

Second Edition

Classical and Computational Solid Mechanics

Y C Fung • Pin Tong
Xiao Hong Chen

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Classical and Computational Solid Mechanics

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DEDICATION

To Luna, Conrad, and Brenda

Y. C. Fung

To my teacher Professor Fung and to Mrs. Fung who have been the constant source of caring, inspiration and guidance; to my parents who were the source of love and discipline; to my wife who has been the source of love, companionship and encouragement; to my children and grandchildren who have been the source of joy and love; to Denny who will be forever in my heart.

Pin Tong

To all my wonderful teachers and mentors who pass the torch of wisdom from generation to generation.

Xiaohong Chen

PREFACE TO THE SECOND EDITION

The objective of this book is to offer students of science and engineering a concise, general, and easy-to-understand account of some of the most important concepts and methods of classical and computational solid mechanics. The classical part is mainly a re-issue of Fung's *Foundations of Solid Mechanics*, with a major addition to the modern theories of plasticity, and a major revision of the theory of large elastic deformation with finite strains. The computational part consists of five new chapters, which focus on numerical methods to solve many major linear and nonlinear boundary-value problems of solid mechanics.

We hold the principle of easy-to-understand for the readers as an objective of our presentation. We believe that to be easily understood, the presentation must be precise, the definitions and hypotheses must be clear, the arguments must be concise and with sufficient details, and the conclusion has to be drawn very carefully. We strive to pay strict attention to these requirements. We believe that the method must be general and the notations should be unified. Hence, we presented the tensor analysis in general coordinates, but kept the indicial notations for tensors in the first twelve chapters. In [Chapters 13–22](#), however, the dyadic notations of tensors and the notations for matrix operations are used to shorten the formulas.

This book was written for engineers who invent and design things for human kind and want to use solid mechanics to help implement their designs and applications. It was written also for engineering scientists who enjoy solid mechanics as a discipline and would like to help develop and advance the subject further. It was further designed to serve those physical and natural scientists and biologists and bioengineers whose activities might be helped by classical and computational solid mechanics. For example, biologists are discovering that the functional behavior of cells depends on the stresses acting on the cell. It is widely recognized that the molecular mechanics of the cell must be developed as soon as possible.

Solid mechanics deals with deformation and motion of “solids.” The displacement that connects the instantaneous position of a particle to its position in an “original” state is of general interest. The preoccupation about particle displacements distinguishes solid mechanics from that of fluids.

This book begins with an introductory chapter containing a brief sketch of history, an outline of some prototypes of theories, and a description of some more complex features of solid mechanics. In [Chapter 2](#), an introduction to tensor analysis is given. The bulk of the text from [Chapters 3 to 12](#) is concerned with the classical theory of elasticity, but the discussion also includes the thermodynamics of solid, thermoelasticity and plasticity. [Chapters 13–16](#) extend the discussion to finite deformation theory, viscoelasticity, viscoplasticity, coupled thermal, mechanical and electric processes in thermodynamic nonequilibrium based on functional and/or state-variable approaches and to their applications to electro-thermo-viscoelastic/plastic problems. Fluid mechanics basically is excluded, but methods that are common to both fluid and solid mechanics are emphasized. Both dynamics and statics are treated; the concepts of wave propagation are introduced in an early stage. Variational calculus is emphasized since it provides a unified point of view and is useful in formulating approximate theories and computational methods. The large deflection theory of plates presented in the concluding section of [Chapter 13](#) illustrates the elegance of the general approach to the large deformation theory.

[Chapters 17 to 22](#) are devoted to computational solid mechanics to deal with linear, nonlinear, and inhomogeneous problems. [Chapter 17](#) develops the incremental theory in considerable detail. It is recognized that the incremental approach is the most practical approach. [Chapter 18](#) is devoted to numerical methods, with the finite element singled out for detailed discussion and its application to elasticity. [Chapter 19](#) presents the calculation methods based on the mixed and hybrid variational principles, illustrating the broadening of the computation power with less restrictive (or weaker) hypotheses in formulating the variational principle. [Chapter 20](#) deals with finite element methods of plates and shells, making the computation methods accessible to the analysis of the structures of aircraft, marine architectures, land vehicles, and shell-like structures in human beings, animals, plants, earth, and space. [Chapter 21](#) deals with finite element modeling of nonlinear elasticity, viscoelasticity, plasticity, viscoplasticity and creep. Finally the book concludes with [Chapter 22](#) on the Meshless Local Petrov–Galerkin and Eshelby–Atluri Methods, alternatives to the conventional finite element methods. Thus, a broad sweep of modern, advanced topics are covered. Since [Chapters 17–21](#) are independent of [Chapters 13–16](#), readers may skip the latter in the first reading.

Overall, this book lays emphasis on general methodology. It prepares the students to tackle new problems. However, as it was said in the original preface of the *Foundations of Solid Mechanics*, no single path can embrace the broad field of mechanics. As in mountain climbing, some routes are safe to travel, others more perilous; some may lead to the summit, others to different vistas of interest; some have popular claims, others are less traveled. In choosing a particular path for a tour through the field, one is influenced by the curriculum, the trends in literature, and the interest in engineering and science. Here, a particular way has been chosen to view some of the most beautiful vistas in classical and computational mechanics. In making this choice, we have aimed at straightforwardness and interest, and practical usefulness in the long run.

Holding the book to a reasonable length did not permit inclusion of many numerical examples, which have to be

supplemented through problems and references. Fortunately, there are many excellent references to meet this demand. We have presented an extensive bibliography in this book, but we suggest that the reader consult the review journal *Applied Mechanics Reviews* (AMR) published by the American Society of Mechanical Engineers International since 1947, for current information. The reader is referred to the periodic in-depth reviews of the literature in specific issues of AMR.

We are indebted to many authors and colleagues as acknowledged in the preface of the *Foundations of Solid Mechanics*. In the preparation of the present edition, we are especially indebted to Professors Satya Atluri and Theodore Pian. We would like to record our gratitude to many colleagues who wrote us to discuss various points and sent us errata in the *Foundations of Solid Mechanics* and *Classical and Computational Solid Mechanics*, especially to Drs. Satya N. Atluri, Pao-Show D. Cheng, Shun Cheng, Ellis H. Dill, Clive L. Dym, J. B. Haddow, Manohar P. Kamat, Hans Krumhaar, T. D. Leko, Howard A. Magrath, Sumio Murakami, Theodore Pian, R. S. Rivlin, William P. Rodden, Bertil Storakers, Howard J. White, Jr. and John C. Yao. We would also like to thank Professors Y. Ohashi, S. Murakami and N. Kamiya for translating the *Foundations* book into Japanese, Professors Oyuang Zhang, Ma Wen-Hua and Wang Kai-Fu for translating the *Foundations* book into Chinese.

We would like to take this opportunity to mention a few of editorial notes:

- (1) The bibliography is given at the end of the book.
- (2) Equations in each section are numbered sequentially. When referring to equations in other sections, we use the format (Sec. no:Eq. no), e.g. (5.6:4).
- (3) Formulas are a concise way of saying lots of things. The most important formulas are marked with a triangular star, ▲. They are worthy of being committed to memory.

Yuan-Cheng Fung
Pin Tong
Xiaohong Chen

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SUBJECT INDEX

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INTRODUCTION

Mechanics is the science of force and motion of matter. Solid mechanics is the science of force and motion of matter in the solid state. Physicists are of course interested in mechanics. The greatest advances in physics in the 20th century are identified with mechanics: the theory of relativity, quantum mechanics, and statistical mechanics. Chemists are interested in the mechanics of chemical reaction, the formation of molecular aggregates, the formation of crystals, or the creation of new materials with desirable properties, or polymerization of larger molecules, etc. Biologists are interested in biomechanics that relates structure to function at all hierarchical levels: from biomolecules to cells, tissues, organs, and individuals. Although a living cell is not a homogeneous continuum, it is a protein factory with internal machinery that moves and functions in an orderly way according to the laws of mechanics. Therefore, all scientists are interested in mechanics and mechanics is developed by scientists continuously.

Engineers, especially aeronautical, mechanical, civil, chemical, materials, biomedical, biotechnological, space, and structural engineers, are real developers and users of fluid and solid mechanics because of their professional needs. They design. They invent. They are concerned about the safety and economy of their products. They want to know the function of their products as precisely as possible. They want results fast. They experiment. They theorize. They test, compute, and validate. To them mechanics is a toy, a bread and butter, a feast or delicacy.

Engineering is quite different from science. Scientists try to understand nature. Engineers try to make things that do not exist in nature. Engineers stress invention. To embody an invention the engineer must put his idea in concrete terms, and design something that people can use. That something can be a device, a gadget, a material, a method, a computing program, an innovative experiment, a new solution to a problem, or an improvement on what is existing. Since a design has to be concrete, it must have its geometry, dimensions, and characteristic numbers. Engineers working on new designs often find that they do not have all the needed information due to insufficient scientific knowledge. Thus they study mathematics, physics, chemistry, biology and mechanics in order to add to the sciences relevant to their profession. Thus engineering sciences are born.

This book is written by engineering scientists, for engineering scientists, and this determines its style. The qualities we want are:

- Easy to read,
- Precise, concise, and practical,
- First priority on the formulation of problems,
- Presenting the classical results as gold standard, and
- Numerical approach as everyday tool to obtain solutions.

If the book is a banquet, this introductory chapter is the hors d'oeuvres.

1.1. HOOKE'S LAW

Historically, the notion of elasticity was first announced in 1676 by Robert Hooke (1635–1703) in the form of an anagram, *ceiiinosssttuv*. He explained it in 1678 as

Ut tensio sic vis,

or “the power of any springy body is in the same proportion with the extension.”¹

As stated in the original form, Hooke's law is not very clear. Our first task is to give it a precise expression. Historically, this was done in two different ways. The first way is to make use of the common notion of “springs,” and consider the load-deflection relationship. The second way is to state it as a tensor equation connecting the stress and strain. Although the second way is the proper way to start a general theory, the first, simpler and more restrictive, is not without interest. In this section, we develop the first alternative as a prototype of the theory of elasticity.

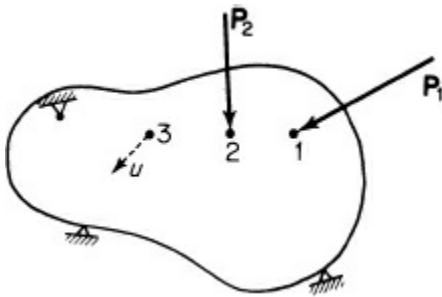


Fig. 1.1:1. Static equilibrium of a body under external forces.

Let us consider the static equilibrium state of a solid body under the action of external forces (Fig. 1.1:1). The body is supported in some manner so that at least three points are fixed in space. We make three basic hypotheses regarding the properties of the body under consideration.

(H1) *The body is continuous and remains continuous under the action of external forces.*

Under this hypothesis the atomistic structure of the body is ignored and the body is idealized into a geometrical copy in Euclidean space whose points are identified with the material particles of the body. Continuity is defined in mathematical sense as an isomorphism of the real number system. Neighboring points remain as neighbors under any loading condition. No cracks or holes may open up in the interior of the body under the action of external load.

A material satisfying this hypothesis is said to be a *continuum*. The study of the deformation or motion of a continuum under the action of forces is called the *continuum mechanics*.

To introduce the second hypothesis, let us consider the action of a set of forces on the body. Let every force be fixed in direction and in point of application, and let the magnitude of all the forces be increased or decreased together: always bearing the same ratio to each other. Let the forces be denoted by $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ and their magnitude by P_1, P_2, \dots, P_n . Then the ratios $P_1 : P_2 : \dots : P_n$ remain fixed. When such a set of forces is applied on the body, the body deforms. Let the displacement at an arbitrary point in an arbitrary direction be measured with respect to a rectangular Cartesian frame of reference fixed with the supports. Let this displacement be denoted by u . Then our second hypothesis is

(H2) *Hooke's law:*

$$(1) \quad u = a_1 P_1 + a_2 P_2 + \dots + a_n P_n,$$

where a_1, a_2, \dots, a_n are constants independent of the magnitude of P_1, P_2, \dots, P_n . The constants a_1, a_2, \dots, a_n depend, of course, on the location of the point at which the displacement component is measured and on the directions and points of application of the individual forces of the loading.

Hooke's law in the form (H2) is one that can be subjected readily to direct experimental examination.

To complete the formulation of the theory of elasticity, we need a third hypothesis:

(H3) *There exists a unique unstressed state of the body, to which the body returns whenever all the external forces are removed.*

A body satisfying these three hypotheses is called a *linear elastic solid*.

A number of deductions can be drawn from these assumptions. We shall list a few important ones.

(A) *Principle of superposition*

By a combination of (H2) and (H3), we can show that Eq. (1) is valid not only for systems of loads for which the ratios $P_1 : P_2 : \dots : P_n$ remain fixed as originally assumed, but also for an arbitrary set of loads $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$. In other words, Eq. (1) holds regardless of the order in which the loads are applied.

Proof. If a proof of the statement above can be established for an arbitrary pair of loads, then the general theorem can be proved by mathematical induction.

Let \mathbf{P}_1 and \mathbf{P}_2 (with magnitudes P_1 and P_2) be a pair of arbitrary loads acting at points 1 and 2, respectively. Let the deflection in a specific direction be measured at a point 3 (see Fig. 1.1:1). According to (H2), if \mathbf{P}_1 is applied alone, then at the point 3 a deflection $u_{31} = c_{31} P_1$ is produced. If \mathbf{P}_2 is applied alone, a deflection $u_{32} = c_{32} P_2$ is produced. If \mathbf{P}_1 and \mathbf{P}_2 are applied together, with the ratio $P_1 : P_2$ fixed, then according to (H2) the deflection can be written as

$$(a) \quad u_3 = c'_{31} P_1 + c'_{32} P_2.$$

The question arises whether $c'_{31} = c_{31}$, $c'_{32} = c_{32}$. The answer is affirmative, as can be shown as follows. After \mathbf{P}_1 and \mathbf{P}_2 are applied, we take away \mathbf{P}_1 . This produces a change in deflection, $-c''_{31}P_1$, and the total deflection becomes

$$(b) \quad u_3 = c'_{31}P_1 + c'_{32}P_2 - c''_{31}P_1.$$

Now only \mathbf{P}_2 acts on the body. Hence, upon unloading \mathbf{P}_2 we shall have

$$(c) \quad u_3 = c'_{31}P_1 + c'_{32}P_2 - c''_{31}P_1 - c_{32}l$$

Now all the loads are removed, and u_3 must vanish according to (H3). Rearranging terms, we have

$$(d) \quad (c'_{31} - c''_{31})P_1 = (c_{32} - c'_{32})P_2.$$

Since the only possible difference of c'_{31} and c''_{31} must be caused by the action of \mathbf{P}_2 , the difference $c'_{31} - c''_{31}$ can only be a function of P_2 (and not of P_1). Similarly, $c_{32} - c'_{32}$ can only be a function of P_1 . If we write Eq. (d) as

$$(e) \quad (c'_{31} - c''_{31})/P_2 = (c_{32} - c'_{32})/P_1,$$

then the left-hand side is a function of P_2 alone, and the right-hand side is a function of P_1 alone. Since P_1 and P_2 are arbitrary numbers, the only possibility for Eq. (e) to be valid is for both sides to be a constant k which is independent of both P_1 and P_2 . Hence,

$$(f) \quad c'_{32} = c_{32} - kP_1.$$

But a substitution of (f) into (a) yields

$$(g) \quad u_3 = c'_{31}P_1 + c_{32}P_2 - kP_1P_2.$$

The last term is nonlinear in P_1 , P_2 , and Eq. (g) will contradict (H2) unless k vanishes. Hence, $k = 0$ and $c'_{32} = c_{32}$. An analogous procedure shows $c'_{31} = c''_{31} = c_{31}$.

Thus the principle of superposition is established for one and two forces. An entirely similar procedure will show that if it is valid for m forces, it is also valid for $m + 1$ forces. Thus, the general theorem follows by mathematical induction.

Q.E.D.

The constants c_{31} , c_{32} , etc., are seen to be of significance in defining the elastic property of the solid body. They are called *influence coefficients* or, more specifically, *flexibility influence coefficients*.

(B) Corresponding forces and displacements and the unique meaning of the total work done by the forces.

Let us now consider a set of external forces $\mathbf{P}_1, \dots, \mathbf{P}_n$ acting on the body and define the set of displacements at the points of application and in the direction of the loads as the displacements "corresponding" to the forces at these points. The reactions at the points of support are considered as external forces exerted on the body and included in the set of forces.

Under the loads $\mathbf{P}_1, \dots, \mathbf{P}_n$, the corresponding displacements may be written as

$$(2) \quad \begin{aligned} u_1 &= c_{11}P_1 + c_{12}P_2 + \dots + c_{1n}P_n, \\ u_2 &= c_{21}P_1 + c_{22}P_2 + \dots + c_{2n}P_n, \\ &\dots \\ u_n &= c_{n1}P_1 + c_{n2}P_2 + \dots + c_{nn}P_n. \end{aligned}$$

If we multiply the first equation by P_1 , the second by P_2 , etc., and add, we obtain

$$(3) \quad \begin{aligned} P_1u_1 + P_2u_2 + \dots + P_nu_n &= c_{11}P_1^2 + c_{12}P_1P_2 + \dots + c_{1n}P_1P_n \\ &+ c_{21}P_1P_2 + c_{22}P_2^2 + \dots + c_{nn}P_n^2. \end{aligned}$$

The quantity above is independent of the order in which the loads are applied. It is the *total work done by the set of forces*.

(C) Maxwell's reciprocal relation

The influence coefficients for corresponding forces and displacements are symmetric.

(4)

$$c_{ij} = c_{ji} .$$

In other words, the displacement at a point i due to a unit load at another point j is equal to the displacement at j due to a unit load at i , provided that the displacements and forces "correspond," i.e., that they are measured in the same direction at each point.

The proof is simple. Consider two forces \mathbf{P}_1 and \mathbf{P}_2 (Fig. 1.1:1). When the forces are applied in the order $\mathbf{P}_1, \mathbf{P}_2$, the work done by the forces is easily seen to be

$$W = (c_{11}P_1^2 + c_{22}P_2^2)/2 + c_{12}P_1P_2 .$$

When the order of application of the forces is interchanged, the work done is

$$W' = (c_{22}P_2^2 + c_{11}P_1^2)/2 + c_{21}P_1P_2 .$$

But according to (B) above, $W = W'$ for arbitrary P_1, P_2 . Hence, $c_{12} = c_{21}$, and the theorem is proved.

(D) Betti-Rayleigh reciprocal theorem

Let a set of loads $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_n$ produce a set of corresponding displacements u_1, u_2, \dots, u_n . Let a second set of loads $\mathbf{P}'_1, \mathbf{P}'_2, \dots, \mathbf{P}'_n$, acting in the same directions and having the same points of application as those of the first, produce the corresponding displacements u'_1, u'_2, \dots, u'_n . Then

$$(5) \quad P_1u'_1 + P_2u'_2 + \dots + P_nu'_n = P'_1u_1 + P'_2u_2 + \dots + P'_nu_n .$$

In other words, in a linear elastic solid, the work done by a set of forces acting through the corresponding displacements produced by a second set of forces is equal to the work done by the second set of forces acting through the corresponding displacements produced by the first set of forces.

A straightforward proof is furnished by writing out the u_i and u'_i in terms of P_i and P'_i , ($i = 1, 2, \dots, n$), with appropriate influence coefficients, comparing the results on both sides of the equation, and utilizing the symmetry of the influence coefficients.

In the form of Eq. (5), the reciprocal theorem can be generalized to include moments and rotations as the corresponding *generalized forces* and *generalized displacements*. An illustration is given in Fig. 1.1:2. These theorems are very useful in practical applications.

(E) Strain energy

Further insight can be gained from the first law of thermodynamics. When a body is thermally isolated and thermal expansions are neglected the first law states that the work done on the body by the external forces in a certain time interval is equal to the increase in the kinetic energy and internal energy in the same interval. If the process is so slow that the kinetic energy can be ignored, the work done is seen to be equal to the change in internal energy.

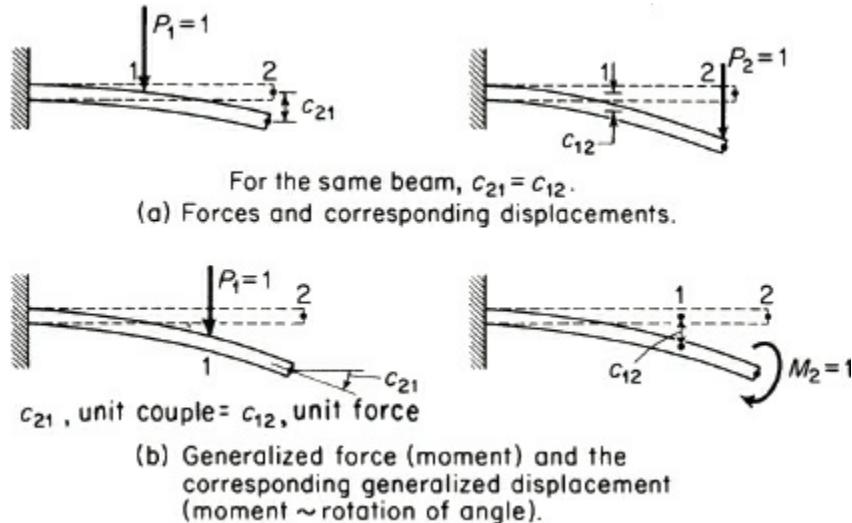


Fig. 1.1:2. Illustration of the reciprocal theorem.

If the internal energy is reckoned as zero in the unstressed state, the stored internal energy shall be called strain energy. Writing U for the strain energy, we have, from (3) and (4),

$$(6) \quad U = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n c_{ij} P_i P_j = \frac{1}{2} \sum_{i=1}^n c_{ii} P_i^2 + \frac{1}{2} \sum_{i \neq j} c_{ij} P_i P_j.$$

If we differentiate Eq. (6) with respect to P_i , we obtain

$$\partial U / \partial P_i = c_{ii} P_i + \sum_{j \neq i} c_{ij} P_j, \quad i = 1, 2, \dots, n.$$

But, the right-hand side is precisely u_i ; hence, we obtain

(F) *Castigliano's theorem*

$$(7) \quad \partial U / \partial P_i = u_i, \quad i = 1, 2, \dots, n.$$

In other words, if a set of loads P_1, \dots, P_n is applied on a perfectly elastic body as described above and the strain energy is expressed as a function of the set P_1, \dots, P_n , then the partial derivative of the strain energy, with respect to a particular load, gives the corresponding displacement at the point of application of that particular load in the direction of that load.

(G) *The principle of virtual work*

On the other hand, for a body in equilibrium under a set of external forces, the principle of virtual work can be applied to show that, *if the strain energy is expressed as a function of the corresponding displacements, then*

$$(8) \quad \partial U / \partial u_i = P_i, \quad i = 1, 2, \dots, n.$$

The proof consists in allowing a virtual displacement δu to take place in the body in such a manner that δu is continuous everywhere but vanishes at all points of loading except under P_i . Due to δu , the strain energy changes by an amount δU , while the virtual work done by the external forces is the product of P_i times the virtual displacement, i.e., $P_i \delta u_i$. According to the principle of virtual work, these two expressions are equal, $\delta U = P_i \delta u_i$. On rewriting it in the differential form, the theorem is established.

The important result (8) is established on the principle of virtual work as applied to a state of equilibrium under the additional assumption that a strain energy function that is a function of displacement exists. It is applicable also to elastic bodies that follow the nonlinear load-displacement relationship.

1.2. PLASTICITY AND VISCOELASTICITY

Most structural metals are nearly linear elastic under small strain, as measurements of load-displacement relationship reveal. When the deflection is small, the structure springs back to its original shape when you release the load. This is elasticity. When the deflection is sufficiently large, a permanent deformation will remain when the load is released. That is plasticity. There is a class of materials, for which the present state of deformation cannot be determined completely unless the entire history of loading is known. This class of materials is named *viscoelasticity*. Details of the two topics will be discussed in [Chapters 6 and 14](#).

1.3. VIBRATIONS

We know vibrations by experience while driving a car, flying an airplane, playing a musical instrument. The trees sway in the wind. A building shakes in an earthquake. Sometimes we want to know if a structure is safe in vibration. Sometimes we want to design a cushion that isolates an instrument from vibrations. A prototype of this kind of problem is shown in [Fig. 1.3:1\(a\)](#). A body with mass M is attached to an initially vertical massless spring, which has a spring constant k , and a damping constant c , and is “built-in” to a “ground” which moves horizontally with a displacement history $s(t)$. Let $x(t)$ denote the horizontal displacement of the mass and a dot over x or s denote a differentiation with respect to time. Then \ddot{x} is the acceleration of the body, $k(x - s)$ is the spring force acting on the body, and $c(\dot{x} - \dot{s})$ is the viscous damping force acting on the body. Newton’s second law requires that

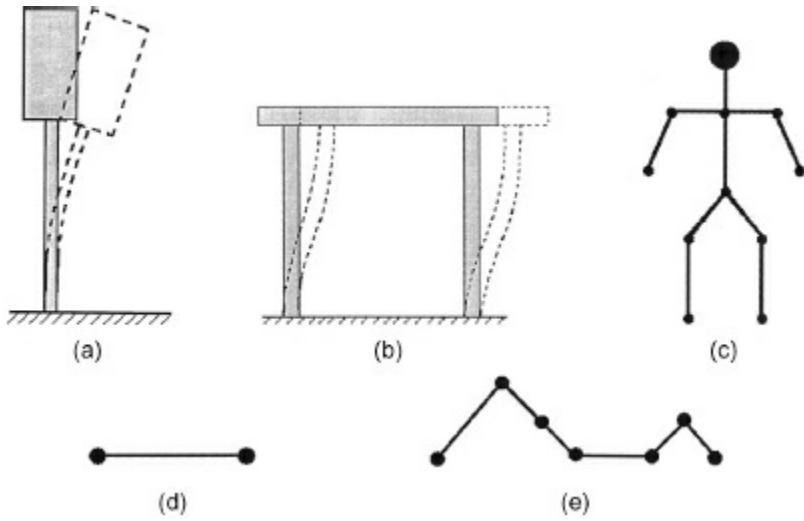


Fig. 1.3:1

$$(1) \quad M\ddot{x} + c(\dot{x} - \dot{s}) + k(x - s) = 0.$$

If we let

$$(2) \quad y = x - s,$$

represent the displacement of the body relative to the ground, then Eq. (1) may be written as

$$(3) \quad M\ddot{y} + c\dot{y} + ky = -M\ddot{s}.$$

The solution of Eq. (3) represents a forced vibration of a damped system. If the forcing function $M\ddot{s}(t)$ were zero, the equation

$$(4) \quad M\ddot{y} + c\dot{y} + ky = 0,$$

describes the free vibration of the damped system. If the damping constant c vanishes, then the free vibration of the undamped system is described by the equation

$$(5) \quad M\ddot{y} + ky = 0.$$

Equation (5) is satisfied by the solution

$$(6) \quad y = A \cos \omega t + B \sin \omega t,$$

in which A and B are arbitrary constants, and

$$(7) \quad \omega = \sqrt{k/M},$$

as can be verified by direct substitution of (6) into (5). When k and M are real values, ω is real. The motion $y(t)$ given by (6) is an oscillation of the body at a circular frequency of ω rad/sec. It can exist without external load. Hence ω is called the *natural frequency* of a *free vibration*. If the damping constant c were zero, and the forcing function \ddot{s} is periodic with the same frequency as the natural one, then the amplitude of the oscillation is unbounded, and we have the phenomenon of *resonance*.

The solution of Eq. (4) is an exponential function of time, since the derivative of an exponential function is another exponential function. Thus

$$(8) \quad y(t) = Ae^{\lambda t} \text{ implies } \dot{y}(t) = A\lambda e^{\lambda t}, \quad \ddot{y}(t) = A\lambda^2 e^{\lambda t}.$$

On substituting (8) into Eq. (4), we have

$$(9) \quad M\lambda^2 + c\lambda + k = 0.$$

If we write

$$(10) \quad k/M = \omega^2, \quad \varepsilon = c/(2\sqrt{kM}),$$

then the two roots of Eq. (9) may be written as λ_1 and λ_2 :

$$(11) \quad \lambda_1 = -\varepsilon\omega + i\omega\sqrt{1-\varepsilon^2}, \quad \lambda_2 = -\varepsilon\omega - i\omega\sqrt{1-\varepsilon^2}.$$

When k and M are real positive numbers, ω is real, and the solution of Eq. (9) under the initial conditions

$$(12) \quad y(0) = y_o, \quad \dot{y}(0) = \dot{y}_o \quad \text{when } t = 0.$$

is

$$(13) \quad y(t) = y_o e^{-\varepsilon\omega t} \cos(\omega\sqrt{1-\varepsilon^2}t) + \frac{\dot{y}_o + y_o\varepsilon\omega}{\omega\sqrt{1-\varepsilon^2}} e^{-\varepsilon\omega t} \sin(\omega\sqrt{1-\varepsilon^2}t).$$

Now, we can return to the solution of Eq. (3). A particular solution of Eq. (3) can be obtained by Laplace or Fourier transformation or other methods. By direct substitution, it can be verified that a particular solution satisfying the initial conditions $y(t) = \dot{y}(t) = 0$ at $t = 0$ is:

$$(14) \quad y(t, \omega, r) = -\frac{1}{\omega\sqrt{1-\varepsilon^2}} \int_0^t \ddot{s}(\xi) e^{-\varepsilon\omega(t-\xi)} \sin(\omega\sqrt{1-\varepsilon^2}(t-\xi)) d\xi.$$

The general solution of Eq. (3) is the sum of the functions given in Eqs. (13) and (14). From this solution we can examine the nature of the forced oscillation and its dependence on the parameters ω , ε , and the frequency spectrum of the forcing function $\ddot{s}(t)$.

The simple solution given by Eqs. (13) and (14) has important applications to the problems of isolation of delicate instruments in shipping, response of buildings to earthquake, impact of an airplane on landing, landing a robot on the moon. A frequently asked question is: what is the maximum absolute value of the displacement y , or the acceleration \ddot{y} as functions of the peak values of the ground displacement s , ground acceleration \ddot{s} , the time course of the ground motion, the system characteristic M , c , k , and the initial conditions y_o , \dot{y}_o i.e.,

$$(15) \quad \begin{aligned} \max |y(t)| / \max |s(t)| &= \text{function of } s(t), M, c, k, y_o, \dot{y}_o \\ \max |\ddot{y}(t)| / \max |\ddot{s}(t)| &= \text{functions of } s(t), M, c, k, y_o, \dot{y}_o. \end{aligned}$$

These functions are called *shock spectra*. Studies of shock spectra are important not only for technological applications, but also for mathematics. Note that although Eq. (3) is linear in $y(t)$, the shock spectra are nonlinear functions of the parameters of the system. Qualitatively, we notice that the most important characteristic time of the free oscillation is $1/\omega$ and that of the excitation $s(t)$ is the length of time it takes to rise from 0 to the peak value, t_r . We call t_r the *rise time of the signal*. Alluring to resonance, we see that a parameter of importance is the ratio of the two characteristic times $1/\omega$ and t_r . Hence the shock spectra are principally functions of ωt_r . See some references in the bibliography at the end of the book.

The picture shown in Fig. 1.3:1(b) may represent, in a very crude way, a concrete floor of a steel frame building during an earthquake. A system shown in Fig. 1.3:1(c) may represent an astronaut in flight. A dumbbell shown in Fig. 1.3:1(d) has been used to model a molecule. It is clear that fairly comprehensive models of dynamic systems can be devised by this approach, i.e., Fig. 1.3:1(e). When there are too many particles of mass and springs of elasticity, however, simplicity may be lost, and one turns naturally to the continuum approach outlined in the next section.

1.4. PROTOTYPE OF WAVE DYNAMICS

A wire with an infinite number of particles of mass connected together elastically is an obvious candidate for the continuum approach. As an example, consider a wire as shown in Fig. 1.4:1. Let an axis x run along the length of the wire, with an origin O chosen at the lower end. When the wire is loaded, each particle in the wire will be displaced longitudinally from its original position by an amount u . We shall consider only axial loading and assume that the plane cross sections remain plane, so that u is parallel to the x -axis and is a function of x . We shall assume further that u is infinitesimal, so that the strain in the wire is



Fig. 1.4:1

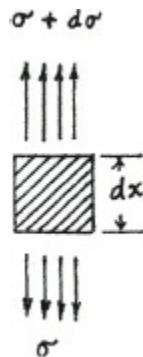


Fig. 1.4:2

$$(1) \quad e = \partial u / \partial x.$$

The change in cross-sectional dimensions of the wire due to the axial load will be ignored. The wire is assumed to be elastic, and is so thin that all stress components other than the axial may be neglected. Then the axial stress is given by Hooke's law

$$(2) \quad \sigma = Ee = E\partial u / \partial x,$$

in which E is a constant called Young's modulus of elasticity. Now consider an element of the wire of a small length dx (see Fig. 1.4:2). The force acting at the lower end is σA , where A is the cross-sectional area of the wire. The force acting at the upper end is $A(\sigma + d\sigma)$. For a continuous function $\sigma(x)$ the differential $d\sigma$ is equal to $(\partial\sigma/\partial x)dx$. The acceleration of the element, $\partial^2 u / \partial t^2$, is caused by the difference in axial forces. The mass of the element is ρAdx , with ρ being the density of the material. Hence, equating the mass times acceleration with the axial load $A d\sigma$, and canceling the nonvanishing factor Adx , we obtain the equation of motion

$$(3) \quad \rho \partial^2 u / \partial t^2 = \partial \sigma / \partial x.$$

A substitution of Eq. (2) yields the wave equation

$$(4) \quad c^2 \partial^2 u / \partial x^2 = \partial^2 u / \partial t^2,$$

in which the constant c is the wave speed:

$$(5) \quad c = \sqrt{E/\rho}.$$

The wave speed c is a characteristic constant of the material.

The general solution of Eq. (4) is

$$(6) \quad u = f(x - ct) + F(x + ct),$$

where f and F are two arbitrary functions. This can be verified by substituting Eq. (6) directly into Eq. (4). The functions $f(x - ct)$ and $F(x + ct)$ represent waves propagating in the positive and negative x directions, respectively. In either case we have

$$(7) \quad c \partial u / \partial x = \pm \partial u / \partial t = \pm v,$$

with $v = \partial u / \partial t$ denoting the particle velocity. A substitution of (7) and (5) into (2) yields the formula

(8)

$$\sigma = \pm Ev/c = \pm \rho cv,$$

where the – sign applies to a wave propagating in the positive x direction, and the + sign applies to a wave in the other direction. This result is remarkable. It says that the stress is equal to the product of the mass density of the material, the velocity of the sound wave, and the longitudinal velocity of the particles of the wire. This result has a very important application to an experiment initiated by John Hopkinson (1872) who hung a steel wire vertically from a ceiling, attached a stopper at the lower end as shown in Fig. 1.4:1, then dropped a massive weight down the wire from a measured height. When the weight struck the stopper, it stretches the wire suddenly. Hopkinson's intention was to measure the strength of the wire with this method. Using different weights dropped from different heights, he found that the minimum height from which a weight had to be dropped to break the wire was, within certain limits, almost independent of the magnitude of the weight, and the diameter of the wire.

Now, when different solid bodies are dropped from a given height, the velocity reached at any given time is independent of the weight. Thus Hopkinson explains his result on the basis of elastic wave propagation. Equation (8) shows that the stress is equal to ρcv . The speed of sound of longitudinal waves in the wire, given by Eq. (5) is 16,000 ft/sec for steel. The velocity v in the wire, however, is not necessarily the largest at the instant of impact at the lower end. To find v , we have to solve Eq. (4) with the boundary conditions of the Hopkinson experiment (Fig. 1.4:1), which has a clamped upper end, and a movable lower end:

$$(9) \quad u = 0 \quad \begin{aligned} &\text{at } x = L \text{ for all } t, \text{ and} \\ &\text{when } t \leq 0 \text{ for all } x, \end{aligned}$$

$$(10) \quad M(dV/dt) = A\sigma \quad \text{at } x = 0 \text{ for all } t,$$

$$(11) \quad V = V_0 \quad \text{at } x = 0 \text{ when } t = 0.$$

Here V_0 is the velocity of the stopper, and M is the combined mass of the dropped weight and the stopper. The mathematical problem is reduced to finding the arbitrary function $f(x - ct)$ and $F(x + ct)$ so that $u(x, t)$ given by Eq. (6) satisfies Eqs. (9)–(11).

Physically, the weight generated an elastic wave $u(x, t)$ in the wire initiated by a velocity V_0 at $x = 0$ when $t = 0$. The wave went up, reflected at the top, went down, reflected at the lower end by the moving weight, and so on. In John Hopkinson's test, a 27-foot long wire and weights ranging from 7 to 41 lb were used, and the absolute maximum of v and σ was reached at the top after a number of reflections. The stress picture was very complicated. Bertram Hopkinson (1905) repeated his father's experiment with a 1 lb weight impacting on the lower end of a 30-foot long No. 10 gauge wire weighing 1.3 lb. Nevertheless, as shown by G. I. Taylor (1946), the maximum tensile stress in Hopkinson's experiment did not occur at the 1st reflection, when the stress was $2\rho cV_0$, but at the 3rd reflection, i.e., the 2nd reflection at the top of the wire, when the tensile stress reached $2.15\rho cV_0$.

If the wire mass were negligible, then the wave velocity is very high and the system of Fig. 1.4:1 would become a mass/spring system of Sec. 1.3.

Thus waves and vibrations are different features of the same dynamic system. Waves deliver extremely important information in a solid body. They are used in geophysics to probe the earth, to study earthquakes and to find oil and gas reserves. In medicine the pulse waves in the artery and the sound waves in the lung are used for diagnosis. In music the piano tuner and violin player listen to the vibration characteristics of their instruments. Some aspects of the general theory are discussed in Chapter 9.

1.5. BIOMECHANICS

Historically, the theory of solid mechanics was first developed along the lines of the linearized theory of elasticity, then it was expanded to the linearized theory of viscoelasticity. Then plasticity came on the scene. The nonlinear theory of large deformation and finite strain was developed. Further expansion was made possible by computational methods. Many significant problems in material science, aerospace structures, metal and plastics industries, geophysics, planetary physics, thermo-nuclear reactors, etc., were solved. Following this trend of expansion, it is natural to consider biological problems.

Many mechanics scientists had contributed to the understanding of physiology, e.g.: Galileo Galilei (1564–1642), William Harvey (1578–1658), René Descartes (1596–1650), Giovanni Alfonso Borelli (1608–1679), Robert Boyle (1627–1691), and Robert Hooke (1635–1703) before Newton (1642–1727), and Leonhard Euler (1707–1783), Thomas Young (1773–1829), Jean Poiseuille (1797–1869), Hermann von Helmholtz (1821–1894) after Newton. In fact, the Greek book *On the Parts of Animals* written by Aristotle (384–322 BC) and the Chinese book *Nei Jing* (or *Internal Classic*) attributed to Huangti (Yellow Emperor), but was believed to be written by anonymous authors in the Warring Period (472–221 BC), contain many concepts of biomechanics.

At the time when the manuscript of the first edition of this book was prepared, the molecule that is responsible for the genetics of the cells had been identified as DNA, and the double helix structure of DNA had been discovered. The beginning of a new age of understanding biology in terms of chemistry, physics and mechanics was recognized by many people.

Biomechanics has the following salient features:

(1) Material Constitution

Every living organism has a solid structure that gives it a unique shape and size, and an internal fluid flow that transports materials and keep the organism alive. Hence biosolid mechanics is inseparable from biofluid mechanics. The cells make new materials. Hence the composition, structure, and ultrastructure of biomaterials change dynamically.

(2) The Constitutive Equations

Blood is a non-Newtonian fluid. Synovial fluid in the knee joint, and body fluid in the abdominal cavity are also non-Newtonian. The bone obeys Hooke's law. The blood vessel, the skin and most other soft tissues in the body do not obey Hooke's law. We must know the constitutive equations of biological tissues, whose determination was a major task of the early biomechanics. When attention is focused on cells, the determination of the constitutive equations of DNA and other molecules becomes a primary task.

(3) Growth and Remodeling of Living Tissues Under Stress

Tissue is made of cells and extracellular matrices. The cell division, growth, hypertrophy, movement, or death are influenced by the cell geometry and the stress and strain in the cell, as well as by the chemical and physical environment. The tissue is, therefore, a changing, living entity. The determination of the growth law is a major task in biology.

(4) The Existence of Residual Stress

Organs such as the blood vessel, heart, esophagus and intestine have large residual strain and stress at the no-load condition. The residual strain changes in life because any birth or death of a cell in a continuum and the building and resorption of extracellular matrix by the cells create residual strains in the continuum. The changes of residual stress and strain cause changes of the stress state *in vivo*. The growth law and the residual stress are coupled.

(5) Concern about the Hierarchy of Sizes

In biology the hierarchy of sizes dominates the scene. Consider human. In studying gait, posture and sports, the length scale of interest is that of the whole body. In the study of the hemodynamics of the heart valves, a characteristic length is the size of the left ventricle. For coronary atherosclerosis studies, a characteristic dimension for hemodynamics in the vessel is the diameter of the coronary arteries, that for the shear stress on the vessel wall is the thickness of the endothelial cell, in the micrometers range, that for the molecular mechanism in the cells must be in the nanometers range. For microcirculation, features must be measured in the scale of the diameters of the red blood cells and the capillary blood vessels. In studying heart muscle contraction we must consider features in length scale of the sarcomere. In considering gene therapy we must think of the functions of the DNA molecule. In determining the active transport phenomenon we must be concerned with ion channels in the cell membrane. At each level of the hierarchy, there are important problems to be solved. Each hierarchy has its own appropriate mechanics. Integration of all the hierarchies is the objective of physiological studies.

(6) Perspectives

Life is motion. The science of motion is mechanics. Hence biology needs mechanics. Molecular biology needs molecular mechanics. Cell biology needs cell mechanics. Tissue biology needs tissue mechanics. Organ biology needs organ mechanics. Medicine, surgery, injury prevention, injury treatment, rehabilitation, and sports need mechanics. Solid and fluid mechanics have much to offer to biology. Yet, still, biomechanics seems to be a satellite, not in the main streams of biology and mechanics. Why? The reason probably lies in certain axiomatic differences between biology and mechanics. Classical continuum mechanics identifies a continuum as an isomorphism of the real number system. Biology studies molecules, cells, tissues, organs, and individuals in an orderly manner. The hierarchies are well defined. How can a biological entity be identified as a continuous body? The answer must lie in the clear recognition of the hierarchical structure. In each hierarchy, one must select a proper length scale as the starting point. No question with a characteristic length smaller than that selected dimension can be discussed. Then, in that hierarchy, one can identify a continuum to study it. This is the first axiom of biomechanics.

Secondly, solids of the classical solid mechanics have fixed structures and materials, an unchanging zero-stress state, and a set of fixed constitutive equations for the materials. Biomechanics, however, has to deal with DNA-controlled changes of materials, structures, zero-stress state, and constitutive equations. All these can be identified phenomenologically by suitably designed experiments. This is the second axiom of biomechanics.

Mechanics and biology can be unified only if these axiomatic differences are recognized. Knowing the theoretical structure of biomechanics, we can then develop classical and computational methods to cover it. Then biology becomes a new source of fresh water to irrigate our field of mechanics.

The present book will not deal with biomechanics any further. Some references are given in the Bibliography at the end of the book.

1.6. HISTORICAL REMARKS

The best-known constitutive equation for a solid is Hooke's law of linear elasticity, which was discovered by Robert Hooke in 1660. This law furnishes the foundation for the mathematical theory of elasticity. By 1821, L. M. Navier (1785–1836) had succeeded in formulating the general equations of the three-dimensional theory of elasticity. All questions of the small strain of elastic bodies were thus reduced to a matter of mathematical calculation. In the same year, 1821, Fresnel (1788–1827) announced his wave theory of light. The concept of transverse oscillations through an elastic medium attracted the attention of Cauchy and Poisson. Augustin Cauchy (1789–1857) developed the concept of stress and strain, and formulated the linear stress-strain relationship that is now called Hooke's law. Simeon Poisson (1781–1840) developed a molecular theory of elasticity and arrived at the same equation as Navier's. Both Navier and Poisson based their analysis on Newtonian conception of the constitution of bodies, and assumed certain laws of intermolecular forces. Cauchy's general reasoning, however, made no use of the hypothesis of material particles. In the ensuing years, with the contributions of G. Green (1793–1841), G. Stokes (1819–1903), Lord Kelvin (1824–1907) and others, the mathematical theory was established. The fundamental questions of continuum mechanics have received renewed attention through the efforts of C. Truesdell, R. Rivlin, W. Noll, J. Erikson, A. Green and others in theories of finite strain and nonlinear constitutive equations. Another focus of development is in the areas of micron and submicron-structures in which strain gradient effects can become dominant. In biomechanical field, the study of the growth and remodeling of living tissues will lead continuum mechanics to another plateau. New developments in continuum mechanics since the late 1940's have been truly remarkable.

The great impact of solid mechanics on civilization, however, is felt through its application to technology. In this respect a fine tradition was established by early masters. Galileo Galilei (1564–1642) considered the question of strength of beams and columns. J. Bernoulli (1654–1705) introduced the simple beam theory. L. Euler (1707–1783) gave the column formula. C. Coulomb (1736–1806) considered the failure criterion. J. Lagrange (1736–1813) gave the equation that governs the bending and vibration of plates. Navier and Poisson gave numerous applications of their general theory to special problems. The best example was set by de Saint-Venant (1797–1886), whose general solution of the problems of torsion and bending of prismatical bars is of great importance in engineering. He solved these problems with completeness that included many numerical coefficients and graphical presentations.

From this auspicious beginning, mechanics have developed into the mainstay of our civilization. Physics and chemistry are dominated by the theory of relativity, quantum mechanics, quantum electrodynamics. Geophysics, meteorology, and oceanography rest on the mechanics of seismic waves, atmospheric and ocean currents, and acoustic waves. Airplanes, ships, rockets, spacecraft, bathyscaphs, automobiles, trains, rails, highways, buildings, internal combustion engines, jet engines, are designed, constructed, and maintained on the basis of fluid and solid mechanics. New materials are invented for their mechanical properties. Artificial heart valves, hearts, kidneys, limbs, skin, pacemakers are mechanical marvels. Fundamental mechanics developed side by side with the developments in science and technology. In the first half of the 20th century, advancements were mainly concentrated on analytical solutions of the differential equations. In the second half of the 20th century, computational methods took center stage. These trends are reflected in this book.

PROBLEMS

1.1. Prove that one can generalize Hooke's law to deal with moments and angles of rotations by considering a concentrated couple as the limiting case of two equal and opposite forces approaching each other but maintaining a constant moment.

1.2. A pin-jointed truss is shown in Fig. P1.2. Every member of the truss is made of the same steel and has the same cross-sectional area. Find the tension or compression in every member when a load P is applied at the point shown in the figure. Note: A joint is said to be a *pin joint* if no moment can be transmitted across it. This problem is statically determinate.

1.3. Find the vertical deflections at a , b , c of the truss in Fig. P1.2 due to a load P at point b . Assume that P is sufficiently small so that the truss remains linear elastic. Use the fact that for a single uniform bar in tension the total change in length of this bar is given by PL/AE , where P is the load in the bar, L is the bar length, A is the cross-sectional area, and E is the Young's modulus of the material. For structural steels E is about 3×10^7 lb/sq in., or 20.7×10^7 kPa. Hint: Use Castigliano's theorem.

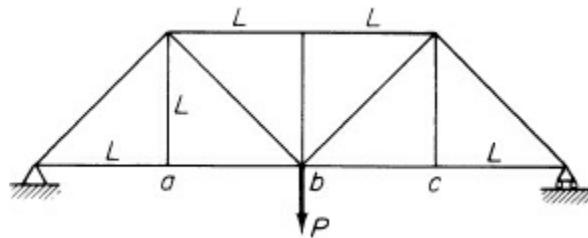


Fig. P1.2

1.4. A “rigid” frame of structural steel shown in Fig. P1.4 is acted on by a horizontal force P . Find the reactions at the points of support a , and d . The cross-sectional area A and the bending rigidity constant EI of the various members are shown in the figure.

By a “rigid” frame is meant a frame whose joints are rigid; e.g., welded. Thus, when we say that corner b is a rigid joint, we mean that the two members meeting at b will retain the same angle at that corner under any loading. The frame under loading will deform. It is useful to sketch the deflection line of the frame. Try to locate the points of inflection on the members. The points of *inflection* are points where the bending moments vanish.

The reactions to be found are the vertical force V , the horizontal force H , and the bending moment M , at each support.

Use engineering beam theory for this problem. In this theory, the change of curvature of the beam is M/EI , where M is the local bending moment and EI is the local bending rigidity. The strain energy per unit length due to bending is, therefore, $M^2/2EI$. The strain energy per unit length due to a tensile force P is $P^2/2EA$. The strain energy due to transverse shear is negligible in comparison with that due to bending.

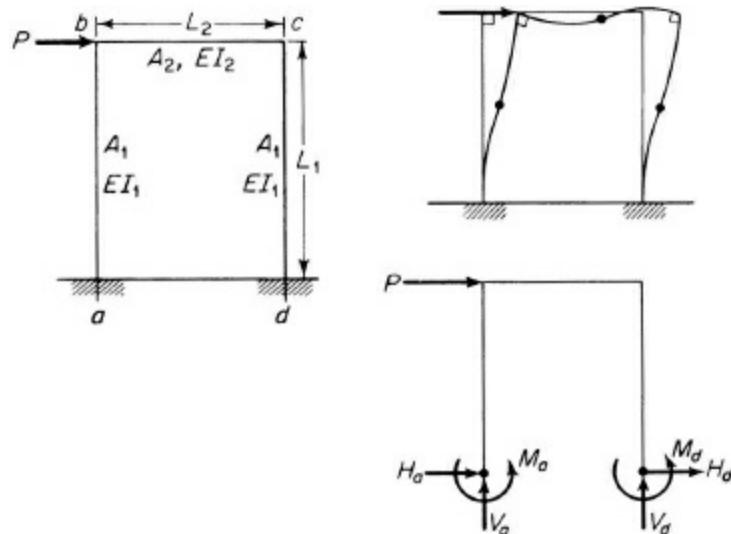


Fig. P1.4

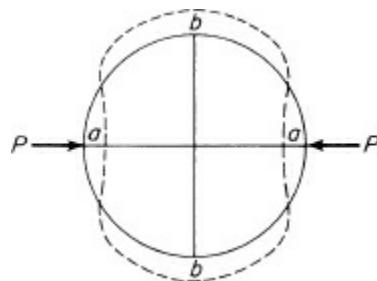


Fig. P1.5

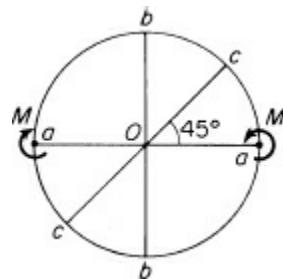


Fig. P1.6

1.5. A circular ring of uniform linear elastic material and uniform cross section is loaded by a pair of equal and opposite forces at the ends of a diameter (Fig. P1.5). Find the change of diameters aa and bb . Note: This is a statically indeterminate problem. You have to determine the bending moment distribution in the ring.

1.6. Compare the changes in diameters aa , bb , and cc when a circular ring of uniform linear elastic material and uniform cross section is subjected to a pair of bending moments at the ends of a diameter (Fig. P1.6).

1.7. Begg's Deformeter: G. E. Beggs used an experimental model to determine the reactions of a statically indeterminate structure. For example, to determine the horizontal reaction H at the right-hand support b of an elastic arch under the load P , he imposes at b a small displacement δ in the horizontal direction and measures at P the deflection δ' in the direction corresponding with P , while preventing the vertical displacement and rotation of the end b , Fig. P1.7. Show that $H = -P\delta'/\delta$. (Use the reciprocal theorem.) [G. E. Beggs, *J. Franklin Institute*, 203 (1927), pp. 375–386.]

1.8. A simply supported, thin elastic beam of variable cross section with bending rigidity $EI(x)$ rests on an elastic foundation with spring constant k and is loaded by a distributed lateral load of intensity $p(x)$ per unit length (Fig. P1.8). Find an approximate expression for the deflection curve by assuming that it can be represented with sufficient accuracy by the expression

$$u(x) = \sum_{n=1}^N a_n \sin(n\pi x/L).$$

Note: This expression satisfies the end conditions for arbitrary coefficients a_n . Use the minimum potential energy theorem.

1.9. Consider a uniform cantilever beam clamped at $x = 0$ (Fig. P1.9). According to Bernoulli–Euler theory of beams, the differential equation governing the deflection of the beam is

$$d^2w/dx^2 = M/(EI)$$

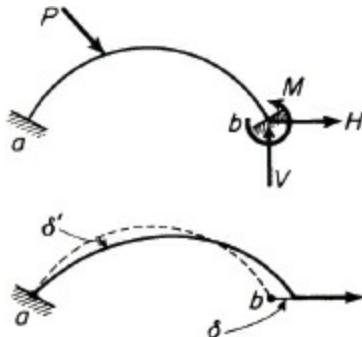


Fig. P1.7

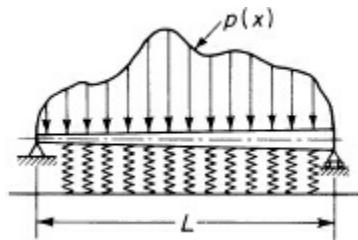


Fig. P1.8

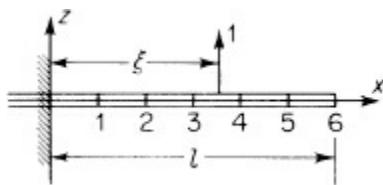


Fig. P1.9

where w is the deflection parallel to the z -axis and M is the bending moment at station x . This equation is valid if the beam is straight and slender and if the loading acts in a plane containing a principal axis of all cross sections of the beam, in which case the deflection w occurs also in that plane.

Use this differential equation and the boundary conditions:

At clamped end: $w = dw/dx = 0$.

At free end: $EI d^2w/dx^2 = EI d^3w/dx^3 = 0$.

Derive the deflection curve of the beam when it is loaded by a unit force located at $x = \xi$.

Ans. For $0 \leq x \leq \xi \leq l$: $w = x^2(3\xi - x)/(6EI)$.

For $0 \leq \xi \leq x \leq l$: $w = \xi^2(3x - \xi)/(6EI)$.

The solution fits the definition of influence coefficients and is known as an influence function.

1.10. Let the beam of Problem 1.9 be divided into six equidistant sections as marked in Fig. P1.9. Compute the influence coefficients c_{ij} , where $i, j = 1, 2, \dots, 6$.

¹Edme Mariotte enunciated the same law independently in 1680.

2

TENSOR ANALYSIS

In attempting to develop the theories outlined in the previous chapter rigorously and succinctly, we shall first learn to use the powerful tool of tensor calculus. At first sight, the mathematical analysis may seem involved, but a little study will soon reveal its simplicity.

2.1. NOTATION AND SUMMATION CONVENTION

Let us begin with the matter of notation. In tensor analysis one makes extensive use of indices. A set of n variables x_1, x_2, \dots, x_n is usually denoted as x_i , $i = 1, \dots, n$. A set of n variables y^1, y^2, \dots, y^n is denoted by y^i , $i = 1, \dots, n$. We emphasize that y^1, y^2, \dots, y^n are n independent variables and *not* the first n powers of the variable y .

Consider an equation describing a plane in a three-dimensional space x^1, x^2, x^3 ,

$$(1) \quad a_1x^1 + a_2x^2 + a_3x^3 = p,$$

where a_i and p are constants. This equation can be written as

$$(2) \quad \sum_{i=1}^3 a_i x^i = p.$$

However, we shall introduce the *summation convention* and write the equation above in the simple form

$$(3) \quad a_i x^i = p.$$

The convention is as follows: *The repetition of an index (whether superscript or subscript) in a term will denote a summation with respect to that index over its range*. The range of an index i is the set of n integer values 1 to n . A lower index i , as in a_i , is called a *subscript*, and upper index i , as in x^i , is called a *superscript*. An index that is summed over is called a *dummy index*; one that is not summed out is called a *free index*.

Since a dummy index just indicates summation, it is immaterial which symbol is used. Thus, $a_i x^i$ may be replaced by $a_j x^j$, etc. This is analogous to the dummy variable in an integral

$$\int_a^b f(x) dx = \int_a^b f(y) dy.$$

The use of index and summation convention may be illustrated by other examples. Consider a unit vector v in a three-dimensional Euclidean space with rectangular Cartesian coordinates x , y , and z . Let the direction cosines α_i be defined as

$$\alpha_1 = \cos(\nu, x), \quad \alpha_2 = \cos(\nu, y), \quad \alpha_3 = \cos(\nu, z),$$

where (ν, x) denotes the angle between v and the x -axis, etc. The set of numbers α_i ($i = 1, 2, 3$) represents the projections of the unit vector on the coordinate axes. The fact that the length of the vector is unity is expressed by the equation

$$(\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 = 1,$$

or simply

$$(4) \quad \alpha_i \alpha_i = 1.$$

As a further illustration, consider a line element (dx, dy, dz) in a three-dimensional Euclidean space with rectangular Cartesian coordinates x , y , z . The square of the length of the line element is

$$(5) \quad ds^2 = dx^2 + dy^2 + dz^2.$$

If we define

$$(6) \quad dx^1 = dx, \quad dx^2 = dy, \quad dx^3 = dz,$$

and

$$(7) \quad \begin{aligned} \delta_{11} &= \delta_{22} = \delta_{33} = 1, \\ \delta_{12} &= \delta_{21} = \delta_{13} = \delta_{31} = \delta_{23} = \delta_{32} = 0. \end{aligned}$$

Then (5) may be written as

$$(8) \quad ds^2 = \delta_{ij} dx^i dx^j,$$

with the understanding that the range of the indices i and j is 1 to 3. Note that there are two summations in the expression above, one over i and one over j . The symbol δ_{ij} as defined in (7) is called the *Kronecker delta*.

The following determinant illustrates another application

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

If we denote the general term in the determinant by a_{ij} and write the determinant as $|a_{ij}|$, then the equation above can be written as

$$(9) \quad |a_{ij}| = e_{rst} a_{r1} a_{s2} a_{t3},$$

where e_{rst} , the *permutation symbol*, is defined by the equations

$$(10) \quad \begin{aligned} e_{111} &= e_{222} = e_{333} = e_{112} = e_{113} = e_{221} = e_{223} = e_{331} = e_{332} = 0, \\ e_{123} &= e_{231} = e_{321} = 1 \quad e_{213} = e_{321} = e_{231} = -1. \end{aligned}$$

In other words, e_{ijk} vanishes whenever the values of any two indices coincide; $e_{ijk} = 1$ when the subscripts permute like 1, 2, 3; and $e_{ijk} = -1$ otherwise.

The Kronecker delta and the permutation symbol are very important quantities which will appear again and again in this book. They are connected by the identity

$$(11) \quad e_{ijk} e_{ist} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks}.$$

This $e\text{-}\delta$ identity is used sufficiently frequently to warrant special attention here. It can be verified by actual trial.

Finally, we shall extend the summation convention to differentiation formulas. Let $f(x^1, x^2, \dots, x^n)$ be a function of n variables x^1, x^2, \dots, x^n . Then its differential shall be written as

$$(12) \quad df = (\partial f / \partial x^i) dx^i.$$

The Kronecker delta and the permutation symbol play important roles in vector or tensor operations in rectangular Cartesian coordinate system. The dot product of two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is a scalar

$$c = x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

which can be written as

$$(13) \quad c = \delta_{ij} x_i y_j = x_i y_i.$$

The vector product of two vectors x and y is the vector $z = x \times y$ whose three components are

$$(14) \quad z_1 = x_2 y_3 - x_3 y_2, \quad z_2 = x_3 y_1 - x_1 y_3, \quad z_3 = x_1 y_2 - x_2 y_1.$$

This can be shortened by writing

$$(15) \quad z_i = e_{ijk} x_j y_k.$$

2.2. COORDINATE TRANSFORMATION

The central point of view of tensor analysis is to study the change of the components of a quantity such as a vector with respect to coordinate transformations.

Let a set of independent variables x_1, x_2, x_3 denote the coordinates of a point in a frame of reference. A transformation from x_1, x_2, x_3 to a set of new variables $\bar{x}_1, \bar{x}_2, \bar{x}_3$ through the equations

$$(1) \quad \bar{x}_i = f_i(x_1, x_2, x_3), \quad i = 1, 2, 3,$$

specifies a transformation of coordinates. The inverse transformation

$$(2) \quad x_i = g_i(\bar{x}_1, \bar{x}_2, \bar{x}_3) \quad i = 1, 2, 3,$$

proceeds in the reverse direction. In order to insure that such a transformation is reversible and in one-to-one correspondence in a certain region R of the variables (x_1, x_2, x_3) , i.e., in order that each set of numbers (x_1, x_2, x_3) defines a unique set of numbers $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, for (x_1, x_2, x_3) in the region R , and vice versa, it is sufficient to meet the following conditions:

- (a) The functions f_i are single-valued, continuous, and possess continuous first partial derivatives in the region R , and
- (b) The *Jacobian determinant* $J = |\partial \bar{x}_i / \partial x_j|$ does not vanish at any point of the region R , i.e.,

$$(3) \quad J = \begin{vmatrix} \frac{\partial \bar{x}_1}{\partial x_1} & \frac{\partial \bar{x}_1}{\partial x_2} & \frac{\partial \bar{x}_1}{\partial x_3} \\ \frac{\partial \bar{x}_2}{\partial x_1} & \frac{\partial \bar{x}_2}{\partial x_2} & \frac{\partial \bar{x}_2}{\partial x_3} \\ \frac{\partial \bar{x}_3}{\partial x_1} & \frac{\partial \bar{x}_3}{\partial x_2} & \frac{\partial \bar{x}_3}{\partial x_3} \end{vmatrix} \neq 0 \quad \text{in } R.$$

Coordinate transformations with the properties (a) and (b) named above are called *admissible transformations*. If the Jacobian is positive everywhere, then a right-handed set of coordinates is transformed into another right-handed set, and the transformation is said to be *proper*. If the Jacobian is negative everywhere, a right-handed set of coordinates is transformed into a left-handed one, and the transformation is said to be *improper*. *In this book, we shall tacitly assume that our transformations are admissible and proper.*

Problem 2.1. Show that, when i, j, k range over 1, 2, 3,

$$(a) \delta_{ij}\delta_{ij} = 3 \quad (b) e_{ijk}e_{jki} = 6 \quad (c) e_{ijk}A_jA_k = 0 \quad (d) \delta_{ij}\delta_{jk} = \delta_{ik}$$

Problem 2.2. Verify the following identity connecting three arbitrary vectors by means of the e - δ identity.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}.$$

Note: The last equation is well known in vector analysis. After identifying the quantities involved as Cartesian tensors, this verification may be construed as a proof of the e - δ identity.

2.3. EUCLIDEAN METRIC TENSOR

The first thing we must know about any coordinate system is how to measure length in that reference system. This information is given by the metric tensor.

Consider a three-dimensional Euclidean space, with the range of all indices 1, 2, 3. Let

$$(1) \quad \theta_i = \theta_i(x_1, x_2, x_3),$$

be an admissible transformation of coordinates from the rectangular Cartesian (in honor of Cartesius, i.e., Descartes) coordinates x_1, x_2, x_3 to some general coordinates $\theta_1, \theta_2, \theta_3$. The inverse transformation

$$(2) \quad x_i = x_i(\theta_1, \theta_2, \theta_3),$$

is assumed to exist, and the point (x_1, x_2, x_3) and $(\theta_1, \theta_2, \theta_3)$ are in one-to-one correspondence.

Consider a line element with three components given by the differentials dx^1, dx^2, dx^3 . Since the coordinates x_1, x_2, x_3 are assumed to be rectangular Cartesian, the length of the element ds is determined by Pythagoras' rule:

$$(3) \blacktrianglequad ds^2 = dx^i dx^i = \delta_{ij} dx^i dx^j.$$

Here ds^2 is the square of ds , not ds with a superscript 2. When a coordinate transformation (1) is effected, we obtain from Eq. (2), according to the ordinary rules of differentiation,

$$(4) \quad dx^i = (\partial x_i / \partial \theta_k) d\theta^k.$$

Substituting (4) into (3), we have

$$ds^2 = (\partial x_i / \partial \theta_k) (\partial x_i / \partial \theta_m) d\theta^k d\theta^m.$$

If we define the functions $g_{km}(\theta_1, \theta_2, \theta_3)$ by

$$(5) \blacktrianglequad g_{km}(\theta_1, \theta_2, \theta_3) = (\partial x_i / \partial \theta_k) (\partial x_i / \partial \theta_m),$$

then the square of the line element in the general $\theta_1, \theta_2, \theta_3$ coordinates takes the form

$$(6) \blacktrianglequad ds^2 = g_{km} d\theta^k d\theta^m.$$

This, of course, stands for

$$(7) \quad \begin{aligned} ds^2 &= g_{11}(d\theta^1)^2 + g_{12}d\theta^1 d\theta^2 + g_{13}d\theta^1 d\theta^3 \\ &+ g_{21}d\theta^1 d\theta^2 + g_{22}(d\theta^2)^2 + g_{23}d\theta^2 d\theta^3 \\ &+ g_{31}d\theta^1 d\theta^3 + g_{32}d\theta^2 d\theta^3 + g_{33}(d\theta^3)^2. \end{aligned}$$

It is apparent from (5) that

$$(8) \blacktrianglequad g_{km} = g_{mk} \quad \text{for each } k \text{ and } m,$$

so the functions g_{km} are symmetric in k and m . The functions g_{km} are called the components of the *Euclidean metric tensor* in the coordinate system $\theta_1, \theta_2, \theta_3$.

Let $\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$ be another general coordinate system. Let

$$(9) \quad \theta_i = \theta_i(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3),$$

be the transformation of coordinates from $\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$ to $\theta_1, \theta_2, \theta_3$. Now

$$(10) \quad d\theta^k = (\partial \theta_k / \partial \bar{\theta}_l) d\bar{\theta}^l.$$

Hence, from (6),

$$(11) \quad ds^2 = g_{km} (\partial \theta_k / \partial \bar{\theta}_l) (\partial \theta_m / \partial \bar{\theta}_n) d\bar{\theta}^l d\bar{\theta}^n.$$

If we define

$$(12) \quad \bar{g}_{ln}(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3) = g_{km}(\theta_1, \theta_2, \theta_3) (\partial \theta_k / \partial \bar{\theta}_l) (\partial \theta_m / \partial \bar{\theta}_n),$$

then (11) assumes a form which is the same as (3) or (6):

$$(13) \quad ds^2 = \bar{g}_{ln} d\bar{\theta}^l d\bar{\theta}^n.$$

Accordingly, we call \bar{g}_{ln} the components of the Euclidean metric tensor in the coordinate system $\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$.

The quadratic differential forms (3), (6), or (13) are of fundamental importance since they define the length of any line element in general coordinate systems. We conclude that if $\theta_1, \theta_2, \theta_3$ and $\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$ are two sets of general coordinates, then Euclidean metric tensors $g_{km}(\theta_1, \theta_2, \theta_3)$ and $\bar{g}_{km}(\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3)$ are related by means of the law of transformation (12).

The law of transformation of the components of a quantity with respect to coordinate transformation is an important property of that quantity. In the following section, we shall see that a quantity shall be called a tensor if and only if it follows certain specific laws of transformation.

All the results above apply as well to the plane (a two-dimensional Euclidean space), as can be easily verified by changing the range of all indices to 1, 2.

PROBLEMS

2.3. Find the components of the Euclidean metric tensor in plane polar coordinates ($\theta_1 = r$, $\theta_2 = \theta$; see Fig. P2.3) and the corresponding expression for the length of a line element.

Ans. Let x_1, x_2 be a set of rectangular Cartesian coordinates.

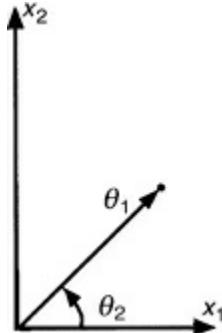


Fig. P2.3

Then

$$x_1 = \theta_1 \cos \theta_2, \quad \theta_1 = \sqrt{(x_1)^2 + (x_2)^2},$$

$$x_2 = \theta_1 \sin \theta_2, \quad \theta_2 = \sin^{-1} \left(\frac{x_2}{\sqrt{(x_1)^2 + (x_2)^2}} \right),$$

$$g_{11} = \cos^2 \theta_2 + \sin^2 \theta_2 = 1,$$

$$g_{12} = (\cos \theta_2)(-\theta_1 \sin \theta_2) + (\sin \theta_2)(\theta_1 \cos \theta_2) = 0 = g_{21},$$

$$g_{22} = (-\theta_1 \sin \theta_2)^2 + (\theta_1 \cos \theta_2)^2 = (\theta_1)^2.$$

The line element is

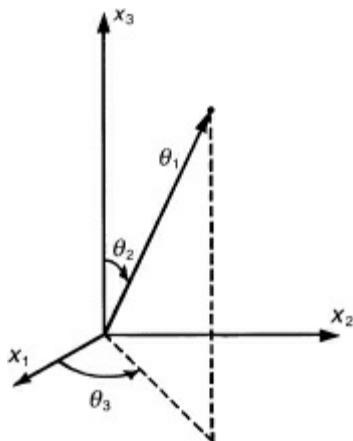


Fig. P2.4

$$ds^2 = (d\theta_1)^2 + (\theta_1)^2(d\theta_2)^2.$$

2.4. Let x_1, x_2, x_3 be rectangular Cartesian coordinates and $\theta_1, \theta_2, \theta_3$ be spherical polar coordinates. (See Fig. P2.4.)

Then

$$\theta_1 = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2},$$

$$\theta_2 = \cos^{-1} \frac{x_3}{\sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}},$$

$$\theta_3 = \tan^{-1}(x_2/x_1),$$

and the inverse transformation is

$$x_1 = \theta_1 \sin \theta_2 \cos \theta_3 ,$$

$$x_2 = \theta_1 \sin \theta_2 \sin \theta_3 ,$$

$$x_3 = \theta_1 \cos \theta_2 .$$

Show that the components of the Euclidean metric tensor in the spherical polar coordinates are

$$g_{11} = 1 , \quad g_{22} = (\theta_1)^2 , \quad g_{33} = (\theta_1)^2 (\sin \theta_2)^2 ,$$

and all other $g_{ij} = 0$. The square of the line element is, therefore,

$$ds^2 = (d\theta_1)^2 + (\theta_1)^2 (d\theta_2)^2 + (\theta_1)^2 (\sin \theta_2)^2 (d\theta_3)^2 .$$

2.5. Show that the length of the line element $(d\theta^1, 0, 0)$ is $\sqrt{g_{11}} |d\theta^1|$; that of the line element $(0, d\theta^2, 0)$ is $\sqrt{g_{22}} |d\theta^2|$.

2.6. Let the angle between the line elements $(d\theta^1, 0, 0)$ and $(0, d\theta^2, 0)$ be denoted by α_{12} . Show that

$$\cos \alpha_{12} = g_{12} / (g_{11} g_{22})^{1/2} .$$

Hint: Find the components of the line elements with respect to a system of rectangular Cartesian coordinates in which we know how to compute the angle between two vectors.

2.4. SCALARS, CONTRAVARIANT VECTORS, COVARIANT VECTORS¹

In nonrelativistic physics there are quantities like mass and length which are independent of reference coordinates, and there are quantities like displacement and velocity whose components do depend on reference coordinates. The former are the scalars; the latter, vectors. Mathematically, we define them according to the way their components change under admissible transformations. In the following discussions we consider a system whose components are defined in the general set of variables θ^i and are functions of $\theta^1, \theta^2, \theta^3$. If the variables θ^i can be changed to $\bar{\theta}^i$ by an admissible and proper transformation, then we can define new components of the system in the new variables $\bar{\theta}^i$. The system will be given various names according to the way in which the new and the old components are related.

A system is called a *scalar* if it has only a single component ϕ in the variables θ^i and a single component $\bar{\phi}$ in the variables $\bar{\theta}^i$, and if ϕ and $\bar{\phi}$ are numerically equal at the corresponding points,

$$(1) \quad \phi(\theta^1, \theta^2, \theta^3) = \bar{\phi}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) .$$

A system is called a *contravariant vector field* or a *contravariant tensor field of rank one*, if it has three components ξ^i in the variables θ^i and three components $\bar{\xi}^i$ in the variables $\bar{\theta}^i$, and if the components are related by the law

$$(2) \quad \bar{\xi}^i(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) = \xi^k(\theta^1, \theta^2, \theta^3)(\partial \bar{\theta}^i / \partial \theta^k) .$$

A contravariant vector is indicated by a superscript, called a *contravariant index*.

A differential $d\theta^i$ is a prototype of a contravariant vector:

$$(3) \quad d\bar{\theta}^i = (\partial \bar{\theta}^i / \partial \theta^k) d\theta^k .$$

Hence, a vector is contravariant if it transforms like a differential.²

A system is called a *covariant vector field*, or a *covariant tensor field of rank one*, if it has three components η_i in the variables $\theta^1, \theta^2, \theta^3$ and three components $\bar{\eta}_i$ in the variables $\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3$, and if the components in these two coordinate systems are related by the law

$$(4) \quad \bar{\eta}_i(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) = \eta_k(\theta^1, \theta^2, \theta^3)(\partial \theta^k / \partial \bar{\theta}^i) .$$

A covariant vector is indicated by a subscript, called a *covariant index*.

A gradient of a scalar potential ϕ transforms like (4),

$$(5) \quad \partial \phi / \partial \bar{\theta}^i = (\partial \phi / \partial \theta^k)(\partial \theta^k / \partial \bar{\theta}^i) .$$

Hence, a vector is covariant if it transforms like a gradient of a scalar potential.

2.5. TENSOR FIELDS OF HIGHER RANK

A system such as the three-dimensional Euclidean metric tensor g_{ij} has nine components when i and j range over 1, 2, 3. Such quantities are given special names when their components in any two coordinate systems are related by specific transformation laws.

Covariant tensor field of rank two, t_{ij} :

$$(1) \quad \bar{t}_{ij}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) = t_{mn}(\theta^1, \theta^2, \theta^3)(\partial\theta^m/\partial\bar{\theta}^i)(\partial\theta^n/\partial\bar{\theta}^j).$$

Contravariant tensor field of rank two, t^{ij} :

$$(2) \quad \bar{t}^{ij}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) = t^{mn}(\theta^1, \theta^2, \theta^3)(\partial\bar{\theta}^i/\partial\theta^m)(\partial\bar{\theta}^j/\partial\theta^n).$$

Mixed tensor field of rank two, \bar{t}_j^i :

$$(3) \quad \bar{t}_j^i(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) = t_n^m(\theta^1, \theta^2, \theta^3)(\partial\bar{\theta}^i/\partial\theta^m)(\partial\theta^n/\partial\bar{\theta}^j)$$

Generalization to tensor fields of higher ranks is immediate. Thus, we call a quantity $\bar{t}_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}$ a *tensor field of rank $r = p + q$, contravariant of rank p and covariant of rank q* , if the components in any two coordinate systems are related by

$$(4) \quad \bar{t}_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} = \frac{\partial\bar{\theta}^{\alpha_1}}{\partial\theta^{k_1}} \cdots \frac{\partial\bar{\theta}^{\alpha_p}}{\partial\theta^{k_p}} \cdot \frac{\partial\theta^{m_1}}{\partial\bar{\theta}^{\beta_1}} \cdots \frac{\partial\theta^{m_q}}{\partial\bar{\theta}^{\beta_q}} t_{m_1 \dots m_q}^{k_1 \dots k_p}.$$

Thus, the location of an index is important in telling whether it is contravariant or covariant. Again, if only rectangular Cartesian coordinates are considered, the distinction disappears.

These definitions can be generalized in an obvious manner if the range of the indices are 1, 2, ..., n .

PROBLEMS

2.7. Show that, if all components of a tensor vanish in one coordinate system, then they vanish in all other coordinate systems which are in one-to-one correspondence with the given system.

This is perhaps the most important property of tensor fields.

2.8. Prove the theorem: *The sum or difference of two tensors of the same type and rank (with the same number of covariant and the same number of contravariant indices) is again a tensor of the same type and rank.*

Thus, any linear combination of tensors of the same type and rank is again a tensor of the same type and rank.

2.9. Theorem. Let $A_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s}$, $B_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s}$ be tensors. The equation

$$A_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s}(\theta^1, \theta^2, \dots, \theta^n) = B_{\alpha_1 \dots \alpha_r}^{\beta_1 \dots \beta_s}(\theta^1, \theta^2, \dots, \theta^n),$$

is a tensor equation; i.e., if this equation is true in some coordinate system, then it is true in all coordinate systems which are in one-to-one correspondence with each other.

Hint: Use the results of the previous problems.

2.6. SOME IMPORTANT SPECIAL TENSORS

If we define the Kronecker delta and the permutation symbol introduced in Sec. 2.1 as components of covariant, contravariant, and mixed tensors of ranks 2 and 3 in rectangular Cartesian coordinates x_1, x_2, x_3 ,

$$(1) \quad \delta_{ij} = \delta^{ij} = \delta_j^i = \delta_i^j \begin{cases} = 0 & \text{when } i \neq j, \\ = 1 & \text{when } i = j, \end{cases}$$

$$(2) \quad e_{ijk} = e^{ijk} \begin{cases} = 0 & \text{when any two indices are equal,} \\ = 1 & \text{when } i, j, k \text{ permute like } 1, 2, 3, \\ = -1 & \text{when } i, j, k \text{ permute like } 1, 3, 2, \end{cases}$$

what will be their components in general coordinates θ^i ? The answer is provided immediately by the tensor transformation laws. Thus,

$$(3) \quad g_{ij} = (\partial x^m / \partial \theta^i)(\partial x^n / \partial \theta^j)\delta_{mn} = (\partial x^m / \partial \theta^i)(\partial x^m / \partial \theta^j),$$

$$(4) \quad g^{ij} = (\partial \theta^i / \partial x^m)(\partial \theta^j / \partial x^n)\delta^{mn} = (\partial \theta^i / \partial x^m)(\partial \theta^j / \partial x^m),$$

$$(5) \quad g_j^i = (\partial \theta^i / \partial x^m)(\partial x^n / \partial \theta^j)\delta_n^m = (\partial \theta^i / \partial x^m)(\partial x^m / \partial \theta^j),$$

$$(6) \quad \varepsilon_{ijk} = (\partial x^r / \partial \theta^i)(\partial x^s / \partial \theta^j)(\partial x^t / \partial \theta^k)e_{rst} = e_{ijk}|\partial x^m / \partial \theta^n| = e_{ijk}\sqrt{g},$$

$$(7) \quad \varepsilon^{ijk} = (\partial \theta^i / \partial x^r)(\partial \theta^j / \partial x^s)(\partial \theta^k / \partial x^t)e^{rst} = e^{ijk}|\partial \theta^n / \partial x^m| = e^{ijk}/\sqrt{g},$$

where g is the value of the determinant $|g_{ij}|$ and is positive for any proper coordinate system, i.e.,

$$(8) \quad g = |g_{ij}| > 0.$$

We see that *the proper generalizations of the Kronecker delta are the Euclidean metric tensors, and those of the permutation symbol are ϵ_{ijk} and ϵ^{ijk} , which are called permutation tensors or alternators.* As defined by Eqs. (6) and (7), the components of ϵ_{ijk} and ϵ^{ijk} do not have values 1, -1, or 0 in general coordinates, they are $(\sqrt{g}, -\sqrt{g}, 0)$ and $(1/\sqrt{g}, -1/\sqrt{g}, 0)$ respectively.

Note that the mixed tensor g_j^i is identical with δ_j^i and is constant in all coordinate systems. From (3) and (4) we see that

$$(9) \quad g_{im}g^{mj} = \delta_i^j.$$

Hence, the determinant

$$|g_{im}g^{mj}| = |g_{ij}| \cdot |g^{ij}| = |\delta_j^i| = 1.$$

Using (3) and (4), we can write

$$(10) \quad g = |g_{ij}| = |\partial x^i / \partial \theta^j|^2, \quad 1/g = |g^{ij}| = |\partial \theta^i / \partial x^j|^2.$$

Since g is positive, Eq. (9) may be solved to give

$$(11) \quad g^{ij} = D^{ij}/g,$$

where D^{ij} is the cofactor of the term g_{ij} in the determinant g . *The tensor g^{ij} is called the associated metric tensor. It is as important as the metric tensor itself in the further development of tensor analysis.*

Problem 2.10. Prove Eqs. (6), (7) and (10). Write down explicitly D^{ij} in Eq. (11) in terms of $g_{11}, g_{12}, \dots, g_{33}$.

2.7. THE SIGNIFICANCE OF TENSOR CHARACTERISTICS

The importance of tensor analysis may be summarized in the following statement. The form of an equation can have general validity with respect to any frame of reference only if every term in the equation has the same tensor characteristics. If this condition is not satisfied, a simple change of the system of reference will destroy the form of the relationship. This form is, therefore, only fortuitous and accidental.

Thus, tensor analysis is as important as dimensional analysis in any formulation of physical relations. Dimensional analysis considers how a physical quantity changes with the particular choice of fundamental units. Two physical quantities cannot be equal unless they have the same dimensions; any physical equation cannot be correct unless it is invariant with respect to change of fundamental units.

Whether a physical quantity should be a tensor or not is a decision for the physicist to make. Why is a force a vector, a stress tensor a tensor? Because we say so! It is our judgement that endowing tensorial character to these quantities is in harmony with the world.

Once we decided upon the tensorial character of a physical quantity, we may take as the components of a tensor field in a given frame of reference any set of functions of the requisite number. A tensor field thus assigned in a given frame of reference then transforms according to the tensor transformation law when admissible transformations are considered. In other words, once the values of the components of a tensor are assigned in one particular coordinate

system, the values of the components in any general coordinate system are fixed.

Why are the tensor transformation laws in harmony with physics? Because tensor analysis is designed so. For example, a tensor of rank one is defined in accordance with the physical idea of a vector. The only point new to the student is perhaps the distinction between contravariance and covariance. In elementary physics, natural laws are studied usually only in rectangular Cartesian coordinates of reference, in which the distinction between the contravariance and covariance disappears. When curvilinear coordinates are used in elementary physics, the vectorial components must be defined specifically in each particular case, and mathematical expressions of physical laws must be derived anew for each particular coordinate system. These derivations are usually quite tedious. Now, what is achieved by the definition of a tensor is a unified treatment, good for any curvilinear coordinates, orthogonal or nonorthogonal. This simplicity is obtained, however, at the expense of recognizing the distinction between contravariance and covariance.

In Sec. 2.14 we discuss the geometric interpretation of the tensor components of a vector in curvilinear coordinates. It will become clear that each physical vector has two tensor images: one contravariant and one covariant, depending on how the components are resolved.

2.8. RECTANGULAR CARTESIAN TENSORS

We have seen that it is necessary to distinguish the contravariant and covariant tensor transformation laws. However, *if only transformations between rectangular Cartesian coordinate systems are considered, the distinction between contravariance and covariance disappears*. To show this, let x_1, x_2, x_3 and $\bar{x}_1, \bar{x}_2, \bar{x}_3$ be two sets of rectangular Cartesian coordinates of reference. The transformation law must be simply

$$(1) \quad \bar{x}_i = \beta_{ij}x_j + a_i,$$

where a_1, a_2, a_3 are constants and β_{ij} are the direction cosines of the angles between unit vectors along the coordinate axes \bar{x}_i and x_j . Thus,

$$(2) \quad \beta_{21} = \cos(\bar{x}_2, x_1),$$

etc. The inverse transform is

$$(3) \quad x_i = \cos(x_i, \bar{x}_j)\bar{x}_j + b_i = \cos(\bar{x}_j, x_i)\bar{x}_j + b_i = \beta_{ji}\bar{x}_j + b_i,$$

where

$$b_i = \beta_{ki}a_k.$$

Hence,

$$(4) \quad \partial x_k / \partial \bar{x}_i = \beta_{ik} = \partial \bar{x}_i / \partial x_k,$$

and the distinction between the transformation laws (2.4:2) and (2.4:4) disappears.

The components of a vector in the two coordinate systems are related by

$$\bar{\eta}_i(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \beta_{ij}\eta_j(x_1, x_2, x_3).$$

The components of a tensor of rank two are related by

$$\bar{\xi}_{ij}(\bar{x}, \bar{x}_2, \bar{x}_3) = \beta_{ik}\beta_{js}\xi_{ks}(x_1, x_2, x_3).$$

When only rectangular Cartesian coordinates are considered, we shall write all indices as subscripts. This convenient practice will be followed throughout this book.

2.9. CONTRACTION

We shall now consider some operations on tensors that generate new tensors.

Let A_{jkl}^i be a mixed tensor so that, in a transformation from the coordinates x^α to \bar{x}^α ($\alpha = 1, 2, \dots, n$), we obtain

$$\bar{A}_{jkl}^i(\bar{x}) = (\partial \bar{x}^i / \partial x^\alpha)(\partial x^\beta / \partial \bar{x}^j)(\partial x^\gamma / \partial \bar{x}^k)(\partial x^\delta / \partial \bar{x}^l)A_{\beta\gamma\delta}^\alpha(x).$$

If we equate the indices i and k and sum, we obtain the set of quantities

$$\begin{aligned}\bar{A}_{jil}^i(\bar{x}) &= (\partial \bar{x}^i / \partial x^\alpha)(\partial x^\beta / \partial \bar{x}^j)(\partial x^\gamma / \partial \bar{x}^l)(\partial x^\delta / \partial \bar{x}^i) A_{\beta\gamma\delta}^\alpha(x) \\ &= (\partial x^\beta / \partial \bar{x}^j)(\partial x^\delta / \partial \bar{x}^l) \delta_\alpha^\gamma A_{\beta\gamma\delta}^\alpha(x) = (\partial x^\beta / \partial \bar{x}^j)(\partial x^\delta / \partial \bar{x}^l) A_{\beta\alpha\delta}^\alpha(x).\end{aligned}$$

Let us write $A_{\beta\alpha\delta}^\alpha$ as $B_{\beta\delta}$. Then the equation above shows that

$$\bar{B}_{jl} = (\partial x^\beta / \partial \bar{x}^j)(\partial x^\delta / \partial \bar{x}^l) B_{\beta\delta}.$$

Hence, $B_{\beta\delta}$ satisfies the tensor transformation law and is therefore a tensor.

The process of equating and summing a covariant and a contravariant index of a mixed tensor is called a *contraction*. It is easy to see that the example above can be generalized to mixed tensors of other ranks. *The result of a contraction is another tensor*. If, as a result of contraction, there is no free index left, the resulting quantity is a scalar.

The following problem shows that, in general, equating and summing two covariant indices or two contravariant indices does not yield a tensor of lower order and is not a proper contraction. However, when only Cartesian coordinates are considered, we write all indices as subscripts and contract them by equating two subscripts and summing.

Problem 2.11. If A_j^i is a mixed tensor of rank two, show that A_i^i is a scalar. If A^{ij} is a contravariant tensor of rank two show that in general A^{ii} is not an invariant. Similarly, if A_{ij} is a covariant tensor of rank two, A_{ii} is not, in general, an invariant.

2.10. QUOTIENT RULE

Consider a set of n^3 functions $A(111)$, $A(112)$, $A(123)$, etc., or $A(i, j, k)$ for short, with the indices i, j, k each ranging over 1, 2, ..., n . Although the set of functions $A(i, j, k)$ has the right number of components, we do not know whether it is a tensor or not. Now, suppose that we know something about the nature of the product of $A(i, j, k)$ with an arbitrary tensor. Then there is a theorem which enables us to establish whether $A(i, j, k)$ is a tensor without going to the trouble of determining the law of transformation directly.

For example, let $\xi^\alpha(x)$ be an arbitrary tensor of rank 1 (a vector). Let us suppose that the product $A(\alpha, j, k)\xi^\alpha$ (summation convention used over α) is known to yield a tensor of the type $A_k^j(x)$,

$$A(\alpha, j, k)\xi^\alpha = A_k^j.$$

Then we can prove that $A(i, j, k)$ is a tensor of the type $A_{ik}^j(x)$.

The proof is very simple. Since $A(\alpha, j, k)\xi^\alpha$ is of type A_k^j , it is transformed into \bar{x} -coordinates as

$$\bar{A}(\alpha, j, k)\bar{\xi}^\alpha = \bar{A}_k^j = \frac{\partial \bar{x}^j}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^k} A_s^r = \frac{\partial \bar{x}^j}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^k} [A(\beta, r, s)\xi^\beta].$$

But $\xi^\beta = (\partial x^\beta / \partial \bar{x}^\alpha)\bar{\xi}^\alpha$. Inserting this in the right-hand side of the equation above and transposing all terms to one side of the equation, we obtain

$$[\bar{A}(\alpha, j, k) - (\partial \bar{x}^j / \partial x^r)(\partial x^s / \partial \bar{x}^k)(\partial x^\beta / \partial \bar{x}^\alpha) A(\beta, r, s)]\bar{\xi}^\alpha = 0.$$

Now $\bar{\xi}^\alpha$ is an arbitrary vector. Hence, the bracket must vanish and we have

$$\bar{A}(\alpha, j, k) = (\partial \bar{x}^j / \partial x^r)(\partial x^s / \partial \bar{x}^k)(\partial x^\beta / \partial \bar{x}^\alpha) A(\beta, r, s),$$

which is precisely the law of transformation of the tensor of the type A_{ik}^j .

The pattern of the example above can be generalized to prove the theorem that, if $[A(i_1, i_2, \dots, i_r)]$ is a set of functions of the variables x^i , and if the product $A(\alpha, i_2, \dots, i_r)\xi^\alpha$ with an arbitrary vector ξ^α be a tensor of the type $A_{k_1 \dots k_p}^{j_1 \dots j_q}(x)$, where $p + q = r$, then the set $A(i_1, i_2, \dots, i_r)$ represents a tensor of the type $A_{\alpha k_1 \dots k_p}^{j_1 \dots j_q}(x)$.

Similarly, if the product of a set of n^2 functions $A(\alpha, j)$ with an arbitrary tensor $B_{\alpha k}$ (and is summed over α) is a covariant tensor of rank 2, then $A(i, j)$ represents a tensor of the type A_j^i .

These and similar theorems that can be derived are called *quotient rules*. Numerous applications of these rules follow.

See, for example, Sec. 3.3, following Eq. (3.3:3); [Sec. 3.12](#), following Eq. (3.12:11); [Sec. 4.1](#), following Eqs. (4.1:9).

2.11. PARTIAL DERIVATIVES IN CARTESIAN COORDINATES

The generation of a tensor of rank one from a tensor of rank zero by differentiation, such as the gradient of a scalar potential, indicates a way of generating tensors of higher rank. But, in general, the set of partial derivatives of a tensor does not behave like a tensor field. However, *if only Cartesian coordinates are considered, then the partial derivatives of any tensor field behave like the components of a tensor field under a transformation from Cartesian coordinates to Cartesian coordinates.* To show this, let us consider two Cartesian coordinates (x_1, x_2, x_3) and $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ related by

$$(1) \quad \bar{x}_i = a_{ij} x_j + b_i,$$

where a_{ij} and b_i are constants. From (1), we have

$$(2) \quad \partial \bar{x}_i / \partial x_j = a_{ij}, \quad \partial^2 \bar{x}_i / (\partial x_j \partial x_k) = 0.$$

Now, if $\xi^i(x_1, x_2, x_3)$ is a contravariant tensor, so that

$$(3) \quad \bar{\xi}^i(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \xi^\alpha(x_1, x_2, x_3) \partial \bar{x}_i / \partial x_\alpha.$$

Then, on differentiating both sides of the equation, one obtains

$$(4) \quad \frac{\partial \bar{\xi}^i}{\partial \bar{x}^j} = \left(\frac{\partial \xi^\alpha}{\partial x_\beta} \frac{\partial x_\beta}{\partial \bar{x}_j} \right) \frac{\partial \bar{x}_i}{\partial x_\alpha} + \xi^\alpha \left(\frac{\partial^2 \bar{x}_i}{\partial x_\alpha \partial x_\beta} \right) \frac{\partial x_\beta}{\partial \bar{x}_j}.$$

When x_i and \bar{x}_i are Cartesian coordinates, the last term vanishes according to (2). Hence,

$$(5) \quad \partial \bar{\xi}^i / \partial \bar{x}_j = (\partial \xi^\alpha / \partial x_\beta) (\partial x_\beta / \partial \bar{x}_j) (\partial \bar{x}^i / \partial x_\alpha).$$

Thus, the set of partial derivatives $\partial \xi^\alpha / \partial x_\beta$ follows the transformation law for a mixed tensor of rank two under a transformation *from Cartesian coordinates to Cartesian coordinates*. However, the presence of the second derivative terms in Eq. (4) which does not vanish in curvilinear coordinates shows that $\partial \xi^\alpha / \partial x_\beta$ are not really the components of a tensor field in general coordinates. A similar situation holds obviously also for tensor fields of higher ranks. See [Sec. 2.12](#) below.

When Cartesian coordinates are used, *we shall use a comma to denote partial differentiation.* Thus,

$$\xi_{i,j} \equiv \partial \xi_i / \partial x_j, \quad \phi_{,i} \equiv \partial \phi / \partial x_i, \quad \sigma_{ij,k} \equiv \partial \sigma_{ij} / \partial x_k.$$

When we restrict ourselves to Cartesian coordinates, $\phi_{,i}$, $\xi_{i,j}$, $\sigma_{ij,k}$ are tensors of rank one, two, three, respectively, provided that ϕ , ξ_i , σ_{ij} are tensors.

Warning: Cartesian tensor equations derived through the use of differentiation are in general not valid in curvilinear coordinates. This important point is discussed in the following five sections.

2.12. COVARIANT DIFFERENTIATION OF VECTOR FIELDS

The generalization of the concept of partial derivatives to the concept of *covariant derivative*, so that the covariant derivative of a tensor field is another tensor field, is the most important milestone in the development of tensor calculus. It is natural to search for such an extension in the form of a correction term that depends on the vector itself. Thus, if ξ^i is a vector, we might expect the combination

$$\partial \xi^i / \partial x^j + \Gamma(i, j, \alpha) \xi^\alpha,$$

to be a tensor. Here the suggested correction is linear function of ξ^i , and $\Gamma(i, j, \alpha)$ is some function with three indices. The success of this scheme hinges on the Euclidean Christoffel symbols, which are certain linear combinations of the derivatives of the metric tensor g_{ij} . This subject is beautiful, and the results are powerful in handling curvilinear coordinates. However, since the topic is not absolutely necessary for the development of solid mechanics, we shall not discuss it in detail, but merely outline below some of the salient results.

We discussed in [Sec. 2.3](#) the metric tensor g_{ij} in a set of general coordinates (x^1, x^2, x^3) , and in [Sec. 2.6](#) the associated metric tensor g^{ij} . By means of these metric tensors, the *Euclidean Christoffel symbols* $\Gamma_{\alpha\beta}^i(x^1, x^2, x^3)$ are defined as follows:

$$(1) \quad \Gamma_{\alpha\beta}^i(x^1, x^2, x^3) = g^{i\sigma}(\partial g_{\sigma\beta}/\partial x^\alpha + \partial g_{\alpha\sigma}/\partial x^\beta - \partial g_{\alpha\beta}/\partial x^\sigma)/2.$$

The $\Gamma_{\alpha\beta}^i$ is not a tensor. It transforms under a coordinate transformation $\bar{x}^i = f^i(x^1, x^2, x^3)$ as follows (see Prob. 2.24, p. 50)

$$(2) \quad \bar{\Gamma}_{\alpha\beta}^i(\bar{x}^1, \bar{x}^2, \bar{x}^3) = \Gamma_{\mu\nu}^\lambda(x^1, x^2, x^3) \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \frac{\partial x^\nu}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^i}{\partial x^\lambda} + \frac{\partial^2 x^\lambda}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} \frac{\partial \bar{x}^i}{\partial x^\lambda}.$$

This equation can be solved for $\partial^2 x^\lambda / \partial \bar{x}^\alpha \partial \bar{x}^\beta$ by multiplying (2) with $\partial x^m / \partial \bar{x}^i$ and sum over i to obtain

$$(3) \quad \partial^2 x^\lambda / (\partial \bar{x}^\alpha \partial \bar{x}^\beta) = \bar{\Gamma}_{\alpha\beta}^i(\bar{x})(\partial x^\lambda / \partial \bar{x}^i) - \Gamma_{\mu\nu}^\lambda(x)(\partial x^\mu / \partial \bar{x}^\alpha)(\partial x^\nu / \partial \bar{x}^\beta).$$

Interchanging the roles of x_i and \bar{x}_i and with suitable changes in indices, we can substitute (3) into Eq. (2.11:4) to obtain

$$\frac{\partial \bar{\xi}^i}{\partial \bar{x}^\alpha} = \frac{\partial \xi^\lambda}{\partial x^\mu} \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^i}{\partial x^\lambda} + \xi^\lambda \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \left[\Gamma_{\lambda\mu}^s(x) \frac{\partial \bar{x}^i}{\partial x^s} - \bar{\Gamma}_{mn}^i(\bar{x}) \frac{\partial \bar{x}^m}{\partial x^\lambda} \frac{\partial \bar{x}^n}{\partial x^\mu} \right],$$

which can be reduced to

$$(4) \quad \partial \bar{\xi}^i / \partial \bar{x}^\alpha + \bar{\Gamma}_{m\alpha}^i \bar{\xi}^m = (\partial \xi^\lambda / \partial x^\mu + \Gamma_{s\mu}^\lambda \xi^s)(\partial x^\mu / \partial \bar{x}^\alpha)(\partial \bar{x}^i / \partial x^\lambda).$$

But this states that the functions $\partial \xi^\lambda / \partial x^\mu + \Gamma_{s\mu}^\lambda \xi^s$ are the components of a mixed tensor of rank two. Hence, the functions

$$(5) \quad \xi^i|_\alpha \equiv \partial \xi^i(x^1, x^2, x^3) / \partial x^\alpha + \Gamma_{\sigma\alpha}^i(x^1, x^2, x^3) \xi^\sigma(x^1, x^2, x^3),$$

are the components of a mixed tensor field of rank two, called the covariant derivative of the contravariant vector ξ^i . We shall use the notation $\xi^i|_\alpha$ for the covariant derivative of ξ^i .

By a slight variation in the derivation, it can be shown that the functions

$$(6) \quad \xi_i|_\alpha \equiv \frac{\partial \xi_i}{\partial x^\alpha} - \Gamma_{i\alpha}^\sigma \xi_\sigma,$$

are the components of a covariant tensor field of rank two whenever ξ_i are the components of a covariant vector field. This is called the covariant derivative of ξ_i , and is denoted by $\xi_i|_\alpha$.

More generally, a long but quite straightforward calculation analogous to the above can be made to establish the covariant derivative of a tensor $T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}$ of rank $p+q$, contravariant of rank p , covariant of rank q :

$$T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} \Big|_\gamma = \partial T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} / \partial x^\gamma + \Gamma_{\sigma\gamma}^{\alpha_1} T_{\beta_1 \beta_2 \dots \beta_q}^{\sigma \alpha_2 \dots \alpha_p} + \dots + \Gamma_{\sigma\gamma}^{\alpha_p} T_{\beta_1 \dots \beta_{q-1} \beta_q}^{\alpha_1 \dots \alpha_{p-1} \sigma} - \Gamma_{\beta_1 \gamma}^\sigma T_{\sigma \beta_2 \dots \beta_q}^{\alpha_1 \alpha_2 \dots \alpha_p} - \dots - \Gamma_{\beta_q \gamma}^\sigma T_{\beta_1 \dots \beta_{q-1} \sigma}^{\alpha_1 \dots \alpha_{p-1} \alpha_p}.$$

This derivative is contravariant of rank p , and covariant of rank $q+1$.

Since the components of the metric tensor g_{ij} are constant in Cartesian coordinates, we see from Eq. (1) that the Euclidean Christoffel symbols are zero in Cartesian coordinates. The covariant derivative of a tensor field reduces to partial derivatives of the tensor field when the tensor field and the operations are evaluated in Cartesian coordinates.

2.13. TENSOR EQUATIONS

The theorems included in the problems at the end of Sec. 2.5 contain perhaps the most important property of tensor fields: if all the components of a tensor field vanish in one coordinate system, they vanish likewise in all coordinate systems which can be obtained by admissible transformations. Since the sum and difference of tensor fields of a given type are tensors of the same type, we deduce that if a tensor equation can be established in one coordinate system, then it must hold for all coordinate systems obtained by admissible transformations.

The last statement affords a powerful method for establishing equations in mathematical physics. For example, if a certain tensor relationship can be shown to be true in rectangular Cartesian coordinates, then it is also true in general curvilinear coordinates in Euclidean space. Thus, once an equation is established in rectangular Cartesian coordinates, the corresponding equation (stating a physical fact, such as a condition of equilibrium, of conservation of energy, etc.) in

any specific curvilinear coordinates in Euclidean space can be obtained by a straightforward “translation” in the language of tensors. The key word here is “tensor.” Every term in the equation must be a tensor. Partial derivatives are tensors only in Cartesian coordinates. They are not tensors in curvilinear coordinates. Covariant differentiation must be used to generate new tensors in curvilinear coordinates.

As an example of the application of these remarks, let us consider the *successive covariant derivatives* of a tensor field. Since the covariant derivative of a tensor field is a tensor field, we can form the covariant derivative of the latter, which is called the *second covariant derivative of the original tensor field*. If T_{kl}^{ij} denotes the original tensor field, we can consider the second covariant derivative $T_{kl}^{ij}|_{\gamma\delta}$. Now, if the space is Euclidean, then it can be described by a rectangular Cartesian coordinate system. In Cartesian coordinates, the Euclidean Christoffel symbols are all zero and the covariant derivatives of a tensor field reduce to partial derivatives of the tensor field. But partial derivatives are *commutative* if they are continuous. Hence, we see that the following equation is true in Cartesian coordinates if every component is continuous,

$$(1) \quad T_{kl}^{ij}|_{\gamma\delta} = T_{kl}^{ij}|_{\delta\gamma},$$

and, therefore, it is true in all coordinates that can be obtained by admissible transformations from a Cartesian system.

We must remark that the commutativeness of the covariant differentiation operation is established above only in Euclidean space. A coordinate system in a more general Riemannian space may not be transformed into Cartesian coordinates and the method of proof used above cannot be applied. In fact, the theorem expressed in Eq. (1) is, in general, untrue in Riemannian space: successive covariant derivatives are, in general, not commutative in the Riemannian space. (See Prob. 2.31 below.)

As a second application we can prove the following theorem: *the covariant derivatives of the Euclidean metric tensor g_{ij} and the associated contravariant tensor g^{ij} are zero:*

$$(2) \quad g_{ij}|_k = 0, \quad g^{ij}|_k = 0.$$

Since $g_{ij}|_k$ and $g^{ij}|_k$ are tensors, the truth of the theorem can be established if we can demonstrate Eq. (2) in one particular coordinate system. But this is exactly the case in Cartesian coordinates, in which g_{ij} and g^{ij} are constants and, hence, their derivative vanish. Thus, the proof is completed. In contrast to Eq. (1), however, it can be proved that Eqs. (2) remain true in Riemannian space.

Further examples are furnished in Probs. 2.17 to 2.23.

To apply this powerful procedure, one must make sure that all quantities involved are tensors. In particular, we must ascertain that all scalars are “absolute” constants. This remark is very important because in physics we also use quantities that transform like *relative tensors*. A *relative tensor of weight w* is an object with components whose transformation law differs from the tensor transformation law by the appearance of the Jacobian to the w^{th} power as a factor. Thus,

$$(3) \quad \bar{\phi}(\theta) = |\partial x_j / \partial \theta_\beta|^w \phi(x),$$

$$(4) \quad \bar{\xi}^i(\theta) = |\partial x_j / \partial \theta_\beta|^w \xi^\alpha(x) (\partial \theta_i / \partial x_\alpha),$$

are the transformation laws for a relative scalar field of weight w and a relative contravariant vector field of weight w , respectively. If $w = 0$, we have the previous notion of a tensor field. Whether an object is a tensor or a relative tensor is often a matter of definition.

As an example, consider the total mass enclosed in a volume expressed in terms of density. Let x_i be rectangular coordinates which are transformed into curvilinear coordinates θ_j . We have

$$(5) \quad M = \iiint_V \rho_0(x_1, x_2, x_3) dx_1 dx_2 dx_3 \\ = \iiint_V \rho_0[x(\theta)] |\partial x_i / \partial \theta_j| d\theta_1 d\theta_2 d\theta_3 = \iiint_V \bar{\rho}(\theta_1, \theta_2, \theta_3) d\theta_1 d\theta_2 d\theta_3.$$

If $\bar{\rho}(\theta)$ in the last term is defined as the density distribution in the θ -coordinates, then it is a relative scalar of weight one. On the other hand, $\rho_0[x(\theta)] = \rho_0(x)$ is an absolute scalar which defines the (physical) density of the medium.

As another example, consider the determinant g of the Euclidean metric tensor g_{ij} whose transformation law is

$$(6) \quad \bar{g}_{\alpha\beta}(\theta) = g_{ij}(x)(\partial x_i / \partial \theta_\alpha)(\partial x_j / \partial \theta_\beta).$$

Let $\bar{g} = |\bar{g}_{\alpha\beta}|$, the determinant of $\bar{g}_{\alpha\beta}$. By a double use of the formula for the product of two determinants when applied to Eq. (6), it is easy to prove that

$$(7) \quad \bar{g}(\theta) = |\partial x_i / \partial \theta_\alpha|^2 g(x),$$

which shows that g is a relative scalar of weight two. It follows that $\sqrt{\bar{g}}$ is a relative scalar of weight one. We note that if x_i are rectangular coordinates, $g = 1$. Hence Eq. (5) shows that $\sqrt{g[x(\theta)]} = |\partial x_i / \partial \theta_j|$, the Jacobian of the transformation. Thus the volume enclosed by a closed surface can be written as

$$(8) \quad V = \iiint dx_1 dx_2 dx_3 = \iiint \left| \frac{\partial x_i}{\partial \theta_\alpha} \right| d\theta_1 d\theta_2 d\theta_3 = \iiint \sqrt{g[x(\theta)]} d\theta_1 d\theta_2 d\theta_3.$$

The last integral shows the importance of \sqrt{g} in mechanics.

The method of tensor equations does not apply to relative tensors. Therefore it is important to properly define all quantities involved in an equation to be tensors.

2.14. GEOMETRIC INTERPRETATION OF TENSOR COMPONENTS

Before we conclude this chapter we shall consider briefly the geometric interpretation of tensor components. For this purpose we must use the concept of *base vectors*. We know that in a three-dimensional Euclidean space any three linearly independent vectors form a *basis* with which any other vectors can be expanded as a linear combination of these three vectors. When a rectangular Cartesian frame of reference is chosen, we can choose as base vectors the unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ parallel to the coordinate axes: thus, if a vector \mathbf{A} has three components (A_1, A_2, A_3) , we can write

$$\mathbf{A} = A_1 \mathbf{i}_1 + A_2 \mathbf{i}_2 + A_3 \mathbf{i}_3 = A_j \mathbf{i}_j$$

In a curvilinear coordinate system in a Euclidean space, we shall introduce the base vectors from the following consideration.

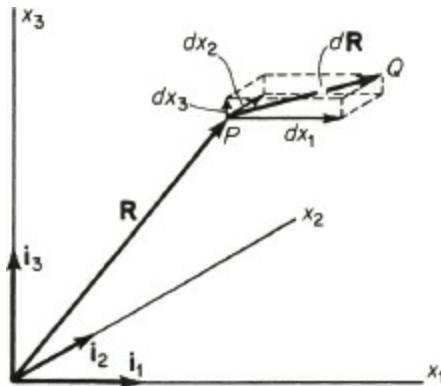


Fig. 2.14:1. Base vectors.

Let $d\mathbf{R}$ denote an infinitesimal vector PQ joining a point $P = (x_1, x_2, x_3)$ to a point $Q = (x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ where x_1, x_2, x_3 are referred to a rectangular Cartesian frame of reference. (See Fig. 2.14:1.) Then, obviously,

$$(1) \quad d\mathbf{R} = dx^r \mathbf{i}_r = dx_r \mathbf{i}^r,$$

where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, or $\mathbf{i}^1, \mathbf{i}^2, \mathbf{i}^3$ denote the base vectors along coordinate axes. Here, since a rectangular Cartesian coordinate system is used, we can assign arbitrarily an index as contravariant or covariant. In the assignment chosen above, dx^r and dx_r are, respectively, contravariant and covariant differentials.

Now let us consider a transformation from the rectangular Cartesian coordinates x^i to general coordinates θ^i .

$$(2) \quad \theta^i = \theta^i(x_1, x_2, x_3).$$

According to the tensor transformation law, when dx^r and dx_r are regarded as tensors of order one, their components in the general coordinates become

$$(3) \quad d\theta^j = (\partial \theta^j / \partial x^i) dx^i, \quad dx^i = (\partial x^i / \partial \theta^j) d\theta^j,$$

$$(4) \quad d\theta_j = (\partial x^i / \partial \theta^j) dx_i, \quad dx_i = (\partial \theta^j / \partial x^i) d\theta_j.$$

Now in (3), $d\theta^i$ can be identified as the usual differential of the variable θ^i , as specified in Eq. (2); hence, the superscript is justified. But in Eq. (4), $d\theta_j$ is not to be identified with the usual differential; $d\theta_j$ is a *covariant* differential that will have a different geometric meaning as will be seen later.

By Eqs. (3) and (4), we may write Eq. (1) as

$$(5) \quad \Delta \quad d\mathbf{R} = g_r d\theta^r = g^r d\theta_r,$$

where

$$(6) \quad g_r = (\partial x^s / \partial \theta^r) \mathbf{i}_s, \quad g^r = (\partial \theta^r / \partial x^s) \mathbf{i}^s.$$

Since g_r and g^r are linear combinations of unit vectors, they are themselves vectors; they are known as the *covariant* and *contravariant base vectors*, respectively, or as the *base vectors* and *reciprocal base vectors*. Equation (5) shows that

$$(7) \quad \Delta \quad g_i = \partial \mathbf{R} / \partial \theta^i.$$

Hence g_i characterizes the change of the position vector \mathbf{R} as θ^i varies. In other words, g_i is *directed tangentially along the coordinate curve θ^i* . These vectors are illustrated in Fig. 2.14:2.

It is easily verified that

$$(8) \quad \Delta \quad g_r \cdot g_s = g_{rs}, \quad g^r \cdot g^s = g^{rs},$$

$$(9) \quad \Delta \quad g^r \cdot g_s = g_s^r = \delta_s^r,$$

$$(10) \quad \Delta \quad g^r = g^{rs} g_s, \quad g_r = g_{rs} g^s,$$

where g_{rs} is the Euclidean metric tensor of the coordinate system, and g^{rs} is the *associated*, or *conjugate metric tensor*. From (9) it is clear that *the contravariant base vectors g^1, g^2, g^3 are, respectively, perpendicular to the planes of $g_2 g_3, g_3 g_1, g_1 g_2$.* (See also Prob. 2.28.) For orthogonal coordinates, that means $g_i \cdot g_j = 0$ for $i \neq j$, it can be shown that g^i is in the same direction as g_i and that $g^i \cdot g^j = 0$ for $i \neq j$.

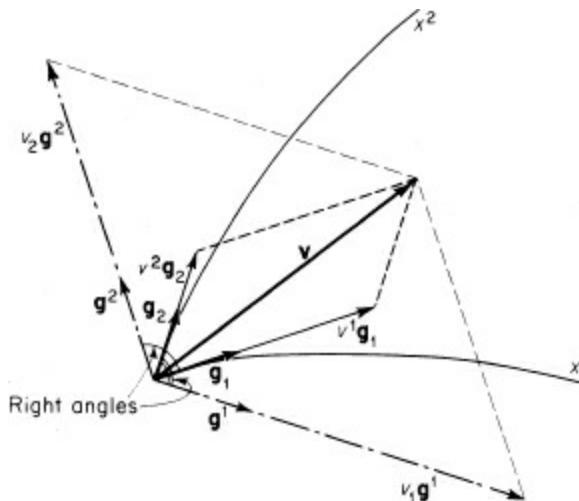


Fig. 2.14:2. Contravariant and covariant components of a vector \mathbf{v} in two dimensions.

We can now deal with the contravariant and covariant components of a vector. Consider the expression $v^r g_r$, where v^r is a contravariant tensor of rank one, and g_r are the covariant base vectors at a generic point. This expression remains *invariant* under coordinate transformations, and, since it is a linear combination of the base vectors g_r , it is a vector, which may be designated \mathbf{v} :

$$(11) \quad \mathbf{v} = v^r g_r.$$

By Eq. (10), replacing g_r by $g_{rs} g^s$, we also have

$$(12) \quad \Delta \quad \mathbf{v} = v^r g_r = v_s g^s,$$

where

$$(13) \quad \mathbf{v}_s = g_{rs} v^r, \quad v^r = g^{rs} v_s.$$

According to Eq. (12), if we represent \mathbf{v} by a directed line, then the contravariant components v^r are the components of \mathbf{v} in the direction of the covariant base vectors, while the covariant components v_r are the components of \mathbf{v} in the direction of the contravariant base vectors. A two-dimensional illustration is shown in Fig. 2.14:2.

Equation (12) justifies naming v^r the *contravariant components* of the vector \mathbf{v} , and v_r the *covariant components* of \mathbf{v} . Thus, in our Euclidean space, the tensors v^r and v_r are two different representations of the same vector \mathbf{v} . Equations (13) establish the process of raising and lowering of indices.

Having recognized the covariant and contravariant base vectors of each coordinate system, we can use them to derive the transformation laws of vectors and tensors from one coordinate system to another. Let g^i, \bar{g}^i and g_r, \bar{g}_r be, respectively, the contravariant and covariant base vectors of two coordinate systems of reference. We define

$$(14) \quad \begin{aligned} \beta_m^r &= \bar{g}_m \cdot g^r = |\bar{g}_{mm} g^{rr}| \cos(\bar{g}_m, g^r), \\ \beta_{.r}^m &= \bar{g}^m \cdot g_r = |\bar{g}^{mm} g_{rr}| \cos(\bar{g}^m, g_r), \end{aligned}$$

in which m and r are not summed. We use a dot in the superscript or subscript to emphasize the difference between β_m^r and $\beta_{.m}^r [= |\bar{g}_{mm} \bar{g}^{rr}| \cos(g_m, \bar{g}^r)]$. In general, $\beta_{.m}^r \neq \beta_r^m \neq \beta_m^r \neq \beta_{.r}^m$. Here the first indices of the superscript and the subscript are associated with the coordinates with base vectors \bar{g} 's and the second indices with g 's.

The distinction between β_m^r and $\beta_{.m}^r$ disappear in Cartesian coordinates. One can easily show that

$$(15) \quad \bar{g}_r = (\bar{g}_r \cdot g^m) g_m = \beta_{.r}^m g_m,$$

$$(16) \quad g_m = (g_m \cdot \bar{g}^k) \bar{g}_k = (\bar{g}^k \cdot g_m) \bar{g}_k = \beta_{.m}^k \bar{g}_k$$

and that the β 's are related by the metric tensors of the two coordinate systems:

$$\beta_m^r = \bar{g}_{mp} \beta_{.q}^p g^{rq}, \quad \beta_{.m}^r = \bar{g}^{rp} \beta_p^q g_{qm}.$$

From Eqs. (6) and (14), it can be seen that

$$(17) \quad \beta_{.r}^m = \bar{g}_r \cdot g^m = \partial x^m / \partial \bar{x}^r, \quad \beta_{.m}^k = \bar{g}^k \cdot g_m = \partial \bar{x}^k / \partial x^m.$$

Thus, Eqs. (15) and (16) establish that \bar{g} 's and g 's follow the tensor transformation law. A substitution of Eq. (16) into Eq. (15) and *vice versa* yield

$$\bar{g}_r = \beta_r^k g_k = \beta_r^k \beta_{.k}^m \bar{g}_m, \quad g_m = \beta_{.m}^k \bar{g}_k = \beta_{.m}^k \beta_k^r g_r$$

which implies that

$$(18) \quad \beta_r^k \beta_{.k}^m = \delta_{rm}. \quad \beta_{.m}^k \beta_k^r = \delta_{rm}.$$

We can now determine the transformation law for the components of \mathbf{v} between different coordinate systems. Dotting the equation

$$\mathbf{v} = \bar{v}^r \bar{g}_r = v^m g_m$$

with \bar{g}^i gives the transformation law

$$(19) \quad \bar{v}^i = (\bar{g}^i \cdot g_m) v^m = \beta_{.m}^i v^m.$$

Similarly, dotting the same equation with g_j gives the inverse transformation

$$(20) \quad v^j = (\bar{g}_r \cdot g^j) \bar{v}^r = \beta_r^j \bar{v}^r.$$

Using the relations of the contravariant base vectors

$$(21) \quad \bar{g}^r = (\bar{g}^r \cdot g_m) g^m = \beta_{.m}^r g^m, \quad g^m = (\bar{g}_r \cdot g^m) \bar{g}^r = \beta_r^m \bar{g}^r,$$

together with $\mathbf{v} = \bar{v}_r \bar{g}^r = v_m g^m$, we obtain

$$(22) \quad \bar{v}_r = (\bar{g}_r \cdot g^m) v_m = \beta_r^m v_m, \quad v_m = (\bar{g}^r \cdot g_m) \bar{v}_r = \beta_{.m}^r \bar{v}_r$$

If both coordinate systems are rectangular Cartesian coordinates, $g_i = g^i$ and $\bar{g}_i = \bar{g}^i$. There is no need to distinguish the super- and subscripts for all quantities and \bar{g} 's, g 's are unit base vectors. Then Eq. (14) reduces to

$$(23) \quad \beta_{rm} = \cos(\bar{g}_r, g_m) = \bar{g}_r \cdot g_m = g_m \cdot \bar{g}_r,$$

that β_{rm} is the direction cosine of the angle between \bar{g}_r and g_m . Equation (18) becomes

$$(24) \quad \beta_{rk}\beta_{mk} = \beta_{kr}\beta_{km} = \delta_{rm},$$

i.e., the transpose of the Cartesian tensor β_{rm} is the inverse of β_{rm} . Equations (19), (20) or (22) become

$$(25) \quad \bar{v}_r = (\bar{g}_r \cdot g_m)v_m = \beta_{rm}v_m, \quad v_m = (\bar{g}_r \cdot g_m)\bar{v}_r = \beta_{rm}\bar{v}_r.$$

Note that in general we still have $\beta_{rm} \neq \beta_{mr}$ for $r \neq m$.

If \bar{v} 's and v 's are the coordinates of the two rectangular Cartesian coordinate systems, Eq. (25) gives the transformation law between the two coordinate systems as stated in Eqs. (2.8:2, 2.8:3) for zero translation.

Similar concept can be generalized to deal with the contravariant and covariant components of tensors of rank two or higher. Consider the expression

$$\mathbf{A} = A^{rs}g_r g_s,$$

where A^{rs} is a contravariant tensor of rank 2. By Eq. (10), replacing g_r by $g_{rm}g_m$, we also have

$$\mathbf{A} = A^{rs}g_r g_s = A_{mn}g^m g^n.$$

The expressions above are invariant under coordinate transformation, i.e.,

$$(26) \quad \mathbf{A} = \bar{A}^{rs}\bar{g}_r\bar{g}_s = A^{mn}g_m g_n = \bar{A}_{rs}\bar{g}^r\bar{g}^s = A_{mn}g^m g^n,$$

in two different coordinate systems of reference. To examine the relations between \bar{A}^{rs} and A^{rs} , we substitute Eq. (16) into Eq. (26) and get

$$(27) \quad \mathbf{A} = \bar{A}^{rs}\bar{g}_r\bar{g}_s = A^{mn}\beta_{.m}^r\beta_{.n}^s\bar{g}_r\bar{g}_s, \quad \text{or} \quad \bar{A}^{rs} = \beta_{.m}^r\beta_{.n}^s A^{mn}.$$

The substitution of Eq. (15) into Eq. (26) yields

$$(28) \quad \mathbf{A} = A^{mn}g_m g_n = \bar{A}^{rs}\beta_r^m\beta_s^n g_m g_n \quad \text{or} \quad A^{mn} = \beta_r^m\beta_s^n \bar{A}^{rs}.$$

Similarly, using the relations of the contravariant base vectors given in Eq. (21), we obtain

$$(29) \quad \bar{A}_{rs} = \beta_r^m\beta_s^n A_{mn} \quad \text{and} \quad A_{mn} = \beta_{.m}^r\beta_{.n}^s \bar{A}_{rs}$$

If both coordinate systems are rectangular Cartesian, Eqs. (27), (28) or (29) becomes

$$(30) \quad \bar{A}_{rs} = \beta_{rm}\beta_{sn} A_{mn} \quad \text{and} \quad A_{mn} = \beta_{rm}\beta_{sn} \bar{A}_{rs}.$$

These formulas tell us how base vectors and tensor components are changed in coordinate transformation. Coordinate transformation occupies a uniquely important position in mechanics for many reasons. One reason is exemplified by Einstein's using it to develop the theory of relativity. Another reason is that the shape of a natural object of interest to science and engineering often has a natural, preferred coordinate system for its description, e.g., the earth is round, the rail is straight, the egg is egg-shaped. Whichever be the reason, we often find it desirable to transform an equation written in one coordinate system to one valid in another coordinate system. For these tasks the method illustrated above can be helpful.

2.15. GEOMETRIC INTERPRETATION OF COVARIANT DERIVATIVES

Consider now a vector field \mathbf{v} defined at every point of space in a region R . Let the vector at the point $P(\theta^1, \theta^2, \theta^3)$ be

$$(1) \quad \mathbf{v}(P) = v^i(P)g_i(P).$$

At a neighboring point $P'(\theta^1 + d\theta^1, \theta^2 + d\theta^2, \theta^3 + d\theta^3)$, the vector becomes

$$\begin{aligned}\mathbf{v}(P') &= \mathbf{v}(P) + d\mathbf{v}(P) \\ &= [v^i(P) + dv^i(P)][\mathbf{g}_i(P) + d\mathbf{g}_i(P)].\end{aligned}$$

On passing to the limit $d\theta^i \rightarrow 0$, we obtain the principal part of the difference

$$(2) \quad d\mathbf{v} = (v^i + dv^i)(\mathbf{g}_i + d\mathbf{g}_i) - v^i \mathbf{g}_i = \mathbf{g}_i dv^i + v^i d\mathbf{g}_i,$$

and the derivative

$$(3) \quad \Delta \quad \partial \mathbf{v} / \partial \theta^j = \mathbf{g}_i \partial v^i / \partial \theta^j + (\partial \mathbf{g}_i / \partial \theta^j) v^i.$$

Thus the derivative of the vector \mathbf{v} is resolved into two parts: one arising from the variation of the components v^i as the coordinates $\theta^1, \theta^2, \theta^3$ are changed, the other arising from the change of the base vector \mathbf{g}_i as the position of the point P^i is changed. It is shown below that, in a Euclidean space,

$$(4) \quad \Delta \quad \partial \mathbf{g}_i / \partial \theta^j = \Gamma_{ij}^\alpha \mathbf{g}_\alpha.$$

Hence

$$(5) \quad \Delta \quad \partial \mathbf{v} / \partial \theta^j = \mathbf{g}_\alpha \partial v^\alpha / \partial \theta^j + v^i \Gamma_{ij}^\alpha \mathbf{g}_\alpha = (\partial v^\alpha / \partial \theta^j + v^i \Gamma_{ij}^\alpha) \mathbf{g}_\alpha = v^\alpha|_j \mathbf{g}_\alpha.$$

Thus, the covariant derivative $v^\alpha|_j$ represents the components of $\partial \mathbf{v} / \partial \theta^j$ referred to the base vectors \mathbf{g}_α .

To establish Eq. (4), we differentiate the equation $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ to obtain

$$\partial g_{ij} / \partial \theta^k = (\partial \mathbf{g}_i / \partial \theta^k) \cdot \mathbf{g}_j + (\partial \mathbf{g}_j / \partial \theta^k) \cdot \mathbf{g}_i.$$

On permuting the indices i, j, k , we can obtain the derivatives $\partial g_{jk} / \partial \theta^i, \partial g_{ki} / \partial \theta^j$. Furthermore, since $\mathbf{g}_i = \partial \mathbf{R} / \partial \theta^i$, we have

$$\frac{\partial \mathbf{g}_i}{\partial \theta^j} = \frac{\partial}{\partial \theta^j} \left(\frac{\partial \mathbf{R}}{\partial \theta^i} \right) = \frac{\partial}{\partial \theta^i} \left(\frac{\partial \mathbf{R}}{\partial \theta^j} \right) = \frac{\partial \mathbf{g}_j}{\partial \theta^i}.$$

Hence, by substitution, it is easy to show that

$$(\partial g_{ik} / \partial \theta^j + \partial g_{jk} / \partial \theta^i - \partial g_{ij} / \partial \theta^k) / 2 = (\partial \mathbf{g}_i / \partial \theta^j) \cdot \mathbf{g}_k.$$

On multiplying the two sides of the equation by $g^{\alpha k}$ and summing over k , the left-hand side becomes Γ_{ij}^α , according to the definition on Christoffel symbols. We then multiply the scalar quantity on both sides of the equation by the vectors \mathbf{g}_α and sum over α to obtain

$$(6) \quad \Gamma_{ij}^\alpha \mathbf{g}_\alpha = [(\partial \mathbf{g}_i / \partial \theta^j) \cdot \mathbf{g}_k] g^{\alpha k} \mathbf{g}_\alpha = [(\partial \mathbf{g}_i / \partial \theta^j) \cdot \mathbf{g}_k] \mathbf{g}^k.$$

The right-hand side, the sum of the vectors \mathbf{g}_k multiplied by the scalar quantities $(\partial \mathbf{g}_i / \partial \theta^j) \cdot \mathbf{g}_k$ is exactly equal to the vector $\partial \mathbf{g}_i / \partial \theta^j$ itself. To see this, we note that since the set of vectors $\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3$ are linearly independent and form a basis of the vector space, $\partial \mathbf{g}_i / \partial \theta^j$ can be expressed as a linear combination

$$(7) \quad \partial \mathbf{g}_i / \partial \theta^j = \lambda_1 \mathbf{g}^1 + \lambda_2 \mathbf{g}^2 + \lambda_3 \mathbf{g}^3,$$

where $\lambda_1, \lambda_2, \lambda_3$ are scalars. If we multiply this equation by \mathbf{g}_1 , we obtain

$$(8) \quad \lambda_1 = (\partial \mathbf{g}_i / \partial \theta^j) \cdot \mathbf{g}_1.$$

Similarly, λ_2, λ_3 can be evaluated. A comparison of Eqs. (6), (8), and (7) thus shows the truth of Eq. (2.15:4):

$$\Gamma_{ij}^\alpha \mathbf{g}_\alpha = \partial \mathbf{g}_i / \partial \theta^j. \quad \text{Q.E.D.}$$

2.16. PHYSICAL COMPONENTS OF A VECTOR

The base vectors \mathbf{g}_r and \mathbf{g}^r are in general not unit vectors. In fact, their lengths are

$$|\mathbf{g}_r| = \sqrt{g_{rr}}, \quad |\mathbf{g}^r| = \sqrt{g^{rr}}, \quad r \text{ not summed.}$$

Let us write Eq. (2.14:12) as

$$(1) \quad \mathbf{v} = \sum_{r=1}^3 v^r \sqrt{g_{rr}} (\mathbf{g}_r / \sqrt{g_{rr}}) = \sum_{r=1}^3 v_r \sqrt{g^{rr}} (\mathbf{g}^r / \sqrt{g^{rr}}).$$

Then, since $\mathbf{g}_r / \sqrt{g_{rr}}$ and $\mathbf{g}^r / \sqrt{g^{rr}}$ are unit vectors, all components $v^r \sqrt{g_{rr}}$ and $v_r \sqrt{g^{rr}}$ (r not summed) will have the same physical dimensions. It is seen that $v^r \sqrt{g_{rr}}$ are the components of \mathbf{v} resolved in the direction of unit vectors $\mathbf{g}_r / \sqrt{g_{rr}}$ which are tangent to the coordinate lines; and that $v_r \sqrt{g^{rr}}$ are the components of \mathbf{v} resolved in the direction of unit vectors $\mathbf{g}^r / \sqrt{g^{rr}}$ which are perpendicular to the coordinate planes. The components

$$v^r \sqrt{g_{rr}}, \quad v_r \sqrt{g^{rr}}, \quad r \text{ not summed},$$

are called the *physical components of the vector \mathbf{v}* . They do not transform according to the tensor transformation law and are not components of tensors. Note that, in general $v^r \sqrt{g_{rr}} \neq v_r \sqrt{g^{rr}}$ (r not summed) except for orthogonal coordinates.

