A General Theory of a Cosserat Surface

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Abstract

This paper is concerned with a general dynamical theory of a Cosserat surface, *i.e.*, a deformable surface embedded in a Euclidean 3-space to every point of which a deformable vector is assigned. These deformable vectors, called directors, are not necessarily along the normals to the surface and possess the property that they remain invariant in length under rigid body motions. An elastic Cosserat surface and other special cases of the theory which bear directly on the classical theory of elastic shells are also discussed.

Contents

1. Introduction

Consider a surface embedded in a Euclidean 3-space to every point of which a vector — not necessarily along the normal to the surface — called a director, is assigned. Such a surface with deformable directors will be called a *Cosserat surface*, and the present paper is concerned with a general dynamical theory of such a surface in continuum mechanics.

The idea of using directors in continuum mechanics evidently goes back to DUHEM (1893), who regarded a body as a collection of points together with directions associated with the points. Theories based on such a model of an oriented medium were further developed by E. & F. COSSERAT (1909). A modern account of the kinematics of oriented bodies characterized by ordinary displacement and the independent deformation of n directors in n-dimensional space, has been given by ERICKSEN & TRUESDELL (1958). An exposition of the kinematics of oriented bodies, together with references to other contributions on the subject up to 1960, is given in the monograph by TRUESDELL & TOUPIN (1960). Related and more recent contributions to the subject include the use of directors by ERICKSEN (1961) in his theory of liquid crystals, the development of a general theory of multipolar continuum mechanics by GREEN & RIVLIN (1964b), a discussion of theories of elasticity with couple-stress by TOUPIN (1964), and a complete dynamical theory of directors by GREEN, NAGHDI & RIVLIN (1965), where the relation of directors to multipolar displacements is also discussed. Additional references may be found in the papers already cited.

As noted previously by ERICKSEN & TRUESDELL (1958), the COSSERATS (1908) had recognized the significance of the idea of oriented bodies in one and two dimensions for the construction of theories of rods and shells and included, in particular, a development of a theory of shells in their book (1909) which, however, is incomplete. A general theory of strain for rods and shells, suggested by the work of the COSSERATS, was developed more recently by ERICKSEN & TRUESDELL in their (1958) paper, where a derivation of the differential equations of equilibrium for shells is also supplied; ERICKSEN & TRUESDELL (1958) do not, however, consider the problem of constitutive equations. A systematic development of a general and complete theory of elastic shells by direct methods (in contrast to the use of the three-dimensional equations of elasticity theory) is not available. In a recent paper SERBIN (1963), however, considers an exact and complete (isothermal) linear theory of elastic shells by a direct method; but in early stages of his analysis he assumes a strain energy function for the linear theory which is too restrictive.

The present paper is concerned with a general theory of a COSSERAT surface which is exact, complete, and fully consistent with dynamical and thermodynamical principles of continuum mechanics. After some geometrical preliminaries, we derive in Section 2 the necessary kinematical results due to the classical (monopolar) surface displacement and the director displacement of a Cosserat surface and also consider, for future use, the nature of these kinematical results under superposed rigid body motions. Next, following GREEN & RIVLIN (1964a, b) and using a principle of balance of energy and an entropy production inequality, valid for a surface embedded in a Euclidean 3-space, we develop in Section 3 a general theory of a Cosserat surface whose deformation is characterized by both the transformation of its coordinates as well as that of its directors [see Eq. (2.6)]. The derivation in Section 3, carried out in a neat vectorial form, leads to an appropriate (local) equation for conservation of mass and various (local) dynamical and thermodynamical equations which hold for any Cosserat surface, as defined here. These equations which (in vector form) display a remarkably simple structure, conceal the relative complexity of the results and hence alternative forms of the basic equations in terms of tensor components are collected in Section 4.

The remaining parts of the paper deal with an elastic Cosserat surface and specialized cases of the general theory. Nonlinear constitutive equations for an elastic Cosserat surface in terms of both the Helmholtz free energy and the internal energy functions are constructed in Section 5. By considering the effect of material symmetries, explicit results are also deduced in Section 5 for an initially isotropic Cosserat surface in terms of joint invariants of temperature and 12 kinematical quantities. Section 6 is concerned with the infinitesimal theory of an elastic Cosserat surface, where the basic field equations and the constitutive equations are systematically and consistently specialized to correspond with a suitable linearization of the kinematical results. Finally, special cases of the general theory involving further simplifying assumptions, motivated by classical treatments of the theories of shells, are discussed in Section 7. These include a special case of the theory of Sections 3 and 5, as well as that of the linearized theory of Section 6, in which the directors are identified with the unit normals to the surface.

Although the use of more general kinematical ingredients would be possible, the general theory of this paper for a Cosserat surface (with deformable directors) is exact and complete from both dynamical and thermodynamical points of view. Almost all of the field equations of Section 3 are either entirely novel or more general than the corresponding previously known results. Even the differential equations of equilibrium in Section 3 (or Section 4) are more general than those derived previously by direct methods. Also, the specialized cases of both the non-linear and the linear theory in which the director is identified with the normal to the surface (Section 7) are rather illuminating and bear directly on the foundations of the classical theory of shells developed under the so-called Kirchhoff-Love hypothesis.

2. Preliminaries. Kinematical results

Let z^i $(i=1,2,3)$ refer to a fixed right-handed Cartesian coordinate system and let x^i denote an arbitrary (real) curvilinear coordinate system defined by the transformation

$$z^i = z^i(x^1, x^2, x^3, t), \qquad \det\left[\frac{\partial z^i}{\partial x^j}\right] > 0, \tag{2.1}$$

and its inverse. Let a surface $\mathscr{s}$, embedded in a Euclidean 3-space, be defined by the equation $x^3=0$ and let x^i be identified as convected normal coordinates with x^α $(\alpha=1,2)$ on $\mathscr{s}$ and x^3 along the normal to $\mathscr{s}$. Also, let the initial values of z^i at time $t=0$ be designated by Z^i and refer to the (initial) undeformed surface by $\mathscr{S}$ and to the deformed surface at time t by $\mathscr{s}$. We can show that the metric tensor g_{ij} of the coordinate system x^i, when evaluated on $x^3=0$, is given by

$$g_{\alpha\beta} = a_{\alpha\beta}, \qquad g_{3\alpha} = 0, \qquad g_{33} = 1, \tag{2.2}$$

where $a_{\alpha\beta}$ is the first fundamental form of $\mathscr{s}$. We can also show that $\det[\partial z^i/\partial x^j] \neq 0$ on $\mathscr{s}$ so that the conjugate tensor g^{ij}, with components $g^{\alpha\beta}=a^{\alpha\beta}$, $g^{3\alpha}=0$, $g^{33}=1$, exists.

The position vector of a point on $\mathscr{S}$ will be denoted by $\boldsymbol{R}$ and its dual on $\mathscr{s}$ by $\boldsymbol{r}$. Thus, if $\boldsymbol{a}_\alpha$ are the base vectors along the x^α-curves on $\mathscr{s}$, and $\boldsymbol{a}_3$ is the unit normal to $\mathscr{s}$, we have

$$\boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha}, \qquad \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = a_{\alpha\beta}, \qquad \boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta^\alpha_\beta, \tag{2.3}$$

$$\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_3 = 0, \qquad \boldsymbol{a}_3 \cdot \boldsymbol{a}_3 = 1, \qquad \boldsymbol{a}_{3,\alpha} \cdot \boldsymbol{a}_3 = 0, \tag{2.4}$$

where δ^α_β is the Kronecker symbol in 2-space and a comma denotes partial differentiation with respect to x^α. We also recall the well-known formulae from differential geometry, namely

$$\begin{aligned} \boldsymbol{a}_{\alpha|\beta} &= b_{\alpha\beta}\,\boldsymbol{a}_3, \\ \boldsymbol{a}_{3,\beta} &= -b^\alpha_\beta\,\boldsymbol{a}_\alpha, \\ b_{\alpha\beta|\gamma} &= b_{\alpha\gamma|\beta}, \end{aligned} \tag{2.5}$$

where $b_{\alpha\beta}$ is the second fundamental form of the surface $\mathscr{s}$ and a single stroke designates covariant differentiation with respect to $a_{\alpha\beta}$. Results similar to those in (2.3) to (2.5) hold also for the surface $\mathscr{S}$ whose base vectors and the first and second fundamental forms will be denoted by $\boldsymbol{A}_i$, $A_{\alpha\beta}$ and $B_{\alpha\beta}$, respectively.

Throughout the paper, Latin indices will have the range 1, 2, 3, whereas Greek indices with the range 1, 2 are used for surface tensors or the components of space tensors.

Let a vector $\boldsymbol{D}$, not necessarily along the normal to the initial surface, be assigned to every point of $\mathscr{S}$. The vector $\boldsymbol{D}$ will be called a director and its dual at time t on $\mathscr{s}$ will be denoted by $\boldsymbol{d}$. In what follows, we regard the motion of a Cosserat surface to be characterized by

$$z^i = z^i(x^\alpha, t), \qquad \boldsymbol{d} = \boldsymbol{d}(x^\alpha, t), \tag{2.6}$$

where $\boldsymbol{d}$ has the property that its components, referred to the base vectors $\boldsymbol{a}^i$, remain invariant when the motion is altered only by superposed rigid body motions. Equivalently, given a surface $\mathscr{S}$ with fundamental forms $A_{\alpha\beta}$, $B_{\alpha\beta}$ and director $\boldsymbol{D}$ at Z^i, we wish to determine a second surface $\mathscr{s}$ and the director $\boldsymbol{d}$ at z^i, according to (2.6), uniquely to within a rigid motion.

Let $\boldsymbol{v}$, a three-dimensional vector field, denote the velocity of $\mathscr{s}$ at time t which, when referred to the base vectors $\boldsymbol{a}_i = \{\boldsymbol{a}_\alpha, \boldsymbol{a}_3\}$ of $\mathscr{s}$, is

$$\boldsymbol{v} = v^i \boldsymbol{a}_i = v^\alpha \boldsymbol{a}_\alpha + v^3 \boldsymbol{a}_3 = v_i \boldsymbol{a}^i. \tag{2.7}$$

Since the coordinate curves on $\mathscr{s}$ are convected, it follows that*

$$\dot{\boldsymbol{a}}_\alpha = \boldsymbol{v}_{,\alpha}, \qquad \boldsymbol{v}_{,\alpha} = (v_{\beta|\alpha} - b_{\beta\alpha} v_3)\, \boldsymbol{a}^\beta + (v_{3,\alpha} + b^\beta_\alpha v_\beta)\, \boldsymbol{a}_3, \tag{2.8}$$

and from $(2.4)_1$ and $(2.4)_2$ we can show that

$$\dot{\boldsymbol{a}}_3 = -(v_{3,\alpha} + b^\beta_\alpha v_\beta)\, \boldsymbol{a}^\alpha, \tag{2.9}$$

where a superposed dot denotes the material derivative with respect to t, holding x^α fixed. The lowering and raising of superscripts and subscripts of space tensor functions such as v^i in (2.7) and c_{ki} in (2.10) can be accomplished by using the metric tensor g_{ij} defined by (2.2).

