Vector space

A vector space V over F is a set on which addition & scalar multiplication are defined and following conditions hold.

Let $x, y, z \in V$ and $a, b \in F$

VS 1 : $\exists!\ x+y \in V$ (closed under addition)

VS 2 : $\exists!\ ax \in V$ (Closed under scalar multiplication)

VS 3 : $x+y = y+x$ (Commutativity of addition)

VS 4 : $(x+y)+z = x+(y+z)$ (associativity of addition)

VS 5 : $\exists!\ \overline{0_V} \in V$ s.t. $x+\overline{0_V} = x$

VS 6 : $\exists!\ (-x) \in V$ s.t. $x+(-x) = \overline{0_V}$

VS 7 : $1 \cdot x = x$

VS 8 : $(ab)x = a(bx)$

VS 9 : $a(x+y) = ax+ay$

VS 10 : $(a+b)x = ax+bx$

Subspace

Let V be a vector space.

Let W be a subset of V.

if W is a vector space with addition and scalar multiplication defined on V

Then W is a subspace of V.

Condition of subspace

any subset of Vector space satisfy VS 3,4,7,8,9,10 which are properties of elements.

To be a vector space VS 1,2,5,6 should be satisfied which are properties about set.

Span

Let V be a vectorspace

Let W be a non-empty subset of V

Then span(W) is a set consisting of all linear combination of vectors in W.

For convinience, we define $\text{span}(\phi) = \{\overline{0_V}\}$

generating set

Let V be a vector space.

Let W be a subset of V

if $\text{span}(W) = V$

Then W is generating set of V

linearly dependent

Let V be a vector space.

Let W be a subset of V.

if all vectors in W satisfies $a_1w_1 + \cdots + a_nw_n = 0$

with not all zero constants a_i

Then W is linearly dependent.

linearly Independent

Let V be a vector space.

Let W be a subset of V

If W is not linearly dependent.

Then W is linearly independent.

linear transformation

Let V & W be vector spaces.

Let T be a function $T: V \to W$

if for $\forall x, y \in V$ & $\forall c \in F$ satisfy $T(cx+y) = cT(x) + T(y)$

Then T is an linear transformation from V to W.

Identity transformation and zero transformation

Let V & W be vector spaces.

Then $I_V : V \to V$ by $I_V(v) = v$ for $\forall v \in V$

Then $T_0 : V \to W$ by $T_0(v) = \overline{0_W}$ for $\forall v \in V$

basis

Let V be a vector space.

Let W be a subset of V.

if W is linearly independent generating set of V.

Then W is basis of V.

null space and range

Let V & W be vector spaces.

Let $T \in \mathcal{L}(V, W)$

Then we define a set null space $N(T) = \{v \in V \mid T(v) = 0\}$

and we define a set range $R(T) = \{T(v) \mid v \in V\} = T(V)$

finite dimensional

Let V be a vector space.

if basis of V has finite number of vectors

Then V is finite dimensional

dimension

Let V be a finite dimensional vector space.

Then $\dim(V)$ is the number of vectors in basis

nullity & rank

Let V & W be finite dimensional vector spaces

Let $T \in \mathcal{L}(V, W)$

Then we define $\text{nullity}(T) = \dim(N(T))$

and we define $\text{rank}(T) = \dim(R(T))$

Left multiplication transformation

Let $A \in M_{mn}(F)$

Then we define a function $L_A : F^n \to F^m$ s.t. $L_A(x) = Ax$ where $x \in F^n$

Coordinate vector

Let V be a finite dimensional vector space.

Let β^V be a ordered basis for V.

For $v \in V$, $\exists!\, a_i$ s.t. $v = \sum a_i \beta_i^V$

Then coordinate vector of v relative to β^V is $[v]_{\beta^V} = a_i$

matrix representation of Linear transformation

Let V & W be vector spaces.

Let β^V & β^W be ordered bases.

Let $T \in \mathcal{L}(V, W)$

Then $\exists!\, A_{ji}$ s.t. $T(\beta_i^V) = \sum_{j=1}^{m} A_{ji} \beta_j^W$ for $i = 1, \cdots, n$

we define matrix representation of T as $A = [T]_{\beta^V}^{\beta^W}$

If $V = W$ & $\beta^V = \beta^W$ $A = [T]_{\beta^V}$

Linear operator for function

Let V & W be vector spaces.