We should remember that the tensor components of a physical quantity which is referred to a particular curvilinear coordinate system may or may not have the same physical dimensions. This difficulty (and it is also a great convenience!) arises because we would like to keep our freedom in choosing arbitrary curvilinear coordinates. Thus, in spherical polar coordinates for a Euclidean space, the position of a point is expressed by a length and two angles. In a four-dimensional space a point may be expressed in three lengths and a time. For this reason, we must distinguish the tensor components from the “physical components,” which must have uniform physical dimensions.

P R O B L E M S

2.12. Show that in an orthogonal n -dimensional coordinates system, we have for each component $i = j$, $g^{ij} = 1/g_{ij}$.

2.13. If a_{ij} is a tensor, and the components $a_{ij} = a_{ji}$, then the tensor a_{ij} is called a *symmetric tensor*. If the components $a_{ij} = -a_{ji}$, then the tensor a_{ij} is said to be *skew-symmetric*, or *antisymmetric*. Show that the symmetry of a tensor with respect to two indices at the same level is conserved under coordinate transformations. Since $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji})$, any covariant (or contravariant) second-order tensor can be written as the sum of a symmetric and a skew-symmetric tensor.

2.14. Show that the scalar product of a symmetric tensor S^{ij} and a skew-symmetric tensor W_{ij} vanishes identically.

2.15. Show that the Cartesian tensor $\omega_{ik} = e_{ijk} u_j$ is skew-symmetric, where u_j is a vector.

2.16. Show that if A_{jk} is a skew-symmetric Cartesian tensor, then the unique solution of the equation $\omega_i = \frac{1}{2}e_{ijk}A_{jk}$ is $A_{mn} = e_{mni}\omega_i$.

2.17. Let $\nabla(\cdot)$ be the del operator $\nabla(\cdot) = (\partial/\partial\theta^r)\mathbf{g}^r$. Show that

$$\text{grad } \phi = \nabla\phi = \left(\frac{\partial\phi}{\partial\theta^r} \right) \mathbf{g}^r, \quad \text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = F^r|_r, \quad \text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \epsilon^{rst} A_s|_r \mathbf{g}_t.$$

Show that these functions are invariant under coordinate transformations.

2.18. Let $g^{\alpha\beta}$ be the associated contravariant Euclidean metric tensor and $\psi(x^1, x^2, x^3)$ be a scalar field. Show that

(a) $g^{\alpha\beta}\psi/\alpha\beta$ is a scalar field.

(b) In rectangular Cartesian coordinates $g\alpha\beta = \delta\alpha\beta$, the scalar $g^{\alpha\beta}\psi/\alpha\beta$ reduces to the form (writing $x^1 = x$, $x^2 = y$, $x^3 = z$)

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 + \partial^2\psi/\partial z^2.$$

(c) Hence, the Laplace equation in curvilinear coordinates with the scalar field $\psi(x^1, x^2, x^3)$ is given by

$$g^{\alpha\beta}(x^1, x^2, x^3)\psi(x^1, x^2, x^3)|_{\alpha\beta} = 0.$$

2.19. Let y^1, y^2, y^3 (or x, y, z) be rectangular Cartesian coordinates and x^1, x^2, x^3 (or r, ϕ, θ) be the spherical polar coordinates. Show that the Laplace equation in spherical polar coordinates, when the unknown function is a scalar field

$\psi(r, \phi, \theta)$, is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{\cot \phi}{r^2} \frac{\partial \psi}{\partial \phi} = 0.$$

2.20. Let $\xi^i(y^1, y^2, y^3)$ be the components of an unknown vector in rectangular Cartesian coordinates. Let each component satisfy Laplace's equation in rectangular coordinates,

$$\partial^2 \xi^i / \partial (y^1)^2 + \partial^2 \xi^i / \partial (y^2)^2 + \partial^2 \xi^i / \partial (y^3)^2 = 0.$$

Show that a generalization of this equation is that $\xi^i(x^1, x^2, x^3)$ in curvilinear coordinates x^1, x^2, x^3 will satisfy the system of three differential equations

$$g^{\alpha\beta} \xi^i |_{\alpha\beta} = 0,$$

where $\xi^i |_{\alpha\beta}$ is the second covariant derivative of $\xi^i(x^1, x^2, x^3)$.

2.21. Prove that

$$F^r |_r = [\partial(\sqrt{g} F^r) / \partial x^r] / \sqrt{g}.$$

Hint: Since $g = g_i \alpha G^{i\alpha}$ (i not summed), G^{ij} is the cofactor of the element g^{ij} in $g = |g_{ij}|$, show that

$$\partial g / \partial x^i = gg^{\alpha\beta} (\partial g_{\alpha\beta} / \partial x^i) \quad \text{and} \quad \partial g / \partial x^i = 2g\Gamma_{\alpha i}^\alpha.$$

Hence,

$$\Gamma_{\alpha i}^\alpha = \partial(\log \sqrt{g}) / \partial x^i.$$

2.22. Prove that the Laplacian of Prob. 2.18 can be written

$$g^{ij} \psi |_{ij} = \{\partial[\sqrt{g} g^{ij} (\partial \psi / \partial x^j)] / \partial x^i\} / \sqrt{g}.$$

Hint: Use the results of Prob. 2.21.

2.23. Show that the covariant differentiation of sums and products follows the usual rules for partial differentiation. Thus,

$$\begin{aligned} (\phi v^r) |_i &= \phi, i v^r + \phi v^r |_i, \\ (v^r v_r) |_i &= (v^r v_r), i = v^r |_i v_r + v^r v_r |_i, \\ (A_{ij} B^{mn}) |_r &= A_{ij} |_r B^{mn} + A_{ij} B^{mn} |_r, \end{aligned}$$

where a comma indicates partial differentiation. Remember that the covariant derivatives of a scalar are the same as the partial derivatives.

2.24. Derive the transformation law for the Euclidean Christoffel symbols $\Gamma_{\alpha\beta}^i(x^1, x^2, x^3)$.

2.25. All the results obtained above can be applied to two-dimensional spaces by assigning the range of indices to be 1 to 2 instead of 1 to 3. Compute the Euclidean Christoffel symbols for the plane polar coordinates ($x^1 = r, x^2 = \theta$)

$$\text{Ans. } \Gamma_{22}^1 = -x^1, \Gamma_{12}^2 = 1/x^1, \text{ all other components} = 0.$$

2.26. Show that Γ_{mn}^r is symmetric in m and n ; i.e., $\Gamma_{mn}^r = \Gamma_{nm}^r$.

2.27. Show that the necessary and sufficient condition that a given curvilinear coordinate system be orthogonal is that $g^{ij} = 0$, if i, j , throughout the domain.

2.28. Prove that $\mathbf{g}_r \times \mathbf{g}_s = \epsilon_{rst} \mathbf{g}^t$, $\mathbf{g}^r \times \mathbf{g}^s = \epsilon^{rst} \mathbf{g}_t$, where ϵ^{rst} , ϵ_{rst} are the permutation tensor of Sec. 2.6. Hence, if we denote the scalar product of the vectors \mathbf{g}_1 and $\mathbf{g}_2 \times \mathbf{g}_3$ by $[\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3]$ or $(\mathbf{g}_1, \mathbf{g}_2 \times \mathbf{g}_3)$, we have

$$[\mathbf{g}_1 \mathbf{g}_2 \mathbf{g}_3] = (\mathbf{g}_1, \mathbf{g}_2 \times \mathbf{g}_3) = \sqrt{g}, \quad [\mathbf{g}^1 \mathbf{g}^2 \mathbf{g}^3] = (\mathbf{g}^1, \mathbf{g}^2 \times \mathbf{g}^3) = 1/\sqrt{g}.$$

2.29. The element of area of a parallelogram with two adjacent edges $d\mathbf{s}_2 = \mathbf{g}_2 d\theta^2$ and $d\mathbf{s}_3 = \mathbf{g}_3 d\theta^3$ is

$$dS_1 = |d\mathbf{s}_2 \times d\mathbf{s}_3| = |\mathbf{g}_2 \times \mathbf{g}_3| d\theta^2 d\theta^3.$$

Show that $dS_1 = \sqrt{(gg^{11})} d\theta^2 d\theta^3$. In general, the element of area dS_i of a parallelogram formed by the elements $g_j d\theta^j$ and $g_k d\theta^k$, (j, k not summed), on the θ^i -surface is $dS_i = \sqrt{(gg^{ii})} d\theta^j d\theta^k$ (i not summed, $i \neq j, k$).

2.30. With reference to Prob. 2.28, show that the volume element

$$dV = ds_1 \cdot (ds_2 \times ds_3) = [g_1 g_2 g_3] d\theta^1 d\theta^2 d\theta^3 = \sqrt{g} d\theta^1 d\theta^2 d\theta^3.$$

2.31. If the space is non-Euclidean, we cannot find a coordinate system in which the metric tensor g_{ij} has constant components everywhere throughout the space. (In a Euclidean space there do exist just such coordinate systems, namely, Cartesian coordinate systems.) In this case, we must compute the successive derivatives $\xi_r|_{st}$ and $\xi_r|_{ts}$ according to the covariant differentiation rules. Show that

$$\xi^i|_{st} - \xi^i|_{ts} = R_{pst}^i \xi^p,$$

where

$$R_{pst}^i = \partial \Gamma_{ps}^i / \partial x^t - \partial \Gamma_{pt}^i / \partial x^s + \Gamma_{ps}^r \Gamma_{rt}^i - \Gamma_{pt}^r \Gamma_{rs}^i.$$

Show that R_{pst}^i is a tensor of rank 4, which is the famous *Riemann–Christoffel curvature tensor*. It is not a zero tensor in a general Riemannian space. Hence, in general,

$$\xi^i|_{st} \neq \xi^i|_{ts},$$

in a Riemannian space.

Note: The results obtained above hold true also for two-dimensional spaces, provided all indices range over 1 and 2 only. A curved shell in a three-dimensional Euclidean space appears, in general, to be a two-dimensional non-Euclidean space to a “two-dimensional” animal who has to measure distances right on the shell surface and is never allowed to leave the shell surface to view the third dimension. For a spherical surface, the nonvanishing components of the two-dimensional Riemann–Christoffel curvature tensor are all equal to a constant, which may be written as 1. For a flat plate, they are zero. For certain hyperboloidal surface all the nonvanishing components of curvature may take on the value –1. Since the spirit of the theory of thin elastic shells is to reduce all the properties of the shells into differential equations describing the middle surface of the shell, an engineer deals with non-Euclidean geometry rather frequently.

2.32. In cylindrical coordinates with unit base vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$, a vector \mathbf{u} and its gradient, gradient transpose and divergent are, respectively,

$$\begin{aligned} \mathbf{u} &= u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z, \\ \nabla \mathbf{u} &= \frac{\partial \mathbf{u}}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \mathbf{u}}{\partial \theta} \mathbf{e}_\theta + \frac{\partial \mathbf{u}}{\partial z} \mathbf{e}_z, \quad (\nabla \mathbf{u})^T = \mathbf{e}_r \frac{\partial \mathbf{u}}{\partial r} + \frac{\mathbf{e}_\theta}{r} \frac{\partial \mathbf{u}}{\partial \theta} + \mathbf{e}_z \frac{\partial \mathbf{u}}{\partial z}, \\ \nabla \cdot \mathbf{u} &= u^i|_i = \mathbf{e}_r \cdot \frac{\partial \mathbf{u}}{\partial r} + \frac{\mathbf{e}_\theta}{r} \cdot \frac{\partial \mathbf{u}}{\partial \theta} + \mathbf{e}_z \cdot \frac{\partial \mathbf{u}}{\partial z} = \frac{1}{r} \frac{\partial r u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}. \end{aligned}$$

Show that, $\nabla \cdot (\nabla \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) = u^i|_{ij} g^j$ in general, and for cylindrical coordinates.

$$\begin{aligned} \nabla(\nabla \cdot \mathbf{u}) &= u^i|_{ij} g^j = \mathbf{e}_r \frac{\partial(\nabla \cdot \mathbf{u})}{\partial r} + \frac{\mathbf{e}_\theta}{r} \frac{\partial(\nabla \cdot \mathbf{u})}{\partial \theta} + \mathbf{e}_z \frac{\partial(\nabla \cdot \mathbf{u})}{\partial z} = \nabla \cdot (\nabla \mathbf{u}), \\ \nabla \cdot (\nabla \mathbf{u})^T &= u^i|_{kj} g^{jk} \mathbf{g}_i = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \mathbf{u}}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \mathbf{u}}{\partial \theta^2} + \frac{\partial^2 \mathbf{u}}{\partial z^2}. \end{aligned}$$

2.33. In spherical coordinates with unit base vectors $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_\theta)$, we have

$$\begin{aligned} \mathbf{u} &= u_r \mathbf{e}_r + u_\phi \mathbf{e}_\phi + u_\theta \mathbf{e}_\theta, \quad \nabla \mathbf{u} = \frac{\partial \mathbf{u}}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \mathbf{u}}{\partial \phi} \mathbf{e}_\phi + \frac{1}{r \sin \phi} \frac{\partial \mathbf{u}}{\partial \theta} \mathbf{e}_\theta, \\ (\nabla \mathbf{u})^T &= \mathbf{e}_r \frac{\partial \mathbf{u}}{\partial r} + \frac{\mathbf{e}_\phi}{r} \frac{\partial \mathbf{u}}{\partial \phi} + \frac{\mathbf{e}_\theta}{r \sin \phi} \frac{\partial \mathbf{u}}{\partial \theta}, \\ \nabla \cdot \mathbf{u} &= u^i|_i = \mathbf{e}_r \cdot \frac{\partial \mathbf{u}}{\partial r} + \frac{\mathbf{e}_\phi}{r} \cdot \frac{\partial \mathbf{u}}{\partial \phi} + \frac{\mathbf{e}_\theta}{r \sin \phi} \cdot \frac{\partial \mathbf{u}}{\partial \theta} \\ &= \frac{1}{r^2} \frac{\partial r^2 u_r}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial \sin \phi u_\phi}{\partial \phi} + \frac{1}{r \sin \phi} \frac{\partial u_\theta}{\partial \theta}. \end{aligned}$$

Show that, $\nabla \cdot (\nabla \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) = u^i|_{ij} g^j$ in general, and for spherical coordinates.

$$\begin{aligned}\nabla(\nabla \cdot \mathbf{u}) &= u^i|_{ij} \mathbf{g}^j = \frac{\partial(\nabla \cdot \mathbf{u})}{\partial r} \mathbf{e}_r + \frac{\partial(\nabla \cdot \mathbf{u})}{r \partial \phi} \mathbf{e}_\phi + \frac{1}{r \sin \phi} \frac{\partial(\nabla \cdot \mathbf{u})}{\partial \theta} \mathbf{e}_\theta = \nabla \cdot (\nabla \mathbf{u}), \\ \nabla \cdot (\nabla \mathbf{u})^T &= u^i|_{kj} g^{jk} \mathbf{g}_i \\ &= \frac{1}{r^2} \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial \mathbf{u}}{\partial r} \right) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial \mathbf{u}}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2 \mathbf{u}}{\partial \theta^2} \right].\end{aligned}$$

These equations are needed in the Navier equations in later chapters.

¹One may pass over the rest of this chapter and proceed directly to [Chapter 3](#) on first reading.

²The variables θ^i and $\bar{\theta}^i$, in general, are not so related [see Eq. (2.3:9)]. Thus, although the differential $d\theta^i$ is a contravariant vector, the set of variables θ^i itself does not transform like a vector. Hence, in this instance, the position of the index of θ^i must be regarded as without significance.

3

STRESS TENSOR

The definitions of stress vector and stress components will be given and the equations of equilibrium will be derived. We shall then show how the stress components change when the frames of reference are changed from one rectangular Cartesian frame of reference to another, and in this way we will prove from physical standpoint that the stress components transform according to the tensor transformation rules. The symmetry of the stress tensor will then be discussed, and the consequences of the symmetry property will be derived. The principal stresses, the stress deviations, the octahedral stresses, and finally, the stress tensor in general curvilinear coordinates, form the material for the remainder of the chapter.

Except for Sec. 3.14 et seq., we shall use only rectangular Cartesian frames of reference, whose coordinate axes will be denoted by x_1, x_2, x_3 and are rectilinear and orthogonal to each other. We shall use subscripts for all components unless stated otherwise.

3.1. STRESSES

Consider a configuration occupied by a body B at some time (Fig. 3.1:1). Imagine a closed surface S within B . We would like to know the interaction between the material exterior to this surface and that in the interior. In this consideration, there arises the basic defining concept of continuum mechanics — the *stress principle* of Euler and Cauchy.

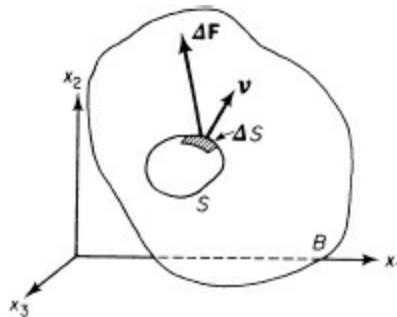


Fig. 3.1:1. Stress principle.

Consider a small surface element of area ΔS on our imagined surface S . Let us draw a unit vector v normal to ΔS , with its direction outward from the interior of S . Then we can distinguish the two sides of ΔS according to the direction of v . Consider the part of material lying on the positive side of the normal. This part exerts a force ΔF on the other part, which is situated on the negative side of the normal. The force ΔF is a function of the area and the orientation of the surface. We introduce the assumption that as ΔS tends to zero, the ratio $\Delta F/\Delta S$ tends to a definite limit $d\mathbf{F}/dS$ and that the moment of the forces acting on the surface ΔS about any point within the area vanishes in the limit. The limiting vector will be written as

$$(1) \quad \overset{\nu}{\mathbf{T}} = d\mathbf{F}/dS,$$

where an overhead ν is introduced to denote the direction of the normal v of the surface ΔS . The limiting vector $\overset{\nu}{\mathbf{T}}$ is called the *stress vector*, or *traction*, and represents the force per unit area acting on the surface.

The assertion that there is defined upon any imagined closed surface S in the interior of a continuum a stress vector field whose action on the material occupying the space interior to S is equipollent to the action of the exterior material upon it, is the stress principle of Euler and Cauchy.

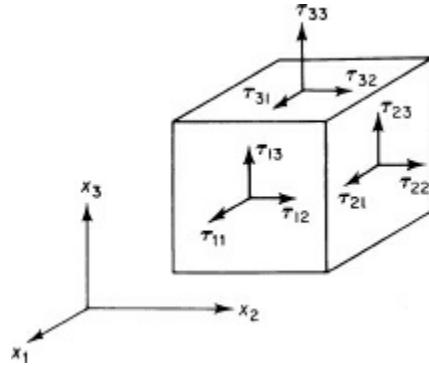


Fig. 3.1:2. Notations of stress components.

Consider now a special case in which the surface ΔS_k is parallel to one of the coordinate planes. Let the normal of ΔS_k be in the positive direction of the x_k -axis. Let the stress vector acting on ΔS_k be denoted by $\overset{k}{\mathbf{T}}$, with three components $\overset{k}{T}_1, \overset{k}{T}_2, \overset{k}{T}_3$ along the direction of the k coordinate axes x_1, x_2, x_3 , respectively. The index i of $\overset{k}{T}_i$ denotes the components of the force, and the symbol k indicates the surface on which the force acts. In this special case, we introduce a new set of symbols for the stress components,

$$(2) \quad \overset{k}{T}_1 = \tau_{k1}, \quad \overset{k}{T}_2 = \tau_{k2}, \quad \overset{k}{T}_3 = \tau_{k3}.$$

If we arrange the components of the tractions acting on the surfaces $k = 1, k = 2, k = 3$ in a square matrix, we obtain

	Components of Stresses		
	1	2	3
Surface normal to x_1	τ_{11}	τ_{12}	τ_{13}
Surface normal to x_2	τ_{21}	τ_{22}	τ_{23}
Surface normal to x_3	τ_{31}	τ_{32}	τ_{33}

This is illustrated in Fig. 3.1:2. The components $\tau_{11}, \tau_{22}, \tau_{33}$ are called *normal stresses*, and the remaining components $\tau_{12}, \tau_{13}, \text{etc.}$, are called *shearing stresses*. Each of these components has the dimension of force per unit area, or that of [Mass] [Length] $^{-1}$ [Time] $^{-2}$.

A great diversity in notations for stress components exists in the literature. The most widely used notations in American literature are, in reference to a system of rectangular Cartesian coordinates x, y, z ,

$$(3) \quad \begin{array}{lll} \sigma_x & \tau_{xy} & \tau_{xz}, \\ \tau_{yx} & \sigma_y & \tau_{yz}, \\ \tau_{zx} & \tau_{zy} & \sigma_z. \end{array}$$

Love writes X_x, Y_x for σ_x and τ_{xy} , and Todhunter and Pearson use $\widehat{x}\widehat{x}, \widehat{x}\widehat{y}$. In Chapters 3–11 we shall use both τ_{ij} and σ_{ij} . We use τ_{ij} to denote stress tensors in general, and we use σ_{ij} to denote the physical components of stress tensors in curvilinear coordinate. In rectangular Cartesian coordinates, the tensor components and the physical components coincide. Hence, we use σ_{ij} for Cartesian stress tensors. Although the lack of uniformity may seem awkward, little confusion will arise, and in many instances different notations actually result in clarity.

It is important to emphasize again that a stress will always be understood to be the force (per unit area) which the part lying on the positive side of a surface element (the side on the positive side of the outer normal) exerts on the part lying on the negative side. Thus, if the outer normal of a surface element points in the positive direction of the x_2 -axis and τ_{22} is positive, the vector representing the component of normal stress acting on the surface element will point in the positive x_2 -direction. But if τ_{22} is positive while the outer normal points in the negative x_2 -axis direction, then the stress vector acting on the element also points to the negative x_2 -axis direction (see Fig. 3.1:3).

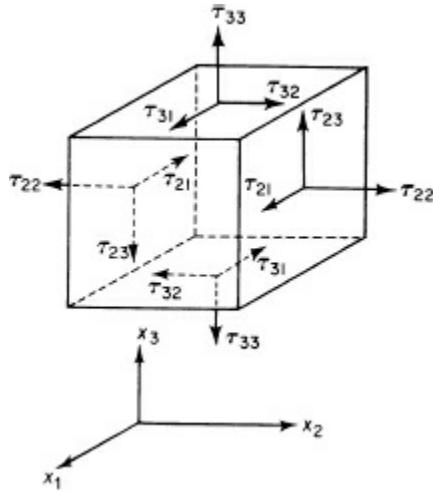


Fig. 3.1:3. Senses of positive stresses.

Similarly, positive values of τ_{21} , τ_{23} will imply shearing stress vectors pointing to positive x_2 , x_3 -axes if the outer normal agrees in sense with x_2 -axis, whereas they point to the negative x_1 , x_3 -direction if the outer normal disagrees in sense with the x_2 -axis, as illustrated in Fig. 3.1:3. A careful study of the figure is essential. Naturally, these rules agree with the usual notions of tension, compression, and shear.

3.2. LAWS OF MOTION

The fundamental laws of mechanics for bodies of all kinds are Euler's equations, which extend Newton's laws of motion for particles. Let the coordinate system x_1 , x_2 , x_3 be an inertial frame of reference. Let the space occupied by a material body at any time t be denoted by $B(t)$. Let \mathbf{r} be the position vector of a particle with respect to the origin of the coordinate system. Let \mathbf{V} be the velocity vector of a particle at the point x_1 , x_2 , x_3 . Then

$$(1) \quad \mathcal{P} = \int_{B(t)} \mathbf{V} \rho dv,$$

is called the *linear momentum* of the body in the configuration $B(t)$, and let

$$(2) \quad \mathcal{H} = \int_{B(t)} \mathbf{r} \times \mathbf{V} \rho dv,$$

is called the *moment of momentum*. In these formulas ρ is the density of the material and the integration is over the volume $B(t)$. *Newton's laws, as stated by Euler for a continuum, assert that the rate of change of linear momentum is equal to the total applied force \mathcal{F} acting on the body,*

$$(3) \quad \dot{\mathcal{P}} = \mathcal{F},$$

and that the rate of change of moment of momentum is equal to the total applied torque \mathcal{L} ,

$$(4) \quad \dot{\mathcal{H}} = \mathcal{L}.$$

The torque \mathcal{L} is taken with respect to the same point as the origin of the position vector \mathbf{r} . It is easy to verify that if (3) holds, then if (4) holds for one choice of origin, it holds for all choices of origin.¹

It is assumed that force and torque are quantities about which we have *a priori* information in certain frames of reference.

On material bodies considered in mechanics of continuous media, two types of external forces act:

1. Body forces, acting on elements of volume of the body.
2. Surface forces or stresses, acting on surface elements.

Examples of body forces are gravitational forces and electromagnetic forces. Examples of surface forces are aerodynamic pressure acting on a body and pressure due to mechanical contact of two bodies.

To specify a body force, we consider a volume bounded by an arbitrary surface S (Fig. 3.2:1). The resultant force

vector contributed by the body force is assumed to be representable in the form of a volume integral

$$\int_B \mathbf{X} dv .$$

The three components of \mathbf{X} , namely, X_1, X_2, X_3 , all of dimensions force per unit volume $M (LT)^{-2}$, are called the body force per unit volume. For example, in a gravitational field,

$$X_i = \rho g_i ,$$

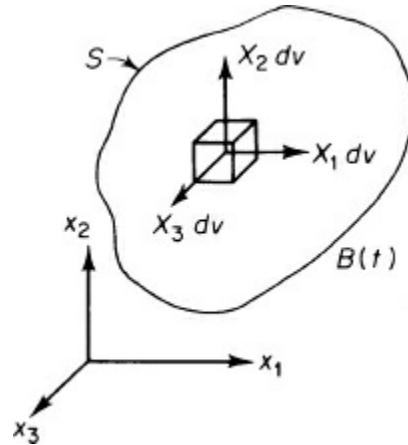


Fig. 3.2:1. Body forces.

where g_i are components of a gravitational acceleration field and ρ is the density (mass per unit volume) at a given point of the body.

The surface force acting on an imagined surface in the interior of a body is the stress vector conceived in Euler and Cauchy's stress principle. According to this concept, the total force acting upon the material occupying the region B interior to a closed surface S is

$$(5) \quad \mathcal{F} = \oint_S \overset{\nu}{\mathbf{T}} dS + \int_B \mathbf{X} dv ,$$

where $\overset{\nu}{\mathbf{T}}$ is the stress vector acting on dS whose outer normal vector is ν . Similarly, the torque about the origin is given by the expression

$$(6) \quad \mathcal{L} = \oint_S \mathbf{r} \times \overset{\nu}{\mathbf{T}} dS + \int_B \mathbf{r} \times \mathbf{X} dv .$$

In the following section we shall make some elementary applications of these equations to obtain the fundamental properties of the stress tensor.

3.3. CAUCHY'S FORMULA

With the equations of motion, we shall first derive a simple result which states that *the stress vector $\mathbf{T}^{(+)}$ representing the action of material exterior to a surface element on the interior is equal in magnitude and opposite in direction to the stress vector $\mathbf{T}^{(-)}$ which represents the action of the interior material on the exterior across the same surface element:*

$$(1) \quad \mathbf{T}^{(-)} = -\mathbf{T}^{(+)} .$$

To prove this, we consider a small "pill box" with two parallel surfaces of area ΔS and thickness δ , as shown in Fig. 3.3:1. When δ shrinks to zero, while ΔS remains small but finite, the volume forces and the linear momentum and its rate of change with time vanish, as well as the contribution of surface forces on the sides of the pill box. The equation of motion (3.2:3) implies, therefore, for small ΔS ,

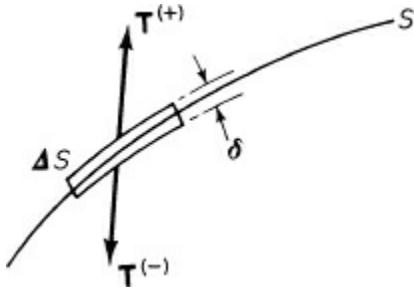


Fig. 3.3:1. Equilibrium of a “pill box” across a surface S .

$$\mathbf{T}^{(+)}\Delta S + \mathbf{T}^{(-)}\Delta S = 0.$$

Equation (1) then follows.

Another way of stating this result is that the stress vector is a function of the normal vector to a surface. When the sense of direction of the normal vector reverses, the stress vector reverses also.

Now we shall show that *knowing the components τ_{ij} , we can write down at once the stress vector acting on any surface with unit outer normal vector v whose components are v_1, v_2, v_3 . This stress vector is denoted by $\overset{\nu}{T}$, with components $\overset{\nu}{T}_1, \overset{\nu}{T}_2, \overset{\nu}{T}_3$ given by Cauchy’s formula*

$$(2) \quad \overset{\nu}{T}_i = v_j \tau_{ji},$$

which can be derived in several ways. We shall give first an elementary derivation.

Let us consider an infinitesimal tetrahedron formed by three surfaces parallel to the coordinate planes and one normal to the unit vector v (see Fig. 3.3:2). Let the area of the surface normal to v be dS . Then the area of the other three surfaces are

$$dS_1 = dS \cos(\nu, \mathbf{x}_1) = v_1 dS = \text{area of surface } \parallel \text{ to the } x_2x_3\text{-plane},$$

$$dS_2 = v_2 dS = \text{area of surface } \parallel \text{ to the } x_3x_1\text{-plane},$$

$$dS_3 = v_3 dS = \text{area of surface } \parallel \text{ to the } x_1x_2\text{-plane},$$

and the volume of the tetrahedron is

$$dv = h dS/3,$$

where h is the height of the vertex P from the base dS . The forces in the positive direction of \mathbf{x}_1 acting on the three coordinate surfaces can be written as

$$(-\tau_{11} + \varepsilon_1)dS_1, \quad (-\tau_{21} + \varepsilon_2)dS_2, \quad (-\tau_{31} + \varepsilon_3)dS_3,$$

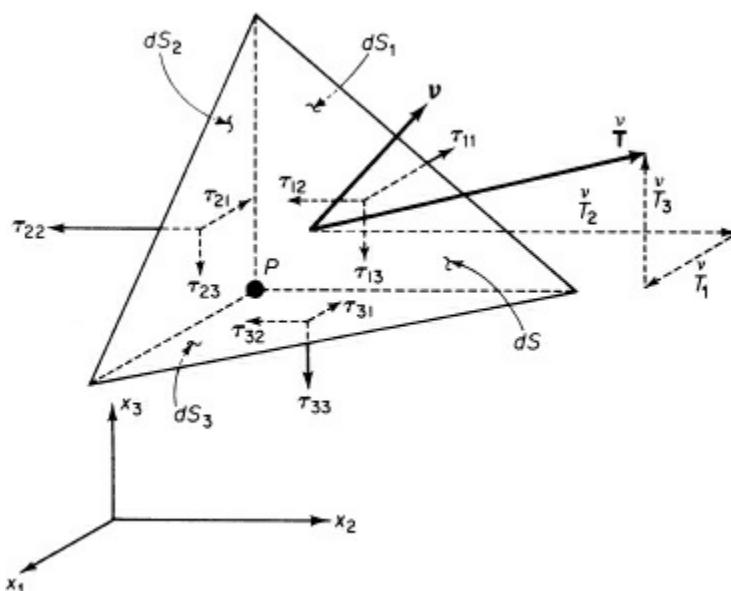


Fig. 3.3:2. Surface tractions on a tetrahedron.

where $\tau_{11}, \tau_{21}, \tau_{31}$ are the stresses at the point P . The negative sign is obtained because the outer normals to the three surfaces are opposite in sense with respect to the coordinate axes, and the ε 's are inserted because the tractions act at points slightly different from P . If we assume that the stress field is continuous, then $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are infinitesimal quantities. On the other hand, the v force acting on the triangle normal to v has a component $(\dot{T}_1 + \varepsilon)dS$ in the x_1 -axis direction, the body force has an x_1 -component equal to $(X_1 + \varepsilon')dv$, and the rate of change of linear momentum has a component $\rho \dot{V}_L dv$. Here \dot{T}_1 and X_1 refer to the point P and $\varepsilon, \varepsilon'$ are again infinitesimal. The first equation of motion is thus

$$(-\tau_{11} + \varepsilon_1)\nu_1 dS + (-\tau_{21} + \varepsilon_2)\nu_2 dS + (-\tau_{31} + \varepsilon_3)\nu_3 dS \\ + (\dot{T}_1 + \varepsilon)dS + (X_1 + \varepsilon')hdS/3 = \rho \dot{V}_1 hdS/3.$$

Dividing through by dS and taking the limit $h \rightarrow 0$, one obtains

$$(3) \quad \dot{T}_1 = \tau_{11}\nu_1 + \tau_{21}\nu_2 + \tau_{31}\nu_3,$$

which is the first component of Eq. (2). Other components follow similarly.

Cauchy's formula assures us that the nine components of stresses τ_{ij} are necessary and sufficient to define the traction across any surface element in a body. Hence the stress state in a body is characterized completely by the set of v quantities τ_{ij} . Since \dot{T}_i is a vector and Eq. (2) is valid for an arbitrary vector v_j , it follows from the quotient rule (Sec. 2.10) that τ_{ij} is a tensor. Henceforth τ_{ij} will be called a stress tensor.

We note again that in the theoretical development up to this point we have assumed, first, that stress can be defined everywhere in a body, and, second, that the stress field is continuous. The same assumption will be made later with respect to strain. These are characteristic assumptions of continuum mechanics. Without these assumptions we can do very little indeed. However, in the further development of the theory, certain mathematical discontinuities will be permitted — often they are very useful tools — but one should always view these discontinuities with great care against the general basic assumptions of continuity of the stress and strain fields.

3.4. EQUATIONS OF EQUILIBRIUM

We shall now transform the equations of motion (3.2:3), (3.2:4) into differential equations. This can be done elegantly by means of Gauss' theorem and Cauchy's formula. But we shall pursue here an elementary course to assure physical clarity.

Consider the static equilibrium state of an infinitesimal parallelepiped with surfaces parallel to the coordinate planes. The stresses acting on the various surfaces are shown in Fig. 3.4:1. The force $\tau_{11}dx_2dx_3$ acts on the left-hand side, the force $(\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} dx_1)dx_2dx_3$ acts on the right-hand side, etc. These expressions are based on the assumption of continuity of the stresses. The body force is $X_i dx_1 dx_2 dx_3$.

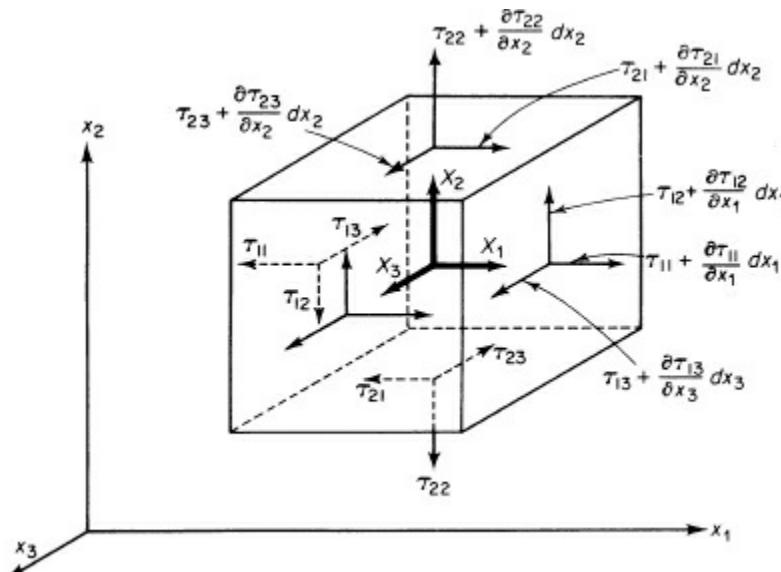


Fig. 3.4:1. Equilibrating stress components on an infinitesimal parallelepiped.

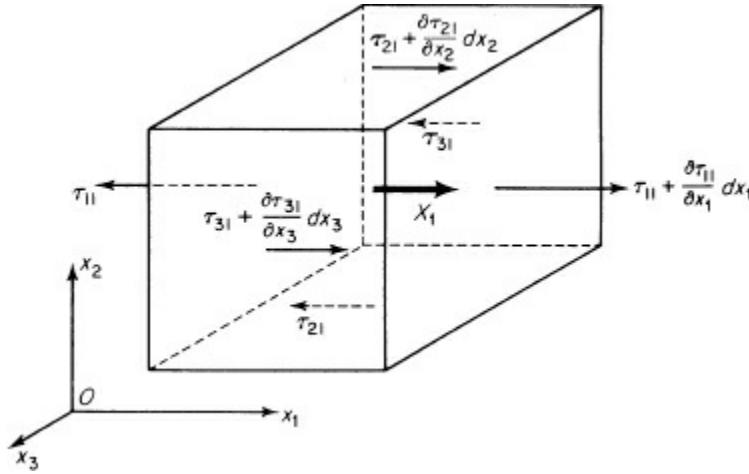


Fig. 3.4:2. Components of tractions in x_1 direction.

Now, the equilibrium of the body demands that the resultant forces vanish. Consider the forces in the x_1 -direction. As shown in Fig. 3.4:2, we have six components of surface forces and one component of body force. The sum is

$$\begin{aligned} & \left(\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} dx_1 \right) dx_2 dx_3 - \tau_{11} dx_2 dx_3 \\ & + \left(\tau_{21} + \frac{\partial \tau_{21}}{\partial x_2} dx_2 \right) dx_3 dx_1 - \tau_{21} dx_3 dx_1 \\ & + \left(\tau_{31} + \frac{\partial \tau_{31}}{\partial x_3} dx_3 \right) dx_1 dx_2 - \tau_{31} dx_1 dx_2 + X_1 dx_1 dx_2 dx_3 = 0. \end{aligned}$$

Dividing by $dx_1 dx_2 dx_3$, we obtain

$$(1) \quad \partial \tau_{11} / \partial x_1 + \partial \tau_{21} / \partial x_2 + \partial \tau_{31} / \partial x_3 + X_1 = 0.$$

A cyclic permutation of subscripts 1, 2, 3 leads to similar equations of equilibrium of forces in x_2 , x_3 -directions. The whole set, written concisely, is

$$(2) \quad \blacktriangle \quad \partial \tau_{ji} / \partial x_j + X_i = 0.$$

This is an important result. A shorter derivation will be given later in Sec. 5.5.

The equilibrium of an element requires also that the resultant moment vanishes. If there do not exist external moments proportional to a volume, the consideration of moments will lead to the important conclusion that *the stress tensor is symmetric*,

$$(3) \quad \blacktriangle \quad \tau_{ij} = \tau_{ji}.$$

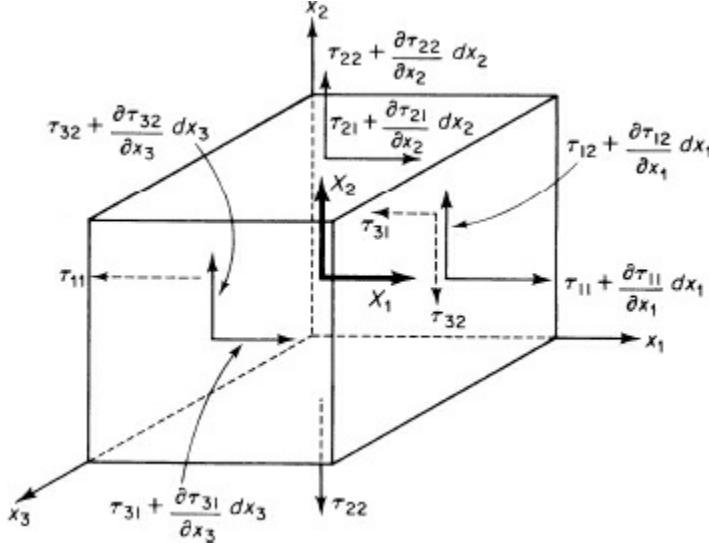


Fig. 3.4:3. Components of tractions that contribute moment about Ox_3 -axis.

This is demonstrated as follows. Referring to Fig. 3.4:3 and considering the moment of all the forces about the axis

Ox_3 , we see that those components of forces parallel to Ox_3 or lying in planes containing Ox_3 do not contribute any moment. The components that do contribute a moment about the Ox_3 -axis are shown in Fig. 3.4:3. Therefore, properly taking care of the moment arm, we have

$$\begin{aligned}
& - \left(\tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} dx_1 \right) dx_2 dx_3 \cdot \frac{dx_2}{2} + \tau_{11} dx_2 dx_3 \cdot \frac{dx_2}{2} \\
& + \left(\tau_{12} + \frac{\partial \tau_{12}}{\partial x_1} dx_1 \right) dx_2 dx_3 \cdot dx_1 - \left(\tau_{21} + \frac{\partial \tau_{21}}{\partial x_2} dx_2 \right) dx_1 dx_3 \cdot dx_2 \\
& + \left(\tau_{22} + \frac{\partial \tau_{22}}{\partial x_2} dx_2 \right) dx_1 dx_3 \cdot \frac{dx_1}{2} - \tau_{22} dx_1 dx_3 \cdot \frac{dx_1}{2} \\
& + \left(\tau_{32} + \frac{\partial \tau_{32}}{\partial x_3} dx_3 \right) dx_1 dx_2 \cdot \frac{dx_1}{2} - \tau_{32} dx_1 dx_2 \cdot \frac{dx_1}{2} \\
& - \left(\tau_{31} + \frac{\partial \tau_{31}}{\partial x_3} dx_3 \right) dx_1 dx_2 \cdot \frac{dx_2}{2} + \tau_{31} dx_1 dx_2 \cdot \frac{dx_2}{2} \\
& - X_1 dx_1 dx_2 dx_3 \cdot \frac{dx_2}{2} + X_2 dx_1 dx_2 dx_3 \cdot \frac{dx_1}{2} = 0.
\end{aligned}$$

On dividing through by $dx_1 dx_2 dx_3$ and passing to the limit $dx_1 \rightarrow 0, dx_2 \rightarrow 0, dx_3 \rightarrow 0$, we obtain

$$(4) \quad \tau_{12} = \tau_{21}.$$

Similar considerations of resultant moments about Ox_1 , Ox_2 lead to the general result given by Eq. (3). Again a shorter derivation will be given later in Sec. 5.5.

It should be noted that if an external moment proportional to the volume does exist, then the symmetry condition does not hold. For example, if there is a moment $c_3 dx_1 dx_2 dx_3$ about the axis Ox_3 , then we obtain in place of Eq. (4) the result

$$(5) \quad \tau_{12} - \tau_{21} + c_3 = 0.$$

Maxwell pointed out that nonvanishing body moments exist in a magnet in a magnetic field and in a dielectric material in an electric field with different planes of polarization. If the electromagnetic field is so intense and the stress level is so low that τ_{12} and c_3 are of the same order of magnitude, then, according to (5), τ_{12} cannot be equated to τ_{21} . In this case, we have to admit the stress tensor τ_{ij} as asymmetric. If c_3 is very much smaller in comparison with τ_{12} , then we can omit c_3 in (5) and consider (4) as valid approximately.

In developing a physical theory, particularly for the purpose of engineering, one of the most important objectives is to obtain the simplest formulation consistent with the desired degree of accuracy. A decision of whether or not we shall treat the stress tensor as symmetric must be based on the purpose of the theory. Since electromagnetic fields pervade the universe, the stress tensor is in general unsymmetric. But, if the theory is formulated for the purpose of a structural or mechanical engineer who studies the stress distribution in a structure or a machine with a view towards assessing its strength, stability, or rigidity, then a stress is important when it is of the order of the yielding stress of the material. Even for a structure which is designed primarily on the basis of stability, such as a column, an arch, or a thin-walled shell, a good design should produce a critical stress of the order of the yielding stress under the critical conditions, for otherwise the material is not economically used. When τ_{ij} is comparable to the yielding stress in magnitude (of order 10,000 to 100,000 lb/sq in. or 70 to 700 mPa for a steel, or 50 to 5,000 lb/sq in. or 0.35 to 35 mPa for a concrete), there are few circumstances in which the assumption of symmetry in stress tensor should cause concern.

However, if one wants to study the influence of a strong electromagnetic field on the propagation of elastic waves, or such influence on some high-frequency phenomenon in the material, then the stress level may be very low and the body moment may be significant. In such problems the stress tensor may not be assumed symmetric.

In the rest of this book the stress tensor will be assumed to be symmetric unless stated otherwise.

Notes on Couple-stresses

If, following Voigt, we assume that across any infinitesimal surface element in a solid the action of the exterior material upon the interior is equipollent to a force *and a couple* (in contrast to the assumption made in Sec. 3.1.) then in addition to the traction \mathbf{T} that acts on the surface we must have also a couple-stress vector \mathbf{M} . These two vectors \mathbf{T} and \mathbf{M} , together, are now equipollent to the action of the exterior upon the interior. Similarly, one might have body couples as pointed out by Maxwell, i.e., couple per unit mass, \mathbf{c} , with components c_i ($i = 1, 2, 3$). If we accept these possibilities,

then we must define a couple-stress tensor \mathcal{M}_{ij} , in addition to the stress tensor τ_{ij} . The tensor \mathcal{M}_{ij} is related to the couple-stress vector by a linear transformation like Eq. (3.3:2):

$$\overset{\nu}{M}_i = \mathcal{M}_{ji} \nu_j .$$

An analysis of the angular momentum then leads to the equation

$$\partial \mathcal{M}_{ji} / \partial x_j + \rho c_i = e_{ijk} \tau_{jk} ,$$

i.e.,

$$\partial \mathcal{M}_{xx} / \partial x + \partial \mathcal{M}_{yx} / \partial y + \partial \mathcal{M}_{zx} / \partial z + \rho c_x = \tau_{yz} - \tau_{zy} , \quad \text{etc.}$$

Thus, the antisymmetric part of the stress tensor is determined by the body couples and the divergence of the couple-stress tensor. When couples of both kinds are absent, the stress tensor must be symmetric.

Couple-stresses and body couples are useful concepts in dealing with sub-micron structures and molecular mechanics of materials, and in the dislocation theory of metals.

3.5. TRANSFORMATION OF COORDINATES

In the previous section, the components of stress τ_{ij} are defined with respect to a rectangular Cartesian system $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. Let us now take a second set of rectangular Cartesian coordinates $\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}'_3$ with the same origin but oriented differently, and consider the stress components in the new reference system (Fig. 3.5:1). Let these coordinates be connected by the linear relations

$$(1) \quad x'_k = \beta_{ki} x_i = (\partial x'_k / \partial x_i) x_i , \quad k = 1, 2, 3 ,$$

where $\beta_{ki} = \cos(\mathbf{x}'_k, \mathbf{x}_i)$ are the direction cosines of the \mathbf{x}/k -axis with respect to the \mathbf{x}' -axis. Since \mathbf{x}'_k -axis a tensor (Sec. 3.3.) we can write down the transformation law at once. However, in order to emphasize the importance of the result we shall insert an elementary derivation based on Cauchy's formula derived in Sec. 3.3., which states that if dS is a surface element whose unit outer normal vector v has components v_p , then the force per unit area acting on dS is a v vector $\overset{\nu}{T}$ with components

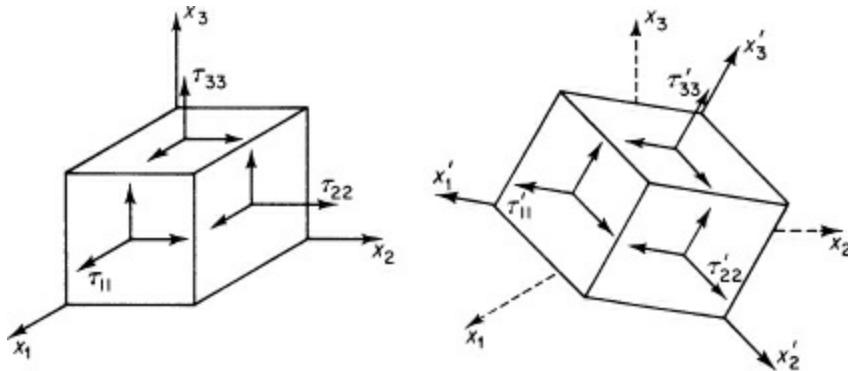


Fig. 3.5:1. Transformation of stress components under rotation of coordinates system.

$$\overset{\nu}{T}_i = \tau_{ji} \nu_j .$$

If the normal v is chosen to be parallel to the axis \mathbf{x}'_k , so that

$$\nu_1 = \beta_{k1} , \quad \nu_2 = \beta_{k2} , \quad \nu_3 = \beta_{k3} ,$$

then denoting $\overset{\nu}{T}_i$ as $\overset{k}{T}_i$, we have

$$\overset{\nu}{T}_i = \overset{k}{T}_i = \tau_{ji} \beta_{kj} .$$

Since the component of the vector $\overset{k}{T} (= \overset{\nu}{T})$ in the direction \mathbf{x}'_m is τ'_{km} , i.e.,

$$\overset{k}{T} = \tau'_{km} \mathbf{g}'_m , \quad \text{then} \quad \overset{k}{T} = \tau'_{km} \mathbf{g}'_m = \overset{k}{T} \mathbf{g}_i = \tau_{ji} \beta_{kj} \mathbf{g}_i ,$$

where \mathbf{g}'_m and g_i are the unit base vectors of the two coordinate systems. Using the relation $\mathbf{g}_i = \beta_{mi}\mathbf{g}'_m$ [Eq. (2.14:16)] and Eq. (1) for rectangular Cartesian coordinates, we obtain

$$\mathbf{T} = \tau'_{km}\mathbf{g}'_m = \tau_{ji}\beta_{kj}\beta_{mi}\mathbf{g}'_m,$$

i.e.,

$$(3) \quad \tau'_{km} = \tau_{ji}\beta_{kj}\beta_{mi} = \tau_{ji}(\partial x'_k / \partial x_j)(\partial x'_m / \partial x_i).$$

If we compare Eq. (3) and Eq. (2.5:2) we see that the stress components transform like a Cartesian tensor of rank two. Thus, the physical concept of stress which is described by τ_{ij} agrees with the mathematical definition of a tensor of rank two in a Euclidean space.

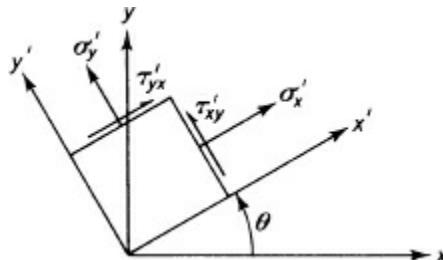


Fig. 3.6:1. Change of coordinates in plane state of stress.

3.6. PLANE STATE OF STRESS

A state of stress in which

$$(1) \quad \tau_{33} = \tau_{31} = \tau_{32} = 0,$$

is called a *plane state of stress* in the x_1x_2 -plane. In this case, the direction cosines between two systems of rectangular Cartesian coordinates can be expressed in terms of a single angle θ , as shown in Fig. 3.6:1. We have,

$$(\beta_{ij}) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Writing x, y and x', y' in place of x_1, x_2 and x'_1, x'_2 ; σ_x for τ_{11} ; τ_{xy} for τ_{12} , etc., we have

$$\sigma'_x = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta,$$

$$\sigma'_y = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2\tau_{xy} \sin \theta \cos \theta,$$

$$\tau'_{xy} = (-\sigma_x + \sigma_y) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta).$$

Since

$$\sin^2 \theta = (1 - \cos 2\theta)/2, \quad \cos^2 \theta = (1 + \cos 2\theta)/2,$$

we may also write

$$(3) \quad \begin{aligned} \sigma'_x &= (\sigma_x + \sigma_y)/2 + (\sigma_x - \sigma_y)(\cos 2\theta)/2 + \tau_{xy} \sin 2\theta, \\ \sigma'_y &= (\sigma_x + \sigma_y)/2 - (\sigma_x - \sigma_y)(\cos 2\theta)/2 - \tau_{xy} \sin 2\theta, \\ \tau'_{xy} &= -(\sigma_x - \sigma_y)(\sin 2\theta)/2 + \tau_{xy} \cos 2\theta. \end{aligned}$$

Note that summing the first two equations of Eq. (3) gives

$$(4) \quad \sigma'_x + \sigma'_y = \sigma_x + \sigma_y,$$

and differentiating them with respect to θ gives

$$(5) \quad \partial \sigma'_x / \partial \theta = -\partial \sigma'_y / \partial \theta = 2\tau'_{xy},$$

$$(6) \quad \tau'_{xy} = 0 \quad \text{when} \quad \tan 2\theta = 2\tau_{xy}/(\sigma_x - \sigma_y).$$

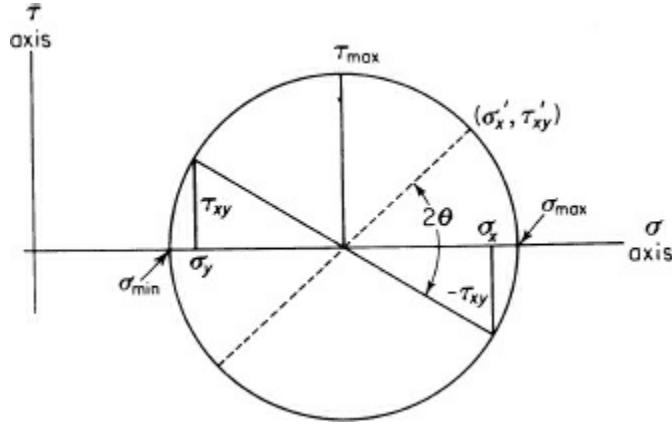


Fig. 3.6:2. Mohr's circle in plane state of stress.

The directions given by the particular values of θ given by (6) are called the *principal directions*; the corresponding normal stresses are called the *principal stresses* (see Sec. 3.7.). Following (5) and (6), the principal stresses are extreme values of the normal stresses,

$$(7) \quad \left. \begin{aligned} \sigma_{\max} \\ \sigma_{\min} \end{aligned} \right\} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

Differentiating τ'_{xy} with respect to θ and setting the derivative to zero, we can find the angle θ at which τ'_{xy} attains its extreme value. This angle is easily seen to be $\pm 45^\circ$ from the principal directions given by (6), and the maximum value of τ'_{xy} is

$$(8) \quad \tau_{\max} = (\sigma_{\max} - \sigma_{\min})/2 = [(\sigma_x - \sigma_y)^2/4 + \tau_{xy}^2]^{1/2}.$$

Figure 3.6:2 is a geometric representation of the relations above. It is the well-known *Mohr's circle*.

Problem 3.1. Find the transformation law for the moments of inertia and products of inertia of an area about a set of rectangular Cartesian coordinates in a plane,

$$I_{xx} = \int y^2 dA, \quad I_{xy} = \int xy dA, \quad I_{yy} = \int x^2 dA,$$

with respect to rotation of the coordinate axes about the origin. Mohr's circle was invented in 1887 for the transformation of the inertia tensor.

Problem 3.2. Let v_i , $i = 1, 2, 3$, be the velocity vector field of a continuum, and let $\overline{v_i v_j}$ be the average value of the product $v_i v_j$ over a period of time. Show that the correlation function $\overline{v_i v_j}$, with components $\overline{u^2}$, \overline{uv} , \overline{uw} , $\overline{v^2}$, etc., in unabridged notations, is a symmetric tensor of the second order.

Problem 3.3. Show that the mass moment of inertia of a set of particles,

$$I_{ij} = \epsilon_{ipq} \epsilon_{jkq} \int x_p x_k dm, \quad i, j = 1, 2, 3,$$

is a tensor, where dm is an element of mass and the integration is extended over the entire set of particles. Write out the matrix of the inertia tensor I_{ij} . Show that I_{ii} (i not summed) is the moment of inertia about the x_i axis, whereas I_{ij} ($i \neq j$) is equal to the negative of the product of inertia about the axes x_i and x_j . Show that for a rigid body rotating at an angular velocity ω_j , the angular momentum vector of the body is $I_{ij} \omega_j$.

3.7. PRINCIPAL STRESSES

The results of the previous section are restricted to the plane state of stress. Let us now generalize them to general three-dimensional problems.

In a general state of stress, the stress vector acting on a surface with outer normal v depends on the direction of v . Let us ask in what direction v the stress vector becomes normal to the surface, on which the shearing stress vanishes. Such a surface is called a *principal plane*, its normal a *principal axis*, and the value of the normal stress acting on the

principal plane is called a *principal stress*.

Let v define a principal axis and let σ be the corresponding principal stress. Then the stress vector acting on the surface normal to v has components σv_i . On the other hand, this same vector is given by the expression $\tau_{ji}v_j$. Hence, writing $v_i = \delta_{ji}v_j$, we have, on equating these two expressions and transposing them to the same side,

$$(1) \quad (\tau_{ji} - \sigma\delta_{ji})v_j = 0.$$

The three equations, $i = 1, 2, 3$, are to be solved for v_1, v_2, v_3 . Since v is a unit vector, we must find a set of nontrivial solutions for which $v_1^2 + v_2^2 + v_3^2 = 1$. Thus, Eq. (1) poses an eigenvalue problem. Since τ_{ij} as a matrix is real and symmetric, we need only to recall a result in the theory of matrices to assert that *there exist three real-valued principal stresses and a set of orthonormal principal axes*. Because of the importance of these results, we shall give the details of the reasoning below.

Equation (1) has a set of nonvanishing solutions v_1, v_2, v_3 if and only if the determinant of the coefficients vanishes, i.e.,

$$(2) \quad |\tau_{ij} - \sigma\delta_{ij}| = 0.$$

Equation (2) is a cubic equation in σ ; its roots are the principal stresses. For each value of the principal stress, a unit normal vector v can be determined. On expanding Eq. (2), we have

$$(3) \quad \begin{aligned} |\tau_{ij} - \sigma\delta_{ij}| &= \begin{vmatrix} \tau_{11} - \sigma & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} - \sigma & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} - \sigma \end{vmatrix} \\ &= -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0, \end{aligned}$$

where

$$(4) \quad \begin{aligned} I_1 &= \tau_{11} + \tau_{22} + \tau_{33}, \\ I_2 &= \begin{vmatrix} \tau_{22} & \tau_{23} \\ \tau_{32} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{13} \\ \tau_{31} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{vmatrix}, \\ I_3 &= \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix}. \end{aligned}$$

On the other hand, if $\sigma_1, \sigma_2, \sigma_3$ are the roots of Eq. (3), which can be written as

$$(\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma - \sigma_3) = 0,$$

it can be seen that the following relations between the roots and the coefficients must hold:

$$(5) \quad I_1 = \sigma_1 + \sigma_2 + \sigma_3, \quad I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1, \quad I_3 = \sigma_1\sigma_2\sigma_3.$$

Since the principal stresses characterize the physical state of stress at a point, they are independent of any coordinates of reference. Hence, Eq. (3) and the coefficients I_1, I_2, I_3 are invariant with respect to the coordinate transformation; I_1, I_2, I_3 are the *invariants* of the stress tensor. The importance of invariants will become evident when physical laws are formulated (see, for example, Chapter 6).

We shall show now that for a symmetric stress tensor the three principal stresses are all real and that the three principal planes are mutually orthogonal. These important properties can be established when the stress tensor is symmetric,

$$(6) \quad \tau_{ij} = \tau_{ji}.$$

The proof is as follows. Let $\overset{1}{\nu}, \overset{2}{\nu}, \overset{3}{\nu}$ be unit vectors in the direction of the principal axes, with components $\overset{1}{\nu}_i, \overset{2}{\nu}_i, \overset{3}{\nu}_i$ ($i = 1, 2, 3$) which are the solutions of Eq. (1) corresponding to the roots $\sigma_1, \sigma_2, \sigma_3$, respectively;

$$(7) \quad \begin{aligned} (\tau_{ij} - \sigma_1 \delta_{ij}) \nu_j^1 &= 0, \\ (\tau_{ij} - \sigma_2 \delta_{ij}) \nu_j^2 &= 0, \\ (\tau_{ij} - \sigma_3 \delta_{ij}) \nu_j^3 &= 0. \end{aligned}$$

Multiplying the first equation by $\frac{2}{\nu_i}$, the second by $\frac{1}{\nu_i}$, summing over i and subtracting the resulting equations, we obtain

$$(8) \quad (\sigma_2 - \sigma_1) \nu_i^1 \nu_i^2 = 0,$$

on account of the symmetry condition (6), which implies that

$$(9) \quad \tau_{ij} \nu_j^1 \nu_i^2 = \tau_{ji} \nu_j^1 \nu_i^2 = \tau_{ij} \nu_j^2 \nu_i^1.$$

The last equality is obtained by interchanging the dummy indices i and j .

Now, if we assume tentatively that Eq. (3) has a complex root, then, since the coefficients in Eq. (3) are real-valued, a complex conjugate root must also exist and the set of roots may be written as

$$\sigma_1 = \alpha + i\beta, \quad \sigma_2 = \alpha - i\beta, \quad \sigma_3,$$

where α, β, σ_3 are real numbers and i stands for the imaginary number $\sqrt{-1}$. In this case, Eq. (7) show that ν_j^1 and ν_j^2 are complex conjugate to each other and can be written as

$$\nu_j^1 \equiv a_j + ib_j, \quad \nu_j^2 \equiv a_j - ib_j,$$

in which a_j and b_j are real numbers and at least one of them is not zero. Therefore,

$$\nu_j^1 \nu_j^2 = (a_j + ib_j)(a_j - ib_j) = a_j^2 + a_j^2 + a_j^2 + b_j^2 + b_j^2 + b_j^2 \neq 0.$$

It follows from (8) that $\sigma_1 - \sigma_2 = 2i\beta = 0$ or $\beta = 0$. This contradicts the original assumption that the roots are complex. Thus, the assumption of the existence of complex roots is untenable, and the roots $\sigma_1, \sigma_2, \sigma_3$ are all real.

When $\sigma_1 \neq \sigma_2 \neq \sigma_3$, Eq. (8) implies

$$(10) \quad \nu_i^1 \nu_i^2 = 0, \quad \nu_i^2 \nu_i^3 = 0, \quad \nu_i^3 \nu_i^1 = 0;$$

i.e., the principal vectors are mutually orthogonal to each other. If $\sigma_1 = \sigma_2 \neq \sigma_3$, we can determine an infinite number of pairs of orthogonal vectors ν_i^1 and ν_i^2 and define ν_i^3 as a vector orthogonal to ν_i^1 and ν_i^2 . If $\sigma_1 = \sigma_2 = \sigma_3$, then any set of orthogonal axes may be taken as the principal axes.

If the reference axes x_1, x_2, x_3 are chosen to coincide with the principal axes, then the matrix of stress components becomes

$$(11) \quad (\tau_{ij}) = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}.$$

3.8. SHEARING STRESSES

We have seen that on an element of surface with a unit outer normal v_i , there acts a traction $\frac{\nu}{T_i}$, ($T_i = \tau_{ij} v_j$). The component of $\frac{\nu}{T_i}$ in the direction of v_i is the normal stress acting on the surface element. Let this normal stress be denoted by $\sigma(n)$. Since the component of a vector in the direction of another vector is given by the scalar product of the two vectors, we obtain

$$(1) \quad \sigma(n) = \tau_{ij} v_i v_j.$$

The magnitude of the resultant shearing stress on a surface element having the normal v_i is given by the equation

$$(2) \quad \tau^2 = |T_i|^2 - \sigma_{(n)}^2,$$

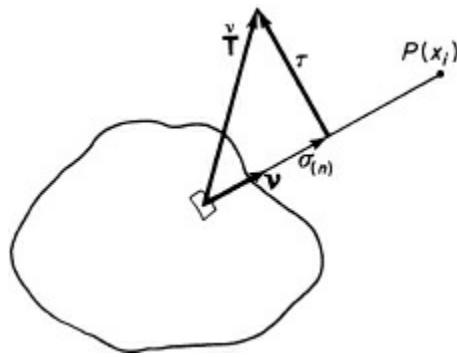


Fig. 3.8:1. Notations.

(see Fig. 3.8:1). Let the principal axes be chosen as the coordinate axes, and let $\sigma_1, \sigma_2, \sigma_3$ be the principal stresses. Then

$$|\overset{\nu}{T}_i|^2 = (\sigma_1\nu_1)^2 + (\sigma_2\nu_2)^2 + (\sigma_3\nu_3)^2,$$

and, from Eq. (1)

$$\sigma_{(n)}^2 = [\sigma_1(\nu_1)^2 + \sigma_2(\nu_2)^2 + \sigma_3(\nu_3)^2]^2.$$

On substituting into Eq. (2) and noting that

$$(\nu_1)^2 - (\nu_1)^4 = (\nu_1)^2[1 - (\nu_1)^2] = (\nu_1)^2[(\nu_2)^2 + (\nu_3)^2],$$

we see that

$$(3) \quad \begin{aligned} \tau^2 &= (\nu_1)^2(\nu_2)^2(\sigma_1 - \sigma_2)^2 + (\nu_2)^2(\nu_3)^2(\sigma_2 - \sigma_3)^2 \\ &\quad + (\nu_3)^2(\nu_1)^2(\sigma_3 - \sigma_1)^2. \end{aligned}$$

If $\nu_1 = \nu_2 = 1/\sqrt{2}$ and $\nu_3 = 0$, then $\tau = \pm(\sigma_1 - \sigma_2)/2$ and $\sigma = (\sigma_1 + \sigma_2)/2$.

Problem 3.4. Show that $\tau_{\max} = (\sigma_{\max} - \sigma_{\min})/2$ and that the plane on which τ_{\max} acts makes an angle of 45° with the directions of the largest and the smallest principal stresses.

3.9. MOHR'S CIRCLES

Let $\sigma_1, \sigma_2, \sigma_3$ be the principal stresses at a point. The stress components acting on any other surface elements can be obtained by the tensor transformation laws, Eq. (3.5:3). Otto Mohr, in papers published in 1882 and 1900, has shown the interesting result that if the normal stress $\sigma_{(n)}$ and the shearing stress τ acting on any surface element be plotted on a plane, with σ and τ as coordinates as shown in Fig. 3.9:1, the locus necessarily falls in a closed domain represented by the shaded area bounded by the three semicircles with centers on the σ -axis. A detailed proof can be found in Westergaard,^{1,2} *Elasticity and Plasticity*, pp. 61–64; or Sokolnikoff,^{1,2} *Elasticity*, p. 52. The practical problem of graphical construction of Mohr's circle from strain-gage data is discussed in Biezeno and Grammel and in Pearson, *Theoretical Elasticity*, p. 64, in Biblio. 1.2.

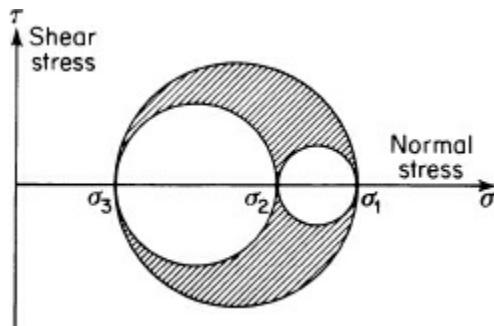


Fig. 3.9:1. Mohr's circles.

3.10. STRESS DEVIATIONS

The tensor

$$(1) \blacktriangle \quad \tau'_{ij} = \tau_{ij} - \sigma_0 \delta_{ij},$$

is called the *stress deviation tensor*, where δ_{ij} is the Kronecker delta and σ_0 is the mean stress

$$(2) \quad \sigma_0 = (\sigma_1 + \sigma_2 + \sigma_3)/3 = (\tau_{11} + \tau_{22} + \tau_{33})/3 = I_1/3,$$

where I_1 is the first invariant of Sec. 3.7. and τ'_{ij} specifies the deviation of the state of stress from the mean stress.

The first invariant of the stress deviation tensor always vanishes:

$$(3) \quad I'_1 = \tau'_{11} + \tau'_{22} + \tau'_{33} = 0.$$

To determine the principal stress deviations, the procedure of Sec. 3.7. may be followed. The determinantal equation

$$(4) \quad |\tau'_{ij} - \sigma' \delta_{ij}| = 0,$$

may be expanded in the form

$$(5) \quad \sigma'^3 - J_2 \sigma' - J_3 = 0.$$

It is easy to verify the following equations relating J_2, J_3 to the invariants I_2, I_3 defined in Sec. 3.7.,

$$(6) \quad J_2 = 3\sigma_0^2 - I_2,$$

$$(7) \quad J_3 = I_3 - I_2 \sigma_0 + 2\sigma_0^3 = I_3 + J_2 \sigma_0 - \sigma_0^3,$$

and the alternative expressions below on account of Eq. (3),

$$\begin{aligned} (8) \quad J_2 &= -\tau'_{11}\tau'_{22} - \tau'_{22}\tau'_{33} - \tau'_{33}\tau'_{11} + (\tau_{12})^2 + (\tau_{23})^2 + (\tau_{31})^2 \\ &= [(\tau'_{11})^2 + (\tau'_{22})^2 + (\tau'_{33})^2]/2 + (\tau_{12})^2 + (\tau_{23})^2 + (\tau_{31})^2 \\ &= [(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2]/6 + (\tau_{12})^2 + (\tau_{23})^2 + (\tau_{31})^2 \\ &= 3\tau_0^2/2. \end{aligned}$$

The τ_0 in the last equation is the *octahedral stress*, which will be defined in the next section.

We note also the simple expressions

$$(9) \blacktriangle \quad J_2 = \tau'_{ij}\tau'_{ij}/2 \quad \text{and} \quad J_3 = \tau'_{ij}\tau'_{jk}\tau'_{ki}/3.$$

It can be easily shown that the principal stress deviations are

$$(10) \quad \sigma'_i = \sigma_i - \sigma_0.$$

Problem 3.5. Show that the principal stresses as given by the three roots of Eq. (5) can be written as

$$\sigma'_1 = \tau_0 \sqrt{2} \cos \alpha, \quad \sigma'_2 = \tau_0 \sqrt{2} \cos \left(\alpha + \frac{2\pi}{3} \right), \quad \sigma'_3 = \tau_0 \sqrt{2} \cos \left(\alpha - \frac{2\pi}{3} \right),$$

where $\cos 3\alpha = J_3 \sqrt{2}/\tau_0^3$, and $J_2 = 3\tau_0^2/2$.

3.11. OCTAHEDRAL SHEARING STRESS

The octahedral shearing stress τ_0 is the resultant shearing stress on a plane that makes the same angle with the three principal directions. Such a plane is called an *octahedral plane*; eight such planes can form an octahedron. See Fig. 3.11:1. The direction cosines v_i of a normal to the octahedral plane relative to the principal axes are such that

$$(\nu_1)^2 + (\nu_2)^2 + (\nu_3)^2 = 1/3.$$

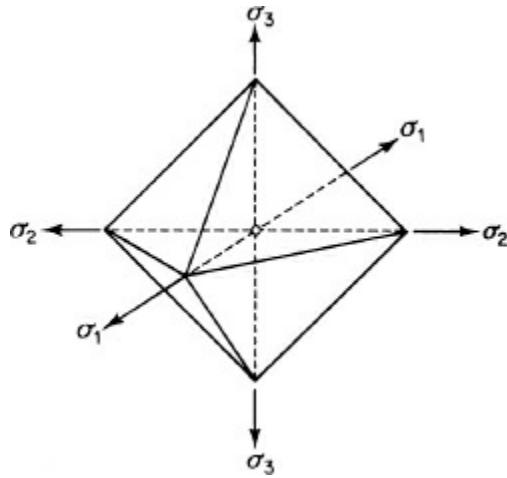


Fig. 3.11:1. Octahedral planes.

Hence, Eq. (3.8:3) gives

$$9\tau_0^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2,$$

which is proportional to the sum of the areas of Mohr's three semicircles. From Eqs. (3.10:6) and (3.10:8), it can be easily verified that the octahedral stress can be expressed in terms of the two invariants I_1 and I_2 of Sec. 3.7.,

$$9\tau_0^2 = 2I_1^2 - 6I_2.$$

The square of the octahedral stress happens to be proportional to the second invariant I_2 of the stress deviation, Eqs. (3.10:6) and (3.10:8). In 1913, Richard von Mises proposed the hypothesis that yielding of some of the most important materials occurs at a constant value of the quantity J_2 . Nadai then introduced the interpretation of J_2 as proportional to the octahedral shearing stress. In this way, J_2 or τ_0 enters into the basic equations of plasticity. Note that the normal stress on the octohedral plane equals the mean stress.

Problem 3.6. If $\sigma_1 > \sigma_2 > \sigma_3$ and σ_1, σ_3 are given, at what values of σ_2 does τ_0 attain its extreme values?

Problem 3.7. Let $\sigma_x = -5c$, $\sigma_y = c$, $\sigma_z = c$, $\tau_{xy} = -c$, $\tau_{yz} = \tau_{zx} = 0$, where $c = 1,000$ lb/sq in. Determine the principal stresses, the principal stress deviations, the direction cosines of the principal directions, the greatest shearing stress, and the octahedral stress.

Problem 3.8. Consider a horizontal beam as shown in Fig. P3.8. According to the usual elementary theory of bending, the "fiber stress" is $\sigma_{xx} = -12M y/bh^3$, where M is the bending moment which is a function of x . Assume this value of σ_{xx} , and assume further that $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$, that the body force is absent, that $\sigma_{xy} = 0$ at the top and bottom of the beam ($y = \pm h/2$), and that $\sigma_{yy} = 0$ at the bottom. Derive σ_{xy} and σ_{yy} from the equations of equilibrium. Compare the results with those derived in elementary mechanics of materials.

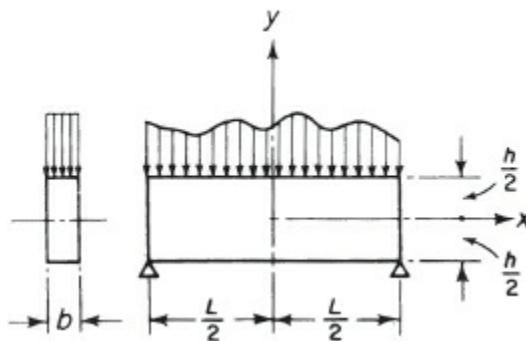


Fig. P3.8

3.12. STRESS TENSOR IN GENERAL COORDINATES

So far we have discussed the stress tensor in rectangular Cartesian co-ordinates, in which there is no necessity to distinguish the contravariant and covariant transformations. The necessary distinction arises in curvilinear coordinates.

Just as a vector in three-dimensional Euclidean space may assume either a contravariant or a covariant form, a stress

tensor can be either contravariant, τ^{ij} , or mixed, τ_j^i , or covariant, τ_{ij} . The tensors τ^{ij} , τ_j^i , and τ_{ij} are related to each other by raising or lowering of the indices by forming inner products with the metric tensors g_{ij} and g^{ij} :

$$(1) \quad \begin{aligned} \tau_j^i &= g_{\alpha j} \tau^{i\alpha} = g_{\alpha j} \tau^{\alpha i}, \\ \tau_{ij} &= g_{i\alpha} \tau_j^\alpha, \\ \tau^{ij} &= g^{i\alpha} \tau_\alpha^j. \end{aligned}$$

The correctness of these tensor relations are again seen by specializing them into rectangular Cartesian coordinates. We have seen that τ_{ij} is symmetric in rectangular Cartesian coordinates. It is easy to see from the tensor transformation law that *the symmetry property remains invariant in coordinate transformation. Hence, $\tau_{ij} = \tau_{ji}$ in all admissible coordinates*,² and we are allowed to write the mixed tensor as τ_j^i and not τ^i_j or τ^i_j . But what is the physical meaning of each of these components?

To clarify the meaning of the stress components in an arbitrary curvilinear coordinates system, let us first consider a geometric relationship. Let us form an infinitesimal tetrahedron whose edges are formed by the coordinate curves PP_1 , PP_2 , PP_3 and the curves P_1P_2 , P_2P_3 , P_3P_1 , as shown in Fig. 3.12:1. Let us write, vectorially,

$$\overline{PP_1} = \mathbf{r}_1, \quad \overline{PP_2} = \mathbf{r}_2, \quad \overline{PP_3} = \mathbf{r}_3.$$

Then

$$\overline{P_1P_2} = \mathbf{r}_2 - \mathbf{r}_1, \quad \overline{P_1P_3} = \mathbf{r}_3 - \mathbf{r}_1, \quad \overline{P_2P_3} = \mathbf{r}_3 - \mathbf{r}_2,$$

and we have

$$(2) \quad \begin{aligned} \overline{P_1P_2} \times \overline{P_1P_3} &= (\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1) \\ &= -\mathbf{r}_1 \times \mathbf{r}_3 - \mathbf{r}_2 \times \mathbf{r}_1 + \mathbf{r}_2 \times \mathbf{r}_3 \\ &= \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1 + \mathbf{r}_1 \times \mathbf{r}_2. \end{aligned}$$

Now the vector product $\mathbf{A} \times \mathbf{B}$ of any two vectors \mathbf{A} and \mathbf{B} is a vector perpendicular to \mathbf{A} and \mathbf{B} , whose positive sense is determined by the *right-hand screw rule* from \mathbf{A} to \mathbf{B} , and whose length is equal to the area of a parallelogram formed by \mathbf{A} , \mathbf{B} as two sides. Hence, if we denote by v , v_1 , v_2 , v_3 the unit vectors normal to the surfaces $P_1P_2P_3$, $P_2P_3P_1$, $P_3P_1P_2$, respectively, and by dS , dS_1 , dS_2 , dS_3 their respective areas, Eq. (2) may be written as

$$(3) \quad \nu dS = \nu_1 dS_1 + \nu_2 dS_2 + \nu_3 dS_3.$$

Now let us recall that in Sec. 2.14., we defined the reciprocal base vectors \mathbf{g}^1 , \mathbf{g}^2 , \mathbf{g}^3 which are perpendicular to the coordinate planes and are of length $\sqrt{g^{11}}$, $\sqrt{g^{22}}$, $\sqrt{g^{33}}$, respectively. We see that the unit vectors v_1 , v_2 , v_3 are exactly $\mathbf{g}^1/\sqrt{g^{11}}$, $\mathbf{g}^2/\sqrt{g^{22}}$, $\mathbf{g}^3/\sqrt{g^{33}}$, respectively. Hence,

$$(4) \quad \nu dS = \sum_{i=1}^3 dS_i \mathbf{g}^i / (g^{ii})^{1/2}.$$

If the unit normal vector v is resolved into its covariant components with respect to the reciprocal base vectors, then

$$(5) \quad \nu = \nu_i \mathbf{g}^i.$$

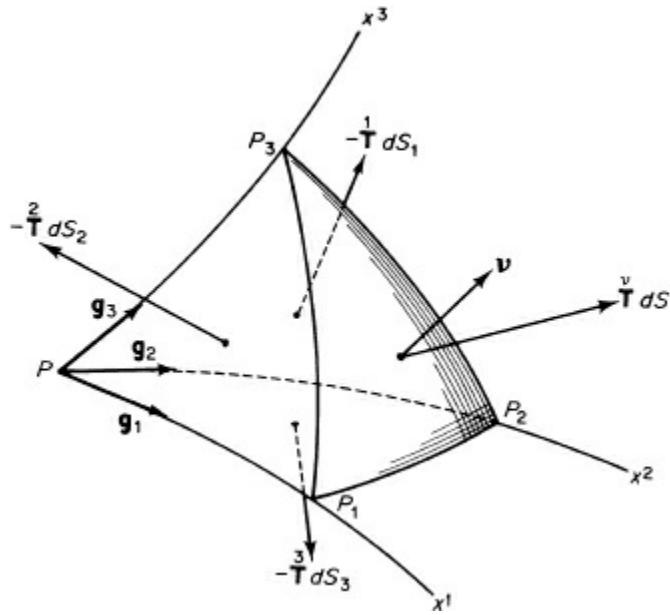


Fig. 3.12:1. Geometric relationship.

We see from the last two equations that

$$(6) \quad \nu_i \sqrt{g^{ii}} dS = dS_i, \quad i \text{ not summed.}$$

This is the desired result.

Let us now consider the forces acting on the external surface of the infinitesimal tetrahedron. In Sec. 3.1., the inside and outside of a volume are distinguished by drawing an outward pointing normal vector. The stress vector $\overset{\nu}{\mathbf{T}}$ is defined as the limit of force acting on the outside surface with a normal v divided by the area of the surface. On the other side, the stress vector is $\underline{\overset{\nu}{\mathbf{T}}}$, see Sec. 3.3., Fig. 3.3:1. Now, for the tetrahedron shown in Fig. 3.12:1, the normal vector v of the triangle $P_1P_2P_3$ is outward, the stress vector is $\overset{\nu}{\mathbf{T}}$, the area v is dS , the force is $\overset{\nu}{\mathbf{T}} dS$. On the triangle $P P_2 P_3$, the normal vector v_i points i inward, the stress vector is $\overset{i}{\mathbf{T}}$, the area is dS_i , the force is, therefore, $-\overset{i}{\mathbf{T}} dS_i$ as shown in Fig. 3.12:1. The forces on the other surfaces are determined similarly. The equation of motion of this infinitesimal tetrahedron is, in the limit,

$$(7) \quad \overset{\nu}{\mathbf{T}} dS = \overset{i}{\mathbf{T}} dS_i.$$

Volume forces and inertia (mass \times acceleration) forces acting on the tetrahedron do not enter into this equation, because they are of higher order of smallness than the surface forces. On substituting (6) into (7) and canceling the nonvanishing factor dS , we have

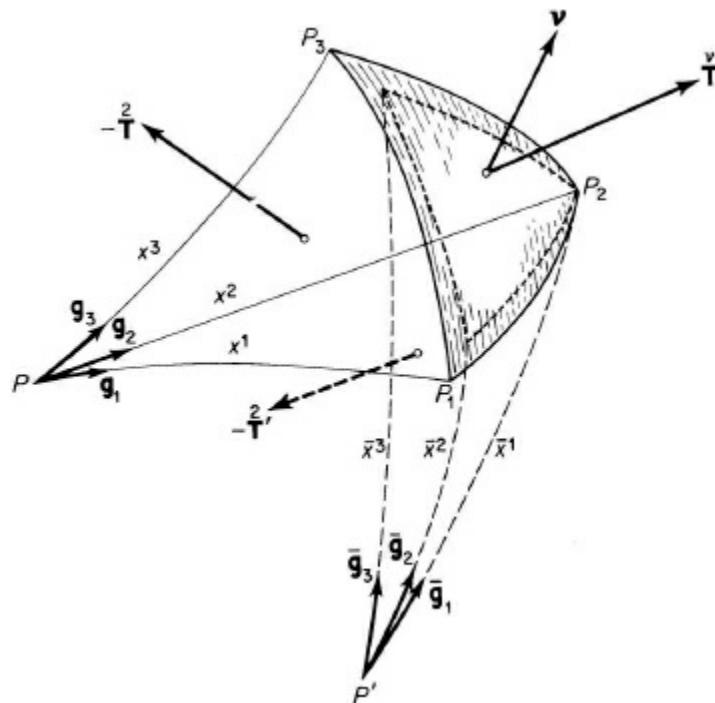


Fig. 3.12:2. Tensions referred to two different elementary tetrahedron with a common surface.

$$(8) \quad \overset{\nu}{\mathbf{T}} = \sum_{i=1}^3 \overset{i}{\mathbf{T}} \nu_i (g^{ii})^{1/2}.$$

If the coordinates x^j are changed to a new set \bar{x}^i while the surface $P_1P_2P_3$ and the unit outer normal v remain unchanged, the stress vector $\overset{\nu}{\mathbf{T}}$ is invariant, but the vectors $\overset{i}{\mathbf{T}}$ will change because they will be associated with the new coordinate surfaces. (See Fig. 3.12:2, in which only one pair of vectors $\overset{2}{\mathbf{T}}, \overset{2}{\mathbf{T}'}$ are shown.) The covariant components ν_i will change also. Inasmuch as $\overset{\nu}{\mathbf{T}}$ is invariant and ν_i is a covariant tensor, Eq. (8) shows that $\overset{i}{\mathbf{T}} \sqrt{g^{ii}}$ transforms according to a contravariant type of transformation.

Resolving the vectors $\overset{i}{\mathbf{T}} \sqrt{g^{ii}}$ into their components with respect to the base vectors \mathbf{g}_i and the reciprocal base vectors \mathbf{g}^i , we have

$$(9) \quad \sqrt{g^{ii}} \overset{i}{\mathbf{T}} = \tau^{ij} \mathbf{g}_j = \tau_j^i \mathbf{g}^j, \quad i \text{ not summed}.$$

On the other hand, the components of $\overset{\nu}{\mathbf{T}}$ may be written as

$$(10) \quad \overset{\nu}{\mathbf{T}} = \overset{\nu}{T}^j \mathbf{g}_j = \overset{\nu}{T}_j \mathbf{g}^j.$$

Substitution of (9) and (10) into (8) shows that

$$(11) \quad \overset{\nu}{T}^j = \tau^{ij} \nu_i, \quad \overset{\nu}{T}_j = \tau_j^i \nu_i.$$

The tensorial character of τ^{ij} and τ_j^i are demonstrated both by (9) and by (11) according to the quotient rule. Following the tensor transformation rules for τ_{ij} , τ_j^i and ν_i , we can rewrite Eq. (11) in the form (12) $\overset{\nu}{T}^j = \tau_j^i \nu^i$, $\overset{\nu}{T}_j = \tau_{ij} \nu^i$ where ν_i are the contravariant components of v .

Let us recapitulate the important points. With the scaling factor $\sqrt{g^{ii}}$, the vectors $\overset{i}{\mathbf{T}} \sqrt{g^{ii}}$ ($i = 1, 2, 3$, not summed) transform according to the con-travariant rule. When the vectors $\overset{i}{\mathbf{T}} \sqrt{g^{ii}}$ are resolved into components along the base vectors \mathbf{g}_j , the tensor components τ^{ij} are obtained. When the resolution is made with respect to the reciprocal base vectors \mathbf{g}^j , then the mixed tensor τ_j^i is obtained. On the other hand, if the stress vector $\overset{\nu}{\mathbf{T}}$ (acting on the triangle $P_1P_2P_3$ with outer normal v) is resolved into the contravariant components $\overset{\nu}{T}^j$ along the base vectors \mathbf{g}_i and the covariant components $\overset{\nu}{T}_j$ along the reciprocal base vectors \mathbf{g}^j , then we obtain Eq. (11).

The contravariant stress tensor τ^{ij} and the mixed stress tensor τ_j^i are related to the stress vectors $\overset{i}{\mathbf{T}}$ by the Eq. (9). The covariant stress tensor τ_{ij} defined in Eq. (1) cannot be so simply related to the vectors $\overset{i}{\mathbf{T}}$ and are therefore of less importance.

3.13. PHYSICAL COMPONENTS OF A STRESS TENSOR IN GENERAL COORDINATES

If we write Eq. (3.12:9) as

$$(1) \quad \overset{i}{\mathbf{T}} = \sum_{j=1}^3 \sqrt{g_{jj}/g^{ii}} \tau^{ij} \mathbf{g}_j / \sqrt{g_{jj}} = \sum_{j=1}^3 \sigma^{ij} \mathbf{g}_j / \sqrt{g_{jj}}, \quad i \text{ not summed}.$$

Then, since $\mathbf{g}_j / \sqrt{g_{jj}}$ (j not summed) are unit vectors along the coordinate curves, the components σ^{ij} are uniform in physical dimensions and represent the *physical components* of the stress vector $\overset{i}{\mathbf{T}}$ in the direction of the unit vectors $\mathbf{g}_j / \sqrt{g_{jj}}$ (j not summed),

$$(2) \quad \sigma^{ij} = (g_{jj}/g^{ii})^{1/2} \tau^{ij}, \quad i, j \text{ not summed}.$$

But σ^{ij} is not a tensor.

On the other hand, if we use the mixed tensor τ_j^i in (3.12:1), we have

$$(3) \quad \overset{i}{\mathbf{T}} = \sum_{j=1}^3 \tau_j^i g^j / \sqrt{g^{ii}} = \sum_{j=1}^3 \tau_j^i \sqrt{g^{jj}/g^{ii}} g^j / \sqrt{g^{jj}}, \quad i \text{ not summed.}$$

Thus

$$(4) \quad \sigma_j^i = (g^{jj}/g^{ii})^{1/2} \tau_j^i, \quad i, j \text{ not summed,}$$

are the physical components of the tensor τ_j^i , of uniform physical dimensions, representing the components of the stress vector $\overset{i}{\mathbf{T}}$ resolved in the directions of the reciprocal base vectors. Note that, in general, $\sigma^{ij} \neq \sigma_j^i \neq \sigma_i^j$, except for orthogonal coordinates. If the coordinates are orthogonal, \mathbf{g}^i and \mathbf{g}_i are in the same direction.

When curvilinear coordinates are used, we like to retain the liberty of choosing coordinates without regard to dimensions. Thus, in cylindrical polar coordinates (r, θ, z) , r and z have the dimensions of length and θ is an angle. The corresponding tensor components of a vector referred to polar coordinates will have different dimensions. For physical understanding it is desirable to employ the physical components, but for the convenience of analysis it is far more expedient to use the tensor components.

3.14. EQUATIONS OF EQUILIBRIUM IN CURVILINEAR COORDINATES

In Sec. 3.4 we discussed the equations of equilibrium in terms of Cartesian tensors in rectangular Cartesian coordinates. To obtain these equations in any curvilinear coordinates, it is only necessary to observe that the equilibrium conditions must be expressed in a tensor equation. Thus the equations of equilibrium must be

$$(1) \quad \tau^{ij}|_j + X^i = 0, \quad \text{in volume,} \quad \tau^{ij}\nu_j = \overset{\nu}{T}^i, \quad \text{on surface.}$$

The truth is at once proved by observing that these are truly tensor equations and that they hold in the special case of rectangular Cartesian coordinates. Hence, they hold in any coordinates that can be derived from the Cartesian coordinates through admissible transformations.

The practical application of tensor analysis in the derivation of the equations of equilibrium in particular curvilinear coordinates will be illustrated in Secs. 4.11 and 4.12. It will be seen that these lengthy equations can be obtained in a routine manner without too much effort. Because the manipulation is routine, chances of error are minimized. This practical application may be regarded as the first dividend to be paid for the long process of learning the tensor analysis.

Problem 3.9. Let us recast the principal results obtained above into tensor equations in general coordinates of reference. In rectangular Cartesian coordinates, there is no difference in contravariant and covariant transformations. Hence, the Cartesian stress tensor may be written as τ_{ij} , or τ^{ij} , or τ_j^i . In general frames of reference, τ_{ij} , τ^{ij} , τ_j^i are different. Their components may have different values. They are different versions of the same physical entity. Now prove the following results in general coordinates.

- (a) The tensors τ^{ij} , τ_{ij} are symmetric if there is no body moment acting on the medium; i.e.,

$$\tau^{ij} = \tau^{ji}, \quad \tau_{ij} = \tau_{ji}.$$

- (b) Principal planes are planes on which the stress vector $\overset{\nu}{\mathbf{T}}$ is parallel to the normal vector ν . If we use contravariant components, we have, on a principal plane, $\overset{\nu}{T}^j = \sigma\nu^j = \sigma g^{ij}\nu_i = \tau^{ij}\nu_i$, where σ is a scalar. If we use covariant components, we have correspondingly $\overset{\nu}{T}_j = \sigma\nu_j = \sigma g_j^i\nu_i = \tau_j^i\nu_i$. Show that σ must satisfy the characteristic determinant equation

$$|\tau_j^i - \sigma\delta_j^i| = 0 \quad \text{or its equivalent} \quad |\tau^{ij} - \sigma g^{ij}| = 0.$$

- (c) The first invariant of the stress tensor is τ_j^i , or $\tau^{ij}g_{ij}$. However, τ_{ii} and τ^{ii} are in general not invariants.

- (d) The stress deviator tensor is defined as

$$s_j^i = \tau_j^i - \tau_\alpha^\alpha g_j^i/3$$

The first invariant of s_j^i is zero. The 2nd invariant has the convenient form $J_2 = s_k^i s_i^k/2$.

- (e) The octahedral shearing stress has the same value in any coordinates system.

¹ The derivatives $\overset{\cdot}{\mathbf{P}}$ and $\overset{\cdot}{\mathbf{H}}$ are *material derivatives*; i.e., the time rate of change of \mathbf{P} and \mathbf{H} of a fixed set of particles (cf. Secs. 5.2 and 5.3).

² Similarly, the contravariant stress tensor τ^{ij} is symmetric, $\tau^{ij} = \tau^{ji}$. It does not make sense, however, to say that the mixed tensor τ^i_j is symmetric, since an equation like $\tau^i_j = \tau^j_i$, with the indices switching roles on the two sides of the equation, is not a tensor equation.

4

ANALYSIS OF STRAIN

In this chapter we shall consider the deformation of a body as a “mapping” of the body from the original state to the deformed state. Strain tensors useful for finite as well as infinitesimal strains are then defined.

4.1. DEFORMATION

In the formulation of continuum mechanics the configuration of a solid body is described by a continuous mathematical model whose geometrical points are identified with the place of the material particles of the body. When such a continuous body changes its configuration under some physical action, we impose the assumption that the change is continuous; i.e., neighborhoods are changed into neighborhoods. Any introduction of new boundary surfaces, such as caused by tearing of a membrane or fracture of a test specimen, must be regarded as an extraordinary circumstance requiring special attention and explanation.

Let a system of coordinates a_1, a_2, a_3 be chosen so that a point P of a body at a certain instant of time is described by the coordinates $a_i (i = 1, 2, 3)$. At a later instant of time, the body is moved (deformed) to a new configuration; the point P is moved to Q with coordinates $x_i (i = 1, 2, 3)$ with respect to a new coordinate system x_1, x_2, x_3 . The coordinate systems a_1, a_2, a_3 and x_1, x_2, x_3 may be curvilinear and need not be the same (Fig. 4.1:1), but they both describe a Euclidean space.

The change of configuration of the body will be assumed to be continuous, and the *point transformation (mapping)* from P to Q is assumed to be one-to-one. The equation of transformation can be written as

$$(1) \quad x_i = \hat{x}_i(a_1, a_2, a_3),$$

which has a unique inverse

$$(2) \quad a_i = \hat{a}_i(x_1, x_2, x_3),$$

for every point in the body. The functions $\hat{x}_i(a_1, a_2, a_3)$ and $\hat{a}_i(x_1, x_2, x_3)$ are assumed to be continuous and differentiable.

We shall be concerned with the description of the strain of the body, i.e., with the stretching and distortion of the body. If P, P', P'' are three neighboring points forming a triangle in the original configuration, and if they are transformed to points Q, Q', Q'' in the deformed configuration, the change in area and angles of the triangle is completely determined if we know the change in length of the sides. But the “location” of the triangle is undetermined by the change of the sides. Similarly, if the change of length between any two arbitrary points of the body is known, the new configuration of the body will be completely defined except for the location of the body in space. In the following discussions our attention will be focused on the strain of the body, because it is the strain that is related to the stress. The description of the change in distance between any two points of the body is the key to the analysis of deformation.

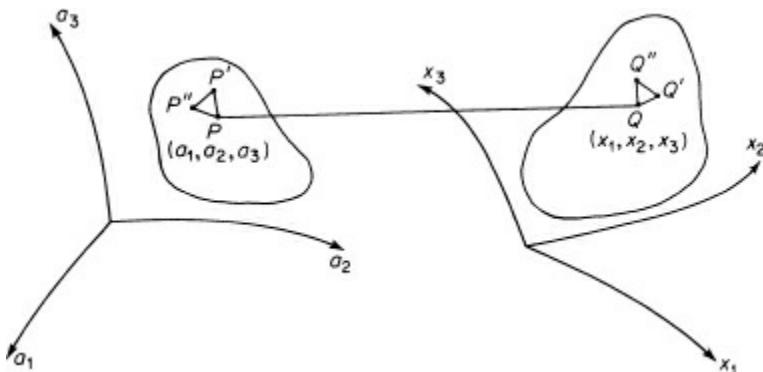


Fig. 4.1:1. deformation of a body.

Consider an infinitesimal line element connecting the point $P(a_1, a_2, a_3)$ to a neighboring point $P'(a_1 + da^1, a_2 + da^2, a_3 + da^3)$.¹ The square of the length ds_0 of PP' in the original configuration is given by

$$(3) \quad ds_0^2 = a_{ij} da^i da^j ,$$

where a_{ij} , evaluated at the point P , is the Euclidean metric tensor for the coordinate system a_i . When P , P' is deformed to the points $Q(x_1, x_2, x_3)$ and $Q'(x_1 + dx^1, x_2 + dx^2, x_3 + dx^3)$, respectively, the square of the length ds of the new element QQ' is

$$(4) \quad ds^2 = g_{ij} dx^i dx^j ,$$

where g_{ij} is the Euclidean metric tensor for the coordinate system x_i .

By Eqs. (1) and (2), we may also write

$$(5) \quad ds_0^2 = a_{ij} \frac{\partial a_i}{\partial x_l} \frac{\partial a_j}{\partial x_m} dx^l dx^m , \quad ds^2 = g_{ij} \frac{\partial x_i}{\partial a_l} \frac{\partial x_j}{\partial a_m} da^l da^m .$$

The difference between the squares of the length elements may be written, after several changes in the symbols for dummy indices, either as

$$(6) \quad ds^2 - ds_0^2 = [g_{\alpha\beta} (\partial x_\alpha / \partial a_i) (\partial x_\beta / \partial a_j) - a_{ij}] da^i da^j ,$$

or as

$$(7) \quad ds^2 - ds_0^2 = [g_{ij} - a_{\alpha\beta} (\partial a_\alpha / \partial x_i) (\partial a_\beta / \partial x_j)] dx^i dx^j .$$

We define the *strain tensors*

$$(8) \quad \blacktriangleleft \quad E_{ij} = \frac{1}{2} \left(g_{\alpha\beta} \frac{\partial x_\alpha}{\partial a_i} \frac{\partial x_\beta}{\partial a_j} - a_{ij} \right) , \quad e_{ij} = \frac{1}{2} \left(g_{ij} - a_{\alpha\beta} \frac{\partial a_\alpha}{\partial x_i} \frac{\partial a_\beta}{\partial x_j} \right) ,$$

so that

$$(9) \quad \blacktriangleleft \quad ds^2 - ds_0^2 = 2E_{ij} da^i da^j , \quad ds^2 - ds_0^2 = 2e_{ij} dx^i dx^j .$$

The strain tensor E_{ij} was introduced by Green and St. Venant, and is called Green's strain tensor. The strain tensor e_{ij} was introduced by Cauchy for infinitesimal strains and by Almansi and Hamel for finite strains, and is known as Almansi's strain tensor. In analogy with a terminology in hydrodynamics, E_{ij} is often referred to as a strain tensor in Lagrangian coordinates and e_{ij} as a strain tensor in Eulerian coordinates.

That E_{ij} and e_{ij} thus defined are tensors in the coordinate systems a_i and x_i , respectively, follows from the quotient rule when it is applied to Eq. (9). The tensorial character of E_{ij} and e_{ij} can also be verified directly from their definitions Eq. (8) by considering further coordinate transformations in either the original configuration (from a_1, a_2, a_3 to $\bar{a}_1, \bar{a}_2, \bar{a}_3$), or the deformed configuration (from x_1, x_2, x_3 to $\bar{x}_1, \bar{x}_2, \bar{x}_3$). The details are left to the reader. The tensors E_{ij} and e_{ij} are obviously symmetric, i.e.,

$$(10) \quad \blacktriangleleft \quad E_{ij} = E_{ji} , \quad e_{ij} = e_{ji} .$$

An immediate consequence of Eq. (9) is the fundamental result that a *necessary and sufficient condition that a deformation of a body be a rigid-body motion (consists merely of translation and rotation without changing distances between particles) is that all the components of the strain tensor E_{ij} or e_{ij} be zero throughout the body*.

In the discussion above we have used two sets of curvilinear coordinates to describe the position of each particle. One, a_1, a_2, a_3 , is used in the original configuration, the other, x_1, x_2, x_3 , is used in the deformed configuration.

Now, there are two particularly favored choices of coordinates:

I. We use the same rectangular Cartesian coordinates for both the original and the deformed configurations of the body. In this case, the metric tensors are extremely simple:

$$(11) \quad g_{ij} = a_{ij} = \delta_{ij} .$$

II. We distort the frame of reference in the deformed configuration in such a way that the coordinates x_1, x_2, x_3 , of a particle have the same numerical values a_1, a_2, a_3 as in the original configuration. In this case, $x_i = a_i$, $\partial x_\alpha / \partial a_i = \delta_{\alpha i}$, $\partial a_\alpha / \partial x_i = \delta_{\alpha i}$, and Eq. (8) are reduced to

$$(12) \quad E_{ij} = e_{ij} = (g_{ij} - a_{ij})/2.$$

Thus all the information about strain is contained in the change of the metric tensor as the frame of reference is distorted from the original configuration to the deformed configuration. In many ways this is the most convenient choice in the study of large deformations. The coordinates so chosen are called *convected* or *intrinsic* coordinates.

In the following sections we wish to discuss the meaning of the individual components of the strain tensor. For this purpose the choice (I) above is the most appropriate.

4.2. STRAIN TENSORS IN RECTANGULAR CARTESIAN COORDINATES

If we use the *same rectangular Cartesian* (rectilinear and orthogonal) coordinate system to describe both the original and the deformed configurations of the body, then

$$(1) \quad g_{ij} = a_{ij} = \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Furthermore, if we introduce the *displacement vector* \mathbf{u} with components

$$(2) \quad u_i = x_i - a_i, \quad i = 1, 2, 3,$$

(see Fig. 4.2:1) then

$$\partial x_\alpha / \partial a_i = \partial u_\alpha / \partial a_i + \delta_{\alpha i}, \quad \partial a_\alpha / \partial x_i = \delta_{\alpha i} - \partial u_\alpha / \partial x_i.$$

The strain tensors reduce to the simple form

$$(3) \quad \Delta \quad \begin{aligned} E_{ij} &= [\delta_{\alpha\beta}(\partial x_\alpha / \partial a_i)(\partial x_\beta / \partial a_j) - \delta_{ij}]/2 \\ &= [\delta_{\alpha\beta}(\partial u_\alpha / \partial a_i + \delta_{\alpha i})(\partial u_\beta / \partial a_j + \delta_{\beta j}) - \delta_{ij}]/2 \\ &= [\partial u_i / \partial a_j + \partial u_j / \partial a_i + (\partial u_\alpha / \partial a_i)(\partial u_\alpha / \partial a_j)]/2, \end{aligned}$$

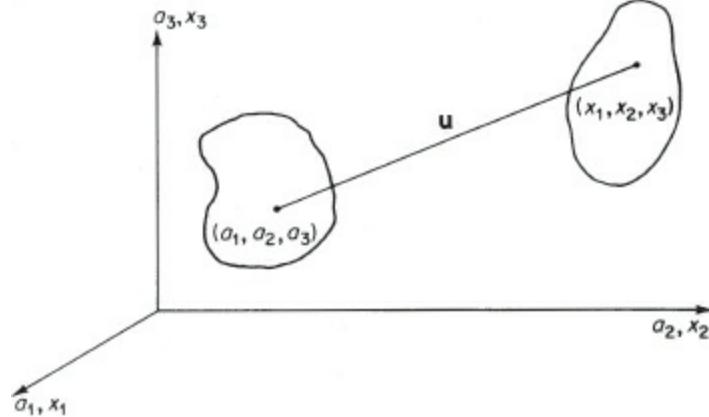


Fig. 4.2:1. Displacement vector.

and

$$(4) \quad \Delta \quad \begin{aligned} e_{ij} &= [\delta_{ij} - \delta_{\alpha\beta}(\partial a_\alpha / \partial x_i)(\partial a_\beta / \partial x_j)]/2 \\ &= [\delta_{ij} - \delta_{\alpha\beta}(-\partial u_\alpha / \partial x_i + \delta_{\alpha i})(-\partial u_\beta / \partial x_j + \delta_{\beta j})]/2 \\ &= [\partial u_i / \partial x_j + \partial u_j / \partial x_i - (\partial u_\alpha / \partial x_i)(\partial u_\alpha / \partial x_j)]/2. \end{aligned}$$

In unabridged notations (x, y, z for x_1, x_2, x_3 ; a, b, c for a_1, a_2, a_3 ; and u, v, w for u_1, u_2, u_3), we have the typical terms

$$\begin{aligned}
E_{aa} &= \partial u / \partial a + [(\partial u / \partial a)^2 + (\partial v / \partial a)^2 + (\partial w / \partial a)^2] / 2, \\
e_{xx} &= \partial u / \partial x - [(\partial u / \partial x)^2 + (\partial v / \partial x)^2 + (\partial w / \partial x)^2] / 2, \\
(5) \quad E_{ab} &= \frac{1}{2} \left[\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a} + \left(\frac{\partial u}{\partial a} \frac{\partial u}{\partial b} + \frac{\partial v}{\partial a} \frac{\partial v}{\partial b} + \frac{\partial w}{\partial a} \frac{\partial w}{\partial b} \right) \right], \\
e_{xy} &= \frac{1}{2} \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right].
\end{aligned}$$

Note that u, v, w are considered as functions of a, b, c , the position of points in the body in unstrained configuration, when the Lagrangian strain tensor is evaluated; whereas they are considered as functions of x, y, z , the position of points in the strained configuration, when the Eulerian strain tensor is evaluated.

If the components of displacement u_i are such that their first derivatives are so small that the squares and products of the partial derivatives of u_i are negligible, then e_{ij} is reduced to Cauchy's *infinitesimal strain tensor*,

$$(6) \quad e_{ij} = (\partial u_j / \partial x_i + \partial u_i / \partial x_j) / 2.$$

In unabridged notation,

$$\begin{aligned}
(7) \quad e_{xx} &= \partial u / \partial x, \quad e_{xy} = (\partial u / \partial y + \partial v / \partial x) / 2, \\
e_{yy} &= \partial v / \partial y, \quad e_{yz} = (\partial v / \partial z + \partial w / \partial y) / 2, \\
e_{zz} &= \partial w / \partial z, \quad e_{zx} = (\partial w / \partial x + \partial u / \partial z) / 2.
\end{aligned}$$

In the infinitesimal displacement case, the distinction between the Lagrangian and Eulerian strain tensors disappears, since then it is immaterial whether the derivatives of the displacements are calculated at the position of a point before or after deformation.

4.3. GEOMETRIC INTERPRETATION OF INFINITESIMAL STRAIN COMPONENTS

Let x, y, z be a set of rectangular Cartesian coordinates. Consider a line element of length dx parallel to the x -axis ($dy = dz = 0$). The change of the square of the length of this element due to deformation is

$$ds^2 - ds_0^2 = 2e_{xx}(dx)^2.$$

Hence,

$$ds - ds_0 = 2e_{xx}(dx)^2 / (ds + ds_0).$$

But $ds = dx$ in this case, and ds_0 differs from ds only by a small quantity of the second order, if we assume the displacements u, v, w and the strain components e_{ij} to be infinitesimal. Hence,

$$(ds - ds_0) / ds = e_{xx},$$

and it is seen that e_{xx} represents the *extension*, or change of length per unit length of a vector parallel to the x -axis. An application of the above discussion to a volume element is illustrated in Fig. 4.3:1, Case 1.

To see the meaning of the component e_{xy} , let us consider a small rectangle in the body with edges dx, dy . It is evident from Fig. 4.3:1, Cases 2, 3, and 4 that the sum $\partial u / \partial y + \partial v / \partial x$ represents the change of angle xOy which was originally a right angle. Thus,

$$e_{xy} = (\partial u / \partial y + \partial v / \partial x) / 2 = (\text{change of angle } xOy) / 2.$$

In engineering usage, the strain components e_{ij} (i, j) doubled, i.e., $2e_{ij}$, are called the *shearing strains*, or *detrusions*. The name is perhaps particularly suggestive in Case 3 of Fig. 4.3:1, which is called the case of *simple shear*.

The quantity

$$\omega_z = (\partial v / \partial x - \partial u / \partial y) / 2,$$

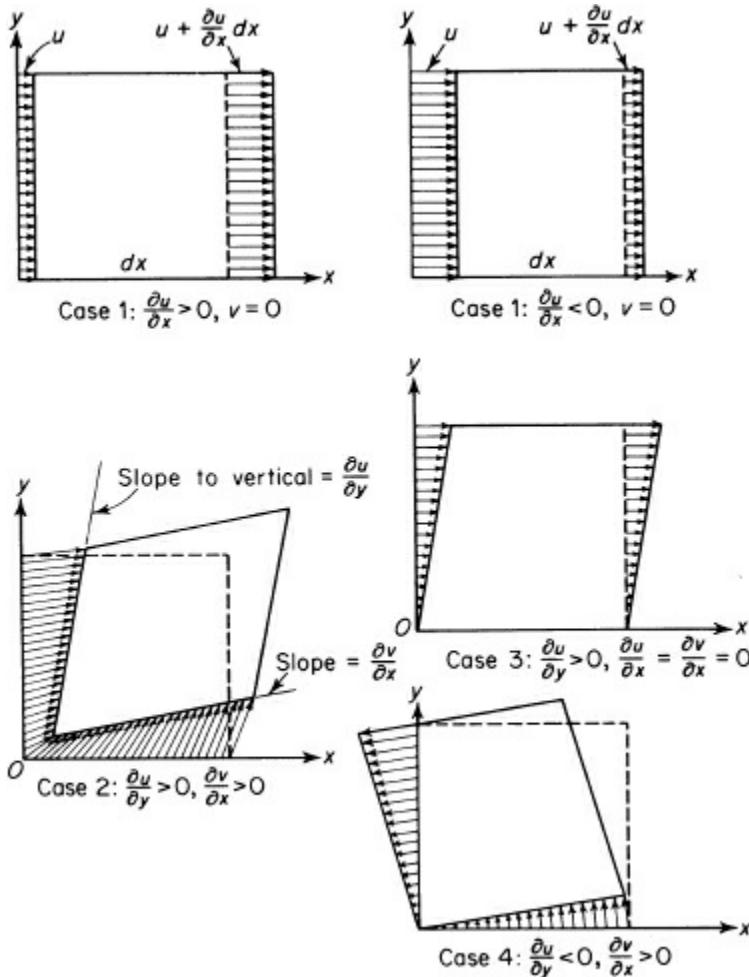


Fig. 4.3:1. Deformation gradients and interpretation of infinitesimal strain components.

is called the *infinitesimal rotation* of the element $dxdy$. This terminology is suggested by Case 4 in Fig. 4.3:1. If

$$\partial v / \partial x = -\partial u / \partial y,$$

then $e_{xy} = 0$ and ω_z is indeed the angle through which the rectangular element is rotated as a rigid body.

4.4. ROTATION

Consider an infinitesimal displacement field $u_i(x_1, x_2, x_3)$. From u_i , form the Cartesian tensor

$$(5) \quad du_i = (\partial u_i / \partial x_j) dx_j.$$

which is antisymmetric, i.e.,

$$(2) \quad \omega_{ij} = -\omega_{ji}.$$

In three dimensions, from an antisymmetric tensor we can always build a *dual vector*,

$$(3) \quad \omega_k = e_{kij} \omega_{ij} / 2, \quad \text{i.e., } \omega = \text{curl } \mathbf{u} / 2,$$

where e_{kij} is the permutation symbol (Secs. 2.1 and 2.6). On the other hand, from Eq. (3) and the $e - \delta$ identity [Eq. (2.1:11)] it follows that $e_{ijk} \omega_k = \frac{1}{2}(\omega_{ij} - \omega_{ji})$. Hence, if ω_{ij} is antisymmetric, the relation Eq. (3) has a unique inverse,

$$(4) \quad \omega_{ij} = e_{ijk} \omega_k.$$

Thus, ω_{ij} may be called the *dual* (antisymmetric) *tensor* of a vector ω_k . We shall call ω_k and ω_{ij} , respectively, the *rotation vector* and *rotation tensor* of the displacement field u_i .

We shall consider the physical meaning of these quantities below.

At the end of Sec. 4.3 we saw that ω_z represents the (infinitesimal) rotation of the element as a rigid body if e_{xy}

vanishes. Now we have the general theorem that *the vanishing of the symmetric strain tensor E_{ij} or e_{ij} is the necessary and sufficient condition for a neighborhood of a particle to be moved like a rigid body*. This follows at once from the definitions of strain tensors, Eqs. (4.1:11) and (4.1:12). For, if a neighborhood of a particle P moves like a rigid body, the length of any element in the neighborhood will not change, so that $ds = ds_0$. It follows that $E_{ij} = e_{ij} = 0$, because, for a symmetric strain tensor E_{ij} , there exists a coordinates system that

$$ds^2 - ds_0^2 = \Lambda_1 da_1^2 + \Lambda_2 da_2^2 + \Lambda_3 da_3^2,$$

where Λ_i are the eigenvalues of E_{ij} . If $ds = ds_0$ for any line element i.e., ds_0 can just equal da_1 , da_2 or da_3 , it implies Λ_i ($i = 1, 2, 3$) are zero which in turn implies that $E_{ij} = 0$. The proof for $e_{ij} = 0$ is the same.

Conversely, if E_{ij} or e_{ij} vanishes at P , the length of line elements joining any two points in a neighborhood of P will not change and the neighborhood moves like a rigid body.

We can show that an infinitesimal displacement field in the neighborhood of a point P can be decomposed into a stretching deformation and a rigidbody rotation. To show this, consider a point P' in the neighborhood of P . Let the coordinates of P and P' be x_i and $x_i + dx_i$, respectively. The relative displacement of P' with respect to P is

$$(5) \quad du_i = (\partial u_i / \partial x_j) dx_j.$$

This can be written as

$$\begin{aligned} du_i &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dx_j + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) dx_j \\ &= e_{ij} dx_j - \omega_{ij} dx_j. \end{aligned}$$

The first term in the right-hand side represents a stretching deformation which involves both stretching and rotation. The second term can be written as

$$(6) \quad -\omega_{ij} dx_j = \omega_{ji} dx_j = -e_{ijk} \omega_k dx_j = (\boldsymbol{\omega} \times \mathbf{dx})_i.$$

Thus the second term is the vector product of $\boldsymbol{\omega}$ and \mathbf{dx} . This is exactly what would have been produced by an infinitesimal rotation $|\boldsymbol{\omega}|$ about an axis through P in the direction of $\boldsymbol{\omega}$.

It should be noted that we have restricted ourselves to infinitesimal displacements. Angular and strain measures for finite displacements are related to the deformation gradient in a more complicated way.

4.5. FINITE STRAIN COMPONENTS

When the strain components are not small, it is no longer possible to give simple geometric interpretations of the components of the strain tensors.

Consider a set of rectangular Cartesian coordinates with respect to which the strain components are defined as in Sec. 4.1. Let a line element before deformation be $da^1 = ds_0$, $da^2 = 0$, $da^3 = 0$. Let the extension E_1 of this element be defined by

$$E_1 = (ds - ds_0) / ds_0,$$

or

$$(1) \quad ds = (1 + E_1) ds_0.$$

From Eq. (4.1:9), we have

$$(2) \quad ds^2 - ds_0^2 = 2E_{ij} da^i da^j = 2E_{11} (da^1)^2.$$

Combining (1) and (2), we obtain

$$(1 + E_1)^2 - 1 = 2E_{11},$$

or

$$(3) \quad E_1 = \sqrt{1 + 2E_{11}} - 1.$$

This reduces to

$$(4) \quad E_1 \doteq E_{11},$$

if E_{11} is small.

To get the physical significance of the component E_{12} , let us consider two line elements \mathbf{ds}_0 and $d\bar{\mathbf{s}}_0$ which are at a right angle in the original state:

$$(5) \quad \begin{aligned} \mathbf{ds}_0 : \quad da^1 &= ds_0, & da^2 &= 0, & da^3 &= 0; \\ d\bar{\mathbf{s}}_0 : \quad d\bar{a}^1 &= 0, & d\bar{a}^2 &= d\bar{s}_0, & d\bar{a}^3 &= 0. \end{aligned}$$

After deformation these line elements become \mathbf{ds} , (dx^i) and $d\bar{\mathbf{s}}$ ($d\bar{x}^i$). Forming the scalar product of the deformed elements, we obtain

$$ds d\bar{s} \cos \theta = dx^k d\bar{x}^k = \frac{\partial x_k}{\partial a_i} da^i \frac{\partial \bar{x}_k}{\partial \bar{a}_j} d\bar{a}^j = \frac{\partial x_k}{\partial a_1} \frac{\partial \bar{x}_k}{\partial \bar{a}_2} ds_0 d\bar{s}_0.$$

The right-hand side is related to the strain component E_{12} according to Eq. (4.1:8), on specializing to rectangular Cartesian coordinates. Hence

$$(6) \quad ds d\bar{s} \cos \theta = 2E_{12} ds_0 d\bar{s}_0.$$

But, from Eqs. (1) and (3), we have

$$ds = \sqrt{1 + 2E_{11}} ds_0, \quad d\bar{s} = \sqrt{1 + 2E_{22}} d\bar{s}_0.$$

Hence, Eq. (6) yields

$$(7) \quad \cos \theta = 2E_{12}/[(1 + 2E_{11})^{1/2}(1 + 2E_{22})^{1/2}].$$

The angle θ is the angle between the line elements ds and $d\bar{s}$ after deformation. The change of angle between the two line elements, which in the original state ds_0 and $d\bar{s}_0$ are orthogonal, is $\alpha_{12} = \pi/2 - \theta$. From Eq. (7) we therefore obtain

$$(8) \quad \sin \alpha_{12} = 2E_{12}/[(1 + 2E_{11})^{1/2}(1 + 2E_{22})^{1/2}].$$

This reduces, in the case of infinitesimal strain, to

$$(9) \quad \alpha_{12} \doteq 2E_{12}.$$

A completely analogous interpretation can be made for the Eulerian strain components. Defining the extension e_1 per unit *deformed* length as

$$(10) \quad e_1 = (ds - ds_0)/ds,$$

we find

$$(11) \quad e_1 = 1 - \sqrt{1 - 2e_{11}}.$$

Furthermore, if the deviation from a right angle between two elements in the original state which after deformation become orthogonal be denoted by β_{12} , we have

$$(12) \quad \sin \beta_{12} = 2e_{12}/[(1 - 2e_{11})^{1/2}(1 - 2e_{22})^{1/2}].$$

These again reduce to the familiar results

$$e_1 \doteq e_{11}, \quad \beta_{12} \doteq 2e_{12},$$

in the case of infinitesimal strain.

4.6. COMPATIBILITY OF STRAIN COMPONENTS

The question of how to determine the displacements u_i when the components of strain tensor are given naturally arises. In other words, how do we integrate the differential equations (in rectangular Cartesian coordinates)

$$(1) \quad e_{ji} = e_{ij} = \frac{1}{2} \left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_\alpha}{\partial x_i} \frac{\partial u_\alpha}{\partial x_j} \right], \quad (i, j = 1, 2, 3)$$

to determine the three unknowns u_1, u_2, u_3 ?

Inasmuch as there are six equations for three unknown functions u_i , the system of Eqs. (1) will not have a single-valued solution in general, if the functions e_{ij} were arbitrarily assigned. One must expect that a solution may exist only if the functions e_{ij} satisfy certain conditions.

Since strain components only determine the relative positions of points in the body, and since any rigid-body motion corresponds to zero strain, we expect the solution u_i can be determined only up to an arbitrary rigid-body motion. But, if e_{ij} were specified arbitrarily, we may expect that something like the cases shown in Fig. 4.6:1 may happen. Here a continuous triangle (portion of material in a body) is given. If we deform it by following an arbitrarily specified strain field starting from the point A , we might end at the points C and D either with a gap between them or with overlapping of material. For a single-valued continuous solution to exist (up to a rigid-body motion), the ends C and D must meet perfectly in the strained configuration. This cannot be guaranteed unless the specified strain field along the edges of the triangle obeys certain conditions.

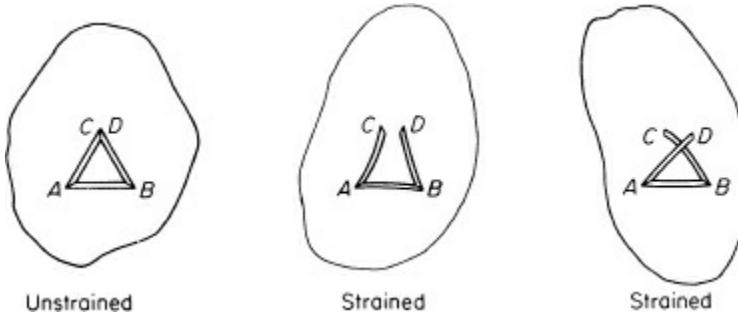


Fig. 4.6:1. Illustrations for the requirement of compatibility.

The conditions of integrability of Eqs. (1) are called the *compatibility conditions*. They are conditions to be satisfied by the strain components e_{ij} and can be obtained by eliminating u_i from Eqs. (1).

The nonlinear differential Eqs. (1) are difficult to handle, so first let us consider the linear infinitesimal strain case

$$(2) \quad e_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i) / 2, \quad \text{i.e., } e_{ij} = (u_{i,j} + u_{j,i}) / 2,$$

where an index i following a comma indicates partial differentiation with respect to x_i . By differentiation of Eq. (2), we have

$$e_{ij,kl} = (u_{i,jkl} + u_{j,ikl}) / 2.$$

Interchanging subscripts, we have

$$e_{kl,ij} = (u_{k,lkj} + u_{l,kij}) / 2, \quad e_{jl,ik} = (u_{j,lik} + u_{l,jik}) / 2, \quad e_{ik,jl} = (u_{i,kjl} + u_{k,ijl}) / 2.$$

From these we verify at once that

$$(3) \quad \blacktriangleleft \quad e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0.$$

This is the equation of compatibility obtained by St. Venant in 1860.

Of the 81 equations represented by Eq. (3), only six are essential. The rest are either identities or repetitions on account of the symmetry of e_{ij} and of e_{kl} . The six equations written in unabridged notations are

$$(4) \quad \blacktriangleleft \quad \begin{aligned} \partial^2 e_{xx} / (\partial y \partial z) &= \partial(-\partial e_{yz} / \partial x + \partial e_{zx} / \partial y + \partial e_{xy} / \partial z) / \partial x, \\ \partial^2 e_{yy} / (\partial z \partial x) &= \partial(-\partial e_{zx} / \partial y + \partial e_{xy} / \partial z + \partial e_{yz} / \partial x) / \partial y, \\ \partial^2 e_{zz} / (\partial x \partial y) &= \partial(-\partial e_{xy} / \partial z + \partial e_{yz} / \partial x + \partial e_{zx} / \partial y) / \partial z, \\ 2\partial^2 e_{xy} / (\partial x \partial y) &= \partial^2 e_{xx} / \partial y^2 + \partial e_{yy} / \partial x^2, \\ 2\partial^2 e_{yz} / (\partial y \partial z) &= \partial^2 e_{yy} / \partial z^2 + \partial e_{zz} / \partial y^2, \\ 2\partial^2 e_{zx} / (\partial z \partial x) &= \partial^2 e_{zz} / \partial x^2 + \partial e_{xx} / \partial z^2. \end{aligned}$$

These conditions are derived for infinitesimal strains referred to rectangular Cartesian coordinates. If the general curvilinear coordinates are used, we define the infinitesimal strain by

$$(5) \quad e_{ij} = (u_i|_j + u_j|i)/2,$$

where $u_i|_j$ is the covariant derivative of u_i , etc. The corresponding compatibility condition is then

$$(6) \quad e_{ij}|_{kl} + e_{kl}|_{ij} - e_{ik}|_{jl} - e_{jl}|_{ik} = 0.$$

If, however, the strain is finite so that e_{ij} is given by Eq. (1) in rectangular Cartesian coordinates, or by

$$(7) \quad e_{ij} = (u_i|_j + u_j|i - u_\alpha|i u^\alpha|_j)/2$$

in general coordinates, then it is necessary to use a new method of derivation. A successful method uses the basic concept that the compatibility conditions say that our body initially situated in a three-dimensional Euclidean space, must remain in the Euclidean space after deformation. A mathematical statement to this effect, expressed in terms of the strain components, gives the compatibility conditions. (See Probs. 2.31 and 4.9. Full expressions can be found in Green and Zerna,^{1,2} *Theoretical Elasticity*, p. 62.)

Let us now return to the question posed at the beginning of this section and inquire whether conditions (3) and (4) are sufficient to assure the existence of a single-valued continuous solution of the differential Eqs. (2) up to a rigid-body motion. The answer is affirmative. Various proofs are available, the simplest having been given by E. Cesaro^{4,1} in 1906. (See notes in [Bibliography 4.1](#).) The proof may proceed as follows.

