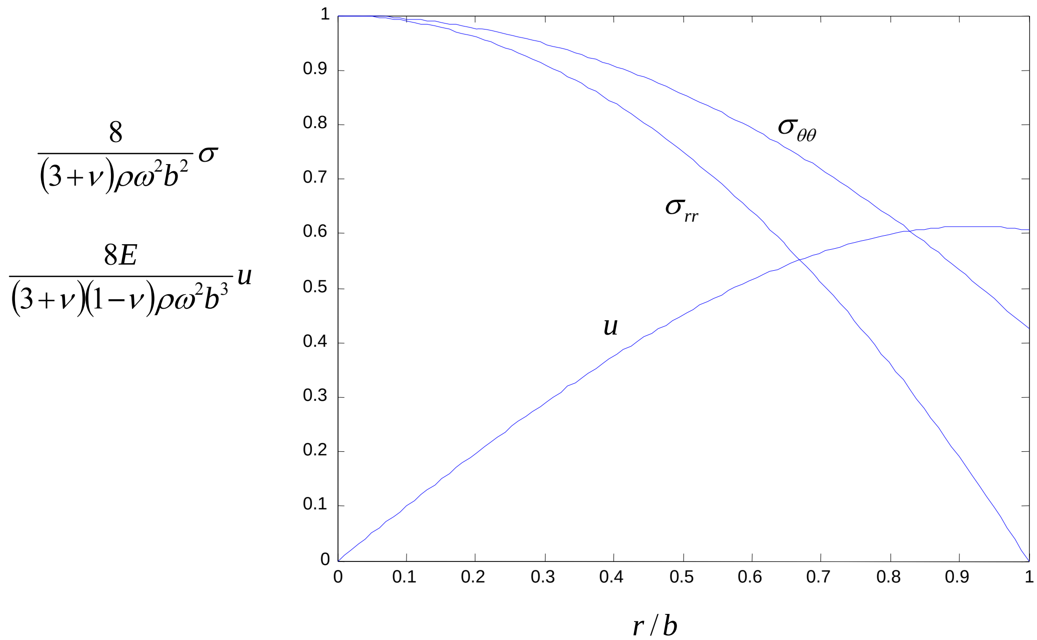


Figure 4.4.2: stresses and displacements in the solid rotating disc

A Hollow Disc

The boundary conditions for the hollow disc are

$$\sigma_{rr}(a) = 0, \qquad \sigma_{rr}(b) = 0 \tag{4.4.10}$$

where a and b are the inner and outer radii respectively. It follows from 4.4.5 that

$$A = -\frac{1}{8}(3+\nu)\rho\omega^2 a^2 b^2, \qquad C = \frac{1}{16}(3+\nu)\rho\omega^2\left(a^2+b^2\right) \tag{4.4.11}$$

and the stresses and displacement are

$$\begin{aligned}
\sigma_{rr}(r) &= \frac{3+\nu}{8}\rho\omega^2\left[a^2+b^2-r^2-\frac{a^2b^2}{r^2}\right]\\
\sigma_{\theta\theta}(r) &= \frac{3+\nu}{8}\rho\omega^2\left[a^2+b^2-\frac{1+3\nu}{3+\nu}r^2+\frac{a^2b^2}{r^2}\right]\\
u(r) &= \frac{3+\nu}{8}\rho\omega^2\frac{1-\nu}{E}r\left[a^2+b^2-\frac{1+\nu}{3+\nu}r^2+\frac{1+\nu}{1-\nu}\frac{a^2b^2}{r^2}\right]
\end{aligned} \tag{4.4.12}$$

which reduce to 4.4.7 when $a = 0$.

Dimensionless stress and displacement are plotted in Fig. 4.4.3 for the case of $\nu = 0.3$ and $a/b = 0.2$. The maximum stress occurs at the inner surface, where

$$\sigma_{\theta\theta}(0) = \frac{3+\nu}{4}\rho\omega^2 b^2\left[1+\frac{1-\nu}{3+\nu}\left(a/b^2\right)\right] \tag{4.4.13}$$

which is approximately twice the solid-disc maximum stress.

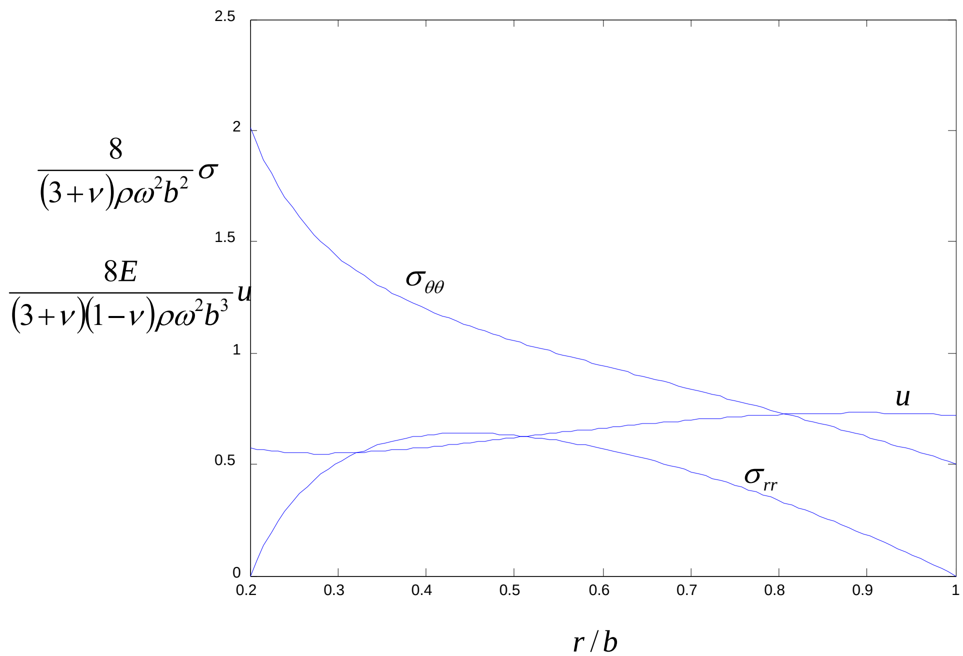


Figure 4.4.3: stresses and displacements in the hollow rotating disc

4.4.2 Problems

1. Derive the full solution equations 4.4.5 for the thin rotating disc, from the displacement solution 4.4.4.

4.4.3 Appendix: Solution to Eqn. 4.4.3

As in §4.3.8, transform Eqn. 4.4.3 using $r = e^t$ into

$$\frac{d^2u}{dt^2} - u = -\frac{1-\nu^2}{E}\rho e^{3t}\omega^2 \tag{4.4.14}$$

The homogeneous solution is given by 4.3.31. Assume a particular solution of the form $u_p = Ae^{3t}$ which, from 4.4.14, gives

$$u_p = -\frac{1}{8}\frac{1-\nu^2}{E}\rho\omega^2 e^{3t} \tag{4.4.15}$$

Adding together the homogeneous and particular solutions and transforming back to r's then gives 4.4.4.

6.1 Plate Theory

6.1.1 Plates

A **plate** is a flat structural element for which the thickness is small compared with the surface dimensions. The thickness is usually constant but may be variable and is measured normal to the **middle surface** of the plate, Fig. 6.1.1

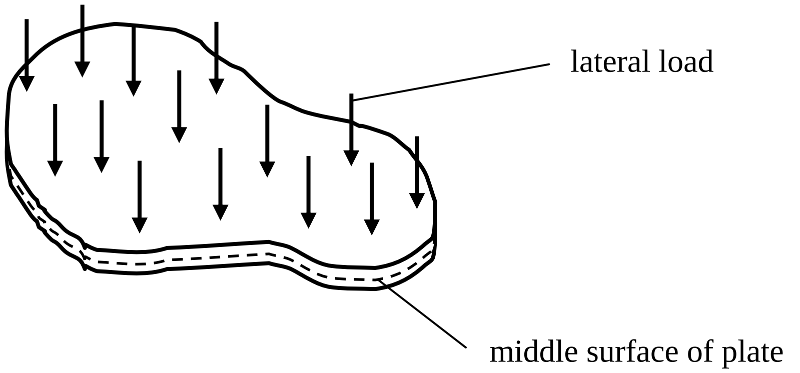


Fig. 6.1.1: A plate

6.1.2 Plate Theory

Plates subjected only to in-plane loading can be solved using two-dimensional plane stress theory[1] (see Book I, §3.5). On the other hand, **plate theory** is concerned mainly with **lateral loading**.

One of the differences between plane stress and plate theory is that in the plate theory the stress components are allowed to vary *through the thickness* of the plate, so that there can be bending moments, Fig. 6.1.2.

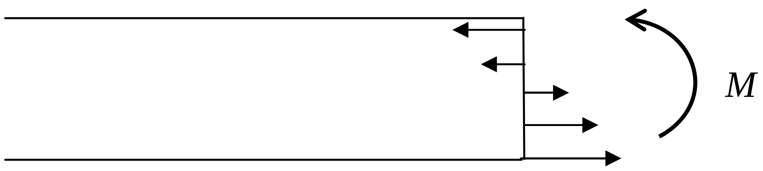


Fig. 6.1.2: Stress distribution through the thickness of a plate and resultant bending moment

Plate Theory and Beam Theory

Plate theory is an approximate theory; assumptions are made and the general three dimensional equations of elasticity are reduced. It is very like the **beam theory** (see Book

---

[1] although if the in-plane loads are compressive and sufficiently large, they can buckle (see §6.7)

I, §7.4) – only with an extra dimension. It turns out to be an accurate theory provided *the plate is relatively thin* (as in the beam theory) but also that *the deflections are small relative to the thickness*. This last point will be discussed further in §6.10.

Things are more complicated for plates than for the beams. For one, the plate not only bends, but torsion may occur (it can twist), as shown in Fig. 6.1.3

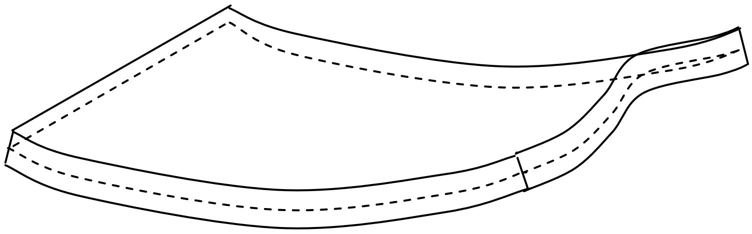


Fig. 6.1.3: torsion of a plate

Assumptions of Plate Theory

Let the plate mid-surface lie in the $x-y$ plane and the z – axis be along the thickness direction, forming a right handed set, Fig. 6.1.4.

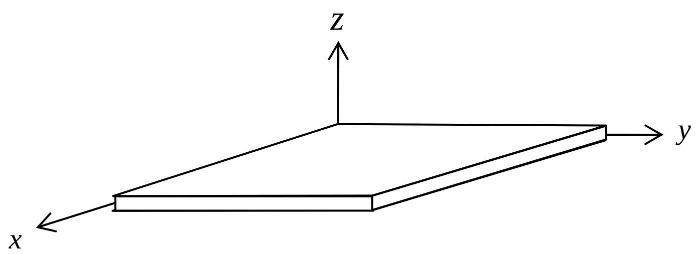


Fig. 6.1.4: Cartesian axes

The stress components acting on a typical element of the plate are shown in Fig. 6.1.5.

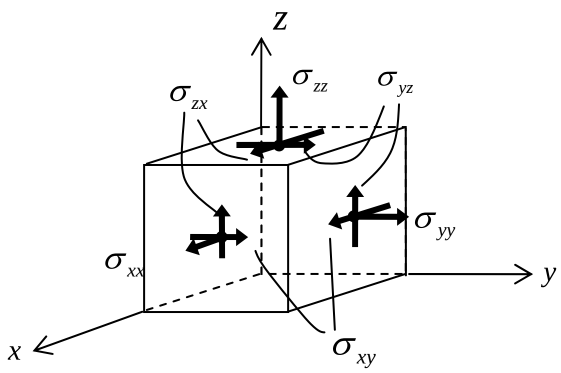


Fig. 6.1.5: stresses acting on a material element

The following assumptions are made:

(i) The mid-plane is a "neutral plane"

The middle plane of the plate remains free of in-plane stress/strain. Bending of the plate will cause material above and below this mid-plane to deform in-plane. The mid-plane plays the same role in plate theory as the neutral axis does in the beam theory.

(ii) Line elements remain normal to the mid-plane

Line elements lying perpendicular to the middle surface of the plate remain perpendicular to the middle surface during deformation, Fig. 6.1.6; this is similar the "plane sections remain plane" assumption of the beam theory.

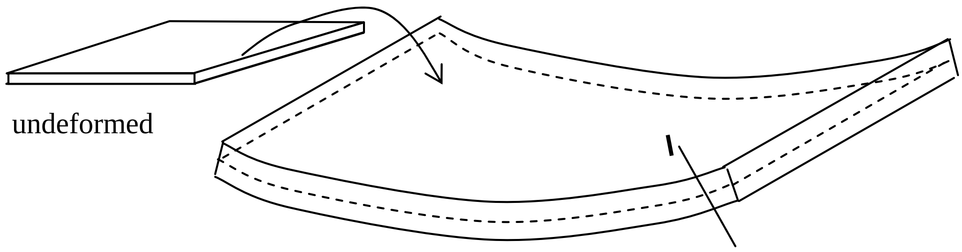


Fig. 6.1.6: deformed line elements remain perpendicular to the mid-plane

(iii) Vertical strain is ignored

Line elements lying perpendicular to the mid-surface do not change length during deformation, so that $\varepsilon_{zz} = 0$ throughout the plate. Again, this is similar to an assumption of the beam theory.

These three assumptions are the basis of the **Classical Plate Theory** or the **Kirchhoff Plate Theory**. The second assumption can be relaxed to develop a more exact theory (see §6.10).

6.1.3 Notation and Stress Resultants

The **stress resultants** are obtained by integrating the stresses through the thickness of the plate. In general there will be

moments M:	2 bending moments and 1 twisting moment
out-of-plane forces V:	2 shearing forces
in-plane forces N:	2 normal forces and 1 shear force

They are defined as follows:

In-plane normal forces and bending moments, Fig. 6.1.7:

$$
\begin{aligned}
N_x &= \int_{-h/2}^{+h/2} \sigma_{xx} dz, \qquad N_y = \int_{-h/2}^{+h/2} \sigma_{yy} dz \\
M_x &= -\int_{-h/2}^{+h/2} z\sigma_{xx} dz, \quad M_y = -\int_{-h/2}^{+h/2} z\sigma_{yy} dz
\end{aligned}
\tag{6.1.1}
$$

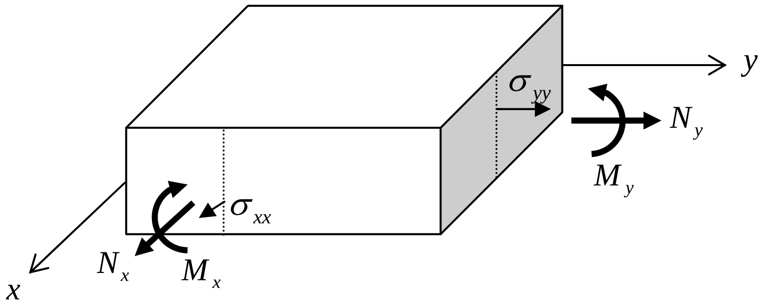


Fig. 6.1.7: in-plane normal forces and bending moments

In-plane shear force and twisting moment, Fig. 6.1.8:

$$
N_{xy} = \int_{-h/2}^{+h/2} \sigma_{xy} dz, \qquad M_{xy} = \int_{-h/2}^{+h/2} z\sigma_{xy} dz
\tag{6.1.2}
$$

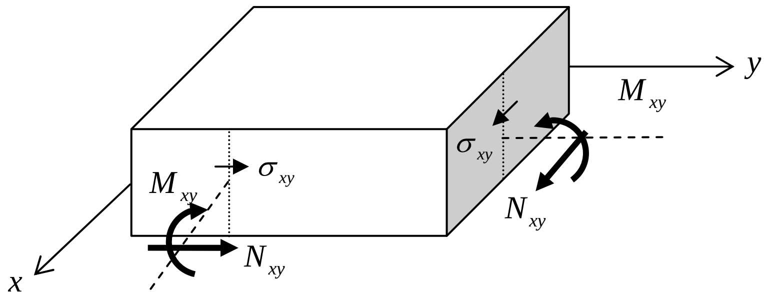


Fig. 6.1.8: in-plane shear force and twisting moment

Out-of-plane shearing forces, Fig. 6.1.9:

$$
V_x = -\int_{-h/2}^{+h/2} \sigma_{zx} dz, \qquad V_y = -\int_{-h/2}^{+h/2} \sigma_{yz} dz
\tag{6.1.3}
$$

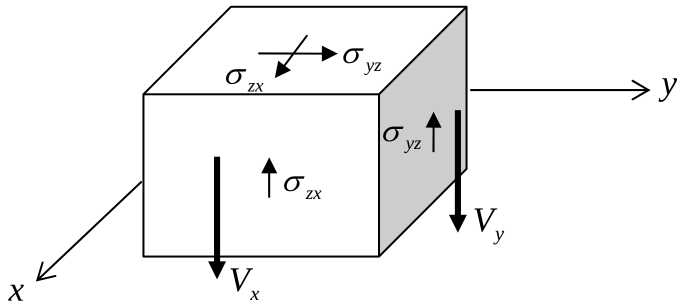


Fig. 6.1.9: out of plane shearing forces

Note that the above "forces" and "moments" are actually forces and moments *per unit length*. This allows one to have moments varying across any section – unlike in the beam theory, where the moments are for the complete beam cross-section. If one considers an element with dimensions Δx and Δy, the actual moments acting on the element are

$$M_x \Delta y, \quad M_y \Delta x, \quad M_{xy} \Delta x, \quad M_{xy} \Delta y \tag{6.1.4}$$

and the forces acting on the element are

$$V_x \Delta y, \quad V_y \Delta x, \quad N_x \Delta y, \quad N_y \Delta x, \quad N_{xy} \Delta x, \quad N_{xy} \Delta y \tag{6.1.5}$$

The in-plane forces, which are analogous to the axial forces of the beam theory, do not play a role in most of what follows. They are useful in the analysis of buckling of plates and it is necessary to consider them in more exact theories of plate bending (see later).

6.2 The Moment-Curvature Equations

6.2.1 From Beam Theory to Plate Theory

In the beam theory, based on the assumptions of plane sections remaining plane and that one can neglect the transverse strain, the strain varies linearly through the thickness. In the notation of the beam, with y positive up, $\varepsilon_{xx} = -y/R$, where R is the **radius of curvature**, R positive when the beam bends "up" (see Book I, Eqn. 7.4.16). In terms of the **curvature** $\partial^2 v/\partial x^2 = 1/R$, where v is the deflection (see Book I, Eqn. 7.4.36), one has

$$\varepsilon_{xx} = -y\frac{\partial^2 v}{\partial x^2} \tag{6.2.1}$$

The beam theory assumptions are essentially the same for the plate, leading to strains which are proportional to distance from the neutral (mid-plane) surface, z, and expressions similar to 6.2.1. This leads again to linearly varying stresses σ_{xx} and σ_{yy} (σ_{zz} is also taken to be zero, as in the beam theory).

6.2.2 Curvature and Twist

The plate is initially undeformed and flat with the mid-surface lying in the $x-y$ plane. When deformed, the mid-surface occupies the surface $w = w(x, y)$ and w is the elevation above the $x-y$ plane, Fig. 6.2.1.

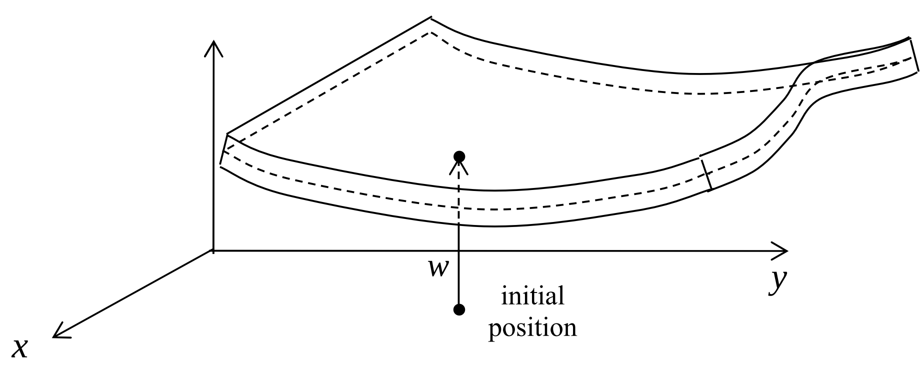


Fig. 6.2.1: Deformed Plate

The slopes of the plate along the x and y directions are $\partial w/\partial x$ and $\partial w/\partial y$.

Curvature

Recall from Book I, §7.4.11, that the curvature in the *x* direction, κ_x, is the rate of change of the slope angle ψ with respect to arc length *s*, Fig. 6.2.2, $\kappa_x = d\psi / ds$. One finds that

$$\kappa_x = \frac{\partial^2 w / \partial x^2}{\left[1 + \left(\partial w / \partial x\right)^2\right]^{3/2}} \tag{6.2.2}$$

Also, the radius of curvature R_x, Fig. 6.2.2, is the reciprocal of the curvature, $R_x = 1/\kappa_x$.

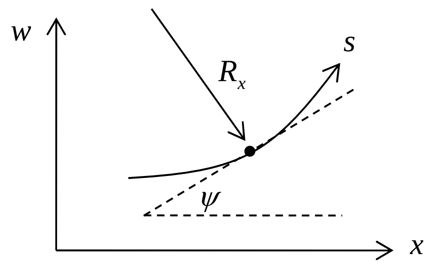


Fig. 6.2.2: Angle and arc-length used in the definition of curvature

As with the beam, *when the slope is small*, one can take $\psi \approx \tan\psi = \partial w / \partial x$ and $d\psi / ds \approx \partial \psi / \partial x$ and Eqn. 6.2.2 reduces to (and similarly for the curvature in the *y* direction)

$$\kappa_x = \frac{1}{R_x} = \frac{\partial^2 w}{\partial x^2}, \qquad \kappa_y = \frac{1}{R_y} = \frac{\partial^2 w}{\partial y^2} \tag{6.2.3}$$

This important assumption of small slope, $\partial w / \partial x, \partial w / \partial y << 1$, means that the theory to be developed will be valid when the deflections are small compared to the overall dimensions of the plate.

The curvatures 6.2.3 can be interpreted as in Fig. 6.2.3, as the unit increase in slope along the *x* and *y* directions.

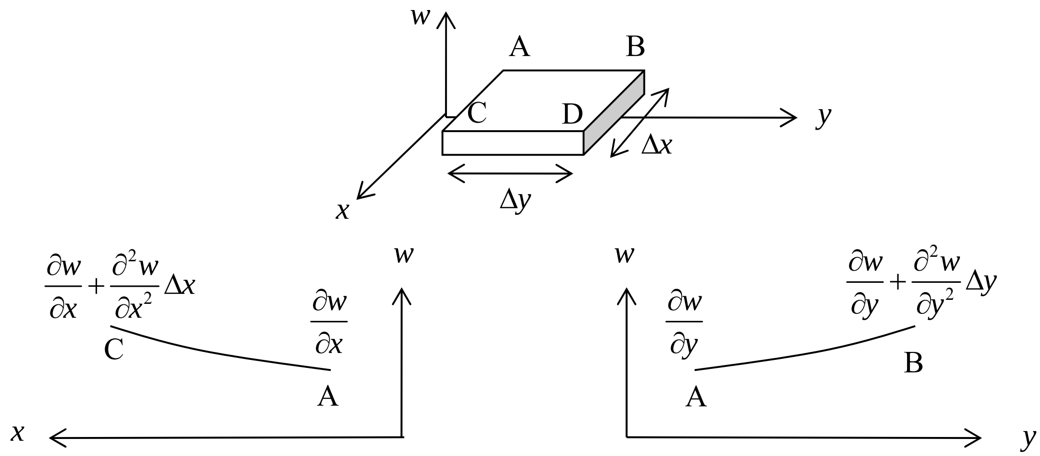


Figure 6.2.3: Physical meaning of the curvatures

Twist

Not only does a plate curve up or down, it can also twist (see Fig. 6.1.3). For example, shown in Fig. 6.2.4 is a plate undergoing a *pure* twisting (constant applied twisting moments and no bending moments).

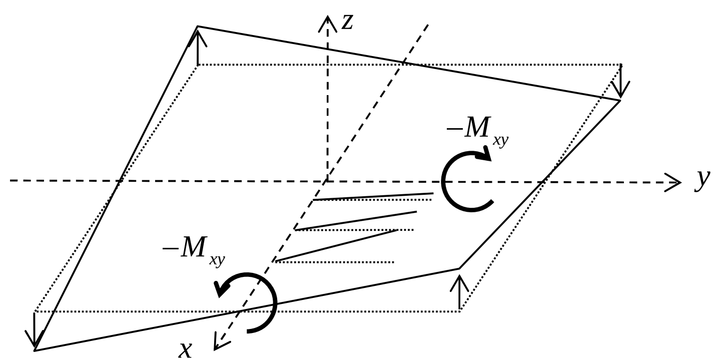


Figure 6.2.4: A twisting plate

If one takes a row of line elements lying in the *y* direction, emanating from the *x* axis, the further one moves along the *x* axis, the more they twist, Fig. 6.2.4. Some of these line elements are shown in Fig. 6.2.5 (bottom right), as veiwed looking down the *x* axis towards the origin (elements along the *y* axis are shown bottom left). If a line element at position *x* has slope $\partial w / \partial y$, the slope at $x + \Delta x$ is $\partial w / \partial y + \Delta x \partial(\partial w / \partial y) / \partial x$. This motivates the definition of the **twist**, defined analogously to the curvature, and denoted by $1/T_{xy}$; it is a measure of the "twistiness" of the plate:

$$\frac{1}{T_{xy}} = \frac{\partial^2 w}{\partial x \partial y} \tag{6.2.4}$$

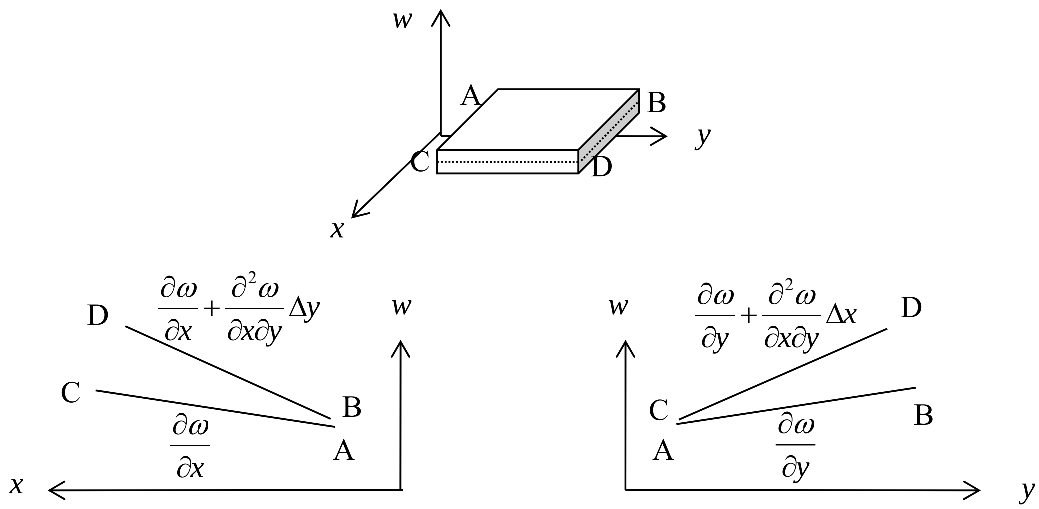


Figure 6.2.5: Physical meaning of the twist

The signs of the moments, radii of curvature and curvatures are illustrated in Fig. 6.2.6. Note that the deflection w may or may not be of the same sign as the curvature. Note also that when $M_x > 0$, $\partial^2\omega/\partial x^2 > 0$, when $M_y > 0$, $\partial^2\omega/\partial y^2 > 0$.

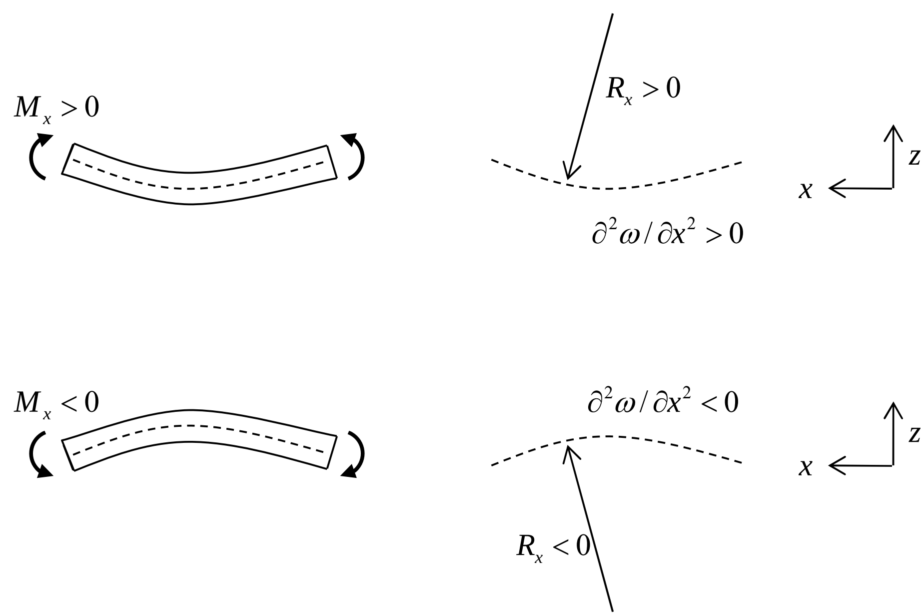


Figure 6.2.6: sign convention for curvatures and moments

On the other hand, for the twist, with the sign convention being used, when $M_{xy} > 0$, $\partial^2\omega/\partial x\partial y < 0$, as depicted in Fig. 6.2.4.

Principal Curvatures

Consider the two Cartesian coordinate systems shown in Fig. 6.2.7, the second ($t-n$) obtained from the first ($x-y$) by a positive rotation θ. The partial derivatives arising in the curvature expressions can be expressed in terms of derivatives with respect to *t* and *n* as follows: with $w = w(x, y)$, an increment in *w* is

$$\Delta w = \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y \tag{6.2.5}$$

Also, referring to Fig. 6.2.7, with $\Delta n = 0$,

$$\Delta x = \Delta t \cos\theta, \qquad \Delta y = \Delta t \sin\theta \tag{6.2.6}$$

Thus

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta \tag{6.2.7}$$

Similarly, for an increment Δn, one finds that

$$\frac{\partial w}{\partial n} = -\frac{\partial w}{\partial x}\sin\theta + \frac{\partial w}{\partial y}\cos\theta \tag{6.2.8}$$

Equations 6.2.7-8 can be inverted to get the inverse relations

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial w}{\partial t}\cos\theta - \frac{\partial w}{\partial n}\sin\theta \\ \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial t}\sin\theta + \frac{\partial w}{\partial n}\cos\theta\end{aligned} \tag{6.2.9}$$

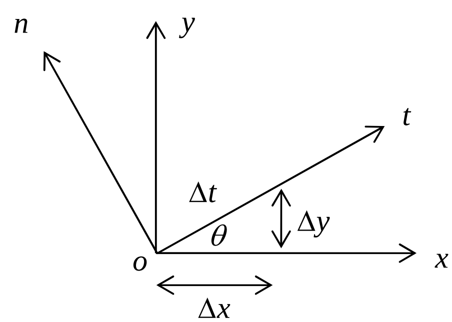


Figure 6.2.7: Two different Cartesian coordinate systems

The relationship between second derivatives can be found in the same way. For example,

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2} &= \left(\frac{\partial}{\partial t}\cos\theta - \frac{\partial}{\partial n}\sin\theta\right)\left(\frac{\partial w}{\partial t}\cos\theta - \frac{\partial w}{\partial n}\sin\theta\right) \\ &= \cos^2\theta\frac{\partial^2 w}{\partial t^2} + \sin^2\theta\frac{\partial^2 w}{\partial n^2} - \sin 2\theta\frac{\partial^2 w}{\partial t\partial n}\end{aligned} \tag{6.2.10}$$

In summary, one has

$$
\begin{aligned}
\frac{\partial^2 w}{\partial x^2} &= \cos^2\theta \frac{\partial^2 \omega}{\partial t^2} + \sin^2\theta \frac{\partial^2 \omega}{\partial n^2} - \sin 2\theta \frac{\partial^2 \omega}{\partial t \partial n} \\
\frac{\partial^2 w}{\partial y^2} &= \sin^2\theta \frac{\partial^2 \omega}{\partial t^2} + \cos^2\theta \frac{\partial^2 \omega}{\partial n^2} + \sin 2\theta \frac{\partial^2 \omega}{\partial t \partial n} \\
\frac{\partial^2 w}{\partial x \partial y} &= -\sin\theta\cos\theta \left(\frac{\partial^2 \omega}{\partial n^2} - \frac{\partial^2 \omega}{\partial t^2} \right) + \cos 2\theta \frac{\partial^2 \omega}{\partial t \partial n}
\end{aligned}
\tag{6.2.11}
$$

and the inverse relations

$$
\begin{aligned}
\frac{\partial^2 w}{\partial t^2} &= \cos^2\theta \frac{\partial^2 \omega}{\partial x^2} + \sin^2\theta \frac{\partial^2 \omega}{\partial y^2} + \sin 2\theta \frac{\partial^2 \omega}{\partial x \partial y} \\
\frac{\partial^2 w}{\partial n^2} &= \sin^2\theta \frac{\partial^2 \omega}{\partial x^2} + \cos^2\theta \frac{\partial^2 \omega}{\partial y^2} - \sin 2\theta \frac{\partial^2 \omega}{\partial x \partial y} \\
\frac{\partial^2 w}{\partial t \partial n} &= \sin\theta\cos\theta \left(\frac{\partial^2 \omega}{\partial y^2} - \frac{\partial^2 \omega}{\partial x^2} \right) + \cos 2\theta \frac{\partial^2 \omega}{\partial x \partial y}
\end{aligned}
\tag{6.2.12}
$$

or[1]

$$
\begin{aligned}
\frac{1}{R_t} &= \cos^2\theta \frac{1}{R_x} + \sin^2\theta \frac{1}{R_y} + \sin 2\theta \frac{1}{T_{xy}} \\
\frac{1}{R_n} &= \sin^2\theta \frac{1}{R_x} + \cos^2\theta \frac{1}{R_y} - \sin 2\theta \frac{1}{T_{xy}} \\
\frac{1}{T_{tn}} &= \sin\theta\cos\theta \left(\frac{1}{R_y} - \frac{1}{R_x} \right) + \cos 2\theta \frac{1}{T_{xy}}
\end{aligned}
\tag{6.2.13}
$$

These equations which transform between curvatures in different coordinate systems have the same structure as the stress transformation equations (and the strain transformation equations), Book I, Eqns. 3.4.8. As with principal stresses/strains, there will be some angle θ for which the twist is zero; at this angle, one of the curvatures will be the minimum and one will be the maximum at that point in the plate. These are called the **principal curvatures**. Similarly, just as the sum of the normal stresses is an invariant (see Book I, Eqn. 3.5.1), the sum of the curvatures is an invariant[2]:

$$
\frac{1}{R_x} + \frac{1}{R_y} = \frac{1}{R_t} + \frac{1}{R_n} \tag{6.2.14}
$$

If the principal curvatures are equal, the curvatures are the same at all angles, the twist is always zero and so the plate deforms locally into the surface of a sphere.

[1] these equations are valid for any continuous surface; Eqns. 6.2.12 are restricted to nearly-flat surfaces.
[2] this is known as Euler's theorem for curvatures

6.2.3 Strains in a Plate

The strains arising in a plate are next examined. A comprehensive strain-state will be first examined and this will then be simpilfied down later to various approximate solutions. Consider a line element parallel to the x axis, of length Δx. Let the element displace as shown in Fig. 6.2.8. Whereas w was used in the previous section on curvatures to denote displacement of the mid-surface, here, for the moment, let $w(x, y, z)$ be the general vertical displacement of any particle in the plate. Let u and v be the corresponding displacements in the x and y directions. Denote the original and deformed length of the element by dS and ds respectively.

The unit change in length of the element (that is, the exact normal strain) is, using Pythagoras' theorem,

$$\varepsilon_{xx} = \frac{ds - dS}{dS} = \frac{|p'q'| - |pq|}{|pq|} = \sqrt{\left(1 + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \qquad (6.2.15)$$

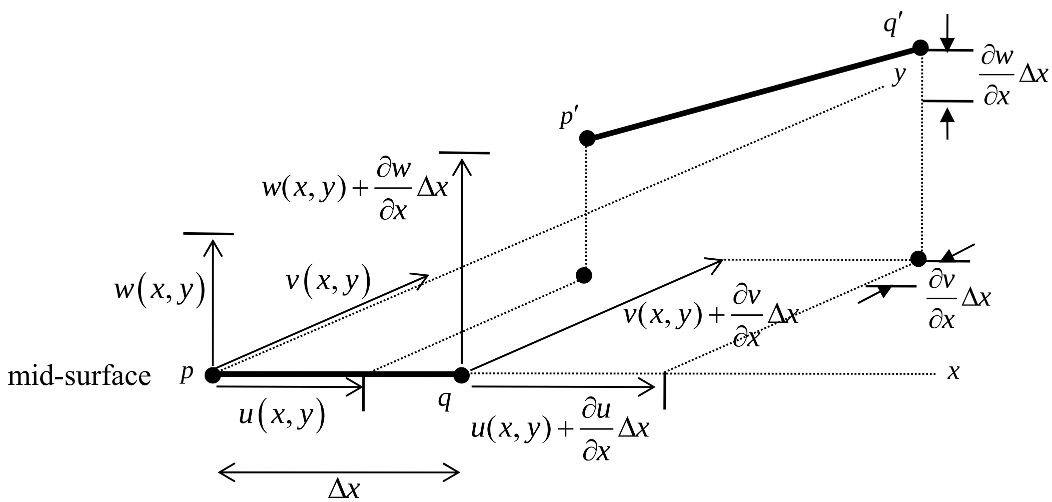


Figure 6.2.8: deformation of a material fibre in the *x* direction

In the plate theory, it will be assumed that the displacement gradients are small:

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial z}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial z}, \quad \frac{\partial w}{\partial z},$$

of order $\varepsilon << 1$ say, so that squares and products of these terms may be neglected.

However, for the moment, the squares and products of the slopes will be retained, as they may be significant, i.e. of the same order as the strains, under certain circumstances:

$$\left(\frac{\partial w}{\partial x}\right)^2, \quad \left(\frac{\partial w}{\partial y}\right)^2, \quad \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

Eqn. 6.2.15 now reduces to

$$\varepsilon_{xx} = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \tag{6.2.16}$$

With $\sqrt{1+x} \approx 1 + x/2$ for $x << 1$, one has (and similarly for the other normal strains)

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \end{aligned} \tag{6.2.17}$$

Consider next the angle change for line elements initially lying parallel to the axes, Fig. 6.2.9. Let θ be the angle $\angle r'p'q'$, so that $\gamma = \pi/2 - \theta$ is the change in the initial right angle $\angle rpq$.

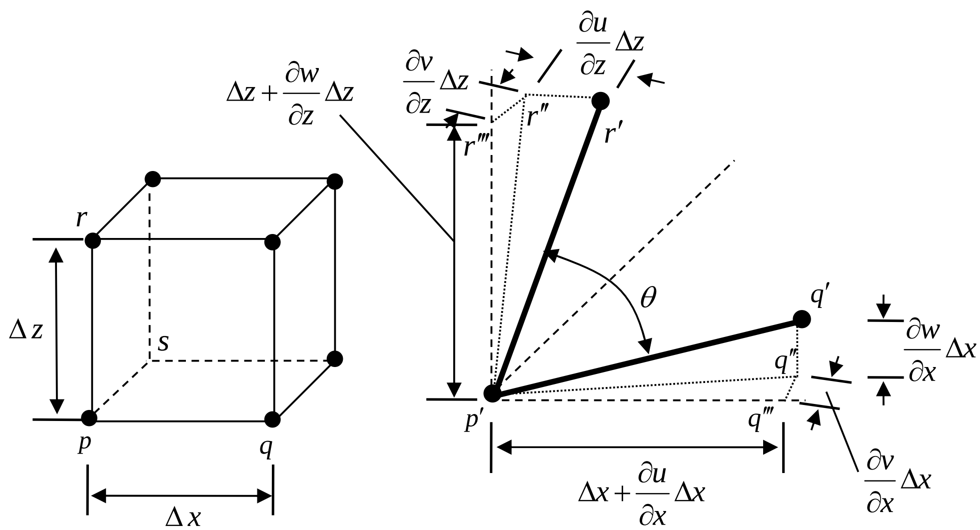


Figure 6.2.9: the deformation of Fig. 6.2.8, showing shear strains

Taking the dot product of the of the vector elements $\overrightarrow{p'q'}$ and $\overrightarrow{p'r'}$:

$$\cos\theta = \frac{|p'q'''||r''r'| + |q'''q''||r'''r''| + |q''q'||p'r'''|}{|p'q'||p'r'|}$$
$$= \frac{\left(\Delta x + \frac{\partial u}{\partial x}\Delta x\right)\left(\frac{\partial u}{\partial z}\Delta z\right) + \left(\frac{\partial v}{\partial x}\Delta x\right)\left(\frac{\partial v}{\partial z}\Delta z\right) + \left(\frac{\partial w}{\partial x}\Delta x\right)\left(\Delta z + \frac{\partial w}{\partial z}\Delta z\right)}{\Delta x\sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}\,\Delta z\sqrt{\left(1+\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2}} \quad (6.2.18)$$

Again, with the displacement gradients $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial z$, $\partial v/\partial z$, $\partial w/\partial z$ of order $\varepsilon << 1$ (and the squares $(\partial w/\partial x)^2$ at most of order $\varepsilon << 1$),

$$\cos\theta = \frac{\Delta x\left(\frac{\partial u}{\partial z}\Delta z\right) + \left(\frac{\partial v}{\partial x}\Delta x\right)\left(\frac{\partial v}{\partial z}\Delta z\right) + \left(\frac{\partial w}{\partial x}\Delta x\right)\Delta z}{\Delta x \Delta z} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} \quad (6.2.19)$$

For small γ, $\gamma \approx \sin\gamma = \cos\theta$, so (and similarly for the other shear strains)

$$\begin{aligned}
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\\
\varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)
\end{aligned} \quad (6.2.20)$$

The normal strains 6.2.17 and the shear strains 6.2.20 are non-linear. They are the starting point for the various different plate theories.

Von Kármán Strains

Introduce now the assumptions of the classical plate theory. The assumption that line elements normal to the mid-plane remain inextensible implies that

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \quad (6.2.21)$$

This implies that $w = w(x, y)$ so that all particles at a given (x, y) through the thickness of the plate experinece the same vertical displacement. The assumption that line elements perpendicular to the mid-plane remain normal to the mid-plane after deformation then implies that $\varepsilon_{xz} = \varepsilon_{yz} = 0$.

The strains now read

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\
\varepsilon_{zz} &= 0 \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
\varepsilon_{xz} &= 0 \\
\varepsilon_{yz} &= 0
\end{aligned} \tag{6.2.22}$$

These are known as the **Von Kármán strains**.

Membrane Strains and Bending Strains

Since $\varepsilon_{xz} = 0$ and $w = w(x, y)$, one has from Eqn. 6.2.20b,

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x} \quad \to \quad u(x, y, z) = -z\frac{\partial w}{\partial x} + u_0(x, y) \tag{6.2.23}$$

It can be seen that the function $u_0(x, y)$ is the displacement in the mid-plane. In terms of the mid-surface displacements u_0, v_0, w_0, then,

$$u = u_0 - z\frac{\partial w_0}{\partial x}, \quad v = v_0 - z\frac{\partial w_0}{\partial y}, \quad w = w_0 \tag{6.2.24}$$

and the strains 6.2.22 may be expressed as

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 - z\frac{\partial^2 w_0}{\partial x^2} \\
\varepsilon_{yy} &= \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 - z\frac{\partial^2 w_0}{\partial y^2} \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right) - z\frac{\partial^2 w_0}{\partial x \partial y}
\end{aligned} \tag{6.2.25}$$

The first terms are the usual small-strains, for the mid-surface. The second terms, involving squares of displacement gradients, are non-linear, and need to be considered when the plate bending is fairly large (when the rotations are about 10 – 15 degrees). These first two terms together are called the **membrane strains**. The last terms, involving second derivatives, are the **flexural** (**bending**) **strains**. They involve the curvatures.

When the bending is not too large (when the rotations are below about 10 degrees), one has (dropping the subscript "0" from *w*)

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \\
\varepsilon_{yy} &= \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2} \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - z\frac{\partial^2 w}{\partial x \partial y}
\end{aligned}
\tag{6.2.26}
$$

Some of these strains are illustrated in Figs. 6.2.10 and 6.2.11; the physical meaning of ε_{xx} is shown in Fig. 6.2.10 and some terms from ε_{xy} are shown in Fig. 6.2.11.

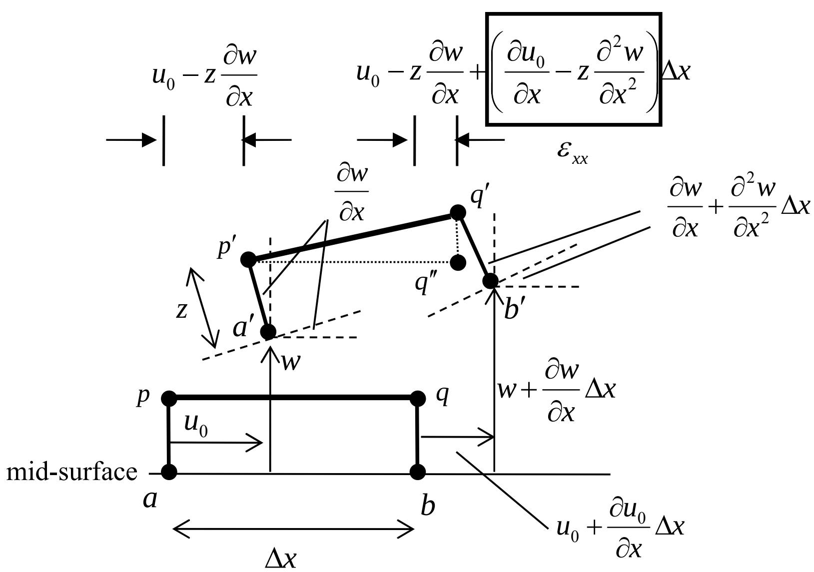


Figure 6.2.10: deformation of material fibres in the *x* direction

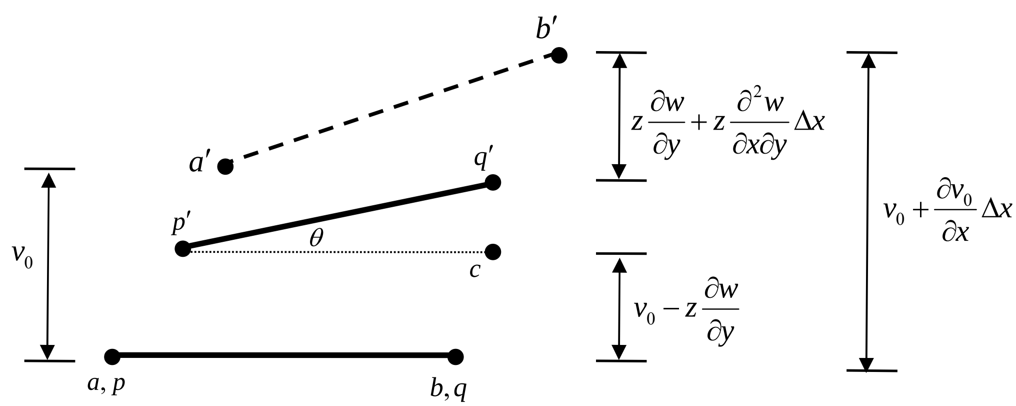


Figure 6.2.11: the deformation of 6.2.10 viewed "from above"; a', b' are the deformed positions of the mid-surface points a, b

Finally, when the mid-surface strains are neglected, according to the final assumption of the classical plate theory, one has

$$\boxed{\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2}, \qquad \varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2}, \qquad \varepsilon_{xy} = -z\frac{\partial^2 w}{\partial x \partial y}} \tag{6.2.27}$$

In summary, when the plate bends "up", the curvature is positive, and points "above" the mid-surface experience negative normal strains and points "below" experience positive normal strains; there is zero shear strain. On the other hand, when the plate undergoes a positive pure twist, so the twisting moment is negative, points "above" the mid-surface experience negative shear strain and points "below" experience positive shear strain; there is zero normal strain. A pure shearing of the plate in the $x-y$ plane is illustrated in Fig. 6.2.12.

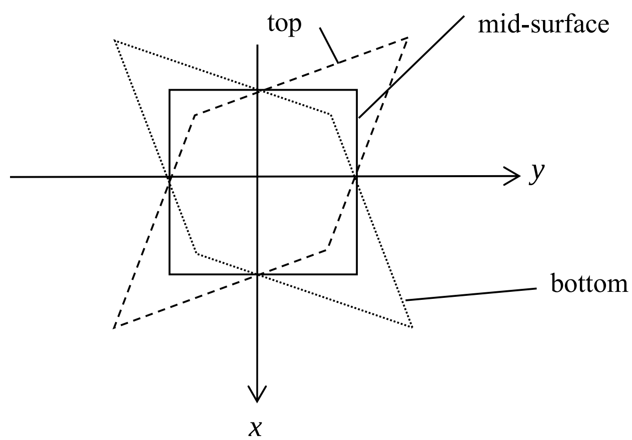


Figure 6.2.12: Shearing of the plate due to a positive twist (neative twisting moment)

Compatibility

Note that the strain fields arising in the plate satisfy the 2D compatibility relation Eqn. 1.3.1:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{6.2.28}$$

This can be seen by substituting Eqn. 6.25 (or Eqns 6.26-27) into Eqn. 6.2.28.

6.2.4 The Moment-Curvature equations

Now that the strains have been related to the curvatures, the moment-curvature relations, which play a central role in plate theory, can be derived.

Stresses and the Curvatures/Twist in a Linear Elastic Plate

From Hooke's law, taking $\sigma_{zz} = 0$,

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\nu}{E}\sigma_{yy}, \quad \varepsilon_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx}, \quad \varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy} \tag{6.2.29}$$

so, from 6.2.27, and solving 6.2.29a-b for the normal stresses,

$$\begin{aligned}
\sigma_{xx} &= -\frac{E}{1-\nu^2} z\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \\
\sigma_{yy} &= -\frac{E}{1-\nu^2} z\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \\
\sigma_{xy} &= -\frac{E}{1+\nu} z\frac{\partial^2 w}{\partial x \partial y}
\end{aligned} \tag{6.2.30}$$

The Moment-Curvature Equations

Substituting Eqns. 6.2.30 into the definitions of the moments, Eqns. 6.1.1, 6.1.2, and integrating, one has

$$\boxed{\begin{aligned}
M_x &= D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \\
M_y &= D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \\
M_{xy} &= -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}
\end{aligned}} \tag{6.2.31}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{6.2.32}$$

Equations 6.2.31 are the **moment-curvature equations** for a plate. The moment-curvature equations are analogous to the beam moment-deflection equation $\partial^2 v / \partial x^2 = M / EI$. The factor D is called the **plate stiffness** or **flexural rigidity** and plays the same role in the plate theory as does the flexural rigidity term EI in the beam theory.

Stresses and Moments

From 6.30-6.31, the stresses and moments are related through

$$\sigma_{xx} = -\frac{M_x z}{h^3/12}, \quad \sigma_{yy} = -\frac{M_y z}{h^3/12}, \quad \sigma_{xy} = +\frac{M_{xy} z}{h^3/12} \tag{6.2.33}$$

Note the similarity of these relations to the beam formula $\sigma = -My/I$ with $I = h^3/12$ times the width of the beam.

6.2.5 Principal Moments

It was seen how the curvatures in different directions are related, through Eqns. 6.2.11-12. It comes as no surprise, examining 6.2.31, that the moments are related in the same way.

Consider a small differential element of a plate, Fig. 6.2.13a, subjected to stresses σ_{xx}, σ_{yy}, σ_{xy}, and corresponding moments M_x, M_y, M_{xy} given by 6.1.1-2. On any perpendicular planes rotated from the orginal $x-y$ axes by an angle θ, one can find the new stresses σ_{tt}, σ_{nn}, σ_{tn}, Fig. 6.2.13b (see Fig. 6.2.5), through the stress transformatrion equations (Book I, Eqns. 3.4.8). Then

$$\begin{aligned} M_t &= -\int z\sigma_{tt}dz = \cos^2\theta\left[-\int z\sigma_{xx}dz\right] + \sin^2\theta\left[-\int z\sigma_{yy}dz\right] + \sin 2\theta\left[-\int z\sigma_{xy}dz\right] \\ &= \cos^2\theta M_x + \sin^2\theta M_y - \sin 2\theta M_{xy} \end{aligned} \tag{6.2.34}$$

and similarly for the other moments, leading to

$$\begin{aligned} M_t &= \cos^2\theta M_x + \sin^2\theta M_y - \sin 2\theta M_{xy} \\ M_n &= \sin^2\theta M_x + \cos^2\theta M_y + \sin 2\theta M_{xy} \\ M_{tn} &= -\cos\theta\sin\theta\left(M_y - M_x\right) + \cos 2\theta M_{xy} \end{aligned} \tag{6.2.35}$$

Also, there exist principal planes, upon which the shear stress is zero (right through the thickness). The moments acting on these planes, M_1 and M_2, are called the **principal moments**, and are the greatest and least bending moments which occur at the element. On these planes, the twisting moment is zero.

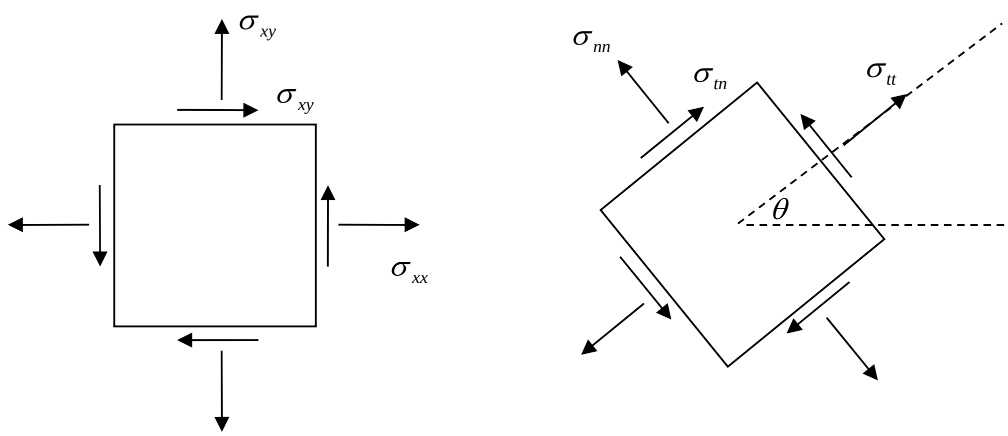


Figure 6.2.13: Plate Element; (a) stresses acting on element, (b) rotated element

Moments in Different Coordinate Systems

From the moment-curvature equations 6.2.31, {▲Problem 1}

$$
\begin{aligned}
M_t &= D\left(\frac{\partial^2\omega}{\partial t^2} + \nu\frac{\partial^2\omega}{\partial n^2}\right) \\
M_n &= D\left(\frac{\partial^2\omega}{\partial n^2} + \nu\frac{\partial^2\omega}{\partial t^2}\right) \\
M_{tn} &= -D(1-\nu)\frac{\partial^2\omega}{\partial t\partial n}
\end{aligned}
\tag{6.2.36}
$$

showing that the moment-curvature relations 6.2.31 hold in all Cartesian coordinate systems.

6.2.6 Problems

1. Use the curvature transformation relations 6.2.11 and the moment transformation relations 6.2.35 to derive the moment-curvature relations 6.2.36.

6.3 Plates subjected to Pure Bending and Twisting

6.3.1 Pure Bending of an Elastic Plate

Consider a plate subjected to bending moments $M_x = M_1 > 0$ and $M_y = M_2 > 0$, with no other loading, as shown in Fig. 6.3.1.

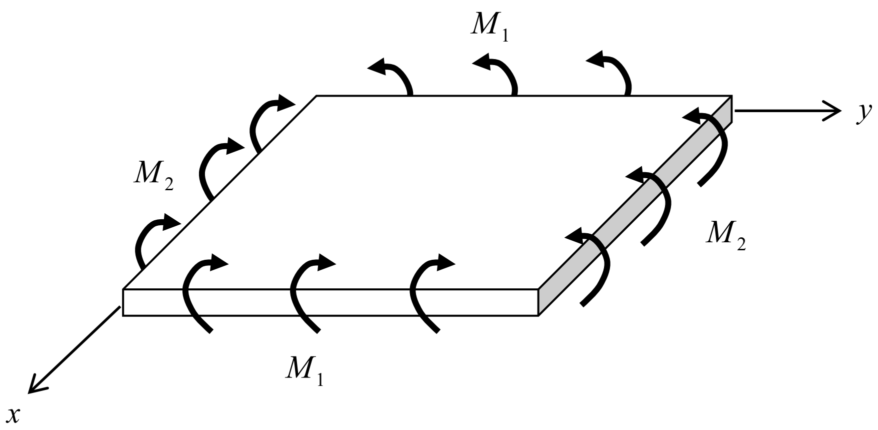


Figure 6.3.1: A plate under Pure Bending

From equilibrium considerations, these moments act at all points within the plate – they are constant throughout the plate. Thus, from the moment-curvature equations 6.2.31, one has the set of coupled partial differential equations

$$\frac{M_1}{D} = \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}, \quad \frac{M_2}{D} = \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}, \quad 0 = \frac{\partial^2 w}{\partial x \partial y} \tag{6.3.1}$$

Solving for the derivatives,

$$\frac{\partial^2 w}{\partial x^2} = \frac{M_1 - \nu M_2}{D(1-\nu^2)}, \quad \frac{\partial^2 w}{\partial y^2} = \frac{M_2 - \nu M_1}{D(1-\nu^2)}, \quad \frac{\partial^2 w}{\partial x \partial y} = 0 \tag{6.3.2}$$

Integrating the first two equations twice gives[1]

$$w = \frac{1}{2}\frac{M_1 - \nu M_2}{D(1-\nu^2)} x^2 + f_1(y)x + f_2(y), \quad w = \frac{1}{2}\frac{M_2 - \nu M_1}{D(1-\nu^2)} y^2 + g_1(x)y + g_2(x) \tag{6.3.3}$$

and integrating the third shows that two of these four unknown functions are constants:

$$\frac{\partial w}{\partial x} = G(x), \quad \frac{\partial w}{\partial y} = F(y) \qquad \to \qquad f_1(y) = A, \; g_1(x) = B \tag{6.3.4}$$

[1] this analysis is similar to that used to evaluate displacements in plane elastostatic problems, §1.2.4

Equating both expressions for w in 6.3.3 gives

$$\frac{1}{2}\frac{M_1 - \nu M_2}{D(1-\nu^2)}x^2 + Ax - g_2(x) = \frac{1}{2}\frac{M_2 - \nu M_1}{D(1-\nu^2)}y^2 + By - f_2(y) \tag{6.3.5}$$

For this to hold, both sides here must be a constant, $-C$ say. It follows that

$$w = \frac{1}{2}\frac{M_1 - \nu M_2}{D(1-\nu^2)}x^2 + \frac{1}{2}\frac{M_2 - \nu M_1}{D(1-\nu^2)}y^2 + Ax + By + C \tag{6.3.6}$$

The three unknown constants represent an arbitrary rigid body motion. To obtain values for these, one must fix three degrees of freedom in the plate. If one supposes that the deflection w and slopes $\partial w/\partial x, \partial w/\partial y$ are zero at the origin $x = y = 0$ (so the origin of the axes are at the plate-centre), then $A = B = C = 0$; all deformation will be measured relative to this reference. It follows that

$$w = \frac{M_2}{2D(1-\nu^2)}\left[\left[(M_1/M_2) - \nu\right]x^2 + \left[1 - \nu(M_1/M_2)\right]y^2\right] \tag{6.3.7}$$

Once the deflection w is known, all other quantities in the plate can be evaluated – the strain from 6.2.27, the stress from Hooke's law or directly from 6.2.30, and moments and forces from 6.1.1-3.

