A NOTE ON THE COSSERAT SURFACE

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SUMMARY

The purpose of this note is to clarify and extend one aspect of some special cases of a Cosserat surface discussed in section 7 of a paper by Green, Naghdi and Wainwright (1) which have a bearing on the classical theory of shells.

LET θ^α ($\alpha = 1, 2$) be convected coordinates on a two-dimensional surface and let $\mathbf{a}_\alpha$, $\mathbf{a}^\alpha$ be covariant and contravariant base vectors at time t associated with θ^α, and $\mathbf{a}_3$ a unit normal vector to the surface. The metric tensors for the surface at time t are $a_{\alpha\beta}$, $a^{\alpha\beta}$, and $b_{\alpha\beta}$ is the second fundamental form. A director $\mathbf{d}$ at time t is associated with every point of the surface and

$$\mathbf{d} = d^i\mathbf{a}_i = d_i\mathbf{a}^i.$$

Italic indices have the values 1, 2, 3. Green, Naghdi and Wainwright (1) obtained basic equations of motion and constitutive equations for an elastic surface. We refer the reader to the previous paper for details and definitions and quote the relevant results here.

The basic equations of motion are

$$\left.\begin{aligned} N^{\beta\alpha}|_\alpha - b^\beta_\alpha N^{3\alpha} + \rho F^\beta &= \rho c^\beta, \\ N^{3\alpha}|_\alpha + b_{\alpha\beta}N^{\beta\alpha} + \rho F^3 &= \rho c^3, \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} M^{\beta\alpha}|_\alpha - b^\beta_\alpha M^{3\alpha} + \rho L^\beta &= m^\beta, \\ M^{3\alpha}|_\alpha + b_{\alpha\beta}M^{\beta\alpha} + \rho L^3 &= m^3, \end{aligned}\right\} \tag{2}$$

and

$$\left.\begin{aligned} N^{3\alpha} + m^3 d^\alpha - m^\alpha d^3 + M^{3\gamma}\lambda^{\cdot\cdot\alpha}_{\cdot\gamma} - M^{\alpha\gamma}\lambda^{\cdot\cdot 3}_{\cdot\gamma} &= 0, \\ N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\beta\alpha} - m^\alpha d^\beta - M^{\alpha\gamma}\lambda^{\cdot\cdot\beta}_{\cdot\gamma}, & \end{aligned}\right\} \tag{3}$$

where a vertical line denotes covariant differentiation and where

$$\lambda_{\beta\alpha} = d_{\beta|\alpha} - b_{\alpha\beta}d_3, \qquad \lambda_{3\alpha} = d_{3,\alpha} + b^\beta_\alpha d_\beta. \tag{4}$$

Greek indices have the values 1, 2.

For an elastic Cosserat surface Green, Naghdi and Wainwright (**1**) obtained constitutive equations which are equivalent to

$$\left.\begin{aligned} S &= -\frac{\partial A}{\partial T}, \\ N'^{\beta\alpha} &= \rho\frac{\partial A}{\partial e_{\alpha\beta}}, \qquad m^{i} = \rho\frac{\partial A}{\partial d_{i}}, \qquad M^{i\alpha} = \rho\frac{\partial A}{\partial \lambda_{i\alpha}}, \end{aligned}\right\} \tag{5}$$

where

$$A = A(T, e_{\alpha\beta}, \lambda_{i\alpha}, d_i, \Lambda_{i\alpha}, D_i), \tag{6}$$

$$2e_{\alpha\beta} = a_{\alpha\beta} - A_{\alpha\beta}, \tag{7}$$

and $A_{\alpha\beta}$, $\Lambda_{i\alpha}$, D_i are the initial value of $a_{\alpha\beta}$, $\lambda_{i\alpha}$, d_i respectively. In evaluating (5), A is regarded as a function of $\frac{1}{2}(e_{\alpha\beta}+e_{\beta\alpha})$.