Let $P_0(x_1^0, x_2^0, x_3^0)$ be a point at which the displacements u_i^0 and the components of rotation ω_{ij}^0 are specified. The displacement u_i at an arbitrary point $\bar{P}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is obtained by a line integral along a continuous rectifiable curve C joining P_0 and \bar{P} ,

$$(8) \quad u_i(\bar{x}_1, \bar{x}_2, \bar{x}_3) = u_i^0 + \int_{P_0}^{\bar{P}} du_i = u_i^0 + \int_{P_0}^{\bar{P}} \frac{\partial u_i}{\partial x_k} dx^k.$$

But

$$(9) \quad \frac{\partial u_i}{\partial x_k} = \frac{1}{2} \left[\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) + \left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i} \right) \right] = e_{ik} - \omega_{ik},$$

according to the definitions of the infinitesimal strain and rotation tensors, e_{ij} and ω_{ij} , respectively, as given by Eqs. (4.2:6) and (4.4:1). Hence, Eq. (8) becomes

$$(10) \quad u_i(\bar{x}_1, \bar{x}_2, \bar{x}_3) = u_i^0 + \int_{P_0}^{\bar{P}} e_{ik} dx^k - \int_{P_0}^{\bar{P}} \omega_{ik} dx^k.$$

We must eliminate ω_{ik} in terms of e_{ik} . To achieve this, the last integral is integrated by parts to yield

$$(11) \quad \begin{aligned} \int_{P_0}^{\bar{P}} \omega_{ik} dx^k &= \int_{P_0}^{\bar{P}} \omega_{ik}(x)(dx^k - d\bar{x}^k) \\ &= (\bar{x}_k - x_k^0)\omega_{ik}^0 + \int_{P_0}^{\bar{P}} (\bar{x}_k - x_k) \frac{\partial \omega_{ik}}{\partial x_l} dx^l, \end{aligned}$$

where ω_{ik}^0 is the value of ω_{ik} at P_0 . In Eq. (11) we have replaced dx^k by $dx^k - d\bar{x}^k$, to have ω_{ik}^0 appear outside the integral instead of ω_{ik} at P . This is permissible, since \bar{P} is fixed with respect to the integration so that $d\bar{x}^k = 0$. Now,

$$\begin{aligned} (12) \quad -\partial \omega_{ik}/\partial x_l &= [\partial^2 u_i/(\partial x_l \partial x_k) - \partial^2 u_k/(\partial x_l \partial x_i)]/2 \\ &= \frac{1}{2} \left(\frac{\partial^2 u_i}{\partial x_k \partial x_l} + \frac{\partial^2 u_l}{\partial x_k \partial x_i} \right) - \frac{1}{2} \left(\frac{\partial^2 u_k}{\partial x_i \partial x_l} + \frac{\partial^2 u_l}{\partial x_i \partial x_k} \right) \\ &= \frac{\partial}{\partial x_k} \frac{1}{2} \left(\frac{\partial u_i}{\partial x_l} + \frac{\partial u_l}{\partial x_i} \right) - \frac{\partial}{\partial x_i} \frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) = \frac{\partial e_{il}}{\partial x_k} - \frac{\partial e_{kl}}{\partial x_i}. \end{aligned}$$

Substitution of Eqs. (11) and (12) into Eq. (10) yields

$$(13) \quad u_i(\bar{x}_1, \bar{x}_2, \bar{x}_3) = u_i^0 - (\bar{x}_k - x_k^0)\omega_{ik}^0 + \int_{P_0}^{\bar{P}} U_{il} dx^l,$$

where

$$(14) \quad U_{il} = e_{il} + (\bar{x}_k - x_k)(\partial e_{il}/\partial x_k - \partial e_{kl}/\partial x_i).$$

Now, if $u_i(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is to be single-valued and continuous, the last integral in Eq. (13) must depend only on the end points P_0 , \bar{P} and must not be dependent on the path of integration C . Therefore, the integrand must be an exact differential. (See Sec. 4.7 below.) The necessary condition that $U_{il}dx^l$ be an exact differential is (see Sec. 4.7)

$$(15) \quad \partial U_{il}/\partial x_m - \partial U_{im}/\partial x_l = 0.$$

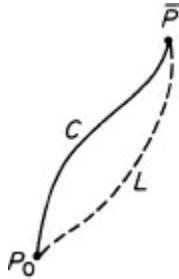


Fig. 4.6:2. Paths of integration.

This condition also suffices in assuring the single-valuedness of the integral if the region is simply connected. However, for a multiply connected region, some additional conditions must be imposed; they will be discussed in the next section.

Now we obtain, from Eqs. (14) and (15),

$$\begin{aligned} & \frac{\partial e_{il}}{\partial x_m} - \delta_{km} \left(\frac{\partial e_{il}}{\partial x_k} - \frac{\partial e_{kl}}{\partial x_i} \right) + (\bar{x}_k - x_k) \frac{\partial}{\partial x_m} \left(\frac{\partial e_{il}}{\partial x_k} - \frac{\partial e_{kl}}{\partial x_i} \right) - \frac{\partial e_{im}}{\partial x_l} \\ & + \delta_{kl} \left(\frac{\partial e_{im}}{\partial x_k} - \frac{\partial e_{km}}{\partial x_i} \right) - (\bar{x}_k - x_k) \frac{\partial}{\partial x_l} \left(\frac{\partial e_{im}}{\partial x_k} - \frac{\partial e_{km}}{\partial x_i} \right) = 0. \end{aligned}$$

The factors not multiplied by $\bar{x}_k - x_k$ cancel each other, and the factors multiplied by $\bar{x}_k - x_k$ form exactly the compatibility condition given by Eq. (3). Hence, Eq. (15) is satisfied if the compatibility conditions (3) are satisfied. Thus, we have proved that the satisfaction of the compatibility conditions (3) is necessary and sufficient for the displacement to be single-valued in a simply connected region. For a multiply connected region, Eq. (3) is necessary, but no longer sufficient. To guarantee single-valuedness of displacement for an assumed strain field, some additional conditions as described in Sec. 4.7 must be imposed.

Problem 4.1. Consider the two-dimensional case in unabridged notations,

$$\partial u/\partial x = e_{xx}, \quad \partial v/\partial y = e_{yy}, \quad \partial u/\partial y + \partial v/\partial x = 2e_{xy}.$$

Prove that in order to guarantee the solutions $u(x, y)$, $v(x, y)$ to be single-valued in a simply connected domain $D(x, y)$, the functions $exx(x, y)$, $eey(x, y)$, $exy(x, y)$ must satisfy the compatibility condition

$$\partial^2 e_{xx}/\partial y^2 + \partial^2 e_{yy}/\partial x^2 = 2\partial^2 e_{xy}/(\partial x \partial y).$$

Prove the sufficiency of this condition in unabridged notations.

4.7. MULTIPLY CONNECTED REGIONS

As necessary conditions, the compatibility equations derived in Sec. 4.6 apply to any continuum. But the sufficiency proof at the end of the preceding section requires that the region be simply connected. For a multiply connected continuum, additional conditions must be imposed.

A region is *simply connected* if any simple closed contour drawn in the region can be shrunk continuously to a point without leaving the region; otherwise the region is said to be *multiply connected*.

Figure 4.7:1(a) illustrates a simply connected region \mathcal{R} in which an arbitrary closed curve C can be shrunk continuously to a point without leaving \mathcal{R} . Figures 4.7:1(b) and 4.7:2 illustrate doubly connected regions in two and three dimensions, respectively. Figures 4.7:1(c) and (d) show how multiply connected regions can be made simply connected by introducing cuts — imaginary boundaries.

The condition (4.6:15) is based on Stokes' theorem, and we shall review the reasoning in preparation for discussing multiply connected regions. In two dimensions, Stokes' theorem for continuous and differentiable functions $P(x, y)$, $Q(x, y)$ may be written as

$$(1) \quad \int_C (Pdx + Qdy) = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy,$$

for a simply connected region \mathcal{R} on a plane x, y bounded by a closed contour C . In three dimensions, Stokes' theorem states, for three functions P, Q, R ,

$$(2) \quad \int_C (Pdx + Qdy + Rdz) = \iint_{\mathcal{R}} \left\{ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \cos(\nu, \mathbf{x}) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \cos(\nu, \mathbf{y}) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cos(\nu, \mathbf{z}) \right\} dS,$$

for a simply connected region \mathcal{R} on a surface S bounded by a closed contour C . The factor $\cos(\nu, \mathbf{x})$ is the cosine of the angle between the x -axis and the normal vector ν normal to the surface S . When the conditions

$$(3) \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y},$$

are imposed, the line integral on the left-hand side of Eq. (2) vanishes for every possible closed contour C ,

$$\int_C (Pdx + Qdy + Rdz) = 0.$$

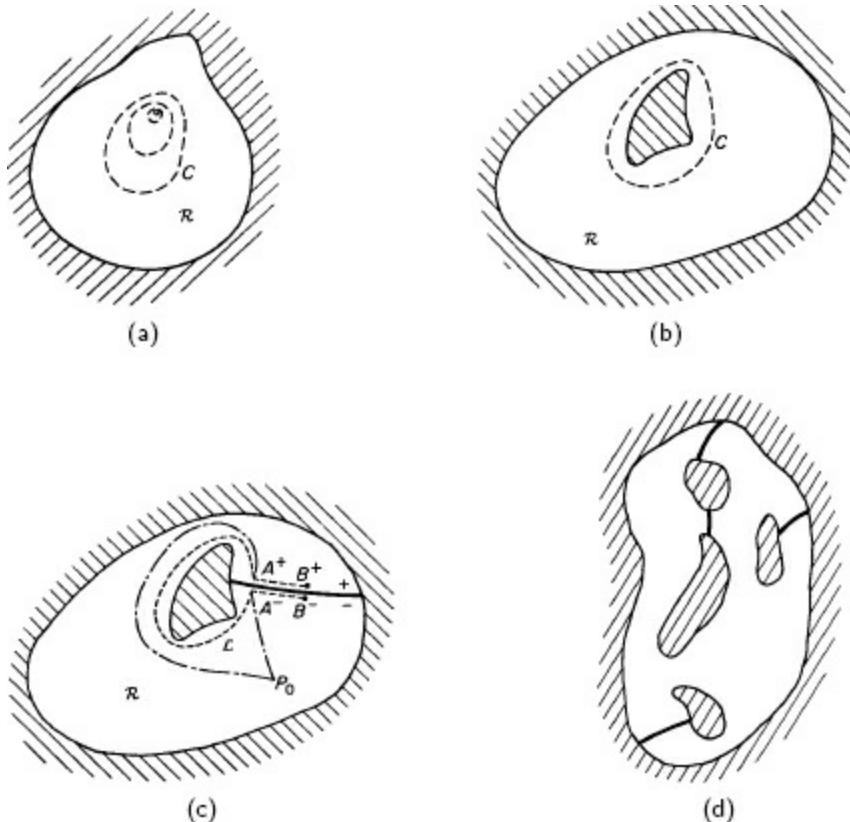


Fig. 4.7:1. Two-dimensional examples of simply and multiply connected regions. In (a), region \mathcal{R} is simply connected; in (b), \mathcal{R} is doubly connected.

When this result is applied to any two possible paths of integration between a fixed point P_0 and a variable point \bar{P} , as in Fig. 4.6:2, we see that the integral

$$(4) \quad F(\bar{P}) = \int_{P_0}^{\bar{P}} (Pdx + Qdy + Rdz),$$

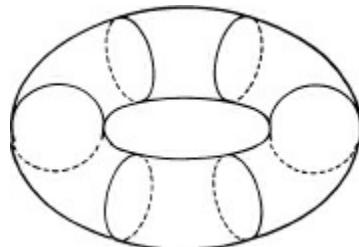


Fig. 4.7:2. A torus doubly connected in three dimensions.

must be a single-valued function of the point \bar{P} , independent of the path of integration; because in Fig. 4.6:2 the curve P_0CP_0 forms a simple contour. Hence,

$$\int_{P_0CP} + \int_{PLP_0} = 0 = \int_{P_0CP} - \int_{P_0LP} .$$

Thus, the last two integrals are equal. This is the result used in Sec. 4.6 to conclude that the compatibility conditions as posed are sufficient for uniqueness of displacements in a simply connected region.

Let us now consider a doubly connected region on a plane as shown in Fig. 4.7:1(b). After a cut is made as in Fig. 4.7:1(c), the region \mathcal{R} is simply connected. Let the two sides of the cut be denoted by (+) and (-) and let the points A^+ and A^- be directly opposite to each other on the two sides of the cut. The line integrals

$$F(A^-) = \int_{P_0}^{A^-} (Pdx + Qdy), \quad F(A^+) = \int_{P_0}^{A^+} (Pdx + Qdy),$$

integrated from a fixed point P_0 along paths in the cut region, are both single-valued. But, since A^+ and A^- are on the opposite sides of the cut, the values of the integrals need not be the same. Let the chain-dot path P_0A^+ in Fig. 4.7:1(c) be deformed into P_0A^- plus a line \mathcal{L} connecting A^- to A^+ , encircling the inner boundary. Then

$$\int_{P_0}^{A^+} = \int_{P_0}^{A^-} + \int_{\mathcal{L}} .$$

Hence,

$$(5) \quad F(A^+) - F(A^-) = \int_{\mathcal{L}} (Pdx + Qdy) .$$

Similar consideration for another pair of arbitrary points B^+ and B^- across the boundary leads to the same result,

$$(6) \quad F(B^+) - F(B^-) = \int_{\mathcal{L}'} (Pdx + Qdy) .$$

The right-hand sides of Eqs. (5) and (6) are the same if P and Q are single-valued in the region \mathcal{R} , because a line \mathcal{L}' connecting B^- to B^+ , encircling the inner boundary, may be deformed into one that goes from B^- to A^- along the cut, then follows \mathcal{L} from A^- to A^+ , and finally moves from A^+ to B^+ along the + side of the cut. If P and Q were single-valued, the integral from B^- to A^- cancels exactly the one from A^+ to B^+ .

From these considerations, it is clear that function $F(\bar{P})$ defined by the line integral (4) will be single-valued in a doubly connected region if the supplementary condition

$$(7) \quad \int_{\mathcal{L}} (Pdx + Qdy) = 0 ,$$

is imposed, where \mathcal{L} is a closed contour that goes from one side of a cut to the other, without leaving the region.

The same consideration can be applied generally to multiply connected regions in two or three dimensions. An $(m + 1)$ -ply connected region can be made simply connected by m cuts. In the cut, simply-connected region, m independent simple contours $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_m$ can be drawn. Each \mathcal{L}_i starts from one side of a cut, and ends on the other side of the same cut. All cuts are thus embraced by the \mathcal{L} 's. Then the single-valuedness of the function $F(\bar{P})$ defined by the line integral (4) can be assured by imposing, in addition to the conditions (3), m supplementary conditions

$$(8) \quad \int_{\mathcal{L}_1} (Pdx + Qdy + Rdz) = \int_{\mathcal{L}_2} (\) = \dots = \int_{\mathcal{L}_m} (\) = 0 .$$

When these results are applied to the compatibility problem of Sec. 4.6 we see that if the body is $(m + 1)$ -ply connected, the single-valuedness of the displacement function $u_i(x_1, x_2, x_3)$ given by Eq. (4.6:13) requires $3m$ supplementary conditions:

$$(9) \quad \Delta \quad \int_{\mathcal{L}_1} U_{il} dx^l = \dots = \int_{\mathcal{L}_m} U_{il} dx^l = 0 , \quad i = 1, 2, 3 ,$$

where U_{il} is given in Eq. (4.6:14), and $\mathcal{L}_1, \dots, \mathcal{L}_m$ in \mathcal{R} as described above.

4.8. MULTIVALUED DISPLACEMENTS

Some problems of thermal stress, initial strain, rigid inclusions, etc., can be formulated in terms of multivalued displacements. For example, if the horseshoe in Fig. 4.8:1(a) is strained so that the faces S_1 and S_2 come into contact and are then welded together, the result is the doubly connected body shown in Fig. 4.8:1(b). If the strain of the body (b) is known, we can use the procedure described in Sec. 4.6 to compute displacements and generally we will obtain the configuration of Fig. 4.8:1(a). In this case a single point P on the welded surface S will open up into two points P_1 and P_2 on the two open ends. Thus the displacement at P is double-valued.

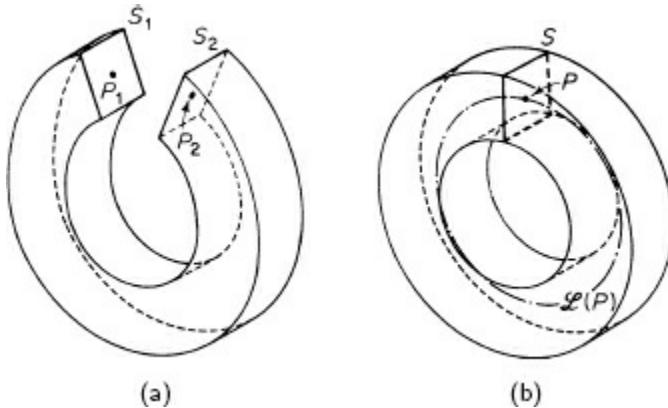


Fig. 4.8:1. Possibility and use of multivalued displacements.

In the example posed here, the strain components and their derivatives may be discontinuous on the two sides of the interface S of Fig. 4.8:1(b). Hence, the value of a line integral around a contour in the body that crosses the interface S will depend on where the crossing point is. In other words,

$$(1) \quad u_i(P_2) - u_i(P_1) = \int_{\mathcal{L}(P)} U_{il} dx^l,$$

depends on the location of P , where $\mathcal{L}(P)$ is a contour passing through the point P as indicated in the figure and U_{il} is given in Eq. (4.6:14) or (4.7:10).

On the other hand, if we have a ring as in Fig. 4.8:1(b) and apply some load on it and ask what is the corresponding deformation, we must first decide whether some crack is allowed to be opened or not. If the ring is to remain integral, the deformation must be such that the displacement field is singlevalued. In this case the strain due to the load must be such that

$$(2) \quad \int_{\mathcal{L}(P)} U_{il} dx^l = 0,$$

for a contour $\mathcal{L}(P)$ as shown in Fig. 4.8:1(b). Of course, Eq. (2) in conjunction with the compatibility conditions (4.6:4) assures that the line integral $\int U_{il} dx^l$ will vanish on any contour in the ring. Thus the point P in Eq. (2) is arbitrary.

These examples show that both single-valued displacement fields and multivalued displacement fields have proper use in answering appropriate question.

4.9. PROPERTIES OF THE STRAIN TENSOR

The symmetric strain tensor e_{ij} has many properties in common with the stress tensor. Thus, the existence of real-valued principal strains and principal planes, the Mohr circle for strain, the octahedral strain, strain deviations, and the strain invariants need no further discussion.

The first strain invariant,

$$(1) \quad e = e_{11} + e_{22} + e_{33} = \text{sum of principal strains},$$

has a simple geometrical meaning in the case of infinitesimal strain. Let a volume element consist of a rectangular parallelepiped with edges parallel to the principal directions of strain. Let the length of the edges be l_1, l_2, l_3 in the

unstrained state. Let e_{11}, e_{22}, e_{33} be the principal strains. In the strained configuration, the edges become of lengths $l_1(1 + e_{11})$, $l_2(1 + e_{22})$, $l_3(1 + e_{33})$ and remain orthogonal to each other. Hence, for small strain the change of volume is

$$\begin{aligned}\Delta V &= l_1 l_2 l_3 (1 + e_{11})(1 + e_{22})(1 + e_{33}) - l_1 l_2 l_3 \\ &\doteq l_1 l_2 l_3 (e_{11} + e_{22} + e_{33}).\end{aligned}$$

Therefore,

$$(2) \quad e = e_{ii} = \Delta V/V.$$

Thus, in the infinitesimal strain theory the first invariant represents the expansion in volume per unit volume. For this reason, e is called the *dilatation*.

If two-dimensional strain state (plane strain) is considered u_3 or $w = 0$, the first invariant $e_{11} + e_{22}$ represents the change of area per unit area of the surface under strain.

For finite strain, the sum of principal strains does not have such a simple interpretation.

P R O B L E M S

4.2. A state of deformation in which all the strain components are constant throughout the body is called a *homogeneous deformation*. What is the equation, of the type $f(x, y, z) = 0$, of a surface which becomes a sphere $x^2 + y^2 + z^2 = r^2$ after a homogeneous deformation? What kind of surface is it? (x, y, z are rectangular Cartesian coordinates.)

4.3. Show that a strain state described in rectangular Cartesian coordinates

$$\begin{aligned}e_{xx} &= k(x^2 + y^2), \quad e_{yy} = k(y^2 + z^2), \quad e_{xy} = k'xyz, \\ e_{xz} &= e_{yz} = e_{zz} = 0,\end{aligned}$$

where k, k' are small constants, is not a possible state of strain for a continuum.

4.4. A solid is heated nonuniformly to a temperature $T(x, y, z)$. If each element has unrestrained thermal expansion, the strain components will be

$$\begin{aligned}e_{xx} &= e_{yy} = e_{zz} = \alpha T, \\ e_{xy} &= e_{yz} = e_{zx} = 0,\end{aligned}$$

where x, y, z , are rectangular Cartesian coordinates and α is the thermal expansion coefficient (a constant). Prove that this can only occur when T is a linear function of x, y, z ; i.e., $T = c_1x + c_2y + c_3z + c_4$, where c_1, \dots, c_4 are constants.

4.5. A soap-film like membrane stretched over a ring is deformed by uniform pressure into a hemisphere of the same radius (Fig. P4.5). In so doing, a point P on the flat surface over the ring is deformed into a point Q on the sphere. Determine a mathematical transformation from P to Q . Note: A “soap-film like” membrane shall be defined as a membrane in which the tension is a constant, isotropic, and independent of the stretching of the membrane.

Let P be referred to plane polar coordinates (r, θ) and Q be referred to spherical polar coordinates (R, ϕ, θ) with the same origin. We assume that $R = a$, $\theta = \theta$. The problem is to determine the angle ϕ as a function of r .

If the thickness of the film is uniform originally, it will not remain so after being blown into spherical shape. Determine the thickness distribution in the final configuration under the assumption of material incompressibility.

4.6. Consider a two-dimensional inflatable structure which consists of an infinitely long tube. When folded flat, the cross section of the tube appears like a line segment AB of length 2 in Fig. P4.6. When blown up with an internal gas pressure p , the tube assumes the form of an ellipse with major axis α and minor axis β .

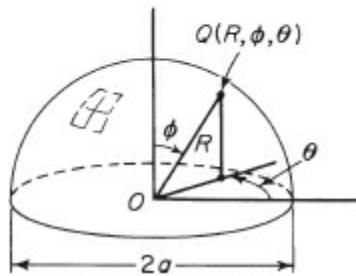
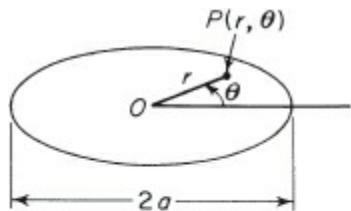


Fig. P4.5

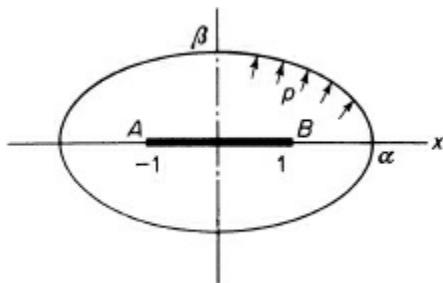


Fig. P4.6

Questions:

- Devise (arbitrarily, but specifically) a law of transformation which transforms the folded tube into the elliptic cylinder.
- Compute the components of membrane strain in the midsurface of the tube referred to the original folded tube (components of Green's strain tensor).
- Compute the corresponding components of strain referred to the blown up tube (components of Almansi's strain tensor).
- For equilibrium, what must be the distribution of membrane tension force T in the blown-up tube? (The internal pressure is uniform.)
- Assume that $T = khe_1$, where T is the membrane tension, e_1 is the strain component of Almansi on the midsurface of the tube wall, h is the wall thickness of the blown up tube, and k is a constant. Determine h .
- If the tube material is incompressible (like natural rubber), what would have to be the initial thickness distribution?

4.10. PHYSICAL COMPONENTS

In Sec. 2.16 we defined the physical components of a vector \mathbf{v} as the components of \mathbf{v} resolved in the directions of a set of unit vectors which are parallel either to the set of base vectors or to the set of reciprocal base vectors. Thus,

$$(1) \quad \begin{array}{ll} \text{Tensor components:} & v^r \quad v_r \\ \text{Physical components:} & v^r \sqrt{g_{rr}} \quad v_r \sqrt{g^{rr}} \\ \text{Reference base vectors:} & g_r \quad g^r \end{array}$$

All the physical components have uniform physical dimensions; but they do not transform conveniently under coordinate transformations.

In Sec. 3.13 we defined the physical components of the stress tensors τ^{ij} as the components of the stress vector $\overset{i}{\mathbf{T}}$

resolved in the directions of unit vectors parallel to the base vectors. If the physical components of τ^{ij} are denoted by σ^{ij} , we have

$$(2) \quad \sigma^{ij} = (g_{jj}/g^{ii})^{1/2} \tau^{ij}, \quad i, j \text{ not summed}.$$

Correspondingly, when $\overset{i}{\mathbf{T}}$ is resolved with respect to the reciprocal base vectors, the physical components are

$$(3) \quad \sigma_j^i = (g^{jj}/g^{ii})^{1/2} \tau_j^i, \quad i, j \text{ not summed}.$$

All components σ^{ij} , σ_j^i have uniform physical dimensions. The physical components of the contravariant, covariant, and mixed tensors are the same if the coordinates are orthogonal.

Now consider the strain tensor e_{ij} , which is covariant in the most natural form of definition (Sec. 4.1). If ds is the length of an element $(d\theta^1, d\theta^2, d\theta^3)$ at a point $(\theta^1, \theta^2, \theta^3)$, and ds_0 is the length of the same element before the deformation takes place, then

$$(4) \quad ds^2 - ds_0^2 = 2e_{ij} d\theta^i d\theta^j.$$

The differential element $(d\theta^i)$ is a vector, whose physical components are $\sqrt{g_{ii}} d\theta^i$, (i not summed). We may rewrite (4) as

$$(5) \quad ds^2 - ds_0^2 = 2 \sum_{i=1}^3 \sum_{j=1}^3 e_{ij} (\sqrt{g_{ii}} d\theta^i) (\sqrt{g_{jj}} d\theta^j) / \sqrt{g_{ii} g_{jj}}.$$

Since all the components $\sqrt{g_{ii}} d\theta^i$, $\sqrt{g_{jj}} d\theta^j$ (i, j not summed) have the dimension of length, we may define the *physical strain components*

$$(6) \quad \epsilon_{ij} = e_{ij} / (g_{ii} g_{jj})^{1/2}, \quad i, j \text{ not summed},$$

which are all dimensionless.

The rules of forming physical components now appear rather complicated. The appearance can be made more systematic if we utilize the relation given in Eq. (2.14:9), i.e.,

$$(7) \quad \mathbf{g}^i \cdot \mathbf{g}_j = \delta_j^i,$$

which implies (8)

$$(8) \quad g^{ii} = (g_{ii})^{-1}, \quad i \text{ not summed}.$$

Then Eqs. (2), (3), and (6) may be written as

$$(9) \quad \sigma^{ij} = \sqrt{g_{ii}} \sqrt{g_{jj}} \tau^{ij},$$

$$(10) \quad \sigma_j^i = \sqrt{g^{jj}} \sqrt{g_{ii}} \tau_j^i,$$

$$(11) \quad \epsilon_{ij} = \sqrt{g^{ii}} \sqrt{g^{jj}} e_{ij},$$

where i, j are not summed. These formulas, similar to Eq. (1), may serve as a pattern for defining the physical components of any tensor in any curvilinear coordinates. In orthogonal curvilinear coordinates, $\sigma^{ij} = \sigma_j^i$.

Let us repeat. The physical components and the tensor components of a physical quantity have the same geometric and physical interpretations, except for physical dimensions. The unification in dimensions is achieved by multiplying the tensor components with appropriate scale factors, which are related to the components of the metric tensor. The physical components do not transform conveniently under general transformation of coordinates. In practical applications, therefore, we generally write basic equations in the tensor form and then substitute the individual tensor components by their physical counterpart, if it is so desired.

We shall illustrate these applications in the following sections.

4.11. EXAMPLE – SPHERICAL COORDINATES

Let $x^1 = r$, $x^2 = \varphi$, $x^3 = \theta$ (Fig. 4.11:1).

Then

$$ds^2 = dr^2 + r^2 d\phi^2 + r^2 \sin^2 \phi d\theta^2,$$

$$g_{11} = 1, \quad g_{22} = r^2,$$

$$g_{33} = r^2 \sin^2 \phi, \quad g_{ij} = 0 \quad \text{if } i \neq j.$$

Thus the coordinate system is orthogonal. From the definition of $g^{\alpha\beta}$, we have

$$g^{11} = 1, \quad g^{22} = 1/r^2,$$

$$g^{33} = \frac{1}{r^2 \sin^2 \phi}, \quad g^{ij} = 0 \quad \text{if } i \neq j.$$

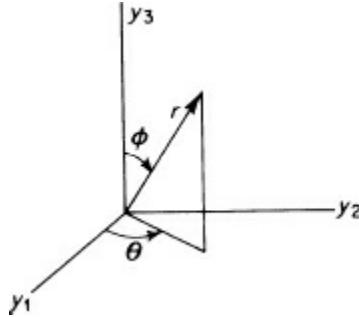


Fig. 4.11:1. Spherical polar coordinates.

The Euclidean Christoffel symbols are

$$\begin{aligned} \Gamma_{22}^1 &= -r, & \Gamma_{33}^1 &= -r \sin^2 \phi, \\ \Gamma_{12}^2 &= \Gamma_{21}^2 = 1/r, & \Gamma_{33}^2 &= -\sin \phi \cos \phi, \\ \Gamma_{13}^2 &= \Gamma_{31}^2 = 1/r, & \Gamma_{23}^3 &= \Gamma_{32}^3 = \cot \phi, \end{aligned}$$

all other $\Gamma_{jk}^i = 0$.

Let u_i be the covariant components of the displacement vector. We have the infinitesimal strain tensor components

$$\epsilon_{ij} = (u_i|_j + u_j|i)/2 = (\partial u_i / \partial x_j + \partial u_j / \partial x_i)/2 - \Gamma_{ij}^\sigma u_\sigma,$$

since $\Gamma_{ij}^\sigma = \Gamma_{ji}^\sigma$. Hence,

$$\begin{aligned} \epsilon_{11} &= \partial u_1 / \partial r, & \epsilon_{22} &= \partial u_2 / \partial \phi + u_1, \\ \epsilon_{33} &= \frac{\partial u_3}{\partial \theta} + r \sin^2 \phi u_1 + \sin \phi \cos \phi u_2, & \epsilon_{12} &= \frac{1}{2} \left(\frac{\partial u_1}{\partial \phi} + \frac{\partial u_2}{\partial r} \right) - \frac{u_2}{r}, \\ \epsilon_{23} &= \frac{1}{2} \left(\frac{\partial u_2}{\partial \theta} + \frac{\partial u_3}{\partial \phi} \right) - \cot \phi u_3, & \epsilon_{31} &= \frac{1}{2} \left(\frac{\partial u_3}{\partial r} + \frac{\partial u_1}{\partial \theta} \right) - \frac{u_3}{r}. \end{aligned}$$

The u_i and ϵ_{ij} are tensor components. Let the corresponding physical components of the displacement vector be written as ξ_r , ξ_ϕ , ξ_θ and that of the strain tensor be written as E_{ij} . Then, since the spherical coordinates are orthogonal,

$$\begin{aligned} \xi_r &= \sqrt{g^{11}} u_1 = u_1, & \xi_\phi &= \sqrt{g^{22}} u_2 = \frac{u_2}{r}, \\ \xi_\theta &= \sqrt{g^{33}} u_3 = \frac{u_3}{r \sin \phi}, & \epsilon_{ij} &= \sqrt{g^{ii} g^{jj}} e_{ij}. \end{aligned}$$

Therefore,

$$\begin{aligned}\epsilon_{11} &= \partial\xi_r/\partial r, \quad \epsilon_{22} = (\partial\xi_\phi/\partial\phi)/r + \xi_r/r, \\ \epsilon_{33} &= (\partial\xi_\theta/\partial\theta)/(r \sin \phi) + \xi_r/r + \cot \phi \xi_\phi/r, \\ \epsilon_{12} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial\xi_r}{\partial\phi} + \frac{1}{r} \frac{\partial(r\xi_\phi)}{\partial r} \right) - \frac{\xi_\phi}{r} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial\xi_r}{\partial\phi} + \frac{\partial\xi_\phi}{\partial r} - \frac{\xi_\phi}{r} \right), \\ e_{23} &= \frac{1}{2r} \left(\frac{1}{\sin \phi} \frac{\partial\xi_\phi}{\partial\theta} + \frac{\partial\xi_\theta}{\partial\phi} - \cot \phi \xi_\theta \right), \quad e_{31} = \frac{1}{2} \left(\frac{1}{r \sin \phi} \frac{\partial\xi_r}{\partial\theta} + \frac{\partial\xi_\theta}{\partial r} - \frac{\xi_\theta}{r} \right).\end{aligned}$$

The equations of equilibrium now become, with F_r, F_ϕ, F_θ denoting the physical components of the body force vector,

$$\begin{aligned}-F_r &= \frac{1}{r^2} \frac{\partial(r^2 \sigma_{rr})}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial(\sin \phi \sigma_{r\phi})}{\partial\phi} + \frac{1}{r \sin \phi} \frac{\partial\sigma_{r\theta}}{\partial\theta} - \frac{\sigma_{\theta\theta} + \sigma_{\phi\phi}}{r}, \\ -F_\phi &= \frac{1}{r^3} \frac{\partial(r^3 \sigma_{r\phi})}{\partial r} + \frac{1}{r \sin \phi} \frac{\partial(\sin \phi \sigma_{\phi\phi})}{\partial\phi} + \frac{1}{r \sin \phi} \frac{\partial\sigma_{\phi\theta}}{\partial\theta} - \frac{\cot \phi}{r} \sigma_{\theta\theta}, \\ -F_\theta &= \frac{1}{r^3} \frac{\partial(r^3 \sigma_{\theta\theta})}{\partial r} + \frac{1}{r \sin^2 \phi} \frac{\partial(\sin^2 \phi \sigma_{\theta\phi})}{\partial\phi} + \frac{1}{r \sin \phi} \frac{\partial\sigma_{\theta\theta}}{\partial\theta}.\end{aligned}$$

4.12. EXAMPLE – CYLINDRICAL POLAR COORDINATES

Letting ξ_r, ξ_θ, ξ_z be the physical components of the displacement and $E_{rr}, E_{r\theta}, \dots, \sigma_{rr}, \sigma_{r\theta}, \dots$, etc., be the physical components of the strain and stress, respectively, we obtain the following results:

$$\begin{aligned}x^1 &= r, \quad x^2 = \theta, \quad x^3 = z, \\ ds^2 &= dr^2 + r^2 d\theta^2 + dz^2, \\ g_{11} &= 1, \quad g_{22} = r^2, \quad g_{33} = 1, \quad \text{all other } g_{ij} = 0, \\ g^{11} &= 1, \quad g^{22} = 1/r^2, \quad g^{33} = 1, \quad \text{all other } g^{ij} = 0, \\ \Gamma_{22}^1 &= -r, \quad \Gamma_{21}^2 = \Gamma_{12}^2 = 1/r, \quad \text{all other } \Gamma_{jk}^i = 0.\end{aligned}$$

Hence,

$$\begin{aligned}\epsilon_{rr} &= \partial\xi_r/\partial r, \quad \epsilon_{\theta\theta} = (\partial\xi_\theta/\partial\theta + \xi_r)/r, \quad \epsilon_{zz} = \partial\xi_z/\partial z, \\ \epsilon_{\theta z} &= [\partial\xi_\theta/\partial z + (\partial\xi_z/\partial\theta)/r]/2, \quad \epsilon_{zr} = (\partial\xi_r/\partial z + \partial\xi_z/\partial r)/2, \\ \epsilon_{r\theta} &= [(\partial\xi_r/\partial\theta)/r + \partial\xi_\theta/\partial r - \xi_\theta/r]/2.\end{aligned}$$

The equations of equilibrium are

$$\begin{aligned}-F_r &= [\partial(r\sigma_{rr})/\partial r]/r + (\partial\sigma_{r\theta}/\partial\theta)/r + \partial\sigma_{rz}/\partial z - \sigma_{\theta\theta}/r, \\ -F_\theta &= [\partial(r^2\sigma_{\theta r})/\partial r]/r^2 + (\partial\sigma_{\theta\theta}/\partial\theta)/r + \partial\sigma_{\theta z}/\partial z, \\ -F_z &= [\partial(r\sigma_{zr})/\partial r]/r + (\partial\sigma_{z\theta}/\partial\theta)/r + \partial\sigma_{zz}/\partial z.\end{aligned}$$

PROBLEMS

4.7. Give proper definitions of strain tensors e_j^i, e_i^j in general frames of reference in terms of e_{ij} . Show that e_{ij}, e^{ij} are symmetric tensors. Show that the principal strains e_1, e_2, e_3 are the roots of the characteristic equation

$$|e_j^i - e\delta_j^i| = 0$$

and that the first invariant is e_i^i .

4.8. The generalization of the expressions of strain tensors in terms of the displacement vector field \mathbf{u} , which represents the displacement of any particle as the body deforms as explained in Sec. 4.2, can be done as follows. The displacement vector \mathbf{u} can be resolved into covariant and contravariant components u_i and u^i , respectively, along base

vectors and reciprocal base vectors defined in the original coordinates a_1, a_2, a_3 . The Green's strain tensor is

$$E_{ij} = (u_j|_i + u_i|_j + u^k|_i u_k|_j)/2,$$

where $u_i|_j$ is the covariant derivative of u_i with respect to a_j .

The displacement vector \mathbf{u} can be resolved also into covariant and contravariant components u_i and u^i along base vectors and reciprocal base vectors defined in the Eulerian coordinates x_1, x_2, x_3 . Then the Almansi strain tensor is

$$e_{ij} = (u_j||_i + u_i||_j - u^k||_i u_k||_j)/2,$$

where a double bar is used to indicate that u_i, u^i and the covariant derivatives are referred to the coordinates x_i .

Prove these statements.

4.9. The condition of compatibility for finite strain can be derived easily if convected coordinates are used [see Eq. (4.1:12)]. By the fact that the space of the deformed body is Euclidean, derive the compatibility conditions. *Hint:* See comments on p. 94 and Prob. 2.31. *Note:* However, the properties $g^{ij}|_k = 0, g_{ij}|_k = 0$ are valid in Riemannian space as well as Euclidean space. The feature that distinguishes a Euclidean space is the vanishing of the Riemann–Christoffel curvature tensor R_{pst}^i of Prob. 2.31. Expressing $R_{pst}^i = 0$ in terms of the metric tensors before and after deformation leads to the compatibility conditions.

¹As remarked before in a footnote in Sec. 2.4, the point transformation between a_i and x_i does not follow tensor transformation law and the position of the indices has no significance. But the *differentials* da^i and dx^i do transform according to the contravariant tensor law and are contravariant vectors.

5

CONSERVATION LAWS

We shall discuss in this chapter the basic laws of conservation of mass and momentum. A rectangular Cartesian frame of reference will be used throughout. All tensors are Cartesian tensors, whose physical and tensor components are the same, unless specified otherwise. Thus stresses can be denoted by σ_{ij} or σ^{ij} .

5.1. GAUSS' THEOREM

Consider a convex region V bounded by a surface S that consists of a finite number of parts whose outer normals form a continuous vector field (Fig. 5.1:1). Let a function $A(x_1, x_2, x_3)$ be defined and differentiable in the region $V + S$. Then, by the usual process of integration, we obtain

$$(1) \quad \iiint_V \frac{\partial A}{\partial x_1} dx_1 dx_2 dx_3 = \iint_S (A^* - A^{**}) dx_2 dx_3,$$

where A^* and A^{**} are the values of A on the surface S at the right and left ends of a line parallel to the x_1 -axis, respectively. The factors $\pm dx_2 dx_3$ in the surface integral in Eq. (1) are the projections of the x_2, x_3 -plane of the areas dS^* and dS^{**} at the ends of a line parallel to the x_1 -axis. Let $v = (v_1, v_2, v_3)$ be the unit vector along the outer normal of S . Then $dx_2 dx_3 = v_1^* dS^*$ at the right end and $dx_2 dx_3 = -v_1^{**} dS^{**}$ at the left end. Therefore, the surface integral in Eq. (1) can be written as

$$\int_S (A^* v_1^* dS^* + A^{**} v_1^{**} dS^{**}) = \int_S A v_1 dS.$$

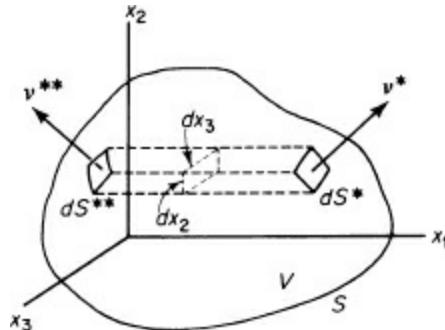


Fig. 5.1:1

Thus, Eq. (1) may be written as

$$\int_V \frac{\partial A}{\partial x_1} dV = \int_S A v_1 dS,$$

where dV and dS denote the elements of V and S , respectively. A similar argument applies to the volume integral of $\partial A / \partial x_2$, or $\partial A / \partial x_3$. In summary, we obtain Gauss' theorem

$$(2) \quad \int_V \frac{\partial A}{\partial x_i} dV = \int_S A v_i dS.$$

This formula holds for any convex regular region or for any region that can be decomposed into a finite number of convex regular regions, as can be seen by summing Eq. (2) over these component regions.

Now let us consider a tensor field $A_{jkl\dots}$. Let the region V with boundary surface S be within the region of definition of $A_{jkl\dots}$. Let every component of $A_{jkl\dots}$ be continuously differentiable. Then Eq. (2) is applicable to every component of the tensor, and we may write

$$(3) \blacktriangle \int_V \frac{\partial}{\partial x_i} A_{jkl\dots} dV = \int_S \nu_i A_{jkl\dots} dS.$$

This is Gauss' theorem in a general form.

Problem 5.1. Show that

$$\begin{aligned} \int \phi_{,i} dV &= \int \phi \nu_i dS \quad \text{or} \quad \int \operatorname{grad} \phi dV = \int \nu \phi dS, \\ \int u_{i,i} dV &= \int u_i \nu_i dS \quad \text{or} \quad \int \operatorname{div} \mathbf{u} dV = \int \nu \cdot \mathbf{u} dS, \\ \int e_{ijk} u_{k,j} dV &= e_{ijk} \int u_{k,j} dV = e_{ijk} \int u_k \nu_j dS = \int e_{ijk} u_k \nu_j dS. \end{aligned}$$

or

$$\int \operatorname{curl} \mathbf{u} dV = \int \nu \times \mathbf{u} dS,$$

where e_{ijk} is the permutation tensor. Verify that these formulas are also valid in two dimensions, in which case the range of indices is 1, 2 and the volume and surface integrals are replaced by surface and line integrals, respectively.

5.2. MATERIAL AND SPATIAL DESCRIPTION OF CHANGING CONFIGURATIONS

We shall speak of a *particle* in the sense of a material particle as we know it in Newtonian particle mechanics. The instantaneous geometric location of a particle will be spoken of as a *point*. A body is composed of particles. To label the particles of a body we choose a Cartesian frame of reference and identify the coordinates of the particles at a time $t = 0$ as (a_1, a_2, a_3) (Fig. 5.2:1). At a later time the particle moves to another point whose coordinates are (x_1, x_2, x_3) , referred to the same coordinate system. The relation

$$(1) \quad x_i = \hat{x}_i(a_1, a_2, a_3, t), \quad i = 1, 2, 3,$$

links the configurations of the body at different instants of time. The functions \hat{x}_i are single-valued continuous functions whose Jacobian does not vanish.

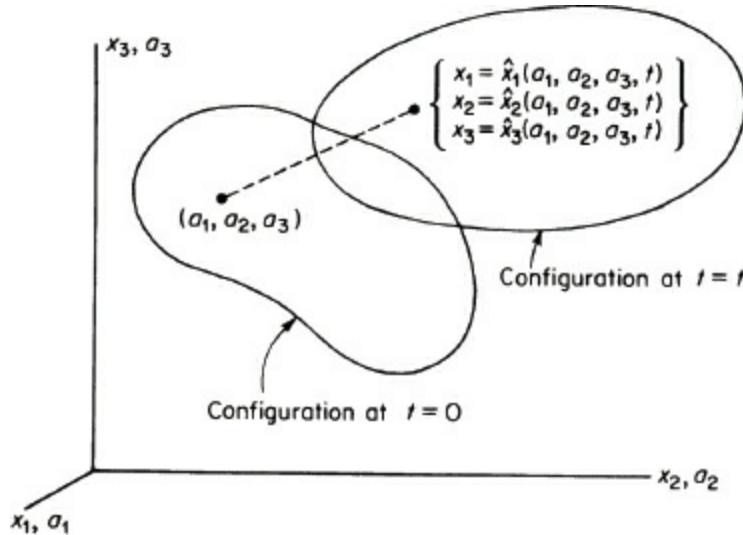


Fig. 5.2:1. Labeling of particles.

A basic property of bodies is that they have mass. In classical mechanics, mass is assumed to be conserved; i.e., the mass of a material body is the same at all times. In continuum mechanics it is further assumed that the mass is a continuous function of volume. In other words, it is assumed that a positive quantity ρ , called *density*, can be defined at every point in the body as the limit

$$(2) \quad \rho(\mathbf{x}) = \lim_{k \rightarrow \infty} \frac{\text{mass of } B_k}{\text{volume of } B_k},$$

where B_k is a suitably chosen infinite sequence of particle sets shrinking down to the point \mathbf{x} ; the symbol \mathbf{x} stands for (x_1, x_2, x_3) . At time $t = 0$, the density at the point (a_1, a_2, a_3) is denoted by $\rho_0(\mathbf{a})$.

The conservation of mass is expressed in the formula

$$(3) \quad \int \rho(\mathbf{x}) dx_1 dx_2 dx_3 = \int \rho_0(\mathbf{a}) da_1 da_2 da_3 ,$$

where the integrals extend over the same particles. Since

$$\int \rho(\mathbf{x}) dx_1 dx_2 dx_3 = \int \rho(\mathbf{x}) \left| \frac{\partial \mathbf{x}_i}{\partial a_j} \right| da_1 da_2 da_3 ,$$

and since this relation must hold for all bodies, we have

$$(4) \quad \begin{aligned} \rho_0(\mathbf{a}) &= \rho(\mathbf{x}) \left| \frac{\partial \mathbf{x}_i}{\partial a_j} \right| , \\ \rho(\mathbf{x}) &= \rho_0(\mathbf{a}) \left| \frac{\partial a_i}{\partial \mathbf{x}_j} \right| , \end{aligned}$$

where $|\partial a_i / \partial \mathbf{x}_j|$ denotes the determinant of the matrix $\{\partial a_i / \partial \mathbf{x}_j\}$. These equations relate the density in different configurations of the body to the transformation that leads from one configuration to the other.

For the particle (a_1, a_2, a_3) whose trajectory is given by Eq. (1), the velocity is

$$(5) \quad v_i(\mathbf{a}, t) = \frac{\partial}{\partial t} x_i(\mathbf{a}, t) ,$$

and the acceleration is

$$(6) \quad \dot{v}_i(\mathbf{a}, t) = \frac{\partial^2}{\partial t^2} x_i(\mathbf{a}, t) = \frac{\partial}{\partial t} v_i(\mathbf{a}, t) ,$$

where \mathbf{a} stands for (a_1, a_2, a_3) and is held constant.

A description of mechanical evolution which uses (a_1, a_2, a_3) and t as independent variables is called a *material description*. In hydrodynamics, traditionally, a different description, called the *spatial description*, is used. In the spatial description, the location (x_1, x_2, x_3) and time t are taken as independent variables. This is convenient because measurements in many kinds of materials are more directly interpreted in terms of what happens at a certain place, rather than following the particles. These two methods of description are commonly designated as the *Lagrangian* and the *Eulerian descriptions*, respectively, although both are due to Euler. The variables a_1, a_2, a_3, t are usually called the *Lagrangian variables*, whereas x_1, x_2, x_3, t are called the *Eulerian variables*; they are related by Eq. (1). For a given particle, it is convenient to speak of (a_1, a_2, a_3) as the *Lagrangian coordinates* of the particle at (x_1, x_2, x_3) .

In a spatial description, the instantaneous motion of the body is described by the velocity vector field $v_i(x_1, x_2, x_3, t)$ associated with the instantaneous location of each particle. The acceleration of the particle is given by the formula

$$(7) \quad \dot{v}_i(\mathbf{x}, t) = \frac{\partial v_i}{\partial t}(\mathbf{x}, t) + v_j \frac{\partial v_i}{\partial x_j}(\mathbf{x}, t) ,$$

where \mathbf{x} again stands for the variables x_1, x_2, x_3 , and every quantity in this formula is evaluated at (\mathbf{x}, t) . The proof follows the fact that a particle located at (x_1, x_2, x_3) at time t is moved to a point with coordinates $x_i + v_i dt$ at the time $t + dt$; and that, according to Taylor's theorem,

$$\begin{aligned} \dot{v}_i(\mathbf{x}, t) dt &= v_i(x_j + v_j dt, t + dt) - v_i(\mathbf{x}, t) \\ &= v_i + \frac{\partial v_i}{\partial t} dt + \frac{\partial v_i(\mathbf{x}, t)}{\partial x_j} v_j dt - v_i , \end{aligned}$$

which reduces to Eq. (7). The first term in Eq. (7) may be interpreted as arising from the time dependence of the velocity field; the second term as the contribution of the motion of the particle in the instantaneous velocity field. Accordingly, these terms are called the *local* and the *convective* parts of the acceleration, respectively.

The same reasoning that led to Eq. (7) is applicable to any function $F(x_1, x_2, x_3, t)$ that is attributable to the moving particles, such as the temperature. A convenient terminology is the *material derivative*, and it is denoted by a dot or the symbol D/Dt . Thus

$$(8) \blacktrianglequad \dot{F} \equiv \frac{DF}{Dt} \equiv \left(\frac{\partial F}{\partial t} \right)_{\mathbf{x}=\text{const.}} + v_1 \frac{\partial F}{\partial x_1} + v_2 \frac{\partial F}{\partial x_2} + v_3 \frac{\partial F}{\partial x_3} \\ \equiv \left(\frac{\partial F}{\partial t} \right)_{\mathbf{a}=\text{const.}},$$

where $\mathbf{a} = (a_1; a_2; a_3)$ is the Lagrangian coordinate of the particle which is located at \mathbf{x} at the time t , connected by Eq. (1).

5.3. MATERIAL DERIVATIVE OF VOLUME INTEGRAL

Consider a volume integral taken over the body

$$(1) \quad I = \int_V A(\mathbf{x}, t) dV,$$

where $A(\mathbf{x}, t)$ denotes a property of the continuum and the integral is evaluated at an instant of time t . We may wish to know how fast the body itself sees its own value of I is changing, so it is of interest to know *the derivative of I with respect to time for a given set of particles*. Now the particle at x_i at the instant t will have the coordinates $x'_i = x_i + v_i dt$ at the time $t+dt$. The boundary S of the body at the instant t will have moved at time $t+dt$ to a neighboring surface S' , which bounds the volume V' (Fig. 5.3:1). The material derivative of I is defined as

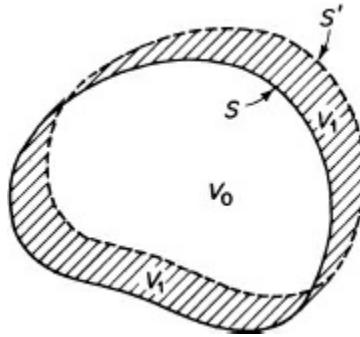


Fig. 5.3:1. Continuous change of the boundary of a region.

$$(2) \quad \frac{DI}{Dt} = \lim_{dt \rightarrow 0} \frac{1}{dt} \left[\int_{V'} A(\mathbf{x}', t+dt) dV' - \int_V A(\mathbf{x}, t) dV \right].$$

Now there are two contributions to the difference on the right-hand side of Eq. (2); one over the region V_0 where V and V' share in common, and another over the region V_1 where V and V' differ. The former contribution to DI/Dt is evidently

$$\int_{V_0} \frac{\partial A}{\partial t} dV.$$

The latter contribution comes from the value of A on the boundary multiplied by the volumes swept by the particles on the boundary in the time interval dt . If v_i is the unit vector along the exterior normal of S , then, since the displacement of a particle on the boundary is $v_i dt$, the volume swept by particles occupying an element of area dS on the boundary S is $dV = v_i v_j dS dt$. The contribution of this element to DI/Dt is $Av_j v_j dS$. The total contribution is obtained by an integration over S . Therefore,

$$(3) \blacktrianglequad \frac{D}{Dt} \int_V A dV = \int_V \frac{\partial A}{\partial t} dV + \int_S Av_j v_j dS.$$

Transforming the last integral by Gauss' theorem and using Eq. (5.2:8), we have

$$\begin{aligned}
 (4) \blacktrianglequad \frac{D}{Dt} \int_V A dV &= \int_V \frac{\partial A}{\partial t} dV + \int_V \frac{\partial}{\partial x_j} (Av_j) dV \\
 &= \int_V \left(\frac{\partial A}{\partial t} + v_j \frac{\partial A}{\partial x_j} + A \frac{\partial v_j}{\partial x_j} \right) dV \\
 &= \int_V \left(\frac{DA}{Dt} + A \frac{\partial v_j}{\partial x_j} \right) dV.
 \end{aligned}$$

This important formula will be used over and over again below. It should be noted that the operation of forming the material derivative and that of spatial integration is noncommutative in general.

5.4. THE EQUATION OF CONTINUITY

The law of conservation of mass has been discussed in Sec. 5.2. With the results of Sec. 5.3, we can now give some alternative forms.

The mass contained in a region V at a time t is

$$(1) \qquad m = \int_V \rho dV,$$

where $\rho = \rho(\mathbf{x}, t)$ is the density field of the continuum. Conservation of mass requires that $Dm/Dt = 0$. The derivative Dm/Dt is given by Eqs. (5.3:3) and (5.3:4), when A is replaced by ρ . Since the result must hold for arbitrary V , the integrand must vanish. Hence, we obtain the following alternative forms of the law of conservation of mass.

$$(2) \blacktrianglequad \int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho v_j \nu_j dS = 0.$$

$$(3) \blacktrianglequad \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0.$$

$$(4) \blacktrianglequad \frac{D\rho}{Dt} + \rho \frac{\partial v_j}{\partial x_j} = 0.$$

These are called the *equations of continuity*. The integral form (2) is useful when the differentiability of ρv_j cannot be assumed.

In problems of statics, these equations are identically satisfied. Then the conservation of mass must be expressed by Eq. (5.2:3), or (5.2:4).

5.5. THE EQUATIONS OF MOTION

Newton's laws of motion state that, in an inertial frame of reference, the material rate of change of the linear momentum of a body is equal to the resultant of applied forces and that the material rate of change of the moment of momentum with respect to the coordinate origin is equal to the resultant moment of applied forces about the same origin.

At an instant of time t , a regular region V of space contains the linear momentum

$$(1) \qquad \mathcal{P}_i = \int_V \rho v_i dV.$$

If the body is subjected to surface traction $\overset{\nu}{T}_i$ and body force per unit volume X_i , the resultant force is

$$(2) \qquad \mathcal{F}_i = \int_S \overset{\nu}{T}_i dS + \int_V X_i dV.$$

According to the stress principle of Euler and Cauchy (Secs. 3.2 and 3.3), $\overset{\nu}{T}_i = \sigma_{ij} \nu_j$, where σ_{ij} is the stress field and ν_j is the unit vector along the exterior normal to the boundary surface S of the region V . Substituting into Eq. (2) and transforming into a volume integral by Gauss' theorem, we have

$$(3) \quad \mathcal{F}_i = \int_V \left(\frac{\partial \sigma_{ij}}{\partial x_j} + X_i \right) dV.$$

Newton's law states that

$$(4) \quad \frac{D}{Dt} \mathcal{P}_i = \mathcal{F}_i.$$

Hence, according to Eq. (5.3:4), with A identified with ρv_i , we have

$$(5) \quad \int_V \left[\frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_i v_j) \right] dV = \int_V \left(\frac{\partial \sigma_{ij}}{\partial x_j} + X_i \right) dV.$$

Since this equation must hold for an arbitrary region V , the integrand on the two sides must be equal. Thus

$$(6) \quad \frac{\partial \rho v_i}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_i v_j) = \frac{\partial \sigma_{ij}}{\partial x_j} + X_i.$$

The left-hand side of Eq. (6) is equal to

$$v_i \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} \right) + \rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right).$$

The first parenthesis vanishes by the equation of continuity (5.4:3), while the second is the acceleration Dv_i/Dt . Hence

$$(7) \quad \blacktriangle \quad \dot{v}_i(\mathbf{x}, t) = \frac{\partial v_i}{\partial t}(\mathbf{x}, t) + v_j \frac{\partial v_i}{\partial x_j}(\mathbf{x}, t),$$

This is the *Eulerian equation of motion* of a continuum. The equation of equilibrium discussed in Sec. 3.4 is obtained by setting all velocity components v_i equal to zero.

If differentiability of the stress field or the momentum field cannot be assumed, we may use Eq. (5.3:3) to compute $D\mathcal{P}/Dt$. Then Eq. (2) and (4) give *Euler's equation in the integral form*,

$$(8) \quad \int_V \frac{\partial \rho v_i}{\partial t} dV = \int_S (\sigma_{ij} - \rho v_i v_j) \nu_j dS + \int_V X_i dV.$$

The corresponding equations of static equilibrium are obtained, of course, by setting all velocity components to zero.

5.6. MOMENT OF MOMENTUM

An application of the law of balance of angular momentum to the particular case of *static equilibrium* leads to the conclusion that stress tensors are symmetric tensors (see Sec. 3.4). We shall now show that no additional restriction to the motion of a continuum is introduced in dynamics by the angular momentum postulate.

At an instant of time t , a body occupying a regular region V of space with boundary S has the moment of momentum [Eq. (3.2:2)]

$$(1) \quad \mathcal{H}_i = \int_V e_{ijk} x_j \rho v_k dV,$$

with respect to the origin. If the body is subjected to surface traction $\overset{\nu}{T}_i$ and body force per unit volume X_i , the resultant moment about the origin is

$$(2) \quad \mathcal{L}_i = \int_V e_{ijk} x_j X_k dV + \int_S e_{ijk} x_j \overset{\nu}{T}_k dS.$$

Introducing Cauchy's formula $\overset{\nu}{T}_k = \sigma_{lk} \nu_l$ into the last integral and transforming the result into a volume integral by Gauss' theorem, we obtain

$$(3) \quad \mathcal{L}_i = \int_V e_{ijk} x_j X_k dV + \int_V (e_{ijk} x_j \sigma_{lk})_{,l} dV.$$

Euler's law states that for any region V

$$(4) \quad \frac{D}{Dt} \mathcal{H}_i = \mathcal{L}_i.$$

Evaluating the material derivative of \mathcal{H}_i according to (5.3:4), and using Eq. (3), we obtain

$$(5) \quad e_{ijk}x_j \frac{\partial}{\partial t}(\rho v_k) + \frac{\partial}{\partial x_l}(e_{ijk}x_j \rho v_k v_l) = e_{ijk}x_j X_k + e_{ijk}(x_j \sigma_{lk})_{,l}.$$

The second term in Eq. (5) can be written as

$$e_{ijk}\rho v_j v_k + e_{ijk}x_j \frac{\partial}{\partial x_l}(\rho v_k v_l) = e_{ijk}x_j \frac{\partial}{\partial x_l}(\rho v_k v_l).$$

The last term in Eq. (5) can be written as $e_{ijk}\sigma_{jk} + e_{ijk}x_j \sigma_{lk,l}$. Hence, Eq. (5) becomes

$$(6) \quad e_{ijk}x_j \left[\frac{\partial}{\partial t}\rho v_k + \frac{\partial}{\partial x_l}(\rho v_k v_l) - X_k - \sigma_{lk,l} \right] - e_{ijk}\sigma_{jk} = 0.$$

The sum in the square brackets vanishes by the equation of motion (5.5:6). Hence, Eq. (6) is reduced to

$$(7) \quad e_{ijk}\sigma_{jk} = 0;$$

i.e., $\sigma_{jk} = \sigma_{kj}$. Thus, if the stress tensor is symmetric, the law of balance of moment of momentum is identically satisfied.

5.7. OTHER FIELD EQUATIONS

The motion of a continuum must be governed further by the law of conservation of energy. If mechanical energy is the only form of energy of interest in a problem, then the energy equation can be obtained by suitable integration of the equation of motion, Eq. (5.5:7). If the interaction of thermal process and mechanical process is significant, then the equation of energy contains a thermal energy term and is an independent equation to be satisfied. We shall discuss the energy equation in Sec. 12.2.

The equations of continuity and motion, Eqs. (5.4:3) and (5.5:7), constitute four equations for ten unknown functions of time and position; namely, the density ρ , the three velocity components v_i (or displacements u_i) and the six independent stress components σ_{ij} . Further restrictions would have to be introduced before the motion of a continuum can be determined. One group of such additional restrictions comes from a statement about the mechanical property of the medium in the form of a specification of stress-strain relationship. These specifications are called *constitutive equations*.

The constitutive equations of elastic and plastic materials are discussed in Chapter 6. The constitutive equations of thermoelastic materials are discussed in Chapter 12. The constitutive equations of solids with large deformation are discussed in Chapter 13. The constitutive equations of viscoelastic and thermoviscoelastic materials are discussed in Chapter 14. The constitutive equations of materials with internal structural changes are discussed in Chapter 15. The constitutive equations of electro-sensitive materials are discussed in Chapter 16. The constitutive equations of plates and shells are discussed in Chapters 13 and 18. A field that is very rich in a variety of constitutive equations is biomechanics. See Fung *Biomechanics: Mechanical Properties of Living Tissues*, 1993, *Biomechanics: Circulation*, 1996, *Biomechanics: Motion, Flow, Stress, and Growth*, 1990, Springer Verlag, and other references listed in the Bibliography.

PROBLEMS

5.2. Express the following statements in tensor equations. Define your symbols.

- (a) The force of gravitational attraction between two particles of inertial masses m_1 and m_2 , respectively, and separated by a distance r , is equal to Gm_1m_2/r^2 and is directed toward each other.
- (b) The components of stress is a linear function of the components of strain.
- (c) A normal vector of a surface is perpendicular to any two line elements tangent to the surface (in particular, to tangents of the parametric curves).

5.3. If the stress-strain law in rectangular Cartesian coordinates is

$$\tau_{ij} = \lambda\theta\delta_{ij} + 2Ge_{ij}, \quad \theta = e_{\alpha\alpha},$$

what is the proper form of tensor relation between τ^{ij} and e_{ij} in general coordinates of reference?

5.4. Discuss the appropriateness, from the tensorial point of view, of the following proposals for the constitutive equations in Cartesian tensors for certain materials.

(a) $\tau_{ij} = P(e_{mn})e_{ij}$, where $P(e_{mn}) = ae_{11} + be_{11}^2$ (a, b , constants).

(b) $\tau_{ij} = P(e_{mn})e_{ij}$, where $P(e_{mn}) = a + b|e_1|^2$ (a, b , constants), and e_1, e_2, e_3 are the three principal strains satisfying the relation $e_1 \geq e_2 \geq e_3$.

(c) $\tau_{ij} = Q(I_1, I_2, I_3)e_{ij}$, where I_1, I_2, I_3 are the first, second, and third invariants of e_{ij} , respectively.

(d) $\tau_{ij} = \alpha\delta_{ij} + \beta e_{ij} + \gamma e_{ik} e_{kj} + \lambda e_{ik} e_{km} e_{mj}$, where $\alpha, \beta, \gamma, \lambda$ are constants.

(e) If in (d), $\alpha, \beta, \gamma, \lambda$ are permitted to depend on the stress components, what kind of combinations of the stress components would be allowed for $\alpha, \beta, \gamma, \lambda$ to be functionally dependent upon?

6

ELASTIC AND PLASTIC BEHAVIOR OF MATERIALS

In this chapter, some commonly used constitutive laws are considered. These laws correspond to isothermal material behavior in small strains and slow rates of flow. The plasticity theories discussed are mathematical formulation of experimental observations, a phenomenological approach. We shall discuss first the strain-rate-independent plasticity theory considering the yield surface, the flow rule, the hardening behavior, and the loading and unloading criteria. We then consider the cyclic loading reversals and strain softening. It is beyond the scope of this book to consider the atomic, crystalline or amorphous structural changes in plastic deformation, and how and why plasticity occurs at the microscopic level. The plasticity considered here is appropriate to most problems in structural engineering, in which excessive plastic flow is undesirable; but it does not meet the needs of solving problems in metal forming, wire drawing, rolling, etc.

We shall use Cartesian tensors, whose physical components are the same as the tensor components, so stresses can be denoted by σ_{ij} or σ^{ij} . The reader should try to put all of the equations in this chapter into general tensor equations in curvilinear coordinates.

6.1. GENERALIZED HOOKE'S LAW

With the introduction of the concepts of stress and strain, Cauchy generalized Hooke's law into the statement that the components of stress are linearly related to the components of strain. As a tensor equation, the generalized Hooke's law may be written in the form

$$(1) \quad \Delta \qquad \sigma^{ij} = D^{ijkl} e_{kl}$$

where σ_{ij} is the stress tensor, e_{ij} is the strain tensor,¹ and D^{ijkl} is the tensor of the *elastic constants*, or *moduli*, of the material and is called the *elastic modulus tensor*. Inasmuch as $\sigma^{ij} = \sigma^{ji}$, we must have

$$(2) \quad D^{ijkl} = D^{jikl}.$$

Furthermore, since $e_{kl} = e_{lk}$ and in Eq. (1) the indices k and l are dummy variables, we can always symmetrize D^{ijkl} with respect to k and l without altering the sum. Hence, without loss of generality, we may assume that

$$(3) \quad D^{ijkl} = D^{ijlk}.$$

According to these symmetry properties, the maximum number of independent elastic constants is 36.

If there exists a strain energy function W ,

$$(4) \quad W = D^{ijkl} e_{ij} e_{kl} / 2,$$

with the property

$$(5) \quad \partial W / \partial e_{ij} = \sigma^{ij},$$

then we can always suppose that the quadratic form Eq. (4) is symmetric, and it follows that

$$(6) \quad D^{ijkl} = D^{klij}.$$

Under the symmetry condition, Eq. (6), the number of independent elastic constants is reduced to 21. The question of the existence of the strain energy function is discussed in [Chapter 12](#).

If the material property possesses further symmetry, the number of independent elastic constants will be reduced, i.e., if the material exhibits symmetry with respect to a plane, the number of independent elastic constants becomes 13. If there is symmetry with respect to three mutually perpendicular planes, the number becomes 9. However, if the material is elastically *isotropic*, i.e., if its elastic properties are identical in all directions, then the number of independent elastic constants reduces to 2.

The study of crystal symmetry is a very interesting subject. Many excellent references exist. See Love,^{1,1} Green and Adkins,^{1,2} Sokolnikoff.^{1,2}

In the remaining chapter, we shall limit our discussion to *isotropic* materials and the use of Cartesian coordinates and, therefore, subscripted indices to indicate tensorial components.

6.2. STRESS-STRAIN RELATIONSHIP FOR ISOTROPIC ELASTIC MATERIALS

For an *isotropic elastic material* in which there is no change of temperature, Hooke's law referred to a set of rectangular Cartesian coordinates may be stated in the form

$$(1) \quad \sigma_{\alpha\alpha} = 3K e_{\alpha\alpha},$$

$$(2) \quad \sigma'_{ij} = 2G e'_{ij},$$

where K and G are constants and σ'_{ij} and e'_{ij} are the *stress deviation* and *strain deviation*, respectively; i.e.,

$$(3) \quad \sigma'_{ij} = \sigma_{ij} - \sigma_{\alpha\alpha}\delta_{ij}/3,$$

$$(3) \quad \sigma'_{ij} = \sigma_{ij} - \sigma_{\alpha\alpha}\delta_{ij}/3,$$

The coefficient 2 included in Eq. (2) makes e'_{ij} a tensor. Before the tensor concept was introduced, it was customary to define the shear strain as $\gamma'_{ij} = 2e'_{ij}$ or $\gamma_{ij} = 2e_{ij}$ for $i \neq j$. We have seen before that $\sigma_{\alpha\alpha}/3$ is the mean stress at a point and that, if the strain were infinitesimal, $e_{\alpha\alpha}$ is the change of volume per unit volume. Both $\sigma_{\alpha\alpha}$ and $e_{\alpha\alpha}$ are *invariants*. Thus, Eq. (1) states that the change of volume of the material is proportional to the mean stress. In the special case of hydrostatic compression,

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p, \quad \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0,$$

we have $\sigma_{\alpha\alpha} = -3p$, and Eq. (1) may be written, in the case of infinitesimal strain, with v and Δv denoting volume and change of volume, respectively,

$$(5) \quad \Delta v/v = -p/K.$$

Thus, the coefficient K is called the *bulk modulus* of the material.

The strain deviation e'_{ij} describes a deformation without volume change. Equation (2) states that the stress deviation is simply proportional to the strain deviation. The constant G is called the *modulus of elasticity in shear*, or *shear modulus*, or the *modulus of rigidity*. In the special case of e_{xy} , $\sigma_{xy} \neq 0$, but all other strain and stress components vanish, we have

$$(6) \quad \sigma_{xy} = 2G e_{xy}.$$

If we substitute Eqs. (3) and (4) into Eq. (2) and make use of Eq. (1), the result may be written in the form

$$(7) \quad \sigma_{ij} = \lambda e_{\alpha\alpha}\delta_{ij} + 2G e_{ij},$$

or

$$(8) \quad e_{ij} = [(1 + \nu)\sigma_{ij} - \nu\sigma_{\alpha\alpha}\delta_{ij}]/E.$$

The constants λ and G are called *Lamé's constants* (Lamé 1852), E the modulus of elasticity, or Young's modulus (Young 1807), and ν Poisson's ratio. In many books the symbol μ is used in place of G . The relationships between these constants are

$$\begin{aligned} \lambda &= 2G\nu/(1 - 2\nu) = G(E - 2G)/(3G - E) = 3K\nu/(1 + \nu) \\ &= K - 2G/3 = E\nu/[(1 + \nu)(1 - 2\nu)] = 3K(3K - E)/(9K - E), \\ G &= \frac{\lambda(1 - 2\nu)}{2\nu} = \frac{3}{2}(K - \lambda) = \frac{E}{2(1 + \nu)} = \frac{3K(1 - 2\nu)}{2(1 + \nu)} = \frac{3KE}{9K - E} \end{aligned}$$

$$(9) \quad \nu = \frac{\lambda}{2(\lambda + G)} = \frac{\lambda}{(3K - \lambda)} = \frac{E}{2G} - 1 = \frac{3K - 2G}{2(3K + G)} = \frac{3K - E}{6K},$$

$$\begin{aligned} E &= \lambda(1 + \nu)(1 - 2\nu)/\nu = 9K(K - \lambda)/(3K - \lambda) = 2G(1 + \nu) \\ &= G(3\lambda + 2G)/(\lambda + G) = 9KG/(3K + G) = 3K(1 - 2\nu), \end{aligned}$$

$$K = \lambda + \frac{2}{3}G = \frac{\lambda(1 + \nu)}{3\nu} = \frac{2G(1 + \nu)}{3(1 - 2\nu)} = \frac{GE}{3(3G - E)} = \frac{E}{3(1 - 2\nu)}.$$

To these we may add the following combinations that appear frequently.

$$(10) \quad G/(\lambda + G) = 1 - 2\nu, \quad \lambda/(\lambda + 2G) = \nu/(1 - \nu).$$

In unabridged notation, Eq. (8) reads

$$\begin{aligned} e_{xx} &= [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]/E, & e_{yy} &= [\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})]/E, \\ (11) \quad e_{zz} &= [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]/E, & e_{xy} &= (1 + \nu)\sigma_{xy}/E = \sigma_{xy}/(2G), \\ e_{yz} &= (1 + \nu)\sigma_{yz}/E = \sigma_{yz}/(2G), & e_{zx} &= (1 + \nu)\sigma_{zx}/E = \sigma_{zx}/(2G). \end{aligned}$$

One can express σ 's in terms of e 's

$$(12) \quad \sigma_{xx} = E \frac{(1 - \nu)e_{xx} + \nu(e_{yy} + e_{zz})}{(1 + \nu)(1 - 2\nu)}, \quad \sigma_{xy} = 2Ge_{xy},$$

with corresponding expressions for σ_{yy} , σ_{zz} , σ_{yz} , σ_{zx} . [Table 6.2:1](#) gives the average values of E , G , and ν at room temperature for several engineering materials which are approximately isotropic.

In 1829, Poisson advanced arguments, later found untenable, that the value of ν should be 1/4, which makes

$$(13) \quad \lambda = G$$

and simplifies the equations of elasticity considerably. Consequently, this assumption is often used, particularly in geophysics, in the study of complicated wave-propagation problems. The special value $\nu = 1/2$ implies that

$$(14) \quad G = E/3, \quad 1/K = 0, \quad \Delta v/v = e_{aa} = 0.$$

Table 6.2:1

	E, 10^6 lb/sq in.	G, 10^6 lb/sq in.	ν	Speed of sound (Dilatational wave) 10^3 ft/sec
Metals:				
Steels	30	11.5	0.29	16.3
Aluminum alloys	10	2.4	0.31	16.5
Magnesium alloys	6.5	2.4	0.35	16.6
Copper (hot rolled)	15.0	5.6	0.33	—
Plastics:				
Cellulose acetate	0.22			0.36
Vinylchloride acetate	0.46			5.1
Phenolic laminates	1.23		0.25	8.2
Glass	8	3.2	0.25	
Concrete	4		0.2	

So far in this chapter we have discussed the simplest of all stress-strain relations for elastic materials — the *Hooke's law*, which are the mechanical properties of a class of materials like steel. There are many things whose mechanical properties cannot be described by the simple elasticity discussed above. For example, almost all biological materials lie beyond the reach of these simple laws. Even rubber, plastics, and metals like aluminum, magnesium, and lead cannot be so described. Many materials obey Hooke's law when the stress and strain are sufficiently small, but yield and flow plastically when a critical condition of yield is reached and maintained. The behavior of materials beyond their elastic limit is complicated. In the rest of this book, we first demonstrate the classical methods that can solve boundary-value problems of bodies whose materials obey Hooke's law; then consider the world beyond: plasticity, thermodynamics, thermoelasticity, irreversible thermodynamics, viscoelasticity, large deformation, elasticity with nonlinear stress-strain

relation, plates and shells, and numerical methods to handle boundary-value problems involving materials with these mechanical properties. The mathematical structure of the theories that renders boundary-value problems solvable by the numerical methods will be proposed.

6.3. IDEAL PLASTIC SOLIDS

Metals obey Hooke's law only in a range of small strain. When a metal is strained beyond an *elastic limit*, the law no longer applies. The behavior of metals beyond their elastic limit is complicated. We shall give a very brief outline of some experimental facts in the next section, and discuss the formulation of the constitutive relations in the plastic regime later. We first present a set of laws which defines the simplest plastic materials — an *ideal plastic solid obeying von Mises' and Tresca's yield criteria and flow rules*. Such laws are a reasonable abstraction of the behavior of certain real materials to be discussed later. At the moment, we shall just exhibit the minimum ingredients that constitute a theory of plasticity.

When the deformation reaches a certain limit, the *elastic limit*, the material starts to deform plastically, and the strain deviation e'_{ij} is no longer given by Hooke's law. In this case we define the *plastic strain increment* de'_{ij} as the actual strain deviation *increment* $de^{(p)}_{ij}$ minus the *elastic strain increment* de'_{ij} computed from Hooke's law as if it would be still applied. For metallic materials experimental evidence indicates that the hydrostatic pressure has little effect on plastic yielding and that the volumetric change of the material is negligible during yielding. Hence, we can assume $de^{(p)}_{ii} = 0$ and write de'_{ij} as the increment of the plastic strain deviation as

$$(1) \quad de^{(p)}_{ij} = de'_{ij} - de'^{(e)}_{ij} = de'_{ij} - d\sigma'_{ij}/(2G), \quad de^{(p)}_{ii} = 0.$$

where $(\cdot)'$ denotes the deviation of the associated quantity in the parenthesis. The total strain deviation increment may now be written as

$$(2) \quad de'_{ij} = de^{(p)}_{ij} + d\sigma'_{ij}/(2G).$$

Equation (2) becomes a rate equation of deformation when all incremental quantities $d(\cdot)$ are replaced by rate quantities $(\cdot)'$,

$$(3) \quad \dot{e}_{ij} = \dot{e}^{(p)}_{ij} + \dot{\sigma}'_{ij}/(2G).$$

The rate of plastic deformation is $\dot{e}^{(p)}_{ij}$. Then the total plastic strain deviation after successive stages of yielding is the algebraic sum of the deformations that occur at all stages:

$$(4) \quad e^{(p)}_{ij} = e^{(p)}_{ij}(0) + \int_0^t \dot{e}^{(p)}_{ij}(t) dt,$$

where $e^{(p)}_{ij}(0)$ is the initial value of $e^{(p)}_{ij}$ at time $t = 0$. A theory of plasticity is formulated by specifying how $e^{(p)}_{ij}$ can be computed.

We consider the yield criteria and flow rules of two *ideal plastic solids*:

Von Mises Criterion. Von Mises² (1913) specified the yield criterion and the stress-strain relation in the elastic and plastic regimes based on the second stress deviation invariant J_2 as defined in Eq. (3.10:8 or 9):

(a) Hook's law holds for the mean stress and strain at all times (i.e., $\sigma_{ii} = 3Ke_{ii}$). Since the plastic strain is incompressible ($e_{ii}^{(p)} = 0$), one can use $e^{(p)}_{ij}$ in place of $e'^{(p)}_{ij}$.

(b) The material is elastic and obeys Hooke's law, i.e., $\dot{e}^{(p)}_{ij} = 0$, as long as $J_2 < k_2$, where k is a constant. For work hardening materials (to be considered later), k depends on strain history.

(c) Yielding occurs (the elastic limit is reached) when and only when

$$(5) \quad J_2 = [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]/6 = k^2.$$

where σ_i are the principal stresses. The rate of plastic strain is proportional to the stress deviation,

$$\dot{e}^{(p)}_{ij} = \sigma'_{ij}/\mu, \quad \mu > 0,$$

where μ is a proportional factor.

(d) The stress state corresponding to $J_2 > k^2$ cannot be realized.

We can use the yield stress σ_Y in simple tension test with $\sigma_1 = \sigma_Y$ and $\sigma_2 = \sigma_3 = 0$ to determine k . From Eq. (5), we obtain

$$k = \sigma_Y / \sqrt{3}.$$

Using the yield stress τ_Y in pure shear test with $\sigma_1 = -\sigma_3 = \tau_Y$, and $\sigma_2 = 0$, we have from Eq. (5)

$$k = \tau_Y \quad \text{and hence} \quad k = \tau_Y = \sigma_Y / \sqrt{3},$$

which gives the yield strength in uniaxial tension $\sqrt{3}$ times that in shear.

Tresca Criterion. Tresca (1868), based his work on metal forming in armory, concluded that the decisive factor for yielding is the maximum shear stress in the material. He proposed the criterion stipulating that plastic yielding will occur when the maximum shear stress τ_{\max} reaches a critical value k of a material, i.e.,

$$(6) \quad \tau_{\max} < k \quad \text{for elastic deformation,}$$

$$(7) \quad \tau_{\max} = k \quad \text{for yielding or during plastic flow,}$$

and $\tau_{\max} > k$ cannot be realized.

In terms of principal stresses, Eq. (6) can be written as

$$(8) \quad \Delta \quad \max [|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|] = 2k, \text{ or } |\sigma_{\max} - \sigma_{\min}| = 2k.$$

Using the result of Problem 3.5 in [Sec. 3.10](#), we can write the equation above in the form,

$$f = \sqrt{J_2} \sin(\alpha + 60^\circ) - k = 0,$$

where

$$3\alpha = \cos^{-1}(\sqrt{2}J_3/\tau_0^3) = \cos^{-1}[\sqrt{2}J_3/(2J_2/3)^{3/2}] \text{ with } -\pi/6 < \alpha < \pi/6,$$

J_2 and J_3 are the second and third stress deviation invariants with J_2 as given in Eq. (6.3:5) and

$$J_3 = \det |\sigma'_{ij}| = \sigma'_{11}\sigma'_{22}\sigma'_{33} + 2\sigma'_{12}\sigma'_{23}\sigma'_{31} - \sigma'_{11}\sigma'_{23}^2 - \sigma'_{22}\sigma'_{31}^2 - \sigma'_{33}\sigma'_{12}^2.$$

It can be shown that $\sqrt{2}J_3/(2J_2/3)^{3/2}$ is always less than or equal to one.

The yield conditions above are not analytic. They can be written in the symmetric form as,

$$(9) \quad \Delta \quad \begin{aligned} f &= [(\sigma_1 - \sigma_2)^2 - 4k^2][(\sigma_2 - \sigma_3)^2 - 4k^2][(\sigma_3 - \sigma_1)^2 - 4k^2] \\ &= 4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6 = 0. \end{aligned}$$

The symmetric form is useful when we cannot tell *a priori* the directions of principal axes and the relative magnitudes of the principal stresses.

To determine k , one can use the yield stress τ_Y in simple tension test with $\sigma_1 = -\sigma_3 = \tau_Y$ and $\sigma_2 = 0$. From Eq. (5), we obtain

$$k = \sigma_Y / 2.$$

Using the yield stress τ_Y in pure shear test with $\sigma_1 = -\sigma_3 = \tau_Y$, and $\sigma_2 = 0$, we have

$$k = \tau_Y \quad \text{and, hence} \quad k = \tau_Y = \sigma_Y / 2,$$

which gives the yield strength in pure shear half of that in uniaxial tension.

Example 1. Comparison of the Tresca and von Mises Criteria

Consider a thin-walled tube subjected to the combined uniaxial tensile stress $\sigma (\geq 0)$ and shear stress τ . The principal stresses are

$$(10) \quad \begin{Bmatrix} \sigma_1 \\ \sigma_3 \end{Bmatrix} = \sigma/2 \pm [(\sigma/2)^2 + \tau^2]^{1/2}, \quad \sigma_2 = 0.$$

The Tresca and von Mises yield functions can be simplified as

$$(11) \quad f = \bar{\sigma} - \sigma_Y,$$

where σ_Y is the yield stress in uniaxial tension, and

$$(12) \quad \bar{\sigma} = \sigma_1 - \sigma_3 = \sqrt{\sigma^2 + 4\tau^2} = 2k \text{ (Tresca),}$$

$$(13) \quad \bar{\sigma} = \sqrt{3J_2} = \sqrt{\frac{(\sigma_1 - \sigma_3)^2 + \sigma_1^2 + \sigma_3^2}{3}} = \sqrt{\sigma^2 + 3\tau^2} = \sqrt{3}k \text{ (von Mises).}$$

Alternatively Eqs. (12) and (13) can be written in the form

$$(12a) \quad (\sigma/\sigma_Y)^2 + 4(\tau/\sigma_Y)^2 = 1 \quad \text{(Tresca),}$$

$$(13a) \quad (\sigma/\sigma_Y)^2 + 3(\tau/\sigma_Y)^2 = 1 \quad \text{(von Mises),}$$

which represent ellipses on $\sigma - \tau$ plane with von Mises' ellipse bounding Tresca's. Taylor and Quinney's^{6.3} (1931) experimental data fall between the two ellipses, but are closer to von Mises.

If k is the yield stress in simple shear ($\sigma = 0, \tau = \kappa$), then $k = \sigma_Y/2$ according to the Tresca criterion Eq. (12a) and $k = \sigma_Y/\sqrt{3}$ according to the von Mises criterion in Eq. (13a).

Example 2. Experimental Investigation of Tresca's and von Mises' Criteria

Lode^{6.3} (1926) investigated the influence of the intermediate principal stress on yielding. He used the *Lode parameter*,

$$(13) \quad \eta = (2\sigma_2 - \sigma_1 - \sigma_3)/(\sigma_1 - \sigma_3)$$

to characterize the stress state where the principal stresses σ_i satisfy

$$(14) \quad \sigma_1 > \sigma_2 > \sigma_3.$$

According to the Tresca criterion

$$(15) \quad (\sigma_1 - \sigma_3)/\sigma_Y = 1,$$

yielding does not depend on η . On the other hand, from Eq. (13), we have

$$\sigma_2 - \sigma_1 = (\sigma_1 - \sigma_3)(\eta - 1)/2, \quad \sigma_2 - \sigma_3 = (\sigma_1 - \sigma_3)(\eta + 1)/2.$$

The von Mises criterion

$$[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]/2 = \sigma_Y^2$$

can be reduced to

$$(16) \quad (\sigma_1 - \sigma_3)/\sigma_Y = 2/\sqrt{3 + \eta^2},$$

which is a function of the Lode parameter η . Lode tested thin-walled tubes of steel, copper, and nickel subjected to a combined loading of a uniaxial tensile force F and an internal pressure p . The induced stresses are

$$\sigma_1 = \sigma_\theta = pR/t, \quad \sigma_2 = \sigma_z = F/(2\pi R t), \quad \sigma_3 = \sigma_r \approx 0,$$

where t is the wall thickness and R the mean tube radius. From Eq. (13), the Lode parameter is

$$\eta = (F - \pi R^2 p)/(\pi R^2 p),$$

with $2\pi R^2 p > F > 0$ to assure that Eq. (14) is satisfied. Experimental results in Fig. 6.3:1 seem to favor the von Mises yield criterion as the results show that the value of $(\sigma_1 - \sigma_3)/\sigma_Y$ depends on η .

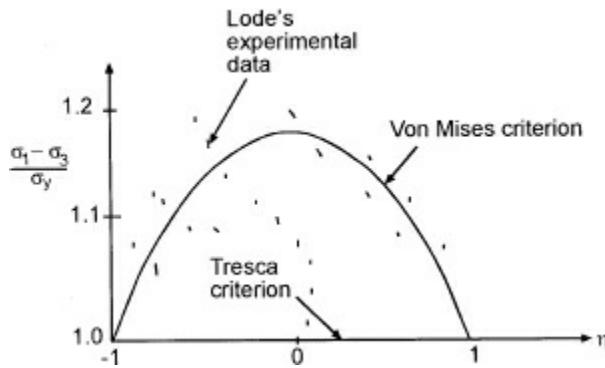


Fig. 6.3:1. Lode's comparison of von Mises and Tresca yield criteria (data from Khan and Huang 1995). Dots are Lode's experimental data.

Only the plastic strain rate is specified by these laws. Under a varying loading program, successive plastic strain increments are added algebraically according to Eq. (4). Such a theory is called an *incremental theory*.

In many applications of the theory of plasticity the rate at which plastic

flow occurs is relatively small. The concern is mostly with the total amount of plastic flow. In these applications we often assume that the plastic deformation is rate insensitive and that we can replace the equation in (c) by the incremental law

$$de_{ij}^{(p)} = \lambda \sigma'_{ij}, \quad \lambda > 0,$$

where λ is a factor of proportionality and not a characteristic constant of the material. The factor λ can be a function of $e_{ij}^{(p)}$ and σ_{ij} , but not $\dot{\sigma}_{ij}$. The sign of λ is determined by the fact that plastic flow involves dissipation of energy. Under a given set of loading conditions λ , the total plastic flow are determined by the total work done by the external load.

Actual materials may exhibit much more complicated plastic behavior as specified above. Many yield conditions and plastic flow rules have been proposed. To prepare for the study of these formulations, we shall review briefly below some basic experimental facts.

Problem 6.1. Apply the yield criteria of von Mises and Tresca to simple tension and to simple shear. Show that, if they give the same yield stress in simple shear, the tensile yield stress predicted by von Mises' criterion is smaller than that predicted by Tresca's criterion by a factor $\sqrt{3}/2$.

Problem 6.2. Show that in no other type of stress is the discrepancy between the predictions of the yield stress by von Mises, and that by Tresca's criteria as large as it is in simple tension, if the two criteria give the same yield stress in simple shear.

6.4. SOME EXPERIMENTAL INFORMATION

Simple Tension Tests. If a rod of a ductile metal is pulled in a testing machine at room temperature, the load applied on the test specimen may be plotted against the elongation,

$$\epsilon = (l - l_0)/l_0,$$

where l_0 is the original length of the rod and l is the length under load. Figure 6.4:1 shows some typical experimental load-elongation relationships. The initial region appears as a straight line, in which the linear elasticity law is expected to hold. Mild steel shows an upper yield point A^* , a lower yield point A , and a flat yield region AB [Fig. 6.4:1(a)] which is caused by many discontinuous steps of microscopic slip along slip planes of the crystals. Most other metals do not have such a flat yield region [Fig. 6.4:1(b)]. Metals such as aluminum, copper and stainless steel exhibit a gradual transition from a linear elastic to a nonlinear plastic behavior without pronounced yield points [Fig. 6.4:1(a)].

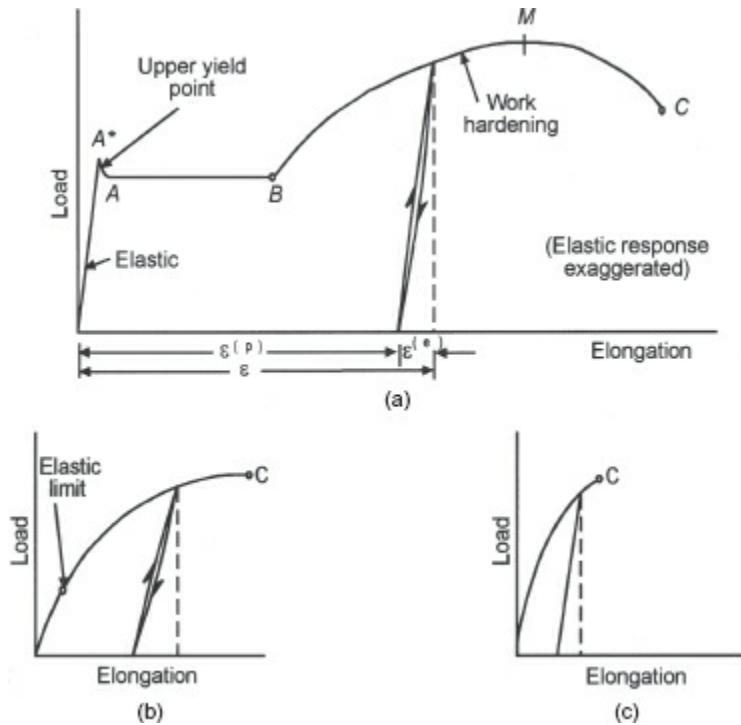


Fig. 6.4:1. Typical load-elongation curves in simple tension tests.

Upon unloading at any stage, the strain is reduced along an elastic unloading line. Reloading retraces the unloading line with minor deviation and then produces further plastic deformation when approximately the previous maximum stress is exceeded. During loading, the test specimen may “neck” at certain strain with the cross section reduced in a small region. Under continued elongation the load reaches a maximum M , called the ultimate load [Fig. 6.4:1(a)] and then drops down. Even though the actual average stress in the neck region (load divided by the true area of the neck) continues to increase. Beyond the ultimate load the metal continues to flow until the specimen breaks at point C shown in Fig. 6.4:1.

Materials like cast iron, titanium carbide, beryllium, and rock, which allow very little plastic deformation before reaching the breaking point C , are called *brittle* materials [Fig. 6.4:1(c)]. A great important fact for geology is that brittle materials like rocks tend to become ductile when subjected to large hydrostatic pressure (negative mean stress) as demonstrated in von Kármán's classical experiments (1911) on marbles.

Simple compression and simple shear tests of cylindrical specimens give load-strain diagrams similar to those of Fig. 6.4:1. Bending of a beam with a shear load is often used to test the material behavior in tension, compression and shear combined in a specific way.

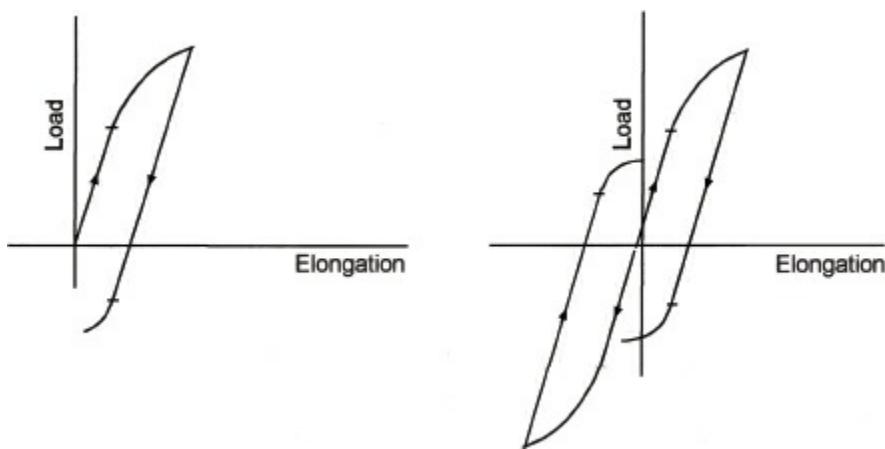


Fig. 6.4:2. Bauschinger effect in a simple tension-elongation test.

Bauschinger Effect. When a metal specimen is subjected to repeated tension-compression loads with deformation exceeding the elastic limit, the load-deflection curve sometimes appears as in Fig. 6.4:2. The tension stroke and the compression stroke are dissimilar. This is referred to as the *Bauschinger effect*, after J. Bauschinger's basic paper on strain hardening published in 1886.

Anisotropy. Plastic deformation in metal is a result of slip along certain crystallographic planes. The process is clearly directional. As a consequence, any initial isotropy, which may have been present is usually destroyed by plastic deformation. From the point of view of the dislocation theory, slip is an irreversible process; every slip produces a new

material. These changes are revealed in the Bauschinger effect and in the anisotropy of materials after plastic deformation.

Time and Temperature Effects. The results described above are obtained by slow application of the load. Higher strain or loading rate can have a pronounced effect on the material behavior in the plastic regime. Increasing loading rate generally decreases the material ductility, and increases the initial and subsequent yield stresses. Under rapid loading, the hereditary nature usually reveals itself. A material with hereditary stress-strain law is called *viscoelasticity* (see Chapter 14). At large strain, the hereditary stress-strain laws for metals are generally nonlinear.

The time-dependent plastic flow is described as *creep*. A simple tension specimen of lead wire under a constant tension shows creep curves as illustrated in Fig. 6.4:3. Following an initial extension, the rate of strain first decreases gradually, then remains nearly constant for a while, and finally accelerates until the specimen breaks. These *three stages of creep* are called the *primary*, *secondary* and *tertiary creeps*, respectively. As shown, creep test performed at constant stress may not have the final stage of accelerated elongation if the material does not deteriorate (Andrade^{6,4}). The creep curve at constant load differs considerably from that at constant stress.

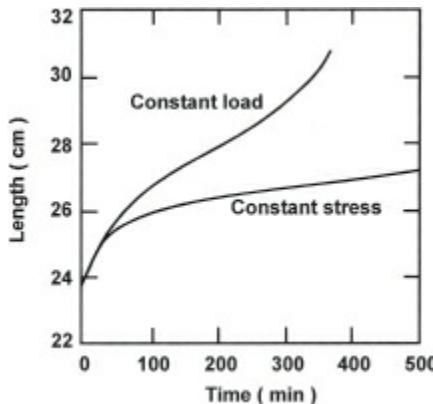


Fig. 6.4:3. Creep test of a lead wire in tension. The initial lengths and initial loads were the same in both tests. (After Andrade, 1910; courtesy of the Royal Society, London.)

The creep phenomenon is important in geophysics and in those engineering problems in which accurate dimensions must be maintained over a long period of time. It is a controlling factor for the design of machinery that operates at high temperature.

Temperature, as strain rate, generally has a very significant influence on material properties. As temperature increases, the ductility of metals usually increases while its stiffness decreases. Plastic deformation generates heat, and rapid application of load usually associates with large temperature gradient, localized plastic flow and anisotropic effects. Continuous metal cutting is a good example of large localized plastic deformation.

Size Effect and Stress Inhomogeneity. Since practically all materials have a nonhomogeneous microscopic structure, plastic properties of a small region may be different from those of a larger region. This consideration becomes important in the question of stress concentration in notched specimens. Size effect is also important for micron and submicron structures in micro electronic applications as the strain gradient can strongly influence the structural behaviors (see Lam *et al.* in Biblio. 6.1).

Combined Stress. In plasticity, the function of mathematics is particularly evident, because one must consider the stress and strain tensors; yet no direct observation of all their components is possible. The task of mathematical theory is to extend the experiences of simple experiments and suggest crucial tests to verify the basic assumptions of the theory.

A relatively simple combined-stress experiment employs a thin-walled circular cylinder. With the combination of axial tension, twisting moment about the cylinder axis, and internal or external pressure, a good approximation to a biaxial state of stress (involving σ_{11} , σ_{12} , σ_{22} with $\sigma_{31} = \sigma_{32} = \sigma_{33} = 0$) can be produced. Much information has been gathered from such experiments showing, among other things, that the shear stress is by far the major cause of yielding.

Bridgman's renowned experiments on the *influence of hydrostatic pressure on yielding* of metals showed that a pressure of one order higher than the yield stress has practically no influence on yielding. Comparing the tensile and shear tests at atmospheric pressure with those under high pressure, he found that the stress-strain curves in the small strain range did not change, but the ductility of the material increased greatly, thus permitting a much larger deformation prior to fracture. The density changes were found to be very small when a metal was subjected to repeated plastic deformation, indicating that the plastic volume change is small.

In the subsequent sections, we shall present some of the best known plasticity theories and derive the mathematical relationships between stress and strain for plastically deformed solids.

6.5. A BASIC ASSUMPTION OF THE MATHEMATICAL THEORY OF PLASTICITY: EXISTENCE OF YIELD FUNCTION

The theory of plasticity describes the mechanical behavior of materials in the plastic range including *energy dissipation, history or path dependent process*, the initial yield surface, its subsequent growth, the constitutive equations for plastic deformation, and the criteria for loading and unloading. Throughout this chapter, we assume that the *plastic deformation is not rate sensitive*, so that the constitutive equations are *invariant with respect to time scale and the rate and incremental forms are equivalent*.

By comparing the laws of linear elasticity and ideal plasticity as described in Secs. 6.1 and 6.3 with the typical experimental features given in Sec. 6.4, we can determine the applicability and limitations of these mathematical laws. We see that near the origin of the stress-strain curves in Fig. 6.4:1, the linear relationship is good in nearly all cases. This is the range over which the material behavior follows the laws of linear elasticity as described in Secs. 6.1 and 6.2. For the case of mild steel represented by Fig. 6.4:1(a), there is a flat region A-B, the feature of ideal plasticity with the load remaining constant while the elongation is increasing under load. The slightly unstable region on the yield curve at the peak A* before the flat range is normally ignored by plasticity laws. The rise beyond point B in Fig. 6.4:1(a), as well as the curved portion of the curves in Figs. 6.4:1(b) and 1(c), obviously cannot be represented by the linear elasticity or ideal plasticity. The features of these curved portions of stress-strain curves are said to be *strain hardening*.³ If the material is unloaded in this region beyond point A, it will follow a path parallel to the initial loading path OA*. As a result only part of the strain, called the *elastic strain* $e^{(e)}$, is recovered, while the other part of the strain will remain as the permanent or *plastic strain* $e^{(p)}$.

To describe the state of stress, it is convenient to represent the stress state by a point in a nine-dimensional *stress space* with axes σ_{ij} ($i, j = 1, 2, 3$). Similarly, a state of strain is a point in a nine-dimensional *strain space* with components e_{ij} . In the theory of plasticity, a state of plastic strain $e_{ij}^{(p)}$ may be so represented also. A program of loading may be regarded as a path in the stress space. The corresponding deformation history is a path in the strain space.

To develop a mathematical theory of plasticity, a basic assumption is made that there exists a continuous scalar *yield function* $f(\sigma_{ij}, T, \xi_i)$, which has the following properties:

- The equation $f(\sigma_{ij}, T, \xi_i) = 0$ represents a closed surface, called the *yield or loading surface*, in the stress space σ_{ij} for a given temperature T and an array of internal variables ξ_1, \dots, ξ_n .
- The plastic strain-rate $\dot{e}_{ij}^{(p)}$ and all internal-variable rates $\dot{\xi}_i$ vanish in the region in which $f(\sigma_{ij}, T, \xi_i) < 0$. This region is called the *elastic region*, which occupies the interior of the yield surface.
- The plastic strain rate $\dot{e}_{ij}^{(p)}$ can be nonzero in the region where $f(\sigma_{ij}, T, \xi_i) = 0$.
- No meaning is associated with $f(\sigma_{ij}, T, \xi_i) > 0$.

There are two types of internal variables ξ_1, \dots, ξ_n . One type consists of “physical” variables, which describe chemical reaction, phase changes, or structural defects. Another type consists of phenomenological variables including $e_{ij}^{(p)}$ itself. Together they characterize the hardening properties of the material and are called *work-hardening parameters*. These parameters may depend on the plastic deformation history, plastic rates, σ_{ij} , T and ξ_i .

Work hardening will be considered later. More examples of the yield criteria will be given in next sections.

6.6. LOADING AND UNLOADING CRITERIA

Let us first clarify what loading and unloading in a plastic state mean. Consider a plastic state at which the yield function introduced in Sec. 6.5 vanishes, i.e.,

$$(1) \quad f(\sigma_{ij}, T, \xi_i) = 0.$$

The time rate of f is

$$(2) \quad \dot{f} = (\partial f / \partial \sigma_{ij}) \dot{\sigma}_{ij} + (\partial f / \partial \xi_i) \dot{\xi}_i + (\partial f / \partial T) \dot{T}.$$

Obviously, $f = 0$ and $\dot{f} < 0$ at a time t would imply $f < 0$, the next instant of time. Such a change leads to an elastic state and is a natural attribute to the term *unloading*. However, we also require that in an unloading process there is no change in the internal-variables and temperature, i.e., $\dot{\xi}_i = \dot{T} = 0$. Hence, by Eq. (2) we stipulate that the criterion for

unloading from a plastic state at constant temperature is

$$(3) \quad (\partial f / \partial \sigma_{ij}) \dot{\sigma}_{ij} < 0, \quad f = 0, \text{ during unloading.}$$

Otherwise it is said to be loading or neutral loading. Thus,

$$(4) \quad (\partial f / \partial \sigma_{ij}) \dot{\sigma}_{ij} = 0, \quad f = 0, \text{ during neutral loading,}$$

$$(5) \quad (\partial f / \partial \sigma_{ij}) \dot{\sigma}_{ij} > 0, \quad f = 0, \text{ during loading.}$$

The function f is also called a *loading function* because of its prominence in these loading criteria.

A simple geometric interpretation of these criteria exists. Since the yield surface, i.e., the loading surface, is assumed to be a closed surface, we can speak of its inside and outside. Then, for a state of stress on the loading surface, loading, unloading, or neutral loading takes place, according to whether the stress increment vector is directed outward, inward, or along the tangent to the loading surface, respectively. Because of this geometric interpretation (also for reasons to be discussed in Sec. 6.9), it is important to obey the sign convention in writing Eq. (1) so that $\partial f / \partial \sigma_{ij}$ be directed outwardly normal to the surface $f = 0$.

6.7. ISOTROPIC STRESS THEORIES OF YIELD FUNCTION

If the yield function of a material depends only on the *invariants* (with respect to rotation of coordinates) of stress, strain, and strain history, then the plastic characteristics of the material is isotropic. The plasticity theory is called an *isotropic stress theory* if the yield function is an isotropic function of the stress alone. In such theory

$$(1) \quad f = f(I_1, I_2, I_3),$$

where I_1, I_2, I_3 are the three invariants of the stress tensor σ_{ij} . Equivalently, we may write Eq. (1) in terms of principal stresses σ_1, σ_2 and σ_3 ,

$$(2) \quad f = f(\sigma_1, \sigma_2, \sigma_3).$$

If the principal stresses are taken as the coordinate axes, the surface $f(\sigma_1, \sigma_2, \sigma_3) = 0$ can be plotted in a three-dimensional stress space.

According to Bridgeman^{6,3} (1923), the plastic deformation of metals essentially is independent of hydrostatic pressure. Thus the yield function is independent of $I_1 (= \sigma_1 + \sigma_2 + \sigma_3)$ and can be written as $f = f(J_2, J_3)$, where J_2, J_3 are the second and third stress deviation invariants. Von Mises simply took $f = J_2 - k^2 = 0$, a circular cylinder, $I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 0$ (Fig. 6.7:1), whose axis equally incline to the $\sigma_1, \sigma_2, \sigma_3$ -axes and is normal to the π -plane. If f involves J_3 , the cross section of the cylinder is no longer circular.

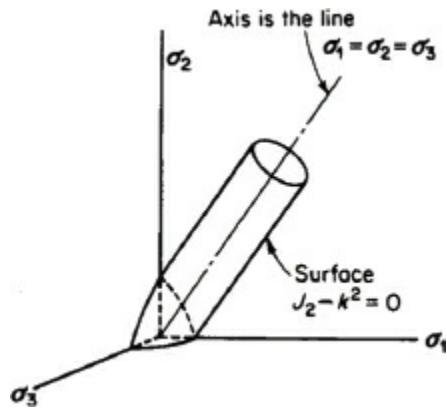


Fig. 6.7:1. A yield surface in the principal stress plane.

The yield functions can be written also in the form $f(\sigma_1 - \sigma_3, \sigma_2 - \sigma_3) = 0$, a two-dimensional surface in the $\sigma_1 - \sigma_3, \sigma_2 - \sigma_3$ coordinates. The von Mises function Eq. (6.3:5)

$$f = \{[(\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3)]/6 - k^2\} = 0$$

represents an ellipse. The Tresca function Eq. (6.3:8) [$\sigma_1 - \sigma_3 = \pm 2k, \sigma_2 - \sigma_3 = \pm 2k$ and $\sigma_1 - \sigma_2 = \pm 2k$] is a hexagon (Fig. 6.7:2).

One can project the yield surface unaffected by hydrostatic pressure on the π -plane. Von Mises' criterion would appear as a circle, and Tresca's criterion as a regular hexagon (Fig. 6.7:3). The π -plane projection will be used extensively below in discussing the flow and the hardening rules in plasticity.

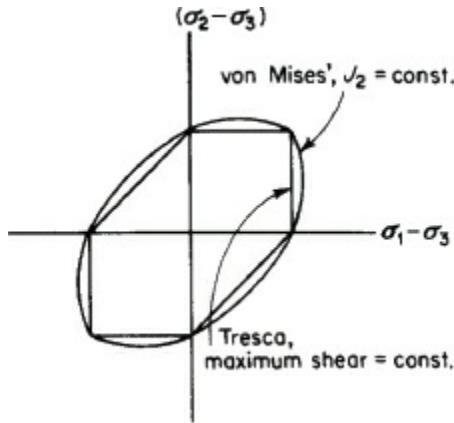


Fig. 6.7:2. Yield surface plotted on the plane of $(\sigma_1 - \sigma_3)$, $(\sigma_2 - \sigma_3)$.

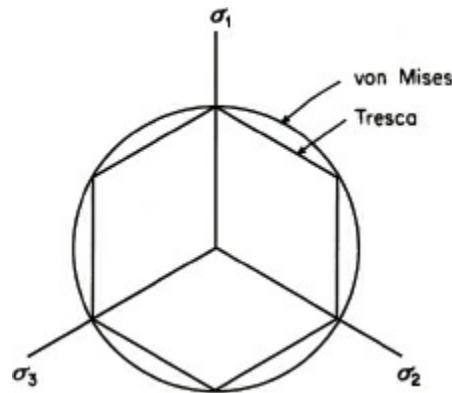


Fig. 6.7:3. Projection of yield surfaces on the π -plane.

6.8. FURTHER EXAMPLES OF YIELD FUNCTIONS

For general anisotropic materials, the initial yield criterion can be expressed in terms of the components σ_{ij} of the stress tensor σ .

Hill's Criterion for Anisotropic Materials. Hill proposed the yield function for orthotropic and incompressible materials without Bauschinger effect,

$$(1) \quad 2f(H_{ijkl}\sigma'_{ij}\sigma'_{kl}) = H_{ijkl}\sigma'_{ij}\sigma'_{kl} - 1 = F_1(\sigma_{yy} - \sigma_{zz})^2 + F_2(\sigma_{zz} - \sigma_{xx})^2 + F_3(\sigma_{xx} - \sigma_{yy})^2 + 2L_1\sigma_{yz}^2 + 2L_2\sigma_{zx}^2 + 2L_3\sigma_{xy}^2 - 1 = 0$$

where F_i and L_i , $i = 1, 2, 3$, are material constants. To determine these six constants, one can measure three tensile yielding stresses T_i in principal directions and three pure shear S_i on each of the orthogonal planes of anisotropy. Then F_i and L_i can be shown to be

$$(2) \quad F_i = (1/T_j^2 + 1/T_k^2 - 1/T_i^2)/2, \quad L_i = 1/(2S_i^2)$$

where i, j, k are the cyclic permutations of 1, 2, 3.

Mohr Coulomb Criterion for Pressure-Sensitive Materials. A yield criterion depending on the mean stress is necessary when it applies to soils, rocks, concrete, or porous materials. One such criterion occurs in the Mohr-Coulomb theory of rupture. This criterion postulates that yield occurs in a body on a plane on which the normal and shear stresses reach a critical combination, as in dry friction between surfaces. The critical condition can be expressed by two bounding curves represented by the equations

$$(3) \quad f = \tau \mp g(\sigma) = 0,$$

where σ and τ are the normal and shear stresses on the failure plane. If the state of stress (σ, τ) is described by points in a shaded area bounded by the three Mohr's circles shown in Fig. 3.9:1 in Chapter 3, then the bounding curves $\tau = \pm g(\sigma)$

can be drawn on the (σ, τ) plane. A material is safe if its state of stress lies within these bounding curves. The state of stress is critical if the bounding curves become tangent to the largest of the three Mohr's circles.

A special case of the Mohr–Coulomb criterion is that the bounding curves are straight lines,

$$(4) \quad g(\sigma) = c - \sigma \tan \phi$$

with c being a material constant and ϕ the angle of internal friction. On the (σ, τ) plane, we can represent the criterion graphically by the straight line BC tangent to the largest Mohr's circle. See Fig. 6.8:1. Let $\sigma_1, \sigma_2, \sigma_3$ be the principal stresses with $\sigma_1 \geq \sigma_2 \geq \sigma_3$, then

$$(\sigma_1 - \sigma_3)/2 = [d - (\sigma_1 + \sigma_3)/2] \sin \phi = [c \cot \phi - (\sigma_1 + \sigma_3)/2] \sin \phi.$$

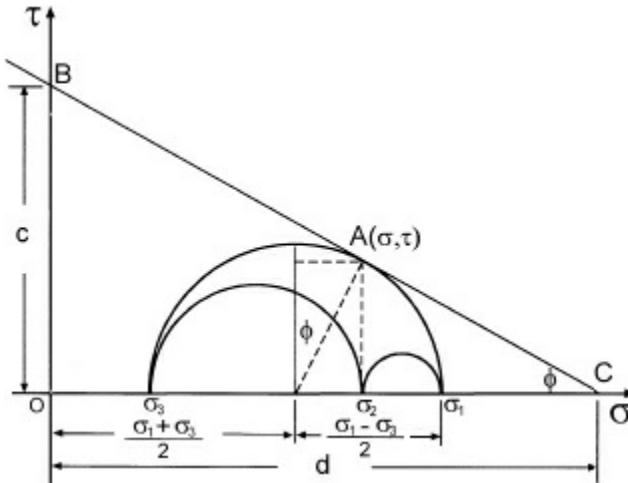


Fig. 6.8:1. Mohr–Coulomb yield criterion.

Hence the yield surface Eq. (3) can be expressed in terms of the maximum and minimum principal stresses σ_1, σ_3 as follows:

$$(5) \quad f = [\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3) \sin \phi]/2 - c \cos \phi = 0.$$

From the Mohr's circle, we have

$$(6) \quad \sigma = [\sigma_1 + \sigma_3 + (\sigma_1 - \sigma_3) \sin \phi]/2,$$

$$(7) \quad \tau = [(\sigma_1 - \sigma_3) \cos \phi]/2.$$

Eliminating σ_1, σ_3 , we can rewrite the yield surface Eq. (5) in the form

$$\tau = c - \sigma \tan \phi,$$

which reduces to the Tresca criterion if $\phi = 0$ and $c = \tau_Y = \sigma_Y/\sqrt{3}$. Let

$$(8) \quad \sigma_t = 2c \cos \phi / (1 + \sin \phi), \quad \sigma_c = 2c \cos \phi / (1 - \sin \phi).$$

Equation (5) becomes

$$(9) \quad \sigma_1/\sigma_t - \sigma_3/\sigma_c = 1,$$

in which σ_t, σ_c are identified as the yield strengths in tension and in compression, respectively. Note that $\sigma_t < \sigma_c$. From Eq. (8), we can express the parameters c, ϕ in terms of σ_t, σ_c :

$$\phi = \sin^{-1}[(\sigma_c - \sigma_t)/(\sigma_c + \sigma_t)] \quad \text{with} \quad 0 \leq \phi < \pi/2, \quad c = \sqrt{\sigma_c \sigma_t}/2.$$

Thus c, ϕ can be obtained from measured values of σ_t, σ_c in uniaxial tension and compression tests.

Equation (9) is a straight line between the positive σ_1 -axis and the negative σ_3 -axis on the octahedral plane for a fixed first stress invariant I_1 . The line intersects the positive σ_1 -axis at

$$(10) \quad \sigma_1 = \sigma_t = (\sigma_1)_0 = [4c \cos \phi + (1 - \sin \phi)I_1]/(3 + \sin \phi) (> 0),$$

which is derived by a substitution of $2\sigma_3 = I_1 - \sigma_1$ into Eq. (5). In other words, at the intersection, $\sigma_2 = \sigma_3$ and $\sigma_1 + \sigma_2$

$\sigma_3 = I_1$. Similarly we find the intersection of the line with the negative σ_3 -axis to be

$$(11) \quad \sigma_3 = -\sigma_c = -(\sigma_3)_0 = -[4c \cos \phi - I_1(1 + \sin \phi)]/(3 - \sin \phi) (< 0),$$

which is obtained by substituting $2\sigma_1 = I_1 - \sigma_3$ in Eq. (5), i.e. $\sigma_2 = \sigma_1$ and $\sigma_1 + \sigma_2 + \sigma_3 = I_1$. It can be shown that $(\sigma_3)_0 \geq (\sigma_1)_0$.

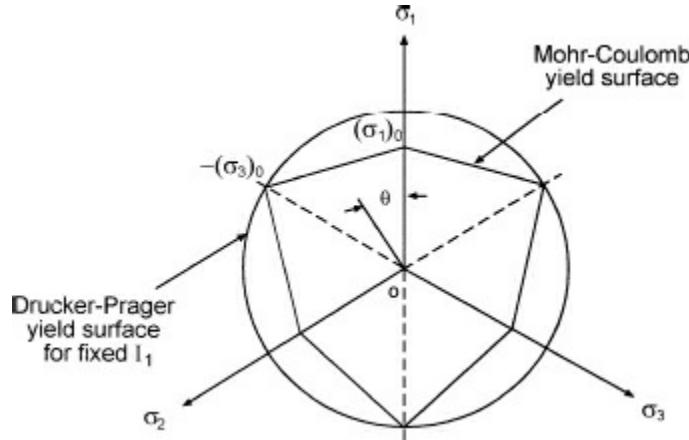


Fig. 6.8.2. Mohr-Coulomb and Drucker-Prager yield surfaces [Eq. (6.8:13)] on the π -plane for $\kappa - c_1 I_1 = (\sigma_3)_0/\sqrt{3}$.

The yield surface is an irregular hexagon on the *octahedral plane* or π -plane as shown in Fig. 6.8.2. The yield locus is symmetric about the σ_1 , σ_2 , σ_3 axes and can be obtained by the symmetry condition.

For two-dimensional stress with $\sigma_3 = 0$, the yield criterion can be written

$$\begin{aligned} \sigma_1 &= \sigma_t && \text{if } \sigma_1 > \sigma_2 > 0, \\ \sigma_2 &= -\sigma_c && \text{if } \sigma_2 < \sigma_1 < 0, \\ \sigma_1/\sigma_t - \sigma_2/\sigma_c &= 1 && \text{if } \sigma_1 > 0 > \sigma_2. \end{aligned}$$

These equations represent an irregular hexagon as shown in Fig. 6.8.3.

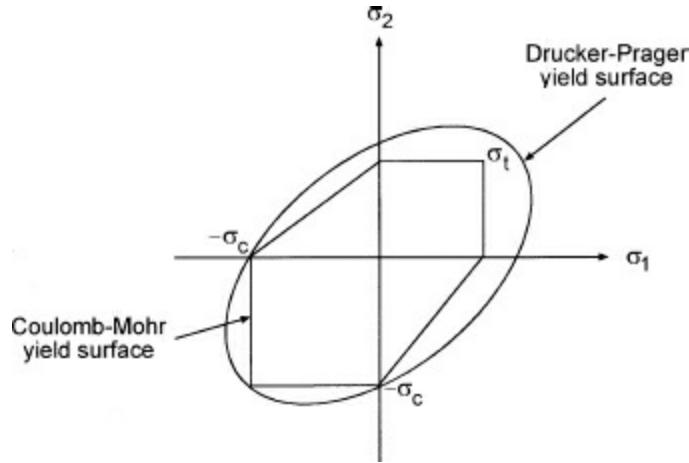


Fig. 6.8.3. Two-dimensional Mohr-Coulomb and Drucker-Prager yield surfaces [Eq. (6.8:13)] for $\kappa - c_1 I_1 = \sigma_c/\sqrt{3}$.

Using the results of Problem 6.3 at the end of this section, we can write the yield surface Eq. (5) in the form

$$(12) \quad f = \frac{I_1}{3} \sin \phi + \sqrt{J_2} \left[\sin(\theta + 60^\circ) + \frac{1}{\sqrt{3}} \cos(\theta + 60^\circ) \sin \phi \right] - c \cos \phi = 0$$

where I_1 is the first stress invariant, J_2 is the second deviatoric stress invariant given by Eq. (3.10:8), and θ is the angle between the principal deviatoric stress axis σ'_1 and the projection vector of the stress state on the octahedral plane. Equation (12) is valid only for $0 \leq \theta \leq 60^\circ$ (in the region between the positive σ_1 -axis and the negative σ_3 -axis where $\sigma_1 \geq \sigma_2 \geq \sigma_3$). For other values of θ , the yield locus is obtained by the symmetric condition about the principal axes σ_1 , σ_2 , σ_3 .

Drucker Prager Criterion. Equation (12) reduces to the *Drucker-Prager yield function*

$$(13) \quad f = \sqrt{J_2} + c_1 I_1 - \kappa = 0,$$

if $b_1 = -c^2$, $b_2 = 2c_1\kappa$, and $b_3 = -\kappa^2$. The yield locus for any octahedral plane (fixed I_1) is a circle for pressure sensitive materials. If the radius of the circle is $(\sigma_3)_0$ as given in Eq. (11) on all octahedral planes, we have

$$\kappa = 4c \cos \phi / [\sqrt{3}(3 - \sin \phi)], \quad c_1 = (\sin \phi + 1) / [\sqrt{3}(3 - \sin \phi)].$$

If the radius equals to $(\sigma_1)_0$ as defined in Eq. (10) on all octahedral planes, we find

$$\kappa = 4c \cos \phi / [\sqrt{3}(\sin \phi + 3)], \quad c_1 = (\sin \phi - 1) / [\sqrt{3}(\sin \phi + 3)].$$

The yield functions discussed up to this point contain no Bauschinger effect. However, the yield function

$$(14) \quad f(H_{ijkl}\sigma_{ij}\sigma_{kl}) = H_{ijkl}(\sigma_{ij} - m e_{ij}^{(p)})(\sigma_{kl} - m e_{kl}^{(p)}) - \kappa^2$$

exhibits the Bauschinger effect, which is controlled by the constant m . The initial anisotropy is not preserved during deformation. Yield functions such as this will be discussed more fully in Sec. 6.12.

All these examples use a single analytic function to represent the entire yield surface. If a yield surface is composed of piecewise smooth surfaces, which meet to form corners, it would be convenient to use a separate expression for each of these piecewise smooth surfaces. This concept leads to Koiter's generalization (see Sec. 6.10).

Problem 6.3. Let σ'_1 , σ'_2 , σ'_3 be the projections of the principal-stress axes σ_1 , σ_2 , σ_3 on an octahedral plane and n'_1 be the unit vector in σ'_1 direction. The components of n'_1 with respect to the principal axes can be expressed in the form $[\cos(90^\circ - \alpha), -a, -a]$, where $90^\circ - \alpha$ is the angle between the σ_1 - and σ'_1 -axes with $\cos \alpha = 1/\sqrt{3}$ and “a” is a constant. (An octahedral plane is the plane whose normal makes equal angles α with each of the principal axes. Also n'_1 makes equal angles with the σ'_2 , σ'_3 -axes.)

(a) Show that $n'_1 = (2, -1, -1)/\sqrt{6}$.

(b) Let S_i be the principal deviatoric stress corresponding to σ_i . Show that

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \frac{I_1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = \frac{I_1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \sqrt{\frac{J_2}{3}} \begin{bmatrix} \cos \theta \\ \cos(120^\circ - \theta) \\ \cos(120^\circ + \theta) \end{bmatrix}$$

where I_1 is the first stress invariant, $J_2 = (S_1^2 + S_2^2 + S_3^2)/2$ is the second deviatoric-stress invariant, and θ is the angle between the σ'_1 -axis and the vector representing the projection of the stress state on the octahedral plane. This result shows that one can choose I_1 , J_2 , θ as the coordinate axes to represent a stress state. [Hint: The components of the projection of a stress state on the octahedral plane with respect to the σ_1 , σ_2 , σ_3 -axes are (S_1, S_2, S_3) and the magnitude of the projection is $\sqrt{S_1^2 + S_2^2 + S_3^2} (= \sqrt{2J_2})$. Also $\sqrt{2J_2} \cos \theta = (S_1, S_2, S_3) \cdot n'_1$.]

Problem 6.4. Show that the line given by Eq. (9) intersects the σ_1 , σ_3 -axes at $(\sigma_1)_0$, $(\sigma_3)_0$ as given in Eq. (10) and (11), respectively. [Hint: At $(\sigma_1)_0$, the deviatoric stress $S_3 = -S_1/2$. Thus, $\sigma_1 = I_1/3 + S_1$ and $\sigma_3 = I_1/3 - S_1/2$ at $\sigma_1 = (\sigma_1)_0$. Show that $S_1 = 4[c \cos \phi - I_1(\sin \phi)/3]/(3 + \sin \phi)$ from Eq. (5). Similarly at $(\sigma_3)_0$, $S_1 = -S_3/2 (< 0)$, $\sigma_1 = I_1/3 - S_3/2$ and $\sigma_3 = I_1/3 + S_3$. Show that $S_3 = 4[-c \cos \phi + I_1(\sin \phi)/3]/(3 - \sin \phi)$.]

Problem 6.5. Show that the yield stresses for the Drucker–Prager criterion in simple shear, tension and compression are, c_2 , $\sqrt{3}c_2/(1+\sqrt{3}c_1)$ and $\sqrt{3}c_2/(1-\sqrt{3}c_1)$, respectively. For this criterion to be physically meaningful, we must have $c_2 > 0$ and $\sqrt{3}c_1 < 1$.

Problem 6.6. The von Mises–Schleicher criterion is

$$f(\sigma_{ij}) = 3J_2 + (\sigma_c - \sigma_t)I_1 - \sigma_c\sigma_t = 0.$$

Show that σ_c and σ_t are, respectively, the compressive and tensile yield stresses in uniaxial tests.

6.9. WORK HARDENING – DRUCKER'S HYPOTHESIS AND DEFINITION

Work hardening in simple tension means that the yield stress monotonically increases with strain. To generalize this concept, Drucker^{6,3} (1951) considers the work done on a material element in equilibrium by an external agency, which slowly applies a set of self-equilibrating forces and then slowly removes them, a process called *stress cycle*. This external agency is entirely separate and distinct from the agency causing the existing state of stress. Removing the additional stress enables the stress of the body to return to the original stress state, but the strain state can be different if plastic

deformation occurred during the stress cycle. *Work hardening then means that positive work is done by the external agency during the application of the stresses, and the net work performed over the cycle of load application and removal is either zero or positive.* Rephrased, work hardening means that *useful net energy over and above the elastic energy cannot be extracted from the material and the system of forces acting upon it.*

Consider a volume of material in a homogeneous state of stress σ_{ij} and strain e_{ij} . Suppose that an external agency applies a small surface traction, which alters the stress at each point by $d\sigma_{ij}$ and the strain by de_{ij} . On removal of the traction, $d\sigma_{ij}$ returns to zero, and a strain $de_{ij}^{(e)}$ recovered. *The material is said to be work-hardening if*

$$(1) \quad d\sigma_{ij}de_{ij} > 0, \quad \text{upon loading;}$$

$$(2) \quad d\sigma_{ij}(de_{ij} - de_{ij}^{(e)}) = d\sigma_{ij}de_{ij}^{(p)} \geq 0, \quad \text{on completing a cycle,}$$

in which $de_{ij}^{(p)}$ ($= de_{ij} - de_{ij}^{(e)}$) denote the *plastic strain increments* not recovered by the process named above.

Let $\bar{\sigma}_{ij}$ be the initial stress at a point inside or on the yield surface and $\sigma_{ij} = \bar{\sigma}_{ij} + d\sigma_{ij}$. Then, the work per unit volume done by external agency is non-negative that

$$(3) \quad (\sigma_{ij} - \bar{\sigma}_{ij})de_{ij}^{(p)} \geq 0.$$

Drucker's hypothesis Eq. (2) or (3) can be satisfied only by materials whose subsequent yield strength increases with deformation. Any material on which an external agency does positive work during an elastic-plastic stress cycle is called a *hardening material*. Otherwise, it is considered to be *nonhardening* or *work-softening*. It can be shown that Eq. (2) or (3) holds for work-softening and perfectly plastic materials (von Mises 1928, Bishop and Hill 1951) under Il'iushin's postulate of plasticity in strain space to be considered in [Sec. 6.14](#).

Equation (3) is also called the *principle of maximum plastic dissipation*. It can be written in the form

$$(4) \quad \sigma_{ij}\dot{e}_{ij}^{(p)} = D(\dot{e}_{ij}^{(p)}, \xi_i) \geq \bar{\sigma}_{ij}\dot{e}_{ij}^{(p)},$$

where $D(\dot{e}_{ij}^{(p)}, \xi_i)$ depends on the plastic strain-rates $\dot{e}_{ij}^{(p)}$ and the internal variables ξ_i only. Equation (4) will be used for limit analysis in [Chapter 10](#).

