Kirchhoff and Mindlin Plates

A plate significantly longer in two directions compared with the third, and it carries load perpendicular to that plane. The theory for plates can be regarded as an extension of beam theory, in the sense that a beam is a 1D specialization of 2D plates. In fact, the Euler-Bernoulli and Timoshenko beam theories both have its counterpart in plate theory:

- Kirchhoff theory for plates = Euler-Bernoulli theory for beams
- Mindlin theory for plates = Timoshenko theory for beams

The Kirchhoff theory assumes that a vertical line remains straight and perpendicular to the neutral plane of the plate during bending. In contrast, Mindlin theory retains the assumption that the line remains straight, but no longer perpendicular to the neutral plane. That means that Kirchhoff theory applies to thin plates, while Mindlin theory applies to thick plates where shear deformation may be significant.

Figure 1 shows the notation for the bending moments and shear forces in plates. The documents posted on this website use a different convention for beams and plates. The BEAM quantities are formulated with ONE index with the following meaning:

- M_y is the bending moment about the y-axis (leading to the notation I_y for the corresponding moment of inertia)
- V_z is the shear force in the direction parallel to the z-axis

Conversely, the PLATE quantities are formulated with TWO indices with the following meaning:

- M_{yy} is the bending moment obtained by integrating the axial stress σ_{yy}
- V_{yz} is the shear force obtained by integrating the shear stress τ_{yz}
- M_{xy} is the "twisting moment" obtained by integrating the shear stress τ_{xy}

The first index of a stress denotes the surface normal of the plane the stress acts on, while the second index is the direction of the stress. Also notice that in plate theory, the moments and shear forces are measured per unit length along the plate edge.

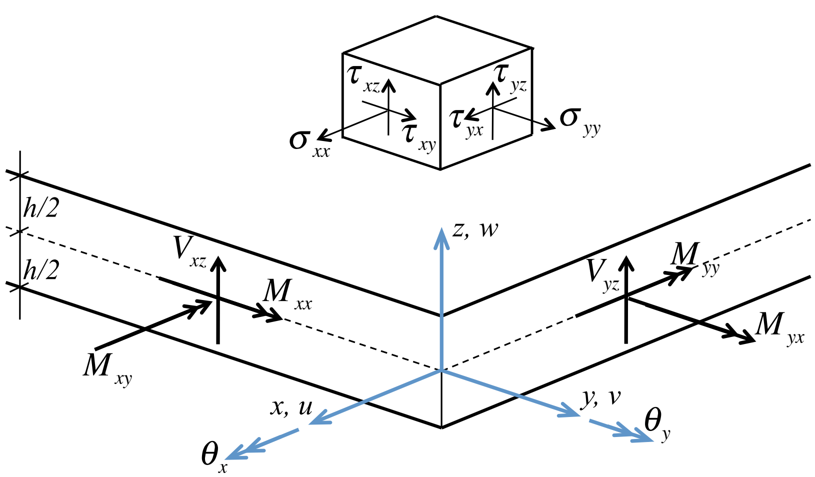

Figure 1: Stresses and stress resultant in a plate element.

Section Integration

Letting the z-axis have its origin at the neutral plane of the plate, the moments are defined by the integrals

$$M_{xx} = \int_{-h/2}^{h/2} z \cdot \sigma_{xx} \, dz \tag{1}$$

$$M_{yy} = \int_{-h/2}^{h/2} z \cdot \sigma_{yy} \, dz \tag{2}$$

$$M_{xy} = \int_{-h/2}^{h/2} z \cdot \tau_{xy} \, dz \tag{3}$$

$$M_{yx} = \int_{-h/2}^{h/2} z \cdot \tau_{yx} \, dz \tag{4}$$

From basic solid mechanics it is known that $\tau_{xy}=\tau_{yx}$, which implies that the twisting moments are equal: $M_{xy}=M_{yx}$. In the Kirchhoff plate theory the postulation from Euler-Bernoulli beams is made that the shear deformation is zero, hence the shear strain, shear stress, and shear forces are omitted from the theory. Of course, the shear forces can later be recovered by equilibrium considerations, but at first V_{xz} and V_{yz} are left out of the Kirchhoff theory. However, in Mindlin theory the shear forces are obtained by integration of the shear stresses:

$$V_{yz} = \int_{-h/2}^{h/2} \tau_{yz} \, dz \tag{5}$$

$$V_{zx} = \int_{-h/2}^{h/2} \tau_{xz} \, dz \tag{6}$$

Equilibrium

Consider the infinitesimal plate element shown in Figure 2. It extends dx in x-direction and dy in y-direction. Its thickness is h and it is subjected to a distributed load of intensity $q(x,y)$ in the z-direction. Equilibrium in the z-direction yields

$$q \cdot dx \cdot dy + \frac{\partial V_{yz}}{dy} \cdot dy \cdot dx + \frac{\partial V_{xz}}{\partial x} \cdot dx \cdot dy = 0 \Rightarrow q + \frac{\partial V_{yz}}{\partial y} + \frac{\partial V_{xz}}{\partial x} = 0 \tag{7}$$

Moment equilibrium about the y-axis, at the left front edge in Figure 2 yields

$$\frac{\partial M_{xx}}{\partial x} \cdot dx \cdot dy + \frac{\partial M_{yx}}{\partial y} \cdot dy \cdot dx - V_{xz} \cdot dy \cdot dx = 0 \Rightarrow V_{xz} = \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yx}}{\partial y} \tag{8}$$

Moment equilibrium about the x-axis, at the right front edge in Figure 2 yields

$$-\frac{\partial M_{yy}}{\partial y}\cdot dy\cdot dx-\frac{\partial M_{xy}}{\partial x}\cdot dx\cdot dy+V_{yz}\cdot dx\cdot dy=0\Rightarrow V_{yz}=\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{xy}}{\partial x} \tag{9}$$

The three equilibrium equations can be combined into one. Partial differentiation of Eq. (8) with respect to *x* and partial differentiation of Eq. (9) with respect to *y*, followed by substitution of those equations into Eq. (7) yields

$$q+\frac{\partial^2 M_{xx}}{\partial x^2}+\frac{\partial^2 M_{yy}}{\partial y^2}+2\cdot\frac{\partial^2 M_{xy}}{\partial x\partial y}=0 \tag{10}$$

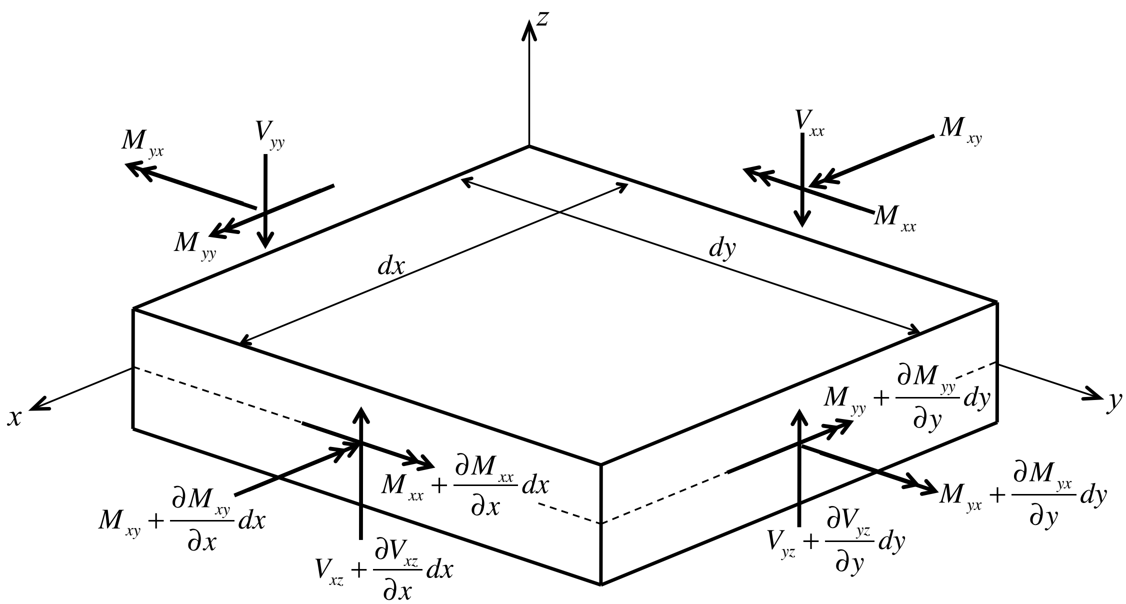

Figure 2: Infinitesimal plate element.