In the special case of equal bending moments, with $M_1 = M_2 = M_o$ say, one has

$$w = \frac{M_o}{2D(1+\nu)}\left(x^2 + y^2\right) \tag{6.3.8}$$

This is the equation of a sphere. In fact, from the relationship between the curvatures and the radius of curvature R,

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = \frac{M_o}{D(1+\nu)} \quad \to \quad R = \frac{D(1+\nu)}{M_o} = \text{constant} \tag{6.3.9}$$

and so the mid-surface of the plate in this case deforms into the surface of a sphere with radius given by 6.3.9, as illustrated in Fig. 6.3.2.

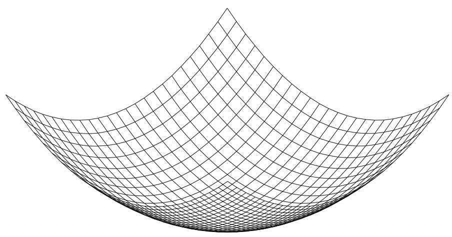
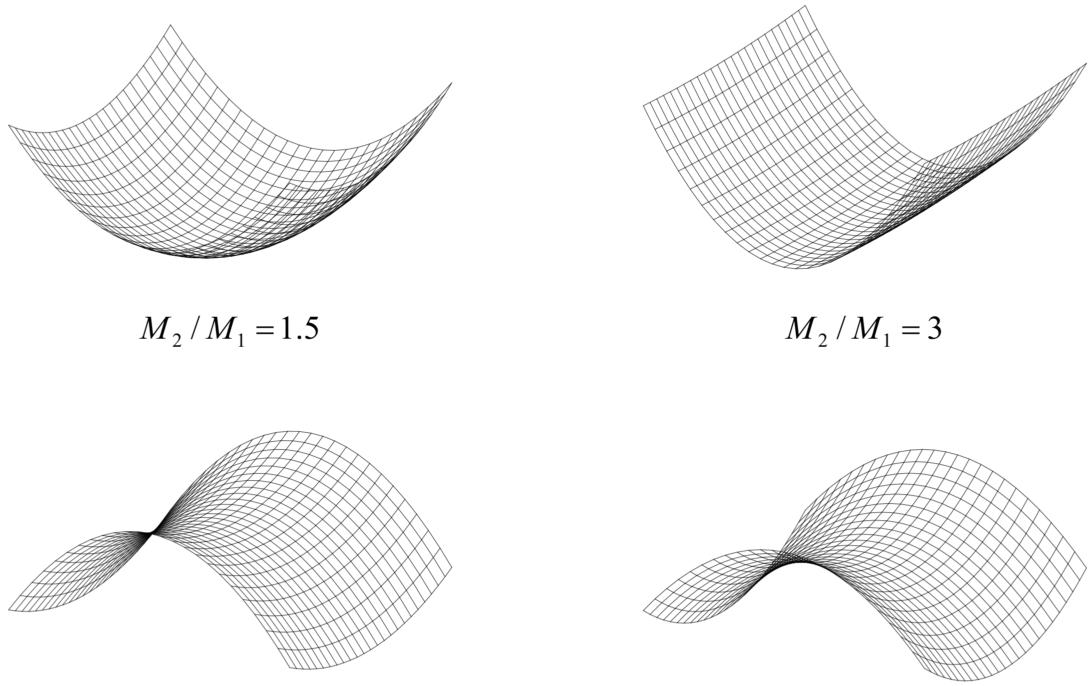


Figure 6.3.2: Deformed plate under Pure Bending with equal moments

The character of the deformed plate is plotted in Fig. 6.3.3 for various ratios M_2 / M_1 (for $\nu = 0.3$).



$M_2 / M_1 = 1.5$ $\qquad$ $M_2 / M_1 = 3$

$M_2 / M_1 = -1.5$ $\qquad$ $M_2 / M_1 = -3$

Figure 6.3.3: Bending of a Plate

When the curvatures $\partial^2 w / \partial x^2$ and $\partial^2 w / \partial y^2$ are of the same sign[2], the deformation is called **synclastic**. When the curvatures are of opposite sign, as in the lower plots of Fig. 6.3.3, the deformation is said to be **anticlastic**.

Note that when there is only one moment, $M_y = 0$ say, there is still curvature in both directions. In this case, one can solve the moment-curvature equations to get

$$\frac{\partial^2 w}{\partial x^2} = \frac{M_x}{\left(1-\nu^2\right)D}, \quad \frac{\partial^2 w}{\partial y^2} = -\nu \frac{\partial^2 w}{\partial x^2}, \qquad w = \frac{M_x}{2\left(1-\nu^2\right)D}\left(x^2 - \nu y^2\right) \tag{6.3.10}$$

which is an anticlastic deformation.

In order to get a pure cylindrical deformation, $w = f(x)$ say, one needs to apply moments M_x and $M_y = \nu M_x$, in which case, from 6.3.6,

[2] or principal curvatures in the case of a more complex general loading

$$w = \frac{M_x}{2D} x^2 \tag{6.3.11}$$

The deformation for $M_2 / M_1 = 3$ in Fig. 6.3.3 is very close to cylindrical, since there $M_x \approx \nu M_y$ for typical values of ν.

6.3.2 Pure Torsion of an Elastic Plate

In pure torsion, one has the twisting moment $M_{xy} = M > 0$ with no other loading, Fig. 6.3.4. From the moment-curvature equations,

$$0 = \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}, \quad 0 = \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}, \quad -\frac{M}{D(1-\nu)} = \frac{\partial^2 w}{\partial x \partial y} \tag{6.3.12}$$

so that

$$\frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0, \quad \frac{\partial^2 w}{\partial x \partial y} = -\frac{M}{D(1-\nu)} \tag{6.3.13}$$

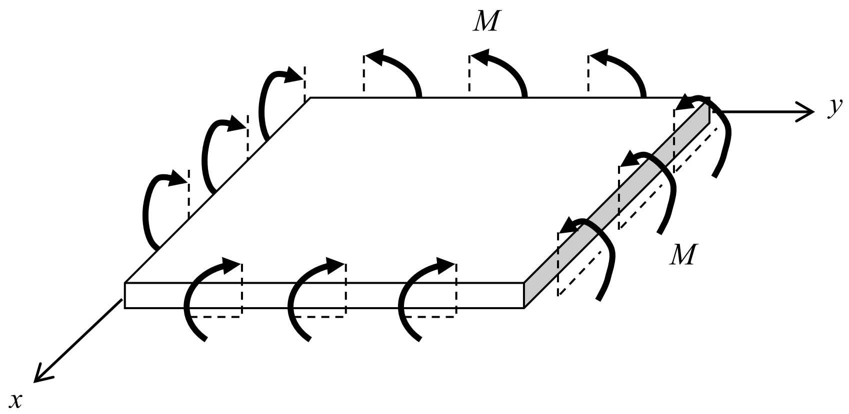


Figure 6.3.4: Twisting of a Plate

Using the same arguments as before, integrating these equations leads to

$$w = -\frac{M}{D(1-\nu)} xy \tag{6.3.14}$$

The middle surface is deformed as shown in Fig. 6.3.5, for a negative M_{xy}. Note that there is no deflection along the lines $x = 0$ or $y = 0$.

The principal curvatures will occur at 45° to the axes (see Eqns. 6.2.13):

$$\frac{1}{R_1} = +\frac{M}{D(1-\nu)}, \qquad \frac{1}{R_2} = -\frac{M}{D(1-\nu)} \tag{6.3.15}$$

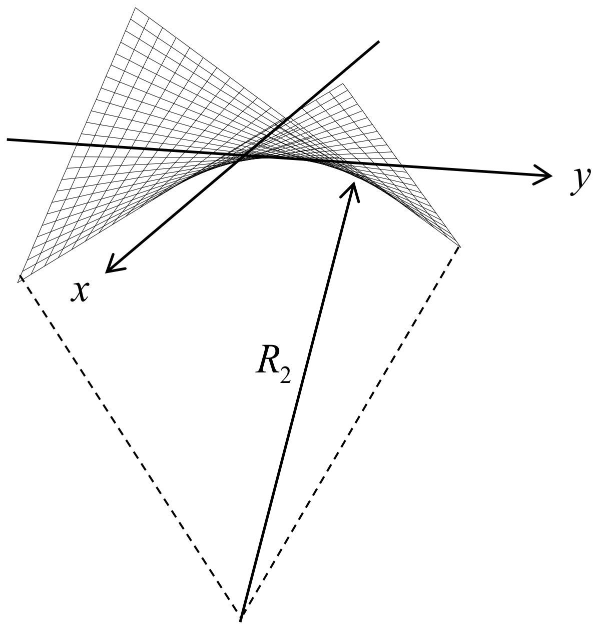


Figure 6.3.5: Deformation for a (negative) twisting moment

6.4 Equilibrium and Lateral Loading

In this section, lateral loads are considered and these lead to shearing forces V_x, V_y, in the plate.

6.4.1 The Governing Differential Equation for Lateral Loads

In general, a plate will at any location be subjected to a lateral *pressure* q, bending moments M_x, M_y, M_{xy} and out-of-plane shear forces V_x and V_y; q is the normal pressure on the upper surface of the plate:

$$\sigma_{zz}(x, y) = \begin{cases} 0, & z = -h/2 \\ -q(x, y), & z = +h/2 \end{cases} \tag{6.4.1}$$

These quantities are related to each other through force equilibrium.

Force Equilibrium

Consider a differential plate element with one corner at $(x, y) = (0,0)$, Fig. 6.4.1, subjected to moments, pressure and shear force. Taking force equilibrium in the vertical direction (neglecting a possible small variation in *q*, since this will only introduce higher order terms):

$$\sum F\Big|_z = +V_y \Delta x - \left(V_y + \frac{\partial V_y}{\partial y}\Delta y\right)\Delta x + V_x \Delta y - \left(V_x + \frac{\partial V_x}{\partial x}\Delta x\right)\Delta y - q\Delta x \Delta y = 0 \tag{6.4.2}$$

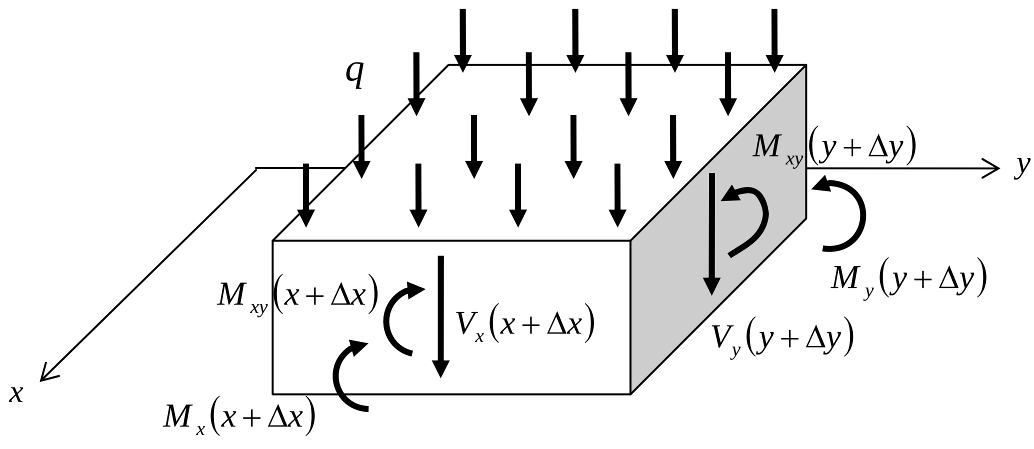


Fig. 6.4.1: a plate element subjected to moments, pressure and shear forces

Eqn. 6.4.2 gives the vertical equilibrium equation

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = -q \tag{6.4.3}$$

This is analogous to the beam theory equation $p = dV / dx$ (see Book I, §7.4.3).

Next, taking moments about the x axis:

$$\begin{aligned}\sum M\big|_x &= M_{xy}\Delta y - \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}\Delta x\right)\Delta y - M_y \Delta x + \left(M_y + \frac{\partial M_y}{\partial y}\Delta y\right)\Delta x \\ &\quad - yV_y\Delta x - (y+\Delta y)\left(V_y + \frac{\partial V_y}{\partial y}\Delta y\right)\Delta x + (y + \Delta y/2)V_x\Delta y \\ &\quad - (y+\Delta y/2)\left(V_x + \frac{\partial V_x}{\partial x}\Delta x\right)\Delta y - (y+\Delta y/2)q\Delta x\Delta y = 0\end{aligned} \qquad (6.4.4)$$

Using 6.4.3, this reduces to (and similarly for moments about the x-axis),

$$\begin{aligned}V_x &= +\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} \\ V_y &= -\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}\end{aligned} \qquad (6.4.5)$$

These are analogous to the beam theory equation $V = dM / dx$ (see Book I, §7.4.3).

Relations directly from the Equations of Equilibrium

The equilibrium relations 6.4.3, 6.4.5 can also be derived directly from the equations of equilibrium, Eqns. 1.1.9, which encompass the force balances:

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} &= 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0\end{aligned} \qquad (6.4.6)$$

Taking the first of these (which ensures equilibrium of forces in the x direction), multiplying by z and integrating over the plate thickness, gives

$$\begin{aligned}&\int_{-h/2}^{+h/2} z\frac{\partial \sigma_{xx}}{\partial x}dz + \int_{-h/2}^{+h/2} z\frac{\partial \sigma_{yx}}{\partial y}dz + \int_{-h/2}^{+h/2} z\frac{\partial \sigma_{zx}}{\partial z}dz = 0 \\ \rightarrow\quad &\frac{\partial}{\partial x}\left[\int_{-h/2}^{+h/2} z\sigma_{xx}dz\right] + \frac{\partial}{\partial y}\left[\int_{-h/2}^{+h/2} z\sigma_{yx}dz\right] + \left[z\sigma_{zx}\right]_{-h/2}^{+h/2} - \int_{-h/2}^{+h/2}\sigma_{zx}dz = 0\end{aligned} \qquad (6.4.7)$$

and, since the shear stress σ_{zx} must be zero over the top and bottom surfaces, one has Eqn. 6.4.5a. Applying a similar procedure to the second equilibrium equation gives Eqn.

6.4.5b. Finally, integrating directly the third equilibrium equation without multiplying across by z, one arrives at Eqn. 6.4.3.

Now, eliminating the shear forces from 6.4.3, 6.4.5 leads to the differential equation

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \tag{6.4.8}$$

This equation is analogous to the equation $\partial^2 M / \partial x^2 = p$ in the beam theory. Finally, substituting in the moment-curvature equations 6.2.31 leads to[1]

$$\boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{q}{D}} \tag{6.4.9}$$

This is sometimes called **the equation of Sophie Germain** after the French investigator who first obtained it in 1815[2]. This partial differential equation is solved subject to the boundary conditions of the problem, i.e. the fixing conditions of the plate (see below). Again, when once an expression for $\omega(x, y)$ is obtained, the strains, stresses, forces and moments follow.

Note that the differential equation 6.4.8 with $q = 0$ is trivially satisfied in the simple pure bending and torsion problems considered earlier.

Eqn. 6.4.9 can be succinctly expressed as

$$\nabla^4 w = -\frac{q}{D} \tag{6.4.10}$$

where ∇^2 is the **Laplacian**, or "del" operator:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{6.4.11}$$

Note that the Laplacian operator (on w) gives the sum of the curvatures in two perpendicular directions and so it is independent of the directions chosen (see Eqn. 6.2.14).

Shear Forces in terms of Deflection

From 6.4.5 and the moment-curvature equations, one has the useful relations

---

[1] note that the moment curvature relations were derived for the case of pure bending; here, as in the beam theory, the possible effect of the shearing forces on the curvature is neglected. This is a valid assumption provided the thickness of the plate is small in comparison with its other dimensions. A more exact theory taking into account the effect of the shear forces on deflection can be developed

[2] Germain submitted her work to the French Academy, which was awarding a prize for anyone who could solve the problem of the vibration of plates; Lagrange was on the Academy awarding committee and corrected some of her work, deriving Eqn. 6.4.9 in its final form

$$V_x = D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \qquad V_y = D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{6.4.12}$$

6.4.2 Stresses in the Plate

The normal and in-plane shear stresses have been expressed in terms of the moments, Eqns. 6.2.33. Note that these stresses are zero over the mid-surface and attain a maximum at the outer surfaces.

Expressions for the remaining stress components can be obtained from the equations of equilibrium as follows: the first of Eqns. 6.4.6 leads, with 6.4.5a, to

$$\begin{aligned} 0 &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \\ &= \frac{\partial}{\partial x}\left[-\frac{12z}{h^3}M_x\right] + \frac{\partial}{\partial y}\left[\frac{12z}{h^3}M_{xy}\right] + \frac{\partial \sigma_{zx}}{\partial z} \\ &= -\frac{12z}{h^3}\left(\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y}\right) + \frac{\partial \sigma_{zx}}{\partial z} \\ &= -\frac{12z}{h^3}V_x + \frac{\partial \sigma_{zx}}{\partial z} \end{aligned} \tag{6.4.13}$$

Integrating now gives (note that V_x is independent of z)

$$\sigma_{zx} = \frac{6}{h^3}V_x z^2 + C \tag{6.4.14}$$

This shear stress must be zero at the upper and lower (free –) surfaces, at $z = \pm h/2$. This condition can be used to determine the arbitrary constant C and one finds that (see Fig. 6.1.9)

$$\sigma_{zx} = -\frac{3V_x}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \tag{6.4.15}$$

The other shear stress, σ_{zy}, can be evaluated in a similar manner: {▲Problem 1}

$$\sigma_{zy} = -\frac{3V_y}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \tag{6.4.16}$$

In some analyses, these shear stresses are taken to be zero, although they can be quite significant.

The only remaining stress component is σ_{zz}. This will never exceed the intensity of the external load on the plate; the lateral load itself, however, is negligibly small in comparison with the in-plane stresses set up by the bending of the plate, and for this reason it is acceptable to disregard σ_{zz}, as has been done, in the plate theory.

6.4.3 Problems

1. Derive the expression for shear stress 6.4.16.

6.5 Plate Problems in Rectangular Coordinates

In this section, a number of important plate problems will be examined using Cartesian coordinates.

6.5.1 Uniform Pressure producing Bending in One Direction

Consider first the case of a plate which bends in one direction only. From 6.3.11 the deflection and moments are

$$w = f(x), \quad M_x(x) = D\frac{d^2 w}{dx^2}, \quad M_y(x) = -\nu D\frac{d^2 w}{dx^2} \tag{6.5.1}$$

The differential equation 6.4.9 reads

$$\frac{d^4 w}{dx^4} = -\frac{q(x)}{D} \tag{6.5.2}$$

The corresponding equation for a beam is $d^4 w / dx^4 = p(x)/EI$. If $p(x)/b = -q(x)$, with b the depth of the beam, with $I = bh^3/12$, the plate will respond more stiffly than the beam by a factor of $1/(1-\nu^2)$, a factor of about 10% for $\nu = 0.3$, since

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)} = \frac{1}{1-\nu^2}\frac{EI}{b} \tag{6.5.3}$$

The extra stiffness is due to the constraining effect of M_y, which is not present in the beam.

6.5.2 Deflection of a Circular Plate by a Uniform Lateral Load

A solution for a circular plate problem is presented next. This problem will be examined again in the section which follows using the more natural polar coordinates.

Consider a circular plate with boundary

$$x^2 + y^2 = a^2, \tag{6.5.4}$$

clamped at its edges and subjected to a uniform lateral load q, Fig. 6.5.1.

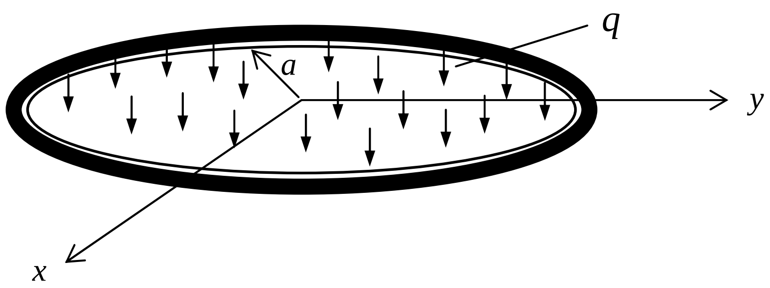


Figure 6.5.1: a clamped circular plate subjected to a uniform lateral load

The differential equation for the problem is given by 6.4.9. The boundary conditions are that the slope and deflection are zero at the boundary:

$$w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad \frac{\partial w}{\partial y} = 0 \qquad \text{along} \quad x^2 + y^2 = a^2 \tag{6.5.5}$$

It will be shown that the deflection

$$w = c(x^2 + y^2 - a^2)^2 \tag{6.5.6}$$

is a solution to the problem. First, this function certainly satisfies 6.5.5. Further, letting

$$f(x, y) = x^2 + y^2 - a^2, \tag{6.5.7}$$

the relevant partial derivatives are

$$\begin{aligned}
&\frac{\partial w}{\partial x} = 4cxf, \quad \frac{\partial w}{\partial y} = 4cyf \\
&\frac{\partial^2 w}{\partial x^2} = 4c\left(2x^2 + f\right), \quad \frac{\partial^2 w}{\partial x \partial y} = 8cxy, \quad \frac{\partial^2 w}{\partial y^2} = 4c\left(2y^2 + f\right) \\
&\frac{\partial^3 w}{\partial x^3} = 24cx, \quad \frac{\partial^3 w}{\partial x^2 \partial y} = 8cy, \quad \frac{\partial^3 w}{\partial x \partial y^2} = 8cx, \quad \frac{\partial^3 w}{\partial y^3} = 24cy \\
&\frac{\partial^4 w}{\partial x^4} = 24c, \quad \frac{\partial^4 w}{\partial x^2 \partial y^2} = 8c, \quad \frac{\partial^4 w}{\partial y^4} = 24c
\end{aligned} \tag{6.5.8}$$

Substituting these into the differential equation now yields

$$c = -\frac{q}{64D} \tag{6.5.9}$$

so the deflection is

$$w = -\frac{q}{64D}(x^2 + y^2 - a^2)^2 \tag{6.5.10}$$

This is plotted in Fig. 6.5.2. The maximum deflection occurs at the plate centre, where

$$w_{\max} = -\frac{qa^4}{64D}. \tag{6.5.11}$$

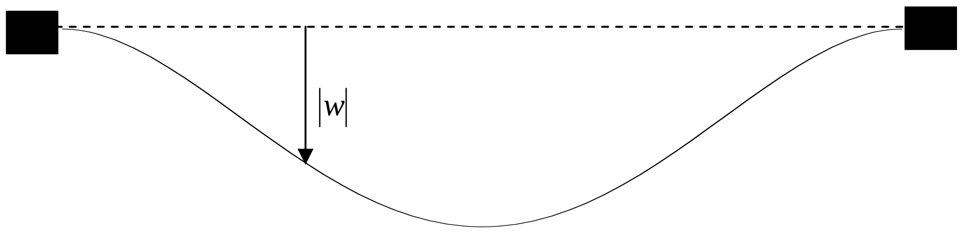


Figure 6.5.2: mid-plane deflection of the clamped circular plate

The curvature $\partial^2 w / \partial x^2$ along a radial line $y = 0$ is displayed in Fig. 6.5.3. The curvature is positive toward the centre of the plate (the plate curves upward) and is negative towards the edge of the plate (the plate curves downward).

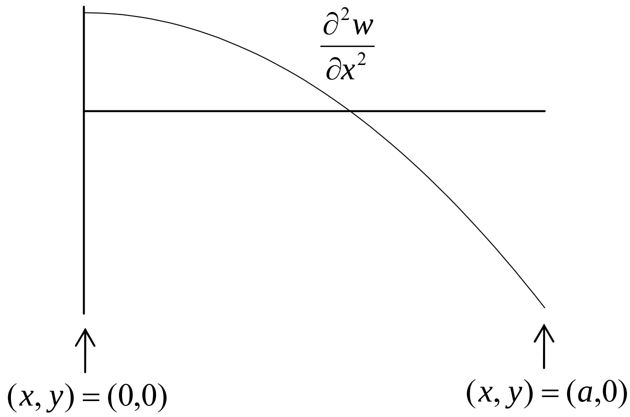


Figure 6.5.3: curvature in the clamped circular plate

The moments occurring in the plate are, from the moment-curvature equations 6.2.31 and 6.5.8,

$$\begin{aligned} M_x &= -\frac{q}{16}\left[(3+\nu)x^2 + (3\nu+1)y^2 - (1+\nu)a^2\right] \\ M_y &= -\frac{q}{16}\left[(3\nu+1)x^2 + (3+\nu)y^2 - (1+\nu)a^2\right] \\ M_{xy} &= +\frac{q}{8}(1-\nu)xy \end{aligned} \tag{6.5.12}$$

The moment M_x along a radial line $y = 0$ is of the same character as the curvature displayed in Fig. 6.5.3.

The out-of-plane shear forces are, from 6.4.5,

$$V_x = -\frac{qx}{2}, \qquad V_y = -\frac{qy}{2} \tag{6.5.13}$$

At the plate centre, the expressions become

$$M_x = M_y = \frac{q}{16}(1+\nu)a^2, \qquad M_{xy} = V_x = V_y = 0 \tag{6.5.14}$$

Stresses in the Plate

From 6.5.12-13 and 6.2.33, 6.4.15-16, the stresses in the plate are

$$\begin{aligned}
\sigma_{xx} &= \frac{3qz}{4h^3}\left[(3+\nu)x^2 + (3\nu+1)y^2 - (1+\nu)a^2\right] \\
\sigma_{yy} &= \frac{3qz}{4h^3}\left[(3\nu+1)x^2 + (3+\nu)y^2 - (1+\nu)a^2\right] \\
\sigma_{xy} &= \frac{3qz}{2h^3}(1-\nu)xy \\
\sigma_{zx} &= \frac{3qx}{4h}\left[1-\left(\frac{z}{h/2}\right)^2\right] \\
\sigma_{zy} &= \frac{3qy}{4h}\left[1-\left(\frac{z}{h/2}\right)^2\right]
\end{aligned} \tag{6.5.15}$$

Converting to polar coordinates (r,θ) through

$$x = r\cos\theta,\ y = r\sin\theta \tag{6.5.16}$$

and using a stress transformation,

$$\begin{aligned}
\sigma_{rr} &= \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} + \sin 2\theta\sigma_{xy} \\
\sigma_{\theta\theta} &= \cos^2\theta\sigma_{xx} + \sin^2\theta\sigma_{yy} - \sin 2\theta\sigma_{xy} \\
\sigma_{r\theta} &= \cos\theta\sin\theta\left(\sigma_{yy} - \sigma_{xx}\right) + \cos 2\theta\sigma_{xy}
\end{aligned} \tag{6.5.17}$$

leads to the axisymmetric stress field {▲Problem 1}

$$\begin{aligned}
\sigma_{rr} &= \frac{3qz}{4h^3}\left[(3+\nu)r^2 - (1+\nu)a^2\right] \\
\sigma_{\theta\theta} &= \frac{3qz}{4h^3}\left[(3\nu+1)r^2 - (1+\nu)a^2\right] \\
\sigma_{r\theta} &= 0
\end{aligned} \tag{6.5.18}$$

At the plate centre,

$$\sigma_{rr} = \sigma_{\theta\theta} = -\frac{3qza^2}{4h^3}(1+\nu) \tag{6.5.19}$$

At the plate edge $r = a$,

$$\sigma_{rr} = \frac{3qza^2}{2h^3}, \qquad \sigma_{\theta\theta} = \frac{3qza^2}{2h^3}\nu \tag{6.5.20}$$

For the shear stress, the traction acting on a surface parallel to the $x-y$ plane can be expressed as (see Fig. 6.5.4)

$$\begin{aligned} \mathbf{t} &= \sigma_{zr}\mathbf{e}_r + \sigma_{z\theta}\mathbf{e}_\theta \\ &= \sigma_{zx}\mathbf{e}_x + \sigma_{zy}\mathbf{e}_y \\ &= \sigma_{zx}(\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta) + \sigma_{zy}(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta) \end{aligned} \tag{6.5.21}$$

where $\mathbf{e}_i$ is a unit vector in the direction *i*. Thus

$$\begin{aligned} \sigma_{zr} &= \cos\theta\sigma_{zx} + \sin\theta\sigma_{zy} = \frac{3qr}{4h}\left[1-\left(\frac{z}{h/2}\right)^2\right] \\ \sigma_{z\theta} &= -\sin\theta\sigma_{zx} + \cos\theta\sigma_{zy} = 0 \end{aligned} \tag{6.5.22}$$

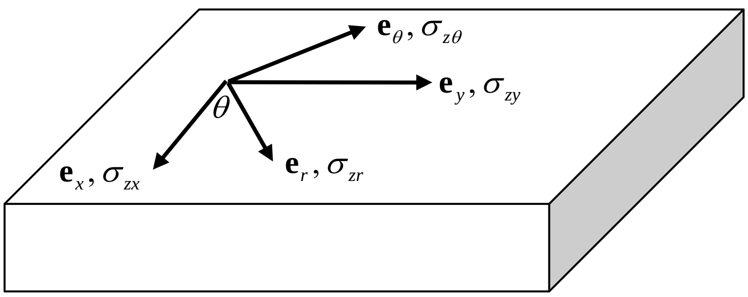


Figure 6.5.4: stress components acting on a surface

Note that the maximum stress in the plate is

$$\sigma_{\max} = \sigma_{rr}(a, h/2) = \frac{3q}{4}\left(\frac{a}{h}\right)^2 \tag{6.5.23}$$

The maximum shear stress, on the other hand, is $\sigma_{zr}(a,0) = 3q/4 \times (a/h)$. Thus the shear stress is of an order h/a smaller than the normal stress.

6.5.3 An Infinite Plate with Sinusoidal Deflection

Consider next the classic plate problem addressed by Navier in 1820. It consists of an infinite plate with an undulating "up/down" sinusoidal deflection, Fig. 6.5.5,

$$w(x,y) = w_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{6.5.24}$$

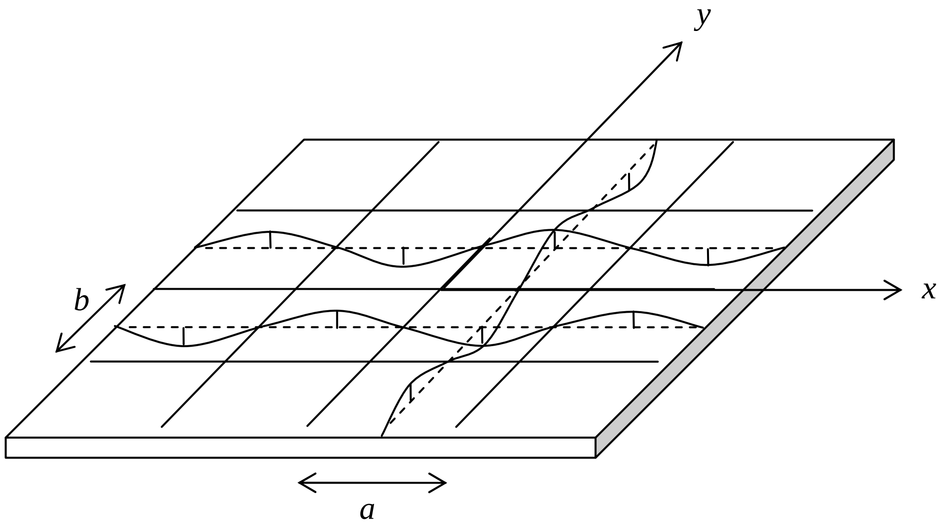


Figure 6.5.5: A plate with sinusoidal deflection

Differentiation of the deflection leads to the curvatures

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2} &= -w_0 \frac{\pi^2}{a^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
\frac{\partial^2 w}{\partial y^2} &= -w_0 \frac{\pi^2}{b^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
\frac{\partial^2 w}{\partial xy} &= w_0 \frac{\pi^2}{ab}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}
\end{aligned} \tag{6.5.25}$$

and hence the pressure

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w}{\partial xy} + \frac{\partial^2 w}{\partial y^2} = \pi^4\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2 w(x,y) \equiv -\frac{q(x,y)}{D} \tag{6.5.26}$$

The pressure thus varies like the deflection. There is no need for supports for the plate since the "up" loads balance the "down" loads.

From the moment-curvature relations,

$$
\begin{aligned}
M_x &= -w_0 D\pi^2\left(\frac{1}{a^2}+\frac{\nu}{b^2}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
M_y &= -w_0 D\pi^2\left(\frac{\nu}{a^2}+\frac{1}{b^2}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
M_{xy} &= -w_0 D(1-\nu)\frac{\pi^2}{ab}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}
\end{aligned}
\tag{6.5.27}
$$

and, from 6.4.12, the shear forces are

$$
\begin{aligned}
V_x &= -w_0 D\pi^3\frac{1}{a}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} \\
V_y &= -w_0 D\pi^3\frac{1}{b}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}
\end{aligned}
\tag{6.5.28}
$$

Note that both q/w and M_x/M_y are constant throughout the plate.

6.5.4 A Simply Supported Plate with Sinusoidal Deflection

Following on from the previous example, consider now a *finite* plate of dimensions a and b with the same sinusoidal deflection 6.5.24, simply supported along the edges $x=0$, $x=a$, $y=0$, $y=b$. In what follows, take w_0 in 6.5.24 to be negative, so that the plate is pushed down towards the centre.

According to 6.5.24 and 6.5.27, the deflection and slope is zero along the supported edges, as required. The vertical reactions at the supports are given by 6.5.28. However, according to Eqn. 6.5.27c, there are varying non-zero twisting moments over the ends of the plate. Thus the solution given by 6.5.24-28 is not quite the solution to the simply supported finite-plate problem, unless one can somehow apply the exact required twisting moments over the edges of the plate.

It turns out, however, that the solution 6.5.24-28 is a correct solution, except in a region close to the edges of the plate. This is explained in what follows.

Twisting Moments over "Free" Surfaces

Consider an element of material of width dy, Fig. 6.5.6. The element is subjected to a twisting moment $M_{xy}dy$, Fig. 6.5.6a. This twisting moment is due to shear stresses acting parallel to the plate surface (see Fig. 6.1.8). This system of horizontal forces can be replaced by the statically equivalent system of vertical forces shown in Fig. 6.5.6b – two *forces* of magnitude M_{xy} separate by a distance dy. Recalling Saint-Venant's principle, the difference between the statically equivalent systems of forces of Fig. 6.5.6a and 6.5.6b will lead to differences in the stress field within the plate only in a small region very close to the plate-edges.

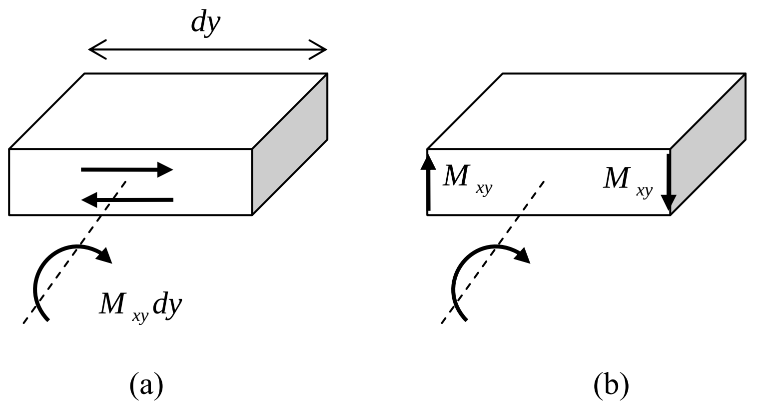


Figure 6.5.6: Equivalent systems of forces leading to the same twisting moment; (a) horizontal forces, (b) vertical forces

Consider next a distribution of twisting moment along the plate edge, Fig. 6.5.7. As can be seen, this distribution is equivalent to a distribution of shearing forces (per unit length) of magnitude

$$\overline{V}_x(y) = -\frac{\partial M_{xy}}{\partial y} \tag{6.5.29}$$

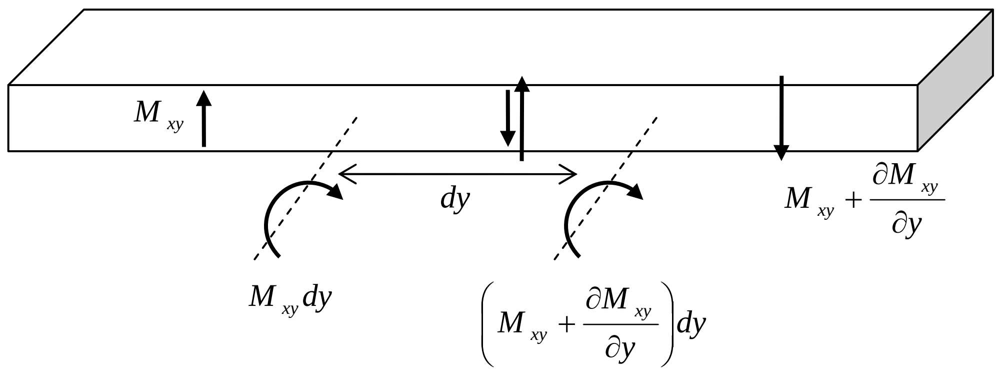


Figure 6.5.7: A distribution of twisting moments along a plate edge

The total vertical reaction along the edges can now be taken to be

$$V_x - \frac{\partial M_{xy}}{\partial y} \tag{6.5.30}$$

(and $V_y - \partial M_{xy} / \partial x$ along the other edges) and this gives a correct solution to the problem. From 6.5.27-28, these reactions are

$$
\begin{aligned}
F_{x0} &= \left(V_x - \frac{\partial M_{xy}}{\partial y} \right)_{(0,y)} = -w_0 D\pi^3 \frac{1}{a}\left[\left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \frac{1-\nu}{b^2} \right] \sin\frac{\pi y}{b} \\
F_{xa} &= \left(V_x - \frac{\partial M_{xy}}{\partial y} \right)_{(a,y)} = +w_0 D\pi^3 \frac{1}{a}\left[\left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \frac{1-\nu}{b^2} \right] \sin\frac{\pi y}{b} \\
F_{y0} &= \left(V_y - \frac{\partial M_{xy}}{\partial x} \right)_{(x,0)} = -w_0 D\pi^3 \frac{1}{b}\left[\left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \frac{1-\nu}{a^2} \right] \sin\frac{\pi x}{a} \\
F_{yb} &= \left(V_y - \frac{\partial M_{xy}}{\partial x} \right)_{(x,b)} = +w_0 D\pi^3 \frac{1}{b}\left[\left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \frac{1-\nu}{a^2} \right] \sin\frac{\pi x}{a}
\end{aligned}
\tag{6.5.31}
$$

Corner Forces

Integrating 6.5.31 over the four edges, the resultant *upward* forces on the four edges (with $w_0 < 0$, they are all four upward) are

$$
\begin{aligned}
+\overline{F}_{x0} &= -\overline{F}_{xa} = -2w_0 D\pi^2 \frac{b}{a}\left[\left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \frac{1-\nu}{b^2} \right] \\
+\overline{F}_{y0} &= -\overline{F}_{yb} = -2w_0 D\pi^2 \frac{a}{b}\left[\left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \frac{1-\nu}{a^2} \right]
\end{aligned}
\tag{6.5.32}
$$

and the resultant of these may be expressed as

$$
F_{\text{up}} = -4w_0 D\pi^2 ab\left[\left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2 + \frac{2(1-\nu)}{a^2 b^2} \right] \tag{6.5.33}
$$

The resultant downward force is, using 6.5.26,

$$
\begin{aligned}
F_{\text{down}} &= \int_0^b \int_0^a q(x,y)dxdy = -w_0 D\pi^4 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2 \int_0^b \int_0^a \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} dxdy \\
&= -4w_0 D\pi^2 ab\left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2
\end{aligned}
\tag{6.5.34}
$$

The difference between F_{up} and F_{down} is due to the re-distributed twisting moment, and is explained a follows: consider again Fig. 6.5.7, where the edge twisting moments have been replaced with a statically equivalent distribution of shear forces. It can be seen that there results shear forces at the ends of the plate-edge (the "corners"), where the shear forces M_{xy} have no neighbouring shear force of opposite sign with which to "cancel out". There are concentrated forces (per unit length) at the plate-corners of magnitude M_{xy}. Examining Fig. 6.5.7, which shows the edge $x = a$, the force $M_{xy}(a,0)$ is positive up whereas the force $M_{xy}(a,b)$ is positive down. There are also contributions to the corner forces at $(a,0)$ and (a,b) from the adjacent edges, shown in Fig. 6.5.8. One finds that the *downward* concentrated forces at the corner are

$$\begin{aligned}
P_{00} &= +2M_{xy}(0,0) = -2w_0 D(1-\nu)\frac{\pi^2}{ab} \\
P_{a0} &= -2M_{xy}(a,0) = -2w_0 D(1-\nu)\frac{\pi^2}{ab} \\
P_{ab} &= +2M_{xy}(a,b) = -2w_0 D(1-\nu)\frac{\pi^2}{ab} \\
P_{0b} &= -2M_{xy}(0,b) = -2w_0 D(1-\nu)\frac{\pi^2}{ab}
\end{aligned} \tag{6.5.35}$$

Adding these to F_{down} of Eqn. 6.5.34 now gives the F_{up} of Eqn. 6.5.33.

Physically, if one applies a pressure to a simply supported plate, the plate will tend to rise at the four corners, in a twisting action. The corner forces 6.5.35 are necessary to keep the corners down and so produce the deflection 6.5.24.

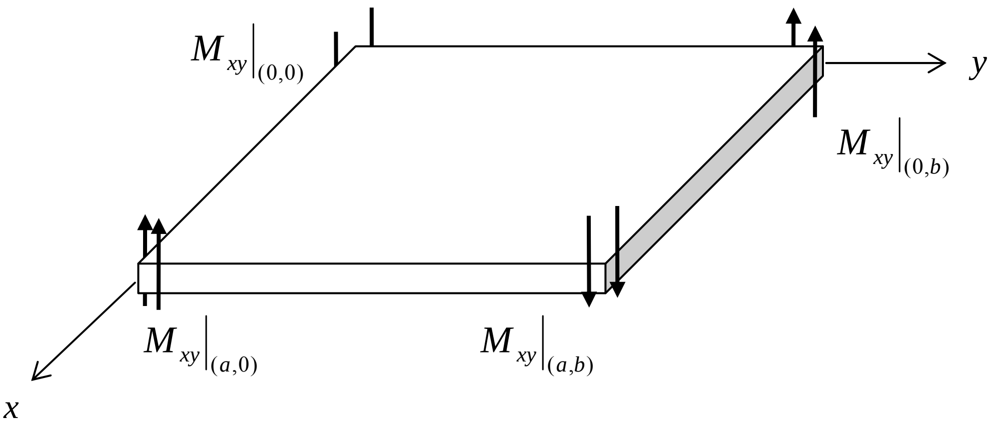


Figure 6.5.8: corner forces in the simply supported plate

The ratio of the resultant downward corner force to the downward force due to the applied pressure, F_{down}, is

$$2(1-\nu)\frac{a^2b^2}{\left(a^2+b^2\right)^2} \tag{6.5.36}$$

For a square plate, this is $(1-\nu)/2$; with $\nu = 0.3$, this is 35%.

6.5.5 A Rectangular Plate Simply Supported at the Edges

The above solution can be used to solve the problem of a simply supported plate loaded by any arbitrary pressure distribution, through the use of Fourier series.

Consider again this plate, whose displacement boundary conditions are

$$w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0$$
$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{(0,y)} = \left.\frac{\partial^2 w}{\partial x^2}\right|_{(a,y)} = 0, \qquad \left.\frac{\partial^2 w}{\partial y^2}\right|_{(x,0)} = \left.\frac{\partial^2 w}{\partial y^2}\right|_{(x,b)} = 0 \tag{6.5.37}$$

Assume the deflection to be of the form

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{6.5.38}$$

with A_{mn} coefficients to be determined. It can be seen that this function satisfies the boundary conditions. Taking the derivatives of this function,

$$\frac{\partial^2 w}{\partial x^2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(-\frac{m^2\pi^2}{a^2}\right) A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{6.5.39}$$

etc., and substituting into the differential equation 6.4.9, gives

$$\pi^4 D \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = -q(x,y) \tag{6.5.40}$$

This can be written compactly in the form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = -q(x,y) \tag{6.5.41}$$

where

$$C_{mn} = \pi^4 D A_{mn}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \tag{6.5.42}$$

It remains to choose the coefficients of the series so as to satisfy the equation identically over the whole area of the plate.

One can evaluate the coefficients as one does for ordinary Fourier series, although here one has a double series and so one proceeds as follows: first, multiply both sides of (6.5.40) by $\sin(k\pi y/b)$ where k is an integer, and integrate over y between the limits $[0,b]$, so that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \sin\frac{m\pi x}{a}\int_0^b \sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}dy = -\int_0^b q(x,y)\sin\frac{k\pi y}{b}dy \tag{6.5.43}$$

Using the orthogonality condition

$$\int_0^b \sin\frac{n\pi y}{b}\sin\frac{k\pi y}{b}dy = \begin{cases} 0, & n \neq k \\ b/2, & n = k \end{cases}, \tag{6.5.44}$$

leads to

$$\frac{b}{2}\sum_{m=1}^{\infty} C_{mk}\sin\frac{m\pi x}{a} = -\int_0^b q(x,y)\sin\frac{k\pi y}{b}dy \tag{6.5.45}$$

Now there are functions of *x* only so, multiplying both sides by $\sin(j\pi x/a)$ and following the same procedure, one has

$$\frac{a}{2}\frac{b}{2}C_{jk} = -\int_0^a \left[\int_0^b q(x,y)\sin\frac{k\pi y}{b}dy\right]\sin\frac{j\pi x}{a}dx \tag{6.5.46}$$

and hence the coefficients C_{mn} are (replacing the dummy subscripts j, k with m, n)

$$C_{mn} = -\frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \tag{6.5.47}$$

Thus the coefficients A_{mn} of the original expression for the deflection $w(x,y)$, 6.5.38, are

$$A_{mn} = -\frac{1}{\pi^4 D}\frac{4}{ab}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}\int_0^a\int_0^b q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \tag{6.5.48}$$

It is now possible to solve for the coefficients given any loading $q(x,y)$ over the plate, and hence evaluate the deflection, moments and stresses in the plate, by taking the derivatives of the infinite series for w.

This solution is due to Navier and is called **Navier's solution** to the rectangular plate problem. A similar solution method has been used by Lévy to solve a more general problem – that of a rectangular plate simply supported on two opposite sides, and any one of the conditions free, simply-supported, or clamped, along the other two opposite sides. For example, considering a square plate, this involves using a trial function for the deflection of the form (compare with 6.5.38)

$$w(x,y) = \sum_{n=1}^{\infty} F_n(y)\sin\frac{n\pi x}{a} \tag{6.5.49}$$

and then attempting to determine the functions $F_n(y)$.

A Uniform Load

In the case of a uniform load $q(x,y) = q$, one has

$$
\begin{aligned}
A_{mn} &= -\frac{4q}{\pi^4 Dab}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}\int_0^a \sin\frac{m\pi x}{a}dx\int_0^b \sin\frac{n\pi y}{b}dy \\
&= -\frac{4q}{\pi^4 Dab}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}\left[\frac{a}{m\pi}\left(1-\cos(m\pi)\right)\right]\left[\frac{b}{n\pi}\left(1-\cos(n\pi)\right)\right] \\
&= \begin{cases} -\dfrac{16q}{\pi^6 Dmn}\left(\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}\right)^{-2} & (m, n = 1, 3, 5, \cdots) \\ 0 & (m, n = 0, 2, 4, \cdots) \end{cases}
\end{aligned}
\tag{6.5.50}
$$

The resulting series in 6.5.50 converges rapidly.

The deflection at the centre of the plate is then

$$
\begin{aligned}
w &= \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} A_{mn}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2} \\
&= -\frac{16qb^4}{\pi^6 D}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{mn}\left(\frac{m^2}{(a/b)^2}+n^2\right)^{-2}(-1)^{(m+n)/2-1}
\end{aligned}
\tag{6.5.51}
$$

For a square plate,

$$
\begin{aligned}
w &= -\frac{16qa^4}{\pi^6 D}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{mn}\left(m^2+n^2\right)^{-2}(-1)^{(m+n)/2-1} \\
&= -\frac{qa^4}{D}\times 0.0040624
\end{aligned}
\tag{6.5.52}
$$

Denoting the area a^2 by A, this is $\omega = -0.0041qA^2/D$. This can be compared with the clamped circular plate; denoting the area there, πa^2, by A, the maximum deflection, Eqn. 6.5.11, gives $\omega = -0.0016qA^2/D$.

Corner Forces

The twisting moment is

$$
M_{xy}(x,y) = -D(1-\nu)\frac{\partial^2\omega}{\partial x\partial y} = -D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}
\tag{6.5.53}
$$

and the four corner forces required to hold the plate down are now

$$
\begin{aligned}
P_{00} &= +2M_{xy}(0,0) = -2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn} \\
P_{a0} &= -2M_{xy}(a,0) = +2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos m\pi \\
P_{0b} &= -2M_{xy}(0,b) = +2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos n\pi \\
P_{ab} &= +2M_{xy}(a,b) = -2D(1-\nu)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mn\pi^2}{ab}\right)A_{mn}\cos m\pi\cos n\pi
\end{aligned}
\tag{6.5.54}
$$

For a uniform load over a square plate, using 6.5.50, the corner forces reduce to

$$
\begin{aligned}
4P &= 4\frac{32q(1-\nu)a^2}{\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{1}{\left(m^2+n^2\right)^2} \\
&= 4\frac{32q(1-\nu)a^2}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\left((2m-1)^2+(2n-1)^2\right)^2} \\
&\approx 4\frac{32q(1-\nu)a^2}{\pi^4}\times 0.2825 \\
&\approx 0.26F_0
\end{aligned}
\tag{6.5.55}
$$

(for $\nu = 0.3$) where $F_0 = qa^2$ is the resultant applied force.

6.5.6 Problems

1. Derive the expressions for the stress components in polar form, for the clamped circular plate under uniform lateral load, Eqn. 6.5.18.

6.6 Plate Problems in Polar Coordinates

6.6.1 Plate Equations in Polar Coordinates

To examine directly plate problems in polar coordinates, one can first transform the Cartesian plate equations considered in the previous sections into ones in terms of polar coordinates.

First, the definitions of the moments and forces are now

$$M_r = -\int_{-h/2}^{+h/2} z\sigma_{rr}dz, \quad M_\theta = -\int_{-h/2}^{+h/2} z\sigma_{\theta\theta}dz, \quad M_{r\theta} = \int_{-h/2}^{+h/2} z\sigma_{r\theta}dz \tag{6.6.1}$$

and

$$V_r = -\int_{-h/2}^{+h/2} \sigma_{zr}dz, \qquad V_\theta = -\int_{-h/2}^{+h/2} \sigma_{z\theta}dz \tag{6.6.2}$$

The strain-curvature relations, Eqns. 6.2.27, can be transformed to polar coordinates using the transformations from Cartesian to polar coordinates detailed in §4.2 (in particular, §4.2.6). One finds that {▲Problem 1}

$$\begin{aligned}
\varepsilon_{rr} &= -z\frac{\partial^2 w}{\partial r^2} \\
\varepsilon_{\theta\theta} &= -z\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) \\
\varepsilon_{r\theta} &= -z\left(-\frac{1}{r^2}\frac{\partial w}{\partial \theta} + \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta}\right)
\end{aligned} \tag{6.6.3}$$

The moment-curvature relations 6.2.31 become {▲Problem 2}

$$\begin{aligned}
M_r &= D\left[\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \\
M_\theta &= D\left[\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + \nu\frac{\partial^2 w}{\partial r^2}\right] \\
M_{r\theta} &= -D(1-\nu)\left(-\frac{1}{r^2}\frac{\partial w}{\partial \theta} + \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta}\right)
\end{aligned} \tag{6.6.4}$$

The governing differential equation 6.4.9 now reads

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 w = -\frac{q}{D} \tag{6.6.5}$$

The shear forces in terms of deflection, Eqn 6.4.12, now read {▲Problem 3}

$$V_r = D\frac{\partial}{\partial r}\left[\frac{\partial^2 w}{\partial r^2} + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right], \quad V_\theta = D\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{\partial^2 w}{\partial r^2} + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \quad (6.6.6)$$

Finally, the stresses are {▲Problem 4}

$$\sigma_{rr} = -\frac{12z}{h^3}M_r, \quad \sigma_{\theta\theta} = -\frac{12z}{h^3}M_\theta, \quad \sigma_{r\theta} = \frac{12z}{h^3}M_{r\theta} \quad (6.6.7)$$

and

$$\sigma_{zr} = -\frac{3V_r}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right], \qquad \sigma_{z\theta} = -\frac{3V_\theta}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \quad (6.6.8)$$

The differential equation 6.6.5 can be solved using a method similar to the Airy stress function method for problems in polar coordinates (the Mitchell solution), that is, a solution is sought in the form of a Fourier series. Here, however, only axisymmetric problems will be considered in detail.

6.6.2 Plate Equations for Axisymmetric Problems

When the loading and geometry of the plate are axisymmetric, the plate equations given above reduce to

$$\begin{aligned} M_r &= D\left[\frac{d^2 w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr}\right] \\ M_\theta &= D\left[\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2 w}{dr^2}\right] \\ M_{r\theta} &= 0 \end{aligned} \quad (6.6.9)$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 w = \frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\} = -\frac{q(r)}{D} \quad (6.6.10)$$

$$V_r = D\frac{d}{dr}\left[\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right], \quad V_\theta = 0 \quad (6.6.11)$$

$$\sigma_{rr} = -\frac{12z}{h^3}M_r, \quad \sigma_{\theta\theta} = -\frac{12z}{h^3}M_\theta, \quad \sigma_{r\theta} = 0 \quad (6.6.12)$$

and

$$\sigma_{zr} = -\frac{3V_r}{2h}\left[1-\left(\frac{z}{h/2}\right)^2\right], \qquad \sigma_{z\theta} = 0 \tag{6.6.13}$$

Note that there is no twisting moment, so the problem of dealing with non-zero twisting moments on free boundaries seen with rectangular plate does not arise here.

6.6.3 Axisymmetric Plate Problems

For uniform q, direct integration of 6.6.10 leads to

$$w = -\frac{qr^4}{64D} + \frac{1}{4}\overline{A}r^2(\ln r - 1) + \frac{1}{4}\overline{B}r^2 + \overline{C}\ln r + \overline{D} \tag{6.6.14}$$

with

$$\begin{aligned}\frac{dw}{dr} &= -\frac{qr^3}{16D} + \frac{1}{4}\overline{A}r(2\ln r - 1) + \frac{1}{2}\overline{B}r + \overline{C}\frac{1}{r}\\ \frac{d^2w}{dr^2} &= -\frac{3qr^2}{16D} + \frac{1}{4}\overline{A}(2\ln r + 1) + \frac{1}{2}\overline{B} - \overline{C}\frac{1}{r^2}\\ \frac{d^3w}{dr^3} &= -\frac{3qr}{8D} + \frac{1}{2}\overline{A}\frac{1}{r} + 2\overline{C}\frac{1}{r^3}\end{aligned} \tag{6.6.15}$$

and

$$V_r = D\left[\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right] = -\frac{qr}{2} + \frac{\overline{A}}{r}D \tag{6.6.16}$$

There are two classes of problem to consider, plates with a central hole and plates with no hole. For a plate with no hole in it, the condition that the stresses remain finite at the plate centre requires that d^2w/dr^2 remains finite, so $\overline{A} = \overline{C} = 0$. Thus immediately one has $V_r = -qr/2$. The boundary conditions at the outer edge $r = a$ give $\overline{B}$ and $\overline{D}$.

1. Solid Plate – Uniform Bending

The simplest case is pure bending of a plate, $M_r = M_0$, with no transverse pressure, $q = 0$. The plate is solid so $\overline{A} = \overline{C} = 0$ and one has $w = \overline{B}r^2/4 + \overline{D}$. The applied moment is

$$M_0 = D\left[\frac{d^2\omega}{dr^2} + \nu\frac{1}{r}\frac{d\omega}{dr}\right] = \frac{1}{2}\overline{B}D[1+\nu] \tag{6.6.17}$$

so $\overline{B} = 2M_0/D(1+\nu)$. Taking the deflection to be zero at the plate-centre, the solution is

$$w = \frac{M_0}{2D(1+\nu)} r^2 \tag{6.6.18}$$

2. Solid Plate Clamped – Uniform Load

Consider next the case of clamped plate under uniform loading. The boundary conditions are that $w = dw/dr = 0$ at $r = a$, leading to

$$\overline{B} = \frac{qa^2}{8D}, \qquad \overline{D} = -\frac{qa^4}{64D} \tag{6.6.19}$$

and hence

$$w = -\frac{q}{64D}\left(r^2 - a^2\right)^2 \tag{6.6.20}$$

which is the same as 6.5.10.

The reaction force at the outer rim is $V_r(a) = -qa/2$. This is a force per unit length; the force acting on an element of the outer rim is $-qa(a\Delta\theta)/2$ and the total reaction force around the outer rim is $-qa^2\pi$, which balances the same applied force.

3. Solid Plate Simply Supported – Uniform Load

For a simply supported plate, $w = 0$ and $M_r = 0$ at $r = a$. Using 6.6.9a, one then has {▲Problem 5}

$$\overline{B} = \frac{3+\nu}{1+\nu}\frac{qa^2}{8D}, \qquad \overline{D} = -\frac{5+\nu}{1+\nu}\frac{qa^4}{64D} \tag{6.6.21}$$

and hence

$$w = -\frac{q}{64D}\left(\frac{5+\nu}{1+\nu}a^2 - r^2\right)\left(a^2 - r^2\right) \tag{6.6.22}$$

The deflection for the clamped and simply supported cases are plotted in Fig. 6.6.1 (for $\nu = 0.3$).