Let T & $U : V \to W$

Then we define $T+U : V \to W$ by $(T+U)(v) = T(v)+U(v)$ for all $v \in V$

and we define $aT : V \to W$ by $(aT)(v) = aT(v)$ for all $v \in V$

Vector space of Linear transformation

Let V & W be vector spaces over F.

Then $\mathcal{L}(V,W)$ is a vector space of all Linear transformation from V into W

Composition of linear transformation & matrix multiplication

Let V, W, Z be vector spaces.

Let $\beta^V, \beta^W, \beta^Z$ be ordered bases.

Let $T \in \mathcal{L}(V,W)$ & $U \in \mathcal{L}(W,Z)$

Let $A = [U]_{\beta^W}^{\beta^Z}$ and $B = [T]_{\beta^V}^{\beta^W}$

Then $UT(\beta_i^V) = U(T(\beta_i^V)) = U\left(\sum_{j=1}^{m} B_{ji}\beta_j^W\right) = \sum_{j=1}^{m} B_{ji}\, U(\beta_j^W) = \sum_{j=1}^{m} B_{ji} \sum_{k=1}^{p} A_{kj}\beta_k^Z$

$= \sum_{k=1}^{p}\left(\sum_{j=1}^{m} A_{kj}B_{ji}\right)\beta_k^Z = \sum_{k=1}^{p} C_{ki}\beta_k^Z$

if we define matrix product AB as $(AB)_{ki} = \sum_{j=1}^{m} A_{kj}B_{ji}$,

$[UT]_{\beta^V}^{\beta^Z} = [U]_{\beta^W}^{\beta^Z}[T]_{\beta^V}^{\beta^W}$

inverse and invertible of Linear transformation

Let V & W be vector spaces.

Let $T \in \mathcal{L}(V,W)$

If $\exists U : W \to V$ s.t. $TU = I_W$ and $UT = I_V$

Then U is an T^{-1} and T is invertible.

Additionally, Let U & T be invertible functions.

Then $(TU)^{-1} = U^{-1}T^{-1}$ and $(T^{-1})^{-1} = T$, in particular T^{-1} is invertible.

a function is invertible if and only if it is both one to one and onto

Inverse and Invertible of matrix

Let $A \in M_{nn}$

if $\exists B \in M_{nn}$ s.t. $AB = BA = I$

Then A is invertible and $A^{-1} = B$ is inverse of A

one to one

a function that each element of the range has a unique preimage

if $f(x) = f(y) \Rightarrow x = y$

equivalently, $x \neq y \Rightarrow f(x) \neq f(y)$

Onto

a function that the range equals the codomain

Isomorphic & Isomorphism

Let V & W be vector spaces.

if $\exists T \in \mathcal{L}(V,W)$ that is invertible.

then V and W are isomorphic and T is isomorphism from V onto W

Standard representation

Let V be a n-dimensional vector space.

Let β^V be an ordered basis.

Then define a function $\phi_{\beta^V} : V \to F^n$ by $\phi_{\beta^V}(v) = [v]_{\beta^V}$

and we call standard representation of V with respect to β^V

Change of coordinate matrix

Let V be a vector space

Let β^{V_1}, β^{V_2} be two ordered bases.

Then define $Q = [I_V]_{\beta^{V_1}}^{\beta^{V_2}}$ and it changes β^{V_1} coordinate $\to \beta^{V_2}$ coordinate

Linear operator

Let $T \in \mathcal{L}(V)$

Then T is linear operator on V.

Similar

Let $A, B \in M_{n n}(F)$

if $\exists Q$ s.t. $B = Q^{-1} A Q$

Then A and B are similar.

Rank of Matrix

Let $A \in M_{n n}(F)$

Then rank(A) to be a rank(L_A)

Determinant of order 2

Let $A \in M_{2\times 2}(F)$

Then $\det(A) = |A| = A_{11}A_{22} - A_{12}A_{21}$

Minor and cofactor of matrix

Let $A \in M_{nn}(F)$

Then Minor $\tilde{A}_{ij}$ is defined from A by deleting row i and column j.

and Cofactor C_{ij} is defined, $C_{ij} = (-1)^{i+j} \det(\tilde{A}_{ij})$

Determinant of order n

Let $A \in M_{nn}(F)$

Then we define determinant of A

$n=1$ $\det(A) = A_{11}$

$n \geq 2$ $\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\tilde{A}_{1j}) = \sum_{j=1}^{n} C_{1j} A_{1j}$