Material Law

For the relatively thin plates it is appropriate to consider the plane stress version of Hooke's law to model the in-plane behaviour:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} \tag{11}$$

Mindlin theory also requires the relationship between the other shear stresses and strains:

$$\tau_{xz}=G\cdot\gamma_{xz} \tag{12}$$

$$\tau_{yz}=G\cdot\gamma_{yz} \tag{13}$$

Kinematics

It is the kinematic relationships that best reveal the difference between the Kirchhoff and Mindlin theories. In Euler-Bernoulli theory for beams and Kirchhoff theory for plates, rotation is equated with the derivative of the lateral displacement. That implies that straight lines remain straight and perpendicular to the neutral axis during bending. This assumption is relaxed in Timoshenko beam theory and Mindlin plate theory. To understand the implications of these assumptions, the fundamental kinematic equations from solid mechanics are first listed:

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ 0 & \partial/\partial z & \partial/\partial y \\ \partial/\partial z & 0 & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{14}$$

Among those six equations, the third is irrelevant here because ε_{zz} is assumed to be zero. To spell out the other equations, it is recognized from Figure 1 that

$$u = z \cdot \theta_y \tag{15}$$

and

$$v = -z \cdot \theta_x \tag{16}$$

Substitution of Eqs. (15) and (16) into Eq. (14) yields the following five remaining equations:

$$\varepsilon_{xx} = z \cdot \theta_{y,x} \tag{17}$$

$$\varepsilon_{yy} = -z \cdot \theta_{x,y} \tag{18}$$

$$\gamma_{xy} = z \cdot \theta_{y,y} - z \cdot \theta_{x,x} \tag{19}$$

$$\gamma_{yz} = -\theta_x + w_{,y} \tag{20}$$

$$\gamma_{zx} = \theta_y + w_{,x} \tag{21}$$

Similar to the Euler-Bernoulli beam theory, in Kirchhoff plate theory it is assumed that

$$\theta_x = w_{,y} \tag{22}$$

and

$$\theta_y = -w_{,x} \tag{23}$$

That implies that in Kirchhoff theory, Eqs. (17) through (19) reads:

$$\varepsilon_{xx} = -z \cdot w_{,xx} \tag{24}$$

$$\varepsilon_{yy} = -z \cdot w_{yy} \tag{25}$$

$$\gamma_{xy} = -z \cdot w_{yx} - z \cdot w_{xy} = -2 \cdot z \cdot w_{xy} \tag{26}$$

while the shear strains γ_{yz} and γ_{zx} are zero.

Differential Equation

For Kirchhoff plates, the combination of stress resultant, material law, and kinematic equations, as well as integration along z, yields the following governing equations:

$$M_{xx} = -D \cdot \left(\frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2} \right) \tag{27}$$

$$M_{yy} = -D \cdot \left(\frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2} \right) \tag{28}$$

$$M_{xy} = -D \cdot (1-\nu) \cdot \frac{\partial^2 w}{\partial x \partial y} \tag{29}$$

where the "plate stiffness, D, comparable with EI for a beam, is defined as:

$$D = \frac{E \cdot h^3}{12 \cdot (1-\nu^2)} \tag{30}$$

Substitution of Eqs. (27), (28), (29) into the equilibrium equation from Eq. (10) yields the fourth order differential equation for plate bending:

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{q}{D} \tag{31}$$

Eqs. (27) and (28) allow a useful interpretation for plates that span in one direction only. A strip of such a plate can be considered as a beam. In other words, when one of the curvatures in the parentheses in Eqs. (27) and (28) is zero then the equations take the form

$$M_{xx} = -D \cdot \frac{\partial^2 w}{\partial x^2} \tag{32}$$

This does indeed lead to the conclusion that D take the place of the bending stiffness EI from beam bending, when a unit-width strip of the plate is considered. The use of D in place of EI essentially accounts for the constrained strain, i.e., "plane strain," in the plate continuing on the sides of the plate strip.