6.10. IDEAL PLASTICITY

According to Drucker's definition of work hardening, *ideal plasticity* is a plastic deformation without work hardening. When plastic deformation occurs, the equality sign prevails in Eq. (6.9:2):

$$(1) \quad d\sigma_{ij}de_{ij}^{(p)} = 0,$$

and that the yield function is unaffected by $de_{ij}^{(p)}$. The differentials $d\sigma_{ij}$ and $de_{ij}^{(p)}$ must be interpreted as in [Sec. 6.9](#).

Applying the definition above, we define the flow rule during yielding of an ideal plastic deformation. The flow rule provides the information concerning the increment or rate of deformation to describe the material behavior.

Now, for an ideal plastic material, we assume that a yield function $f(\sigma_{ij})$ exists, which is a function of the stresses σ_{ij} and not of the strains $de_{ij}^{(p)}$, such that $f(\sigma_{ij}) \leq 0$ prevails; and

$$(2) \quad \dot{e}_{ij}^{(p)} \neq 0 \quad \text{only if} \quad f(\sigma_{ij}) = 0.$$

Since f is assumed to be a function of σ_{ij} only, any change in stresses during plastic flow must satisfy the relation

$$(3) \quad df = (\partial f / \partial \sigma_{ij})d\sigma_{ij} = 0.$$

Equation (3) is often called the *consistency condition for ideal plasticity*. A comparison of Eqs. (1) and (3) shows that

$$(4) \quad \Delta \quad de_{ij}^{(p)} = d\Lambda(\partial f / \partial \sigma_{ij})|\partial f / \partial \sigma_{kl}|^{-1} = n_{ij}d\Lambda,$$

where $d\Lambda$ is a scalar (not a material constant), which may vary during deformation, and $n_{ij} = (\partial f / \partial \sigma_{ij})/|\partial f / \partial \sigma_{kl}| = (\partial f / \partial \sigma_{ij})/[(\partial f / \partial \sigma_{kl})(\partial f / \partial \sigma_{kl})]^{1/2}$ is a unit normal to the yield surface. We can also write Eq. (4) in the rate form:

$$(5) \quad \dot{e}_{ij}^p = \dot{\Lambda}(\partial f / \partial \sigma_{ij}) / |\partial f / \partial \sigma_{kl}| = \dot{\Lambda} n_{ij} .$$

The sign of $\dot{\Lambda}$ is restricted by the condition that plastic flow always involves dissipation of mechanical energy, a condition which may be written as

$$(6) \quad \dot{W} = \sigma_{ij} \dot{e}_{ij}^{(p)} > 0 .$$

Equation (4) gives *the rule of plastic flow* (plastic strain increments) in ideal plasticity. It is a prototype of *the theory of plastic potential* developed by von Mises^{6,3} (1928). The general theory is discussed in Sec. 6.11 *et seq.*

Prager's Geometric Interpretation. The formula $f(\sigma_{ij}) = 0$ defines a surface in the nine-dimensional stress space with σ_{ij} ($i, j = 1, 2, 3$) as coordinates. The outward normal vector to this surface has the components $\partial f / \partial \sigma_{ij}$. Equation (5) states thus that the vector of plastic deformation rate $\dot{e}_{ij}^{(p)}$ is normal to the surface $f = 0$ in the stress space. So, during unloading,

$$f = 0, \quad (\partial f / \partial \sigma_{ij}) d\sigma_{ij} < 0 ,$$

the stress increment $d\sigma_{ij}$ is pointing inward from the yield surface, while during loading or neutral loading,

$$f = 0, \quad (\partial f / \partial \sigma_{ij}) d\sigma_{ij} = 0 ,$$

i.e., the stress increment $d\sigma_{ij}$ is on the tangential plane at a stress point on the yield surface. Because the yield surface is fixed for ideal plasticity, $d\sigma_{ij}$ cannot point outward.

Incremental Stress-Strain Relationship in Plastic Flow. Using the basic assumption that the total strain increment can be decomposed into an elastic and a plastic component,

$$(7) \quad de_{ij} = de_{ij}^{(e)} + de_{ij}^{(p)} ,$$

together with the flow rule and the consistency condition, we can determine $d\Lambda$ in Eqs. (4) and (5) in terms of the total strain increment and the current state of stress. We have

$$(8) \quad d\sigma_{ij} = D_{ijkl}(de_{ij} - de_{ij}^p) = D_{ijkl}(de_{ij} - n_{ij}d\Lambda) .$$

Multiplying both sides of Eq. (8) by $\partial f / \partial \sigma_{ij}$, we obtain

$$(9) \quad (\partial f / \partial \sigma_{ij}) d\sigma_{ij} = (\partial f / \partial \sigma_{ij}) D_{ijkl}(de_{ij} - n_{ij}d\Lambda) .$$

Using the consistency condition $(\partial f / \partial \sigma_{ij}) d\sigma_{ij} = 0$, we find

$$(10) \quad \Delta \quad d\Lambda = (\partial f / \partial \sigma_{ij}) D_{ijkl} de_{kl} [D_{mnrs} (\partial f / \partial \sigma_{mn}) n_{rs}]^{-1} .$$

A substitution of Eq. (10) into Eq. (8) establishes the *general stress-strain increment relationship*

$$(11) \quad d\sigma_{ij} = D_{ijkl} [de_{kl} - de_{kl}^{(p)}] = D_{ijkl}^{ep} de_{kl} ,$$

where

$$(12) \quad D_{ijkl}^{ep} = D_{ijkl} - D_{ijtu} \frac{\partial f}{\partial \sigma_{tu}} \frac{\partial f}{\partial \sigma_{qp}} D_{qpkl} \left(D_{mnrs} \frac{\partial f}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{rs}} \right)^{-1} .$$

For isotropic materials, $D_{ijkl} = \lambda \delta_{ij} \delta_{kl} + G(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$, Eq. (11) reduces to

$$(13) \quad d\sigma_{ij} = \lambda de_{\alpha\alpha} \delta_{ij} + 2G de_{ij} - (\lambda n_{kk} \delta_{ij} + 2G n_{ij})(\lambda n_{\beta\beta} de_{\alpha\alpha} + 2G n_{kl} de_{kl}) / (\lambda n_{mm} n_{ss} + 2G) ,$$

where λ is the Lamé constant, and recall that

$$(14) \quad n_{ij} = (\partial f / \partial \sigma_{ij}) / |\partial f / \partial \sigma_{kl}| .$$

For materials with plastic deformation insensitive to hydrostatic pressure, we have $n_{kk} = 0$ and Eq. (13) reduces to

$$(15) \quad d\sigma_{ij} = \lambda de_{\alpha\alpha} \delta_{ij} + 2G de_{ij} - 2G n_{ij} n_{kl} de_{kl} .$$

W. T. Koiter^{6.3} (1953) generalized this theory of plasticity by allowing the yield limit to be specified by a set of yield functions,

$$f_1(\sigma_{ij}), f_2(\sigma_{ij}), \dots, f_n(\sigma_{ij}).$$

A state of stress is said to be below the yield limit if all these functions are negative. For a state of stress at the *yield limit*, at least one yield function vanishes, while none has a value greater than zero.

If the functions $f_h = \dots = f_m = 0$, ($1 \leq h \leq m \leq n$) whereas all other f 's are negative, then the Koiter generalization of the flow rule given in Eq. (5) is

$$(16) \quad \dot{e}_{ij}^{(p)} = (n_{ij})_h \dot{\Lambda}_h + \dots + (n_{ij})_m \dot{\Lambda}_m, \quad (h, \dots, m \text{ not summed}),$$

where $(n_{ij})_h = (\partial f_h / \partial \sigma_{ij}) / |\partial f_h / \partial \sigma_{kl}|$ (h not summed) and $d\Lambda_h, \dots, d\Lambda_m$ are nonnegative proportional factors. Thus, in the case of ideal plasticity, the basic concept leads at once to a general incremental stress-strain relationship.

Example 1. Under von Mises' criterion, $f = J_2 - k$, where k is independent of plastic deformation, we can derive the flow rule below according to Eq. (5):

$$(17) \quad \dot{e}_{ij}^{(p)} = \dot{\Lambda} \sigma'_{ij} (\sigma'_{kl} \sigma'_{kl})^{-1/2},$$

which is the rule presented in Sec. 6.3.

Example 2. Tresca's yield condition can be expressed in terms of Koiter's generalized plastic potential by defining f 's as the followings:

$$(18) \quad \begin{aligned} f_1 &= \sigma_2 - \sigma_3 - 2k, & f_2 &= \sigma_3 - \sigma_1 - 2k, \\ f_3 &= \sigma_1 - \sigma_2 - 2k, & f_4 &= -(\sigma_2 - \sigma_3) - 2k, \\ f_5 &= -(\sigma_3 - \sigma_1) - 2k, & f_6 &= -(\sigma_1 - \sigma_2) - 2k, \end{aligned}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses. A state of stress is below the yield limit if f_1, \dots, f_6 are all negative. Yielding occurs when one or more of the f 's are equal to zero. None of the f 's can have a positive value. When yielding occurs, the flow rule is given by Eq. (7). For example, if $f_1 = 0$, while all the other f 's are negative, then

$$(19) \quad \dot{e}_2^{(p)} = \dot{\Lambda}_1, \quad \dot{e}_3^{(p)} = -\dot{\Lambda}_1, \quad \dot{e}_1^{(p)} = 0,$$

where $\dot{\Lambda}_1 > 0$, and e_1, e_2, e_3 are the principal strains corresponding to $\sigma_1, \sigma_2, \sigma_3$. The principal axes of the strain tensor coincide with those of the stress tensor under Tresca's condition.

Remarks. Formal application of the method of derivation above to a simple load-deflection experiment may appear difficult. If we twist a tube of ideal plastic material in torsion, the yield condition according to von Mises' criterion is reached when the shearing stress $\tau = \kappa$ the yield stress. Plastic flow will continue with no possibility of increasing τ and κ . Hence, if we limit ourselves to torsion and apply Eq. (1) and (2), we would have obtained $d\tau = 0$, which yields no useful information. To deduce anything significant we would have to consider adding other loads, for example, tension or internal pressure in the tube. These additional varieties of loads will alter the plastic flow, thus providing nontrivial changes $d\sigma_{ij}$ and $de_{ij}^{(e)}$ to which the derivation above applies.

Problem 6.7. A plane strain condition is defined as $e_{33} = e_{31} = e_{32} = 0$. This condition requires $de_{3i}^{(p)} = -de_{3i}^{(e)}$, for $i = 1, 2, 3$. Consider the von Mises criterion with the associated flow rule $(\sigma'_{kl} \sigma'_{kl})^{-1/2} de_{ij}^{(p)} = \sigma'_{ij} d\Lambda$. Show that $\sigma_3 = v(\sigma_1 + \sigma_2)$, where σ_i are the principal stresses.

6.11. FLOW RULE FOR WORK HARDENING MATERIALS

In this section, the Drucker's hypothesis discussed in Sec. 6.9 is taken to define work-hardening, and von Mises' plastic potential theory is taken as the framework to derive the flow rule. Von Mises^{6.3} (1928) suggested that there exists a *plastic potential function* $h(\sigma_{ij})$ so that the plastic strain rate $\dot{e}_{ij}^{(p)}$ could be derived from

$$(1) \quad \dot{e}_{ij}^{(p)} = \lambda (\partial h / \partial \sigma_{ij}) |\partial h / \partial \sigma'_{kl}|^{-1},$$

where λ is a positive scalar factor. If the plastic potential is the same as the yield function: $h = f$, Eq. (1) is called the *associated flow rule*. On the other hand, if $h \neq f$, the flow rule is called *non-associated*. In the preceding section we have

seen that $h = f$ for an ideal plasticity body. Now, we shall consider the more general case of work hardening materials by allowing the plastic potential to be a function of not only stresses but also temperature T and internal variables ξ . We shall show that under Drucker's hypothesis and the assumption that the elastic moduli D_{ijkl} are independent of the plastic deformation, the yield function f itself is the plastic potential. Experimental observations show that the associated flow rule characterizes the plastic deformation of metals quite well, but the nonassociated flow rule provides a better representation for the plastic deformation of porous materials such as rocks, concrete, and soils.

Consequences of Drucker's Hypothesis. We shall prove the following consequences of Drucker's hypothesis from which the term work-hardening is defined:

- A. The yield surface and all subsequent loading surfaces must be *convex*.
- B. The plastic strain increment vector must be *normal to the loading surface*

at a regular point, and it must lie between the adjacent normals to the loading surface at a corner of the surface.

- C. The rate of change of plastic strain must be a *linear function of the rate of change of the stress*.

Proof of A. To facilitate the proof,⁴ let us think of an increment of stress $d\sigma_{ij}$ as the components of a vector $d\mathbf{S}$ in the nine-dimensional stress space, and the corresponding plastic strains $de_{ij}^{(p)}$ as the components of a vector $d\mathbf{E}$ in the same (stress) space. Then, by Eq. (6.9:2), we have

$$(2) \quad d\mathbf{S} \cdot d\mathbf{E} = |d\mathbf{S}| |d\mathbf{E}| \cos \psi > 0,$$

which implies $-\pi/2 \leq \psi \leq \pi/2$, i.e. the angle between $d\mathbf{S}$ and $d\mathbf{E}$ is acute.

Now let P be a regular (smooth) point on the yield surface and $d\mathbf{E}$ a plastic strain vector at P . According to Eq. (2), all loading stress increments $d\mathbf{S}$ ending at P , that produces $d\mathbf{E}$, must form an acute angle with $d\mathbf{E}$. If we represent a hyper-plane normal to $d\mathbf{E}$ by AB in Fig. 6.11:1, then the vectors $d\mathbf{S}$ ($= d\sigma_{ij}$) must all originate in one side of AB and so are the initial points of $d\mathbf{S}$ representing different stress states. However, $d\mathbf{S}$ are loading vectors (since $d\mathbf{E}$ exists), whose directions are bounded by the tangent plane of the yield surface (see Sec. 6.6). Hence, the hyper-plane AB must be tangent to the loading surface. Since $d\mathbf{E}$ is normal to AB, it is also normal to the yield surface at the point of tangency. Furthermore, since the yield surface lies on one side of the tangent plane, the yield surface is convex at P . Finally, since P is an arbitrary point on the yield surface, the convexity of the entire yield surface is established.

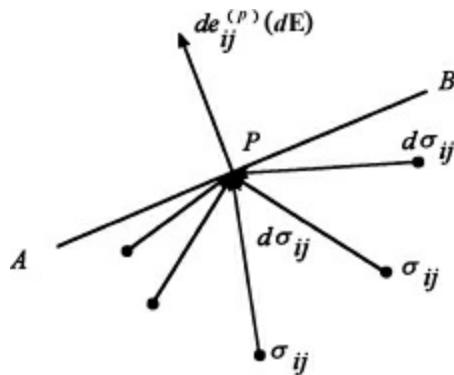


Fig. 6.11:1. Possible stress increments corresponding to a plastic strain increment $d\mathbf{E}$.

Proof of B. Since at a regular point on a surface there is a unique tangent plane, the hyper-plane is unique at a regular point of the yield surface. Thus, the direction of $d\mathbf{E}$ normal to the hyper-plane is also unique. In other words, at a smooth point of the yield surface the direction of $d\mathbf{E}$ (i.e. $de_{ij}^{(p)}$) is independent of the direction of $d\sigma_{ij}$. At a corner on the yield surface, there can have been more than one limiting tangent plane; convexity must still hold, but the direction of strain increment $de_{ij}^{(p)}$ may depend on the direction of the loading $d\sigma_{ij}$.

Proof of C. The proof is given in Eq. (14), (18) and (19) below. To derive these equations, we use the formulas for normality and consistency.

Normality Condition. Under Drucker's hypothesis, the *normality of the plastic strain rate vector* at a smooth point of the loading surface requires

$$(3) \quad \Delta \quad \dot{e}_{ij}^{(p)} = \dot{\Lambda} n_{ij} = \dot{\Lambda} (\partial f / \partial \sigma_{ij}) / |\partial f / \partial \sigma_{kl}|$$

where $\dot{\Lambda}$ is a proportional scalar factor, which can be a function of stress, strain, strain history, temperature and internal variables, and n_{ij} is a unit normal to the yield surface at the loading point. Since the work done by an external agency during loading must be positive, it is easy to show that $\dot{\Lambda}$ is nonnegative. Equation (1) is similar to Eq. (6.10:5); thus, the loading function h plays the role of *plastic potential*. We transform Eq. (1) to the form of Eq. (3) so that the *flow equation* becomes

$$(4) \quad \dot{e}_{ij}^{(p)} = \dot{\Lambda} h_{ij} = \dot{\Lambda} (\partial h / \partial \sigma_{ij}) / |\partial h / \partial \sigma_{kl}|,$$

where h_{ij} is a unit normal to the plastic potential $h(\sigma_{ij}, T, \xi_i)$ at a smooth point of the surface. Comparing Eq. (3) and (4), we obtain

$$(5) \quad h_{ij} = n_{ij}, \quad \text{and} \quad h = h(\sigma_{ij}, T, \xi_i).$$

In other words, Eq. (4) is an associated flow rule.

Consistency Condition. As plastic flow proceeds, the subsequent loading surfaces pass through the loading point, we must have

$$(6) \quad f(\sigma_{ij}, T, \xi_i) = 0.$$

Expressing the *internal variable-rates* in the form

$$(7) \quad \dot{\xi}_i = \dot{\Lambda} h_i,$$

in which h_i is a known function of σ_{ij} , T , and ξ_i , we must have, during loading,

$$(8) \quad \dot{f} = (\partial f / \partial \sigma_{ij}) \dot{\sigma}_{ij} + \dot{\Lambda} (\partial f / \partial \xi_i) h_i = 0,$$

or

$$(9) \quad \dot{\Lambda} = n_{ij} \dot{\sigma}_{ij} / K^p, \quad K^p = -h_i (\partial f / \partial \xi_i) / |(\partial f / \partial \sigma_{kl})|,$$

where K^p is called the *plastic modulus*. Prager named Eq. (8) the consistency condition, which means that loading from a plastic state leads to another one. Combining Eq. (9) with (4), we find the *flow rule* in the form

$$(10) \quad \dot{e}_{ij}^{(p)} = h_{ij} n_{kl} \dot{\sigma}_{kl} / K^p, \quad \dot{\kappa} = (\partial \kappa / \partial e_{ij}^{(p)}) \dot{e}_{ij}^{(p)} = (\partial \kappa / \partial e_{ij}^{(p)}) \dot{\Lambda} h_{ij}.$$

Since $\dot{e}_{ij}^{(p)}$ during unloading, [$n_{ij} \dot{\sigma}_{ij} < 0$, see Eq. (6.6:3)], one may write the *flow rule* in a slightly different form

$$(11) \quad \dot{e}_{ij}^{(p)} = \frac{h_{ij} n_{kl} \dot{\sigma}_{kl}}{K^p} H(n_{mn} \dot{\sigma}_{mn}).$$

where H is the *Heaviside function* defined as

$$\begin{aligned} H(x) &= 1, & \text{if } x \geq 0, \\ &= 0, & \text{if } x < 0. \end{aligned}$$

which is the unit step function denoted by $\mathbf{1}(x)$ elsewhere in the book.

Incremental Strain-Stress Relation. Consider the yield surface

$$(12) \quad f(\sigma_{ij}, e_{ij}^{(p)}, \kappa) = 0.$$

The internal variables are $e_{ij}^{(p)}$ and κ , with the latter being a function of $e_{ij}^{(p)}$ also. The flow rule is given by Eq. (4) or (11) and the equation for the internal variable-rate is given by the second equation of Eq. (10). The consistency equation is

$$\dot{f} = (\partial f / \partial \sigma_{ij}) \dot{\sigma}_{ij} + [\partial f / \partial e_{rs}^{(p)} + (\partial f / \partial \kappa) (\partial \kappa / \partial e_{rs}^{(p)})] h_{rs} \dot{\Lambda} = 0.$$

We obtain $\dot{\Lambda}$ in the form of Eq. (9) with

$$(13) \quad K^p = -[\partial f / \partial e_{rs}^{(p)} + (\partial f / \partial \kappa) (\partial \kappa / \partial e_{rs}^{(p)})] h_{rs} / |\partial f / \partial \sigma_{kl}|,$$

for the plastic modulus. These results were first given by Prager^{6.3} (1948) and Drucker^{6.3} (1959). Equation (10) proves the linearity statement (item C) given above. Using

$$de_{ij} = D_{ijkl}^{-1} d\sigma_{kl} + de_{ij}^{(p)}$$

and Eq. (10) leads to the *incremental strain-stress relation* (the constitutive equations) for plastic deformation

$$(14) \quad \begin{aligned} de_{ij} &= [D_{ijkl}^{-1} + h_{ij}n_{kl}/K^p]d\sigma_{kl}, \quad \text{or} \\ d\sigma_{kl} &= [D_{klrs}^{-1} + n_{kl}h_{rs}/K^p]^{-1}de_{rs} = D_{klrs}^{ep}de_{rs} \end{aligned}$$

where D_{ijkl}^{-1} , often denoted by C_{ijkl} and called the *elastic flexibility tensor*, are the components of the inverse of the fourth-order elastic modulus tensor D_{ijkl} . Equation (14) breaks down for ideal plasticity because

$$\partial f/\partial \xi_m = 0 \quad \text{i.e.} \quad \partial f/\partial e_{mn}^{(p)} = \partial \kappa/\partial e_{mn}^{(p)} = 0,$$

and K^p becomes zero. In this case we have to use the incremental stress-strain relation given in Eqs. (6.10:11) and (6.10:12).

Koiter's Generalization. For a loading surface composed of individual smooth surfaces $f_r = 0$, which meet and form corners, Koiter^{6.3} (1953) shows that if the surfaces act independently, the total plastic deformation is the sum of contributions from certain of the f_r 's that

$$(15) \quad \dot{e}_{ij}^{(p)} = \sum_{r=1}^n C_r \dot{\Lambda}_r (n_{ij})_r (n_{kl})_r \dot{\sigma}_{kl},$$

$$(16) \quad \begin{aligned} C_r &= 0 \quad \text{if} \quad f_r \leq 0, \quad \text{or} \quad (n_{kl})_r \dot{\sigma}_{kl} < 0, \\ C_r &= 1 \quad \text{if} \quad f_r = 0, \quad \text{and} \quad (n_{kl})_r \dot{\sigma}_{kl} \geq 0, \end{aligned}$$

where $\dot{\Lambda}_r$ are positive functions associated with f_r as defined in Eq. (9) and $(n_k)_r$ are the components of the unit outward normal to f_r at the loading point. Equations (15) and (16) specify the condition of yielding and loading.

It should be remarked that the properties deduced above, namely, the convexity, normality, and linearity, follow Drucker's hypothesis — which is often interpreted as a statement that the material is stable. Hence, these properties holds only for stable materials. Unstable materials, like mild steel at its upper yield point (the point A* in Fig. 6.4:1), do exist. Other engineering materials such as rocks, concrete, and soils exhibit softening phenomena for which

$$d\sigma_{ij} de_{ij} < 0 \quad \text{and} \quad d\sigma_{ij} de_{ij}^{(p)} < 0.$$

A formulation of the plasticity theory in the strain space is needed to describe the softening behavior.

The normality of $de_{ij}^{(p)}$ and the convexity of the yield surface do not hold under Drucker's postulate, if D_{ijkl} is a function of $e_{ij}^{(p)}$, i.e., there is elasticplastic coupling during the process of plastic deformation. This is also the case for softening materials. Il'iushin (1960) showed that Drucker's postulate results in

$$\begin{aligned} d\sigma_{ij} [\sigma_{kl} dD_{ijkl}^{-1}(e_{mn}^{(p)}) + de_{ij}^{(p)}] &> 0, \\ de_{ij}^{(p)} &= (\partial f/\partial \sigma_{ij}) d\Lambda - \sigma_{kl} dD_{ijkl}^{-1}(e_{mn}^{(p)}). \end{aligned}$$

Example. If we choose $f = J_2 - \kappa$, where J_2 is the second invariant of the stress deviation, and κ is a hardening parameter, which depends on plastic deformation, then the flow rule at yielding is

$$\begin{aligned} \dot{e}_{ij}^{(p)} &= \hat{G} \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{kl}} \dot{\sigma}_{kl} = \hat{G} \left(\frac{\partial f}{\partial J_2} \frac{\partial J_2}{\partial \sigma'_{ij}} \right) \left(\frac{\partial f}{\partial \sigma'_{kl}} \frac{d\sigma'_{kl}}{dt} \right) \\ &= \hat{G} \sigma'_{ij} \sigma'_{kl} \dot{\sigma}_{kl} = \hat{G} \sigma'_{ij} \dot{J}_2, \quad \dot{J}_2 \geq 0, \end{aligned}$$

where from Eqs. (4), (9) and (13),

$$\hat{G} = 1/[\|\partial f/\partial \sigma_{mn}\|^2 K^p] = 1/[(\partial \kappa/\partial e_{rs}^{(p)}) \sigma'_{rs}].$$

If we set $f = F(J_2, J_3) - \kappa$, assume $\partial \kappa/\partial e_{mn}^{(p)} \neq 0$ for some m and n , and note that $\partial \sigma_{kk}/\partial \sigma_{ij} = \delta_{ij}$ and $t_{ij} = \sigma'_{ik} \sigma'_{kj} - 2J_2 \delta_{ij}/3$, then the flow rule is

$$\dot{e}_{ij}^{(p)} = \hat{G} [(\partial F/\partial J_2) \sigma'_{ij} + (\partial f/\partial J_3) (\sigma'_{ik} \sigma'_{kj} - 2J_2 \delta_{ij}/3)] \dot{F}, \quad \dot{F} \geq 0.$$

Note that the yield-function of a work-hardening material depends on the plastic strain $e_{ij}^{(p)}$ in a significant manner, in contrast to an ideal plastic material, whose yield function is independent of the plastic strain.

6.12. SUBSEQUENT LOADING SURFACES — ISOTROPIC AND KINEMATIC HARDENING RULES

We have discussed yield surfaces and flow rules thus far. Now we consider the third aspect: the determination of subsequent loading surfaces as plastic flow proceeds, i.e., to determine how the internal variables including the plastic deformation $e_{ij}^{(p)}$ enter into the loading surface

$$(1) \quad f(\sigma_{ij}, T, \xi_i) = 0.$$

Laws governing this aspect are called *hardening rules*.

If we assume that plastic deformation is independent of hydrostatic pressure and that the plastic flow is incompressible, then the yield surfaces in the principal stress space ($\sigma_1, \sigma_2, \sigma_3$) are cylinders (not necessary with circular cross-section) of infinite length with axis $\sigma_1 = \sigma_2 = \sigma_3$ (Fig. 6.7:1). The plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$ is called the π -plane, which is perpendicular to the axis. Hence, the yield surfaces can be represented by their cross sections on this plane (see Sec. 6.7). The cross-sectional curves are closed, convex, and piecewise smooth, but change in size and shape during plastic deformation. For materials involving volume change in the plastic deformation, the yield surface is not parallel to the $\sigma_1 = \sigma_2 = \sigma_3$ axis.

We illustrate several proposed hardening rules in Fig. 6.12:1:

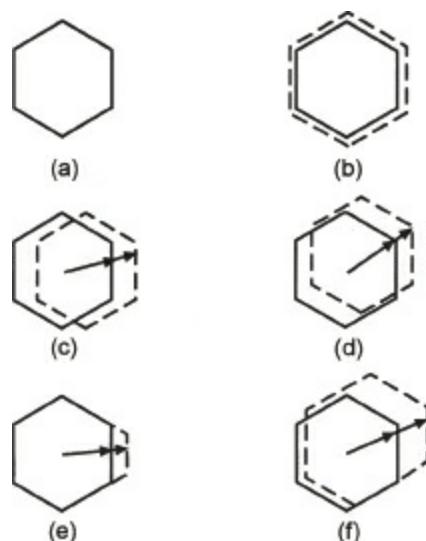


Fig. 6.12:1. Several hardening rules. (a) Initial yield condition (Tresca), (b) isotropic hardening, (c) kinematic hardening (Prager), (d) Kinematic hardening (Ziegler's modification), (e) independently acting plane loading surfaces, (f) interdependent plane loading surfaces. (From Naghdi^{6.3} 1960.)

Isotropic Hardening. Isotropic hardening assumes that the material remains isotropic during plastic loading and that the subsequent yield surface is a uniform expansion of the initial yield surface with the same center. Figure 6.12:1(a) shows a Tresca initial yield surface of a regular hexagon on the π -plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$ and the subsequent expansion of the yield surface [Fig. 6.12:1(b)].

In the general case, if the effect of hydrostatic pressure on yield can be neglected, the subsequent yield surfaces can be expressed in the form

$$(2) \quad f = f^*(J_2, J_3) - \kappa = 0,$$

where κ is an internal variable that characterizes the hardening of the material. The *strain-hardening hypothesis* assumes that κ is a monotonically increasing function, which depends only on the *effective plastic strain* but not the strain path. An effective or equivalent plastic strain, which is an internal variable, can be defined as

$$(3) \quad \epsilon_e^p = \int d\epsilon_e^p,$$

where

$$(4) \blacktriangle \quad d\epsilon_e^p = \sqrt{2de_{ij}^{(p)}de_{ij}^{(p)}/3}.$$

Then κ is a monotonically increasing function of ϵ_e^p . We have

$$(5) \quad \epsilon_e^p = e^p$$

where e^p is the total uniaxial plastic strain for uniaxial loading.

Using Eq. (6.11:4) for $\dot{\epsilon}_{ij}^{(p)}$ and (6.11:9) for $\dot{\Lambda}$ in Eq. (4) lead to

$$(6) \quad \dot{\varepsilon}_e^p = \sqrt{2de_{ij}^{(p)}de_{ij}^{(p)}/3} = \sqrt{2\dot{\Lambda}^2 h_{ij}h_{ij}/3} = \sqrt{2/3}\dot{\Lambda} = \sqrt{2/3}n_{kl}\dot{\sigma}_{kl}/K^p.$$

The *internal variable-rate* defined in Eq. (6.11:7) becomes

$$(6a) \quad \dot{\kappa} = \dot{\xi}_i = \dot{\Lambda}h_i = \varepsilon_e^p d\kappa/d\varepsilon_e^p = \sqrt{2/3}\dot{\Lambda}(d\kappa/d\varepsilon_e^p), \quad h_i = \sqrt{2/3}(d\kappa/d\varepsilon_e^p).$$

For the yield function as defined in Eq. (2), Eq. (6.11:13) gives

$$(7) \quad K^p = -\sqrt{2/3}(\partial f/\partial \kappa)(\partial \kappa/\partial e_e^{(p)})/|\partial f/\partial \sigma_{kl}|,$$

where $\partial f/\partial \kappa = -1$. Once K^p is determined, we can calculate $\dot{\epsilon}_e^p$ and $e_{ij}^{(p)}$ from Eqs. (6) and (6.11:11). The calculation of K^p will be discussed later.

Another commonly used parameter for characterizing isotropic hardening is the *total plastic work* defined by

$$(8) \quad W_p = \int \sigma_{ij}de_{ij}^{(p)} = \int \sigma'_{ij}de_{ij}^{(p)}.$$

Assuming κ as a function of W_p is called the *work-hardening hypothesis*. The function $\kappa(W_p)$ or $\sigma_Y(W_p)$ can also be determined from a uniaxial tension test, in which case we have,

$$(9) \quad K^p = -h_{mn}\sigma'_{mn}(\partial f/\partial \kappa)(\partial \kappa/\partial W_p)/|\partial f/\partial \sigma_{kl}|.$$

For the von Mises yield function $f = \sqrt{J_2} - \kappa$, we have $\kappa(\epsilon_e^p) = \sigma_Y(\epsilon_e^p)/\sqrt{3}$, where $\sigma_Y(\epsilon_e^p)$ is the tensile yield stress. The yield stress can be determined from the uniaxial curve of σ versus $e^p (= \epsilon_e^p)$ in simple tension and compression. For *linear hardening materials*,

$$(10) \quad \sigma_Y = \sigma_Y^0 + E^p\epsilon_e^p$$

where E^p is a material constant, from Eq. (8) we find

$$(11) \quad W^p = (\sigma_Y^0 + E^p\epsilon_e^p/2)\epsilon_e^p.$$

Then

$$(12) \quad \sigma_Y^2 = (\sigma_Y^0)^2 + 2E^p(\sigma_Y^0 + E^p\epsilon_e^p/2)\epsilon_e^p = (\sigma_Y^0)^2 + 2E^pW_p.$$

In this case we have, from Eqs. (7) and (9),

$$\frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial e_e^p} = -\frac{1}{\sqrt{3}} \frac{\partial \sigma_Y}{\partial e_e^p} = -\frac{E^p}{\sqrt{3}}, \quad \frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial W_p} = -\frac{1}{\sqrt{3}} \frac{\partial \sigma_Y}{\partial W_p} = -\frac{E^p}{\sqrt{3}\sigma_Y}.$$

Example of Isotropic Hardening. Consider a thin-walled tube subjected to stretching and torsion, with tensile stress $\sigma(\geq 0)$ and shear stress τ . The Tresca and von Mises yield functions can be expressed in the form $f = \bar{\sigma} - \sigma_Y$, with

$$(13a) \quad \bar{\sigma} = \sigma_1 - \sigma_3 = \sqrt{(\sigma_{zz} - \sigma_{\theta\theta})^2 + 4\sigma_{z\theta}^2} \quad (\text{Tresca}),$$

and (13b):

$$\bar{\sigma} = \sqrt{\frac{(\sigma_{zz} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{rr})^2 + (\sigma_{rr} - \sigma_{zz})^2}{2} + 3(\sigma_{z\theta}^2 + \sigma_{r\theta}^2 + \sigma_{rz}^2)} \quad (\text{von Mises}).$$

For $\sigma_{zz} = \sigma$, $\sigma_{z\theta} = \tau$, $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{rz} = \sigma_{r\theta} = 0$, Eqs. (13a) and (13b) reduces to Eqs. (6.3:12–13). The yield surface becomes

$$f = \bar{\sigma} - \sigma_Y = \sqrt{\sigma^2 + \alpha\tau^2} - \sigma_Y = 0,$$

where $\alpha = 4$ for the case of Tresca, and $\alpha = 3$ for the case of von Mises. As to the flow rule, we obtain

$$(14) \quad \dot{e}_{zz}^p = a\dot{\Lambda}\frac{\partial f}{\partial\sigma_{zz}} = \dot{\Lambda}\frac{\sigma}{b}, \quad \dot{e}_{rz}^p = \dot{e}_{r\theta}^p = 0, \quad \dot{e}_{z\theta}^p = \frac{a\dot{\Lambda}}{2}\frac{\partial f}{\partial\sigma_{z\theta}} = \dot{\Lambda}\frac{\alpha\tau}{2b},$$

where a is a normalization factor, $b = \sqrt{\sigma^2 + \alpha^2\tau^2/2}$ and

$$\begin{aligned} \dot{e}_{\theta\theta}^p &= -\dot{e}_{zz}^p, & \dot{e}_{rr}^p = 0 & \text{for the Tresca criterion,} \\ \dot{e}_{\theta\theta}^p &= \dot{e}_{rr}^p = -\dot{e}_{zz}^p/2, & & \text{for the von Mises criterion.} \end{aligned}$$

Note that the factor 1/2 for $\dot{e}_{z\theta}^p$ in Eq. (14) is due to the fact that we have treated $\sigma_{z\theta}$ and $\sigma_{\theta z}$ as one rather than two independent variables in the yield function.

From Eq. (7), we have

$$K^p = \sqrt{2/3}(d\sigma_Y/de_e^p)\bar{\sigma}/b.$$

Then, $\dot{\Lambda}$ can be derived from Eqs. (6) and (7)

$$\dot{\Lambda} = \sqrt{\frac{3}{2}}\left(\frac{d\sigma_Y}{de_e^p}\right)^{-1}\frac{\sigma\dot{\sigma} + \alpha\tau\dot{\tau}}{\bar{\sigma}} = \sqrt{\frac{3}{2}}\left(\frac{d\sigma_Y}{de_e^p}\right)^{-1}\dot{\bar{\sigma}} = \sqrt{\frac{3}{2}}\dot{e}_e^p$$

where $d\sigma_Y/de_e^p$ is the slope of the uniaxial tension σ - e_p (stress-plastic strain) curve evaluating at $e^p = E^p$. The equivalent plastic strain $e^p = e_e^p$ is obtained from Eqs. (3) and (4),

$$e_e^p = \int de_e^p = \int \sqrt{\frac{2}{3}}d\Lambda = \int \left(\frac{d\sigma_Y}{de_e^p}\right)^{-1}\frac{\sigma d\sigma + \alpha\tau d\tau}{\bar{\sigma}} = \int \left(\frac{d\sigma_Y}{de_e^p}\right)^{-1}d\bar{\sigma}.$$

For linear hardening materials, Eq. (10) gives

$$d\sigma_Y/de^p = E^p.$$

Finally, for isotropic materials, we have

$$\dot{e}_{zz} = \dot{\sigma}/E + \dot{e}_{zz}^p = \dot{\sigma}/E + \dot{\Lambda}\sigma/b, \quad 2\dot{e}_{z\theta} = \dot{\tau}/G + 2\dot{e}_{z\theta}^p = \dot{\tau}/G + \dot{\Lambda}\alpha\tau/b.$$

When the stress path is given, we can obtain the strain history by integrating these equations. This solution is complete because the system is statically determinate that the stress field can be determined directly from the applied load.

Lee and Zaveni^{6.6} (1978) and Chaboche^{6.6} (1977) introduced a nonlinear evolution equation for the isotropic hardening parameter κ :

$$d\kappa = b(\kappa_s - \kappa)d\epsilon_e^p$$

where b and κ_s are material constants. This equation can be integrated to obtain

$$\kappa(\epsilon_e^p) = \kappa_s + (\kappa_0 - \kappa_s)e^{-b\epsilon_e^p}.$$

For cyclic loading, ϵ_e^p increases monotonically and κ approaches κ_s after a number of cycles. The number of cycles required for κ to reach the steady state value depends on the cyclic strain magnitude.

Kinematic Hardening. The kinematic hardening model assumes that during plastic loading the yield surface translates in the stress space without rotation and without change in size and shape. Prager illustrates in Fig. 6.12:1(c) *kinematic hardening* of an initial yield surface translated in the π -plane. Prager^{6.3} (1955) used a mechanical model shown in Fig. 6.12:2 to explain this rule. The initial yield surface is regarded as a planar rigid frame lying on the π -plane. The loading point on the π -plane is represented by a frictionless pin. If the pin engages the frame, it may push the frame around. Under the frictionless assumption, any motion imparted by the pin to the frame must be normal to the edge in contact. However, when a corner of the frame is caught by the pin, the pin may carry the frame in the direction lying within a certain angle. Rotation of the frame is supposed to be prevented by some mechanism. If the pin disengages and

moves away from the frame, the frame stays put, and the change represents an unloading. It is obvious that none of the flow and hardening rules deduced from Drucker's hypothesis is violated. In fact, the Bauschinger effect is represented very simply, and the development of anisotropy due to plastic deformation appears most naturally.

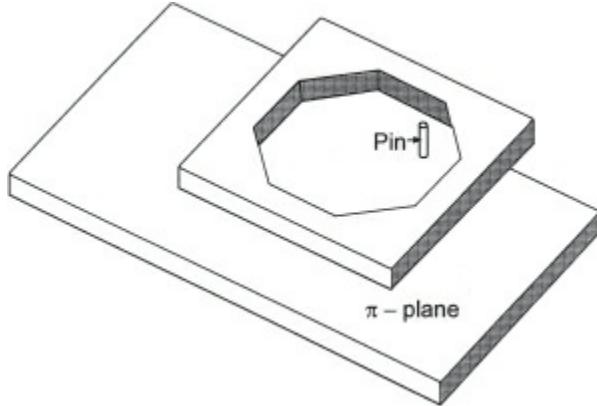


Fig. 6.12:2. A mechanical model used in explaining the kinematic hardening rule.

With some variation, Prager^{6.3} (1955) was able to represent various models of plasticity: rigid, perfectly plastic, rigid work-hardening elastic, etc. Almost simultaneously, Ishlinskii^{6.3} introduced a similar concept independently. Further developments were made by Prager,^{6.3} Boyce,^{6.3} Hodge^{6.3} and others. Hodge^{6.3} (1956) points out that the concept of kinematic hardening can be applied in the nine-dimensional stress space.

Suppose that the initial yield surface is $f(\sigma_{ij}) = 0$. The subsequent yield surface can be expressed as

$$(15) \quad f(\sigma_{ij} - \alpha_{ij}) = 0,$$

where α_{ij} , usually called the *back stress*, represents the translation of the center of the initial yield surface during the process of plastic loading. The formulation of a kinematic hardening model is to specify the evolution of α_{ij} in terms of $\dot{\epsilon}_{ij}^{(p)}$, σ_{ij} , and/or α_{ij} itself. Prager's linear kinematic hardening model assumes that

$$(16) \quad \dot{\alpha}_{ij} = c\dot{\epsilon}_{ij}^{(p)} = c\dot{\Lambda}h_{ij}$$

where c is a constant (Shield and Ziegler,^{6.3} 1958). According to this model the yield surface moves in the direction of the plastic strain rate, i.e., the direction normal to the yield surface at the loading point. From Eqs. (15), (16), (6.11:4) and (6.11:9) with $\dot{\Lambda}h_i(\partial f/\partial \xi_i) = (\partial f/\partial \alpha_{ij})\dot{\alpha}_{ij}$ we have

$$(17) \quad K^p = ch_{ij}n_{ij}.$$

If the material obeys the associated flow rule, $h_{ij} = n_{ij}$, we have $K_p = c$.

Prager's linear kinematic model does not give consistent results for 2-dimensional and 3-dimensional cases. It also introduces a transverse softening in simple tension or compression as shown below. In a uniaxial tension test with load in the 1-direction, the plastic strain increments are

$$(18) \quad de_1^{(p)} = de^p, \quad de_2^{(p)} = de_3^{(p)} = -de^p/2.$$

From Eq. (16), we obtain

$$(19) \quad d\alpha_1 = cde^p, \quad d\alpha_2 = d\alpha_3 = -d\alpha_1/2 = -cde^p/2.$$

The yield surface moves toward the negative 2- and 3-directions causing transverse softening, although no load is applied along these directions.

Ziegler^{6.3} (1959) modified Prager's rule Eq. (16) by replacing it with

$$(20) \quad \dot{\alpha}_{ij} = \dot{\mu}(\sigma_{ij} - \alpha_{ij})$$

where $\dot{\mu} > 0$. Geometrically, this means the yield surface moves along the direction of $\sigma_{ij} - \alpha_{ij}$, which is the radial vector joining the instantaneous center α_{ij} and the loading point σ_{ij} . Ziegler's modified kinematic hardening is shown in Fig. 6.12:1(d). Determining $\dot{\mu}$ would involve the yield criterion.

Armstrong and Frederick (1966) introduced a nonlinear term in Prager's model Eq. (16),

$$(21) \quad \dot{\alpha}_{ij} = c\dot{e}_{ij}^p - \gamma\alpha_{ij}\dot{e}_e^p = \dot{\Lambda}(ch_{ij} - \sqrt{2/3}\gamma\alpha_{ij}).$$

The use of Eq. (21) as the internal variable-rate and

$$\partial f/\partial\alpha_{ij} = -\partial f/\partial\sigma_{ij},$$

reduce Eq. (6.11:9) to

$$(22) \quad K^p = (ch_{ij} - \sqrt{2/3}\gamma\alpha_{ij})n_{ij}.$$

This model was further developed by Chaboche^{6.6} (1977, 1986).

Combined Isotropic and Kinematic Hardening. The yield surface of a combined isotropic and kinematic hardening model can be expressed in the form

$$(23) \quad f(\sigma_{ij} - \alpha_{ij}, \kappa) = 0,$$

where α_{ij} and κ are internal variables. For $\dot{\alpha}_{ij}$ in the form of Eq. (21) and κ as a function of the equivalent plastic strain, Eq. (6.11:9) gives

$$(24) \quad K^p = ch_{ij}n_{ij} - \sqrt{2/3}[\alpha_{ij}n_{ij}\gamma + (\partial f/\partial\kappa)(\partial\kappa/\partial\epsilon_e^{(p)})/|\partial f/\partial\sigma_{ij}|].$$

If the material obeys the associated flow rule, $h_{ij} = n_{ij}$, Eq. (24) becomes

$$(25) \quad K^p = c - \sqrt{2/3}[\alpha_{ij}n_{ij}\gamma + (\partial f/\partial\kappa)(\partial\kappa/\partial\epsilon_e^{(p)})/|\partial f/\partial\sigma_{ij}|].$$

From Eqs. (6.11:3), (6.11:11) and (6.11:14), reproduced here for clarity,

$$n_{ij} = \frac{\partial f}{\partial\sigma_{ij}} / \left| \frac{\partial f}{\partial\sigma_{kl}} \right|, \quad \dot{e}_{ij}^p = \frac{n_{ij}n_{kl}\dot{\sigma}_{kl}}{K^p}, \quad d\epsilon_{ij} = (D_{kl}^{-1} + \frac{n_{kl}n_{ij}}{K^p})d\sigma_{kl},$$

we can determine the plastic strain rate and incremental stress strain relation.

More complicated hardening rules involve translation, expansion and distortion of the yield surface simultaneously. Plastic deformation can cause a linear segment to move [Fig. 6.12:1(e)], or change the loading surface in some interdependent manner [Fig. 6.12:1(f)], and include simultaneous expansion of the yield surface (Hodge^{6.3} 1957). Budiansky^{6.3} (1959) considered the possibility of creating corners in subsequent yield surfaces at the point of loading. See Hill^{6.2}, Nadai^{6.2} and papers by Naghdi, Phillips, etc., in Biblos. 6.2–6.6.

Example of Combined Isotropic and Kinematic Hardening. For the von Mises criterion, the yield surface is

$$f = (\sigma'_{ij} - \alpha'_{ij})(\sigma'_{ij} - \alpha'_{ij})/2 - \sigma_Y^2/3 = 0,$$

where σ'_{ij} and α'_{ij} are the deviators of σ_{ij} and α_{ij} , respectively. Then

$$\frac{\partial f}{\partial\kappa} \frac{d\kappa}{d\epsilon_e^p} = -\frac{2\sigma_Y}{3} \frac{d\sigma_Y}{d\epsilon_e^p}, \quad \left| \frac{\partial f}{\partial\sigma_{ij}} \right| = |\sigma'_{ij} - \alpha'_{ij}| = \sqrt{\frac{2}{3}}\sigma_Y, \quad n_{ij} = \sqrt{\frac{3}{2}} \frac{\sigma'_{ij} - \alpha'_{ij}}{\sigma_Y},$$

and

$$(26) \quad d\epsilon_e^p = \sqrt{2/3}n_{ij}d\sigma_{ij}/K^p = (\sigma'_{ij} - \alpha'_{ij})d\sigma_{ij}/(K^p\sigma_Y),$$

Eq. (25) becomes

$$(27) \quad K^p = c - \gamma(\sigma'_{ij} - \alpha'_{ij})\alpha'_{ij}/\sigma_Y + 2(d\sigma_Y/d\epsilon_e^p)/3.$$

Finally we obtain the strain-stress relation for the von Mises criterion with combination of isotropic and kinematic hardening,

$$d\epsilon_{ij} = [D_{ijkl}^{-1} + 3(\sigma'_{ij} - \alpha'_{ij})(\sigma'_{kl} - \alpha'_{kl})/(2K^p\sigma_Y^2)]d\sigma_{kl}.$$

We shall illustrate the determination of the yield stress σ_Y (as functions of ϵ_e^p) and the material constants (c, γ) from data measured from uniaxial tests for the von Mises criterion. For uniaxial tension in the 1-direction, on the yield surface, we have with $d\epsilon^p = d\epsilon_e^p$,

$$(28) \quad \begin{aligned} de_{11}^p &= de^p, & de_{22}^p = de_{33}^p &= -de^p/2, & de_e^p &= |de^p| = |\sigma_Y|, \\ \alpha'_{11} &= 2\alpha/3, & \alpha'_{22} = \alpha'_{33} &= -\alpha/3, & \sigma_{11} &= \sigma, \\ \sigma_{22} = \sigma_{33} &= 0, & \sigma'_{11} - \alpha'_{11} &= 2(\sigma - \alpha)/3 = \pm 2\sigma_Y/3, \\ \sigma'_{22} - \alpha'_{22} &= \sigma'_{33} - \alpha'_{33} = -(\sigma - \alpha)/3 = \mp \sigma_Y/3. \end{aligned}$$

All other components of e_{ij} , σ_{ij} and α_{ij} are zero. The upper and lower signs in the last two equations of Eq. (28) correspond to loading (tensile plastic flow) and reverse loading (compressive plastic flow), respectively. The yield function can be simplified to become

$$|\sigma - \alpha| = \sigma_Y.$$

From Eqs. (21), (26) and (27), it follows that

$$(29) \quad d\alpha = 3cde^p/2 - \gamma\alpha|de^p|,$$

$$(30) \quad K^p de_e^p = 2d\sigma/3,$$

$$(31) \quad d\sigma/de_e^p = 3K^p/2 = 3c/2 \mp \gamma\alpha + d\sigma_Y/de_e^p.$$

Equation (31) shows the different effect of the nonlinear term $\gamma\alpha$ on the tensile and compressive plastic flow. Since $\gamma > 0$, the reverse plastic flow has a higher hardening modulus.

Equation (31) reduces to Prager's linear kinematic hardening if $\gamma = (d\sigma_Y/d\epsilon_e^{(p)}) = 0$. In this case, the yield function is simply

$$(32) \quad \sigma = \alpha \pm \sigma_Y = ce^p \pm \sigma_Y.$$

In general, Eq. (29) can be integrated to give

$$(33) \quad \alpha(e^p) = [\pm 3c + (2\gamma\alpha_0 \mp 3c)e^{\mp\gamma(e^p - e_0^p)}]/(2\gamma).$$

The yield condition becomes

$$(34) \quad \sigma = \alpha(e^p) \pm \sigma_Y = [\pm 3c + (2\gamma\alpha_0 \mp 3c)e^{\mp\gamma(e^p - e_0^p)}]/(2\gamma) \pm \sigma_Y,$$

where the upper and lower signs are, respectively, for tensile and compressive loadings that makes e^p change to e_0^p .

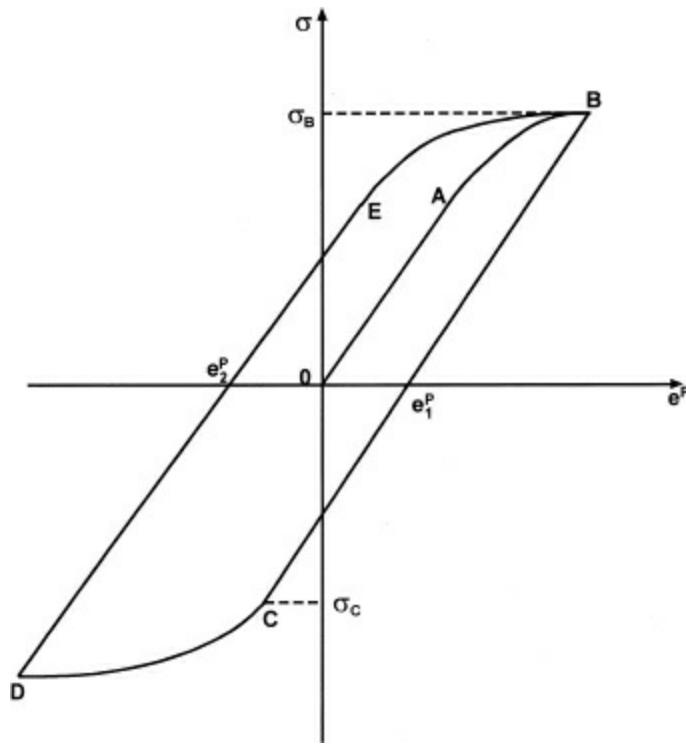


Fig. 6.12:3. A schematic loading path for evaluation of c , γ and σ_Y .

Equations (32) or (34) can be used to determine the material constants (c , γ) and the yield stress σ_Y as function of e_e^p . Consider a load path OABCDE shown in Fig. 6.12:3, where unloading occurs at points B and D, and yielding starts at

points A, C and E. Equation (34) gives

$$(35) \quad \sigma_B = \alpha(e_1^p) + \sigma_Y(e_1^p), \quad \sigma_E = \alpha(e_2^p) + \sigma_Y(e_2^p),$$

for the tensile loading curve EB,

$$(36) \quad \sigma_C = \alpha(e_1^p) - \sigma_Y(e_1^p), \quad \sigma_D = \alpha(e_2^p) - \sigma_Y(e_2^p),$$

for the compressive loading curve CD, where e_1^p and e_2^p are the values of the plastic strain along BC and DE, respectively, and the subscripts B, C, D and E denote the value of σ at those points. Similarly from Eq. (33), along OAB $\alpha_0 = e_0^p = 0$ and along BCD $\alpha_0 = \alpha(e_1^p)$, $e_0^p = e_1^p$, we find

$$(37) \quad \alpha(e_1^p) = 3c(1 - e^{-\gamma e_1^p})/(2\gamma),$$

$$(38) \quad \begin{aligned} \alpha(e_2^p) &= -3c/(2\gamma) + [\alpha(e_1^p) + 3c/(2\gamma)]e^{\gamma(e_2^p - e_1^p)} \\ &= 3c[-1 + (2 - e^{-\gamma e_1^p})e^{\gamma(e_2^p - e_1^p)}]/(2\gamma). \end{aligned}$$

The following relations are derived for c , γ from Eqs. (35)–(38),

$$(39) \quad 3c(1 - e^{-\gamma e_1^p})/(2\gamma) = (\sigma_B + \sigma_C)/2,$$

$$(40) \quad 3c[-1 + (2 - e^{-\gamma e_1^p})e^{\gamma(e_2^p - e_1^p)}]/(2\gamma) = (\sigma_D + \sigma_E)/2.$$

One then determines, from Eqs. (31) and (33) with $\alpha_0 = e_0^p = 0$,

$$(41) \quad d\sigma_Y/de_e^p = (d\sigma/de^p)|_{e^p=e_e^p} - 3ce^{-\gamma e_e^p}/2$$

as a function of e_e^p where $\frac{d\sigma}{de^p}|_{e^p=e_e^p}$ is the measured curve AB. Recall that e^p is the plastic strain measured in the uniaxial tension test. Equation (41) is used to evaluate K^p in Eq. (25) or Eq. (27) for multi-axial loading analyses.

In the derivation c and γ are assumed to be constant. In the more general formulation, c and γ can be functions of plastic deformation.

6.13. MROZ'S, DAFALIAS AND POPOV'S, AND VALANIS' PLASTICITY THEORIES

Hardening is one of the most important mechanical properties in plasticity. The theories in Secs. 6.3–6.12 characterize such behavior by the evolution of *hardening parameters* (e.g., κ and α_{ij}), including the isotropic hardening and three types of kinematic hardening (Prager, Ziegler, and Armstrong and Frederick). These models offer a reasonable description of hardening for monotonically proportional loading without unloading. The material behaviors under cyclic loading are much too complex to be modeled by these hardening rules. Based on experimental evidence, Drucker and Palgen (1981) identified five basic features of cyclic plasticity:

- (1) Plastic strain accumulates, the *cycle creep or ratcheting effect*, in the direction of mean stress for stress cycles.
- (2) The mean stress progressively relaxes to zero for strain cycles with nonzero mean.
- (3) The elastic-plastic transition is usually smooth.
- (4) Material hardens or softens toward a stabilized state with kinematic hardening only.
- (5) Extensive plastic loading eliminates most, if not all, of the past effects.

It can be shown, under strain cycles, that isotropic hardening will lead to an elastic state eventually and thus is not capable of simulating the hysteresis loop. Prager's and Ziegler's kinematic hardening models cannot simulate the ratcheting effect.

We shall focus on cyclic loading reversal in the following subsections.

Mroz's Multisurface Model. Mroz^{6,6} (1969) introduced a multi-surface model to describe the cyclic effect and the nonlinearity of stress-strain loops.

The model uses a number of yield surfaces

$$(1) \quad f_i(\sigma_{kl} - \alpha_{kl}, \kappa_i) = 0 \quad i = 0, 1, 2, \dots, n \text{ (not summed)}$$

to simulate nonlinear hardening and smooth transition from elastic to plastic deformation. Each of the yield surfaces has its unique plastic modulus κ_i and back stress α_{ij} to represent a combination of linear isotropic hardening and kinematic hardening. For a virgin material, the yield surfaces are concentric and similar of different size. Mroz approximates the nonlinear stress-strain curve of plastic deformation as piecewise linear. During loading f_0 yields first. As loading continues, f_0 expands and translates while other surfaces remain stationary. When f_0 touches f_1 at a point A, f_0 and f_1 will move in unison for further loading if the load point stays at point A. The plastic deformation continues until f_1 touches f_2 at the same contact point. When f_0, f_1 , and f_2 are in contact, they will move together under further loading if the load point remains at A. This process continues. If the load point moves away from A, but remain on the surface $f_0 = 0$, then f_0 may move away from f_1 for further loading. In this case, f_1, f_2 will remain stationary until f_0 touches f_1 again. Then the process will proceed as just described. Note that upon unloading, the loading vector moves away or is already away from f_0 , and all yield surfaces remain stationary. This means that unloading occurs simultaneously for all yield surfaces and that the contacted surfaces remain in contact. During reloading f_0 will yield first again and the process will continue as described before.

With a sufficient number of yield surfaces, Mroz's model can describe nonlinear stress-strain loops, the Bauschinger effect, and cyclic hardening or softening of the material. The model requires extensive computational effort in implementation. For more details interested readers are referred to Mroz (1969, 1976), Mroz *et al.* (1979).

Dafalias and Popov's Two-Surface Model. Mroz's model cannot simulate ratcheting and mean stress relaxation in strain cyclic loading with nonzero mean. Uniaxial experimental data indicate that, even under complex loading, the stress strain curves tend to converge with definite bounding lines. This suggests the existence of *bounding surface(s)*. The plastic modulus $K^p (= \frac{d\sigma}{de^p})$ depends on the distance from the stress-strain curve to the bounding line as well as the recent loading history. Dafalias and Popov^{6,6} (1975) proposed a two-surface model to account for these effects. One surface is similar to the conventional yield surface and the other is a bounding surface. For uniaxial loading, the *yield function* f and *bounding function* \bar{f} are in the following forms

$$(2) \quad f = (\sigma - \alpha)^2 - \sigma_Y^2 = 0, \quad \bar{f} = (\bar{\sigma} - \bar{\alpha})^2 - \bar{\sigma}_Y^2 = 0,$$

where σ and $\bar{\sigma}$ are the stresses on f and \bar{f} , α and $\bar{\alpha}$ their centers, while σ_Y and $\bar{\sigma}_Y$ their respective sizes. Their incremental relations are

$$(3) \quad de^p = d\sigma/K^p, \quad d\bar{\sigma} = \bar{K}^p de^p = \bar{K}^p d\sigma/K^p \quad (\text{flow rule}),$$

$$(4) \quad d\alpha = K^\alpha de^p = K^\alpha d\sigma/K^p, \quad d\bar{\alpha} = \bar{K}^\alpha de^p = \bar{K}^\alpha d\sigma/K^p \quad (\text{kinematic hardening}).$$

Using the *consistency equation* $df = 0$ and $d\bar{f} = 0$, we obtain

$$(5) \quad K^\alpha = K^p - d\sigma_Y/de^p, \quad \bar{K}^\alpha = \bar{K}^p - d\bar{\sigma}_Y/de^p.$$

Dafalias and Popov^{6,6} assumed that K_p varies continuously, as opposed to the piecewise-constant plastic moduli in Mroz's model, in the form

$$(6) \quad K^p = K^p(\delta_{in}, \delta)$$

where $\delta = |\bar{\sigma} - \sigma|$ is the distance between the present state σ on the yield surface and the stress state $\bar{\sigma}$ on the bounding surface, with the same outward normal. The value of δ at the initiation of a new loading process is δ_{in} measuring how far the material state is from the state represented by the bounds. This is used to reflect the effect of loading history. To specify K_p , Dafalias and Popov^{6,6} (1976) proposed that

$$(7) \quad K^p(\delta_{in}, \delta, \epsilon_e^p) = \bar{K}^p(\epsilon_e^p) + a\delta / [(1 + b\delta_{in})(\delta_{in} - \delta)],$$

where a and b are constant and \bar{K}^p is the limiting value of the bounds defined as $\partial\bar{\sigma}_Y/\partial e^p$. These quantities are determined experimentally. Note that $\bar{K}^p(\delta_{in}, \delta, \epsilon_e^p) = \infty$ is specified to give a smooth elastic plastic transition. We can now calculate de^p , $d\alpha$ and $d\bar{\alpha}$ for a given $d\sigma$ if the isotropic hardening characteristics are $\partial\sigma_Y/\partial e^p$ and $\partial\bar{\sigma}_Y/\partial e^p$.

To generalize into the multiaxial case, the yield surface f and the bounding surface \bar{f} are written in the following forms:

$$(8) \quad f(\sigma_{ij} - \alpha_{ij}, \kappa) = F(\sigma_{ij} - \alpha_{ij}) - \kappa = 0,$$

$$(9) \quad \bar{f}(\bar{\sigma}_{ij} - \bar{\alpha}_{ij}, \bar{\kappa}) = \bar{F}(\bar{\sigma}_{ij} - \bar{\alpha}_{ij}) - \bar{\kappa} = 0,$$

where $\bar{\sigma}_{ij}$ is a stress on the bounding surface corresponding to the stress state σ_{ij} on the yield surface. When unloading occurs, σ_{ij} and $\bar{\sigma}_{ij}$ move away from the yield and bounding surfaces in unison. If f and \bar{f} are similar, then $\bar{\sigma}_{ij} - \bar{\alpha}_{ij} = \eta(\sigma_{ij} - \alpha_{ij})$. The flow rules are

$$(10) \quad d\epsilon_{ij}^p = n_{ij}n_{kl}d\sigma_{kl}/K^p = n_{ij}n_{kl}d\bar{\sigma}_{kl}/\bar{K}^p$$

which gives the relation between $d\sigma_{ij}$ and $d\bar{\sigma}_{ij}$. The model assumes that

$$d\bar{\alpha}_{ij} - d\alpha_{ij} = (\bar{\sigma}_{ij} - \sigma_{ij})d\mu, \quad \Lambda \partial f/\partial \sigma_{ij} = \bar{\Lambda} \partial \bar{f}/\partial \sigma_{ij}.$$

Using Eq. (7), the flow rules (10), and the consistency equations derived from Eqs. (8) and (9), we can express $d\bar{\sigma}_{ij}$, $d\mu$, $d\alpha_{ij}$, $d\bar{\alpha}_{ij}$ in terms of $d\sigma_{ij}$, \bar{K}^p , $d\bar{\kappa}/d\epsilon_e^p$ and $d\kappa/d\epsilon_e^p$. The last three quantities are usually determined by experiments. In Eq. (7), $\delta = |\bar{\sigma}_{ij} - \sigma_{ij}|$. The two-surface model is smooth at the elastic-plastic transition point, but produces an inconsistency in the uniaxial load-unload-reload situation if the unloading step is very small (Chaboche 1986). This model, like the model before, requires extensive computational effort in implementation. Interested readers are referred to papers by Dafalias and Popov (1975, 1976), Dafalias (1984), and Chaboche (1986).

Valanis' Endochronic Theory. The plasticity theories discussed so far are based on the concept that the yield surface divides the elastic and plastic domains with a sharp demarcation. However, many materials do not exhibit a sharp yield point. Some materials such as aluminum and stainless steel have a nonlinear uniaxial stress-strain curve almost from the start. Valanis^{6,6} (1971, 1980) proposed a plastic theory in a convolution integral form that the present state of the material depends on the present values and the past history of observable variables. This gives rise to a hereditary theory, called the *endochronic theory*, which assumes

$$(11) \quad \sigma'_{ij} = 2G \int_0^z \rho(z-z')(d\epsilon_{ij}^p/dz')dz',$$

in which the yield surface is not *a priori* defined but rather a derivable result of the theory. The kernel function ρ is a material function of the intrinsic time that

$$(12) \quad dz = d\zeta/f(\zeta) \quad \text{with} \quad d\zeta = |d\epsilon_{ij}^p| \quad \text{and} \quad f(0) = 1,$$

where $f(\zeta)$ is a nonnegative function, called the intricate time scale.

Let

$$(13) \quad \rho(z) = \rho_0\delta(z) + \rho_1(z),$$

where $\delta(z)$ is the Dirac delta function, ρ_0 is a material constant and $\rho_1(z)$ is a nonsingular function. Substituting Eqs. (12) and (13) into Eq. (11) gives

$$(14) \quad \sigma'_{ij} = S_Y^0 f(\zeta)(d\epsilon_{ij}^p/d\zeta) + \alpha_{ij},$$

where

$$(15) \quad S_Y^0 = 2G\rho_0,$$

$$(16) \quad \alpha_{ij} = 2G \int_0^z \rho_1(z-z') \frac{d\epsilon_{ij}^p}{dz'} dz'.$$

The assumption of the kernel function $\rho(z)$ in the form of Eq. (13) effectively divides the constitutive equation, Eq. (14) in two parts: the part represented by the first term on the right hand side of the equation depends on the present values of observable variables, such as S_Y^0 , and a set of internal-state variables, which characterize both S_Y^0 and $d\epsilon_{ij}^p/d\zeta$. The second term α_{ij} is equivalent to the back stress introduced previously, which depends on the past histories of these variables.

From Eqs. (12) and (14), we obtain

$$(17) \quad (\sigma'_{ij} - \alpha_{ij})(\sigma'_{ij} - \alpha_{ij}) - [S_Y^0 f(\zeta)]^2 = 0,$$

$$(18) \quad d\epsilon_{ij}^p = (\sigma'_{ij} - \alpha_{ij})d\zeta/[S_Y^0 f(\zeta)].$$

Note that

$$(19) \quad (\sigma'_{ij} - \alpha_{ij})d\epsilon_{ij}^p = S_Y^0 f(\zeta) d\zeta > 0.$$

Equations (17) and (18) correspond to the yield function and the flow rule, respectively. Equation (17) shows the characteristics of combined isotropic and kinematic hardening. If $f(\zeta) = 1$, Eq. (17) corresponds to the von Mises yield surface. In this case, the plastic strain increment de_{ij}^p is normal to the yield surface.

Different plasticity theories can be derived through different choices of the kernel function $\rho_1(z)$ and the intrinsic time scale $f(\zeta)$. Valanis (1980) proved that general form for $\rho_1(z)$ is given by

$$(20) \quad \rho_1(z) = \sum_{j=1}^{\infty} \rho_{1j}(z) \exp(-\xi_j z),$$

where ρ_{1j} , ξ_j are material constants to be determined experimentally. Two commonly used forms of $f(\zeta)$ for isotropic hardening are

$$(21) \quad f(\zeta) = 1 + \gamma\zeta,$$

and

$$(22) \quad f(\zeta) = \alpha + (1 - \alpha)e^{-\beta\zeta},$$

where α , β , γ , are material constants. The second form reaches a limit asymptotically as ζ increases and is therefore called the *saturated form*. Many features of the endochronic model are described in detail in Watanabe and Atluri^{6,6} (1986).

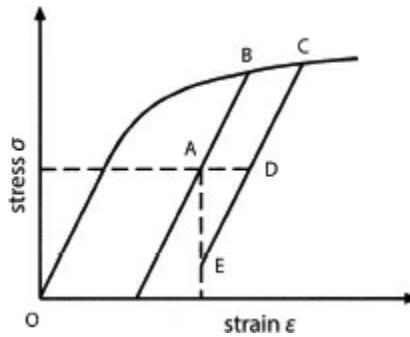


Fig. 6.14:1. Closed strain and stress cycles.

6.14. STRAIN SPACE FORMULATIONS

In the remaining chapter, Gibbs dyadic notations will be used to simply writing. Consider tensors $\mathbf{a} (= a_{ij})$ and $\mathbf{B} (= B_{ijkl})$. Then $\mathbf{a} : \mathbf{a} = a_{ij} a_{ij}$ and $\mathbf{B} : \mathbf{B} = B_{ijkl} B_{ijkl}$ are scalar, while $\mathbf{B} : \mathbf{a} = B_{ijkl} a_{kl}$ and $\mathbf{a} : \mathbf{B} = a_{ij} B_{ijkl}$ are second order tensors. We have thus far discussed the plasticity theories formulated in the stress space only. The approach has two inherent drawbacks: For ideal plastic materials, a different approach is needed to derive the incremental stress-strain relation; and for softening materials, confusion can arise between further plastic loading and elastic unloading (Naghdi and Trapp 1975a, 1975b). Il'iushin (1961) introduced a strain space formulation to avoid these drawbacks. Similar to Drucker's postulate, Il'iushin hypothesized that the work done by external forces on a material over a closed strain cycle C_ϵ is nonnegative,

$$W = \int_{C_\epsilon} \sigma : d\epsilon \geq 0,$$

with $\sigma (= \sigma_{ij})$ and $\epsilon (= e_{ij})$. In other words, plastic deformation occurs during a closed strain cycle if the work of external forces over the cycle is positive. A closed strain cycle is defined as a closed path in the strain space that the strain starts from an equilibrium and compatible state ϵ_0 , experiences a change due to external loading, and then reverts to ϵ_0 . At the end of the cycle the stress may differ from that at the initial state. As seen in Fig. 6.14:1, the work done (area ABCDE) by the external forces over a closed strain cycle is larger than that (area ABCD) over a closed stress cycle. The two cycles are the same only if no plastic deformation occurs.

Let $\sigma_e^p (= \mathbf{D} : \mathbf{e}^p)$ denote the *equivalent plastic stress tensor*. We have

$$(1) \quad \sigma = \mathbf{D} : \epsilon^e = \mathbf{D} : (\epsilon - \epsilon^p) = \mathbf{D} : \epsilon - \sigma_e^p,$$

where σ , ϵ , ϵ^e , ϵ^p are the stress, the total strain, the elastic strain, and the plastic strain tensors, respectively, and \mathbf{D} is the fourth-ordered elastic stiffness tensor. In the incremental form with $d\sigma^p = \mathbf{D} : de^p$, then

$$d\sigma = \mathbf{D} : (d\varepsilon - d\varepsilon^p) + (d\mathbf{D}) : \varepsilon^e = \mathbf{D} : d\varepsilon - d\sigma^p + (d\mathbf{D}) : \varepsilon^e.$$

Based on the hypothesis, Il'iushin showed that the vector $d\sigma^p - d\mathbf{D} : \varepsilon^e$ is normal to the yield surface with

$$(2) \quad d\sigma^p - d\mathbf{D} : \varepsilon^e = \mathbf{D} : d\varepsilon - d\sigma = \eta(\partial\hat{f}/\partial\varepsilon) = \hat{\Lambda}\hat{\mathbf{n}}$$

If the elastic stiffness tensor is independent of deformation, we have

$$(3) \quad d\sigma^p = \eta(\partial\hat{f}/\partial\varepsilon) = \hat{\Lambda}\hat{\mathbf{n}}, \quad \text{and} \quad \hat{\Lambda} = \eta|\partial\hat{f}/\partial\varepsilon|$$

where $f(\sigma, \alpha, \kappa) = \hat{f}(\varepsilon, \sigma_e^p, \alpha_e, \kappa) = 0$ is the *yield surface*, η , $\hat{\Lambda}$ *scalar factors*, $\hat{\mathbf{n}}$ is a unit normal to the yield surface defined as are positive

$$(4) \quad \hat{\mathbf{n}} = (\partial\hat{f}/\partial\varepsilon)/|\partial\hat{f}/\partial\varepsilon|,$$

α_e is the *back strain tensor* governing kinematic hardening, while κ is a parameter associated with isotropic hardening. Equation (3) is the *flow rule* and called the *normality condition in the strain space formulation* equivalent to Eq. (6.11:4) in the stress space formulation. Equation (2) or (3) provide a form of constitutive equation for σ^p , and in turn for ε^p .

The yield function forms a closed surface in strain space enclosing an elastic region with the following features.

- No change in plastic deformation occurs as long as $\hat{f} < 0$.
- Change in plastic deformation occurs when $\hat{f} = 0$.
- No meaning is associated with $\hat{f} > 0$.

Assuming constant \mathbf{D} with $d\sigma^p = \mathbf{D} : d\varepsilon^p$, we obtain the yield surfaces by substituting Eq. (1) into the yield function in stress space:

$$(5) \quad f(\sigma, \alpha, \kappa) = f(\mathbf{D} : \varepsilon - \sigma^p, \alpha, \kappa) = \hat{f}(\varepsilon, \sigma^p, \alpha_e, \kappa) = 0$$

where $\alpha_e = \mathbf{D}^{-1} : \alpha$, \mathbf{D}^{-1} is the inverse of \mathbf{D} , and α_e and κ can be functions of σ_e^p . For linear kinematic hardening,

$$(6) \quad d\alpha_e = \mathbf{g} : d\sigma^p$$

where \mathbf{g} is a fourth-ordered tensor. If κ is a function of the *equivalent plastic stress* σ_e^p , the *increment* is

$$(7) \quad d\sigma_e^p = \sqrt{2d\sigma_{ij}^p d\sigma_{ij}^p / 3} = \sqrt{2/3}|d\sigma_{kl}^p| = \sqrt{2(\hat{\Lambda}\hat{n}_{ij})(\hat{\Lambda}\hat{n}_{ij})/3} = \sqrt{2/3}\hat{\Lambda},$$

then

$$(8) \quad d\kappa = (d\kappa/d\sigma_e^p)d\sigma_e^p = \sqrt{2/3}(d\kappa/d\sigma_e^p)\hat{\Lambda}.$$

If $\mathbf{g} = c\mathbf{D}^{-1} : \mathbf{D}^{-1}$, Eq. (6) reduces to Prager's model given in Eq. (6.12:16). For isotropic materials, $[\mathbf{D}^{-1} : \mathbf{D} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2]$

$$\begin{aligned} D_{ijkl} &= 2G[(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2 + \nu\delta_{ij}\delta_{kl}/(1-2\nu)], \\ D_{ijkl}^{-1} &= [(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2 - \nu\delta_{ij}\delta_{kl}/(1-2\nu)]/(2G) \\ g_{ijkl} &= c[(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2 + \nu(2-\nu)\delta_{ij}\delta_{kl}/(1+\nu)^2]/4G^2, \\ (\alpha_e)_{ij} &= c\sigma_{ij}^p/4G^2, \quad \sigma_{ij}^p = 2G\varepsilon_{ij}^p \end{aligned}$$

The *consistency equation* can be derived from Eq. (5), i.e.,

$$(9) \quad \blacktriangle \quad d\hat{f} = \frac{\partial\hat{f}}{\partial\varepsilon} : d\varepsilon + \frac{\partial\hat{f}}{\partial\sigma^p} : d\sigma^p + \frac{\partial\hat{f}}{\partial\alpha_e} : d\alpha_e + \frac{\partial\hat{f}}{\partial\kappa} d\kappa = 0,$$

A substitution of Eqs. (6) through (8) into Eq. (9) yields

$$(10) \quad \hat{\Lambda} = \hat{\mathbf{n}} : d\varepsilon / \hat{L}^p,$$

where

$$(11) \quad \hat{L}^p = -\left(\frac{\partial \hat{f}}{\partial \sigma^p} : \hat{n} + g : \frac{\partial \hat{f}}{\partial \alpha_e} : \hat{n} + \sqrt{\frac{2}{3} \frac{\partial f}{\partial \kappa} \frac{d\kappa}{d\sigma_e^p}}\right) / \left|\frac{\partial \hat{f}}{\partial \varepsilon}\right|.$$

The *flow rule* Eq. (3) can now be written as

$$(12) \quad \Delta \quad d\sigma^p = (\hat{n} : d\varepsilon)\hat{n}/\hat{L}^p \quad \text{or} \quad d\varepsilon^p = (\hat{n} : d\varepsilon)\mathbf{D}^{-1} : \hat{n}/\hat{L}^p.$$

The *incremental stress-strain relation* becomes

$$(13) \quad \Delta \quad d\sigma = (\mathbf{D} - \hat{n}\hat{n}/\hat{L}^p) : d\varepsilon.$$

Using the yield surface Eq. (5) and noting that

$$\frac{\partial \hat{f}}{\partial \varepsilon} = \mathbf{D} : \frac{\partial f}{\partial \sigma}, \quad \frac{\partial \hat{f}}{\partial \sigma^p} = -\frac{\partial f}{\partial \sigma}, \quad \frac{\partial \hat{f}}{\partial \alpha_e} = \mathbf{D} : \frac{\partial f}{\partial \alpha} = -\mathbf{D} : \frac{\partial f}{\partial \sigma},$$

we can express quantities *in strain space* in term of those *in stress space*:

$$(14) \quad \hat{n} = \mathbf{D} : \mathbf{n} / (\mathbf{n} : \mathbf{D} : \mathbf{D} : \mathbf{n})^{1/2} \quad (\text{unit normal to yield surface})$$

$$(15) \quad \hat{\Lambda} = (\mathbf{n} : \mathbf{D} : \mathbf{D} : \mathbf{n})^{1/2} (\mathbf{n} : \mathbf{D} : d\varepsilon) / L^p \quad (\text{scalar factor})$$

$$(16) \quad d\sigma^p = (\mathbf{n} : \mathbf{D} : d\varepsilon)\mathbf{D} : \mathbf{n} / L^p \quad (\text{incremental plastic stress: flow rule})$$

$$(17) \quad d\sigma = [\mathbf{D} - (\mathbf{D} : \mathbf{n})(\mathbf{D} : \mathbf{n}) / L^p] : d\varepsilon = \mathbf{D}^{ep} : d\varepsilon \quad (\text{incremental stress-strain relation})$$

where \mathbf{n} is a unit outward normal to the yield surface in the stress space and

$$(18) \quad L^p = (\mathbf{n} : \mathbf{D} : \mathbf{D} : \mathbf{n})\hat{L}^p = \mathbf{n} : \mathbf{D} : \mathbf{n} + K^p$$

$$(19) \quad K^p = \mathbf{n} : \mathbf{D} : g : \mathbf{D} : \mathbf{n} - \sqrt{\mathbf{n} : \mathbf{D} : \mathbf{D} : \mathbf{n}} \sqrt{\frac{2}{3} \frac{\partial f}{\partial \kappa} \frac{d\kappa}{d\sigma_e^p}} / \left|\frac{\partial f}{\partial \sigma}\right|.$$

For isotropic materials

$$L^p = 2G + K^p, \quad K^p = c - (\partial f / \partial \kappa)(d\kappa / d\sigma_e^p) \sqrt{2/3} / |\partial f / \partial \sigma_{ij}|.$$

It can be shown that K^p has the same definition as Eq. (6.11:9) in the stress space formulation. For ideal plasticity, $K^p = 0$ causes the breakdown of the flow rule in the stress space formulation. There is no such problem in the strain space formulation, since L^p is non-zero.

Example. For isotropic von Mises materials, the yield surface in stress space is

$$(20) \quad f = (\sigma' - \alpha) : (\sigma' - \alpha)/2 - \sigma_Y^2/3 = 0$$

with σ' being the deviatoric stress and $\alpha = c\varepsilon^p$ for Prager's linear kinematic hardening model Eq. (6.12:16). For infinitesimal deformation

$$\sigma' = 2G(\varepsilon' - \varepsilon^p) = 2G\varepsilon' - \sigma^p$$

where ε' is the deviatoric strain, the corresponding yield surface in strain space can be written as

$$(21) \quad \hat{f}(\varepsilon - \hat{\alpha}, \kappa) = f(\sigma' - \alpha, \kappa) = 2G^2(\varepsilon' - \hat{\alpha}) : (\varepsilon' - \hat{\alpha}) - \sigma_Y^2/3 = 0,$$

where $\hat{\alpha}_{ii} = \sigma_{ii}^p = 0$ and

$$2G\hat{\alpha} = \sigma^p + 2G\alpha_e = \sigma^p + \alpha = \sigma^p + c\varepsilon^p = [1 + c/(2G)]\sigma^p.$$

Note that the material exhibits linear kinematic hardening in strain space even if there is no kinematic hardening in the stress space, i.e., $\alpha = 0$. Then

$$\begin{aligned}\partial \hat{f} / \partial \varepsilon &= 4G^2(\varepsilon' - \hat{\alpha}'), \quad \hat{n} = \sqrt{6} G(\varepsilon - \hat{\alpha}) / \sigma_Y, \\ \frac{\partial f}{\partial \sigma} : \frac{\partial f}{\partial \sigma} &= \frac{2}{3} \sigma_Y^2, \quad \frac{\partial f}{\partial \varepsilon'}(d\varepsilon' - d\hat{\alpha}) - \frac{2\sigma_Y}{3} \frac{d\sigma_Y}{d\sigma_e^p} d\sigma_e^p = 0, \\ d\sigma &= 2Gd\varepsilon + \lambda\delta d\varepsilon_{kk} - 6G^2(\varepsilon' - \hat{\alpha})(\varepsilon' - \hat{\alpha}):d\varepsilon / (\hat{L}^p \sigma_Y^2), \\ d\hat{\alpha} &= (1 + \frac{c}{2G}) \frac{\hat{\Lambda}}{2G} \hat{n}, \quad \hat{L}^p = (1 + \frac{c}{2G} + \frac{2}{3} \frac{d\sigma_Y}{d\sigma_e^p}) \frac{1}{2G}.\end{aligned}$$

These equations are valid for both hardening and ideal plastic materials. For further details, readers are referred to the references: Il'iushin 1960, 1961, Naghdi and Trapp 1975, 1975(a), 1981, and Yoder and Iwan 1981.

6.15. DEFORMATION THEORY OF PLASTICITY

We have thus far discussed the incremental theories of plasticity in which the plastic deformation is dependent on the loading path. Hencky^{6.3} and Il'iushin^{6.6} proposed a *deformation theory of plasticity* by assuming that the total plastic strain ε^p is proportional to the total deviatoric stress σ' :

$$(1) \quad \varepsilon^p = \Lambda \sigma' / |\sigma'|$$

where Λ is a proportional factor. Defining the effective stress σ_e and plastic strain ε_e^p , respectively, as:

$$\sigma_e = (3\sigma' : \sigma'/2)^{1/2}, \quad \varepsilon_e^p = (2\varepsilon^p : \varepsilon^p/3)^{1/2},$$

we find

$$\varepsilon^p = \sigma' (3\varepsilon_e^p / |\sigma'| / 2), \quad \text{and} \quad \Lambda = 3\varepsilon_e^p / 2.$$

The theory also assumes that σ_e is a unique function of ε^p that

$$\sigma_e = \sigma_e(\varepsilon_e^p).$$

The total strain is the sum of the total elastic and plastic strains. Then we can write the strain-stress relation as

$$\varepsilon = \varepsilon^e + \varepsilon^p = \sigma' / (2G) + \sigma_{kk} \delta / (9K) + 3\varepsilon_e^p \sigma' / (2|\sigma'|)$$

where δ is the Kronecker delta tensor and K the elastic bulk modulus.

The application of the deformation theory is usually limited since plastic deformation is in general path dependent. However, for proportional loading paths, i.e., the stresses increase in ratio,

$$\sigma = \alpha \sigma^o$$

where α is a monotonically increasing factor and σ^o a reference stress state, a deformation theory is basically a nonlinear stress strain theory. We find

$$\sigma_e = \alpha \sigma_e^o, \quad \text{and} \quad \sigma' = \alpha \sigma'^o.$$

The incremental Prandtl–Reuss equation can be written as

$$d\varepsilon^p = 3\sigma' d\varepsilon_e^p / (2\sigma_e) = 3\sigma'^o d\varepsilon_e^p / (2\sigma_e^o),$$

and integrated to become

$$\varepsilon^p = 3\sigma' \varepsilon_e^p / (2\sigma_e) = 3\sigma'^o \varepsilon_e^p / (2\sigma_e^o).$$

When a loading path deviates significantly from the proportional loading path, such as cyclic loading cases, serious errors may result.

6.16. FINITE DEFORMATION

Considerable attention has been given to the study of finite elastic-plastic deformation in recent years. For finite deformation, it is necessary to distinguish between stress and strain measures defined with reference to the deformed and the undeformed states of a continuum. In formulating a plasticity model, stress and strain rates are required.

Unfortunately, depending on the definition, a rate may or may not be objective (independence of coordinate transformation) in large deformation. A key element in finite plasticity formulation is the choice of appropriate objective rates that will correctly reflect the underlying physics of the plastic flow. The choice of rates also affects strongly the form of constitutive equations. In general, finite deformation will also produce elastic-plastic coupling, thus may make the yield surface nonconvex. The subject of the best choice of rates is still a topic of research. Detailed discussion of finite plasticity theory is beyond the scope of this book.

6.17. PLASTIC DEFORMATION OF CRYSTALS

The underlying physics of plastic flow for metals is the micro-slip of crystallographic planes. Micro-slip in crystals or grains is highly directional and nonuniform. Dislocation theories and the plasticity theories for single and poly-crystals have provided the basis to quantify plastic deformation at the microscopic level and explain how and why plastic deformation occurs. The approach is based on the movements of atoms and the deformation of crystals and grains along slip planes. The macroscopic movements of metals are the aggregates of single and poly-crystal responses to applied loads. This physical theory is a very important aspect of plasticity theory. Interested readers are referred to the references listed in [Bibliography 6.9](#) at the end of the book for details.

PROBLEMS

6.8. A solid is assumed to obey the von Mises criterion with isotropic hardening. If the virgin curve in uniaxial tension can be described in the infinitesimal deformation range by $\sigma_Y = F(\varepsilon^P)$, determine

$$d\varepsilon_{ij}^P = \Lambda n_{ij}$$

when κ is assumed to depend on either ε^P , or W_p .

6.9. If the yield function is

$$f = \sqrt{J_2} - \kappa = \sqrt{J_2(\sigma' - \alpha)} - \kappa = \sqrt{J_2(\sigma' - c\varepsilon^P)} - \kappa = 0,$$

where κ and c are constant. Show that

$$\dot{\alpha} = c(\sigma' - \alpha)[(\sigma' - \alpha) : \dot{\sigma}'] / \{2\kappa^2[c + 2(d\kappa/d\varepsilon_e^P)/\sqrt{3}]\}.$$

6.10. From the proceeding result, determine the rate equations for α_{ij} and $\dot{\varepsilon}_{ij}^P$.

6.11. Consider testing a certain metallic material in two ways: by torsion of a hollow cylindrical specimen as shown in Fig. P6.11(a), and by uniaxial stretching of a tension specimen as shown in Fig. P6.11(b). The torsional test shows that the load-deflection curve is linear for a shearing stress below 25,000 lb/sq in. and that yielding occurs at the stress 25,000 lb/sq in. If the criterion of yielding for this material is von Mises' $J^2 = k^2$, what is the expected value of the tensile stress at which yielding will occur in the tension test?

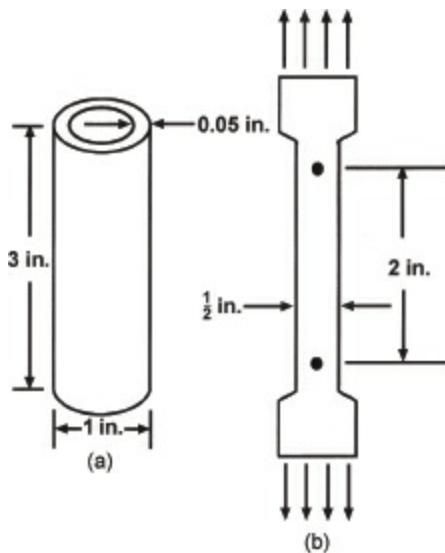


Fig. P6.11

6.12. Consider an isotropic, incompressible, but nonlinearly elastic material. Let ε'_{ij} , σ'_{ij} be the strain deviation and stress deviation, respectively. If the material were elastic,

$$e'_{ij} = \sigma'_{ij}/(2G) \quad \text{or} \quad \mathbf{e}' = \sigma'/(2G),$$

which can be written in the matrix form

$$\mathbf{e}' = \begin{bmatrix} e'_{11} & e'_{12} & e'_{13} \\ e'_{21} & e'_{22} & e'_{23} \\ e'_{31} & e'_{32} & e'_{33} \end{bmatrix} = \frac{1}{2G} \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \frac{1}{2G} \sigma'.$$

Generalize such a relation to a nonlinear elastic material by expressing the matrix \mathbf{e}' as a power series in σ' .

6.13. Prove, by an application of the Cayley–Hamilton theorem, that the most general nonlinear elasticity law of this type can be reduced to the form

$$\mathbf{e}' = \mathbf{P} : \sigma' + \mathbf{Q} : (\sigma' \cdot \sigma'). \quad (\text{Prager})$$

6.14. In soil mechanics, Coulomb's yield condition is widely used. Consider a two-dimensional problem. If σ_1 and σ_2 are the principal stresses in the plane of concern, Coulomb's yield condition is specified by the yield functions

$$\begin{aligned} f_1 &= \sigma_1(1 + \sin \phi) - \sigma_2(1 - \sin \phi) - 2c \cos \phi, \\ f_2 &= -\sigma_1(1 - \sin \phi) + \sigma_2(1 + \sin \phi) - 2c \cos \phi, \end{aligned}$$

where c = cohesion and ϕ = angle of internal friction. The soil is elastic if both f_1 and f_2 are negative, and it is plastic when f_1 or f_2 , or both vanish.

(a) Show that the flow rule derived from this condition by means of Koiter's theory of generalized plastic potential requires that the rate of the planar dilatation $\dot{e}_1^{(p)} + \dot{e}_2^{(p)}$ and the maximum shear rate $|\dot{e}_1^{(p)} - \dot{e}_2^{(p)}|$ have the constant ratio $\sin \varphi$ for all states of stress at the yield limit, except the state $\sigma_1 = \sigma_2 = c \cot \varphi$.

(b) Generalize the results above to obtain the generalized plastic potentials for three-dimensional problems.

(c) According to your generalization of (b), is there any general relationship between the three-dimensional dilatation and the maximum shear rate for stress states at the yield limit?

6.15. For the yield surface as defined in Eq. (6.14:5), show that

$$b = \sqrt{\mathbf{n} : \mathbf{D} : \mathbf{D} : \mathbf{n}} = 1/\sqrt{\hat{\mathbf{n}} : \mathbf{D}^{-1} : \mathbf{D}^{-1} : \hat{\mathbf{n}}}.$$

6.16. For the equivalent plastic stress ($d\sigma_e^p$) and strain ($d\varepsilon_e^p$) as defined in Eqs. (6.14:7) and (6.12:4), respectively, show that

$$d\sigma_e^p = d\varepsilon_e^p \sqrt{\mathbf{n} : \mathbf{D} : \mathbf{D} : \mathbf{n}}.$$

6.17. For $\alpha = \mathbf{D} : \alpha_e$, $d\alpha_e = \mathbf{g} : d\sigma^p$, $d\varepsilon^p$ as defined in Eq. (6.12:4), and the yield function as defined in Eq. (6.14:5), show that

(a) $d\alpha = \mathbf{D} : \mathbf{g} : \mathbf{D} : d\varepsilon^p$; and

(b) Eq. (6.14:19) becomes

$$K^p = -[(\partial f / \partial \alpha) : \mathbf{D} : \mathbf{g} : \mathbf{D} : \mathbf{n} + (\partial f / \partial \kappa)(d\kappa / d\varepsilon_e^p)\sqrt{2/3}] / |\partial f / \partial \sigma|,$$

which is the same as that given in Eq. (6.11:13) in the stress space formulation.

¹If the displacement ui is infinitesimal, $eij = (ui,j + uj,i)/2$. If ui is finite, eij is the Almansi strain tensor of Sec. 4.1. See also Sec. 13.3.

²It is usually believed that von Mises first suggested this criterion. But it was in fact first published by Huber in 1904. There is even evidence that Maxwell came up with the idea in 1856.

³The material most qualified to be called *elastic* is probably natural rubber. But the stress-strain relationship of natural rubber is not linear. For some rubbers this can be corrected by relating the stress to Eulerian (Almansi) strain or, equivalently, by relating the Kirchhoff stress to Green's strain. In such cases, the material is said to be *linearly elastic* with respect to finite strain. In most cases the nonlinearity in the stress-strain relationship remains, although the material returns to its original state along the same stress-strain curve when all the loads are removed. In this case, the material is said to be *nonlinearly elastic*. In this chapter, we discuss the strain hardening only under the assumption of small strain. Finite deformation is discussed in Chapter 13.

⁴The following explanation follows that of P. M. Naghdi^{6.3} (1960).

7

LINEARIZED THEORY OF ELASTICITY

In this chapter we shall discuss the classical theory of elasticity. We shall discuss the general structure of the theory and illustrate the applications of the linearized theory by a few examples. *Rectangular Cartesian* coordinates of reference will be used throughout. The coordinates will be denoted by x_1, x_2, x_3 or x, y, z unless stated otherwise.

7.1. BASIC EQUATIONS OF ELASTICITY

An elastic body has a unique *zero-stress state*, to which the body returns when all stresses vanish. All stresses, strains, and particle displacements are measured from this *zero-stress state* in which their values are zero.

There are two ways to describe a deformed body: the *material* and the *spatial* (see Sec. 5.2). Consider the spatial description. The motion of a continuum is described by the instantaneous velocity field $v_i(x_1, x_2, x_3, t)$. To describe the strain in the body, a displacement field $u_i(x_1, x_2, x_3, t)$ is specified which describes the displacement of a particle located at x_1, x_2, x_3 at time t from its position in the natural state. Various strain measures can be defined for the displacement field. The Almansi strain tensor is expressed in terms of $u_i(x_1, x_2, x_3, t)$ according to Eq. (4.2:4),

$$(1) \quad e_{ij} = [\partial u_i / \partial x_j + \partial u_j / \partial x_i - (\partial u_k / \partial x_i)(\partial u_k / \partial x_j)]/2.$$

The particle displacements u_i are functions of time and position. The particle velocity is given by the material derivative of the displacement,

$$(2) \quad v_i = \partial u_i / \partial t + v_j \partial u_i / \partial x_j.$$

The particle acceleration is given by the material derivative of the velocity (5.2:7),

$$(3) \quad \alpha_i = \partial v_i / \partial t + v_j \partial v_i / \partial x_j.$$

The motion of the body must obey the equation of continuity (5.4:3)

$$(4) \quad \partial \rho / \partial t + \partial \rho v_i / \partial x_i = 0,$$

and the equation of motion (5.5:7)

$$(5) \quad \rho \alpha_i = \partial \sigma_{ij} / \partial x_j + X_i.$$

In addition to the field Eq. (4) and (5), the linear elasticity theory is based on Hooke's law [Eq. (6.2:70)], for homogeneous isotropic materials,

$$(6) \quad \sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2G e_{ij},$$

where λ and G are constants independent of the spatial coordinates.¹

The famous nonlinear terms in Eq. (1)–(3) are sources of major difficulty in the theory of elasticity. To make some progress, we are forced to *linearize* by considering small displacements and small velocities, i.e., by restricting ourselves to values of u_i, v_i so small that the nonlinear terms in Eq. (1)–(3) may be neglected. In such a linearized theory, we have

$$(7) \quad e_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i)/2, \quad v_i = \partial u_i / \partial t, \quad \alpha_i = \partial v_i / \partial t.$$

Unless stated otherwise, all discussions below are subjected to this linearized restriction. Many useful results can be obtained from this linearized theory.

Equations (1)–(6) or (4)–(8) together are 22 equations for the 22 unknowns $\rho, u_i, v_i, \alpha_i, e_{ij}, \sigma_{ij}$. In the infinitesimal displacement theory we may eliminate σ_{ij} by substituting Eq. (6) into Eq. (5) and using Eq. (7) to obtain the well-known *Navier's equation*,

$$(8) \quad \Delta G u_{i,jj} + (\lambda + G) u_{j,ji} + X_i = \rho \partial^2 u_i / \partial t^2.$$

This can be written in the form

$$(9) \quad \blacktriangleleft \quad G\nabla^2 u_i + (\lambda + G)e_{,i} + X_i = \rho \partial^2 u_i / \partial t^2$$

where

$$(10) \quad e = u_{i,i}, \quad \nabla^2 u_i = u_{i,jj}.$$

The quantity e is the *divergence* of the displacement vector u_i ; ∇^2 is the *Laplace operator*. If we write x, y, z instead of x_1, x_2, x_3 , we have

$$(11) \quad e = \partial u / \partial x + \partial v / \partial y + \partial w / \partial z, \quad \nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2.$$

Love^{1,2} writes Eq. (10) in the form,

$$(12) \quad G\nabla^2(u, v, w) + (\lambda + G)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)e + (X, Y, Z) = \rho \frac{\partial^2(u, v, w)}{\partial t^2},$$

which is a shorthand for three equations of the type

$$(13) \quad \blacktriangleleft \quad G\nabla^2 u + (\lambda + G)\partial e / \partial x + X = \rho \partial^2 u / \partial t^2.$$

This can also be written as

$$(14) \quad \blacktriangleleft \quad G[\nabla^2 u + (\partial e / \partial x) / (1 - 2\nu)] + X = \rho \partial^2 u / \partial t^2.$$

If we introduce the rotation vector

$$(15) \quad \omega \equiv \frac{1}{2} \boldsymbol{\nabla} \times \mathbf{u} \equiv \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right),$$

and use the identity

$$(16) \quad \nabla^2(u, v, w) = (\partial / \partial x, \partial / \partial y, \partial / \partial z)e - 2\boldsymbol{\nabla} \times \omega,$$

then Eq. (9) may be written as

$$(17) \quad (\lambda + 2G)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)e - 2G\boldsymbol{\nabla} \times \omega + (X, Y, Z) = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}.$$

7.2. EQUILIBRIUM UNDER ZERO BODY FORCE

Consider the conditions of static equilibrium of an elastic body. If the body force vanishes, $X_i = 0$, then the divergence of Eq. (7.1:8) becomes

$$Gu_{i,jji} + (\lambda + G)u_{j,jii} = 0,$$

or

$$(1) \quad (u_{j,j}),_{ii} = 0.$$

In unabridged notations for rectangular Cartesian coordinates, this is

$$(2) \quad (\partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2)e = 0, \quad \text{or} \quad \nabla^2 e = 0.$$

Equation (2) is a Laplace equation. A function satisfying Eq. (2) is called a *harmonic function*. Thus, *the dilatation e is a harmonic function when the body force vanishes*. But

$$(3\lambda + 2G)e = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 3\sigma,$$

where σ is the mean stress. Hence, *the mean stress is also a harmonic function*:

$$(3) \quad \nabla^2 \sigma = 0.$$

If we put $X = 0$, $\partial^2 u / \partial t^2 = 0$, and operate on Eq. (7.1:13) with the Laplacian ∇^2 , we have

$$(\lambda + G)\partial(\nabla^2 e)/\partial x + G\nabla^2 \nabla^2 u = 0.$$

With Eq. (2), this implies that

$$(4) \quad \nabla^4 u = 0,$$

which is called a *biharmonic* equation, and its solution is called a *biharmonic function*. In rectangular Cartesian coordinates,

$$(5) \quad \nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + 2\frac{\partial^4}{\partial y^2 \partial z^2} + 2\frac{\partial^4}{\partial z^2 \partial x^2}.$$

Hence, the displacement component u is biharmonic. Similarly, the components v, w are biharmonic. It follows that when the body force is zero, each of the strain and stress components being linear combination of the first derivatives of u, v, w , are all biharmonic functions:

$$(6) \quad \nabla^4 \sigma_{ij} = 0,$$

$$(7) \quad \nabla^4 e_{ij} = 0.$$

7.3. BOUNDARY VALUE PROBLEMS

Navier's Eq. (7.1:8) combines Hooke's law and the equation of motion. It is to be solved for appropriate boundary and initial conditions. The boundary conditions that occur are usually one of two kinds:

A. *Specified displacements*. The components of displacement u_i are prescribed on the boundary.

B. *Specified surface tractions*. The components of surface traction $\overset{\nu}{T}_i$ are assigned on the boundary.

In most problems of elasticity, the boundary conditions are such that over part of the boundary displacements are specified, whereas over another part the surface tractions are specified. Let the region occupied by an elastic body be denoted by V . Let the boundary surface of V be denoted by S . We separate S into two parts: S_u , where displacements are specified; and S_σ , where surface tractions are specified. Therefore, on S_σ ,

$$\overset{\nu}{T}_i = \sigma_{ij}\nu_j = \text{a prescribed function},$$

where ν_j is a unit vector along the outer normal to the surface S_σ . By Hooke's law, this may be written as

$$[\lambda u_{k,k}\delta_{ij} + G(u_{i,j} + u_{j,i})]\nu_j = \text{a prescribed function}.$$

Hence, over the entire surface, the boundary conditions are that either u_i , or a combination of the first derivatives of u_i , are prescribed.

In dynamic problems, a set of initial conditions on u_i or σ_{ij} must be specified in the region V and on the boundary S .

The question arises whether a boundary-value problem posed in this way has a solution, and whether the solution is unique or not. The question has two parts. First, do we expect a unique solution on physical grounds? Second, does the specific mathematical problem have a unique solution? In continuum mechanics, there are many occasions in which we do not expect a unique solution to exist. For example, when a thin-walled spherical shell is subjected to a uniform external pressure, the phenomenon of buckling may occur when the pressure exceeds certain specific value. At the buckling load, the shell may assume several different forms of deformation, some stable, some unstable. On the other hand, our everyday experience about the physical world tells us that the vast majority of mechanical cause-and-effect relationships are unique. Theoretically, the physical question is partly answered by thermodynamics. But the mathematical question must be answered by the theory of partial differential equations. A satisfactory theory must bring harmony between the mathematical formulation and the physical world.

In the preceding discussions we have taken the displacements u_i as the basic unknown variables. In problems of static equilibrium, however, it is customary to use an alternate procedure. The equations of equilibrium are first solved for the stresses σ_{ij} . We then use Hooke's law to obtain the strain e_{ij} . This solution will not be unique. In fact, an infinite set of solutions will be found. The correct one is then singled out by the conditions of compatibility. Only the one solution that satisfies both the equations of equilibrium and the equations of compatibility corresponds to a continuous displacement field. This procedure becomes very attractive when stress functions are introduced, which yield general solutions of the equations of equilibrium (see Sec. 9.2).

By means of Hooke's law, the compatibility equation

$$(1) \quad e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$$

can be expressed directly in terms of stress components. On substituting

$$e_{ij} = (1 + \nu)\sigma_{ij}/E - \nu\theta\delta_{ij}/E, \quad \text{where } \theta = \sigma_{kk},$$

into Eq. (1), we obtain

$$(2) \quad \begin{aligned} & \sigma_{ij,kl} + \sigma_{kl,ij} - \sigma_{ik,jl} - \sigma_{jl,ik} \\ &= \nu(\delta_{ij}\theta_{,kl} + \delta_{kl}\theta_{,ij} - \delta_{ik}\theta_{,jl} - \delta_{jl}\theta_{,ik})/(1 + \nu). \end{aligned}$$

Since only six of the 81 equations represented by Eq. (1) are linearly independent, the same must be true for Eq. (2). If we combine Eq. (2) linearly by setting $k = l$ and summing, we obtain

$$(3) \quad \begin{aligned} & \sigma_{ij,kk} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} \\ &= \nu(\delta_{ij}\theta_{,kk} + 3\theta_{,ij} - \delta_{ik}\theta_{,jk} - \delta_{jk}\theta_{,ik})/(1 + \nu), \end{aligned}$$

which is a set of nine equations of which six are independent because of the symmetry in i and j . Hence, the number of independent equations is not reduced, and Eq. (2) and (3) are equivalent. Since $\sigma_{kk} = \theta$ and $\sigma_{ij,kk} = \nabla^2\sigma_{ij}$, if we use the equation of equilibrium to replace, say, $\sigma_{ik,jk}$ by $-X_{i,j}$, we can write Eq. (3) as

$$(4) \quad \nabla^2\sigma_{ij} + \theta_{,ij}/(1 + \nu) - \nu\delta_{ij}\nabla^2\theta/(1 + \nu) = -(X_{i,j} + X_{j,i}),$$

where X_i is the body force per unit volume. In dynamic problems the inertia force should be included in X_i . With a contraction $i = j$, Eq. (4) furnishes a relation between $\nabla^2\theta$ and $X_{i,i}$. If this is used to transform the third term in Eq. (4), we obtain

$$(5) \quad \nabla^2\sigma_{ij} + \theta_{,ij}/(1 + \nu) = -\nu\delta_{ij}X_{kk}/(1 - \nu) - (X_{i,j} + X_{j,i}).$$

Written out *in extenso*, these are

$$(6) \quad \begin{aligned} \nabla^2\sigma_{xx} + \frac{1}{1 + \nu}\frac{\partial^2\theta}{\partial x^2} &= -\frac{\nu}{1 - \nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial X}{\partial x}, \\ \nabla^2\sigma_{yy} + \frac{1}{1 + \nu}\frac{\partial^2\theta}{\partial y^2} &= -\frac{\nu}{1 - \nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial Y}{\partial y}, \\ \nabla^2\sigma_{zz} + \frac{1}{1 + \nu}\frac{\partial^2\theta}{\partial z^2} &= -\frac{\nu}{1 - \nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial Z}{\partial z}, \\ \nabla^2\sigma_{yz} + [\partial^2\theta/(\partial y\partial z)]/(1 + \nu) &= -(\partial Y/\partial z + \partial Z/\partial y), \\ \nabla^2\sigma_{zx} + [\partial^2\theta/(\partial z\partial x)]/(1 + \nu) &= -(\partial Z/\partial x + \partial X/\partial z), \\ \nabla^2\sigma_{xy} + [\partial^2\theta/(\partial x\partial y)]/(1 + \nu) &= -(\partial X/\partial y + \partial Y/\partial x). \end{aligned}$$

These equations were obtained by Michell in 1900, and, for the case in which body forces are absent, by Beltrami in 1892. They are known as the *Beltrami–Michell compatibility equations*.

For a simply connected region, satisfaction of the Beltrami–Michell equations implies that the stress system σ_{ij} is derivable from a continuous displacement field. If the region concerned is multiply connected, additional conditions in the form of certain line integrals must be satisfied (see Sec. 4.7).

7.4. EQUILIBRIUM AND UNIQUENESS OF SOLUTIONS

Consider the problem of determining the state of stress and strain in a body of a given shape which is held strained by body forces X_i and surface tractions $\overset{\nu}{T}_i$. Let us assume that a function $W(e_{11}, e_{12}, \dots, e_{33})$, called *the strain energy function of the elastic material*, exists and has the property

$$(1) \quad \partial W / \partial e_{ij} = \sigma_{ij} .$$

For example, if the material obeys Hooke's law [Eq. (6.1:1)], then

$$(2) \quad W = D_{ijkl} e_{ij} e_{kl} / 2 .$$

The existence and the positive definiteness of the strain energy function for an elastic body are discussed in [Chapter 12](#). In [Sec. 12.4](#), it is shown that W is positive definite in the neighborhood of the natural state. A natural state of a material is a stable state in which the material can exist by itself in thermodynamic equilibrium. Limiting ourselves to consider materials which have a unique natural state in the theory of elasticity, we are assured of a positive definite strain energy function in the neighborhood of the natural state.

The equations of equilibrium $\sigma_{jj} + X_i = 0$ may be written in the form

$$(3) \quad (\partial W / \partial e_{ij})_{,j} + X_i = 0 .$$

The boundary conditions over the boundary surface $S_u + S_\sigma$ are:

(4a) Over S_u , the values of u_i are given,

(4b) Over S_σ , the tractions $\vec{T}_i = (\partial W / \partial e_{ij}) \nu_j$ are specified

(see [Sec. 7.3](#)). If S_σ constitutes the entire surface of the body, it is obvious that equilibrium would be impossible unless the system of body forces and surface tractions satisfy the conditions of static equilibrium for the body as a whole. In the same case, the displacement will be indeterminate to the extent of a possible rigid-body motion.

We shall prove the following theorem due to Kirchhoff. *If either the surface displacements or the surface tractions are given, the solution of the problem of equilibrium of an elastic body as specified by Eqs. (1)–(4) is unique in the sense that the state of stress (and strain) is determinate without ambiguity, provided that the magnitude of the stress (and strain) is so small that the strain energy function exists and remains positive definite.*

Proof. Since a strained state must be either unique or nonunique, a proof can be constructed by showing that an assumption of nonuniqueness leads to contradiction. Assume that there exist two sets of displacements \vec{u}'_i and \vec{u}''_i which define two states of strain, both satisfying Eq. (3) and the boundary conditions (4a) and (4b). Then the difference $u_i \equiv u'_i - u''_i$ satisfies the equation

$$(5) \quad (\partial W / \partial e_{ij})_{,j} = 0$$

and the boundary conditions that

$$(6) \quad u_i = 0 \text{ on } S_u \text{ and } \frac{\partial W}{\partial e_{ij}} \nu_j = 0 \text{ on } S_\sigma .$$

From Eq. (5) we have

$$\int_V u_i \left(\frac{\partial W}{\partial e_{ij}} \right)_{,j} dV = 0 ,$$

which, on integrating by parts, becomes

$$(7) \quad \int_S u_i \frac{\partial W}{\partial e_{ij}} \nu_j dS - \int_V \frac{\partial W}{\partial e_{ij}} u_{i,j} dV = 0 .$$

The first surface integral vanishes because of the boundary conditions (6). The second volume integral may be written as

$$\int_V \frac{\partial W}{\partial e_{ij}} e_{ij} dV .$$

Now, when W is a homogeneous quadratic function of e_{ij} [Eq. (2)], the integral above is equal to $\int 2W dV$. Since W is assumed to be positive definite, the integral $\int W dV$ cannot vanish unless W vanishes, which in turn implies that $e_{ij} = 0$ everywhere. Hence, $u_i \equiv u'_i - u''_i$ corresponds to the natural, unstrained state of the body. Therefore, the states of strain (and stress) defined by \vec{u}'_i and \vec{u}''_i are the same, contrary to the assumption. Hence, the state of strain (and stress) is unique.

We must note that the uniqueness theorem is proved only in the neighborhood of the natural state. In fact when the strain energy function fails to remain positive definite, multi-valued solutions may be possible.

Problem 7.1. Prove Neumann's theorem that the solution $ui(x, t)$, for x in $V + Su + S\sigma$ and $t \geq 0$, of the following system of equations, is unique.

$$(8) \quad \partial(\partial W/\partial e_{ij})/\partial x_j + X_i - \rho \partial u_i/\partial t^2 = 0, \quad \text{for } x \text{ in } V, t \geq 0,$$

$$(9) \quad u_i = f_i(x, t), \quad \text{for } x \text{ on } S_u, t \geq 0,$$

$$(10) \quad \frac{\partial W}{\partial e_{ij}} v_j = g_i(x, t), \quad \text{for } x \text{ on } S_\sigma, t \geq 0,$$

$$(11) \quad u_i = h_i(x), \quad \dot{u}_i = k_i(x), \quad \text{when } t = 0, x \text{ in } V + S,$$

$$(12) \quad e_{ij} = (u_{i,j} + u_{j,i})/2,$$

where X_i, f_i, g_i, h_i, k_i are preassigned functions and $W(e_{ij})$ is a positive definite quadratic form. Hint: Multiply Eq. (8) by $\partial u_i/\partial t$ and integrate over V and $(0, t)$. Note that the kinetic energy is positive definite. (Love, *Elasticity*,^{1,2} p. 176.)

Notes on Possible Loss of Uniqueness

The uniqueness theorem of Kirchhoff is the foundation for the method of potentials ([Chapters 8 and 9](#)). For, when uniqueness of solution is established, one needs to find only a solution of a given boundary-value problem: that solution is *the* solution.

But it is essential for our theory to be able to violate the uniqueness of solution in one way or another. We know that elastic columns can buckle, thin shells may collapse, airplane wings can flutter, machinery can become unstable in one sense or other. The word "stability" has many meanings; to define a stability problem we must define the sense of the word stability. A large class of practical stability problems is connected with the loss of uniqueness of solution. Under certain circumstances two or more solutions may become possible; some of these may be dangerous from engineering point of view or undesirable for the function of the machinery; and the circumstances are said to cause instability.

The uniqueness theorem can be violated by violating any one of its assumptions. Referring to Kirchhoff's theorem, we have two possibilities:

- (a) Loss of positive-definiteness of the strain energy function, $W(e_{ij})$.
- (b) Basic changes in the equations of equilibrium (or of motion).

The first possibility arises when the material becomes unstable, as yielding or flow occurs (cf. [Secs. 6.3 and 6.5](#)). It is relevant to the plastic buckling of columns, plates, and shells.

The second possibility may arise in a variety of forms. The most important are those due to

- (1) finite deformation,
- (2) nonconservative forces, and
- (3) forces that are functionals of the deformation or history of deformation.

Most buckling problems can be understood only if we realize the basic changes in the equation of equilibrium introduced by finite deformation. For finite deformations the equation of equilibrium (or of motion) is given by Eq. (13.10:7) or (13.10:8). These equations are basically nonlinear because the strains and rotations depend on the stresses. The corresponding equations of equilibrium or motion for a plate are given in [Sec. 13.15](#), e.g., Eqs. (13.15:10) to (13.15:14). The situation with a column is similar. The linearized versions of these equations retain the basic features of these large-deflection equations, so that these problems generally become eigenvalue problems or bifurcation problems.

Nonconservative forces generally depend on the deformation of the structure in a certain specific manner. For example, consider an axial load applied to the end of a cantilever column. If this load is fixed in direction, it is conservative. If this load is not fixed in direction, but may rotate in the process of buckling, then it is nonconservative. The case (3) listed above is a special but broad class of nonconservative forces. It occurs commonly if a solid body is placed in a flowing fluid. The aerodynamic or hydrodynamic pressure acting on the body depends on the deformation of the body, and on the local and whole body velocity and acceleration. If the wake and vorticity are important, the aerodynamic pressure will depend also on the history of deformation. This is commonly the case of an aircraft lifting

surface. Under these loadings the v terms X_i and $\overset{\nu}{T}_i$ in Eqs. (7.4:3), (7.4:4), and (7.4:8), (7.4:10) are functions or functionals of $u_i(x_1, x_2, x_3, t)$.

In any of the cases mentioned above the basic equations differ from those assumed in the Kirchhoff theorem. Loss of uniqueness does not necessarily follow, but it becomes a possibility and must be investigated.

7.5. SAINT-VENANT'S THEORY OF TORSION

To illustrate the applications of the theory of elasticity, we shall consider the problem of torsion of a cylindrical body. A cylindrical shaft, with an axis z , is acted on at its ends by a distribution of shearing stresses whose resultant force is zero but whose resultant moment is a torque T . The lateral surface of the shaft is stress free. See Fig. 7.5:1.

If the shaft is a circular cylinder, it is very easy to show that all plane cross sections normal to the z -axis remain plane, and the deformation consists of relative rotation θ of the cross sections. The rate of rotation per unit axial length $d\theta/dz$ is proportional to the torque T , with a proportionality constant equal to the product of the shear modulus G of the shaft material, and the polar moment of inertia J of the shaft cross-sectional area:

$$(1) \quad GJ(d\theta/dz) = T.$$

The only nonvanishing component of stress is the shear in cross sections perpendicular to z , whose magnitude is

$$(2) \quad \tau = Tr/J,$$

where r is the radius vector from the central axis z .

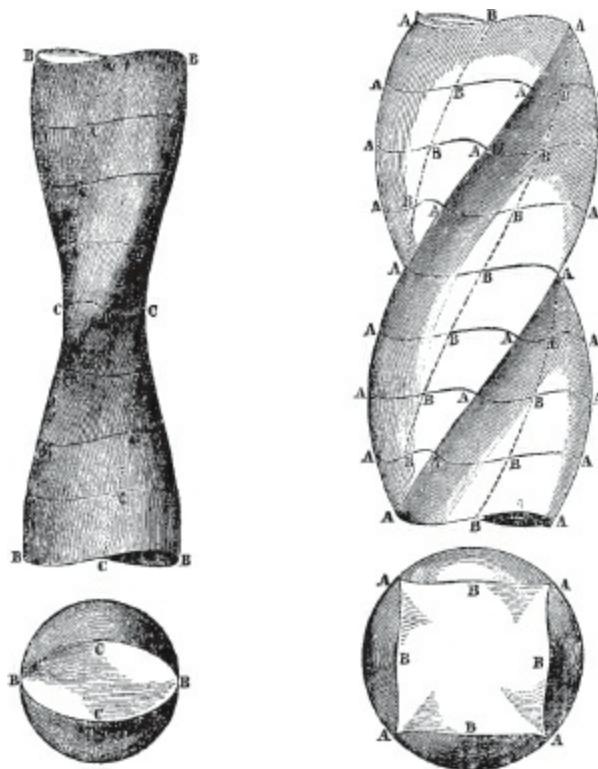


Fig. 7.5:1. Torsion of bars of elliptic and square cross section, as drawn by Saint-Venant.

If the cross section of the shaft is not circular, a plane cross section does not remain plane; it warps, as is shown in Fig. 7.5:1. The problem is to calculate the stress distribution and the deformation of the shaft.

This is an important problem in engineering; for shafts are used to transmit torques and they are seen everywhere. The celebrated solution to the problem is due to Barre de Saint-Venant (1855), who used the so-called *semi-inverse method*, i.e., a method in which one guesses at part of the solution, tries to determine the rest rationally so that all the differential equations and boundary conditions are satisfied. The torsion problem is not simple. Saint-Venant, guided by the solution of the circular shaft, made a brilliant guess and showed that an exact solution to a well-defined problem can be obtained.

We shall consider, then, a cylindrical shaft with an axis z , with the ends at $z = 0$ and $z = L$. A set of rectangular Cartesian coordinates x, y, z will be used, with the x, y -plane perpendicular to the axis of the shaft. The displacement components in the x, y, z -directions will be written as u, v, w , respectively. Saint-Venant assumed that, as the shaft twists, the plane cross sections are warped but the *projections* on the x, y -plane rotate as a rigid body; i.e.,

$$(3) \quad u = -\alpha zy, \quad v = \alpha zx, \quad w = \alpha \varphi(x, y),$$

where $\varphi(x, y)$ is some function of x and y , called the *warping function*, and α is the angle of twist per unit length of the bar and is assumed to be very small ($\ll 1$). We rely on the function $\varphi(x, y)$ to satisfy the differential equations of equilibrium (without body force)

$$(4) \quad \begin{aligned} \partial \sigma_{xx}/\partial x + \partial \sigma_{xy}/\partial y + \partial \sigma_{xz}/\partial z &= 0, \\ \partial \sigma_{xy}/\partial x + \partial \sigma_{yy}/\partial y + \partial \sigma_{yz}/\partial z &= 0, \\ \partial \sigma_{zx}/\partial x + \partial \sigma_{zy}/\partial y + \partial \sigma_{zz}/\partial z &= 0, \end{aligned}$$

the boundary conditions on the lateral surface of the cylinder

$$(5) \quad \sigma_{xx}\nu_x + \sigma_{xy}\nu_y = 0, \quad \sigma_{yx}\nu_x + \sigma_{yy}\nu_y = 0, \quad \sigma_{zx}\nu_x + \sigma_{zy}\nu_y = 0,$$

and the boundary conditions at the ends $z = 0$ and $z = L$:

$$(6) \quad \begin{aligned} \sigma_{zz} &= 0 \\ \sigma_{zx}, \sigma_{zy} &\text{ equivalent to a torque } T. \end{aligned}$$

The constants ν_x, ν_y are the direction cosines of the exterior normal to the lateral surface ($\nu_z = 0$).

From Eq. (3), we obtain the stresses according to Hooke's law.

$$(7) \quad \begin{aligned} \sigma_{yz} &= \alpha G(\partial \varphi / \partial y + x), & \sigma_{xz} &= \alpha G(\partial \varphi / \partial x - y), \\ \sigma_{xy} &= \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0. \end{aligned}$$

A substitution of these values into Eq. (4) shows that the equilibrium equations will be satisfied if $\varphi(x, y)$ satisfies the equation

$$(8) \quad \partial^2 \varphi / \partial x^2 + \partial^2 \varphi / \partial y^2 = 0$$

throughout the cross section of the cylinder. To satisfy the boundary conditions (5), we must have

$$(9) \quad \partial u / \partial y + \partial v / \partial x = 0, \quad \lambda(\partial u / \partial x + \partial v / \partial y) + 2G \partial v / \partial y = 0,$$

where C is the boundary of the cross section R (Fig. 7.5.2). But

$$\begin{aligned} (\partial \varphi / \partial x) \cos(\nu, x) + (\partial \varphi / \partial y) \cos(\nu, y) \\ = \partial \varphi / \partial \nu; \end{aligned}$$

hence, the boundary condition (9) can be written as

$$(10) \quad \partial \varphi / \partial \nu = y \cos(\nu, x) - x \cos(\nu, y) \quad \text{on } C.$$

The boundary conditions (6) are satisfied at the ends $z = 0$ and $z = L$ if

$$(11) \quad \iint_R \sigma_{zx} dx dy = 0, \quad \iint_R \sigma_{zy} dx dy = 0,$$

$$(12) \quad \iint_R (x \sigma_{zy} - y \sigma_{zx}) dx dy = T.$$

We can show that Eq. (11) are readily satisfied if φ satisfies Eq. (8) and (10); because

$$\begin{aligned} \iint_R \sigma_{zx} dx dy &= \alpha G \iint_R \left(\frac{\partial \varphi}{\partial x} - y \right) dx dy \\ &= \alpha G \iint_R \left\{ \frac{\partial}{\partial x} \left[x \left(\frac{\partial \varphi}{\partial x} - y \right) \right] + \frac{\partial}{\partial y} \left[x \left(\frac{\partial \varphi}{\partial y} + x \right) \right] \right\} dx dy, \end{aligned}$$

since φ satisfies Eq. (8). On applying Gauss' theorem to the last integral, it becomes a line integral on the boundary C of the region R :

$$\alpha G \int_C x \left[\frac{\partial \varphi}{\partial \nu} - y \cos(x, \nu) + x \cos(y, \nu) \right] ds,$$

which vanishes on account of Eq. (10). Similarly, the second of Eq. (11) is satisfied. Finally, the last condition (12) requires that

$$(13) \quad T = \alpha G \iint_R \left(x^2 + y^2 + x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} \right) dx dy.$$

Writing J for the integral

$$(14) \quad J \equiv \iint_R \left(x^2 + y^2 + x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} \right) dx dy,$$

we have

$$(15) \quad T = \alpha G J.$$

This merely shows that the torque T is proportional to the angle of twist per unit length α , with a proportionality constant GJ , which is usually called the *torsional rigidity* of the shaft. The J represents the polar moment of inertia of the section when the cross section is circular and the warping function φ is zero. However, it is conventional to retain the notation GJ for torsional rigidity, even for noncircular cylinders.

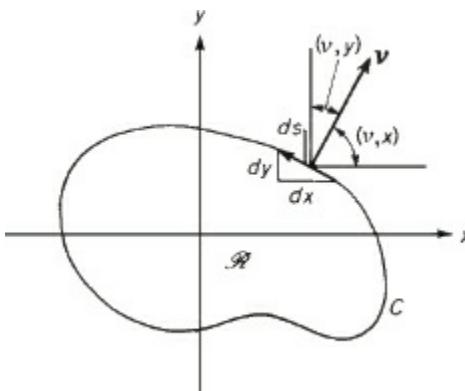


Fig. 7.5:2. Notations.

Thus, we see that the problem of torsion is reduced to the solution of Eq. (8) and (10). The solution will yield a stress system σ_{zx} , σ_{zy} . If the end sections of the shaft are free to warp, and if the stresses prescribed on the end sections are exactly the same as those given by the solution, then an exact solution is obtained, and the solution is unique (see Sec. 7.4). If the distribution of stresses acting on the end sections, while equipollent to a torque T , does not agree exactly with that given by Eq. (7), then only an approximate solution is obtained. According to a principle proposed by Saint-Venant, the error in the approximation is significant only in the neighborhood of the end section (see Secs. 10.11–10.13).

Equation (8) is a *potential* equation; its solutions are called *harmonic functions*. The same equation appears in hydrodynamics. The boundary condition (10) is similar to that for the velocity potential in hydrodynamics with prescribed velocity efflux over the boundary. In the hydrodynamics problem, the condition for the existence of a solution φ is that the total flux of fluid across the boundary must vanish. Translated to our problem, the condition for the existence of a solution φ is that the integral of the normal derivative of the function φ , calculated over the entire boundary C , must vanish. This follows from the identity

$$(16) \quad \int_C \frac{\partial \varphi}{\partial \nu} ds = \iint_R \operatorname{div}(\operatorname{grad} \varphi) dx dy = \iint_R \nabla^2 \varphi dx dy$$

and from the fact that $\nabla^2 \varphi = 0$. This condition is satisfied in our case by Eq. (10), as can be easily shown. Therefore, our problem is reduced to the solution of a potential problem (called Neumann's problem) subjected to the boundary condition Eq. (10).

An alternate approach was proposed by Prandtl, who takes the stress components as the principal unknowns. If we assume that only σ_{xz} , σ_{yz} differ from zero, then all the equations of equilibrium (4) are satisfied if

$$(17) \quad \partial \sigma_{xz} / \partial x + \partial \sigma_{yz} / \partial y = 0.$$

Prandtl observes that this equation is identically satisfied if σ_{xz} and σ_{yz} are derived from a *stress function* $\psi(x, y)$ so that

$$(18) \quad \sigma_{xz} = \partial \psi / \partial y, \quad \sigma_{yz} = -\partial \psi / \partial x.$$

This corresponds to the stream function in hydrodynamics, if σ_{xz} and σ_{yz} were identified with velocity components. Although ψ can be arbitrary as far as equilibrium conditions are concerned, the stress system (18) must satisfy the boundary conditions (5) and (6), and the compatibility conditions. From Eq. (7.3:6), we see that compatibility requires that (in the absence of body force),

$$\nabla^2 \sigma_{yz} = 0, \quad \nabla^2 \sigma_{zx} = 0,$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Hence

$$(19) \quad \partial(\nabla^2 \psi)/\partial x = 0, \quad \partial(\nabla^2 \psi)/\partial y = 0.$$

It follows that

$$(20) \quad \nabla^2 \psi = \text{const.}$$

Of the boundary conditions (5), only the last equation is not identically satisfied. If we note from Fig. 7.5:2, that

$$(21) \quad \nu_x = \cos(\nu, x) = dy/ds, \quad \nu_y = \cos(\nu, y) = -dx/ds,$$

we can write the last of Eq. (5) as

$$(22) \quad (\partial \psi / \partial y)(dy/ds) + (\partial \psi / \partial x)(dx/ds) = \partial \psi / ds = 0, \quad \text{on } C.$$

Hence ψ must be a constant along the boundary curve C . For a simply connected region, no loss of generality is involved in setting

$$(23) \quad \psi = 0, \quad \text{on } C.$$

If the cross section occupies a region R that is multi-connected, additional conditions of compatibility must be imposed (see Sec. 4.7).