Since, for each i, $\dot{\boldsymbol{a}}_i$ may be expressed as a linear combination of $\boldsymbol{a}^i$, we may write

$$\dot{\boldsymbol{a}}_i = c_{ki}\, \boldsymbol{a}^k, \qquad c_{ki} = \boldsymbol{a}_k \cdot \dot{\boldsymbol{a}}_i. \tag{2.10}$$

Let η_{ki} and ψ_{ki} stand, respectively, for the symmetric and antisymmetric parts of c_{ki}. Then,

$$c_{ki} = \eta_{ki} + \psi_{ki}, \tag{2.11}$$

where

$$2\eta_{ki} = \overline{\boldsymbol{a}_k \cdot \boldsymbol{a}_i}^{\,\cdot} = \boldsymbol{a}_k \cdot \dot{\boldsymbol{a}}_i + \boldsymbol{a}_i \cdot \dot{\boldsymbol{a}}_k = \dot{a}_{ki} = 2\eta_{ik}, \tag{2.12}$$

$$2\psi_{ki} = \boldsymbol{a}_k \cdot \dot{\boldsymbol{a}}_i - \boldsymbol{a}_i \cdot \dot{\boldsymbol{a}}_k = -2\psi_{ik}. \tag{2.13}$$

From (2.12) and (2.13) together with $(2.8)_1$, $(2.4)_1$ and $(2.4)_2$, we conclude that

$$\begin{aligned} 2\eta_{\alpha\beta} &= \boldsymbol{a}_\alpha \cdot \boldsymbol{v}_{,\beta} + \boldsymbol{a}_\beta \cdot \boldsymbol{v}_{,\alpha} \\ &= v_{\alpha|\beta} + v_{\beta|\alpha} - 2 b_{\alpha\beta} v_3, \\ \eta_{3\alpha} &= \eta_{\alpha 3} = 0, \qquad \eta_{33} = 0, \end{aligned} \tag{2.14}$$

* See, *e.g.*, GREEN & ZERNA (1954).

and

$$
\begin{gathered}
2\psi_{\alpha\beta} = -2\psi_{\beta\alpha} = v_{\alpha|\beta} - v_{\beta|\alpha}, \\
\psi_{\alpha 3} = -\psi_{3\alpha} = -\boldsymbol{a}_3 \cdot \boldsymbol{v}_{,\alpha} = -(v_{3,\alpha} + b_\alpha^\beta v_\beta), \\
\psi_{33} = 0,
\end{gathered} \tag{2.15}
$$

where $\eta_{\alpha\beta}$ and $\psi_{\alpha\beta}$ may be referred to as the *surface deformation rate* tensor and the *surface spin* tensor, respectively. We also note that in view of (2.11), $(2.10)_1$ may also be expressed in the form

$$
\dot{\boldsymbol{a}}_i = (\eta_{ki} + \psi_{ki})\, \boldsymbol{a}^k. \tag{2.16}
$$

To obtain suitable expressions for the rate of change of the reciprocal base vectors, we recall that the reciprocal base vectors are given by

$$
\boldsymbol{a}^\alpha = a^{\alpha\lambda} \boldsymbol{a}_\lambda, \qquad \boldsymbol{a}^3 = \boldsymbol{a}_3. \tag{2.17}
$$

Hence, with the use of (2.12) to (2.17), and the result

$$
\dot{a}^{\alpha\beta} = -a^{\alpha\lambda} a^{\beta\nu} \dot{a}_{\lambda\nu},
$$

we obtain

$$
\begin{aligned}
\dot{\boldsymbol{a}}^\alpha &= \overline{a^{\alpha\lambda} \boldsymbol{a}_\lambda}^{\,\boldsymbol{\cdot}} \\
&= a^{\alpha\lambda} (\eta_{k\lambda} + \psi_{k\lambda})\, \boldsymbol{a}^k - 2 a^{\alpha\lambda} a^{\beta\nu} \eta_{\lambda\nu} \boldsymbol{a}_\beta \\
&= a^{\alpha\lambda} (\psi_{k\lambda} - \eta_{k\lambda})\, \boldsymbol{a}^k,
\end{aligned} \tag{2.18}
$$

and

$$
\dot{\boldsymbol{a}}^3 = \dot{\boldsymbol{a}}_3 = \psi_{k3}\, \boldsymbol{a}^k. \tag{2.19}
$$

We now introduce further kinematical results in terms of the director displacement $\boldsymbol{d}$ and its derivatives. The vector $\boldsymbol{d}$ referred to the base vectors $\boldsymbol{a}^i$ is

$$
\boldsymbol{d} = d_i \boldsymbol{a}^i = d_\alpha \boldsymbol{a}^\alpha + d_3 \boldsymbol{a}^3 \tag{2.20}
$$

and the director velocity $\dot{\boldsymbol{d}}$, with the aid of (2.18) and (2.19), can be put in the form

$$
\boldsymbol{w} = \dot{\boldsymbol{d}} = \boldsymbol{\Gamma} + d_i \dot{\boldsymbol{a}}^i = \boldsymbol{\Gamma} + d^i \psi_{ki} \boldsymbol{a}^k + d^\alpha \boldsymbol{\eta}_\alpha, \tag{2.21}
$$

where

$$
\boldsymbol{\eta}_\alpha = -\eta_{k\alpha} \boldsymbol{a}^k, \qquad \boldsymbol{\Gamma} = \dot{d}_i \boldsymbol{a}^i. \tag{2.22}
$$

Also,

$$
\begin{gathered}
\boldsymbol{d}_{,\alpha} = \lambda_{i\alpha} \boldsymbol{a}^i \\
\boldsymbol{w}_{,\alpha} = \dot{\boldsymbol{d}}_{,\alpha} = \boldsymbol{\Gamma}_{:\alpha} + \lambda^i_{.\alpha} \psi_{ki} \boldsymbol{a}^k + \lambda^\beta_{.\alpha} \boldsymbol{\eta}_\beta,
\end{gathered} \tag{2.23}
$$

where

$$
\begin{gathered}
\lambda_{\beta\alpha} = \boldsymbol{a}_\beta \cdot \boldsymbol{d}_{,\alpha} = d_{\beta|\alpha} - b_{\beta\alpha} d_3, \qquad \lambda^\beta_{.\alpha} = a^{\beta\nu} \lambda_{\nu\alpha}, \\
\lambda_{3\alpha} = \boldsymbol{a}_3 \cdot \boldsymbol{d}_{,\alpha} = d_{3,\alpha} + b_\alpha^\beta d_\beta, \qquad \lambda^3_{.\alpha} = \lambda_{3\alpha}, \\
\lambda_{i\alpha} = \boldsymbol{a}_i \cdot \boldsymbol{d}_{,\alpha}, \qquad \dot{\lambda}_{i\alpha} = \boldsymbol{a}_i \cdot \boldsymbol{\Gamma}_{:\alpha}.
\end{gathered} \tag{2.24}
$$

For future reference, we also calculate the rate of change of an element of area $d\sigma$ of the surface $\mathscr{s}$. Since

$$d\sigma=(\boldsymbol{a}_1\times\boldsymbol{a}_2\cdot\boldsymbol{a}_3)\,dx^1dx^2=a^{\frac{1}{2}}\,dx^1dx^2$$

and $a=|a_{\alpha\beta}|$, by (2.12), we have

$$\begin{aligned}\dot{\overline{d\sigma}}&=\tfrac{1}{2}a^{-\frac{1}{2}}\,\dot{a}\,dx^1dx^2\\&=\eta^\alpha_\alpha\,d\sigma=(v^\alpha{}_{|\alpha}-b^\alpha_\alpha v_3)\,d\sigma.\end{aligned}\tag{2.25}$$

In the remainder of this section, we introduce further kinematical results which will be utilized in Section 3. In particular, we consider a second motion of $\mathscr{s}$ which differs from the previous motion only by a superposed rigid motion when the medium has the same orientation as in the first motion. Let the velocity vector at time t, corresponding to the second motion, be denoted by $\boldsymbol{v}^*$. Then

$$\begin{aligned}\boldsymbol{v}^*&=\boldsymbol{v}+[\boldsymbol{v}_0+\boldsymbol{\omega}\times(\boldsymbol{r}-\boldsymbol{r}_0)]\\&=\boldsymbol{v}+[\boldsymbol{b}+\boldsymbol{\omega}\times\boldsymbol{r}],\end{aligned}\tag{2.26}$$

where the square bracket represents an arbitrary velocity due to rigid motion, and $\boldsymbol{b}=(\boldsymbol{v}_0-\boldsymbol{\omega}\times\boldsymbol{r}_0)$, $\boldsymbol{v}_0$ and $\boldsymbol{\omega}$ are vector functions which depend on t only.

If in (2.12) and (2.13) we replace $\dot{\boldsymbol{a}}_i$ by $[\dot{\boldsymbol{a}}_i+\boldsymbol{\omega}\times\boldsymbol{a}_i]$, then it can be easily verified that

$$\eta^*_{ki}=\eta_{ki},\qquad \psi^*_{ki}=\psi_{ki}-\Omega_{ki},\tag{2.27}$$

where

$$\Omega_{ki}=\varepsilon_{kim}\,\omega^m=a^{\frac{1}{2}}\,e_{kim}\,\omega^m,\tag{2.28}$$

and the ε-system is related to the permutation symbols e_{kim}, e^{kim}, through

$$\varepsilon_{kim}=a^{\frac{1}{2}}\,e_{kim},\qquad \varepsilon^{kim}=a^{-\frac{1}{2}}\,e^{kim}.\tag{2.29}$$

Also, by (2.26), $(2.3)_1$ and (2.28), we have

$$\begin{aligned}\boldsymbol{v}^*_{,\alpha}&=\boldsymbol{v}_{,\alpha}+\boldsymbol{\omega}\times\boldsymbol{a}_\alpha\\&=\boldsymbol{v}_{,\alpha}-\Omega_{k\alpha}\,\boldsymbol{a}^k.\end{aligned}\tag{2.30}$$

When the motion is specified by (2.26), then the kinematical results corresponding to (2.20) to (2.24) become

$$d^*_i=d_i,\qquad \lambda^*_{i\alpha}=\lambda_{i\alpha},\tag{2.31}$$

$$\boldsymbol{\Gamma}^*=\boldsymbol{\Gamma},\qquad \boldsymbol{\Gamma}^*_{:\alpha}=\boldsymbol{\Gamma}_{:\alpha},\tag{2.32}$$

$$\boldsymbol{w}^*=\boldsymbol{w}+\boldsymbol{\omega}\times\boldsymbol{d}=\boldsymbol{w}+d^i\,\Omega_{ik}\,\boldsymbol{a}^k,\tag{2.33}$$

$$\boldsymbol{w}^*_{,\alpha}=\boldsymbol{w}_{,\alpha}+\boldsymbol{\omega}\times\boldsymbol{d}_{,\alpha}=\boldsymbol{w}_{,\alpha}+\lambda^i{}_{.\alpha}\,\Omega_{ik}\,\boldsymbol{a}^k.\tag{2.34}$$