In section 7 of (**1**) the special case

$$D_\alpha = 0, \qquad d_\alpha = 0, \qquad D_3 = 1, \qquad d_3 = 1, \tag{8}$$

was considered and yielded formulae which are similar to those developed from three-dimensional elasticity theory by other means. In particular,

$$N'^{\beta\alpha} = \rho\frac{\partial A}{\partial e_{\alpha\beta}}, \qquad M^{(\alpha\beta)} = \rho\frac{\partial A}{\partial \kappa_{\alpha\beta}}, \tag{9}$$

where

$$\kappa_{\alpha\beta} = \kappa_{\beta\alpha} = -b_{\alpha\beta}, \tag{10}$$

and

$$A = A(T, e_{\alpha\beta}, \kappa_{\alpha\beta}, B_{\alpha\beta}). \tag{11}$$

The tensor $B_{\alpha\beta}$ is the initial curvature tensor and $M^{(\alpha\beta)}$ is the symmetric part of $M^{\alpha\beta}$. The special case (8), however, really corresponds to a shell which is inextensible along normals to the surface and is, to some extent, artificial. The results can have a wider interpretation if A in (11) is regarded as a general function of its arguments and not as the *value* of A given by (6) when we adopt the special values (8) for d_i. In the rest of this note we consider a special case of our general theory of a Cosserat surface which corresponds to an approximate theory of shells derived from the three-dimensional theory of elasticity, but without the restriction implied by (8).

The first stage in the development of the required special case is to adopt part of the condition (8), namely

$$D_\alpha = 0, \qquad D_3 = 1, \qquad d_\alpha = 0. \tag{12}$$

Then, from (4), we have

$$\left.\begin{aligned} &\lambda_{[\alpha\beta]} = \tfrac{1}{2}(\lambda_{\alpha\beta}-\lambda_{\beta\alpha}) = 0, \qquad \Lambda_{[\alpha\beta]} = 0, \\ &\lambda_{(\alpha\beta)} = \tfrac{1}{2}(\lambda_{\alpha\beta}+\lambda_{\beta\alpha}) = d_3\kappa_{\alpha\beta}, \qquad \Lambda_{(\alpha\beta)} = -B_{\alpha\beta}, \\ &\lambda_{3\alpha} = d_{3,\alpha}, \qquad \Lambda_{3\alpha} = 0, \end{aligned}\right\} \tag{13}$$

where $\kappa_{\alpha\beta}$ is defined in (10). Equations (3) reduce to

$$N^{3\alpha} = d^3(m^\alpha + M^{3\gamma}b^\alpha_\gamma) + M^{\alpha\gamma}d_{3,\gamma}, \tag{14}$$

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\beta\alpha} + d_3 M^{\alpha\gamma}b^\beta_\gamma. \tag{15}$$

Combining equations $(2)_1$ and (14) we obtain

$$M^{*\beta\alpha}|_\alpha + \rho L^{*\beta} = N^{3\beta}, \tag{16}$$

where

$$M^{*\beta\alpha} = d_3 M^{\beta\alpha}, \qquad L^{*\beta} = d_3 L^\beta. \tag{17}$$

Using (12) the Helmholtz function A in (6) becomes a function A^* of the form

$$A^* = A^*(T, e_{\alpha\beta}, \kappa_{\alpha\beta}, \lambda_{3\alpha}, d_3, B_{\alpha\beta}). \tag{18}$$

The reduction to (18) may be a limiting process in which certain elastic coefficients in the form (6) may tend to infinity as $d_\alpha \to 0$. As a result the quantities m^β and $M^{*[\alpha\beta]} = d_3 M^{[\alpha\beta]}$ may either be given by constitutive equations or be indeterminate. An examination of the linear theory of a Cosserat plate (2) suggests that $M^{*[\alpha\beta]}$ will still have a constitutive equation but that m^β is indeterminate and we assume that the same result holds here. In this case $M^{*[\alpha\beta]}$ is the value of $\rho d_3\, \partial A/\partial\lambda_{[\alpha\beta]}$ at the value $\lambda_{[\alpha\beta]} = 0$ and we shall restrict further discussion to the case when this is zero, so that

$$M^{*[\alpha\beta]} = 0. \tag{19}$$

The quantity m^β is now determined by $(2)_1$ and has no constitutive equation. Moreover, from (5), (6), (13) and (18)

$$\rho\frac{\partial A^*}{\partial\kappa_{\alpha\beta}} = \rho\frac{\partial A}{\partial\lambda_{(\alpha\beta)}}d_3 = M^{(\alpha\beta)}d_3 = M^{*(\alpha\beta)}, \tag{20}$$

$$\rho\frac{\partial A^*}{\partial d_3} = \rho\frac{\partial A}{\partial d_3} + \rho\frac{\partial A}{\partial\lambda_{(\alpha\beta)}}\kappa_{\alpha\beta} = m^3 - b_{\alpha\beta}M^{(\alpha\beta)}. \tag{21}$$