It remains to examine the boundary conditions (6). The first, $\sigma_{zz} = 0$, follows the starting assumption. The other conditions are stated in Eq. (11) and (12). Now,

$$\iint_R \sigma_{zx} dx dy = \iint_R \frac{\partial \psi}{\partial y} dx dy.$$

By Gauss' theorem, this is $\int_C \psi v_y ds$, and it vanishes on account of Eq. (23). Similarly, the resultant force in the y -direction vanishes. Thus, Eq. (11) are satisfied. Finally, Eq. (12) requires that

$$T = - \iint_R \left(x \frac{\partial \psi}{\partial x} + y \frac{\partial \psi}{\partial y} \right) dx dy,$$

which can be transformed by Gauss' theorem as follows:

$$(24) \quad \begin{aligned} T &= - \iint_R \left\{ \frac{\partial}{\partial x}(x\psi) + \frac{\partial}{\partial y}(y\psi) - 2\psi \right\} dx dy \\ &= - \int_C \{x\psi \cos(\nu, x) + y\psi \cos(\nu, y)\} ds + \iint_R 2\psi dx dy. \end{aligned}$$

If R is a simply connected region, the line integral vanishes by the boundary condition (23). Hence,

$$(24a) \quad T = 2 \iint_R \psi dx dy.$$

Thus, all differential equations and boundary conditions concerning stresses are satisfied if ψ obeys Eq. (20), (23), and (24). But there remains an indeterminate constant in Eq. (20). This constant has to be determined by boundary conditions on displacements. We have, from Eq. (3) and (7),

$$(25) \quad \partial w / \partial x = \sigma_{zx}/G + \alpha y, \quad \partial w / \partial y = \sigma_{zy}/G - \alpha x.$$

Differentiating with respect to y and x , respectively, and subtracting, we get

$$(26) \quad (\partial \sigma_{zx} / \partial y - \partial \sigma_{zy} / \partial x)/G = -2\alpha.$$

Hence, a substitution from Eq. (18) gives

$$(27) \quad \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = -2G\alpha.$$

In this way, the problem of torsion is reduced to the solution of the Poisson Eq. (27) with boundary condition (23).

With either of the two approaches outlined above, the problem of torsion is reduced to standard problems in the theory of potentials in two dimensions. Such potential problems occur also in the theory of hydrodynamics, gravitation, static electricity, steady flow of heat, etc. A great deal is known about these potential problems; many special solutions have been worked out in detail and general methods of solution are available. The most powerful tool for potential theory in two dimensions comes from the theory of functions of a complex variable. Since this branch of applied mathematics is probably well-known to the readers in their study of other branches of physics, we shall not elaborate further. Many excellent books in this beautiful field exist (see, for example, Courant,^{5,1} Courant and Hilbert,^{10,1} Kellogg,^{5,1} etc.) The complex variable method and the associated singular integral equations approach are developed in the monumental works of Muskhelishvili;^{1,2} a shorter account is given by Sokolnikoff.^{1,2} Other methods of solution and detailed examples can be found in the classical books^{1,2} of Love, Sokolnikoff, Southwell, Timoshenko and Goodier, Trefftz, Sechler, Green and Zerna. Goodier's article in Flügge's^{1,2} *Handbook of Engineering Mechanics* contains results for various cross sections commonly used in engineering.

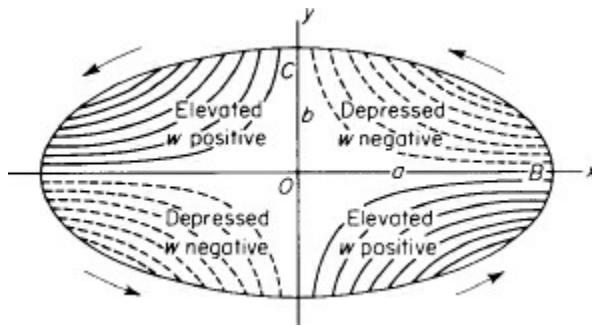


Fig. 7.5x:3. Lines of constant warping, according to Saint-Venant.

Example. Bars with Elliptical Cross Section

Let the boundary of the cross section (Fig. 7.5:3) be given by the equation

$$x^2/a^2 + y^2/b^2 - 1 = 0.$$

Then Eq. (27) and (23) are satisfied by

$$\psi = -a^2 b^2 G\alpha (x^2/a^2 + y^2/b^2 - 1)/(a^2 + b^2).$$

Equation (24) gives the relation between the torque and the rate of twist $\alpha = d\theta/dz$.

$$T = \pi a^3 b^3 G(d\theta/dz)/(a^2 + b^2).$$

The stresses are given by Eq. (18). Note that the curves $\psi(x, y) = \text{const.}$ have an interesting meaning. The slope dy/dx of the tangent to such a curve is determined by the formula

$$\partial\psi/\partial x + (\partial\psi/\partial y)(dy/dx) = 0.$$

Hence, according to Eq. (18), we have

$$(28) \quad dy/dx = \sigma_{zy}/\sigma_{zx}.$$

Thus, at each point of the curve $\psi(x, y) = \text{const.}$, the stress vector $(\sigma_{zx}, \sigma_{zy})$ is directed along the tangent to the curve. The curves $\psi(x, y) = \text{const.}$ are called the *lines of shearing stress*. The magnitude of the tangential stress is

$$(29) \quad \tau = (\sigma_{zx}^2 + \sigma_{zy}^2)^{1/2} = [(\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2]^{1/2}.$$

Hence, τ is equal to the absolute value of the gradient of the surface $z = \psi(x, y)$. The maximum stress occurs where the gradient is the largest.

In the present example, the lines of shearing stress are concentric ellipses. It is easy to see that the spacing of the $\psi = \text{const.}$ lines are closest at the end of the minor axis. The maximum shearing stress occurs there and is given by Eq. (29) to be

$$\tau_{\max} = 2G\alpha a^2 b/(a^2 + b^2).$$

A general theorem can be proved that the points at which the maximum shearing stress occurs lie on the boundary curve of the cross section.

The warping function $\varphi(x, y)$ is easily shown to be

$$\varphi = -(a^2 - b^2)xy/(a^2 + b^2).$$

Contour lines of constant displacement along the z -axis, $w = \alpha\varphi(x, y) = \text{const.}$, are hyperbolas as shown in Figs. 7.5:3, which were taken from Saint-Venant's original publication. The solid lines in the figure indicate where w is positive, the dotted lines where w is negative.

7.6. SOAP FILM ANALOGY

As remarked before, Eqs. (7.5:8) and (7.5:27) occur in many other physical theories entirely unrelated to the torsion problem. For example, if we consider a thin film of liquid, such as that of a soap bubble, we see that the predominant force acting in the film is the surface tension, which may be considered to be constant. The equation of equilibrium of an element of soap film is

$$T/R_1 + T/R_2 = p,$$

where R_1, R_2 are the principal radii of curvature and p is the pressure per unit area normal to the film. The derivation of this equation is simple and can be understood from elementary considerations as illustrated in Fig. 7.6:1. Now if we take a tube whose cross-sectional shape is the same as that of the shaft whose torsional property is questioned, cut a plane section, spread a soap film over it under a small pressure p . If the film deflection is sufficiently small, the mean curvature of the film is given by the sum of the second derivatives of the deflection surface. Thus,

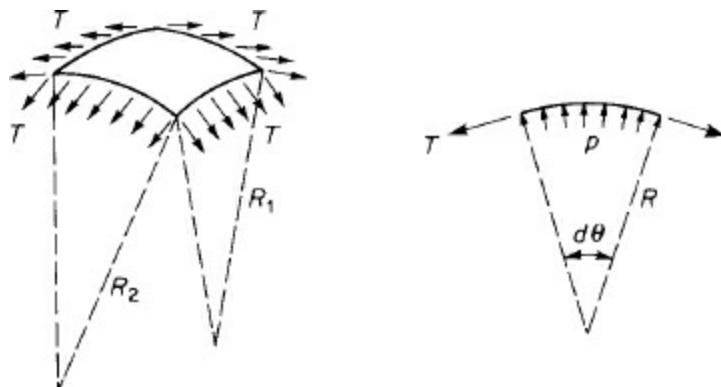


Fig. 7.6:1. Equilibrium of a soap film.

$$T(\partial^2 w / \partial x^2 + \partial^2 w / \partial y^2) = p, \quad w = 0 \text{ on boundary},$$

where w denotes the deflection of the film measured from the plane of the cross section and x, y are a set of rectangular coordinates. These equations are identical with Eqs. (7.5:23) and (7.5:27). Thus, we obtain Prandtl's *soap film analogy*. The gradient of the soap film is proportional to the shear stress in torsion. The volume under the film and above the cross section is proportional to the total torque.

The value of an analogy lies in its power of suggestion. Most people can visualize the shape of a soap film, perhaps because of their long experience with it. Thus, with the soap film analogy it is very easy to explain the stress concentration at an re-entrant corner in a cross section, such as the points marked by P in Fig. 7.6:2.

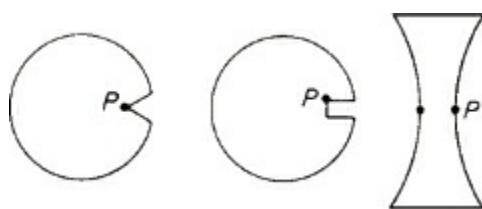


Fig. 7.6:2. Re-entrant corners suggest stress concentration.

Economical and efficient use of materials to transmit forces is an important objective in engineering; and the problem of avoiding the weakest links — points where stress concentrations occur — is obviously of great interest.

7.7. BENDING OF BEAMS

When a shaft is used to transmit bending moments and transverse shear, it is called a *beam*. Since beams are used in every engineering structure, the theory of beams is of great importance. The long history of the development of man's understanding of the action of the beam is a fascinating subject well-recorded in Timoshenko's book. 1.1 Modern investigation began with Galileo, but it is again to the credit of Saint-Venant that the problem is solved within the general theory of elasticity.

Consider first the pure bending of a prismatic beam (Fig. 7.7:1). Let the beam be subjected to two equal and opposite couples M acting in one of its principal planes.² Let the origin of the coordinates be taken at the centroid of a cross section, and let the x, z -plane be the principal plane of bending. The usual elementary theory of bending assumes that the stress components are

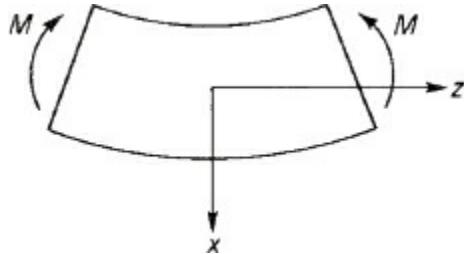


Fig. 7.7:1. Pure bending of a prismatic beam.

$$(1) \quad \sigma_{zz} = Ex/R, \quad \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0,$$

in which R is the radius of curvature of the beam after bending. It is easily verified that the stress system (1) satisfies all the equations of equilibrium (7.5:4) and compatibility (7.3:6). The boundary conditions on the lateral surface of the beam are also satisfied. If the normal stress at the ends of the beam is linearly distributed as in Eq. (1), and if the bending moment is

$$(2) \quad M = \iint \sigma_{zz} x dx dy = \iint \frac{1}{R} Ex^2 dx dy = \frac{EI}{R},$$

where I is the moment of inertia of the cross section of the beam with respect to the neutral axis parallel to the y -axis, then every condition is satisfied, and Eq. (1) gives an exact solution.

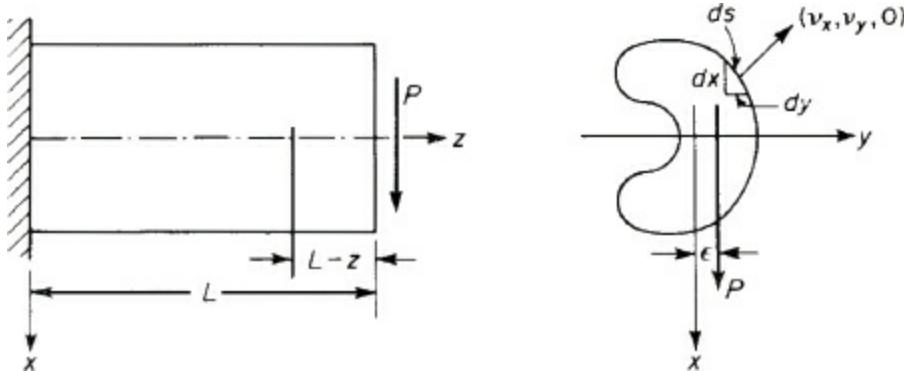


Fig. 7.7:2. Cantilever beam loaded at one end.

Consider next the same prismatic beam loaded by a lateral force P at the end $z = L$, and clamped at the end $z = 0$ (Fig. 7.7:2). For a beam so loaded, the resultant shear P has to be resisted by the shearing stresses σ_{zx} , σ_{zy} . Hence, the stress system (1) will not suffice.

Let the force P be parallel to one of the principal axes of the cross section of the beam. (An arbitrary force can be resolved into components parallel to the principal axes, and the action of each component may be considered separately.) Let P be parallel to the x -axis and let the x, z -plane be a principal plane. Saint-Venant, using his semi-inverse method, assumes that

$$(3) \quad \sigma_{zz} = -P(L-z)x/I, \quad \sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0,$$

leaving σ_{zx} , σ_{zy} undetermined. The first term is given by the elementary theory, $P(L-z)$ being the bending moment at section z . Now we must see how the equilibrium, compatibility, and boundary conditions can be satisfied. In the absence of body force, the equilibrium equations (7.5:4) require that

$$(4) \quad \partial\sigma_{zx}/\partial z = 0, \quad \partial\sigma_{zy}/\partial z = 0, \quad \partial\sigma_{zx}/\partial x + \partial\sigma_{zy}/\partial y + Px/I = 0.$$

The Beltrami–Michell compatibility conditions (7.3:6) require that

$$(5) \quad \nabla^2 \sigma_{yz} = 0, \quad \nabla^2 \sigma_{xz} + \frac{1}{1+\nu} \frac{P}{I} = 0,$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The stress-free boundary condition on the lateral surface requires that [Eq. (7.5:5)]

$$(6) \quad \sigma_{zx} \cos(\nu, x) + \sigma_{zy} \cos(\nu, y) = 0, \quad \text{on } C.$$

The end condition at $z = L$ requires that

$$(7) \quad \sigma_{zz} = 0, \quad \iint \sigma_{zy} dx dy = 0, \quad \iint \sigma_{zx} dx dy = P.$$

The end condition at $z = 0$ is concerned with the conditions of clamping, and it is usually stated in the form

$$(8) \quad u = \partial u / \partial z = 0, \quad \text{at} \quad x = y = z = 0.$$

The method of solving Eq. (4) through (7) is similar to that of Sec. 7.5. Equations (4) imply that both σ_{xz} and σ_{yz} are independent of z . The last equation of Eq. (4) may be written as

$$(9) \quad \partial[\sigma_{zx} + Px^2/(2I) - f(y)] / \partial x + \partial\sigma_{zy} / \partial y = 0,$$

where $f(y)$ is a function of y only. Equation (9) can be satisfied identically if the stresses σ_{xz} , σ_{yz} are derived from a stress function $\psi(x, y)$ such that

$$(10) \quad \sigma_{zx} = \partial\psi / \partial y - Px^2/(2I) + f(y), \quad \sigma_{zy} = -\partial\psi / \partial x.$$

Equations (5) imply that

$$(11) \quad \partial(\nabla^2 \psi) / \partial x = 0, \quad \partial(\nabla^2 \psi) / \partial y = \nu P / [(1 + \nu)I] - d^2 f / dy^2.$$

Hence,

$$(12) \quad \partial^2 \psi / \partial x^2 + \partial^2 \psi / \partial y^2 = \nu Py / [(1 + \nu)I] - df / dy + c.$$

The integration constant c has a very simple physical meaning. Consider the rotation ω of an element of area in the plane of a cross section.

$$(13) \quad \omega = (\partial v / \partial x - \partial u / \partial y) / 2.$$

The rate of change of this rotation in the z -axis direction is

$$(14) \quad \begin{aligned} \partial\omega / \partial z &= [\partial(\partial v / \partial x - \partial u / \partial y) / \partial z] / 2 \\ &= \frac{1}{2} \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) - \frac{1}{2} \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{xz}}{\partial y} \\ &= \frac{1}{2G} \left(\frac{\partial\sigma_{yz}}{\partial x} - \frac{\partial\sigma_{xz}}{\partial y} \right) = -\frac{1}{2G} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{df}{dy} \right). \end{aligned}$$

In deriving the last line, Hooke's law and Eq. (10) are used. Hence, Eq. (12) leads to

$$(15) \quad -2G\partial\omega / \partial z = \nu Py / [(1 + \nu)I] + c.$$

This shows that c represents a constant rate of rotation, i.e., it corresponds to a rigid-body rotation of a cross section (the same kind as in the torsion problem). It can be shown that by a proper shifting of the load P parallel to itself in the plane $z = L$, the torsional deformation can be eliminated so that $c = 0$. (This leads to the concept of *shear center*, the point through which P must act so that $c = 0$.) In the following discussion we shall assume that P acts through the shear center. The more general problem can be solved obviously by a linear superposition of the solutions of bending and of torsion.

On setting $c = 0$, the boundary condition (6) now requires that [see Fig. 7.5:2 and Eq. (7.5:21)]

$$(16) \quad \begin{aligned} (\partial\psi / \partial y)(dy/ds) + (\partial\psi / \partial x)(dx/ds) &= \partial\psi / \partial s \\ &= [Px^2 / (2I) - f(y)]dy/ds. \end{aligned}$$

From these equations, the value of ψ can be determined up to an integration constant which does not contribute anything to the stress system. The function $f(y)$ is arbitrary; it was introduced by Timoshenko to simplify the solution in case the boundary curve of the cross section can be written in the form

$$(17) \quad C : Px^2/(2I) - f(y) = 0.$$

This would be the case, for example, if C is a circle or an ellipse. In such a case, we choose $f(y)$ according to Eq. (17). Then the boundary condition may be written as

$$(18) \quad \psi = 0 \quad \text{on } C.$$

It remains to show that the load at the end $z = L$ is equipollent to a shear P , i.e., that Eq. (7) are satisfied. This is easily done. For example, using the last equation of (4) and Eq. (6), we have

$$\begin{aligned} \iint \sigma_{xz} dxdy &= \iint \left[\sigma_{xz} + x \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} \right) + \frac{Px^2}{I} \right] dxdy \\ &= P + \int_C x[\sigma_{xz} \cos(\nu, x) + \sigma_{yz} \cos(\nu, y)] ds = P, \\ \iint \sigma_{yz} dxdy &= \iint \left[\sigma_{yz} + y \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} \right) + \frac{Pxy}{I} \right] dxdy \\ &= \iint y[\sigma_{xz} \cos(\nu, x) + \sigma_{yz} \cos(\nu, y)] ds = 0, \end{aligned}$$

since $\iint xydxdy$ vanishes because the x -axis is assumed to be a principal axis.

Thus, all the equations are satisfied if a solution $\psi(x, y)$ is found from Eq. (12) (with $c = 0$) and (16) or (18). This reduces the beam problem to a standard problem in two-dimensional potential theory.

If a solution is obtained and the moment of the shearing stresses is computed and set equal to $P\epsilon$

$$(19) \quad \iint (x\sigma_{yz} - y\sigma_{xz}) dxdy = P\epsilon,$$

the constant ϵ will give the location of the shear center. The applied load must have an eccentricity ϵ in order to obtain a bending without twisting (see discussion about the constant c above).

Example. Beam of Circular Cross Section of Radius r

The boundary curve C is given by the equation

$$x^2 + y^2 = r^2.$$

Hence, we take

$$f(y) = P(r^2 - y^2)/(2I).$$

Equation (12) (with $c = 0$) becomes

$$\psi = -(1 + 2\nu)P(x^2 + y^2 - r^2)y/[8(1 + \nu)I].$$

It is easily verified that the solution that satisfies the boundary condition (18) is

$$\psi = -\frac{(1 + 2\nu)}{8(1 + \nu)} \frac{P}{I} (x^2 + y^2 - r^2)y.$$

The stress components are

$$\sigma_{xz} = \frac{3 + 2\nu}{8(1 + \nu)} \frac{P}{I} \left(r^2 - x^2 - \frac{1 - 2\nu}{3 + 2\nu} y^2 \right), \quad \sigma_{zy} = -\frac{1 + 2\nu}{4(1 + \nu)} \frac{Pxy}{I}.$$

A large number of examples can be found in Love,^{1,1} Sokolnikoff,^{1,2} Timoshenko,^{1,2} etc. An extensive discussion of the shear center can be found in Sechler's and Sokolnikoff's books.^{1,2} There are several possible definitions of shear center and center of twist [Treffitz (1935), Goodier (1944) and Wein-stein (1947)]; an outline and comparison of various definitions can be found in Fung's *Aeroelasticity*^{10,5} (1957) pp. 471–475.

7.8. PLANE ELASTIC WAVES

As a further illustration of the theory of elasticity, let us consider some simple types of waves in an elastic medium.

The displacement components u_1, u_2, u_3 (or in unabridged notations, u, v, w) will be assumed to be infinitesimal so that all equations are linearized. The basic field equation, in the absence of body force, is Navier's Eq. (7.1:8)

$$(1) \quad \rho \partial^2 u_i / \partial t^2 = G u_{i,jj} + (\lambda + G) u_{j,ji} .$$

We shall first verify that the motion

$$(2) \quad u = A \sin[2\pi(x \pm ct)/l], \quad v = w = 0,$$

where A, l, c are constants, is possible if c assumes the special value c_L ,

$$(3) \quad \Delta \quad C_L = [(\lambda + 2G)/\rho]^{1/2} = \{E(1 - \nu)/[(1 + \nu)(1 - 2\nu)\rho]\}^{1/2} .$$

This can be verified at once by substituting Eq. (2) into (1). The pattern of motion expressed by Eq. (2) is unchanged when $x \pm c_L t$ remains constant. Hence, if the negative sign were taken, the pattern would move to the right with a velocity c_L as the time t increases. The constant c_L is called the *phase velocity* of the wave motion. In Eq. (2), l is the *wave length*, as can be seen from the sinusoidal pattern of u as a function of x , at any instant of time. The particle velocity represented by Eq. (2) is in the same direction as that of the wave propagation (namely, the x -axis). Such a motion is said to constitute a train of *longitudinal waves*. Since at any instant of time the wave crests lie in parallel planes, the motion represented by Eq. (2) is called a train of *plane waves*.

Next, let us consider the motion

$$(4) \quad u = 0, \quad v = A \sin[2\pi(x \pm ct)/l], \quad w = 0,$$

which represents a train of plane waves of wave length l propagating in the x -axis direction with a phase velocity c . When Eq. (4) are substituted into Eq. (1), it is seen that c must assume the value c_T ,

$$(5) \quad \Delta \quad C_T = (G/\rho)^{1/2} .$$

The particle velocity (in the y -direction) represented by Eq. (4) is perpendicular to the direction of wave propagation (x -direction). Hence, it is said to be a *transverse wave*. The speeds c_L and c_T are called the characteristic *longitudinal wave speed* and *transverse wave speed*, respectively. They depend on the elastic constants and the density of the material. The ratio c_T/c_L depends on Poisson's ratio only,

$$(6) \quad \Delta \quad C_T = C_L \{(1 - 2\nu)/[2(1 - \nu)]\}^{1/2} .$$

If $\nu = 0.25$, then $c_L = \sqrt{3}c_T$.

Similar to Eq. (4), the following example represents a transverse wave in which the particles move in the z -axis direction.

$$(7) \quad u = 0, \quad v = 0, \quad w = A \sin[2\pi(x \pm c_T t)/l] .$$

The plane parallel to which the particles move [such as the x, y -plane in Eq. (4), or the x, z -plane in Eq. (7)], is called the *plane of polarization*.

Table 6.2:1, Sec. 6.2, gives a brief list of the longitudinal wave velocities of some common media. It is very interesting to see that most metals and alloys have approximately the same wave velocities.

Plane waves as described above may exist only in an unbounded elastic continuum. In a finite body, a plane wave will be reflected when it hits a boundary. If there is another elastic medium beyond the boundary, refracted waves occur in the second medium. The features of reflection and refraction are similar to those in acoustics and optics; the main difference is that, in general, an incident longitudinal wave will be reflected and refracted in a combination of longitudinal and transverse waves, and an incident transverse wave will also be reflected in a combination of both types of waves. The details can be worked out by a proper combination of these waves so that the boundary conditions are satisfied. See [Sec. 8.14](#).

7.9. RAYLEIGH SURFACE WAVES

In an elastic body, it is possible to have another type of wave, which is propagated over the surface and which penetrates only a little into the interior of the body. These waves are similar to waves produced on a smooth surface of water when a stone is thrown into it. They are called *surface waves*. The simplest is the *Rayleigh wave* that occurs on the free surface of a homogeneous, isotropic, semi-infinite solid. It is an important type of wave because the largest disturbances caused by an earthquake recorded on a distant seismogram are usually those of Rayleigh waves.

The criterion for surface waves is that the amplitude of the displacement in the medium diminishes exponentially with increasing distance from the boundary.

Let us demonstrate the existence of Rayleigh waves in the simple two-dimensional case. Consider an elastic half-space $y \geq 0$. The surface $y = 0$ is stress-free. Let us consider displacements represented by the real part of the following expressions:

$$(1) \quad u = Ae^{-by} \exp[ik(x - ct)], \quad v = Be^{-by} \exp[ik(x - ct)], \quad w = 0,$$

where i is the imaginary unit $\sqrt{-1}$, k is the wave number (a constant), and A and B are complex constants. The coefficient b is supposed to be real and positive so that the amplitude of the waves decreases exponentially with increasing y , and tends to zero as $y \rightarrow \infty$.

We would first see if the displacements given by Eq. (1) can satisfy the equations of motion, which, in view of the definitions of c_L and c_T by Eqs. (7.8:3) and (7.8:5), can be written as

$$(2) \quad \partial^2 u_i / \partial t^2 = c_T^2 u_{i,jj} + (c_L^2 - c_T^2) u_{j,ji}.$$

Substituting Eq. (1) into (2), cancelling the common exponential factor and rearranging terms, we obtain the equations

$$(3) \quad \begin{aligned} & [c_T^2 b^2 + (c^2 - c_L^2) k^2] A - i(c_L^2 - c_T^2) b k B = 0, \\ & -i(c_L^2 - c_T^2) b k A + [c_L^2 b^2 + (c^2 - c_T^2) k^2] B = 0. \end{aligned}$$

The condition for the existence of a nontrivial solution is the vanishing of the determinant of the coefficients, which may be written in the form

$$(4) \quad [c_L^2 b^2 - (c_L^2 - c^2) k^2][c_T^2 b^2 - (c_T^2 - c^2) k^2] = 0.$$

This gives the following roots for b .

$$(5) \quad b' = k(1 - c^2/c_L^2)^{1/2}, \quad b'' = k(1 - c^2/c_T^2)^{1/2}.$$

The assumption that b is real requires that $c < c_T < c_L$. Corresponding to b' and b'' , respectively, the ratio B/A can be solved from Eq. (3).

$$(6) \quad (B/A)' = -b'/(ik), \quad (B/A)'' = ik/b''.$$

Hence, a general solution of the type (1), satisfying the equations of motion, may be written as

$$(7) \quad \begin{aligned} u &= (A'e^{-b'y} + A''e^{-b''y}) e^{ik(x-ct)}, \\ v &= [-b'A'e^{-b'y}/(ik) + ikA''e^{-b''y}/b''] e^{ik(x-ct)}, \quad w = 0. \end{aligned}$$

The constants A' , A'' , k , and c can be so chosen to satify the boundary conditions on the free surface $y = 0$

$$(8) \quad \sigma_{yx} = \sigma_{yy} = \sigma_{yz} = 0, \quad \text{on } y = 0.$$

By Hooke's law, and in view of Eq. (7), conditions (8) are equivalent to

$$(9) \quad \partial u / \partial y + \partial v / \partial x = 0, \quad \lambda(\partial u / \partial x + \partial v / \partial y) + 2G\partial v / \partial y = 0,$$

on $y = 0$. On substituting Eq. (7) into Eq. (9), setting $y = 0$, omitting the common factor $\exp[ik(x - ct)]$, and writing

$$G = \rho c_T^2, \quad \lambda = \rho(c_L^2 - 2c_T^2),$$

we obtain the results

$$(10) \quad -2b'A' - \frac{b'^2 + k^2}{b''} A'' = 0, \quad [(c_L^2 - 2c_T^2) - c_L^2 \frac{b'^2}{k^2}] A' - 2c_T^2 A'' = 0.$$

This can be written more symmetrically, by Eq. (5), as

$$(11) \quad 2b'A' + (2 - c^2/c_T^2)k^2 A''/b'' = 0, \quad (2 - c^2/c_T^2)A' + 2A'' = 0.$$

For a nontrivial solution, the determinant of the coefficients of AI , AII must vanish, yielding the characteristic equation for c

$$(12) \quad (2 - c^2/c_T^2)^2 = 4(1 - c^2/c_L^2)^{1/2}(1 - c^2/c_T^2)^{1/2}.$$

The quantity c^2/c_T^2 can be factored out after rationalization, and Eq. (12), called the *Rayleigh equation*, takes the form

$$(13) \quad \frac{c^2}{c_T^2} \left[\frac{c^6}{c_T^6} - 8\frac{c^4}{c_T^4} + c^2 \left(\frac{24}{c_T^2} - \frac{16}{c_L^2} \right) - 16 \left(1 - \frac{c_T^2}{c_L^2} \right) \right] = 0.$$

If $c = 0$, Eq. (7) are independent of time, and from Eq. (11) we have A' , A'' and $u = v = 0$. Hence, this solution is of no interest. The second factor in Eq. (13) is negative for $c = 0$, $c_T < c_L$, and is positive for $c = c_T$. There is always a root c of Eq. (13) in the range $(0, c_T)$. Hence, surface waves can exist with a speed less than c_T .

For an incompressible solid $c_L \rightarrow \infty$, Eq. (13) becomes

$$(14) \quad c^6/c_T^6 - 8c^4/c_T^4 + 24c^2/c_T^2 - 16 = 0.$$

This cubic equation in c^2 has a real root at $c^2 = 0.91275c_T^2$, corresponding to surface waves with speed $c \approx 0.95538c_T$. The other two roots for this case are complex and do not represent surface waves.

If the Poisson ratio is $\frac{1}{4}$, so that $\lambda = G$ and $c_L = \sqrt{3}c_T$, Eq. (13) becomes

$$(15) \quad c^6/c_T^6 - 8c^4/c_T^4 + 56(c^2/c_T^2)/3 - 32/3 = 0.$$

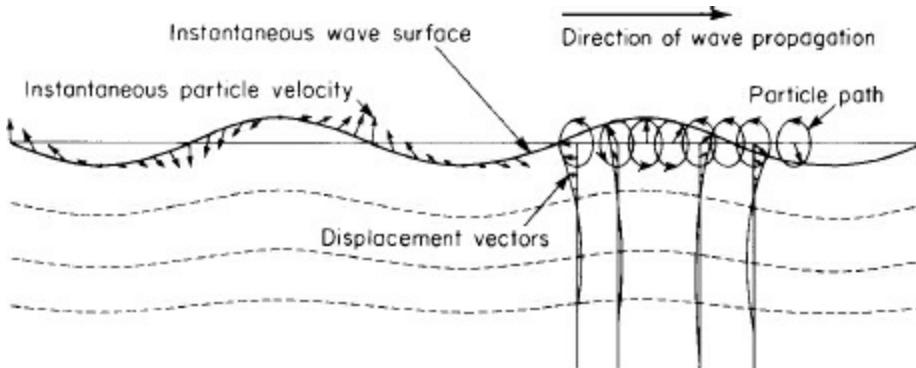


Fig. 7.9:1. Schematic drawing for Rayleigh surface waves.

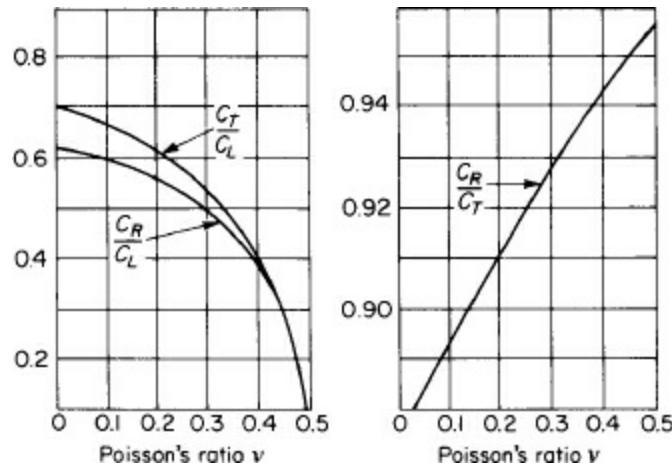


Fig. 7.9:2. Ratios c_R/c_L , c_R/c_T and c_T/c_L , where c_R = Rayleigh wave speed, c_L = the longitudinal wave speed, c_T = the transverse wave speed.

This equation has three real roots: $c^2/c_T^2 = 4$, $2 + 2/\sqrt{3}$, $2 - 2/\sqrt{3}$. The last root alone can satisfy the condition that b' and b'' be real for surface waves. The other two roots correspond to complex b' and b'' and do not represent surface waves. In fact, these extraneous roots do not satisfy Eq. (12); they arise from the rationalization process of squaring. The last root corresponds to the velocity

$$(16) \quad c_R = 0.9194c_T,$$

which corresponds, in turn, to the displacements

$$(17) \quad \begin{aligned} v &= A'(e^{-0.8475ky} - 0.5773e^{-0.3933ky}) \cos k(x - c_R t), \\ v &= A'(-0.8475e^{-0.8475ky} + 1.4679e^{-0.3933ky}) \sin k(x - c_R t), \end{aligned}$$

where A' is taken to be a real number, which may depend on k . From Eq. (17), it is seen the particle motion for Rayleigh waves is elliptical retrograde in contrast to the elliptical direct orbit for surface waves on water (see Fig. 7.9:1). The vertical displacement is about 1.5 times the horizontal displacement at the surface. Horizontal motion vanishes at a depth of 0.192 of a wavelength and reverses sign below this.

Figure 7.9:2 shows Knopoff's calculated results for the ratios c_R/c_T , c_R/c_L for Rayleigh waves as functions of Poisson's ratio ν .

7.10. LOVE WAVES

In the Rayleigh waves examined in the previous section the material particles move in the plane of propagation. Thus, in Rayleigh waves over the half-space $y \geq 0$ along the surface $y = 0$, propagating in the x -direction, the z -component of displacement w vanishes. It may be shown that surface waves with displacements perpendicular to the direction of propagation (the so-called *SH waves*) is impossible in a homogeneous half-space. However, *SH* surface waves are observed as prominently on the Earth's surface as other surface waves. Love showed that a theory sufficient to include *SH* surface waves can be constructed by having a homogeneous layer of a medium M_1 of uniform thickness H_1 , overlying a homogeneous half-space of another medium M .

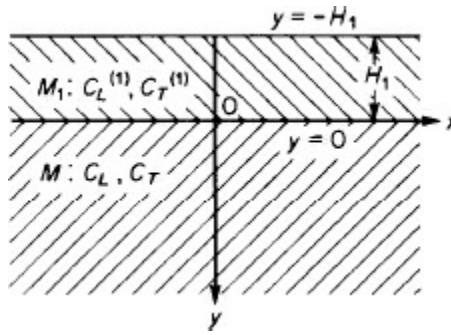


Fig. 7.10:1. A layered half-space.

Using axes as in Fig. 7.10:1, we take $u = v = 0$, and

$$(1) \quad w = A \exp[-k(1 - c^2/c_T^2)^{1/2}y] \exp[ik(x - ct)]$$

in M , and

$$(2) \quad w = \{A_1 \exp[-k\sqrt{1 - (c/c_T^{(1)})^2}y] + A'_1 \exp[k\sqrt{1 - (c/c_T^{(1)})^2}y]\} \exp[ik(x - ct)]$$

in M_1 . It is easily verified that these equations satisfy the Navier's equations. If $c < c_T$, then $w \rightarrow 0$ as $y \rightarrow \infty$, as desired.

The boundary conditions are that w and σ_{xy} must be continuous across the surface $y = 0$, and σ_{zy} zero at $y = -H_1$. On applying these conditions to Eq. (1) and (2), we obtain

$$(3) \quad A = A_1 + A'_1,$$

$$(4) \quad GA[1 - (c/c_T)^2]^{1/2} = G_1(A_1 - A'_1)[1 - (c/c_T^{(1)})^2]^{1/2},$$

$$(5) \quad A_1 \exp\{kH_1[1 - (c/c_T^{(1)})^2]^{1/2}\} = A'_1 \exp\{-kH_1[1 - (c/c_T^{(1)})^2]^{1/2}\}.$$

Eliminating A from Eq. (3) and (4), and then using Eq. (5) to eliminate A_1 and A'_1 , we have

$$\frac{G[1 - (c/c_T)^2]^{1/2}}{G_1[1 - (c/c_T^{(1)})^2]^{1/2}} = \frac{A_1 - A'_1}{A_1 + A'_1} = \tanh\{kH_1[1 - (c/c_T^{(1)})^2]^{1/2}\}.$$

Hence, we have

$$(6) \quad G \left(1 - \frac{c^2}{c_T^2} \right)^{1/2} - G_1 \left[\left(\frac{c}{c_T^{(1)}} \right)^2 - 1 \right]^{1/2} \tanh \left\{ k H_1 \left[\left(\frac{c}{c_T^{(1)}} \right)^2 - 1 \right]^{1/2} \right\} = 0$$

as the equation to give the *SH* surface wave velocity c in the present conditions.

If $c_T^{(1)} < c_T$, Eq. (6) yields a real value of c which lies in the range $c_T^{(1)} < c < c_T$ and depends on k and H_1 (as well as on G , G_1 , c_T , and $c_T^{(1)}$), because for c in this range the values of the left-hand-side terms in Eq. (6) are real and opposite in sign. Thus, *SH* surface waves can occur under the stated boundary conditions, provided the shear velocity $c^{(1)}$ in the upper layer is less than that in the medium M . These waves are called *Love waves*.

Love waves of general shape may be derived by superposing harmonic Love waves of the type (2) with different k . The dependence of the wave speed c on the wave number k introduces a dispersion phenomenon which will be considered later.

P R O B L E M S

7.2. Derive Navier's equation in spherical polar coordinates.

7.3. From data given in various handbooks or websites, determine the longitudinal and shear wave speeds in the following materials:

(a) Gases: air at sea level, and at 100,000 ft altitude.

(b) Metals: iron, a carbon steel, a stainless steel, copper, bronze, brass, nickel, aluminum, an aluminum alloy, titanium, titanium carbide, beryllium, beryllium oxide.

(c) Rocks and soils: a granite, a sandy loam.

(d) Wood: spruce, mahogany, balsa.

(e) Plastics: lucite, a foam rubber.

7.4. Sketch the instantaneous wave surface, particle velocities, and particle paths of a Love wave.

7.5. Investigate plane wave propagations in an anisotropic elastic material. Apply the results to a cubic crystal. *Note:*

$$\rho \partial^2 u_i / \partial t^2 = D_{ijkl} \partial^2 u_l / (\partial x_j \partial x_k), \quad u_l = A_l \exp[-i(\omega t - k_j x_j)]$$

where $\mathbf{k}(k_1, k_2, k_3)$ is the wave number vector normal to the wave front.

7.6. Determine the stress field in a rotating, gravitating sphere of uniform density.

¹ The corresponding equations based on the material description are the following: velocity and acceleration, Eqs. (5.2:3) and (5.2:6); strain measure, Green's strain tensor, Eq. (4.2:5); the equation of continuity, Eq. (5.2:3); stress tensors, Sec. 14.7; the equations of motion, Eqs. (14.10:1–8); the stress-strain laws, see Sec. 14.11; in particular, Eqs. (14.11:6) and (14.11:7). It can be seen that the kinematical relations appear simpler in the material description, but the equations of motion appear more complicated.

² A principal plane is one that contains the principal axes of the moment of area of the cross sections of the beam.

8

SOLUTION OF PROBLEMS IN LINEARIZED THEORY OF ELASTICITY BY POTENTIALS

In this chapter we shall consider the method of potentials in treating static and dynamic problems of an isotropic elastic body subjected to forces on the body boundary surface. The forces are independent of the deformation. The problem is to determine the stresses and displacements at every point in the body under appropriate boundary conditions. We assume that the displacements are infinitesimal and Hooke's law holds, so that the basic equations are those listed in Sec. 7.1.

In both static and dynamic problems, we start with Navier's equation and try to determine a continuous and twice differentiable solution u_i under appropriate boundary conditions.

To establish the solution, we will discuss potentials related to displacements including scalar and vector potentials, the Galerkin vectors, and the Papkovich–Neuber functions. Potentials that generate systems of equilibrating stresses are the Airy stress-functions, the Maxwell–Morera stress functions, etc. (see Chapter 9). Applications will be illustrated in this and subsequent chapters.

8.1. SCALAR AND VECTOR POTENTIALS FOR DISPLACEMENT VECTOR FIELDS

It is well-known (sometimes referred to as Helmholtz' theorem) that *any analytic vector field $\mathbf{u}(u_1, u_2, u_3)$ can be expressed in the form*

$$(1) \quad \blacktriangle u_i = \phi_{,i} + e_{ijk}\psi_{k,j}$$

i.e.,

$$(2a) \quad \blacktriangle \mathbf{u} = \operatorname{grad} \phi + \operatorname{curl} \psi$$

involving three equations of the type

$$(2b) \quad \blacktriangle u_1 = \partial\phi/\partial x_1 + \partial\psi_3/\partial x_2 - \partial\psi_2/\partial x_3,$$

where ϕ is a scalar function and ψ is a vector field with three components ψ_1, ψ_2, ψ_3 . The requirement (1) leaves ψ_i indeterminate to the extent that its divergence is arbitrary. For definiteness, we may impose the condition

$$(3) \quad \psi_{i,i} = 0.$$

Then ϕ is called the *scalar potential* and ψ the *vector potential* of the vector field \mathbf{u} . We shall prove this theorem presently by showing that the functions ϕ and ψ_1, ψ_2, ψ_3 so proposed can be found.

Applying Eq. (1) to the elastic displacement vector u_i , we shall see that *the dilatation e can be derived from the scalar potential ϕ and that the rotation can be derived from the vector potential ψ .* Note first that if Eq. (1) is effected, then

$$(4) \quad u_{i,i} = \phi_{,ii} + e_{ijk}\psi_{k,ji}.$$

The last term vanishes when summed in pairs since $\psi_{k,ji} = \psi_{k,ij}$ but $e_{ijk} = -e_{jik}$. Hence,

$$(5) \quad e = u_{i,i} = \phi_{,ii}.$$

$$\begin{aligned} e_{ijk}u_{k,j} &= e_{ijk}\phi_{,kj} + e_{ijk}e_{klm}\psi_{m,lj} = e_{kij}e_{klm}\psi_{m,lj} \quad [\text{by symmetry of } \phi_{,kj}] \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\psi_{m,lj} \quad [\text{by the } e - \delta \text{ identity and (Eq. 2.1:11)}] \end{aligned}$$

$$(6) \quad = -\psi_{i,jj} + \psi_{j,ij} = -\psi_{i,jj} \quad [\text{by virtue of Eq. (3)}].$$

The curl of u_i , the left-hand side of Eq. (6), is twice the rotation ω_i , hence

$$(7) \quad \omega_i = e_{ijk}u_{k,j}/2, \quad \psi_{i,jj} = -2\omega_i.$$

This shows that each component of the vector ψ_i is related to a component of the rotation vector field.

Now, particular solutions of Eqs. (5) and (7) are known in the theory of Newtonian potentials. They are

$$(8) \quad \phi(x_1, x_2, x_3) = -\frac{1}{4\pi} \iiint \frac{e(x'_1, x'_2, x'_3)}{r} dx'_1 dx'_2 dx'_3,$$

$$(9) \quad \bar{\psi}_i(x_1, x_2, x_3) = \frac{1}{2\pi} \iiint \frac{\omega_i(x'_1, x'_2, x'_3)}{r} dx'_1 dx'_2 dx'_3,$$

where the integration extends over the entire body and

$$r^2 = (x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2.$$

But $\bar{\psi}_i$ given in Eq. (9) does not always satisfy Eq. (3). To overcome this difficulty, we note that the curl of the gradient of a scalar function vanishes, and a general vector potential is

$$(10) \quad \psi_i = \bar{\psi}_i + \theta_{,i},$$

where θ is a scalar function. The added term $\theta_{,i}$ contributes nothing to u_i . However, θ can be determined in such a manner that

$$(11) \quad \theta_{,ii} = -\bar{\psi}_{i,i}$$

by an integral analogous to Eq. (8). With θ so determined, the function ψ_i given by Eq. (10) will satisfy the requirement Eq. (3).

The general solution of Eq. (5) is the sum of Eq. (8) and a harmonic function ϕ^* . The general solution of Eq. (7) is the sum of Eq. (10) and a set of harmonic functions ψ_i^* . Thus, Eq. (1), considered as a set of partial differential equations for the unknown functions ϕ and ψ , can be solved.

Love shows (*Mathematical Theory of Elasticity*,^{1.1} p. 48), through integration by parts of Eqs. (8) and (9), that the following resolution meets all the potentials requirements:

$$(12) \quad \phi = -\frac{1}{4\pi} \iiint u_i(x'_1, x'_2, x'_3) \frac{\partial}{\partial x_i} \left(\frac{1}{r} \right) dx'_1 dx'_2 dx'_3,$$

$$(13) \quad \psi_i = \frac{1}{4\pi} \iiint e_{ijk} u_k(x'_1, x'_2, x'_3) \frac{\partial}{\partial x_j} \left(\frac{1}{r} \right) dx'_1 dx'_2 dx'_3.$$

Pendse (Phil. Mag. Ser. 7, 39: 862–867, 1948) pointed out, however, that the use of scalar and vector potentials is not always free from ambiguity.

8.2. EQUATIONS OF MOTION IN TERMS OF DISPLACEMENT POTENTIALS

Let the displacement vector field u_1, u_2, u_3 be represented by a scalar potential $\phi(x_1, x_2, x_3, t)$ and a triple of vector potentials $\psi_i(x_1, x_2, x_3, t)$, $i = 1, 2, 3$, so that

$$(1) \quad u_i = \partial\phi/\partial x_i + e_{ijk}\partial\psi_k/\partial x_j, \quad \psi_{i,i} = 0.$$

We shall assume that the body force is zero. [If this is not the case, the body force term can be removed by a particular integral such as Kelvin's, see Eq. (8.8:9). See also Sec. 8.18.] We can write the Navier equation of linear elasticity for a homogeneous, isotropic body,

$$(2a) \quad \rho \partial^2 u_i / \partial t^2 = G u_{i,jj} + (\lambda + G) u_{j,ji}$$

$$(2) \quad \begin{aligned} \rho \partial(\partial^2 \phi / \partial t^2) / \partial x_i + \rho e_{ijk} \partial(\partial^2 \psi_k / \partial t^2) / \partial x_j \\ = (\lambda + G) \partial \nabla^2 \phi / \partial x_i + G \partial \nabla^2 \phi / \partial x_i + G e_{ijk} \partial \nabla^2 \psi_k / \partial x_j. \end{aligned}$$

The fact that $\psi_{i,i} = 0$, [Eq. (8.1:3)] is used here. Furthermore, since the displacements are assumed to be infinitesimal, any change in the density ρ is an infinitesimal of the first order, as can be seen easily from the equation of the continuity [Eq. (5.4:4)]. Since in Eq. (2a), ρ is multiplied by small quantities of the first order, u_i , it can be treated as a constant. Now, it is easy to see that Eq. (2) will be satisfied if the functions ϕ and ψ_i are solutions of the equations

$$(3) \quad \nabla^2 \phi = (\partial^2 \phi / \partial t^2) / c_L^2, \quad \nabla^2 \psi_k = (\partial^2 \psi_k / \partial t^2) / c_T^2,$$

where

$$(4) \quad c_L = [(\lambda + 2G)/\rho]^{1/2}, \quad c_T = (G/\rho)^{1/2}.$$

These are wave equations. They indicate that two types of disturbances with velocities c_L and c_T may be propagated through an elastic solid. A comparison with Eqs. (7.8:3) and (7.8:5) shows that c_L and c_T are the speeds of plane longitudinal and transverse waves, respectively. These waves are also referred to as *dilatational* and *distorsional* waves. The latter are also known as *shear*, *equivoluminal*, or *rotational* waves. If we write e for the dilatation $u_{i,i}$ and $\omega(\omega_1, \omega_2, \omega_3)$ for the rotation, then, by Eqs. (8.1:5) and (8.1:7), we have, from Eq. (3),

$$(5) \quad \nabla^2 e = (\partial^2 e / \partial t^2) / c_L^2, \quad \nabla^2 \omega_k = (\partial^2 \omega_k / \partial t^2) / c_T^2.$$

Thus, by introducing the potentials ϕ and ψ_1, ψ_2, ψ_3 , we have reduced the problems of linear elasticity to that of solving the wave equations.

If static equilibrium is considered, then all derivatives with respect to time vanish, and we see that Eq. (2) is satisfied if

$$(6) \quad \nabla^2 \phi = \text{const.}, \quad \nabla^2 \psi_i = \text{const.}$$

In the following sections we shall consider some applications to statics based on Eqs. (6).

Of course, not all solutions of Eq. (2) are given by Eqs. (3) and (6). We shall now consider a more general solution. Such a generalization often becomes important when we wish to examine the limiting case of a dynamic problem as the loading tends to a steady state. For example, if we consider a load suddenly applied at $t = 0$ on a body and then kept constant afterwards, we may question whether the steady-state solution is given by Eqs. (3) with $\partial\phi/\partial t, \partial\psi_k/\partial t$ terms set to zero. A quick comparison with Eq. (6) shows that this may not be the case if the constants in Eq. (6) do not vanish.

Let us differentiate Eq. (2) with respect to x_m .

$$(7) \quad \begin{aligned} & \rho \partial^2 (\partial^2 \phi / \partial t^2) / (\partial x_m \partial x_i) + \rho e_{ijk} \partial^2 (\partial^2 \psi_k / \partial t^2) / (\partial x_m \partial x_j) \\ &= (\lambda + G) \partial^2 \nabla^2 \phi / (\partial x_m \partial x_i) + G \partial^2 \nabla^2 \phi / (\partial x_m \partial x_i) \\ &+ G e_{ijk} \partial^2 \nabla^2 \psi_k / (\partial x_m \partial x_j). \end{aligned}$$

A contraction of m with i yields

$$(8) \quad \nabla^2 [\nabla^2 \phi - (\partial^2 \phi / \partial t^2) / c_L^2] = 0.$$

A multiplication of (7) with e_{imb} summing over m and summing over i , and using Eq. (8.1:3), yields, on the other hand,

$$(9) \quad \nabla^2 [\nabla^2 \psi_k - (\partial^2 \psi_k / \partial t^2) / c_T^2] = 0.$$

Hence,

$$(10) \quad \nabla^2 \phi - (\partial^2 \phi / \partial t^2) / c_L^2 = \Phi, \quad \nabla^2 \Phi = 0,$$

$$(11) \quad \nabla^2 \psi_k - (\partial^2 \psi_k / \partial t^2) / c_T^2 = \Psi_k, \quad \nabla^2 \Psi_k = 0,$$

where $\Phi, \Psi_1, \Psi_2, \Psi_3$, are harmonic functions. The Φ and Ψ 's are not entirely independent, but are connected through Eq. (2)

$$(12) \quad c_L^2 (\partial \Phi / \partial x_i) / c_T^2 + e_{ijk} \partial \Psi_k / \partial x_j = 0.$$

Equations (10)–(12) give the general solution of (2). Equations (3) are obtained by setting $\Phi = \Psi_k = 0$; Eqs. (6), by taking Φ and Ψ_k to be constants and by letting time derivatives vanish.

8.3. STRAIN POTENTIAL

In this section we shall consider the static equilibrium of elastic bodies in the restricted case in which the components of displacements can be derived from a scalar function $\phi(x_1, x_2, x_3)$ so that

$$(1) \quad 2G u_i = \partial \phi / \partial x_i.$$

The function ϕ is called Lamé's *strain potential*. With Eq. (1), the dilatation e , the strain tensor e_{ij} , and the stress tensor σ_{ij} are

$$(2) \quad 2Ge = 2Gu_{i,i} = \phi_{,ii},$$

$$(3) \quad e_{ij} = (u_{i,j} + u_{j,i})/2 = \phi_{,ij}/(2G),$$

$$(4) \quad \sigma_{ij} = \lambda\delta_{ij}e + 2Ge_{ij} = \lambda\delta_{ij}e + \phi_{,ij}.$$

In the absence of body force, the function ϕ must satisfy Eqs. (8.2:6). Since our objective is to obtain some solution, not necessarily general, the constant in (8.2:6) will be chosen as zero. Then

$$(5) \quad \nabla^2\phi = 0,$$

a *harmonic function* by definition. Under this choice, we have

$$(6) \quad \sigma_{ij} = \phi_{,ij}.$$

In unabridged notations with respect to rectangular Cartesian coordinates x, y, z , Eqs. (2)–(6) are

$$(7) \quad 2Gu = \partial\phi/\partial x, \quad 2Gv = \partial\phi/\partial y, \quad 2Gw = \partial\phi/\partial z,$$

$$(8) \quad 2Ge = \nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0,$$

$$(9) \quad \sigma_{xx} = \partial^2\phi/\partial x^2, \quad \sigma_{xy} = \partial^2\phi/(\partial x\partial y), \quad \text{etc.}$$

Corresponding formulas for curvilinear coordinates can be obtained by converting Eqs. (1)–(6) into general tensor equations. The results for cylindrical coordinates r, θ, z are as follows, where $(\xi_r, \xi_\theta, \xi_z)$ and $(\sigma_{rr}, \sigma_{r\theta}, \text{etc.})$ denote the physical components of displacement and stress, respectively.

$$(10) \quad 2G\xi_r = \partial\phi/\partial r, \quad 2G\xi_\theta = (\partial\phi/\partial\theta)/r, \quad 2G\xi_z = \partial\phi/\partial z,$$

$$(11) \quad \nabla^2\phi = \partial^2\phi/\partial r^2 + (\partial\phi/\partial r)/r + (\partial^2\phi/\partial\theta^2)/r^2 + \partial^2\phi/\partial z^2 = 0,$$

$$(12) \quad \sigma_{rr} = \partial^2\phi/\partial r^2, \quad \sigma_{\theta\theta} = (\partial\phi/\partial r)/r + (\partial^2\phi/\partial\theta^2)/r^2, \quad \sigma_{zz} = \partial^2\phi/\partial z^2,$$

$$(13) \quad \sigma_{r\theta} = \partial^2(\phi/r)/(\partial r\partial\theta), \quad \sigma_{\theta z} = \partial^2(\phi/r)/(\partial\theta\partial z), \quad \sigma_{zz} = \partial^2\phi/(\partial z\partial r).$$

Harmonic functions are well-known. For example, it can easily be verified that the functions

$$(14) \quad A(x^2 - y^2) + Bxy, \quad Cr^n \cos n\theta, \quad C \log(r/a), \\ C\theta, \quad \theta = \tan^{-1}(y/x), \quad r^2 = x^2 + y^2;$$

$$(15) \quad C/R, \quad C \log(R+z), \quad C \log[(R_1 + z - c)(R_2 - z - c)/r^2], \\ R^2 = x^2 + y^2 + z^2, \quad R_1^2 = r^2 + (z - c)^2, \quad R_2^2 = r^2 + (z + c)^2,$$

are harmonic functions. Combinations of them can solve problem of elasticity. Any combination of Eqs. (14) corresponds to a plane strain condition.

To the stress systems derived from these potentials, we may add the uniform stress distributions given by

$$(16) \quad \phi = Cr^2 = C(x^2 + y^2) \quad \text{or} \quad \phi = CR^2 = C(x^2 + y^2 + z^2),$$

which are special solutions of Eq. (8.2:6). With Eqs. (16), Eqs. (8)–(9) and (11)–(13) must be modified because $e \neq 0$.

Example. Hollow Spheres Subjected to Internal and External Pressure

Let a hollow sphere of inner radius a and outer radius b be subjected to an internal pressure p and an external pressure q . Because of the spherical symmetry of the problem, it will be advantageous to use spherical coordinates R, θ, φ . All shearing stresses $\sigma_{R\theta}, \sigma_{\varphi\theta}$, etc., vanish on account of symmetry and the solution is a function of R only.

The solution is furnished by the strain potential (15) and (16),

$$\phi = C/R + DR^2, \quad R^2 = x^2 + y^2 + z^2.$$

Hence, at the point $(R, 0, 0)$,

$$\sigma_{RR} = \sigma_{xx} = \frac{\partial^2 \phi}{\partial x^2} = \frac{2C}{R^3} + 2D, \quad \sigma_{\theta\theta} = \sigma_{yy} = \frac{\partial^2 \phi}{\partial y^2} = -\frac{C}{R^3} + 2D.$$

The constants C and D are easily determined from the boundary conditions,

$$\sigma_{RR} = -p \quad \text{when } R = a, \quad \sigma_{RR} = -q \quad \text{when } R = b.$$

The results are

$$\begin{aligned}\sigma_{RR} &= -p[(b/R)^3 - 1]/[(b/a)^3 - 1] - q[1 - (a/R)^3]/[1 - (a/b)^3], \\ \sigma_{\theta\theta} &= p[(b/R)^3 + 2]/[2(b/a)^3 - 2] - q[2 + (a/R)^3]/[2 - 2(a/b)^3].\end{aligned}$$

Problem 8.1. Find the solution for a solid sphere subjected to a uniform external pressure q .

Problem 8.2. Let the inner wall of the hollow spherical shell, at $R = a$, be rigid (a condition realized by filling the hole with some incompressible material). When the outer surface $R = b$ is subjected to a uniform pressure q , what are the stresses at the inner wall?

Problem 8.3. Show that the function $\phi = C\theta$ solves the problem of a circular disk subjected to uniformly distributed tangential shear on the circumference.

8.4. GALERKIN VECTOR

We have shown in Sec. 8.2 that when the displacement field u_i is expressed in terms of scalar and vector potentials, ϕ and ψ_1, ψ_2, ψ_3 ,

$$(1) \quad 2Gu_i = \phi_{,i} + e_{ijk}\psi_{k,j}, \quad \psi_{i,i} = 0$$

respectively, the equations of static equilibrium for a homogeneous isotropic linear elastic medium are satisfied if

$$(2) \quad \nabla^2\phi = \text{const.}, \quad \nabla^2\psi_k = \text{const.}$$

Thus, a broad class of problems in elastic equilibrium is solved by determining the four functions $\phi, \psi_1, \psi_2, \psi_3$. However, this may not be the general solution. The general solution (8.2:10–12) is more complex.

In searching for other solutions of equal generality, Galerkin introduced, in papers published in 1930, displacement potential functions which satisfy biharmonic equations. Papkovich noted in 1932 that Galerkin's functions are components of a vector. It is perhaps simpler to introduce the Galerkin vector by considering the vector potential ψ_k itself as being generated by another vector field \tilde{F}_i ,

$$(3) \quad \psi_k = -e_{klm}c\tilde{F}_{m,l},$$

where c is a constant. Then Eq. (1) becomes

$$(4) \quad 2Gu_i = \phi_{,i} - e_{ijk}e_{klm}c\tilde{F}_{m,lj}.$$

This can be simplified by means of the e - δ identity [Eq. (2.1:11)] into

$$(5) \quad 2Gu_i = \phi_{,i} - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})c\tilde{F}_{m,lj} = \phi_{,i} + c\tilde{F}_{i,jj} - c\tilde{F}_{j,ji}.$$

But $c\tilde{F}_{j,ji}$ is a scalar function, and can be specified arbitrarily without disturbing the definition (3). This suggests the representation¹

$$(6) \quad 2Gu_i = cF_{i,jj} - F_{j,ji}.$$

By a judicious choice of the constant c , we can simplify the Navier Eq. (7.1:8),

$$(\lambda + G)u_{j,jk} + Gu_{k,mm} + X_k = 0,$$

where X_k represents the body force per unit volume. On substituting Eq. (6) into the Navier equation, and remembering that $\lambda + G = G/(1 - 2\nu)$, we obtain

$$(cF_{i,jji} - F_{j,jii}),_k / (1 - 2\nu) + (cF_{k,jj} - F_{j,jk}),_{mm} + 2X_k = 0.$$

With a proper change of dummy indices, this is

$$[(c-1)/(1-2\nu)-1]F_{j,jiik}+cF_{k,jjii}+2X_k=0.$$

The coefficient of the first term vanishes if $c = 2(1-\nu)$. Therefore, we conclude that *the basic equation of elasticity is satisfied when*

$$(7) \quad \Delta \quad 2Gu_i = 2(1-\nu)F_{i,jj} - F_{j,ji},$$

if F_i satisfies the equation

$$(8) \quad \Delta \quad F_{i,jjm} = -X_i/(1-\nu).$$

The vector F_i is called the Galerkin vector. If $X_i = 0$, Eq. (8) is biharmonic and its solutions are called biharmonic functions. Thus, to solve a problem in static equilibrium is to determine the three functions F_1, F_2, F_3 .

From Eq. (7), we can derive expressions for the stresses in terms of F_i . In unabridged notation, we have the following results:

$$(9) \quad \nabla^4 F_1 = -X_1/(1-\nu), \quad \nabla^2 A \equiv A_{,ii} \equiv \partial^2 A / \partial x_1^2 + \partial^2 A / \partial x_2^2 + \partial^2 A / \partial x_3^2,$$

$$\nabla \cdot \mathbf{F} \equiv F_{i,i} \equiv \partial F_1 / \partial x_1 + \partial F_2 / \partial x_2 + \partial F_3 / \partial x_3,$$

$$(10) \quad 2Gu_1 = 2(1-\nu)\nabla^2 F_1 - \partial F_{j,j} / \partial x_1,$$

$$(11) \quad 2Gu_{i,i} = (1-2\nu)\nabla^2 F_{j,j},$$

$$(12) \quad \sigma_{11} = 2(1-\nu)\partial\nabla^2 F_1 / \partial x_1 + \nu\nabla^2 F_{j,j} - \partial^2 F_{j,j} / \partial x_1^2,$$

$$(13) \quad \sigma_{12} = (1-\nu)(\partial\nabla^2 F_1 / \partial x_2 + \partial\nabla^2 F_2 / \partial x_1) - \partial^2 F_{j,j} / (\partial x_1 \partial x_2),$$

$$(14) \quad \sigma_{ii} = (1+\nu)\nabla^2 F_{j,j}.$$

Other components $u_2, u_3, \sigma_{22}, \sigma_{33}$, etc., are obtained by cyclic permutation of the subscripts 1, 2, 3.

Problem 8.4. Determine the body forces, stresses, and displacements defined by the following Galerkin vectors in rectangular Cartesian coordinates: (a) $F_1 = F_2 = 0, F_3 = R^2$, (b) $F_1 = yR^2, F_2 = -xR^2, F_3 = 0$, and (c) $F_1 = F_2 = 0, F_3 = z^4$, where $R^2 = x^2 + y^2 + z^2$.

Problem 8.5. Show that the Lamé strain potential ϕ can be identified with the divergence $-F_{i,b}$ if the body force vanishes and F_i is harmonic.

8.5. EQUIVALENT GALERKIN VECTORS

Let \bar{F}_i and \tilde{F}_i be two equivalent Galerkin vectors defining the same displacements. Then their difference

$$(1) \quad F_i = \bar{F}_i - \tilde{F}_i$$

is a biharmonic vector and corresponds to zero displacement, where u_i is defined in Eq. (8.4:7). A general form of F_i can be obtained, as in Sec. 8.4,

$$(2) \quad F_i = (1-2\nu)f_{i,jj} + f_{j,ji}, \quad \text{provided that } f_{i,jjkk} = 0.$$

As an application, let us consider a Galerkin vector whose third component does not vanish,

$$(3) \quad \bar{F}_1, \bar{F}_2, \bar{F}_3 \neq 0.$$

An equivalent \tilde{F}_i with $\tilde{F}_3 = 0$ can be formed if the body force $X_i = 0$. A particular solution is obtained by taking a special vector f_i

$$(4) \quad f_1 = f, \quad f_2 = f_3 = 0.$$

Then, from Eqs. (2) and (3),

$$(5) \quad F_1 = \bar{F}_1 - \tilde{F}_1 = [(1-2\nu)\nabla^2 + \partial^2 / \partial x^2]f,$$

$$(6) \quad F_2 = \bar{F}_2 - \tilde{F}_2 = \partial^2 f / (\partial x \partial y),$$

$$(7) \quad F_3 = \bar{F}_3 - \bar{\bar{F}}_3 = \bar{F}_3 = \partial^2 f / (\partial x \partial z), \quad \bar{\bar{F}}_3 = 0.$$

Now, since $X_i = 0$ implies that \bar{F}_3 is biharmonic, it will be possible to determine a biharmonic function f that satisfies Eq. (7). When f is so determined, we could compute \bar{F}_1 and \bar{F}_2 according to Eqs. (5) and (6). Thus, *when the body forces are zero, any Galerkin vector has an equivalent Galerkin vector with a zero component in the direction of z .*

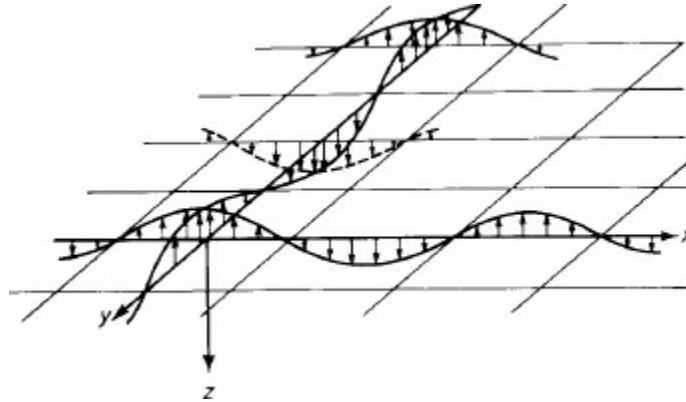


Fig. 8.6:1. Sinusoidally distributed load acting on a semi-infinite solid.

8.6. EXAMPLE – VERTICAL LOAD ON THE HORIZONTAL SURFACE OF A SEMI-INFINITE SOLID

Consider a semi-infinite homogeneous, isotropic, linear elastic solid which occupies the space $z \geq 0$ as shown in Fig. 8.6:1, and is subjected to a sinusoidally distributed vertical load on the boundary surface $z = 0$, so that the boundary conditions are

$$(1) \quad \sigma_{zz} = C \cos(\pi x/l) \cos(\pi y/L), \quad \sigma_{zx} = \sigma_{zy} = 0, \quad \text{at } z = 0;$$

The boundary condition at $z = \infty$ is the vanishing of all stress components.

The problem can be solved by taking a Galerkin vector with one nonvanishing component,

$$(2) \quad F_1 = F_2 = 0, \quad F_3 = Z.$$

The function Z must be biharmonic. It can be easily verified that, if ψ is a harmonic function, then the following function is biharmonic (cf. Sec. 8.11):

$$(3) \quad Z = (A + Bcz)\psi.$$

The form of the 1st boundary condition of (1) suggests that we try

$$(4) \quad \psi = \cos(\pi x/l) \cos(\pi y/L)f(z).$$

On substituting (4) into the Laplace equation $\nabla^2\psi = 0$, one finds that

$$(5) \quad f(z) = e^{-cz}, \quad \text{where } c = [(\pi/l)^2 + (\pi/L)^2]^{1/2}.$$

The other solution e^{cz} is rejected by the boundary condition at ∞ .

The stresses corresponding to the Galerkin vector above can be easily derived. Note that

$$(6) \quad \begin{aligned} \nabla^2 Z &= 2Bc\partial\psi/\partial z = -2Bc^2\psi, & \partial Z/\partial z &= -cA\psi + Bc(1-cz)\psi, \\ (\partial^2/\partial x^2 + \partial^2/\partial y^2)Z &= -(A + Bcz)c^2\psi. \end{aligned}$$

Hence, by Eqs. (8.4:13),

$$(7) \quad \sigma_{zx} = \partial[(2\nu B - A - Bcz)c^2\psi]/\partial x.$$

The 2nd and 3rd boundary conditions of (1), that σ_{zx} and σ_{zy} vanish at the surface $z = 0$, are satisfied if $A = 2\nu B$. Then, Eq. (8.4:12) yields

$$(8) \quad \sigma_{zz} = Bc^3(1 + cz)\psi = Bc^3(1 + cz)\cos(\pi x/l)\cos(\pi y/L)e^{-cz}.$$

A comparison with the 1st boundary condition of (1) shows that all boundary conditions are satisfied by taking

$$(9) \quad B = \mathcal{A}/c^3, \quad A = 2\nu\mathcal{A}/c^3.$$

The resulting displacements and stresses are as follows, where we write, for convenience,

$$(10) \quad \alpha = \pi/l, \quad \beta = \pi/L,$$

$$u_x = \mathcal{A}\alpha(-1 + 2\nu + cz)(\sin \alpha x \cos \beta y)e^{-cz}/(2Gc^2),$$

$$u_y = \mathcal{A}\beta(-1 + 2\nu + cz)(\cos \alpha x \sin \beta y)e^{-cz}/(2Gc^2),$$

$$u_z = \mathcal{A}[2(1 - \nu) + cz](\cos \alpha x \cos \beta y)e^{-cz}/(2Gc),$$

$$(11) \quad \sigma_{xx} = \mathcal{A}(\alpha^2 cz - \alpha^2 - 2\nu\beta^2)\psi/c^2, \quad \sigma_{yy} = \mathcal{A}(\beta^2 cz - \beta^2 - 2\nu\alpha^2)\psi/c^2,$$

$$\sigma_{zz} = -\mathcal{A}(1 + cz)\psi, \quad \sigma_{xy} = \mathcal{A}\alpha\beta(1 - 2\nu - cz)(\sin \alpha x \sin \beta y)e^{-cz}/c^2,$$

$$\sigma_{yz} = -\mathcal{A}\beta z(\cos \alpha x \sin \beta y)e^{-cz}, \quad \sigma_{zx} = -\mathcal{A}\alpha z(\sin \alpha x \cos \beta y)e^{-cz}.$$

Thus, stresses are attenuated exponentially as the depth z is increased; the rate of attenuation c depends on the wave lengths l and L .

Solutions of this form may be superposed together to produce further solutions. The method of Fourier series may be used to obtain periodic loadings on the surface $z = 0$, and the method of Fourier integral may be used to obtain more general loadings.

8.7. LOVE'S STRAIN FUNCTION

A Galerkin vector that has only one nonvanishing component F_3 is called *Love's strain function* and shall be denoted by

$$(1) \quad F_3 = Z.$$

In many applications it may be desired to express Z both in rectangular coordinates x, y, z and in cylindrical coordinates r, θ, z . In both cases,

$$(2) \quad \nabla^4 Z = -X_z/(1 - \nu),$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

and X_z is the body force per unit volume in the z -direction — the only nonvanishing component that can be treated by such a strain function.

On putting $F_1 = F_2 = 0$ and $F_3 = Z$ in Eqs. (8.4:10–14), we obtain the following physical components of displacements and stresses:

In rectangular coordinates:

$$2Gu_x = -\frac{\partial^2 Z}{\partial x \partial z}, \quad 2Gu_y = -\frac{\partial^2 Z}{\partial y \partial z}, \quad 2Gu_z = [2(1 - \nu)\nabla^2 - \frac{\partial^2}{\partial z^2}]Z,$$

$$\theta = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = (1 + \nu)\partial(\nabla^2 Z)/\partial z,$$

$$(3) \quad \sigma_{xx} = \partial(\nu \nabla^2 Z - \partial^2 Z/\partial x^2)/\partial z,$$

$$\sigma_{yy} = \partial(\nu \nabla^2 Z - \partial^2 Z/\partial y^2)/\partial z, \quad \sigma_{zz} = \partial[(2 - \nu)\nabla^2 Z - \partial^2 Z/\partial z^2]/\partial z,$$

$$\sigma_{xy} = -\partial^3 Z/(\partial x \partial y \partial z),$$

$$\sigma_{zx} = \partial[(1 - \nu)\nabla^2 Z - \partial^2 Z/\partial z^2]/\partial x, \quad \sigma_{zy} = \partial[(1 - \nu)\nabla^2 Z - \partial^2 Z/\partial z^2]/\partial y.$$

In cylindrical coordinates:

$$\begin{aligned}
2G\xi_r &= -\partial^2 Z / (\partial r \partial z), \quad 2G\xi_z = 2(1-\nu) \nabla^2 Z - \partial^2 Z / \partial z^2, \\
2G\xi_\theta &= -[\partial^2 Z / (\partial \theta \partial z)]/r, \quad \theta = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = (1+\nu) \partial(\nabla^2 Z) / \partial z, \\
(4) \quad \sigma_{rr} &= \frac{\partial}{\partial z} (\nu \nabla^2 - \frac{\partial^2}{\partial r^2}) Z, \quad \sigma_{\theta\theta} = \frac{\partial}{\partial z} (\nu \nabla^2 - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}) Z, \quad \sigma_{zz} = \partial[(2-\nu) \nabla^2 Z - \partial^2 Z / \partial z^2] / \partial z, \\
\sigma_{r\theta} &= -\partial^3 (Z/r) / (\partial r \partial \theta \partial z), \\
\sigma_{\theta z} &= \{\partial[(1-\nu) \nabla^2 Z - \partial^2 Z / \partial z^2] / \partial \theta\} / r, \quad \sigma_{zr} = \partial[(1-\nu) \nabla^2 Z - \partial^2 Z / \partial z^2] / \partial r.
\end{aligned}$$

Love introduced, in 1906, the strain function Z as a function of r and z only in treating solids of revolution under axis-symmetric loading.

8.8. KELVIN'S PROBLEM – A SINGLE FORCE ACTING IN THE INTERIOR OF AN INFINITE SOLID

Let a force $2P$ be applied at the origin in the direction of z (Fig. 8.8:1). This concentrated force may be regarded as the limit of a system of loads applied on the surface of a small cavity at the origin. The boundary conditions of the problem are: (1) At infinity, all stresses vanish. (2) At the origin, the stress singularity is equivalent to a concentrated force of magnitude $2P$ in the z -direction.

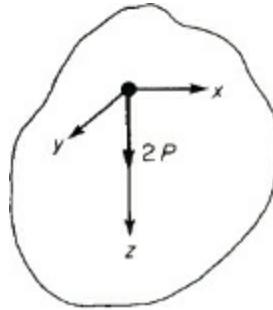


Fig. 8.8:1. Kelvin's problem.

The symmetry of the problem suggests the use of cylindrical coordinates, and Love's strain function $Z(r, z)$ suggests itself. Since body force is absent, Z must be a biharmonic function whose third partial derivatives should define stresses that vanish at infinity, but have a singularity at the origin. One such function is

$$(1) \quad Z = BR = B(z^2 + r^2)^{1/2}$$

(the function $\frac{1}{R}$ is harmonic, hence $R = R^2 \frac{1}{R}$ is biharmonic, see [Theorem 2, Sec. 8.11](#)), for which

$$(2) \quad \partial Z / \partial r = Br / R, \quad \partial^2 Z / \partial r^2 = B(1 - r^2 / R^2) / R = Bz^2 / R^3,$$

$$\partial Z / \partial z = Bz / R, \quad \partial^2 Z / \partial z^2 = B(1 - z^2 / R^2) / R = Br^2 / R^3, \quad \nabla^2 Z = 2B / R.$$

Therefore,

$$\begin{aligned}
(3) \quad 2G\xi_r &= Brz / R^3, \quad 2G\xi_\theta = 0, \quad 2G\xi_z = B(4 - 4\nu - r^2 / R^2) / R, \\
\sigma_{rr} &= Bz(1 - 2\nu - 3r^2 / R^2) / R^3, \quad \sigma_{\theta\theta} = B(1 - 2\nu)z / R^3, \quad \sigma_{r\theta} = \sigma_{z\theta} = 0, \\
\sigma_{zz} &= -Bz(1 - 2\nu + 3z^2 / R^2) / R^3, \quad \sigma_{rz} = -Br(1 - 2\nu + 3z^2 / R^2) / R^3.
\end{aligned}$$

These stresses are singular at the origin and vanish at infinity, and have the correct symmetry. The stress singularity is equivalent to a vertical force. Thus, Eq. (1) is indeed the desired solution if this force can be made equal to $2P$ by a proper choice of the constant B . To determine B , we consider a cylinder with a cavity at its center at the origin and with bases at $z = \pm a$ (Fig. 8.8:2). Since this cylinder is in equilibrium, the resultant of surface tractions on the outer surface must balance the load $2P$ acting on the surface of the cavity at the origin. Therefore, we must have, when the radius of this cylinder tends to infinity,

$$2P = \int_0^\infty 2\pi r dr (-\sigma_{zz})_{z=a} + \int_0^\infty 2\pi r dr (\sigma_{zz})_{z=-a} \\ + \lim_{r \rightarrow \infty} \int_{-a}^a 2\pi r dz (\sigma_{rz}).$$

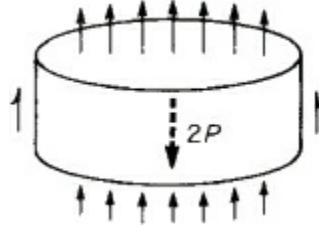


Fig. 8.8:2. Boundary used in evaluating the integration constant.

The values of the 1st and the 2nd integrals are seen to be the same, and the 3rd integral vanishes in the limit. Noting that $rdr = RdR$, we have the solution

$$(4) \quad \frac{P}{4\pi(1-\nu)} = \int_a^\infty \frac{(-\sigma_{zz})_{z=a}}{2(1-\nu)} RdR = B \int_a^\infty \left(1 - 2\nu + \frac{3a^2}{R^2}\right) \frac{aRdR}{2(1-\nu)R^3} = B.$$

An especially simple situation results when Poisson's ratio is $\frac{1}{2}$. Then the factor $(1 - 2\nu)$ vanishes, and the stresses become

$$(5) \quad \sigma_{rr} = -\frac{3Pr^2z}{2\pi R^5}, \quad \sigma_{zz} = -\frac{3Pz^3}{2\pi R^5}, \quad \sigma_{\theta\theta} = \sigma_{r\theta} = \sigma_{z\theta} = 0, \quad \sigma_{rz} = -\frac{3Prz^2}{2\pi R^5}.$$

These results appear even simpler if the stresses are resolved in directions of spherical coordinates. In any meridional plane (see Fig. 8.8:3), at a point whose coordinates are (r, z) in cylindrical and (R, φ) in spherical coordinates, we find

$$(6) \quad \begin{aligned} \sigma_{RR} &= \sigma_{rr} \sin^2 \varphi + \sigma_{zz} \cos^2 \varphi + 2\sigma_{rz} \sin \varphi \cos \varphi \\ &= \sigma_{rr} r^2/R^2 + \sigma_{zz} z^2/R^2 + 2\sigma_{rz} rz/R^2 = -3Pz/(2a\pi R^3). \end{aligned}$$

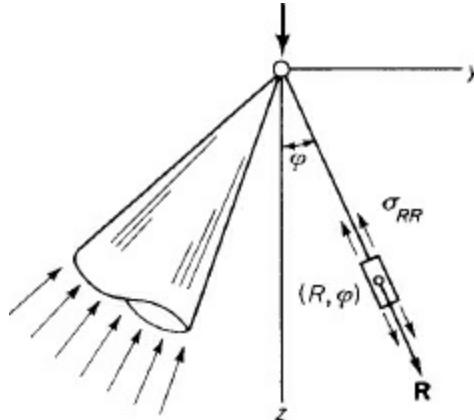


Fig. 8.8:3. Simple interpretation of the solution to Kelvin's problem when Poisson's ratio is 1/2.

Similarly, we have

$$(7) \quad \sigma_{\varphi\varphi} = \sigma_{R\varphi} = \sigma_{\theta\theta} = 0,$$

and, by symmetry,

$$(8) \quad \sigma_{R\theta} = \sigma_{\varphi\theta} = 0.$$

Hence, σ_{RR} , $\sigma_{\varphi\varphi}$, $\sigma_{\theta\theta}$ are principal stresses, and the only nonvanishing component is σ_{RR} . If the solid were divided into many cones extending from a common vertex at the origin, each of these cones would transmit its own radial force without reaction from the adjacent cones.

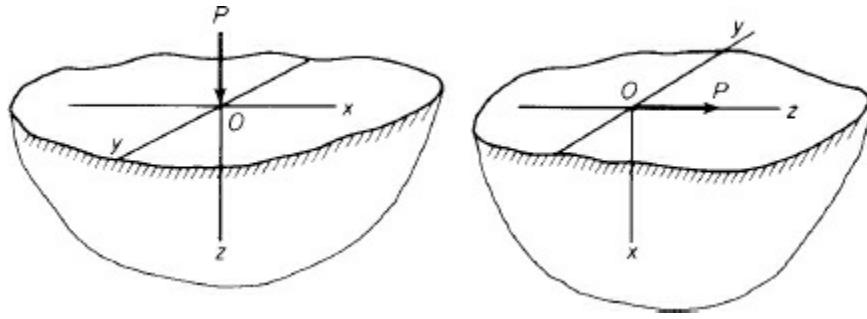


Fig. 8.8:4. Boussinesq and Cerruti's problems.

In the special case of Poisson's ratio $\nu = 1/2$, it is evident that the solution of the Kelvin problem also furnishes the solution to Boussinesq's problem of a normal force and Cerruti's problem of a tangential force acting on the boundary of a semi-infinite solid (see Fig. 8.8:4). When $\nu = 1/2$, Eqs. (5)–(8) hold for Boussinesq's problem when the solid occupies the space $z \geq 0$, and for Cerruti's problem when the solid occupies the space $x \geq 0$.

The observation that the solutions of Boussinesq's and Cerruti's problems take on such simple form when Poisson's ratio is $\frac{1}{2}$ led Westergaard to consider a method of solution for these problems by perturbation of Poisson's ratio, a method to be described in the next section.

Originally, Lord Kelvin, Boussinesq, and Cerruti obtained solutions to the problems which now are adorned by their names (in 1848, 1878, 1882, respectively) by an extension of the method of singularities in the theory of Newtonian potentials. Lord Kelvin (1848) discovered that a particular solution of Navier's Eq. (7.1:8) is

$$(9) \quad u_i(x) = A \iiint \left[B \frac{X_i(\xi)}{r} - (x_j - \xi_j) X_j(\xi) \frac{\partial}{\partial x_i} \left(\frac{1}{r} \right) \right] d\xi_1 d\xi_2 d\xi_3,$$

where

$$r = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{1/2}, \\ A = (\lambda + G)/[8\pi G(\lambda + 2G)], \quad B = (\lambda + 3G)/(\lambda + G).$$

In this formula x stands for (x_1, x_2, x_3) , and ξ for (ξ_1, ξ_2, ξ_3) . The quantity r is the distance between the field point x to the variable point ξ . The functions $X_i(\xi)$ are the components of the body force X ; expressed in terms of the variables of integration ξ . If an idealized case is considered in which a small sphere is isolated in the body, in which the body force concentrates, the limiting case will provide a solution of the Kelvin problem named above. A combination of such a solution with some other singular solutions of Navier's equation yields solutions to the Boussinesq and Cerruti problems. A general theory for this adaptation was given by Betti (1872), who showed how to express the dilatation and rotation at every point in a solid body by surface integrals containing explicitly the surface tractions and surface displacements. A lucid exposition of this approach can be found in Love, *Elasticity*,^{1,1} Chapter X.

The corresponding problems of specific normal and tangential displacements at the origin on a semi-infinite solid were also solved by Boussinesq (1888). There are numerous ways of arriving at these results. For references, see Love,^{1,1} p. 243.

Problem 8.6. Investigate the system of stresses corresponding to

- (a) Strain potential $\phi = \log(R + z)$, $R^2 = x^2 + y^2 + z^2$.
- (b) Love's strain functions $Z = \log(R + z)$, and (c) $Z = x \log(R + z)$.

Find a strain potential that is equivalent to (b).

8.9. PERTURBATION OF ELASTICITY SOLUTIONS BY A CHANGE OF POISSON'S RATIO

In many problems of elasticity, the effect of Poisson's ratio is relatively unimportant. In two-dimensional problems without body force, Poisson's ratio has no effect on the stress distribution at all. In fact, this is the basic justification of most photoelasticity studies, in which the selection of materials for the models can be made regardless of elastic constants.

In three-dimensional problems, the effect of Poisson's ratio generally cannot be disregarded, but the examples of

Boussinesq's and Cerruti's problems in Sec. 8.8 show that particularly simple solutions can be obtained when Poisson's ratio assumes a certain particular value. This leads one to consider the effect of Poisson's ratio. Westergaard proposes to obtain a solution by first solving a problem for a specific value of Poisson's ratio m , and then determining the necessary corrections when the actual value of ν is used. The value of the shear modulus G is considered fixed.

Consider a body bounded by a surface S . Let u_i be the solution of a problem satisfying the Navier equation (in Cartesian coordinates).

$$(1) \quad e_{,i}/(1-2\nu) + u_{i,\alpha\alpha} + X_i/G = 0, \quad e = u_{\alpha,\alpha},$$

and assumes on the boundary S surface traction $\frac{\nu}{T_i}$ or displacement u_i .

Let u_i^* be a solution of another problem, satisfying, for a different value of Poisson's ratio m , but the same value of G , the equation

$$(2) \quad e_{,i}^*/(1-2m) + u_{i,\alpha\alpha}^* + X_i^*/G = 0, \quad e^* = u_{\alpha,\alpha}^*, \quad T_i^*, \quad \text{or } u_i^* \text{ on } S.$$

This problem shall be assumed to have been solved, and is the starting point for further investigation on the effect of variation of Poisson's ratio.

Subtracting Eq. (2) from Eq. (1), and writing

$$(3) \quad u_i^{**} = u_i - u_i^*, \quad e_{,i}^{**} = e_{,i} - e_{,i}^*, \quad X_i^{**} = X_i - X_i^*,$$

etc., we obtain

$$(4) \quad \left(\frac{1}{1-2\nu} - \frac{1}{1-2m}\right)e_{,i}^* + \frac{1}{1-2\nu}e_{,i}^{**} + u_{i,\alpha\alpha}^{**} + \frac{X_i^{**}}{G} = 0,$$

and

$$(5) \quad T_i^{**} = \frac{\nu}{T_i} - \frac{\nu}{T_i^*}, \quad \text{or } u_i^{**} = u_i - u_i^* \text{ on } S.$$

Equations (4) and (5) define another problem in the linear theory of elastic equilibrium. Any solution of Eqs. (4) and (5), added to a solution of Eq. (2), which corresponds to Poisson's ratio m , will yield a solution of Eq. (1) with the value of Poisson's ratio equal to ν .

To make this concept useful, let us consider a class of problems such as Boussinesq's, in which the boundary conditions take the form of specified surface tractions; e.g.,

$$(6) \quad \sigma_{33}^*, \sigma_{31}^*, \sigma_{32}^* \text{ are specified on } S: \quad x_3 = \text{const.}$$

Then it is expedient to consider a problem to solve Eq. (4) with

$$(7) \quad \sigma_{33}^{**} = \sigma_{31}^{**} = \sigma_{32}^{**} = 0 \quad \text{on } S: \quad x_3 = \text{const.}$$

When such a particular solution is found, it can be added to the solution of Eq. (2) to form a solution of Eq. (1) with the same specified boundary values of σ_{33} , σ_{31} , σ_{32} as listed in Eq. (6).

Westergaard^{1,2} shows how to construct a general solution in which σ_{33}^{**} , σ_{31}^{**} , and σ_{32}^{**} vanish identically throughout the elastic body. The method is based on the assumption that the components of displacement are derived from a scalar function φ according to the rule

$$(8) \quad 2Gu_1^{**} = \varphi_{,1}, \quad 2Gu_2^{**} = \varphi_{,2}, \quad 2Gu_3^{**} = -\varphi_{,3}.$$

Westergaard calls this a "twinned gradient," the third component $-\varphi_{,3}$ being regarded as the "twin" of the ordinary gradient $\varphi_{,3}$. Now

$$(9) \quad \sigma_{ij} = 2G[\nu e_{\alpha\alpha} \delta_{ij}/(1-2\nu) + e_{ij}], \quad \sigma_{ij}^* = 2G[m e_{\alpha\alpha}^* \delta_{ij}/(1-2m) + e_{ij}^*].$$

Hence,

$$(10) \quad \sigma_{ij}^{**} = \sigma_{ij} - \sigma_{ij}^* = 2G\{\delta_{ij}[(\nu-m)e_{\alpha\alpha}^*/(1-2m) + \nu e_{\alpha\alpha}^{**}]/(1-2\nu) + e_{ij}^{**}\}.$$

From Eq. (8), we have

$$(11) \quad e_{23}^{**} = 0, \quad e_{31}^{**} = 0.$$

Hence,

$$(12) \quad \sigma_{23}^{**} = 0, \quad \sigma_{31}^{**} = 0.$$

Furthermore,

$$(13) \quad 2Ge_{\alpha\alpha}^{**} = \varphi_{,11} + \varphi_{,22} - \varphi_{,33} = \nabla^2\varphi - 2\varphi_{,33},$$

and, from Eq. (10),

$$(14) \quad \sigma_{33}^{**} = \frac{2G(\nu - m)}{(1 - 2\nu)(1 - 2m)}e_{\alpha\alpha}^* + \frac{\nu}{1 - 2\nu}(\nabla^2\varphi - 2\varphi_{,33}) - \varphi_{,33}.$$

Hence, if we choose to have

$$(15) \quad \sigma_{33}^{**} \equiv 0,$$

then φ must satisfy the equation

$$(16) \quad \nu \nabla^2\varphi - \varphi_{,33} = -2G(\nu - m)e_{\alpha\alpha}^*/(1 - 2m).$$

With φ so chosen, Eq. (4) will have to yield the required value of the body force X_i^{**} in order to keep the body in equilibrium. However, it is simpler to proceed as follows. Compute first the stress σ_{11}^{**} ,

$$(17) \quad \sigma_{11}^{**} = \sigma_{11}^{**} - \sigma_{33}^{**} = 2G(e_{11}^{**} - e_{33}^{**}) = \varphi_{,11} + \varphi_{,33},$$

$\sigma_{33}^{**} = 0$. Then the only nonvanishing components of stress are

$$(18) \quad \sigma_{11}^{**} = \nabla^2\varphi - \varphi_{,22}, \quad \sigma_{22}^{**} = \nabla^2\varphi - \varphi_{,11}, \quad \sigma_{12}^{**} = \varphi_{,12}.$$

The equations of equilibrium expressed in terms of stresses show at once that the components of the required body force are

$$(19) \quad -X_1^{**} = (\nabla^2\varphi)_{,1}, \quad -X_2^{**} = (\nabla^2\varphi)_{,2}.$$

The solution is established if a function φ can be found that satisfies Eqs. (16) and (19) simultaneously.

In a majority of significant problems, both the original and the final body forces are zero; i.e.,

$$(20) \quad X_i = X_i^* = X_i^{**} = 0.$$

Then (16) and (19) can be satisfied if

$$(21) \quad \nabla^2\varphi = 0 \text{ and } \varphi_{,33} = 2G(\nu - m)e_{\alpha\alpha}^*/(1 - 2m) = (\nu - m)\sigma_{\alpha\alpha}^*/(1 + m).$$

But $e_{\alpha\alpha}^*$ is a harmonic function. Hence, these two equations can be satisfied simultaneously, and the method is established.

The formulas given above are valid as long as ν and m are not exactly equal to 1/2. In case $m \rightarrow 1/2$, the dilatation e^* tends to zero, but the bulk modulus tends to infinity. The sum of normal stresses

$$(22) \quad \sigma_{ii}^* = 2G(1 + m)e_{ii}^*/(1 - 2m)$$

remains finite as $m \rightarrow 1/2$. It is easy to show that all the formulas in this section remain valid when $m = 1/2$, provided that the e_{ii}^* term is replaced by the σ_{ii}^* term according to Eq. (22), as in the last equation of Eq. (21).

8.10. BOUSSINESQ'S PROBLEM

A load P acts at the origin of coordinates and perpendicular to the plane surface of a semi-infinite solid occupying the space $z \geq 0$ (Fig. 8.8:4). When Poisson's ratio is $\frac{1}{2}$, the problem has the simple solution as stated in Eqs. (8.8:5) which gives

$$(1) \quad \theta^* = \sigma_{ii} = -3Pz/(2\pi R^3), \quad R^2 = z^2 + r^2 = x^2 + y^2 + z^2.$$

If Poisson's ratio is ν , and $m = 1/2$, Westergaard's Eq. (8.9:21) becomes,

$$(2) \quad \nabla^2 \varphi = 0 \quad \text{and} \quad \varphi_{,zz} = (1 - 2\nu)Pz/(2\pi R^3),$$

in this case. Integrating, we have

$$(3) \quad \varphi_{,z} = -(1 - 2\nu)P/(2\pi R), \quad \varphi = -[(1 - 2\nu)P \log(R + z)]/(2\pi).$$

It can be shown that φ is harmonic; therefore the problem is solved.

On substituting Eq. (3) into Eqs. (8.9:8–14), and adding to the solution for $m = 1/2$, we obtain the final results, in which the only nonvanishing components are

$$(4) \quad \xi_r = \frac{P}{4\pi GR} \left[\frac{rz}{R^2} - \frac{(1 - 2\nu)r}{R + z} \right], \quad \xi_z = \frac{P}{4\pi GR} \left[2(1 - \nu) + \frac{z^2}{R^2} \right],$$

$$(5) \quad \theta = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} = -\frac{(1 + \nu)Pz}{\pi R^3}, \quad \sigma_{rz} = -\frac{3Prz^2}{2\pi R^5}, \quad \sigma_{zz} = -\frac{3Pz^3}{2\pi R^5},$$

$$\sigma_{rr} = \frac{P}{2\pi R^2} \left[-\frac{3r^2 z}{R^3} + \frac{(1 - 2\nu)R}{R + z} \right], \quad \sigma_{\theta\theta} = \frac{(1 - 2\nu)P}{2\pi R^2} \left(\frac{z}{R} - \frac{R}{R + z} \right),$$

where $r^2 = x^2 + y^2$, $R^2 = r^2 + z^2 = x^2 + y^2 + z^2$.

Problem 8.7. Solve the Boussinesq problem by a combination of a Galerkin vector

$$F_1 = F_2 = 0, \quad F_3 = BR, \quad R = (r^2 + z^2)^{1/2}$$

and a Lamé strain potential $\Phi = c \log(R + z)$. Show that $c = -(1 - 2\nu)B$, $B = P/2\pi$.

Problem 8.8. Solve Cerruti's problem by the method of "twinned gradient" (Reference, Westergaard,^{1,2} p. 142).

8.11. ON BIHARMONIC FUNCTIONS

We have seen that many problems in elasticity are reduced to the solution of biharmonic equations with appropriate boundary conditions. It will be useful to consider the mathematical problem in some detail.

We shall consider the equation

$$(1) \quad \nabla^2 \nabla^2 u = 0,$$

where ∇^2 is the Laplace operator given in Eq. (7.1:11), and Probs. 2.32 and 2.33 for Cartesian, cylindrical and spherical coordinates, respectively, except for the case that r and φ of the spherical coordinates is replaced by R and φ here.

A regular solution of Eq. (1) in a region \mathcal{R} is one that is four times continuously differentiable in \mathcal{R} . A regular solution of Eq. (1) is called a . Since Eq. (1) is obtained by repeated operation of the *Laplace operator* (2), and the regular solution of the equation $\nabla^2 u = 0$ is called a *harmonic function*, it is expected that biharmonic functions are closely connected with harmonic functions. In fact, we have the following theorems due to Almansi.^{8,1}

Theorem 1. If u_1, u_2 are two functions, harmonic in a region $\mathcal{R}(x, y, z)$, then

$$(2) \quad u = xu_1 + u_2$$

is biharmonic in \mathcal{R} . Conversely, if u is a given biharmonic function in a region \mathcal{R} , and if every line parallel to the x -axis intersects the boundary of \mathcal{R} in at most two points, then there exist two harmonic functions u_1 and u_2 in \mathcal{R} , so that u can be represented in the form of Eq. (2).

Proof. The first part of the theorem can be verified directly according to the identity

$$(3) \quad \nabla^2(\phi\psi) = \phi\nabla^2\psi + \psi\nabla^2\phi + 2 \left(\frac{\partial\phi}{\partial x} \frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial z} \frac{\partial\psi}{\partial z} \right).$$

To prove the converse theorem, we note that the theorem is established if we can show that there exist a function u_1 such that

$$(4) \quad (a) \nabla^2 u_1 = 0, \quad (b) \nabla^2(xu_1 - u) = 0.$$

By virtue of Eq. (4a), the second equation can be written as

$$(5) \quad \nabla^2 u = \nabla^2(xu_1) = 2\partial u_1/\partial x,$$

which has a particular solution

$$(6) \quad \bar{u}_1(x, y, z) = \int_{x_0}^x \frac{1}{2} \nabla^2 u(\xi, y, z) d\xi,$$

where x_0 is an arbitrary point in the region \mathcal{R} . This particular solution does not necessarily satisfy Eq. (4a). However, since u is biharmonic, we have

$$(7) \quad \frac{\partial}{\partial x} \nabla^2 \bar{u}_1 = \nabla^2 \frac{\partial \bar{u}_1}{\partial x} = \frac{1}{2} \nabla^4 u = 0.$$

Hence $\nabla^2 \bar{u}_1$ is a function $v(y, z)$ of the variables y, z only. Now let us determine a function $\bar{u}_1(y, z)$ so that

$$(8) \quad (\partial^2/\partial y^2 + \partial^2/\partial z^2) \bar{u}_1 = -v(y, z);$$

for example, by

$$(9) \quad \bar{u}_1(y, z) = -\frac{1}{2\pi} \iint (\log r) \cdot v(\eta, \zeta) d\eta d\zeta,$$

where $r^2 = (y - \eta)^2 + (z - \zeta)^2$ and the integral extends through the region \mathcal{R} . Then the function $u_1 = \bar{u}_1 + \bar{u}_1$ satisfies both conditions (5) and (4a) and the theorem is proved.

By a slight change in the proof it can be shown that [Theorem 1](#) holds as well in the two-dimensional case. Similarly, we have the following

Theorem 2. If u_1, u_2 are two harmonic functions in a three-dimensional region \mathcal{R} , then

$$(10) \quad u = (R^2 - R_0^2)u_1 + u_2,$$

is biharmonic in \mathcal{R} , where $R^2 = x^2 + y^2 + z^2$ and R_0 is an arbitrary constant. Conversely, if u is a given biharmonic function in a region \mathcal{R} , and if \mathcal{R} is such that, with an origin inside \mathcal{R} , each radius vector intersects the boundary of \mathcal{R} in at most one point, then two harmonic functions u_1, u_2 can be determined so that Eq. (10) holds.

Proof. The proof of the first part again follows by direct calculation. Since u_1, u_2 are harmonic, an application of the identity (3) yields

$$(11) \quad \begin{aligned} \nabla^2 u &= u_1 \nabla^2 R^2 + 4 \left(x \frac{\partial u_1}{\partial x} + y \frac{\partial u_1}{\partial y} + z \frac{\partial u_1}{\partial z} \right) = 6u_1 + 4R \frac{\partial u_1}{\partial R}, \\ \nabla^2 \nabla^2 u &= 6\nabla^2 u_1 + 8(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)u_1 = 0. \end{aligned}$$

To prove the converse theorem, we note that the theorem is established if we can determine a function u_1 with the properties

$$(12) \quad (a) \quad \nabla^2 u_1 = 0, \quad (b) \quad \nabla^2 [u - (R^2 - R_0^2)u_1] = 0.$$

Equation (12b) can be simplified, by virtue of Eq. (12a), into

$$(13) \quad \nabla^2 u = 6u_1 + 4R \partial u_1 / \partial R.$$

An integral of this differential equation is

$$(14) \quad u_1 = R^{-3/2} \int_0^R \rho^{1/2} \nabla^2 u d\rho / 4.$$

It will now be shown that this integral indeed satisfies the condition (12a) and, hence, is the desired function. The demonstration will be simpler if the spherical coordinates are used. From Eq. (14),

$$(15) \quad \nabla^2 u_1 = \left\{ \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial}{\partial R} \right) + \left[\frac{1}{R^2 \sin \varphi} \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial}{\partial \varphi} \right) + \frac{1}{R^2 \sin^2 \varphi} \frac{\partial^2}{\partial \theta^2} \right] \right\} \left(R^{-3/2} \int_0^R \frac{1}{4} \rho^{1/2} \nabla^2 u d\rho \right).$$

The part $(1/\sin^2 \varphi)[\sin \varphi \partial(\sin \varphi \partial/\partial \varphi)/\partial \varphi + \partial^2/\partial \theta^2]$ of ∇^2 can be taken under the sign of integration. Since u is biharmonic $\nabla^2 \nabla^2 u = 0$, we have

$$\left[\frac{1}{\sin \varphi} \frac{\partial}{\partial \varphi} \left(\sin \varphi \frac{\partial}{\partial \varphi} \right) + \frac{1}{\sin^2 \varphi} \frac{\partial^2}{\partial \theta^2} \right] \nabla^2 u = -\frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial}{\partial \rho} \right) \nabla^2 u.$$

Therefore, (15) may be written as

$$\nabla^2 u_1 = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial}{\partial R} \int_0^R \frac{\rho^{1/2} \nabla^2 u}{4R^{3/2}} d\rho \right) - \int_0^R \frac{\rho^{1/2}}{4R^{7/2}} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial \nabla^2 u}{\partial \rho} \right) d\rho.$$

On carrying out the indicated differentiation in the first term and integrating the second term twice by parts, we obtain finally

$$\begin{aligned} \nabla^2 u_1 &= \frac{1}{R^2} \frac{\partial}{\partial R} \left(\frac{R \nabla^2 u}{4} - \frac{3}{8} \int_0^R \frac{\rho^{1/2} \nabla^2 u}{R^{1/2}} d\rho \right) + \frac{1}{8} \int_0^R \frac{\rho^{3/2}}{R^{7/2}} \frac{\partial \nabla^2 u}{\partial \rho} d\rho \\ &- \frac{1}{4R} \frac{\partial \nabla^2 u}{\partial R} = \frac{1}{R^2} \left(\frac{3}{16} \int_0^R \frac{\rho^{1/2} \nabla^2 u}{R^{3/2}} d\rho - \frac{1}{8} \nabla^2 u + \frac{R}{4} \frac{\partial \nabla^2 u}{\partial R} \right) \\ &- \frac{1}{4R} \frac{\partial \nabla^2 u}{\partial R} + \frac{\nabla^2 u}{8R^2} - \frac{3}{16} \int_0^R \frac{\rho^{1/2} \nabla^2 u}{R^{7/2}} d\rho = 0. \end{aligned}$$

Hence, u_1 given by Eq. (14) satisfies all the requirements, and the theorem is proved.

Theorem 2 holds also in the two-dimensional case when R^2 is replaced by $r^2 = x^2 + y^2$. The proof is analogous to the above. It is also evident that the choice of x in **Theorem 1** is incidental. The theorem holds when x is replaced by y or z .

Special representation of biharmonic functions in two-dimensions by means of analytic functions of a complex variable will be discussed in [Sec. 9.5](#).

Problem 8.9. *Yih's solution of multiple Bessel equations.* Harmonic functions in cylindrical polar coordinates may assume the form $Z(r) e^{i\alpha x} e^{i\beta \theta}$, where $Z(r)$ is a Bessel function. Now consider the hyper-Bessel equation:

$$[d^2/dr^2 + (1/r)d/dr - p^2/r^2 + k^2]^n f = 0, \quad n, \text{ positive integer}.$$

With the help of Almansi's theorems discussed above, show that if p (taken to be positive for convenience) is not an integer, the solutions are $r^m J_{\pm(p+m)}(kr)$, in which $m = 0, 1, 2, \dots, n-1$; otherwise they are $r^m J_{p+m}(kr)$ and $r^m N_{p+m}(kr)$, with m ranging over the same integers. The symbols J and N stand for the Bessel function and the Neumann function, respectively. (C. S. Yih, *Quart. Appl. Math.*, **13**, 4, 462–463, 1956.)

Problem 8.10. *Generation of biharmonic functions in cylindrical coordinates.* If $u_1(z, x)$ is a harmonic function in a space $\mathcal{R}(x, y, z)$, then

$$v_1 = \frac{1}{2\pi} \int_0^{2\pi} u_1(z, r \cos \theta) d\theta,$$

obtained by turning the space around the z -axis, is also harmonic. By selecting $u(z, x)$ as the real part of $(z + ix)^n$, in which n is a positive integer, show that the following functions are harmonic in cylindrical coordinates (r, θ, z) , where $r^2 = x^2 + y^2$.

$$\begin{aligned} \psi_2 &= z^2 - r^2/2, & \psi_3 &= z^3 - 3zr^2/2, & \psi_4 &= z^4 - 3z^2r^2 + 3r^4/8, \\ \psi_5 &= z^5 - 5z^3r^2 + 15zr^4/8, & \psi_6 &= z^6 - 5z^4r^2/2 + 45z^2r^4/8 - 5r^6/16. \end{aligned}$$

Show that $z\psi_n$ and $(z^2 + r^2)\psi_n$ are biharmonic ($n = 2, 3, \dots$). Note: If $u(z, x)$ is a biharmonic function, then the process indicated above generates a biharmonic function $v(z, r)$.

8.12. NEUBER–PAPKOVICH REPRESENTATION

In Sec. 8.4, the displacement field u_i is represented by the Galerkin vector (F_1, F_2, F_3) , in the form

$$(1) \quad 2Gu_i = 2(1-\nu)\nabla^2F_i - F_{j,ji} \quad i = 1, 2, 3.$$

If we set

$$(2) \quad \nabla^2F_i = \Phi_i/[2(1-\nu)], \quad F_{j,j} = \Psi,$$

Eq. (1) becomes

$$(3) \quad 2Gu_i = \Phi_i - \Psi_{,i}.$$

Consider first the case in which the body force is absent. Since in the absence of body force the Galerkin vector must satisfy the biharmonic equation, we see that Φ and Ψ satisfy the equations

$$(4) \quad \nabla^2\Phi_i = 0, \quad \nabla^4\Psi = 0.$$

Hence the Φ 's are the harmonic and the Ψ is biharmonic. They are related through Eq. (2) by

$$(5) \quad \nabla^2\Psi = \Phi_{j,j}/[2(1-\nu)].$$

On noting that $\nabla^2(x_j\Phi_j) = 2\Phi_{j,j}$, we see that a particular solution of this $\frac{1}{4(1-\nu)}x_j\Phi_j$. Hence, the general solution can be written as

$$(6) \quad \Psi = x_j\Phi_j/(1-\nu) + \phi_0,$$

where ϕ_0 is an arbitrary harmonic function. On substituting into Eq. (3), we obtain

$$(7) \quad 2Gu_i = [(3-4\nu)\Phi_i - x_j\Phi_{j,i}]/[4(1-\nu)] - \phi_{0,i}.$$

If we define

$$(8) \quad \phi_i = \Phi_i/[4(1-\nu)], \quad \kappa = 3-4\nu,$$

we get

$$(9) \quad \Delta \quad 2Gu_i = \kappa\phi_i - x_j\phi_{j,i} - \phi_{0,i}.$$

This formula expresses u_i in terms of four harmonic functions $\phi_0, \phi_1, \phi_2, \phi_3$. It was given independently by Papkovich (1932) and Neuber (1934) by different methods. The connection with the Galerkin vector was pointed out by Mindlin (1936). *The special importance of the Neuber–Papkovich solution lies in its strict similarity to a general solution in two dimensions (Sec. 9.6), in which case a well-known procedure exists for the determination of the harmonic functions involved from specified boundary conditions.*

The question of whether all four of the harmonic functions are independent, or whether one of them may be eliminated so that the general solution of the three-dimensional Navier's equation involves only three independent harmonic functions, has been a subject of much discussion. See Sokolnikoff, *Elasticity*,^{1,2} 2nd ed. (1956), p. 331, and Naghdi^{8,1} (1960).

If the body force is not zero, a general solution of Navier's equation can be obtained by adding a particular integral to the solution [see Eq. (8.8:9)].

Two other classical methods of solving Navier's equations for elastostatic problems need mentioned. The first is Betti's method referred to in Sec. 8.8 (Love^{1,2}). The second is the integral transformation method (Fourier, Laplace, Hankel, Mellin, Stieltjes, etc., Sneddon^{9,2} 1951 and Flügge^{1,4} 1962).

8.13. REFLECTION AND REFRACTION OF PLANE P AND S WAVES

So far we have considered only static problems. We shall now consider some dynamic problems in order to illustrate the use of displacement potentials in dynamics.

According to Sec. 8.2, when the displacements are represented by a scalar potential ϕ and vector potentials ψ_1, ψ_2, ψ_3

through the expression

$$u_i = \partial\phi/\partial x_i + e_{ijk}\partial\psi_k/\partial x_j ,$$

a broad class of solution is obtained if ϕ and ψ_k satisfy the wave equations

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c_L^2} \frac{\partial^2\phi}{\partial t^2} , \quad \frac{\partial^2\psi_k}{\partial x^2} + \frac{\partial^2\psi_k}{\partial y^2} + \frac{\partial^2\psi_k}{\partial z^2} = \frac{1}{c_T^2} \frac{\partial^2\psi_k}{\partial t^2} .$$

The functions ϕ and ψ_1, ψ_2, ψ_3 define dilatational and distorsional waves, which are called, in seismology, the *primary*, or *P*(or “push”), waves, and the secondary, or *S*(or “shake”), waves. If we consider plane waves, as in Sec. 7.8, we see that the *S* waves are polarized. If an *S*-wave train propagates along the *x*-axis in the *x, z*-plane, (*z* “vertical,” *x* “horizontal”), and the material particles move in the *z*-direction (vertical), then we speak of *SV* waves. If the *S* waves propagate along the *x, z*-plane but the particles move in the *y*-direction (“horizontal”), then we speak of *SH* waves.

Consider a homogeneous isotropic elastic medium occupying a half-space $z \geq 0$. Since an elastic medium has two characteristic wave speeds, plane *P* waves hitting the free boundary $z = 0$ are reflected into plane *P* waves and plane *S* waves. Similarly, incident *SV* waves are reflected as both *P* and *SV* waves. If two elastic media are in contact with a “welded” interface, then incident *P* waves will be reflected in the first medium into *P* and *S* waves, and also refracted in the second medium in *P* and *S* waves. A similar statement holds for incident *SV* waves. The *SH* waves behave simpler. A train of incident *SH* waves will not generate *P* waves at the interface, so it is reflected and refracted in *SH* waves.

We shall show that *the laws of reflection and refraction are the same Snell's law as in optics*. Thus, if we have two homogeneous isotropic elastic media *M* and *M*₁, of infinite extent and in “welded” contact at the plane $z = 0$, as shown in Fig. 8.13:1, and if the directions of advance of the waves are all parallel to the *x, z*-plane as illustrated by rays in the figure, then for incident *SV* waves,

$$(1) \quad \frac{c_T}{\cos f_0} = \frac{c_T}{\cos f} = \frac{c_L}{\cos e} = \frac{c_T^{(1)}}{\cos f_1} = \frac{c_L^{(1)}}{\cos e_1} , \quad (\therefore f = f_0) .$$

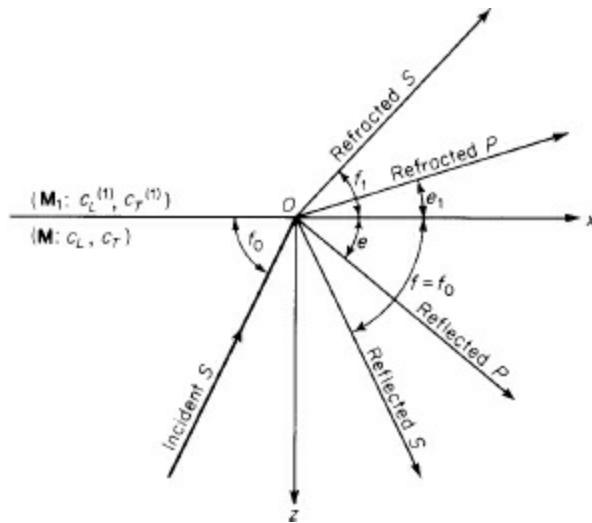


Fig. 8.13:1. Reflection of a *SV* ray incident against a plane boundary.

In this equation, c_L, c_T are, respectively, the longitudinal and transverse wave speeds of the medium *M*, and $c_L^{(1)}, c_T^{(1)}$ are the corresponding speeds of the medium *M*₁. The angle f_0 between the ray of the incident waves and the plane boundary is called the *angle of emergence* of the waves. Its complement is called the *angle of incidence*. Similarly, for incident *SH* waves, we have

$$(2) \quad c_T/\cos f_0 = c_T/\cos f = c_T^{(1)}/\cos f_1 , \quad (\therefore f = f_0) ,$$

and for incident *P* waves,

$$(3) \quad \frac{c_L}{\cos e_0} = \frac{c_L}{\cos e} = \frac{c_T}{\cos f} = \frac{c_L^{(1)}}{\cos e_1} = \frac{c_T^{(1)}}{\cos f_1}$$

(so that $e = e_0$). If we consider a half-space *M* with a free surface $z = 0$, we have these same equations, with the $c_L^{(1)}, c_T^{(1)}$ terms, which are now irrelevant, omitted, of course.

These results are easily proved. Let us work out the case of *SV* waves emerging against a free boundary. Other cases

are similar.

In *SV* waves emerging at an angle f_0 from a free plane boundary, the wave front has a normal in the direction of a unit vector whose direction cosines are $(\cos f_0, 0, \sin f_0)$; but in the incident *SV* waves the normal to the wave front has a different direction, with direction cosines $(\cos f_0, 0, -\sin f_0)$. This change of direction excites a reflected *P* wave. We assume, therefore, that

$$(4) \quad \begin{aligned} \phi &= \Phi(x \cos e + z \sin e - c_L t), \quad \psi_1 = \psi_3 = 0, \\ \psi_2 &= \psi = \Psi_0(x \cos f_0 - z \sin f_0 - c_T t) + \Psi(x \cos f + z \sin f - c_T t). \end{aligned}$$

The displacements and stresses are

$$(5) \quad u = \partial\phi/\partial x - \partial\psi/\partial z, \quad w = \partial\phi/\partial z + \partial\psi/\partial x,$$

$$(6) \quad \begin{aligned} \sigma_{zz} &= \lambda(\partial^2\phi/\partial x^2 + \partial^2\phi/\partial z^2) + 2G[\partial^2\phi/\partial z^2 + \partial^2\psi/(\partial x\partial z)] \\ \sigma_{zx} &= G[2\partial^2\phi/(\partial x\partial z) + \partial^2\psi/\partial x^2 - \partial^2\psi/\partial z^2]. \end{aligned}$$

The boundary conditions are

$$(7) \quad \text{at } z = 0, \quad \sigma_{zz} = \sigma_{zx} = 0.$$

On substituting Eqs. (4) and (6) into Eq. (7), we have

$$(8) \quad (\lambda + 2G \sin^2 e)\Phi''(x \cos e - c_L t) - 2G[\cos f_0 \sin f_0 \Psi_0''(x \cos f_0 - c_T t) \\ - \cos f \sin f \Psi''(x \cos f - c_T t)] = 0,$$

$$(9) \quad 2 \cos e \sin e \Phi''(x \cos e - c_L t) + (\cos^2 f_0 - \sin^2 f_0) \Psi_0''(x \cos f_0 - c_T t) \\ + (\cos^2 f - \sin^2 f) \Psi''(x \cos f - c_T t) = 0.$$

These equations can be satisfied for all values of x and t only if the arguments of the various Φ and Ψ functions are in a constant ratio. Hence,

$$(10) \quad c_T/\cos f_0 = c_T/\cos f = c_L/\cos e \quad \text{Q.E.D.}$$

When Eqs. (10) are satisfied, the functional relationships between Ψ_0 , Ψ , and Φ are given by Eqs. (8) and (9). A detailed study of such functional relationships yields information about the partitioning of energy among the various components of reflected and refracted waves — an important subject whose details can be found in Ewing, Jardetzky, and Press.^{7,4}

8.14. LAMB'S PROBLEM — LINE LOAD SUDDENLY APPLIED ON ELASTIC HALF-SPACE

Lamb,^{7,4} in a classic paper published in 1904, considered the disturbance generated in a semi-infinite medium by an impulsive force applied along a line or at a point on the surface or inside the medium. Lamb's solution, as well as the extensions thereof, has been studied by Nakano, Lapwood, Pekeris, Cagniard, Garvin, Chao, and others. In this section we shall consider only the problem of a line load suddenly applied on the surface (Fig. 8.14:1).

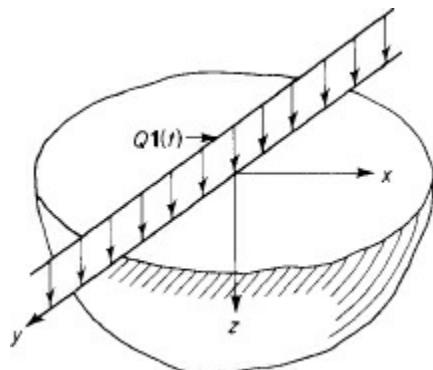


Fig. 8.14:1. A suddenly applied line load on an elastic half-space.

Consider a semi-infinite body of homogeneous isotropic linear elastic material occupying the space $z \geq 0$. For time $t < 0$, the medium is stationary. At $t = 0$, a concentrated load is suddenly applied normal to the free surface $z = 0$, along a line

coincident with the y -axis. The boundary conditions are therefore two-dimensional. We may assume the deformation state to be plane strain. The displacement v vanishes, and u and w are independent of y . Under this assumption, only one component of the vector potential is required. According to Sec. 8.2, the displacements are represented by Eq. (8.13:5).

Navier's equations of motion are satisfied if ϕ and ψ satisfy

$$(1) \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c_r^2} \frac{\partial^2 \phi}{\partial t^2}, \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{c_r^2} \frac{\partial^2 \psi}{\partial t^2},$$

which are wave equations. The stress components are

$$(2) \quad \begin{aligned} \sigma_{yz} &= \sigma_{xy} = 0, & \sigma_{yy} &= \nu(\sigma_{xx} + \sigma_{zz}), \\ \sigma_{zx} &= G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = G\left(2\frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2}\right), \\ \sigma_{zz} &= \lambda e + 2G\frac{\partial w}{\partial z} = \lambda\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) + 2G\left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z}\right), \\ \sigma_{xx} &= \lambda e + 2G\frac{\partial u}{\partial x} = \lambda\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right) + 2G\left(\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial z}\right). \end{aligned}$$

The boundary conditions on the surface $z = 0$ are

$$(3) \quad (\sigma_{zx})_{z=0} = (\sigma_{zy})_{z=0} = 0, \quad (\sigma_{zz})_{z=0} = -Q\delta(x)\mathbf{1}(t),$$

where $\mathbf{1}(t)$ is the unit-step function

$$(4) \quad \mathbf{1}(t) = 0 \quad \text{for } t < 0, \quad \mathbf{1}(t) = 1 \quad \text{for } t > 0,$$

while $\delta(x)$ is the Dirac delta function, i.e., one whose value is zero everywhere except in the neighborhood of $x = 0$ where it becomes infinitely large in such a way that $\int_{-\infty}^{\infty} \delta(x)dx = 1$. It is seen that the surface stress given by Eq. (3) is equivalent $-\infty \delta(x)dx = 1$. It is seen that the surface stress given by Eq. (3) concentrated load Q per unit length suddenly applied on the to a line $x = 0, z = 0$ and maintained constant afterwards.

For the conditions at infinity (a) all displacements and stresses remain finite, and (b) at large distances from the point of application of the load the disturbance consists of outgoing waves. These are called the *finiteness and radiation* conditions, respectively.

In the present problem, the disturbance is propagated outward at a finite velocity, so conditions (a) and (b) are equivalent to the statement that there exists an outgoing wave front, beyond which the medium is undisturbed. The question arises whether the boundary conditions (3) and the finiteness and radiation conditions, in the absence of any other disturbances in the medium, will determine a unique solution of our problem. For a *point* load, the answer is obviously affirmative, because for a suddenly applied point load, the wave front will be at a finite distance from the point of application of the load at any finite time. Hence, if we take a volume sufficiently large so that it includes the wave front in its interior, we have a finite body over whose entire surface the surface tractions are specified. Neumann's uniqueness theorem (Problem 7.1, Sec. 7.4) then guarantees a unique solution. For a *line* load, we do not have such a simple and general proof. In fact, difficulty may arise in two-dimensional problems. (For example, in the corresponding static problem — a static line load acting on the surface — the displacement at infinity is logarithmically divergent. See Sec. 9.4, Example 2.) However, for the present problem, a unique solution can be determined if we assume the deformation to be truly two-dimensional.

The significance of the last assumption is made clear by the following remarks. Note that a cylindrical body subjected to surface forces uniform along the generators *may* have an internal stress state that is not uniform along the axis. For example, transient axial waves may be superposed without disturbing the lateral boundary conditions. In other words, a seemingly two-dimensional problem may actually be three-dimensional. Such an occasion also occurs in fluid mechanics. A nontrivial example in hydrodynamics is the flow around a circular cylinder, with a velocity field uniform at infinity and normal to the cylinder axis. At supercritical Reynolds numbers, the three-dimensionality of the flow in the wake, i.e., variation along the cylinder axis, is very pronounced and becomes a predominant feature.

The boundary conditions (3) can be written in a different form. The unit-step function $\mathbf{1}(t)$ has no Fourier transform. But if we consider it to be the limiting case of the function $e^{-\beta t}\mathbf{1}(t)$, which has a Fourier transform, then the Fourier integral theorem

$$(5) \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \int_{-\infty}^{\infty} f(\xi) e^{-ik\xi} d\xi,$$

which is valid for an arbitrary function $f(x)$ that is square-integrable in the Lebesgue sense, yields the representation

$$(6) \quad \mathbf{1}(t) = \frac{1}{2\pi} \lim_{\beta \rightarrow 0} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\beta + i\omega} d\omega.$$

Equation (6) is obtained on substituting $e^{-\beta\xi}\mathbf{1}(\xi)$ for $f(\xi)$ in Eq. (5) and changing k to ω . Similarly, $\delta(x)$ has no Fourier transform. But, considering the delta function as the limit of a square wave

$$\mathbf{1}(x + \epsilon/2) - \mathbf{1}(x - \epsilon/2)]/\epsilon$$

as $\epsilon \rightarrow 0$, we can use Eq. (5) to obtain the representation

$$(7) \quad \delta(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi\epsilon} \int_{-\infty}^{\infty} \frac{\sin k\epsilon}{k} e^{ikx} dk.$$

Therefore, the second condition in Eq. (3) may be written

$$(8) \quad [\sigma_{zz}]_{z=0} = \frac{-Q}{4\pi^2} \lim_{\substack{\epsilon \rightarrow 0 \\ \beta \rightarrow 0}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin k\epsilon}{k\epsilon} \frac{1}{\beta + i\omega} e^{i(\omega t + kx)} d\omega dk.$$

From this, we see that if we can obtain an elementary solution of Eqs. (1) satisfying the boundary conditions

$$(9) \quad [\sigma_{xz}]_{z=0} = 0, \quad [\sigma_{zz}]_{z=0} = Ze^{i(\omega t + kx)},$$

then by the principle of superposition the solution to the original problem with boundary conditions (3) can be obtained by setting

$$(10) \quad Z(k, \omega) = -Q \sin k\epsilon / [4\pi^2 k\epsilon (\beta + i\omega)],$$

integrating with respect to k and ω both from $-\infty$ to ∞ , and then passing to the limit $\beta \rightarrow 0, \epsilon \rightarrow 0$.

The solution of the elementary problem is obtained by assuming

$$(11) \quad \phi = Ae^{-\nu z + ikx + i\omega t}, \quad \psi = Be^{-\nu' z + ikx + i\omega t},$$

which satisfy the wave Eqs. (1) if

$$(12) \quad \nu^2 = k^2 - k_\alpha^2, \quad \nu'^2 = k^2 - k_\beta^2, \quad k_\alpha = \omega/c_L, \quad k_\beta = \omega/c_T.$$

On substituting Eq. (11) into Eqs. (2) and (9), we obtain

$$(13) \quad -2i\nu k A - (2k^2 - k_\beta^2) B = 0, \quad (2k^2 - k_\beta^2) A - 2i\nu' k B = Z(k, \omega)/G.$$

Solving these equations for A and B , we obtain

$$(14) \quad A = (2k^2 - k_\beta^2)Z(k, \omega)/[GF(k)], \quad B = -2i\nu k Z(k, \omega)/[GF(k)],$$

where $F(k)$ is the Rayleigh function defined as

$$(15) \quad F(k) \equiv (2k^2 - k_\beta^2)^2 - 4k^2 \nu \nu'$$

A solution of the problem is obtained by substituting Eqs. (14) and (10) into Eq. (11) and integrating with respect to ω and k from $-\infty$ to ∞ . In so doing, an appropriate branch of the multi-valued functions ν and ν' must be chosen so that the conditions at infinity are satisfied.

The evaluation of these integrals is a formidable task. Direct integration or numerical integration are exceedingly difficult. Lamb uses the method of contour integration. The variables of integration k and ω are replaced by complex variables, and the contours of integration are deformed in such a way that some explicit results are obtained. Details can be found in Lamb's paper. For alternative approaches, some very elegant, see book by Ewing, Jardetzky and Press, and papers by Cagniard, Pekeris and Lifson listed in [Biblio. 7.4](#) at the end of this book. See also papers listed in [Biblio. 9.2](#).

P R O B L E M S

8.11. Derive potentials to solve the following equations:

Example: $\partial u / \partial y - \partial v / \partial x = 0$ is solved by taking $u = \partial \phi / \partial x$, $v = \partial \phi / \partial y$.