3. Theory of a Cosserat surface

Let σ, the area of $\mathscr{s}$ at time t, be bounded by a closed curve c and let $\boldsymbol{\nu}=\nu_\alpha\,\boldsymbol{a}^\alpha$ be the outward unit normal to c lying in the surface. If $\boldsymbol{N}$ is a three-dimensional vector field and if, for all arbitrary velocity fields $\boldsymbol{v}$, the scalar $\boldsymbol{N}\cdot\boldsymbol{v}$ represents a rate of work per unit length of c, then N is a curve force vector measured per unit length*. Similarly, if $\boldsymbol{M}$ is a three-dimensional vector field and if, for all arbitrary director velocity fields $\boldsymbol{w}$, the scalar $\boldsymbol{M}\cdot\boldsymbol{w}$ is a rate of work per unit length, then $\boldsymbol{M}$ will be called the director force vector, per unit length of c. Also, let $\boldsymbol{F}$ and $\boldsymbol{L}$ stand, respectively, for the three-dimensional fields of *assigned force* and *director force*, per unit mass of $\mathscr{s}$, at time t, such that $\boldsymbol{F}\cdot\boldsymbol{v}$ and $\boldsymbol{L}\cdot\boldsymbol{w}$, for all arbitrary $\boldsymbol{v}$ and $\boldsymbol{w}$, respectively, represent rate of work per unit area of $\mathscr{s}$. When referred to the base vectors $\boldsymbol{a}_i$, these vector fields can be expressed as

$$N = N^i \boldsymbol{a}_i = N^\alpha \boldsymbol{a}_\alpha + N^3 \boldsymbol{a}_3, \tag{3.1}$$

$$\boldsymbol{M} = M^i \boldsymbol{a}_i = M^\alpha \boldsymbol{a}_\alpha + M^3 \boldsymbol{a}_3, \tag{3.2}$$

and

$$\boldsymbol{F} = F^i \boldsymbol{a}_i, \qquad \boldsymbol{L} = L^i \boldsymbol{a}_i. \tag{3.3}$$

If ρ is the mass density at time t per unit area of $\mathscr{s}$ and U is the internal energy per unit mass, then the equation of balance of energy may be written as**

$$\frac{D}{Dt}\int_\sigma [\tfrac{1}{2}\rho v^2 + \rho U]\, d\sigma = \int_\sigma \rho[r + \boldsymbol{F}\cdot\boldsymbol{v} + \bar{\boldsymbol{L}}\cdot\boldsymbol{w}]\, d\sigma + \int_c [\boldsymbol{N}\cdot\boldsymbol{v} + \boldsymbol{M}\cdot\boldsymbol{w}]\, dc - \int_c h\, dc; \tag{3.4}$$

where v denotes the magnitude of $\boldsymbol{v}$, $\bar{\boldsymbol{L}}$ is the difference of the assigned director force per unit mass $\boldsymbol{L}$ and the inertia terms due to the director displacement $\boldsymbol{d}$, r is the heat supply function per unit mass per unit time, h is the flux of heat across c per unit length per unit time, and D/Dt stands for material derivative.

After carrying out the indicated differentiation in (3.4) and recalling (2.25), we obtain

$$\begin{aligned}\int_\sigma [\rho\boldsymbol{v}\cdot\dot{\boldsymbol{v}} + \rho\dot{U}]\, d\sigma + \int_\sigma \{\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + U\}[\dot{\rho} + \rho(v^\alpha{}_{|\alpha} - b^\alpha_\alpha v_3)]\, d\sigma \\ = \int_\sigma \rho[r + \boldsymbol{F}\cdot\boldsymbol{v} + \bar{\boldsymbol{L}}\cdot\boldsymbol{w}]\, d\sigma + \int_c [\boldsymbol{N}\cdot\boldsymbol{v} + \boldsymbol{M}\cdot\boldsymbol{w} - h]\, dc,\end{aligned} \tag{3.5}$$

where a superposed dot stands for D/Dt. Now assume that $\rho, U, r, h, \boldsymbol{F} - \dot{\boldsymbol{v}}, \bar{\boldsymbol{L}}, \boldsymbol{N}, \boldsymbol{M}$ and $\boldsymbol{w}$ remain unchanged under superposed uniform rigid body translational velocities***. Thus, in the notation of (2.26), if we replace $\boldsymbol{v}$ by $(\boldsymbol{v} + \boldsymbol{b})$ in (3.5), then by subtraction we deduce

$$\begin{aligned}\boldsymbol{b}\cdot\{\int_\sigma (\rho\dot{\boldsymbol{v}} - \rho\boldsymbol{F})\, d\sigma - \int_c \boldsymbol{N}\, dc + \int_\sigma \boldsymbol{v}[\dot{\rho} + \rho(v^\alpha{}_{|\alpha} - b^\alpha_\alpha v_3)]\, d\sigma\} + \\ + \tfrac{1}{2}(\boldsymbol{b}\cdot\boldsymbol{b})\int_\sigma [\dot{\rho} + \rho(v^\alpha{}_{|\alpha} - b^\alpha_\alpha v_3)]\, d\sigma = 0\end{aligned} \tag{3.6}$$

which holds for all arbitrary $\boldsymbol{b}$. By replacing $\boldsymbol{b}$ by $\beta\,\boldsymbol{b}$, β being a scalar, it then follows from (3.6) that

$$\int_\sigma [\dot{\rho} + \rho(v^\alpha{}_{|\alpha} - b^\alpha_\alpha v_3)]\, d\sigma = 0 \tag{3.7}$$

---

* N is called a curve force vector, in analogy to a corresponding surface force vector in a three-dimensional continuum.

** In this form of (3.4) the inertia term in $\boldsymbol{I}$ contains also a contribution $-\frac{1}{2}\boldsymbol{w}[\dot{\rho} + \rho(v^\alpha{}_{|\alpha} - b^\alpha_\alpha v_3)]$ which eventually vanishes in view of (3.8).

*** The operator D/Dt is unaltered by superposed rigid body motions.

for all arbitrary areas σ. From (3.7) we deduce that

$$\frac{D\rho}{Dt}+\rho(v^{\alpha}{}_{|\alpha}-b^{\alpha}_{\alpha}v_3)=\dot{\rho}+\rho\,\eta^{\alpha}_{\alpha}=0 \tag{3.8}$$

which is the (local) equation for conservation of mass. Also, using (3.8), we obtain from (3.6) the integral form of the equations of motion, namely

$$\int_{\sigma}(\rho\,\dot{\boldsymbol{v}}-\rho\boldsymbol{F})\,d\sigma-\int_{c}\boldsymbol{N}\,dc=0. \tag{3.9}$$

Over a curve with a unit normal $\boldsymbol{v}=v_{\alpha}\,\boldsymbol{a}^{\alpha}$, the (physical) curve force vector is $\boldsymbol{N}$. If $\boldsymbol{n}^{\alpha}$ are the (physical) force vectors over each coordinate line, then the application of (3.9) to a curvilinear triangle yields

$$\boldsymbol{N}=\sum_{\alpha}v_{\alpha}\,\boldsymbol{n}^{\alpha}(a^{\alpha\alpha})^{\frac{1}{2}}=\boldsymbol{N}^{\alpha}v_{\alpha},\qquad \boldsymbol{N}^{\alpha}=\boldsymbol{n}^{\alpha}(a^{\alpha\alpha})^{\frac{1}{2}}. \tag{3.10}$$

Thus, $\boldsymbol{N}^{\alpha}$ transforms as a contravariant surface vector and we can put*

$$\boldsymbol{N}^{\alpha}=N^{\gamma\alpha}\boldsymbol{a}_{\gamma}+N^{3\alpha}\boldsymbol{a}_3, \tag{3.11}$$

and by (3.1)

$$N^{\gamma}=N^{\gamma\alpha}v_{\alpha},\qquad N^{3}=N^{3\alpha}v_{\alpha}, \tag{3.12}$$

where $N^{\gamma\alpha}$ and $N^{3\alpha}$ are surface tensors under transformation of surface coordinates. Substituting (3.10) into (3.9), making the usual smoothness assumptions and applying Stokes' theorem to the line integral, we obtain

$$\int_{\sigma}[\rho(\dot{\boldsymbol{v}}-\boldsymbol{F})-\boldsymbol{N}^{\alpha}{}_{|\alpha}]\,d\sigma=0 \tag{3.13}$$

which holds for arbitrary σ. Hence, from the vanishing of the integrand follow the equations of motion

$$\boldsymbol{N}^{\alpha}{}_{|\alpha}+\rho\boldsymbol{F}=\rho\,\dot{\boldsymbol{v}}. \tag{3.14}$$

Returning to (3.5), since

$$\int_{c}\boldsymbol{N}\cdot\boldsymbol{v}\,dc=\int_{\sigma}[\boldsymbol{N}^{\alpha}{}_{|\alpha}\cdot\boldsymbol{v}+\boldsymbol{N}^{\alpha}\cdot\boldsymbol{v}_{,\alpha}]\,d\sigma,$$

then with the use of (3.14) and the continuity equation (3.8), the energy equation becomes

$$\int_{\sigma}\rho\,\dot{U}\,d\sigma=\int_{\sigma}[\rho(r+\overline{\boldsymbol{L}}\cdot\boldsymbol{w})+\boldsymbol{N}^{\alpha}\cdot\boldsymbol{v}_{,\alpha}]\,d\sigma+\int_{c}[\boldsymbol{M}\cdot\boldsymbol{w}-h]\,dc. \tag{3.15}$$

If h^{α} is the flux of heat across the x^{α}-curves and $\boldsymbol{m}^{\alpha}$ is the (physical) director force vector over these curves, then application of (3.15) to a curvilinear triangle gives

$$\overline{\boldsymbol{M}}\cdot\boldsymbol{w}-\bar{h}=0, \tag{3.16}$$

where we have set

$$\bar{h}=h-q^{\alpha}v_{\alpha},\qquad q^{\alpha}=h^{\alpha}(a^{\alpha\alpha})^{\frac{1}{2}}, \tag{3.17}$$

$$\overline{\boldsymbol{M}}=\boldsymbol{M}-\boldsymbol{M}^{\alpha}v_{\alpha},\qquad \boldsymbol{M}^{\alpha}=\boldsymbol{m}^{\alpha}(a^{\alpha\alpha})^{\frac{1}{2}}. \tag{3.18}$$

* The order of indices in $N^{\gamma\alpha}$, $N^{3\alpha}$ differs from that used, *e.g.*, by GREEN & NAGHDI (1965).

Also, after substituting (3.16) into (3.15) and applying the result to an arbitrary surface area, we have

$$\rho r - q^{\alpha}{}_{|\alpha} - \rho \dot{U} + \boldsymbol{m} \cdot \boldsymbol{w} + \boldsymbol{N}^{\alpha} \cdot \boldsymbol{v}_{,\alpha} + \boldsymbol{M}^{\alpha} \cdot \boldsymbol{w}_{,\alpha} = 0, \tag{3.19}$$

where

$$\boldsymbol{m} = \boldsymbol{M}^{\alpha}{}_{|\alpha} + \rho \overline{\boldsymbol{L}}. \tag{3.20}$$

We now use the usual argument about uniform superposed rigid body angular velocities, the continuum occupying the same position at time t. We assume that ρ, r, q^{α}, U, $\boldsymbol{m}$, $\boldsymbol{N}^{\alpha}$, $\boldsymbol{M}^{\alpha}$, $\bar{h}$, are unaltered under such uniform superposed rigid body motion and then with the use of (2.27), (2.33) and (2.34), equations (3.16) and (3.19) yield

$$\overline{\boldsymbol{M}} \cdot \boldsymbol{\Gamma} + \overline{\boldsymbol{M}} \cdot \boldsymbol{\eta}_{\alpha} d^{\alpha} - \bar{h} = 0, \tag{3.21}$$

$$\rho r - q^{\alpha}{}_{|\alpha} - \rho \dot{U} + \boldsymbol{m} \cdot (\boldsymbol{\Gamma} + \boldsymbol{\eta}_{\alpha} d^{\alpha}) - \boldsymbol{N}^{\alpha} \cdot \boldsymbol{\eta}_{\alpha} + \boldsymbol{M}^{\alpha} \cdot (\boldsymbol{\Gamma}_{:\alpha} + \lambda^{\beta}_{.\alpha} \boldsymbol{\eta}_{\beta}) = 0, \tag{3.22}$$

together with

$$\boldsymbol{d} \times \overline{\boldsymbol{M}} = 0, \tag{3.23}$$

and

$$\boldsymbol{N}^{\alpha} \times \boldsymbol{a}_{\alpha} + (\boldsymbol{M}^{\alpha} \times \boldsymbol{d})_{|\alpha} + \rho \overline{\boldsymbol{L}} \times \boldsymbol{d} = 0. \tag{3.24}$$