Recovery of Shear Forces: Kirchhoff's Shear Force

The anomaly of beam theory due to Navier's hypothesis carries over to plate theory. Instead of including shear stresses and strains in the theory, the shear forces are recovered only after solving the differential equation. The equilibrium in Eqs. (8) and (9) are employed for this purpose. However, these two equations are only one part of the total shear force in plate theory, as explained next.

According to the theory above, three stress resultants act along the edge of a plate: bending moment, shear force, and twisting moment. The number of unknowns in the general solution to the differential equation is insufficient to prescribe that many boundary conditions. This leads to a closer examination of the twisting moment, and subsequently its inclusion into the total shear force. To this end, consider the twisting moment M_{xy} and its variation in the y-direction. Figure 1 may be of help in visualizing this. Next, imagine that within each infinitesimal segment of length dy the twisting moment M_{xy} gives rise to a force pair. Let the forces be dy apart; because the moment within a length dy is $M_{xy} \cdot dy$, each force is M_{xy}. When the twisting moment varies along y, then there will be a "surplus" of the force M_{xy} within each infinitesimal segment. This surplus is same as the change of M_{xy} within a length dy, namely $\partial M_{xy} / \partial y$. The total shear forces are then:

$$V_{xz} + \frac{\partial M_{xy}}{\partial y} = -D \cdot \left(\frac{\partial^3 w}{\partial x^3} + (2-\nu) \frac{\partial^3 w}{\partial x \partial y^2} \right) \tag{33}$$

$$V_{yz} + \frac{\partial M_{yx}}{\partial y} = -D \cdot \left(\frac{\partial^3 w}{\partial y^3} + (2-\nu) \frac{\partial^3 w}{\partial x^2 \partial y} \right) \tag{34}$$

The force-interpretation of the twisting moment leads to another conclusion. When M_{xy} and M_{yx} and varies along the plate edge there is a net shear force at the corner of the plate. For example, when a square plate is bending under uniform downward loading then the corners will experience uplift. This is due to the net unbalanced concentrated force equal to $2M_{xy}$ at the corner. This shear force is known as Kirchhoff's shear force and the corner uplift is referred to as the Kirchhoff effect.

Navier's Solution

This solution for thin plates was presented to the French Academy in 1820 by Claude-Louis Navier and is explained in detail in one of Timoshenko's books (Timoshenko and Woinowsky-Krieger 1959). A simply supported rectangular plate with length a in the x-direction and length b in the y-direction is considered. Navier's solution stems from the preliminary consideration of one "sine pillow" as loading on the plate:

$$q(x,y) = \alpha \cdot \sin\left(\frac{\pi x}{a} \right) \cdot \sin\left(\frac{\pi y}{b} \right) \tag{35}$$

where α is the maximum amplitude of the load, at the middle of the plate. Furthermore, the following trial solution satisfies the boundary conditions that require zero displacement and bending moment on the edges:

$$w(x,y) = C \cdot \sin\left(\frac{\pi x}{a} \right) \cdot \sin\left(\frac{\pi y}{b} \right) \tag{36}$$

Substitution of Eqs. (35) and (36) into the differentiation equation in Eq. (31) yields:

$$\begin{aligned}&\left(C\cdot\frac{\pi^4}{a^4}\cdot\sin\left(\frac{\pi x}{a}\right)\cdot\sin\left(\frac{\pi y}{b}\right)\right)+\left(C\cdot\frac{\pi^4}{b^4}\cdot\sin\left(\frac{\pi x}{a}\right)\cdot\sin\left(\frac{\pi y}{b}\right)\right)\\&+2\cdot\left(C\cdot\frac{\pi^4}{a^2b^2}\cdot\sin\left(\frac{\pi x}{a}\right)\cdot\sin\left(\frac{\pi y}{b}\right)\right)=\frac{\alpha}{D}\cdot\sin\left(\frac{\pi x}{a}\right)\cdot\sin\left(\frac{\pi y}{b}\right)\end{aligned}\tag{37}$$