- (a) $\partial u / \partial x + \partial v / \partial y = 0$.
- (b) Plane stress: $\partial \sigma_x / \partial x + \partial \sigma_{xy} / \partial y = 0$, $\partial \sigma_{xy} / \partial x + \partial \sigma_y / \partial y = 0$.
- (c) In the theory of membrane stresses in a flat plate:

$$\partial N_x / \partial x + \partial N_{xy} / \partial y = 0, \quad \partial N_{xy} / \partial x + \partial N_y / \partial y = 0.$$

- (d) In the theory of bending of plates: $\partial Q_x / \partial x + \partial Q_y / \partial y = 0$,

$$\partial M_x / \partial x + \partial M_{xy} / \partial y = Q_x, \quad \partial M_{xy} / \partial x + \partial M_y / \partial y = Q_y.$$

8.12. A train of plane wave of wave length L and phase velocity c can be represented as

$$\phi = A \exp[i2\pi(\nu_1 x + \nu_2 y + \nu_3 z \pm ct)/L]$$

where A is a constant and ν 's are the direction cosines of a vector normal to the wave front. Superpose these waves and derive expressions for trains of:

- (a) cylindrical waves, and
- (b) spherical waves, such as those generated by a point source.

Ans. (a) $B J_0(kr) e^{-\nu_3 z} e^{i\omega t}$, $\nu^2 = k^2 - k^2$, $r^2 = x^2 + y^2$.

(b) $C \{ \exp[\pm i(k_\alpha R - \omega t)] \} / R$, $R^2 = x^2 + y^2 + z^2$.

Here B , C are constants, $k_\alpha = \omega/c$, and k is a parameter. The analysis may be simplified by means of contour integrations, regarding some variable of integration as complex numbers.

8.13. High pressure vessels of steel will be designed on the basis of von Mises' yield criterion. Consider spherical and cylindrical tanks of outer radius b and inner radius a . Compare the maximum internal pressure p at which yielding occurs.

Ans. $J_2 - k^2 = 0$, k = yield stress in simple shear.

Sphere: $p_{yield} = 2(1 - a^3/b^3)k/\sqrt{3}$; Cylinder: $p_{yield} = (1 - a^2/b^2)k$.

If $b/a = 1.1$, $p_{yield} = 0.2872k$ for sphere. = $0.1736k$ for cylinder.

8.14. Consider a spherical fluid gyroscope which consists of a hollow metallic sphere filled with a dense fluid. This sphere is rotated at an angular velocity ω about its polar axis. In a steady-state rotation (ω = constant), what are the stresses and displacements in the shell due to the fluid pressure and the centrifugal forces acting on the shell?

8.15. An infinitely long circular cylindrical hole of radius a is drilled in an infinite elastic medium. A pressure load is suddenly applied in the hole and starts to travel at a constant speed, so that the boundary conditions on the surface of the hole are, at $r = a$,

$$\sigma_{rr} = p \mathbf{1}(t - |z|/c), \quad \sigma_{rz} = \sigma_{r\theta} = 0.$$

The medium is initially quiescent. Determine the response of the medium.

8.16. Shock tubes are common tools used in aerodynamic research. A shock tube consists of a long cylindrical shell, closed at both ends. Near one end a thin diaphragm is inserted normal to the tube axis. On one side of the diaphragm the air is evacuated; on the other side, gas at high pressure is stored. In operation, the diaphragm is suddenly split, the onrushing gas from the high pressure side creates a shock front that travels down the evacuated tube.

Discuss the transient elastic response of the shock tube wall after the bursting of the diaphragm with the shock wave moving along the tube.

8.17. An infinite elastic medium contains a spherical cavity of radius a . A sinusoidally fluctuating pressure acts on the surface of the cavity. Determine the displacement field in the medium.

¹ In order that Eq. (5) and (6) both represent the same displacement field, we may choose $F_i = \tilde{F}_i + H_i$, where H_i are arbitrary harmonic functions ($H_{ijj} = 0$) and demand that $\tilde{F}_{j,j} = (\phi + H_{j,j})/(c - 1)$.

9

TWO-DIMENSIONAL PROBLEMS IN LINEARIZED THEORY OF ELASTICITY

The application of the Airy stress function reduces elastostatic problems in plane stress and plane strain for isotropic materials to boundary-value problems of a biharmonic equation. A general method of solution using the theory of functions of a complex variable is available. We shall discuss this method briefly and illustrate its utility in solving a few important problems.

Throughout this chapter x, y, z represent a set of rectangular Cartesian coordinates, with respect to which the displacement components are written as u, v, w , the strain components are e_{xx}, e_{xy} etc. [or $e_{ij} = (u_{i,j} + u_{j,i})/2$], and the stress components are σ_{xx}, σ_{xy} , etc. When curvilinear coordinates are used, we retain the notations of Secs. 4.10–4.12, in which u_i and e_{ij} denote the tensor components of the displacement and the strain, respectively; whereas ξ_i, ϵ_{ij} denote the physical components of these tensors. See Secs. 4.10–4.12.

The methods presented in this chapter, as those discussed in the two preceding chapters, are for the linearized theory of elasticity. We assume that the deformation gradient and velocities are small that the nonlinear terms in Eqs. (1)–(3) of Sec. 7.1 are negligible. In other words, the convective acceleration, the convective velocity, and the products of deformation gradients are negligibly small compared with the retained linear terms in the acceleration, velocity, and strain tensor, respectively. There are beautiful methods and results in the linearized theory. There are spectacular panorama in the nonlinear theory. We believe, however, that the best way to learn the nonlinear theory is to master the linearized theory first. This is what this book tries to do.

9.1. PLANE STATE OF STRESS OR STRAIN

If the stress components $\sigma_{zz}, \sigma_{zx}, \sigma_{zy}$ vanish everywhere,

$$(1) \quad \sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0,$$

the state of stress is said to be *plane stress* parallel to the x, y -plane. In this case, for isotropic materials,

$$(2) \quad \begin{aligned} e_{xx} &= (\sigma_{xx} - \nu\sigma_{yy})/E, & e_{yy} &= (\sigma_{yy} - \nu\sigma_{xx})/E, \\ e_{zz} &= -\nu(\sigma_{xx} + \sigma_{yy})/E, & e_{xy} &= \sigma_{xy}/(2G), & e_{xz} &= e_{yz} = 0, \end{aligned}$$

$$(3) \quad \begin{aligned} \sigma_{xx} &= E(e_{xx} + \nu e_{yy})/(1 - \nu^2), & \sigma_{yy} &= E(e_{yy} + \nu e_{xx})/(1 - \nu^2), \\ \sigma_{xy} &= Ee_{xy}/(1 + \nu), \end{aligned}$$

$$(4) \quad \sigma_{xx} + \sigma_{yy} = E(e_{xx} + e_{yy})/(1 - \nu),$$

$$(5) \quad e_{xx} + e_{yy} = \partial u / \partial x + \partial v / \partial y.$$

Substituting Eq. (3) into the equation of equilibrium (7.1:5), we obtain the basic equations for plane stress,

$$(6) \quad \begin{aligned} \blacktriangle & G \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + G \frac{1+\nu}{1-\nu} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + X = \rho \frac{\partial^2 u}{\partial t^2}, \\ & G \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + G \frac{1+\nu}{1-\nu} \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + Y = \rho \frac{\partial^2 v}{\partial t^2}. \end{aligned}$$

If the z -component of displacement w vanishes everywhere, and if the displacements u, v are functions of x, y only, the body is said to be in *plane strain* state parallel to the x, y -plane. In plane strain we have

$$(7) \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0, \quad \text{and} \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}), \quad (\text{since } e_{zz} = 0).$$

The basic equation (7.1:8) becomes, in plane strain,

$$(8) \quad \begin{aligned} G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{1-2\nu}G\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + X &= \rho\frac{\partial^2 u}{\partial t^2}, \\ G\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{1}{1-2\nu}G\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + Y &= \rho\frac{\partial^2 v}{\partial t^2}. \end{aligned}$$

If v is replaced by $v/(1+\nu)$ in Eq. (8), then it assumes the form (6). Hence any problem of a plane state of strain may be solved as a problem of a plane state of stress after replacing the true value of v by the “apparent value” $v/(1+\nu)$.¹ Conversely, any plane stress problem may be solved as a problem of plane strain by replacing the true value of v by an apparent value $v/(1-\nu)$.¹

The strain state in a long cylindrical body acted on by loads that are normal to the axis of the cylinder and uniform in the axial direction often can be approximated by a plane strain state. A constant axial strain e_{zz} may be imposed on a plane strain state without any change in stresses in the x, y -plane. Hence a minor extension of the definition of plane strain can be formulated by requiring that e_{zz} be a constant, that u and v be functions of x, y only, and that w be a linear function of z only.

The state of stress in a thin flat plate acted on by forces parallel to the midplane of the plate is approximately plane stress. However, since in general e_{zz} does not vanish, the displacements u, v, w are functions of z , and the problem is not truly two-dimensional. In fact, it can be shown that the general state of plane stress, satisfying Eq. (1), the equations of equilibrium, and the Beltrami–Michell compatibility conditions (7.3:6), is one in which the stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ are parabolically distributed throughout the thickness of the plate, i.e., of the form $f(x, y) + g(x, y)z^2$. (See Timoshenko and Goodier,^{1,2} p. 241.) However, the part proportional to z^2 can be made as small as we please compared with the first term, by restricting ourselves to plates which are sufficiently thin (with the ratio $h/L \rightarrow 0$, where h is the plate thickness and L is a characteristic dimension of the plate).

9.2. AIRY STRESS FUNCTIONS FOR 2-D PROBLEMS

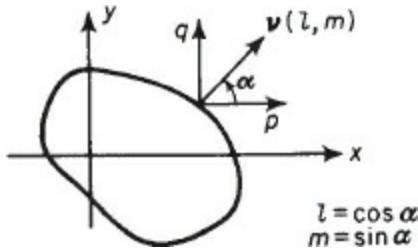


Fig. 9.2:1. Notations.

For plane stress or plane strain problems, we may try to find general stress systems that satisfy the equations of equilibrium and compatibility and then determine the solution to a particular problem by the boundary conditions.

Let x, y be a set of rectangular Cartesian coordinates. For plane stress and plane strain problems in the x, y -plane, the equations of equilibrium (3.4:2) are specialized into

$$(1) \quad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = -X,$$

$$(2) \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = -Y,$$

with the boundary conditions

$$(3) \quad l\sigma_{xx} + m\sigma_{xy} = p, \quad m\sigma_{yy} + l\sigma_{xy} = q,$$

where l, m are the direction cosines of the outer normal to the boundary curve and where p, q are surface tractions acting on the boundary surface.

The strain components are,

(a) *In the plane strain case:*

$$(4) \quad e_{xx} = \frac{\sigma_{xx} - \nu\sigma_{yy}}{E}, \quad e_{yy} = \frac{\sigma_{yy} - \nu\sigma_{xx}}{E}, \quad e_{xy} = \frac{\sigma_{xy}}{2G} = \frac{1+\nu}{E}\sigma_{xy}.$$

(b) *In the plane stress case:*

$$(5) \quad e_{xx} = [(1 - \nu^2)\sigma_{xx} - \nu(1 + \nu)\sigma_{yy}]/E, \\ e_{yy} = [(1 - \nu^2)\sigma_{yy} - \nu(1 + \nu)\sigma_{xx}]/E, \quad e_{xy} = (1 + \nu)\sigma_{xy}/E.$$

In view of what was discussed in the preceding section, for very thin plates we may assume σ_{xx} , σ_{yy} , σ_{xy} to be independent of z . Then the plane stress problem becomes truly two-dimensional, as well as the plane strain problem.

The compatibility conditions are as follows (see Sec. 4.6):

$$(6) \quad \begin{aligned} \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} &= 2\frac{\partial^2 e_{xy}}{\partial x \partial y}, \quad \frac{\partial^2 e_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x} \left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{xz}}{\partial y} + \frac{\partial e_{xy}}{\partial z} \right), \\ \frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} &= 2\frac{\partial^2 e_{yz}}{\partial y \partial z}, \quad \frac{\partial^2 e_{yy}}{\partial x \partial z} = \frac{\partial}{\partial y} \left(\frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{xz}}{\partial y} + \frac{\partial e_{xy}}{\partial z} \right), \\ \frac{\partial^2 e_{zz}}{\partial x^2} + \frac{\partial^2 e_{xx}}{\partial z^2} &= 2\frac{\partial^2 e_{xz}}{\partial z \partial x}, \quad \frac{\partial^2 e_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z} \left(\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{xz}}{\partial y} - \frac{\partial e_{xy}}{\partial z} \right). \end{aligned}$$

On substituting Eq. (4) into the first equation of Eq. (6), we obtain, in the plane stress case,

$$(7) \quad \partial^2(\sigma_{xx} - \nu\sigma_{yy})/\partial y^2 + \partial^2(\sigma_{yy} - \nu\sigma_{xx})/\partial x^2 = 2(1 + \nu)\partial^2\sigma_{xy}/(\partial x \partial y).$$

Differentiating Eq. (1) with respect to x and Eq. (2) with respect to y and adding we obtain

$$(8) \quad \partial^2\sigma_{xx}/\partial x^2 + \partial^2\sigma_{yy}/\partial y^2 + \partial X/\partial x + \partial Y/\partial y = -2\partial^2\sigma_{xy}/(\partial x \partial y).$$

Eliminating σ_{xy} between Eqs. (7) and (8), we obtain

$$(9) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_{xx} + \sigma_{yy}) = -(1 + \nu) \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right).$$

Similarly, in the plane strain case, we have

$$(10) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_{xx} + \sigma_{yy}) = - \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \frac{1}{1 - \nu}.$$

Equations (1), (2), (3), and (9) or (10) define the plane problems in terms of the stress components σ_{xx} , σ_{yy} , σ_{xy} . If the boundary conditions of a problem are such that surface tractions are all known, then the problem can be solved in terms of stresses, with no need to mention displacements unless they are desired. Even in a mixed boundary-value problem in which part of the boundary has prescribed displacements, it still may be advantageous to solve for the stress state first. These practical considerations lead to the method of Airy stress function.²

Airy's method is based on the observation that the left hand side of Eqs. (1) and (2) appears as the divergence of a vector. In hydrodynamics we are familiar with the fact that the conservation of mass, expressed in the equation of continuity

$$(11) \quad \partial u / \partial x + \partial v / \partial y = 0,$$

where u , v are components of the velocity vector, can be derived from an arbitrary stream function $\psi(x, y)$:

$$(12) \quad u = \partial \psi / \partial y, \quad v = -\partial \psi / \partial x.$$

In other words, if u , v are derived from an arbitrary $\psi(x, y)$ according to Eq. (12), then Eq. (11) is satisfied identically.

Let us use the same technique for Eq. (1) and (2). These equations can be put into the form of Eq. (11) if we assume that the body forces can be derived from a potential V , so that

$$(13) \quad X = -\partial V / \partial x, \quad Y = -\partial V / \partial y.$$

A substitution of Eq. (13) into Eqs. (1) and (2) results in

$$(14) \quad \partial(\sigma_{xx} - V)/\partial x + \partial\sigma_{xy}/\partial y = 0, \quad \partial\sigma_{xy}/\partial x + \partial(\sigma_{yy} - V)/\partial y = 0.$$

Now, as in Eq. (11), these equations are identically satisfied if we introduce two stream functions Ψ and χ in such a way that

$$(15) \quad \sigma_{xx} - V = \frac{\partial \Psi}{\partial y}, \quad \sigma_{xy} = -\frac{\partial \Psi}{\partial x}, \quad \sigma_{xy} = -\frac{\partial \chi}{\partial y}, \quad \sigma_{yy} - V = \frac{\partial \chi}{\partial x}.$$

In other words, a substitution of Eq. (15) into Eq. (14) reduces Eq. (14) into an identity in Ψ and χ . Now, Eqs. (15) can be combined if we let

$$(16) \quad \chi = \partial \Phi / \partial x, \quad \Psi = \partial \Phi / \partial y,$$

i.e.,

$$(17) \quad \Delta \quad \sigma_{xx} - V = \partial^2 \Phi / \partial y^2, \quad \sigma_{xy} = -\partial^2 \Phi / (\partial x \partial y), \quad \sigma_{yy} - V = \partial^2 \Phi / \partial x^2.$$

It is readily verified that if σ_{xx} , σ_{xy} , σ_{yy} are derived from an arbitrary function $\Phi(x, y)$ according to Eq. (17), then Eqs. (14) are identically satisfied. The function $\Phi(x, y)$ is called the *Airy stress function*, in deference to its inventor, the famous astronomer.

An arbitrary function $\Phi(x, y)$ generates stresses that satisfy the equations of equilibrium, but Φ is not entirely arbitrary: it is required to generate only those stress fields that satisfy the condition of compatibility. Since the compatibility condition is given by Eqs. (9) or (10), a substitution gives the requirement that, in the plane stress case,

$$(18) \quad \Delta \quad \frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = -(1 - \nu) \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right),$$

and that, in the plane strain case,

$$(19) \quad \Delta \quad \frac{\partial^4 \Phi}{\partial x^4} + 2 \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = -\frac{(1 - 2\nu)}{(1 - \nu)} \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right).$$

If the body forces vanish, then, in both plane stress and plane strain, Φ is governed by the equation

$$(20) \quad \Delta \quad \partial^4 \Phi / \partial x^4 + 2\partial^4 \Phi / (\partial x^2 \partial y^2) + \partial^4 \Phi / \partial y^4 = 0.$$

A regular solution of Eq. (20) is called a *biharmonic function*. Solution of plane elasticity problems by biharmonic functions will be discussed in the following sections.

How about the other five compatibility conditions in Eqs. (6) left alone so far? In the case of plane strain it is clear that they are identically satisfied. In the case of plane stress, however, they cannot be satisfied in general if we assume σ_{xx} , σ_{yy} , σ_{xy} to be independent of z . For, under such an assumption these compatibility conditions imply that

$$(21) \quad \partial^2 e_{zz} / \partial x^2 = \partial^2 e_{zz} / \partial y^2 = \partial^2 e_{zz} / (\partial x \partial y) = 0.$$

Hence, e_{zz} , and hence $\sigma_{xx} + \sigma_{yy}$, must be a linear function of x and y [$e_{zz} = -\nu(\sigma_{xx} + \sigma_{yy})/E$], which is the exception rather than the rule in the solution of plane stress problems. Hence, in general, the assumption that plane stress state is two-dimensional, so that σ_{xx} , σ_{yy} , σ_{xy} are functions of x , y only, cannot be true; and the solutions obtained under this assumption cannot be exact. However, as we have discussed previously (Sec. 9.1), they are close approximations for thin plates.

The stress-function method can be extended to three dimensions. The crucial observation is simply that the equation of equilibrium represents a vector divergence of the stress tensor. We are familiar with the stream function in hydrodynamics. In three-dimensions we need a triple of stream functions. Similarly, a generalization of Airy's procedure to equations of equilibrium in three dimensions requires a *tensor* of stress functions. Finzi8.1 (1934) showed that a general solution to the equations

$$(22) \quad \sigma_{ij,j} = 0, \quad \sigma_{ij} = \sigma_{ji}$$

is

$$(23) \quad \sigma_{ij} = e_{imr} e_{jns} \phi_{rs,mn},$$

where ϕ_{rs} stands for the components of a symmetric second-order tensor of stress functions, while e_{imr} is the usual permutation symbol (Sec. 2.1). Specialization by taking $\phi_{rs} = 0$ ($r \neq s$), yields Maxwell's stress functions, while taking $\phi_{rr} = 0$ (no sum), yields Morera's stress function, see Sec. 10.9, Eq. (10.9:18), *et seq.* If all elements ϕ_{rs} , except ϕ_{33} , are assumed to vanish, then (23) degenerates into Airy's solution of the two-dimensional equilibrium equations. An elegant proof of Finzi's result that is applicable to n -dimensional Euclidean space was given by Dorn and Schild.^{8.1}

Finzi^{8.1} (1934) obtained further a beautiful extension to the equations of *motion* of a continuum, with an arbitrary

density field, by introducing a fourth dimension. Finzi's arbitrary tensor yields by differentiation a motion and a stress field that satisfy the equation of motion.

For a curved space (non-Euclidean), Truesdell^{13.1} obtained related results by methods of calculus of variations. The curved space problem arises naturally in the intrinsic theory of thin shells or membranes. The two-dimensional surface is, to a two-dimensional observer who is not allowed to leave the surface, a non-Euclidean space (imbedded, of course, in a three-dimensional Euclidean space).

Example 1. The following polynomials of second and third degree are obviously biharmonic.

$$\Phi_2 = a_2x^2 + b_2xy + c_2y^2,$$

$$\Phi_3 = a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3.$$

By adjusting the constants a_2 , a_3 , etc., many problems in which the stresses are linearly distributed on rectangular boundaries can be solved. Examples of such problems are shown in Fig. 9.2:2.

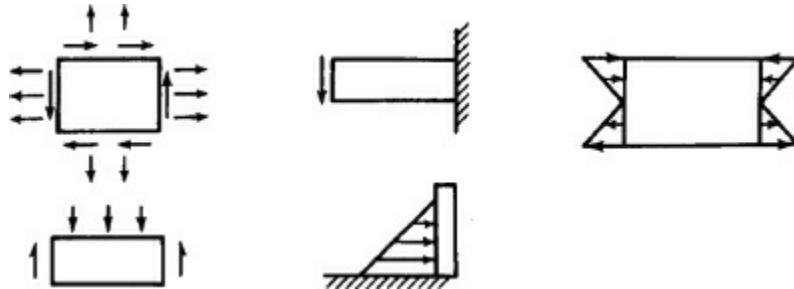


Fig. 9.2:2. Examples of problems solvable by simple polynomials.

Example 2. Consider a rectangular beam (Fig. 9.2:3) supported at the ends and subjected to the surface tractions

$$\text{on } y = c \quad \sigma_{xy} = 0, \quad \sigma_{yy} = -B \sin \alpha x,$$

$$\text{on } y = -c \quad \sigma_{xy} = 0, \quad \sigma_{yy} = -A \sin \alpha x.$$

Fig. 9.2:3. A beam subjected to a sinusoidally distributed loading.

Other edge conditions are unspecified at the beginning, and the body forces are absent.

Solution. Let

$$\Phi = \sin \alpha x f(y),$$

where $f(y)$ is a function of y only. A substitution into Eq. (20) yields

$$\alpha^4 f(y) - 2\alpha^2 f''(y) + f^{iv}(y) = 0.$$

The general solution is

$$f(y) = C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y.$$

Hence,

$$\begin{aligned} \Phi &= \sin \alpha x (C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y), \\ \sigma_{xx} &= \sin \alpha x [C_1 \alpha^2 \cosh \alpha y + C_2 \alpha^2 \sinh \alpha y \\ &\quad + C_3 (\alpha^2 \cosh \alpha y + 2\alpha \sinh \alpha y) + C_4 (\alpha^2 \sinh \alpha y + 2\alpha \cosh \alpha y)], \\ \sigma_{yy} &= -\alpha^2 \sin \alpha x [C_1 \cosh \alpha y + C_2 \sinh \alpha y + \dots], \\ \sigma_{xy} &= -\alpha \cos \alpha x [C_1 \alpha \sinh \alpha y + C_2 \alpha \cosh \alpha y \\ &\quad + C_3 (\cosh \alpha y + y \alpha \sinh \alpha y) + C_4 (\sinh \alpha y + y \alpha \cosh \alpha y)]. \end{aligned}$$

On application of the boundary conditions $\sigma_{xy} = 0$ on $y = \pm c$, we can express C_3 and C_4 in terms of C_1 and C_2 . The other boundary conditions then yield the constants

$$C_1 = \frac{A+B}{\alpha^2} \frac{\sinh \alpha c + \alpha c \cosh \alpha c}{\sinh 2\alpha c + 2\alpha c}, \quad C_3 = \frac{(A-B) \cosh \alpha c}{\alpha(\sinh 2\alpha c - 2\alpha c)},$$

$$C_2 = -\frac{A-B}{\alpha^2} \frac{\cosh \alpha c + \alpha c \sinh \alpha c}{\sinh 2\alpha c - 2\alpha c}, \quad C_4 = -\frac{(A+B) \sinh \alpha c}{\alpha(\sinh 2\alpha c + 2\alpha c)}.$$

The details can be found in Timoshenko and Goodier,^{1.2} p. 48.

9.3. AIRY STRESS FUNCTION IN POLAR COORDINATES

For two-dimensional problems with circular boundaries, polar coordinates can be used to advantage. Let ξ_r, ξ_θ, ξ_z denote physical components of the displacement, and let $\epsilon_{rr}, \epsilon_{r\theta}, \dots, \sigma_{rr}, \sigma_{r\theta}, \dots$, etc., be the physical components of the strain and stress, respectively. The general equations in cylindrical polar coordinates are given in Sec. 4.12. For *plane stress* problems, we assume that

$$(1) \quad \sigma_{zz} = \sigma_{zr} = \sigma_{z\theta} = 0.$$

For *plane strain* problems, we assume that

$$(2) \quad \xi_z = 0,$$

and that all derivatives with respect to z vanish. In both cases, the strain components are defined as follows:

$$(3) \quad \epsilon_{rr} = \frac{\partial \xi_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta} + \frac{\xi_r}{r}, \quad \epsilon_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial \xi_r}{\partial \theta} + \frac{\partial \xi_\theta}{\partial r} - \frac{\xi_\theta}{r} \right).$$

Furthermore, in plane stress,

$$(4) \quad \epsilon_{rr} = \frac{1}{E} (\sigma_{rr} - \nu \sigma_{\theta\theta}), \quad \epsilon_{\theta\theta} = \frac{1}{E} (\sigma_{\theta\theta} - \nu \sigma_{rr}), \quad \epsilon_{r\theta} = \frac{1+\nu}{E} \sigma_{r\theta},$$

and in plane strain,

$$(5) \quad e_{rr} = [(1-\nu^2)\sigma_{rr} - \nu(1+\nu)\sigma_{\theta\theta}]/E, \\ e_{\theta\theta} = [(1-\nu^2)\sigma_{\theta\theta} - \nu(1+\nu)\sigma_{rr}]/E, \quad e_{r\theta} = (1+\nu)\sigma_{r\theta}/E.$$

and the equations of equilibrium become

$$(6) \quad [\partial(r\sigma_{rr})/\partial r]/r + (\partial\sigma_{r\theta}/\partial\theta)/r - \sigma_{\theta\theta}/r + F_r = 0, \\ [\partial(r^2\sigma_{r\theta})/\partial r]/r^2 + (\partial\sigma_{\theta\theta}/\partial\theta)/r + F_\theta = 0.$$

If the body forces F_r, F_θ are zero Eqs. (6) are satisfied identically if the stresses are derived from a function $\Phi(r, \theta)$:

$$(7) \quad \Delta \quad \sigma_{rr} = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right),$$

as can be verified by direct substitution. The function $\Phi(r, \theta)$ is the *Airy stress function*. The compatibility conditions (9.2:9) and (9.2:10) are, if body forces are zero,

$$(8) \quad (\partial^2/\partial x^2 + \partial^2/\partial y^2)(\sigma_{xx} + \sigma_{yy}) = 0.$$

The sum $\sigma_{xx} + \sigma_{yy}$ is an invariant with respect to rotation of coordinates. Hence,

$$(9) \quad \sigma_{xx} + \sigma_{yy} = \sigma_{rr} + \sigma_{\theta\theta}.$$

The Laplace operator is transformed as

$$(10) \quad \partial^2/\partial x^2 + \partial^2/\partial y^2 = \partial^2/\partial r^2 + (\partial/\partial r)/r + (\partial^2/\partial \theta^2)/r^2.$$

Hence, on substituting Eq. (7) into Eq. (9) and using Eq. (10), Eq. (8) is transformed into

$$(11) \quad \Delta \quad \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \right) = 0.$$

This is the compatibility equation to be satisfied by the Airy stress function $\Phi(r, \theta)$. If one can find a solution of Eq.

(11) that also satisfies the boundary conditions, then the problem is solved; since by Kirchhoff's uniqueness theorem, such a solution is unique.

Axially Symmetric Problems. If Φ is a function of r alone and is independent of θ , all derivatives with respect to θ vanish and Eq. (11) becomes

$$(12) \quad d^4\Phi/dr^4 + 2(d^3\Phi/dr^3)/r - (d^2\Phi/dr^2)/r^2 + (d\Phi/dr)/r^3 = 0.$$

This is a homogeneous differential equation which can be reduced to a linear differential equation with constant coefficients by introducing a new variable t such that $r = e^t$. The general solution is

$$(13) \quad \Phi = A \log r + Br^2 \log r + Cr^2 + D,$$

which corresponds to

$$(14) \quad \begin{aligned} \sigma_{rr} &= A/r^2 + B(1 + 2 \log r) + 2C, \\ \sigma_{\theta\theta} &= -A/r^2 + B(3 + 2 \log r) + 2C, \quad \sigma_{r\theta} = 0. \end{aligned}$$

The solutions of all problems of symmetrical stress distribution and no body forces can be obtained from this.

The displacement components corresponding to stresses given by Eq. (14) may be obtained as follows. Consider the *plane stress* case. Substituting Eq. (14) into the first of the Eqs. (4) and (3), we obtain

$$\frac{\partial \xi_r}{\partial r} = \frac{1}{E} \left[\frac{(1+\nu)A}{r^2} + 2(1-\nu)B \log r + (1-3\nu)B + 2(1-\nu)C \right],$$

from which, by integration,

$$(15) \quad \xi_r = \frac{1+\nu}{E} \left[-\left(\frac{A}{r} + Br \right) + 2r(B \log r + C) \frac{1-\nu}{1+\nu} \right] + f(\theta),$$

where $f(\theta)$ is an arbitrary function of θ only. From Eq. (14) and the second of the Eqs. (4) and (3), we obtain

$$\partial \xi_\theta / \partial \theta = 4Br/E - f(\theta).$$

Hence, by integration,

$$(16) \quad \xi_\theta = \frac{4Br\theta}{E} - \int_0^\theta f(\theta) d\theta + f_1(r),$$

where $f_1(r)$ is a function of r only. Finally, from the last of Eqs. (14), Eqs. (4) and (3), we find, since $\sigma_{r\theta} = \epsilon_{r\theta} = 0$,

$$\frac{1}{r} \frac{df(\theta)}{d\theta} + \frac{df_1(r)}{dr} + \frac{1}{r} \int_0^\theta f(\theta) d\theta - \frac{1}{r} f_1(r) = 0.$$

Multiplying throughout by r , we find that the first and the third term are functions of θ only and the other two terms are functions of r only. Hence, the only possibility for the last equation to be satisfied is

$$\frac{df(\theta)}{d\theta} + \int_0^\theta f(\theta) d\theta = \alpha, \quad r \frac{df_1(r)}{dr} - f_1(r) = -\alpha,$$

where α is an arbitrary constant. The solutions are

$$f(\theta) = \alpha \sin \theta + \gamma \cos \theta, \quad f_1(r) = \beta r + \alpha,$$

where α, β, γ are arbitrary constants. Substituting back into Eqs. (15) and (16), we obtain, for the *plane stress case*,

$$(17) \quad \begin{aligned} \xi_r &= [-(1+\nu)A/r + 2(1-\nu)Br \log r \\ &\quad - B(1+\nu)r + 2C(1-\nu)r]/E + \alpha \sin \theta + \gamma \cos \theta, \\ \xi_\theta &= 4Br\theta/E + \alpha \cos \theta - \gamma \sin \theta + \beta r \end{aligned}$$

The arbitrary constants $A, B, C, \alpha, \beta, \gamma$ are to be determined from the boundary conditions of each special problem. The corresponding expressions for the *plane strain case* are

$$(18) \quad \begin{aligned} \xi_r &= [-(1+\nu)(A/r + Br) + 2B(1-\nu-2\nu^2)r\log r \\ &\quad + 2C(1-\nu-2\nu^2)r]/E + \alpha\sin\theta + \gamma\cos\theta, \\ \xi_\theta &= 4Br\theta(1-\nu^2)/E + \alpha\cos\theta - \gamma\sin\theta + \beta r. \end{aligned}$$

Example 1. Uniform Pressure Acting on a Solid Cylinder

The boundary conditions are, at $r = a$,

$$\sigma_{rr} = -p_0, \quad \sigma_{r\theta} = 0.$$

There should be no singularity in the solid. A glance at Eq. (14) shows that the problem can be solved by taking

$$A = B = 0, \quad C = -p_0/2.$$

Example 2. Circular Cylindrical Tube Subjected to Uniform Internal and External Pressure

For this problem (Fig. 9.3:1), the boundary conditions are

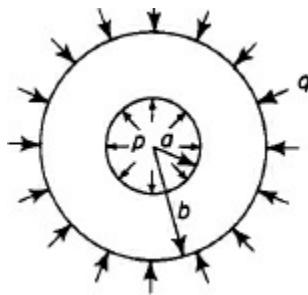


Fig. 9.3:1. Circular tube.

$$(19) \quad \begin{aligned} \sigma_{rr} &= -p & \text{at } r = a, \\ \sigma_{rr} &= -q & \text{at } r = b. \end{aligned}$$

The form of σ_{rr} in Eq. (14) suggests that the boundary conditions can be satisfied by fixing two of the constants, say A and C . From Eq. (19) alone, however, there is no reason to reject the term involving B ; but inspection of the displacements given by Eq. (18) shows that if $B \neq 0$ the circumferential displacement ξ_θ will have a nonvanishing term

$$(20) \quad \xi_\theta = 4Br\theta(1-\nu^2)/E$$

which is zero when $\theta = 0$ but becomes $\xi_\theta = (1-\nu^2)8\pi Br/E$ when one traces a circuit around the axis of symmetry and returns to the same point after turning around an angle 2π . Thus the displacement given by (20) is not *single-valued*. Such a *multi-valued* expression for a displacement is physically impossible in a full cylindrical tube. Hence, $B = 0$.

It is simple to derive the constants A and C in the expression

$$\sigma_{rr} = A/r^2 + 2C$$

such that Eq. (19) is satisfied. Thus we obtain Lamé's formulas for the stresses,

$$(21) \quad \begin{aligned} \sigma_{rr} &= -p(b^2/r^2 - 1)/(b^2/a^2 - 1) - q(1 - a^2/r^2)/(1 - a^2/b^2), \\ \sigma_{\theta\theta} &= p(b^2/r^2 + 1)/(b^2/a^2 - 1) - q(1 + a^2/r^2)/(1 - a^2/b^2). \end{aligned}$$

It is interesting to note that $\sigma_{rr} + \sigma_{\theta\theta}$ is constant throughout the cylinder. If $q = 0$ and $p > 0$, σ_{rr} is always a compressive stress and $\sigma_{\theta\theta}$ is always a tensile stress, the maximum value of which occurs at the inner radius and is always numerically greater than the internal pressure p .

Example 3. Pure Bending of a Curved Bar

The coefficients A , B , C may be chosen to satisfy the conditions $\sigma_{rr} = 0$ for $r = a$ and $r = b$, and

$$\int_a^b \sigma_{\theta\theta} dr = 0, \quad \int_a^b \sigma_{\theta\theta} r dr = -M, \quad \text{and} \quad \sigma_{r\theta} = 0,$$

at the boundary $\theta = \text{const.}$ (see Fig. 9.3:2). See Timoshenko and Goodier,^{1,2} p. 61; Sechler,^{1,2} p. 143.

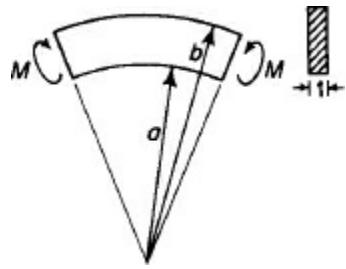


Fig. 9.3:2. Pure bending of a curved bar.

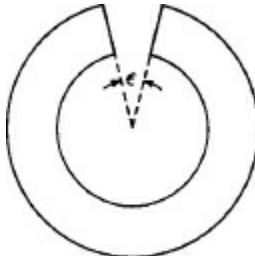


Fig. 9.3:3. Welding of a ring.

Example 4. Initial Stress or Residual Stress in a Welded Ring

If the ends of an opening in a ring, as shown in Fig. 9.3:3, are joined together and welded, the initial stresses can be obtained from the expressions in Eq. (14) by taking

$$(22) \quad B = \epsilon E / (8\pi),$$

with $\alpha = \beta = \gamma = 0$. See Eq. (17) for circumferential displacements.

Problem 9.1. Determine the constant C in the stress function

$$\Phi = C[r^2(\alpha - \theta) + r^2 \sin \theta \cos \theta - r^2 \cos^2 \theta \tan \alpha]$$

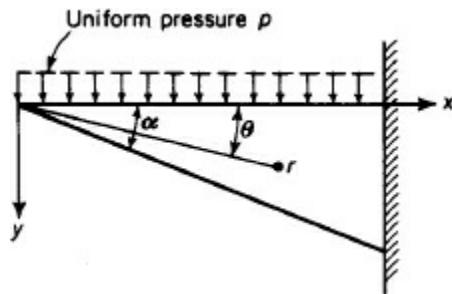


Fig. P9.1

required to satisfy the conditions on the upper and lower edges of the triangular plate shown in Fig. P9.1. Determine the components of displacement for points on the upper edge.

9.4. GENERAL CASE

By direct substitution, it can be verified that a solution of the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \right) = 0$$

is (J. H. Michell 1899)

$$(1) \quad \Phi = a_0 \log r + b_0 r^2 + c_0 r^2 \log r + d_0 r^2 \theta + a'_0 \theta \\ + (a''_1 r \theta \sin \theta)/2 + (a_1 r + b_1 r^3 + a'_1 r^{-1} + b'_1 r \log r) \cos \theta \\ + (c''_1 r \theta \cos \theta)/2 + (c_1 r + d_1 r^3 + c'_1 r^{-1} + d'_1 r \log r) \sin \theta \\ + \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a'_n r^{-n} + b'_n r^{-n+2}) \cos n\theta \\ + \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c'_n r^{-n} + d'_n r^{-n+2}) \sin n\theta.$$

By adjusting the coefficients a_0, b_0, a_1, \dots , etc., a number of important problems can be solved. The general form of Eq. (1) as a Fourier series in θ and power series in r provides a powerful means for solving problems involving circular and radial boundaries. The individual terms of the series provide beautiful solutions to several important engineering problems. Many examples are given in Timoshenko and Goodier,^{1,2} pp. 73–130 and Sechler,^{1,2} pp. 149–171.

Example 1. Bending of a Curved Bar by a Force at the End

Consider a bar of a narrow rectangular cross section and with a circular axis, as shown in Fig. 9.4:1, loaded by a force P in the radial direction. The bending moment at any cross section is proportional to $\sin \theta$. Since the elementary beam theory suggests that the normal stress $\sigma_{\theta\theta}$ is proportional to the bending moment, it is reasonable to try a solution for which $\sigma_{\theta\theta}$, and hence Φ , is proportional to $\sin \theta$. Such a solution can be obtained from the term involving $\sin \theta$ in Eq. (1). It can be verified that the solution is

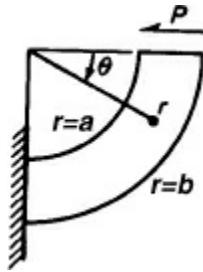


Fig. 9.4:1. Bending of a curved bar.

$$(2) \quad \Phi = (d_1 r^3 + c'_1 r^{-1} + d'_1 r \log r) \sin \theta,$$

with

$$d_1 = P/(2N), \quad c'_1 = -Pa^2b^2/(2N), \quad d'_1 = -P(a^2 + b^2)/N,$$

where $N = a^2 - b^2 + (a^2 + b^2) \log(b/a)$. Note that the term $c_1 r \sin \theta$ in Eq. (1) does not contribute to the stresses and thus is not used here. The stresses are

$$(3) \quad \begin{aligned} \sigma_{rr} &= P[r + a^2b^2/r^3 - (a^2 + b^2)/r](\sin \theta)/N, \\ \sigma_{\theta\theta} &= P[3r - a^2b^2/r^3 - (a^2 + b^2)/r](\sin \theta)/N \\ \sigma_{r\theta} &= -P[r + a^2b^2/r^3 - (a^2 + b^2)/r](\cos \theta)/N. \end{aligned}$$

If the boundary stress distribution were exactly as prescribed by the equations above, namely,

$$(4) \quad \begin{aligned} \sigma_{rr} &= \sigma_{r\theta} = 0 \quad \text{for } r = a \quad \text{and} \quad r = b, \\ \sigma_{\theta\theta} &= 0, \quad \sigma_{r\theta} = -\frac{P}{N} \left[r + \frac{a^2b^2}{r^3} - \frac{1}{r}(a^2 + b^2) \right] \quad \text{for } \theta = 0, \\ \sigma_{r\theta} &= 0, \quad \sigma_{\theta\theta} = \frac{P}{N} \left[3r - \frac{a^2b^2}{r^3} - (a^2 + b^2)\frac{1}{r} \right] \quad \text{for } \theta = \frac{\pi}{2}, \end{aligned}$$

then an exact solution is obtained.

An examination of the exact solution shows that a commonly used engineering approximation in the elementary beam theory, that plane cross sections of a beam remain plane during bending, gives satisfactory results.

Example 2. Concentrated Force at a Point on the Edge of a Semi-Infinite Plate

Consider a concentrated vertical load P acting on a horizontal straight boundary AB of an infinitely large plate (Fig. 9.4:2). The distribution of the load along the thickness of the plate is uniform. The plate thickness is assumed to be unity,

so that P is the load per unit thickness.

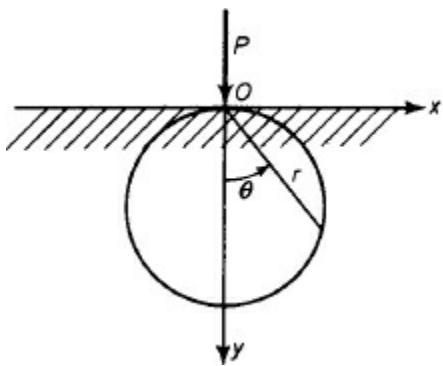


Fig. 9.4:2. Boussinesq-Flamant problem.

This is the static counterpart of Lamb's problem (Sec. 8.14). Its solution was obtained by Flamant (1892) from Boussinesq's (1885) three-dimensional solution (Sec. 8.10). In contrast to the dynamic or the three-dimensional cases, the solution to the present case is given simply by the Airy stress function

$$(5) \quad \Phi = -Pr \theta(\sin \theta)/\pi,$$

which gives

$$(6) \quad \sigma_{rr} = -2P(\cos \theta)/(\pi r), \quad \sigma_{\theta\theta} = 0, \quad \sigma_{r\theta} = 0.$$

From Eq. (6), it is seen that an element at a distance r from the point of application of the load is subjected to a simple compression in the radial direction. Equation (6) shows also that the locus of points where the radial stress is a constant (σ_{rr}) is a circle $r = (-2P/\pi\sigma_{rr}) \cos \theta$, which is tangent to the x -axis, as shown in Fig. 9.4:2.

Using the relations (9.3:3) and (9.3:4) and Eq. (6), we can determine the displacement field ξ_r , ξ_θ . If the constraint is such that the points on the y -axis have no lateral displacement ($\xi_\theta = 0$ when $\theta = 0$), then it can be shown that the elastic displacements are

$$(7) \quad \begin{aligned} \xi_r &= -P[2 \log r \cos \theta + (1 - \nu)\theta \sin \theta]/(\pi E) + B \cos \theta, \\ \xi_\theta &= P\{[2\nu + 2 \log r + (1 - \nu)] \sin \theta - (1 - \nu)\theta \cos \theta\}/(\pi E) - B \sin \theta. \end{aligned}$$

The constant B can be fixed by fixing a point, say, $\xi_r = 0$ at $\theta = 0$ and $r = a$. But the characteristic logarithmic singularity at ∞ cannot be removed. This is a peculiarity of the two-dimensional problem. The corresponding three-dimensional or dynamic cases do not have such logarithmically infinite displacements at infinity.

PROBLEMS

9.2. Verify Eqs. (6) and (7) and show that Eq. (5) yields the exact solution of the problem posed in Example 2.

9.3. Find the stresses in a semi-infinite plate ($-\infty < x < \infty, 0 \leq y < \infty$) due to a shear load of intensity $\tau \cos \alpha x$ acting on the edge $y = 0$, where τ and α are given constants. Hint: Consider $\Phi = (Ae^{\alpha y} + Be^{-\alpha y} + Cy e^{\alpha y} + Dy e^{-\alpha y}) \sin \alpha x$.

9.4. Show that the function $(M_t/2\pi)\theta$, where M_t is a constant, is a stress function. Consider some boundary value problems which may be solved by such a stress function and give a physical meaning to the constant M_t .

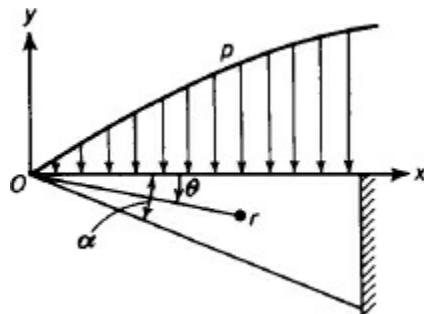


Fig. P9.5

9.5. Consider a two-dimensional wedge of perfectly elastic material as shown in Fig. P9.5. If one side of the wedge (θ

$= 0$) is loaded by a normal pressure distribution $p(r) = P r^m$, where P and m are constants, while the other side ($\theta = \alpha$) is stress-free, show that the problem can be solved by an Airy stress function $\Phi(r, \theta)$ expressed in the following form. If $m \neq 0$ (m may be > 0 or < 0),

$$(1) \quad \Phi = r^{m+2} [a \cos(m+2)\theta + b \sin(m+2)\theta + c \cos m\theta + d \sin m\theta].$$

If $m = 0$,

$$(2) \quad \Phi = Kr^2 [-\tan \alpha \cos^2 \theta + (\sin 2\theta)/2 + \alpha - \theta].$$

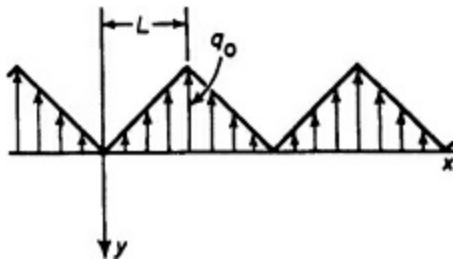
Determine the constants a, b, c, d, K . Discuss the *boundedness* (i.e., whether they are zero, finite, or infinite) of the stresses $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}$; the slope $\partial v/\partial r$, and the second derivative $\partial^2 v/\partial r^2$, in the neighborhood of the tip of the wedge ($r \rightarrow 0$). The symbol v denotes the component of displacement in the direction of increasing θ .

When the wedge angle α is small, v is approximately equal to the vertical displacement. This problem is of interest to the question of curling up of the sharp leading edge of a supersonic wing. (Fung, *J. Aeronautical Sciences*, **20**, 9, 1953).

9.6. A semi-infinite plate ($-\infty < x < \infty, 0 \leq y < \infty$) is subjected to the edge conditions

$$\begin{aligned} \sigma_{yy} &= \frac{q_0}{2} - \frac{4q_0}{\pi^2} \left(\cos \frac{\pi x}{L} + \frac{1}{3^2} \cos \frac{3\pi x}{L} + \frac{1}{5^2} \cos \frac{5\pi x}{L} + \dots \right), \\ \sigma_{xy} &= 0, \end{aligned}$$

on the edge $y = 0$ (Fig. P9.6). Find the Airy stress function that solves the problem.



Obtain expression for σ_{xx} and σ_{yy} . Sketch σ_{yy} as a function of x on the line $y = d$, where $d > L$.

9.5. REPRESENTATION OF TWO-DIMENSIONAL BIHARMONIC FUNCTIONS BY ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

It is well-known that the real and imaginary parts of any analytic function of a complex variable $z = x + iy$ is harmonic, where $i = \sqrt{-1}$. Thus, if $f(z)$

is an analytic function of z , and u and v are the real and imaginary parts of $f(z)$, we have

$$(1) \quad f(z) = u + iv,$$

$$(2) \quad \frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial f}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y},$$

$$(3) \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

$$(4) \quad \nabla^2 u = 0, \quad \nabla^2 v = 0, \quad \nabla^2 f = 0,$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, is the two-dimensional Laplace operator. Equation (3) is the well-known Cauchy–Riemann relation.

An alternative proof that an analytic function is harmonic is as follows. Let $\bar{z} = x - iy$ be the complex conjugate of z , then the differential operators in the x, y coordinates are related to those in the z, \bar{z} coordinates by

$$(5) \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = i(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}),$$

or

$$(6) \quad 2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \quad 2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

Equation (6) gives the Laplace operator in the z, \bar{z} coordinates as

$$(7) \quad 4 \frac{\partial^2}{\partial z \partial \bar{z}} = \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla^2.$$

Obviously any analytic function ψ of z or \bar{z} satisfies Eq. (7), i.e., $\nabla^2 \psi(z) = 0$ and $\nabla^2 \psi(\bar{z}) = 0$; so do the real and imaginary parts of $\psi(z)$ and $\psi(\bar{z})$.

Repeating the operation of Eq. (7) gives the biharmonic operator

$$(8) \quad \nabla^2 \nabla^2 = 16 \partial^4 / (\partial z^2 \partial \bar{z}^2).$$

Thus any biharmonic function can be written in the form of the real or imaginary part of $\bar{z}\psi(z) + \chi(z)$ or $z\psi(\bar{z}) + \chi(\bar{z})$, where $\psi(z), \chi(z)$ are arbitrary analytic functions, because

$$(9) \quad \nabla^2 \nabla^2 [\bar{z}\psi(z) + \chi(z)] = \nabla^2 \nabla^2 [z\psi(\bar{z}) + \chi(\bar{z})] = 0.$$

Problem 9.7.

(a) Show that $x - iy$ is not an analytic function of $z = x + iy$.

(b) Determine the real functions of x and y which are real and imaginary parts of the complex functions z^n and $\tanh z$.

(c) Determine the real functions of r and θ which are the real and imaginary parts of the complex function $z \ln z$. Note that $z = re^{i\theta}$.

(d) $z = x + iy, \bar{z} = x - iy, a = \alpha + i\beta, \bar{a} = \alpha - i\beta$, where x, y, α, β are real numbers. Express the real and imaginary parts of the following functions explicitly in terms of x, y, α, β to get acquainted with the notations $f(z), \bar{f}(\bar{z}), f(\bar{z})$ and $\bar{f}(z)$:

$$\begin{array}{llll} 1. f(z) = az, & \bar{f}(\bar{z}) = \bar{a}\bar{z}, & f(\bar{z}) = a\bar{z}, & \bar{f}(z) = \bar{a}z. \\ 2. f(z) = e^{iaz}, & \bar{f}(\bar{z}) = e^{-i\bar{a}\bar{z}}, & f(\bar{z}) = e^{i\bar{a}\bar{z}}, & \bar{f}(z) = e^{-i\bar{a}z}. \\ 3. f(z) = az^n, & \bar{f}(\bar{z}) = \bar{a}\bar{z}^n, & f(\bar{z}) = a\bar{z}^n, & \bar{f}(z) = \bar{a}z^n. \end{array}$$

Show that the complex conjugate of $f(z)$ is $\bar{f}(\bar{z})$ in these examples.

(e) Show that the complex conjugate of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is $\bar{f}(\bar{z}) = \sum_{n=0}^{\infty} \bar{a}_n \bar{z}^n$.

(f) Show that the derivative of $\bar{f}(\bar{z})$ with respect to \bar{z} is equal to the complex conjugate of $df(z)/dz$.

(g) Find a function $v(x, y)$ of two real variables x, y , such that $\ln(x^2 + y^2) + iv(x, y)$ is an analytic function of a complex variable $x + iy$. (Use the Cauchy–Riemann differential equations).

9.6. KOLOSOFF–MUSKHELISHVILI METHOD

The representation of biharmonic functions by analytic functions leads to a general method of solving problems in plane stress and plane strain. Consider a region — simply or multiply connected — on the x, y -plane bounded by contours (Fig. 9.6:1). The interior of the region represents a disk of unit thickness.

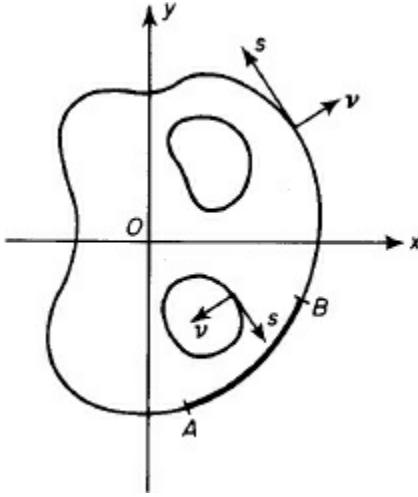


Fig. 9.6:1. Notations.

Surface traction is applied on the boundaries of this region. Body forces are assumed to be zero to simplify the general solution. If body forces actually exist they can be represented by a particular solution so that the general problem is reduced to one without body force.

On introduction of the Airy stress function $\Phi(x, y)$ (Sec. 9.2), the plane stress and plane strain problems are reduced to the solution of a biharmonic equation subjected to appropriate boundary conditions. According to the results of Sec. 9.5, the Airy stress function $\Phi(x, y)$ can be written as

$$(1) \quad \Delta \quad 2\Phi(x, y) = 2\operatorname{Re}[\bar{z}\psi(z) + \chi(z)] = \bar{z}\psi(z) + z\bar{\psi}(\bar{z}) + \chi(z) + \bar{\chi}(\bar{z})$$

where $\operatorname{Re}(.)$ denotes the real part of the quantity in the parenthesis, and $\psi(z)$ and $\chi(z)$ are analytic functions, called the *complex stress functions*. The formula (1) is due to the French mathematician Goursat.

The problem in plane elasticity is now simply to determine the complex stress functions $\psi(z)$ and $\chi(z)$ that satisfy the boundary conditions. For this purpose, we must express all the stresses and displacements in terms of $\psi(z)$ and $\chi(z)$. We shall show that

$$(2) \quad \Delta \quad \sigma_{xx} + \sigma_{yy} = 2[\psi'(z) + \bar{\psi}'(\bar{z})],$$

$$(3) \quad \Delta \quad \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\bar{z}\psi''(z) + \chi''(z)],$$

$$(4) \quad \Delta \quad 2G(u + iv) = \kappa\psi(z) - z\bar{\psi}'(\bar{z}) - \bar{\chi}'(\bar{z}),$$

where a prime denotes differentiation with respect to the independent variable, i.e., $\psi'(z) = d\psi(z)/dz$ and $\bar{\psi}'(\bar{z}) = d\bar{\psi}(\bar{z})/d\bar{z}$,

$$(5) \quad \kappa = 3 - 4\nu \quad \text{for plane strain},$$

$$(6) \quad \kappa = (3 - \nu)/(1 + \nu) \quad \text{for plane stress},$$

and ν is Poisson's ratio. In addition, we have the following expressions for the resultant forces F_x, F_y and the moment M about the origin of the surface traction on an arc AB of the boundary from point A to point B :

$$(7) \quad \Delta \quad F_x + iF_y = \int_A^B \left(T_x + iT_y \right) ds = -i[\psi(z) + z\bar{\psi}'(\bar{z}) + \bar{\chi}'(\bar{z})]_A^B$$

$$(8) \quad M = \operatorname{Re} [\chi(z) - \bar{z}\bar{\chi}'(\bar{z}) - z\bar{z}\bar{\psi}'(\bar{z})]_A^B.$$

The physical quantities $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, u, v$, etc., are of course real-valued. If the complex-valued functions on the right-hand side of Eqs. (2)–(4) are known, a separation into real and imaginary parts determines all the stresses and displacements.

The derivation of these relations is as follows. According to Eqs. (1) and (9.2:17), for vanishing body force, we have

$$(9) \quad \sigma_{xx} + \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 4 \frac{\partial^2 \Phi}{\partial z \partial \bar{z}} = 2[\psi'(z) + \bar{\psi}'(\bar{z})].$$

$$(10) \quad \sigma_{xx} - i\sigma_{xy} = i \frac{\partial}{\partial y} \left(\frac{\partial \Phi}{\partial x} - i \frac{\partial \Phi}{\partial y} \right) = 2i \frac{\partial}{\partial y} \frac{\partial \Phi}{\partial z}.$$

$$(11) \quad \sigma_{yy} + i\sigma_{xy} = \frac{\partial}{\partial x} \left(\frac{\partial \Phi}{\partial x} - i \frac{\partial \Phi}{\partial y} \right) = 2 \frac{\partial}{\partial x} \frac{\partial \Phi}{\partial z}.$$

Taking the difference of Eqs. (10) and (11) and using Eq. (1) lead to Eq. (3).

To derive Eq. (4), we have to consider plane stress and plane strain separately. For plane stress, Hooke's law states

$$(12) \quad e_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) = \frac{1}{E} \left(\frac{\partial^2 \Phi}{\partial y^2} - \nu \frac{\partial^2 \Phi}{\partial x^2} \right),$$

$$(13) \quad e_{yy} = \frac{\partial v}{\partial y} = \frac{1}{E} (\sigma_{yy} - \nu \sigma_{xx}) = \frac{1}{E} \left(\frac{\partial^2 \Phi}{\partial x^2} - \nu \frac{\partial^2 \Phi}{\partial y^2} \right),$$

$$(14) \quad e_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2G} \sigma_{xy} = -\frac{1}{2G} \frac{\partial^2 \Phi}{\partial x \partial y}.$$

Note that

$$(15) \quad \frac{d\psi(z)}{dz} + \frac{d\bar{\psi}(\bar{z})}{d\bar{z}} = \frac{\partial}{\partial x} [\psi(x+iy) + \bar{\psi}(x-iy)] = 2 \frac{\partial}{\partial x} \operatorname{Re}(\psi),$$

$$(16) \quad \frac{d\psi(z)}{dz} - \frac{d\bar{\psi}(\bar{z})}{d\bar{z}} = \frac{1}{i} \frac{\partial}{\partial y} [\psi(x+iy) - \bar{\psi}(x-iy)] = 2 \frac{\partial}{\partial y} \operatorname{Im}(\psi).$$

where $\operatorname{Re}(\cdot)$ and $\operatorname{Im}(\cdot)$ denote the real and imaginary parts of the quantity in the parenthesis. From Eqs. (9) and (15) and from Eqs. (9) and (16), we have

$$(17) \quad \frac{\partial^2 \Phi}{\partial y^2} = 4 \frac{\partial}{\partial x} \operatorname{Re}(\psi) - \frac{\partial^2 \Phi}{\partial x^2}, \quad \frac{\partial^2 \Phi}{\partial x^2} = 4 \frac{\partial}{\partial y} \operatorname{Im}(\psi) - \frac{\partial^2 \Phi}{\partial y^2},$$

respectively. A substitution of Eq. (17) into Eq. (12) yields

$$\frac{\partial u}{\partial x} = [4 \frac{\partial \operatorname{Re}(\psi)}{\partial x} - (1+\nu) \frac{\partial^2 \Phi}{\partial x^2}] / E.$$

An integration with respect to x gives

$$(18) \quad u = [4 \operatorname{Re}(\psi) - (1+\nu) \frac{\partial \Phi}{\partial x}] / E + f(y),$$

where $f(y)$ is an arbitrary function of y . Similarly, from Eq. (13), we obtain

$$(19) \quad v = [4 \operatorname{Im}(\psi) - (1+\nu) \frac{\partial \Phi}{\partial y}] / E + g(x),$$

where $g(x)$ is an arbitrary function of x . Substituting Eqs. (18) and (19) into Eq. (14), and noticing that $\frac{\partial \operatorname{Re}(\psi)}{\partial y} = -\frac{\partial \operatorname{Im}(\psi)}{\partial x}$ we obtain

$$f'(y) + g'(x) = 0$$

Hence, $f(y) = \alpha y + \beta$, $g(x) = -\alpha x + \gamma$, where α, β, γ are constants. The forms of f and g represent rigid body motions and can be disregarded in the analysis of deformation. If we set $f = g = 0$ in Eqs. (18) and (19), using

$$\frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y} = 2 \frac{\partial \Phi}{\partial \bar{z}} = \psi(z) + z \bar{\psi}'(\bar{z}) + \bar{\chi}'(\bar{z})$$

and combining them properly, we obtain Eqs. (4) and (5). The plane strain case is similar, and Eq. (6) can be obtained without further analysis by applying a rule derived in Sec. 9.1, namely, replacing ν in Eq. (5) by an apparent value $\nu/(1-\nu)$.

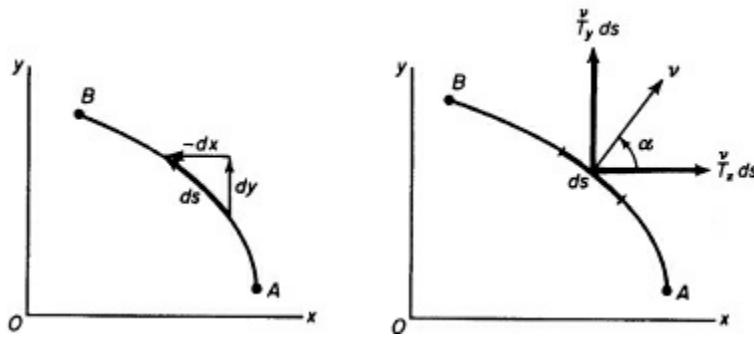


Fig. 9.6.2. Notations.

To prove formulas (7) and (8), let AB be a continuous arc, ds an arc length, v the unit vector normal to the arc, and $\frac{\nu}{T_x}$, $\frac{\nu}{T_y}$ be, respectively, the x 's and y 's components of the traction acting on the arc (Fig. 9.6.2). We note that

$$(20) \quad \nu_x = \cos \alpha = dy/ds, \quad -\nu_y = -\sin \alpha = dx/ds,$$

$$(21) \quad \frac{\nu}{T_x} = \sigma_{xx} \cos \alpha + \sigma_{xy} \sin \alpha, \quad \frac{\nu}{T_y} = \sigma_{yy} \sin \alpha + \sigma_{xy} \cos \alpha.$$

Hence by Eq. (9.2:17),

$$(22) \quad \frac{\nu}{T_x} = \frac{\partial^2 \Phi}{\partial y^2} \frac{dy}{ds} + \frac{\partial^2 \Phi}{\partial x \partial y} \frac{dx}{ds} = \frac{d}{ds} \left(\frac{\partial \Phi}{\partial y} \right), \quad \frac{\nu}{T_y} = -\frac{d}{ds} \left(\frac{\partial \Phi}{\partial x} \right).$$

Therefore, the resultant of forces acting on the arc AB can be written as:

$$(23) \quad F_x + iF_y = \int_A^B (\frac{\nu}{T_x} + i\frac{\nu}{T_y}) ds = (\partial \Phi / \partial y - i \partial \Phi / \partial x) \Big|_A^B \\ = -i[\partial \Phi / \partial x + i \partial \Phi / \partial y] \Big|_A^B = -2i[\partial \Phi / \partial z] \Big|_A^B$$

The moment about the origin O of the forces acting on the arc AB is

$$(24) \quad M = \int_A^B (x \frac{\nu}{T_y} + y \frac{\nu}{T_x}) ds = - \int_A^B \left(x \frac{d}{ds} \frac{\partial \Phi}{\partial x} + y \frac{d}{ds} \frac{\partial \Phi}{\partial y} \right) ds \\ = \int_A^B \left(\frac{\partial \Phi}{\partial x} \frac{dx}{ds} + \frac{\partial \Phi}{\partial y} \frac{dy}{ds} \right) ds - \left[x \frac{\partial \Phi}{\partial x} + y \frac{\partial \Phi}{\partial y} \right]_A^B \\ = \Phi \Big|_A^B - Rl(2z \partial \Phi / \partial z) \Big|_A^B.$$

A substitution of Eq. (1) into Eqs. (23) and (24) verifies Eqs. (7) and (8).

Equations (7) and (8), applied to a closed contour show that if $\psi(z)$ and $\chi(z)$ are single-valued, the resultant forces and moment of the surface traction acting on the contour vanish, since the functions in the brackets return to their initial values when the circuit is completed. If the resultant force of traction acting on a closed boundary does not vanish, the value of the functions in the brackets must not return to their initial values when the circuit is completed. For instance, the function $\ln z = \ln r + i\theta$ does not return to its original value on completing a circuit around the origin, since θ increases by 2π . Thus, if $\psi(z) = C \ln z$, or $\chi(z) = Dz \ln z$, where C and D are complex constants, Eq. (7) will yield a nonzero value of $F_x + iF_y$. Similarly, $\chi(z) = D \ln z$ yields a nonzero value of M according to Eq. (8) if D is imaginary, but a zero value if D is real.

The examples below will show that a number of problems in plane stress and plane strain can be solved by simple stress functions. By reducing the problem into the determination of two analytic functions of a complex variable, Muskhelishvili and his school have devised several general methods of approach. The principal tools for these general approaches are the conformal mapping and Cauchy integral equations. The details are well recorded in Muskhelishvili's books.

Example 1. Circular Hole in a Large Plate

Consider a large plate with a circular hole of radius R at the origin. The plate is subjected to uniform remote stresses σ_{xx0} , σ_{yy0} , σ_{xy0} . The traction on the surface of the hole is zero. From Eq. (7), the traction free condition can be written as

$$(25) \quad -i(F_x - iF_y) = \bar{\psi}(\bar{z}) + \bar{z} d\psi(z)/dz + \varphi(z) = 0,$$

at $z = Re^{i\theta}$, where ϕ is a complex stress function defined as

$$(26) \quad \varphi(z) = d\chi(z)/dz,$$

which is introduced to save writing. If we define

$$(27) \quad \varphi(z) = -\bar{\psi}(R^2/z) - R^2[d\psi(z)/dz]/z,$$

then the resultant force on a circular arc connecting points P and Q is

$$-i(F_x - iF_y) = [\bar{\psi}(\bar{z}) - \bar{\psi}(R^2/z) + (\bar{z} - R^2/z)d\psi(z)/dz] \Big|_P^Q$$

which is zero at the hole ($z\bar{z} = R^2$) for arbitrary $\psi(z)$.

We can use the remote stress conditions to determine $\psi(z)$. Assume

$$(28) \quad \psi = Az + B/z + C/z^2 + \dots,$$

where A, B, C, \dots are complex constants. From Eqs. (2) and (3), we have

$$\begin{aligned} (\sigma_{xx} + \sigma_{yy})/2 &= A - B/z^2 - 2C/z^3 + \bar{A} - \bar{B}/\bar{z}^2 - 2\bar{C}/\bar{z}^3 + \dots, \\ (\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy})/2 &= (\bar{z} - R^2/z)\psi''(z) + [\bar{\psi}'(R^2/z) + \psi'(z)]R^2/z^2 \\ &= (\bar{z} - R^2/z)(2B/z^3 - 6C/z^4) \\ &\quad + (\bar{A} - \bar{B}z^2/R^4 - 2\bar{C}z^3/R^6 + A - B/z^2 - 2C/z^3]R^2/z^2 + \dots. \end{aligned}$$

For $\sigma_{xx} + \sigma_{yy} = \sigma_{xx0} + \sigma_{yy0}$ and $\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = \sigma_{yy0} - \sigma_{xx0} + 2i\sigma_{xy0}$ as z approaches infinite, we obtain

$$(29) \quad A + \bar{A} = (\sigma_{xx0} + \sigma_{yy0})/2, \quad B = (\sigma_{xx0}/2 - \sigma_{yy0}/2 + i\sigma_{xy0})/R^2, \quad C = \dots = 0.$$

The imaginary part of A is indeterminate since $\psi = \text{Im}(A)z$ represents a rigid body motion, and can be set to zero in the analysis of deformation. Thus, we have

$$\psi = (\sigma_{xx0} + \sigma_{yy0})z/4 - [(\sigma_{yy0} - \sigma_{xx0})/2 - i\sigma_{xy0}]R^2/z,$$

$\psi(z)$ can be calculated from Eq. (27) and then

$$\begin{aligned} (\sigma_{xx} + \sigma_{yy})/2 &= 2A - B/z^2 - \bar{B}/\bar{z}^2, \\ (\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy})/2 &= 2B\bar{z}/z^3 + (2A - \bar{B}z^2/R^4 + 3B/z^2)R^2/z^2. \end{aligned}$$

At $z = \pm R$, one can show that $\sigma_{yy} = 3\sigma_{yy0} - \sigma_{xx0}$, which is the well-known result of stress concentration at a hole in a large plate for $\sigma_{xx0} = 0$.

Example 2. Interaction of an Elliptical Hole and a Concentrated Load (Tong 1984)

Consider a large plate with an elliptical hole of major and minor axes a, b along the x, y axes at the origin. A concentrated force $F_0 (= F_{x0} + iF_{y0})$ is applied at $z = z_0$. We introduce the transformation

$$(30) \quad z = R(\zeta + m/\zeta), \quad \text{where} \quad R = (a + b)/2, \quad m = (a - b)/(a + b)$$

which maps the unit circle at the origin of the ζ -plane to the ellipse in the z -plane. Similar to Eq. (25), the traction free condition on an ellipse is

$$(31) \quad -i(F_x - iF_y) = \bar{\psi}(\bar{\zeta}) + \bar{z}(\bar{\zeta})(d\zeta/dz)(d\psi(z)/d\zeta) + \varphi(\zeta) = 0$$

on the unit circle in the ζ -plane where

$$d\zeta/dz = 1/(dz/d\zeta) = 1/[R(1 - m/\zeta^2)].$$

Let

$$(32) \quad \varphi(\zeta) = -\bar{\psi}(1/\zeta) - \bar{z}(1/\zeta)(d\zeta/dz)(d\psi(z)/d\zeta),$$

then, according to Eq. (7), the resultant force on an arc AB is

$$(33) \quad -i(F_x - iF_y) = \bar{\psi}(\bar{\zeta}) - \bar{\psi}\left(\frac{1}{\zeta}\right) + \left[\bar{z}(\bar{\zeta}) - \bar{z}\left(\frac{1}{\zeta}\right)\right] \frac{d\zeta}{dz} \frac{d\psi(z)}{d\zeta} \Big|_A^B,$$

which equals to zero on the unit circle for arbitrary $\psi(\zeta)$.

We shall choose $\psi(\zeta)$ to give a net force F_0 at $z = z_0$ (or $\zeta = \zeta_0$) and zero stress at infinite. Assume $\psi(\zeta)$ in the form

$$(34) \quad \psi(\zeta) = A \ln(\zeta - \zeta_0) + \psi_1(\zeta),$$

where $\psi_1(\zeta)$ is analytic and bounded outside the unit circle. From Eq. (4),

$$\begin{aligned} 2G(u - iv) &= \kappa \bar{\psi}(\bar{\zeta}) + \bar{\psi}(1/\zeta) - [\bar{z}(\bar{\zeta}) - \bar{z}(1/\zeta)](d\zeta/dz)[d\psi(z)/d\zeta] \\ &= \kappa [\bar{A} \ln(\bar{\zeta} - \bar{\zeta}_0) + \bar{\psi}_1(\bar{\zeta})] + \bar{A} \ln(1/\zeta - \bar{\zeta}_0) + \bar{\psi}_1(1/\zeta) \\ &\quad - [\bar{z}(\bar{\zeta}) - \bar{z}(1/\zeta)](d\zeta/dz)[A/(\zeta - \zeta_0) + \bar{\psi}'_1(\zeta)]. \end{aligned}$$

To assure that $u - iv$ is single-valued and has no singularity other than $\ln |\zeta - \zeta_0|$ outside the unit circle, we require that

$$\bar{\psi}_1\left(\frac{1}{\zeta}\right) = \kappa \bar{A} \ln(\zeta - \zeta_0) + \frac{A}{\zeta - \zeta_0} \left[\bar{z}(\bar{\zeta}_0) - \bar{z}\left(\frac{1}{\zeta_0}\right) \right] / R \left(1 - \frac{m}{\zeta_0^2}\right)$$

or

$$\psi_1(\zeta) = \kappa A \ln\left(\frac{1}{\zeta} - \bar{\zeta}_0\right) + \frac{\bar{A}\zeta}{1 - \zeta\bar{\zeta}_0} \left(\zeta_0 + \frac{m}{\zeta_0} - \frac{1}{\bar{\zeta}_0} - m\bar{\zeta}_0\right) / \left(1 - \frac{m}{\zeta_0^2}\right).$$

Clearly $\psi_1(\zeta)$ is analytic and bounded outside the unit circle. To determine A , we examine Eq. (33) in a small neighborhood of ζ_0 , i.e., $\zeta = \zeta_0 + \delta e^{i\theta}$,

$$F_{x0} + iF_{y0} = -i[(A \ln \delta + i\theta) - \kappa(A \ln \delta - i\theta) + f(\delta, \theta)] \Big|_0^{2\pi} = 2\pi(\kappa + 1)A$$

where $f(\delta, \theta)$ is a single-valued function related to ψ_1 . Therefore

$$A = F_0/[2\pi(\kappa + 1)].$$

One can then evaluate all the stresses and displacements according to Eqs. (2)–(4).

For $b = 0$, m equals 1 and the ellipse becomes a crack of length $2a$. Then

$$d\zeta/dz = 2\zeta^2/[a(\zeta^2 - 1)] = 1/(2ar e^{i\theta})^{1/2} + O(1),$$

as $r \rightarrow 0$, where $re^{i\theta} = z \mp a$, which gives rise to a $1/\sqrt{r}$ stress singularity at the crack tips $z = \pm a$ or $\zeta = \pm 1$. The stress intensity factors, a measure of the intensity of the singularity, are defined as

$$\begin{aligned} K_I - iK_{II} &= \lim_{|z| \rightarrow a^+} \{[\pi(z^2 - a^2)/a]^{1/2}(\sigma_{yy} - i\sigma_{xy})\} \\ &= \lim_{|z| \rightarrow a^+} \{2[\pi(z^2 - a^2)/a]^{1/2}(d\psi/dz)\} = 2[\pi/a]^{1/2} \lim_{|\zeta| \rightarrow 1^+} (d\psi/d\zeta) \end{aligned}$$

which is proportional to the coefficients of the terms of singular stress. In the equation above, z and ζ approach the limit along the real axis. The $1/\sqrt{r}$ stress singularity at crack tip is the fundamental characteristics of linear fracture mechanics. Interested readers are referred to books on the subject. From Eqs. (34) and (35), we obtain

$$K_I + iK_{II} = \frac{1}{(1+\kappa)\sqrt{a\pi}} \left[\left(\frac{1}{1-\bar{\zeta}_0} - \frac{\kappa}{1-\zeta_0} \right) \bar{F}_0 + \frac{\zeta_0(\zeta_0\bar{\zeta}_0 - 1)(\bar{\zeta}_0 - \zeta_0)}{\bar{\zeta}_0(1-\zeta_0)^2(\zeta_0^2 - 1)} F_0 \right].$$

PROBLEMS

9.8. Discuss the conformal mapping specified by $\zeta = \frac{1}{2}(z + \frac{1}{z})$. Obtain the inverse transformation. This transformation is the basis of airplane wing theory.

9.9. Consider a multiply connected region in the x , y -plane. Express the condition for the single-valuedness of

displacements in terms of stresses.

9.10. If $\partial\Phi/\partial x + i\partial\Phi/\partial y = \psi(z) + z\bar{\psi}'(\bar{z}) + \chi'(\bar{z})$ is to be single-valued, how arbitrary are the functions $\psi(z)$, $\chi(z)$?

9.11. Consider a doubly connected region R bounded by two concentric circles. If the stress is single-valued in R , how arbitrary are the functions $\psi(z)$, $\chi(z)$ in the Airy stress function? If both stresses and displacements are single-valued in R , how arbitrary are $\psi(z)$, $\chi(z)$?

[Muskhelishvili,^{1,2} p. 116–128.]

9.12. Show that, if all stress and displacement components σ_{ij} and u_i vanish on an arc AB , then the stresses and displacements vanish identically in the entire region R containing the arc AB .

[Muskhelishvili,^{1,2} p. 132.]

9.13. Show that the transformation $z = e^{\zeta}$ defines a transformation from rectangular coordinates to polar coordinates.

9.14. A ring ($a < r < b$) is subjected to uniformly distributed shearing stress in the circumferential direction on the inner and outer surfaces as in Fig. P9.14. The couple of the shearing forces over the circumference is of magnitude M . Determine the stress and displacement components by means of complex potentials

$$\psi(z) = 0, \quad \chi(z) = A \log z,$$

where A is a constant which can be complex-valued.

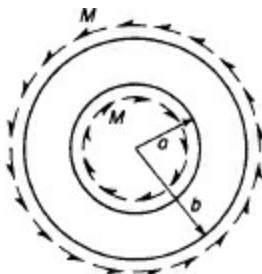


Fig. P9.14. Twisting of a disk.

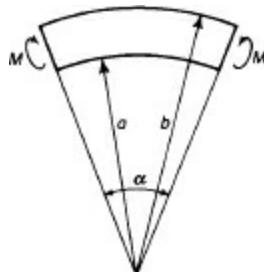


Fig. P9.15. Pure bending of a curved bar.

9.15. A curved bar is subjected to a bending moment M at each end. The bar is defined by circular arcs of radius a and b and radial lines with an opening angle α , ($\alpha < 2\pi$) as in Fig. P9.15. Show that the problem can be solved by complex potentials of the form

$$\psi(z) = Az \log z + Bz, \quad \chi(z) = C \log z.$$

Determine the constants A , B , and C .

9.16. Consider a rectangular plate of sides $2l$ and $2L$, and small thickness $2h$. Loads are applied on the edges, Fig. P9.16. There is no body force. Determine the stresses and displacements in the plate with the suggested stress functions $\psi(z)$ and $\chi(z)$. Express the constants a , b in terms of the external loads.

(a) All-round tension, Fig. P9.16(a):

$$\psi(z) = az, \quad \chi(z) = 0.$$

(b) Uniaxial tension at an angle α to the x -axis, Fig. P9.16(b):

$$\psi(z) = az, \quad \chi'(z) = -2aze^{-2i\alpha}.$$

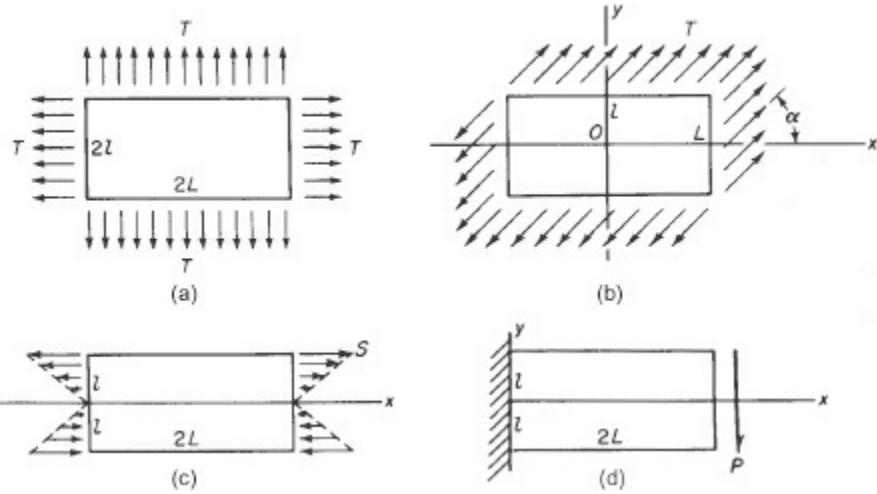


Fig. P9.16

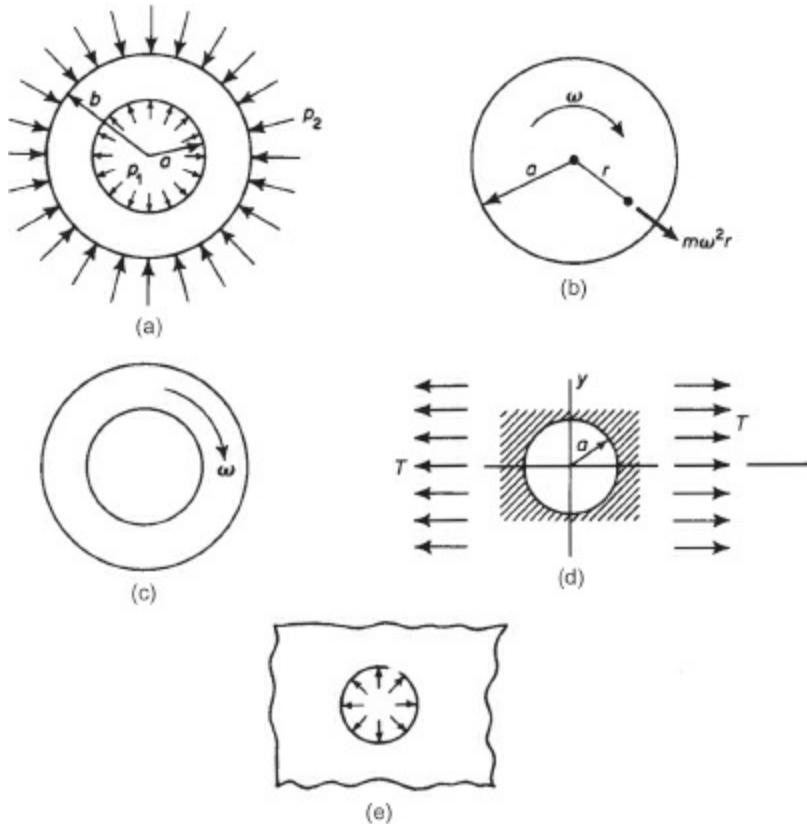


Fig. P9.17

(c) Pure bending, Fig. P9.16(c):

$$\psi(z) = aiz^2, \quad \chi(z) = -aiz^3/3.$$

(d) Bending with shear, Fig. P9.16(d):

$$\psi(z) = 2aiz^3, \quad \chi'(z) = -4ai(z^3 + 6zb^2).$$

Note: In comparison with the bending of a beam in the three-dimensional case discussed in Sec. 7.7, the example in (d) shows how much simpler is the two-dimensional problem.

9.17. Consider plane-stress or plane-strain states, parallel to the x, y -plane, of elastic bodies occupying regions bounded by circles. Determine the stresses and displacements for the following physical problems with the suggested stress functions (Fig. P9.17):

(a) Cylinder under internal pressure p_1 and external pressure p_2 [Fig. P9.17(a)]

$$\psi(z) = Az, \quad \chi'(z) = B/z.$$

(b) A disk of radius a rotating about its axis at a constant angular speed ω [Fig. P9.17(b)]: the centrifugal force

constitutes a body forces per unit mass = $\omega^2 r$, which has a potential $V = -\frac{1}{2}\omega^2 r^2$. Find a particular integral to Eqs. (9.2:18) and (9.2:19).

(c) Rotating hollow disk [Fig. P9.17(c)]. Solve by combining (a) and (b).

(d) Large plate under tension and containing an unstressed circular hole [Fig. P9.17(d)]: $\psi(z) = T z/4 + A/z$, $\chi'(z) = -T z/2 + B/z + C/z^3$. Show that $A = -B = T a^2/2$, $C = T a^4/2$.

(e) Large plate containing a circular hole under uniform pressure [Fig. P9.17(e)]:

$$\psi(z) = 0, \quad \chi'(z) = A/z.$$

¹This substitution refers only to the field Eqs. (6) and (8). The boundary conditions, the stress-strain relationship, and the shear modulus G are not to be changed.

²For problems in which displacements are prescribed over the entire boundary, the displacement potential or other devices of the preceding chapter should be tried first.

10

VARIATIONAL CALCULUS, ENERGY THEOREMS, SAINT-VENANT'S PRINCIPLE

There are at least three important reasons for taking up the calculus of variations in the study of continuum mechanics.

1. Because basic minimum principles exist, which are among the most beautiful of theoretical physics.

2. The field equations (ordinary or partial differential equations) and the associated boundary conditions of many problems can be derived from variational principles. In formulating an approximate theory, the shortest and clearest derivation is usually through variational calculus.

3. The computational methods of solution of variational problems are the most powerful tools for obtaining numerical results in practical problems of engineering importance.

In this chapter we shall discuss several variational principles and their applications. A brief introduction to the calculus of variations is furnished below. Those readers who are familiar with the mathematical techniques of the calculus of variations may skip over the first six sections.

10.1. MINIMIZATION OF FUNCTIONALS

The calculus of variations is concerned with the minimization of functionals. If $u(x)$ is a function of x , defined for x in the interval (a, b) , and if I is a quantity defined by the integral

$$I = \int_a^b [u(x)]^2 dx,$$

then the value of I depends on the function $u(x)$ as a whole. We may indicate this dependence by writing $I[u(x)]$. Such a quantity I is said to be a *functional* of $u(x)$. Physical examples of functionals are the total kinetic energy of a flow field and the strain energy of an elastic body.

The basic problem of the calculus of variations may be illustrated by the following example. Let us consider a functional $J[u]$ defined by the integral

$$(1) \quad J[u] = \int_a^b F(x, u, u') dx .$$

We shall give our attention to *all* functions $u(x)$ which are continuous and differentiable, with continuous derivatives $u'(x)$ and $u''(x)$ in the interval $a \leq x \leq b$, and satisfying the boundary conditions

$$(2) \quad u(a) = u_0, \quad u(b) = u_1,$$

where u_0 and u_1 are given numbers. We assume that $F(x, u, u')$ in Eq. (1) is continuous and differentiable with respect to x , and all such u , and u' , up to all second-order partial derivatives, which are themselves continuous.

Among all functions $u(x)$ satisfying these continuity and boundary conditions, we find a special one $u(x) = y(x)$, with the property that $J[u]$ attains a minimum when $u(x) = y(x)$, in a sufficiently small neighborhood of $y(x)$. The neighborhood (h) of $y(x)$ is defined as follows. If h is a positive quantity, a function $u(x)$ is said to lie in the neighborhood (h) of $y(x)$ if the inequality

$$(3) \quad |y(x) - u(x)| < h$$

holds for all x in (a, b) . The situation is illustrated in Fig. 10.1:1.

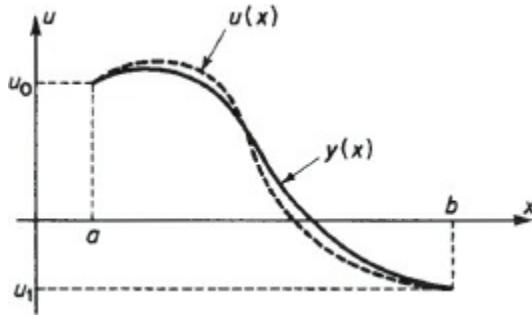


Fig. 10.1:1. Functions $u(x)$ and $y(x)$.

Let us assume that the problem posed above has a solution, designated by $y(x)$; i.e., there exists a function $y(x)$ such that the inequality

$$(4) \quad J[y] \leq J[u]$$

holds for all functions $u(x)$ in a sufficiently small neighborhood (h) of $y(x)$. Let us exploit the necessary consequences of this assumption.

Let $\eta(x)$ be an arbitrary function with the properties that $\eta(x)$ and its derivatives $\eta'(x)$, $\eta''(x)$ are continuous in the interval $a \leq x \leq b$, and that

$$(5) \quad \eta(a) = \eta(b) = 0.$$

Then the function

$$(6) \quad u(x) = y(x) + \epsilon\eta(x)$$

satisfies all the continuity conditions and boundary values specified at the beginning of this section. In fact, any function $u(x)$ satisfying these conditions can be represented with some function $\eta(x)$ in this manner. For a sufficiently small $\delta > 0$, this function $u(x)$ belongs, for all ϵ with $|\epsilon| < \delta$, to a prescribed neighborhood (h) of $y(x)$. Now we introduce the function

$$(7) \quad \phi(\epsilon) = J[y + \epsilon\eta] = \int_a^b F[x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)]dx.$$

Since $y(x)$ is assumed to be known, $\phi(\epsilon)$ is a function of ϵ for any specific $\eta(x)$. According to (4), the inequality

$$(8) \quad \phi(0) \leq \phi(\epsilon)$$

must hold for all ϵ with $|\epsilon| < \delta$. In other words, $\phi(\epsilon)$ attains a minimum at $\epsilon = 0$. The function $\phi(\epsilon)$ is differentiable with respect to ϵ . Therefore, the necessary condition for $\phi(\epsilon)$ to attain a minimum at $\epsilon = 0$ must follow,

$$(9) \quad \phi'(0) = 0,$$

where a prime indicates a differentiation with respect to ϵ . Now, a differentiation under the sign of integration yields

$$(10) \quad \phi'(\epsilon) = \int_a^b [F_u(x, y + \epsilon\eta, y' + \epsilon\eta')\eta(x) + F_{u'}(x, y + \epsilon\eta, y' + \epsilon\eta')\eta'(x)]dx.$$

where F_u , $F_{u'}$ indicates $\partial F/\partial u$, $\partial F/\partial u'$, respectively. Integrating the last integral by parts, we obtain

$$(11) \quad \begin{aligned} \phi'(\epsilon) &= \int_a^b \left[F_u(x, y + \epsilon\eta, y' + \epsilon\eta') - \frac{d}{dx} F_{u'}(x, y + \epsilon\eta, y' + \epsilon\eta') \right] \eta(x) dx \\ &\quad + F_{u'}(x, y + \epsilon\eta, y' + \epsilon\eta') \eta(x) \Big|_a^b \end{aligned}$$

The last term vanishes according to Eq. (5). $\phi'(\epsilon)$ must vanish at $\epsilon = 0$, at which u equals y . Hence, we obtain the equation

$$(12) \quad 0 = \phi'(0) = \int_a^b \left[F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') \right] \eta(x) dx,$$

which must be valid for an arbitrary function $\eta(x)$.

Equation (12) leads immediately to Euler's differential equation, by virtue of the following "fundamental lemma of the calculus of variations:"

Lemma. Let $\psi(x)$ be a continuous function in $a \leq x \leq b$. If the relation

$$(13) \quad \int_a^b \psi(x)\eta(x)dx = 0$$

holds for all functions $\eta(x)$ which vanish at $x = a$ and b and are continuous together with their first $2n$ derivatives, where n is a positive integer, then $\psi(x) = 0$.

Proof. This lemma is easily proved indirectly. We shall show first that $\psi(x) = 0$ in the open interval $a < x < b$. Let us suppose that this statement is not true, that $\psi(x)$ is different from zero, say positive, at $x = \xi$, where ξ lies in the open interval. Then, according to the continuity of $\psi(x)$, there must exist an interval $\xi - \delta \leq x \leq \xi + \delta$ (with $a < \xi - \delta, \xi + \delta < b, \delta > 0$), in which $\psi(x)$ is positive (Fig. 10.1:2). Now we take the function

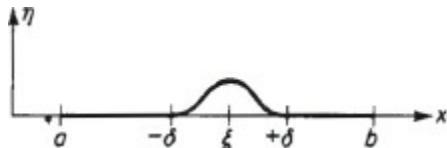


Fig. 10.1:2. The function $\eta(x)$.

$$(14) \quad \eta(x) = \begin{cases} (x - \xi + \delta)^{4n}(x - \xi - \delta)^{4n} & \text{in } \xi - \delta \leq x \leq \xi + \delta, \\ 0 & \text{elsewhere,} \end{cases}$$

where n is a positive integer. The function $\eta(x)$ satisfies the continuity and boundary conditions specified above. The choice of $\eta(x)$ in Eq. (14) makes $\int_a^b \psi(x)\eta(x)dx > 0$, in contradiction to the hypothesis. Thus, the hypothesis is untenable, and $\psi(x) = 0$ in $a < x < b$. According to the continuity of $\psi(x)$, we get $\psi(x) = 0$ also in $a \leq x \leq b$. This lemma can be extended to hold equally well for multiple integrals.

From Eq. (12), it follows immediately from the lemma that the necessary condition for the functional $J[y + \epsilon\eta]$ in Eq. (7) to have an extremum is that the factor in front of $\eta(x)$ must vanish:

$$(15a) \quad \Delta \quad F_y(x, y, y') - dF_{y'}(x, y, y')/dx = 0, \quad a \leq x \leq b.$$

This is a differential equation that $y(x)$ must satisfy and is known as Euler's equation. Written out *in extenso*, we have

$$(15b) \quad \Delta \quad \left(\frac{d^2y}{dx^2} \frac{\partial^2}{\partial y'^2} + \frac{dy}{dx} \frac{\partial^2}{\partial y' \partial y} + \frac{\partial^2}{\partial y' \partial x} - \frac{\partial}{\partial y} \right) F(x, y, y') = 0.$$

Should the problem be changed to finding the necessary condition for $J[u]$ to attain a maximum in a sufficiently small neighborhood h of $y(x)$, the same result would be obtained. Hence, the result: *The validity of Euler's differential Eq. (15) is a necessary condition for a function $y(x)$ to furnish an extremum of the functional $J[u]$ in a sufficiently small neighborhood (h) of $y(x)$.*

The satisfaction of Euler's equation is a necessary condition for $J[u]$ to attain an extremum; but it is not a sufficient condition. The question of sufficiency is rather involved; an interested reader must refer to treatises on calculus of variations such as those listed in [Bibliography 10.1](#).¹

It is customary to call $\epsilon\eta(x)$ the *variation* of $u(x)$ and write

$$(16) \quad \epsilon\eta(x) = \delta u(x).$$

It is also customary to define the first variation of the functional $J[u]$ as

$$(17) \quad \delta J = \epsilon\phi'(\epsilon).$$

On multiplying both sides of Eq. (11) by ϵ , it can be written as

$$(18) \quad \delta J = \int_a^b [F_u(x, u, u') - \frac{dF_{u'}(x, u, u')}{dx}] \delta u(x) dx + F_{u'}(x, u, u') \delta u(x) \Big|_a^b.$$

This is analogous to the notation of the differential calculus, in which the expression $df = \epsilon f'(x)$ for an arbitrary small parameter ϵ is called the "differential of the function $f(x)$." It is obvious that δJ depends on the function $u(x)$ and its

variation $\delta u(x)$. Thus, a necessary condition for $J[u]$ to attain an extremum when $u(x) = y(x)$ is the vanishing of the first variation δJ for all variations δu with $\delta u(a) = \delta u(b) = 0$.

Example 1. $J[u] = \int_a^b (1 + u'^2) dx = \min, u(a) = 0, u(b) = 1$.

The Euler equation is

$$dF_{y'}/dx = d(2y')/dx = 2y'' = 0.$$

Hence $y(x)$ is a straight line passing through the points $(a, 0)$ and $(b, 1)$.

Example 2. Which curve minimizes the following functional?

$$I[u] = \int_0^{\pi/2} [(u')^2 - u^2] dx, \quad u(0) = 0, \quad u(\pi/2) = 1. \quad \text{Ans. } u = y = \sin x.$$

Example 3. Minimum Surface of Revolution

Find $y(x)$ that minimizes the functional

$$I[u] = 2\pi \int_a^b u \sqrt{1 + u'^2} dx, \quad u(a) = A, \quad u(b) = B.$$

Ans. The Euler equation can be integrated to give

$$y\sqrt{1 + y'^2} - yy'/\sqrt{1 + y'^2} = y/\sqrt{1 + y'^2} = c_1.$$

Let $y' = \sinh t$, then $y = c_1 \cosh t$ and

$$dx = dy/y' = c_1 \sinh t dt / \sinh t = c_1 dt, \quad x = c_1 t + c_2.$$

Hence, the minimum surface is obtained by revolving a curve with the following parametric equations about the x -axis,

$$x = c_1 t + c_2, \quad y = c_1 \cosh t, \quad \text{or} \quad y = c_1 \cosh[(x - c_2)/c_1],$$

which is a family of catenaries. The constants c_1, c_2 can be determined from the values of y at the end points.

10.2. FUNCTIONAL INVOLVING HIGHER DERIVATIVES OF THE DEPENDENT VARIABLE

In an analogous manner one can treat the variational problem connected with the functional

$$(1) \quad J[u] = \int_a^b F(x, u, u^{(1)}, \dots, u^{(n)}) dx$$

involving $u(x)$ and its successive derivatives $u^{(1)}(x), u^{(2)}(x), \dots, u^{(n)}(x)$, for $a \leq x \leq b$. To state the problem concisely we denote by D the set of all real functions $u(x)$ with the following properties:

$$(2a) \quad u(x), \dots, u^{(2n)}(x) \text{ continuous in } a \leq x \leq b,$$

$$(2b) \quad u(a) = \alpha_0, \quad u^{(\nu)}(a) = \alpha_\nu, \quad \nu = 1, \dots, n-1,$$

$$(2c) \quad u(b) = \beta_0, \quad u^{(\nu)}(b) = \beta_\nu, \quad \nu = 1, \dots, n-1,$$

where $\alpha_0, \beta_0, \alpha_\nu$ and β_ν are given numbers. A function $u(x)$ which possesses these continuity properties and boundary values is said to be *admissible* or in the set D . We assume that the function $F(x, u, u^{(1)}, \dots, u^{(n)})$ has continuous partial derivatives up to the order n with respect to the $n+2$ arguments $x, u, u^{(1)}, \dots, u^{(n)}$ for u in the set D . We seek a function $u(x) = y(x)$ in D so that $J[u]$ is a minimum (or a maximum) for $u(x) = y(x)$, with respect to a sufficiently small neighborhood (h) of $y(x)$.

Let $\eta(x)$ be an arbitrary function with the properties

$$(3) \quad \begin{aligned} & \eta(x), \eta^{(1)}(x), \dots, \eta^{(2n)}(x) \text{ continuous in } a \leq x \leq b, \\ & \eta(a) = \dots = \eta^{(n-1)}(a) = \eta(b) = \dots = \eta^{(n-1)}(b) = 0. \end{aligned}$$

Then the function $u(x) = y(x) + \epsilon\eta(x)$ belongs to the set D for all ϵ . For sufficiently small ϵ 's this function belongs to a prescribed neighborhood (h) of $y(x)$. Once again, we introduce the functional

$$(4) \quad \phi(\epsilon) = J[y + \epsilon\eta] = \int_a^b F(x, y + \epsilon\eta, y^{(1)} + \epsilon\eta^{(1)}, \dots, y^{(n)} + \epsilon\eta^{(n)}) dx.$$

The assumption that $y(x)$ minimizes $J[u]$ leads to the necessary condition $\phi'(0) = 0$. So we obtain by differentiating under the sign of integration, integrating by parts, and using Eqs. (3), the result

$$(5) \quad \begin{aligned} 0 = \phi'(0) &= \int_a^b \left\{ \sum_{\nu=0}^n \left[\frac{\partial}{\partial y^{(\nu)}} F(x, y, y^{(1)}, \dots, y^{(n)}) \right] \eta^{(\nu)}(x) \right\} dx \\ &= \int_a^b \left\{ \sum_{\nu=0}^n (-1)^\nu \frac{d^\nu}{dx^\nu} \left[\frac{\partial}{\partial y^{(\nu)}} F(x, y, y^{(1)}, \dots, y^{(n)}) \right] \right\} \eta(x) dx, \end{aligned}$$

which holds for an arbitrary function $\eta(x)$ satisfying Eq. (3). According to the lemma proved in Sec. 10.1, we obtain the Euler equation

$$(6) \quad \Delta \quad \sum_{\nu=0}^n (-1)^\nu d^\nu [\partial F(x, y^{(1)}, \dots, y^{(n)}) / \partial y^{(\nu)}] / dx^\nu = 0,$$

which is a necessary condition for the function $u = y(x)$ to minimize (or maximize) the functional $J[u]$ given in Eq. (1).

The notations δu , δJ , etc., introduced in Sec. 10.1, can be extended to this case by obvious changes.

Example. Find the extremal of $J[u] = \int_0^1 (1 + u''^2) dx$ satisfying the boundary conditions

$$u(0) = 0, \quad u'(0) = 1, \quad u(1) = 1, \quad u'(1) = 1. \quad \text{Ans. } u = y(x) = x.$$

10.3. SEVERAL UNKNOWN FUNCTIONS

The method used in the previous sections can be extended to more complicated functionals. For example, let

$$(1) \quad J[u_1, u_2, \dots, u_m] = \int_a^b F(x, u_1, \dots, u_m; u'_1, \dots, u'_m) dx$$

be a functional depending on m functions $u_1(x), \dots, u_m(x)$. We assume that $F(x, u_1, \dots, u_m)$, $u_1(x), \dots, u_m(x)$ are twice differentiable, and that the boundary values of u_1, \dots, u_m are given at $x = a$ and b . We seek a special set of functions $u_\mu(x) = y_\mu(x)$, $\mu = 1, \dots, m$, in order that $J[u_1, \dots, u_m]$ attains a minimum (or a maximum) when $u_\mu(x) = y_\mu(x)$, with respect to a sufficiently small neighborhood of the $y_\mu(x)$; i.e., for all $u_\mu(x)$ satisfying the relation

$$|y_\mu(x) - u_\mu(x)| < h_\mu, \quad h_\mu > 0, \quad \mu = 1, 2, \dots, m.$$

Again it is easy to obtain the necessary conditions. Let the set of functions $y_1(x), \dots, y_m(x)$ be a solution of the variational problem. Let $\eta_1(x), \dots, \eta_m(x)$ be an arbitrary set of functions with the properties

$$(2) \quad \begin{aligned} & \eta_\mu(x), \eta'_\mu(x), \eta''_\mu(x) \text{ continuous in } a \leq x \leq b, \\ & \eta_\mu(a) = \eta_\mu(b) = 0, \quad \mu = 1, \dots, m. \end{aligned}$$

Then consider the function

$$(3) \quad \begin{aligned} \phi(\epsilon_1, \dots, \epsilon_m) &= J[y_1 + \epsilon_1 \eta_1, \dots, y_m + \epsilon_m \eta_m] \\ &= \int_a^b F(x, y_1 + \epsilon_1 \eta_1, \dots, y_m + \epsilon_m \eta_m; y'_1 + \epsilon_1 \eta'_1, \dots, y'_m + \epsilon_m \eta'_m) dx. \end{aligned}$$

Since $y_1(x), \dots, y_m(x)$ minimize (or maximize) $J[y_1 + E\eta_1, \dots, y_m + E\eta_m]$, the following inequality must hold for sufficiently small ϵ_1, \dots, E_m :

$$(4) \quad \phi(\epsilon_1, \dots, \epsilon_m) \geq \phi(0, \dots, 0), \quad [\text{or } \phi(\epsilon_1, \dots, \epsilon_m) \leq \phi(0, \dots, 0)].$$

The corresponding necessary conditions are

$$(5) \quad \frac{\partial \phi}{\partial \epsilon_\mu} = 0, \quad \text{for } \epsilon_1 = \epsilon_2 = \dots = \epsilon_m = 0, \quad \mu = 1, \dots, m,$$

which, again with F_{yu} , $F_{y'_u}$ denoting $\partial F/\partial y_u$, $\partial F/\partial y'_u$, lead to

$$(6) \quad \int_a^b [F_{y_\mu}(x, y_1, \dots, y_m, y'_1, \dots, y'_m) \eta_\mu(x) + F_{y'_\mu}(x, y_1, \dots, y_m, y'_1, \dots, y'_m) \eta'_\mu(x)] dx = 0, \quad \mu = 1, \dots, m.$$

Using integration by parts and the conditions (2), we obtain

$$(7) \quad \int_a^b [F_{y_\mu}(x, y_1, \dots, y_m, y'_1, \dots, y'_m) - \frac{d}{dx} F_{y'_\mu}(x, y_1, \dots, y_m, y'_1, \dots, y'_m)] \eta_\mu(x) dx + F_{y'_\mu}(x, y_1, \dots, y_m, y'_1, \dots, y'_m) \eta_\mu(x) \Big|_a^b = 0, \quad \mu = 1, \dots, m,$$

Equation (7) must hold for any set of functions $\eta_\mu(x)$ satisfying (2). By the lemma of Sec. 10.1 we have the following Euler equations which must be satisfied by $y_1(x), \dots, y_m(x)$ minimizing (or maximizing) the functional (1):

$$(8) \quad \Delta \quad F_{y_\mu}(x, y_1, \dots, y_m; y'_1, \dots, y'_m) - dF_{y'_\mu}(x, y_1, \dots, y_m, y'_1, \dots, y'_m)/dx = 0, \quad \mu = 1, \dots, m.$$

As a generalization of the variational notations defined in Sec. 10.1, the expression δJ given below is called the *first variation of the functional*:

$$(9) \quad \delta J = \sum_{\mu=1}^m (\partial \phi / \partial \epsilon_\mu) \epsilon_\mu.$$

Variational problems for functionals involving higher derivatives of u_1, \dots, u_m can be treated in the same way.

Example. Find the extremal of the functional

$$J[y, z] = \int_0^{\pi/2} (y'^2 + z'^2 + 2yz) dx, \\ y(0) = 0, \quad y(\pi/2) = 1, \quad z(0) = 0, \quad z(\pi/2) = -1.$$

The Euler equations are

$$y'' - z = 0, \quad z'' - y = 0.$$

Eliminating z , we have $y^{IV} - y = 0$. Hence,

$$y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x, \\ z = y'' = c_1 e^x + c_2 e^{-x} - c_3 \cos x - c_4 \sin x.$$

From the boundary conditions we obtain the solution

$$y = \sin x, \quad z = -\sin x.$$

10.4. SEVERAL INDEPENDENT VARIABLES

Consider the functional

$$(1) \quad J[u] = \iint_G F(x, y, u, u_x, u_y) dx dy,$$

where G is a finite, closed domain of the x, y -plane with a boundary curve C which has a piecewise continuously turning tangent, $u(x, y)$ is a continuous twice differentiable function of x, y in G , and u_x, u_y denote partial derivatives $\partial u / \partial x, \partial u / \partial y$ respectively. The function $F(x, y, u, u_x, u_y)$ is assumed to be twice continuously differentiable with respect to its

five arguments. Let D be the set of all admissible functions $u(x, y)$ with the properties:

$$(2) \quad D : [(a) u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yy}(x, y) \\ \text{continuous in } G; (b) u(x, y) \text{ prescribed on } C.]$$

We now seek a special function $u(x, y) = v(x, y)$ in the set D , which minimizes (or maximizes) the functional $J[u]$.

Let $v(x, y)$ be a solution of this variational problem. Let $\eta(x, y)$ denote an arbitrary function with the properties

$$(3) \quad \begin{cases} (a) \eta(x, y) \text{ has continuous derivatives up to the 2nd order in } G, \\ (b) \eta(x, y) = 0 \text{ for } (x, y) \text{ on the boundary } C. \end{cases}$$

A consideration of the function

$$(4) \quad \phi(\epsilon) = J[v + \epsilon\eta],$$

which attains an extremum when $\epsilon = 0$, leads to the necessary condition

$$(5) \quad d\phi(0)/d\epsilon = 0,$$

i.e., explicitly, with subscripts to F denoting partial derivatives of F ,

$$(6) \quad 0 = \iint_G \{F_v(x, y, v, v_x, v_y)\eta + F_{v_x}(x, y, v, v_x, v_y)\eta_x \\ + F_{v_y}(x, y, v, v_x, v_y)\eta_y\} dx dy.$$

The last two terms can be simplified by Gauss' theorem after rewriting Eq. (6) as²

$$0 = \iint_G \left\{ F_v \cdot \eta + \frac{\hat{\partial}}{\partial x} (F_{v_x} \cdot \eta) - \eta \frac{\hat{\partial}}{\partial x} F_{v_x} + \frac{\hat{\partial}}{\partial y} (F_{v_y} \cdot \eta) - \eta \frac{\hat{\partial}}{\partial y} F_{v_y} \right\} dx dy.$$

An application of Gauss' theorem to the sum of the second and fourth terms in the integrand gives

$$(7) \quad 0 = \iint_G (F_v - \frac{\hat{\partial} F_{v_x}}{\partial x} - \frac{\hat{\partial} F_{v_y}}{\partial y}) \eta(x, y) dx dy + \int_C [F_{v_x} n_1(s) + F_{v_y} n_2(s)] \eta ds.$$

Here $n_1(s)$ and $n_2(s)$ are the components of the unit outward-normal vector $\mathbf{n}(s)$ of C .

The line integral in Eq. (7) vanishes according to Eq. (3). The function $\eta(x, y)$ in the surface integral is arbitrary. The generalized lemma of Sec. 10.1 then leads to the Euler equations

$$(8) \quad F_v - \hat{\partial} F_{v_x}/\partial x - \hat{\partial} F_{v_y}/\partial y = 0, \quad \text{i.e.,}$$

$$(9) \quad \Delta \quad \frac{\partial F}{\partial v} - \frac{\partial^2 F}{\partial v_x \partial x} - \frac{\partial^2 F}{\partial v_y \partial y} - \frac{\partial^2 F}{\partial v_x \partial v} \frac{\partial v}{\partial x} - \frac{\partial^2 F}{\partial v_y \partial v} \frac{\partial v}{\partial y} \\ - \frac{\partial^2 F}{\partial v_x^2} \frac{\partial^2 v}{\partial x^2} - 2 \frac{\partial^2 F}{\partial v_x \partial v_y} \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 F}{\partial v_y^2} \frac{\partial^2 v}{\partial y^2} = 0.$$

Equation (9) is a necessary condition for a function $v(x, y)$ in the set D minimizing (or maximizing) the functional (1).

In the same way one may treat variational problems connected with functionals which involve more than two independent variables, higher derivatives, and several unknown functions.

Example 1. $J[u] = \iint_D (u_x^2 + u_y^2) dx dy$, with a boundary condition that u is equal to an assigned function $f(x, y)$ on the boundary C of the domain D . The Euler equation is the Laplace equation $v_{xx} + v_{yy} = 0$.

Example 2. $J[u] = \iint_D (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2)^2 dx dy = \min$. Then u must satisfy the biharmonic equation

$$\partial^4 u / \partial x^4 + 2\partial^4 u / (\partial x^2 \partial y^2) + \partial^4 u / \partial y^4 = 0.$$

10.5. SUBSIDIARY CONDITIONS – LAGRANGE MULTIPLIERS

In many problems, we are interested in the extremum of a function or functional under certain subsidiary conditions. As an elementary example, let us consider a function $f(x, y)$ defined for all (x, y) in a domain G . Suppose that we are interested in the extremum of $f(x, y)$, not for all points in G , but only for those points (x, y) in G which satisfy the relation

$$(1) \quad \phi(x, y) = 0.$$

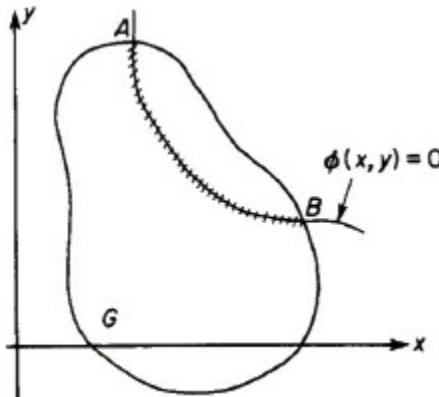


Fig. 10.5:1. Illustrating a subsidiary condition.

Thus, if the domain G and the curve $\phi(x, y) = 0$ are as shown in Fig. 10.5:1, then we are interested in finding the extremum of $f(x, y)$ among all points that lie on the segment of the curve AB . For conciseness of expression, we shall designate such a subdomain, the segment AB , by a symbol G_ϕ .

Let us find the necessary conditions for $f(x, y)$ to attain an extreme value at a point (\bar{x}, \bar{y}) in G_ϕ , with respect to all points of a sufficiently small neighborhood of (\bar{x}, \bar{y}) belonging to G_ϕ .

Let us assume that the function $\phi(x, y)$ has continuous partial derivatives with respect to x and y , $\phi_x = \partial\phi/\partial x$, $\phi_y = \partial\phi/\partial y$, and that at the point (\bar{x}, \bar{y}) not both derivatives are zero; say,

$$(2) \quad \phi_y(\bar{x}, \bar{y}) \neq 0.$$

Then, according to a fundamental theorem on implicit functions, there exists a neighborhood of \bar{x} , say $\bar{x} - \delta \leq x \leq \bar{x} + \delta$ ($\delta > 0$), where the equation $\phi(x, y) = 0$ can be solved uniquely in the form

$$(3) \quad y = g(x).$$

The function $g(x)$ so defined is single-valued and differentiable, and $\phi[x, g(x)] = 0$ is an identity in x . Hence

$$(4) \quad 0 = \phi_x[x, g(x)] + \phi_y[x, g(x)]dg(x)/dx.$$

Now, let us consider the problem of the extremum value of $f(x, y)$ in G_ϕ . According to (3), in a sufficiently small neighborhood of (\bar{x}, \bar{y}) , y is an implicit function of x , and $f(x, y)$ becomes

$$(5) \quad \mathcal{F}(x) = f[x, g(x)].$$

If $\mathcal{F}(x)$ attains an extreme value at \bar{x} , then the first derivative of $\mathcal{F}(x)$ vanishes at \bar{x} , i.e.,

$$(6) \quad \frac{d\mathcal{F}}{dx}(\bar{x}) = f_x(\bar{x}, \bar{y}) + f_y(\bar{x}, \bar{y})\frac{dg}{dx}(\bar{x}) = 0.$$

But the derivative dg/dx can be eliminated between Eqs. (6) and (4). The result, together with Eq. (1), constitutes the necessary condition for an extremum of $f(x, y)$ in G_ϕ .

The formalism will be more elegant by introducing a number λ , called *Lagrange multiplier*, defined by

$$(7) \quad \lambda = -f_y(\bar{x}, \bar{y})/\phi_y(\bar{x}, \bar{y}).$$

A combination of Eqs. (4), (6), and (7) gives

$$(8) \quad f_x(\bar{x}, \bar{y}) + \lambda\phi_x(\bar{x}, \bar{y}) = 0,$$

while Eq. (7) may be written as

$$(9) \quad f_y(\bar{x}, \bar{y}) + \lambda\phi_y(\bar{x}, \bar{y}) = 0.$$

Equations (1), (8), and (9) are necessary conditions for the function $f(x, y)$ to attain an extreme at a point (\bar{x}, \bar{y}) , where $\phi_x^2 + \phi_y^2 > 0$. These conditions constitute three equations for the three “unknowns” x , y , and λ .

These results can be summarized by *introducing a new function*

$$(10) \quad F(x, y; \lambda) = f(x, y) + \lambda\phi(x, y).$$

If the function $f(x, y)$ has an extreme value at the point (\bar{x}, \bar{y}) with respect to G_ϕ , and if

$$(11) \quad [\phi_x(\bar{x}, \bar{y})]^2 + [\phi_y(\bar{x}, \bar{y})]^2 > 0,$$

then there exists a certain number $\bar{\lambda}$ so that the three partial derivatives of $F(x, y; \lambda)$ with respect to x , y , and λ are zero at $(\bar{x}, \bar{y}, \bar{\lambda})$:

$$(12) \quad \frac{\partial F}{\partial x}(\bar{x}, \bar{y}, \bar{\lambda}) = f_x(\bar{x}, \bar{y}) + \bar{\lambda}\phi_x(\bar{x}, \bar{y}) = 0,$$

$$\frac{\partial F}{\partial y}(\bar{x}, \bar{y}, \bar{\lambda}) = f_y(\bar{x}, \bar{y}) + \bar{\lambda}\phi_y(\bar{x}, \bar{y}) = 0, \quad \frac{\partial F}{\partial \lambda}(\bar{x}, \bar{y}, \bar{\lambda}) = \phi(\bar{x}, \bar{y}) = 0.$$

If there exist points (x', y') in G with $\phi(x', y') = 0$ and $\phi_x(x', y') = \phi_y(x', y') = 0$, additional considerations are necessary.

In the formulation (10) and (12), the theorem can be generalized to the case of n variables x_1, \dots, x_n and several subsidiary conditions $0 = \phi_1(x_1, \dots, x_n) = \dots = \phi_m(x_1, \dots, x_n)$, $m < n$.

The application of the Lagrange multiplier method to the minimization of functionals follows a similar reasoning.

As an example, let us consider the classical problem of geodesics: to find the line of minimal length lying on a given surface $\phi(x, y, z) = 0$ and joining two given points on this surface (Fig. 10.5:2). Here we must minimize the functional

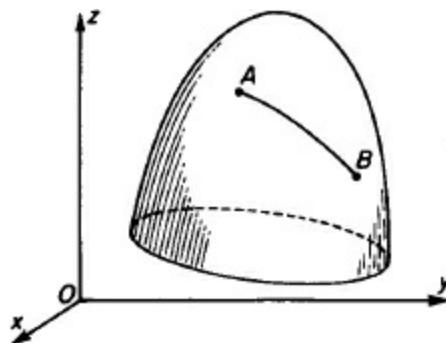


Fig. 10.5:2. A problem of geodesics.

$$(13) \quad l = \int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2} dx,$$

with $y(x)$, $z(x)$ satisfying the condition $\phi(x, y, z) = 0$. This problem was solved in 1697 by Johann Bernoulli, but a general method of solution was given by L. Euler and J. Lagrange.

We consider a new functional

$$(14) \quad I^*[y, z, \lambda] = \int_{x_0}^{x_1} \left[\sqrt{1 + y'^2 + z'^2} + \lambda(x)\phi(x, y, z) \right] dx$$

and use the method of Sec. 10.3 to determine the functions $y(x)$, $z(x)$, and $\lambda(x)$ that minimizes I^* . The necessary conditions are

$$(15) \quad \begin{aligned} \phi(x, y, z) &= 0, & \lambda \partial \phi / \partial y - d[y' / (1 + y'^2 + z'^2)^{1/2}] / dx &= 0, \\ \lambda \partial \phi / \partial z - d[z' / (1 + y'^2 + z'^2)^{1/2}] / dx &= 0. \end{aligned}$$

This system of equations determines the functions $y(x)$, $z(x)$, and $\lambda(x)$.

10.6. NATURAL BOUNDARY CONDITIONS

In previous sections we considered variational problems in which the admissible functions have prescribed values on the boundary. We shall now consider problems in which no boundary values are prescribed for the admissible functions. Such problems lead to the *natural boundary conditions*.

Consider again the functional (10.1:1), but now omit the boundary condition (10.1:2). Following the arguments of Sec. 10.1, we obtain the necessary condition (10.1:12),

$$(1) \quad 0 = \int_a^b (F_y - dF_{y'}/dx)\eta(x)dx + F_{y'}\eta(x)|_a^b.$$

This equation must hold for all arbitrary functions $\eta(x)$ with $\eta(a) = \eta(b) = 0$. This leads at once to the Euler equation (10.1:15)

$$(2) \quad F_y(x, y, y') - dF_{y'}/dx = 0, \quad \text{in } a \leq x \leq b.$$

In contrast to Sec. 10.1, however, the last term $F_{y'}\eta(x)|_a^b$, does not vanish by prescription. Hence, by Eqs. (1) and (2), we must have

$$(3) \quad (F_{y'}\cdot\eta)|_{x=b} - (F_{y'}\cdot\eta)|_{x=a} = 0$$

for all functions $\eta(x)$. But now $\eta(a)$ and $\eta(b)$ are arbitrary. Taking two functions $\eta_1(x)$ and $\eta_2(x)$ with

$$(4) \quad \eta_1(a) = 1, \quad \eta_1(b) = 0; \quad \eta_2(a) = 0, \quad \eta_2(b) = 1;$$

we get, from (3),

$$(5) \quad F_{y'}[a, y(a), y'(a)] = 0, \quad F_{y'}[b, y(b), y'(b)] = 0.$$

The conditions (5) are called the *natural boundary conditions* of our problem. They are the boundary conditions which must be satisfied by the function $y(x)$ if the functional $J[u]$ reaches an extremum at $u(x) = y(x)$, provided that $y(a)$ and $y(b)$ are entirely arbitrary.

Thus, if the first variation of a functional $J[u]$ vanishes at $u(x) = y(x)$, and if the boundary values of $u(x)$ at $x = a$ and b are arbitrary, then $y(x)$ must satisfy not only the Euler equation but also the natural boundary conditions which, in general, involve the derivatives of $y(x)$. In contrast to the natural boundary conditions, the conditions $u(a) = \alpha$, $u(b) = \beta$ [the specified boundary values of $u(x)$] are called *rigid boundary conditions*.

The concept of natural boundary conditions is also important in a more general type of variational problem in which boundary values occur explicitly in the functionals. It can be generalized to functionals involving several dependent and independent variables and higher derivatives of the dependent variables.

The idea of deriving natural boundary conditions for a physical problem is of great importance and will be illustrated again and again later, e.g. in Secs. 10.8 and 11.2.

10.7. THEOREM OF MINIMUM POTENTIAL ENERGY UNDER SMALL VARIATIONS OF DISPLACEMENTS

Let a body be in *static equilibrium* under the action of specified body and surface forces (Fig. 10.7:1). The boundary surface S shall be assumed to consist of two parts, S_σ and S_u , with the following boundary conditions.

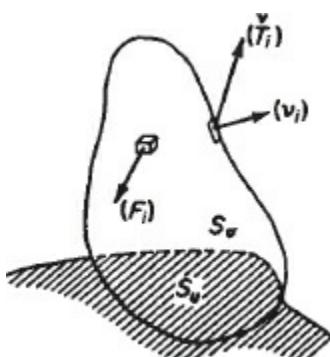


Fig. 10.7:1. Notations.

Over S_σ : The surface traction $\overset{\nu}{T}_i$ is prescribed.

Over S_u : The displacement u_i is prescribed.

We assume that there exists a system of displacements u_1, u_2, u_3 satisfying Navier's equations of equilibrium and the given boundary conditions. Consider an arbitrary displacement $u_i + \delta u_i$ consistent with the imposed constraints $\delta u_i = 0$ over S_u , arbitrary over S_σ , and triply differentiable of magnitude that the material remains elastic. Such an arbitrary displacement δu_i is called *virtual displacement*.

Assume that static equilibrium prevails. The *virtual work* done by the body force F_i per unit volume and the surface force $\overset{\nu}{T}_i^*$ per unit area is:

$$\int_V F_i \delta u_i dv + \int_S \overset{\nu}{T}_i^* \delta u_i dS .$$

On substituting $\overset{\nu}{T}_i^* = \sigma_{ij} \nu_j$ and applying Gauss' theorem, we have

$$(1) \quad \begin{aligned} \int_S \overset{\nu}{T}_i^* \delta u_i dS &= \int_S \sigma_{ij} \nu_j \delta u_i dS = \int_V (\sigma_{ij} \delta u_i)_{,j} dv \\ &= \int_V \sigma_{ij,j} \delta u_i dv + \int_V \sigma_{ij} \delta u_{i,j} dv . \end{aligned}$$

According to the equation of equilibrium the first integral on the right-hand side is equal to $-\int F_i \delta u_i dv$. On account of the symmetry of σ_{ij} , the second integral may be written as

$$\int_V \sigma_{ij} \frac{1}{2} (\delta u_{i,j} + \delta u_{j,i}) dv = \int_V \sigma_{ij} \delta e_{ij} dv .$$

Therefore, Eq. (1) becomes

$$(2) \quad \Delta \quad \int_V F_i \delta u_i dv + \int_{S_\sigma} \overset{\nu}{T}_i^* \delta u_i dS = \int_V \sigma_{ij} \delta e_{ij} dv .$$

This equation expresses the *principle of virtual work*. The surface integral needs only be integrated over S_σ , since δu_i vanishes over the surface S_u where boundary displacements are given.

If the *strain-energy function* $W(e_{11}, e_{12}, \dots)$ exists, so that $\sigma_{ij} = \partial W / \partial e_{ij}$ (Sec. 6.1), then it can be introduced into Eq. (2). Since

$$(3) \quad \int_V \sigma_{ij} \delta e_{ij} dv = \int_V \frac{\partial W}{\partial e_{ij}} \delta e_{ij} dv = \delta \int_V W dv ,$$

the principle of virtual work can be stated as

$$(4) \quad \Delta \quad \delta \int_V W dv - \int_V F_i \delta u_i dv - \int_{S_\sigma} \overset{\nu}{T}_i^* \delta u_i dS = 0 .$$

Further simplification is possible if the body force F_i and the surface tractions $\overset{\nu}{T}_i^*$ are *conservative* so that

$$(5) \quad F_i = -\partial G / \partial u_i , \quad \overset{\nu}{T}_i^* = -\partial g / \partial u_i .$$

The functions $G(u_1, u_2, u_3)$ and $g(u_1, u_2, u_3)$ are called the potential of F_i and $\overset{\nu}{T}_i^*$, respectively. In this case,

$$(6) \quad - \int_V F_i \delta u_i dv - \int_{S_\sigma} \overset{\nu}{T}_i^* \delta u_i dS = \delta \int_V G dv + \delta \int_S g dS .$$

Then Eq. (4) may be written as

$$(7) \quad \Delta \quad \delta \mathcal{V} = 0 \quad \text{where} \quad \mathcal{V} \equiv \int_V (W + G) dv + \int_{S_\sigma} g dS ,$$

which is called the *potential energy of the system*. This equation states that the potential energy has a stationary value

in the admissible variations δu_i of the displacements u_i in the equilibrium state. It can also be stated that, *of all displacements satisfying the given boundary conditions, those which satisfy the equations of equilibrium are distinguished by a stationary (extreme) value of the potential energy*. For rigid-body motion, W vanishes. Note that the linearity of the stress-strain relationship has not been invoked in the derivation, so that *this principle is valid for nonlinear, as well as linear, stress-strain law, as long as the body remains elastic*.

That this stationary value is a minimum in the neighborhood of the natural, unstrained state follows the assumption that the strain energy function is positive definite in such a neighborhood (see Secs. 12.3 and 12.9). This can be shown by comparing the potential energy \mathcal{V} of the actual displacements u_i with the energy \mathcal{V}' of another system of displacements $u_i + \delta u_i$ satisfying the condition $\delta u_i = 0$ over S_u . We have

$$(8) \quad \mathcal{V}' - \mathcal{V} = \int_V [W(e_{11} + \delta e_{11}, \dots) - W(e_{11}, \dots)] dv - \int_V F_i \delta u_i dv - \int_S T_i^* \delta u_i dS.$$

Expanding $W(e_{11} + \delta e_{11}, \dots)$ into a power series, we have

$$(9) \quad W(e_{11} + \delta e_{11}, \dots) = W(e_{11}, \dots) + \frac{\partial W}{\partial e_{ij}} \delta e_{ij} + \frac{\partial^2 W}{\partial e_{ij} \partial e_{kl}} \frac{\delta e_{ij} \delta e_{kl}}{2} + \dots$$

A substitution into Eq. (8) yields, up to the second order in δe_{ij} ,

$$(10) \quad \mathcal{V}' - \mathcal{V} = \int_V \frac{\partial W}{\partial e_{ij}} \delta e_{ij} dv - \int_V F_i \delta u_i dv - \int_S T_i^* \delta u_i dS + \int_V \frac{1}{2} \frac{\partial^2 W}{\partial e_{ij} \partial e_{kl}} \delta e_{ij} \delta e_{kl} dv.$$

The sum of the terms in the first line vanishes on account of Eq. (4). The sum in the second line is positive for sufficiently small values of strain δe_{ij} as can be seen as follows. Let us set $e_{ij} = 0$ in Eq. (9). The constant term $W(0)$ is immaterial. The linear term must vanish because $\partial W / \partial e_{ij} = \sigma_{ij}$, which must vanish as $e_{ij} \rightarrow 0$. Hence, up to the second order,

$$(11) \quad [\partial^2 W / (\partial e_{ij} \partial e_{kl})] \delta e_{ij} \delta e_{kl} / 2 = W(\delta e_{ij}).$$

Therefore, Eq. (10) becomes

$$(12) \quad \mathcal{V}' - \mathcal{V} = \int W(\delta e_{ij}) dv.$$

If $W(\delta e_{ij})$ is positive definite, then the last line in Eq. (11) is positive, and

$$(13) \quad \mathcal{V}' - \mathcal{V} \geq 0$$

and that \mathcal{V} is a minimum is proved. Accordingly, our principle is called the *principle of minimum potential energy*. The equality sign holds only if all δe_{ij} vanish, i.e., if the virtual displacements consist of a virtual rigid-body motion. If there were three or more points of the body fixed in space, such a rigid-body motion would be excluded, and \mathcal{V} is a strong minimum; otherwise it is a weak minimum.