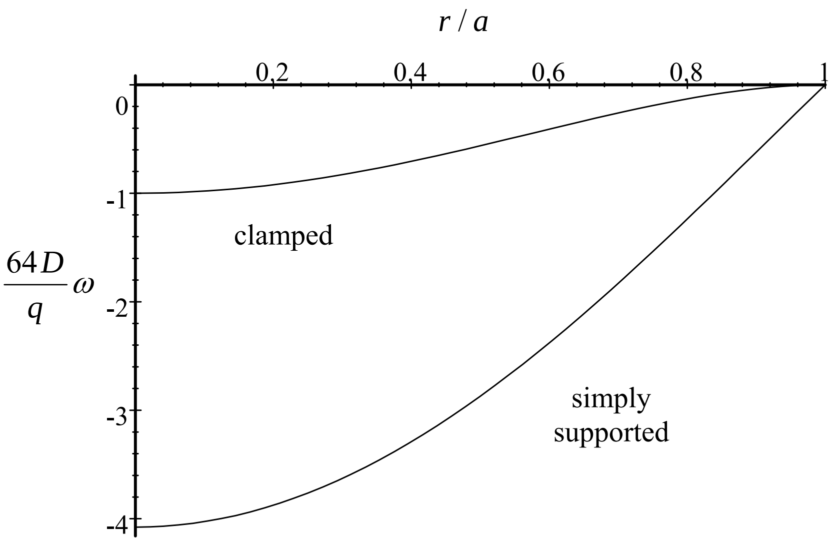


Figure 6.6.1: deflection for a circular plate under uniform loading

4. Solid Plate with a Central Concentrated Force

Consider now the case of a plate subjected to a single concentrated force F at $r = 0$. The resultant shear force acting on any cylindrical portion of the plate with radius r about the plate-centre is $2\pi r V_r(r)$. As $r \to 0$, one must have an infinite V_r so that this resultant is finite and equal to the applied force F. An infinite shear force implies infinite stresses. It is possible for the stresses at the centre of the plate to be infinite. However, although the stresses and strain might be infinite, the displacements, which are obtained from the strains through integration, can remain, and should remain, finite. Although the solution will be "unreal" at the plate-centre, one can again use Saint-Venant's principle to argue that the solution obtained will be valid everywhere except in a small region near where the force is applied.

Thus, seek a solution which has finite displacement in which case, by symmetry, the slope at $r = 0$ will be zero. From the general axisymmetric solution 6.6.15a,

$$\left.\frac{dw}{dr}\right|_{r=0} = \overline{C}\left.\frac{1}{r}\right|_{r=0} \tag{6.6.23}$$

so $\overline{C} = 0$.

From 6.6.16

$$2\pi r V_r\big|_{r=0} = 2\pi \overline{A} D \equiv F \tag{6.6.24}$$

Thus $\overline{A} = F / 2\pi D$ and the moments and shear force become infinite at the plate-centre.

The other two constants can be obtained from the boundary conditions. For a clamped plate, $w = dw/dr = 0$, and one finds that {▲Problem 7}

$$w = \frac{F}{16\pi D}\left[\left(a^2 - r^2\right) + 2r^2 \ln\left(r/a\right)\right] \tag{6.6.25}$$

This solution results in $\ln(r/a)$ terms in the expressions for moments, giving logarithmically infinite in-plane stresses at the plate-centre.

5. Plate with a Hole

For a plate with a hole in it, there will be four boundary conditions to determine the four constants in Eqn. 6.6.14. For example, for a plate which is simply supported around the outer edge $r = b$ and free on the inner surface $r = a$, one has

$$\begin{aligned} &M_r(a) = 0, \quad F_r(a) = 0 \\ &w(b) = 0, \quad\ \ M_r(b) = 0 \end{aligned} \tag{6.6.26}$$

6.6.4 Problems

1. Use the expressions 4.2.11-12, which relate second partial derivatives in the Cartesian and polar coordinate systems, together with the strain transformation relations 4.2.17, to derive the strain-curvature relations in polar coordinates, Eqn. 6.6.3.

2. Use the definitions of the moments, 6.6.1, and again relations 4.2.11-12, together with the stress transformation relations 4.2.18, to derive the moment-curvature relations in polar coordinates, Eqn. 6.6.4.

3. Derive Eqns. 6.6.6.

4. Use 6.2.33, 6.4.15-16 to derive the stresses in terms of moments and shear forces, Eqns. 6.6.7-8.

5. Solve the simply supported solid plate problem and hence derive the constants 6.6.21.

6. Show that the solution for a simply supported plate (with no hole), Eqn. 6.6.22, can be considered a superposition of the clamped solution, Eqn. 6.6.20, and a pure bending, by taking an appropriate deflection at the plate-centre in the pure bending case.

7. Solve for the deflection in the case of a clamped solid circular plate loaded by a single concentrated force, Eqn. 6.6.25.

6.7 In-Plane Forces and Plate Buckling

In the previous sections, only bending and twisting moments and out-of-plane shear forces were considered. In this section, in-plane forces are considered also. The in-plane forces will give rise to in-plane membrane strains, but here it is assumed that these are uncoupled from the bending strains. In other words, the membrane strains can be found from a separate plane stress analysis of the mid-surface and the bending of the plate does not affect these membrane strains. The possible effect of the in-plane forces on the bending strains is the main concern here.

6.7.1 Equilibrium for In-plane Forces

Start again with the equations of equilibrium, Eqns. 6.4.6. Integrating the first and second through the thickness of the plate (this time without multiplying first by z), and using the definitions of the in-plane forces 6.1.1-6.1.2, leads to

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0 \end{aligned} \tag{6.7.1}$$

6.7.2 The Governing Differential Equation

Consider an element of the deflected plate, Fig. 6.7.1. Only a deflection in the y direction, $\partial \omega / \partial y$, is considered for clarity. Resolving the components of the in-plane forces into horizontal and vertical components:

$$\begin{aligned} \sum F_H &= -N_y \Delta x + \left(N_y + \frac{\partial N_y}{\partial y} \Delta y \right) \Delta x - N_{xy} \Delta y + \left(N_{xy} + \frac{\partial N_{xy}}{\partial x} \Delta x \right) \Delta y \\ \sum F_V &= -N_y \frac{\partial w}{\partial y} \Delta x + \left(N_y \frac{\partial w}{\partial y} + \frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} \right) \Delta x \right) \Delta x \\ &\qquad\qquad - N_{xy} \frac{\partial w}{\partial y} \Delta y + \left(N_{xy} \frac{\partial w}{\partial y} + \frac{\partial}{\partial x} \left(N_{xy} \frac{\partial w}{\partial y} \right) \Delta x \right) \Delta y \end{aligned} \tag{6.7.2}$$

These reduce to

$$\sum F_H = \left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \Delta y \Delta x$$

$$\begin{aligned} \sum F_V &= \left[\frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial x}\left(N_{xy} \frac{\partial w}{\partial y} \right) \right] \Delta x \Delta y \\ &= \left[\left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} \right) \frac{\partial w}{\partial y} + \left(N_y \frac{\partial^2 w}{\partial y^2} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \right] \Delta x \Delta y \end{aligned} \qquad (6.7.3)$$

Using 6.7.1, one has

$$\sum F_H = 0, \qquad \sum F_V = \left(N_y \frac{\partial^2 w}{\partial y^2} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \Delta x \Delta y \qquad (6.7.4)$$

Considering also a deflection $\partial \omega / \partial x$, one has for the resultant vertical force :

$$\sum F_V = \left(N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) \Delta x \Delta y \qquad (6.7.5)$$

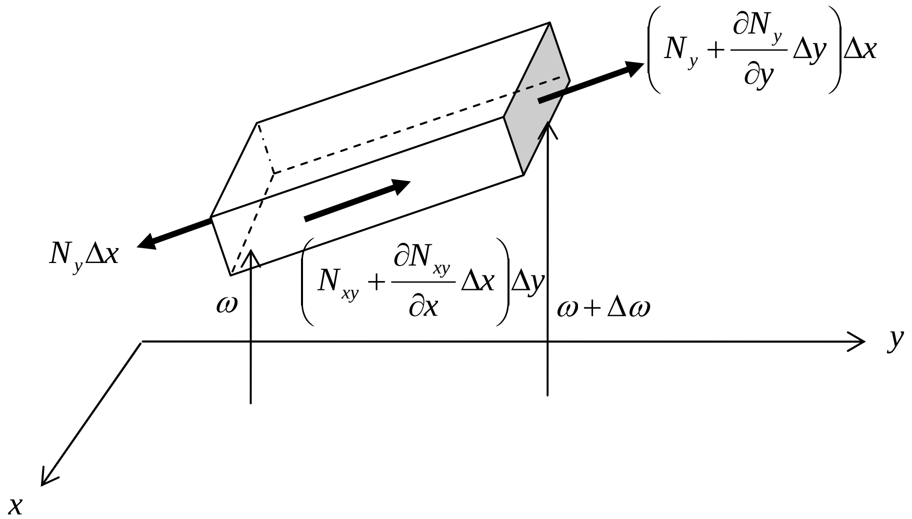


Figure 6.7.1: In-plane forces acting on a plate element

When the in-plane forces were neglected, the vertical stress resisted by bending and shear force was $\sigma_{zz} = -q$. Here, one has an additional stress given by 6.7.5, and so the governing differential equation 6.4.7 becomes

$$\frac{\partial^4 \omega}{\partial x^4} + 2\frac{\partial^4 \omega}{\partial x^2 \partial y^2} + \frac{\partial^4 \omega}{\partial y^4} = \frac{1}{D}\left(-q + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) \qquad (6.7.6)$$

6.7.3 Buckling of Plates

When compressive in-plane forces are applied to a plate, the plate will at first remain flat and simply be compressed. However, when the in-plane forces reach a critical level, the plate will bend and the deflection will be given by the solution to 6.7.6. For example, consider the case of a simply supported plate subjected to a uniform in-plane compression N_x only, Fig. 6.7.2, in which case 6.7.6 reduces to

$$\frac{\partial^4 \omega}{\partial x^4} + 2\frac{\partial^4 \omega}{\partial x^2 \partial y^2} + \frac{\partial^4 \omega}{\partial y^4} = \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2} \tag{6.7.7}$$

Following Navier's method from §6.5.5, assume a buckled shape

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{6.7.8}$$

so that 6.7.7 becomes

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + \frac{N_x}{D}\frac{m^2\pi^2}{a^2}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0 \tag{6.7.9}$$

Disregarding the trivial $A_{mn} = 0$, this can be satisfied by taking

$$N_x = -\frac{Da^2\pi^2}{m^2}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \tag{6.7.10}$$

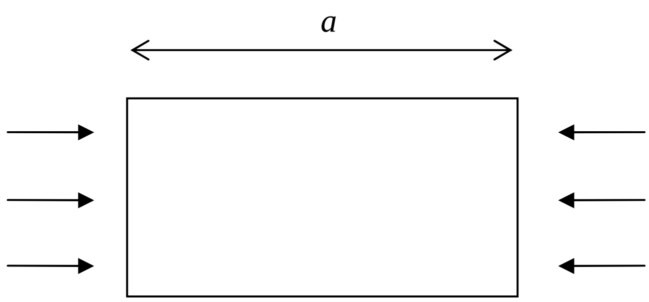


Figure 6.7.2: In-plane compression of a plate

The lowest in-plane force N_x which will deflect the plate is sought. Clearly, the smallest value on the right hand side of 6.7.10 will be when $n = 1$. This means that the buckling modes as given by 6.7.8 will be of the form

$$\sin\frac{m\pi x}{a}\sin\frac{\pi y}{b} \tag{6.7.11}$$

so that the plate will only ever buckle with one half-wave in the direction perpendicular to loading.

When $a \le b$, the smallest value occurs when $m = 1$, in which case the critical in-plane force is

$$\left(N_x\right)_{\mathrm{cr}} = -\frac{D\pi^2}{b^2}\left(\frac{b}{a} + \frac{a}{b}\right)^2 \tag{6.7.12}$$

When a/b is very small, the plate is loaded along the relatively long edges and the critical load is much higher than for a square plate.

The deflection (buckling mode) corresponding to this critical load is

$$w(x, y) = A_{11} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \tag{6.7.13}$$

Note that the amplitude A_{11} cannot be determined from the analysis[1].

As a/b increases above unity, the value of m at which the applied load is a minimum increases. When a/b reaches just over $\sqrt{2}$, the critical buckling load occurs for $m = 2$, for which

$$\left(N_x\right)_{cr} = -\frac{D\pi^2}{b^2}\left(\frac{2b}{a} + \frac{a}{2b}\right)^2 \tag{6.7.14}$$

and corresponding buckling more

$$w(x, y) = A_{21} \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b} \tag{6.7.15}$$

The plate now buckles in two half-waves, as if the centre-line were simply supported and there were two smaller separate plates buckling similarly.

As a/b increases further, so too does m. For a very long, thin, plate, $m \approx a/b$, and so the plate subdivides approximately into squares, each buckling in a half-wave.

[1] this is a consequence of assuming small deflections; it can be determined when the deflections are not assumed to be small

6.8 Plate Vibrations

In this section, the problem of a vibrating circular plate will be considered. Vibrating plates will be re-examined again in the next section, using a strain energy formulation.

6.8.1 Vibrations of a Clamped Circular Plate

When a plate vibrates with velocity $\partial w / \partial t$, the third equation of equilibrium, Eqn. 6.6.2c becomes the equation of motion

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \tag{6.8.1}$$

With this adjustment, the term q is replaced with $q + \rho h \partial^2 w / \partial t^2$ in the relevant equations; the acceleration term is treated as a transverse load of intensity $\rho h \partial^2 w / \partial t^2$.

Regarding the circular plate, one has from the axisymmetric governing equation 6.6.10 (with $q = 0$),

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \right)^2 w = -\frac{\rho h}{D} \frac{\partial^2 w}{\partial t^2} \tag{6.8.2}$$

Assume a solution of the form

$$w(r,t) = W(r)\cos(\omega t + \phi) \tag{6.8.3}$$

Substituting into 6.8.2 gives

$$\left[\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \right)^2 - k^4 \right] W = 0 \tag{6.8.4}$$

where

$$k^2 = \omega \sqrt{\frac{\rho h}{D}} \tag{6.8.5}$$

Eqn. 6.8.4 gives the two differential equations

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k^2 \right) W = 0, \qquad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - k^2 \right) W = 0 \tag{6.8.6}$$

The solution to these equations are

$$W = C_1 J_0(kr) + C_2 Y_0(kr), \qquad W = C_3 I_0(kr) + C_4 K_0(kr) \tag{6.8.7}$$

where J_0 and Y_0 are, respectively, the Bessel functions of order zero of the first kind and of the second kind; I_0 and K_0 are, respectively, the Modified Bessel functions of order zero of the first kind and of the second kind[1]. These functions are plotted in Fig. 6.8.1 below. For a solid plate with no hole at $r = 0$, one requires that $C_2 = C_4 = 0$, since Y_0 and K_0 become unbounded as $r \to 0$. The general solution is thus

$$W(r) = \bar{A} J_0(kr) + \bar{B} I_0(kr) \tag{6.8.8}$$

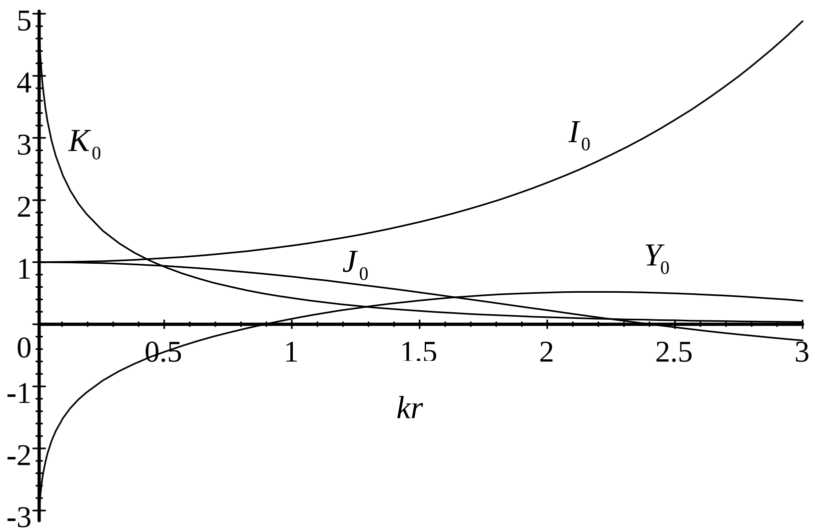


Figure 6.8.1: Bessel Functions

For a clamped plate, the boundary conditions give

$$\begin{aligned} W(a) &= \bar{A} J_0(ka) + \bar{B} I_0(ka) = 0 \\ \left.\frac{dW}{dr}\right|_{r=a} &= \bar{A} J_0'(ka) + \bar{B} I_0'(ka) = 0 \end{aligned} \tag{6.8.9}$$

where the dash means $J_0'(x) = dJ_0(x)/dx$ and $I_0'(x) = dI_0(x)/dx$. Using the relations

$$J_0'(x) = -J_1(x), \quad I_0'(x) = +I_1(x) \tag{6.8.10}$$

where J_1, I_1 are Bessel functions of order one, one has

$$\frac{J_0(ka)}{I_0(ka)} = -\frac{J_1(ka)}{I_1(ka)} \tag{6.8.11}$$

[1] by *definition*, these Bessel functions are the solution of the differential equations 6.8.6.

The roots ka give the frequencies of vibration of the plate. The function

$$J_0(ka)I_1(ka)+I_0(ka)J_1(ka) \tag{6.8.12}$$

is plotted in Fig. 6.8.2 below. The smallest root is found to be 3.1962. Eqn. 6.8.5 then gives for the frequency,

$$\omega = \alpha \frac{1}{a^2}\sqrt{\frac{D}{\rho h}} \tag{6.8.13}$$

where $\alpha = 10.2158$.

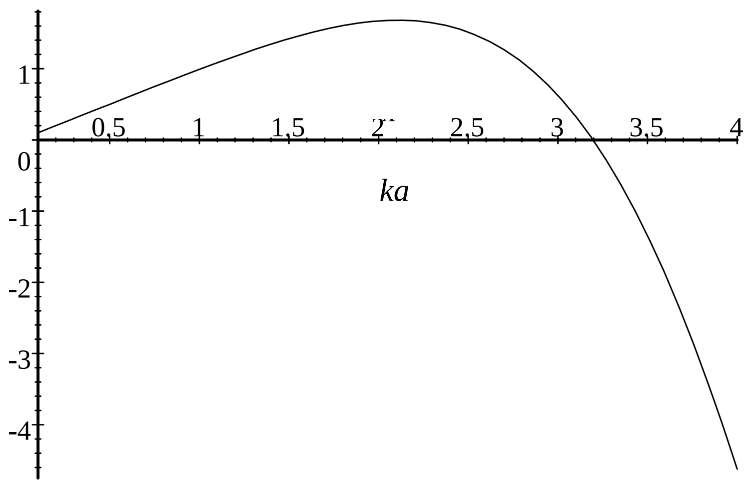


Figure 6.8.2: The Function 6.8.12

Further roots ka of 6.8.12 are given in Table 6.8.1. For each of these roots there is a corresponding frequency ω given by Eqn. 6.8.13, for which the value of α is also tabulated.

	ka	α	nodal circle
1	3.1962	10.2158	
2	6.3064	39.7711	0.3790
3	9.4395	89.1041	0.2548, 0.5833

Table 6.8.1: Roots of Eqn. 6.8.11, frequency factors and nodal circle roots

From 6.8.3, 6.8.8-9, the solution for the deflection is

$$w(r,t) = \overline{A}\left[J_0(kr) - \frac{J_0(ka)}{I_0(ka)}I_0(kr)\right]\cos(\omega t + \phi) \tag{6.8.14}$$

These are an infinite number of deflections, each one corresponding to a root ka. The actual deflection will be a superposition of these individual solutions.

The term inside the square brackets gives the mode shape of the plate during the vibration. The first three (normalized) mode shapes, corresponding to the first three roots, are shown in Fig. 6.8.3.

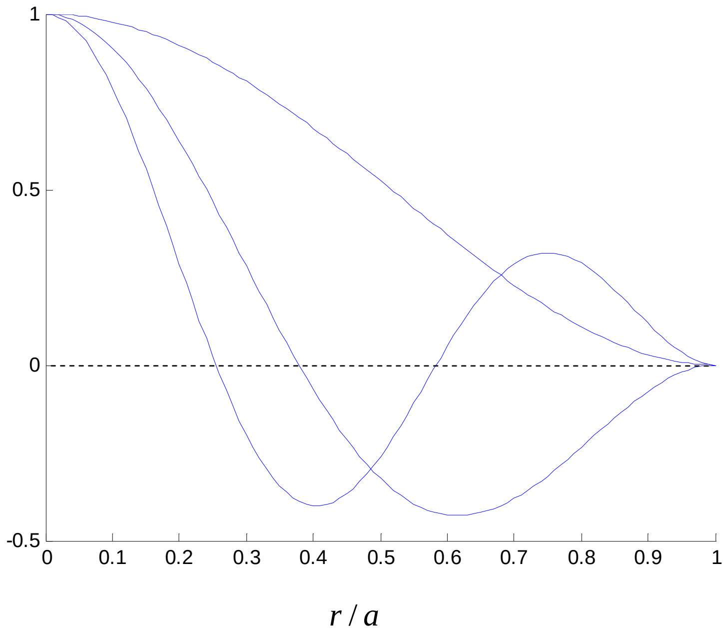


Figure 6.8.3: Mode shapes for the Clamped Circular Plate

The point r/a where these mode-shapes change sign are the positions of the so-called **nodal circles**. These roots of the mode shapes are given in the last column of Table 6.8.1

The General Problem

For circular plates not constrained to an axisymmetric response, one must use the more general differential equation 6.6.5

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 w=-\frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2} \tag{6.8.15}$$

This time, instead of 6.8.3, assume a solution of the form

$$w(r,\theta,t)=\sum W_n(r)\cos(n\theta)\sin(\omega t+\phi) \tag{6.8.16}$$

Then 6.8.4-6 become

$$\left[\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{d}{\partial r}-\frac{n^2}{r^2}\right)^2-k^4\right]W=0 \tag{6.8.17}$$

where k is again given by 6.8.5, and 6.8.6 becomes

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}+k^2\right)W=0, \qquad \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{n^2}{r^2}-k^2\right)W=0 \tag{6.8.18}$$

The solution to these equations are

$$W=C_1J_n(kr)+C_2Y_n(kr), \qquad W=C_3I_n(kr)+C_4K_n(kr) \tag{6.8.19}$$

where one now has Bessel functions of order n. Proceeding as before, one now needs to find roots of the equation

$$J_n(ka)I_{n+1}(ka)+I_n(ka)J_{n+1}(ka)=0 \tag{6.8.20}$$

and the deflection is

$$w(r,t)=\sum\overline{A}\left[J_n(kr)-\frac{J_n(ka)}{I_n(ka)}I_n(kr)\right]\cos(n\theta)\sin(\omega t+\phi) \tag{6.8.21}$$

The solution for $n=0$ has been given already. For other values of n, there are n so-called **nodal diameters**. For example, for $n=1$ there is one nodal diameter along $\theta=\pm\pi/2$, along which the deflection is zero. The roots of 6.8.20 for this case are given in Table 6.8.2, together with the nodal circle locations.

	ka	α	nodal circle
1	4.6109	21.2604	
2	7.7993	60.8287	0.4897
3	10.9581	120.0792	0.3497, 0.6390

Table 6.8.2: Roots of Eqn. 6.8.20 (*n*=1), frequency factors and nodal circle roots

The mode shapes for half the plate for this case of one nodal diameter are shown in Fig. 6.8.4, corresponding to the first two roots in Table 6.8.2. The frequencies corresponding to these solutions are again given by 6.8.13 with the frequency factor α given in the table.

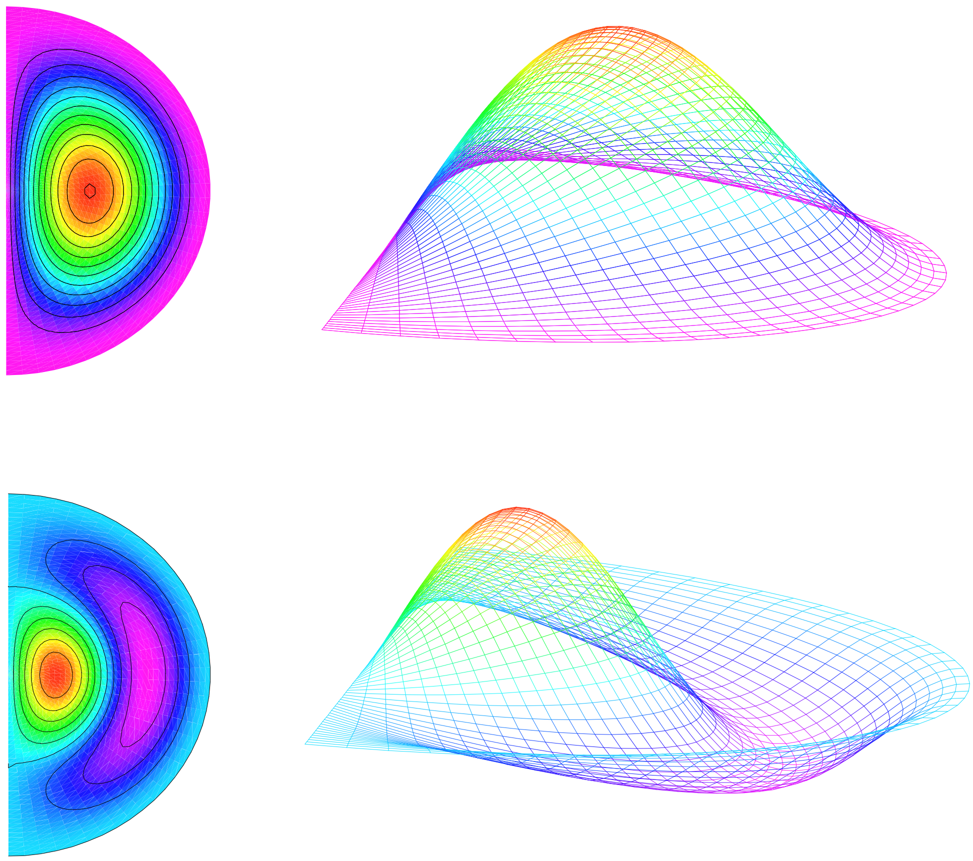


Figure 6.8.4: Mode shapes for the case of one nodal diameter

6.9 Strain Energy in Plates

6.9.1 Strain Energy due to Plate Bending and Torsion

Here, the elastic strain energy due to plate bending and twisting is considered.

Consider a plate element bending in the *x* direction, Fig. 6.9.1. The radius of curvature is $R = \partial^2 w / \partial x^2$. The strain energy due to bending through an angle $\Delta\theta$ by a moment $M_x \Delta y$ is

$$\Delta U = \frac{1}{2}(M_x \Delta y)\frac{\partial^2 w}{\partial x^2}\Delta x \tag{6.9.1}$$

Considering also contributions from M_y and M_{xy}, one has

$$\Delta U = \frac{1}{2}\left(M_x \frac{\partial^2 w}{\partial x^2} - 2M_{xy}\frac{\partial^2 w}{\partial x \partial y} + M_y \frac{\partial^2 w}{\partial y^2}\right)\Delta x \Delta y \tag{6.9.2}$$

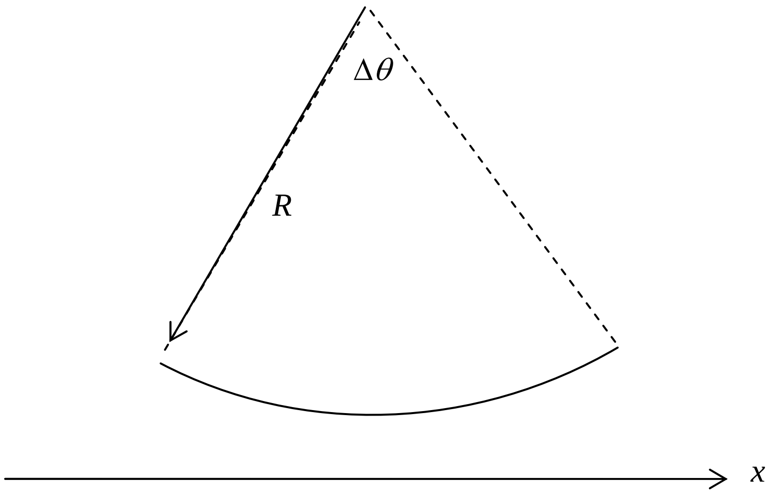


Figure 6.9.1: a bending plate element

Using the moment-curvature relations, one has

$$\begin{aligned}\Delta U &= \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]\Delta x \Delta y \\ &= \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right)\right]\Delta x \Delta y\end{aligned} \tag{6.9.3}$$

This can now be integrated over the complete plate surface to obtain the total elastic strain energy.

6.9.2 The Principle of Minimum Potential Energy

Plate problems can be solved using the principle of minimum potential energy (see Book I, §8.6). Let $V = -W_{ext}$ be the potential energy of the loads, equivalent to the negative of the work done by those loads, and so the potential energy of the system is $\Pi(w) = U(w) + V(w)$. The solution is then the deflection which minimizes $\Pi(w)$.

When the load is a uniform lateral pressure q, one has

$$\Delta V = -\Delta W_{ext} = +q\, w(x,y)\Delta x \Delta x \tag{6.9.4}$$

and

$$\Delta\Pi = \left\{ \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) \right] + qw \right\} \Delta x \Delta y \tag{6.9.5}$$

As an example, consider again the simply supported rectangular plate subjected to a uniform load q. Use the same trial function 6.5.38 which satisfies the boundary conditions:

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{6.9.6}$$

Substituting into 6.9.5 and integrating over the plate gives

$$\begin{aligned} \Pi = \int_0^b\int_0^a & \left\{ \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \left[\pi^4\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} \right.\right. \\ & \left. - 2(1-\nu)\left(\frac{m^2 n^2 \pi^4}{a^2 b^2}\left(\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{b} - \cos^2\frac{m\pi x}{a}\cos^2\frac{n\pi y}{b} \right) \right) \right] \\ & \left. + q\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \right\} dxdy \end{aligned} \tag{6.9.7}$$

Carrying out the integration leads to

$$\Pi = \frac{D}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \left[\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \times \frac{1}{4}ab \right] + q\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} A_{mn} \times \frac{4ab}{mn\pi^2} \tag{6.9.8}$$

To minimize the total potential energy, one sets

$$\frac{\partial \Pi}{\partial A_{mn}} = DA_{mn} \frac{ab\pi^4}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + q\frac{4ab}{mn\pi^2} = 0$$
$$\rightarrow \quad A_{mn} = -\frac{16q}{\pi^6 Dmn}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2} \tag{6.9.9}$$

which is the same result as 6.5.50.

6.9.3 Strain Energy in Polar Coordinates

For circular plates, one can transform the strain energy expression 6.9.3 into polar coordinates, giving {▲Problem 1}

$$\Delta U = \frac{D}{2}\left\{\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 - 2(1-\nu)\times \right.$$
$$\left.\left[\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) - \left(\frac{1}{r^2}\frac{\partial w}{\partial \theta} - \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta}\right)^2\right]\right\}\Delta x \Delta y \tag{6.9.10}$$

For an axisymmetric problem, the strain energy is

$$\Delta U = \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)\right]\Delta x \Delta y \tag{6.9.11}$$

6.9.4 Vibration of Plates

For vibrating plates, one needs to include the kinetic energy of the plate. The kinetic energy of a plate element of dimensions $\Delta x, \Delta y$ and moving with velocity $\partial \omega / \partial t$ is

$$\Delta K = \frac{1}{2}\rho h\left(\frac{\partial w}{\partial t}\right)^2 \Delta x \Delta y \tag{6.9.12}$$

According to Hamilton's principle, then, the quantity to be minimized is now $U(w) + V(w) - K(w)$.

Consider again the problem of a circular plate undergoing axisymmetric vibrations. The potential energy function is

$$D\pi\int_0^a\left[\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)\right]rdr - \pi\rho h\int_0^a\left(\frac{\partial w}{\partial t}\right)^2 rdr \tag{6.9.13}$$

Assume a solution of the form

$$w(r,t) = W(r)\cos(\omega t + \phi) \tag{6.9.14}$$

Substituting this into 6.9.13 leads to

$$D\pi\int_0^a \left[\left(\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr}\right)^2 - 2(1-\nu)\frac{d^2W}{dr^2}\left(\frac{1}{r}\frac{dW}{dr}\right)\right] rdr - \pi\rho h\omega\int_0^a W^2 rdr \tag{6.9.15}$$

Examining the clamped plate, assume a solution, an assumption based on the known static solution 6.6.20, of the form

$$W(r) = A\left(a^2 - r^2\right)^2 \tag{6.9.16}$$

Substituting this into 6.9.15 leads to

$$\begin{aligned} &32D\pi A^2 \int_0^a \left[2\left(a^4 - 4a^2r^2 + 4r^4\right) - (1-\nu)\left(a^4 - 4a^2r^2 + 3r^4\right)\right] rdr \\ &\qquad - \pi\rho\omega h A^2 \int_0^a r\left(a^2 - r^2\right)^4 dr \end{aligned} \tag{6.9.17}$$

Evaluating the integrals leads to

$$\pi A^2\left(\frac{32}{3}Da^6 - \frac{1}{10}\rho h\omega a^{10}\right) \tag{6.9.18}$$

Minimising this function, setting $\partial / \partial A\{\ \} = 0$, then gives

$$\omega = \alpha\frac{1}{a^2}\sqrt{\frac{D}{\rho h}}, \qquad \alpha = \sqrt{\frac{320}{3}} \approx 10.328 \tag{6.9.19}$$

This simple one-term solution is very close to the exact result given in Table 6.8.1, 10.2158. The result 6.9.19 is of course greater than the actual frequency.

6.9.5 Problems

1. Derive the strain energy expression in polar coordinates, Eqn. 6.9.10.

6.10 Limitations of Classical Plate Theory

The validity of the classical plate theory depends on a number of factors:
1. the curvatures are small
2. the in-plane plate dimensions are large compared to the thickness
3. membrane strains can be neglected

The second and third of these points are discussed briefly in what follows.

6.10.1 Moderately Thick Plates

As with beam theory, and as mentioned already, it turns out that the solutions based on the classical theory agree well with the full elasticity solutions (away from the edges of the plate), provided the plate thickness is small relative to its other linear dimensions. When the plate is relatively thick, one is advised to use a more exact theory, for example one of the shear deformation theories:

Shear deformation Theories

The **Mindlin plate theory (**or **moderately thick plate theory** or **shear deformation theory**) was developed in the mid-1900s to allow for possible transverse shear strains. In this theory, there is the added complication that vertical line elements before deformation do *not* have to remain perpendicular to the mid-surface after deformation, although they do remain straight. Thus shear strains ε_{yz} and ε_{zx} are generated, constant through the thickness of the plate.

The classical plate theory is inconsistent in the sense that elements are assumed to remain perpendicular to the mid-plane, yet equilibrium requires that stress components σ_{xz}, σ_{yz} still arise (which would cause these elements to deform). The theory of thick plates is more consistent, but it still makes the assumption that $\sigma_{zz} = 0$. Note that *both* are approximations of the exact three-dimensional equations of elasticity.

As an indication of the error involved in using the classical plate theory, consider the problem of a simply supported square plate subjected to a uniform pressure. According to Eqn. 6.5.52, the central deflection is $w/(qa^4/1000D) = 4.062$. The shear deformation theory predicts 4.060 (for $a/h = 100$), 4.070 (for $a/h = 50$), 4.111 (for $a/h = 20$) and 4.259 (for $a/h = 10$). This trend holds in general; the classical theory is good for thin plates but under-predicts deflections (and over-predicts buckling loads and natural frequencies) in relatively thick plates.

An important difference between the thin plate and thick plate theories is that in the former the moments are related to the curvatures through (using M_x for illustration)

$$M_x = D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \qquad (6.10.1)$$

This is only an approximate relation (although it turns out to be exact in the case of pure bending). The thick plate theory predicts that, in the case of a uniform lateral load q, the relationship is given by

$$M_x = D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) + h^2 q \frac{8 + \nu + \nu^2}{20(1-\nu)} \tag{6.10.2}$$

The thin plate expression will be approximately equal to the thick plate expression when the thickness h is very small, since in that case $h^2 \to 0$, or when the ratio of load to stiffness, q / D, is small.

Some solutions for circular plates using the various theories are presented next for comparison.

Comparison of Solutions for Circular Plates

1. Uniform load q, clamped:

Both thin and thick plate theories give

$$\omega = -\frac{q}{64D}\left(a^2 - r^2\right)^2 \tag{6.10.3}$$

2. Uniform load q, simply supported:

$$\omega = -\frac{q}{64D}\left(a^2 - r^2\right)\left(\frac{5+\nu}{1+\nu}a^2 - r^2 + \eta\right), \qquad \eta = \begin{cases} 0 & \text{thin} \\ +\frac{8}{5}\dfrac{8+\nu+\nu^2}{1-\nu^2}h^2 & \text{thick} \end{cases} \tag{6.10.4}$$

3. Concentrated central load, clamped (Eqn. 6.6.25):

$$\omega = -\frac{F}{16\pi D}\left(\left(a^2 - r^2\right) + 2r^2 \ln(r/a) + \eta\right)$$

$$\eta = \begin{cases} 0 & \text{thin} \\ 0 & \text{thick} \\ -\frac{4}{5}\dfrac{2-\nu}{1-\nu}h^2 \ln(r/a) & \text{exact} \end{cases} \tag{6.10.5}$$

4. Concentrated central load Q, simply supported:

$$\omega = -\frac{F}{16\pi D}\left(\frac{3+\nu}{1+\nu}\left(a^2 - r^2\right) + 2r^2\ln(r/a) + \eta\right)$$

$$\eta = \begin{cases} 0 & \text{thin} \\ +\tfrac{2}{5}\dfrac{8+\nu}{1+\nu}\dfrac{h^2}{a^2}\left(a^2 - r^2\right) & \text{thick} \\ -\tfrac{4}{5}\dfrac{2-\nu}{1-\nu}h^2\ln(r/a) - \tfrac{1}{5}\dfrac{2\nu}{1+\nu}\dfrac{h^2}{a^2}\left(a^2 - r^2\right) & \text{exact} \end{cases} \tag{6.10.6}$$

Higher-order deformation Theories

A further theory known as **third-order plate theory** has also been developed. This allows for the displacements to vary not only linearly (the previously described Mindlin theory is also called the **first order shear deformation theory**), but as cubic functions; the in-plane strains are cubic ($\sim z^3$) and the shear strains are quadratic. This allows the line elements normal to the mid-surface not only to rotate, but also to deform and not necessarily remain straight.

6.10.2 Large Deflections

Consider now the assumption that the membrane stresses may be neglected. To investigate the validity of this, consider an initially circular plate of diameter *d*, clamped at the edges, and deformed into a spherically shaped surface, Fig. 6.10.1.

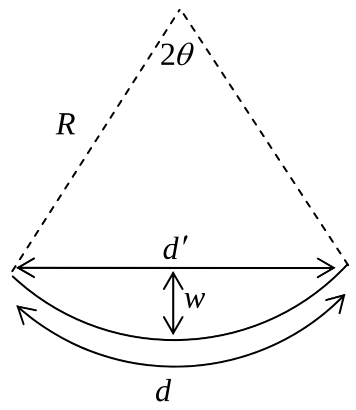


Figure 6.10.1: a deformed circular plate

Considering a beam, the length of the neutral axis before and after deformation can safely be taken to be equal, even when the beam deforms as in Fig. 6.10.1, i.e. one can take $d = d'$. The reason for this is that the "supports" are assumed to move slightly to accommodate any small deflection; thus the neutral axis of a beam remains strain-free and hence stress-free.

Consider next the plate. Suppose that the supports could move slightly to accommodate the deformation of the plate so that the curved length in Fig. 6.10.1 was equal in length to the original diameter. One then sees that a compressive circumferential strain is set up in the plate mid-surface, of magnitude $(d - d')/d$. To quantify this, note that $\theta = d/2R$ and $\sin\theta = d'/2R$. Thus $d'/2R = d/2R - (d/2R)^3/3! + (d/2R)^5/5! - \cdots$. Then

$$\varepsilon^0_{\theta\theta} = \frac{1}{24}\left(\frac{d}{R}\right)^2 - \frac{1}{1920}\left(\frac{d}{R}\right)^2 + \cdots \tag{6.10.7}$$

With $\cos\theta = 1 - w/R$ and $\cos\theta = 1 - \theta^2/2! + \cdots$, one also has

$$R = \frac{1}{8}\frac{d^2}{w} - \frac{1}{384}\frac{1}{w}\frac{d^4}{R^2} + \cdots \tag{6.10.8}$$

so that, approximately,

$$\varepsilon^0_{\theta\theta} = \frac{8w^2}{3d^2} \tag{6.10.9}$$

The maximum bending strain occurs at $z = h/2$, where $\varepsilon_{rr} = (h./2)(1/R)$, so

$$\varepsilon_{rr} = \frac{4hw}{d^2} \tag{6.10.10}$$

One can conclude from this rough analysis that, in order that the membrane strains can be safely ignored, the deflection *w* must be small when compared to the thickness *h* of the plate[1]. The corollary of this is that when there are large deflections, the middle surface will strain and take up the load as in a stretching membrane. For the bending of circular plates, one usually requires that $w < 0.5h$ in order that the membrane strains can be safely ignored without introducing considerable error. For example, a uniformly loaded clamped plate deflected to $w = h$ experiences a maximum membrane stress of approximately 20% of the maximum bending stress.

When the deflections are large, the membrane strains need to be considered. This means that the von Kármán strains, Eqns. 6.2.22, 6.2.25, must be used in the analysis. Further, the in-plane forces, for example in the bending Eqn. 6.7.6, are now an unknown of the problem. Some approximate solutions of the resulting equations have been worked out, for example for uniformly loaded circular and rectangular plates.

[1] except in some special cases, for example when a plate deforms into the surface of a cylinder

7 3D Elasticity

7.1 Vectors, Tensors and the Index Notation

The equations governing three dimensional mechanics problems can be quite lengthy. For this reason, it is essential to use a short-hand notation called the **index notation**[1]. Consider first the notation used for vectors.

7.1.1 Vectors

Vectors are used to describe physical quantities which have both a magnitude and a direction associated with them. Geometrically, a vector is represented by an arrow; the arrow defines the direction of the vector and the magnitude of the vector is represented by the length of the arrow. Analytically, in what follows, vectors will be represented by lowercase bold-face Latin letters, e.g. **a**, **b**.

The **dot product** of two vectors **a** and **b** is denoted by $\mathbf{a} \cdot \mathbf{b}$ and is a scalar defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta . \qquad (7.1.1)$$

θ here is the angle between the vectors when their initial points coincide and is restricted to the range $0 \le \theta \le \pi$.

Cartesian Coordinate System

So far the short discussion has been in **symbolic notation**[2], that is, no reference to 'axes' or 'components' or 'coordinates' is made, implied or required. Vectors exist independently of any coordinate system. The symbolic notation is very useful, but there are many circumstances in which use of the component forms of vectors is more helpful – or essential. To this end, introduce the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ having the properties

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \qquad (7.1.2)$$

so that they are mutually perpendicular, and

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1, \qquad (7.1.3)$$

so that they are unit vectors. Such a set of orthogonal unit vectors is called an **orthonormal** set, Fig. 7.1.1. This set of vectors forms a **basis,** by which is meant that any other vector can be written as a **linear combination** of these vectors, i.e. in the form

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \qquad (7.1.4)$$

where a_1, a_2 and a_3 are scalars, called the **Cartesian components** or **coordinates** of **a** along the given three directions. The unit vectors are called **base vectors** when used for

[1] or **indicial** or **subscript** or **suffix** notation
[2] or **absolute** or **invariant** or **direct** or **vector** notation

this purpose. The components a_1, a_2 and a_3 are measured along lines called the x_1, x_2 and x_3 **axes**, drawn through the base vectors.

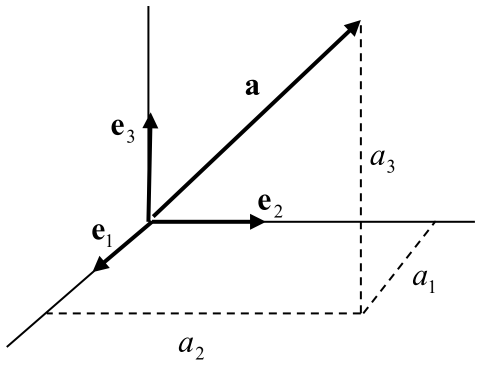


Figure 7.1.1: an orthonormal set of base vectors and Cartesian coordinates

Note further that this orthonormal system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is **right-handed**, by which is meant $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ (or $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ or $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$).

In the index notation, the expression for the vector **a** in terms of the components a_1, a_2, a_3 and the corresponding basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is written as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i \mathbf{e}_i \tag{7.1.5}$$

This can be simplified further by using Einstein's **summation convention**, whereby the summation sign is dropped and it is understood that for a repeated index (i in this case) a summation over the range of the index (3 in this case[3]) is implied. Thus one writes $\mathbf{a} = a_i \mathbf{e}_i$. This can be further shortened to, simply, a_i.

The dot product of two vectors **u** and **v**, referred to this coordinate system, is

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3) \cdot (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \\
&= u_1 v_1 (\mathbf{e}_1 \cdot \mathbf{e}_1) + u_1 v_2 (\mathbf{e}_1 \cdot \mathbf{e}_2) + u_1 v_3 (\mathbf{e}_1 \cdot \mathbf{e}_3) \\
&\quad + u_2 v_1 (\mathbf{e}_2 \cdot \mathbf{e}_1) + u_2 v_2 (\mathbf{e}_2 \cdot \mathbf{e}_2) + u_2 v_3 (\mathbf{e}_2 \cdot \mathbf{e}_3) \\
&\quad\quad + u_3 v_1 (\mathbf{e}_3 \cdot \mathbf{e}_1) + u_3 v_2 (\mathbf{e}_3 \cdot \mathbf{e}_2) + u_3 v_3 (\mathbf{e}_3 \cdot \mathbf{e}_3) \\
&= u_1 v_1 + u_2 v_2 + u_3 v_3
\end{aligned} \tag{7.1.6}$$

The dot product of two vectors written in the index notation reads

$$\boxed{\mathbf{u} \cdot \mathbf{v} = u_i v_i} \qquad \textbf{Dot Product} \tag{7.1.7}$$

[3] 2 in the case of a two-dimensional space/analysis

The repeated index i is called a **dummy index**, because it can be replaced with any other letter and the sum is the same; for example, this could equally well be written as $\mathbf{u} \cdot \mathbf{v} = u_j v_j$ or $u_k v_k$.

Introduce next the **Kronecker delta symbol** δ_{ij}, defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{7.1.8}$$

Note that $\delta_{11} = 1$ but, using the index notation, $\delta_{ii} = 3$. The Kronecker delta allows one to write the expressions defining the orthonormal basis vectors (7.1.2, 7.1.3) in the compact form

$$\boxed{\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}} \qquad \textbf{Orthonormal Basis Rule} \tag{7.1.9}$$

Example

Recall the equations of motion, Eqns. 1.1.9, which in full read

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 &= \rho a_1 \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 &= \rho a_2 \\ \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 &= \rho a_3 \end{aligned} \tag{7.1.10}$$

The index notation for these equations is

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho a_i \tag{7.1.11}$$

Note the dummy index j. The index i is called a **free index**; if one term has a free index i, then, to be consistent, all terms must have it. One free index, as here, indicates three separate equations.

7.1.2 Matrix Notation

The symbolic notation $\mathbf{v}$ and index notation $v_i \mathbf{e}_i$ (or simply v_i) can be used to denote a vector. Another notation is the **matrix notation**: the vector $\mathbf{v}$ can be represented by a 3×1 matrix (a **column vector**):

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Matrices will be denoted by square brackets, so a shorthand notation for this matrix/vector would be $[\mathbf{v}]$. The elements of the matrix $[\mathbf{v}]$ can be written in the index notation v_i.

Note the distinction between a vector and a 3×1 matrix: the former is a mathematical object independent of any coordinate system, the latter is a representation of the vector in a particular coordinate system – matrix notation, as with the index notation, relies on a particular coordinate system.

As an example, the dot product can be written in the matrix notation as

$$[\mathbf{u}^{\mathrm{T}}][\mathbf{v}] = [u_1 \quad u_2 \quad u_3]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

"short"
matrix notation

"full"
matrix notation

Here, the notation $[\mathbf{u}^{\mathrm{T}}]$ denotes the 1×3 matrix (the **row vector**). The result is a 1×1 matrix, $u_i v_i$.

The matrix notation for the Kronecker delta δ_{ij} is the identity matrix

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, for example, in both index and matrix notation:

$$\delta_{ij} u_j = u_i \qquad [\mathbf{I}][\mathbf{u}] = [\mathbf{u}] \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{7.1.12}$$

Matrix – Matrix Multiplication

When discussing vector transformation equations further below, it will be necessary to multiply various matrices with each other (of sizes 3×1, 1×3 and 3×3). It will be helpful to write these matrix multiplications in the short-hand notation.

First, it has been seen that the dot product of two vectors can be represented by $[\mathbf{u}^{\mathrm{T}}][\mathbf{v}]$ or $u_i v_i$. Similarly, the matrix multiplication $[\mathbf{u}][\mathbf{v}^{\mathrm{T}}]$ gives a 3×3 matrix with element form $u_i v_j$ or, in full,

$$\begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix}$$

This operation is called the **tensor product** of two vectors, written in symbolic notation as $\mathbf{u}\otimes\mathbf{v}$ (or simply **uv**).

Next, the matrix multiplication

$$[\mathbf{Q}][\mathbf{u}] \equiv \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

is a 3×1 matrix with elements $([\mathbf{Q}][\mathbf{u}])_i \equiv Q_{ij}u_j$. The elements of $[\mathbf{Q}][\mathbf{u}]$ are the same as those of $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}]$, which can be expressed as $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}])_i \equiv u_j Q_{ij}$.

The expression $[\mathbf{u}][\mathbf{Q}]$ is meaningless, but $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$ {▲Problem 4} is a 1×3 matrix with elements $([\mathbf{u}^{\mathrm{T}}][\mathbf{Q}])_i \equiv u_j Q_{ji}$.

This leads to the following rule:

1. if a vector pre-multiplies a matrix $[\mathbf{Q}]$ $\rightarrow$ the vector is the transpose $[\mathbf{u}^{\mathrm{T}}]$
2. if a matrix $[\mathbf{Q}]$ pre-multiplies the vector $\rightarrow$ the vector is $[\mathbf{u}]$
3. if summed indices are "beside each other", as the j in $u_j Q_{ji}$ or $Q_{ij}u_j$
   $\rightarrow$ the matrix is $[\mathbf{Q}]$
4. if summed indices are not beside each other, as the j in $u_j Q_{ij}$
   $\rightarrow$ the matrix is the transpose, $[\mathbf{Q}^{\mathrm{T}}]$

Finally, consider the multiplication of 3×3 matrices. Again, this follows the "beside each other" rule for the summed index. For example, $[\mathbf{A}][\mathbf{B}]$ gives the 3×3 matrix {▲Problem 8} $([\mathbf{A}][\mathbf{B}])_{ij} = A_{ik}B_{kj}$, and the multiplication $[\mathbf{A}^{\mathrm{T}}][\mathbf{B}]$ is written as $([\mathbf{A}^{\mathrm{T}}][\mathbf{B}])_{ij} = A_{ki}B_{kj}$. There is also the important identity

$$([\mathbf{A}][\mathbf{B}])^{\mathrm{T}} = [\mathbf{B}^{\mathrm{T}}][\mathbf{A}^{\mathrm{T}}] \qquad (7.1.13)$$

Note also the following:

(i) if there is no free index, as in $u_i v_i$, there is one element

(ii) if there is one free index, as in $u_j Q_{ji}$, it is a 3×1 (or 1×3) matrix

(iii) if there are two free indices, as in $A_{ki} B_{kj}$, it is a 3×3 matrix

7.1.3 Vector Transformation Rule

Introduce two Cartesian coordinate systems with base vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$ and common origin *o*, Fig. 7.1.2. The vector $\mathbf{u}$ can then be expressed in two ways:

$$\mathbf{u} = u_i \mathbf{e}_i = u'_i \mathbf{e}'_i \tag{7.1.14}$$

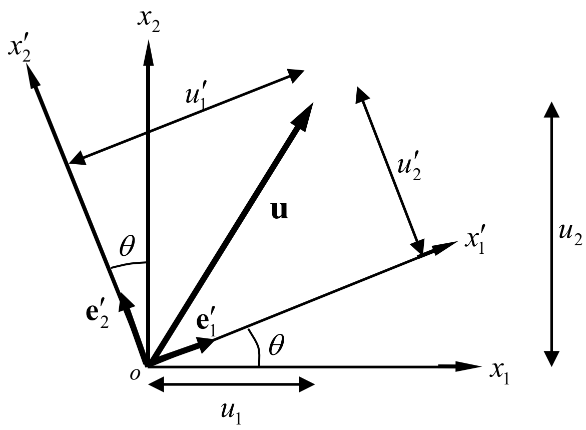


Figure 7.1.2: a vector represented using two different coordinate systems

Note that the x'_i coordinate system is obtained from the x_i system by a *rotation* of the base vectors. Fig. 7.1.2 shows a rotation θ about the x_3 axis (the sign convention for rotations is positive counterclockwise).

Concentrating for the moment on the two dimensions $x_1 - x_2$, from trigonometry (refer to Fig. 7.1.3),

$$\begin{aligned}\mathbf{u} &= u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 \\ &= \left[|OB| - |AB|\right]\mathbf{e}_1 + \left[|BD| + |CP|\right]\mathbf{e}_2 \\ &= \left[\cos\theta\, u'_1 - \sin\theta\, u'_2\right]\mathbf{e}_1 + \left[\sin\theta\, u'_1 + \cos\theta\, u'_2\right]\mathbf{e}_2\end{aligned} \tag{7.1.15}$$

and so

$$\begin{aligned} u_1 &= \cos\theta\, u'_1 - \sin\theta\, u'_2 \\ u_2 &= \sin\theta\, u'_1 + \cos\theta\, u'_2 \end{aligned} \tag{7.1.16}$$

vector components in first coordinate system

vector components in second coordinate system

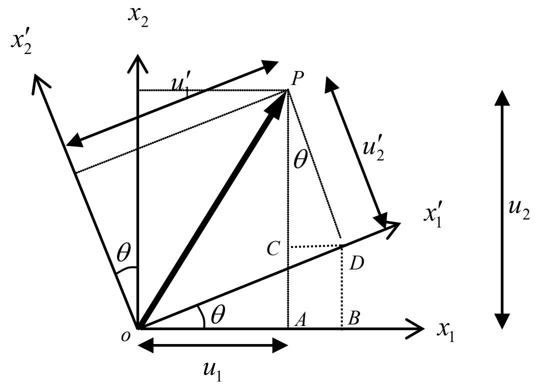


Figure 7.1.3: geometry of the 2D coordinate transformation

In matrix form, these transformation equations can be written as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \tag{7.1.17}$$

The 2×2 matrix is called the **transformation matrix** or **rotation matrix** $[\mathbf{Q}]$. By pre-multiplying both sides of these equations by the inverse of $[\mathbf{Q}]$, $[\mathbf{Q}^{-1}]$, one obtains the transformation equations transforming from $[u_1 \quad u_2]^{\mathrm{T}}$ to $[u_1' \quad u_2']^{\mathrm{T}}$:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{7.1.18}$$

It can be seen that the components of $[\mathbf{Q}]$ are the **directions cosines**, i.e. the cosines of the angles between the coordinate directions:

$$Q_{ij} = \cos(x_i, x_j') = \mathbf{e}_i \cdot \mathbf{e}_j' \tag{7.1.19}$$

It is straight forward to show that, in the full three dimensions, Fig. 7.1.4, the components in the two coordinate systems are also related through

$$\boxed{\begin{array}{ll} u_i = Q_{ij} u_j' & \quad \ldots \quad [\mathbf{u}] = [\mathbf{Q}][\mathbf{u}'] \\ u_i' = Q_{ji} u_j & \quad \ldots \quad [\mathbf{u}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] \end{array}} \quad \textbf{Vector Transformation Rule} \tag{7.1.20}$$

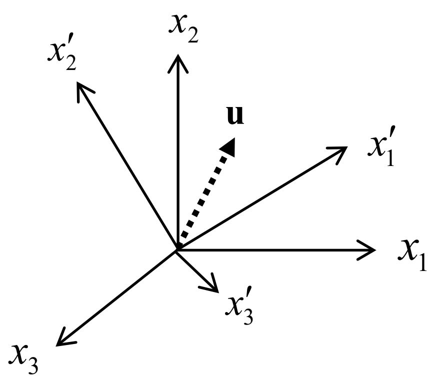


Figure 7.1.4: two different coordinate systems in a 3D space

Orthogonality of the Transformation Matrix $[\mathbf{Q}]$

From 7.1.20, it follows that

$$\begin{aligned} u_i &= Q_{ij} u_j' & \ldots \quad [\mathbf{u}] &= [\mathbf{Q}][\mathbf{u}'] \\ &= Q_{ij} Q_{kj} u_k & \ldots \quad &= [\mathbf{Q}][\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] \end{aligned} \tag{7.1.21}$$

and so

$$Q_{ij} Q_{kj} = \delta_{ik} \qquad \ldots \qquad [\mathbf{Q}][\mathbf{Q}^{\mathrm{T}}] = [\mathbf{I}] \tag{7.1.22}$$

A matrix such as this for which $[\mathbf{Q}^{\mathrm{T}}] = [\mathbf{Q}^{-1}]$ is called an **orthogonal matrix**.

Example

Consider a Cartesian coordinate system with base vectors $\mathbf{e}_i$. A coordinate transformation is carried out with the new basis given by

$$\begin{aligned} \mathbf{e}_1' &= a_1^{(1)} \mathbf{e}_1 + a_2^{(1)} \mathbf{e}_2 + a_3^{(1)} \mathbf{e}_3 \\ \mathbf{e}_2' &= a_1^{(2)} \mathbf{e}_1 + a_2^{(2)} \mathbf{e}_2 + a_3^{(2)} \mathbf{e}_3 \\ \mathbf{e}_3' &= a_1^{(3)} \mathbf{e}_1 + a_2^{(3)} \mathbf{e}_2 + a_3^{(3)} \mathbf{e}_3 \end{aligned}$$

What is the transformation matrix?

<u>Solution</u>

The transformation matrix consists of the direction cosines $Q_{ij} = \cos(x_i, x_j') = \mathbf{e}_i \cdot \mathbf{e}_j'$, so

$$[\mathbf{Q}] = \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & a_1^{(3)} \\ a_2^{(1)} & a_2^{(2)} & a_2^{(3)} \\ a_3^{(1)} & a_3^{(2)} & a_3^{(3)} \end{bmatrix}$$

■

7.1.4 Tensors

The concept of the **tensor** is discussed in detail in Book III, where it is indispensable for the description of large-strain deformations. For small deformations, it is not so necessary; the main purpose for introducing the tensor here (in a rather non-rigorous way) is that it helps to deepen one's understanding of the concept of stress.