We complete the general theory by stating an entropy production inequality in the form

$$\int_{\sigma} \rho \dot{S}\, d\sigma - \int_{\sigma} \rho \frac{r}{T}\, d\sigma + \int_{c} \frac{h}{T}\, dc \geqq 0 \tag{3.25}$$

which holds for all arbitrary surfaces, and where S is the entropy per unit mass and $T(>0)$ is the temperature. If we apply (3.25) to an arbitrary curvilinear triangle bounded by coordinate curves on $\mathscr{s}$, we obtain

$$h - \nu_{\alpha} q^{\alpha} \geqq 0. \tag{3.26}$$

In the special case when $\bar{h}$ does not depend explicitly on the director velocity $\boldsymbol{w}$, it may be shown from (3.16) or (3.21) that $\bar{h} = 0$ and then, by $(3.17)_1$,

$$h = q^{\alpha} \nu_{\alpha}. \tag{3.27}$$

Inspection of equations (3.21) and (3.22) suggests that for a complete theory of a given Cosserat surface constitutive equations are required for U, $\boldsymbol{N}^{\alpha}$, $\boldsymbol{M}^{\alpha}$, $\boldsymbol{m}$, $\overline{\boldsymbol{M}}$, $\bar{h}$ and q^{α} and these can be reduced to a canonical form with the aid of invariance conditions for each equation which keeps the left-hand sides of (3.21) and (3.22) unaltered by all superposed rigid body motions. It may be noted here that following the discussion given by GREEN & RIVLIN (1964b), we can assume that $\bar{h}$ is a scalar and $\overline{\boldsymbol{M}}$ is an (invariant) vector under transformation of surface coordinates. It then follows from (3.17), (3.18) and (3.20) that q^{α} are components of a contravariant vector, $\boldsymbol{m}$ is a vector, and $\boldsymbol{M}^{\alpha}$ transforms as a contravariant vector under transformation of surface coordinates. Also, if we set

$$\boldsymbol{M}^{\alpha} = M^{i\alpha} \boldsymbol{a}_i = M^{\gamma\alpha} \boldsymbol{a}_{\gamma} + M^{3\alpha} \boldsymbol{a}_3, \tag{3.28}$$

then $M^{\gamma\alpha}$ and $M^{3\alpha}$ are surface tensors.

4. Alternative forms of the basic field equations

The elegance and the simplicity of the derivation in the previous section does not display the relative complexity of the results. For this reason and for future reference, we obtain here the basic equations of Section 3 in tensor components.

To obtain the equations of motion in component form, consider the scalar product of (3.14) with $\boldsymbol{a}^\beta$ and again with $\boldsymbol{a}_3$ and deduce

$$(\boldsymbol{a}^\beta \cdot \boldsymbol{N}^\alpha)_{|\alpha} - \boldsymbol{N}^\alpha \cdot \boldsymbol{a}^\beta{}_{|\alpha} + \rho(\boldsymbol{F} - \dot{\boldsymbol{v}}) \cdot \boldsymbol{a}^\beta = 0,$$

$$(\boldsymbol{a}_3 \cdot \boldsymbol{N}^\alpha)_{|\alpha} - \boldsymbol{N}^\alpha \cdot \boldsymbol{a}_{3|\alpha} + \rho(\boldsymbol{F} - \dot{\boldsymbol{v}}) \cdot \boldsymbol{a}_3 = 0,$$

or equivalently

$$\begin{aligned} N^{\beta\alpha}{}_{|\alpha} - b^\beta_\alpha N^{3\alpha} + \rho F^\beta &= \rho c^\beta, \\ N^{3\alpha}{}_{|\alpha} + b_{\alpha\beta} N^{\beta\alpha} + \rho F^3 &= \rho c^3, \end{aligned} \tag{4.1}$$

where (3.11) and (2.5) have been used and where $c^i = \{c^\beta, c^3\}$ are the components of acceleration. In an entirely similar manner, we obtain from (3.20) and (3.28) the equations

$$\begin{aligned} M^{\beta\alpha}{}_{|\alpha} - b^\beta_\alpha M^{3\alpha} + \rho \bar{L}^\beta &= m^\beta, \\ M^{3\alpha}{}_{|\alpha} + b_{\alpha\beta} M^{\beta\alpha} + \rho \bar{L}^3 &= m^3, \end{aligned} \tag{4.2}$$

where $\bar{L}^i = \bar{\boldsymbol{L}} \cdot \boldsymbol{a}^i$ are the components of $\bar{\boldsymbol{L}}$, the difference of the assigned director force per unit mass and the inertia terms due to the director displacement $\boldsymbol{d}$, and $m^i = \boldsymbol{m} \cdot \boldsymbol{a}^i$ for which, in general, constitutive equations are required.

We now deduce the component forms of (3.23) and (3.24). With $\bar{M}^i = \bar{\boldsymbol{M}} \cdot \boldsymbol{a}^i$, from (3.23) follows the result $\varepsilon_{ijk}\, d^j\, \bar{M}^k = 0$, or equivalently

$$\varepsilon_{\alpha\beta}\, d^\alpha\, \bar{M}^\beta = 0, \qquad \bar{M}^3 d^\alpha = \bar{M}^\alpha d^3, \tag{4.3}$$

where $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta 3} = a^{\frac{1}{2}} e_{\alpha\beta}$ and $e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$. Similarly, from (3.24) we have

$$\varepsilon_{jim}[\delta^j_\alpha N^{i\alpha} + d^j m^i + \lambda^j{}_{.\alpha} M^{i\alpha}] = 0,$$

or equivalently

$$\begin{aligned} \varepsilon_{\beta\alpha}[N^{\beta\alpha} + m^\beta d^\alpha + M^{\beta\gamma} \lambda^\alpha_{.\gamma}] &= 0, \\ N^{3\alpha} + (m^3 d^\alpha - m^\alpha d^3) + M^{3\gamma} \lambda^\alpha_{.\gamma} - M^{\alpha\gamma} \lambda^3_{.\gamma} &= 0, \end{aligned} \tag{4.4}$$

where $\lambda^\alpha_{.\gamma}$ and $\lambda^3_{.\gamma}$ are given by (2.24).

In addition, equation (3.21) and the energy equation (3.22) in component forms may be written as

$$\bar{M}^i \dot{d}_i - \bar{M}^\alpha d^\beta \eta_{\alpha\beta} - \bar{h} = 0, \tag{4.5}$$

$$\rho r - q^\alpha{}_{|\alpha} - \rho \dot{U} + N'^{\beta\alpha} \eta_{\alpha\beta} + m^i \dot{d}_i + M^{i\alpha} \dot{\lambda}_{i\alpha} = 0, \tag{4.6}$$

where

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\beta\alpha} - m^\alpha d^\beta - M^{\alpha\gamma} \lambda^\beta_{.\gamma}. \tag{4.7}$$

Also, it is convenient to introduce here the Helmholtz free energy function per unit mass as

$$A = U - TS \tag{4.8}$$

and express the energy equation (4.6) in the form

$$\rho r - q^{\alpha}{}_{|\alpha} - \rho(T\dot{S} + \dot{T}S) - \rho\dot{A} + N'^{\beta\alpha}\eta_{\alpha\beta} + m^i \dot{d}_i + M^{i\alpha}\dot{\lambda}_{i\alpha} = 0. \tag{4.9}$$

Thus, in addition to the (local) equation (3.8) for conservation of mass and the entropy production inequality (3.25), the remaining basic equations and symmetry conditions are given by (4.1), (4.2), (4.5), (4.6) or (4.7), and (4.3) and (4.4). These equations are the counterparts of those given in vector form by (3.14), (3.20), (3.21), (3.22), and (3.23) and (3.24), respectively. From the discussion concerning the required constitutive equations at the end of Section 3, it is clear that $N'^{\alpha\beta}$ is a tensor and the left-hand sides of (4.5) and (4.6), as well as (4.9) are scalars. Also, in the construction of the constitutive equations we assume that A (or U), $N'^{\alpha\beta}$, m^i, $M^{i\alpha}$, $\bar{h}$, and q^{α} are all unaltered under superposed rigid body motions so that, in view of the behaviour of the kinematic variables, the left-hand sides of (4.5) and (4.9), or (4.6), remain unaltered by such motions.

5. An elastic Cosserat surface

We define an elastic Cosserat surface as one for which the following constitutive equations hold for all time t:

$$\begin{aligned} A &= A(T, e_{\alpha\beta}, \kappa_{i\alpha}, \delta_i, \Lambda_{i\alpha}, D_i), \\ S &= S(T, e_{\alpha\beta}, \kappa_{i\alpha}, \delta_i, \Lambda_{i\alpha}, D_i), \end{aligned} \tag{5.1}$$

$$q^{\alpha} = q^{\alpha}(T, T_{,\gamma}, e_{\gamma\delta}, \kappa_{i\gamma}, \delta_i, \Lambda_{i\gamma}, D_i) \tag{5.2}$$

and

$$\begin{aligned} \bar{h} &= \bar{h}(T, e_{\alpha\beta}, \kappa_{i\alpha}, \delta_i, \Lambda_{i\alpha}, D_i, v_{\alpha}), \\ \bar{M}^i &= \bar{M}^i(T, e_{\alpha\beta}, \kappa_{i\alpha}, \delta_i, \Lambda_{i\alpha}, D_i, v_{\alpha}), \end{aligned} \tag{5.3}$$

as well as

$$\begin{aligned} N'^{\alpha\beta} &= N'^{\alpha\beta}(T, e_{\gamma\delta}, \kappa_{i\gamma}, \delta_i, \Lambda_{i\gamma}, D_i), \\ m^i &= m^i(T, e_{\alpha\beta}, \kappa_{j\alpha}, \delta_j, \Lambda_{j\alpha}, D_j), \\ M^{i\alpha} &= M^{i\alpha}(T, e_{\gamma\delta}, \kappa_{j\gamma}, \delta_j, \Lambda_{j\gamma}, D_j), \end{aligned} \tag{5.4}$$

where we have set

$$2e_{\alpha\beta} = a_{\alpha\beta} - A_{\alpha\beta}, \tag{5.5}$$

$$\kappa_{i\alpha} = \lambda_{i\alpha} - \Lambda_{i\alpha}, \tag{5.6}$$

$$\delta_i = d_i - D_i, \tag{5.7}$$

and where $\Lambda_{i\alpha} = \boldsymbol{A}_i \cdot \boldsymbol{D}_{,\alpha}$ is the value of $\lambda_{i\alpha}$ in the reference state*. We note here that

$$\dot{e}_{\alpha\beta} = \eta_{\alpha\beta}, \quad \dot{\kappa}_{i\alpha} = \dot{\lambda}_{i\alpha}, \quad \dot{\delta}_i = \dot{d}_i, \tag{5.8}$$

and $e_{\alpha\beta}$, $\kappa_{i\alpha}$, δ_i are unaltered under arbitrary superposed rigid body motions.