To recapitulate, we remark again that the variational principle (2) is generally valid; Eq. (4) is established whenever the strain energy function $W(e_{11}, e_{12}, \dots)$ exists; and Eq. (7) is established when the potential energy \mathcal{V} can be meaningfully defined, but the fact that \mathcal{V} is a minimum for “actual” displacements is established only in the neighborhood of the stable natural state, where W is positive definite.

Conversely, we may show that the variational principle gives the equations of elasticity. In fact, starting from Eq. (7) and varying u_i , we have

$$(14) \quad \delta \mathcal{V} = \int_V \frac{\partial W}{\partial e_{ij}} \delta e_{ij} dv - \int_V F_i \delta u_i dv - \int_S T_i^* \delta u_i dS = 0.$$

$$\int_V \frac{\partial W}{\partial e_{ij}} \delta e_{ij} dv = \frac{1}{2} \int_V \sigma_{ij} (\delta u_{i,j} + \delta u_{j,i}) dv = - \int_V \sigma_{ij,j} \delta u_i dv + \int_S \sigma_{ij} \nu_j \delta u_i dS.$$

Hence,

$$(15) \quad \int_V (\sigma_{ij,j} + F_i) \delta u_i dv + \int_S (\sigma_{ij} \nu_j - \overset{\nu}{T}_i^*) \delta u_i dS = 0,$$

which can be satisfied for arbitrary δu_i if

$$(16) \quad \sigma_{ij,i} + F_i = 0 \quad \text{in } V$$

$$(17) \quad \delta u_i = 0 \quad \text{on } S_u \text{ (rigid boundary condition),}$$

$$(18) \quad \overset{\nu}{T}_i^* = \sigma_{ij} \nu_j \quad \text{on } S_\sigma \text{ (natural boundary condition).}$$

The one-to-one correspondence between the differential equations of equilibrium and the variational equation is thus demonstrated; for we have first derived Eq. (7) from the equation of equilibrium and then have shown that, conversely, Eqs. (16)–(18) necessarily follow Eq. (7).

The commonly encountered external force systems in elasticity are conservative systems in which the body force F_i and surface tractions $\overset{\nu}{T}_i^*$ are independent of the elastic deformation of the body. In this case, \mathcal{V} is more commonly written as

$$(19) \quad \mathcal{V} \equiv \int_V W dv - \int_V F_i u_i dv - \int_{S_\sigma} \overset{\nu}{T}_i^* u_i dS.$$

A branch of mechanics in which the external forces are in general nonconservative is the theory of aeroelasticity. In aeroelasticity one is concerned with the interaction of aerodynamic forces and elastic deformation. The aerodynamic forces depend on the flow and the deformation of the entire body, not just the local deformation; thus in general it cannot be derived from a potential function. The principle of virtual work, in the form of Eq. (4), is still applicable to aeroelasticity.

Problem 10.1. The minimum potential energy principle states that elastic equilibrium is equivalent to the condition $J = \min$, where

$$(a) \quad J = \int_V [W(e_{ij}) - F_i u_i] dV - \int_{S_\sigma} \overset{\nu}{T}_i^* u_i dS, \quad (\text{varying } u_i);$$

$$(b) \quad e_{ij} = (u_{i,j} + u_{j,i})/2.$$

By the method of Lagrange multipliers, the subsidiary condition (b) can be incorporated into the functional J , and we are led to consider the variational equation

$$(c) \quad \delta J' = 0, \quad (\text{varying } e_{ij}, u_i, \lambda_{ij} \text{ independently}),$$

where

$$(d) \quad J' = J + \int_V \lambda_{ij} \left[e_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i}) \right] dV.$$

Since the quantity in [] is symmetric in i, j , we may restrict the Lagrange multipliers λ_{ij} to be symmetric, $\lambda_{ij} = \lambda_{ji}$, so that only six independent multipliers are needed. Derive the Euler equations for (c) and show that, $-\lambda_{ij}$ should be interpreted as the stress tensor.

Note: Once Lagrange's multipliers are employed, the phrase "minimum conditions" used in the principle of minimum potential energy has to be replaced by "stationary conditions."

10.8. EXAMPLE OF APPLICATION: STATIC LOADING ON A BEAM – NATURAL AND RIGID END CONDITIONS

As an illustration of the application of the minimum potential energy principle in formulating approximate theories of elasticity, let us consider the approximate theory of bending of a slender beam under static loading. Let the beam be perfectly straight lying along the x -axis before the application of external loading, which consists of a distributed lateral load $p(x)$ per unit length, a bending moment M_0 and a shearing force Q_0 at the end $x = 0$, and a moment M_l and a shear Q_l at the end $x = l$ (Fig. 10.8:1). We assume that the principal axes of inertia of every cross section of the beam lie in two mutually orthogonal principal planes and that the loading p, M, Q are applied in one of the principal planes. In

accordance with the approximate beam theory, we assume that every plane cross section of the beam remains plane during bending. If the deformation is small, then the analysis given in Sec. 7.7 applies.

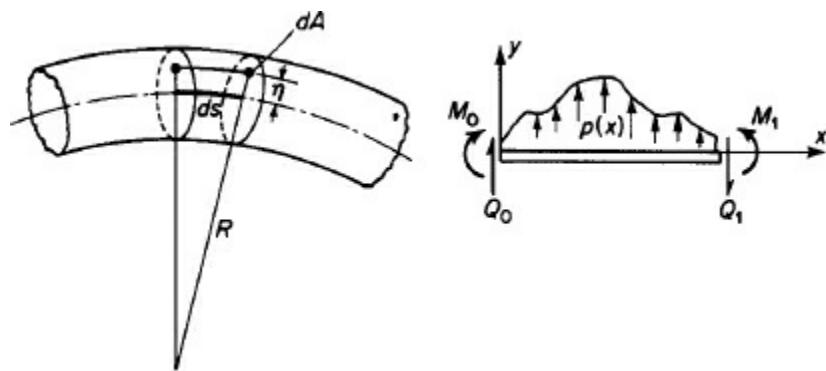


Fig. 10.8:1. Applications to a simple beam.

Let ds be the arc length along the longitudinal neutral axis. Under these assumptions, we see that when the neutral axis of the beam is bent from the initial straight line into a curve with radius of curvature R , the length of a filament, initially ds and parallel to the neutral axis, is altered by the bending in the ratio $1 : (1 + \eta/R)$, where η is the distance between the filament and the neutral axis. The strain is η/R , and the force acting on the filament is $E\eta dA/R$, where dA is the cross section of the filament. The resultant moment of these forces about the neutral axis is

$$M = \int_A \eta E \frac{\eta}{R} dA = \frac{E}{R} \int_A \eta^2 dA = \frac{EI}{R},$$

where I is the moment of inertia of the area of the beam cross section.

The bending moment acting on the cross sections at the ends of a beam segment of length ds does work if the cross sections rotate relative to each other. Let θ be the angle between a tangent to the neutral axis and the x -axis. In a segment of length ds , the tangent rotates by an amount $d\theta = ds/dR$, where R is the radius of curvature of the neutral axis (Fig. 10.8:1). Thus the angle through which the cross sections rotate relative to each other is ds/R . Hence, the work to bend a segment ds of the beam from a curvature of zero to a curvature $1/R$ is

$$Md\theta/2 = EI ds/(2R^2),$$

where the factor $\frac{1}{2}$ is added since the mean work is half the value of the product of the final moment and angle of rotation. Hence, integrating throughout the beam, we have the strain energy

$$(1) \quad U = \frac{1}{2} \int_0^l EI \frac{ds}{R^2}.$$

If we now assume that the deflection of the beam is *infinitesimal* so that if y denotes the lateral deflection, the curvature $1/R$ is approximated by d^2y/dx^2 , and ds is approximated by dx , then

$$(1a) \quad U = \frac{1}{2} \int_0^l EI \left(\frac{d^2y}{dx^2} \right)^2 dx.$$

The potential energy of the external loading is, with the sign convention specified in Fig. 10.8:1 and with y_0 , y_l , $(dy/dx)_0$, $(dy/dx)_l$, denoting the value of y and dy/dx at $x = 0$, l , respectively,

$$(2) \quad - \int p(x)y(x)dx + M_0 \left(\frac{dy}{dx} \right)_0 - M_l \left(\frac{dy}{dx} \right)_l - Q_0 y_0 + Q_l y_l.$$

Hence, the total potential energy is

$$(3) \quad \mathcal{V} = \frac{1}{2} \int_0^l EI \left(\frac{d^2y}{dx^2} \right)^2 dx - \int_0^l pydx + M_0 \left(\frac{dy}{dx} \right)_0 - M_l (dy/dx)_l - Q_0 y_0 + Q_l y_l.$$

At equilibrium, the variation of \mathcal{V} with respect to the virtual displacement δy must vanish. Hence,

$$(4) \quad \delta\mathcal{V} = \int_0^l EI \frac{d^2y}{dx^2} \delta \left(\frac{d^2y}{dx^2} \right) dx - \int_0^l p \delta y dx + M_0 \delta(dy/dx)_0 - M_l \delta(dy/dx)_l - Q_0 \delta y_0 + Q_l \delta y_l = 0.$$

Integrating the first term by parts twice and collecting terms, we obtain

$$(5) \quad \delta\mathcal{V} = \int_0^l [d^2(EI d^2y/dx^2)/dx^2 - p] \delta y dx + [(EI d^2y/dx^2)_l - M_l] \delta(dy/dx)_l - [(EI d^2y/dx^2)_0 - M_0] \delta(dy/dx)_0 - \{d[EI(d^2y/dx^2)]/dx\}_l - Q_l \delta y_l + \{d[EI(d^2y/dx^2)]/dx\}_0 - Q_0 \delta y_0 = 0.$$

Since δy is arbitrary in the interval $(0, l)$, we obtain the differential equation of the beam

$$(6) \quad d^2[EI(d^2y/dx^2)]/dx^2 - p = 0, \quad 0 \leq x \leq l.$$

In order that the remaining terms in (4) may vanish, it is sufficient to have the end conditions

$$(6a) \quad \text{Either } EI(d^2y/dx^2)_l - M_l = 0 \quad \text{or} \quad \delta(dy/dx)_l = 0.$$

$$(6b) \quad \text{Either } EI(d^2y/dx^2)_0 - M_0 = 0 \quad \text{or} \quad \delta(dy/dx)_0 = 0.$$

$$(6c) \quad \text{Either } \{d[EI(d^2y/dx^2)]/dx\}_l - Q_l = 0 \quad \text{or} \quad \delta y_l = 0.$$

$$(6d) \quad \text{Either } \{d[EI(d^2y/dx^2)]/dx\}_0 - Q_0 = 0 \quad \text{or} \quad \delta y_0 = 0.$$

If the deflection y_0 is prescribed at the end $x = 0$, then $\delta y_0 = 0$. If the slope $(dy/dx)_0$ is prescribed at the end $x = 0$, then $\delta(dy/dx)_0 = 0$. These are called *rigid boundary conditions*. On the other hand, if the value of y_0 is unspecified and perfectly free, then δy_0 is arbitrary and we must have

$$(7) \quad \{d[EI(d^2y/dx^2)]/dx\}_0 - Q_0 = 0$$

as an end condition for a free end subjected to a shear load Q_0 ; otherwise $\delta\mathcal{V}$ cannot vanish for arbitrary variations δy_0 . Equation (7) is called a *natural boundary condition*. Similarly, all the left-hand equations in Eqs. (6a)–(6d) are natural boundary conditions, and all the right-hand equations are rigid boundary conditions.

The distinction between natural and rigid boundary conditions assumes great importance in the application of the direct methods of solution of variational problems; the assumed functions in the direct methods must satisfy the rigid boundary conditions. See [Sec. 11.8](#).

It is worthwhile to consider the following question. The reader must be familiar with the fact that in the engineering beam theory, the end conditions often considered are:

$$(8a) \quad \text{Clamped end: } y = 0, \quad dy/dx = 0,$$

$$(8b) \quad \text{Free end: } EI(d^2y/dx^2) = 0, \quad d[EI(d^2y/dx^2)]/dx = 0,$$

$$(8c) \quad \text{Simply supported end: } y = 0, \quad EI(d^2y/dx^2) = 0,$$

where $y(x)$ is the deflection function of the beam. May we ask why

$$(9a) \quad y = 0, \quad \frac{d}{dx} \left(EI \frac{d^2y}{dx^2} \right) = 0; \quad \text{and} \quad (9b) \quad dy/dx = 0, \quad EI \frac{d^2y}{dx^2} = 0,$$

the two remaining combinations, never considered?

A possible answer is that (9a) and (9b) cannot be realized easily in the laboratory. But a more satisfying answer is that they are not proper sets of boundary conditions. If the conditions (9a) or (9b) were imposed, then, according to Eq. (4), it cannot at all be assured that the equation $\delta\mathcal{V} = 0$ will be satisfied. Thus, a basic physical law might be violated. These boundary conditions are, therefore, inadmissible.

From the point of view of differential equation theory, one may feel that (9a) or (9b) are legitimate for Eq. (5). Nevertheless, they are ruled out by the minimum potential energy principle on physical grounds. In fact, in the differential equation theory Eq. (5) and the end conditions (8) are known to be a so-called *self-adjoint differential system*, whereas Eqs. (5) and (9) form a *nonself-adjoint differential system*. Great difference in mathematical character exists

between these two categories. For example, a free vibration problem of a nonself-adjoint system may not have eigenvectors, or may have complex eigenvalues or complex eigenvectors.

Two other conceivable admissible boundary conditions are to require

$$(10) \quad dy/dx = 0, \quad d[EI(d^2y/dx^2)]/dx = 0, \quad \text{at } x = 0.$$

Such an end, with zero slope and zero shear, cannot be easily established in the laboratory. Similarly, it is conceivable that one may require that at the end $x = 0$, the following ratios hold:

$$(11) \quad \delta(dy/dx) : \delta y = c, \quad \text{a constant,}$$

$$\{d[EI(d^2y/dx^2)]/dx - Q_0\} : [EI(d^2y/dx^2) - M_0] = c, \quad \text{the same constant,}$$

which are also admissible, but are unlikely to be encountered in practice.

10.9. THE COMPLEMENTARY ENERGY THEOREM UNDER SMALL VARIATIONS OF STRESSES

In contrast to the previous sections let us now consider the variation of stresses in order to investigate whether the “actual” stresses satisfy a minimum principle. We pose the problem as in Sec. 10.7 with a body held in equilibrium under the body force per unit volume, F_i and surface tractions per unit area, $\overset{\nu}{T}_i^*$, over the boundary S_σ , whereas over the boundary S_u the displacements are prescribed. Let σ_{ij} be the “actual” stress field which satisfies the equations of equilibrium and boundary conditions

$$(1) \quad \sigma_{ij,j} + F_i = 0 \quad \text{in } V, \quad \sigma_{ij}\nu_j = \overset{\nu}{T}_i^* \quad \text{on } S_\sigma.$$

Let us now consider a system of variations of stresses which also satisfy the equations of equilibrium and the stress boundary conditions

$$(2) \quad (\delta\sigma_{ij})_{,j} + \delta F_i = 0 \quad \text{in } V, \quad (\delta\sigma_{ij})\nu_j = \delta \overset{\nu}{T}_i^* \quad \text{on } S_\sigma,$$

$$\delta\sigma_{ij} \quad \text{are arbitrary on } S_u.$$

In contrast to the previous sections, we shall now consider the *complementary virtual work*,

$$\int_V u_i \delta F_i dv + \int_S u_i \delta \overset{\nu}{T}_i^* dS,$$

which, by virtue of Eq. (2) and through integration by parts,

$$\begin{aligned} &= - \int_V u_i (\delta\sigma_{ij})_{,j} dv + \int_S u_i (\delta\sigma_{ij}) \nu_j dS \\ &= \int_V (\delta\sigma_{ij}) u_{i,j} dv - \int_S u_i \nu_j (\delta\sigma_{ij}) dS + \int_S u_i (\delta\sigma_{ij}) \nu_j dS \\ &= \int_V (\delta\sigma_{ij})(u_{i,j} + u_{j,i}) dv / 2 = \int_V e_{ij} \delta\sigma_{ij} dv. \end{aligned}$$

Hence,

$$(3) \quad \Delta \quad \int_V e_{ij} \delta\sigma_{ij} dv = \int_V u_i \delta F_i dv + \int_S u_i \delta \overset{\nu}{T}_i^* dS.$$

This equation may be called the *principle of virtual complementary work*. Now, we introduce the *complementary strain energy* W_c ³ a function of the stress components $\sigma_{11}, \sigma_{12}, \dots$, which has the property that,

$$(4) \quad \partial W_c / \partial \sigma_{ij} = e_{ij}$$

then the complementary virtual work may be written as

$$(5) \quad \int_V u_i \delta F_i dv + \int_S u_i \delta \overset{\nu}{T}_i^* dS = \int_V \frac{\partial W_c}{\partial \sigma_{ij}} \delta\sigma_{ij} dv = \delta \int_V W_c dv.$$

Since the volume and u_i are not varied, the result above can be written as

$$(6) \quad \Delta \quad \delta \mathcal{V}^* = 0,$$

where \mathcal{V}^* is a function of the stresses $\sigma_{11}, \sigma_{12}, \dots$, the traction $\overset{\nu}{T}_i$ and the body force per unit volume F_i , is defined as the *complementary energy*

$$(7) \quad \Delta \quad \mathcal{V}^*(\sigma_{11}, \dots, F_i) \equiv \int_V W_c dv - \int_V u_i F_i dv - \int_S u_i \overset{\nu}{T}_i dS.$$

In practice, we would like to compare stress fields which all satisfy the equations of equilibrium, but not necessarily the conditions of compatibility. In other words, we would have $\delta F_i = 0$ in V and $\delta \overset{\nu}{T}_i = 0$ on S_σ . In this case $\delta \sigma_{ij}$ and, hence, $\delta \overset{\nu}{T}_i$ are arbitrary only on that portion of the boundary where displacements are prescribed, S_u . Therefore, only a surface integral over S_u is left in the left-hand side of Eq. (5) and we have

$$(8) \quad \Delta \quad \mathcal{V}^*(\sigma_{11}, \dots, \sigma_{33}) \equiv \int_V W_c dv - \int_{S_u} u_i \overset{\nu}{T}_i dS.$$

Therefore, we have the following theorem

Theorem. *Of all stress tensor fields σ_{ij} that satisfy the equation of equilibrium and the boundary conditions where stresses are prescribed, the “actual” one is distinguished by a stationary (extreme) value of the complementary energy $\mathcal{V}^*(\sigma_{11}, \dots, \sigma_{33})$ as given by Eq. (8).*

In this formulation, the linearity of the stress-strain relationship is *not* required, only the existence of the complementary strain energy function is assumed. However, if the stress-strain law were *linear*, and the material is isotropic and obeys Hooke’s law, then

$$(9) \quad W_c(\sigma_{11}, \sigma_{12}, \dots, \sigma_{33}) = [-\nu(\sigma_{\alpha\alpha})^2 + (1 + \nu)\sigma_{ij}\sigma_{ij}]/(2E).$$

We must remark that the variational Eqs. (10.7:2) and Eq. (3) of the present section are applicable even if the body is *not elastic*, for which the energy functional cannot be defined. These variational equations are used in the analysis of inelastic bodies.

Before proceeding further, we consider the concept of *complementary work* and *complementary strain energy*. Let an elastic bar be subjected to a tensile load. The relationship between the load P and the elongation of the bar u is given by a unique curve as shown in Fig. 10.9:1. Then the work W is the area between the displacement axis and the curve, while the complementary work W_c is that included between the force axis and the curve. Thus, the two areas complement each other in the rectangular area (force) · (displacement), which would be the work if the force were acting with its full intensity from the beginning of the displacement. Naturally, W and W_c are equal if the material follows Hooke’s law.

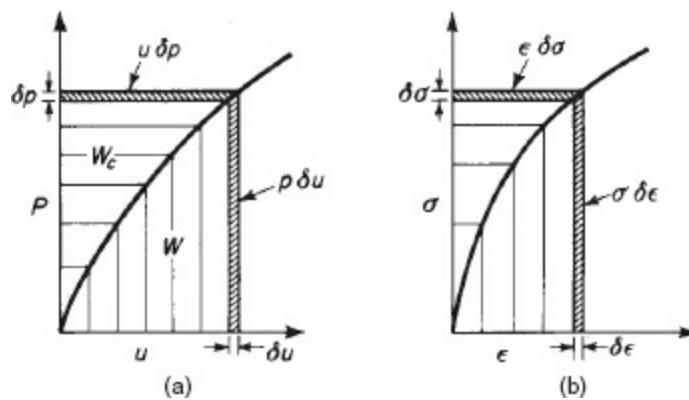


Fig. 10.9:1. Complementary work and strain energy.

The principle of minimum potential energy was formulated by Gibbs; and many applications were shown by Rayleigh. The complementary energy concept was introduced by Engesser; its applications were developed by Westergaard. Kirchhoff made the minimum potential energy theorem the foundation of the approximate theories of plates and shells. Argyris has made the complementary energy theorem the starting point for practical methods of analysis of complex elastic structures using modern digital computers. See [Biblio. 10.2](#).

In the neighborhood of the natural state, the extreme value of the complementary energy \mathcal{V}^* is a minimum. A natural state means a state of stable thermodynamic existence (see [Sec. 12.3](#)). In the neighborhood of a natural state, the

thermodynamic potential per unit volume $\rho\Phi$ can be approximated by a homogeneous quadratic form of stresses. For an isotropic material, $-\rho\Phi$ is given by Eq. (9), in which W_c is positive definite.

The proof that \mathcal{V}^* is a minimum is analogous to that in Sec. 10.7. Comparing $\mathcal{V}^*(\sigma_{ij} + \delta\sigma_{ij})$ with $\mathcal{V}^*(\sigma_{ij})$ gives, for linear isotropic elastic materials,

$$\begin{aligned}
(10) \quad & \mathcal{V}^*(\sigma_{11} + \delta\sigma_{11}, \dots) - \mathcal{V}^*(\sigma_{11}, \dots) \\
&= \int_V [W_c(\sigma_{11} + \delta\sigma_{11}, \dots) - W_c(\sigma_{11}, \dots)] dv \\
&\quad - \int_{S_u} [\bar{T}_i(\sigma_{11} + \delta\sigma_{11}, \dots) - \bar{T}_i(\sigma_{11}, \dots)] u_i dS \\
&= \int_V \left[\left(\frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{\alpha\alpha} \delta_{ij} \right) \delta\sigma_{ij} + W_c(\delta\sigma_{11}, \dots) \right] dv \\
&\quad - \int_{S_u} (\delta\sigma_{ij}) \nu_j u_i dS \\
&= \int_V e_{ij} \delta\sigma_{ij} dv - \int_{S_u} (\delta\sigma_{ij}) \nu_j u_i dS + \int W_c(\delta\sigma_{11}, \dots) dv \\
&= \delta\mathcal{V}^* + \int_V W_c(\delta\sigma_{11}, \delta\sigma_{12}, \dots) dv,
\end{aligned}$$

where

$$(11) \quad W_c(\delta\sigma_{ij}, \dots) = [(1+\nu)\delta\sigma_{ij}\delta\sigma_{ij} - \nu(\delta\sigma_{ii})^2]/(2E) \geq 0.$$

This result is also valid for general elastic materials with $W_c(\delta\sigma_{11}, \dots)$ being positive definite for infinitesimal variations $\delta\sigma_{ij}$. Hence, when $\delta\mathcal{V}^* = 0$,

$$(12) \quad \mathcal{V}^*(\sigma_{11} + \delta\sigma_{11}, \dots) - \mathcal{V}^*(\sigma_{11}, \dots) \geq 0,$$

and that $\mathcal{V}^*(\sigma_{11}, \dots)$ is a minimum is proved. The converse theorem reads:

Let \mathcal{V}^ be the complementary energy defined by Eq. (8). If the stress tensor field σ_{ij} is such that $\delta\mathcal{V}^* = 0$ for all variations of stresses $\delta\sigma_{ij}$ which satisfy the equations of equilibrium in the body and on the boundary where surface tractions are prescribed, then σ_{ij} also satisfies the equations of compatibility. In other words, the conditions of compatibility are the Euler equation for the variational equation $\delta\mathcal{V}^* = 0$.*

Southwell^{1,2} (1936) gave the proof through the application of Maxwell and Morera stress functions. Consider the variational equation

$$(13) \quad \delta\mathcal{V}^* = \int_V e_{ij} \delta\sigma_{ij} dv - \int_{S_u} u_i \delta \bar{T}_i dS = 0.$$

The variations $\delta\sigma_{ij}$ are subjected to the restrictions

$$(14) \quad (\delta\sigma_{ij})_{,j} = 0 \text{ in } V; \quad (\delta\sigma_{ij}) \nu_j = 0 \text{ on } S_\sigma.$$

To accommodate the restrictions (14) into Eq. (13), we make use of the celebrated result that the equations of equilibrium (14) are satisfied formally, as can be easily verified, by taking

$$\begin{aligned}
(15) \quad & \delta\sigma_{11} = \phi_{22,33} + \phi_{33,22} - 2\phi_{23,23}, \quad \delta\sigma_{22} = \phi_{33,11} + \phi_{11,33} - 2\phi_{31,31}, \\
& \delta\sigma_{33} = \phi_{11,22} + \phi_{22,11} - 2\phi_{12,12}, \quad \delta\sigma_{23} = \phi_{13,12} + \phi_{12,13} - \phi_{11,23} - \phi_{23,11}, \\
& \delta\sigma_{31} = \phi_{12,23} + \phi_{23,12} - \phi_{22,31} - \phi_{31,22}, \\
& \delta\sigma_{12} = \phi_{23,31} + \phi_{31,23} - \phi_{33,12} - \phi_{12,33},
\end{aligned}$$

where $\phi_{ij} = \phi_{ji}$ are arbitrary stress functions. On setting $\phi_{12} = \phi_{23} = \phi_{31} = 0$, we obtain the solutions proposed by James Clerk Maxwell. On taking $\phi_{11} = \phi_{22} = \phi_{33} = 0$, we obtain the solutions proposed by G. Morera.

Let us use the Maxwell system of arbitrary stress functions for the variations $\delta\sigma_{ij}$. Equation (13) may be written as

$$(16) \quad \delta\mathcal{V}^* = \int_V [e_{11}(\phi_{22,33} + \phi_{33,22}) + e_{22}(\phi_{33,11} + \phi_{11,33}) \\ + e_{33}(\phi_{11,22} + \phi_{22,11}) - 2e_{23}\phi_{11,23} \\ - 2e_{31}\phi_{22,31} - 2e_{12}\phi_{33,12}]dv - \int_{S_u} u_i \delta \overset{\nu}{T}_i dS = 0.$$

Integrating by parts twice, we obtain

$$(17) \quad \delta\mathcal{V}^* = \int_V [(e_{22,33} + e_{33,22} - 2e_{23,23})\phi_{11} + (e_{33,11} + e_{11,33} - 2e_{31,31})\phi_{22} \\ + (e_{11,22} + e_{22,11} - 2e_{12,12})\phi_{33}]dv + \text{a surface integral} = 0.$$

Inasmuch as the stress functions $\phi_{11}, \phi_{22}, \phi_{33}$ are arbitrary in the volume V , the Euler equations are

$$(18) \quad e_{22,33} + e_{33,22} - 2e_{23,23} = 0,$$

etc., which are Saint-Venant's compatibility equations (see Sec. 4.6). The treatment of the surface integral is cumbersome, but it says only that over S_u the values of u_i are prescribed and the stresses are arbitrary; the derivation concerns a certain relationship between ϕ_{ij} and their derivatives to be satisfied on the boundary.

Similarly, the use of Morera system of arbitrary stress functions leads to the other set of Saint-Venant's compatibility equations [Eq. (4.6:4)].

$$(19) \quad e_{11,23} = -e_{23,11} + e_{31,21} + e_{12,31}, \quad \text{etc.}$$

Introducing the stress functions in $\delta\mathcal{V}^*$ (valid for general elastic materials)

$$(20) \quad \delta V^* = \int_V e_{ij} \delta \sigma_{ij} dv + \int_{S_u} u_i \delta \overset{\nu}{T}_i dS = 0,$$

we obtain directly the following Beltrami–Michell compatibility equations for linear isotropic elastic materials [$Ee_{ij} = (1 + \nu)\sigma_{ij} - \nu\sigma_{\alpha\alpha}\delta_{ij}$]:

$$(21) \quad \nabla^2 \sigma_{ij} + \sigma_{\alpha\alpha,ij}/(1 + \nu) + \nu\delta_{ij}F_{k,k}/(1 - \nu) + F_{i,j} + F_{j,i} \equiv 0, \quad \text{in } V.$$

In concluding this section, we remark once more that when we consider the variation of the stress field of a body in equilibrium, the principle of virtual complementary work has broad applicability. The introduction of the complementary energy functional \mathcal{V}^* , however, limits the principle to elastic bodies. That \mathcal{V}^* actually is a minimum with respect to all admissible variations of stress field is established only if the complementary strain energy function is positive definite.

There are many fascinating applications of the minimum complementary energy principle. In Sec. 10.11, we shall consider its application in proving Saint-Venant's principle in Zanaboni's formulation.

Problem 10.2. The principle of virtual complementary work states that

$$(22) \quad \int_V e_{ij} \delta \sigma_{ij} dv - \int_{S_u} u_i^* \delta \overset{\nu}{T}_i dS = 0$$

under the restrictions that

$$(23) \quad \delta \sigma_{ij,j} = 0 \text{ in } V, \quad \delta \sigma_{ij}\nu_j = 0 \text{ on } S_\sigma,$$

$$(24) \quad u_i = u_i^* \text{ prescribed on } S_u, \text{ but } \delta \overset{\nu}{T}_i = \delta \sigma_{ij}\nu_j \text{ arbitrary on } S_u.$$

Using Lagrange multipliers, we may restate this principle as

$$(25) \quad \int_V e_{ij} \delta \sigma_{ij} dv - \int_{S_u} u_i^* \delta \overset{\nu}{T}_i dS + \int_V \lambda_i \delta \sigma_{ij,j} dv - \int_{S_\sigma} \mu_i \delta \sigma_{ij}\nu_j dS = 0.$$

The six equations (23), $i = 1, 2, 3$, require six Lagrange multipliers λ_i, μ_i which are functions of (x_1, x_2, x_3) . Show that the Euler equations for Eq. (25) yield the physical interpretation

$$(26) \quad \lambda_i = u_i, \quad \mu_i = u_i.$$

10.10. VARIATIONAL FUNCTIONALS FREQUENTLY USED IN COMPUTATIONAL MECHANICS

We shall introduce hybrid and mixed variational principles frequently used in finite element methods, which are powerful numerical tools for solving linear and non-linear problems. Finite element methods approximate field variables in terms of unknown parameters in subregions called “*elements*” and derive approximate solutions for the whole domain by enforcing certain relations among the unknowns. Hybrid or mixed principles are used to establish these relations. The details will be discussed in [Chapters 18–21](#).

Variational principles can be classified into *irreducible*, *hybrid*, or *mixed principle* based on the nature of the unknown functions of the functional. If the unknown functions (called *field variables*) cannot be eliminated from the functional until the problem solution has been obtained, then the functional is said to be *irreducible*. For example, in the minimum potential energy principle, the displacement \mathbf{u} has three components u_i . In most cases, none of the components can be eliminated from the functional *a priori*. Thus the principle is irreducible. So is the complementary energy theorem based on σ_{ij} . On the other hand, a functional of u_i and σ_{ij} is reducible if the stresses can be eliminated from the functional using the constitutive law. Such a functional is said to be *mixed*. Further, in finite element analysis a continuum is divided into *elements*. One may choose the field variables on the boundary of the element different from those over the element. Then the functional and the associated variational principle are said to be *hybrid*.

In subsequent discussion in this chapter we shall use interchangeably F_i and b_i for the body force; ∂V , S for the boundaries of V ; v_i , n_i for unit normal to a boundary surface; ∂V_u , S_u and ∂V_σ , S_σ for displacement and traction prescribed boundaries with prescribed values \bar{u}_i , u_i^* and \bar{T}_i , T_i^* , respectively.

Reissner–Hellinger Principle. For materials satisfy Hooke’s law, the Reissner–Hellinger principle uses the functional

$$(1) \quad \Pi_R(\sigma_{ij}, u_i) = \int_V \left(\sigma_{ij} e_{ij} - \frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} - b_i u_i \right) dV - \int_{\partial V_\sigma} \bar{T}_i u_i dS,$$

where V is the volume of the domain, C_{ijkl} are the *elastic flexibility tensor* of rank 4, σ_{ij} and e_{ij} are Cartesian stress and strain tensors of rank 2, u_i are the displacements, b_i are the body force components, and ∂V_σ is the portion of the boundaries of V over which traction \bar{T}_i prescribed. The strains are $e_{ij} = (u_{i,j} + u_{j,i})/2$, and the prescribed displacements over $\partial V_u (= \partial V - \partial V_\sigma)$ are $u_i = \bar{u}_i$ in which \bar{u}_i are known functions. The independent field variables σ_{ij} and u_i are limited by the *admissibility conditions* that either the displacements or the traction $\bar{T}_i (= n_j \sigma_{ij})$ are C^0 continuous⁴ over any internal surfaces and that $u_i = \bar{u}_i$ on ∂V_u . Since $u_i = \bar{u}_i$ is part of the admissibility requirements, which must be satisfied *a priori* and is called a *rigid condition*. The first variation of Π_R with respect to the field variables is

$$\begin{aligned} \delta \Pi_R(\sigma_{ij}, u_i) &= \int_V \sigma_{ij} \delta e_{ij} dV - \int_V b_i \delta u_i dV \\ &\quad + \int_V (e_{ij} - C_{ijkl} \sigma_{kl}) \delta \sigma_{ij} dV - \int_{\partial V_\sigma} \bar{T}_i \delta u_i dS. \end{aligned}$$

The necessary condition for Π_R to be stationary is that $\delta \Pi_R$ vanishes for all admissible variations of the independent field variables. Using the Gauss theorem for the first integral yields the stationarity condition

$$\begin{aligned} (2) \quad \delta \Pi_R(\sigma_{ij}, u_i) &= - \int_V (\sigma_{ij,j} + b_i) \delta u_i dV + \int_V (e_{ij} - C_{ijkl} \sigma_{kl}) \delta \sigma_{ij} dV \\ &\quad + \int_{\partial V_\sigma} (n_j \sigma_{ij} - \bar{T}_i) \delta u_i dS = 0, \end{aligned}$$

where n ’s are the components of a unit normal to the element boundaries. Since $\delta \Pi_R$ must be zero for arbitrary admissible δu_i and $\delta \sigma_{ij}$, the integrands of all three integrals must vanish which implies.

$$(3) \quad \sigma_{ij,j} + b_i = 0 \quad \text{equations of equilibrium in } V,$$

$$(4) \quad e_{ij} = C_{ijkl} \sigma_{kl} \quad \text{strain-stress relations in } V,$$

$$(5) \quad n_j \sigma_{ij} = \bar{T}_i \quad \text{traction boundary conditions on } \partial V_\sigma.$$

The functional Π_R has both stresses and displacements as independent fields. Since the stresses can be expressed

readily in terms of the displacements from Eq. (4), the functional is not irreducible. The variational formulation based on such a functional is a *mixed principle*.

If the domain V is divided into subdomains, called *elements*, for admissible σ_{ij} , u_i we can write Π_R in the form (Pian and Tong 1969)

$$(6) \quad \Pi_R = \sum_{\text{all elements}} \Pi_{R_e}(\sigma_{ij}, u_i),$$

where Π_{R_e} is the corresponding functional for an element

$$(7a) \quad \Pi_{R_e}(\sigma_{ij}, u_i) = \int_{V_e} \left(\sigma_{ij} e_{ij} - \frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} - b_i u_i \right) dV - \int_{\partial V_{\sigma_e}} \bar{T}_i u_i dS$$

with V_e and ∂V_{σ_e} being, respectively, the element volume and the portion of element boundaries over which traction is prescribed. For isotropic materials, Eq. (7a) reduces to

$$(7b) \quad \Pi_{R_e}(\sigma_{ij}, u_i) = \int_{V_e} \left[\sigma_{ij} e_{ij} - \frac{1}{2} \left(\frac{1}{2G} \sigma_{ij} \sigma_{ij} + \frac{2G - 3K}{18GK} \sigma_{ii} \sigma_{jj} \right) - b_i u_i \right] dV - \int_{\partial V_{\sigma_e}} \bar{T}_i u_i dS$$

In finite element applications to be discussed in later chapters, one assumes admissible u 's and σ 's in terms of unknown parameters within each element and uses the stationarity condition Eq. (2) to establish the equations for the unknown parameters. Note that $(2G - 3K)/(9K) = -\nu/(1 + \nu)$.

Hybrid Stress Principle. One can relax the admissibility requirements at ∂V_e by introducing Lagrangian multipliers \hat{u}_i , \hat{T}_i along ∂V_e of each element and turn Π_{R_e} of Eqs. (7) to a four-field hybrid principle (Tong and Pian 1969, Pian and Tong 1972). Here we consider a simpler case with stresses satisfying the equilibrium equations (3) and traction conditions (5), we obtain a two-field hybrid stress functional

$$(8) \quad \Pi_{HS} = \sum \Pi_{HS_e}(\sigma_{ij}, \hat{u}_i),$$

where

$$\Pi_{HS_e}(\sigma_{ij}, \hat{u}_i) = \int_{\partial V_e} n_j \sigma_{ij} \hat{u}_i dS - \frac{1}{2} \int_{V_e} C_{ijkl} \sigma_{ij} \sigma_{kl} dV - \int_{\partial V_{\sigma_e}} \bar{T}_i \hat{u}_i dS,$$

which is the basis for the hybrid stress model proposed by Pian (1964). The admissibility conditions are: \hat{u}_i are defined along the common boundaries of any two adjacent elements and satisfy the prescribed displacement condition $u_i = \bar{u}_i$ on S_u ; and the stresses satisfy the equilibrium equations Eq. (3). The first variation of Π_{HS_e} is

$$\begin{aligned} \delta \Pi_{HS_e} = & - \int_{V_e} C_{ijkl} \sigma_{kl} \delta \sigma_{ij} dV + \int_{\partial V_e} n_j \hat{u}_i \delta \sigma_{ij} dS \\ & + \int_{\partial V_{\sigma_e}} (n_j \sigma_{ij} - \bar{T}_i) \delta \hat{u}_i dS + \int_{\partial V_e - \partial V_{\sigma_e}} n_j \sigma_{ij} \delta \hat{u}_i dS. \end{aligned}$$

Thus the stationarity condition $\delta \Pi_{HS} = 0$ requires

$$\begin{aligned} & - \int_{V_e} C_{ijkl} \sigma_{kl} \delta \sigma_{ij} dV + \int_{\partial V_e} n_j \hat{u}_i \delta \sigma_{ij} dS = 0, \\ & \int_{\partial V_e} (n_j \sigma_{ji} - \bar{T}_i) \delta \hat{u}_i dS = 0, \text{ and } \sum_{\text{all elements}} \int_{\partial V_e - \partial V_{\sigma_e}} n_j \sigma_{ji} \delta \hat{u}_i dS = 0. \end{aligned}$$

The last two equalities above enforce the traction boundary condition (5) on ∂V_{σ_e} and the inter-element equilibrium, respectively. From Sec. 10.9, the first equality gives the Beltrami–Michell compatibility equations as the Euler equation with \hat{u}_i as the element boundary displacements for every element. The compatibility equations for isotropic materials are given in Eq. (10.9-21).

An alternative proof of stress compatibility is as follows. For given \hat{u}_i , let u_i be the elastic solution in V_e with $u_i = \hat{u}_i$ on ∂V_e . Then the first equality above can be written as

$$\int_{V_e} \left[\frac{1}{2}(u_{i,j} + u_{j,i}) - C_{ijkl}\sigma_{kl} \right] \delta\sigma_{ij} dV = 0$$

For arbitrary equilibrium $\delta\sigma_{ij}$, the stresses σ_{kl} must satisfy the strain-stress relations

$$e_{ij} = (u_{i,j} + u_{j,i})/2 = C_{ijkl}\sigma_{kl}.$$

Since e_{ij} are derived from displacement u 's, it implies that σ_{kl} satisfy the Beltrami–Michell compatibility equations. The variational formulation associated with Π_{HS} is called the *hybrid stress principle*, which forms the basis for the *hybrid stress finite element model*.

Hybrid Displacement Principle. If the stresses satisfy the strain stress relations above, one can express the stresses in terms of the strains or displacements and obtain the stress strain relations

$$(9) \quad \sigma_{ij} = D_{ijkl}e_{kl} = D_{ijkl}(u_{k,l} + u_{l,k})/2.$$

One can then derive a three-field hybrid functional (Tong 1970)

$$(10) \quad \Pi_{HD} = \sum_{\text{all elements}} \Pi_{HDe}(u_i, \hat{u}_i, \hat{T}_i),$$

where

$$\begin{aligned} \Pi_{HDe}(u_i, \hat{u}_i, \hat{T}_i) &= \int_{\partial V_e} \hat{T}_i (\hat{u}_i - u_i) dS \\ &\quad + \int_{V_e} \left(\frac{1}{2} D_{ijkl} e_{ij} e_{kl} - b_i u_i \right) dV - \int_{\partial V_{\sigma e}} \hat{T}_i \hat{u}_i dS, \end{aligned}$$

in which e 's are in terms of u 's. The admissibility conditions require that \hat{u}_i are common along the common boundary of any two adjacent elements and satisfy the prescribed displacement condition $u_i = \bar{u}_i$ over ∂V_u . The stationarity condition of Π_{HD} is

$$\begin{aligned} \delta\Pi_{HD}(u_i, \hat{u}_i, \hat{T}_i) &= \sum_{\text{all elements}} \left\{ - \int_{V_e} [(D_{ijkl}e_{kl}),_j + b_i] \delta u_i dV \right. \\ &\quad + \int_{\partial V_e} (n_j D_{ijkl} e_{kl} - \hat{T}_i) \delta u_i dS + \int_{\partial V_e} (\hat{u}_i - u_i) \delta \hat{T}_i dS \\ &\quad \left. + \int_{\partial V_{\sigma e}} (\hat{T}_i - \bar{T}_i) \delta \hat{u}_i dS + \int_{\partial V_e - \partial V_{\sigma e}} \hat{T}_i \delta \hat{u}_i dS \right\} = 0. \end{aligned}$$

The vanish of the first two integrals gives

$$(11) \quad (D_{ijkl}e_{kl}),_j + b_i = 0 \quad \text{in } V_e, \quad \text{and} \quad \hat{T}_i = n_j D_{ijkl} e_{kl} \quad \text{on } \partial V_e,$$

which are the equilibrium equations (3) and the stress traction relations (5), with stresses in terms of strains (9). Equation (11) is *Navier's equation*. The zero condition of the last three integrals of $\delta\Pi_{HD}$ implies the $\hat{u}_i = u_i$ on ∂V_e , the prescribed boundary traction conditions on $\partial V_{\sigma e}$, and the continuity of \hat{T}_i across interelement boundaries. The variational formulation forms the basis for the *hybrid displacement finite element model*.

The functional Π_{HD} can be further simplified by requiring u 's to satisfy Navier's equation and the 2nd equation of (11) *a priori*. Applying the Gauss theorem, one obtains a two-field hybrid functional

$$(12) \quad \Pi_{HD1} = \sum_{\text{all elements}} H_{HD1e}(u_i, \hat{u}_i),$$

where

$$\Pi_{HD1e}(u_i, \hat{u}_i) = \int_{\partial V_e} n_j D_{ijkl} e_{kl} \hat{u}_i dS - \frac{1}{2} \int_{V_e} D_{ijkl} e_{ij} e_{kl} dV - \int_{\partial V_{\sigma e}} \bar{T}_i \hat{u}_i dS,$$

in which e 's are strains in terms of the displacements. If the body force is zero, Π_{HD1e} reduces to

$$\Pi_{HD1e}(u_i, \hat{u}_i) = \int_{\partial V_e} n_j D_{ijkl} e_{kl} \left(\hat{u}_i - \frac{1}{2} u_i \right) dS - \int_{\partial V_{\sigma e}} \bar{T}_i \hat{u}_i dS,$$

which involves integration over ∂V_e only. One can show that the stationarity condition of Π_{HD1} gives the matching conditions $\hat{u}_i = u_i$ on ∂V_e , the prescribed boundary traction conditions $\hat{T}_i = \bar{T}_i^{\nu}$ on $\partial V_{\sigma e}$, and the continuity of $n_j D_{ijkl} e_{kl} (= \hat{T}_i)$ across interelement boundaries. The formulation is very useful for problems involving singularities.

Pian (1995, 1998) reviewed various variational principles of mixed/hybrid formulations. Interested readers are referred to published literature.

Incompressible or Nearly Incompressible Materials. If the material is incompressible, one cannot use the principle of minimum potential energy directly because certain components of the elastic modulus tensor D_{ijkl} becomes infinite. However, the Reissner or hybrid stress functional is still valid which can be seen from Eq. (7b) for isotropic materials as the bulk modulus $K \rightarrow \infty$.

We can establish a modified potential energy principle valid for incompressible or nearly incompressible materials. Consider an isotropic material with the elastic modulus tensor being

$$D_{ijkl} = G \left[\left(\frac{K}{G} - \frac{2}{3} \right) \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right] = 2G \left(\frac{\nu \delta_{ij} \delta_{kl}}{1 - 2\nu} + \frac{\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}}{2} \right).$$

Here K is the bulk modulus, which becomes infinite if the material is incompressible, i.e., when $\nu = 0.5$. Using the *contact transformation*,

$$D_{ijkl} e_{ij} e_{kl} / 2 + p e_{kk} = G e'_{ij} e'_{ij} - p^2 / (2K),$$

where p is the hydrostatic pressure and $e'_{ij} (= e_{ij} - \delta_{ij} e_{kk} / 3)$ is the deviatoric strain, we can write the strain energy density as

$$(13) \quad D_{ijkl} e_{ij} e_{kl} / 2 = G e'_{ij} e'_{ij} - p e_{kk} - p^2 / (2K),$$

which is a function of both u 's and p . The *deviatoric energy* ($= G e'_{ij} e'_{kl}$) is bounded for all real materials.

We can now construct a mixed functional with both u 's and p as independent field variables. The functional is suitable for applications to incompressible or nearly incompressible materials (Tong 1969).

$$(14) \quad \Pi_{Me}(u_i, p) = \int_{V_e} \left(G e'_{ij} e'_{ij} - p e_{kk} - \frac{p^2}{2K} - b_i u_i \right) dV - \int_{\partial V_{\sigma e}} \bar{T}_i u_i dS.$$

Penalty Functional. A simple way to relax constraints in the integral formulation is to introduce penalty function in the functional. For instance, if a material is incompressible, one can require that

$$(15) \quad e_{kk} = 0$$

as a constraint equation. Equation (14) becomes

$$(16) \quad \Pi_{Me}(u_i) = \int_{V_e} (G e'_{ij} e'_{ij} - b_i u_i) dV - \int_{\partial V_{\sigma e}} \bar{T}_i u_i dS$$

with Eq. (15) as the constraint condition. One can relax the constraint by using the functional

$$(17) \quad \Pi_{Me}(u_i, p) = \int_{V_e} (G e'_{ij} e'_{ij} + p e_{kk} - b_i u_i) dV - \int_{\partial V_{\sigma e}} \bar{T}_i u_i dS,$$

or introducing a *penalty function* and expressing the functional in the form

$$(18) \quad \Pi_{Me}(u_i, p) = \int_{V_e} \left(G e'_{ij} e'_{ij} + \frac{c}{2} e_{kk} e_{jj} - b_i u_i \right) dV - \int_{\partial V_{\sigma e}} \bar{T}_i u_i dS,$$

where c is an assumed positive constant of the order G or larger. The term associated with c is the penalty function, which enforces incompressibility when $c \rightarrow \infty$.

10.11. SAINT-VENANT'S PRINCIPLE

In 1855, Saint-Venant enunciated the “principle of the elastic equivalence of statically equipollent systems of loads.” According to this principle, the strains (produced in a body by the application, to a small part of its surface, of a system

of forces statically equivalent to zero force and zero couple) are negligible at distances large compared with the linear dimensions of the part. When a couple is applied at the ends of a long shaft, the shear stress distribution at a distance from the ends large compared with the cross-sectional dimension of the shaft is practically independent of the exact distribution of the surface tractions of which the couple is the resultant. Such a principle is nearly always applied, consciously or unconsciously, when we simplify or idealize a problem in mathematical physics. For example, in a simple tension test of a material, we clamp the ends of a test specimen in the jaws of a testing machine and assume that the action on the central part of the bar is nearly the same as if the forces were uniformly applied at the ends.

The justification of the principle is largely empirical and, as such, its interpretation is not entirely clear.

To mathematically formulate Saint-Venant's principle (Zanaboni^{10.3} 1937, Locatelli^{10.3} 1940, 1941), consider Fig. 10.11:1. A system of static equilibrium forces P with zero resultant force and couple is applied to a region of the body enclosed in a small sphere B . Otherwise the body is free. Let S' and S'' be two arbitrary nonintersecting cross sections, both outside of B , with S'' farther away from B . If we know the stresses in the body, we can calculate the tractions acting on the surfaces S' and S'' . Let the body be divided into two parts at S' , and let the system of surface traction acting on the surface S' be denoted by R' which is surely a system of forces in equilibrium (see Fig. 10.11:2). Then a convenient measure of the magnitude of the traction R' is the total strain energy that would be induced in the two parts should they be loaded by R' alone. Let this strain energy be denoted by $U_{R'}$. We have,



Fig. 10.11:1

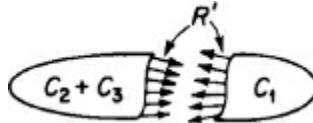


Fig. 10.11:2

$$U_{R'} = \int_V W(\sigma_{ij}^{(R)}) dv$$

where W is the strain energy density function, and the stresses $\sigma_{ij}^{(R)}$ are the loading system R' . Similarly, let the stresses at the section S'' be measured by the strain energy $U_{R''}$, which would have been induced by the traction R'' acting on the surface S'' over the two parts of the body severed at S'' . Both $U_{R'}$ and $U_{R''}$ are positive and vanish only if R', R'' are zero identically. Now we shall formulate Saint-Venant's principle (Zanaboni^{10.3} 1937).

Let S' and S'' be two nonintersecting sections both outside a sphere B . If the section S'' lies at a greater distance than the section S' from the sphere B in which a system of self-equilibrating forces P acts on the body, then

$$(1) \quad U_{R''} < U_{R'} .$$

In this form, the diminishing influence of the self-equilibrating system of loading P as the distance from B increases is expressed by the functional U_R , which is a special measure of the stresses induced at any section outside B . The reason for the choice of U_R as a measure is its positive definiteness character and the simplicity with which the theorem can be proved. Further sharpening of the principle will be discussed later.

In order to prove the Theorem (1), we first derive an auxiliary principle. Let a self-equilibrating system of forces P be applied to a limited region B at the surface of an otherwise free elastic body C_1 (Fig. 10.11:3). Let U_1 be the strain energy produced by P in C_1 . Let us now consider an enlarged body $C_1 + C_2$ by affixing to C_1 an additional body C_2 across a surface S which does not intersect the region B . When P is applied to the enlarged body $C_1 + C_2$, the strain energy induced is denoted by U_{1+2} . Then the lemma states that

$$(2) \quad U_{1+2} < U_1 .$$

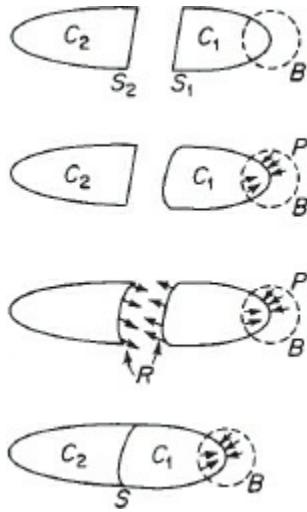


Fig. 10.11:3

Proof of the Lemma. To compute U_{1+2} , we imagine that the stresses in the enlarged body $C_1 + C_2$ are built up by the following steps (see Fig. 10.11:3). First the load P is applied to C_1 . The face S_1 of C_1 is deformed. Next a system of surface tractions R is applied to C_1 and C_2 on the surfaces of separation, S_1 and S_2 . R will be so chosen that the deformed surfaces S_1 and S_2 fit each other exactly, so that displacements of material points in C_1 and C_2 are continuous as well as the stresses. Now C_1 and C_2 can be brought together and welded, and S becomes merely an interface. The result is the same if C_1 and C_2 were joined in the unloaded state and the combined body $C_1 + C_2$ is loaded by P .

The strain energy U_{1+2} is the sum of the work done by the forces in the above stages. In the first stage, the work done by P is U_1 . In the second stage, the work done by R on C_2 is U_{R2}^* ; the work done by R on C_1 consists of two parts, U_{R1}^* if C_1 were free, and U_{PR}^* , the work done by the system of loads P due to the deformation caused by R . Hence,

$$(3) \quad U_{1+2} = U_1 + U_{R1}^* + U_{R2}^* + U_{PR}^*.$$

Now the system of forces R represents the internal normal and shear stresses acting on the interface S of the body $C_1 + C_2$. It is therefore determined by the minimum complementary energy theorem. Consider a special variation of stresses in which all the actual forces R are varied in the ratio $1 : (1 + \epsilon)$, where ϵ may be positive or negative. The work U_{R1}^* will be changed to $(1 + \epsilon)^2 U_{R1}^*$, because the load and deformation will both be changed by a factor $(1 + \epsilon)$. Similarly, U_{R2}^* is changed to $(1 + \epsilon)^2 U_{R2}^*$. But U_{PR}^* is only changed to $(1 + \epsilon) U_{PR}^*$, because the load P is fixed, while the deformation is varied by a factor $(1 + \epsilon)$. Hence, U_{1+2} is changed to

$$(4) \quad U'_{1+2} = U_1 + (1 + \epsilon)^2 U_{R1}^* + (1 + \epsilon)^2 U_{R2}^* + (1 + \epsilon) U_{PR}^*.$$

The difference between Eqs. (4) and (3) is

$$\Delta U_{1+2} = \epsilon(2U_{R1}^* + 2U_{R2}^* + U_{PR}^*) + \epsilon^2(U_{R1}^* + U_{R2}^*).$$

For U_{1+2} to be a minimum, ΔU_{1+2} must be positive regardless of the sign of ϵ . This is satisfied if

$$(5) \quad 2U_{R1}^* + 2U_{R2}^* + U_{PR}^* = 0.$$

On substituting Eqs. (5) into (3), we obtain

$$(6) \quad U_{1+2} = U_1 - (U_{R1}^* + U_{R2}^*).$$

Since U_{R1}^* and U_{R2}^* are positive definite, we see that Lemma (2) is proved.

Proof of Zanaboni theorem. Now we shall prove Saint-Venant's principle embodied in Eq. (1). Consider an elastic body consisting of three parts $C_1 + C_2 + C_3$ loaded by P in B , as shown in Fig. 10.11:1. Let this body be regarded first as a result of adjoining $C_2 + C_3$ to C_1 with an interface force system R' , and then as a result of adjoining C_3 to $C_1 + C_2$ with an interface force R'' . We have, by repeated use of Eq. (6),

$$U_{1+(2+3)} = U_1 - (U_{R'1}^* + U_{R'(2+3)}^*),$$

$$U_{(1+2)+3} = U_{1+2} - (U_{R''(1+2)}^* + U_{R''3}^*) = U_1 - (U_{R1}^* + U_{R2}^*) - (U_{R''(1+2)}^* + U_{R''3}^*).$$

Equating these expressions, we obtain

$$U_{R'1}^* + U_{R'(2+3)}^* = U_{R1}^* + U_{R2}^* + U_{R''(1+2)}^* + U_{R''3}^*,$$

or, since U_{R1}^* and U_{R2}^* are essentially positive quantities,

$$(7) \quad U_{R'1}^* + U_{R'(2+3)}^* > U_{R''(1+2)}^* + U_{R''3}^*.$$

This is Eq. (1), on writing $U_{R'}$ for $U_{R'1}^* + U_{R'(2+3)}^*$, etc.

Q.E.D.

10.12. SAINT-VENANT'S PRINCIPLE — BOUSSINESQ- VON MISES-STERNBERG FORMULATION

The Saint-Venant principle, as enunciated in terms of the strain energy functional, does not yield any detailed information about individual stress components at any specific point in an elastic body. However, such information is clearly desired. To sharpen the principle, it may be stated as follows (von Mises 1945). “*If the forces acting upon a body are restricted to several small parts of the surface, each included in a sphere of radius E, then the strains and stresses produced in the interior of the body at a finite distance from all those parts are smaller in order of magnitude when the forces for each single part are in equilibrium than when they are not.*”

The classical demonstration of this principle is due to Boussinesq (1885), who considered an infinite body filling the half-space $z \geq 0$ and subjected to several concentrated forces, each of magnitude F , normal to the boundary $z = 0$. If these normal forces are applied to points in a small circle B with diameter ϵ , Boussinesq proved that the largest stress component at a point P which lies at a distance R from B is (cf. Secs. 8.8 and 8.10).

- (1) of order F/R^2 if the resultant of the forces is of order F ,
- (2) of order $(\epsilon/R)(F/R^2)$ if the resultant of the forces is zero,
- (3) of order $(\epsilon/R)^2(F/R^2)$ if both the resultant force and the resultant moment vanish.

These relative orders of magnitude were believed to have general validity until von Mises (1945) showed that a modification is necessary.

Consider the half-space $z \geq 0$ again. Let forces of magnitude F tangent to the boundary $z = 0$ be applied to points in a small circle B of diameter ϵ . Making use of the well-known Cerruti solution (Sec. 8.8), von Mises obtained the following results for the four cases illustrated in Fig. 10.12:1. The order of magnitude of the largest stress component at a point P which lies at a distance R from B is, (1) of order $\sigma_0 = F/R^2$ in case (a), (2) of order $(\epsilon/R)\sigma_0$ in case (b), (3) of order $(\epsilon/R)\sigma_0$ in case (c), and (4) of order $(\epsilon/R)^2\sigma_0$ in case (d).

The noteworthy case is (c), which is drastically different from what one would expect from an indiscriminating generalization of Boussinesq's result, for in this case the forces are in static equilibrium, with zero moment about any axes, but all one could expect is a stress magnitude of order no greater than $(\epsilon/R)\sigma_0$, not $(\epsilon/R)^2\sigma_0$. Von Mises found that in this case the order of magnitude of the largest stress component is reduced to $(\epsilon/R)^2\sigma_0$ if and only if the external forces acting upon a small part of the surface are such as to remain in equilibrium when all the forces are turned through an arbitrary angle. (Such a case is called *astatic equilibrium*.)

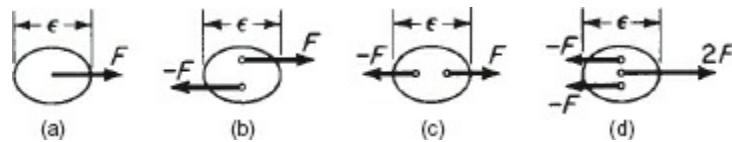


Fig. 10.12:1. von Mises' examples in which forces tangential to the surface of an elastic half-space are applied in a small area.

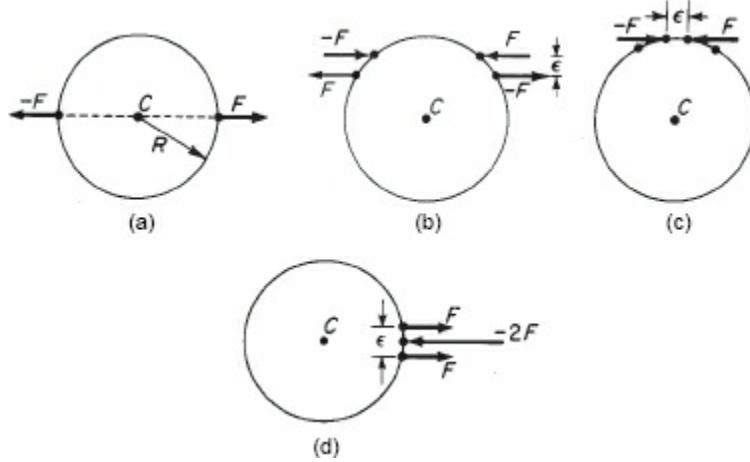


Fig. 10.12:2. von Mises' examples of a circular disk subjected to loads on the circumference.

Von Mises examined the stresses in a circular disk due to loads on the circumference, and reached a similar conclusion. (See Fig. 10.12:2.)

These examples show that Saint-Venant's principle, as stated in the traditional form at the beginning of this section, does not hold true.

Accordingly, von Mises^{10.3} proposed, and later Sternberg^{10.3} proved (1954), the following mathematical statement of the Saint-Venant principle:

Let a body be acted on by surface tractions which are finite and are distributed over a number of regions all no greater than a sphere of diameter ϵ . Consider an interior point x whose distance to any of these loading areas is no less than a characteristic length which will be taken as unity. ϵ is nondimensionalized with respect to this characteristic length. Then, as $\epsilon \rightarrow 0$, the order of magnitude of the strain components at x is as follows:

$$e(x, \epsilon) = O(\epsilon^\rho)$$

where

- (a) If the tractions have nonvanishing vector sums in at least one area, then in general, $\rho \geq 2$. (Note that the surface traction is assumed to be finite, so the resultant force $\rightarrow 0$ as ϵ^2 , since the area on which the surface tractions act $\rightarrow 0$ as ϵ^2 .)
- (b) If the resultant of the surface tractions in every loading area vanishes, then $\rho \geq 3$.
- (c) If, in addition, the resultant moment of the surface tractions in every loading area also vanishes, then still we can be assured only of $\rho \geq 3$.
- (d) $\rho \geq 4$ in case of astatic equilibrium in every loading area, which may be described by the 12 scalar conditions

$$\int_{S(\epsilon)} T_i^* dS = 0, \quad \int_{S(\epsilon)} T_i^* x_j dS = 0, \quad (i, j = 1, 2, 3)$$

for each loading area $S(\epsilon)$, where T_i^* is the specified surface traction over $S(\epsilon)$, and x_j is the coordinate of the point of application of T_i^* .

If the tractions applied to $S(\epsilon)$ are parallel to each other and not tangential to the surface, then if they are in equilibrium they are also in astatic equilibrium, and the condition $\rho \geq 4$ prevails.

If, instead of prescribing finite surface traction, we consider finite forces being applied which remain finite as $\epsilon \rightarrow 0$, then the exponent ρ should be replaced by 0, 1, 2 in the cases named above, as illustrated in Figs. 10.12:1 and 10.12:2.

The theorem enunciated above does not preclude the validity of a stronger Saint-Venant principle for bodies, such as thin plates or shells or long rods. With respect to perturbations that occur at the edges of a thin plate or shell, a significant result was obtained by Friedrichs^{10.3} (1950) in the form of a so-called boundary-layer theory. With respect to lateral loads on shells, Naghdi^{10.3} (1960) obtained similar results.

10.13. PRACTICAL APPLICATIONS OF SAINT-VENANT'S PRINCIPLE

Saint-Venant's principle has its analogy in hydrodynamics associated with the elliptic nature of partial differential equations. If the differential equations were hyperbolic and two-dimensional, local disturbances may be propagated far along the characteristics without attenuation. Then the concept of the Saint-Venant's principle will not apply. For example, in the problem of the response of an elastic half-space to a line load traveling at supersonic speeds over the free surface, the governing equations are hyperbolic, and we know that any fine structure of the surface pressure distribution is propagated to infinity.

On the other hand, one feels intuitively that the validity of Saint-Venant's principle is not limited to linear elastic solid or infinitesimal displacements. One expects it to apply in the case of rubber for finite strain or to steel even when yielding occurs. Although no precise proof is available, Goodier^{10,3} (1937) has argued on the basis of energy as follows:

Let a solid body be loaded in a small area whose linear dimensions are of order ϵ , with tractions of zero resultant force and couple. The tractions impart energy to the solid through the relative displacements of the points in the small loaded area. Let the tractions be of order p , the slope of the stress strain curve of the material be of order E . (The stress-strain relationship does not have to be linear), and one element of the loaded area be regarded as fixed in position and orientation. Then the strain is of order p/E and the displacements of points within the area are of order $(p\epsilon/E)$. The work done by the traction acting on an element dS is of order $(p\epsilon pdS/E)$. The total work is, therefore, $p^2\epsilon^3/E$. Since a stress of order p implies a strain energy of order p^2/E per unit volume, the region in which the stress is of order p must have a volume comparable with ϵ^3 . Hence, the influence of tractions cannot be appreciable at a distance from the loaded area which are large compared with ϵ .

Goodier's argument can be extended to bodies subjected to limited plastic deformation. In fact, the argument provides an insight to practical judgement of how local self-equilibrating tractions should influence the strain and stress in the interior of a body.

An engineer needs to know not only the order of magnitude comparison such as stated in von Mises-Sternberg theorem; he needs to know also how numerically trustworthy Saint-Venant's principle is to his particular problem. Hoff^{10,3} (1945) has considered several interesting examples, two of which are given in Figs. 10.13:1 and 10.13:2 and will be explained below.

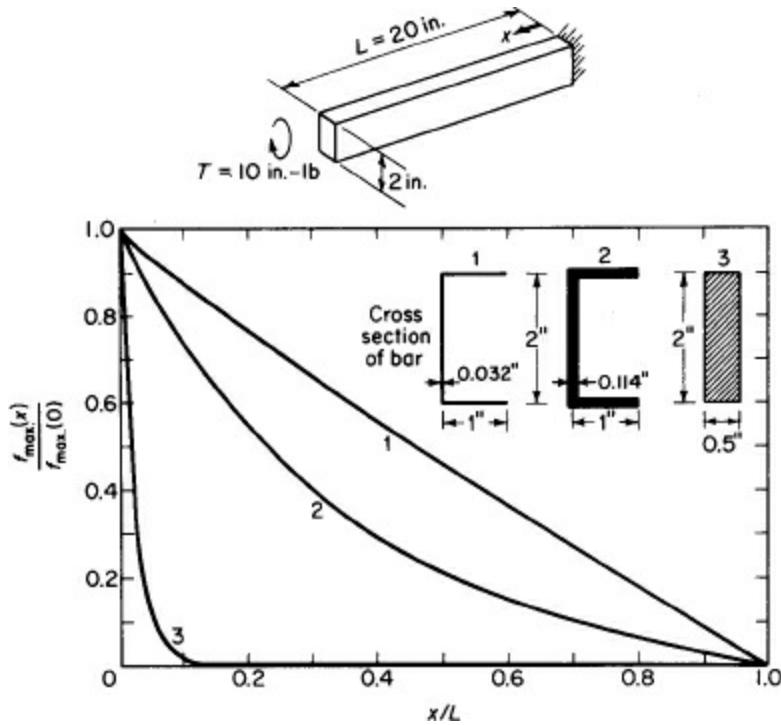


Fig. 10.13:1. Hoff's illustration of Saint-Venant's principle.

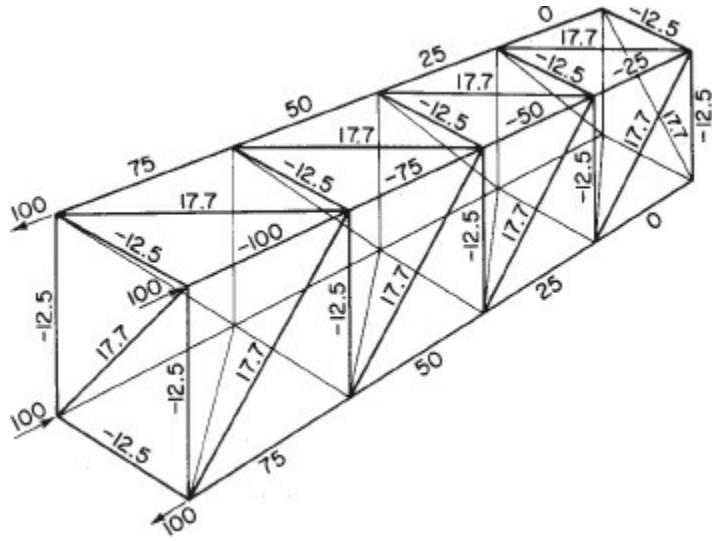


Fig. 10.13:2. Hoff's example illustrating the slow decay of self-equilibrating forces in a space framework.

In the first example, the torsion of beams with different cross-sections is considered. One end of the beam is clamped, where cross-sectional warping is prevented. The other end is free, where a torque is applied by means of shear stresses distributed according to the requirements of the theory of pure torsion. The difference between the prescribed end conditions at the clamped end from those assumed in Saint-Venant's torsion theory (Sec. 7.5) may be stated in terms of a system of self-equilibrating tractions that act at the clamped end. Timoshenko has given approximate solutions to these problems. Hoff's example refers to beams with dimensions as shown in the figure and subjected to a torque of 10 in. lb. Due to the restrictions of warping, normal stresses are introduced in the bar in addition to the shearing stresses of Saint-Venant's torsion. For the rectangular beam, the maximum normal stress at the fixed end is equal to 157 lb/sq in. For the other two thin-walled channel sections, the maximum normal stresses at the fixed end are 1230 and 10,900 lb/sq in., respectively, for the thicker and thinner sections. In Fig. 10.13:1 curves are shown for the ratio of maximum normal stress $f_{\max}(x)$ in any section x divided by the maximum normal stress $f_{\max}(0)$ in the fixed-end section, plotted against the ratio distance x of the section from the fixed end of the bar, divided by the total length L of the bar. Inspection of the curves reveals that, while in the case of the solid rectangular section the normal stress caused by the restriction to warping at the fixed end is highly localized, it has appreciable values over the entire length of the channel section bars. Consequently, reliance on Saint-Venant's principle in the calculation of stresses caused by torsion is entirely justified with the bar of rectangular section. In contrast, stresses in the thin-walled section bars depend largely upon the end conditions.

Hoff's second example refers to pin-jointed space frameworks. In one case a statically determinate framework is considered, to one end of which a set of four self-equilibrating concentrated loads are applied, as shown in Fig. 10.13:2. The figures written on the elements of the framework represent the forces acting in the bars measured in the same units as the applied loads. Negative sign indicates compression. It can be seen that the effect of the forces at one end of the structure is still noticeable at the other end.

Hoff's examples show that Saint-Venant's principle works only if there is a possibility for it to work; in other words, only if there exist paths for the internal forces to follow in order to balance one another within a short distance of the region at which a group of self-equilibrating external forces is applied. This point of view is in agreement with Goodier's reasoning.

Hoff's examples are not in conflict with the von Mises-Sternberg theorem. The latter asserts the order of magnitude comparison for the stress and strain as the size of the region of self-equilibrating loading shrinks to zero, and does not state how the stresses are propagated. On the other hand, Goodier's reasoning does not provide a definitive theorem, but points out the basic reason for Saint-Venant's principle and can be used in estimating the practical efficiency of the principle.

10.14. EXTREMUM PRINCIPLES FOR PLASTICITY

In this section we shall derive the principles of minimum potential energy and minimum complementary energy for displacement and stress increments in elastic-plastic bodies (Handelman 1944, Markov 1947, Greenberg 1949, and Hill 1950). We consider a body V subjected to prescribed body forces in V , traction \bar{T}_i^v on ∂V_σ , and displacements on ∂V_u . Suppose that u_i and σ_{ij} are the elastic-plastic solution of the static problem and that du_i and $d\sigma_{ij}$ are the increment solution for infinitesimal increments db_i in V , $d\bar{T}_i^v$ on ∂V_σ and $d\bar{u}_i$ on ∂V_u . Then $d\sigma_{ij}$ satisfy

$$(1) \quad d\sigma_{ij,j} + db_i = 0 \quad \text{the equilibrium equation in } V, \\ n_j d\sigma_{ji} = d\bar{T}_i \quad \text{the traction conditions on } \partial V_\sigma,$$

and can be expressed in terms of displacement increments du_j through the strain increments de_{kl} [Eq. (6.10:11) or (6.11:14)]

$$(2) \quad \begin{aligned} d\sigma_{ij} &= D_{ijkl} de_{kl} && \text{for } f < 0 \\ &= D_{ijkl}^{ep} de_{kl} && \text{for } f > 0, \end{aligned}$$

where D_{ijkl}^{ep} is given in Eq. (6.11:8) or (6.11:14) and f is the yield function.

Extremum Principle for Displacement Increment. Let du_i be the incremental solution over a volume V . Let $d\hat{u}_j$ be kinematically admissible displacements obeying the internal constraints, if any, and satisfying $d\hat{u}_j = d\bar{u}_j$ on ∂V_u . The corresponding strain and stress increments are $(d\hat{e}_{ij}, d\hat{\sigma}_{ij})$. Consider the following functional for displacement increments

$$(3) \quad \Pi_p(du_i) = \frac{1}{2} \int_V d\sigma_{ij} de_{ij} dV - \int_V db_i du_i dV - \int_{\partial V_\sigma} d\bar{T}_i^\nu du_i dS.$$

We can show that

$$\begin{aligned} \Pi_p(d\hat{u}) - \Pi_p(du) &= \frac{1}{2} \int_V (d\hat{\sigma}_{ij} d\hat{e}_{ij} - d\sigma_{ij} de_{ij}) dV - \int_V db_i (d\hat{u}_i - du_i) dV \\ &\quad - \int_{\partial V_\sigma} d\bar{T}_i^\nu (d\hat{u}_i - du_i) dS. \end{aligned}$$

By taking into account of the relation

$$\int_{\partial V_\sigma} d\bar{T}_i^\nu (d\hat{u}_i - du_i) dS = \int_V [d\sigma_{ij} (d\hat{e}_{ij} - de_{ij}) + d\sigma_{ij,j} (d\hat{u}_i - du_i)] dV,$$

the difference can be written as

$$(4) \quad \Pi_p(d\hat{u}_i) - \Pi_p(du_i) = \frac{1}{2} \int_V (d\hat{\sigma}_{ij} d\hat{e}_{ij} + d\sigma_{ij} de_{ij} - 2d\sigma_{ij} d\hat{e}_{ij}) dV.$$

With the substitution of $de_{ij} = D_{ijkl}^{-1} d\sigma_{kl} + de_{ij}^p$ and $d\hat{e}_{ij} = D_{ijkl}^{-1} d\hat{\sigma}_{kl} + d\hat{e}_{ij}^p$, the integrand becomes

$$D_{ijkl}^{-1} (d\hat{\sigma}_{ij} - d\sigma_{ij}) (d\hat{\sigma}_{kl} - d\sigma_{kl}) + d\hat{\sigma}_{ij} d\hat{e}_{ij}^p + d\sigma_{ij} de_{ij}^p - 2d\sigma_{ij} d\hat{e}_{ij}^p.$$

The first term is obviously non-negative due to the positive definiteness of D_{ijkl} . For perfectly plastic materials,

$$d\hat{\sigma}_{ij} d\hat{e}_{ij}^p = d\sigma_{ij} de_{ij}^p = 0, \quad d\sigma_{ij} d\hat{e}_{ij}^p = d\hat{\Lambda} n_{ij} d\sigma_{ij} \leq 0,$$

[see Eqs. (6.10:3, 6.10:4)] because $d\hat{\Lambda} \geq 0$ and $n_{ij} d\sigma_{ij} \leq 0$. Consequently, the remaining terms of the integrand are never negative, so that

$$(5) \quad \Pi_p(d\hat{u}_i) \geq \Pi_p(du_i).$$

The equality holds only if $d\hat{\sigma}_{ij} = d\sigma_{ij}$.

For work-hardening materials, let

$$(6) \quad df = n_{kl} d\sigma_{kl}.$$

with $df < 0$ denoting elastic unloading. From Eq. (6.11:11), the flow equation can be written as

$$(7) \quad K^p de_{ij}^p = H(df) n_{ij} df, \quad K^p d\hat{e}_{ij}^p = H(\hat{df}) n_{ij} \hat{df}.$$

The sum of the remaining terms of the integrand becomes

$$(8) \quad R = [(d\hat{f})^2 H(\hat{df}) + (df)^2 H(df) - 2(d\hat{f})(df)H(\hat{df})]/K^p.$$

We can show that

$$(9) \quad \begin{aligned} R &= 0 && \text{if } df \leq 0 \text{ and } d\hat{f} \leq 0, \\ &= (d\hat{f} - df)^2/K^p \geq 0 && \text{if } df > 0 \text{ and } d\hat{f} > 0, \\ &= d\hat{f}(d\hat{f} - 2df)/K^p > 0 && \text{if } df \leq 0 \text{ and } d\hat{f} > 0, \\ &= (df)^2/K^p > 0 && \text{if } df > 0 \text{ and } d\hat{f} \leq 0. \end{aligned}$$

Consequently, the integrand is never negative except when $d\hat{\sigma}_{ij} = d\sigma_{ij}$, then $\Pi_p(d\hat{u}_i) = \Pi_p(du_i)$. In other words, Eq. (5) also holds for work-hardening materials. $d\hat{\sigma}_{ij} = d\sigma_{ij}$ implies that $d\hat{e}_{ij} = de_{ij}$ and therefore $d\hat{u}_i = du_i$.

The extremum principle for the functional Eq. (3) is also called the *principle of minimum potential energy for the displacement increment*.

Extremum Principle for Stress. The complementary extremum principle concerns a statically and plastically admissible stress increment field. A stress field is statically admissible if it satisfies the equilibrium equations in V and the traction condition on ∂V_σ . A stress field is plastically admissible if it obeys the yield criterion everywhere in V . The *complementary functional for the stress increments* $d\sigma_{ij}$ is

$$(10) \quad \Pi_c(d\sigma_{ij}) = \frac{1}{2} \int_V d\sigma_{ij} de_{ij} dV - \int_{\partial V_u} n_j d\sigma_{ij} d\bar{u}_i dS.$$

Expressing de_{ij} in terms of $d\sigma_{ij}$ and de_{ij}^p , we can rewrite Eq. (10) as

$$\Pi_c(d\sigma_{ij}) = \frac{1}{2} \int_V (D_{ijkl}^{-1} d\sigma_{ij} d\sigma_{kl} + d\sigma_{ij} de_{ij}^p) dV - \int_{\partial V_u} n_j d\sigma_{ij} d\bar{u}_i dS,$$

where

$$(11) \quad \begin{aligned} d\sigma_{ij} de_{ij}^p &= 0, && \text{for perfectly plastic materials,} \\ &= (df)^2 H(df)/K^p, && \text{for work-hardening materials.} \end{aligned}$$

and $df (= n_{ij} d\sigma_{ij})$ is defined in Eq. (6). Clearly the functional is a function of $d\sigma_{ij}$ only. Let $d\sigma_{ij}$ be the plastic solution of the increment and $d\hat{\sigma}_{ij}$ be a statically admissible plastic stress field. Then

$$(12) \quad \Pi_c(d\hat{\sigma}_{ij}) - \Pi_c(d\sigma_{ij}) = \frac{1}{2} \int_V (d\hat{\sigma}_{ij} d\hat{e}_{ij} + d\sigma_{ij} de_{ij} - 2d\hat{\sigma}_{ij} de_{ij}^p) dV.$$

In deriving Eq. (12), we have used the relation that

$$\int_V (d\hat{\sigma}_{ij} - d\sigma_{ij}) de_{ij} dV = \int_{\partial V_u} n_i (d\hat{\sigma}_{ij} - d\sigma_{ij}) d\bar{u}_j dS$$

for any two admissible stress fields with a continuous displacement $du_j = d\bar{u}_j$ on ∂V_u . The integrand is similar to that in Eq. (4) and may be shown to be positive definite by the same method. Therefore, we have

$$(13) \quad \Pi_c(d\hat{\sigma}_{ij}) \geq \Pi_c(d\sigma_{ij}).$$

The equality holds only if $d\hat{\sigma}_{ij} = d\sigma_{ij}$.

It can be shown that

$$\Pi_c(d\sigma_{ij}) = -\Pi_p(du_i),$$

which implies

$$(14) \quad \Pi_c(d\hat{\sigma}_{ij}) \geq \Pi_c(d\sigma_{ij}) = -\Pi_p(du_i) \geq -\hat{\Pi}_p(d\hat{u}_{ij}).$$

The extremum principle for Eq. (10) is also called the *principle of minimum complementary energy for $d\sigma_{ij}$* . The stationary condition of $\delta\Pi_p = 0$ or $\delta\Pi_c = 0$ for functional in Eqs. (3) and (10) are useful for constructing “weak-form” increment solutions.

Rigid-Plastic Materials. For rigid-plastic materials, the elastic strain-increment is zero, and therefore,

$$(15) \quad de_{ij} = (du_{i,j} + du_{j,i})/2 = de_{ij}^p.$$

The principle of extremum plastic work becomes

$$(16) \quad \sigma_{ij} de_{ij} = D_p(de_{ij}) \geq \hat{\sigma}_{ij} de_{ij},$$

where the strain increment de_{ij} is normal to the yield surface at the stress point σ_{ij} in the stress space and $\hat{\sigma}_{ij}$ is any stress laying inside or at another point on the yield surface. Note that $D_p(de_{ij})$ is a function of de_{ij} only.

Let us define another functional

$$(17) \quad \Pi(du_i) = \int_V D_p(de_{ij}) dV - \int_V b_i du_i dV - \int_{\partial V_\sigma} \frac{\nu}{T_i} du_i dS$$

as a function of the velocity field solution du_i . Let $d\hat{u}_i$ be a kinematically admissible with its associated stress $\hat{\sigma}_{ij}$ satisfying the yield criterion in part or all of V [i.e., $f(\hat{\sigma}_{ij}, T, \xi_i) < 0$ or $f = 0, \hat{e}_{ij} = \hat{n}_{ij} d\Lambda$]. Then

$$\begin{aligned} (18) \quad \Pi(d\hat{u}_i) - \Pi(du_i) &= \int_V [D_p(d\hat{e}_{ij}) - D_p(de_{ij}) - \sigma_{ij}(d\hat{e}_{ij} - de_{ij})] dV \\ &= \int_V [D_p(d\hat{e}_{ij}) - \sigma_{ij} d\hat{e}_{ij}] dV. \end{aligned}$$

By the maximum principle of plastic dissipation the integrand is nonnegative. Therefore,

$$(19) \quad \Pi(d\hat{u}_i) \geq \Pi(du_i).$$

10.15. LIMIT ANALYSIS

We define the *critical state of a body* as a large increase in plastic deformation, much larger than the elastic deformation, with little or no increase in load. In other words, it is a state of impending plastic collapse or incipient plastic flow in which $de_{ij} \neq 0$ under constant load (i.e., $db_i = d\frac{\nu}{T_i} = d\bar{u}_i = 0$). Therefore,

$$\begin{aligned} (1) \quad \int_V db_i du_i dV + \int_{\partial V_\sigma} d\frac{\nu}{T_i} du_i dS &= \int_V d\sigma_{ij} de_{ij} dV \\ &= \int_V (D_{ijkl}^{-1} d\sigma_{ij} d\sigma_{kl} + d\sigma_{ij} de_{ij}^p) dV = 0. \end{aligned}$$

The positive definiteness of $D_{ijkl}^{-1} d\sigma_{ij} d\sigma_{kl}$ combined with Drucker's inequality implies $d\sigma_{ij} = 0$. This in turns implies $de_{ij}^e = 0$ and $de_{ij} = de_{ij}^p$, i.e., at impending plastic collapse the plastic flow is rigid-plastic. This result was first noted by Drucker *et al.* (1951). It makes possible to use rigid-plastic formulation to establish the upper and lower bounds in limit analysis.

Lower Bound Theorem. Consider the rigid-plastic deformation with known body force \bar{b}_i in V , traction $\frac{\nu}{T_i}$ on ∂V_σ and increment $d\bar{u}_i = 0$ on ∂V_u . Let the applied loads and prescribed displacements be respectively written in the form $P\bar{b}_i$, $P\bar{T}_i$ and $Pd\bar{u}_i$, where P is a load parameter. We further let $(du_i, de_{ij}$ and $\sigma_{ij})$ be the actual solution at plastic collapse associated with the load parameter P_l . We have

$$\int_V D_p(de_{ij}) dV = \int_V \sigma_{ij} de_{ij} dV = P_l \left(\int_V \bar{b}_i du_i dV + \int_{\partial V_\sigma} \frac{\nu}{T_i} du_i dS \right).$$

If a statically admissible stress $\bar{\sigma}_{ij}$ is in equilibrium with $P_{LB}\bar{b}_i$ and $P_{LB}\frac{\nu}{T_i}$, then

$$(2) \quad \int_V \bar{\sigma}_{ij} de_{ij} dV = P_{LB} \left(\int_V \bar{b}_i du_i dV + \int_{\partial V_\sigma} \frac{\nu}{T_i} du_i dS \right).$$

Using Eq. (2) and the principle of maximum plastic work for rigid-plastic deformation, $D_p(de_{ij}) \geq \bar{\sigma}_{ij} de_{ij}$, we obtain a lower bound of the collapse load

$$(3) \quad P_{LB} = \int_V d\bar{\sigma}_{ij} de_{ij} dV / \left(\int_V \bar{b}_i du_i dV + \int_{\partial V_\sigma} \bar{T}_i du_i dS \right)$$

$$\leq \int_V D_p(de_{ij}) dV / \left(\int_V \bar{b}_i du_i dV + \int_{\partial V_\sigma} \bar{T}_i du_i dS \right) = P_l.$$

Since $d\bar{u}_i$ and de_{ij} generally not known, we cannot use Eq. (2) directly to evaluate P_{LB} .

Upper Bound Theorem. For a kinematically admissible increment $d\bar{u}_i$ with $d\bar{u}_i = 0$ on ∂V_u , we have

$$(4) \quad \int_V \sigma_{ij} d\bar{e}_{ij} dV = P_l \left(\int_V \bar{b}_i d\bar{u}_i dV + \int_{\partial V_\sigma} \bar{T}_i d\bar{u}_i dS \right).$$

where σ_{ij} is the solution at impending collapse and P_l is the limit load. Let

$$(5) \quad P_{UB} = \int_V D_p(d\bar{e}_{ij}) dV / \left(\int_V \bar{b}_i d\bar{u}_i dV + \int_{\partial V_\sigma} \bar{T}_i d\bar{u}_i dS \right).$$

Then using Eq. (4) and $D_p(d\bar{e}_{ij}) \geq \sigma_{ij} d\bar{e}_{ij}$ for rigid-plastic deformation, we conclude that

$$(6) \quad P_{UB} = P_l \int_V D_p(d\bar{e}_{ij}) dV / \int_V \sigma_{ij} d\bar{e}_{ij} dV \geq P_l.$$

If one has found a kinematically admissible velocity field having its strain rate associated with a statically admissible stress field everywhere⁵ in the domain, then one has a complete solution. This solution predicts the correct collapse load using limit analysis. However, this may not be the exact solution, because the solution may not be unique.

Lower and Upper Bound Loci for Multiple Load Parameters. The results given in Eqs. (3) and (4) can be generalized to include several parameters P_I that

$$(7) \quad \begin{aligned} F_I &= \sum_I P_I (\bar{b}_i)_I, && \text{prescribed body force in } V, \\ \bar{T}_i &= \sum_I P_I (\bar{T}_i)_I, && \text{prescribed traction on } \partial V_\sigma. \\ d\bar{u}_i &= \sum_I P_I (d\bar{u}_i)_I, && \text{prescribed displacement increment on } \partial V_u. \end{aligned}$$

For a kinematically admissible velocity field $d\bar{u}_i$, we define the generalized velocities

$$(8) \quad d\bar{v}_I = \int_V (\bar{b}_i)_I d\bar{u}_i dV + \int_{\partial V_\sigma} (\bar{T}_i)_I d\bar{u}_i dS.$$

Let $(du_i, de_{ij}, \sigma_{ij})$ be the solution at collapse with loading parameters $(P_I)_I$, then

$$\int_V \sigma_{ij} de_{ij} dV = \sum_I (P_I)_I d\bar{v}_I.$$

For a statically admissible plastic stress field $\bar{\sigma}_{ij}$, we have

$$(9) \quad \begin{aligned} \int_V \sigma_{ij} de_{ij} dV &= \sum_I (P_{LB})_I \left[\int_V (\bar{b}_i)_I du_i dV + \int_{\partial V_\sigma} (\bar{T}_i)_I du_i dS \right] \\ &= \sum_I (P_{LB})_I dv_I. \end{aligned}$$

The extremum of plastic dissipation implies that

$$(10) \quad \sum_I (P_I)_I dv_I \geq \sum_I (P_{LB})_I dv_I$$

The right hand side of the inequality gives a lower bound locus of the collapse loads.

For a kinematically admissible increment du_i^* and its associated strainincrement de_{ij}^* and stresses σ_{ij}^* , we have

$$(11) \quad \begin{aligned} \int_V D_p(de_{ij}^*) dV &= \sum_I (P_{UB})_I \left[\int_V (\bar{b}_i)_I d\bar{u}_i dV + \int_{\partial V_\sigma} (\bar{T}_i)_I d\bar{u}_i dS \right] \\ &= \sum_I (P_{UB})_I dv_I \\ \int_V \sigma_{ij} de_{ij}^* dV &= \sum_I (P_l)_I dv_I^* \end{aligned}$$

Then

$$(12) \quad \sum_I (P_l)_I dv_I^* \leq \sum_I (P_{UB})_I dv_I^*$$

gives an upper locus of the collapse loads, which can be computed explicitly from the admissible velocity field du_i^* . More general theorems of limit analysis have introduced by Salencon (1977). Interested readers are referred to the literature.

We shall consider a multi-parameter example of a sandwich beam consist of two equal thin flanges of cross-sectional area A and length L . A web of negligible longitudinal strength separates the flanges at a distance d apart. The beam is under an axial force P and a bending moment M , which are treated as two independent loading parameters.

For lower bound analysis, the statically admissible stresses are

$$(13) \quad \sigma_u = P/(2A) + M/(Ad), \quad \sigma_l = P/(2A) - M/(Ad),$$

where σ_u and σ_l are the stress in the upper and lower flanges, respectively. The yield criterion requires that both $|\sigma_u|$ and $|\sigma_l|$ are less or equal to σ_Y ,

$$(14) \quad |P/P_l \pm M/M_l| \leq 1,$$

which is the lower bound locus with $P_l = 2A\sigma_Y$ and $M_l = \sigma_Y Ad$. Since the problem is statically determinant, the lower bound locus is the actual limit locus with P_l and M_l being the limit loads for simple tensile and pure bending, respectively.

For upper bound analysis, we use the elongation increments of the upper and lower flanges $d\Delta u_u$ and $d\Delta u_l$ as the kinematically admissible increment. Then the plastic dissipation is

$$\int_V D_p(de_{ij}^*) dV = A\sigma_Y (|d\Delta u_u| + |d\Delta u_l|),$$

and the external work is

$$\sum_I (P_{UB})_I \int_{\partial V} (\bar{T}_i)_I du_i^* dS = A \left[\left(\frac{P}{2A} + \frac{M}{Ad} \right) d\Delta u_u + \left(\frac{P}{2A} - \frac{M}{Ad} \right) d\Delta u_l \right].$$

By equating the plastic dissipation and the external work, we obtain the upper bound locus

$$(P/P_l + M/M_l)d\Delta u_u + (P/P_l - M/M_l)d\Delta u_l = |d\Delta u_u| + |d\Delta u_l|.$$

P R O B L E M S

10.3. Derive the differential equation and permissible boundary conditions for a membrane stretched over a simply connected regular region A by minimizing the functional I with respect to the displacement w ; $p(x, y)$ being a given function

$$I = \frac{1}{2} \iint_A (w_x^2 + w_y^2) dx dy - \iint_A p w dx dy.$$

10.4. Obtain Euler's equation for the functional

$$I' = \frac{1}{2} \iint_A [w_x^2 + w_y^2 + (\nabla^2 w + p)^2 - 2pw] dx dy.$$

$$Ans. \quad \nabla^4 w - \nabla^2 w + \nabla^2 p - p = 0.$$

Note: Compare I' with I of Prob. 10.3. Since $\Delta^2 w + p = 0$ for a membrane, the solution of Prob. 10.3 also satisfies the

present problem. But the integrand of I' contains higher derivatives of w and the equation $\delta I' = 0$ is a sharper condition than $\delta I = 0$. These examples illustrate Courant's method of sharpening a variational problem and accelerating the convergence of an approximating sequence in the direct method of solution of variational problems.

10.5. Bateman's principle in fluid mechanics states that in a flow of an ideal, nonviscous fluid (compressible or incompressible) in the absence of external body forces, the "pressure integral," the integral of pressure over the entire fluid volume, is an extremum. Consider the special case of a steady, irrotational flow of an incompressible fluid, for which the Bernoulli's equation gives

$$p = \text{const} - \rho V^2/2 = \text{const} - p[(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2]/2,$$

where ϕ is the velocity potential. Derive the field equation governing ϕ and the corresponding natural boundary conditions according to Bateman's principle. [H. Bateman, *Proc. Roy. Soc. London, A* **125** (1929) 598–618.]

10.6. Let

$$T = \sum_{\mu,\nu=1}^m P_{\mu\nu}(t, q_1, \dots, q_m) \dot{q}_\mu(t) \dot{q}_\nu(t), \quad U = U(t, q_1, \dots, q_m),$$

and

$$L = T - U,$$

where $\dot{q}_\mu(t) = dq_\mu/dt$, and $q_\mu(t_0) = a_\mu$, $q_\mu(t_1) = b_\mu$, ($\mu = 1, \dots, m$); a_μ, b_μ are given numbers. Derive the Euler differential equations for the variational problem connected with the functional

$$J[q_1, \dots, q_m] = \int_{t_0}^{t_1} L dt.$$

Prove that

$$d(T + U)/dt = 0,$$

if $\partial U/\partial t = 0$, $\partial P_{\mu\nu}/\partial t = 0$, $\mu, \nu = 1, \dots, m$. Hint: Use Euler's relation for homogeneous functions.

10.7. Find the curves in the x, y -plane such that

$$\int_a^b \sqrt{2E - n^2 y^2} ds, \quad ds^2 = dx^2 + dy^2$$

is stationary, where E and n are constants and a and b are the fixed end points.

10.8. Let D be the set of all functions $u(x)$ with the following properties:

$$D : \left\{ \begin{array}{l} (1) \ u(x) = a_1 \sin \pi x + a_2 \sin 2\pi x, \quad (2) \ a_1^2 + a_2^2 > 0, \text{ and} \\ (3) \ a_1, a_2 \text{ are real, arbitrary numbers.} \end{array} \right\}$$

Consider the functional, for all functions $u(x)$ in D ,

$$J(u) = \int_0^1 [u'(x)]^2 dx / \int_0^1 [u(x)]^2 dx.$$

Questions:

(a) Give necessary conditions for a function $u^*(x)$ in D so that $u^*(x)$ furnishes an extremum of the functional J .

(b) Show that there exist exactly two functions

$$u_1(x) = a_{11} \sin \pi x + a_{12} \sin 2\pi x, \quad u_2(x) = a_{21} \sin \pi x + a_{22} \sin 2\pi x,$$

notwithstanding a constant factor, which satisfy these necessary conditions mentioned in (a).

(c) Show that for one of the functions mentioned in (b), say, $u_1(x)$, the inequality

$$J[u_1] \leq J[u]$$

holds for all functions $u(x)$ in D . Then show that for the other solution, $u_2(x)$, the inequality

$$J[u] \leq J[u_2]$$

holds for all functions $u(x)$ in D .

- (d) Does there exist any relation between $J[u_1], J[u_2]$ and the eigenvalues of

$$-u''(x) = \lambda u(x), \quad 0 \leq x \leq 1,$$

with $u(0) = u(1) = 0$?

10.9. Clapeyron's theorem states that, if a linear elastic body is in equilibrium under a given system of body forces F_i and surface forces T_i , then the strain energy of deformation is equal to one-half the work that would be done by the external forces (of the equilibrium state) acting through the displacements u_i from the unstressed state to the state of equilibrium, i.e.,

$$\int_V F_i u_i dv + \int_S T_i u_i ds = 2 \int_V W dv.$$

Demonstrate Clapeyron's Theorem for:

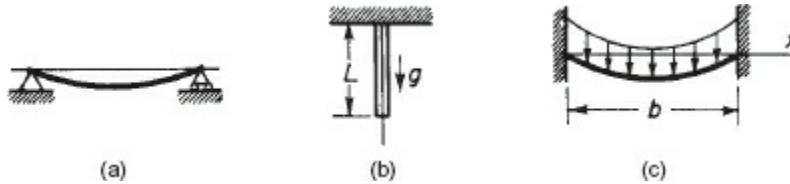


Fig. P10.9

- (a) A simply supported beam under its own weight [Fig. P10.9(a)].

- (b) A rod under its own weight [Fig. P10.9(b)],

$$u(x) = g\rho L^2 [(x/L)^2 / 2 - x/L] / E.$$

- (c) A strip of membrane of infinite length and width b [Fig. P10.9(c)], under a constant pressure p_0 , for which

$$w = w_0(1 - 4x^2/b^2), \quad w_0/h = [3(1 - \nu^2)64p_0b^4/(Eh^4)]^{1/3}.$$

10.10. The “poker chip” problem (Max Williams). To obtain a nearly triaxial tension test environment for polymer materials, a “poker chip” test specimen (a short circular cylinder) is glued between two circular cylinders (Fig. P10.10). When the cylinder is subjected to simple tension, the center of the poker chip is subjected to a triaxial tension stress field. Let the elastic constants of the media be E_1, G_1, ν_1 and E_2, G_2, ν_2 , as indicated in Fig. P10.10, with $E_1 \ll E_2$, $\nu_1 \approx 0.5$. Assume cylindrical symmetry, and obtain approximate expressions of the stress field by the following methods.

- (a) Use the complementary energy theorem and a stress field satisfying stress boundary conditions and the equations of equilibrium.
- (b) Use the potential energy theorem and assumed displacements satisfying displacement boundary conditions.
- (c) Experimental results suggest that the following displacement field is reasonable:

$$w = w_0 z/h, \quad u = (1 - z^2/h^2)g(r), \quad -h \leq z \leq h,$$

where $g(r)$ is as yet an unknown function of r . Obtain the governing equation for $g(r)$ and its appropriate solution for this problem by attempting to satisfy the following equilibrium equations in some average sense with respect to the z -direction:

$$\begin{aligned} \partial\sigma_{rr}/\partial r + \partial\sigma_{rz}/\partial z + (\sigma_{rr} - \sigma_{\theta\theta})/r &= 0 \\ \partial\sigma_{rz}/\partial r + \partial\sigma_{zz}/\partial z + \sigma_{rz}/r &= 0. \end{aligned}$$

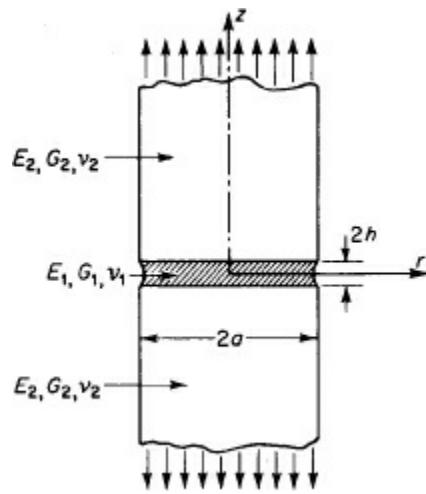


Fig. P10.10. Poker chip specimen.

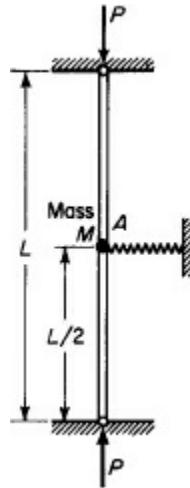


Fig. P10.11

10.11. Consider a pin-ended column of length L and bending stiffness EI subjected to an end thrust P . A spring is attached at the middle of the column as shown in Fig. P10.11. When the column is straight the spring tension is zero. If the column deflects by an amount Δ , the spring exerts a force R on the column:

$$R = K\Delta - \alpha\Delta^3, \quad (K > 0, \alpha > 0).$$

Derive the equation of equilibrium of the system. It is permissible to use the Euler–Bernoulli approximation for a beam, for which the strain energy per unit length is

$$EI \cdot (\text{curvature})^2/2 \quad \text{and} \quad \text{bending moment} = EI \cdot (\text{curvature}).$$

10.12. For the column of Problem 10.11 under the axial load P , is the solution unique? Under what situation is the solution nonunique? What are the possible solutions when the uniqueness is lost?

10.13. A linear elastic beam of bending stiffness EI is supported at four equidistant points $ABCD$, Fig. P10.13. The supports at A and D are pin-ended. At B and C , the beam rests on two identical nonlinear pillars. The characteristic of the pillars may be described as “hardening” and is expressed by the equation

$$K\Delta = R - \beta R^3 \quad (K > 0, \beta > 0),$$

where R is the reaction of the pillar, Δ is the downward deflection of the beam at the point of attachment of the beam to the pillar, K and β are constants.

A load P is applied to the beam at the midspan point. Find the reactions at the supports B and C . One of the two minimum principles (of potential energy and of complementary energy) is easier to apply to this problem. Solve the problem by a variational method with the appropriate minimum principle.

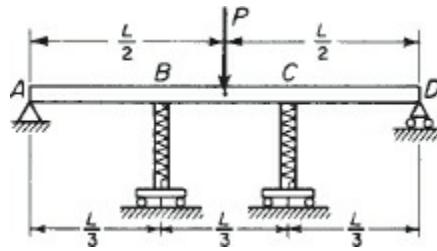


Fig. P10.13

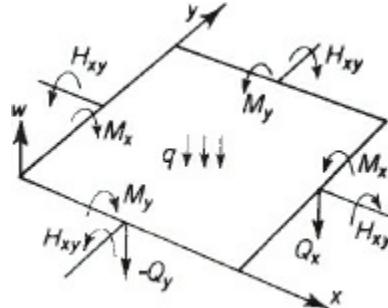


Fig. P10.14

10.14. Consider a square plate loaded in the manner shown in Fig. P10.14. Derive Euler's equation and the natural boundary conditions for V to be a minimum when $w(x, y)$ is varied.

$$\begin{aligned} V &= U + A, \\ U &= \frac{D}{2} \int_0^1 \int_0^1 [(w_{xx} + w_{yy})^2 - 2(1-\nu)(w_{xx}w_{yy} - w_{xy}^2)] dx dy, \\ A &= \int_0^1 \int_0^1 q w dx dy - \int_0^1 \left(M_x \frac{\partial w}{\partial x} \right) \Big|_{x=0}^{x=1} dy + \int_0^1 (Q_x w) \Big|_{x=0}^{x=1} dy \\ &\quad + \int_0^1 \left(H_{xy} \frac{\partial w}{\partial y} \right) \Big|_{x=0}^{x=1} dy - \int_0^1 \left(M_y \frac{\partial w}{\partial y} \right) \Big|_{y=0}^{y=1} dx \\ &\quad + \int_0^1 (Q_y w) \Big|_{y=0}^{y=1} dx + \int_0^1 \left(H_{xy} \frac{\partial w}{\partial x} \right) \Big|_{y=0}^{y=1} dx. \end{aligned}$$

10.15. Consider a material with the elastic modulus tensor satisfying

$$D_{ijkk} = D_{kkij} = 3K\delta_{ij} = E\delta_{ij}/(1-2\nu).$$

The bulk modulus K is infinite while $D_{ijkl} - \delta_{ij}\delta_{kl}K$ is bounded if the material is incompressible. Show that the *contact transformation* gives

$$D_{ijkl}e_{ij}e_{kl}/2 - pe_{kk} = D'_{ijkl}e'_{ij}e'_{kl}/2 - p^2/(2K),$$

where $D'_{ijkl} = D_{ijkl} - K\delta_{ij}\delta_{kl}$, p is the hydrostatic pressure and e'_{ij} is the deviatoric strain and that the potential energy of the element is

$$\Pi_{M_e}(u_i, p) = \int_{V_e} \left(\frac{1}{2} D'_{ijkl} e'_{ij} e'_{kl} + p e_{kk} - \frac{p^2}{2K} - b_i u_i \right) dV - \int_{\partial V_{\sigma e}} \bar{T}_i u_i dS.$$

The deviatoric energy $D'_{ijkl}e'_{ij}e'_{kl}/2$ is bounded.

¹ Any consideration of the sufficient conditions requires the concept of the second variations and the examination of its positive or negative definiteness. Several conditions are known; they are similar to, but more complex than, the corresponding conditions for maxima or minima of ordinary functions. It is useful to remember that many subtleties exist in the calculus of variations. A physicist or an engineer rarely worries about the mathematical details. His minimum principles are established on physical grounds and the existence of a solution is usually taken for granted. Mathematically, however, many examples can be constructed to show that a functional may not have a maximum or a minimum, or that a solution of the Euler equation may not minimize the functional. An engineer should be aware of these possibilities.

²The symbol $(\hat{\partial}/\partial x)F_{vx}$ means that F_{vx} is considered as a function of x and y , e.g.,

$$\frac{\hat{\partial}}{\partial x} F_{vx} = \frac{\partial}{\partial x} F_{vx} + \frac{\partial F_{vx}}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial F_{vx}}{\partial v_x} \frac{\partial v_x}{\partial x} + \frac{\partial F_{vx}}{\partial v_y} \frac{\partial v_y}{\partial x}.$$

³The Gibbs' thermodynamic potential (Sec. 12.3) per unit volume, $\rho\Phi$, is equal to the negative of the complementary strain energy function. If the stress-strain law were linear, then $W_c(\sigma_{ij})$ and $W(e_{ij})$ are equal: $-\rho\Phi = W_c = W$ (for linear stress-strain law).

⁴A continuous function $u(x, y, z)$ in V is said to be C^0 continuous. If u and all of its partial derivatives up to order n are continuous in V , u is said to be C^n continuous.

⁵In the rigid region where $d\bar{e}_{ij} = 0$, the question of association does not arise.

11

HAMILTON'S PRINCIPLE, WAVE PROPAGATION, APPLICATIONS OF GENERALIZED COORDINATES

In dynamics, the counterpart of the minimum potential energy theorem is Hamilton's principle. In this chapter we shall discuss this important principle and its applications to vibrations and wave propagations in beams.

Toward the end of the chapter a brief discussion of computational methods of solving variational problems is given. The basic idea is to apply the concept of generalized coordinates to obtain approximate solutions for a continuous system by reducing it to one with a finite number of degrees of freedom. Several important methods — those of Euler, Rayleigh–Ritz–Galerkin, and Kantrovich — will be outlined. Much more about computational mechanics is given later in [Chapters 17–22](#).

11.1. HAMILTON'S PRINCIPLE

For an oscillating body, with displacements u_i so small that the acceleration is given by $\partial^2 u_i / \partial t^2$ in Eulerian coordinates [Sec. 2.1](#) ([Sec. 5.2](#)), the equation of small motion is

$$(1) \quad \sigma_{ij,j} + F_i = \rho \partial^2 u_i / \partial t^2,$$

where ρ is the density of the material and F_i is the body force per unit volume. Let us again consider virtual displacements δu_i as specified in [Sec. 10.7](#), but instead of a body in static equilibrium we now consider a vibrating body. The variations δu_i must vanish over the boundary surface S_u , where values of displacements are prescribed; but are arbitrary, triply differentiable over the domain V ; and are arbitrary also over the boundary surface S_σ , where surface tractions are prescribed. The total boundary surface of the volume V is S , and $S = S_u + S_\sigma$.

The virtual work done by the body and surface forces is, as in [Sec. 10.7](#),

$$\int_V F_i \delta u_i dv + \int_S \overset{\nu}{T}_i \delta u_i dS.$$

The last integral can be transformed on introducing $\overset{\nu}{T}_i = \sigma_{ij} \nu_j$ and using Gauss' theorem, as in [Sec. 10.7](#):

$$\int_S \overset{\nu}{T}_i \delta u_i dS = \int_S \sigma_{ij} \nu_j \delta u_i dS = \int_V (\sigma_{ij} \delta u_i)_{,j} dv = \int_V \sigma_{ij,j} \delta u_i dv + \int_V \sigma_{ij} \delta u_{i,j} dv.$$

By Eq. (1), and by the symmetry of σ_{ij} , the right hand side is equal to

$$\int_V \left(\rho \frac{\partial^2 u_i}{\partial t^2} - F_i \right) \delta u_i dv + \int_V \sigma_{ij} \delta e_{ij} dv.$$

Therefore, we obtain the *variational equation of motion*

$$(2) \quad \Delta \quad \int_V \sigma_{ij} \delta e_{ij} dv = \int_V \left(F_i - \rho \frac{\partial^2 u_i}{\partial t^2} \right) \delta u_i dv + \int_S \overset{\nu}{T}_i \delta u_i dS.$$

As before, this general equation can be stated more concisely if we introduce various levels of restrictions. Thus, if the body is perfectly elastic and a strain energy function W exists, then the variational equation of motion can be written as

$$(3) \quad \Delta \quad \delta \int_V W dv = \int_V \left(F_i - \rho \frac{\partial^2 u_i}{\partial t^2} \right) \delta u_i dv + \int_S \overset{\nu}{T}_i \delta u_i dS.$$

The variations δu_i are assumed to vanish over the part of the boundary S_u where surface displacements are prescribed. Hence, the limit for the surface integral can be replaced by S_σ . If the variations δu_i were identified with the actual displacements $(\partial u_i / \partial t) dt$, then the result above states that, in an arbitrary time interval, the sum of the energy of

deformation and the kinetic energy increases by an amount that is equal to the work done by the external forces during the same time interval.

If the virtual displacements δu_i are regarded as functions of time and space, not to be identified with the actual displacements, and Eq. (3) is integrated with respect to time between two arbitrary instants t_0 and t_1 , an important variational principle for the moving body can be derived:

$$(4) \int_{t_0}^{t_1} \int_V \delta W dudt = \int_{t_0}^{t_1} \left[\int_V F_i \delta u_i dv + \int_V \nu T_i \delta u_i dS - \int_V \rho (\partial^2 u_i / \partial t^2) \delta u_i dv \right] dt.$$

Calling the last term J , inverting the order of integration, and integrating by parts, we obtain

$$(5) \quad J = \int_V \rho \frac{\partial u_i}{\partial t} \delta u_i dv \Big|_{t_0}^{t_1} - \int_V dv \int_{t_0}^{t_1} \frac{\partial u_i}{\partial t} \left(\rho \frac{\partial \delta u_i}{\partial t} + \frac{\partial \rho}{\partial t} \delta u_i \right) dt.$$

The $\partial \rho / \partial t$ term can be ignored because $\partial \rho / \partial t = -\partial \rho \dot{u}_i / \partial x_i$ according to the equation of continuity, and thus the term $\delta u_i \partial \rho / \partial t$ is an order of magnitude smaller than other terms in this equation. Let us now impose the restriction that at the time t_0 and t_1 , the variations δu_i are zero, i.e.,

$$(6) \quad \delta u_i(t_0) = \delta u_i(t_1) = 0 \quad \text{at all points in } V.$$

Then

$$(7) \quad J = - \int_{t_0}^{t_1} \int_V \rho \frac{\partial u_i}{\partial t} \delta \frac{\partial u_i}{\partial t} dv dt = -\frac{1}{2} \int_{t_0}^{t_1} \delta \left(\int_V \rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} dv \right) dt = -\int_{t_0}^{t_1} \delta K dt,$$

where

$$(8) \quad K = \frac{1}{2} \int_V \rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} dv$$

is the kinetic energy of the moving body. Therefore, under the assumption (6), Eq. (4) becomes,

$$(9) \quad \Delta \quad \int_{t_0}^{t_1} \delta(U - K) dt = \int_{t_0}^{t_1} \int_V F_i \delta u_i dv dt + \int_{t_0}^{t_1} \int_{S_\sigma} \nu T_i \delta u_i dS dt,$$

where U represents the total strain energy of the body,

$$U = \int_V W dv.$$

If the external forces acting on the body are such that the sum of the integrals on the right-hand side of Eq. (9) represents the variation of a single function — *the potential energy of the loading* — A ,

$$(10) \quad \int_V F_i \delta u_i dv + \int_{S_\sigma} \nu T_i \delta u_i dS = -\delta A,$$

then Eq. (9) can be written as

$$(11) \quad \Delta \quad \delta \int_{t_0}^{t_1} (U - K + A) dt = \delta \int_{t_0}^{t_1} L dt = 0.$$

The term L (or sometimes $-L$) is called the *Lagrangian function* and Eq. (11) represents *Hamilton's principle*, which states that:

The time integral of the Lagrangian function over a time interval t_0 to t_1 is an extremum for the “actual” motion with respect to all admissible virtual displacements which vanish, first, at instants of time t_0 and t_1 at all points of the body, and, second, over S_u , where the displacements are prescribed, throughout the entire time interval.

To formulate this principle in another way, let us call $u(x_1, x_2, x_3; t)$ a dynamic path. Then Hamilton's principle states that *among all dynamic paths that satisfy the boundary conditions over S_u at all times and that start and end with the actual values at two arbitrary instants of time t_0 and t_1 at every point of the body, the “actual” dynamic path is distinguished by making the Lagrangian function an extremum.*

In rigid body dynamics the term U drops out, and we obtain Hamilton's principle in the familiar form. The symbol A replaces the usual symbol V in books on dynamics because we have used V for something else.

Note that the potential energy — A of the external loads exists and is a linear function of the displacements if the loads are independent of the elastic displacements, as is commonly the case. In aeroelastic problems, however, the aerodynamic loading is sensitive to the small surface displacements u_i ; moreover, it depends on the time history of the displacements and cannot be derived from a potential. Hence, in aeroelasticity we are generally forced to use variational form (9) of Hamilton's principle.

In some applications of the direct method of calculation, it is even desirable to liberalize the variations δu_i at the instants t_0 and t_1 and use Hamilton's principle in the variational form (4) which cannot be expressed elegantly as the minimum of a well-defined functional. On the other hand, such a formulation will be accessible to the direct methods of solution. Using Eq. (5), (7), and (10), we can rewrite Eq. (4) in the form:

$$(12) \quad \int_{t_0}^{t_1} \delta(U - K + A) dt = \int_{t_0}^{t_1} \int_V F_i \delta u_i dv dt + \int_{t_0}^{t_1} \int_S T_i^\nu \delta u_i dS dt - \int_V \rho \frac{\partial u_i}{\partial t} \delta u_i dv \Big|_{t_0}^{t_1}.$$

Here U is the total strain energy, K is the total kinetic energy, A is the potential energy for the conservative external forces, F_i and T_i^ν are, respectively, the external body and surface forces that are not included in A , and δu_i are the virtual displacements.

Problem 11.1. Prove the converse theorem that, for a conservative system, the variational Eq. (11) leads to the equation of motion

$$\rho \partial^2 u_i / \partial t^2 = F_i + \partial(\partial W / \partial e_{ij}) / \partial x_j$$

and the boundary conditions

$$\text{either } \delta u_i = 0 \text{ or } (\partial W / \partial e_{ij}) \nu_j = T_i^\nu.$$

11.2. EXAMPLE OF APPLICATION – EQUATION OF VIBRATION OF A BEAM

As an example of the application of Hamilton's principle in the formulation of approximate theories in elasticity, let us consider the free, lateral vibration of a straight simple beam. We assume that the beam possesses principal planes and that the vibration takes place in one of the principal planes, and let y denote the small deflection of the neutral axis of the beam from its initial, straight configuration. In Sec. 10.8, it is shown that the strain energy of the beam is, for small deflections,

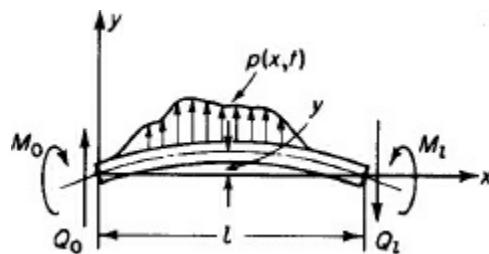


Fig. 11.2:1. Application to a beam.

$$(1) \quad U = \frac{1}{2} \int_0^l EI \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx,$$

where E is Young's modulus of the beam material, I is the cross-sectional moment of inertia, and l is the length of the beam.

The kinetic energy of the beam is derived partly from the translation, parallel to y , of the elements composing it, and partly from the rotation of the same elements about an axis perpendicular to the neutral axis and the plane of vibration. The former part is

$$\frac{1}{2} \int_0^l m \left(\frac{\partial y}{\partial t} \right)^2 dx,$$

where m is the mass per unit length of the beam. The latter part is, for each element dx , the product of moment of inertia times one-half of the square of the angular velocity. Let I_p denote the mass moment of inertia about the neutral axis per unit length of the beam. The angular velocity is $\partial^2 y / \partial t \partial x$ and the kinetic energy of the beam is

$$(2) \quad K = \frac{1}{2} \int_0^l m \left(\frac{\partial y}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^l I_p \left(\frac{\partial^2 y}{\partial x \partial t} \right)^2 dx.$$

If the beam is loaded by a distributed lateral load of intensity $p(x, t)$ per unit length and moment and shear M and Q , respectively, at the ends as shown in Fig. 11.2:1, then the potential energy of the external loading is

$$(3) \quad A = - \int_0^l p(x, t) y(x) dx - M_l \left(\frac{\partial y}{\partial x} \right)_l + M_0 \left(\frac{\partial y}{\partial x} \right)_0 + Q_l y_l - Q_0 y_0.$$

The equation of motion is given by Hamilton's principle:

$$(4) \quad \delta \int_{t_0}^{t_1} (U - K + A) dt = 0;$$

i.e.,

$$(5) \quad \delta \int_{t_0}^{t_1} \left\{ \int_0^l \left[\frac{1}{2} EI \left(\frac{\partial^2 y}{\partial x^2} \right)^2 - \frac{1}{2} m \left(\frac{\partial y}{\partial t} \right)^2 - \frac{1}{2} I_p \left(\frac{\partial^2 y}{\partial x \partial t} \right)^2 - py \right] dx \right. \\ \left. - M_l (\partial y / \partial x)_l + M_0 (\partial y / \partial x)_0 + Q_l y_l - Q_0 y_0 \right\} dt = 0.$$

Following the usual procedure of the calculus of variations, noting that the virtual displacement must be so specified that $\delta y = 0$ at t_0 and t_1 , and, hence, $\partial(\delta y) / \partial x = \delta(\partial y / \partial x) = 0$ at t_0 and t_1 , we obtain

$$\int_{t_0}^{t_1} \left\{ \int_0^l \left[EI \frac{\partial^2 y}{\partial x^2} \frac{\partial^2 \delta y}{\partial x^2} - m \frac{\partial y}{\partial t} \frac{\partial \delta y}{\partial t} - I_p \frac{\partial^2 y}{\partial x \partial t} \frac{\partial^2 \delta y}{\partial x \partial t} - p \delta y \right] dx \right. \\ \left. - M_l \delta (\partial y / \partial x)_l + M_0 \delta (\partial y / \partial x)_0 + Q_l \delta y_l - Q_0 \delta y_0 \right\} dt = 0.$$

Integrating by parts, we obtain

$$(6) \quad \int_{t_0}^{t_1} \int_0^l \left[\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right) + m \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2}{\partial x \partial t} \left(I_p \frac{\partial^2 y}{\partial x \partial t} \right) - p(x, t) \right] \delta y dx dt \\ + \int_{t_0}^{t_1} \left[EI \frac{\partial^2 y}{\partial x^2} - M \right] \delta \left(\frac{\partial y}{\partial x} \right)_0^l dt \\ - \int_{t_0}^{t_1} \left[\frac{\partial}{\partial x} \left(EI \frac{\partial^2 y}{\partial x^2} \right) - \frac{\partial}{\partial t} \left(I_p \frac{\partial^2 y}{\partial x \partial t} \right) - Q \right] \delta y \Big|_0^l dt = 0.$$

Hence, the Euler equation of motion is

$$(7) \quad \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right) + m \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2}{\partial x \partial t} \left(I_p \frac{\partial^2 y}{\partial x \partial t} \right) = p(x, t),$$

and a proper set of boundary conditions at each end is

$$(18a) \quad \partial(EI \partial \psi / \partial x) / \partial x + k(\partial y / \partial x - \psi) - I_p \partial^2 \psi / \partial t^2 = 0,$$

$$(18b) \quad m \partial^2 y / \partial t^2 - \partial[k(\partial y / \partial x - \psi)] / \partial x - p = 0.$$

These are equations governing the motion of a beam including the effect of the rotary inertia, due to Lord Rayleigh, and known as Rayleigh's equations. If the rotary inertia is neglected and if the beam were uniform, then the governing equation is simplified into:

$$(9) \quad \frac{\partial^2 y}{\partial t^2} + c_0^2 R^2 \frac{\partial^4 y}{\partial x^4} = p/m,$$

where

$$(10) \quad c_0^2 = E/\rho, \quad R^2 = I/A.$$

The constant c_0 has the dimension of speed and can be identified as the phase velocity of longitudinal waves in a uniform bar.¹ R is the radius of gyration of the cross section. A is the cross-sectional area, so that $m = \rho A$.

In the special case of a uniform beam of infinite length free from lateral loading, $p = 0$, Eq. (9) becomes

$$(11) \quad \frac{\partial^2 y}{\partial t^2} + c_0^2 R^2 \frac{\partial^4 y}{\partial x^4} = 0.$$

It admits a progressive wave solution of phase velocity c in the form

$$(12) \quad y = a \sin[2\pi(x - ct)/\lambda], \quad \text{where } c = \pm 2\pi c_0 R / \lambda.$$

This shows that the phase velocity depends on the wave length and tends to infinity for very short waves. According to Eq. (12), the group velocity (see Sec. 11.3) also tends to infinity as the wave length tends to zero. Since group velocity is the speed at which energy is transmitted, this result is physically unreasonable. If Eq. (12) were correct, then the effect of a suddenly applied concentrated load will be felt at once everywhere in the beam, as the Fourier representation for a concentrated load contains harmonic components with infinitesimal wave length, and hence infinite wave speed. Thus, Eq. (11) cannot be very accurate in describing the effect of impact loads on a beam.

The infinite wave speed can be removed by the inclusion of the rotary inertia. However, the speed versus wave length relationship obtained from Rayleigh's Eq. (7) for a uniform beam of circular cross section with radius a , as shown in Fig. 11.2:2, still deviates appreciably from Pochhammer and Chree's results, which were derived from the exact three-dimensional linear elasticity theory. A much better approximation is obtained by including the shear deflection of the beam first shown by Timoshenko.

To incorporate the shear deformation, we note that the slope of the deflection curve depends not only on the rotation of cross sections of the beam but also on the shear. Let ψ denote the slope of the deflection curve when the shearing force is neglected and β the angle of shear at the neutral axis in the same cross section. Then the total slope is

$$(13) \quad \frac{\partial y}{\partial x} = \psi + \beta.$$

The strain energy due to bending, Eq. (1), must be replaced by

$$(14) \quad \frac{1}{2} \int_0^l EI \left(\frac{\partial \psi}{\partial x} \right)^2 dx,$$

because the internal bending moment does no work when shear deformation takes place (see Fig. 11.2:3). The strain energy due to shearing strain β must

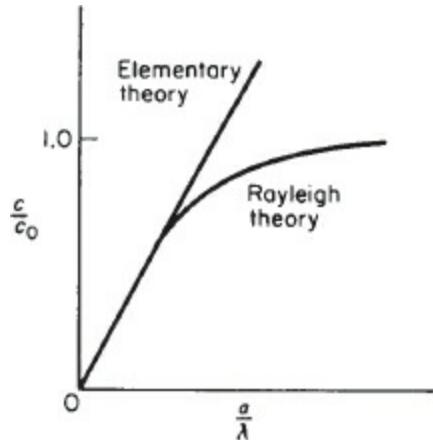


Fig. 11.2:2. Phase velocity curves for flexural elastic waves in a circular cylinder of radius a .

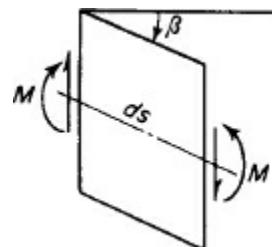


Fig. 11.2:3. A Timoshenko beam element.

be a quadratic function of β if linear elasticity is assumed. We write

$$(15) \quad \frac{1}{2} \int_0^l k \beta^2 dx = \frac{1}{2} \int_0^l k \left(\frac{\partial y}{\partial x} - \psi \right)^2 dx$$

for the strain energy for shear. The kinetic energy is

$$(16) \quad K = \frac{1}{2} \int_0^l m \left(\frac{\partial y}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^l I_p \left(\frac{\partial \psi}{\partial t} \right)^2 dx,$$

because the translational velocity is $\partial y / \partial t$, but the angular velocity is $\partial \psi / \partial t$. Hence, Hamilton's principle states that

$$(17) \quad \delta \int_{t_0}^{t_1} \int_0^l \frac{1}{2} [EI \left(\frac{\partial \psi}{\partial x} \right)^2 + k \left(\frac{\partial y}{\partial x} - \psi \right)^2 - m \left(\frac{\partial y}{\partial t} \right)^2 - I_p \left(\frac{\partial \psi}{\partial t} \right)^2] dx dt + \delta A = 0,$$

where A is given by Eq. (3) except that $\partial y / \partial x$ at the ends is to be replaced by ψ . The virtual displacements now consist of δy and $\delta \psi$, which must vanish at t_0 and t_1 and also where displacements are prescribed. On carrying out the calculations, the following two Euler equations are obtained:

$$(18a) \quad \partial(EI \partial \psi / \partial x) / \partial x + k(\partial y / \partial x - \psi) - I_p \partial^2 \psi / \partial t^2 = 0,$$

$$(18b) \quad m \partial^2 y / \partial t^2 - \partial[k(\partial y / \partial x - \psi)] / \partial x - p = 0.$$

The appropriate boundary conditions are, at each end of the beam,

$$(19a) \quad \text{either } EI \partial \psi / \partial x = M \quad \text{or } \delta \psi = 0,$$

$$(19b) \quad \text{either } k(\partial y / \partial x - \psi) = -Q \quad \text{or } \delta y = 0.$$

These are the differential equation and boundary conditions of the so-called *Timoshenko beam theory*.