A **second-order tensor**[4] **A** may be *defined* as an operator that acts on a vector **u** generating another vector **v**, so that $\mathbf{T}(\mathbf{u}) = \mathbf{v}$, or

$$\boxed{\mathbf{Tu} = \mathbf{v}} \qquad \textbf{Second-order Tensor} \qquad (7.1.23)$$

The second-order tensor **T** is a **linear operator**, by which is meant

$$\begin{aligned} \mathbf{T}(\mathbf{a}+\mathbf{b}) &= \mathbf{Ta}+\mathbf{Tb} && \ldots \text{ distributive} \\ \mathbf{T}(\alpha\mathbf{a}) &= \alpha(\mathbf{Ta}) && \ldots \text{ associative} \end{aligned}$$

for scalar α. In a Cartesian coordinate system, the tensor **T** has nine components and can be represented in the matrix form

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

The rule 7.1.23, which is expressed in symbolic notation, can be expressed in the index and matrix notation when **T** is referred to particular axes:

$$u_i = T_{ij}v_j \qquad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad [\mathbf{u}] = [\mathbf{T}][\mathbf{v}] \qquad (7.1.24)$$

Again, one should be careful to distinguish between a tensor such as **T** and particular matrix representations of that tensor. The relation 7.1.23 is a **tensor relation**, relating vectors and a tensor and is valid in all coordinate systems; the matrix representation of this tensor relation, Eqn. 7.1.24, is to be sure valid in all coordinate systems, but the entries in the matrices of 7.1.24 depend on the coordinate system chosen.

---

[4] to be called simply a tensor in what follows

Note also that the transformation formulae for vectors, Eqn. 7.1.20, is not a tensor relation; although 7.1.20 looks similar to the tensor relation 7.1.24, the former relates the components of a vector to the components of the *same* vector in different coordinate systems, whereas (by definition of a tensor) the relation 7.1.24 relates the components of a vector to those of a different vector in the same coordinate system.

For these reasons, the notation $u_i = Q_{ij}u'$ in Eqn. 7.1.20 is more formally called **element form**, the Q_{ij} being elements of a matrix rather than components of a tensor. This distinction between element form and index notation should be noted, but the term "index notation" is used for both tensor and matrix-specific manipulations in these notes.

Example

Recall the strain-displacement relations, Eqns. 1.2.19, which in full read

$$\begin{aligned} &\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \\ &\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \quad \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \quad \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \end{aligned} \tag{7.1.25}$$

The index notation for these equations is

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{7.1.26}$$

This expression has two free indices and as such indicates nine separate equations. Further, with its two subscripts, ε_{ij}, the strain, is a tensor. It can be expressed in the matrix notation

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \partial u_1/\partial x_1 & \frac{1}{2}(\partial u_1/\partial x_2 + \partial u_2/\partial x_1) & \frac{1}{2}(\partial u_1/\partial x_3 + \partial u_3/\partial x_1) \\ \frac{1}{2}(\partial u_2/\partial x_1 + \partial u_1/\partial x_2) & \partial u_2/\partial x_2 & \frac{1}{2}(\partial u_2/\partial x_3 + \partial u_3/\partial x_2) \\ \frac{1}{2}(\partial u_3/\partial x_1 + \partial u_1/\partial x_3) & \frac{1}{2}(\partial u_3/\partial x_2 + \partial u_2/\partial x_3) & \partial u_3/\partial x_3 \end{bmatrix}$$

7.1.5 Tensor Transformation Rule

Consider now the tensor definition 7.1.23 expressed in two different coordinate systems:

$$\begin{aligned} u_i &= T_{ij}v_j \qquad [\mathbf{u}] = [\mathbf{T}][\mathbf{v}] \qquad \text{in} \quad \{x_i\} \\ u_i' &= T_{ij}'v_j' \qquad [\mathbf{u}'] = [\mathbf{T}'][\mathbf{v}'] \qquad \text{in} \quad \{x_i'\} \end{aligned} \tag{7.1.27}$$

From the vector transformation rule 7.1.20,

$$u_i' = Q_{ji} u_j \qquad [\mathbf{u}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}]$$
$$v_i' = Q_{ji} v_j \qquad [\mathbf{v}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{v}] \tag{7.1.28}$$

Combining 7.1.27-28,

$$Q_{ji} u_j = T_{ij}' Q_{kj} v_k \qquad [\mathbf{Q}^{\mathrm{T}}][\mathbf{u}] = [\mathbf{T}'][\mathbf{Q}^{\mathrm{T}}][\mathbf{v}] \tag{7.1.29}$$

and so

$$Q_{mi} Q_{ji} u_j = Q_{mi} T_{ij}' Q_{kj} v_k \qquad [\mathbf{u}] = [\mathbf{Q}][\mathbf{T}'][\mathbf{Q}^{\mathrm{T}}][\mathbf{v}] \tag{7.1.30}$$

(Note that $Q_{mi} Q_{ji} u_j = \delta_{mj} u_j = u_m$.) Comparing with 7.1.24, it follows that

$$\boxed{\begin{aligned} T_{ij} &= Q_{ip} Q_{jq} T_{pq}' \quad \ldots \quad [\mathbf{T}] = [\mathbf{Q}][\mathbf{T}'][\mathbf{Q}^{\mathrm{T}}] \\ T_{ij}' &= Q_{pi} Q_{qj} T_{pq} \quad \ldots \quad [\mathbf{T}'] = [\mathbf{Q}^{\mathrm{T}}][\mathbf{T}][\mathbf{Q}] \end{aligned}} \quad \textbf{Tensor Transformation Rule} \tag{7.1.31}$$

7.1.6 Problems

1. Write the following in index notation: $|\mathbf{v}|$, $\mathbf{v} \cdot \mathbf{e}_1$, $\mathbf{v} \cdot \mathbf{e}_k$.
2. Show that $\delta_{ij} a_i b_j$ is equivalent to $\mathbf{a} \cdot \mathbf{b}$.
3. Evaluate or simplify the following expressions:
   (a) δ_{kk} (b) $\delta_{ij}\delta_{ij}$ (c) $\delta_{ij}\delta_{jk}$
4. Show that $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$ is a 1×3 matrix with elements $u_j Q_{ji}$ (write the matrices out in full)
5. Show that $([\mathbf{Q}][\mathbf{u}])^{\mathrm{T}} = [\mathbf{u}^{\mathrm{T}}][\mathbf{Q}^{\mathrm{T}}]$
6. Are the three elements of $[\mathbf{Q}][\mathbf{u}]$ the same as those of $[\mathbf{u}^{\mathrm{T}}][\mathbf{Q}]$?
7. What is the index notation for $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$?
8. Write out the 3×3 matrices $[\mathbf{A}]$ and $[\mathbf{B}]$ in full, i.e. in terms of A_{11}, A_{12}, etc. and verify that $[\mathbf{AB}]_{ij} = A_{ik} B_{kj}$ for $i = 2$, $j = 1$.
9. What is the index notation for
   (a) $[\mathbf{A}][\mathbf{B}^{\mathrm{T}}]$
   (b) $[\mathbf{v}^{\mathrm{T}}][\mathbf{A}][\mathbf{v}]$ (there is no ambiguity here, since $([\mathbf{v}^{\mathrm{T}}][\mathbf{A}])[\mathbf{v}] = [\mathbf{v}^{\mathrm{T}}]([\mathbf{A}][\mathbf{v}])$)
   (c) $[\mathbf{B}^{\mathrm{T}}][\mathbf{A}][\mathbf{B}]$
10. The angles between the axes in two coordinate systems are given in the table below.

	x_1	x_2	x_3
x_1'	135°	60°	120°
x_2'	90°	45°	45°
x_3'	45°	60°	120°

Construct the corresponding transformation matrix $[\mathbf{Q}]$ and verify that it is orthogonal.

11. Consider a two-dimensional problem. If the components of a vector **u** in one coordinate system are

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

what are they in a second coordinate system, obtained from the first by a positive rotation of 30°? Sketch the two coordinate systems and the vector to see if your answer makes sense.

12. Consider again a two-dimensional problem with the same change in coordinates as in Problem 11. The components of a 2D tensor in the first system are

$$\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$$

What are they in the second coordinate system?

7.2 Analysis of Three Dimensional Stress and Strain

The concept of traction and stress was introduced and discussed in Book I, §3. For the most part, the discussion was confined to two-dimensional states of stress. Here, the fully three dimensional stress state is examined. There will be some repetition of the earlier analyses.

7.2.1 The Traction Vector and Stress Components

Consider a traction vector **t** acting on a surface element, Fig. 7.2.1. Introduce a Cartesian coordinate system with base vectors $\mathbf{e}_i$ so that one of the base vectors is a normal to the surface and the origin of the coordinate system is positioned at the point at which the traction acts. For example, in Fig. 7.1.1, the $\mathbf{e}_3$ direction is taken to be normal to the plane, and a superscript on **t** denotes this normal:

$$\mathbf{t}^{(\mathbf{e}_3)} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3 \qquad (7.2.1)$$

Each of these components t_i is represented by σ_{ij} where the first subscript denotes the direction of the normal and the second denotes the direction of the component to the plane. Thus the three components of the traction vector shown in Fig. 7.2.1 are $\sigma_{31}, \sigma_{32}, \sigma_{33}$:

$$\mathbf{t}^{(\mathbf{e}_3)} = \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3 \qquad (7.2.2)$$

The first two stresses, the components acting tangential to the surface, are shear stresses whereas σ_{33}, acting normal to the plane, is a normal stress.

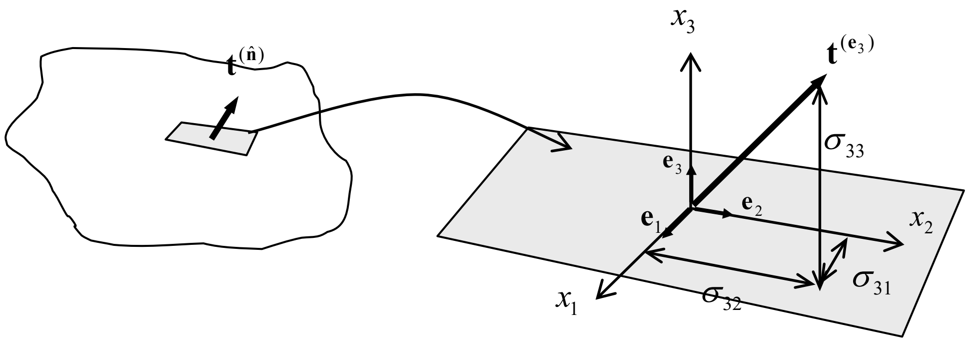


Figure 7.2.1: components of the traction vector

Consider the three traction vectors $\mathbf{t}^{(\mathbf{e}_1)}, \mathbf{t}^{(\mathbf{e}_2)}, \mathbf{t}^{(\mathbf{e}_3)}$ acting on the surface elements whose outward normals are aligned with the three base vectors $\mathbf{e}_j$, Fig. 7.2.2a. The three (or six) surfaces can be amalgamated into one diagram as in Fig. 7.2.2b.

In terms of stresses, the traction vectors are

$$\begin{aligned} \mathbf{t}^{(\mathbf{e}_1)} &= \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3 \\ \mathbf{t}^{(\mathbf{e}_2)} &= \sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{23}\mathbf{e}_3 \qquad \text{or} \qquad \mathbf{t}^{(\mathbf{e}_i)} = \sigma_{ij}\mathbf{e}_j \\ \mathbf{t}^{(\mathbf{e}_3)} &= \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3 \end{aligned} \tag{7.2.3}$$

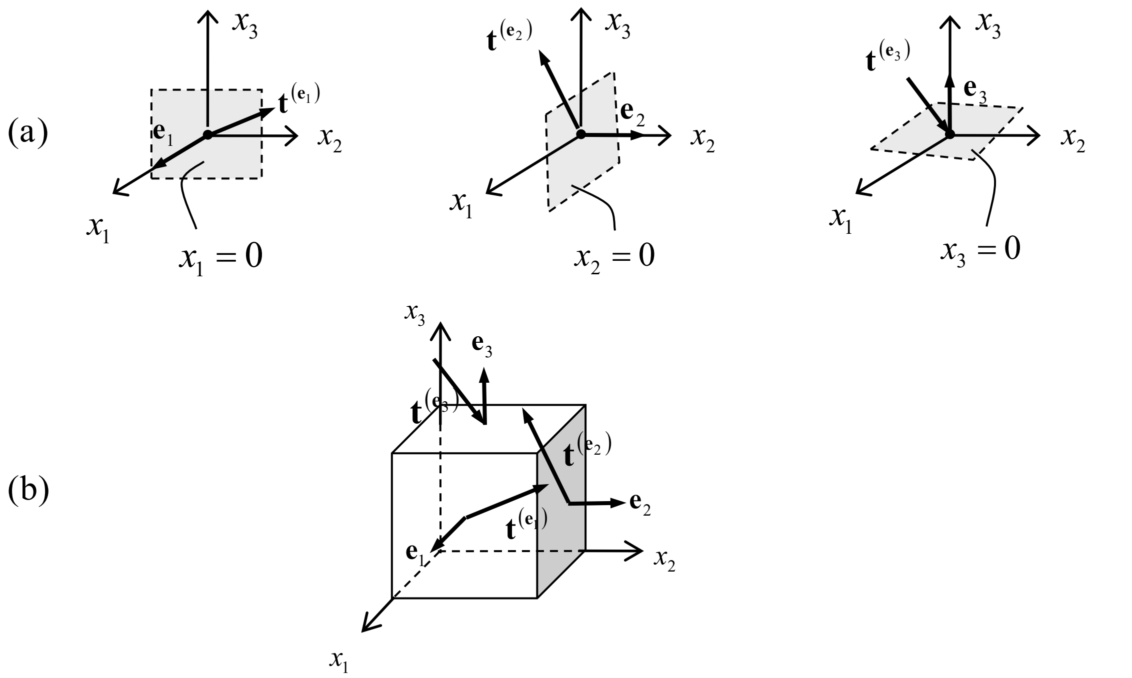


Figure 7.2.2: the three traction vectors acting at a point; (a) on mutually orthogonal planes, (b) the traction vectors illustrated on a box element

The components of the three traction vectors, i.e. the stress components, can now be displayed on a box element as in Fig. 7.2.3. Note that the stress components will vary slightly over the surfaces of an elemental box of finite size. However, it is assumed that the element in Fig. 7.2.3 is small enough that the stresses can be treated as constant, so that they are the stresses acting *at* the origin.

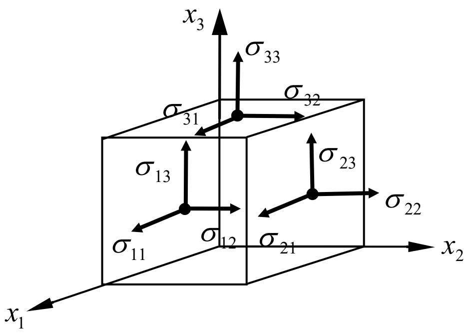


Figure 7.2.3: the nine stress components with respect to a Cartesian coordinate system

The nine stresses can be conveniently displayed in 3×3 matrix form:

$$\left[\sigma_{ij}\right]=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} \tag{7.2.4}$$

It is important to realise that, if one were to take an element at some different orientation to the element in Fig. 7.2.3, but at the *same material particle*, for example aligned with the axes x_1', x_2', x_3' shown in Fig. 7.2.4, one would then have different tractions acting and the nine stresses would be different also. The stresses acting in this new orientation can be represented by a new matrix:

$$\left[\sigma_{ij}'\right]=\begin{bmatrix}\sigma_{11}' & \sigma_{12}' & \sigma_{13}'\\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}'\\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}'\end{bmatrix} \tag{7.2.5}$$

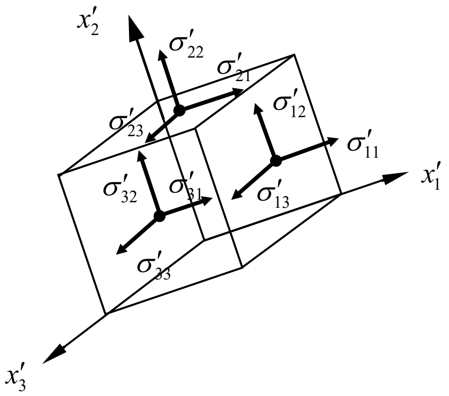


Figure 7.2.4: the stress components with respect to a Cartesian coordinate system different to that in Fig. 7.2.3

7.2.2 Cauchy's Law

Cauchy's Law, which will be proved below, states that the normal to a surface, $\mathbf{n} = n_i \mathbf{e}_i$, is related to the traction vector $\mathbf{t}^{(\mathbf{n})} = t_i \mathbf{e}_i$ acting on that surface, according to

$$t_i = \sigma_{ji} n_j \tag{7.2.6}$$

Writing the traction and normal in vector form and the stress in 3×3 matrix form,

$$\left[t_i^{(\mathbf{n})}\right]=\begin{bmatrix}t_1^{(\mathbf{n})}\\ t_2^{(\mathbf{n})}\\ t_3^{(\mathbf{n})}\end{bmatrix},\quad \left[\sigma_{ij}\right]=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix},\quad \left[n_i\right]=\begin{bmatrix}n_1\\ n_2\\ n_3\end{bmatrix} \tag{7.2.7}$$

and Cauchy's law in matrix notation reads

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{7.2.8}$$

Note that it is the transpose stress matrix which is used in Cauchy's law. Since the stress matrix is symmetric, one can express Cauchy's law in the form

$$\boxed{t_i = \sigma_{ij} n_j} \quad \textbf{Cauchy's Law} \tag{7.2.9}$$

Cauchy's law is illustrated in Fig. 7.2.5; in this figure, positive stresses σ_{ij} are shown.

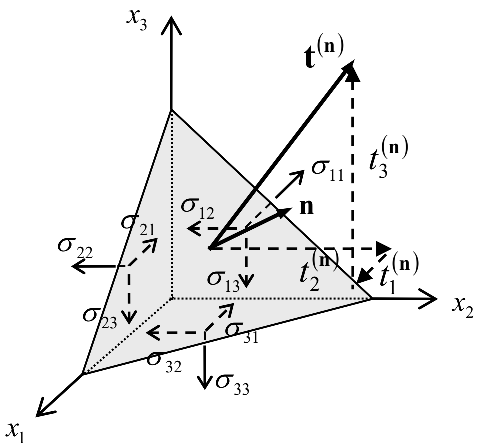


Figure 7.2.5: Cauchy's Law; given the stresses and the normal to a plane, the traction vector acting on the plane can be determined

Normal and Shear Stress

It is useful to be able to evaluate the normal stress σ_N and shear stress σ_S acting on any plane, Fig. 7.2.6. For this purpose, note that the stress acting normal to a plane is the projection of $\mathbf{t}^{(\mathbf{n})}$ in the direction of $\mathbf{n}$,

$$\sigma_N = \mathbf{n} \cdot \mathbf{t}^{(\mathbf{n})} \tag{7.2.10}$$

The magnitude of the shear stress acting on the surface can then be obtained from

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} \tag{7.2.11}$$

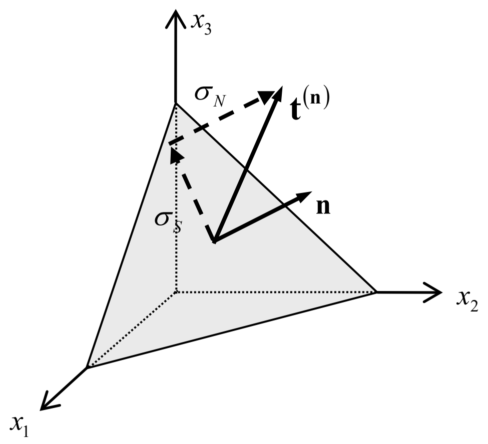


Figure 7.2.6: the normal and shear stress acting on an arbitrary plane through a point

Example

The state of stress at a point with respect to a Cartesian coordinates system $0x_1x_2x_3$ is given by:

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

Determine:

(a) the traction vector acting on a plane through the point whose unit normal is $\mathbf{n} = (1/3)\mathbf{e}_1 + (2/3)\mathbf{e}_2 - (2/3)\mathbf{e}_3$

(b) the component of this traction acting perpendicular to the plane

(c) the shear component of traction on the plane

<u>Solution</u>

(a) From Cauchy's law,

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 9 \\ -3 \end{bmatrix}$$

so that $\mathbf{t}^{(\mathbf{n})} = (-2/3)\mathbf{e}_1 + 3\mathbf{e}_2 - \hat{\mathbf{e}}_3$.

(b) The component normal to the plane is

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (-2/3)(1/3) + 3(2/3) + (2/3) = 22/9 \approx 2.4.$$

(c) The shearing component of traction is

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} = \left\{\left[\left(-\tfrac{2}{3}\right)^2 + (3)^2 + (-1)^2\right] - \left[\left(\tfrac{22}{9}\right)^2\right]\right\}^{1/2} \approx 2.1$$

■

Proof of Cauchy's Law

Cauchy's law can be proved using force equilibrium of material elements. First, consider a tetrahedral free-body, with vertex at the origin, Fig. 7.2.7. It is required to determine the traction **t** in terms of the nine stress components (which are all shown positive in the diagram).

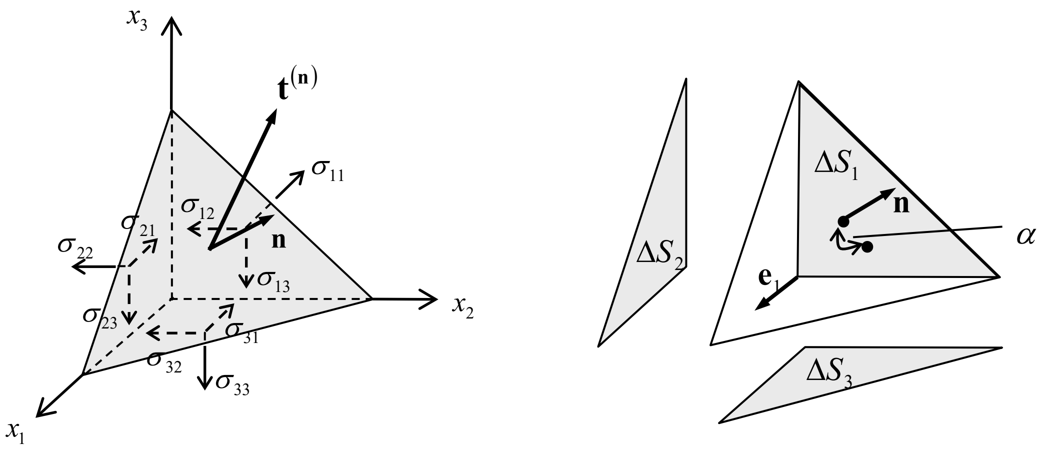


Figure 7.2.7: proof of Cauchy's Law

The components of the unit normal, n_i, are the direction cosines of the normal vector, i.e. the cosines of the angles between the normal and each of the coordinate directions:

$$\cos(\mathbf{n}, \mathbf{e}_i) = \mathbf{n} \cdot \mathbf{e}_i = n_i \tag{7.2.12}$$

Let the area of the base of the tetrahedron, with normal **n**, be ΔS. The area ΔS_1 is then $\Delta S \cos\alpha$, where α is the angle between the planes, as shown to the right of Fig. 7.2.7; this angle is the same as that between the vectors **n** and $\mathbf{e}_1$, so $\Delta S_1 = n_1 \Delta S$, and similarly for the other surfaces:

$$\Delta S_i = n_i \Delta S \tag{7.2.13}$$

The resultant surface force on the body, acting in the x_i direction, is then

$$\sum F_i = t_i \Delta S - \sigma_{ji} \Delta S_j = t_i \Delta S - \sigma_{ji} n_j \Delta S \tag{7.2.14}$$

For equilibrium, this expression must be zero, and one arrives at Cauchy's law.

<u>Note:</u>
As proved in Book III, this result holds also in the general case of accelerating material elements in the presences of body forces.

7.2.3 The Stress Tensor

Cauchy's law 7.2.9 is of the same form as 7.1.24 and so by definition the stress is a tensor. Denote the stress tensor in symbolic notation by $\boldsymbol{\sigma}$. Cauchy's law in symbolic form then reads

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} \tag{7.2.15}$$

Further, the transformation rule for stress follows the general tensor transformation rule 7.1.31:

$$\boxed{\begin{array}{ll} \sigma_{ij} = Q_{ip}Q_{jq}\sigma'_{pq} & \cdots \quad [\boldsymbol{\sigma}] = [\mathbf{Q}][\boldsymbol{\sigma}'][\mathbf{Q}^{\mathrm{T}}] \\ \sigma'_{ij} = Q_{pi}Q_{qj}\sigma_{pq} & \cdots \quad [\boldsymbol{\sigma}'] = [\mathbf{Q}^{\mathrm{T}}][\boldsymbol{\sigma}][\mathbf{Q}] \end{array}} \quad \textbf{Stress Transformation Rule} \tag{7.2.16}$$

As with the normal and traction vectors, the components and hence matrix representation of the stress changes with coordinate system, as with the two different matrix representations 7.2.4 and 7.2.5. However, there is only one stress tensor $\boldsymbol{\sigma}$ at a point. Another way of looking at this is to note that an infinite number of planes pass through a point, and on each of these planes acts a traction vector, and each of these traction vectors has three (stress) components. *All* of these traction vectors taken together define the complete **state of stress** at a point.

Example

The state of stress at a point with respect to an $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

(a) What are the stress components with respect to axes $0x'_1x'_2x'_3$ which are obtained from the first by a 45° rotation (positive counterclockwise) about the x_2 axis, Fig. 7.2.8?

(b) Use Cauchy's law to evaluate the normal and shear stress on a plane with normal $\mathbf{n} = (1/\sqrt{2})\mathbf{e}_1 + (1/\sqrt{2})\mathbf{e}_3$ and relate your result with that from (a)

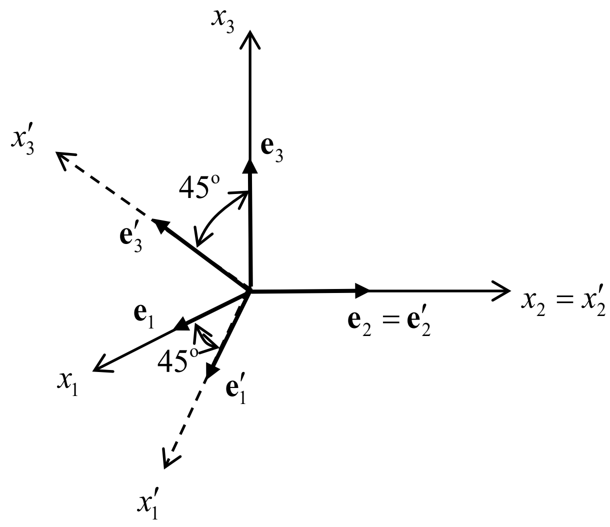


Figure 7.2.8: two different coordinate systems at a point

<u>Solution</u>

(a) The transformation matrix is

$$[Q_{ij}] = \begin{bmatrix} \cos(x_1, x_1') & \cos(x_1, x_2') & \cos(x_1, x_3') \\ \cos(x_2, x_1') & \cos(x_2, x_2') & \cos(x_2, x_3') \\ \cos(x_3, x_1') & \cos(x_3, x_2') & \cos(x_3, x_3') \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$ as expected. The rotated stress components are therefore

$$\begin{bmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{3}{\sqrt{2}} & \frac{1}{2} \\ \frac{3}{\sqrt{2}} & 3 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{3}{2} \end{bmatrix}$$

and the new stress matrix is symmetric as expected.

(b) From Cauchy's law, the traction vector is

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_2^{(\mathbf{n})} \\ t_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

so that $\mathbf{t}^{(\mathbf{n})} = (\sqrt{2})\mathbf{e}_1 - (1/\sqrt{2})\mathbf{e}_2 + (1/\sqrt{2})\mathbf{e}_3$. The normal and shear stress on the plane are

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = 3/2$$

and

$$\sigma_S = \sqrt{\left|\mathbf{t}^{(\mathbf{n})}\right|^2 - \sigma_N^2} = \sqrt{3 - (3/2)^2} = \sqrt{3}/2$$

The normal to the plane is equal to $\mathbf{e}_3'$ and so σ_N should be the same as σ_{33}' and it is. The stress σ_S should be equal to $\sqrt{(\sigma_{31}')^2 + (\sigma_{32}')^2}$ and it is. The results are displayed in Fig. 7.2.9, in which the traction is represented in different ways, with components $\left(t_1^{(\mathbf{n})}, t_2^{(\mathbf{n})}, t_3^{(\mathbf{n})}\right)$ and $\left(\sigma'_{31}, \sigma'_{32}, \sigma'_{33}\right)$.

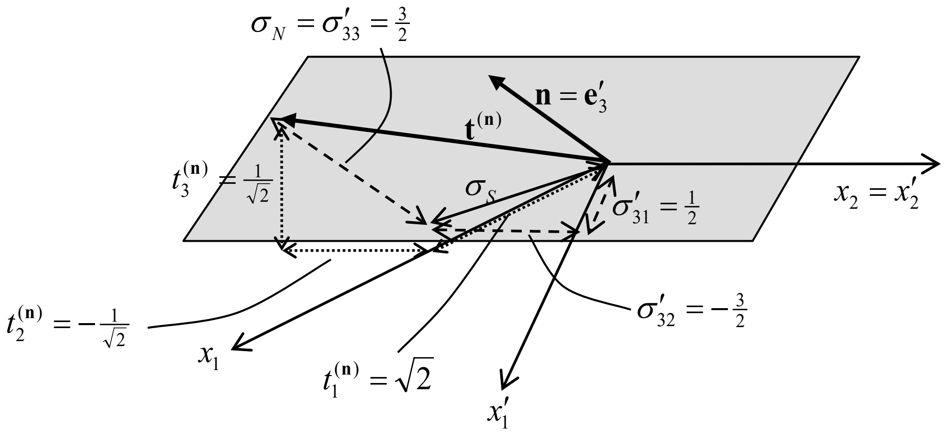


Figure 7.2.9: traction and stresses acting on a plane

Isotropic State of Stress

Suppose the state of stress in a body is

$$\sigma_{ij} = \sigma_0 \delta_{ij} \qquad [\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix} \tag{7.2.17}$$

One finds that the application of the stress tensor transformation rule yields the very same components no matter what the new coordinate system {▲Problem 3}. In other words, no shear stresses act, no matter what the orientation of the plane through the point. This is termed an **isotropic state of stress**, or a **spherical state of stress**. One example of isotropic stress is the stress arising in a fluid at rest, which cannot support shear stress, in which case

$$[\boldsymbol{\sigma}] = -p[\mathbf{I}] \tag{7.2.18}$$

where the scalar p is the fluid **hydrostatic pressure**. For this reason, an isotropic state of stress is also referred to as a **hydrostatic state of stress**.

7.2.4 Principal Stresses

For certain planes through a material particle, there are traction vectors which act normal to the plane, as in Fig. 7.2.10. In this case the traction can be expressed as a scalar multiple of the normal vector, $\mathbf{t}^{(\mathbf{n})} = \sigma \mathbf{n}$.

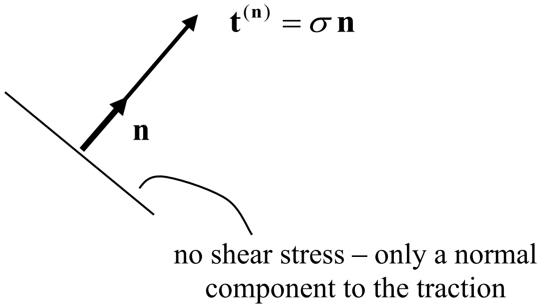


Figure 7.2.10: a purely normal traction vector

From Cauchy's law then, for these planes,

$$\boldsymbol{\sigma}\,\mathbf{n} = \sigma\,\mathbf{n}, \quad \sigma_{ij} n_j = \sigma n_i, \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{7.2.19}$$

This is a standard **eigenvalue problem** from Linear Algebra: given a matrix $\lfloor \sigma_{ij} \rfloor$, find the **eigenvalues** σ and associated **eigenvectors n** such that Eqn. 7.2.19 holds.
To solve the problem, first re-write the equation in the form

$$(\boldsymbol{\sigma} - \sigma\mathbf{I})\mathbf{n} = \mathbf{0}, \quad (\sigma_{ij} - \sigma\delta_{ij})n_j = 0, \quad \left\{ \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \sigma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7.2.20}$$

or

$$\begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7.2.21}$$

This is a set of three homogeneous equations in three unknowns (if one treats σ as known). From basic linear algebra, this system has a solution (apart from $n_i = 0$) if and only if the determinant of the coefficient matrix is zero, i.e. if

$$\det(\boldsymbol{\sigma} - \sigma\mathbf{I}) = \det \begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix} = 0 \tag{7.2.22}$$

Evaluating the determinant, one has the following cubic **characteristic equation** of the stress tensor $\boldsymbol{\sigma}$,

$$\boxed{\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0} \qquad \textbf{Characteristic Equation} \tag{7.2.23}$$

and the **principal scalar invariants** of the stress tensor are

$$
\begin{aligned}
I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
I_2 &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \\
I_3 &= \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{23}\sigma_{31}
\end{aligned}
\qquad (7.2.24)
$$

(I_3 is the determinant of the stress matrix.) The characteristic equation 7.2.23 can now be solved for the eigenvalues σ and then Eqn. 7.2.21 can be used to solve for the eigenvectors **n**.

Now another theorem of linear algebra states that the eigenvalues of a real (that is, the components are real), symmetric matrix (such as the stress matrix) are all real and further that the associated eigenvectors are mutually orthogonal. This means that the three roots of the characteristic equation are real and that the three associated eigenvectors form a mutually orthogonal system. This is illustrated in Fig. 7.2.11; the eigenvalues are called **principal stresses** and are labelled $\sigma_1, \sigma_2, \sigma_3$ and the three corresponding eigenvectors are called **principal directions**, the directions in which the principal stresses act. The planes on which the principal stresses act (to which the principal directions are normal) are called the **principal planes**.

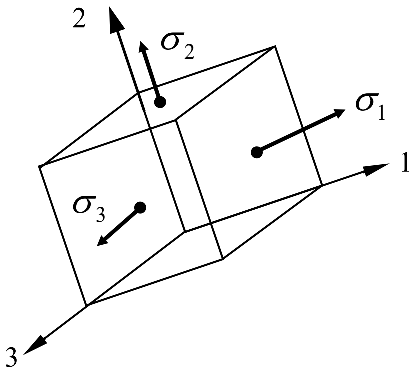


Figure 7.2.11: the three principal stresses acting at a point and the three associated principal directions 1, 2 and 3

Once the principal stresses are found, as mentioned, the principal directions can be found by solving Eqn. 7.2.21, which can be expressed as

$$
\begin{aligned}
(\sigma_{11} - \sigma)n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 &= 0 \\
\sigma_{21}n_1 + (\sigma_{22} - \sigma)n_2 + \sigma_{23}n_3 &= 0 \\
\sigma_{31}n_1 + \sigma_{32}n_2 + (\sigma_{33} - \sigma)n_3 &= 0
\end{aligned}
\qquad (7.2.25)
$$

Each principal stress value in this equation gives rise to the three components of the associated principal direction vector, n_1, n_2, n_3. The solution also requires that the magnitude of the normal be specified: for a unit vector, $\mathbf{n} \cdot \mathbf{n} = 1$. The directions of the normals are also chosen so that they form a right-handed set.

Example

The stress at a point is given with respect to the axes $Ox_1x_2x_3$ by the values

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}.$$

Determine (a) the principal values, (b) the principal directions (and sketch them).

<u>Solution:</u>
(a)
The principal values are the solution to the characteristic equation

$$\begin{vmatrix} 5-\sigma & 0 & 0 \\ 0 & -6-\sigma & -12 \\ 0 & -12 & 1-\sigma \end{vmatrix} = (-10+\sigma)(5-\sigma)(15+\sigma) = 0$$

which yields the three principal values $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$.
(b)
The eigenvectors are now obtained from Eqn. 7.2.25. First, for $\sigma_1 = 10$,

$$\begin{aligned} -5n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 - 16n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 - 9n_3 &= 0 \end{aligned}$$

and using also the equation $n_1^2 + n_2^2 + n_3^2 = 1$ leads to $\mathbf{n}_1 = -(3/5)\mathbf{e}_2 + (4/5)\mathbf{e}_3$. Similarly, for $\sigma_2 = 5$ and $\sigma_3 = -15$, one has, respectively,

$$\begin{aligned} 0n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 - 11n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 - 4n_3 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 20n_1 + 0n_2 + 0n_3 &= 0 \\ 0n_1 + 9n_2 - 12n_3 &= 0 \\ 0n_1 - 12n_2 + 16n_3 &= 0 \end{aligned}$$

which yield $\mathbf{n}_2 = \mathbf{e}_1$ and $\mathbf{n}_3 = (4/5)\mathbf{e}_2 + (3/5)\mathbf{e}_3$. The principal directions are sketched in Fig. 7.2.12. Note that the three components of each principal direction, n_1, n_2, n_3, are the direction cosines: the cosines of the angles between that principal direction and the three coordinate axes. For example, for σ_1 with $n_1 = 0, n_2 = -3/5, n_3 = 4/5$, the angles made with the coordinate axes x_1, x_2, x_3 are, respectively, 90, 126.87° and 36.87°.

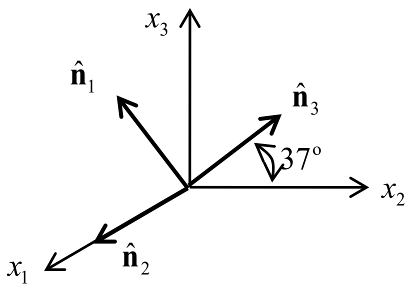


Figure 7.2.12: principal directions

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Invariants

The principal stresses $\sigma_1, \sigma_2, \sigma_3$ are independent of any coordinate system; the $0x_1x_2x_3$ axes to which the stress matrix in Eqn. 7.2.19 is referred can have any orientation – the same principal stresses will be found from the eigenvalue analysis. This is expressed by using the symbolic notation for the problem: $\boldsymbol{\sigma}\mathbf{n} = \sigma\mathbf{n}$, which is independent of any coordinate system. Thus the principal stresses are intrinsic properties of the stress state at a point. It follows that the functions I_1, I_2, I_3 in the characteristic equation Eqn. 7.2.23 are also independent of any coordinate system, and hence the name principal scalar invariants (or simply **invariants**) of the stress.

The stress invariants can also be written neatly in terms of the principal stresses:

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{7.2.26}$$

Also, if one chooses a coordinate system to coincide with the principal directions, Fig. 7.2.12, the stress matrix takes the simple form

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{7.2.27}$$

Note that when two of the principal stresses are equal, one of the principal directions will be unique, but the other two will be arbitrary – one can choose any two principal directions in the plane perpendicular to the uniquely determined direction, so that the three form an orthonormal set. This stress state is called **axi-symmetric**. When all three principal stresses are equal, one has an isotropic state of stress, and all directions are principal directions – the stress matrix has the form 7.2.27 no matter what orientation the planes through the point.

Example

The two stress matrices from the Example of §7.2.3, describing the stress state at a point with respect to different coordinate systems, are

$$\left[\sigma_{ij}\right] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}, \qquad \left[\sigma'_{ij}\right] = \begin{bmatrix} 3/2 & 3/\sqrt{2} & 1/2 \\ 3/\sqrt{2} & 3 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 3/2 \end{bmatrix}$$

The first invariant is the sum of the normal stresses, the diagonal terms, and is the same for both as expected:

$$I_1 = 2 + 3 + 1 = \tfrac{3}{2} + 3 + \tfrac{3}{2} = 6$$

The other invariants can also be obtained from either matrix, and are

$$I_2 = 6, \;\; I_3 = -3$$

■

7.2.5 Maximum and Minimum Stress Values

Normal Stresses

The three principal stresses include the maximum and minimum normal stress components acting at a point. To prove this, first let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors *in the principal directions*. Consider next an arbitrary unit normal vector $\mathbf{n} = n_i \mathbf{e}_i$. From Cauchy's law (see Fig. 7.2.13 – the stress matrix in Cauchy's law is now with respect to the principal directions 1, 2 and 3), the normal stress acting on the plane with normal $\mathbf{n}$ is

$$\sigma_N = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{n}, \qquad \sigma_N = \sigma_{ij} n_j n_i \tag{7.2.28}$$

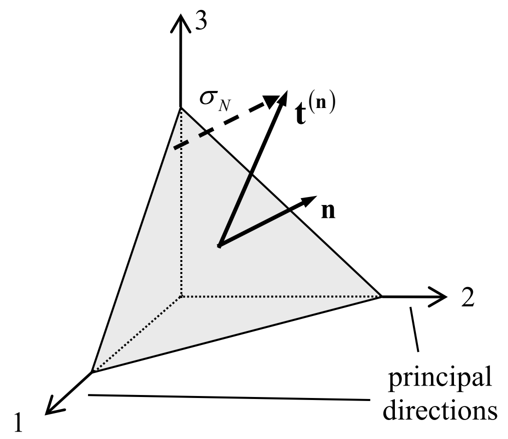

Figure 7.2.13: normal stress acting on a plane defined by the unit normal n

Thus

$$\sigma_N = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \right\} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{7.2.29}$$

Since $n_1^2 + n_2^2 + n_3^2 = 1$ and, without loss of generality, taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, one has

$$\sigma_1 = \sigma_1 \left(n_1^2 + n_2^2 + n_3^2 \right) \geq \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 = \sigma_N \tag{7.2.30}$$

Similarly,

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \geq \sigma_3 \left(n_1^2 + n_2^2 + n_3^2 \right) \geq \sigma_3 \tag{7.2.31}$$

Thus the maximum normal stress acting at a point is the maximum principal stress and the minimum normal stress acting at a point is the minimum principal stress.

Shear Stresses

Next, it will be shown that the maximum shearing stresses at a point act on planes oriented at 45° to the principal planes and that they have magnitude equal to half the difference between the principal stresses. First, again, let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors in the principal directions and consider an arbitrary unit normal vector $\mathbf{n} = n_i \mathbf{e}_i$. The normal stress is given by Eqn. 7.2.29,

$$\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{7.2.32}$$

Cauchy's law gives the components of the traction vector as

$$\begin{bmatrix} t_1^{(\mathbf{n})} \\ t_1^{(\mathbf{n})} \\ t_1^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 n_1 \\ \sigma_2 n_2 \\ \sigma_3 n_3 \end{bmatrix} \tag{7.2.33}$$

and so the shear stress on the plane is, from Eqn. 7.2.11,

$$\sigma_S^2 = \left(\sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2\right) - \left(\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2\right)^2 \tag{7.2.34}$$

Using the condition $n_1^2 + n_2^2 + n_3^2 = 1$ to eliminate n_3 leads to

$$\sigma_S^2 = \left(\sigma_1^2 - \sigma_3^2\right)n_1^2 + \left(\sigma_2^2 - \sigma_3^2\right)n_2^2 + \sigma_3^2 - \left[(\sigma_1 - \sigma_3)n_1^2 + (\sigma_2 - \sigma_3)n_2^2 + \sigma_3\right]^2 \tag{7.2.35}$$

The stationary points are now obtained by equating the partial derivatives with respect to the two variables n_1 and n_2 to zero:

$$\begin{aligned} \frac{\partial(\sigma_S^2)}{\partial n_1} &= n_1(\sigma_1 - \sigma_3)\left\{\sigma_1 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3)n_1^2 + (\sigma_2 - \sigma_3)n_2^2\right]\right\} = 0 \\ \frac{\partial(\sigma_S^2)}{\partial n_2} &= n_2(\sigma_2 - \sigma_3)\left\{\sigma_2 - \sigma_3 - 2\left[(\sigma_1 - \sigma_3)n_1^2 + (\sigma_2 - \sigma_3)n_2^2\right]\right\} = 0 \end{aligned} \tag{7.2.36}$$

One sees immediately that $n_1 = n_2 = 0$ (so that $n_3 = \pm 1$) is a solution; this is the principal direction $\mathbf{e}_3$ and the shear stress is by definition zero on the plane with this normal. In this calculation, the component n_3 was eliminated and σ_S^2 was treated as a function of the variables (n_1, n_2). Similarly, n_1 can be eliminated with (n_2, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_1$, and n_2 can be eliminated with (n_1, n_3) treated as the variables, leading to the solution $\mathbf{n} = \mathbf{e}_2$. Thus these solutions lead to the minimum shear stress value $\sigma_S^2 = 0$.

A second solution to Eqn. 7.2.36 can be seen to be $n_1 = 0, n_2 = \pm 1/\sqrt{2}$ (so that $n_3 = \pm 1/\sqrt{2}$) with corresponding shear stress values $\sigma_S^2 = \frac{1}{4}(\sigma_2 - \sigma_3)^2$. Two other solutions can be obtained as described earlier, by eliminating n_1 and by eliminating n_2. The full solution is listed below, and these are evidently the maximum (absolute value of the) shear stresses acting at a point:

$$\begin{aligned}
\mathbf{n} &= \left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_2 - \sigma_3\right| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_3 - \sigma_1\right| \\
\mathbf{n} &= \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right), \quad \sigma_S = \frac{1}{2}\left|\sigma_1 - \sigma_2\right|
\end{aligned} \tag{7.2.37}$$

Taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the maximum shear stress at a point is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) \tag{7.2.38}$$

and acts on a plane with normal oriented at 45° to the 1 and 3 principal directions. This is illustrated in Fig. 7.2.14.

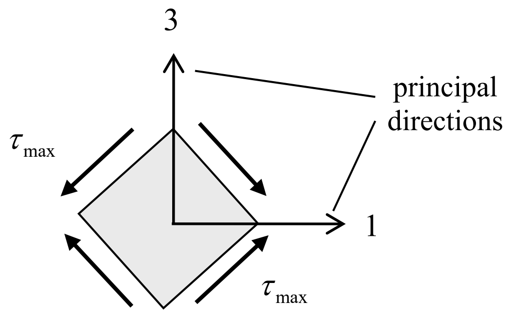

Figure 7.2.14: maximum shear stress at a point

Example

Consider the stress state examined in the Example of §7.2.4:

$$[\sigma_{ij}] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & 1 \end{bmatrix}$$

The principal stresses were found to be $\sigma_1 = 10, \sigma_2 = 5, \sigma_3 = -15$ and so the maximum shear stress is

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{25}{2}$$

One of the planes upon which they act is shown in Fig. 7.2.15 (see Fig. 7.2.12)

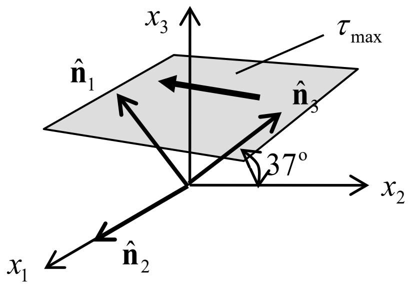

Figure 7.2.15: maximum shear stress

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7.2.6 Mohr's Circles of Stress

The Mohr's circle for 2D stress states was discussed in Book I, §3.5.5. For the 3D case, following on from section 7.2.5, one has the conditions

$$\begin{aligned} \sigma_N &= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \\ \sigma_S^2 + \sigma_N^2 &= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 \\ n_1^2 + n_2^2 + n_3^2 &= 1 \end{aligned} \tag{7.2.39}$$

Solving these equations gives

$$\begin{aligned} n_1^2 &= \frac{(\sigma_N - \sigma_2)(\sigma_N - \sigma_3) + \sigma_S^2}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \\ n_2^2 &= \frac{(\sigma_N - \sigma_3)(\sigma_N - \sigma_1) + \sigma_S^2}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \\ n_3^2 &= \frac{(\sigma_N - \sigma_1)(\sigma_N - \sigma_2) + \sigma_S^2}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \end{aligned} \tag{7.2.40}$$

Taking $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and noting that the squares of the normal components must be positive, one has that

$$\begin{aligned} (\sigma_N - \sigma_2)(\sigma_N - \sigma_3) + \sigma_S^2 &\geq 0 \\ (\sigma_N - \sigma_3)(\sigma_N - \sigma_1) + \sigma_S^2 &\leq 0 \\ (\sigma_N - \sigma_1)(\sigma_N - \sigma_2) + \sigma_S^2 &\geq 0 \end{aligned} \tag{7.2.41}$$

and these can be re-written as

$$\begin{aligned} \sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_2 + \sigma_3)\right]^2 &\geq \left[\tfrac{1}{2}(\sigma_2 - \sigma_3)\right]^2 \\ \sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right]^2 &\leq \left[\tfrac{1}{2}(\sigma_1 - \sigma_3)\right]^2 \\ \sigma_S^2 + \left[\sigma_N - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right]^2 &\geq \left[\tfrac{1}{2}(\sigma_1 - \sigma_2)\right]^2 \end{aligned} \tag{7.2.42}$$

If one takes coordinates (σ_N, σ_S), the equality signs here represent circles in (σ_N, σ_S) stress space, Fig. 7.2.16. Each point (σ_N, σ_S) in this stress space represents the stress on a particular plane through the material particle in question. Admissible (σ_N, σ_S) pairs are given by the conditions Eqns. 7.2.42; they must lie inside a circle of centre $\left(\frac{1}{2}(\sigma_1 + \sigma_3), 0\right)$ and radius $\frac{1}{2}(\sigma_1 - \sigma_3)$. This is the large circle in Fig. 7.2.16. The points must lie outside the circle with centre $\left(\frac{1}{2}(\sigma_2 + \sigma_3), 0\right)$ and radius $\frac{1}{2}(\sigma_2 - \sigma_3)$ and also outside the circle with centre $\left(\frac{1}{2}(\sigma_1 + \sigma_2), 0\right)$ and radius $\frac{1}{2}(\sigma_1 - \sigma_2)$; these are the two smaller circles in the figure. Thus the admissible points in stress space lie in the shaded region of Fig. 7.2.16.

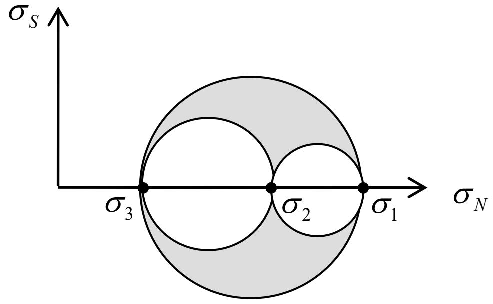


Figure 7.2.16: admissible points in stress space

7.2.7 Three Dimensional Strain

The strain ε_{ij}, in symbolic form $\boldsymbol{\varepsilon}$, is a tensor and as such it follows the same rules as for the stress tensor. In particular, it follows the general tensor transformation rule 7.2.16; it has principal values ε which satisfy the characteristic equation 7.2.23 and these include the maximum and minimum normal strain at a point. There are three principal strain invariants given by 7.2.24 or 7.2.26 and the maximum shear strain occurs on planes oriented at 45° to the principal directions.

7.2.8 Problems

1. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & -2 \end{bmatrix}$$

Use Cauchy's law to determine the traction vector acting on a plane trough this point whose unit normal is $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$. What is the normal stress acting on the plane? What is the shear stress acting on the plane?

2. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

What are the stress components with respect to axes $0x_1'x_2'x_3'$ which are obtained from the first by a 45° rotation (positive counterclockwise) about the x_3 axis

3. Show, in both the index and matrix notation, that the components of an isotropic stress state remain unchanged under a coordinate transformation.

4. Consider a two-dimensional problem. The stress transformation formulae are then, in full,

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Multiply the right hand side out and use the fact that the stress tensor is symmetric ($\sigma_{12} = \sigma_{21}$ - not true for all tensors). What do you get? Look familiar?

5. The state of stress at a point with respect to a $0x_1x_2x_3$ coordinate system is given by

$$[\sigma_{ij}] = \begin{bmatrix} 5/2 & -1/2 & 0 \\ -1/2 & 5/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Evaluate the principal stresses and the principal directions. What is the maximum shear stress acting at the point?

7.3 Governing Equations of Three Dimensional Elasticity

7.3.1 Hooke's Law and Lamé's Constants

Linear elasticity was introduced in Book I, §6. The three-dimensional Hooke's law for isotropic linear elastic solids (Book I, Eqns. 6.1.9) can be expressed in index notation as

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \tag{7.3.1}$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{7.3.2}$$

are the Lamé constants (μ is the Shear Modulus). Eqns. 7.3.1 can be inverted to obtain (see Book I, Eqns. 6.1.8) {▲Problem 1}

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\delta_{ij}\sigma_{kk} \tag{7.3.3}$$

7.3.2 Navier's Equations

The governing equations of elasticity are Hooke's law (Eqn. 7.3.1), the equations of motion, Eqn. 1.1.9 (see Eqns. 7.1.10-11),

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho a_i \tag{7.3.4}$$

and the strain-displacement relations, Eqn. 1.2.19 (see Eqns. 7.1.25-26),

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{7.3.5}$$

Substituting 7.3.5 into 7.3.1 and then into 7.3.4 leads to the 3D Navier's equations {▲Problem 2}

$$\boxed{(\lambda + \mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + b_i = \rho a_i} \qquad \textbf{Navier's Equations} \tag{7.3.6}$$

These reduce to the 2D plane strain Navier's equations, Eqns. 3.1.4, by setting $u_3 = 0$ and $\partial / \partial x_3 = 0$. They do not reduce to the plane stress equations since the latter are only an approximate solution to the equations of elasticity which are valid only in the limit as the thickness of the thin plate of plane stress tends to zero.

In terms of the dilatation and the rotation

The dilatation (unit volume change) is the scalar invariant

$$e = \varepsilon_{ii} = \operatorname{div}\mathbf{u} = \nabla \cdot \mathbf{u} = \frac{\partial u_j}{\partial x_j} \tag{7.3.7}$$

The gradient of this scalar is then the vector

$$\text{grad div}\mathbf{u} \equiv \nabla(\nabla \cdot \mathbf{u}) \equiv \frac{\partial^2 u_j}{\partial x_i \partial x_j}. \tag{7.3.8}$$

The Lapacian of the displacement vector is defined to be the vector

$$\nabla^2 \mathbf{u} \equiv \frac{\partial^2 u_i}{\partial x_j \partial x_j}. \tag{7.3.9}$$

Navier's equations can thus be expressed in the vector notation

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \mathbf{b} = \rho \ddot{\mathbf{u}} \tag{7.3.10}$$

or, in full, and in terms of the dilatation,

$$\begin{aligned}
(\lambda + \mu)\frac{\partial e}{\partial x_1} + \mu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) + b_1 &= \rho \frac{\partial u_1}{\partial t^2} \\
(\lambda + \mu)\frac{\partial e}{\partial x_2} + \mu\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}\right) + b_2 &= \rho \frac{\partial u_2}{\partial t^2} \\
(\lambda + \mu)\frac{\partial e}{\partial x_3} + \mu\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2}\right) + b_1 &= \rho \frac{\partial u_3}{\partial t^2}
\end{aligned} \tag{7.3.11}$$

Using the vector relation

$$\nabla^2 \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla \times (\nabla \times \mathbf{a}), \tag{7.3.12}$$

Naviers equations can also be expressed in the form

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u}) + \mathbf{b} = \rho \ddot{\mathbf{u}} \tag{7.3.13}$$

Now the curl of the displacement is the rotation vector

$$\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{u} \tag{7.3.14}$$

In full, these are (see Eqns. 1.2.20):

$$\omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right), \quad \omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right), \quad \omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{7.3.15}$$

For example, as illustrated in Fig. 7.3.1, the third component is the (angle of) rotation of material about the x_3 axis.

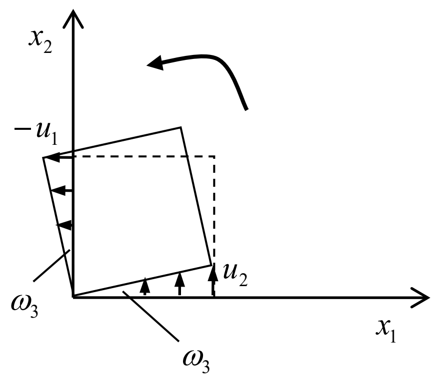


Figure 7.3.1: a rotation

Navier's equations can thus be expressed as

$$(\lambda + 2\mu)\nabla e - 2\mu\nabla \times \boldsymbol{\omega} + \mathbf{b} = \rho\ddot{\mathbf{u}} \tag{7.3.16}$$

7.3.3 Problems

1. Invert Eqns. 7.3.1 to get 7.3.3.

2. Derive the 3D Navier's equations from 7.3.6 from 7.3.1, 7.3.4 and 7.3.5

7.4 Elastodynamics

In this section, solutions to the Navier's equations are derived, which show that two different types of wave travel through elastic solids. The (gravitational) body force will be neglected, as it is a relatively small term in most practical applications of wave propagation through elastic solids.

7.4.1 Solutions to Navier's equations

Navier's equations are given by Eqn. 7.3.6 (and 7.3.10)

$$\begin{gathered}(\lambda+\mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}=\rho a_i \\ (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}=\rho\ddot{\mathbf{u}}\end{gathered} \tag{7.4.1}$$

It was seen that Navier's equations could also be expressed in terms of the dilataion and rotation, Eqn. 7.3.16:

$$\begin{gathered}(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u})-\mu\nabla\times(\nabla\times\mathbf{u})=\rho\ddot{\mathbf{u}} \\ (\lambda+2\mu)\nabla e-2\mu\nabla\times\boldsymbol{\omega}=\rho\ddot{\mathbf{u}}\end{gathered} \tag{7.4.2}$$

Navier's equations can be reduced to the three-dimensional wave equation in one of two ways.

P Waves

First, taking the divergence of Eqn. 7.4.2 leads to

$$(\lambda+2\mu)\nabla\cdot\nabla e-\mu\nabla\cdot\left(\nabla\times(\nabla\times\mathbf{u})\right)=\rho\nabla\cdot\ddot{\mathbf{u}} \tag{7.4.3}$$

But the divergence of the curl of any vector is zero:

$$\begin{aligned}\text{div curl}\mathbf{a}&=\text{div}\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ a_1 & a_2 & a_3\end{vmatrix} \\ &=\frac{\partial}{\partial x_1}\left(\frac{\partial a_3}{\partial x_2}-\frac{\partial a_2}{\partial x_3}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial a_1}{\partial x_3}-\frac{\partial a_3}{\partial x_1}\right)+\frac{\partial}{\partial x_3}\left(\frac{\partial a_2}{\partial x_1}-\frac{\partial a_1}{\partial x_2}\right)=0\end{aligned} \tag{7.4.4}$$

and so one has

$$(\lambda+2\mu)\nabla^2 e=\rho\ddot{e} \tag{7.4.5}$$

In full, this is the three dimensional wave equation:

$$\frac{\partial^2 e}{\partial x_1^2} + \frac{\partial^2 e}{\partial x_2^2} + \frac{\partial^2 e}{\partial x_3^2} = \frac{1}{c_L^2}\frac{\partial^2 e}{\partial t^2} \tag{7.4.6}$$

where the wave speed is

$$c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} \tag{7.4.7}$$

This solution predicts that a wave propagates through an elastic solid at wave speed c_L, and the wave excites volume changes within the material it passes, as characterised by the dilatation *e*.

Such waves are called **P-waves** (for reasons which will be explained later).