The same sine product appears in all terms; hence, they cancel, and rearranging yields the unknown constant in the solution:

$$C=\frac{\alpha}{D\cdot\pi^4\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^2}\tag{38}$$

Navier extended this approach by using multiple sine functions to describe the load, expressed as a series expansion:

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}\cdot\sin\left(\frac{m\pi x}{a}\right)\cdot\sin\left(\frac{n\pi y}{b}\right)\tag{39}$$

The coefficient q_{mn} is in general different for each n and m and essentially describes the magnitude of the load. To determine q_{mn}, i.e., to link Eq. (39) with actual load, e.g., uniformly distributed load, it is useful to express Eq. (39) as an integral instead of a sum. This is cleverly done by multiplying Eq. (39) by an identical sine product, only with different "counters" n and m, and integrating the result:

$$\begin{aligned}&\int_0^b\int_0^a q(x,y)\cdot\sin\left(\frac{\tilde{m}\pi x}{a}\right)\cdot\sin\left(\frac{\tilde{n}\pi y}{b}\right)dx\,dy\\&=\int_0^b\int_0^a\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}\cdot\sin\left(\frac{m\pi x}{a}\right)\cdot\sin\left(\frac{n\pi y}{b}\right)\right)\cdot\sin\left(\frac{\tilde{m}\pi x}{a}\right)\cdot\sin\left(\frac{\tilde{n}\pi y}{b}\right)dx\,dy\end{aligned}\tag{40}$$

Because

$$\begin{aligned}&\int_0^a\sin\left(\frac{m\pi x}{a}\right)\cdot\sin\left(\frac{\tilde{m}\pi x}{a}\right)dx=0\quad\text{when}\quad m\neq\tilde{m}\\&\int_0^a\sin\left(\frac{m\pi x}{a}\right)\cdot\sin\left(\frac{\tilde{m}\pi x}{a}\right)dx=\frac{a}{2}\quad\text{when}\quad m=\tilde{m}\end{aligned}\tag{41}$$

Eq. (40) simplifies to

$$\int_0^b\int_0^a q(x,y)\cdot\sin\left(\frac{\tilde{m}\pi x}{a}\right)\cdot\sin\left(\frac{\tilde{n}\pi y}{b}\right)dx\,dy=\frac{ab}{4}q_{mn}\tag{42}$$

from which q_{mn} is solved for specific load distributions $q(x,y)$. For the case of uniformly distributed load, $q(x,y)=q_o$, Eq. (42) yields:

$$q_{mn} = \frac{4q_o}{ab} \cdot \int_0^b \int_0^a \sin\left(\frac{\tilde{m}\pi x}{a}\right) \cdot \sin\left(\frac{\tilde{n}\pi y}{b}\right) dx\, dy = \begin{cases} \dfrac{16q_o}{\pi^2 mn} & m = n = \text{odd} \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

For the case of a point load with value P positioned at $x=\xi$ and $y=\eta$ Eq. (42) yields a sum over all m and n, now both odd and even:

$$q_{mn} = \frac{4P}{ab} \cdot \sin\left(\frac{m\pi\xi}{a}\right) \cdot \sin\left(\frac{n\pi\eta}{b}\right) \tag{44}$$

Now to the displacement solution. The solution in Eqs. (36) and (38) contained *one* sine term. Navier's solution contains many:

$$w(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q_{mn}}{D \cdot \pi^4 \cdot \left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)^2} \cdot \sin\left(\frac{m\pi x}{a}\right) \cdot \sin\left(\frac{n\pi y}{b}\right) \tag{45}$$

with the coefficient q_{mn} determined above.