* The variables $e_{\alpha\beta}$, $\kappa_{i\alpha}$ and δ_i in the constitutive equations (5.1) to (5.4) are introduced for convenience. Alternatively, the kinematic variables in the constitutive equations may be taken as $e_{\alpha\beta}$, $\lambda_{i\alpha}$, d_i together with the initial values $\Lambda_{i\alpha}$ and D_i. Dependence on the initial metric tensor $A_{\alpha\beta}$ is also understood.

Recalling that equation (3.16) holds for all arbitrary values of $\boldsymbol{w}$ and since, in view of (5.3), the coefficients $\bar{h}$ and $\overline{\boldsymbol{M}}$ are independent of the director velocity, it follows that*

$$\bar{h}=0, \quad \overline{\boldsymbol{M}}=0. \tag{5.9}$$

Hence, by (3.17),

$$h=q^{\alpha} v_{\alpha}, \tag{5.10}$$

and by (3.18) and (3.28),

$$\boldsymbol{M}=\boldsymbol{M}^{\alpha} v_{\alpha}=M^{i} \boldsymbol{a}_{i}, \quad M^{i}=M^{i\alpha} v_{\alpha}, \tag{5.11}$$

so that $(4.3)_1$ and $(4.3)_2$ are identically satisfied.

Using (5.10) in equation (3.25), making the usual smoothness assumptions and transforming the line integral into a surface integral, we obtain

$$\rho T \dot{S}-\rho r+q^{\alpha}{}_{|\alpha}-\frac{q^{\alpha} T_{,\alpha}}{T} \geqq 0, \tag{5.12}$$

since (3.25) holds for all arbitrary surfaces σ. From (5.12) and (4.9), we have

$$-\rho \dot{T} S-\rho \dot{A}+N'^{\beta\alpha} \eta_{\alpha\beta}+m^{i} \dot{d}_{i}+M^{i\alpha} \dot{\lambda}_{i\alpha}-\frac{q^{\alpha} T_{,\alpha}}{T} \geqq 0. \tag{5.13}$$

Introducing $(5.1)_1$ into (5.13) and taking account of $(5.1)_2$, (5.2), (5.4) and (5.8), we deduce that

$$\begin{aligned} -\rho\left(S+\frac{\partial A}{\partial T}\right) \dot{T}+\left(N'^{\beta\alpha}-\rho \frac{\partial A}{\partial e_{\alpha\beta}}\right) \eta_{\alpha\beta}+\left(m^{i}-\rho \frac{\partial A}{\partial \delta_{i}}\right) \dot{d}_{i}+ \\ +\left(M^{i\alpha}-\rho \frac{\partial A}{\partial \kappa_{i\alpha}}\right) \dot{\lambda}_{i\alpha}-\frac{q^{\alpha} T_{,\alpha}}{T} \geqq 0 \end{aligned} \tag{5.14}$$

which holds for all arbitrary values of $\dot{T}$, $\eta_{\alpha\beta}$, $\dot{d}_i$ and $\dot{\lambda}_{i\alpha}$ at time t. By considering a homogeneous temperature distribution varying with time, it follows from (5.14) that**

$$S=-\frac{\partial A}{\partial T}, \tag{5.15}$$

$$N'^{\beta\alpha}=\rho \frac{\partial A}{\partial e_{\alpha\beta}}, \quad m^{i}=\rho \frac{\partial A}{\partial \delta_{i}}, \quad M^{i\alpha}=\rho \frac{\partial A}{\partial \kappa_{i\alpha}}, \tag{5.16}$$

$$-q^{\alpha} T_{,\alpha} \geqq 0, \tag{5.17}$$

and the heat supply function r is determined from (4.9).

If desired, expressions for $N'^{\beta\alpha}$, m^i and $M^{i\alpha}$ may also be obtained in terms of the internal energy U in which case, through (5.15), T is expressed as a function of S and the kinematic variables $e_{\alpha\beta}$, $\kappa_{i\alpha}$, δ_i, $\Lambda_{i\alpha}$ and D_i. Thus, with

$$U=U(S, e_{\alpha\beta}, \kappa_{i\alpha}, \delta_{i}, \Lambda_{i\alpha}, D_{i}), \tag{5.18}$$

* The results (5.9) may also be deduced from (3.21).

** See COLEMAN & NOLL (1963). To avoid ambiguity in evaluating $\partial A/\partial e_{\alpha\beta}$, the tensor $e_{\alpha\beta}$ is understood to stand for $\frac{1}{2}(e_{\alpha\beta}+e_{\beta\alpha})$.

instead of (5.15) and (5.16), we have

$$T = \frac{\partial U}{\partial S}, \tag{5.19}$$

$$N'^{\beta\alpha} = \rho \frac{\partial U}{\partial e_{\alpha\beta}}, \quad m^i = \rho \frac{\partial U}{\partial \delta_i}, \quad M^{i\alpha} = \rho \frac{\partial U}{\partial \kappa_{i\alpha}}. \tag{5.20}$$

The above nonlinear constitutive equations are valid for an elastic Cosserat surface which is anisotropic in some preferred state, usually taken to be the initial undeformed state. In the remainder of this section, we omit a detailed discussion of the heat conduction vector q^α, but consider briefly a more explicit form of the free energy function for a Cosserat surface which initially has holohedral isotropy, *i.e.*, isotropy with a center of symmetry, to which the work of RIVLIN (1955) and ADKINS (1960) can be adapted. A discussion of the effect of material symmetries in restricting the form of constitutive equations is given by GREEN & ADKINS (1960). References to more general developments on symmetry restrictions may be found in a recent paper by WINEMAN & PIPKIN (1964) who have shown that the results of RIVLIN, SPENCER, ADKINS and others can be used, with only minor changes, for any type of dependence of the functions on their arguments and not necessarily polynomial dependence.

Consider the tangent plane and the normal at each point $\boldsymbol{R}$ of $\mathscr{S}$ with which point, regarded as the origin of a Cartesian coordinate system, we associate a triad of mutually orthogonal unit vectors $\boldsymbol{H}_m$ such that $\boldsymbol{H}_\alpha$ are in the tangent plane and $\boldsymbol{H}_3$ is directed along the normal. If a typical vector $\boldsymbol{T}$ is decomposed in such a coordinate system along the local unit vectors $\boldsymbol{H}_m$ into tangential components and a normal component, then under a change from one coordinate system to another the tangential components transform like the components of a 2-vector under the orthogonal group and the third component remains invariant. The symmetry of the material, defined by preferred directions in the initial undeformed state is then characterized by $\boldsymbol{H}_\alpha$ and the appropriate group of (2-dimensional) transformations which specify the equivalent positions of the vectors $\boldsymbol{H}_\alpha$ from one system to another and the constitutive relations must then be form-invariant under each transformation of this group. We consider here symmetries for which the associated group of transformations is a sub-group of the full orthogonal group.

When an elastic Cosserat surface is initially isotropic with a center of symmetry and assuming that the energy function A is a polynomial in the arguments indicated in (5.1)*, then we may use the works of RIVLIN (1955) and ADKINS (1960)** to express A as a function of the joint invariants of the arguments in (5.1). Although, in principle, there is no difficulty in writing down all the invariants, to avoid undue complications we limit ourselves in the rest of this Section to the special case when A does not depend explicitly on $A_{i\alpha}$ and D_i. For convenience, we introduce the

* Since the coordinate system used is convected, the invariance of (5.1), as well as (5.15) and (5.16), under superposed rigid motions is automatically fulfilled.

** See, especially, table (5.5) and the statement (ii) on p. 268 of the paper by ADKINS (1960).

notations

$$
\begin{aligned}
&e^{\alpha}_{\beta}=A^{\alpha\gamma}e_{\gamma\beta}, \\
&\kappa^{\alpha}_{\cdot\beta}=A^{\alpha\gamma}\kappa_{\gamma\beta}, \qquad \kappa_{\alpha}^{\cdot\beta}=A^{\beta\gamma}\kappa_{\alpha\gamma}, \\
&\kappa_{3}^{\cdot\alpha}=A^{\alpha\gamma}\kappa_{3\gamma}, \qquad \delta^{\alpha}=A^{\alpha\gamma}\delta_{\gamma},
\end{aligned}
\tag{5.21}
$$

and consider the 2×2 matrices

$$
\begin{gathered}
I, \quad J, \quad J^T, \\
K=u\,u^T, \quad P=u\,v^T, \quad P^T=v\,u^T, \quad Q=v\,v^T,
\end{gathered}
\tag{5.22}
$$

where u and v are column vectors, the notation u^T stands for the transpose of u, and

$$
I=e^{\alpha}_{\beta}, \quad J=\kappa^{\alpha}_{\cdot\beta}, \quad u=\kappa_{3}^{\cdot\alpha}, \quad v=\delta^{\alpha}. \tag{5.23}
$$

Then, by forming the invariants of the seven matrices in (5.22) and omitting the redundant elements, A may be expressed as a function of T, d_3 and the following twenty-four joint invariants:

$$
\begin{aligned}
&\operatorname{tr} I, \quad \operatorname{tr} J, \quad \operatorname{tr} K, \quad \operatorname{tr} P, \quad \operatorname{tr} Q, \\
&\operatorname{tr} I^2, \quad \operatorname{tr} J^2, \quad \operatorname{tr} IJ, \quad \operatorname{tr} JJ^T, \\
&\operatorname{tr} IK, \quad \operatorname{tr} IP, \quad \operatorname{tr} IQ, \\
&\operatorname{tr} JK, \quad \operatorname{tr} JQ, \quad \operatorname{tr} JP, \quad \operatorname{tr} JP^T, \\
&\operatorname{tr} IJJ^T, \\
&\operatorname{tr} IJK, \quad \operatorname{tr} IJQ, \quad \operatorname{tr} IJP, \quad \operatorname{tr} IJP^T, \\
&\operatorname{tr} JJ^TK, \quad \operatorname{tr} JJ^TQ, \quad \operatorname{tr} JJ^TP.
\end{aligned}
\tag{5.24}
$$

The above invariants apply when A does not depend explicitly on $A_{i\alpha}$ and D_i. In the general case, in addition to (5.24), invariants arising from the functions in (5.21) and $A_{\alpha\beta}$, $A_{3\beta}$, D_α should also be included.