S Waves

A second wave equation can be derived from Navier's equations; taking the curl of 7.4.2 leads to:

$$(\lambda + 2\mu)\nabla \times (\nabla e) - 2\mu \nabla \times (\nabla \times \boldsymbol{\omega}) = \rho \nabla \times \ddot{\mathbf{u}} \tag{7.4.8}$$

Now the curl of the gradient of any scalar is zero:

$$\text{curl grad}e = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial / \partial x_1 & \partial / \partial x_2 & \partial / \partial x_3 \\ \partial e / \partial x_1 & \partial e / \partial x_2 & \partial e / \partial x_3 \end{vmatrix} = 0 \tag{7.4.9}$$

Further, using again the vector identity Eqn. 7.3.12,

$$-\nabla \times (\nabla \times \boldsymbol{\omega}) = \nabla^2 \boldsymbol{\omega} - \nabla(\nabla \cdot \boldsymbol{\omega}) \tag{7.4.10}$$

From Eqn. 7.4.4, the second term on the right is again zero, so one has the alternative wave equation:

$$\mu \nabla^2 \boldsymbol{\omega} = \rho \ddot{\boldsymbol{\omega}} \tag{7.4.11}$$

which in full reads

$$\frac{\partial^2 \omega_i}{\partial x_1^2} + \frac{\partial^2 \omega_i}{\partial x_2^2} + \frac{\partial^2 \omega_i}{\partial x_3^2} = \frac{1}{c_T^2}\frac{\partial^2 \omega_i}{\partial t^2} \tag{7.4.12}$$

where the wave speed is

$$c_T = \sqrt{\frac{\mu}{\rho}} \tag{7.4.13}$$

This solution predicts that a wave propagates through an elastic solid at wave speed c_T, and the wave excites rotation within the material it passes, as characterised by the rotation vector $\boldsymbol{\omega}$.

Such waves are called **S-waves**.

7.4.2 Longitudinal and Transverse Waves

Consider now a sinusoidal wave of the general form

$$\begin{aligned} \mathbf{u} &= \mathbf{a}\sin\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right) \\ &= \mathbf{a}\sin\left[\mathbf{k}\cdot\left(\mathbf{x} - \mathbf{c}t\right)\right] \end{aligned} \tag{7.4.14}$$

This is a displacement with amplitude **a**, wave vector **k** and angular frequency ω. The wave vector indicates the direction of the propagation of the wave, through the wave velocity vector:

$$\mathbf{c} = \frac{\omega}{|\mathbf{k}|^2}\mathbf{k} \tag{7.4.15}$$

In substituting Eqn. 7.4.14 into Navier's equations 7.4.1, make the substitution

$$\begin{aligned} \xi &= k_m x_m - \omega t \\ \frac{\partial \xi}{\partial x_i} &= k_m \frac{\partial x_m}{\partial x_i} = k_m \delta_{mi} = k_i \\ \frac{\partial u_i}{\partial x_j} &= \frac{\partial}{\partial x_j} a_i \sin\xi = k_j a_i \cos\xi \\ \frac{\partial^2 u_i}{\partial x_j \partial x_j} &= \frac{\partial}{\partial x_j} k_j a_i \cos\xi = -k_j k_j a_i \sin\xi \\ \frac{\partial^2 u_i}{\partial x_i \partial x_j} &= \frac{\partial}{\partial x_i} k_j a_i \cos\xi = -k_i k_j a_i \sin\xi \\ \frac{\partial^2 u_i}{\partial t^2} &= \frac{\partial}{\partial t^2} a_i \sin\xi = a_i \sin\xi \end{aligned} \tag{7.4.1}$$

and using the relations

$$(\lambda+\mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}=\rho a_i$$

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}=\rho\ddot{\mathbf{u}} \tag{7.4.1}$$

7.4.3 Description in terms of Potentials

The above analysis can be expressed in terms of vector potentials, as described next.

It can be shown that any vector **u** can be written in the form[1]

$$\mathbf{u}=\nabla\phi+\mathrm{curl}\mathbf{a} \tag{7.4.17}$$

where ϕ is a **scalar potential** and **a** is a **vector potential**. These two terms in the displacement field can be examined separately. The most general displacement field can be obtained by adding both solutions together.

P Waves (Irrotational Waves)

First looking at the scalar potential term, suppose that the displacement is given by $\mathbf{u}=\nabla\phi$. Now, as in Eqn. 7.4.9, the curl of $\nabla\phi$ is zero, and so $\mathrm{curl}\mathbf{u}=\mathbf{0}$.Thus the rotation $\boldsymbol{\omega}=\mathbf{0}$. Thus there is *no rotation* of material particles.

Taking the displacement field $\mathbf{u}=\nabla\phi$, writing it in index notation, $u_j=\partial\phi/\partial x_j$, and substituting into Navier's equations, leads to the three-dimensional wave equation:

$$\frac{\partial^2 u_i}{\partial x_k \partial x_k}=\frac{1}{c_L^2}\frac{\partial^2 u_i}{\partial t^2} \tag{7.4.4}$$

This displacement field thus corresponds to stress waves travelling at speed c_L, causing material to strain but not to rotate. These **irrotational waves** are also called **waves of dilatation**.

Equivoluminal Waves

Consider now the displacement field $\mathbf{u}=\mathrm{curl}\mathbf{a}$. If one can find a vector **a** such that $\mathbf{u}=\mathrm{curl}\mathbf{a}$, then it follows that $e=\nabla\cdot\mathbf{u}=0$. Thus the condition that the displacement field be divergence-free implies that there is *no volume change*. There can be normal strains only so long as their sum is zero.

Taking $\varepsilon_{kk}=\partial u_k/\partial x_k$ and substituting into Navier's equations then leads immediately to

[1] from the **Helmholtz theory**

$$\mu \frac{\partial^2 u_i}{\partial x_k \partial x_k} = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{7.4.6}$$

or the three-dimensional wave equation:

$$\frac{\partial^2 u_i}{\partial x_k \partial x_k} = \frac{1}{c_T^2} \frac{\partial^2 u_i}{\partial t^2}, \qquad c_T = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{2\rho(1+\nu)}} \tag{7.4.7}$$

This displacement field thus corresponds to stress waves travelling at speed c_T, causing material to shear. These equivoluminal waves are also called **shear waves** or **waves of distortion**.

In summary, when an event such as an explosion occurs, two different types of wave emerge, irrotational waves which result in irrotational displacement fields, and equivoluminal waves which result in equivoluminal displacements. These waves travel at different speeds.

7.4.4 Plane Waves

At a sufficient distance from any initial disturbance, a stress wave will travel in a plane. It can be assumed that all material particles will displace either parallel to the direction of wave propagation (**longitudinal waves**) or perpendicular to this direction (**transverse waves**).

Let the wave travel in the x_1 direction.

Irrotational (p / longitudinal) Plane Waves

Consider particles which displace in the direction of wave propagation according to $\mathbf{u} = u_1(x_1, t)\mathbf{e}_1$. This is an irrotational wave since $\mathrm{curl}\mathbf{u} = \mathbf{0}$, and the stress wave is governed by the one-dimensional wave equation

$$\frac{\partial^2 u_1}{\partial x_1^2} = \frac{1}{c_L^2} \frac{\partial^2 u_1}{\partial t^2} \tag{7.4.8}$$

These longitudinal plane waves are also called **p-waves**[2].

[2] p stands for "primary"

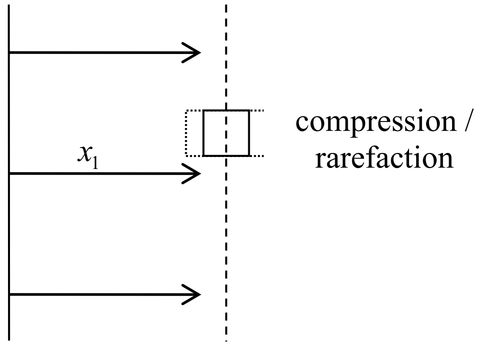


Figure 7.4.2: a longitudinal wave

Equivoluminal (s / transverse / shear) Plane Waves

Consider particles which displace according to $\mathbf{u} = u_2(x_1, t)\mathbf{e}_2$. This is an equivoluminal wave since $\nabla \cdot \mathbf{u} = 0$, and the stress wave is governed by the one-dimensional wave equation

$$\frac{\partial^2 u_2}{\partial x_1^2} = \frac{1}{c_T^2} \frac{\partial^2 u_2}{\partial t^2} \tag{7.4.9}$$

These transverse/shear waves are also called **s-waves**[3].

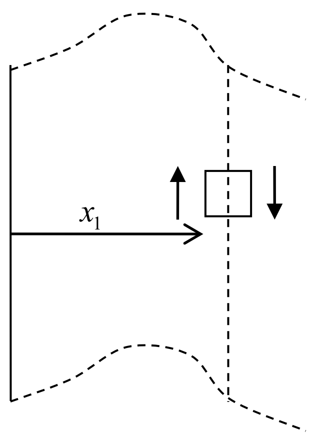
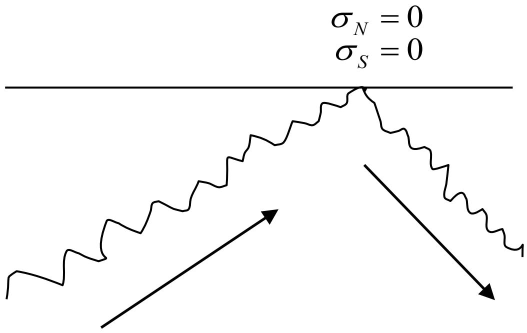


Figure 7.4.3: a transverse wave

7.4.5 Vibration Analysis

[3] s stands for "secondary"

A vibration analysis can be carried out in exactly the same way as in Chapter 2, only the wave speeds in the 1D wave equations 7.4.8 and 7.4.9 are now different from the 1D speed $\sqrt{E/\rho}$. The particular solutions, forced vibration and resonance theory of Chapter 2 can again be applied here. The analysis here is appropriate for thin plates “infinitely wide” in the x_2, x_3 directions, Fig. 7.4.4. The figure shows longitudinal vibration, but one can also have transverse vibration where the particles displace perpendicular to the x_1 axis.



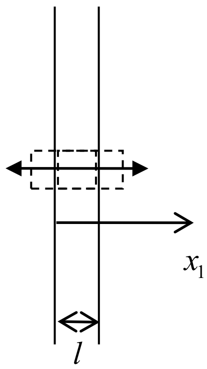


Figure 7.4.4: stretch vibration of a plate

7.4.6 Waves at Boundaries

Plane waves exist in unbounded elastic continua. In a finite body, a plane wave will be *reflected* when it hits a free surface. In this case, one needs to solve Navier’s equations subject to the boundary conditions of zero normal and shear stress at the free surface. Waves of both types will in general be reflected for any single type of incident wave.

Similarly, when a wave meets an interface between two different materials, there will be reflection and refraction. The boundary conditions are that the displacements are continuous and the normal and shear stresses are continuous, Fig. 7.4.5

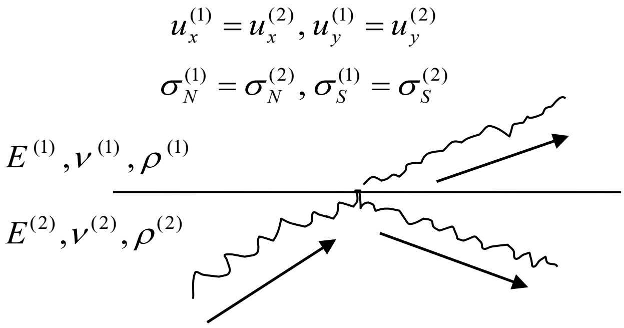


Figure 7.4.5: reflection and refraction of a wave at an interface

7.4.7 Waves at Boundaries

The waves discussed thus far are **body waves**. When a free surface exists, for example the surface of the earth, another type of wave motion is possible; these are the **Rayleigh waves** and travel along the surface very much like water waves. It can be shown that the speed of Rayleigh waves is between 90% and 95% of c_T, depending on the value of Poisson's ratio. Similar types of waves can propagate along the interface between two different materials.

7.4.8 Problems

1. Consider the motion
$$u_1 = \bar{u}\sin\frac{2\pi}{l}(x_1 - ct),\, u_2 = 0,\, u_3 = 0\,,$$
What are the strains in the material? What are the corresponding stresses? What is the volume change in the material? What is the name (or names) given to the type of wave which causes this kind of motion?

2. Consider the motion
$$u_1 = 0,\, u_2 = \bar{u}\sin\frac{2\pi}{l}(x_1 - ct),\, u_3 = 0\,,$$
What are the strains in the material? What are the corresponding stresses? What is the volume change in the material? What is the name (or names) given to the type of wave which causes this kind of motion?

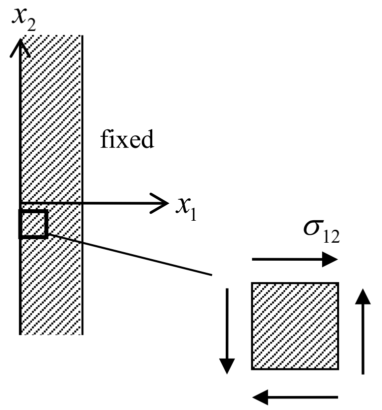
3. Derive an expression for the ratio c_L / c_T in terms of the material's Poisson's ratio only. Which is the faster, the longitudinal or transverse wave?

4. Show that the motion
$$u_1 = 0,\, u_2 = 0,\, u_3 = \bar{u}\cos(px_2)\cos\frac{2\pi}{l}(x_1 - ct)\,,$$
is equivoluminal.

5. Consider the motion
$$u_1 = \bar{u}[\sin\beta(x_3 - ct) + \alpha\sin\beta(x_3 + ct)],\, u_2 = 0,\, u_3 = 0$$
(i) what kind of elastic stress wave does this involve ? (Sketch the plane of the wave and its direction of propagation.)
(ii) what are the strains and stresses.
(iii) use the equations of motion to determine the wave speed. Is it what you expected?
(iv) Suppose that the plane $x_3 = 0$ is a free surface. Determine α.
(v) Suppose also that $x_3 = h$ is a free surface. Determine β.

6. Consider a plate with left face $(x_1 = 0)$ subjected to a forced displacement $\mathbf{u} = \alpha \sin \Omega t \mathbf{e}_1$ and the right face $(x_1 = l)$ free.
   (i) find the "thickness-stretch" vibration of the plate. What are the natural frequencies?
   (ii) When does resonance occur?

7. Consider a plate with left face $(x_1 = 0)$ subjected to a traction $\mathbf{t} = -\alpha \cos \Omega t \mathbf{e}_2$ and the right face $(x_1 = l)$ fixed, as shown in the figure below.
   (i) find the "thickness-shear" vibration of the plate. What are the natural frequencies?
   (ii) When does resonance occur?



[note: assume a displacement $\mathbf{u} = u_2(x_1, t)\mathbf{e}_2$; as with transverse waves, this will satisfy the 1-d wave equation with *c* being the transverse wave speed. Use the traction to obtain an expression for the shear stress σ_{12} over the left hand face. When applying the stress boundary condition, you will need the strain-displacement expression and stress-strain law,

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \qquad \varepsilon_{12} = \frac{1}{2\mu}\sigma_{12}$$

8. Consider the case of $\mathbf{u} = \alpha(\cos \Omega t \mathbf{e}_2 + \sin \Omega t \mathbf{e}_3)$ over the left face $(x_1 = 0)$ with the right face $(x_1 = l)$ fixed. Derive an expression for the particular solution and show that it represents circular motion of the particles in the $x_2 - x_3$ plane.
   [hint: evaluate the particular solutions for u_2 and u_3 separately and then show that $u_2^2 + u_3^2 = r^2$ for some *r* (independent of time)]

8.1 Introduction to Plasticity

8.1.1 Introduction

The theory of linear elasticity is useful for modelling materials which undergo small deformations and which return to their original configuration upon removal of load. Almost all real materials will undergo some **permanent** deformation, which remains after removal of load. With metals, significant permanent deformations will usually occur when the stress reaches some critical value, called the **yield stress**, a material property.

Elastic deformations are termed **reversible;** the energy expended in deformation is stored as elastic strain energy and is completely recovered upon load removal. Permanent deformations involve the dissipation of energy; such processes are termed **irreversible**, in the sense that the original state can be achieved only by the expenditure of more energy.

The **classical theory of plasticity** grew out of the study of metals in the late nineteenth century. It is concerned with materials which initially deform elastically, but which deform **plastically** upon reaching a yield stress. In metals and other crystalline materials the occurrence of plastic deformations at the micro-scale level is due to the motion of dislocations and the migration of grain boundaries on the micro-level. In sands and other granular materials plastic flow is due both to the irreversible rearrangement of individual particles and to the irreversible crushing of individual particles. Similarly, compression of bone to high stress levels will lead to particle crushing. The deformation of micro-voids and the development of micro-cracks is also an important cause of plastic deformations in materials such as rocks.

A good part of the discussion in what follows is concerned with the plasticity of metals; this is the 'simplest' type of plasticity and it serves as a good background and introduction to the modelling of plasticity in other material-types. There are two broad groups of metal plasticity problem which are of interest to the engineer and analyst. The first involves relatively small plastic strains, often of the same order as the elastic strains which occur. Analysis of problems involving small plastic strains allows one to design structures optimally, so that they will not fail when in service, but at the same time are not stronger than they really need to be. In this sense, plasticity is seen as a material **failure**[1].

The second type of problem involves very large strains and deformations, so large that the elastic strains can be disregarded. These problems occur in the analysis of metals manufacturing and forming processes, which can involve extrusion, drawing, forging, rolling and so on. In these latter-type problems, a simplified model known as **perfect plasticity** is usually employed (see below), and use is made of special **limit theorems** which hold for such models.

Plastic deformations are normally **rate independent,** that is, the stresses induced are independent of the rate of deformation (or rate of loading). This is in marked

---

[1] two other types of failure, *brittle fracture*, due to dynamic crack growth, and the *buckling* of some structural components, can be modelled reasonably accurately using elasticity theory (see, for example, Part I, §6.1, Part II, §5.3)

contrast to classical **Newtonian fluids** for example, where the stress levels are governed by the *rate* of deformation through the viscosity of the fluid.

Materials commonly known as "plastics" are not plastic in the sense described here. They, like other polymeric materials, exhibit **viscoelastic** behaviour where, as the name suggests, the material response has both elastic and viscous components. Due to their viscosity, their response is, unlike the plastic materials, **rate-dependent**. Further, although the viscoelastic materials can suffer irrecoverable deformation, they do not have any critical yield or threshold stress, which is the characteristic property of plastic behaviour. When a material undergoes plastic deformations, i.e. irrecoverable and at a critical yield stress, and these effects *are* rate dependent, the material is referred to as being **viscoplastic**.

Plasticity theory began with Tresca in 1864, when he undertook an experimental program into the extrusion of metals and published his famous yield criterion discussed later on. Further advances with yield criteria and plastic flow rules were made in the years which followed by Saint-Venant, Levy, Von Mises, Hencky and Prandtl. The 1940s saw the advent of the classical theory; Prager, Hill, Drucker and Koiter amongst others brought together many fundamental aspects of the theory into a single framework. The arrival of powerful computers in the 1980s and 1990s provided the impetus to develop the theory further, giving it a more rigorous foundation based on thermodynamics principles, and brought with it the need to consider many numerical and computational aspects to the plasticity problem.

8.1.2 Observations from Standard Tests

In this section, a number of phenomena observed in the material testing of metals will be noted. Some of these phenomena are simplified or ignored in some of the standard plasticity models discussed later on.

At issue here is the fact that any model of a component with complex geometry, loaded in a complex way and undergoing plastic deformation, must involve material parameters which can be obtained in a straight forward manner from simple laboratory tests, such as the tension test described next.

The Tension Test

Consider the following key experiment, the **tensile test**, in which a small, usually cylindrical, specimen is gripped and stretched, usually at some given rate of stretching (see Part I, §5.2.1). The force required to hold the specimen at a given stretch is recorded, Fig. 8.1.1. If the material is a metal, the deformation remains elastic up to a certain force level, the yield point of the material. Beyond this point, permanent plastic deformations are induced. On unloading only the elastic deformation is recovered and the specimen will have undergone a permanent elongation (and consequent lateral contraction).

In the elastic range the force-displacement behaviour for most engineering materials (metals, rocks, plastics, but not soils) is linear. After passing the elastic limit (point A in Fig. 8.1.1), the material "gives" and is said to undergo plastic **flow**. Further increases in load are usually required to maintain the plastic flow and an increase in displacement; this phenomenon is known as **work-hardening** or **strain-hardening**. In some cases, after an initial plastic flow and hardening, the force-displacement curve decreases, as in some soils; the material is said to be **softening**. If the specimen is unloaded from a plastic state (B) it will return along the path BC shown, parallel to the original elastic line. This is **elastic recovery**. The strain which remains upon unloading is the permanent plastic deformation. If the material is now loaded again, the force-displacement curve will re-trace the unloading path CB until it again reaches the plastic state. Further increases in stress will cause the curve to follow BD.

Two important observations concerning the above tension test (on most metals) are the following:
(1) after the onset of plastic deformation, the material will be seen to undergo negligible volume change, that is, it is **incompressible**.
(2) the force-displacement curve is more or less the same regardless of the rate at which the specimen is stretched (at least at moderate temperatures).

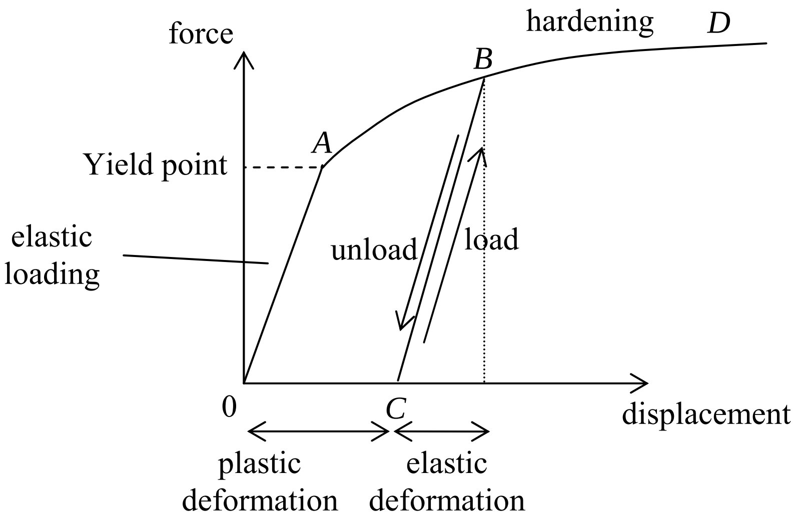


Figure 8.1.1: force/displacement curve for the tension test

Nominal and True Stress and Strain

There are two different ways of describing the force F which acts in a tension test. First, normalising with respect to the *original* cross sectional area of the tension test specimen A_0, one has the **nominal stress** or **engineering stress**,

$$\sigma_n = \frac{F}{A_0} \tag{8.1.1}$$

Alternatively, one can normalise with respect to the *current* cross-sectional area A, leading to the **true stress**,

$$\sigma = \frac{F}{A} \tag{8.1.2}$$

in which F and A are both changing with time. For very small elongations, within the elastic range say, the cross-sectional area of the material undergoes negligible change and both definitions of stress are more or less equivalent.

Similarly, one can describe the deformation in two alternative ways. Denoting the original specimen length by l_0 and the current length by l, one has the **engineering strain**

$$\varepsilon = \frac{l - l_0}{l_0} \tag{8.1.3}$$

Alternatively, the **true strain** is based on the fact that the "original length" is continually changing; a small change in length dl leads to a **strain increment** $d\varepsilon = dl / l$ and the total strain is *defined* as the accumulation of these increments:

$$\varepsilon_t = \int_{l_0}^{l} \frac{dl}{l} = \ln\left(\frac{l}{l_0}\right) \tag{8.1.4}$$

The true strain is also called the **logarithmic strain** or **Hencky strain**. Again, at small deformations, the difference between these two strain measures is negligible. The true strain and engineering strain are related through

$$\varepsilon_t = \ln(1 + \varepsilon) \tag{8.1.5}$$

Using the assumption of constant volume for plastic deformation and ignoring the very small elastic volume changes, one has also {▲Problem 3}

$$\sigma = \sigma_n \frac{l}{l_0}. \tag{8.1.6}$$

The stress-strain diagram for a tension test can now be described using the true stress/strain or nominal stress/strain definitions, as in Fig. 8.1.2. The shape of the nominal stress/strain diagram, Fig. 8.1.2a, is of course the same as the graph of force versus displacement (change in length) in Fig. 8.1.1. *A* here denotes the point at which the maximum force the specimen can withstand has been reached. The *nominal stress* at *A* is called the **Ultimate Tensile Strength** (UTS) of the material. After this point, the specimen "necks", with a very rapid reduction in cross-sectional area somewhere about the centre of the specimen until the specimen ruptures, as indicated by the asterisk.

Note that, during loading into the plastic region, *the yield stress increases*. For example, if one unloads and re-loads (as in Fig. 8.1.1), the material stays elastic up until a stress higher than the original yield stress *Y*. In this respect, the stress-strain curve can be regarded as a yield stress versus strain curve.

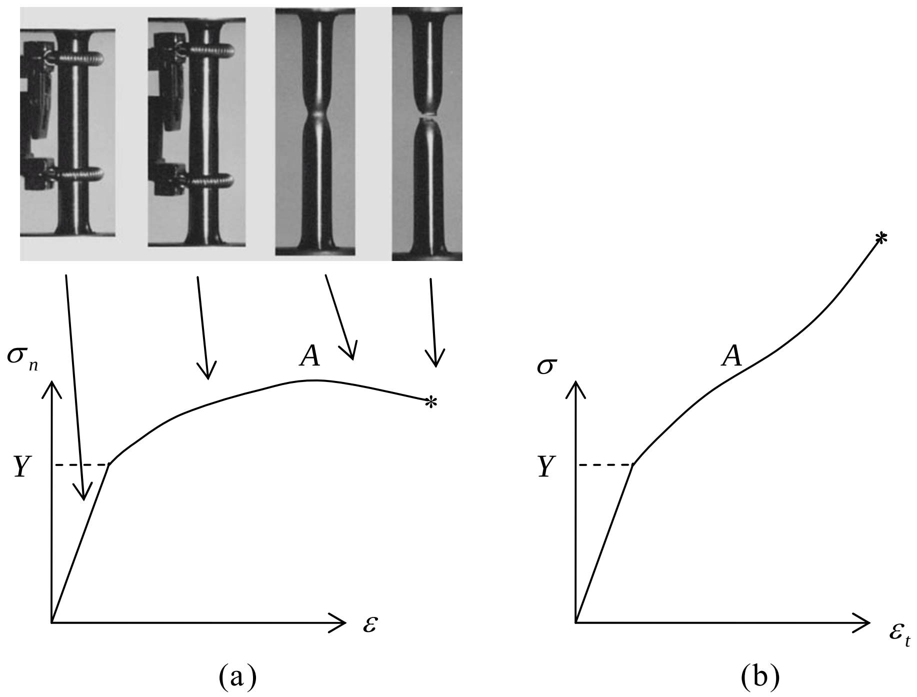


Figure 8.1.2: typical stress/strain curves; (a) engineering stress and strain, (b) true stress and strain

Compression Test

A compression test will lead to similar results as the tensile stress. The yield stress in compression will be approximately the same as (the negative of) the yield stress in tension. If one plots the true stress versus true strain curve for both tension and compression (absolute values for the compression), the two curves will more or less coincide. This would indicate that the behaviour of the material under compression is broadly similar to that under tension. If one were to use the nominal stress and strain, then the two curves would not coincide; this is one of a number of good reasons for using the *true* definitions.

The Bauschinger Effect

If one takes a virgin sample and loads it in tension into the plastic range, and *then* unloads it and continues on into compression, one finds that the yield stress in compression is *not* the same as the yield strength in tension, as it would have been if the specimen had not first been loaded in tension. In fact the yield point in this case will be significantly *less* than the corresponding yield stress in tension. This reduction in yield stress is known as the **Bauschinger effect**. The effect is illustrated in Fig. 8.1.3. The solid line depicts the response of a real material. The dotted lines are two extreme cases which are used in plasticity models; the first is the **isotropic hardening** model, in which the yield stress in tension and compression are maintained equal, the second being **kinematic hardening**, in which the total elastic range is maintained constant throughout the deformation.

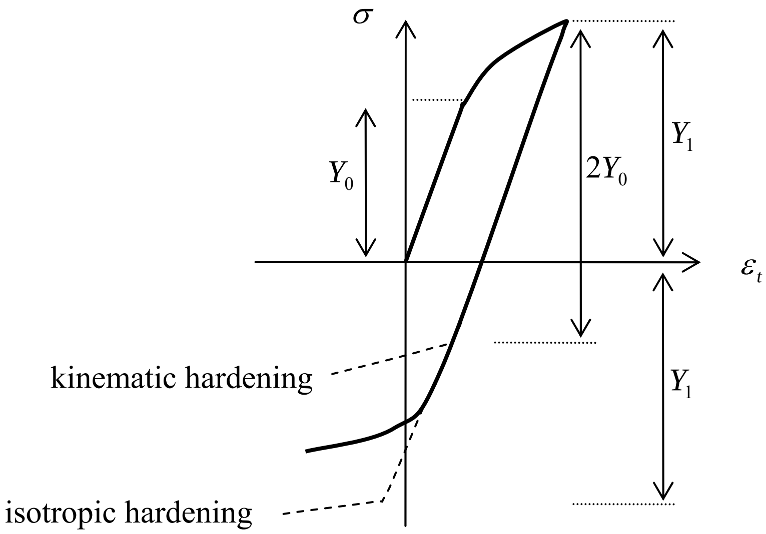


Figure 8.1.3: The Bauschinger effect

The presence of the Bauschinger effect complicates any plasticity theory. However, it is not an issue provided there are no reversals of stress in the problem under study.

Hydrostatic Pressure

Careful experiments show that, for metals, the yield behaviour is independent of hydrostatic pressure. That is, a stress state $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$ has negligible effect on the yield stress of a material, right up to very high pressures. Note however that this is not true for soils or rocks.

8.1.3 Assumptions of Plasticity Theory

Regarding the above test results then, in formulating a basic plasticity theory with which to begin, the following assumptions are usually made:

(1) the response is independent of rate effects
(2) the material is incompressible in the plastic range
(3) there is no Bauschinger effect
(4) the yield stress is independent of hydrostatic pressure
(5) the material is isotropic

The first two of these will usually be very good approximations, the other three may or may not be, depending on the material and circumstances. For example, most metals can be regarded as isotropic. After large plastic deformation however, for example in rolling, the material will have become anisotropic: there will be distinct material directions and asymmetries.

Together with these, assumptions can be made on the type of hardening and on whether elastic deformations are significant. For example, consider the hierarchy of models illustrated in Fig. 8.1.4 below, commonly used in theoretical analyses. In (a) both the elastic and plastic curves are assumed linear. In (b) work-hardening is neglected and the yield stress is constant after initial yield. Such **perfectly-plastic** models are particularly appropriate for studying processes where the metal is worked at a high temperature – such as hot rolling – where work hardening is small. In many areas of applications the strains involved are large, e.g. in metal working processes such as extrusion, rolling or drawing, where up to 50% reduction ratios are common. In such cases the elastic strains can be neglected altogether as in the two models (c) and (d). The **rigid/perfectly-plastic** model (d) is the crudest of all – and hence in many ways the most useful. It is widely used in analysing metal forming processes, in the design of steel and concrete structures and in the analysis of soil and rock stability.

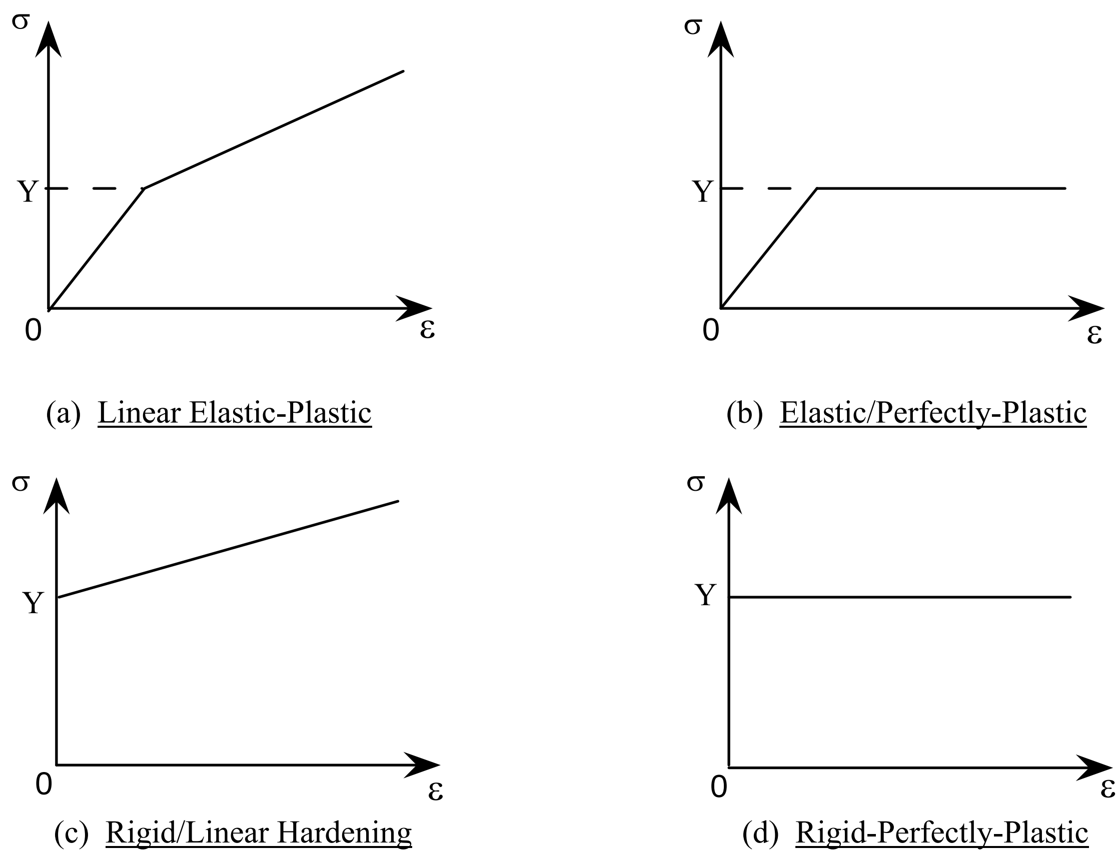


Figure 8.1.4: Simple models of elastic and plastic deformation

8.1.4 The Tangent and Plastic Modulus

Stress and strain are related through $\sigma = E\varepsilon$ in the elastic region, E being the Young's modulus, Fig. 8.1.5. The **tangent modulus** K is the slope of the stress-strain curve in the plastic region and will in general change during a deformation. At any instant of strain, the *increment* in stress $d\sigma$ is related to the *increment* in strain $d\varepsilon$ through[2]

$$d\sigma = Kd\varepsilon \tag{8.1.7}$$

[2] the symbol ε here represents the true strain (the subscript t has been dropped for clarity); as mentioned, when the strains are small, it is not necessary to specify which strain is in use since all strain measures are then equivalent

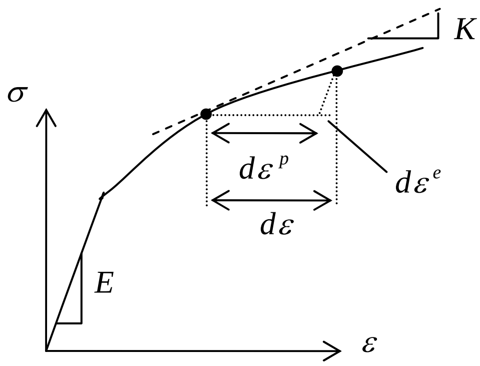


Figure 8.1.5: The tangent modulus

After yield, the strain increment consists of both elastic, ε^e, and plastic, $d\varepsilon^p$, strains:

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p \tag{8.1.8}$$

The stress and plastic strain increments are related by the **plastic modulus** *H*:

$$d\sigma = H\, d\varepsilon^p \tag{8.1.9}$$

and it follows that {▲Problem 4}

$$\frac{1}{K} = \frac{1}{E} + \frac{1}{H} \tag{8.1.10}$$

8.1.5 Friction Block Models

Some additional insight into the way plastic materials respond can be obtained from friction block models. The rigid perfectly plastic model can be simulated by a Coulomb friction block, Fig. 8.1.6. No strain occurs until σ reaches the yield stress *Y*. Then there is movement – although the amount of movement or plastic strain cannot be determined without more information being available. The stress cannot exceed the yield stress in this model:

$$|\sigma| \le Y \tag{8.1.11}$$

If unloaded, the block stops moving and the stress returns to zero, leaving a permanent strain, Fig. 8.1.6b.

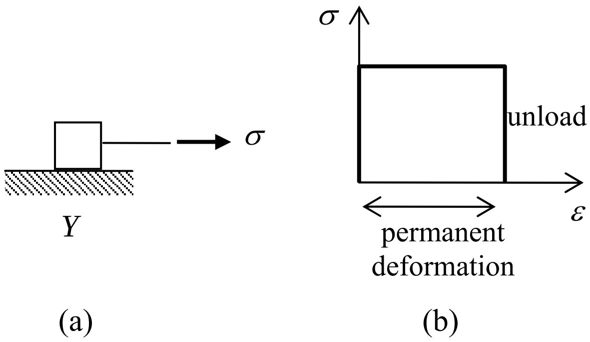


Figure 8.1.6: (a) Friction block model for the rigid perfectly plastic material, (b) response of the rigid-perfectly plastic model

The linear elastic perfectly plastic model incorporates a free spring with modulus *E* in series with a friction block, Fig. 8.1.7. The spring stretches when loaded and the block also begins to move when the stress reaches *Y*, at which time the spring stops stretching, the maximum possible stress again being *Y*. Upon unloading, the block stops moving and the spring contracts.

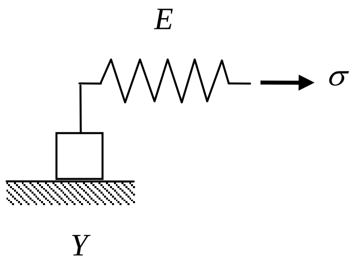


Figure 8.1.7: Friction block model for the elastic perfectly plastic material

The linear elastic plastic model with linear strain hardening incorporates a second, hardening, spring with stiffness *H*, in parallel with the friction block, Fig. 8.1.8. Once the yield stress is reached, an ever increasing stress needs to be applied in order to keep the block moving – and elastic strain continues to occur due to further elongation of the free spring. The stress is then split into the yield stress, which is carried by the moving block, and an **overstress** $\sigma - Y$ carried by the hardening spring.

Upon unloading, the block "locks" – the stress in the hardening spring remains constant whilst the free spring contracts. At zero stress, there is a negative stress taken up by the friction block, equal and opposite to the stress in the hardening spring.

The slope of the elastic loading line is *E*. For the plastic hardening line,

$$\varepsilon = \varepsilon^e + \varepsilon^p = \frac{\sigma}{E} + \frac{\sigma - Y}{H} \qquad \rightarrow \qquad K = \frac{d\sigma}{d\varepsilon} = \frac{EH}{E+H} \tag{8.1.12}$$

It can be seen that *H* is the plastic modulus.

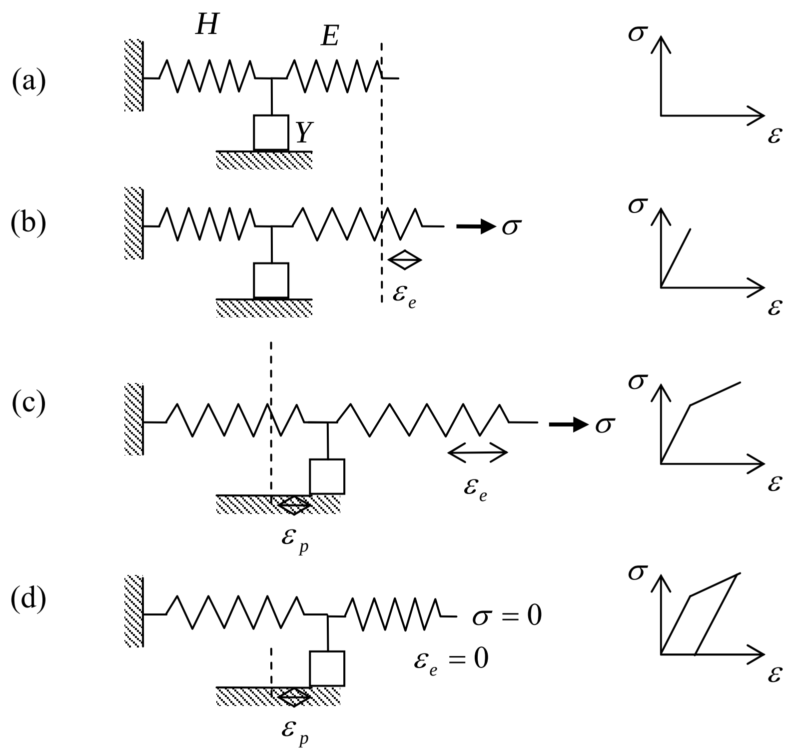


Figure 8.1.8: Friction block model for a linear elastic-plastic material with linear strain hardening; (a) stress-free, (b) elastic strain, (c) elastic and plastic strain, (d) unloading

8.1.6 Problems

1. Give two differences between plastic and viscoelastic materials.

2. A test specimen of initial length $0.01\,\text{m}$ is extended to length $0.0101\,\text{m}$. What is the percentage difference between the engineering and true strains (relative to the engineering strain)? What is this difference when the specimen is extended to length $0.015\,\text{m}$?

3. Derive the relation 8.1.6, $\sigma / \sigma_n = l / l_0$.

4. Derive Eqn. 8.1.10.

5. Which is larger, H or K? In the case of a perfectly-plastic material?

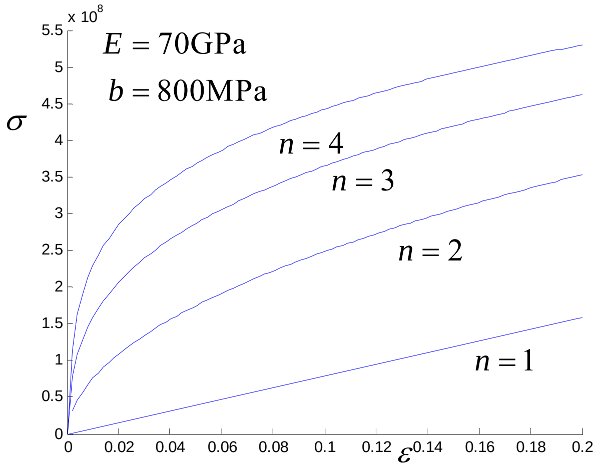
6. The **Ramberg-Osgood** model of plasticity is given by

$$\varepsilon = \varepsilon^e + \varepsilon^p = \frac{\sigma}{E} + \left(\frac{\sigma}{b} \right)^n$$

where E is the Young's modulus and b and n are model constants (material parameters) obtained from a curve-fitting of the uniaxial stress-strain curve.

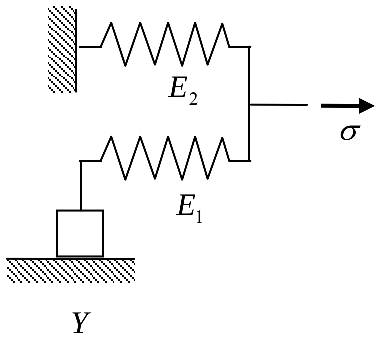
(i) Find the tangent and plastic moduli in terms of plastic strain ε^p (and the material constants).

(ii) A material with model parameters $n = 4$, $E = 70\,\text{GPa}$ and $b = 800\,\text{MPa}$ is strained in tension to $\varepsilon^p = 0.02$ and is subsequently unloaded and put into compression. Find the stress at the initiation of compressive yield assuming isotropic hardening.

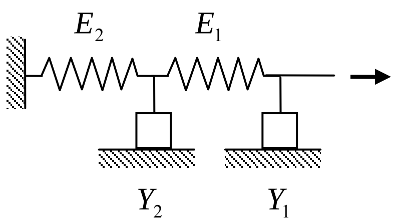


[Note that the yield stress is actually zero in this model, although the plastic strain at relatively low stress levels is small for larger values of *n*.]

7. Consider the plasticity model shown below.
   (i) What is the elastic modulus?
   (ii) What is the yield stress?
   (iii) What are the tangent and plastic moduli?
   Draw a typical loading and unloading curve.



8. Draw the stress-strain diagram for a cycle of loading and unloading to the rigid - plastic model shown here. Take the maximum load reached to be $\sigma_{\max} = 4Y_1$ and $Y_2 = 2Y_1$. What is the permanent deformation after complete removal of the load? [Hint: split the cycle into the following regions: (a) $0 \le \sigma \le Y_1$, (b) $Y_1 \le \sigma \le 3Y_1$, (c) $3Y_1 \le \sigma \le 4Y_1$, then unload, (d) $4Y_1 \le \sigma \le 3Y_1$, (e) $3Y_1 \le \sigma \le 2Y_1$, (f) $2Y_1 \le \sigma \le 0$.]



8.2 Stress Analysis for Plasticity

This section follows on from the analysis of three dimensional stress carried out in §7.2. The plastic behaviour of materials is often independent of a **hydrostatic stress** and this feature necessitates the study of the **deviatoric stress**.

8.2.1 Deviatoric Stress

Any state of stress can be decomposed into a **hydrostatic** (or **mean**) **stress** $\sigma_m \mathbf{I}$ and a **deviatoric stress s**, according to

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \tag{8.2.1}$$

where

$$\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \tag{8.2.2}$$

and

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(2\sigma_{11} - \sigma_{22} - \sigma_{33}) & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \frac{1}{3}(2\sigma_{22} - \sigma_{11} - \sigma_{33}) & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \frac{1}{3}(2\sigma_{33} - \sigma_{11} - \sigma_{22}) \end{bmatrix} \tag{8.2.3}$$

In index notation,

$$\sigma_{ij} = \sigma_m \delta_{ij} + s_{ij} \tag{8.2.4}$$

In a completely analogous manner to the derivation of the principal stresses and the principal scalar invariants of the stress matrix, §7.2.4, one can determine the principal stresses and principal scalar invariants of the deviatoric stress matrix. The former are denoted s_1, s_2, s_3 and the latter are denoted by J_1, J_2, J_3. The characteristic equation analogous to Eqn. 7.2.23 is

$$s^3 - J_1 s^2 - J_2 s - J_3 = 0 \tag{8.2.5}$$

and the deviatoric invariants are (compare with 7.2.24, 7.2.26)[1]

[1] unfortunately, there is a convention (adhered to by most authors) to write the characteristic equation for stress with a $+I_2\sigma$ term and that for deviatoric stress with a $-J_2 s$ term; this means that the formulae for J_2 in Eqn. 8.2.5 are the negative of those for I_2 in Eqn. 7.2.24

$$
\begin{aligned}
J_1 &= s_{11} + s_{22} + s_{33} \\
&= s_1 + s_2 + s_3 \\
J_2 &= -\left(s_{11}s_{22} + s_{22}s_{33} + s_{33}s_{11} - s_{12}^2 - s_{23}^2 - s_{31}^2\right) \\
&= -\left(s_1 s_2 + s_2 s_3 + s_3 s_1\right) \\
J_3 &= s_{11}s_{22}s_{33} - s_{11}s_{23}^2 - s_{22}s_{31}^2 - s_{33}s_{12}^2 + 2s_{12}s_{23}s_{31} \\
&= s_1 s_2 s_3
\end{aligned}
\tag{8.2.6}
$$

Since the hydrostatic stress remains unchanged with a change of coordinate system, the principal directions of stress coincide with the principal directions of the deviatoric stress, and the decomposition can be expressed with respect to the principal directions as

$$
\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \tag{8.2.7}
$$

Note that, from the definition Eqn. 8.2.3, the first invariant of the deviatoric stress, the sum of the normal stresses, is zero:

$$
J_1 = 0 \tag{8.2.8}
$$

The second invariant can also be expressed in the useful forms {▲Problem 3}

$$
J_2 = \tfrac{1}{2}\left(s_1^2 + s_2^2 + s_3^2\right), \tag{8.2.9}
$$

and, in terms of the principal stresses, {▲Problem 4}

$$
J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]. \tag{8.2.10}
$$

Further, the deviatoric invariants are related to the stress tensor invariants through {▲Problem 5}

$$
J_2 = \tfrac{1}{3}\left(I_1^2 - 3I_2\right), \quad J_3 = \tfrac{1}{27}\left(2I_1^3 - 9I_1 I_2 + 27I_3\right) \tag{8.2.11}
$$

A State of Pure Shear

The stress state at a point is one of **pure shear** if for *any* one coordinate axes through the point one has only shear stress acting, i.e. the stress matrix is of the form

$$
\left[\sigma_{ij}\right] = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & 0 \end{bmatrix} \tag{8.2.12}
$$

Applying the stress transformation rule 7.2.16 to this stress matrix and using the fact that the transformation matrix $\mathbf{Q}$ is orthogonal, i.e. $\mathbf{QQ}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$, one finds that the first invariant is zero, $\sigma'_{11} + \sigma'_{22} + \sigma'_{33} = 0$. Hence the deviatoric stress is one of pure shear.

8.2.2 The Octahedral Stresses

Examine now a material element subjected to principal stresses $\sigma_1, \sigma_2, \sigma_3$ as shown in Fig. 8.2.1. By definition, no shear stresses act on the planes shown.

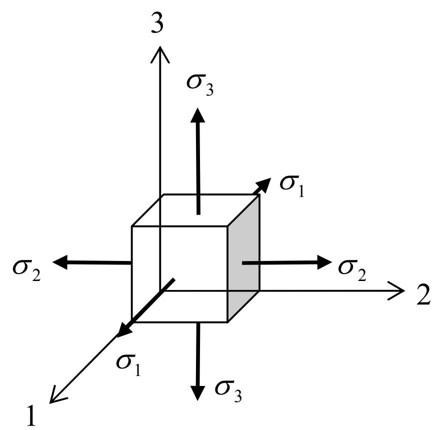


Figure 8.2.1: stresses acting on a material element

Consider next the **octahedral plane**; this is the plane shown shaded in Fig. 8.2.2, whose normal $\mathbf{n}_a$ makes equal angles with the principal directions. It is so-called because it cuts a cubic material element (with faces perpendicular to the principal directions) into a triangular plane and eight of these triangles around the origin form an octahedron.

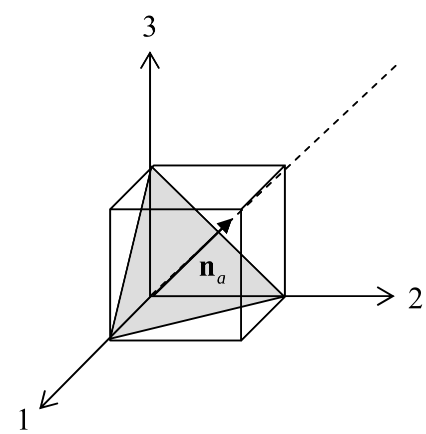


Figure 8.2.2: the octahedral plane

Next, a new Cartesian coordinate system is constructed with axes parallel and perpendicular to the octahedral plane, Fig. 8.2.3. One axis runs along the unit normal $\mathbf{n}_a$;

this normal has components $\left(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\right)$ with respect to the principal axes. The angle θ_0 the normal direction makes with the 1 direction can be obtained from $\mathbf{n}_a \cdot \mathbf{e}_1 = \cos\theta_0$, where $\mathbf{e}_1 = (1,0,0)$ is a unit vector in the 1 direction, Fig. 8.2.3. To complete the new coordinate system, any two perpendicular unit vectors which lie in (parallel to) the octahedral plane can be chosen. Choose one which is along the projection of the 1 axis down onto the octahedral plane. The components of this vector are {▲Problem 6} $\mathbf{n}_c = \left(\sqrt{2/3}, -1/\sqrt{6}, -1/\sqrt{6}\right)$. The final unit vector $\mathbf{n}_b$ is chosen so that it forms a right hand Cartesian coordinate system with $\mathbf{n}_a$ and $\mathbf{n}_c$, i.e. $\mathbf{n}_a \times \mathbf{n}_b = \mathbf{n}_c$. In summary,

$$\mathbf{n}_a = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{n}_b = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}, \quad \mathbf{n}_c = \frac{1}{\sqrt{6}}\begin{bmatrix}2\\-1\\-1\end{bmatrix} \tag{8.2.13}$$

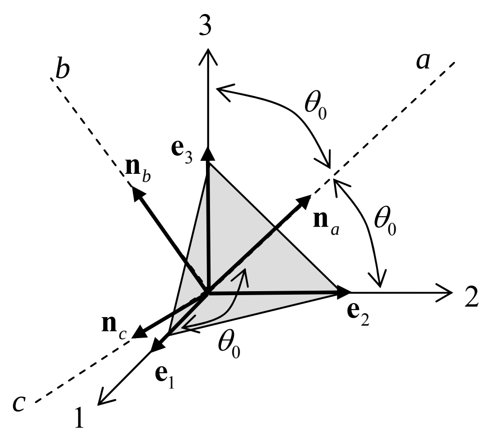


Figure 8.2.3: a new Cartesian coordinate system

To express the stress state in terms of components in the a, b, c directions, construct the stress transformation matrix:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{e}_1 \cdot \mathbf{n}_a & \mathbf{e}_1 \cdot \mathbf{n}_b & \mathbf{e}_1 \cdot \mathbf{n}_c \\ \mathbf{e}_2 \cdot \mathbf{n}_a & \mathbf{e}_2 \cdot \mathbf{n}_b & \mathbf{e}_2 \cdot \mathbf{n}_c \\ \mathbf{e}_3 \cdot \mathbf{n}_a & \mathbf{e}_3 \cdot \mathbf{n}_b & \mathbf{e}_3 \cdot \mathbf{n}_c \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \tag{8.2.14}$$

and the new stress components are

$$\begin{aligned} \begin{bmatrix} \sigma_{aa} & \sigma_{ab} & \sigma_{ac} \\ \sigma_{ba} & \sigma_{bb} & \sigma_{bc} \\ \sigma_{ca} & \sigma_{cb} & \sigma_{cc} \end{bmatrix} &= \mathbf{Q}^T \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \mathbf{Q} \\ &= \begin{bmatrix} \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) & -\frac{1}{\sqrt{6}}(\sigma_2 - \sigma_3) & \frac{1}{3\sqrt{2}}(2\sigma_1 - \sigma_2 - \sigma_3) \\ -\frac{1}{\sqrt{6}}(\sigma_2 - \sigma_3) & \frac{1}{2}(\sigma_2 + \sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2 - \sigma_3) \\ \frac{1}{3\sqrt{2}}(2\sigma_1 - \sigma_2 - \sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2 - \sigma_3) & \frac{1}{6}(4\sigma_1 + \sigma_2 + \sigma_3) \end{bmatrix} \end{aligned} \tag{8.2.15}$$

Now consider the stress components acting on the octahedral plane, $\sigma_{aa}, \sigma_{ab}, \sigma_{ac}$, Fig. 8.2.4. Recall from Cauchy's law, Eqn. 7.2.9, that these are the components of the traction vector $\mathbf{t}^{(\mathbf{n}_a)}$ acting on the octahedral plane, with respect to the (a,b,c) axes:

$$\mathbf{t}^{(\mathbf{n}_a)} = \sigma_{aa}\mathbf{n}_a + \sigma_{ab}\mathbf{n}_b + \sigma_{ac}\mathbf{n}_c \tag{8.2.16}$$

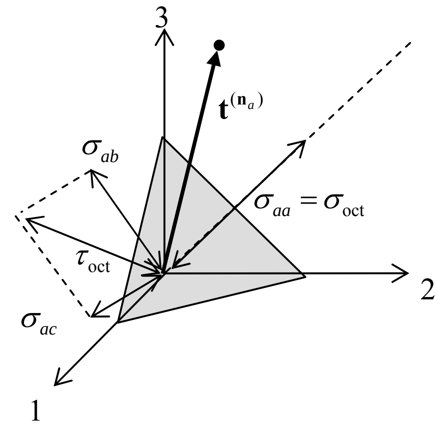


Figure 8.2.4: the stress vector $\boldsymbol{\sigma}$ and its components

The magnitudes of the normal and shear stresses acting on the octahedral plane are called the **octahedral normal stress** σ_{oct} and the **octahedral shear stress** τ_{oct}. Referring to Fig. 8.2.4, these can be expressed as {▲Problem 7}

$$\begin{aligned} \sigma_{oct} &= \sigma_{aa} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}I_1 \\ \tau_{oct} &= \sqrt{\sigma_{ab}^2 + \sigma_{ac}^2} \\ &= \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sqrt{\frac{2J_2}{3}} \end{aligned} \tag{8.2.17}$$

The octahedral normal and shear stresses on all 8 octahedral planes around the origin are the same.

Note that the octahedral normal stress is simply the hydrostatic stress. This implies that the deviatoric stress has no normal component in the direction $\mathbf{n}_a$ and only contributes to shearing on the octahedral plane. Indeed, from Eqn. 8.2.15,

$$\begin{bmatrix} s_{aa} & s_{ab} & s_{ac} \\ s_{ba} & s_{bb} & s_{bc} \\ s_{ca} & s_{cb} & s_{cc} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{6}}(\sigma_2 - \sigma_3) & \frac{1}{3\sqrt{2}}(2\sigma_1 - \sigma_2 - \sigma_3) \\ -\frac{1}{\sqrt{6}}(\sigma_2 - \sigma_3) & -\frac{1}{6}(2\sigma_1 - \sigma_2 - \sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2 - \sigma_3) \\ \frac{1}{3\sqrt{2}}(2\sigma_1 - \sigma_2 - \sigma_3) & \frac{1}{2\sqrt{3}}(\sigma_2 - \sigma_3) & \frac{1}{6}(2\sigma_1 - \sigma_2 - \sigma_3) \end{bmatrix} \tag{8.2.18}$$

The σ 's on the right here can be replaced with s 's since $\sigma_i - \sigma_j = s_i - s_j$.

8.2.3 Problems

1. What are the hydrostatic and deviatoric stresses for the uniaxial stress $\sigma_{11} = \sigma_0$?
   What are the hydrostatic and deviatoric stresses for the state of pure shear $\sigma_{12} = \tau$?
   In both cases, verify that the first invariant of the deviatoric stress is zero: $J_1 = 0$.