Lévy's Solution

Navier's solution is a conceptually straightforward application of a double trigonometric series. However, the series does not converge fast; thus, high-order derivatives of w may be inaccurate. Around 1899 Levy suggested another approach (Timoshenko and Woinowsky-Krieger 1959). Again a simply supported rectangular plate with length a in the x-direction and length b in the y-direction is considered, but now the coordinate system is shifted as shown in Figure 3.

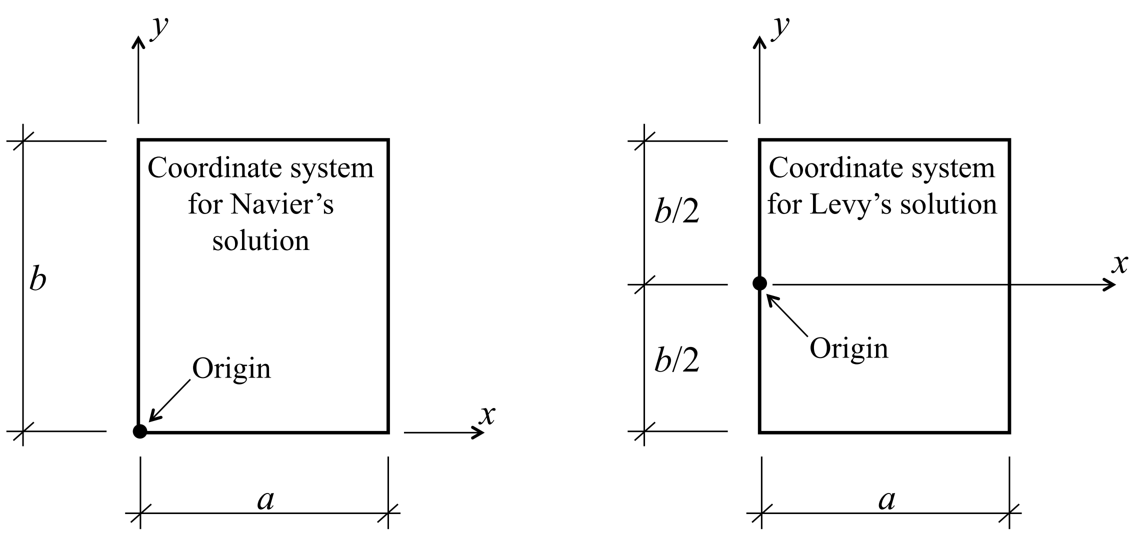

Figure 3: Coordinate systems for plate solutions.

Levy formulated a solution that first focuses on the x-direction span from x=0 to x=a consisting of a "homogeneous solution," w_h, and a "particular solution," w_p:

$$w(x,y)=\underbrace{\sum_{m=1,3,5\ldots}^{\infty} h_m(y)\cdot\sin\left(\frac{m\pi x}{a}\right)}_{w_h}+\underbrace{\frac{q(x,y)}{24D}\cdot\left(x^4-2ax^3+a^3x\right)}_{w_p} \tag{46}$$

where h_m is a function that depends on y only, and m must be odd due to symmetry. The particular solution, w_p, satisfies the differential equation in Eq. (31) and also the boundary conditions at the two edges x=0 and x=a, namely zero displacement and moment/curvature. Now $h_m(y)$ must be formulated such that it satisfies the homogeneous version of the differential equation in Eq. (31), and such that w=w_h+w_p satisfies the full differential equation and all boundary conditions. Applying those two conditions, and reformulating the particular solution as the series expansion

$$w_p=\frac{q}{24D}\cdot\left(x^4-2ax^3+a^3x\right)=\frac{4qa^4}{\pi^5 D}\sum_{m=1,3,5\ldots}^{\infty}\frac{1}{m^5}\sin\left(\frac{m\pi x}{a}\right) \tag{47}$$

yields the solution (Timoshenko and Woinowsky-Krieger 1959)