Equations $(2)_2$ and (21) may then be combined to yield

$$m^{*3} = \rho\frac{\partial A^*}{\partial d_3}, \qquad m^{*3} = M^{3\alpha}|_\alpha + \rho L^3. \tag{22}$$

Also

$$M^{3\alpha} = \rho\frac{\partial A^*}{\partial\lambda_{3\alpha}}. \tag{23}$$

Next we take a special case of (18) in which A^* does not depend on $\lambda_{3\alpha}$. Thus

$$A^* = A^*(T, e_{\alpha\beta}, \kappa_{\alpha\beta}, d_3, B_{\alpha\beta}) \tag{24}$$

and, from (23) and $(22)_2$,

$$M^{3\alpha} = 0, \qquad m^{*3} = \rho L^3. \tag{25}$$

The basic equations have now been reduced to (1), $(9)_1$, (15), (16), (17), (19), (20), $(22)_1$, (24), (25) with m^β being determined by $(2)_1$.

We assume that equation $(22)_1$, in which A^* is given by (24), may be solved to yield d_3 as a single-valued function of m^{*3}, $e_{\alpha\beta}$, $\kappa_{\alpha\beta}$, T and $B_{\alpha\beta}$. It follows that

$$A^*(T, e_{\alpha\beta}, \kappa_{\alpha\beta}, d_3, B_{\alpha\beta}) = A'(T, e_{\alpha\beta}, \kappa_{\alpha\beta}, m^{*3}, B_{\alpha\beta}). \tag{26}$$

Then

$$\left.\begin{aligned} \rho\frac{\partial A'}{\partial e_{\alpha\beta}} &= \rho\frac{\partial A^*}{\partial e_{\alpha\beta}} + m^{*3}\frac{\partial d_3}{\partial e_{\alpha\beta}}, \\ \rho\frac{\partial A'}{\partial \kappa_{\alpha\beta}} &= \rho\frac{\partial A^*}{\partial \kappa_{\alpha\beta}} + m^{*3}\frac{\partial d_3}{\partial \kappa_{\alpha\beta}}, \\ \rho\frac{\partial A'}{\partial T} &= \rho\frac{\partial A^*}{\partial T} + m^{*3}\frac{\partial d_3}{\partial T}, \\ \rho\frac{\partial A'}{\partial m^{*3}} &= m^{*3}\frac{\partial d_3}{\partial m^{*3}}. \end{aligned}\right\} \tag{27}$$

Now suppose that the director body force normal to the shell is zero and that the director inertia in this direction can be neglected. Then $\mathit{L}^3 = 0$ and, from (25),

$$m^{*3} = 0. \tag{28}$$

If the coefficients of m^{*3} in (27) are finite when $m^{*3} = 0$ then, recalling $(5)_1$, $(9)_1$ and (20), we have

$$N'^{\beta\alpha} = \rho\frac{\partial A'}{\partial e_{\alpha\beta}}, \qquad M^{*(\alpha\beta)} = \rho\frac{\partial A'}{\partial \kappa_{\alpha\beta}}, \qquad S = -\frac{\partial A'}{\partial T}, \tag{29}$$

and A' in (26) reduces to

$$A' = A'(T, e_{\alpha\beta}, \kappa_{\alpha\beta}, B_{\alpha\beta}). \tag{30}$$

Also, from (15), (17) and (19), we have

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\beta\alpha} + M^{*(\alpha\gamma)}b^\beta_\gamma. \tag{31}$$

The basic equations are now (1), (16), (19), (28), (29), (30), (31), and $(2)_1$ merely determines m^β in the form

$$m^\beta = M^{\beta\alpha}|_\alpha + \rho\mathit{L}^\beta. \tag{32}$$

Also, equation $(22)_1$, with $m^{*3} = 0$, determines d_3.

Membrane theory follows by setting $\mathit{L}^{*\beta} = 0$, and assuming that A' is independent of $\kappa_{\alpha\beta}$ and $B_{\alpha\beta}$.

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REFERENCES

1. A. E. GREEN, P. M. NAGHDI and W. L. WAINWRIGHT, *Arch. ration. Mech. Analysis* **20** (1965) 287–308.
2. A. E. GREEN and P. M. NAGHDI, *Proc. Camb. phil. Soc. math. phys. Sci.* **63** (1967) 537–550.