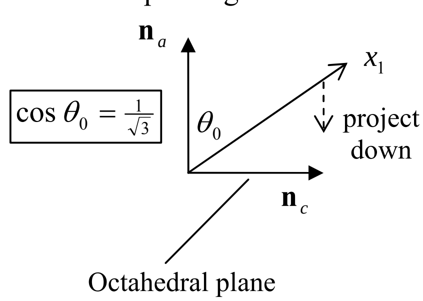
2. For the stress state $\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 1 \\ 4 & 1 & 3 \end{bmatrix}$, calculate
   (a) the hydrostatic stress
   (b) the deviatoric stresses
   (c) the deviatoric invariants

3. The second invariant of the deviatoric stress is given by Eqn. 8.2.6,
   $$J_2 = -(s_1 s_2 + s_2 s_3 + s_3 s_1)$$
   By squaring the relation $J_1 = s_1 + s_2 + s_3 = 0$, derive Eqn. 8.2.9,
   $J_2 = \frac{1}{2}\left(s_1^2 + s_2^2 + s_3^2\right)$

4. Use Eqns. 8.2.9 (and your work from Problem 3) and the fact that $\sigma_1 - \sigma_2 = s_1 - s_2$, etc. to derive 8.2.10,
   $$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

5. Use the fact that $J_1 = s_1 + s_2 + s_3 = 0$ to show that
   $$\begin{aligned} I_1 &= 3\sigma_m \\ I_2 &= (s_1 s_2 + s_2 s_3 + s_3 s_1) + 3\sigma_m^2 \\ I_3 &= s_1 s_2 s_3 + \sigma_m (s_1 s_2 + s_2 s_3 + s_3 s_1) + \sigma_m^3 \end{aligned}$$
   Hence derive Eqns. 8.2.11,
   $$J_2 = \frac{1}{3}\left(I_1^2 - 3I_2\right), \quad J_3 = \frac{1}{27}\left(2I_1^3 - 9I_1 I_2 + 27 I_3\right)$$

6. Show that a unit normal $\mathbf{n}_c$ in the octahedral plane in the direction of the projection of the 1 axis down onto the octahedral plane has coordinates $\left(\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$, Fig. 8.2.3. To do this, note the geometry shown below and the fact that when the 1 axis is projected down, it remains at equal angles to the 2 and 3 axes.



7. Use Eqns. 8.2.15 to derive Eqns. 8.2.17.

8. For the stress state of problem 2, calculate the octahedral normal stress and the octahedral shear stress

8.3 Yield Criteria in Three Dimensional Plasticity

The question now arises: a material yields at a stress level Y in a uniaxial tension test, but when does it yield when subjected to a complex three-dimensional stress state?

Let us begin with a very general case: an anisotropic material with different yield strengths in different directions. For example, consider the material shown in Fig. 8.3.1. This is a composite material with long fibres along the x_1 direction, giving it extra strength in that direction – it will yield at a higher tension when pulled in the x_1 direction than when pulled in other directions.

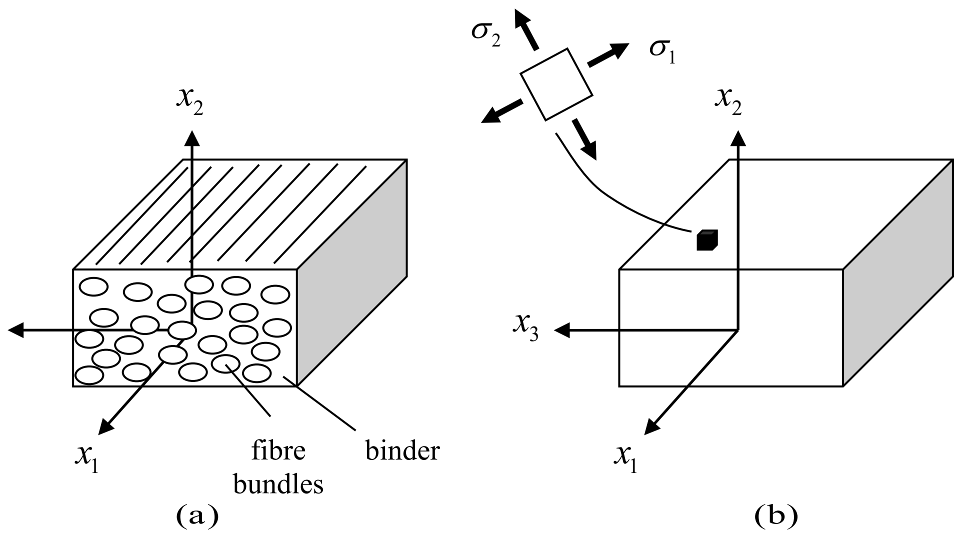


Figure 8.3.1: an anisotropic material; (a) microstructural detail, (b) continuum model

We can assume that yield will occur at a particle when some combination of the stress components reaches some critical value, say when

$$F(\sigma_{11},\sigma_{12},\sigma_{13},\sigma_{22},\sigma_{23},\sigma_{33}) = k \; . \tag{8.3.1}$$

Here, F is some function of the 6 independent components of the stress tensor and k is some material property which can be determined experimentally[1].

Alternatively, it is very convenient to express yield criteria in terms of principal stresses. Let us suppose that we know the principal stresses everywhere, $(\sigma_1,\sigma_2,\sigma_3)$, Fig. 8.3.1. Yield must depend somehow on the microstructure – on the orientation of the axes x_1, x_2, x_3, but this information is not contained in the three numbers $(\sigma_1,\sigma_2,\sigma_3)$. Thus we express the yield criterion in terms of principal stresses in the form

$$F(\sigma_1,\sigma_2,\sigma_3,\mathbf{n}_i) = k \tag{8.3.2}$$

[1] F will no doubt also contain other parameters which need to be determined experimentally

where $\mathbf{n}_i$ represent the principal directions – these give the orientation of the principal stresses relative to the material directions x_1, x_2, x_3.

If the material is isotropic, the response is independent of any material direction – independent of any "direction" the stress acts in, and so the yield criterion can be expressed in the simple form

$$F(\sigma_1, \sigma_2, \sigma_3) = k \tag{8.3.3}$$

Further, since it should not matter which direction is labelled '1', which '2' and which '3', F must be a symmetric function of the three principal stresses.

Alternatively, since the three principal invariants of stress are independent of material orientation, one can write

$$F(I_1, I_2, I_3) = k \tag{8.3.4}$$

or, more usually,

$$F(I_1, J_2, J_3) = k \tag{8.3.5}$$

where J_2, J_3 are the non-zero principal invariants of the deviatoric stress. With the further restriction that the yield stress is independent of the hydrostatic stress, one has

$$F(J_2, J_3) = k \tag{8.3.6}$$

8.3.1 The Tresca and Von Mises Yield Conditions

The two most commonly used and successful yield criteria for isotropic metallic materials are the **Tresca** and **Von Mises** criteria.

The Tresca Yield Condition

The Tresca yield criterion states that a material will yield if the maximum shear stress reaches some critical value, that is, Eqn. 8.3.3 takes the form

$$\max\left\{\frac{1}{2}|\sigma_1 - \sigma_2|, \frac{1}{2}|\sigma_2 - \sigma_3|, \frac{1}{2}|\sigma_3 - \sigma_1|\right\} = k \tag{8.3.7}$$

The value of k can be obtained from a simple experiment. For example, in a tension test, $\sigma_1 = \sigma_0, \sigma_2 = \sigma_3 = 0$, and failure occurs when σ_0 reaches Y, the yield stress in tension. It follows that

$$k = \frac{Y}{2}. \tag{8.3.8}$$

In a shear test, $\sigma_1 = \tau, \sigma_2 = 0, \sigma_3 = -\tau$, and failure occurs when τ reaches τ_Y, the yield stress of a material in pure shear, so that $k = \tau_Y$.

The Von Mises Yield Condition

The Von Mises criterion states that yield occurs when the principal stresses satisfy the relation

$$\sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{6}} = k \tag{8.3.9}$$

Again, from a uniaxial tension test, one finds that the *k* in Eqn. 8.3.9 is

$$k = \frac{Y}{\sqrt{3}}. \tag{8.3.10}$$

Writing the Von Mises condition in terms of *Y*, one has

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = Y \tag{8.3.11}$$

The quantity on the left is called the **Von Mises Stress**, sometimes denoted by σ_{VM}. When it reaches the yield stress in pure tension, the material begins to deform plastically. In the shear test, one again finds that $k = \tau_Y$, the yield stress in pure shear.

Sometimes it is preferable to work with arbitrary stress components; for this purpose, the Von Mises condition can be expressed as {▲Problem 2}

$$(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right) = 6k^2 \tag{8.3.12}$$

The piecewise linear nature of the Tresca yield condition is sometimes a theoretical advantage over the quadratic Mises condition. However, the fact that in many problems one often does not know which principal stress is the maximum and which is the minimum causes difficulties when working with the Tresca criterion.

The Tresca and Von Mises Yield Criteria in terms of Invariants

From Eqn. 8.2.10 and 8.3.9, the Von Mises criterion can be expressed as

$$f(J_2) \equiv J_2 - k^2 = 0 \tag{8.3.13}$$

Note the relationship between $\sqrt{J_2}$ and the octahedral shear stress, Eqn. 8.2.17; the Von Mises criterion can be interpreted as predicting yield when the octahedral shear stress reaches a critical value.

With $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the Tresca condition can be expressed as

$$f(J_2, J_3) \equiv 4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6 = 0 \tag{8.3.14}$$

but this expression is too cumbersome to be of much use.

Experiments of Taylor and Quinney

In order to test whether the Von Mises or Tresca criteria best modelled the real behaviour of metals, G I Taylor & Quinney (1931), in a series of classic experiments, subjected a number of thin-walled cylinders made of copper and steel to combined tension and torsion, Fig. 8.3.2.

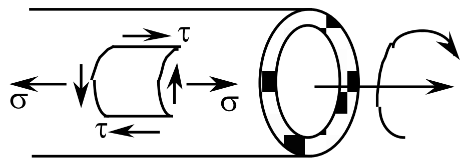


Figure 8.3.2: combined tension and torsion of a thin-walled tube

The cylinder wall is in a state of plane stress, with $\sigma_{11} = \sigma$, $\sigma_{12} = \tau$ and all other stress components zero. The principal stresses corresponding to such a stress-state are (zero and) {▲Problem 3}

$$\tfrac{1}{2}\sigma \pm \sqrt{\tfrac{1}{4}\sigma^2 + \tau^2} \tag{8.3.15}$$

and so Tresca's condition reduces to

$$\sigma^2 + 4\tau^2 = 4k^2 \quad \text{or} \quad \left(\frac{\sigma}{Y}\right)^2 + \left(\frac{\tau}{Y/2}\right)^2 = 1 \tag{8.3.16}$$

The Mises condition reduces to {▲Problem 4}

$$\sigma^2 + 3\tau^2 = 3k^2 \quad \text{or} \quad \left(\frac{\sigma}{Y}\right)^2 + \left(\frac{\tau}{Y/\sqrt{3}}\right)^2 = 1 \tag{8.3.17}$$

Thus both models predict an elliptical **yield locus** in (σ, τ) **stress space**, but with different ratios of principal axes, Fig. 8.3.3. The origin in Fig. 8.3.3 corresponds to an unstressed state. The horizontal axes refer to uniaxial tension in the absence of shear, whereas the vertical axis refers to pure torsion in the absence of tension. When there is a combination of σ and τ, one is off-axes. If the combination remains "inside" the yield locus, the material remains elastic; if the combination is such that one reaches anywhere along the locus, then plasticity ensues.

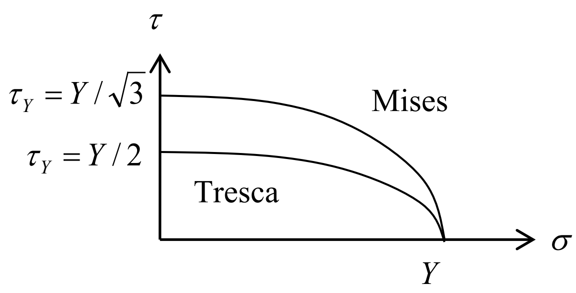


Figure 8.3.3: the yield locus for a thin-walled tube in combined tension and torsion

Taylor and Quinney, by varying the amount of tension and torsion, found that their measurements were closer to the Mises ellipse than the Tresca locus, a result which has been repeatedly confirmed by other workers[2].

2D Principal Stress Space

Fig. 8.3.3 gives a geometric interpretation of the Tresca and Von Mises yield criteria in (σ, τ) space. It is more usual to interpret yield criteria geometrically in a **principal stress space**. The Taylor and Quinney tests are an example of plane stress, where one principal stress is zero. Following the convention for plane stress, label now the two non-zero principal stresses σ_1 and σ_2, so that $\sigma_3 = 0$ (even if it is not the minimum principal stress). The criteria can then be displayed in (σ_1, σ_2) 2D principal stress space. With $\sigma_3 = 0$, one has

$$\begin{aligned} &\text{Tresca:} && \max\left\{|\sigma_1 - \sigma_2|, |\sigma_2|, |\sigma_1|\right\} = Y \\ &\text{Von Mises:} && \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = Y^2 \end{aligned} \qquad (8.3.18)$$

These are plotted in Fig. 8.3.4. The Tresca criterion is a hexagon. The Von Mises criterion is an ellipse with axes inclined at 45^0 to the principal axes, which can be seen by expressing Eqn. 8.3.18b in the canonical form for an ellipse:

[2] the maximum difference between the predicted stresses from the two criteria is about 15%. The two criteria can therefore be made to agree to within ± 7.5% by choosing *k* to be half-way between $Y/2$ and $Y/\sqrt{3}$

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \begin{bmatrix}\sigma_1 & \sigma_2\end{bmatrix}\begin{bmatrix}1 & -1/2 \\ -1/2 & 1\end{bmatrix}\begin{bmatrix}\sigma_1 \\ \sigma_2\end{bmatrix}$$

$$= \begin{bmatrix}\sigma_1 & \sigma_2\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/2 & 0 \\ 0 & 3/2\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sigma_1 \\ \sigma_2\end{bmatrix}$$

$$= \frac{1}{2}\left(\frac{\sigma_1+\sigma_2}{\sqrt{2}}\right)^2 + \frac{3}{2}\left(\frac{\sigma_2-\sigma_1}{\sqrt{2}}\right)^2 = Y^2$$

$$\to \quad \left(\frac{\sigma_1'}{\sqrt{2}Y}\right)^2 + \left(\frac{\sigma_2'}{\sqrt{2/3}Y}\right)^2 = 1$$

where σ_1', σ_2' are coordinates along the new axes; the major axis is thus $\sqrt{2}Y$ and the minor axis is $\sqrt{2/3}Y$.

Some stress states are shown in the stress space: point *A* corresponds to a uniaxial tension, *B* to a equi-biaxial tension and *C* to a pure shear τ.

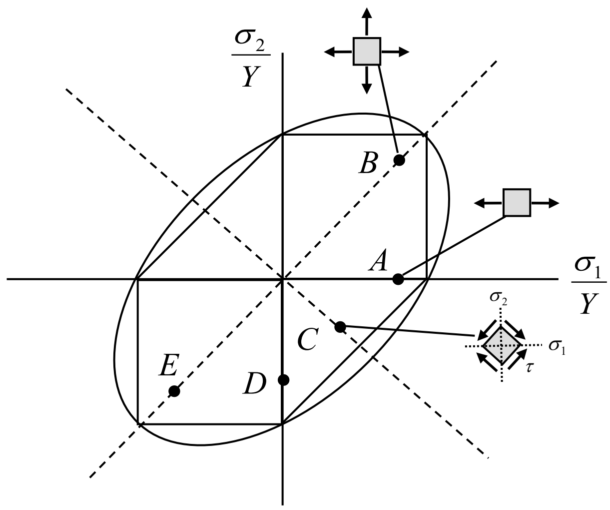


Figure 8.3.4: yield loci in 2D principal stress space

Again, points inside these loci represent an elastic stress state. Any combination of principal stresses which push the point out to the yield loci results in plastic deformation.

8.3.2 Three Dimensional Principal Stress Space

The 2D principal stress space has limited use. For example, a stress state that might start out two dimensional can develop into a fully three dimensional stress state as deformation proceeds.

In three dimensional principal stress space, one has a **yield surface** $f(\sigma_1, \sigma_2, \sigma_3) = 0$, Fig. 8.3.5 [3]. In this case, one can draw a line at equal angles to all three principal stress axes, the **space diagonal**. Along the space diagonal $\sigma_1 = \sigma_2 = \sigma_3$ and so points on it are in a state of hydrostatic stress.

Assume now, for the moment, that *hydrostatic stress does not affect yield* and consider some arbitrary point A, $(\sigma_1, \sigma_2, \sigma_3) = (a, b, c)$, on the yield surface, Fig. 8.3.5. A pure hydrostatic stress σ_h can be superimposed on this stress state without affecting yield, so any other point $(\sigma_1, \sigma_2, \sigma_3) = (a + \sigma_h, b + \sigma_h, c + \sigma_h)$ will also be on the yield surface. Examples of such points are shown at B, C and D, which are obtained from A by moving along a line parallel to the space diagonal. The yield behaviour of the material is therefore specified by a yield *locus* on a plane perpendicular to the space diagonal, and the yield surface is generated by sliding this locus up and down the space diagonal.

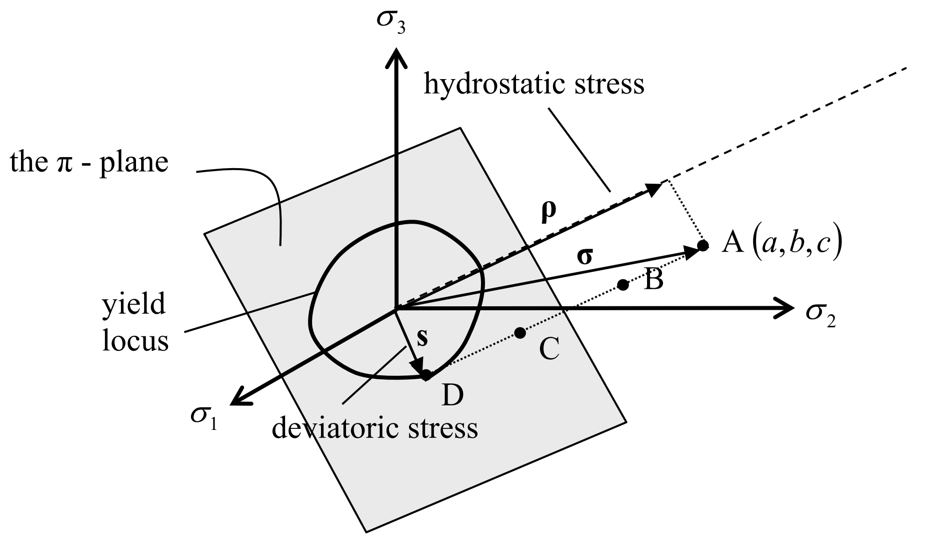


Figure 8.3.5: Yield locus/surface in three dimensional stress-space

The π-plane

Any surface in stress space can be described by an equation of the form

$$f(\sigma_1, \sigma_2, \sigma_3) = \text{const} \tag{8.3.19}$$

and a normal to this surface is the gradient vector

$$\frac{\partial f}{\partial \sigma_1}\mathbf{e}_1 + \frac{\partial f}{\partial \sigma_2}\mathbf{e}_2 + \frac{\partial f}{\partial \sigma_3}\mathbf{e}_3 \tag{8.3.20}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit vectors along the stress space axes. In particular, any plane perpendicular to the space diagonal is described by the equation

[3] as mentioned, one has a six dimensional stress space for an anisotropic material and this cannot be visualised

$$\sigma_1 + \sigma_2 + \sigma_3 = \text{const} \tag{8.3.21}$$

Without loss of generality, one can choose as a representative plane the **π – plane**, which is defined by $\sigma_1 + \sigma_2 + \sigma_3 = 0$. For example, the point $(\sigma_1, \sigma_2, \sigma_3) = (1,-1,0)$ is on the π – plane and, with yielding independent of hydrostatic stress, is equivalent to points in principal stress space which differ by a hydrostatic stress, e.g. the points $(2,0,1)$, $(0,-2,-1)$, etc.

The stress state at any point A represented by the vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ can be regarded as the sum of the stress state at the corresponding point on the π – plane, D, represented by the vector $\mathbf{s} = (s_1, s_2, s_3)$ together with a hydrostatic stress represented by the vector $\boldsymbol{\rho} = (\sigma_m, \sigma_m, \sigma_m)$:

$$(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 - \sigma_m, \sigma_2 - \sigma_m, \sigma_3 - \sigma_m) + (\sigma_m, \sigma_m, \sigma_m) \tag{8.3.22}$$

The components of the first term/vector on the right here sum to zero since it lies on the π – plane, and this is the deviatoric stress, whilst the hydrostatic stress is $\sigma_m = (\sigma_1 + \sigma_2 + \sigma_3)/3$.

Projected view of the π-plane

Fig. 8.3.6a shows principal stress space and Fig. 8.3.6b shows the π – plane. The heavy lines $\sigma_1', \sigma_2', \sigma_3'$ in Fig. 8.3.6b represent the projections of the principal axes down onto the π – plane (so one is "looking down" the space diagonal). Some points, A, B, C in stress space and their projections onto the π – plane are also shown. Also shown is some point D *on* the π – plane. It should be kept in mind that the deviatoric stress vector **s** in the projected view of Fig. 8.3.6b is in reality a three dimensional vector (see the corresponding vector in Fig. 8.3.6a).

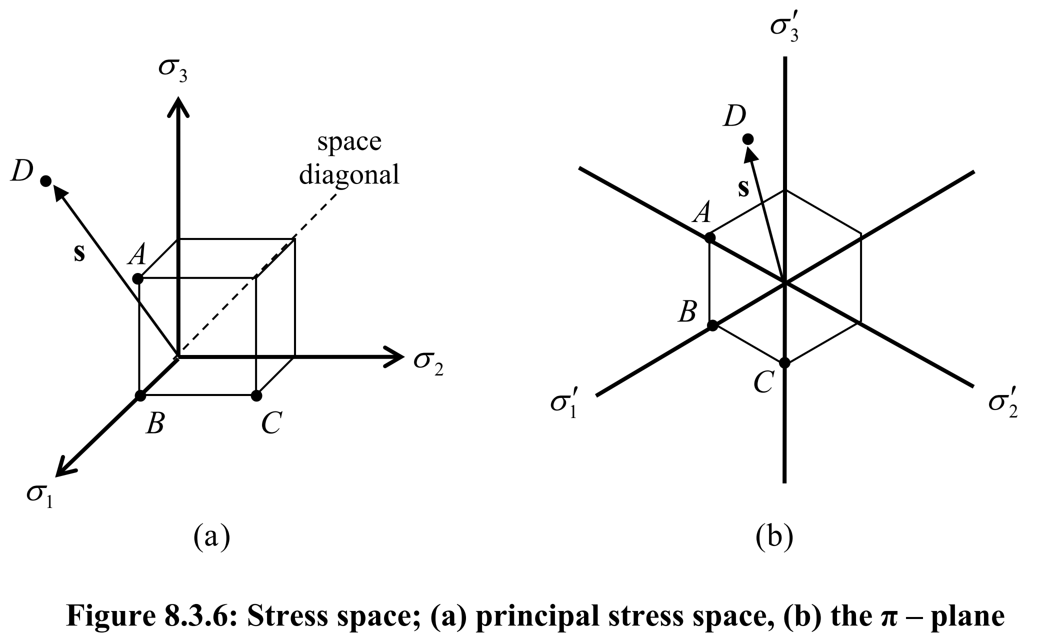


Figure 8.3.6: Stress space; (a) principal stress space, (b) the π – plane

Consider the more detailed Fig. 8.3.7 below. Point *A* here represents the stress state $(2,-1,0)$, as indicated by the arrows in the figure. It can also be "reached" in different ways, for example it represents $(3,0,1)$ and $(1,-2,-1)$. These three stress states of course differ by a hydrostatic stress. The actual $\pi-$plane value for *A* is the one for which $\sigma_1+\sigma_2+\sigma_3=0$, i.e. $(\sigma_1,\sigma_2,\sigma_3)=(s_1,s_2,s_3)=\left(\frac{5}{3},-\frac{4}{3},-\frac{1}{3}\right)$. Points *B* and *C* also represent multiple stress states {▲Problem 7}.

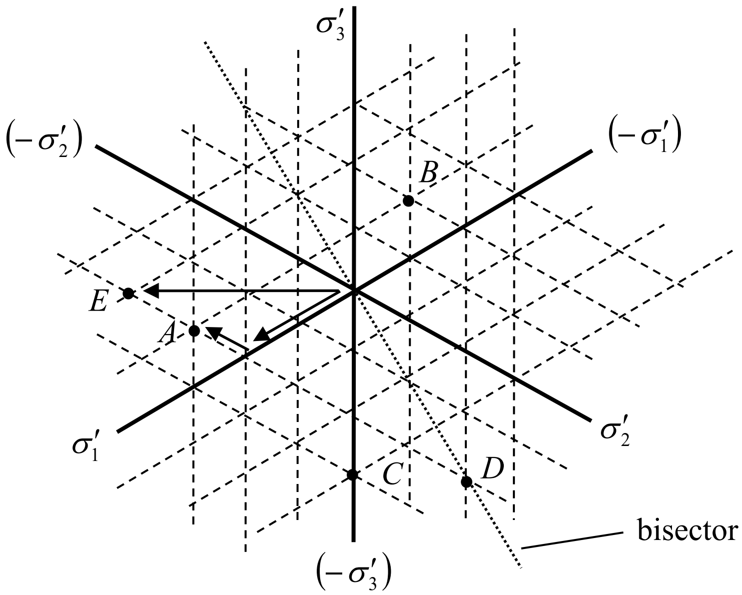


Figure 8.3.7: the π-plane

The bisectors of the principal plane projections, such as the dotted line in Fig. 8.3.7, represent states of pure shear. For example, the $\pi-$plane value for point *D* is $(0,2,-2)$, corresponding to a pure shear in the $\sigma_2-\sigma_3$ plane.

The dashed lines in Fig. 8.3.7 are helpful in that they allow us to plot and visualise stress states easily. The distance between each dashed line along the directions of the projected axes represents one unit of principal stress. Note, however, that these "units" are not consistent with the *actual* magnitudes of the deviatoric vectors in the $\pi-$plane. To create a more complete picture, note first that a unit vector along the space diagonal is $\mathbf{n}_\rho=\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]$, Fig. 8.3.8. The components of this normal are the direction cosines; for example, a unit normal along the '1' principal axis is $\mathbf{e}_1=[1,0,0]$ and so the angle θ_0 between the '1' axis and the space diagonal is given by $\mathbf{n}_\rho\cdot\mathbf{e}_1=\cos\theta_0=\frac{1}{\sqrt{3}}$. From Fig. 8.3.8, the angle θ between the '1' axis and the $\pi-$plane is given by $\cos\theta=\sqrt{\frac{2}{3}}$, and so a length of σ_1 units gets projected down to a length $s=|\mathbf{s}|=\sqrt{\frac{2}{3}}\sigma_1$.

For example, point *E* in Fig. 8.3.7 represents a pure shear $(\sigma_1,\sigma_2,\sigma_3)=(2,-2,0)$, which is on the $\pi-$plane. The length of the vector out to *E* in Fig. 8.3.7 is $2\sqrt{3}$ "units". To convert to actual magnitudes, multiply by $\sqrt{\frac{2}{3}}$ to get $s = 2\sqrt{2}$, which agrees with $s = |\mathbf{s}| = \sqrt{s_1^2 + s_2^2 + s_3^2} = \sqrt{2^2 + 2^2} = 2\sqrt{2}$.

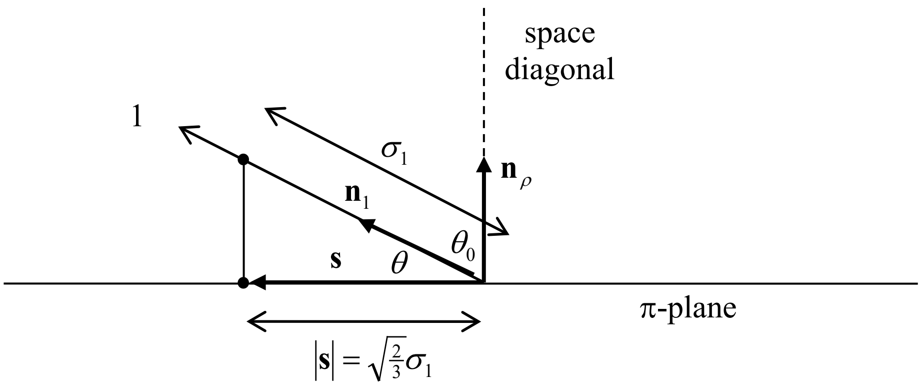


Figure 8.3.8: principal stress projected onto the π-plane

Typical π-plane Yield Loci

Consider next an arbitrary point (a,b,c) on the $\pi-$plane *yield locus*. If the material is isotropic, the points (a,c,b), (b,a,c), (b,c,a), (c,a,b) and (c,b,a) are also on the yield locus. If one assumes the same yield behaviour in tension as in compression, e.g. neglecting the Bauschinger effect, then so also are the points $(-a,-b,-c)$, $(-a,-c,-b)$, etc. Thus 1 point becomes 12 and one need only consider the yield locus in one 30º sector of the $\pi-$plane, the rest of the locus being generated through symmetry. One such sector is shown in Fig. 8.3.9, the axes of symmetry being the three projected principal axes and their (pure shear) bisectors.

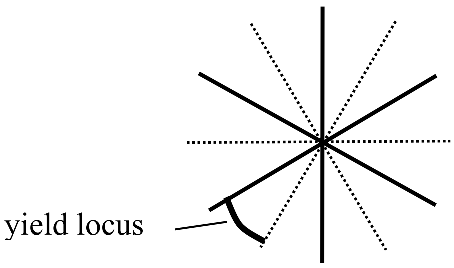


Figure 8.3.9: A typical sector of the yield locus

The Tresca and Von Mises Yield Loci in the π-plane

The Tresca criterion, Eqn. 8.3.7, is a regular hexagon in the $\pi-$plane as illustrated in Fig. 8.3.10. Which of the six sides of the locus is relevant depends on which of $\sigma_1, \sigma_2, \sigma_3$ is the maximum and which is the minimum, and whether they are tensile or compressive.

For example, yield at the pure shear $\sigma_1 = Y/2, \sigma_2 = 0, \sigma_3 = -Y/2$ is indicated by point A in the figure.

Point B represents yield under uniaxial tension, $\sigma_1 = Y$. The distance oB, the "magnitude" of the hexagon, is therefore $\sqrt{\tfrac{2}{3}}Y$; the corresponding point on the $\pi-$plane is $(s_1, s_2 s_3) = \left(\tfrac{2}{3}Y, -\tfrac{1}{3}Y, -\tfrac{1}{3}Y\right)$.

A criticism of the Tresca criterion is that there is a sudden change in the planes upon which failure occurs upon a small change in stress at the sharp corners of the hexagon.

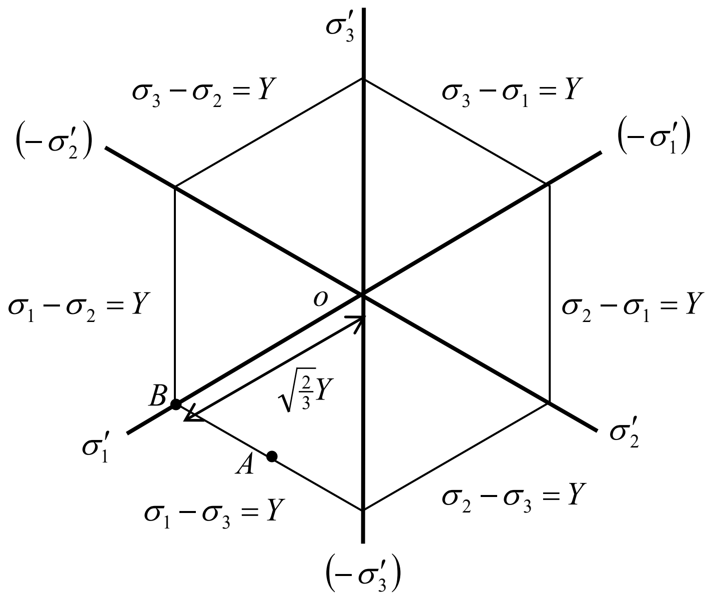


Figure 8.3.10: The Tresca criterion in the π-plane

Consider now the Von Mises criterion. From Eqns. 8.3.10, 8.3.13, the criterion is $\sqrt{J_2} = Y/\sqrt{3}$. From Eqn. 8.2.9, this can be re-written as

$$\sqrt{s_1^2 + s_2^2 + s_3^2} = \sqrt{\frac{2}{3}}Y \qquad (8.3.23)$$

Thus, the magnitude of the deviatoric stress vector is constant and one has a circular yield locus with radius $\sqrt{\tfrac{2}{3}}Y = \sqrt{2}k$, which circumscribes the Tresca hexagon, as illustrated in Fig. 8.3.11.

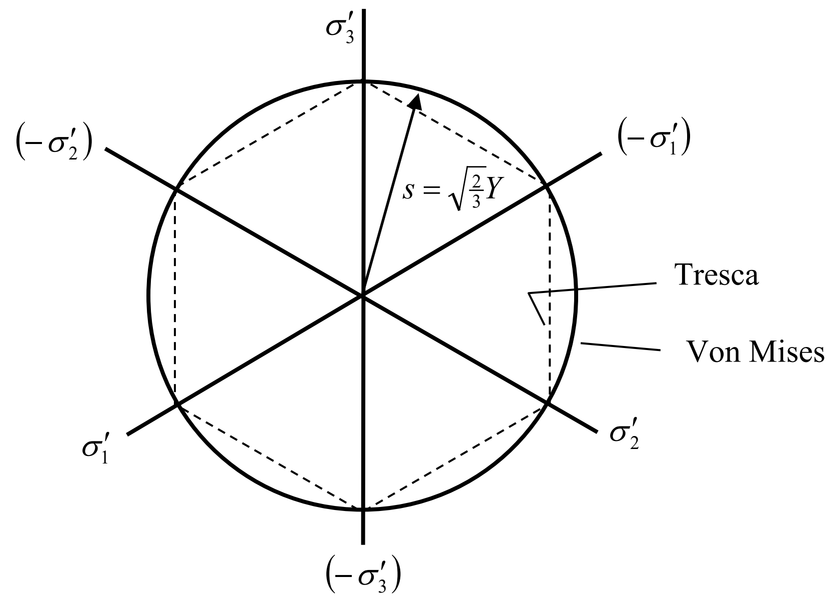


Figure 8.3.11: The Von Mises criterion in the π-plane

The yield surface is a circular cylinder with axis along the space diagonal, Fig. 8.3.12. The Tresca surface is a similar hexagonal cylinder.

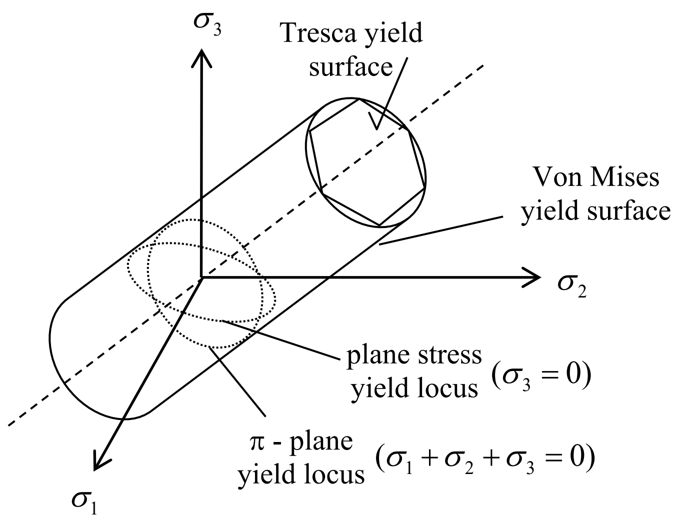


Figure 8.3.12: The Von Mises and Tresca yield surfaces

8.3.3 Haigh-Westergaard Stress Space

Thus far, yield criteria have been described in terms of principal stresses $(\sigma_1, \sigma_2, \sigma_3)$. It is often convenient to work with (ρ, s, θ) coordinates, Fig. 8.3.13; these cylindrical coordinates are called **Haigh-Westergaard coordinates**. They are particularly useful for describing and visualising geometrically pressure-dependent yield-criteria.

The coordinates (ρ, s) are simply the magnitudes of, respectively, the hydrostatic stress vector $\boldsymbol{\rho} = (\sigma_m, \sigma_m, \sigma_m)$ and the deviatoric stress vector $\mathbf{s} = (s_1, s_2, s_3)$. These are given by ($|\mathbf{s}|$ can be obtained from Eqn. 8.2.9)

$$\rho = |\boldsymbol{\rho}| = \sqrt{3}\sigma_m = I_1/\sqrt{3}, \qquad s = |\mathbf{s}| = \sqrt{2J_2} \tag{8.3.24}$$

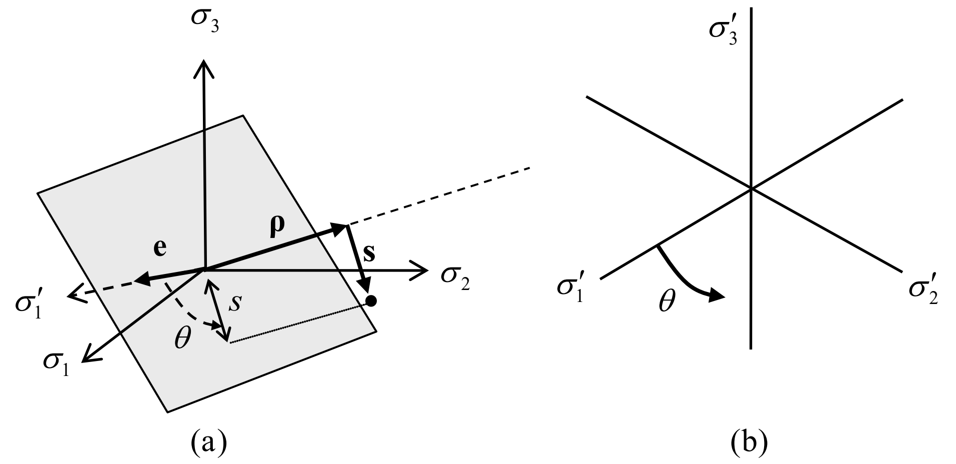


Figure 8.3.13: A point in stress space

θ is measured from the σ_1' (s_1) axis in the $\pi-$plane. To express θ in terms of invariants, consider a unit vector $\mathbf{e}$ in the $\pi-$plane in the direction of the σ_1' axis; this is the same vector $\mathbf{n}_c$ considered in Fig. 8.2.4 in connection with the octahedral shear stress, and it has coordinates $(\sigma_1, \sigma_2, \sigma_2) = \left(\sqrt{\tfrac{2}{3}}, -\tfrac{1}{\sqrt{6}}, -\tfrac{1}{\sqrt{6}}\right)$, Fig. 8.3.13. The angle θ can now be obtained from $\mathbf{s}\cdot\mathbf{e} = s\cos\theta$ {▲Problem 9}:

$$\cos\theta = \frac{\sqrt{3}s_1}{2\sqrt{J_2}} \tag{8.3.25}$$

Further manipulation leads to the relation {▲Problem 10}

$$\cos 3\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}} \tag{8.3.26}$$

Since J_2 and J_3 are invariant, it follows that $\cos 3\theta$ is also. Note that J_3 enters through $\cos 3\theta$, and does not appear in ρ or s; it is J_3 which makes the yield locus in the π-plane non-circular.

From Eqn. 8.3.25 and Fig. 8.3.13b, the deviatoric stresses can be expressed in terms of the Haigh-Westergaard coordinates through

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \frac{2}{\sqrt{3}} \sqrt{J_2} \begin{bmatrix} \cos\theta \\ \cos(2\pi/3 - \theta) \\ \cos(2\pi/3 + \theta) \end{bmatrix} \tag{8.3.27}$$

The principal stresses and the Haigh-Westergaard coordinates can then be related through {▲Problem 12}

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \rho \\ \rho \\ \rho \end{bmatrix} + \sqrt{\frac{2}{3}} s \begin{bmatrix} \cos\theta \\ \cos(\theta - 2\pi/3) \\ \cos(\theta + 2\pi/3) \end{bmatrix} \tag{8.3.28}$$

In terms of the Haigh-Westergaard coordinates, the yield criteria are

$$\begin{aligned} &\text{Von Mises:} \quad f(s) = \tfrac{1}{2} s^2 - k^2 = 0 \\ &\text{Tresca} \quad f(s,\theta) = \sqrt{2} s \sin\left(\theta + \tfrac{\pi}{3}\right) - Y = 0 \end{aligned} \tag{8.3.29}$$

8.3.4 Pressure Dependent Yield Criteria

The Tresca and Von Mises criteria are independent of hydrostatic pressure and are suitable for the modelling of plasticity in metals. For materials such as rock, soils and concrete, however, there is a strong dependence on the hydrostatic pressure.

The Drucker-Prager Criteria

The **Drucker-Prager** criterion is a simple modification of the Von Mises criterion, whereby the hydrostatic-dependent first invariant I_1 is introduced to the Von Mises Eqn. 8.3.13:

$$f(I_1, J_2) \equiv \alpha I_1 + \sqrt{J_2} - k = 0 \tag{8.3.30}$$

with α is a new material parameter. On the $\pi-$plane, $I_1 = 0$, and so the yield locus there is as for the Von Mises criterion, a circle of radius $\sqrt{2}k$, Fig. 8.3.14a. Off the $\pi-$plane, the yield locus remains circular but the radius changes. When there is a state of pure hydrostatic stress, the magnitude of the hydrostatic stress vector is {▲Problem 13} $\rho = |\boldsymbol{\rho}| = k/\sqrt{3}\alpha$, with $s = |\mathbf{s}| = 0$. For large pressures, $\sigma_1 = \sigma_2 = \sigma_3 < 0$, the I_1 term in Eqn. 8.3.30 allows for large deviatoric stresses. This effect is shown in the **meridian plane** in Fig. 8.3.14b, that is, the (ρ, s) plane which includes the σ_1 axis.

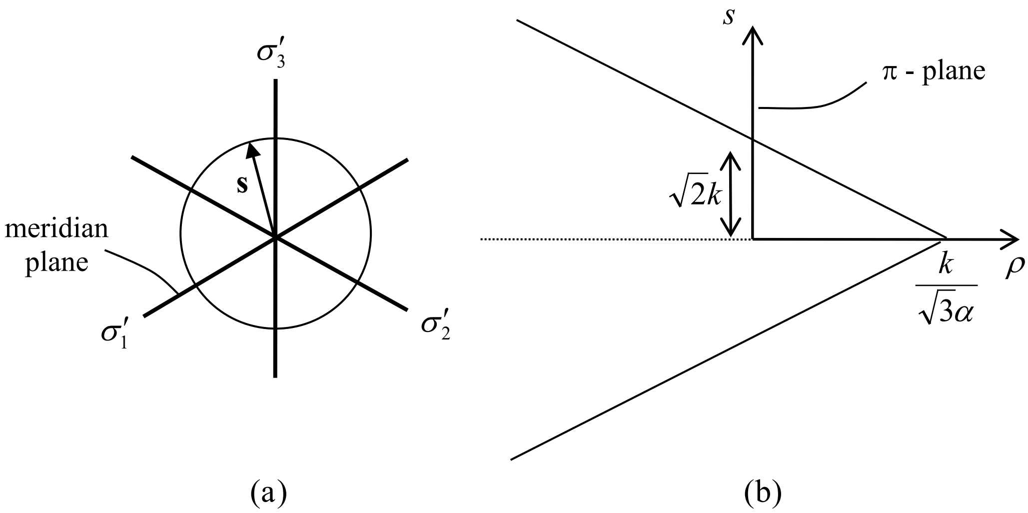


Figure 8.3.14: The Drucker-Prager criterion; (a) the π-plane, (b) the Meridian Plane

The Drucker-Prager surface is a right-circular cone with apex at $\rho = k/\sqrt{3}\alpha$, Fig. 8.3.15. Note that the plane stress locus, where the cone intersects the $\sigma_3 = 0$ plane, is an ellipse, but whose centre is off-axis, at some $(\sigma_1 < 0, \sigma_2 < 0)$.

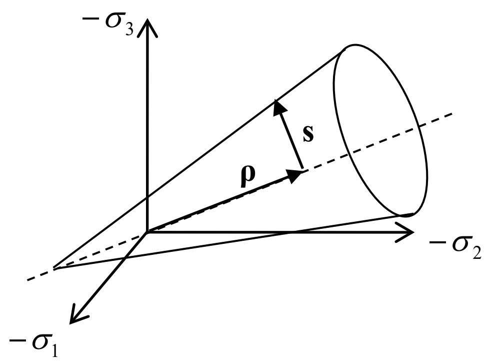


Figure 8.3.15: The Drucker-Prager yield surface

In terms of the Haigh-Westergaard coordinates, the yield criterion is

$$f(\rho, s) = \sqrt{6}\alpha\rho + s - \sqrt{2}k = 0 \tag{8.3.31}$$

The Mohr Coulomb Criteria

The Mohr-Coulomb criterion is based on Coulomb's 1773 friction equation, which can be expressed in the form

$$|\tau| = c - \sigma_n \tan\phi \tag{8.3.32}$$

where c, ϕ are material constants; c is called the **cohesion**[4] and ϕ is called the **angle of internal friction**. τ and σ_n are the shear and normal stresses acting on the plane where failure occurs (through a shearing effect), Fig. 8.3.16, with $\tan\phi$ playing the role of a coefficient of friction. The criterion states that the larger the pressure $-\sigma_n$, the more shear the material can sustain. Note that the Mohr-Coulomb criterion can be considered to be a generalised version of the Tresca criterion, since it reduces to Tresca's when $\phi = 0$ with $c = k$.

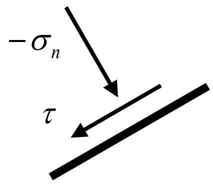


Figure 8.3.16: Coulomb friction over a plane

This criterion not only includes a hydrostatic pressure effect, but also allows for different yield behaviours in tension and in compression. Maintaining isotropy, there will now be three lines of symmetry in any deviatoric plane, and a typical sector of the yield locus is as shown in Fig. 8.3.17 (compare with Fig. 8.3.9)

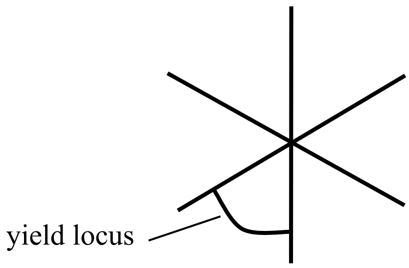


Figure 8.3.17: A typical sector of the yield locus for an isotropic material with different yield behaviour in tension and compression

Given values of c and ϕ, one can draw the failure locus (lines) of the Mohr-Coulomb criterion in (σ_n, τ) stress space, with intercepts $\tau = \pm c$ and slopes $\mp \tan\phi$, Fig. 8.3.18. Given some stress state $\sigma_1 \geq \sigma_2 \geq \sigma_3$, a Mohr stress circle can be drawn also in (σ_n, τ) space (see §7.2.6). When the stress state is such that this circle reaches out and touches the failure lines, yield occurs.

[4] $c = 0$ corresponds to a cohesionless material such as sand or gravel, which has no strength in tension

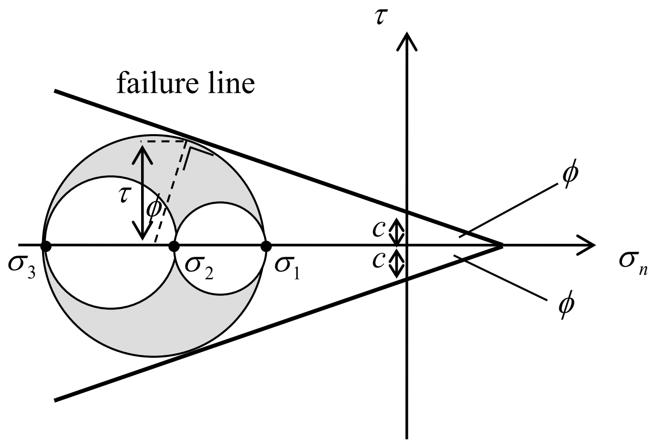


Figure 8.3.18: Mohr-Coulomb failure criterion

From Fig. 8.3.18, and noting that the large Mohr circle has centre $\left(\frac{1}{2}(\sigma_1+\sigma_3),0\right)$ and radius $\frac{1}{2}(\sigma_1-\sigma_3)$, one has

$$\begin{aligned}\tau &= \frac{\sigma_1-\sigma_3}{2}\cos\phi \\ \sigma_n &= \frac{\sigma_1+\sigma_3}{2}+\frac{\sigma_1-\sigma_3}{2}\sin\phi\end{aligned} \tag{8.3.33}$$

Thus the Mohr-Coulomb criterion in terms of principal stresses is

$$(\sigma_1-\sigma_3)=2c\cos\phi-(\sigma_1+\sigma_3)\sin\phi \tag{8.3.34}$$

The strength of the Mohr-Coulomb material in uniaxial tension, f_{Yt}, and in uniaxial compression, f_{Yc}, are thus

$$f_{Yt}=\frac{2c\cos\phi}{1+\sin\phi},\quad f_{Yc}=\frac{2c\cos\phi}{1-\sin\phi} \tag{8.3.35}$$

In terms of the Haigh-Westergaard coordinates, the yield criterion is

$$f(\rho,s,\theta)=\sqrt{2}\rho\sin\phi+\sqrt{3}s\sin\left(\theta+\tfrac{\pi}{3}\right)+s\cos\left(\theta+\tfrac{\pi}{3}\right)\sin\phi-\sqrt{6}c\cos\phi=0 \tag{8.3.36}$$

The Mohr-Coulomb yield surface in the π – plane and meridian plane are displayed in Fig. 8.3.19. In the π – plane one has an irregular hexagon which can be constructed from two lengths: the magnitude of the deviatoric stress in uniaxial tension at yield, s_{t0}, and the corresponding (larger) value in compression, s_{c0}; these are given by:

$$s_{t0}=\frac{\sqrt{6}f_{Yc}(1-\sin\phi)}{3+\sin\phi},\quad s_{c0}=\frac{\sqrt{6}f_{Yc}(1-\sin\phi)}{3-\sin\phi} \tag{8.3.37}$$

In the meridian plane, the failure surface cuts the $s = 0$ axis at $\rho = \sqrt{3}c \cot\phi$ {▲Problem 14}.

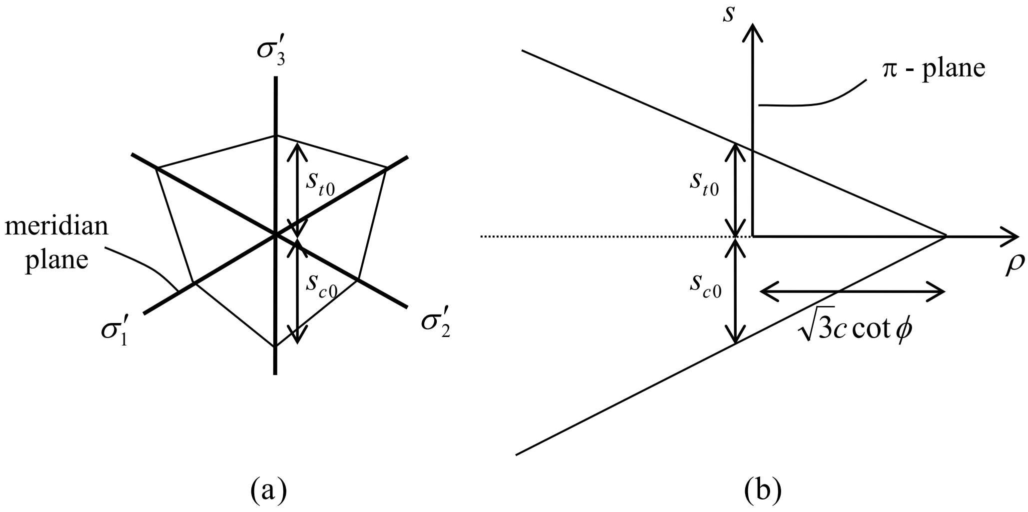


Figure 8.3.19: The Mohr-Coulomb criterion; (a) the π-plane, (b) the Meridian Plane

The Mohr-Coulomb surface is thus an irregular hexagonal pyramid, Fig. 8.3.20.

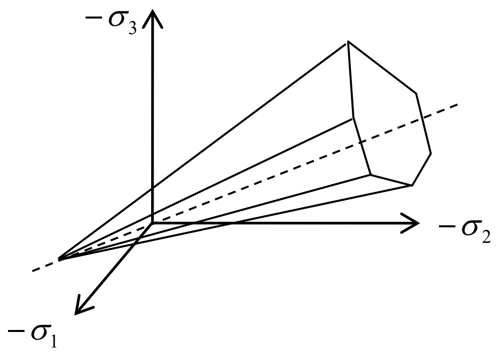


Figure 8.3.20: The Mohr-Coulomb yield surface

By adjusting the material parameters α, k, c, ϕ, the Drucker-Prager cone can be made to match the Mohr-Coulomb hexagon, either inscribing it at the minor vertices, or circumscribing it at the major vertices, Fig. 8.3.21.

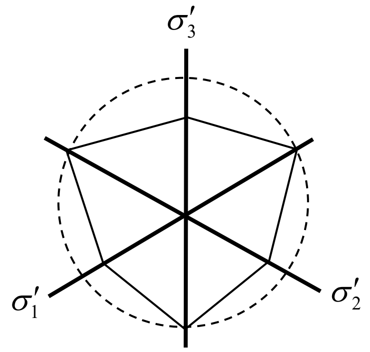


Figure 8.3.21: The Mohr-Coulomb and Drucker-Prager criteria matched in the π-plane

Capped Yield Surfaces

The Mohr-Coulomb and Drucker-Prager surfaces are open in that a pure hydrostatic pressure can be applied without affecting yield. For many geomaterials, however, for example soils, a large enough hydrostatic pressure will induce permanent deformation. In these cases, a **closed (capped) yield surface** is more appropriate, for example the one illustrated in Fig. 8.3.22.

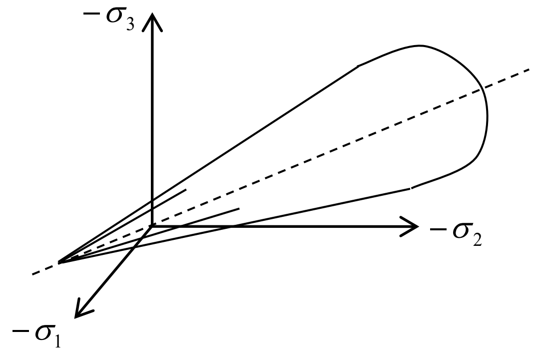


Figure 8.3.22: a capped yield surface

An example is the **modified Cam-Clay** criterion:

$$3J_2 = -\tfrac{1}{3}I_1 M^2\left(2p_c + \tfrac{1}{3}I_1\right) \text{ or } s = \sqrt{-\tfrac{2}{3\sqrt{3}}\rho M^2\left(2p_c + \tfrac{1}{\sqrt{3}}\rho\right)}, \quad \rho < 0 \tag{8.3.38}$$

with M and p_c material constants. In terms of the standard geomechanics notation, it reads

$$q^2 = M^2 p\left(2p_c - p\right) \tag{8.3.39}$$

where

$$p = -\frac{1}{3}I_1 = -\frac{1}{\sqrt{3}}\rho, \qquad q = \sqrt{3J_2} = \sqrt{\frac{3}{2}}s \tag{8.3.40}$$

The modified Cam-Clay locus in the meridian plane is shown in Fig. 8.3.23. Since s is constant for any given ρ, the locus in planes parallel to the π - plane are circles. The material parameter p_c is called the **critical state pressure**, and is the pressure which carries the maximum deviatoric stress. M is the slope of the dotted line shown in Fig. 8.3.23, known as the **critical state line**.

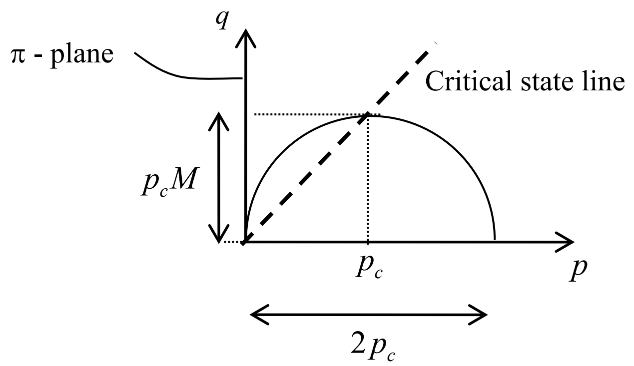


Figure 8.3.23: The modified Cam-Clay criterion in the Meridian Plane

8.3.5 Anisotropy

Many materials will display anisotropy. For example metals which have been processed by rolling will have characteristic material directions, the tensile yield stress in the direction of rolling being typically 15% greater than that in the transverse direction. The form of anisotropy exhibited by rolled sheets is such that the material properties are symmetric about three mutually orthogonal planes. The lines of intersection of these planes form an orthogonal set of axes known as the **principal axes of anisotropy**. The axes are (a) in the rolling direction, (b) normal to the sheet, (c) in the plane of the sheet but normal to rolling direction. This form of anisotropy is called **orthotropy** (see Part I, §6.2.2). Hill (1948) proposed a yield condition for such a material which is a natural generalisation of the Mises condition:

$$\begin{aligned} f(\sigma_{ij}) = F\left(\sigma_{22}-\sigma_{33}\right)^2 + G\left(\sigma_{33}-\sigma_{11}\right)^2 + H\left(\sigma_{11}-\sigma_{22}\right)^2 \\ + 2L\sigma_{23}^2 + 2M\sigma_{31}^2 + 2N\sigma_{12}^2 - 1 = 0 \end{aligned} \tag{8.3.41}$$

where F, G, H, L, M, N are material constants. One needs to carry out 6 tests: uniaxial tests in the three coordinate directions to find the uniaxial yield strengths $(\sigma_Y)_x, (\sigma_Y)_y, (\sigma_Y)_z$, and shear tests to find the shear strengths $(\tau_Y)_{xy}, (\tau_Y)_{yz}, (\tau_Y)_{zx}$. For a uniaxial test in the x direction, Eqn. 8.3.41 reduces to $G + H = 1/(\sigma_Y)_x^2$. By considering the other simple uniaxial and shear tests, one can solve for the material parameters:

$$F=\frac{1}{2}\left[\frac{1}{(\sigma_Y)_y^2}+\frac{1}{(\sigma_Y)_z^2}-\frac{1}{(\sigma_Y)_x^2}\right] \qquad L=\frac{1}{2}\frac{1}{(\tau_Y)_{yz}^2}$$
$$G=\frac{1}{2}\left[\frac{1}{(\sigma_Y)_z^2}+\frac{1}{(\sigma_Y)_x^2}-\frac{1}{(\sigma_Y)_y^2}\right] \qquad M=\frac{1}{2}\frac{1}{(\tau_Y)_{zx}^2} \tag{8.3.42}$$
$$H=\frac{1}{2}\left[\frac{1}{(\sigma_Y)_x^2}+\frac{1}{(\sigma_Y)_y^2}-\frac{1}{(\sigma_Y)_z^2}\right] \qquad N=\frac{1}{2}\frac{1}{(\tau_Y)_{xy}^2}$$

The criterion reduces to the Mises condition 8.3.12 when

$$F=G=H=\frac{L}{3}=\frac{M}{3}=\frac{N}{3}=\frac{1}{6k^2} \tag{8.3.43}$$

The 1, 2, 3 axes of reference in Eqn. 8.3.41 are the principal axes of anisotropy. The form appropriate for a general choice of axes can be derived by using the usual stress transformation formulae. It is complicated and involves cross-terms such as $\sigma_{11}\sigma_{23}$, etc.