6. Infinitesimal theory

The theory of an elastic Cosserat surface, where the displacements are infinitesimal, can be deduced as a special case of the general theory of Sections 4 and 5. Let

$$
\boldsymbol{r}=\boldsymbol{R}+\varepsilon\boldsymbol{u}, \qquad \boldsymbol{u}=u^i\boldsymbol{A}_i, \qquad \boldsymbol{v}=\varepsilon\dot{\boldsymbol{u}}, \tag{6.1}
$$

$$
\boldsymbol{d}=\boldsymbol{D}+\varepsilon\boldsymbol{\delta}^*, \qquad \boldsymbol{\delta}^*=\delta^*_i\boldsymbol{A}^i, \qquad \boldsymbol{w}=\varepsilon\dot{\boldsymbol{\delta}}^*, \tag{6.2}
$$

where ε is a non-dimensional parameter. By $(6.1)_1$ (2.3) and (2.4), we can show that

$$
\begin{gathered}
\boldsymbol{a}_3=\boldsymbol{A}_3+\varepsilon\boldsymbol{\beta}+O(\varepsilon^2), \\
\boldsymbol{\beta}=\beta^\alpha\boldsymbol{A}_\alpha=\beta_\alpha\boldsymbol{A}^\alpha, \qquad \beta_\alpha=-(u_{3,\alpha}+B^\gamma_\alpha u_\gamma).
\end{gathered}
\tag{6.3}
$$

From (2.3), (2.4), (5.5) to (5.7), and (6.1) to (6.3) we have

$$
e_{\alpha\beta}=\tfrac{1}{2}(a_{\alpha\beta}-A_{\alpha\beta})=\varepsilon\left[\tfrac{1}{2}(u_{\alpha|\beta}+u_{\beta|\alpha})-B_{\alpha\beta}u_3\right]+O(\varepsilon^2), \tag{6.4}
$$

$$
\kappa_{\beta\alpha}=\varepsilon\left[\boldsymbol{\delta}^*_{,\alpha}\cdot\boldsymbol{A}_\beta+\boldsymbol{u}_{,\beta}\cdot\boldsymbol{D}_{,\alpha}\right]+O(\varepsilon^2), \tag{6.5}
$$

$$
\kappa_{3\alpha}=\varepsilon\left[\boldsymbol{\delta}^*_{,\alpha}\cdot\boldsymbol{A}_3+\boldsymbol{D}_{,\alpha}\cdot\boldsymbol{\beta}\right]+O(\varepsilon^2), \tag{6.6}
$$

where

$$\boldsymbol{\delta}^*_{,\alpha}\cdot \boldsymbol{A}_\gamma=\delta^*_{\gamma|\alpha}-B_{\gamma\alpha}\delta^*_3,\qquad \boldsymbol{\delta}^*_{,\alpha}\cdot \boldsymbol{A}_3=\delta^*_{3,\alpha}+B^\gamma_\alpha\delta^*_\gamma$$

$$\boldsymbol{u}_{,\beta}\cdot \boldsymbol{D}_{,\alpha}=(u^\gamma{}_{|\beta}-B^\gamma_\beta u_3)(D_{\gamma|\alpha}-B_{\alpha\gamma}D_3)+(u_{3,\beta}+B^\gamma_\beta u_\gamma)(D_{3,\alpha}+B^\nu_\alpha D_\nu),\tag{6.7}$$

$$\boldsymbol{D}_{,\alpha}\cdot \boldsymbol{\beta}=(D_{\gamma|\alpha}-B_{\gamma\alpha}D_3)\beta^\gamma,$$

and in (6.4) and (6.7), and throughout the present section, a stroke denotes covariant differentiation with respect to $A_{\alpha\beta}$ of the undeformed surface, in contrast to the meaning of a stroke in earlier sections. The infinitesimal theory which we wish to consider here is such that the displacements and the director displacements, as well as their surface and time derivatives, remain small of the order of ε. We also assume that all kinematical quantities such as $e_{\alpha\beta}$ and $\kappa_{i\alpha}$ and their rates are of the order of ε; the curve and director forces such as $N^{i\alpha}$ and $M^{i\alpha}$, expressed in suitable non-dimensional form, are of the order of ε; and that $(T-T_0)/T_0$ and $(S-S_0)/S_0$ are of the order of ε, where T_0 and S_0 refer to a standard temperature and entropy of the initial undeformed surface.

In the following, we retain only terms of the order ε and after the approximations, without loss in generality, we set $\varepsilon=1$ and obtain*

$$\begin{aligned}
e_{\alpha\gamma}&=\tfrac{1}{2}(u_{\alpha|\gamma}+u_{\gamma|\alpha})-B_{\alpha\gamma}u_3,\\
\kappa_{\gamma\alpha}&=(\delta^*_{\gamma|\alpha}-B_{\alpha\gamma}\delta^*_3)+\big[(u^\nu{}_{|\gamma}-B^\nu_\gamma u_3)(D_{\nu|\alpha}-B_{\alpha\nu}D_3)+\\
&\qquad+(u_{3,\gamma}+B^\beta_\gamma u_\beta)(D_{3,\alpha}+B^\nu_\alpha D_\nu)\big],\\
\kappa_{3\alpha}&=(\delta^*_{3,\alpha}+B^\gamma_\alpha\delta^*_\gamma)+(D_{\gamma|\alpha}-B_{\gamma\alpha}D_3)\beta^\gamma,\qquad \beta_\alpha=-(u_{3,\alpha}+B^\gamma_\alpha u_\gamma),
\end{aligned}\tag{6.8}$$

$$\begin{aligned}
\dot e_{\alpha\gamma}&=\tfrac{1}{2}(v_{\alpha|\gamma}+v_{\gamma|\alpha})-B_{\alpha\gamma}v_3,\\
\dot\kappa_{\gamma\alpha}&=(w_{\gamma|\alpha}-B_{\alpha\gamma}w_3)+\big[(v^\nu{}_{|\gamma}-B^\nu_\gamma v_3)(D_{\nu|\alpha}-B_{\alpha\nu}D_3)+\\
&\qquad+(v_{3,\gamma}+B^\beta_\gamma v_\beta)(D_{3,\alpha}+B^\nu_\alpha D_\nu)\big],\\
\dot\kappa_{3\alpha}&=(w_{3,\alpha}+B^\gamma_\alpha w_\gamma)+(D_{\gamma|\alpha}-B_{\gamma\alpha}D_3)\dot\beta^\gamma,
\end{aligned}\tag{6.9}$$

and

$$v_i=\dot u_i,\qquad w_i=\dot\delta^*_i,\qquad \dot\beta_\alpha=-(v_{3,\alpha}+B^\gamma_\alpha v_\gamma).\tag{6.10}$$

Also, now in all of the basic field equations (4.1) to (4.7), $b_{\alpha\beta}$, d_i and $\lambda_{i\alpha}$ must be replaced to order ε by $B_{\alpha\beta}$, D_i and $\Lambda_{i\alpha}=\boldsymbol{A}_i\cdot\boldsymbol{D}_{,\alpha}$, respectively; all tensors in these equations are referred to the initial undeformed surface, and covariant differentiation is with respect to $A_{\alpha\beta}$. In particular, for the infinitesimal theory, the energy equation (4.9) assumes the form

$$\rho_0 r-Q^\alpha{}_{|\alpha}-\rho_0(T\dot S+\dot T S)-\rho_0\dot A+N'^{\beta\alpha}\dot e_{\alpha\beta}+m^i\dot\delta_i+M^{i\alpha}\dot\kappa_{i\alpha}=0\tag{6.11}$$

* Recalling (5.7), we have

$$\begin{aligned}
\delta_\alpha&=\delta^*_\alpha+D_\lambda(u^\lambda|_\alpha-B^\lambda_\alpha u_3)-D_3\beta_3,\\
\delta_3&=\delta^*_3+D_\alpha\beta^\alpha.
\end{aligned}$$

and equation (5.10) and the constitutive equations (5.16) and (5.17) become, respectively,

$$h = {}_0\nu^\alpha Q_\alpha, \tag{6.12}$$

$$N'^{\beta\alpha} = \rho_0 \frac{\partial A}{\partial e_{\alpha\beta}}, \quad m^i = \rho_0 \frac{\partial A}{\partial \delta_i}, \quad M^{i\alpha} = \rho_0 \frac{\partial A}{\partial \kappa_{i\alpha}}, \tag{6.13}$$

$$-Q^\alpha T_{,\alpha} \geqq 0, \tag{6.14}$$

where ρ_0 is the initial mass density, Q_α are the components of the heat flux per unit length (in the undeformed surface) per unit time, and ${}_0\nu^\alpha$ are the components of the unit outward normal to the x^α-curves on $\mathscr{S}$.

If the surface is initially homogeneous, free from curve and director forces, and in the state of rest at a constant temperature T_0 and entropy S_0, then to the order of approximation considered, it is sufficient to express $\rho_0 A$ in (6.13) as a quadratic function of $e_{\alpha\beta}$, $\kappa_{i\alpha}$, δ_i and T, where T is now the temperature difference from T_0. Thus, if $S_0 = 0$,

$$\begin{aligned} \rho_0 A = {} & {}_1C^{\alpha\beta\gamma\delta} e_{\alpha\beta} e_{\gamma\delta} + {}_2C^{\alpha\beta\gamma\delta} \kappa_{\alpha\beta} \kappa_{\gamma\delta} + {}_3C^{\alpha\beta\gamma\delta} e_{\alpha\beta} \kappa_{\gamma\delta} + \\ & + {}_1C^{\alpha\beta\gamma} \kappa_{3\alpha} \kappa_{\beta\gamma} + {}_2C^{\alpha\beta\gamma} e_{\alpha\beta} \delta_\gamma + \\ & + {}_3C^{\alpha\beta\gamma} e_{\alpha\beta} \kappa_{3\gamma} + {}_4C^{\alpha\beta\gamma} \delta_\alpha \kappa_{\beta\gamma} + \\ & + {}_1C^{\alpha\beta} \delta_\alpha \delta_\beta + {}_2C^{\alpha\beta} \kappa_{3\alpha} \kappa_{3\beta} + {}_3C^{\alpha\beta} \delta_\alpha \kappa_{3\beta} + \\ & + {}_4C^{\alpha\beta} e_{\alpha\beta} \delta_3 + {}_5C^{\alpha\beta} \kappa_{\alpha\beta} \delta_3 + {}_4C'^{\alpha\beta} e_{\alpha\beta} T + {}_5C'^{\alpha\beta} \kappa_{\alpha\beta} T + \\ & + {}_1C^\alpha \delta_\alpha \delta_3 + {}_2C^\alpha \kappa_{3\alpha} \delta_3 + {}_1C'^\alpha \delta_\alpha T + {}_2C'^\alpha \kappa_{3\alpha} T + \\ & + C(\delta_3)^2 + C' T^2 + C'' \delta_3 T, \end{aligned} \tag{6.15}$$

where the coefficients

$$C,\ C',\ {}_nC^\alpha,\ {}_nC'^\alpha,\ {}_nC^{\alpha\beta},\ {}_nC'^{\alpha\beta},\ {}_nC^{\alpha\beta\gamma},\ {}_nC^{\alpha\beta\gamma\delta} \qquad (n = 1, 2, \ldots)$$

are constants* and some of them satisfy certain symmetry conditions, *e.g.*,

$${}_1C^{\alpha\beta\gamma\delta} = {}_1C^{\beta\alpha\gamma\delta} = {}_1C^{\alpha\beta\delta\gamma} = {}_1C^{\gamma\delta\alpha\beta}, \tag{6.16}$$

$${}_2C^{\alpha\beta\gamma\delta} = {}_2C^{\gamma\delta\alpha\beta}, \quad {}_3C^{\alpha\beta\gamma\delta} = {}_3C^{\beta\alpha\gamma\delta}, \quad {}_2C^{\alpha\beta\gamma} = {}_2C^{\beta\alpha\gamma}. \tag{6.17}$$

We now restrict our attention to an elastic Cosserat surface possessing isotropy with a center of symmetry. In this case, a tensor basis is given by** $A^{\alpha\beta}$ and since there are no holohedral isotropic tensors of odd order, it follows that all odd order coefficients in (6.15) must vanish. Hence

$$\begin{gathered} {}_1C^{\alpha\beta\gamma} = {}_2C^{\alpha\beta\gamma} = {}_3C^{\alpha\beta\gamma} = {}_4C^{\alpha\beta\gamma} = 0, \\ {}_1C^\alpha = {}_2C^\alpha = {}_1C'^\alpha = {}_2C'^\alpha = 0. \end{gathered} \tag{6.18}$$

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* In general, these coefficients will be functions of the initial values $A_{i\alpha}$ and D_i. But, to avoid undue length, we restrict our attention throughout this section to the special case when A does not depend explicitly on $A_{i\alpha}$ and D_i.