$$w(x,y)=\frac{4qa^4}{\pi^5 D}\sum_{m=1,3,5\ldots}^{\infty}\frac{1}{m^5}\left(\begin{array}{l} 1-\dfrac{\dfrac{m\pi b}{2a}\cdot\tanh\left(\dfrac{m\pi b}{2a}\right)+2}{2\cdot\cosh\left(\dfrac{m\pi b}{2a}\right)}\cosh\left(\dfrac{m\pi y}{a}\right)+ \\ \cdots+\dfrac{\left(\dfrac{m\pi b}{2a}\right)}{2\cdot\cosh\left(\dfrac{m\pi b}{2a}\right)}\cdot\dfrac{2y}{b}\cdot\sinh\left(\dfrac{m\pi y}{a}\right)\end{array}\right)\cdot\sin\left(\frac{m\pi x}{a}\right) \tag{48}$$

Assuming $b \geq a$ the maximum deflection at the middle of the plate, i.e., at x=a/2 and y=0, can be expressed relative to the maximum deflection of a comparable beam:

$$w_{\max}=\frac{5qa^4}{384D}-\frac{4qa^4}{\pi^5 D}\sum_{m=1,3,5\ldots}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{m^5}\cdot\frac{\left(\dfrac{m\pi x}{a}\right)\cdot\tanh\left(\dfrac{m\pi x}{a}\right)+2}{2\cdot\cosh\left(\dfrac{m\pi x}{a}\right)} \tag{49}$$

Edge Moments

The solutions presented above are for simply supported plates. Those solutions can be superimposed with the solution for a plate with distributed moments along the edges to enforce other boundary conditions. Using Levy's coordinate system in Figure 3, Timoshenko presents in Article 41 of his book on Plates and Shells (Timoshenko and Woinowsky-Krieger 1959) a solution for a simply supported plate subjected to uniformly distributed moment, M_0, about the x-axis along the edges y=±b/2:

$$w(x,y)=\frac{2M_0a^2}{\pi^3 D}\sum_{m=1,3,5...}^{\infty}\frac{1}{m^3\cdot\cosh\left(\frac{m\pi b}{2a}\right)}\cdot\gamma_m\cdot\sin\left(\frac{m\pi x}{a}\right) \tag{50}$$

where

$$\gamma_m=\frac{m\pi b}{2a}\cdot\tanh\left(\frac{m\pi b}{2a}\right)\cdot\cosh\left(\frac{m\pi y}{a}\right)-\frac{m\pi y}{a}\cdot\sinh\left(\frac{m\pi y}{a}\right) \tag{51}$$

A more general solution with varying intensity of the edge moments is

$$w(x,y)=\frac{a^2}{2\pi^2 D}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{m\pi x}{a}\right)}{m^2\cdot\cosh\left(\frac{m\pi b}{2a}\right)}\cdot E_m\cdot\gamma_m \tag{52}$$

where the edge moment, now varying in the x-direction, is expressed as

$$f(x)=\sum_{m=1,3,5...}^{\infty}E_m\cdot\sin\left(\frac{m\pi x}{a}\right) \tag{53}$$

Two Edges Clamped, Two Edges Simply Supported

For a uniformly loaded plate, simply supported along the two edges x=(0,a) and clamped along the two edges y=($\pm b$/2), Timoshenko finds the following edge moment factors to be substituted into Eq. (52):

$$E_m=\frac{4qa^2}{\pi^3 m^3}\cdot\frac{\frac{m\pi b}{2a}-\tanh\left(\frac{m\pi b}{2a}\right)\cdot\left(1+\frac{m\pi b}{2a}\cdot\tanh\left(\frac{m\pi b}{2a}\right)\right)}{\frac{m\pi b}{2a}-\tanh\left(\frac{m\pi b}{2a}\right)\cdot\left(\frac{m\pi b}{2a}\cdot\tanh\left(\frac{m\pi b}{2a}\right)-1\right)} \tag{54}$$

The total solution is the sum of Eq. (48) and Eq. (52). The solution for other boundary conditions can be obtained in a similar manner, although it can get a bit mathematically messy.

References

Timoshenko, S. P., and Woinowsky-Krieger, S. (1959). *Theory of Plates and Shells*. McGraw-Hill.