8.3.6 Problems

1. A material is to be loaded to a stress state
$$[\sigma_{ij}]=\begin{bmatrix}50 & -30 & 0\\ -30 & 90 & 0\\ 0 & 0 & 0\end{bmatrix}\text{ MPa}$$
What should be the minimum uniaxial yield stress of the material so that it does not fail, according to the
(a) Tresca criterian
(b) Von Mises criterion
What do the theories predict when the yield stress of the material is 80MPa?

2. Use Eqn. 8.2.6, $J_2=s_{12}^2+s_{23}^2+s_{31}^2-(s_{11}s_{22}+s_{22}s_{33}+s_{33}s_{11})$ to derive Eqn. 8.3.12, $(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)=6k^2$, for the Von Mises criterion.

3. Use the plane stress principal stress formula $\sigma_{1,2}=\dfrac{\sigma_{11}+\sigma_{22}}{2}\pm\sqrt{\left(\dfrac{\sigma_{11}-\sigma_{22}}{2}\right)^2+\sigma_{12}^2}$ to derive Eqn. 8.3.15 for the Taylor-Quinney tests.

4. Derive Eqn. 8.3.17 for the Taylor-Quinney tests.

5. Describe the states of stress represented by the points D and E in Fig. 8.3.4. (The complete stress states can be visualised with the help of Mohr's circles of stress, Fig. 7.2.17.)

6. Suppose that, in the Taylor and Quinney tension-torsion tests, one has $\sigma = Y/2$ and $\tau = \sqrt{3}Y/4$. Plot this stress state in the 2D principal stress state, Fig. 8.3.4. (Use Eqn. 8.3.15 to evaluate the principal stresses.) Keeping now the normal stress at $\sigma = Y/2$, what value can the shear stress be increased to before the material yields, according to the von Mises criterion?

7. What are the $\pi-$plane principal stress values for the points *B* and *C* in Fig. 8.3.7?

8. Sketch on the $\pi-$plane Fig. 8.3.7 a line corresponding to $\sigma_1 = \sigma_2$ and also a region corresponding to $\sigma_1 > \sigma_2 > 0 > \sigma_3$

9. Using the relation $\mathbf{s}\cdot\mathbf{e} = s\cos\theta$ and $s_2 + s_3 = -s_1$, derive Eqn. 8.3.25, $\cos\theta = \dfrac{\sqrt{3}s_1}{2\sqrt{J_2}}$.

10. Using the trigonometric relation $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and Eqn. 8.3.25, $\cos\theta = \dfrac{\sqrt{3}s_1}{2\sqrt{J_2}}$, show that $\cos 3\theta = \dfrac{3\sqrt{3}s_1}{2J_2^{3/2}}\left(s_1^2 - J_2\right)$. Then using the relations 8.2.6, $J_2 = -\left(s_1 s_2 + s_2 s_3 + s_3 s_1\right)$, with $J_1 = 0$, derive Eqn. 8.3.26, $\cos 3\theta = \dfrac{3\sqrt{3}}{2}\dfrac{J_3}{J_2^{3/2}}$

11. Consider the following stress states. For each one, evaluate the space coordinates (ρ, s, θ) and plot in the $\pi-$plane (see Fig. 8.3.13b):
    (a) triaxial tension: $\sigma_1 = T_1 > \sigma_2 = \sigma_3 = T_2 > 0$
    (b) triaxial compression: $\sigma_3 = -p_1 < \sigma_1 = \sigma_2 = -p_2 < 0$ (this is an important test for geomaterials, which are dependent on the hydrostatic pressure)
    (c) a pure shear $\sigma_{xy} = \tau$: $\sigma_1 = +\tau,\ \sigma_2 = 0,\ \sigma_3 = -\tau$
    (d) a pure shear $\sigma_{xy} = \tau$ in the presence of hydrostatic pressure p:
    $\sigma_1 = -p + \tau,\ \sigma_2 = -p,\ \sigma_3 = -p - \tau$, i.e. $\sigma_1 - \sigma_2 = \sigma_2 - \sigma_3$

12. Use relations 8.3.24, $\rho = I_1/\sqrt{3},\ s = \sqrt{2J_2}$ and Eqns. 8.3.27 to derive Eqns. 8.3.28.

13. Show that the magnitude of the hydrostatic stress vector is $\rho = |\boldsymbol{\rho}| = k/\sqrt{3}\alpha$ for the Drucker-Prager yield criterion when the deviatoric stress is zero

14. Show that the magnitude of the hydrostatic stress vector is $\rho = \sqrt{3}c\cot\phi$ for the Mohr-Coulomb yield criterion when the deviatoric stress is zero

15. Show that, for a Mohr-Coulomb material, $\sin\phi = (r-1)/(r+1)$, where $r = f_{Yc}/f_{Yt}$ is the compressive to tensile strength ratio

16. A sample of concrete is subjected to a stress $\sigma_{11} = \sigma_{22} = -p,\ \sigma_{33} = -Ap$ where the constant $A > 1$. Using the Mohr-Coulomb criterion and the result of Problem 15, show that the material will not fail provided $A < f_{Yc}/p + r$

8.4 Elastic Perfectly Plastic Materials

Once yield occurs, a material will deform plastically. Predicting and modelling this plastic deformation is the topic of this section. For the most part, in this section, the material will be assumed to be perfectly plastic, that is, there is no work hardening.

8.4.1 Plastic Strain Increments

When examining the strains in a plastic material, it should be emphasised that one works with *increments in strain* rather than a total accumulated strain. One reason for this is that when a material is subjected to a certain stress state, the corresponding strain state could be one of many. Similarly, the strain state could correspond to many different stress states. Examples of this state of affairs are shown in Fig. 8.4.1.

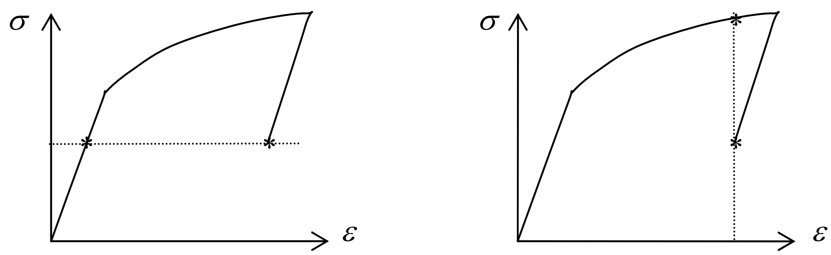


Figure 8.4.1: stress-strain curve; (a) different strains at a certain stress, (b) different stress at a certain strain

One cannot therefore make use of stress-strain relations in plastic regions (except in some special cases), since there is no unique relationship between the current stress and the current strain. However, one can relate the current stress to the current **increment in strain**, and these are the "stress-strain" laws which are used in plasticity theory. The total strain can be obtained by summing up, or integrating, the strain increments.

8.4.2 The Prandtl-Reuss Equations

An increment in strain $d\varepsilon$ can be decomposed into an elastic part $d\varepsilon^e$ and a plastic part $d\varepsilon^p$. If the material is isotropic, it is reasonable to suppose that the principal plastic strain increments $d\varepsilon_i^p$ are proportional to the principal deviatoric stresses s_i:

$$\frac{d\varepsilon_1^p}{s_1} = \frac{d\varepsilon_2^p}{s_2} = \frac{d\varepsilon_3^p}{s_3} = d\lambda \geq 0 \qquad (8.4.1)$$

This relation only gives the ratios of the plastic strain increments to the deviatoric stresses. To determine the precise relationship, one must specify the positive scalar $d\lambda$ (see later). Note that the plastic volume constancy is inherent in this relation: $d\varepsilon_1^p + d\varepsilon_2^p + d\varepsilon_3^p = 0$.

Eqns. 8.4.1 are in terms of the principal deviatoric stresses and principal plastic strain increments. In terms of Cartesian coordinates, one has

$$\frac{d\varepsilon_{xx}^p}{s_{xx}} = \frac{d\varepsilon_{yy}^p}{s_{yy}} = \frac{d\varepsilon_{zz}^p}{s_{zz}} = \frac{d\varepsilon_{xy}^p}{s_{xy}} = \frac{d\varepsilon_{xz}^p}{s_{xz}} = \frac{d\varepsilon_{yz}^p}{s_{yz}} = d\lambda \qquad (8.4.2)$$

or, succinctly,

$$d\varepsilon_{ij}^p = s_{ij}\, d\lambda \qquad (8.4.3)$$

These equations are often expressed in the alternative forms

$$\frac{d\varepsilon_{xx}^p - d\varepsilon_{yy}^p}{s_{xx} - s_{yy}} = \frac{d\varepsilon_{yy}^p - d\varepsilon_{zz}^p}{s_{yy} - s_{zz}} = \cdots \quad = \frac{d\varepsilon_{xx}^p - d\varepsilon_{yy}^p}{\sigma_{xx} - \sigma_{yy}} = \frac{d\varepsilon_{yy}^p - d\varepsilon_{zz}^p}{\sigma_{yy} - \sigma_{zz}} = \cdots \quad = d\lambda \qquad (8.4.4)$$

or, dividing by dt to get the **rate** equations,

$$\frac{\dot{\varepsilon}_{xx}^p - \dot{\varepsilon}_{yy}^p}{s_{xx} - s_{yy}} = \cdots = \frac{\dot{\varepsilon}_{xx}^p - \dot{\varepsilon}_{yy}^p}{\sigma_{xx} - \sigma_{yy}} = \cdots = \dot{\lambda} \qquad (8.4.5)$$

In terms of actual stresses, one has, from 8.2.3,

$$\begin{aligned}
d\varepsilon_{xx}^p &= \tfrac{2}{3} d\lambda \left[\sigma_{xx} - \tfrac{1}{2}\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
d\varepsilon_{yy}^p &= \tfrac{2}{3} d\lambda \left[\sigma_{yy} - \tfrac{1}{2}\left(\sigma_{zz} + \sigma_{xx}\right)\right] \\
d\varepsilon_{zz}^p &= \tfrac{2}{3} d\lambda \left[\sigma_{zz} - \tfrac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
d\varepsilon_{xy}^p &= d\lambda \sigma_{xy} \\
d\varepsilon_{yz}^p &= d\lambda \sigma_{yz} \\
d\varepsilon_{zx}^p &= d\lambda \sigma_{zx}
\end{aligned} \qquad (8.4.6)$$

This *plastic* stress-strain law is known as a **flow rule**. Other flow rules will be considered later on. Note that one cannot propose a flow rule which gives the plastic strain increments as explicit functions of the stress, otherwise the yield criterion might not be met (in particular, when there is strain hardening); one must include the to-be-determined scalar plastic multiplier λ. The plastic multiplier is determined by ensuring the stress-state lies on the yield surface during plastic flow.

The full elastic-plastic stress-strain relations are now, using Hooke's law,

$$
\begin{aligned}
d\varepsilon_{xx} &= \frac{1}{E}\left[d\sigma_{xx} - \nu\left(d\sigma_{yy} + d\sigma_{zz}\right)\right] + \frac{2}{3}d\lambda\left[\sigma_{xx} - \frac{1}{2}\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
d\varepsilon_{yy} &= \frac{1}{E}\left[d\sigma_{yy} - \nu\left(d\sigma_{xx} + d\sigma_{zz}\right)\right] + \frac{2}{3}d\lambda\left[\sigma_{yy} - \frac{1}{2}\left(\sigma_{zz} + \sigma_{xx}\right)\right] \\
d\varepsilon_{zz} &= \frac{1}{E}\left[d\sigma_{zz} - \nu\left(d\sigma_{xx} + d\sigma_{yy}\right)\right] + \frac{2}{3}d\lambda\left[\sigma_{zz} - \frac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
d\varepsilon_{xy} &= \frac{1+\nu}{E}d\sigma_{xy} + d\lambda\sigma_{xy} \\
d\varepsilon_{yz} &= \frac{1+\nu}{E}d\sigma_{yz} + d\lambda\sigma_{yz} \\
d\varepsilon_{zx} &= \frac{1+\nu}{E}d\sigma_{zx} + d\lambda\sigma_{zx}
\end{aligned}
\qquad (8.4.7)
$$

or

$$
d\varepsilon_{ij} = \frac{1+\nu}{E}d\sigma_{ij} - \frac{\nu}{E}\delta_{ij}d\sigma_{kk} + d\lambda s_{ij}
$$

These expressions are called the **Prandtl-Reuss equations**. If the first, elastic, terms are neglected, they are known as the **Lévy-Mises** equations.

8.4.3 Application: Plane Strain Compression of a Block

Consider the plane strain compression of a thick block, Fig. 8.4.2. The block is subjected to an increasing pressure $\sigma_{xx} = -p$, is constrained in the *z* direction, so $\varepsilon_{zz} = 0$, and is free to move in the *y* direction, so $\sigma_{yy} = 0$.

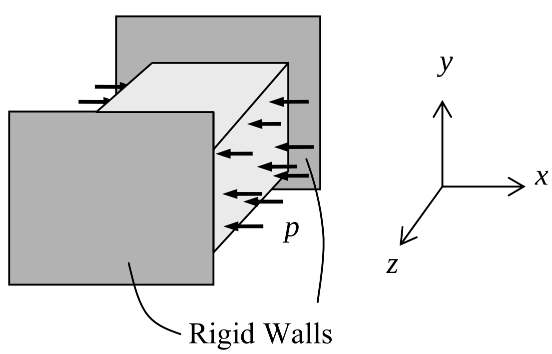


Figure 8.4.2: Plane strain compression of a thick block

The solution to the *elastic* problem is obtained from 8.4.7 (disregarding the plastic terms). One finds that {▲Problem 1}

$$
\sigma_{xx} = -p, \quad \sigma_{zz} = -\nu p, \quad \varepsilon_{xx} = -\frac{p}{E}\left(1-\nu^2\right), \quad \varepsilon_{yy} = +\frac{p}{E}\nu\left(1+\nu\right) \qquad (8.4.8)
$$

and all other stress and strain components are zero. In this elastic phase, the principal stresses are clearly

$$\sigma_1 = \sigma_{yy} = 0 > \sigma_2 = \sigma_{zz} > \sigma_3 = \sigma_{xx} \tag{8.4.9}$$

The Prandtl-Reuss equations are

$$\begin{aligned} d\varepsilon_{xx} &= \frac{1}{E}\left[d\sigma_{xx} - \nu d\sigma_{zz}\right] + \frac{2}{3}d\lambda\left[\sigma_{xx} - \frac{1}{2}\sigma_{zz}\right] \\ d\varepsilon_{yy} &= -\frac{\nu}{E}\left(d\sigma_{xx} + d\sigma_{zz}\right) - \frac{1}{3}d\lambda\left(\sigma_{xx} + \sigma_{zz}\right) \\ d\varepsilon_{zz} &= \frac{1}{E}\left[d\sigma_{zz} - \nu d\sigma_{xx}\right] + \frac{2}{3}d\lambda\left[-\frac{1}{2}\sigma_{xx} + \sigma_{zz}\right] \end{aligned} \tag{8.4.10}$$

The magnitude of the plastic straining is determined by the multiplier $d\lambda$. This can be evaluated by noting that plastic deformation proceeds so long as the stress state remains on the yield surface, the so-called **consistency condition**. By definition, a perfectly plastic material is one whose yield surface remains unchanged during deformation.

A Tresca Material

Take now the Tresca yield criterion, which states that yield occurs when $\sigma_{xx} = -Y$, where Y is the uniaxial yield stress (in compression). Assume further perfect plasticity, so that $\sigma_{xx} = -Y$ holds during all subsequent plastic flow. Thus, with $d\sigma_{xx} = 0$, and since $d\varepsilon_{zz} = 0$, 8.4.10 reduce to

$$\begin{aligned} d\varepsilon_{xx} &= -\frac{\nu}{E}d\sigma_{zz} - \frac{2}{3}d\lambda\left[Y + \frac{1}{2}\sigma_{zz}\right] \\ d\varepsilon_{yy} &= -\frac{\nu}{E}d\sigma_{zz} + \frac{1}{3}d\lambda\left(Y - \sigma_{zz}\right) \\ 0 &= \frac{1}{E}d\sigma_{zz} + \frac{2}{3}d\lambda\left[\frac{1}{2}Y + \sigma_{zz}\right] \end{aligned} \tag{8.4.11}$$

Thus

$$d\lambda = -\frac{3}{E}\frac{d\sigma_{zz}}{2\sigma_{zz} + Y} \tag{8.4.12}$$

and, eliminating $d\lambda$ from Eqns. 8.4.11 {▲Problem 2},

$$\begin{aligned} Ed\varepsilon_{xx} &= -\nu d\sigma_{zz} + Y\frac{1}{\sigma_{zz} + Y/2}d\sigma_{zz} + \frac{1}{2}\frac{\sigma_{zz}}{\sigma_{zz} + Y/2}d\sigma_{zz} \\ Ed\varepsilon_{yy} &= -\nu d\sigma_{zz} - \frac{Y}{2}\frac{1}{\sigma_{zz} + Y/2}d\sigma_{zz} + \frac{1}{2}\frac{\sigma_{zz}}{\sigma_{zz} + Y/2}d\sigma_{zz} \end{aligned} \tag{8.4.13}$$

Using the relation

$$\int \frac{x}{x+a}\,dx = x - a\ln(x+a) \tag{8.4.14}$$

and the initial (yield point) conditions, i.e. Eqns. 8.4.8 with $p = Y$, one can integrate 8.4.13 to get {▲Problem 2}

$$\begin{aligned} \frac{E}{Y}\varepsilon_{xx} &= -\frac{3}{4}\ln\left(\frac{1-2\nu}{1+2\sigma_{zz}/Y}\right) + \frac{1}{2}(1-2\nu)\frac{\sigma_{zz}}{Y} - \frac{1}{2}(2-\nu) \\ \frac{E}{Y}\varepsilon_{yy} &= +\frac{3}{4}\ln\left(\frac{1-2\nu}{1+2\sigma_{zz}/Y}\right) + \frac{1}{2}(1-2\nu)\frac{\sigma_{zz}}{Y} + \frac{3}{2}\nu \end{aligned}, \qquad \frac{\sigma_{zz}}{Y} < -\nu \tag{8.4.15}$$

The stress-strain curves are shown in Fig. 8.4.3 below for $\nu = 0.3$. Note that, for a typical metal, $E/Y \sim 10^3$, and so the strains are very small right through the plastic compression; the plastic strains are of comparable size to the elastic strains. There is a rapid change of stress and then little change once σ_{zz} has approached close to its limiting value of $-Y/2$.

The above plastic analysis was based on σ_{xx} remaining the minimum principal stress. This assumption has proved to be valid, since σ_{zz} remains between 0 and $-Y$ in the plastic region.

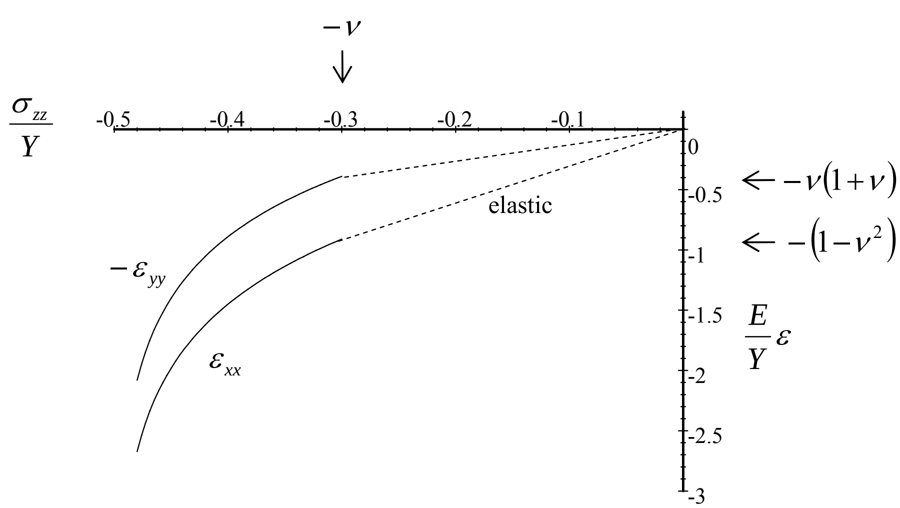


Figure 8.4.3: Stress-strain results for plane strain compression of a thick block for $\nu = 0.3$

A Von Mises Material

Slightly different results are obtained with the Von Mises yield criterion, Eqn. 8.4.11, which for this problem reads

$$\sigma_{xx}^2 - \sigma_{xx}\sigma_{zz} + \sigma_{zz}^2 = Y^2 \tag{8.4.16}$$

The Prandtl-Reuss equations can be solved by making the substitution

$$\sigma_{xx} = -\frac{2Y}{\sqrt{3}}\cos\theta \tag{8.4.17}$$

in the plastic region. In what follows, use is made of the trigonometric relations

$$\begin{aligned} \sin\left(\frac{\pi}{6} - \theta\right) &= \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta \\ \cos\left(\frac{\pi}{6} - \theta\right) &= \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta \end{aligned} \tag{8.4.18}$$

From Eqn. 8.4.16,

$$\sigma_{zz} = -\frac{2Y}{\sqrt{3}}\sin\left(\frac{\pi}{6} - \theta\right) \tag{8.4.19}$$

Substituting into 8.4.10 then leads to

$$\begin{aligned} \frac{E}{Y}d\varepsilon_{xx} &= \frac{2}{\sqrt{3}}\left\{\left[\sin\theta - \nu\cos\left(\frac{\pi}{6} - \theta\right)\right]d\theta - \frac{1}{\sqrt{3}}Ed\lambda\cos\left(\frac{\pi}{6} - \theta\right)\right\} \\ \frac{E}{Y}d\varepsilon_{yy} &= \frac{2}{\sqrt{3}}\left\{-\nu\left(\sin\theta + \cos\left(\frac{\pi}{6} - \theta\right)\right)d\theta - \frac{1}{3}Ed\lambda\left(-\cos\theta - \sin\left(\frac{\pi}{6} - \theta\right)\right)\right\} \\ \frac{E}{Y}d\varepsilon_{zz} &= \frac{2}{\sqrt{3}}\left\{\left[\cos\left(\frac{\pi}{6} - \theta\right) - \nu\sin\theta\right]d\theta + \frac{1}{\sqrt{3}}Ed\lambda\sin\theta\right\} \end{aligned} \tag{8.4.20}$$

Using $d\varepsilon_{zz} = 0$ leads to

$$d\lambda = -\frac{\sqrt{3}}{E}\frac{\cos\left(\frac{\pi}{6} - \theta\right) - \nu\sin\theta}{\sin\theta}d\theta \tag{8.4.21}$$

and

$$\frac{E}{Y}d\varepsilon_{xx} = \frac{2}{\sqrt{3}}\left\{(1 - 2\nu)\cos\left(\frac{\pi}{6} - \theta\right) + \frac{3}{4}\operatorname{cosec}\theta\right\}d\theta \tag{8.4.22}$$

An integration gives

$$\frac{E}{Y}\varepsilon_{xx} = -\frac{2}{\sqrt{3}}(1 - 2\nu)\sin\left(\frac{\pi}{6} - \theta\right) + \frac{\sqrt{3}}{2}\ln\left|\tan\frac{\theta}{2}\right| + C \tag{8.4.23}$$

To determine the constant of integration, consider again the conditions at first yield. Suppose the block first yields when σ_{xx} reaches σ_{xx}^{Y}. Then $\sigma_{zz}^{Y} = \nu\sigma_{xx}^{Y}$ and

$$\sigma_{xx}^{Y} = -\frac{Y}{\sqrt{1-\nu+\nu^2}} \tag{8.4.24}$$

Note that in this case it is predicted that first yield occurs when $\sigma_{xx} < -Y$. From Eqn. 8.4.17, the value of θ at first yield is

$$\cos\theta^{Y} = \frac{\sqrt{3}}{2\sqrt{1-\nu+\nu^2}} \quad \text{or} \quad \tan\theta^{Y} = \frac{1-2\nu}{\sqrt{3}} \tag{8.4.25}$$

Thus, with $\varepsilon_{xx} = \sigma_{xx}^{Y}\left(1-\nu^2\right)/E$ at yield,

$$C = -\sqrt{1-\nu+\nu^2} - \frac{\sqrt{3}}{2}\ln\left|\tan\frac{\theta^{Y}}{2}\right| \tag{8.4.26}$$

and so

$$-\frac{E}{Y}\varepsilon_{xx} = \frac{2}{\sqrt{3}}(1-2\nu)\sin\left(\frac{\pi}{6}-\theta\right) + \frac{\sqrt{3}}{2}\ln\left|\tan\frac{\theta^{Y}}{2}\cot\frac{\theta}{2}\right| + \sqrt{1-\nu+\nu^2} \tag{8.4.27}$$

This leads to a similar stress-strain curve as for the Tresca criterion, only now the limiting value of σ_{zz} is $-Y/\sqrt{3} \approx -0.58Y$.

8.4.4 Application: Combined Tension/Torsion of a thin walled tube

Consider now the combined tension/torsion of a thin-walled tube as in the Taylor/Quinney tests. The only stresses in the tube are $\sigma_{xx} = \sigma$ due to the tension along the axial direction and $\sigma_{xy} = \tau$ due to the torsion. The Prandtl-Reuss equations reduce to

$$\begin{aligned} d\varepsilon_{xx} &= \frac{1}{E}d\sigma_{xx} + \frac{2}{3}d\lambda\sigma_{xx} \\ d\varepsilon_{yy} &= d\varepsilon_{zz} = -\frac{\nu}{E}d\sigma_{xx} - \frac{1}{3}d\lambda\sigma_{xx} \\ d\varepsilon_{xy} &= \frac{1+\nu}{E}d\sigma_{xy} + d\lambda\sigma_{xy} \end{aligned} \tag{8.4.28}$$

Consider the case where the tube is twisted up to the yield point. Torsion is then halted and tension is applied, holding the angle of twist constant. In that case, during the tension, $d\varepsilon_{xy} = 0$ and so {▲Problem 3}

$$d\varepsilon_{xx} = \frac{1}{E}d\sigma - \frac{2}{3}\frac{1+\nu}{E}\frac{d\tau}{\tau}\sigma \tag{8.4.29}$$

If one takes the Von Mises criterion, then $\sigma^2 + 3\tau^2 = Y^2$ (see Eqn. 8.3.17). Assuming perfect plasticity, one has {▲Problem 4},

$$d\varepsilon_{xx} = \frac{1}{E}d\sigma + \frac{2}{3}\frac{1+\nu}{E}\frac{\sigma^2 d\sigma}{Y^2 - \sigma^2} \tag{8.4.30}$$

Using the relation

$$\int \frac{x^2}{a^2 - x^2}dx = -x + \frac{a}{2}\ln\left(\frac{a+x}{a-x}\right), \tag{8.4.31}$$

an integration leads to {▲Problem 5}

$$\frac{E}{Y}\varepsilon_{xx} = \frac{1}{3}\left\{(1-2\nu)\frac{\sigma}{Y} + (1+\nu)\ln\left(\frac{1+\sigma/Y}{1-\sigma/Y}\right)\right\} \tag{8.4.32}$$

This result is plotted in Fig. 8.4.4. Note that, with $\sigma^2 + 3\tau^2 = Y^2$, as σ increases (rapidly) to its limiting value Y, τ decreases from its yield value of $Y/\sqrt{3}$ to zero.

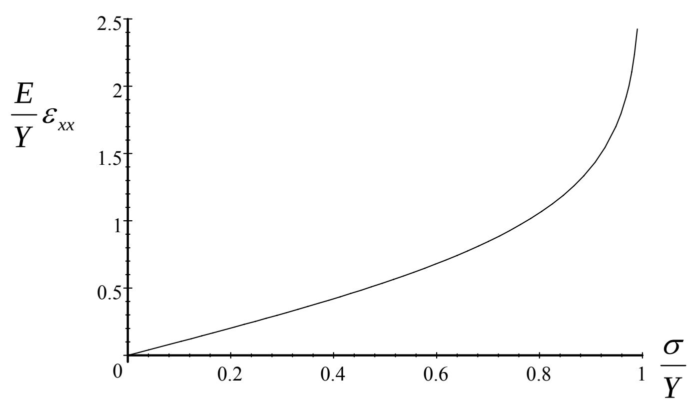


Figure 8.4.4: Stress-strain results for combined tension/torsion of a thin walled tube for $\nu = 0.3$

8.4.5 The Tresca Flow Rule

The flow rule used in the preceding applications was the Prandtl-Reuss rule 8.4.7. Many other flow rules have been proposed. For example, the Tresca flow rule is simply (for $\sigma_1 > \sigma_2 > \sigma_3$)

$$\begin{aligned} d\varepsilon_1^p &= +d\lambda \\ d\varepsilon_2^p &= 0 \\ d\varepsilon_3^p &= -d\lambda \end{aligned} \qquad (8.4.33)$$

This flow rule will be used in the next section, which details the classic solution for the plastic deformation and failure of a thick cylinder under internal pressure.

A unifying theory of flow rules will be presented in a later section, in which the reason for the name "Treca flow rule" will become clear.

8.4.6 Problems

1. Derive the elastic strains for the plane strain compression of a thick block, Eqns. 8.4.8.

2. Derive Eqns. 8.4.13 and 8.4.15

3. Derive Eqn. 8.4.29

4. Use Eqn. 8.3.17 to show that $\sigma d\tau / \tau = -\sigma^2 d\sigma / (Y^2 - \sigma^2)$ and hence derive Eqn. 8.4.30

5. Derive Eqns. 8.4.32

6. Does the axial stress-stress curve of Fig. 8.4.4 differ when the Tresca criterion is used?

7. Consider the uniaxial straining of a perfectly plastic isotropic Von Mises metallic block. There is only one non-zero strain, ε_{xx}. One only need consider two stresses, σ_{xx}, σ_{yy} since $\sigma_{zz} = \sigma_{yy}$ by isotropy.
   (i) Write down the two relevant Prandtl-Reuss equations
   (ii) Evaluate the stresses and strains at first yield
   (iii) For plastic flow, show that $d\sigma_{xx} = d\sigma_{yy}$ and that the plastic modulus is
   $$\frac{d\sigma_{xx}}{d\varepsilon_{xx}} = \frac{E}{3(1-2\nu)}$$

8. Consider the combined tension-torsion of a thin-walled cylindrical tube. The tube is made of a perfectly plastic Von Mises metal and Y is the uniaxial yield strength in tension. The only stresses are $\sigma_{xx} = \sigma$ and $\sigma_{xy} = \tau$ and the Prandtl-Reuss equations reduce to

$$
\begin{aligned}
d\varepsilon_{xx} &= \frac{1}{E}d\sigma_{xx} + \frac{2}{3}d\lambda\sigma_{xx} \\
d\varepsilon_{yy} &= d\varepsilon_{zz} = -\frac{\nu}{E}d\sigma_{xx} - \frac{1}{3}d\lambda\sigma_{xx} \\
d\varepsilon_{xy} &= \frac{1+\nu}{E}d\sigma_{xy} + d\lambda\sigma_{xy}
\end{aligned}
$$

The axial strain is increased from zero until yielding occurs (with $\varepsilon_{xy} = 0$). From first yield, the axial strain is held constant and the shear strain is increased up to its final value of $(1+\nu)Y/\sqrt{3}E$

(i) Write down the yield criterion in terms of σ and τ only and sketch the yield locus in $\sigma - \tau$ space
(ii) Evaluate the stresses and strains at first yield
(iii) Evaluate $d\lambda$ in terms of $\sigma, d\sigma$
(iv) Relate $\sigma, d\sigma$ to $\tau, d\tau$ and hence derive a differential equation for shear strain in terms of τ only
(v) Solve the differential equation and evaluate any constant of integration
(vi) Evaluate the shear stress when ε_{xy} reaches its final value of $(1+\nu)Y/\sqrt{3}E$.
Taking $\nu = 1/2$, put in the form $\tau = \alpha Y$ with α to 3 d.p.

8.5 The Internally Pressurised Cylinder

8.5.1 Elastic Solution

Consider the problem of a long thick hollow cylinder, with internal and external radii a and b, subjected to an internal pressure p. This can be regarded as a plane problem, with stress and strain independent of the axial direction z. The solution to the axisymmetric elastic problem is (see §4.3.5)

$$\begin{aligned}\sigma_{rr} &= -p\frac{b^2/r^2-1}{b^2/a^2-1}\\ \sigma_{\theta\theta} &= +p\frac{b^2/r^2+1}{b^2/a^2-1}\\ \sigma_{zz} &= +p\frac{1}{b^2/a^2-1}\times\begin{cases}2\nu, & \text{plane strain}\\ 0, & \text{open end}\\ 1, & \text{closed end}\end{cases}\end{aligned} \tag{8.5.1}$$

There are no shear stresses and these are the principal stresses. The strains are (with constant axial strain $\bar{\varepsilon}_{zz}$),

$$\begin{aligned}\bar{\varepsilon}_{zz} &= \frac{1}{E}\left[\sigma_{zz}-\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right)\right] \quad \to \quad \sigma_{zz}=E\bar{\varepsilon}_{zz}+\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right)\\ \varepsilon_{rr} &= \frac{1}{E}\left[\sigma_{rr}-\nu\left(\sigma_{\theta\theta}+\sigma_{zz}\right)\right]=-\nu\bar{\varepsilon}_{zz}+\frac{1+\nu}{E}\left[\left(1-\nu\right)\sigma_{rr}-\nu\sigma_{\theta\theta}\right]\\ \varepsilon_{\theta\theta} &= \frac{1}{E}\left[\sigma_{\theta\theta}-\nu\left(\sigma_{rr}+\sigma_{zz}\right)\right]=-\nu\bar{\varepsilon}_{zz}+\frac{1+\nu}{E}\left[-\nu\sigma_{rr}+\left(1-\nu\right)\sigma_{\theta\theta}\right]\end{aligned} \tag{8.5.2}$$

with

$$\bar{\varepsilon}_{zz}=\frac{p}{E}\frac{1}{b^2/a^2-1}\times\begin{cases}0, & \text{plane strain}\\ -2\nu, & \text{open end}\\ 1-2\nu, & \text{closed end}\end{cases} \tag{8.5.3}$$

Axial Force

The axial force in the elastic tube can be calculated through

$$\begin{aligned}P &= \int_0^{2\pi}\int_a^b \sigma_{zz}\,r\,dr\,d\theta = \int_0^{2\pi}\int_a^b\left[E\bar{\varepsilon}_{zz}+\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right)\right]r\,dr\,d\theta\\ &= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right)+\nu\int_0^{2\pi}\int_a^b\left(\sigma_{rr}+\sigma_{\theta\theta}\right)r\,dr\,d\theta \qquad ,\\ &= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right)+2\nu\pi p a^2\end{aligned} \tag{8.5.4}$$

which is the result expected:

$$P = \pi a^2 p \times \begin{cases} 2\nu, & \text{plane strain} \\ 0, & \text{open end} \\ 1, & \text{closed end} \end{cases} \tag{8.5.5}$$

i.e., consistent with the result obtained from a simple consideration of the axial stress in 8.5.1c.

8.5.2 Plastic Solution

The pressure is now increased so that the cylinder begins to deform plastically. It will be assumed that the material is isotropic and elastic perfectly-plastic and that it satisfies the Tresca criterion.

First Yield

It can be seen from 8.5.1 that $\sigma_{\theta\theta} > \sigma_{zz} \geq 0 > \sigma_{rr}$ (σ_{rr} does equal zero on the outer surface, $r = b$, but that is not relevant here, as plastic flow will begin on the inner wall), so the intermediate stress is σ_{zz}, and so the Tresca criterion reads

$$\left|\sigma_{rr} - \sigma_{\theta\theta}\right| = \sigma_{\theta\theta} - \sigma_{rr} = 2p\frac{b^2/r^2}{b^2/a^2 - 1} \equiv 2k \tag{8.5.6}$$

This expression has its maximum value at the inner surface, $r = a$, and hence it is here that plastic flow first begins. From this, plastic deformation begins when

$$p_{flow} = k\left(1 - \frac{a^2}{b^2}\right), \tag{8.5.7}$$

irrespective of the end conditions.

Confined Plastic Flow and Collapse

As the pressure increases above p_{flow}, the plastic region spreads out from the inner face; suppose that it reaches out to $r = c$. With the material perfectly plastic, the material in the annulus $a < r < c$ satisfies the yield condition 8.5.6 at all times. Consider now the equilibrium of this *plastic* material. Since this is an axi-symmetric problem, there is only one equilibrium equation:

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = 0\,. \tag{8.5.8}$$

It follows that

$$\frac{d\sigma_{rr}}{dr} - \frac{2k}{r} = 0 \quad \to \quad \sigma_{rr} = 2k\ln r + C_1 . \tag{8.5.9}$$

The constant of integration can be obtained from the pressure boundary condition at $r = a$, leading to

$$\sigma_{rr} = -p + 2k\ln(r/a) \qquad (a \le r \le c) \ \textbf{Plastic} \tag{8.5.10}$$

The stresses in the elastic region are again given by the elastic stress solution 8.5.1, only with *a* replaced by *c* and the pressure *p* is now replaced by the pressure exerted by the plastic region at $r = c$, i.e. $p - 2k\ln(c/a)$.

The precise location of the boundary *c* can be obtained by noting that the elastic stresses must satisfy the yield criterion at $r = c$. Since in the elastic region,

$$\sigma_{\theta\theta} - \sigma_{rr} = 2(p - 2k\ln(c/a))\frac{b^2/r^2}{b^2/c^2 - 1} \qquad (c \le r \le b) \ \textbf{Elastic} \tag{8.5.11}$$

one has from $(\sigma_{\theta\theta} - \sigma_{rr})_{r=c} = 2k$ that

$$(1 - c^2/b^2) + 2\ln(c/a) = \frac{p}{k} \tag{8.5.12}$$

Fig. 8.5.1 shows a plot of Eqn. 8.5.12.

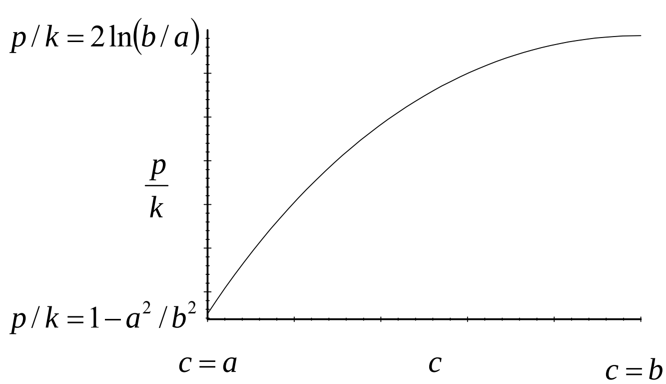


Figure 8.5.1: Extent of the plastic region $r = c$ during confined plastic flow

The complete cylinder will become plastic when *c* reaches *b*, or when the pressure reaches the **collapse pressure** (or **ultimate pressure**)

$$p_U = 2k\ln(b/a) . \tag{8.5.13}$$

This problem illustrates a number of features of elastic-plastic problems in general. First, **confined plastic flow** occurs. This is where the plastic region is surrounded by an elastic region, and so the plastic strains are of the same order as the elastic strains. It is only when the pressure reaches the collapse pressure does catastrophic failure occur.

Stress Field

As discussed after Eqn. 8.5.10, the stresses in the elastic region are derived from Eqns. 8.5.1 with the pressure given by $p - 2k\ln(c/a)$ and a replaced with c:

$$\begin{aligned} \sigma_{rr} &= -k\frac{c^2}{b^2}\left(\frac{b^2}{r^2} - 1\right) \\ \sigma_{\theta\theta} &= +k\frac{c^2}{b^2}\left(\frac{b^2}{r^2} + 1\right), \qquad c \le r \le b \quad \textbf{Elastic} \\ \sigma_{zz} &= +2k\nu\frac{c^2}{b^2} + E\bar{\varepsilon}_{zz} \end{aligned} \tag{8.5.14}$$

For the plastic region, the radial and hoop stresses can be obtained from 8.5.10 and 8.5.6. The Tresca flow rule, 8.4.33, implies that $d\varepsilon_{zz}^p = 0$ and so the axial strain $\bar{\varepsilon}_{zz}$ is purely elastic. Thus the elastic Hooke's Law relation $\bar{\varepsilon}_{zz} = \left[\sigma_{zz} - \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right]/E$ holds also in the plastic region, and

$$\begin{aligned} \sigma_{rr} &= -k\left(1 - \frac{c^2}{b^2} + 2\ln\frac{c}{r}\right) \\ \sigma_{\theta\theta} &= +k\left(1 + \frac{c^2}{b^2} - 2\ln\frac{c}{r}\right) \qquad , \quad a \le r \le c \quad \textbf{Plastic} \\ \sigma_{zz} &= +2k\nu\left(\frac{c^2}{b^2} - 2\ln\frac{c}{r}\right) + E\bar{\varepsilon}_{zz} \end{aligned} \tag{8.5.15}$$

Note that, as in the fully elastic solution, the radial and tangential stresses are independent of the end conditions, i.e. they are independent of $\bar{\varepsilon}_{zz}$. Unlike the elastic case, the axial stress is not constant in the plastic zone, and in fact it is tensile and compressive in different regions of the cylinder, depending on the end-conditions.

Axial Force and Strain

The total axial force can be obtained by integrating the axial stresses over the elastic zone and the plastic zone:

$$\begin{aligned} P &= \int_0^{2\pi}\int_a^c \sigma_{zz}^p r dr d\theta + \int_0^{2\pi}\int_c^b \sigma_{zz}^e r dr d\theta \\ &= 2\pi\left\{-4k\nu\ln\frac{c}{r}\int_a^c \sigma_{zz}^p r dr + \left(2k\nu\frac{c^2}{b^2} + E\bar{\varepsilon}_{zz}\right)\int_a^c \sigma_{zz}^p r dr + \left(2k\nu\frac{c^2}{b^2} + E\bar{\varepsilon}_{zz}\right)\int_c^b r dr\right\} \\ &= E\bar{\varepsilon}_{zz}\pi\left(b^2 - a^2\right) + 2k\pi\nu a^2\left[\left(1 - c^2/b^2\right) + \ln(c/a)\right] \\ &= E\bar{\varepsilon}_{zz}\pi\left(b^2 - a^2\right) + 2\nu\pi p a^2 \end{aligned} \tag{8.5.16}$$

the last line coming from Eqn. 8.5.12.

A neat alternative way of deriving this result for this problem is as follows: first, the equation of equilibrium 8.5.8, which of course applies in both elastic and plastic zones, can be used to write

$$\left(\sigma_{rr}+\sigma_{\theta\theta}\right)r = r\left(\sigma_{\theta\theta}-\sigma_{rr}\right)+2r\sigma_{rr} = r^2\frac{d\sigma_{rr}}{dr}+2r\sigma_{rr} = \frac{d}{dr}\left(r^2\sigma_{rr}\right) \tag{8.5.17}$$

Using the same method as used in Eqn. 8.5.4, but this time using 8.5.17 and the boundary conditions on the radial stresses at the inner and outer walls:

$$\begin{aligned} P &= \int_0^{2\pi}\int_a^b \left[E\bar{\varepsilon}_{zz}+\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right)\right] r\,dr\,d\theta \\ &= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right)+\nu 2\pi\left[r^2\sigma_{rr}\right]_a^b \\ &= E\bar{\varepsilon}_{zz}\pi\left(b^2-a^2\right)+2\nu\pi pa^2 \end{aligned} \tag{8.5.18}$$

which is the same as 8.5.16.

This axial force is the same as Eqn. 8.5.4. In other words, although σ_{zz} in general varies in the plastic zone, the axial force is independent of the plastic zone size *c*.

For the open cylinder, $P=0$, for plane strain, $\bar{\varepsilon}_{zz}=0$, and for the closed cylinder, $P=pa^2$. Unsurprisingly, since 8.5.16 (or 8.5.18) is the same as the elastic version, 8.5.4, one sees that the axial strains are as for the elastic solution, Eqn. 8.5.3. In terms of *c* and *k*, and using Eqn. 8.5.12, the axial strain is

$$\bar{\varepsilon}_{zz} = \frac{k}{E}\frac{1}{b^2/a^2-1}\left[1-\frac{c^2}{b^2}+2\ln(c/a)\right]\times\begin{cases} 0, & \text{plane strain} \\ -2\nu, & \text{open end} \\ 1-2\nu, & \text{closed end}\end{cases} \tag{8.5.19}$$

This can now be substituted into Eqns. 8.5.14 and 8.5.15 to get explicit expressions for the axial stresses. As an example, the stresses are plotted in Fig. 8.5.2 for the case of $b/a=2$, and $c/b=0.6, 0.8$.

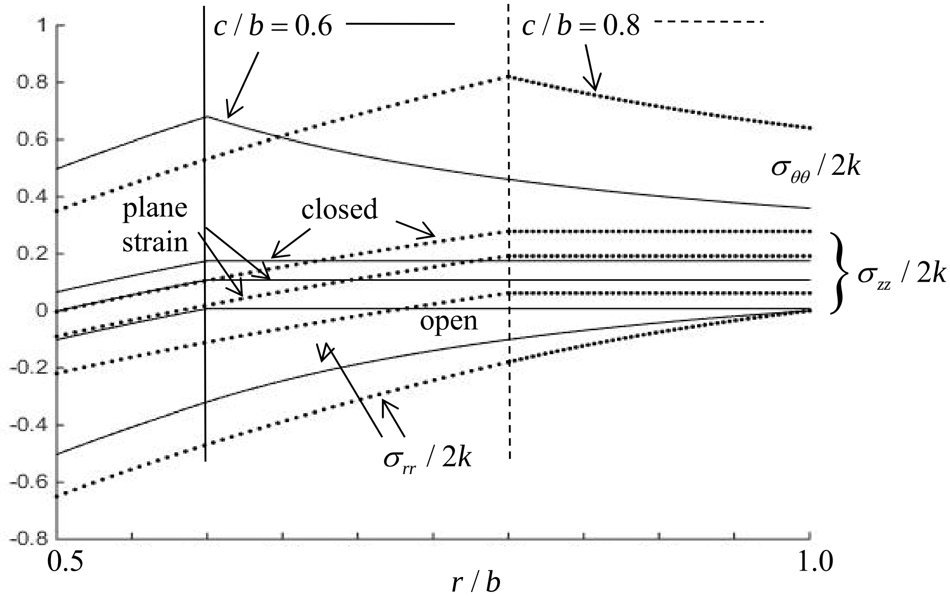


Figure 8.5.2: Stress field in the cylinder for the case of $b/a = 2$ and $c/b = 0.6, 0.8$

Strains and Displacement

In the elastic region, the strains are given by Hooke's law

$$\begin{aligned} \varepsilon_{rr} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}\right] - \nu\bar{\varepsilon}_{zz} \\ \varepsilon_{\theta\theta} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right] - \nu\bar{\varepsilon}_{zz} \end{aligned} \tag{8.5.20}$$

From Eqns. 8.5.14,

$$\begin{aligned} \varepsilon_{rr} &= k\frac{1+\nu}{E}\frac{c^2}{b^2}\left[(1-2\nu) - \frac{b^2}{r^2}\right] - \nu\bar{\varepsilon}_{zz} \\ \varepsilon_{\theta\theta} &= k\frac{1+\nu}{E}\frac{c^2}{b^2}\left[(1-2\nu) + \frac{b^2}{r^2}\right] - \nu\bar{\varepsilon}_{zz} \end{aligned}, \quad c \le r \le b \quad \textbf{Elastic} \tag{8.5.21}$$

From the definition of strain

$$\begin{aligned} \varepsilon_{rr} &= \frac{du_r}{dr} \\ \varepsilon_{\theta\theta} &= \frac{u_r}{r} \end{aligned} \tag{8.5.22}$$

Substituting Eqn. 8.5.21a into Eqn. 8.5.22a (or Eqn. 8.5.21b into Eqn. 8.5.22b) gives the displacement in the elastic region:

$$u_r = k\frac{1+\nu}{E}\frac{c^2}{b^2}\left((1-2\nu)r + \frac{b^2}{r}\right) - \nu r\bar{\varepsilon}_{zz}, \quad c \le r \le b \quad \textbf{Elastic} \tag{8.5.23}$$

Now for a Tresca material, from 8.4.33, the increments in plastic strain are $d\varepsilon_{zz}^p = 0$ and $d\varepsilon_{rr}^p = -d\varepsilon_{\theta\theta}^p$, so, from the elastic Hooke's Law,

$$\begin{aligned}\varepsilon_{rr} + \varepsilon_{\theta\theta} &= \varepsilon_{rr}^e + \varepsilon_{\theta\theta}^e \\ &= \frac{1}{E}\left[(1-\nu)(\sigma_{rr} + \sigma_{\theta\theta}) - 2\nu\sigma_{zz}\right] \\ &= \frac{1}{E}\left[(1-\nu)(\sigma_{rr} + \sigma_{\theta\theta}) - 2\nu\left(E\bar{\varepsilon}_{zz} + \nu\right)(\sigma_{rr} + \sigma_{\theta\theta})\right] \\ &= \frac{(1+\nu)(1-2\nu)}{E}(\sigma_{rr} + \sigma_{\theta\theta}) - 2\nu\bar{\varepsilon}_{zz}\end{aligned} \tag{8.5.24}$$

Using 8.5.22 and 8.5.17, one has

$$\frac{d}{dr}(ru_r) = \frac{(1+\nu)(1-2\nu)}{E}\frac{d}{dr}\left(r^2\sigma_{rr}\right) - 2\nu r\bar{\varepsilon}_{zz}, \tag{8.5.25}$$

which integrates to

$$u_r = \frac{(1+\nu)(1-2\nu)}{E}r\sigma_{rr} - \nu r\bar{\varepsilon}_{zz} + \frac{C}{r} \tag{8.5.26}$$

Equations 8.5.24-26 are valid in both the elastic and plastic regions. The constant of integration can be obtained from the condition $\sigma_{rr} = 0$ at $r = b$, where u_r equals the elastic displacement 8.5.23, and so $C = 2k(1-\nu^2)c^2 / E$ and

$$u_r = \frac{(1+\nu)(1-2\nu)}{E}r\sigma_{rr} + 2k\frac{(1-\nu^2)c^2}{Er} - \nu r\bar{\varepsilon}_{zz}, \quad a \le r \le c \quad \textbf{Plastic} \tag{8.5.27}$$

8.5.3 Unloading

Residual Stress

Suppose that the cylinder is loaded beyond p_{flow} but not up to the collapse pressure, to a pressure p_0 say. It is then unloaded completely. After unloading the cylinder is still subjected to a stress field – these stresses which are locked into the cylinder are called **residual stresses**. If the unloading process is fully elastic, the new stresses are obtained by subtracting 8.5.1 from 8.5.14-15. Using Eqn. 8.5.7, 8.5.12 {▲Problem 1},

$$\begin{aligned}\sigma_{rr} &= -k\left(\frac{c^2}{a^2}-\frac{p_0}{p_{flow}}\right)\left(\frac{a^2}{r^2}-\frac{a^2}{b^2}\right)\\ \sigma_{\theta\theta} &= +k\left(\frac{c^2}{a^2}-\frac{p_0}{p_{flow}}\right)\left(\frac{a^2}{r^2}+\frac{a^2}{b^2}\right), \qquad c \le r \le b \qquad \textbf{Elastic} \qquad (8.5.28)\\ \sigma_{zz} &= +2k\nu\left(\frac{c^2}{a^2}-\frac{p_0}{p_{flow}}\right)\frac{a^2}{b^2}\end{aligned}$$

$$\begin{aligned}\sigma_{rr} &= -k\left[\frac{p_0}{p_{flow}}\left(1-\frac{a^2}{r^2}\right)-2\ln\frac{r}{a}\right]\\ \sigma_{\theta\theta} &= -k\left[\frac{p_0}{p_{flow}}\left(1+\frac{a^2}{r^2}\right)-2-2\ln\frac{r}{a}\right], \qquad a \le r \le c \qquad \textbf{Plastic} \qquad (8.5.29)\\ \sigma_{zz} &= -2k\nu\left[\frac{p_0}{p_{flow}}-1-2\ln\frac{r}{a}\right]\end{aligned}$$

Consider as an example the case $a=1$, $b=2$, with $c=1.5$, for which

$$\frac{p_{flow}}{2k}=0.375, \qquad \frac{p_0}{2k}=0.624 \qquad (8.5.30)$$

The residual stresses are as shown in Fig. 8.5.3.

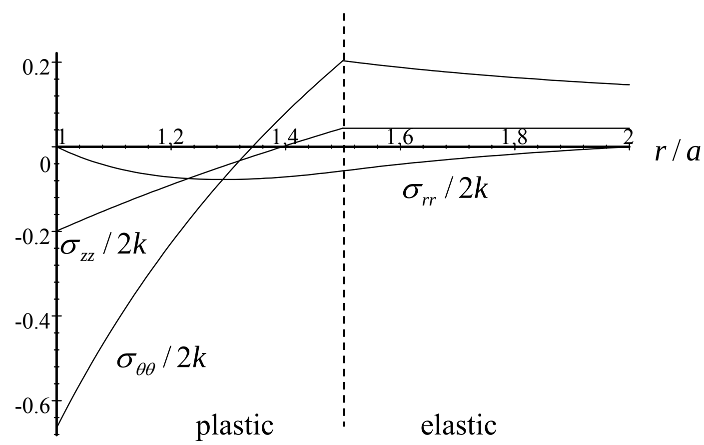


Figure 8.5.3: Residual stresses in the unloaded cylinder for the case of $a=1, b=2, c=1.5$

Note that the axial strain, being purely elastic, is completely removed, and the axial stresses, as given by Eqns. 8.5.28c, 8.5.29c, are independent of the end condition.

There is the possibility that if the original pressure p_0 is very large, the unloading will lead to compressive yield. The maximum value of $|\sigma_{\theta\theta} - \sigma_{rr}|$ occurs at $r = a$ {▲Problem 2}:

$$\left|\sigma_{\theta\theta} - \sigma_{rr}\right|_{r=a} = 2k\left(\frac{p_0}{p_{flow}} - 1\right) \tag{8.5.31}$$

and so, neglecting any Bauschinger effect, yield will occur if $p_0 \geq 2p_{flow}$. Yielding in compression will not occur right up to the collapse pressure p_U if the wall ratio b/a is such that $p_0|_{c=b} = p_U < 2p_{flow}$, or when {▲Problem 3}

$$\ln\frac{b}{a} < 1 - \frac{a^2}{b^2} \tag{8.5.32}$$

The largest wall ratio for which the unloading is completely elastic is $b/a \approx 2.22$. For larger wall ratios, a new plastic zone will develop at the inner wall, with $\sigma_{\theta\theta} - \sigma_{rr} = -2k$.

Shakedown

When the cylinder is initially loaded, plasticity begins at a pressure $p = p_{flow}$. If it is loaded to some pressure p_0, with $p_{flow} < p_0 < 2p_{flow}$, then unloading will be completely elastic. When the cylinder is reloaded again it will remain elastic up to pressure p_0. In this way, it is possible to strengthen the cylinder by an initial loading; theoretically it is possible to increase the flow pressure by a factor of 2. This maximum possible new flow pressure is called the **shakedown pressure** $p_s = \min(2p_{flow}, p_U)$. **Shakedown** is said to have occurred when any subsequent loading/unloading cycles are purely elastic. The strengthening of the cylinder is due to the compressive residual hoop stresses at the inner wall – similar to the way a barrel can be strengthened with hoops. This method of strengthening is termed **autofrettage**, a French term meaning "self-hooping".

8.5.4 Validity of the Solution

One needs to check whether the assumption of the ordering of the principal stresses, $\sigma_{\theta\theta} > \sigma_{zz} > \sigma_{rr}$, holds through the deformation in the plastic region. It can be confirmed that the inequality $\sigma_{zz} \geq \sigma_{rr}$ always holds. For the inequality $\sigma_{\theta\theta} \geq \sigma_{zz}$, consider the inequality $\sigma_{\theta\theta} - \sigma_{zz} \geq 0$. The quantity on the left is a minimum when $r = a$, where it equals {▲Problem 4}

$$k\left[(1-2\nu)\left(1 + \frac{c^2}{b^2} - 2\ln\frac{c}{a}\right) + 2\nu - \frac{1}{b^2/a^2 - 1}\left(1 - \frac{c^2}{b^2} + 2\ln\frac{c}{a}\right)\begin{bmatrix} 0 \\ -2\nu \\ 1-2\nu \end{bmatrix}\right] \tag{8.5.33}$$

This quantity must be positive for all values of c up to the maximum value b, where it takes its minimum value, and so one must have

$$2(1-2\nu)\left(1-\ln\frac{b}{a}\right)+2\nu-\frac{1}{b^2/a^2-1}\left(2\ln\frac{b}{a}\right)\begin{bmatrix}0\\-2\nu\\1-2\nu\end{bmatrix}\geq 0 \qquad (8.5.34)$$

The solution is thus valid only for limited values of b/a. For $\nu = 0.3$, one must have $b/a < 5.42$ (closed ends), $b/a < 5.76$ (plane strain), $b/a < 6.19$ (open ends). For higher wall ratios, the axial stress becomes equal to the hoop stress. In this case, a solution based on large changes in geometry is necessary for higher pressures.

8.5.5 Problems

1. Derive Eqns. 8.5.28-29.
2. Use Eqns. 8.5.29a,b to show that the maximum value of $|\sigma_{\theta\theta} - \sigma_{rr}|$ occurs at $r = a$, where it equals $2k(p_0 / p_{flow} - 1)$, Eqn. 8.5.31.
3. Derive Eqn. 8.5.32.
4. Use Eqns. 8.5.15, 8.5.12 to derive Eqn. 8.5.33.

Hardening

In the applications discussed in the preceding two sections, the material was assumed to be perfectly plastic. The issue of hardening (softening) materials is addressed in this section.

8.6.1 Hardening

In the one-dimensional (uniaxial test) case, a specimen will deform up to yield and then generally harden, Fig. 8.6.1. Also shown in the figure is the perfectly-plastic idealisation. In the perfectly plastic case, once the stress reaches the yield point (A), plastic deformation ensues, so long as the stress is maintained at Y. If the stress is reduced, elastic unloading occurs. In the hardening case, once yield occurs, the stress needs to be continually increased in order to drive the plastic deformation. If the stress is held constant, for example at B, no further plastic deformation will occur; at the same time, no elastic unloading will occur. Note that this condition cannot occur in the perfectly-plastic case, where there is one of plastic deformation or elastic unloading.

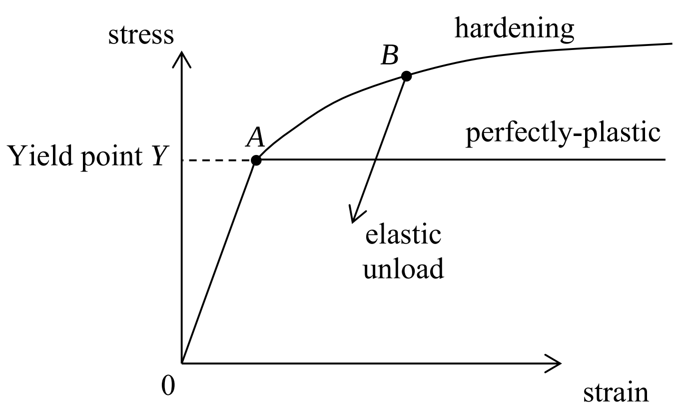


Figure 8.6.1: uniaxial stress-strain curve (for a typical metal)

These ideas can be extended to the multiaxial case, where the initial yield surface will be of the form

$$f_0(\sigma_{ij}) = 0 \tag{8.6.1}$$

In the perfectly plastic case, the yield surface remains unchanged.. In the more general case, the yield surface may change size, shape and position, and can be described by

$$f(\sigma_{ij}, \mathrm{K}_i) = 0 \tag{8.6.2}$$

Here, K_i represents one or more **hardening parameters**, which change during plastic deformation and determine the evolution of the yield surface. They may be scalars or higher-order tensors. At first yield, the hardening parameters are zero, and $f(\sigma_{ij}, 0) = f_0(\sigma_{ij})$.