** See SMITH & RIVLIN (1957).

Moreover, the remaining coefficients in (6.15) must be homogeneous, linear functions of products of $A^{\alpha\beta}$ so that, for example, ${}_1C^{\alpha\beta\gamma\delta}$ may be written as

$$ {}_1C^{\alpha\beta\gamma\delta}=\alpha_1 A^{\alpha\beta} A^{\gamma\delta}+\alpha_2 A^{\alpha\gamma} A^{\beta\delta}+\alpha_3 A^{\alpha\delta} A^{\beta\gamma}. \tag{6.19} $$

But by $(6.16)_1$, $\alpha_2=\alpha_3$ and the same conclusion may be reached by $(6.16)_2$. Similar arguments can be applied to other coefficients in (6.15). In particular, it can be shown that ${}_3C^{\alpha\beta\gamma\delta}$, as in ${}_1C^{\alpha\beta\gamma\delta}$, has only two independent constants but ${}_2C^{\alpha\beta\gamma\delta}$ involves three independent constants. Thus, the free energy $\rho_0 A$ may be finally written in the form

$$ \begin{aligned} \rho_0 A=&\tfrac{1}{2}[\alpha_1 A^{\alpha\beta} A^{\gamma\delta}+\alpha_2(A^{\alpha\gamma} A^{\beta\delta}+A^{\alpha\delta} A^{\beta\gamma})]\, e_{\alpha\beta} e_{\gamma\delta}+\\ &+\tfrac{1}{2}\alpha_3 A^{\alpha\beta}\delta_\alpha\delta_\beta+\tfrac{1}{2}\alpha_4(\delta_3)^2+\alpha_4'\delta_3 T+\tfrac{1}{2}\alpha_4'' T^2+\\ &+\tfrac{1}{2}[\alpha_5 A^{\alpha\beta} A^{\gamma\delta}+\alpha_6 A^{\alpha\gamma} A^{\beta\delta}+\alpha_7 A^{\alpha\delta} A^{\beta\gamma}]\,\kappa_{\alpha\beta}\kappa_{\gamma\delta}+\\ &+\tfrac{1}{2}\alpha_8 A^{\alpha\beta}\kappa_{3\alpha}\kappa_{3\beta}+\alpha_9 A^{\alpha\beta} e_{\alpha\beta}\delta_3+\alpha_9' A^{\alpha\beta} e_{\alpha\beta} T+\\ &+[\alpha_{10} A^{\alpha\beta} A^{\gamma\delta}+\alpha_{11}(A^{\alpha\gamma} A^{\beta\delta}+A^{\alpha\delta} A^{\beta\gamma})]\, e_{\alpha\beta}\kappa_{\gamma\delta}+\\ &+\alpha_{12} A^{\alpha\beta}\kappa_{\alpha\beta}\delta_3+\alpha_{12}' A^{\alpha\beta}\kappa_{\alpha\beta} T+\\ &+\alpha_{13} A^{\alpha\beta}\delta_\alpha\kappa_{3\beta}, \end{aligned} \tag{6.20} $$

and then by (6.13), we also have

$$ \begin{aligned} N'^{\alpha\beta}=&[\alpha_1 A^{\alpha\beta} A^{\gamma\delta}+\alpha_2(A^{\alpha\gamma} A^{\beta\delta}+A^{\alpha\delta} A^{\beta\gamma})]\, e_{\gamma\delta}+\\ &+[\alpha_{10} A^{\alpha\beta} A^{\gamma\delta}+\alpha_{11}(A^{\alpha\gamma} A^{\beta\delta}+A^{\alpha\delta} A^{\beta\gamma})]\,\kappa_{\gamma\delta}+\\ &+\alpha_9 A^{\alpha\beta}\delta_3+\alpha_9' A^{\alpha\beta} T, \end{aligned} \tag{6.21} $$

$$ \begin{aligned} m^\alpha&=\alpha_3 A^{\alpha\gamma}\delta_\gamma+\alpha_{13} A^{\alpha\gamma}\kappa_{3\gamma},\\ m^3&=\alpha_4\delta_3+\alpha_4' T+\alpha_9 A^{\alpha\beta} e_{\alpha\beta}+\alpha_{12} A^{\alpha\beta}\kappa_{\alpha\beta}, \end{aligned} \tag{6.22} $$

and

$$ \begin{aligned} M^{\alpha\beta}=&[\alpha_5 A^{\alpha\beta} A^{\gamma\delta}+\alpha_6 A^{\alpha\gamma} A^{\beta\delta}+\alpha_7 A^{\alpha\delta} A^{\beta\gamma}]\,\kappa_{\gamma\delta}+\\ &+[\alpha_{10} A^{\alpha\beta} A^{\gamma\delta}+\alpha_{11}(A^{\alpha\gamma} A^{\beta\delta}+A^{\alpha\delta} A^{\beta\gamma})]\, e_{\gamma\delta}+\\ &+\alpha_{12} A^{\alpha\beta}\delta_3+\alpha_{12}' A^{\alpha\beta} T,\\ M^{3\alpha}=&\alpha_8 A^{\alpha\gamma}\kappa_{3\gamma}+\alpha_{13} A^{\alpha\gamma}\delta_\gamma, \end{aligned} \tag{6.23} $$

where the coefficients $\alpha_1, \alpha_2 \ldots \alpha_{13}, \alpha_4', \alpha_9'$ and α_{12}' are constants.

7. Special cases of the general theory

We consider in this section some special cases of the general nonlinear theory of Sections 3 (or 4) and 5, as well as that of Section 6, by introducing simplifying assumptions which are motivated by the existing classical theories of elastic shells and plates. It may be recalled here that the known complete theories of shells and plates, both linear and nonlinear, are developed from the equations of elasticity theory in 3-space under certain assumptions and by integration of these equations along the x^3-coordinate and this, in general, requires additional approximations*.

* For a discussion of classical theories of shells and plates, see Naghdi (1963a).

When the director is absent and only curve forces are admitted, *i.e.*, when both $\boldsymbol{M}=\boldsymbol{L}=0$ in (3.4), the resulting theory may be called the *membrane* theory. The basic field equations of the membrane theory follow from the results of the general theory and include, in particular, the kinematical results (2.7) to (2.9), (2.14) and (2.15), and the equation (3.8) for conservation of mass. Also, in this case, since $\overline{M}^i=0$ and $M^{i\alpha}=0$, it follows that (4.3) is identically satisfied and, by (4.2) and (4.5), we have $m^i=0$ and $\bar{h}=0$, respectively. Equations (4.4) and (4.7) then yield

$$N'^{\beta\alpha}=N'^{\alpha\beta}=N^{\beta\alpha}, \quad N^{3\alpha}=0, \tag{7.1}$$

so that (4.1) and (4.9), the remaining relevant equations of Section 4 reduce to

$$N^{\beta\alpha}{}_{|\alpha}+\rho F^\beta=\rho c^\beta, \quad b_{\alpha\beta} N^{\beta\alpha}+\rho F^3=\rho c^3, \tag{7.2}$$

$$\rho r-q^\alpha{}_{|\alpha}-\rho(T\dot{S}+\dot{T}S)-\rho \dot{A}+N^{\beta\alpha}\eta_{\alpha\beta}=0. \tag{7.3}$$

The constitutive equations of the elastic membrane theory in terms of A are given by (5.15) and $(5.16)_1$ and A depends only on T and $e_{\alpha\beta}$. For a surface which is initially isotropic with a center of symmetry,

$$A=A(T, I_1, I_2), \quad \text{where} \quad I_1=\operatorname{tr} I \quad \text{and} \quad I_2=\operatorname{tr} I^2.$$

It may be noted here that the theory of elastic membranes as given in GREEN & ADKINS (1960), and developed from the equations of nonlinear elasticity in 3-space, is slightly more general than the above nonlinear membrane theory.

We now specialize the general theory to the case for which $\boldsymbol{D}=\boldsymbol{A}_3$ and $\boldsymbol{d}=\boldsymbol{a}_3$, so that*

$$\begin{aligned} D_\alpha&=0, \quad D_3=1, \\ d_\alpha&=0, \quad d_3=1. \end{aligned} \tag{7.4}$$

Then, from (2.22) and (2.24) and their duals follow the results

$$\begin{aligned} \Lambda_{\beta\alpha}&=-B_{\beta\alpha}, \quad \Lambda_{3\alpha}=0, \\ \lambda_{\beta\alpha}&=-b_{\beta\alpha}, \quad \lambda_{3\alpha}=0, \end{aligned} \tag{7.5}$$

and by (2.21), (2.22), (2.9) and (5.6), we have

$$\boldsymbol{w}=\dot{\boldsymbol{a}}_3=-(v_{3,\alpha}+b_\alpha^\beta v_\beta)\,\boldsymbol{a}^\alpha, \tag{7.6}$$

$$\kappa_{\beta\alpha}=-(b_{\beta\alpha}-B_{\beta\alpha}), \quad \kappa_{3\alpha}=0. \tag{7.7}$$

In view of $(7.4)_2$, $(4.3)_1$ is identically satisfied and $(4.3)_2$ gives

$$\overline{M}^\alpha=0. \tag{7.8}$$

By (7.8), $(7.4)_2$ and (4.5), we also have $\bar{h}=0$ so that (3.27) holds**. Thus, under the assumption (7.4), the kinematical results are modified only by (7.6) and (7.7),

* We can also examine a more general case in which only the initial value of the director is along the normal to the undeformed surface, *i.e.*, when $\boldsymbol{D}=\boldsymbol{A}_3$ only, but we do not consider this here.

** The vanishing of $\overline{M}^i$ and $\bar{h}$, in the case of an elastic Cosserat surface (Section 5) follow without any special assumption such as (7.4). Here, the results $\overline{M}^\alpha=0$, $\bar{h}=0$, as well as others between (7.4) and (7.20) are deduced directly from the general theory of Sections 3 and 4 by specialization and without reference to constitutive equations.

and the basic equations of Section 4 are given by (4.1) and (4.2) while equations (4.4), (4.7) and (4.9) become

$$N^{3\alpha}=m^{\alpha}+M^{3\gamma}\,b_{\gamma}^{\alpha}, \tag{7.9}$$

$$N'^{\alpha\beta}=N'^{\beta\alpha}=N^{\beta\alpha}+M^{\alpha\gamma}\,b_{\gamma}^{\beta}, \tag{7.10}$$

and

$$\rho\, r-q^{\alpha}{}_{|\alpha}-\rho(T\dot{S}+\dot{T}S)-\rho\,\dot{A}+N'^{\beta\alpha}\,\eta_{\alpha\beta}+M^{\beta\alpha}\,\dot{\kappa}_{\alpha\beta}=0, \tag{7.11}$$

where, by (7.7), $\dot{\kappa}_{\alpha\beta}=-\dot{b}_{\alpha\beta}$. If we combine (7.9) with $(4.2)_1$, we obtain

$$N^{3\beta}=M^{\beta\alpha}{}_{|\alpha}+\rho\,\bar{L}^{\beta}. \tag{7.12}$$

Then, using (7.12), we have from $(4.1)_1$ and $(4.1)_2$, respectively,

$$N^{\beta\alpha}{}_{|\alpha}-b_{\lambda}^{\beta}\,M^{\lambda\alpha}{}_{|\alpha}+\rho\,F^{\beta}=\rho\,\bar{c}^{\beta}, \tag{7.13}$$

and

$$M^{\beta\alpha}{}_{|\alpha\beta}+b_{\alpha\beta}\,N^{\beta\alpha}+\rho\,F^{3}=\rho\,\bar{c}^{3}, \tag{7.14}$$

where we have set

$$\begin{aligned}\bar{c}^{\beta}&=c^{\beta}+b_{\lambda}^{\beta}\,\bar{L}^{\lambda},\\ \rho\,\bar{c}^{3}&=\rho\,c^{3}-(\rho\,\bar{L}^{\beta})_{|\beta}.\end{aligned} \tag{7.15}$$