The description of how the yield surface changes with plastic deformation, Eqn. 8.6.2, is called the **hardening rule**.

Strain Softening

Materials can also **strain soften**, for example soils. In this case, the stress-strain curve "turns down", as in Fig. 8.6.2. The yield surface for such a material will in general decrease in size with further straining.

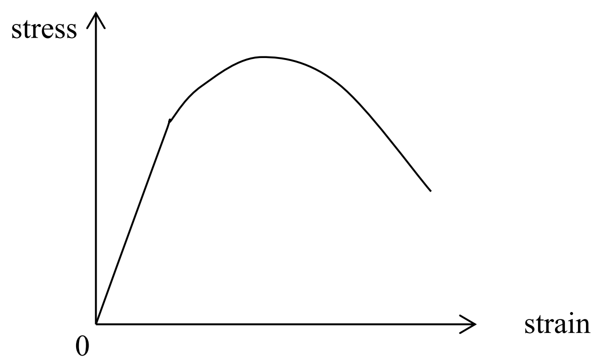


Figure 8.6.2: uniaxial stress-strain curve for a strain-softening material

8.6.2 Hardening Rules

A number of different hardening rules are discussed in this section.

Isotropic Hardening

Isotropic hardening is where the yield surface remains the same shape but expands with increasing stress, Fig. 8.6.3.

In particular, the yield function takes the form

$$f(\sigma_{ij}, \mathrm{K}_i) = f_0(\sigma_{ij}) - \mathrm{K} = 0 \tag{8.6.3}$$

The shape of the yield function is specified by the initial yield function and its size changes as the hardening parameter K changes.

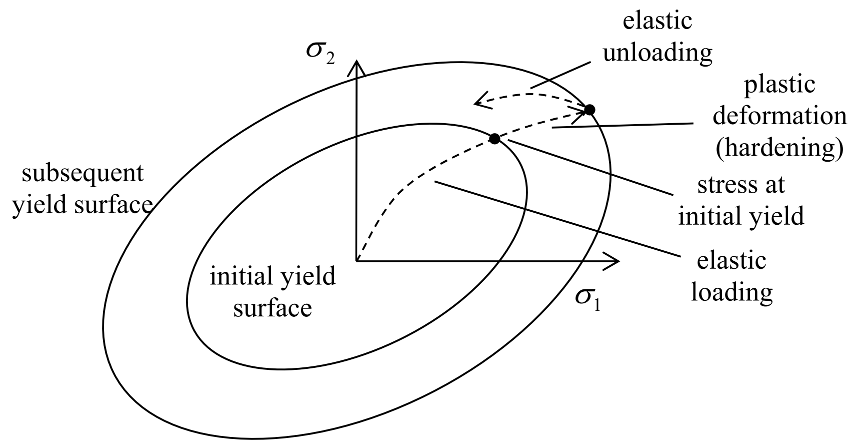


Figure 8.6.3: isotropic hardening

For example, consider the Von Mises yield surface. At initial yield, one has

$$\begin{aligned} f_0(\sigma_{ij}) &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}-Y \\ &= \sqrt{3J_2}-Y \\ &= \sqrt{\tfrac{3}{2}s_{ij}s_{ij}}-Y \end{aligned} \tag{8.6.4}$$

where Y is the yield stress in uniaxial tension. Subsequently, one has

$$f(\sigma_{ij},\mathrm{K}_i)=\sqrt{3J_2}-Y-\mathrm{K}=0 \tag{8.6.5}$$

The initial cylindrical yield surface in stress-space with radius $\sqrt{\frac{2}{3}}Y$ (see Fig. 8.3.11) develops with radius $\sqrt{\frac{2}{3}}(Y+\mathrm{K})$. The details of how the hardening parameter K actually changes with plastic deformation have not yet been specified.

As another example, consider the Drucker-Prager criterion, Eqn. 8.3.30, $f_0(\sigma_{ij})=\alpha I_1+\sqrt{J_2}-k=0$. In uniaxial tension, $I_1=Y$, $\sqrt{J_2}=Y/\sqrt{3}$, so $k=(\alpha+1/\sqrt{3})Y$. Isotropic hardening can then be expressed as

$$f(\sigma_{ij},\mathrm{K}_i)=\frac{1}{\alpha+1/\sqrt{3}}\left(\alpha I_1+\sqrt{J_2}\right)-Y-\mathrm{K}=0 \tag{8.6.6}$$

Kinematic Hardening

The isotropic model implies that, if the yield strength in tension and compression are initially the same, i.e. the yield surface is symmetric about the stress axes, they remain equal as the yield surface develops with plastic strain. In order to model the Bauschinger effect, and similar responses, where a hardening in tension will lead to a softening in a subsequent compression, one can use the **kinematic hardening** rule. This is where the yield surface remains the same shape and size but merely translates in stress space, Fig. 8.6.4.

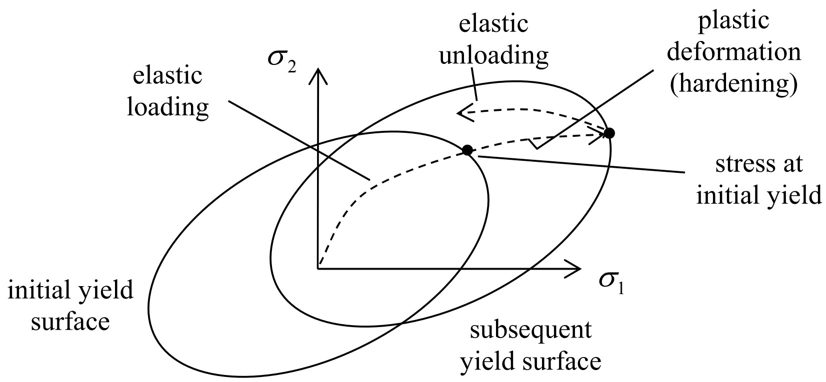


Figure 8.6.4: kinematic hardening

The yield function now takes the general form

$$f(\sigma_{ij}, \mathrm{K}_i) = f_0(\sigma_{ij} - \alpha_{ij}) = 0 \tag{8.6.7}$$

The hardening parameter here is the stress α_{ij}, known as the **back-stress** or **shift-stress**; the yield surface is shifted relative to the stress-space axes by α_{ij}, Fig. 8.6.5.

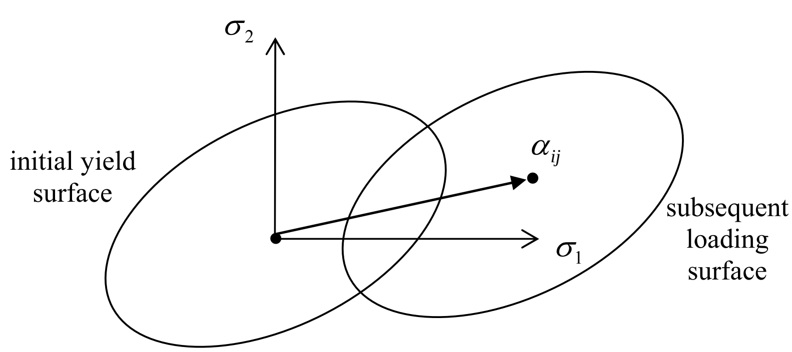


Figure 8.6.5: kinematic hardening; a shift by the back-stress

For example, again considering the Von Mises material, one has, from 8.6.4, and using the deviatoric part of $\boldsymbol{\sigma} - \boldsymbol{\alpha}$ rather than the deviatoric part of $\boldsymbol{\sigma}$,

$$f\left(\sigma_{ij}, \mathrm{K}_i\right) = \sqrt{\tfrac{3}{2}(s_{ij} - \alpha_{ij}^d)(s_{ij} - \alpha_{ij}^d)} - Y = 0 \tag{8.6.8}$$

where $\boldsymbol{\alpha}^d$ is the deviatoric part of $\boldsymbol{\alpha}$. Again, the details of how the hardening parameter α_{ij} might change with deformation will be discussed later.

Other Hardening Rules

More complex hardening rules can be used. For example, the **mixed hardening** rule combines features of both the isotropic and kinematic hardening models, and the loading function takes the general form

$$f(\sigma_{ij}, \mathrm{K}_i) = f_0(\sigma_{ij} - \alpha_{ij}) - \mathrm{K} = 0 \tag{8.6.9}$$

The hardening parameters are now the scalar K and the tensor α_{ij}.

8.6.3 The Flow Curve

In order to model plastic deformation and hardening in a complex three-dimensional geometry, one will generally have to use but the data from a simple test. For example, in the uniaxial tension test, one will have the data shown in Fig. 8.6.6a, with stress plotted against plastic strain. The idea now is to define a scalar **effective stress** $\hat{\sigma}$ and a scalar **effective plastic strain** $\hat{\varepsilon}^p$, functions respectively of the stresses and plastic strains in the loaded body. The following hypothesis is then introduced: a plot of effective stress against effective plastic strain follows the same **universal plastic stress-strain curve** as in the uniaxial case. This assumed universal curve is known as the **flow curve**.

The question now is: how should one define the effective stress and the effective plastic strain?

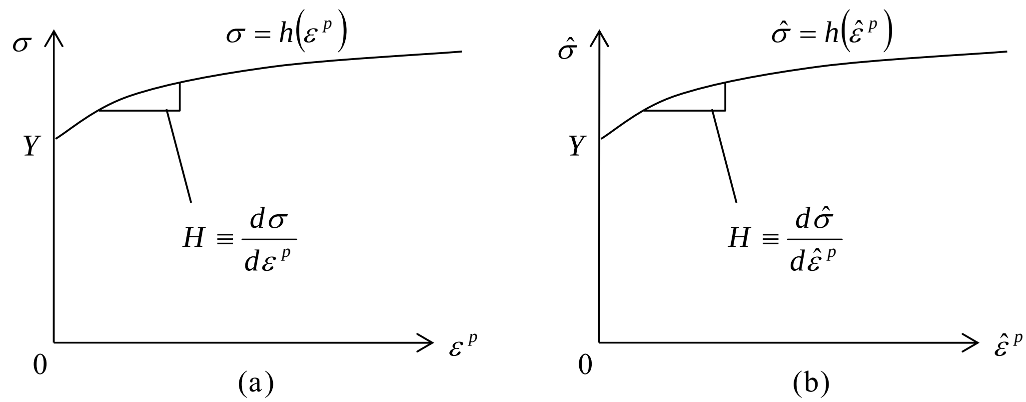


Figure 8.6.6: the flow curve; (a) uniaxial stress – plastic strain curve, (b) effective stress – effective plastic strain curve

8.6.4 A Von Mises Material with Isotropic Hardening

Consider a Von Mises material. Here, it is appropriate to define the effective stress to be

$$\hat{\sigma}(\sigma_{ij}) = \sqrt{3J_2} \tag{8.6.10}$$

This has the essential property that, in the uniaxial case, $\hat{\sigma}(\sigma_{ij}) = Y$. (In the same way, for example, the effective stress for the Drucker-Prager material, Eqn. 8.6.6, would be $\hat{\sigma}(\sigma_{ij}) = \left(\alpha I_1 + \sqrt{J_2}\right)/(\alpha + 1/\sqrt{3})$.)

For the effective plastic strain, one possibility is to define it in the following rather intuitive, non-rigorous, way. The deviatoric stress **s** and plastic strain (increment) tensor $d\boldsymbol{\varepsilon}^p$ are of a similar character. In particular, their traces are zero, albeit for different physical reasons; $J_1 = 0$ because of independence of hydrostatic pressure, $d\varepsilon_{ii}^p = 0$ because of material incompressibility in the plastic range. For this reason, one chooses the effective plastic strain (increment) $d\hat{\varepsilon}^p$ to be a similar function of $d\varepsilon_{ij}^p$ as $\hat{\sigma}$ is of the s_{ij}. Thus, in lieu of $\hat{\sigma} = \sqrt{\frac{3}{2} s_{ij} s_{ij}}$, one chooses $d\hat{\varepsilon}^p = C\sqrt{d\varepsilon_{ij}^p d\varepsilon_{ij}^p}$. One can determine the constant C by ensuring that the expression reduces to $d\hat{\varepsilon}^p = d\varepsilon_1^p$ in the uniaxial case. Considering this uniaxial case, $d\varepsilon_{11}^p = d\varepsilon_1^p$, $d\varepsilon_{22}^p = d\varepsilon_{33}^p = -\frac{1}{2} d\varepsilon_1^p$, one finds that

$$\begin{aligned} d\hat{\varepsilon}^p &= \sqrt{\tfrac{2}{3} d\varepsilon_{ij}^p d\varepsilon_{ij}^p} \\ &= \frac{\sqrt{2}}{3}\sqrt{\left(d\varepsilon_1^p - d\varepsilon_2^p\right)^2 + \left(d\varepsilon_2^p - d\varepsilon_3^p\right)^2 + \left(d\varepsilon_3^p - d\varepsilon_1^p\right)^2} \end{aligned} \tag{8.6.11}$$

Let the hardening in the uniaxial tension case be described using a relationship of the form (see Fig. 8.6.6)

$$\sigma = h(\varepsilon^p) \tag{8.6.12}$$

The slope of this flow curve is the plastic modulus, Eqn. 8.1.9,

$$H = \frac{d\sigma}{d\varepsilon^p} \tag{8.6.13}$$

The effective stress and effective plastic strain for any conditions are now assumed to be related through

$$\hat{\sigma} = h(\hat{\varepsilon}^p) \tag{8.6.14}$$

and the effective plastic modulus is given by

$$H = \frac{d\hat{\sigma}}{d\hat{\varepsilon}^p} \tag{8.6.15}$$

Isotropic Hardening

Assuming isotropic hardening, the yield surface is given by Eqn. 8.6.5, and with the definition of the effective stress, Eqn. 8.6.10,

$$f(\sigma_{ij}, \mathrm{K}_i) = \hat{\sigma} - Y - \mathrm{K} = 0 \tag{8.6.16}$$

Differentiating with respect to the effective plastic strain,

$$H = \frac{\partial \hat{\sigma}}{\partial \hat{\varepsilon}^p} = \frac{\partial \mathrm{K}}{\partial \hat{\varepsilon}^p} \tag{8.6.17}$$

One can now see how the hardening parameter evolves with deformation: K here is a function of the effective plastic strain, and its functional dependence on the effective plastic strain is given by the plastic modulus H of the universal flow curve.

Loading Histories

Each material particle undergoes a plastic strain history. One such path is shown in Fig. 8.6.7. At point q, the plastic strain is $\varepsilon_i^p(q)$. The effective plastic strain at q must be evaluated through an integration over the complete history of deformation:

$$\hat{\varepsilon}^p(q) = \int_0^q d\hat{\varepsilon}^p = \int_0^q \sqrt{\tfrac{2}{3} d\varepsilon_i^p d\varepsilon_i^p} \tag{8.6.18}$$

Note that the effective plastic strain at q is not simply $\sqrt{\tfrac{2}{3}\varepsilon_i^p(q)\varepsilon_i^p(q)}$, hence the definition of an effective plastic strain *increment* in Eqn. 8.6.11.

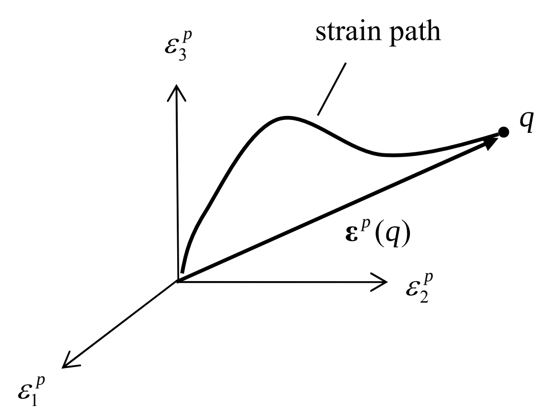


Figure 8.6.7: plastic strain space

Prandtl-Reuss Relations in terms of Effective Parameters

Using the Prandtl-Reuss (Levy-Mises) flow rule 8.4.1, and the definitions 8.6.10-11 for effective stress and effective plastic strain, one can now express the plastic multiplier as{▲Problem 1}

$$d\lambda = \frac{3}{2}\frac{d\hat{\varepsilon}^p}{\hat{\sigma}} \tag{8.6.19}$$

and the plastic strain increments, Eqn. 8.4.6, now read

$$\begin{aligned}
d\varepsilon_{xx}^p &= \left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\left[\sigma_{xx} - \tfrac{1}{2}\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
d\varepsilon_{yy}^p &= \left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\left[\sigma_{yy} - \tfrac{1}{2}\left(\sigma_{zz} + \sigma_{xx}\right)\right] \\
d\varepsilon_{zz}^p &= \left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\left[\sigma_{zz} - \tfrac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
d\varepsilon_{xy}^p &= \tfrac{3}{2}\left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\sigma_{xy} \\
d\varepsilon_{yz}^p &= \tfrac{3}{2}\left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\sigma_{yz} \\
d\varepsilon_{zx}^p &= \tfrac{3}{2}\left(d\hat{\varepsilon}^p / \hat{\sigma}\right)\sigma_{zx}
\end{aligned} \tag{8.6.20}$$

or

$$d\varepsilon_{ij}^p = \frac{3}{2}\frac{d\hat{\varepsilon}^p}{\hat{\sigma}} s_{ij}\,. \tag{8.6.21}$$

Knowledge of the plastic modulus, Eqn. 8.6.15, now makes equations 8.6.21 complete.

Note here that the plastic modulus in the Prandtl-Reuss equations is conveniently expressible in a simple way in terms of the effective stress and plastic strain increment, Eqn. 8.6.19. It will be shown in the next section that this is no coincidence, and that the Prandtl-Reuss flow-rule is indeed naturally associated with the Von-Mises criterion.

8.6.5 Application: Combined Tension/Torsion of a thin walled tube with Isotropic Hardening

Consider again the thin-walled tube under combined tension and torsion. The Von Mises yield function in terms of the axial stress σ and the shear stress τ is, as in §8.3.1, $f_0\left(\sigma_{ij}\right) = \sqrt{\sigma^2 + 3\tau^2} - Y = 0$. This defines the ellipse of Fig. 8.3.2. Subsequent yield surfaces are defined by

$$\begin{aligned}
f\left(\sigma_{ij}, \mathrm{K}_i\right) &= \sqrt{\sigma^2 + 3\tau^2} - Y - K \\
&= f_0\left(\sigma_{ij}\right) - K \\
&= \hat{\sigma} - \left(Y + K\right) \\
&= 0
\end{aligned} \tag{8.6.22}$$

Whereas the initial yield surface is the ellipse with major and minor axes Y and $Y/\sqrt{3}$, subsequent yield ellipses have axes $Y + K$ and $(Y + K)/\sqrt{3}$, Fig. 8.6.8.

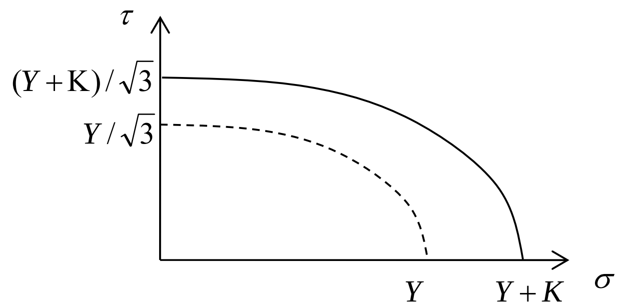


Figure 8.6.8: expansion of the yield locus (ellipse) for a thin-walled tube under isotropic hardening

The Prandtl-Reuss equations in terms of effective stress and effective plastic strain, 8.6.20-21, reduce to

$$\begin{aligned} d\varepsilon_{xx} &= \frac{1}{E}d\sigma + \frac{d\hat{\varepsilon}^{p}}{\hat{\sigma}}\sigma \\ d\varepsilon_{yy} = d\varepsilon_{zz} &= -\frac{\nu}{E}d\sigma - \frac{1}{2}\frac{d\hat{\varepsilon}^{p}}{\hat{\sigma}}\sigma \\ d\varepsilon_{xy} &= \frac{1+\nu}{E}d\tau + \frac{3}{2}\frac{d\hat{\varepsilon}^{p}}{\hat{\sigma}}\tau \end{aligned} \tag{8.6.23}$$

Consider the case where the material is brought to first yield through tension only, in which case the Von Mises condition reduces to $\sigma = Y$. Let the material then be subjected to a twist whilst maintaining the axial stress constant. The expansion of the yield surface is then as shown in Fig. 8.6.9.

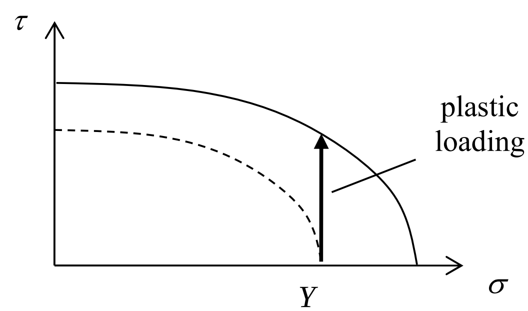


Figure 8.6.9: expansion of the yield locus for a thin-walled tube under constant axial loading

Introducing the plastic modulus, then, one has

$$\begin{aligned}
d\varepsilon_{xx} &= \frac{1}{H}\frac{d\hat{\sigma}}{\hat{\sigma}}Y \\
d\varepsilon_{yy} = d\varepsilon_{zz} &= -\frac{1}{2}\frac{1}{H}\frac{d\hat{\sigma}}{\hat{\sigma}}Y \\
d\varepsilon_{xy} &= \frac{1+\nu}{E}d\tau + \frac{3}{2}\frac{1}{H}\frac{d\hat{\sigma}}{\hat{\sigma}}\tau
\end{aligned} \tag{8.6.24}$$

Using $\hat{\sigma} = \sqrt{Y^2 + 3\tau^2}$,

$$\begin{aligned}
d\varepsilon_{xx} &= \frac{Y}{H}\frac{\tau d\tau}{\tau^2 + Y^2/3} \\
d\varepsilon_{yy} = d\varepsilon_{zz} &= -\frac{1}{2}\frac{Y}{H}\frac{\tau d\tau}{\tau^2 + Y^2/3} \\
d\varepsilon_{xy} &= \frac{1+\nu}{E}d\tau + \frac{3}{2}\frac{1}{H}\frac{\tau^2 d\tau}{\tau^2 + Y^2/3}
\end{aligned} \tag{8.6.25}$$

These equations can now be integrated. If the material is **linear hardening**, so H is constant, then they can be integrated exactly using

$$\int \frac{x}{x^2+a^2}dx = \frac{1}{2}\ln\left(x^2+a^2\right), \qquad \int \frac{x^2}{x^2+a^2}dx = x - a\arctan\left(\frac{x}{a}\right) \tag{8.6.26}$$

leading to {▲Problem 2}

$$\begin{aligned}
\frac{E}{Y}\varepsilon_{xx} &= 1 + \frac{1}{2}\frac{E}{H}\ln\left(1+3\frac{\tau^2}{Y^2}\right) \\
\frac{E}{Y}\varepsilon_{yy} = \frac{E}{Y_0}\varepsilon_{zz} &= -\frac{E}{4H}\ln\left(1+3\frac{\tau^2}{Y^2}\right) \\
\frac{E}{Y}\varepsilon_{xy} &= (1+\nu)\left(\frac{\tau}{Y}\right) + \frac{3}{2}\frac{E}{H}\left[\frac{\tau}{Y} - \frac{1}{\sqrt{3}}\arctan\left(\sqrt{3}\frac{\tau}{Y}\right)\right]
\end{aligned} \tag{8.6.27}$$

Results are presented in Fig. 8.6.10 for the case of $\nu = 0.3$, $E/H = 10$. The axial strain grows logarithmically and is eventually dominated by the faster-growing shear strain.

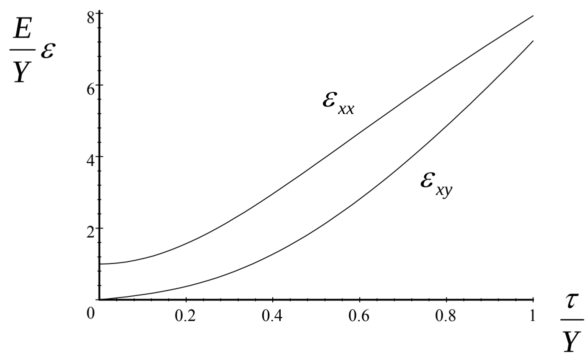


Figure 8.6.10: Stress-strain curves for thin-walled tube with isotropic linear strain hardening

8.6.6 Kinematic Hardening Rules

A typical uniaxial kinematic hardening curve is shown in Fig. 8.6.11a (see Fig. 8.1.3). During cyclic loading, the elastic zone always remains at $2Y$. Depending on the stress history, one can even have the situation shown in Fig. 8.6.11b, where yielding occurs upon unloading, even though the stress is still tensile.

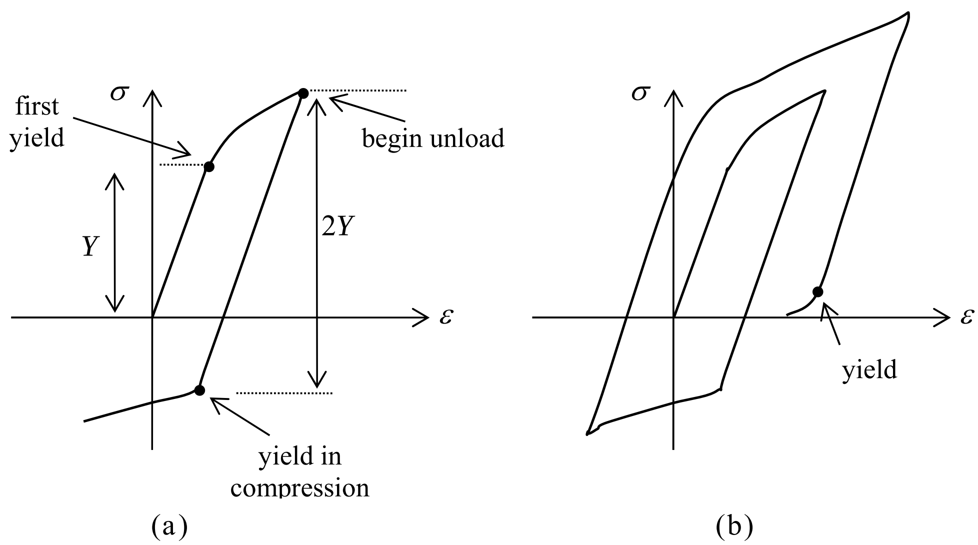


Figure 8.6.11: Kinematic Hardening; (a) load-unload, (b) cyclic loading

The multiaxial yield function for a kinematic hardening Von Mises is given by Eqn. 8.6.8,

$$f\left(\sigma_{ij}, \mathrm{K}_i\right) = \sqrt{\tfrac{3}{2}(s_{ij} - \alpha_{ij}^d)(s_{ij} - \alpha_{ij}^d)} - Y = 0$$

The deviatoric shift stress α_{ij}^d describes the shift in the centre of the Von Mises cylinder, as viewed in the π-plane, Fig. 8.6.12. This is a generalisation of the uniaxial case, in that the radius of the Von Mises cylinder remains constant, just as the elastic zone in the uniaxial case remains constant (at $2Y$).

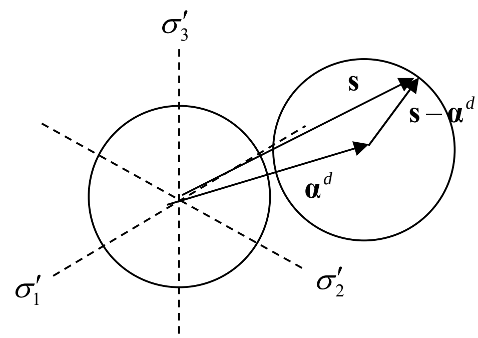


Figure 8.6.12: The Von Mises cylinder shifted in the π-plane

One needs to specify, by specifying the evolution of the hardening paremter $\boldsymbol{\alpha}$, how the yield surface shifts with deformation. In the multiaxial case, one has the added complication that the direction in which the yield surface shifts in stress space needs to be specified. The simplest model is the **linear kinematic** (or **Prager's**) **hardening rule**. Here, the back stress is assumed to depend on the plastic strain according to

$$\alpha_{ij} = c\varepsilon_{ij}^p \quad \text{or} \quad d\alpha_{ij} = cd\varepsilon_{ij}^p \tag{8.6.28}$$

where c is a material parameter, which might change with deformation. Thus the yield surface is translated in the same direction as the plastic strain increment. This is illustrated in Fig. 8.6.13, where the principal directions of stress and plastic strain are superimposed.

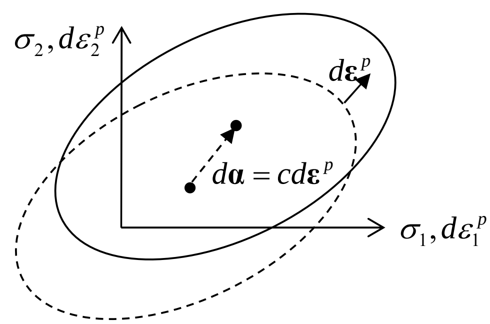


Figure 8.6.13: Linear kinematic hardening rule

One can use the uniaxial (possibly cyclic) curve to again define a universal plastic modulus H. Using the effective plastic strain, one can relate the constant c to H. This will be discussed in §8.8, where a more general formulation will be used.

Ziegler's hardening rule is

$$d\alpha_{ij} = da(\varepsilon_{ij}^p)(\sigma_{ij} - \alpha_{ij}) \tag{8.6.29}$$

where a is some scalar function of the plastic strain. Here, then, the loading function translates in the direction of $\sigma_{ij} - \alpha_{ij}$, Fig. 8.6.14.

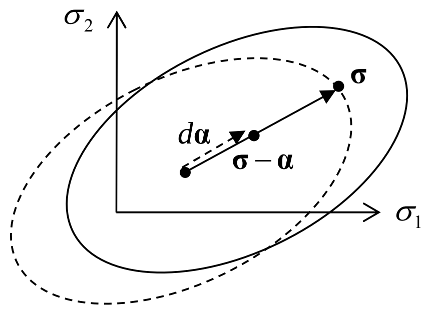


Figure 8.6.14: Ziegler's kinematic hardening rule

8.6.7 Strain Hardening and Work Hardening

In the models considered above, the hardening parameters have been functions of the plastic strains. For example, in the Von Mises isotropic hardening model, the hardening parameter K is a function of the effective plastic strain, $\hat{\varepsilon}^p$. Hardening expressed in this way is called **strain hardening**.

Another means of generalising the uniaxial results to multiaxial conditions is to use the **plastic work** (per unit volume), also known as the **plastic dissipation**,

$$dW^p = \sigma_{ij} d\varepsilon_{ij}^p \tag{8.6.30}$$

The total plastic work is the area under the stress – plastic strain curve of Fig. 8.6.6a,

$$W^p = \int \sigma_{ij} d\varepsilon_{ij}^p \tag{8.6.31}$$

A plot of stress against the plastic work can therefore easily be generated, as in Fig. 8.6.15.

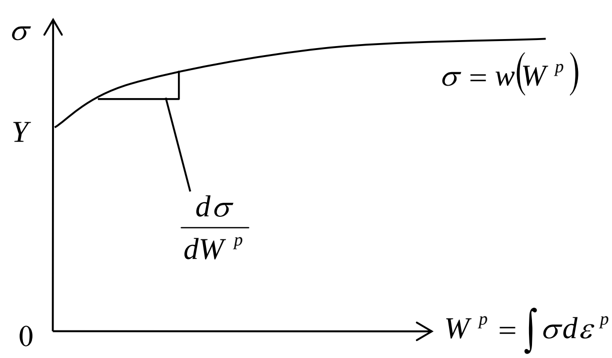


Figure 8.6.15: uniaxial stress – plastic work curve (for a typical metal)

The stress is now expressed in the form (compare with Eqn. 8.6.12)

$$\sigma = w\left(W^p\right) = w\left(\int \sigma d\varepsilon^p\right) \tag{8.6.32}$$

Again defining an effective stress $\hat{\sigma}$, the universal flow curve to be used for arbitrary loading conditions is then (compare with Eqn. 8.6.14)

$$\hat{\sigma} = w\left(W^p\right) \tag{8.6.33}$$

where now W^p is the plastic work during the multiaxial deformation. This is known as a **work hardening** formulation.

Equivalence of Strain and Work Hardening for the Isotropic Hardening Von Mises Material

Consider the Prandtl-Reuss flow rule, Eqn. 8.4.1, $d\varepsilon_i^p = s_i d\lambda$ (other flow rules will be examined more generally in §8.7). In this case, working with principal stresses, the plastic work increment is (see Eqns. 8.2.7-10)

$$\begin{aligned} dW^p &= \sigma_i d\varepsilon_i^p \\ &= \sigma_i s_i d\lambda \\ &= \frac{1}{3}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right] d\lambda \end{aligned} \tag{8.6.34}$$

Using the Von Mises effective stress 8.6.10, and Eqn. 8.6.19,

$$\begin{aligned} dW^p &= \tfrac{2}{3}\hat{\sigma}^2 d\lambda \\ &= \hat{\sigma}\, d\hat{\varepsilon}^p \end{aligned} \tag{8.6.35}$$

where $\hat{\varepsilon}^p$ is the very same effective plastic strain as used in the strain hardening isotropic model, Eqn. 8.6.11. Although true for the Von Mises yield condition, this will not be so in general.

8.6.8 Problems

1. Staring with the definition of the effective plastic strain, Eqn. 8.6.11, and using Eqn. 8.4.1, $d\varepsilon_i^p = d\lambda s_i$, derive Eqns. 8.6.19, $d\lambda = \dfrac{3}{2}\dfrac{d\hat{\varepsilon}^p}{\hat{\sigma}}$

2. Integrate Eqns. 8.6.25 and use the initial (first yield) conditions to get Eqns. 8.6.27.

3. Consider the combined tension-torsion of a thin-walled cylindrical tube. The tube is made of an isotropic hardening Von Mises metal with uniaxial yield stress Y. The strain-hardening is linear with plastic modulus *H*. The tube is loaded, keeping the ratio $\sigma/\tau = \sqrt{3}$ at all times throughout the elasto-plastic deformation, until $\sigma = Y$.

(i) Show that the stresses and strains at first yield are given by

$$\sigma^Y = \frac{1}{\sqrt{2}}Y, \quad \tau^Y = \frac{1}{\sqrt{6}}Y, \quad \varepsilon_{xx}^Y = \frac{1}{\sqrt{2}}\frac{Y}{E}, \quad \varepsilon_{xy}^Y = \frac{1+\nu}{\sqrt{6}}\frac{Y}{E}$$

(ii) The Prandtl-Reuss equations in terms of the effective stress and effective plastic strain are given by Eqns. 8.6.23. Eliminate τ from these equations (using $\sigma/\tau = \sqrt{3}$).

(iii) Eliminate the effective plastioc strain using the plastic modulus.

(iv) The effective stress is defined as $\hat{\sigma} = \sqrt{\sigma^2 + 3\tau^2}$ (see Eqn. 8.6.22). Eliminate the effective stress to obtain

$$d\varepsilon_{xx} = \frac{1}{E}d\sigma + \frac{1}{H}d\sigma$$

$$d\varepsilon_{xy} = \frac{1}{\sqrt{3}}\frac{1+\nu}{E}d\sigma + \frac{\sqrt{3}}{2}\frac{1}{H}d\sigma$$

(v) Integrate the differential equations and evaluate any constants of integration

(vi) Hence, show that the strains at the final stress values $\sigma = Y$, $\tau = Y/\sqrt{3}$ are given by

$$\frac{E}{Y}\varepsilon_{xx} = 1 + \frac{E}{H}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$\frac{E}{Y}\varepsilon_{xy} = \frac{1+\nu}{\sqrt{3}} + \frac{\sqrt{3}}{2}\frac{E}{H}\left(1 - \frac{1}{\sqrt{2}}\right)$$

(vii) Sketch the initial yield (elliptical) locus and the final yield locus in (σ, τ) space and the loading path.

(viii) Plot σ against ε_{xx}.

8.7 Associated and Non-associated Flow Rules

Recall the Levy-Mises flow rule, Eqn. 8.4.3,

$$d\varepsilon_{ij}^{p} = d\lambda s_{ij} \tag{8.7.1}$$

The plastic multiplier can be determined from the hardening rule. Given the hardening rule one can more generally, instead of the particular flow rule 8.7.1, write

$$d\varepsilon_{ij}^{p} = d\lambda G_{ij}\,, \tag{8.7.2}$$

where G_{ij} is some function of the stresses and perhaps other quantities, for example the hardening parameters. It is symmetric because the strains are symmetric.

A wide class of material behaviour (perhaps all that one would realistically be interested in) can be modelled using the general form

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial g}{\partial \sigma_{ij}}\,. \tag{8.7.3}$$

Here, *g* is a scalar function which, when differentiated with respect to the stresses, gives the plastic strains. It is called the **plastic potential**. The flow rule 8.7.3 is called a **non-associated flow rule**.

Consider now the sub-class of materials whose plastic potential *is* the yield function, $g = f$:

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}\,. \tag{8.7.4}$$

This flow rule is called an **associated flow-rule**, because the flow rule is associated with a particular yield criterion.

8.7.1 Associated Flow Rules

The yield surface $f(\sigma_{ij}) = 0$ is displayed in Fig 8.7.1. The axes of principal stress and principal plastic strain are also shown; the material being isotropic, these are taken to be coincident. The normal to the yield surface is in the direction $\partial f / \sigma_{ij}$ and so the associated flow rule 8.7.4 can be interpreted as saying that *the plastic strain increment vector is normal to the yield surface*, as indicated in the figure. This is called the **normality rule**.

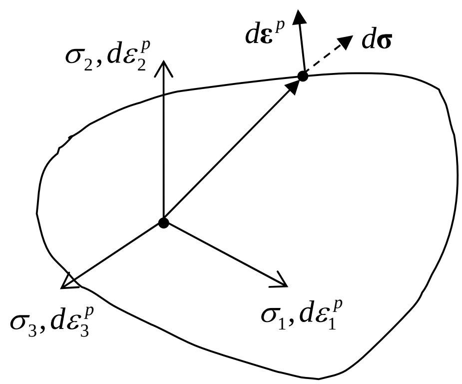


Figure 8.7.1: Yield surface

The normality rule has been confirmed by many experiments on metals. However, it is found to be seriously in error for soils and rocks, where, for example, it overestimates plastic volume expansion. For these materials, one must use a non-associative flow-rule.

Next, the Tresca and Von Mises yield criteria will be discussed. First note that, to make the differentiation easier, the associated flow-rule 8.7.4 can be expressed in terms of principal stresses as

$$d\varepsilon_i^p = d\lambda \frac{\partial f}{\partial \sigma_i}. \tag{8.7.5}$$

Tresca

Taking $\sigma_1 > \sigma_2 > \sigma_3$, the Tresca yield criterion is

$$f = \frac{\sigma_1 - \sigma_3}{2} - k \tag{8.7.6}$$

One has

$$\frac{\partial f}{\partial \sigma_1} = +\frac{1}{2}, \quad \frac{\partial f}{\partial \sigma_2} = 0, \quad \frac{\partial f}{\partial \sigma_3} = -\frac{1}{2} \tag{8.7.7}$$

so, from 8.7.5, the flow-rule associated with the Tresca criterion is

$$\begin{bmatrix} d\varepsilon_1^p \\ d\varepsilon_2^p \\ d\varepsilon_3^p \end{bmatrix} = d\lambda \begin{bmatrix} +\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}. \tag{8.7.8}$$

This is the flow-rule of Eqns. 8.4.33. The plastic strain increment is illustrated in Fig. 8.7.2 (see Fig. 8.3.9). All plastic deformation occurs in the $1-3$ plane. Note that 8.7.8 is independent of stress.

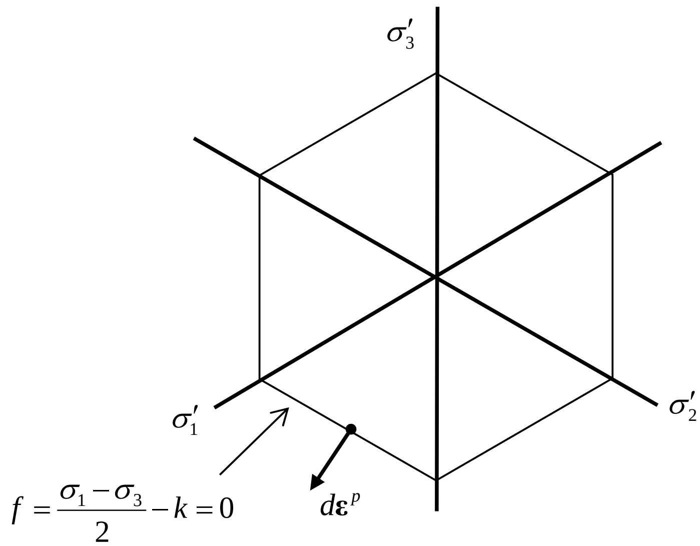


Figure 8.7.2: The plastic strain increment vector and the Tresca criterion in the π-plane (for the associated flow-rule)

Von Mises

The Von Mises yield criterion is $f = J_2 - k^2 = 0$. With

$$\frac{\partial J_2}{\partial \sigma_1} = \frac{\partial}{\partial \sigma_1} \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = \frac{2}{3}\left[\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right] \quad (8.7.9)$$

one has

$$\begin{bmatrix} d\varepsilon_1^p \\ d\varepsilon_2^p \\ d\varepsilon_3^p \end{bmatrix} = d\lambda \begin{bmatrix} \frac{2}{3}\left(\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right) \\ \frac{2}{3}\left(\sigma_2 - \frac{1}{2}(\sigma_1 + \sigma_3)\right) \\ \frac{2}{3}\left(\sigma_3 - \frac{1}{2}(\sigma_1 + \sigma_2)\right) \end{bmatrix}. \quad (8.7.10)$$

This is none other than the Levy-Mises flow rule 8.4.6[1].

The associative flow-rule is very appealing, connecting as it does the yield surface to the flow-rule. Many attempts have been made over the years to justify this rule, both

[1] note that if one were to use the alternative but equivalent expression $f = \sqrt{J_2} - k = 0$, one would have a $1/2\sqrt{J_2}$ term common to all three principal strain increments, which could be "absorbed" into the $d\lambda$ giving the same flow-rule 8.7.10

mathematically and physically. However, it should be noted that the associative flow-rule is not a law of nature by any means. It is simply very convenient. That said, it does agree with experimental observations of many plastically deforming materials, particularly metals.

In order to put the notion of associative flow-rules on a sounder footing, one can define more clearly the type of material for which the associative flow-rule applies; this is tied closely to the notion of stable and unstable materials.

8.7.2 Drucker's Postulate

Stress Cycles

First, consider the one-dimensional loading of a hardening material. The material may have undergone any type of deformation (e.g. elastic or plastic) and is now subjected to the stress σ^*, point A in Fig. 8.7.3. An *additional* load is now applied to the material, bringing it to the current yield stress σ at point B (if σ^* is below the yield stress) and then plastically (greatly exaggerated in the figure) through the infinitesimal increment $d\sigma$ to point C. It is conventional to call these additional loads the **external agency**. The external agency is then removed, bringing the stress back to σ^* and point D. The material is said to have undergone a **stress cycle**.

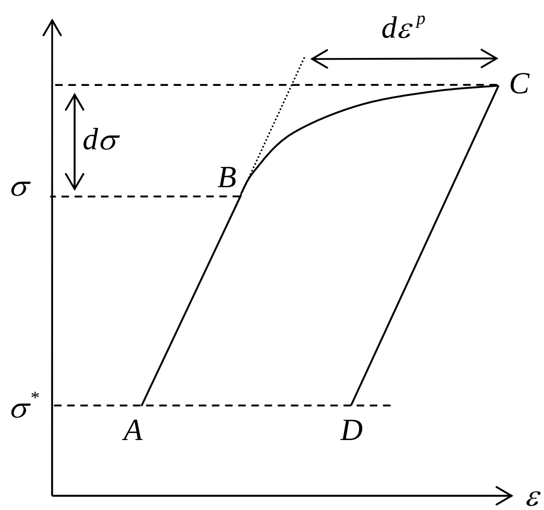


Figure 8.7.3: A stress cycle for a hardening material

Consider now a softening material, Fig. 8.7.4. The external agency first brings the material to the current yield stress σ at point B. To reach point C, the loads must be reduced. This cannot be achieved with a stress (force) control experiment, since a reduction in stress at B will induce elastic unloading towards A. A strain (displacement) control must be used, in which case the stress required to induce the (plastic) strain will be seen to drop to $\sigma + d\sigma$ ($d\sigma < 0$) at C. The stress cycle is completed by unloading from C to D.

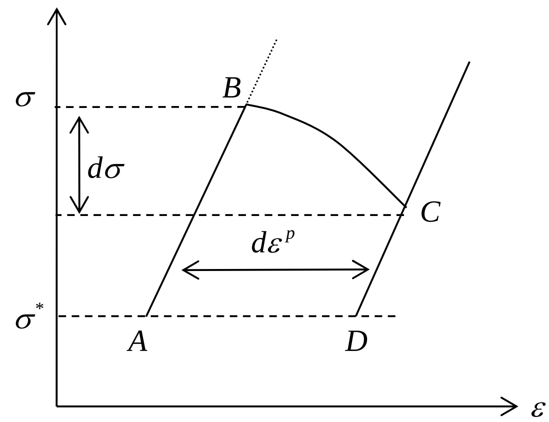


Figure 8.7.4: A stress cycle for a softening material

Suppose now that $\sigma = \sigma^*$, so the material is at point B, on the yield surface, before action by the external agency. It is now not possible for the material to undergo a stress cycle, since the stress cannot be increased. This provides a means of distinguishing between strain hardening and softening materials:

Strain-hardening … Material can always undergo a stress-cycle
Strain-softening … Material cannot always undergo a stress-cycle

Drucker's Postulate

The following statements define a **stable material**: (these statements are also known as **Drucker's postulate**):

(1) Positive work is done by the external agency during the application of the loads
(2) The net work performed by the external agency over a stress cycle is nonnegative

By this definition, it is clear that a strain hardening material is stable (and satisfies Drucker's postulates). For example, considering plastic deformation ($\sigma = \sigma^*$ in the above), the work done during an increment in stress is $d\sigma d\varepsilon$. The work done by the external agency is the area shaded in Fig. 8.7.5a and is clearly positive (note that the work referred to here is not the *total* work, $\int_{\varepsilon}^{\varepsilon+d\varepsilon} \sigma d\varepsilon$, but only that part which is done by the external agency[2]). Similarly, the net work over a stress cycle will be positive.

On the other hand, note that plastic loading of a softening (or perfectly plastic) material results in a non-positive work, Fig. 8.7.5b.

[2] the laws of thermodynamics insist that the total work is positive (or zero) in a complete cycle.

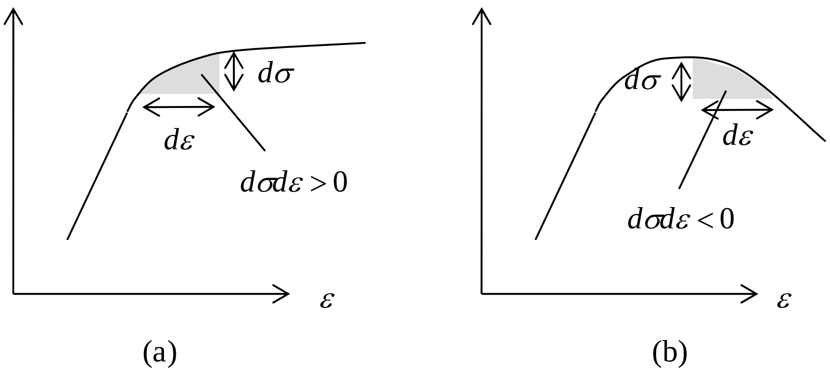


Figure 8.7.5: Stable (a) and unstable (b) stress-strain curves

The work done (per unit volume) by the additional loads during a stress cycle A-B-C-D is given by:

$$W = \int_{A-B-C-D} \left(\sigma(\varepsilon) - \sigma^*\right) d\varepsilon \tag{8.7.11}$$

This is the shaded work in Fig. 8.7.6. Writing $d\varepsilon = d\varepsilon^e + d\varepsilon^p$ and noting that the elastic work is recovered, i.e. the net work due to the elastic strains is zero, this work is due to the plastic strains,

$$W = \int_{B-C} \left(\sigma(\varepsilon) - \sigma^*\right) d\varepsilon^p \tag{8.7.12}$$

With $d\sigma$ infinitesimal, this equals

$$W = \left(\sigma - \sigma^*\right) d\varepsilon^p + \tfrac{1}{2} d\sigma d\varepsilon^p \tag{8.7.13}$$

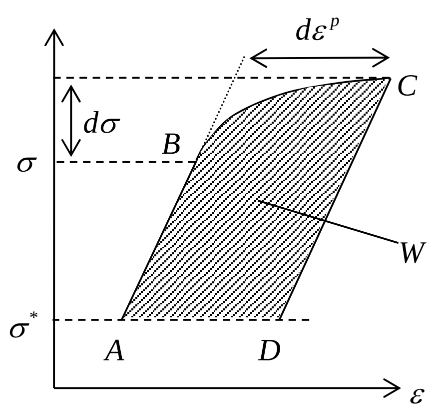


Figure 8.7.6: Work *W* done during a stress cycle of a strain-hardening material

The requirement (2) of a stable material is that this work be non-negative,

$$W = \left(\sigma - \sigma^*\right) d\varepsilon^p + \tfrac{1}{2} d\sigma d\varepsilon^p \geq 0 \tag{8.7.14}$$

Making $\sigma - \sigma^* >> d\sigma$, this reads

$$\left(\sigma - \sigma^*\right) d\varepsilon^p \geq 0 \tag{8.7.15}$$

On the other hand, making $\sigma = \sigma^*$, it reads

$$d\sigma d\varepsilon^p \geq 0 \tag{8.7.16}$$

The three dimensional case is illustrated in Fig. 8.7.7, for which one has

$$\left(\sigma_{ij} - \sigma_{ij}^*\right) d\varepsilon_{ij}^p \geq 0, \quad d\sigma_{ij} d\varepsilon_{ij}^p \geq 0 \tag{8.7.17}$$

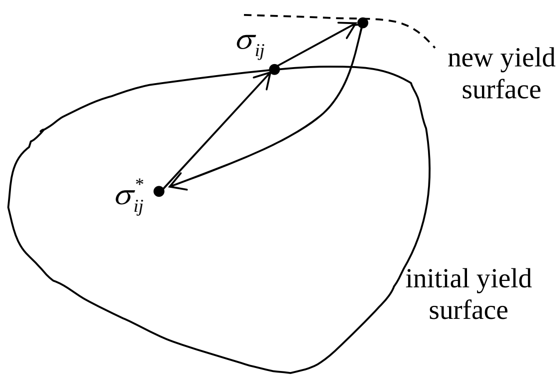


Figure 8.7.7: Stresses during a loading/unloading cycle

8.7.3 Consequences of the Drucker's Postulate

The criteria that a material be stable have very interesting consequences.

Normality

In terms of vectors in principal stress (plastic strain increment) space, Fig. 8.7.8, Eqn. 8.7.17 reads

$$\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*\right) \cdot d\boldsymbol{\varepsilon}^p \geq 0 \tag{8.7.18}$$

These vectors are shown with the solid lines in Fig. 8.7.8. Since the dot product is non-negative, the angle between the vectors $\boldsymbol{\sigma} - \boldsymbol{\sigma}^*$ and $d\boldsymbol{\varepsilon}^p$ (with their starting points coincident) must be less than 90°. This implies that the plastic strain increment vector must be normal to the yield surface since, if it were not, an initial stress state $\boldsymbol{\sigma}^*$ could be found for which the angle was greater than 90° (as with the dotted vectors in Fig. 8.7.8). Thus a consequence of a material satisfying the stability requirements is that the **normality rule** holds, i.e. the flow rule is associative, Eqn. 8.7.4.

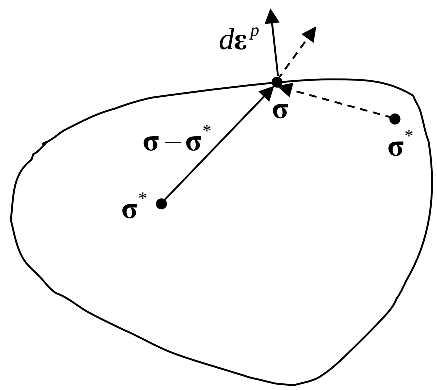


Figure 8.7.8: Normality of the plastic strain increment vector

When the yield surface has sharp corners, as with the Tresca criterion, it can be shown that the plastic strain increment vector must lie within the cone bounded by the normals on either side of the corner, as illustrated in Fig. 8.7.9.

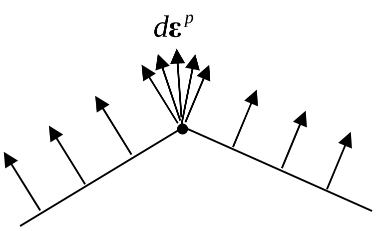


Figure 8.7.9: The plastic strain increment vector for sharp corners

Convexity

Using the same arguments, one cannot have a yield surface like the one shown in Fig. 8.7.10. In other words, the yield surface is **convex**: the entire elastic region lies to one side of the tangent plane to the yield surface[3].

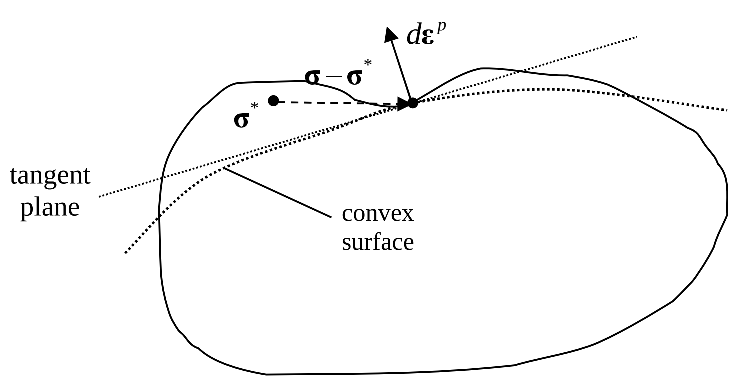


Figure 8.7.10: A non-convex surface

[3] note that when the plastic deformation affects the elastic response of the material, it can be shown that the stability postulate again ensures normality, but that the convexity does not necessarily hold

In summary then, Drucker's Postulate, which is satisfied by a stable, strain-hardening material, implies normality (associative flow rule) and convexity[4].

8.7.4 The Principle of Maximum Plastic Dissipation

The rate form of Eqn. 8.7.18 is

$$\left(\sigma_{ij} - \sigma_{ij}^*\right)\dot{\varepsilon}_{ij}^p \geq 0 \tag{8.7.19}$$

The quantity $\sigma_{ij}\dot{\varepsilon}_{ij}^p$ is called the **plastic dissipation**, and is a measure of the *rate* at which energy is being dissipated as deformation proceeds.

Eqn. 8.7.19 can be written as

$$\sigma_{ij}\dot{\varepsilon}_{ij}^p \geq \sigma_{ij}^*\dot{\varepsilon}_{ij}^p \quad \text{or} \quad \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}}^p \geq \boldsymbol{\sigma}^*\cdot\dot{\boldsymbol{\varepsilon}}^p \tag{8.7.20}$$

and in this form is known as the **principle of maximum plastic dissipation**: of all possible stress states σ_{ij}^* (within or on the yield surface), the one which arises is that which requires the maximum plastic work.

Although the principle of maximum plastic dissipation was "derived" from Drucker's postulate in the above, it is more general, holding also for the case of perfectly plastic and softening materials. To see this, disregard stress cycles and consider a stress state σ^* which is at or below the current (yield) stress σ, and apply a strain $d\varepsilon > 0$. For a perfectly plastic material, $\sigma - \sigma^* \geq 0$ and $d\varepsilon = d\varepsilon^p > 0$. For a softening material, again $\sigma - \sigma^* \geq 0$ and $d\varepsilon^e < 0$, $d\varepsilon^p > d\varepsilon > 0$.

It follows that the normality rule and convexity hold also for the perfectly plastic and softening materials which satisfy the principle of maximum plastic dissipation.

In summary:

Drucker's postulate leads to the Principle of maximum plastic dissipation
For hardening materials
    Principle of maximum plastic dissipation leads to Drucker's postulate
For softening materials
    Principle of maximum plastic dissipation does not lead to Drucker's postulate

Finally, note that, for many materials, hardening and softening, a non-associative flow rule is required, as in Eqn. 8.7.3. Here, the plastic strain increment is no longer normal to the yield surface and the principle of maximum plastic dissipation does not hold in general. In this case, when there is hardening, i.e. the stress increment is directed out from the yield surface, it is easy to see that one can have $d\sigma_{ij}d\varepsilon_{ij}^p < 0$, Fig. 8.7.11, contradicting the stability postulate (1),. With hardening, there is no

[4] it also ensures the uniqueness of solution to the boundary value elastoplastic problem

obvious instability, and so it could be argued that the use of the term "stability" in Drucker's postulate is inappropriate.

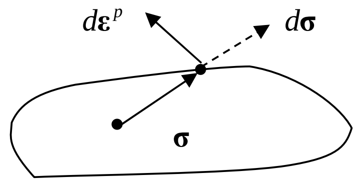


Figure 8.7.11: plastic strain increment vector not normal to the yield surface; non-associated flow-rule

8.7.5 Problems

1. Derive the flow-rule associated with the Drucker-Prager yield criterion $f = \alpha I_1 + \sqrt{J_2} - k$

2. Derive the flow-rule associated with the Mohr-Coulomb yield criterion, i.e. with $\sigma_1 > \sigma_2 > \sigma_3$,
$$\frac{\alpha\sigma_1 - \sigma_3}{2} = k$$
Here,
$$\alpha = \frac{1+\sin\phi}{1-\sin\phi} > 1, \qquad 0 < \phi < \frac{\pi}{2}$$
Evaluate the volumetric plastic strain increment, that is
$$\frac{\Delta V^p}{\Delta V} = d\varepsilon_1^p + d\varepsilon_2^p + d\varepsilon_3^p,$$
and hence show that the model predicts dilatancy (expansion).

3. Consider the plastic potential
$$g = \frac{\beta\sigma_1 - \sigma_3}{2} - k$$
Derive the non-associative flow-rule corresponding to this potential. Hence show that compaction of material can be modelled by choosing an appropriate value of β.