Hence, instead of the equations of motion (4.1) and (4.2), we have (7.12), (7.13) and (7.14), as well as the three equations given by $(4.2)_2$ and (7.9) or equivalently

$$m^{3}=M^{3\alpha}{}_{|\alpha}+b_{\alpha\beta}\,M^{\beta\alpha}+\rho\,\bar{L}^{3}, \tag{7.16}$$

and

$$m^{\alpha}=N^{3\alpha}-M^{3\gamma}\,b_{\gamma}^{\alpha}. \tag{7.17}$$

It is convenient now to decompose $N^{\alpha\beta}$ and $M^{\alpha\beta}$ into their respective symmetric and anti-symmetric parts. Thus,

$$N^{\alpha\beta}=N^{(\alpha\beta)}+N^{[\alpha\beta]},\quad N^{(\alpha\beta)}=\tfrac{1}{2}(N^{\alpha\beta}+N^{\beta\alpha}),\quad N^{[\alpha\beta]}=\tfrac{1}{2}(N^{\alpha\beta}-N^{\beta\alpha}), \tag{7.18}$$

and with a similar notation for $M^{(\alpha\beta)}$ and $M^{[\alpha\beta]}$. As is evident from the energy equation (7.11), since $\kappa_{3\alpha}=0$, $\delta_i=0$ and $\kappa_{\alpha\beta}$ is now a symmetric tensor, under the assumption (7.4), there are no constitutive equations for $M^{3\alpha}$, m^i, $M^{[\alpha\beta]}$ and these, as well as $\overline{M}^3$, will remain indeterminate.

In the case of an elastic Cosserat surface, in addition to (7.8), we also have $\overline{M}^3=0$ so that (5.11) holds. Moreover, under the assumption (7.4), the constitutive equations (5.16) for an elastic Cosserat surface reduce to*

$$N'^{\alpha\beta}=\rho\frac{\partial A}{\partial e_{\alpha\beta}},\qquad M^{(\alpha\beta)}=\rho\frac{\partial A}{\partial\kappa_{\alpha\beta}}. \tag{7.19}$$

In order to provide a determinate theory, we now assume that

$$M^{[\alpha\beta]}=0 \tag{7.20}$$

* The tensors $e_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ in (7.19) are understood to stand for $\frac{1}{2}(e_{\alpha\beta}+e_{\beta\alpha})$ and $\frac{1}{2}(\kappa_{\alpha\beta}+\kappa_{\beta\alpha})$, respectively.

and then $N^{\alpha\beta}$ is determined from (7.10) and $(7.19)_1$. We also note here that by (7.10) and (7.20), the anti-symmetric part of $N^{\alpha\beta}$ may be determined through

$$N^{[\alpha\beta]}=\tfrac{1}{2}\{b_\lambda^\beta M^{(\lambda\alpha)}-b_\lambda^\alpha M^{(\lambda\beta)}\}. \tag{7.21}$$

If we introduce (7.18) and (7.21) into (7.13) and (7.14), then these equations may also be put in the form*

$$N^{(\beta\alpha)}{}_{|\beta}-\tfrac{1}{2}[b_\beta^\alpha M^{(\lambda\beta)}]_{|\lambda}+\tfrac{1}{2}[b_\beta^\lambda M^{(\beta\alpha)}]_{|\lambda}-b_\lambda^\alpha M^{(\beta\lambda)}{}_{|\beta}+\rho(F^\beta-\bar{c}^\beta)=0, \tag{7.22}$$

$$b_{\alpha\beta}N^{(\alpha\beta)}+M^{(\alpha\beta)}{}_{|\alpha\beta}+\rho(F^3-\bar{c}^3)=0. \tag{7.23}$$

The system of equations (2.7), (2.14), (5.5), (7.5), (7.7), (7.12), (7.20) to (7.23) together with (7.11), (5.1) and (5.2) form a determinate system for $N^{\alpha\beta}$, $M^{\alpha\beta}$ and $N^{3\alpha}$. The five unknowns $M^{3\alpha}, m^\alpha, m^3$ remain indeterminate; however, they are related by the three equations (7.16) and (7.17). We complete our present discussion of an elastic Cosserat surface, by observing that under the assumption (7.4) and by (5.1), the free energy A assumes the form

$$A=A(T, e_{\alpha\beta}, \kappa_{\alpha\beta}, B_{\alpha\beta}). \tag{7.24}$$

When the surface is initially isotropic with a center of symmetry, following the discussion concerning material symmetry at the end of Section 5, A may be expressed as a function of T and the joint invariants

$$\begin{array}{lll} \operatorname{tr} I, & \operatorname{tr} J, & \operatorname{tr} L \\ \operatorname{tr} I^2, & \operatorname{tr} J^2, & \operatorname{tr} L^2, \\ \operatorname{tr} IJ, & \operatorname{tr} IL, & \operatorname{tr} JL, \\ \operatorname{tr} IJL, & & \end{array} \tag{7.25}$$

where the matrices I and J are defined as in (5.23) and (5.21), and the 2×2 symmetric matrix L is defined by

$$L=B_\beta^\alpha=A^{\alpha\gamma}B_{\gamma\beta}. \tag{7.26}$$

If the free energy A does not depend explicitly on $B_{\alpha\beta}$, then the invariants corresponding to those in (7.25) may also be obtained as a special case of the joint invariants listed in (5.24).

The above special theory, deduced under the assumption (7.4), may be regarded as comparable to the classical theory of shells founded under the so-called Kirchhoff-Love hypothesis. Previously, a nonlinear theory of shells under the Kirchhoff-Love hypothesis, using the equations of nonlinear elasticity in 3-space, was given by NAGHDI & NORDGREN (1963) and their results may be brought into correspondence with those given here between (7.4) and (7.24), provided that the components $M^{\alpha\beta}$ of the director force $\boldsymbol{M}$ are identified with the so-called "stress-couple" resultants.

* This form of the equations of motion is equivalent to those obtained previously by NAGHDI (1963b, 1965) in another context. It may be of interest to note that Eq. (7.23) holds even without the use of (7.20), since $M^{\beta\alpha}{}_{|\alpha\beta}=M^{\beta\alpha}{}_{|\beta\alpha}$.

The linearized theory corresponding to the nonlinear theory for which (7.4) holds may be obtained directly as a special case of the infinitesimal theory of Section 6. Thus, with

$$\boldsymbol{D}=\boldsymbol{A}_3, \quad \boldsymbol{d}=\boldsymbol{a}_3, \tag{7.27}$$

it follows from (6.1) to (6.3) that

$$\begin{gathered}\boldsymbol{\delta}^*=\boldsymbol{\beta}, \quad \delta^*_\alpha=\beta_\alpha, \quad \delta^*_3=0, \\ w=\dot{\beta}_\alpha, \quad w_3=0.\end{gathered} \tag{7.28}$$

In view of (7.27) and (7.28), equations (6.8) and (6.9) simplify and may be written as*

$$\begin{aligned}&e_{\alpha\gamma}=e_{\gamma\alpha}=\tfrac{1}{2}(u_{\alpha|\gamma}+u_{\gamma|\alpha})-B_{\alpha\gamma}u_3, \\ &\kappa_{\alpha\gamma}=\kappa_{\gamma\alpha}=-u_{3|\gamma\alpha}-B^\lambda_{\gamma|\alpha}u_\lambda-B^\lambda_\gamma u_{\lambda|\alpha}-B^\lambda_\alpha u_{\lambda|\gamma}+B_{\alpha\lambda}B^\lambda_\gamma u_3, \\ &\kappa_{3\alpha}=0,\end{aligned} \tag{7.29}$$

and

$$\begin{aligned}&\dot{e}_{\alpha\gamma}=\tfrac{1}{2}(v_{\alpha|\gamma}+v_{\gamma|\alpha})-B_{\alpha\gamma}v_3, \\ &\dot{\kappa}_{\alpha\gamma}=-v_{3|\gamma\alpha}-B^\lambda_{\gamma|\alpha}v_\lambda-B^\lambda_\gamma v_{\lambda|\alpha}-B^\lambda_\alpha v_{\lambda|\gamma}+B_{\alpha\lambda}B^\lambda_\gamma v_3,\end{aligned} \tag{7.30}$$

where in (7.29) and (7.30), and all equations that follow in the remainder of this section, covariant differentiation is understood to be with respect to $A_{\alpha\beta}$. By an argument entirely analogous to that discussed for the nonlinear theory subject to the assumption (7.4), we can show that $\overline{M}^\alpha=0$, $\bar{h}=0$ and hence (6.12) holds. Also, the energy equation (6.11) now becomes

$$\rho_0 r-Q^\alpha{}_{|\alpha}-\rho_0(T\dot{S}+\dot{T}S)-\rho_0\dot{A}+N'^{\alpha\beta}\dot{e}_{\alpha\beta}+M^{(\alpha\beta)}\dot{\kappa}_{\alpha\beta}=0. \tag{7.31}$$

For the linearized theory of an elastic Cosserat surface under consideration, $\overline{M}^3=0$ and again, in order to obtain a determinate theory, we set $M^{[\alpha\beta]}=0$ and conclude that $N^{[\alpha\beta]}$ is given by (7.21) with $b_{\alpha\beta}$ replaced by $B_{\alpha\beta}$. Similarly, the relevant equations of motion are (7.22) and (7.23) except that $b_{\alpha\beta}$ must be replaced by $B_{\alpha\beta}$. The unknowns $M^{3\alpha}, m^\alpha, m^3$ will remain indeterminate, but they are related by equations corresponding to (7.16) and (7.17) with $b_{\alpha\beta}$ replaced by $B_{\alpha\beta}$. The free energy $\rho_0 A$, in this case, is a function of T, $B_{\alpha\beta}$, $e_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ given by (7.29) and the constitutive equations (6.13) reduce to

$$N'^{\alpha\beta}=\rho_0\frac{\partial A}{\partial e_{\alpha\beta}}, \quad M^{(\alpha\beta)}=\rho_0\frac{\partial A}{\partial \kappa_{\alpha\beta}}. \tag{7.32}$$

The stress $N^{\alpha\beta}$ is obtained from (7.10) with b^α_γ replaced by B^α_γ. We also note that for a surface which initially is isotropic with a center of symmetry, $\rho_0 A$ and the explicit forms of (7.32) may be obtained from (6.20) to (6.23) by setting α_3, α'_4, α_8, α_9, α_{12} and α_{13} equal to zero and since $\kappa_{\alpha\beta}=\kappa_{\beta\alpha}$ and $M^{[\alpha\beta]}=0$, without loss

* It is perhaps interesting to note that these expressions are the same as those used previously by NAGHDI (1963b, 1965) and GREEN & NAGHDI (1965) in entirely different developments of the linear theory of elastic shells.

in generality, we may also put $\alpha_6=\alpha_7$. Further development of the above linear theory is possible, but we do not pursue this further. Results (6.20)–(6.23) hold only when A does not depend explicitly on $B_{\alpha\beta}$. More generally, A is a quadratic function of $e_{\alpha\beta}$, $\kappa_{\alpha\beta}$, T and a function of $B_{\alpha\beta}$. This function can be obtained from the invariants in (7.25) when the surface is isotropic with a centre of symmetry.

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