For a uniform beam, EI , k , m , etc., are constants, and the function ψ can be eliminated from the equations above to obtain the well-known *Timoshenko equation for lateral vibration of prismatic beams*,

$$(20) \quad EI \partial^4 \psi / \partial x^4 + m \partial^2 y / \partial t^2 - (I_p + EIm/k) \partial^4 \psi / (\partial x^2 \partial t^2) + (I_p m/k) \partial^4 \psi / \partial t^4 = p + (I_p/k) \partial^2 p / \partial t^2 - (EI/k) \partial^2 p / \partial x^2.$$

So far we have not discussed the constants m , I_p , and k . For a beam of uniform material, $m = \rho A$, $I_p = \rho A R^2$, where ρ is the mass density of the beam material, A is the cross-sectional area, and R is the radius of gyration of the cross section about an axis perpendicular to the plane of motion and through the neutral axis. But k depends on the distribution of shearing stress in the beam cross section. Timoshenko writes

$$(21) \quad k = k'AG,$$

where G is the shear modulus of elasticity and k' is a numerical factor depending on the shape of the cross section, and ascertains that according to the elementary beam theory, $k' = \frac{2}{3}$ for a rectangular cross section. The use of such a value of k is, however, a subject of controversy in the literature. Mindlin^{11.1} suggests that the value of k can be so selected that the solution of Eq. (20) be made to agree with certain solution of the exact three-dimensional equations of Pochhammer (1876) and Chree (1889) (see Love,^{11.1} *Elasticity*, 4th edition, pp. 287–292). Indeed, I_p , which arises in the assumption of plane sections remain plane in bending, may also be regarded, when such an assumption is relaxed, as an empirical factor to be determined by comparison with exact solutions.

For a uniform beam free from lateral loadings, Eq. (20) can be written as

$$(22) \quad \frac{\partial^4 y}{\partial x^4} - \left(\frac{1}{c_0^2} + \frac{1}{c_Q^2} \right) \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{1}{c_0^2 c_Q^2} \frac{\partial^4 y}{\partial t^4} + \frac{1}{c_0^2 R^2} \frac{\partial^2 y}{\partial t^2} = 0,$$

where

$$(23) \quad c_0^2 = EI/I_p = E/\rho, \quad c_Q^2 = k'G/\rho, \quad R^2 = I/A.$$

If the beam is of infinite length, a substitution of the solution of the form (12) into (22) yields the equation for the wave speed c

$$(24) \quad 1 - (c^2/c_0^2 + c^2/c_Q^2) + c^4/(c_0^2 c_Q^2) - [c^2/(c_0^2 R^2)] \lambda^2/(2\pi)^2 = 0.$$

The solution c/c_0 versus λ has two branches, corresponding to two *modes* of motion (two different shear-to-bending deflection ratios for the same wave-length). They are plotted in Fig. 11.2:4 for a circular beam of radius a . The exact solution of Pochhammer and Chree for Poisson's ratio $\nu = 0.29$ are also plotted for comparison. The Timoshenko theory agrees reasonably well with the exact theory in the first mode, but wide discrepancy occurs in the second mode. The approximate theory gives no information about higher modes: an infinite number of which exist in the exact theory.

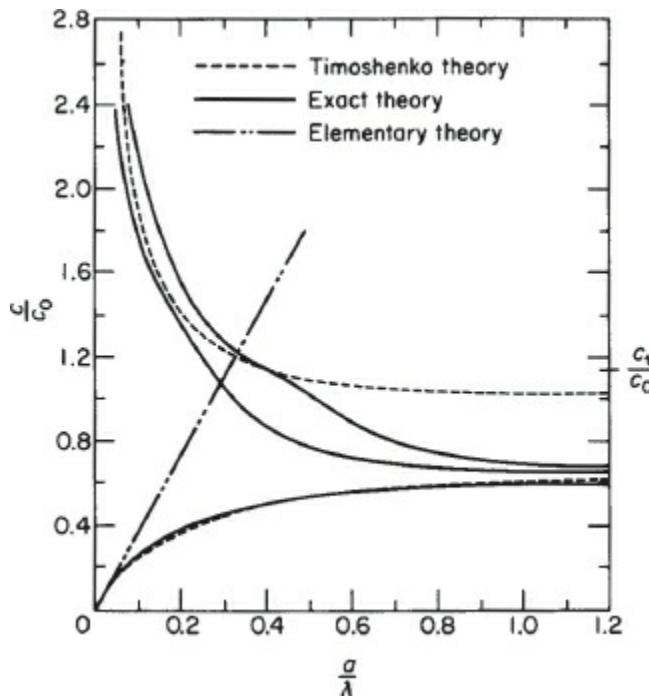


Fig. 11.2:4. Phase velocity curves for flexural elastic waves in a solid circular cylinder of radius a . (From Abramson,^{11.1} *J. Acoust. Soc. Am.*, 1957.)

The equations derived above are appropriate for determining the free-vibration modes and frequencies of a beam. The effects of rotary inertia and shear are unimportant if the wavelength of the vibration mode is large compared with the cross-sectional dimensions of the beam; but these effects become more important with a decrease of wavelength, i.e., with an increase in the vibration frequency. For a uniform beam with rectangular cross section and simply supported at both ends, with $E = 8G/3$ and $k' = 2/3$, we find that the shear deflection and rotary inertia reduce the natural frequencies. If the wavelength is ten times larger than the depth of the beam, the correction on the frequency due to rotary inertia alone is about 0.4%, and the correction due to rotary inertia and shear together will be about 2%. For a survey of literature, see Abramson, Plass, and Ripperger.^{11.1}

PROBLEMS

11.2. Consider the free longitudinal vibration of a rod of uniform cross section and length L , as shown in Fig. P11.2. Assume that plane cross sections remain plane, that only axial stresses uniformly distributed over the cross section are present, and that radial displacements are negligible (i.e., the only displacement component u in the x -direction is nonvanishing). Derive expressions for the potential and kinetic energies and show that the equation of motion is

$$(25) \quad \frac{\partial^2 u}{\partial x^2} - (\frac{\partial^2 u}{\partial t^2})/c_0^2 = 0, \quad c_0^2 = E/\rho.$$

Show that the general solution is of the form

$$(26) \quad u = f(x - c_0 t) + F(x + c_0 t),$$

where f and F are two arbitrary functions.

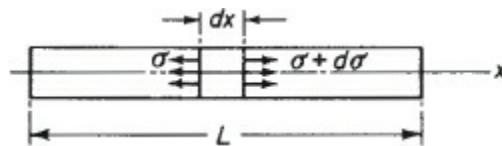


Fig. P11.2. Longitudinal vibration of a rod.

11.3. Consider the same problem as above, but now incorporate approximately the transverse inertia associated with the lateral expansion or contraction connected with axial compression and extension, respectively. Assume (Love) that the radial displacement v is proportional to the radial coordinate r , measured from a centroidal axis, and to the axial

strain $\partial u / \partial x$,

$$(27) \quad v = -\nu r \partial u / \partial x,$$

where ν is Poisson's ratio. Derive expressions of the kinetic and potential energies and obtain the equation of motion according to Hamilton's principle,

$$(28) \quad \rho [\partial^2 u / \partial t^2 - (\nu R)^2 \partial^4 u / (\partial x^2 \partial t^2)] - E \partial^2 u / \partial x^2 = 0,$$

where R is the polar radius of gyration of the cross section. The natural boundary condition at the end $x = 0$, if that end is subjected to a stress $\sigma_0(t)$, is

$$(29) \quad \rho \nu^2 R^2 \partial^3 u / (\partial x \partial t^2) + E \partial u / \partial x = \sigma_0(t) \quad \text{at } x = 0.$$

Note that, according to the last equation, the familiar proportionality between axial stress σ and axial strain $\partial u / \partial x$ does not exist in this theory.

Comparison of the dispersion curves obtained from the elementary theory (Problem 11.2), the Love theory, the Pochhammer–Chree “exact” theory, and the Mindlin/Herrmann^{11.1} approximate theory are shown in Fig. P11.3. The last-mentioned theory accounts for the strain energy associated with the transverse displacement v , of which the most important contribution comes from the shearing strain caused by the lateral expansion of the cross section near a wave front.

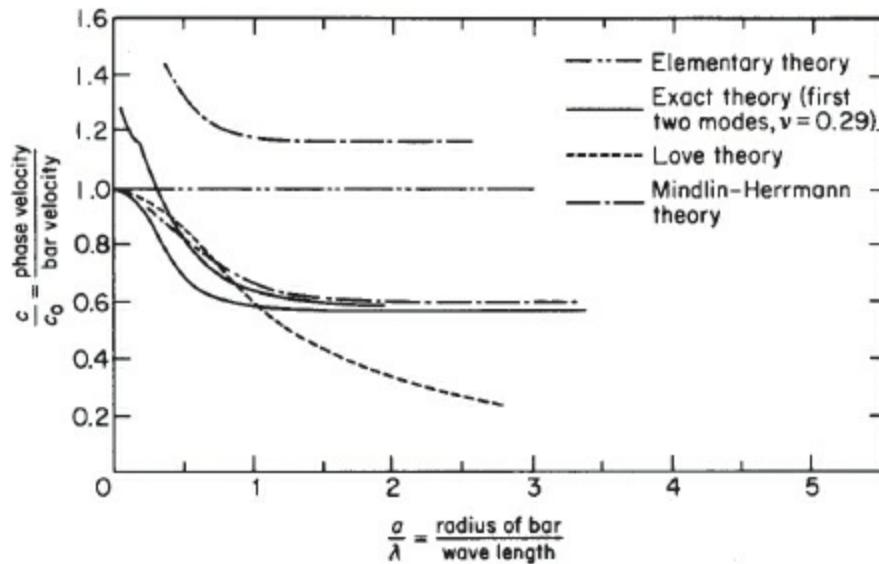


Fig. P11.3. Phase velocity curves for longitudinal elastic waves in a solid circular cylinder of radius a . (After Abramson *et al.*, *Adv. Applied Mech.*, 5, 1958.)

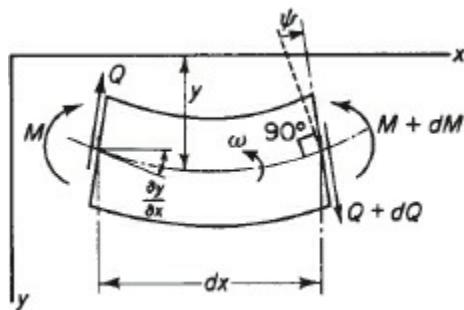


Fig. P11.4. Element of a beam in bending.

11.4. The method of derivation of the various forms of equations of motion of beams as presented above has the advantage of being straightforward, but it does not convey the physical concepts as clearly as in an elementary derivation. Hence, rederive the basic equations by considering the forces that act on an element of length dx , as shown in Fig. P11.2 and Fig. P11.4. Obtain the following equations, and then derive the wave equations by proper reductions.

Longitudinal waves, elementary theory (Fig. P11.2):

$$(30) \quad \begin{aligned} \partial \sigma_{xx} / \partial x &= \rho \partial^2 u / \partial t^2 \text{ (motion)}; & \sigma_{xx} &= E \epsilon_{xx} \text{ (material behavior)}; \\ \partial^2 u / (\partial x \partial t) &= \partial \epsilon_{xx} / \partial t \text{ (eq. of strain)}, \end{aligned}$$

where σ_{xx} = axial stress, E_{xx} = axial strain, $\partial u/\partial t$ = axial particle velocity, x = axial coordinate, t = time, E = modulus of elasticity, and ρ = mass density.

Flexural waves, Timoshenko theory (Fig. P11.4):

$$(31) \quad \partial M/\partial x - Q = \rho I \partial \omega/\partial t \text{ (rotational)}, \quad \partial Q/\partial x = \rho A \partial v/\partial t \text{ (transverse)},$$

$$(32) \quad \partial K/\partial t = \partial \omega/\partial x \text{ (bending)}, \quad \partial \beta/\partial t = \partial v/\partial x + \omega \text{ (shear)},$$

$$(33) \quad M = EIK \text{ (bending)}, \quad Q = GA_s \beta \text{ (shear material behavior)},$$

where M = moment, Q = shear force, K = axial rate of change of section angle $= -\partial \psi/\partial x$, β = shear strain $= \partial y/\partial x - \psi$, ω = angular velocity of section $= -\partial \psi/\partial t$, v = transverse velocity $= \partial y/\partial t$, I = section moment of inertia, A = section area, and A_s = area parameter defined by $\int \int \gamma(z) dA = \beta A_s$ where $\gamma(z)$ is the shear strain at a point z in the cross section, and G = shear modulus.

11.3. GROUP VELOCITY

Since we have been concerned in the preceding sections about wave propagations in beams, it seems appropriate to make a digression to explain the concept of *group velocity* as distinguished from the *phase velocity*. We have seen that for certain equations a solution of the following form exists:

$$(1) \quad u = a \sin(\mu x - vt).$$

If x is increased by $2\pi/\mu$, or t by $2\pi/v$, the sine function takes the same value as before, so that $\lambda = 2\pi/\mu$ is the wavelength and $T = 2\pi/v$ is the period of oscillation. If $\mu x - vt = \text{constant}$, i.e. $x = \text{const.} + vt/\mu$, the argument of the sine function remains constant in time; which means that the whole waveform is displaced towards the right with a velocity $c = v/\mu$. Thus c is called the phase velocity, and Eq. (1) may be exhibited as

$$(2) \quad u = a \sin[2\pi(x - ct)/\lambda].$$

If the phase velocity c depends on the wavelength λ , the wave is said to exhibit *dispersion*. Our examples in the previous section show that dispersion exists in both longitudinal and flexural waves in rods and beams.

What happens when two sine waves of the same amplitude but slightly different wavelengths and frequencies are superposed? Let these two waves be characterized by two sets of slightly different values μ, v and μ', v' . The resultant of the superposed waves is

$$u + u' = A [\sin(\mu x - vt) + \sin(\mu' x - v't)].$$

Using the well-known formula of

$$\sin \alpha + \sin \beta = 2 \sin[(\alpha + \beta)/2] \cos[(\alpha - \beta)/2],$$

we have

$$(3) \quad u + u' = 2A \sin\left(\frac{\mu + \mu'}{2}x - \frac{v + v'}{2}t\right) \cos\left(\frac{\mu - \mu'}{2}x - \frac{v - v'}{2}t\right).$$

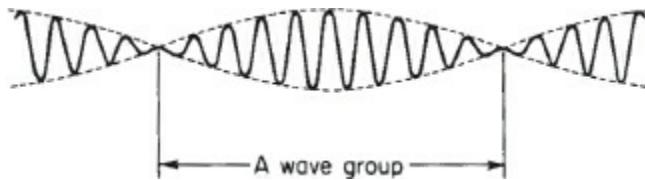


Fig. 11.3:1. An illustration of a wave group.

This expression represents the well-known phenomenon of "beats." The sine factor represents a wave whose wave number and frequency are equal to the mean of μ, μ' and v, v' , respectively. The cosine factor, which varies very slowly when $\mu - \mu', v - v'$ are small, may be regarded as a varying amplitude, as shown in Fig. 11.3:1. The "wave group" ends wherever the cosine factor becomes zero. The velocity of advance of these points is called the *group velocity* $U [= (v - v')/(\mu - \mu')]$. For long groups (or slow beats), the group velocity may be written with sufficient accuracy as

$$(4) \quad U = dv/d\mu.$$

In terms of the wavelength $\lambda (= 2\pi/\mu)$ and the phase velocity c , we have

(5)

$$U = d(\mu c)/d\mu = c - \lambda dc/d\lambda.$$

From the fact that no energy can travel past the nodes, one can infer that the rate of transfer of energy is identical with the group velocity. This fact is capable of rigorous proof for single trains of waves.

The most familiar examples of propagation of wave groups are perhaps the water waves. It has often been noticed that when an isolated group of waves, of sensibly the same length, advancing over relatively deep water, the velocity of the group as a whole is less than that of the individual waves composing it. If attention is fixed on a particular wave, it is seen to advance through the group, gradually dying out as it approaches the front, while its former place in the group is occupied in succession by other waves which have come forward from the rear. Another familiar example is the wave train set up by ships. The explanation as presented above seems to have been first given by Stokes (1876). Other derivations and interpretations of the group velocity concept can be found in Lamb, *Hydrodynamics* (New York: Dover Pub., 1879), Secs. 236 and 237.

If a concentrated lateral load is suddenly and impulsively applied on an infinitely long beam, the disturbance is propagated out along the beam by flexural waves. The initial loading may be regarded as composed of an infinite number of sine-wave components of all wavelengths but of the same amplitude, with proper phase relationship, so that they reinforce each other in the limited region where the force is applied, but cancel each other everywhere outside the region of load application. As time increases, these sine waves propagate with their own phase velocities, and the pattern of interference changes with time. Thus, at time t and at a point which is at a distance x from the initial loading, only a group of waves in the neighborhood of a specific wavelength can be seen, and the energy of this wave group is propagated at the group velocity U .

From the dispersion curves of Fig. 11.2:4 for the phase velocity of flexural elastic waves in a circular cylinder of radius a , the group wave velocity curves of Fig. 11.3:2 for the two lowest modes can be constructed. These curves are obtained by Abramson^{11.1} for the Pochhammer–Chree theory and are important in physical considerations. The number c_1 in Fig. 11.3:2 is the dilatational wave speed (see Sec. 7.8) given by the formula

$$c_1 = \sqrt{(1-\nu)E/[(1+\nu)(1-2\nu)\rho]}.$$

The number $\sqrt{E/\rho}$. The group velocity is denoted by c_g . The ratio c_1/c_0 depends on the Poisson's ratio ν :

$$\nu = 0.25, \quad c_1/c_0 = 1.095; \quad \nu = 0.30, \quad c_1/c_0 = 1.16.$$

Note that the phase velocities in the higher branches exceed the dilatational wave velocity (at long wavelengths), but the group velocities do not. Similar group velocity curves can be constructed for longitudinal waves from Fig. P11.3. Note that the greatest possible velocity of energy transmission is the dilatational velocity, for both flexural and longitudinal waves.

From Fig. 11.3:2 for a circular cylinder of radius a , it can be seen that for the first branch the group velocity reaches a maximum when a/λ is between 0.25 and 0.30, where λ is the wavelength. Hence, when a pulse is propagated along a beam in accordance with the first branch, Fourier components of wavelengths about 0.25 or 0.30 times the radius will be found at the head of the pulse. A detailed study of the transmission of a pulse along the beam can be found in R. M. Davies' paper.^{11.2}

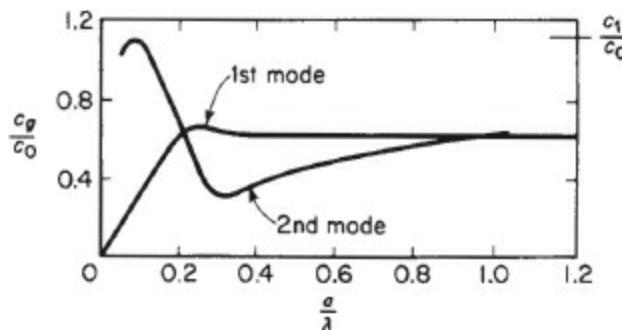


Fig. 11.3:2. Group velocity curves for flexural elastic waves in a solid circular cylinder of radius a . (After Abramson, *J. Acoust. Soc. Am.*, 1957.)

Remarks. Elastic substances are not inherently dispersive. The dispersion effect appears in waves traveling through rods and beams because of wave reflections from the boundaries. Therefore, the dispersion in an elastic medium is simply an interface phenomenon and not a physical property of the material. The phenomenon is analogous to that in electrical wave guides. In this light, we can explain the existence of phase velocities greater than the dilatational velocity, although no disturbance can be propagated with a velocity greater than that. The point is that energy is propagated at the group velocity which cannot exceed the dilatational velocity. The phase velocities are merely velocities of propagation in the axial direction of the loci of constant phase in specific mode patterns and their values are thus not

limited by physical considerations.

11.4. HOPKINSON'S EXPERIMENT

In Sec. 1.6, we mentioned that John Hopkinson published in 1872 an account of an interesting experiment, the explanation of which will help us understand the nature of elastic wave propagation and its significance in engineering. Hopkinson wanted to measure the strength of steel wires when they were suddenly stretched by a falling weight. A ball-shaped weight pierced by a hole was threaded on the wire and was dropped from a known height so that it struck a clamp attached to the bottom of the wire (Fig. 11.4:1). For a given weight he expected to find a critical height beyond which the falling weight would break the wire. Using different weights dropped from different heights, however, Hopkinson found that the minimum height from which a weight had to be dropped to break the wire was nearly *independent* of the size of the weight!



Fig. 11.4:1. A sketch of Hopkinson's experiment.

Now, when different weights are dropped from a given height, the velocity reached at the end is independent of the size of the weight. Hopkinson's result suggests that in breaking the wire it is the velocity of the loading end that counts. Following this lead, Hopkinson explains his result on the basis of elastic wave propagation (ignoring the effects of plasticity). Assuming that the stress state in the wire is approximately one-dimensional, we see from Eqs. (11.2:26), or (11.2:30), that the stress in the wire is proportional to the particle velocity:

$$(1) \quad \sigma_{xx} = \rho c_0 \partial u / \partial t,$$

where $c_0 = \sqrt{E/\rho}$ is the speed of sound of longitudinal waves in the wire. For steel, c_0 is about 16,000 ft/sec. The largest particle velocity in the wire, however, is not reached at the instant of the blow, but is reached at the top after the elastic waves propagate up and down the wire several times. When this largest particle velocity induces a stress equal to the ultimate stress of the wire, the wire breaks.

When the weight hits the clamp in Hopkinson's test, the end of the wire acquires a particle velocity V_0 equal to that of the weight and clamp. A steep-fronted tension wave is generated and propagated up the wire. In the meantime, the weight and the clamp are slowed down exponentially by the tensile load imposed by the wire:

$$(2) \quad M \partial V / \partial t = A \sigma_{xx} = A \rho c_0 V,$$

where M is the mass of the weight and clamp, V is its velocity, and A is the cross-sectional area of the wire. The elastic wave, on reaching the fixed end at the top, is reflected as a tension wave of twice the intensity of the incident wave. The reflected wave, going through the tail of the incident pulse, is reflected again at the lower end as a compression wave, and so on. If the stress at the first reflection (equal to $2\rho c_0 V_0$), is sufficient to break the wire, then fracture will be expected to take place near the top. In a systematic test of gradually increasing the height of drop h (i.e., gradually increasing V_0), this breaking at the first reflection does not happen. In J. Hopkinson's experiments, the head of the stress waves was able to travel the length of wire several times before the weight M was decelerated sufficiently, and the stress wave pattern in the wire was very complicated. Bertram Hopkinson (1905), in repeating his father's experiment, used smaller weights so that the rate of decay was rapid. Nevertheless, as shown by G. I. Taylor^{11,2} (1946), the maximum tensile stress in B. Hopkinson's experiment did not occur at the first reflection, when the stress was $2\rho c_0 V_0$, but at the third reflection, i.e., the second reflection at the top of the wire, when the tensile stress reached $2.15\rho c_0 V_0$.

To work out the details, we have to determine the function $f(x - c_0 t)$ and $F(x + c_0 t)$ of Eq. (11.2:26) to satisfy the initial conditions $u = \partial u / \partial t = 0$ at $t = 0$, and the boundary conditions $u = 0$ at the top; $\partial u / \partial t = V$, the velocity of the weight and clamp, at the bottom. The velocity V is determined by Eq. (2). The full solution is given in Taylor's paper.^{11,2}

Incidentally, Hopkinson found that the tensile strength of steel wires under rapid loading is much greater than that under a static load.

11.5. GENERALIZED COORDINATES

We shall show the actual construction of the minimizing sequence in solving a variational problem. Our aim is to obtain efficient approximate solutions. For this purpose, we shall discuss first the idea of generalized coordinates and the definitions of the “best approximations.”

Let us consider a function $u(x)$ of a real variable x , defined over an interval (a, b) . If $u(x)$ can be represented by a power series

$$(1) \quad u(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

or by a Fourier series

$$(2) \quad u(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

or, more generally, by a series of the form

$$(3) \quad u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where $\{\phi_n(x)\}$ is a given set of functions, then $u(x)$ can be specified by the coefficients $\{a_n\}$, $\{b_n\}$, or $\{c_n\}$. These coefficients may be regarded as the generalized coordinates of $u(x)$ referred to the bases $\{x^n\}$, $\{\sin nx\}$, $\{\phi_n\}$, respectively. In Eq. (3), each term $\phi_n(x)$ represents a degree-of-freedom, and the coefficients c_n specifies the extent to which $\phi_n(x)$ participates in the function $u(x)$. For example, if $u(x)$ represents a disturbed state of an elastic string which is anchored at the points $x = 0$ and π , then Eq. (3) states that $u(x)$ can be obtained by a superposition of successive normal modes $\phi_n = \sin nx$, and that the n^{th} mode participates with an amplitude c_n .

In the dynamics of a vibrating string the displacement $u(x, t)$ may be written as

$$(4) \quad u(x) = \sum_{n=1}^{\infty} q_n(t) \sin nx.$$

The quantity $q_n(t)$ is called a degree-of-freedom (DOF), known as the generalized coordinate with respect to the mode $\sin nx$. The possibility of generalizing this terminology to cases in which \mathbf{x} and \mathbf{u} represent vectors in vector spaces of dimensions m and k , respectively, is obvious.

The fundamental idea of approximate methods of solution is to reduce a continuum problem (with infinite number of DOFs) into one of finite number of DOFs. The reduced problem is solved by approximate means. Then a limiting process is used to extend the restricted solution to that of the original problem. In the following discussion we shall describe the ideas about the first step – reducing a given problem to one of finite DOFs. The second step – the proof of the validity of the limiting process – is the main mathematical problem. Examples of the mathematical theories which deal with the second step are the theories of Taylor’s series, Fourier series, eigenfunction expansions, etc. The theories are beyond the scope of our discussion, and the reader is referred to [Biblio. 11.3](#).

11.6. APPROXIMATE REPRESENTATION OF FUNCTIONS

Let a function $f(x_1, x_2, \dots, x_n)$ of n variables x_1, \dots, x_n be defined in a domain $\mathcal{R}(x_1, x_2, \dots, x_n)$. To represent $f(x_j)$ “as well as possible” by a linear combination of given functions $\phi_n(x_j)$, $n = 1, 2, \dots, N$, say, $Q_N(x_j)$, where

$$(1) \quad Q_N(x_j) = \sum_{n=1}^N a_n \phi_n(x_j),$$

the question arises as to the meaning of the term “as well as possible.” As soon as the “best approximation” is defined, a definite procedure for calculating a_n can be devised.

The definitions of the “best approximation” may be classified into two classes. In the first class the concept of *norm* is introduced.² For any function $g(x_j)$ in the domain $\mathcal{R}(x_j)$, a real positive number, called the *norm of $g(x_j)$* , and denoted by $\|g(x_j)\|$, is defined, with the stipulation that $\|g(x_j)\| = 0$ only for $g(x_j) = 0$ in $\mathcal{R}(x_j)$. Then the best approximation is taken to mean that

$$(2) \quad \|f(x_j) - \sum_{n=1}^N a_n \phi_n(x_j)\| = \min.$$

In the second class, a set of linear homogeneous expressions L_k , $k = 1, 2, \dots, N$, defined for functions in the domain \mathcal{R}

(x_j) , is selected, and the best approximation is taken to mean that

$$(3) \quad L_k(\varepsilon_N) = 0, \quad k = 1, 2, \dots, N,$$

where ε_N is the “error”

$$(4) \quad \varepsilon_N \equiv \varepsilon_N(x_j) \equiv f(x_j) - \sum_{n=1}^N a_n \phi_n(x_j).$$

Examples of the first class are

(A) *Absolute error method:* In this method the norm for any function $g(x_j)$ in $\mathcal{R}(x_j)$ is defined to be the maximum absolute value of $g(x_j)$ in $\mathcal{R}(x_j)$:

$$(5) \quad \|g(x_j)\| = \max_{x \text{ in } \mathcal{R}} |g(x_j)|.$$

Hence, the best approximation is defined by requiring that

$$(6) \quad \|\varepsilon_N\| = \max_{x \text{ in } \mathcal{R}} |\varepsilon_N(x_j)| = \min.$$

This method was used in the early 19th century, but is not in favor now because it is difficult to apply.

(B) *Least squares method:* In this method we take a positive integrable function $W(x_j)$ and define the norm

$$(7) \quad \|g(x_j)\| = \left(\int_{\mathcal{R}} W g^2 dv \right)^{1/2},$$

where $W(x_j)$ is a *weighting function*. The volume integral, which may be defined in Riemann or Lebesgue sense, is assumed to exist, with dv being a volume element. The constants a_n are to be determined from

$$(8) \quad \frac{1}{2} \frac{\partial \|\varepsilon_N\|^2}{\partial a_k} = - \int_{\mathcal{R}} \varepsilon_N W \phi_k dv = 0, \quad k = 1, 2, \dots, N.$$

Examples of the second class are

(C) *Collocation method:* We take N points P_k in the domain $\mathcal{R}(x_j)$ and let

$$(9) \quad L_k(\varepsilon_N) = \varepsilon_N(P_k).$$

Hence Eq. (3) implies that the constants a_n are so chosen that $f(x_j)$ is represented exactly at N points. A substituting leads to the equations

$$(10) \quad \sum a_N \phi_N(P_k) = f(P_k), \quad k = 1, 2, \dots, N.$$

If the determinant Δ of coefficients of a_n does not vanish,

$$\Delta = \begin{vmatrix} \phi_1(P_1) & \cdots & \phi_1(P_N) \\ \vdots & \cdots & \vdots \\ \phi_N(P_1) & \cdots & \phi_N(P_N) \end{vmatrix} \neq 0,$$

the constants a_n can be uniquely determined.

(D) *Orthogonality method:* We choose a set of N linearly independent functions $g_k(x_j)$ and let

$$(11) \quad L_k(\varepsilon_N) = \int_{\mathcal{R}} \varepsilon_N g_k dv, \quad k = 1, 2, \dots, N.$$

If we choose $g_k = W \phi_k$ then we get the conditions

$$(12) \quad \int_{\mathcal{R}} \varepsilon_N W \phi_k dv = 0$$

identical with the least squares method.

(E) *Subregion method:* As a generalization of the collocation method, we subdivide the region $\mathcal{R}(x_j)$ into N subregions $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_N$ and define

$$(13) \quad L_k(\epsilon) = \int_{\mathcal{R}_k} \epsilon dv, \quad k = 1, \dots, N.$$

then follow the steps of the collocation method.

11.7. APPROXIMATE SOLUTION OF DIFFERENTIAL EQUATIONS

If we try to approximate the solution $u(x_j)$ of a differential equation defined over a domain $\mathcal{R}(x_1, \dots, x_N)$ with certain boundary conditions by an expression

$$(1) \quad Q_N(x_j) \equiv \sum_{n=1}^N a_n \phi_n(x_j),$$

we may take one of the following approaches: (A) each $\phi_n(x_j)$ satisfies the boundary conditions, (B) each $\phi_n(x_j)$ satisfies the differential equation. In either case an error function can be defined, and it is required to be “as small as possible.”

For example, consider the case (A). Let the differential equation be

$$(2) \quad \mathcal{L}\{u(x_j)\} = 0.$$

A substitution of Eq. (1) into Eq. (2) gives the error

$$(3) \quad \epsilon_N(x_j) \equiv \mathcal{L}\left\{\sum_{n=1}^N a_n \phi_n(x_j)\right\},$$

which must be minimized. The definition of the “best approximation” can be varied, but all methods of the previous section can be used here.

A possible generalization of Eq. (1) to the form

$$Q_N(x_j) = w(x_1, \dots, x_N; a_1, \dots, a_N),$$

which depends on the N parameters a_n , seems evident, although a mathematical theory of such a generalization is by no means simple.

11.8. DIRECT METHODS OF VARIATIONAL CALCULUS

Ideas similar to those presented in the preceding sections can be applied to variational problems. Let \mathcal{D} be the class of admissible functions $u(x_j)$ of n variables x_1, x_2, \dots, x_n , or x_j for short, defined over a domain $\mathcal{R}(x)$ and satisfying all the “rigid” boundary conditions (cf. Sec. 10.8). Let $y(x_j)$, belonging to \mathcal{D} , minimize the functional $I[u]$. The following approximate representations of $y(x_j)$ are suggested:

Method of Finite Differences. We choose a set of points P_k , $k = 1, 2, \dots, N$, in \mathcal{R} and let

$$(1) \quad y(P_k) = y_k, \quad P_k \text{ in } \mathcal{R}, k = 1, 2, \dots, N.$$

The derivatives of $y(x_j)$ are then replaced by expressions involving successive finite differences of y_1, y_2, \dots, y_N , according to the calculus of finite differences, and the integrations are replaced by a finite summation. Then the functional $I[y]$ becomes a function $\Psi(y_1, y_2, \dots, y_N)$. We choose the values y_1, \dots, y_N so that $\Psi(y_j)$ has an extremum; i.e.,

$$(2) \quad \frac{\partial \Psi}{\partial y_k} = 0, \quad k = 1, 2, \dots, N$$

This method was used by Euler.

Rayleigh–Ritz Method. We set

$$(3) \quad y_N(x_j) \equiv \sum_{n=1}^N a_n \phi_n(x_j),$$

where $\phi_n(x_j)$ are known functions which are so chosen that $y_N(x_j)$ are admissible, i.e., they belong to the class \mathcal{D} . When Eq. (3) is substituted into $I[u]$, the latter becomes a function $\Psi(a_j)$ of the coefficients a_j . These coefficients a_j are then so chosen that $\Psi(a_j)$ has an extremum; i.e.,

$$(4) \quad \partial\Psi/\partial a_j = 0, \quad j = 1, 2, \dots, N.$$

Kantorovich's Method. We choose a set of coordinate functions

$$\phi_1(x_1, x_2, \dots, x_n), \phi_2(x_1, x_2, \dots, x_n), \dots, \phi_N(x_1, x_2, \dots, x_n)$$

and try an approximate solution in the form

$$(5) \quad y_N \equiv \sum_{k=1}^N a_k(x_1) \phi_k(x_1, x_2, \dots, x_n),$$

where the coefficients $a_k(x_1)$ are no longer constants, but are unknown functions of one of the independent variables. The functional $I[u]$ is reduced to a new functional $\Psi[a_1(x_1), a_2(x_1), \dots, a_N(x_1)]$, which depends on N functions of one independent variable,

$$(6) \quad a_1(x_1), a_2(x_1), \dots, a_N(x_1).$$

The functions (6) must be chosen to minimize the functional Ψ (Sec. 10.3).

It should be recognized that Kantorovich's method is a generalization of a method commonly used in the classical treatment of small oscillations of an elastic body, in which the displacements are represented in the form

$$(7) \quad u(x, t) = \sum_{k=1}^N q_k(t) \phi_k(x),$$

where $\phi_k(x)$, depending solely on the spatial coordinates, are the bases of the generalized coordinates. The Euler equations of the energy integral, according to Hamilton's principle, are the Lagrangian equations of motion.

Galerkin's Method. Galerkin's idea of minimization of errors by orthogonalizing with respect to a set of given functions is best illustrated by an example. Consider the simple beam discussed in Sec. 10.8. The minimization of the functional V , Eq. (10.8.3), leads to the variational equation

$$(8) \quad 0 = \delta V = \int_0^l \left[\frac{d^2}{dx^2} \left(EI \frac{d^2y}{dx^2} \right) - p \right] \delta y dx + \left[EI \frac{d^2y}{dx^2} - M \right] \delta \left(\frac{dy}{dx} \right) \Big|_0^l - \left[\frac{d}{dx} \left(EI \frac{d^2y}{dx^2} \right) - Q \right] \delta y \Big|_0^l.$$

Here EI , and p are known functions of x ; M , Q are unspecified if an end is "rigid" [where $\delta y = \delta(dy/dx) = 0$]; or are given numbers if an end is "natural" [where δy and $\delta(dy/dx)$ are arbitrary].

Let us consider a clamped-clamped beam (both ends "rigid"). If we assume

$$(9) \quad y_N = \sum_{k=1}^N a_k \phi_k(x),$$

where

$$(10) \quad \phi_k = d\phi_k/dx = 0 \quad \text{at} \quad x = 0 \text{ and } l, \quad k = 1, 2, \dots, N,$$

then, in general, Eq. (8) cannot be satisfied. The quantity

$$(11) \quad d^2[EI(d^2y_n/dx^2)]/dx^2 - p = \epsilon_n(x)$$

may be called an *error*. Now we demand that the coefficients a_k be so chosen that Eq. (8) be satisfied when δy is identified with any of the functions $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$:

$$(12) \quad \int_0^l \epsilon_n(x) \phi_k(x) dx = 0, \quad k = 1, \dots, N.$$

In other words, we demand that the error be orthogonal to $\phi_k(x)$.

If some of the end conditions are “natural,” we still represent the approximate solution in the form (9) and require $y_n(x)$ to satisfy only the “rigid” boundary conditions wherever they apply. In this case, Eq. (8) leads to the following in place of Eq. (12).

$$(13) \int_0^l \epsilon_n(x)\phi_k(x)dx + [EI\frac{d^2y_n}{dx^2} - M]\frac{d\phi_k}{dx}\Big|_0^l - \left[\frac{d}{dx}(EI\frac{d^2y_n}{dx^2} - Q)\right]\phi_k\Big|_0^l = 0,$$

where $k = 1, 2, \dots, N$. Equations (12) or (13) are sets of N linear equations to solve for the N unknowns constants a_1, \dots, a_N .

Trefftz’s Method. Consider again Eq. (8). In the Trefftz method, the approximate solution (9) is so chosen that $\phi_k(x)$ satisfies the Euler equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2\phi_k}{dx^2}\right) - p = 0$$

but not necessarily the boundary conditions. We now select the coefficients a_k in such a way that Eq. (8) is satisfied if the variations δy were limited to the set of functions ϕ_k :

$$\left(EI\frac{d^2y_n}{dx^2} - M\right)\frac{d\phi_k}{dx}\Big|_0^l - \left[\frac{d}{dx}\left(EI\frac{d^2y_n}{dx^2}\right) - Q\right]\phi_k\Big|_0^l = 0.$$

P R O B L E M S

11.5. Saint-Venant’s problem of torsion of a shaft with a cross section occupying a region R on the x, y plane bounded by a curve (or curves) C , is discussed in Sec. 7.5. Show that the Saint-Venant’s theory is equivalent to finding a function $\psi(x, y)$ (Prandtl’s stress function, p. 167) which minimizes the functional

$$I = \iint_R \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 - 4G\alpha\psi \right] dx dy,$$

under the boundary condition that $d\psi/ds = 0$ on C . If the region R is simply connected, we may take $\psi = 0$ on C . If R is multiply connected, we take ψ equal to a different constant on each boundary. The torque is then given by Eq. (7.5:24).

Consider a shaft of rectangular cross section. Obtain approximate solutions to the torsion problem by two different direct methods outlined in Sec. 11.8. Compare the values of the torsional rigidity of the shaft obtained in each method. (Sokolnikoff,^{1,2} 2nd edition, Chapter 7, pp. 400, 416, 423, 430, 437, 442.)

11.6. Consider the pin-ended column described in Problem 10.11. Assume that the column and the spring are massless but that a lumped mass M is situated on the column at the point A . At time $t = 0$, the column is plucked so that the deflection at A is $\Delta_0 \neq 0$. By Hamilton’s principle, or otherwise, find the equation of motion of the mass. Integrate the equation of motion to determine the motion of the mass in the case in which the lateral spring is not there ($K = 0, \alpha = 0$).

11.7. Let the load P of Problems 10.11 and 11.6 be an oscillatory load,

$$P = P_0 + P_1 \cos \omega t.$$

Could the column become unstable (i.e., with the amplitude of motion unbounded as $t \rightarrow \infty$)? In particular, if $P_0 = P_{cr}$, the buckling load, is there a chance that by properly selecting P_1 and ω , the column remains stable? Discuss the case of no lateral spring ($K = 0, \alpha = 0$) in greater detail.

11.8. Consider longitudinal wave propagation in a slender rod which in various segments is made of different materials. It is desired to transmit the wave through the rod with as little distortion as possible. How should the material constants be matched? (This is a practical problem which often occurs in instrumentation, such as piezoelectric sensing, etc.) Hint: Consider transmission of harmonic waves without reflection at the interfaces between segments. If quantities pertaining to the n^{th} segment are indicated by a subscript n , show that we must have $\rho_n c_n = \text{const.}$ for all n or, in another form, $E_n \rho_n = \text{const.}$

11.9. A traveling load moves at a constant speed over a beam on an elastic foundation. Determine the steady-state elastic responses of the beam.

11.10. Consider the longitudinal vibrations of a slender elastic rod of variable cross-sectional area $A(x)$. Determine the speed of propagation of longitudinal waves, assuming that the rod is so slender that the variation of the elastic

displacement over the cross section is negligible. If one end of the beam is subjected to a harmonic longitudinal oscillation with the boundary condition $(u)x=0 = ae^{i\omega t}$, whereas the other end is free, determine the beam response for the following cases:

- (a) $A(x) = \text{const.};$
- (b) $A(x) = A_0x^2, (0 < a \leq x \leq b);$
- (c) $A(x) = A_0e^{2x/L}, (0 \leq x \leq L).$

¹See Problem 11.2, p. 329.

²We follow Collatz^{11,3} (1959) in the subsequent exposition.

12

THERMODYNAMICS AND THERMOELASTICITY

In this chapter we shall consider the fundamental principles of thermodynamics and illustrate their applications by examples.

12.1. LAWS OF THERMODYNAMICS

In classical thermodynamics, a collection of matter to be studied is called a *system*. Only *closed systems*, which do not exchange matter with their surroundings, will be considered in this chapter. Occasionally, we make further restriction that no interactions between the system and its surroundings occur. The system is said to be *isolated*.

When all information required to completely characterize the system is available for the purpose at hand, it is said that the *state of the system* is known. For example, for a homogeneous elastic body at rest, a complete description of its thermodynamic state requires the specification of its material content, i.e., the quantity of each chemical substance contained; its geometry in the natural or unstrained state; its deviation from the natural state, or its strain and stress fields; and, if some physical properties depend on temperature, one extra independent quantity which fixes the body's hotness or coldness. These quantities are called *state variables*. If a certain state variable can be expressed as a single-valued function of a set of other state variables, then the functional relationship is said to be an *equation of state*, and the variable so described is called a *state function*. The selection of a particular set of independent state variables is important for each problem, but the choice is to a certain extent arbitrary.

If the values of the state variables of a system are independent of time, the system is said to be in *thermodynamic equilibrium*. If the state variables vary with time, then the system is said to undergo a *process*. The number of state variables required to describe a process may be larger than that required to describe the system at thermodynamic equilibrium. For example, in describing the flow of a fluid we may need to know the viscosity.

A system is said to be *thermally insulated* if it is surrounded by an insulating wall that no change can be produced in the system by an external agency except by movement of the wall or by long-range forces such as gravitation. Any process taking place in the system is called *adiabatic*.

A system is said to be *homogeneous* if the state variables does not depend on space coordinates. The classical thermodynamics concerns with the conditions

of equilibrium within a system composed of homogeneous parts ("phases"). Nonhomogeneous systems requires additional hypotheses, which will be considered in [Sec. 12.4](#).

The first step in the formulation of thermodynamics is to introduce the concept of temperature. It is postulated that *if two systems are each in thermal equilibrium with a third system, they are in thermal equilibrium with each other*.

It can be shown then the condition of thermal equilibrium between several systems is a certain single-valued function of the thermodynamic states of the systems, which may be called the temperature T: any one of the systems being used as a "thermometer" reading the temperature T on a suitable scale. The temperature whose existence is thus postulated is measured on a scale by an arbitrary choice of the thermometer called the *empirical temperature*.

The first law of thermodynamics can be formulated as follows. If a thermally insulated system can be taken from a state I to a state II by alternative paths, the work done on the system has the same value for every such (adiabatic) path. From this we can deduce that there exists a single-valued function of the state of a system, called its energy, such that for an adiabatic process the increase of the energy is equal to the work done on the system. Thus,

$$(1) \quad \Delta \text{ energy} = \text{work done} \quad (\text{adiabatic process}).$$

Note that for this definition of energy it is necessary and sufficient that it be possible by an adiabatic process to change the system either from state I to II or from state II to I.

We now define the heat Q absorbed by a system as the increase in energy of the system, less the work done on the system. Thus,

$$(2) \quad Q = \Delta \text{ energy} - \text{work done} \quad (\text{all processes}),$$

$$(3) \quad \Delta \text{ energy} = Q + \text{work done}.$$

If this is regarded as a statement of the conservation of energy and is compared with (1), we observe that the energy of a system can be increased either by work done on it or by absorption of heat.

It is customary to identify several types of energy which make up the total, e.g., the kinetic energy E^k and the internal energy E^i . Thus,

$$(4) \quad \text{energy} = E^k + E^i.$$

We now formulate the *second law of thermodynamics* for a homogeneous system as follows. There are two single-valued functions of state, T the absolute temperature, and S the entropy, with the following properties:

- I. T is positive and is a function of the empirical temperature only.
- II. The entropy of a system is equal to the sum of entropies of its parts.
- III. The entropy of a system can change in two distinct ways, namely, by interaction with the surroundings and by changes taking place inside the system. Symbolically, we may write

$$(5) \quad dS = d_e S + d_i S,$$

where dS denotes the increase of entropy of the system, $d_e S$ is the part of increase due to interaction with the surroundings, and $d_i S$ the part due to changes taking place inside the system. Then, if dQ denotes the heat absorbed by the system from its surroundings, we have

$$(6) \quad d_e S = dQ/T.$$

The change $d_i S$ is never negative. If $d_i S$ is zero the process is said to be reversible. If $d_i S$ is positive the process is said to be irreversible:

$$(7) \quad d_i S > 0 \text{ (irreversible process), and } d_i S = 0 \text{ (reversible process).}$$

The remaining case, $d_i S < 0$, never occurs in nature.

The absolute temperature T and the entropy S are two fundamental quantities. We will not attempt to define them in terms of other quantities regarded as simpler. They are defined merely by their properties expressed in the second law. Lord Kelvin has shown how it is possible to calibrate any thermometer into absolute temperature scale. (See Problem 12.3.) The scale of the absolute temperature is fixed by defining the temperature at equilibrium between liquid water and ice at a pressure of 1 atm at 273.16 K, i.e., 273.16 degrees on Kelvin's scale.

These postulates form the basis of the classical thermodynamics. The justification is the empirical fact that all conclusions derived from these assumptions are in agreement with the experimentally observed behavior of systems in nature at the macroscopic scale. The form in which these principles are enunciated above is essentially that used by Born (1921) and Prigogine (1947).

It is important to familiarize ourselves with the concept that entropy is an attribute to a material body, just as its mass or its electric charges are. A pound of oxygen has a definite amount of entropy, which can be changed by changing the temperature and specific volume of the gas. A pound of steel at a given temperature and in a given state of strain has a definite amount of entropy. It is instructive to consider Problems 12.5 and 12.6 to obtain some intuitive feeling about entropy.

PROBLEMS

12.1. Calculate the work performed by 10g of hydrogen expanding isothermally at 20°C from 1 to 0.5 atm of pressure.
Note: The equation of state for a system composed of m grams of a gas whose molecular weight is M is given approximately by $pv = (m/M)RT$, where R is a universal constant (same for all gases); $R = 8.314 \times 10^7 \text{ ergs/degree}$, or 1.986 cal/degree .

12.2. Calculate the energy variation of a system which performs 8×10^8 ergs of work and absorbs 60 cal of heat.
(Note: 1 cal = 4.185×10^7 ergs.)

12.3. To calibrate an empirical temperature scale against the absolute temperature scale, let us consider a heat engine. A working substance is assumed which undergoes a cycle of operations involving changes of temperature, volume, and pressure and which is eventually brought back to its original condition. In a Carnot cycle, an ideal gas is

first brought to contact with a large heat source which is maintained at a constant empirical temperature T_1 . The gas at the equilibrium temperature T_1 is allowed to expand (isothermally) and absorbs an amount of heat Q_1 , from the heat source. Then it is removed and insulated from the heat source and is allowed to expand adiabatically to perform work, lowering its empirical temperature to a value T_2 . Next it is put in contact with a heat sink of temperature T_2 and is compressed isothermally to release an amount of heat Q_2 . Finally, it is recompressed adiabatically back to its original state. If every process in the Carnot cycle is reversible, the engine is said to be reversible, otherwise it is irreversible.

From the second law of thermodynamics prove that:

- (a) When the temperatures of the source and sink are given, a reversible engine is the most efficient type of engine, i.e., it yields the maximum amount of mechanical work for a given abstraction of heat from the source. Note: Efficiency = $(Q_1 - Q_2)/Q_1$.
- (b) The efficiency of a reversible engine does not depend on its construction or material, but only on the temperatures of the source and sink.
- (c) In a reversible Carnot engine, we must have

$$Q_2/Q_1 = f(T_1, T_2),$$

a universal function of the two temperatures T_1 and T_2 . Show that

$$f(T_1, T_2) = f(T_0, T_2)/f(T_0, T_1)$$

where T_0 , T_1 , and T_2 are arbitrary temperatures. If T_0 is kept constant, we may consider $f(T_0, T)$ as a function of the temperature T only. Hence, we may write

$$Kf(T_0, T) = \theta(T)$$

where K is a constant. Thus,

$$Q_1/\theta(T_1) = Q_2/\theta(T_2)$$

Kelvin (1848) perceived that $\theta(T_1)$ and $\theta(T_2)$ may be taken to define an absolute temperature scale. Identify this with the T introduced above.

- (d) The theory of the Joule–Thomson experiment is used in practice to calibrate a gas thermometer to absolute temperature scale.

12.4. By considering an ideal gas as the working fluid of a Carnot cycle, show that the gas thermometer reads exactly the absolute temperature.

12.5. Derive the following formula for the entropy of one mole of an ideal gas.

$$\mathcal{S} = C_v \log T + R \log V + a$$

where C_v is the specific heat at constant volume, R is the universal gas constant, and a is a constant of integration.

12.6. Prove the following analogy. The work that one can derive from a waterfall is proportional to the product of the height of the waterfall and the quantity of the water flow. The work that can be derived from a heat engine performing a Carnot cycle is proportional to the product of the temperature difference between the heat source and sink, and the quantity of entropy flow.

12.2. ENERGY EQUATION

We shall now express the equation of balance of energy, Eqs. (12.1:3) and (12.1:4), in a form more convenient for continuum mechanics in Cartesian coordinates. This section is not limited to homogeneous systems.

For a body of particles in a configuration occupying a region V bounded by a surface S , the *kinetic energy* is

$$(1) \quad E^k = \int_V \rho \hat{k} dv = \int_V \rho \mathbf{v} \cdot \mathbf{v} dv / 2,$$

where \hat{k} is the kinetic energy per unit mass, \mathbf{v} the velocity vector of a particle occupying an element of volume dv , and ρ the material density.

Let \hat{e} be the internal energy per unit mass, the *internal energy* is then

$$(2) \quad E^i = \int_V \rho \hat{e} dv .$$

The heat input into the body must be imparted through the boundary if there is no heat source in the body. A new vector, the *heat flux* $\mathbf{h} [= (h_1, h_2, h_3)]$ is defined as follows. Let $d\mathbf{a}$ be a surface element in the body with unit outer normal \mathbf{v} . Then the rate at which heat is transmitted across the surface $d\mathbf{a}$ in the direction of \mathbf{v} is assumed to be representable as $h_i v_i d\mathbf{a}$. To be specific, we insist on defining the heat flux in the case of a moving medium that the surface element be composed of the same particles. The *rate of heat input* is therefore

$$(3) \quad \dot{Q} = - \int_{\partial V} \nu \cdot \mathbf{h} d\mathbf{a} = - \int_V \nabla \cdot \mathbf{h} dv .$$

The rate at which work is done on the body by the external forces is

$$(4) \quad P = \int_{\partial V} \tilde{\mathbf{T}} \cdot \mathbf{v} d\mathbf{a} + \int_V \rho \mathbf{b} \cdot \mathbf{v} dv ,$$

where $\tilde{\mathbf{T}}$ is the surface traction and \mathbf{b} is the body force per unit mass.

The first law states that

$$(5) \quad \dot{E}^k + \dot{E}^i = \dot{Q} + P ,$$

where the dot denotes the material derivative D/Dt .

Formula for computing the material derivative of an integral has been given in Eq. (5.3:4). Since the volume is arbitrary, Eq. (5) leads to the local energy balance equation with the use of Cauchy's formula (3.3:2). Thus,

$$(6) \quad \rho \frac{D}{Dt} (\hat{e} + \frac{v_i v_i}{2}) + (\hat{e} + \frac{v_i v_i}{2}) (\frac{D\rho}{Dt} + \rho v_{i,i}) = -h_{i,i} + (\tau_{ij} v_j)_{,i} + \rho b_i v_i .$$

On substituting the equation of continuity and the equations of motion

$$D\rho/Dt + \rho \nabla \cdot \mathbf{v} = 0, \quad \rho D\mathbf{v}/Dt = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{b} ,$$

respectively, into Eq. (6), we obtain

$$(7) \quad \rho D\hat{e}/Dt = -\nabla \cdot \mathbf{h} + \boldsymbol{\tau} : [\nabla \mathbf{v} + (\nabla \mathbf{v})^T]/2 = -\nabla \cdot \mathbf{h} + \boldsymbol{\tau} : \mathbf{D} ,$$

where \mathbf{D} is the symmetric part of the velocity gradient $\nabla \mathbf{v}$, called the rate of deformation tensor, and $\boldsymbol{\tau} : \mathbf{D}$ is the inner product of $\boldsymbol{\tau}$ and \mathbf{D} . The antisymmetric part of $\nabla \mathbf{v}$ has no contributions since $\boldsymbol{\tau}$ is symmetric [Eq. (3.4:3)].

In classical thermodynamics we are concerned with the small neighborhood of thermodynamic equilibrium. It suffices to consider infinitesimal strain, which is imposed very slowly. In this case we may write Eq. (7) in a form commonly used in thermodynamics books:

$$(8) \quad \rho d\hat{e} = dQ + \sigma_{ij} de_{ij} .$$

Equation (7) gives precise meaning to Eq. (8). If there is no internal entropy production in the process, $d_i S = 0$, then the second law gives $dQ = T \rho d\hat{s}$, where \hat{s} is the specific entropy, or the entropy per unit mass. Hence,

$$(9) \quad \Delta \quad d\hat{e} = T d\hat{s} + \sigma_{ij} de_{ij} / \rho .$$

Problem 12.7. In the case of a fluid under hydrostatic pressure, so that

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \quad \sigma_{12} = \sigma_{23} = \sigma_{31} = 0 ,$$

show that

$$(10) \quad d\hat{e} = T d\hat{s} - p d\mathcal{V} ,$$

where $\mathcal{V} = 1/\rho$ is the specific volume, i.e., the volume per unit mass of the fluid.

12.3. STABILITY CONDITIONS OF THERMODYNAMIC SYSTEMS

The second law of thermodynamics stating that $d_i S < 0$ never occurs in nature connects the theory with the direction of motion of events in the world. The law endows the arrow to time of the proverb “time is an arrow.”

Let us compare a system in equilibrium with systems whose state variables are slightly different from the equilibrium state. All systems are subjected to the same boundary conditions. We use Δ to denote the variations including the first, second, and higher-order infinitesimal terms. From the second law the stability condition prohibits the system from leaving the current state spontaneously, if

$$(1) \quad d_i S < 0.$$

Thereby, the system is said to be in *stable equilibrium*, i.e., a neighboring disturbed state tends to return to the equilibrium state.

We shall derive the entropy maximum or energy minimum principle for stable equilibrium. For an adiabatic process, no heat is transferred between a system and its surroundings, i.e., $d_e S = 0$. The stability criterion becomes

$$(2) \quad \Delta S < 0.$$

In other words, the entropy S of an adiabatically insulated system in stable equilibrium is the maximum with respect to all the neighboring states which have the same energy (*concavity* of the entropy surface), i.e.,

$$(3) \quad S = \max \quad (E^i = \text{const.}).$$

This is the celebrated Gibbs condition of thermodynamic equilibrium. Gibbs gave an alternative, more directly applicable to continuum mechanics, statement: *for the equilibrium of any isolated system it is necessary and sufficient that in all possible variations in the state of the system which do not alter its entropy, the variation of its energy shall either vanish or be positive*. Thus the second law indicates a trend in nature with the end point being thermodynamic equilibrium, a state in which no state variable varies with time ([Sec. 12.1](#)).

Based on the first law, we can rewrite the condition (1) as

$$(4) \quad \Delta E^i - T \Delta S > 0.$$

For an isentropic process, the entropy in a system remains constant, that is, $\Delta S = 0$. The stability criterion becomes

$$(5) \quad (\Delta E^i)_S > 0.$$

In other words, the internal energy E^i of a system in stable equilibrium is the minimum with respect to all the neighboring states which have the same entropy (*convexity* of the internal energy surface), that is,

$$(6) \quad E^i = \min \quad (S = \text{const.}).$$

For an isothermal process, the temperature remains constant, i.e., $\Delta T = 0$. With the introduction of the Helmholtz free energy of the system

$$(7) \quad H = E^i - TS,$$

which is the Legendre transformation of the internal energy with respect to entropy, the stability criterion becomes

$$(8) \quad (\Delta H)_T > 0.$$

In other words, the Helmholtz free energy H of a system in stable equilibrium is the minimum with respect to all the neighboring states which have the same temperature (*convexity* of the Helmholtz free energy surface),

$$(9) \quad H = \min \quad (T = \text{const.}).$$

Since the zero-stress state of a solid body is a stable equilibrium state, it follows from Eq. (6) or (9) that $\Delta E^i = E^i - E_0^i$ under isentropic conditions or $\Delta H = H - H_0$ under isothermal conditions is positive definite, and vanishes only in the trivial case of the zero-stress state.

The strain energy function can be identified as the internal energy difference under isentropic conditions or Helmholtz free energy difference under isothermal conditions. Therefore, the strain energy function is positive definite in the neighborhood of the zero-stress state, from which the following important theorems are deduced:

1. The uniqueness of the solution in elastostatics and dynamics ([Sec. 7.4](#)).
2. The minimum potential energy theorem ([Sec. 10.7](#)).

3. The minimum complementary energy theorem ([Sec. 10.9](#)).
4. Saint-Venant's principle (in a certain sense) ([Sec. 10.11](#)).

12.4. IRREVERSIBLE THERMODYNAMICS

The classical thermodynamics ([Sec. 12.1](#)) deals essentially with equilibrium conditions of a uniform system (or a heterogeneous system with uniform phases). It contains more than a statement about the trend toward thermodynamic equilibrium for irreversible processes. To bring the irreversibility to a sharper focus, we like to write down the definite “equations of evolution” (or of “motion”), to describe in precise terms the way an irreversible process evolves. To make the theory useful to continuum mechanics, we extend the formulation to include nonuniform systems. These two items are the main objectives of the theory of irreversible thermodynamics. Since these objectives are beyond the scope of classical thermodynamics, new hypotheses must be introduced, whose justification can only be sought by comparing any theoretical deductions with experiments.

The *first assumption* is that the entropy is a function of state in irreversible processes as well as in reversible processes. In [Sec. 12.1](#), we considered reversible processes and showed that the value of the entropy of a system can be computed when the values of the state variables are known (except for an integration constant which, in general, is of no importance). (Near zero absolute temperature, the additive constant of entropy does become important, but in that case its value can be fixed according to Nernst's theorem — the third law of thermodynamics.) We now assume that this same function of state defines entropy even if the system is not in equilibrium. The motivation for this assumption is analogous to the mass distribution of any mechanical system: once the mass of a particle in a continuum is determined, we assume that it remains constant no matter how fast the particle moves.

Just as the constancy of mass must be subjected to relativistic restrictions, the validity of the assumption of entropy as a state function must be restricted to relatively mild processes. Prigogine investigated this question by comparing the assumption with statistical mechanics results for some particular models, such as the Chapman–Enskog theory of nonuniform gases. He shows that the domain of validity of the assumption extends throughout the domain of validity of linear phenomenological laws (Fourier's law of heat conduction, Fick's law of mass diffusion, etc.). In the case of chemical reactions, he shows that the reaction rates must be sufficiently slow so as not to perturb Maxwell–Boltzmann equilibrium distribution of velocities of each component to an appreciable extent. This means, for example, that the temperature changes over the length of one mean free path must be much smaller than the absolute temperature itself, and the like for other properties.

The *second assumption* consists of an extension of the second law of thermodynamics locally to every portion of a continuum, whether it is uniform or not, and may be explained as follows. We notice first that entropy is an extensive quantity, so it must be subjected to a conservation law: for a given set of particles occupying a domain V the total change of entropy must be equal to the total amount of entropy transferred to these particles through the boundary, plus the entropy produced inside this domain. Denoting the boundary of this domain by ∂V , we have the entropy balance equation in the global form

$$(1) \quad D\left(\int_V \rho \hat{s} dv\right)/Dt = - \int_{\partial V} \mathbf{j}_s \cdot d\mathbf{a} + D\left(\int_V \rho_i \hat{s} dv\right)/Dt,$$

where \hat{s} is the specific entropy per unit mass, $i\hat{s}$ is the entropy source (internal entropy produced per unit mass), \mathbf{j}_s is the entropy flux vector, so that $\mathbf{j}_s \cdot d\mathbf{a}$ represents the outflow. The material derivative D/Dt is taken with respect to a given set of particles. Transforming the second term into a volume integral and using Eq. (5.3:4) to reduce the material derivative of an integral, we obtain

$$\rho D\hat{s}/Dt + \hat{s}(D\rho/Dt + \rho \nabla \cdot \mathbf{v}) = -\nabla \cdot \mathbf{j}_s + \rho D_i\hat{s}/Dt + i\hat{s}(D\rho/Dt + \rho \nabla \cdot \mathbf{v}),$$

since the domain V is arbitrary. The sum in the parentheses vanishes by the equation of continuity, Eq. (5.4:4). Hence, we have the entropy balance equation in the local form

$$(2) \quad \Delta \quad \rho D\hat{s}/Dt = -\nabla \cdot \mathbf{j}_s + \rho D_i\hat{s}/Dt.$$

We can now state the second hypothesis (an extension of the second law of thermodynamics) as follows.

$$(3) \quad \Delta \quad \mathbf{j}_s = \mathbf{h}/T, \quad \mathbf{h} = \text{heat flux vector},$$

$$(4) \quad \Delta \quad D_i\hat{s}/Dt = 0, \quad \text{reversible processes},$$

$$(5) \quad \Delta \quad D_i\hat{s}/Dt > 0, \quad \text{irreversible processes},$$

$$(6) \quad D_i\hat{s}/Dt < 0, \quad \text{never occurs in nature}.$$

In Eq. (3), \mathbf{h} is the heat flux vector, the heat flow per unit area, and T is the local absolute temperature. In a moving medium, \mathbf{h} is defined convectively with respect to an element of surfaces composed of the same sets of particles.

When Eq. (3)–(5) are compared with Eqs. (12.1:6) and (12.1:7), we see that the entropy production is now required to be nonnegative everywhere in the system. Such a formulation may be called a local formulation of the second law in contrast to the global formulation of the classical thermodynamics.

Corresponding to \mathbf{j}_s we have the points of the body assigned as an *entropy displacement field* φ (Biot^{8,1} 1956) for time $t > t_0$ with $\varphi = 0$ at $t = t_0$. This is just like assigning the points as a material displacement vector \mathbf{u} . The two vector fields \mathbf{u} and φ connect the state of the body for $t > t_0$ with the natural state at t_0 . The reversible change of mass distribution, $-\operatorname{div}(\rho\mathbf{u})$, spreads the gradient tensor $u_{i,j}$, while the entropy displacement, $-\operatorname{div}(\varphi)$, spreads $\varphi_{i,j}$. Introducing the notation analogous to that of Eq. (12.1:5), we may write the rate of reversible change of entropy as

$$(7) \quad \rho D_e \hat{s} / Dt = -\nabla \cdot \mathbf{j}_s, \quad \mathbf{j}_s = \dot{\varphi}.$$

Further development of irreversible thermodynamics requires a detailed description of the entropy production $D_i \hat{s} / Dt$.

Problem 12.8. Show the entropy production per unit volume for heat transfer is

$$(8) \quad \Delta \quad \rho D_i \hat{s} / Dt = -h_{xi} (\partial T / \partial x_i) / T^2 = -(\mathbf{h} \cdot \nabla T) / T^2,$$

where $\mathbf{h} [= (hx_i)]$ is the heat flux vector per unit area across the surface.

12.5. PHENOMENOLOGICAL RELATIONS – ONSAGER PRINCIPLE

In the preceding section it was shown that in heat conduction the entropy production can be written as the product of a “generalized force” $-\nabla T / T$ and the corresponding flux \mathbf{h}/T . This is true for other irreversible processes such as diffusion and chemical reaction. In chemical reaction the “force” and “flux” are often termed affinity and reaction rate. In general, for an irreversible process, we may write,

$$(1) \quad D_i \hat{s} / Dt = \sum_k J_k X_k > 0,$$

where J_k denotes the k^{th} generalized flux and X_k the corresponding generalized force. For example, in heat conduction

$$(2) \quad J_k = h_k / T, \quad X_k = -(\partial T / \partial x_k) / T, \quad k = 1, 2, 3.$$

At thermodynamic equilibrium all processes stop and we have simultaneously for all irreversible processes

$$(3) \quad J_k = 0, \quad X_k = 0, \quad \text{at equilibrium}.$$

It is natural to assume, at least in the neighborhood of an equilibrium state, that the relations between generalized forces and generalized fluxes are linear. Fourier’s law of heat conduction is such an example. This kind of laws are called phenomenological relations. For such relations, an important symmetry among the coefficients is embodied in the so-called Onsager principle.

Let us consider heat conduction again. It is well known that, in anisotropic crystals, a symmetry exists in the heat conduction coefficient matrix, which could not be explained by crystallographic symmetry properties. It was found that when we write

$$(4) \quad h_{xi} = L_{ij} \partial T / \partial x_j,$$

in which the matrix \mathbf{L} is always symmetric, i.e., $L_{ij} = L_{ji}$ for arbitrary orientations of the coordinate axes, and irrespective of the crystallographic axes. Considerable effort has been spent to detect any deviation from symmetry. The most ingenious experiments are those of Soret (1879) and Voigt (1903) verifying the symmetry with great accuracy.

A similar symmetry property occurs in other phenomenological relations describing interferences between several simultaneous irreversible processes. For example, consider the case of one-dimensional thermo-diffusion for which two phenomenological relations may be written as

$$(5) \quad J_i = \sum_{j=1}^2 L_{ij} X_j, \quad i = 1, 2$$

where J_1 is the heat flux, J_2 the mass flux of a particular component in a mixture, X_1 the temperature gradient, and X_2

the concentration gradient of that particular component. Then L_{11} is the heat conductivity, L_{22} the diffusion coefficient, and L_{12} and L_{21} are coefficients describing the interference of the two irreversible processes of heat conduction and diffusion. The coefficient L_{21} associates with the appearance of a concentration gradient when a temperature gradient is imposed, called the *Soret effect* (discovered in 1879); and L_{12} with the inverse phenomenon, the *Dufour effect*, which describes the appearance of temperature difference when a concentration gradient exists. The first equation ($i = 1$) in Eq. (5) is a modification of Fourier's law of heat conduction to include the Dufour effect. The second equation ($i = 2$) in Eq. (5) is a modification of Fick's law of diffusion to include the Soret effect. In this case, we have $L_{12} = L_{21}$ (Mortimer and Eyring 1980, Landau and Lifshitz 1997).

Other linear phenomenological laws verified by experiments include Ohm's law between electrical current and potential gradient, Newton's law between shearing force and velocity gradient, the chemical reaction law between reaction rate and chemical affinity, etc. The reciprocal phenomena of thermoelectricity arising from the interference of heat conduction and electrical conduction include the *Peltier effect* (discovered in 1834), which refers to the evolution or absorption of heat at junctions of different metals resulting from the flow of an electric current, the *Seebeck effect* (discovered in 1822), which relates to the electromotive force developed in a circuit made up of different conducting elements, when not all of the junctions are at the same temperature, and the *Thomson effect* (discovered by Lord Kelvin in 1851), which refers to the reversible heat absorption when an electric current flows in a homogeneous conductor with temperature gradient. For further information see Disalvo (1999).

The simultaneous action of several irreversible processes may cause interferences as typified by the examples named above. As a general form of phenomenological laws, let J_k ($k = 1, \dots, m$) be the generalized fluxes (heat flow, electric current, chemical reaction rate, etc.), and X_k ($k = 1, \dots, m$) the generalized forces (temperature gradient, electric potential gradient, chemical affinity, etc.). Then a linear phenomenological law is

$$(6) \quad J_k = \sum_{l=1}^m L_{kl} X_l, \quad k = 1, 2, \dots, m.$$

To this general form, the Onsager principle states a symmetry property of the coefficients L_{kl} as follows: With a proper choice of the fluxes J_k and the forces X_l , the entropy production per unit time can be written as

$$(7) \quad D_i \hat{s}/Dt = \sum_{k=1}^m J_k X_k,$$

and if the flux and forces are related by linear phenomenological relations,

$$(8) \quad J_k = \sum_{l=1}^m L_{kl} X_l,$$

then the matrix of the coefficients \mathbf{L} is symmetric; i.e., $L_{ij} = L_{ji}$, called the Onsager reciprocal relations. Substituting Eq. (8) into Eq. (7), we obtain a quadratic form of positive definite,

$$(9) \quad D_i \hat{s}/Dt = \sum_{k=1}^m \sum_{l=1}^m L_{kl} X_l X_k > 0.$$

When all X 's are zero, the entropy production vanishes.

Onsager (1931) derived this principle from statistical mechanics considerations under the assumption of microscopic reversibility; i.e., the symmetry of all mechanical equations of motion of individual particles with respect to time. It furnishes an explanation of the observed relation (5) of the heat conduction for anisotropic crystals. The statistical derivation may be found in Onsager's papers, or in Prigogine and De Groot's books.

Casimir (1945) classified irreversible processes into two classes depending on $J_k(t)$ being an even or odd function of time. Heat flow, mass flow, and so on are odd functions of time. From the point of view of the kinetic theory of matter, \mathbf{J} is even or odd depending on whether \mathbf{J} does or does not change sign when particle velocity reverses. So far, we have considered cases in which generalized flows are so chosen that they are all odd. In the more general case, Onsager's principle should be generalized to state that if the irreversible processes i and k are both even or both odd, then

$$(10) \quad L_{ik} = L_{ki}.$$

If one is even and the other is odd, then

$$(11) \quad L_{ik} = -L_{ki}.$$

A further modification is necessary if there exist forces depending explicitly on the velocity, such as Lorentz forces, Coriolis forces, and so on. For example, in the presence of a magnetic field, \mathbf{H} , two odd or two even processes are related in the *Onsager–Casimir theorem*, by:

$$(12) \quad L_{ik}(\mathbf{H}) = L_{ki}(-\mathbf{H}).$$

For simultaneous action of several irreversible processes a general phenomenological law may be written in the form discussed in this section. The Onsager–Casimir theorem simplifies these laws. A further simplification is that of certain symmetry requirements may dictate whether certain interference coefficients should vanish or not. This is *Curie's symmetry principle* (P. Curie, Oeuvres, Société Française de Physique 1908), stating that macroscopic causes always have fewer elements of symmetry than the effects they produce. An interference or the irreversible processes is possible only if this general principle is satisfied. For example, chemical affinity (a scalar) cannot produce a directed heat flow (a vector) and the interference coefficient between the heat flow and chemical affinity must vanish.

12.6. BASIC EQUATIONS OF THERMOECHANICS

Thermomechanical systems are subjected to the same general conservation laws with regard to mass and momentum as discussed in Chapter 5. However, the law of conservation of energy contains both mechanical and thermal energies. The change of thermal energy is related to the change of entropy. Thus, a complete description of the evolution of a system requires the knowledge of entropy production. Phenomenological relations, together with the Onsager principle, provide sufficient details about the entropy production. These laws, taken together, determine the evolution of the system.

To demonstrate the procedure, we consider a solid body occupying a domain V with boundary ∂V in Cartesian coordinates. Let \mathbf{x} denote the instantaneous position of a particle. We write the equations in the instantaneous configuration, i.e., in Eulerian coordinates. The conservation of mass is the equation of continuity (5.4:3)

$$(1) \quad D\rho/Dt + \rho\nabla \cdot \mathbf{v} = 0.$$

The conservation of momentum [Eulerian equation of motion (5.5:7)], Cauchy's formula (3.3:2), and the conservation of moment of momentum [the symmetry of the stress tensor (3.4:3)] are

$$(2) \quad \rho\dot{\mathbf{v}} = \nabla \cdot \boldsymbol{\tau} + \rho\mathbf{b} \text{ in } V, \quad \overset{\nu}{\mathbf{T}} = \boldsymbol{\nu} \cdot \boldsymbol{\tau} \text{ on } \partial V, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^T \text{ in } V + \partial V.$$

The conservation of energy is given by Eq. (12.2:7)

$$(3) \quad \rho\dot{\hat{e}} = -\nabla \cdot \mathbf{h} + \boldsymbol{\tau} : [\nabla \mathbf{v} + (\nabla \mathbf{v})^T]/2 = -\nabla \cdot \mathbf{h} + \boldsymbol{\tau} : \mathbf{D}.$$

In these equations, ρ is the mass density and

$$(4) \quad (\cdot) = D(\cdot)/Dt = \partial(\cdot)/\partial t + v_j \partial(\cdot)/\partial x_j,$$

denotes the material derivative, \mathbf{b} is the body force per unit mass, $\overset{\nu}{\mathbf{T}}$ the surface traction per unit area, \hat{e} the internal energy per unit mass, $\boldsymbol{\tau}$ the stress tensor, \mathbf{h} the heat flux vector, and the indices i, j range over 1, 2, 3.

On introducing the Helmholtz free energy per unit mass

$$(5) \quad \hat{h} = \hat{e} - T\hat{s},$$

from Eq. (12.2:7), we obtain the rate of change of entropy

$$(6) \quad \rho\dot{\hat{s}} = -\nabla \cdot (\mathbf{h}/T) + \mathbf{h} \cdot \nabla(1/T) + \boldsymbol{\tau} : \mathbf{D}/T - \rho(\dot{\hat{h}} + \dot{\hat{s}}\dot{T})/T.$$

Comparison with the entropy balance equation (12.4:2) yields

$$(7) \quad \rho_i\dot{\hat{s}} = -\nabla \cdot (\mathbf{h}/T - \mathbf{j}_s) + \mathbf{h} \cdot \nabla(1/T) + (\boldsymbol{\tau} : \mathbf{D} - \rho\dot{\hat{h}} - \rho\dot{\hat{s}}\dot{T})/T.$$

From (12.4:4–5), we have the entropy production inequality

$$(8) \quad i\dot{\hat{s}} = 0 \text{ for reversible processes,} \quad i\dot{\hat{s}} > 0 \text{ for irreversible processes.}$$

This is all we can deduce from the general consideration of mass, momentum, moment of momentum, energy and entropy. Equations (1)–(8), taken together, still do not define the displacement and temperature fields uniquely. To complete the formulation, we must add a constitutive law in order to have sufficient differential equations for the boundary value problem.

12.7. ENERGY DENSITY FUNCTION AND DISSIPATION POTENTIAL FOR HYPER-THERMOELASTICITY

The concept of simple materials was originally proposed to describe the mechanical behavior (Noll 1958, 1972) and then generalized to the thermomechanical behavior (Truesdell and Noll 2004). As a special class of thermomechanically simple materials, hyper-thermoelasticity is a type of constitutive model for which the responses can be derived from an energy density function and a dissipation potential, which depend on up to the first-order gradients of displacement and temperature fields.

In place of the first assumption in Sec. 12.4, the Helmholtz free energy per unit mass is assumed to be a function of deformation, temperature and temperature gradient. For infinitesimal thermoelasticity,

$$(1) \quad \hat{h} = \hat{h}(e_{ij}, \theta, \theta_{,i}), \quad e_{ij} = (u_{i,j} + u_{j,i})/2, \quad \theta = T - T_0,$$

where e_{ij} are the strains, θ the temperature difference, and T_0 the absolute temperature at the reference state.

Since the entropy production inequality (12.6:8) should be always valid, it is necessary and sufficient that the state equations fulfill the following conditions of thermodynamic admissibility:

$$(2) \quad \begin{aligned} \partial \hat{h} / \partial \theta_{,i} &= 0, \quad \tau_{ij} = \rho \partial \hat{h} / \partial e_{ij}, \quad \dot{s} = -\partial \hat{h} / \partial \theta, \quad \mathbf{j}_s = \mathbf{h}/T, \\ i \dot{\hat{s}} &= [\mathbf{h} \cdot \nabla(1/T)]/\rho = \mathbf{h} \cdot [-(\nabla T)/T]/(\rho T) \geq 0. \end{aligned}$$

From equations above, the Helmholtz free energy \hat{h} is independent of the temperature gradient, and so are the derived variables like stress and entropy; the entropy flux \mathbf{j}_s equals the heat flux divided by the absolute temperature; and the entropy production $i \dot{\hat{s}}$ is the inner product of the thermodynamic flux \mathbf{h}/T and the thermodynamic force $-\nabla T/T$. To satisfy the entropy production inequality, the transport law of heat conduction may be derived from the Fourier dissipation potential, a quadratic scalar function of the thermodynamic force $(-\nabla T/T)$, i.e.,

$$(3) \quad \mathcal{D}_{th} = k_{ij}(-T_{,i}/T)(-T_{,j}/T)/2,$$

where $k_{ij} = k_{ji}$ is the thermal conductivity coefficient matrix.

Hence, the thermodynamic flux for heat conduction depends linearly on the corresponding thermodynamic force with satisfaction of the Onsager principle, which is consistent with the discussion in Sec. 12.5, that is,

$$(4) \quad h_i/T = \partial \mathcal{D}_{th} / \partial (-T_{,i}/T) = k_{ij}(-T_{,j}/T).$$

Substituting Eq. (4) into Eqs. (12.6:7–8) yields

$$(5) \quad \rho_i \dot{\hat{s}} = \mathbf{h} \cdot \nabla(1/T) = -\mathbf{h} \cdot \nabla T/T^2 = k_{ij}(-T_{,i})(-T_{,j})/T^2 = 2\mathcal{D}_{th} \geq 0.$$

As discussed in Sec. 12.5, in order that the quadratic form (5) be positive definite, k_{ij} must be so. A substitution of entropy \dot{s} from Eq. (2) into Eq. (12.6:6) yields the heat transfer equation in terms of the Helmholtz free energy:

$$(6) \quad \rho \partial(-\partial \hat{h} / \partial \theta) / \partial t = -(\partial h_i / \partial x_i)/T.$$

In the infinitesimal theory, we have $(\cdot)' \approx \partial(\cdot)/\partial t$.

On the other hand, the Gibbs free energy may be introduced alternatively through the Legendre transformation

$$(7) \quad \rho \hat{g} = \rho \hat{h} - \tau_{ij} e_{ij}.$$

The Gibbs free energy per unit mass is assumed to be a function of stress, temperature, and temperature gradient.

For infinitesimal thermoelasticity, we have

$$(8) \quad \hat{g} = \hat{g}(\tau_{ij}, \theta, \theta_{,i}).$$

Thus, the rate of change of entropy (12.6:6) becomes

$$(9) \quad \rho \dot{\hat{s}} = -\nabla \cdot (\mathbf{h}/T) + \mathbf{h} \cdot \nabla(1/T) - (\mathbf{e} : \dot{\sigma} + \rho \dot{\hat{g}} + \rho \dot{s} \dot{T})/T.$$

Comparison with the entropy balance equation (12.4:2) yields

$$(10) \quad \rho_i \dot{\hat{s}} = -\nabla \cdot (\mathbf{h}/T - \mathbf{j}_s) + \mathbf{h} \cdot \nabla(1/T) - (\mathbf{e} : \dot{\sigma} + \rho \dot{\hat{g}} + \rho \dot{s} \dot{T})/T.$$

In order that the entropy production inequality (12.6:8) is always valid, it is necessary and sufficient that state equations fulfill the following conditions of thermodynamic admissibility:

$$(11) \quad \partial\hat{g}/\partial\theta_{,i} = 0, \quad e_{ij} = -\rho\partial\hat{g}/\partial\tau_{ij}, \quad \hat{s} = -\partial\hat{g}/\partial\theta, \quad \mathbf{j}_s = \mathbf{h}/T,$$

$$\rho_i\dot{\hat{s}} = \mathbf{h} \cdot \nabla(1/T) = \mathbf{h} \cdot (-\nabla T)/T^2 \geq 0.$$

From equations above, the Gibbs free energy is independent of the temperature gradient, and so are the derived variables like strain and entropy; the 4th and 5th equations are identical to the corresponding equations of Eq. (2). Consequently, the transport law of heat conduction may be derived from the Fourier dissipation potential in the same way.

A substitution of \hat{S} in Eq. (11) into Eq. (9) yields the heat transfer equation in terms of the Gibbs free energy:

$$(12) \quad \rho\partial(-\partial\hat{g}/\partial\theta)/\partial t = -(\partial h_i/\partial x_i)/T_0.$$

12.8. COUPLED THERMOELASTIC CONSTITUTIVE RELATIONS

Expansion of the Helmholtz free energy density with respect to infinitesimal strain and temperature deviation up to the second-order terms gives

$$(1) \quad \Delta \quad \rho\hat{h} = \rho\hat{h}_0 - \rho\hat{s}_0\theta + \tau_{ij}^0 e_{ij} + C_{ijkl}e_{ij}e_{kl}/2 - \rho C_v\theta^2/(2T_0) - \beta_{ij}e_{ij}\theta,$$

where $C_{ijkl} = C_{klji}$ is a tensor of rank 4. Since e_{ij} and τ_{ij} are symmetric tensors, we also have $C_{ijkl} = C_{jikl} = C_{ijlk}$ and $\beta_{ij} = \beta_{ji}$. Thus, for an anisotropic material there are 21 independent constants C 's and 6 β 's. For an isotropic material, the numbers reduce to two C 's and one β .

A substitution of Eq. (1) into Eq. (12.7:2) yields the following coupled thermoelastic constitutive equations for an anisotropic material:

$$(2) \quad \Delta \quad \tau_{ij} = \rho\partial\hat{h}/\partial e_{ij} = \tau_{ij}^0 + C_{ijkl}e_{kl} - \beta_{ij}\theta,$$

$$(3) \quad \rho\hat{s} = -\rho\partial\hat{h}/\partial\theta = \rho\hat{s}_0 + \beta_{ij}e_{ij} + \rho C_v\theta/T_0.$$

Substituting Eq. (3) into Eq. (12.7:6) yields the coupled heat transfer equation:

$$(4) \quad -\partial h_i/\partial x_i = \rho C_v\partial\theta/\partial t + T_0\beta_{ij}\partial e_{ij}/\partial t.$$

Finally, on substituting h_i of Eq. (12.7:4) into Eq. (4), we obtain

$$(5) \quad \Delta \quad \partial(k_{ij}\partial\theta/\partial x_j)/\partial x_i = \rho C_v\partial\theta/\partial t + T_0\beta_{ij}\partial e_{ij}/\partial t.$$

Alternatively, expansion of the Gibbs free energy density with respect to the stress and temperature deviation up to the second-order terms gives

$$(6) \quad \Delta \quad \rho\hat{g} = \rho\hat{g}_0 - \rho s_0\theta - \tau_{ij}e_{ij}^0 - S_{ijkl}\tau_{ij}\tau_{kl}/2 - \rho C_p\theta^2/(2T_0) - \alpha_{ij}\tau_{ij}\theta,$$

where $S_{ijkl} = S_{klji}$ is a tensor of rank 4 and α_{ij} a tensor rank 2. Since e_{ij} and τ_{ij} are symmetric, we have $S_{ijkl} = S_{jikl} = S_{ijlk}$ and $\alpha_{ij} = \alpha_{ji}$. Thus, for an anisotropic material there are 21 independent constants S 's and 6 α 's. The numbers reduce to two S 's and one α if the material is isotropic.

A substitution of Eq. (6) into Eq. (12.7:11) yields the following coupled thermoelastic constitutive equations for an anisotropic material:

$$(7) \quad \Delta \quad e_{ij} = -\rho\partial\hat{g}/\partial\tau_{ij} = e_{ij}^0 + S_{ijkl}\tau_{kl} + \alpha_{ij}\theta,$$

$$(8) \quad \rho\hat{s} = -\rho\partial\hat{g}/\partial\theta = \rho s_0 + \alpha_{ij}\tau_{ij} + \rho C_p\theta/T_0.$$

Substituting Eq. (8) into Eq. (12.7:12) yields the coupled heat transfer equation:

$$(9) \quad -\partial h_i/\partial x_i = \rho C_p\partial\theta/\partial t + T_0\alpha_{ij}\partial\tau_{ij}/\partial t.$$

Finally, on substituting h_i of Eq. (12.7:4) into Eq. (9), we obtain

$$(10) \quad \Delta \quad \partial(k_{ij}\partial\theta/\partial x_j)/\partial x_i = \rho C_p\partial\theta/\partial t + T_0\alpha_{ij}\partial\tau_{ij}/\partial t.$$

These coupled thermoelastic constitutive equations are the general form of the *Duhamel–Neumann law*. In these equations, the indices range over 1, 2, 3; C_{ijkl} is the elastic modulus tensor of rank 4; β_{ij} the thermal modulus tensor of

rank 2; C_v the heat capacity per unit mass at constant volume; $\alpha_{ij} = C_{ijkl}^{-1}\beta_{kl}$ the thermal expansion tensor of rank 2; $S_{ijkl} = C_{ijkl}^{-1}$ the elastic compliance tensor of rank 4; and $C_p = \rho^{-1}T_0\beta_{ij}C_{ijkl}^{-1}\beta_{kl} + C_v$ the heat capacity per unit mass at constant pressure.

For an isotropic material, Eq. (2), (3) and (5) reduce to

$$(11) \quad \Delta \quad \tau_{ij} = \tau_{ij}^0 + \lambda e_{kk}\delta_{ij} + 2Ge_{ij} - \beta\theta\delta_{ij},$$

$$(12) \quad \rho\hat{s} = \rho\hat{s}_0 + \beta e_{ii} + \rho C_v\theta/T_0,$$

$$(13) \quad \Delta \quad \partial(k\partial\theta/\partial x_i)/\partial x_i = \rho C_v \partial\theta/\partial t + T_0\beta \partial e_{ii}/\partial t.$$

The corresponding Helmholtz free energy function is

$$(14) \quad \Delta \quad \rho\hat{h} = \rho\hat{h}_0 - \rho\hat{s}_0\theta + \tau_{ij}^0 e_{ij} + Ge_{ij}e_{ij} + \lambda e^2/2 - \rho C_v\theta^2/(2T_0) - \beta e\theta,$$

where $e = e_{ii}$. For isotropic media, Eq. (7), (8) and (10) reduce to

$$(15) \quad \Delta \quad e_{ij} = e_{ij}^0 + (1+\nu)\tau_{ij}/E - \nu\tau_{kk}\delta_{ij}/E + \alpha\theta\delta_{ij},$$

$$(16) \quad \rho\hat{s} = \rho\hat{s}_0 + \alpha\tau_{ii} + \rho C_p\theta/T_0.$$

$$(17) \quad \Delta \quad \partial(k\partial\theta/\partial x_i)/\partial x_i = \rho C_p \partial\theta/\partial t + T_0\alpha \partial\tau_{ii}/\partial t.$$

The corresponding Gibbs free energy function is

$$(18) \quad \Delta \quad \rho\hat{g} = \rho\hat{g}_0 - \rho s_0\theta - \tau_{ij}e_{ij}^0 - \frac{1+\nu}{2E}\tau_{ij}\tau_{ij} + \frac{\nu\tau^2}{2E} - \frac{\rho C_p\theta^2}{2T_0} - \alpha\tau\theta,$$

where $\tau = \tau_{ii}$. The relations between the material constants based on the Helmholtz and Gibbs free energies are $\alpha = \beta(1-2\nu)/E$ and those given in Eqs. (6.2:9) and (6.2:10).

12.9. STRAIN AND COMPLEMENTARY ENERGY FUNCTIONS

In elasticity theory, one defines the strain energy W as a function of the strains e_{ij} with the property that¹

$$(1) \quad \partial W/\partial e_{ij} = \tau_{ij}, \quad (i, j = 1, 2, 3).$$

As a counterpart to Eq. (1), the complementary energy W_c is a function of the stresses τ_{ij} with property

$$(2) \quad \partial W_c/\partial \tau_{ij} = e_{ij}, \quad (i, j = 1, 2, 3).$$

Comparing Eq. (1) with Eq. (12.8:2), we see that W can be identified with the Helmholtz free energy difference $\rho(\hat{h} - \hat{h}_0)$ that,

$$(3) \quad W = \rho(\hat{h} - \hat{h}_0).$$

Comparing Eq. (2) with Eq. (12.8:7), we see that $-W_c$ equals the Gibbs free energy difference $\rho(\hat{g} - \hat{g}_0)$:

$$(4) \quad W_c = -\rho(\hat{g} - \hat{g}_0).$$

The strain energy function is a minimum at the zero-stress state (see Sec. 12.3), which implies that, when $T = T_0$, the quadratic forms $W = C_{ijkl}e_{ij}e_{kl}/2$ and $W_c = S_{ijkl}\tau_{ij}\tau_{kl}/2$ must be positive definite.

For an isotropic material, W and W_c are reduced to

$$(5) \quad W = Ge_{ij}e_{ij} + \lambda e^2/2,$$

$$(6) \quad W_c = [(1+\nu)\tau_{ij}\tau_{ij} - \nu\tau^2]/(2E).$$

Introducing the strain deviation $e'_{ij} = e_{ij} - e\delta_{ij}/3$, we rewrite Eq. (5) as

$$(7) \quad W = Ke^2/2 + 2GJ_2,$$

where $J_2 = e'_{ij}e'_{ij}/2$ is the second invariant of the strain deviation. The two terms in Eq. (7) are independent of each other. Hence, the necessary and sufficient condition for W to be positive definite is

$$(8) \quad K = \lambda + 2G/3 > 0, \quad G > 0.$$

Since the bulk modulus K and the shear modulus G are related to Young's modulus E and Poisson's ratio ν with $K = E/[3(1-2\nu)]$ and $G = E/[2(1+\nu)]$, an equivalent set of conditions are

$$(9) \quad E > 0, \quad -1 < \nu < 1/2.$$

This result is obtained under the assumption that the stress-strain relationship is linear and isotropic, that the strain energy function is positive definite, and, tacitly, that the material is homogeneous and without hierarchical internal structure. The same thermodynamic restrictions on material properties can be obtained from the positive definiteness of the complementary energy function.

The values of E and ν of some common materials are given in Sec. 6.2, Table 6.2:1. Common steels have ν about 0.25, aluminum alloys have ν about 0.32; ν for beryllium is near zero (0.01– 0.06), and for lead is near 0.45. For some polymer foams ν can be greater than 1 or smaller than -1, see papers by Lakes, and Lee and Lakes listed in Bibliography. Smart materials have fine ultrastructures whose states influence the overall mechanical properties of the materials. Hence their state variables are not limited to stress, strain, and temperature. Therefore, they are not bounded by Eq. (9), which is derived under a restricted set of state variables. This raises the question of the condition when the ultrastructure can be ignored. There is no simple answer, especially in a field like biomechanics, in which organs, tissues, cells, subcellular units, and molecules form successive hierarchies. In fact, investigations of questions at the borderlines of neighboring hierarchies could be most rewarding.

A thermoelasticity problem involves appropriate boundary and initial conditions. Several simple examples will be discussed below. A proof of the existence and uniqueness of solution under suitable continuity conditions can be constructed analogous to that discussed in Sec. 7.4; an explicit proof in the case of isotropic materials can be found in Boley and Weiner, 1997, pp. 38–40.

Problem 12.9. State the conditions which must be satisfied by the constants C_{ijkl} in order that the quadratic form $W = C_{ijkl}e_{ij}e_{kl}/2$ be positive definite.

Problem 12.10. Let $T = T_0$, derive Clapeyron's formula for a linear elastic material:

$$W = \sigma_{ij}e_{ij}/2.$$

Problem 12.11. In conventional structural analyses, honeycomb sandwich material is often treated as a homogeneous orthotropic material, provided that the dimensions of the structure are much larger than the dimensions of the individual honeycomb cell. Suggest a stress-strain law for an orthotropic material. Determine explicitly the limitations imposed on the elastic constants that appear in the stressstrain law by the first and second laws of thermodynamics.

12.10. THERMAL EFFECTS: KELVIN'S FORMULA

An adiabatic expansion of a gas is accompanied by a drop in its temperature. Similarly, a solid body changes its temperature when the strain of the body is altered adiabatically. For steel, there is a fall of temperature when the body is strained to expand adiabatically.

Equation (12.8:4) gives the relationship between the rate of change of the temperature and the strain with the heat conduction. If heat conduction is prevented, the left-hand side of Eq. (12.8:4) vanishes, we obtain at once

$$(1) \quad \partial\theta/\partial t = -T_0\beta_{ij}(\partial e_{ij}/\partial t)/(\rho C_v),$$

which is Kelvin's formula for the change of temperature of an insulated body due to strain rate.

If an elastic modulus is measured on a completely insulated sample, or if the strain changes so rapidly that the heat does not have time to escape, the measured value is called the adiabatic modulus. On the other hand, it is called the isothermal modulus if the modulus is measured in such a way that the temperature is kept uniform and constant throughout the process.

The coefficients C_{ijkl} in the Duhamel–Neumann law (12.8:2) is the isothermal modulus of elasticity at $\theta = 0$:

$$(2) \quad \tau_{ij} = C_{ijkl}e_{kl} - \beta_{ij}\theta.$$

If a deformation is adiabatic, and no heat conduction takes place, then Eq. (1) gives

$$(3) \quad \rho C_v \theta + T_0 \beta_{ij} e_{ij} = \text{const.}$$

The constant of integration is zero if $e_{ij} = 0$ when $\theta = 0$. On solving Eq. (3) for θ and substituting into Eq. (2), we obtain in an adiabatic process

$$(4) \quad \tau_{ij} = [C_{ijkl} + T_0 \beta_{ij} \beta_{kl} / (\rho C_v)] e_{kl} = C'_{ijkl} e_{kl}.$$

Thus, the adiabatic modulus of elasticity is related to the isothermal modulus by the equation

$$(5) \quad C'_{ijkl} = C_{ijkl} + T_0 \beta_{ij} \beta_{kl} / (\rho C_v).$$

Problem 12.12. Derive the relationship of the temperature and the elastic constant $(\partial p / \partial V)T$ for an ideal gas, and compare qualitatively the temperature change in an adiabatic expansion of a gas with the temperature change of a similar process in a solid.

Problem 12.13. Show that Young's modulus of steel determined by an adiabatic process such as the propagation of elastic waves or the longitudinal vibration of a rod is higher than that determined by a slow static test in which the strains are maintained constant for a sufficiently long time for the temperature to become uniform.

Problem 12.14. For steel at $T = 274.7^\circ\text{K}$, or nearly 1.6°C , $\rho = 7.0 \text{ gm/cm}^3$, $\alpha = 1.23 \times 10^{-5}$ per K, $C_v = 0.102 \text{ cal/(gm}\cdot^\circ\text{C)}$, show that $dT = -0.125^\circ\text{C}$ if a wire is subjected adiabatically to a sudden increment of tensile stress of $1.09 \times 10^9 \text{ dyne/sq cm}$. Over a century ago, Joule [*Phil. Trans. Roy. Soc. London*, 149 (1859), p. 91] gave an observed value $dT = -0.1620^\circ\text{C}$ for the above case by experiments on cylindrical bars.

Problem 12.15. Consider simple tension and derive the following relationship between the adiabatic and isothermal Young's modulus, E' and E ,

$$1/E' = 1/E - \alpha^2 T_0 / (\rho C_v J),$$

which shows that the adiabatic modulus E' is always greater than the isothermal modulus E . In the equation above α is the linear coefficient of expansion, T_0 is the equilibrium temperature, J is the mechanical equivalent of heat, C_v is the specific heat per unit mass at zero strain rate, ρ is the density of the material.

Problem 12.16. Show that the theoretical value of the ratio E'/E is of order 1.003 for steel and copper at room temperature.

12.11. UNCOUPLED, QUASI-STATIC THERMOELASTIC THEORY

The basic equations given in Secs. 12.6–12.8 combine the theory of elasticity with heat conduction under transient conditions. Boundary-value problems involving these equations are rather difficult to solve. Fortunately, in most engineering applications it is possible to omit the mechanical coupling term in the heat transfer Eq. (12.8:4) or (12.8:9) and the inertia term in the equation of motion (12.6:2) without significant error. When these simplifications are introduced, the theory is referred to as an uncoupled, quasi-static theory and degenerates into heat conduction and thermoelasticity as two separate problems.

A plausible argument for small thermoelastic coupling is as follows. The change of temperature in an elastic body due to adiabatic straining is, in general, very small. If the interaction between strain and temperature is ignored, then the only effects of elasticity on the temperature distribution are those of change in dimensions of the body under investigation. The change in body dimension is of the order of the product of the linear dimension of the body L , the temperature rise θ , and the coefficient of thermal expansion α . If $L = 1 \text{ in.}$, $\theta = 1000^\circ\text{F}$, $\alpha = 10^{-5}$ per $^\circ\text{F}$, the change in dimension is 10^{-2} in. , which is negligible in problems of heat conduction. Again, if the temperature rise from 0 to 1000°F were achieved in a time interval of 0.1 sec, then the acceleration is of the order $10^{-2}/(0.1)^2 = 1 \text{ in./sec}^2$.

The change of stress due to this acceleration may be estimated from the equation of equilibrium $\Delta\sigma_{xx} \sim = \sigma x \rho d^2 u / dt^2$. If the specific gravity of the material is ten and the material is 1 in. thick, we have

$$\Delta\sigma_{xx} = [1/(12 \times 32.2)] \times 10 \times 62.4 / (12 \times 144) \cong 0.001 \text{ lb/sq in.}$$

This stress is negligible in most structural problems in which the stress magnitude is of the order of the yielding stress or ultimate stress of the material. Some examples of the coupled theory are presented in Chapter 2 of Boley and Weiner's book (1997). It points out that the thermomechanical coupling is important in the problem of internal friction of metals.

To consider the heat conduction problem, we denote the temperature above a reference temperature by θ , the space coordinates by \mathbf{x} , and time by t . In an uncoupled theory for an isotropic material, Eq. (12.8:13) becomes,

$$(1) \quad \partial(k\partial\theta/\partial x_i)/\partial x_i = \rho C_v \partial\theta/\partial t.$$

In the following discussion we assume that k and C_v are constant. Then,

$$(2) \quad k\nabla^2\theta = \rho C_v \partial\theta/\partial t$$

where ∇^2 is the Laplace operator defined in Prob. 2.32 for different coordinate systems.

For the special case of steady heat flow, $\partial\theta/\partial t$ vanishes, then

$$(3) \quad \nabla^2\theta = 0.$$

Numerous examples of solution of these equations can be found in books on the conduction of heat in solids, e.g., Carslaw and Jaeger (1959).

In the uncoupled, quasi-static theory, the stress and strain fields are computed for each instantaneous temperature distribution $\theta(\mathbf{x})$ according to Eqs. (12:6:1), (12.6:2), (12.7:1), (12.8:2) or (12.8:7), with appropriate boundary conditions. For an isotropic material, we have, with $(1 - 2\nu)\beta = \alpha E$,

$$(4) \quad \tau_{ij} = \lambda e_{kk}\delta_{ij} + 2Ge_{ij} - \beta\theta\delta_{ij}, \quad e_{ij} = (1 + \nu)\tau_{ij}/E - \nu\tau_{kk}\delta_{ij}/E + \alpha\theta\delta_{ij}.$$

The stress equilibrium equations and strain compatibility conditions are

$$(5) \quad \tau_{ij,j} + \rho b_i = 0, \quad e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0.$$

We first observe that if $\tau_{ij} = 0$, then

$$(6) \quad e_{ij} = \alpha\theta\delta_{ij},$$

and the 2nd equation of Eq. (5) is reduced to

$$(7) \quad \theta_{,kl}\delta_{ij} + \theta_{,ij}\delta_{kl} - \theta_{,jl}\delta_{ik} - \theta_{,ik}\delta_{jl} = 0,$$

which is satisfied if

$$(8) \quad \theta_{,ij} = 0, \quad i, j = 1, 2, 3.$$

The solution is

$$(9) \quad \theta = a_0 + a_1x_1 + a_2x_2 + a_3x_3,$$

with arbitrary coefficients a_i . Hence, if θ is a linear function of spatial coordinates x_1, x_2, x_3 , and if the displacements on the boundary are unrestrained, then it is possible to satisfy the compatibility condition without involving stresses. Generally, for an arbitrary temperature field with the strain field corresponding to thermal expansion alone, Eq. (6) will not be compatible, in which case thermal stresses must be involved.

Substituting Eq. (4) into Eq. (5), we obtain the generalized Navier's equation

$$(10) \quad Gu_{i,\mu\mu} + (\lambda + G)u_{\mu,\mu i} + \rho b_i - \beta\theta_{,i} = 0.$$

This is convenient to use if the boundary displacements \bar{u}_i is specified:

$$(11) \quad u_i = \bar{u}_i(\mathbf{x}) \quad \text{on the boundary}.$$

On the other hand, if traction \bar{T}_i is specified on the boundary, we have

$$(12) \quad \bar{T}_i = \nu_j \tau_{ij} = \nu_j [\lambda u_{\mu,\mu}\delta_{ij} + Gu_{i,j} + Gu_{j,i} - \beta\delta_{ij}\theta] = \bar{T}_i.$$

By comparing Eq. (10)–(12) with the corresponding equations in linear elasticity (Secs. 7.1 and 7.3), we see that the effect of the temperature change θ is equivalent to replacing the body force ρb_i in Navier's equations by $\rho b_i - \beta\theta_{,i}$ and the surface traction \bar{T}_i by $\bar{T}_i + \beta\theta\nu_i$. Thus the displacement \mathbf{u} produced by a temperature change θ are the same as that by the body forces $-\beta\theta_{,i}$ and the normal traction $\beta\theta$ acting on the surface of a body of the same shape but throughout which the temperature is uniform. These facts can be stated in a theorem.

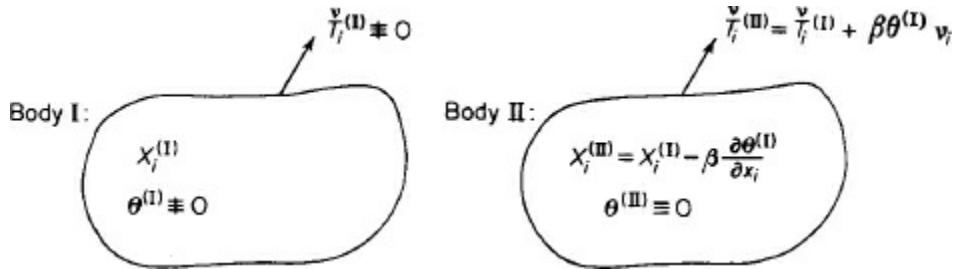


Fig. 12.11:1. Duhamel–Neumann analogy.

Theorem: Duhamel–Neumann Analogy. Consider two bodies of exactly the same shape but with boundary conditions prescribed as shown in Fig. 12.11:1. Then

$$u_i^{(I)}(\mathbf{x}, t) = u_i^{(II)}(\mathbf{x}, t), \quad \tau_{ij}^{(I)} = \tau_{ij}^{(II)} - \beta \theta^{(I)} \delta_{ij}.$$

By introducing suitable particular integrals, the thermal stress problem is reduced to the solution of the homogeneous Navier's equations. Let us assume the solution and the conservative body forces of Eq. (10) in the form

$$(13) \quad u_i = \partial \phi / \partial x_i, \quad \rho b_i = -\partial P / \partial x_i,$$

where ϕ is a displacement potential, then Eq. (10) may be written as

$$(14) \quad (\lambda + 2G)\phi_{,i\mu\mu} = P_{,i} + \beta \theta_{,i}.$$

An integration yields

$$(15) \quad \phi_{,\mu\mu} = (P + \beta \theta) / (\lambda + 2G).$$

The constant of integration may be absorbed in the potential P . Since Eq. (15) is linear its solution may be written in the form

$$(16) \quad \phi = \phi^{(p)} + \phi^{(c)},$$

where $\phi^{(p)}$ is a particular solution satisfying Eq. (15), and $\phi^{(c)}$ is the complementary solution satisfying the equation

$$(17) \quad \nabla^2 \phi^{(c)} = 0.$$

A particular solution can be taken in the form of the gravitational potential due to a distribution of matter of density $(P + \beta \theta) / (\lambda + 2G)$, i.e.,

$$(18) \quad \phi^{(p)}(\mathbf{x}) = - \int_V \{ [P(\mathbf{x}') + \beta \theta(\mathbf{x}')] / |\mathbf{x} - \mathbf{x}'| \} dV' / [4\pi(\lambda + 2G)].$$

Once $\phi^{(p)}$ is determined, the boundary conditions for the complementary function $\phi^{(c)}$ can be derived.

There are problems which cannot be solved by the method of scalar potential. However, the particular solution $\phi^{(p)}$ can still be used. From $\phi^{(p)}$, the displacements $\mathbf{u}(p)$ are computed according to Eq. (13). Then

$$(19) \quad u_i = u_i^{(p)} + u_i^{(c)}.$$

The equations governing $\mathbf{u}(c)$ are the homogeneous equations

$$(20) \quad (\lambda + G)\partial e^{(c)} / \partial x_i + G \nabla^2 u_i^{(c)} = 0, \quad e^{(c)} = u_{j,j}^{(c)}, \quad i = 1, 2, 3.$$

The boundary conditions for $\mathbf{u}(c)$ can be derived, of course, from the original boundary conditions by subtracting the contributions of the particular solution on the boundary.

Problem 12.17. Consider a flat plate of arbitrary shape clamped at all edges with its unloaded midsurface being the x, y -plane. The temperature is uniform over the plane of the plate but varies through the thickness, i.e., $\theta = \theta(z)$. Show that the plate will remain flat, i.e., no bending is introduced except possibly near the edges. Determine the thermal stresses in the plate. This example shows the extreme importance of the edge constraints on the thermal stress problem.

Problem 12.18. For $b_i = 0$, show that the heat conduction equation is

$$k \nabla^2 \theta = \rho C_v \partial \theta / \partial t,$$

and if $\theta \rightarrow 0$ as $t \rightarrow \infty$, a particular solution ϕ of Eq. (15) is

$$\phi^{(p)}(\mathbf{x}, t) = \alpha k(1 + \nu) \int_t^\infty \theta(\mathbf{x}, \xi) d\xi / [\rho C_v (1 - \nu)].$$

12.12. PLANE STRAIN (PLANE STRESS)

Consider a long cylindrical body $V(x, y)$ with boundary B (Fig. 12.12:1) in plane strain with temperature distribution independent of z . Then we have

$$(1) \quad \begin{aligned} u &= u(x, y), \quad v = v(x, y), \quad w = 0, \quad \theta = \theta(x, y), \\ e_{xz} &= e_{yz} = e_{zz} = 0, \quad \tau_{xz} = \tau_{yz} = 0, \\ \tau_{zz} &= \lambda(e_{xx} + e_{yy}) - \beta\theta = \nu(\tau_{xx} + \tau_{yy}) - \alpha E\theta. \end{aligned}$$

Duhamel's analogy can be derived explicitly in this case. For the body $V + B$ subjected to a temperature rise $\theta(x, y)$, zero body force and zero surface traction over B_σ and zero prescribed displacements on B_u , the stress field is the same as that given by the imposition of a hydrostatic pressure

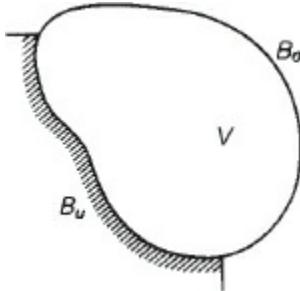


Fig. 12.12:1. Notations.

$$(2) \quad \tau_{xx} = \tau_{yy} = -\beta\theta, \quad \tau_{xy} = 0,$$

and that given by the same body kept at the uniform reference temperature, but subjected to a body force $X = -\beta(\partial\theta/\partial x)$, $Y = -\beta(\partial\theta/\partial y)$, and a tensile traction $\beta\theta$ over B_σ . The last-mentioned problem is solved as follows. We introduce the Airy stress function Φ as

$$(3) \quad \tau_{xx} = \partial^2\Phi/\partial y^2 + \beta\theta, \quad \tau_{yy} = \partial^2\Phi/\partial x^2 + \beta\theta, \quad \tau_{xy} = -\partial^2\Phi/\partial x\partial y,$$

to satisfy the equilibrium equations

$$(4) \quad \partial\tau_{xx}/\partial x + \partial\tau_{xy}/\partial y - \beta\partial\theta/\partial x = 0, \quad \partial\tau_{xy}/\partial x + \partial\tau_{yy}/\partial y - \beta\partial\theta/\partial y = 0.$$

Combining Eq. (4) above with $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, we obtain

$$(5) \quad \nabla^2(\tau_{xx} + \tau_{yy}) = \nabla^2\nabla^2\Phi + 2\beta\nabla^2\theta.$$

The compatibility equation is

$$(6a) \quad \partial^2e_{xx}/\partial y^2 + \partial^2e_{yy}/\partial x^2 = 2\partial^2e_{xy}/(\partial x\partial y),$$

or in terms of the stresses for plane strain is

$$(6b) \quad (1 - \nu) \left[\frac{\partial^2\tau_{xx}}{\partial y^2} + \frac{\partial^2\tau_{yy}}{\partial x^2} \right] - \nu \left(\frac{\partial^2\tau_{xx}}{\partial x^2} + \frac{\partial^2\tau_{yy}}{\partial y^2} \right) = 2\frac{\partial^2\tau_{xy}}{\partial x\partial y}.$$

Substituting Eq. (3) into Eqs. (6b) yields

$$(7) \quad \nabla^4\Phi = \nabla^2\nabla^2\Phi = -(1 - 2\nu)\beta\nabla^2\theta/(1 - \nu).$$

A substitution of Eq. (7) into Eq. (5) yields

$$(8) \quad \nabla^2(\tau_{xx} + \tau_{yy}) = \beta\nabla^2\theta/(1 - \nu).$$

Since

$$(9) \quad k\nabla^2\theta = \rho C_v \partial\theta/\partial t.$$

and $(1 - 2\nu)\beta = \alpha E$, Eq. (7) can be written as,

$$(10) \quad \nabla^4\Phi = -\alpha E \rho C_v (\partial\theta/\partial t)/[k(1 - \nu)].$$

Example. Consider the heat flow along a long cylinder with temperature distribution $\theta(x, y)$ independent of t , then $\nabla^2\theta = 0$ and Eq. (10) becomes

$$(11) \quad \nabla^4\Phi = 0.$$

Let the boundary surface of the cylinder be unrestrained. According to Duhamel analogy, the boundary condition is a tensile traction $\beta\theta$. Hence, according to Eq. (3), the boundary conditions can be satisfied by taking

$$(12) \quad \partial^2\Phi/\partial y^2 = 0, \quad \partial^2\Phi/\partial x^2 = 0, \quad \partial^2\Phi/\partial x\partial y = 0 \quad \text{on the boundary}.$$

A solution is evidently $\Phi = 0$. For this solution, a superposition of Eq. (2) and (3) yields $\tau_{xx} = \tau_{yy} = \tau_{xy} = 0$, whereas, from Eq. (1),

$$(13) \quad \tau_{zz} = -\alpha E\theta.$$

If the region $V(x, y)$ occupied by the cylinder is simply connected, and if plane strain condition can be assumed, we can quote the uniqueness theorem to say that this is the solution. Hence, we conclude that if a cylinder is simply connected and unrestrained on the surface, then under steady two-dimensional heat flow and plane strain condition, there will be no thermal stress in any surface element parallel to the cylinder axis.

Problem 12.19. Plane Stress. Consider a body of plane stress with

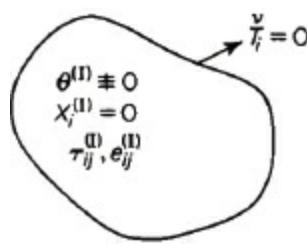
$$\begin{aligned} \tau_{13} &= \tau_{23} = \tau_{33} = 0, \\ e_{33} &= -\nu(\partial u_i/\partial x_i)/(1 - \nu) + (1 + \nu)\alpha\theta/(1 - \nu), \quad e_{13} = e_{23} = 0, \end{aligned}$$

where $i = 1, 2$. Prove the following Duhamel analogy (Fig. 12.12:2):

$$e_{ij}^{(I)} = e_{ij}^{(II)}, \quad \tau_{ij}^{(I)} = \tau_{ij}^{(II)} - \alpha E \theta^{(I)} \delta_{ij}/(1 - \nu).$$

It is seen that the role of the parameter β is played by the parameter $\alpha E/(1 - \nu)$. If Problem II in plane stress is expressed in terms of stresses and Airy stress functions

Problem I: Heating



Problem II: Nonheating

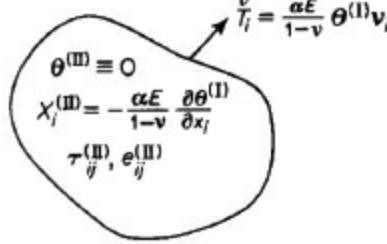


Fig. 12.12:2. Duhamel analogy in the case of plane stress.

so that

$$\tau_{xx} = \frac{\partial^2\Phi}{\partial y^2} + \frac{\alpha E\theta}{1 - \nu}, \quad \tau_{yy} = \frac{\partial^2\Phi}{\partial x^2} + \frac{\alpha E\theta}{1 - \nu}, \quad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y}.$$

Using the compatibility equation (6a) and the strain/stress relations

$$e_{xx} = (\tau_{xx} - \nu\tau_{yy})/E, \quad e_{yy} = (\tau_{yy} - \nu\tau_{xx})/E, \quad e_{xy} = (1 + \nu)\tau_{xy}/E,$$

show that

$$\nabla^4\Phi = -\alpha E \nabla^2\theta.$$

Problem 12.20. Consider a thin plate parallel to the x, y -plane subjected to a temperature distribution $T(x, y)$ which is independent of z . The Duhamel analogy can be applied either in the three-dimensional form or in the plane stress form with the vanishing of $\tau_{zz}, \tau_{zx}, \tau_{zy}$, introduced explicitly at the outset. Demonstrate the similarity and differences of these

two analogies.

12.13. VARIATIONAL PRINCIPLE FOR UNCOUPLED THERMOELASTICITY

In uncoupled thermoelasticity theory, the temperature is determined by heat conduction and the influence of the latent heat due to change of strain is ignored. Variational principles similar to those of Chapter 10 can be derived. The only difference between the linear elasticity of Chapters 7–9, valid in isothermal or isentropic conditions, and the thermoelasticity of the present chapter lies in the difference in the stress-strain law. Furthermore, in linear thermoelasticity concerning infinitesimal displacement, even this difference in stress-strain is hidden in the following expressions which apply to both elasticity and thermoelasticity:

$$(1) \quad \partial W / \partial e_{ij} = \tau_{ij}, \quad \partial W_c / \partial \tau_{ij} = e_{ij}.$$

Here $W(e_{ij}, T)$ and $W_c(\tau_{ij}, T)$ are the strain energy and the complementary strain energy functions, respectively, with $W(e_{ij}, T)$ expressed in terms of strains and the latter in terms of stresses. For general thermoelasticity, W is given by Eq. (12.9:3) and W_c by (12.9:4).

Following these remarks we can derive the following theorems: Let \mathcal{U} be the total strain energy of a body which occupies V with boundary B ,

$$(2) \quad \mathcal{U} = \int_V W(e_{ij}, T) dv.$$

The body is subjected to a body force ρb_i in V , and surface tractions \bar{T}_i on B_σ ($B_\sigma \in B$). Both ρb_i and \bar{T}_i are functions of time and space. Let \mathcal{K} be the kinetic energy of the body and \mathcal{A} the external loading potential:

$$(3) \quad \mathcal{K} = \int_V \rho (\partial u_i / \partial t) (\partial u_i / \partial t) dv / 2,$$

$$(4) \quad \mathcal{A} = - \int_V \rho b_i u_i dv - \int_{B_\sigma} \bar{T}_i u_i dS.$$

Then *Hamilton's principle* states that of all displacements u_i that satisfy the boundary condition

$$(5) \quad u_i = \bar{u}_i(x) \quad \text{over} \quad B_u (= B - B_\sigma),$$

the one that also satisfies the equations of motion minimizes the integral

$$(6) \quad \int_{t_0}^{t_1} (\mathcal{U} - \mathcal{K} + \mathcal{A}) dt = \text{minimum},$$

where t_0 and t_1 are two arbitrary instants, and the admissible variations δu_i are triply differentiable and satisfy the conditions

$$(7) \quad \delta u_i = 0 \quad \text{on} \quad B_u \quad \text{in} \quad t_0 \leq t \leq t_1,$$

$$(8) \quad \delta u_i = 0 \quad \text{at} \quad t = t_0 \quad \text{and} \quad t = t_1 \quad \text{in} \quad V + B.$$

If we consider static equilibrium, then the principle of minimum potential energy for thermoelasticity states that of all displacements, continuous and triply differentiable in a body $V + B$ and satisfying the specified boundary displacements \bar{u}_i over the surface B_u , the one that also satisfies the condition of equilibrium minimizes the potential

$$(9) \quad \int_V W(e_{ij}, T) dv - \int_V \rho b_i u_i dv - \int_{B_\sigma} \bar{T}_i u_i dS = \text{minimum}.$$

Similarly, on varying the stresses τ_{ij} , the complementary energy theorem in thermoelasticity states that the stress system τ_{ij} that satisfies the equilibrium equations minimizes the complementary energy with respect to variations of τ_{ij} ,

$$(10) \quad \int_V W_c(\tau_{ij}, T) dv - \int_V \rho b_i u_i dv - \int_{B_u} \bar{T}_i u_i dS = \text{minimum}.$$

The variations $\delta \tau_{ij}$ are subjected to the conditions $(\delta \tau_{ij})_j = 0$ in V and $\delta \tau_{ij} v_j = 0$ over B_σ , where surface tractions are specified.

The basic equations for heat conduction, Eqs. (12.7:4) and (12.8:4) omitting thermoelastic coupling, are

$$(11) \quad h_i = -k_{ij} \partial \theta / \partial x_j,$$

$$(12) \quad -\partial h_i / \partial x_i = \rho C_v \partial \theta / \partial t,$$

where C_v is the specific heat per unit mass at constant volume, ρ is the density, k_{ij} are the coefficients of heat conduction, $\theta = T - T_0$ is the difference of local temperature T from a uniform reference temperature T_0 , and $\mathbf{h} [= (h_1 \ h_2 \ h_3)]$ is the heat flux vector. A problem in heat conduction for a body occupying a volume V with boundary B is to find a continuously differentiable function $\theta(\mathbf{x}, t)$ which satisfies Eq. (11) and (12), the boundary conditions

$$(13) \quad \theta = \bar{\theta}(\mathbf{x}, t) \quad \text{on } B_1,$$

$$(14) \quad h_n = \kappa(\mathbf{x}, t)[\theta_0(\mathbf{x}, t) - \theta(\mathbf{x}, t)] \quad \text{on } B - B_1,$$

and the initial condition at $t = 0$

$$(15) \quad \theta = \bar{\theta}(\mathbf{x}, 0) \quad \text{in } V \text{ and } B,$$

where $h_n (= h_i \nu_i)$ is the outward normal component of the heat flux vector on the boundary, $\bar{\theta}(\mathbf{x}, t)$ is the prescribed temperature, and $\theta(\mathbf{x}, t)$ is that just outside the boundary layer. Equation (14) represents Newton's law of surface heat transfer, and κ is the heat transfer coefficient which can depend on \mathbf{x} .

We shall now prove *Biot's variational principle* which states that if C_v is constant, then Eq. (11)–(14) are the necessary conditions for the following variational equation to hold for an arbitrary variation of the vector H_i :

$$(16) \quad \delta \int_V \rho C_v \theta^2 dv / 2 + \int_V \lambda_{ij} \dot{H}_j \delta H_i dv + \int_{B_1} \bar{\theta} \nu_i \delta H_i dS = 0,$$

under the stipulations that

$$(17) \quad \rho C_v \theta = -H_{i,i}$$

and that $\delta H_i = 0$ over the boundary $B - B_1$, where the heat flux is specified that $\partial H_i / \partial t = h_i$ satisfies Eq. (14). The matrix $(\lambda_{ij}) = (k_{ij})^{-1}$, the thermal resistivity matrix, is the inverse of the conductivity matrix (k_{ij}) .

Proof. First, we have

$$\begin{aligned} \delta \int_V \rho C_v \theta^2 dv / 2 &= \int_V H_{i,i} \delta H_{j,j} / (\rho C_v) dv \quad (\text{by substitution}) \\ &= \int_B \frac{H_{i,i} \nu_j}{\rho C_v} \delta H_j dS - \int_V \left(\frac{H_{i,i}}{\rho C_v} \right)_{,j} \delta H_j dv = - \int_B \theta \nu_j \delta H_j dS + \int_V \theta_{,j} \delta H_j dv. \end{aligned}$$

To satisfy the above equation for arbitrary δH_i in V and on B_θ , the necessary conditions are

$$\begin{aligned} \theta_{,i} + \lambda_{ij} \dot{H}_j &= 0, \quad \text{or} \quad \dot{H}_i = -k_{ij} \theta_{,j} \text{ in } V, \\ \theta &= \bar{\theta} \quad \text{on } B_\theta, \\ \nu_j \delta H_j &= 0 \quad \text{on } B - B_\theta. \end{aligned}$$

The first equation, together with $\partial H_i / \partial t = h_i$, is exactly Eq. (11). The second equation is the same as Eq. (13), the last equation is in accordance with the stipulation stated.

Applications of this variational principle will be shown later in Sec. 12.15. Note that the vector H_i is related to the heat flux h_i :

$$(18) \quad h_i = H_i.$$

If we impose the condition that

$$(19) \quad H_i = 0 \quad \text{when } \theta = 0,$$

then Eq. (12) and (17) are consistent since C_v is a constant. Comparing Eq. (18) with Eq. (12.4:7), it is seen that H_i is proportional to the entropy displacement introduced by Biot.

Second, the special way in which the second term in Eq. (16) is posed should be noted. Biot denotes this term by $\delta\mathcal{D}$ and calls it the variation of dissipation function. The justification of this terminology becomes clear when generalized coordinates q_1, \dots, q_n are introduced so that $H_j = H_j(q_1, q_2, \dots, q_n; \mathbf{x})$. Then

$$\dot{H}_j = (\partial H_j / \partial q_k) \dot{q}_k.$$

Hence,

$$(20) \quad \partial \dot{H}_j / \partial \dot{q}_k = \partial H_j / \partial q_k.$$

Therefore,

$$\begin{aligned} \int_V \lambda_{ij} \dot{H}_j \delta H_i dv &= \int_V \lambda_{ij} \dot{H}_j (\partial H_i / \partial q_k) \delta q_k dv = \delta q_k \int_V \lambda_{ij} \dot{H}_j (\partial \dot{H}_i / \partial \dot{q}_k) dv \\ &= \delta q_k \partial \left(\int_V \lambda_{ij} \dot{H}_j \dot{H}_i dv \right) / \partial \dot{q}_k / 2. \end{aligned}$$

It is then clear that if we define the dissipation function as

$$(21) \quad \mathcal{D} = \int_V \lambda_{ij} \dot{H}_j \dot{H}_i dv / (2T_0),$$

then

$$(22) \quad \int_V \lambda_{ij} \dot{H}_j \delta H_i dv = T_0 (\partial \mathcal{D} / \partial \dot{q}_k) \delta q_k.$$

We will discuss the application of generalized coordinates in [Sec. 12.15](#).

It may appear unnatural to introduce the vector H_i rather than vary the temperature θ directly. A little reflection shows that the thermal evolution is determined by heat flow. The components of the heat flow vector are capable of independent variations. Hence, as a proper choice of variables, we use the components of the heat flow vector. Equivalently, the three components of the temperature gradient may be used also, but then the analysis will be purely formal; the functional do not have as simple an interpretation as those encountered above.

Problem 12.21. Derive the Euler equations for the functional in Eq. (6), (9), and (10) and compare the results with the basic equations of [Sec. 12.6](#).

12.14. VARIATIONAL PRINCIPLE FOR COUPLED THERMOELASTICITY

A variational equation corresponding to the basic equations of [Secs. 12.6–12.8](#) is given by Biot under the assumptions that the elastic constants C_{ijkl} , the thermal stress constants β_{ij} , the specific heat per unit mass at constant strain C_v , and the heat conductivity matrix (k_{ij}) or its inverse (λ_{ij}), are independent of the temperature and time, and that the temperature change θ is small compared to T_0 , the absolute temperature of the reference state. Under these assumptions, Eq. (12.8:4) can be integrated to give

$$(1) \quad -H_{i,i} = \rho C_v \theta + T_0 \beta_{ij} e_{ij},$$

where

$$\partial H_i / \partial t \cong h_i, \quad H_i = 0 \text{ when } \theta = e_{ij} = 0.$$

The vector H_i so defined is proportional to the entropy displacement. Biot's principle now states that the thermoelastic equations are necessary conditions for the following variational equation to hold for arbitrary δu_i and δH_i , except that $\delta u_i = 0$ over B_u , the portion of the boundary where the displacement u_i is prescribed, and $\delta H_i = 0$ where the heat flow is prescribed:

$$\begin{aligned}
(2) \quad \delta\mathcal{V} + \delta\mathcal{D} &= \delta \int_V (C_{ijkl}e_{kl}e_{ij} + \rho C_v \theta^2 / T_0) dv / 2 + \int_V \lambda_{ij} \dot{H}_j \delta H_i dv / T_0 \\
&= \int_V [C_{ijkl}e_{kl}\delta e_{ij} - \theta \delta(\beta_{ij}e_{ij} + H_{i,i}/T_0)] dv + \int_V \lambda_{ij} \dot{H}_j \delta H_i dv / T_0 \\
&= \int_V [(C_{ijkl}e_{kl} - \beta_{ij}\theta)\delta e_{ij} - (\theta \delta H_i)_{,i} / T_0 + (\theta_{,i} + \lambda_{ij}\dot{H}_j)\delta H_i / T_0] dv \\
&= \int_V (\rho b_i - \rho \ddot{u}_i) \delta u_i dv + \int_{B_\sigma} \bar{T}_i \delta u_i dS - \int_{B_\theta} \bar{\theta} \nu_i \delta H_i / T_0 dS,
\end{aligned}$$

which is a variational principle of two fields u_i and H_i . In the equation above, we have used the following the heat conduction equation

$$-h_{i,i} = T_0 \beta_{ij} \dot{e}_{ij} + \rho C_v \dot{\theta} = -\dot{H}_{i,i}, \quad \text{or} \quad \rho C_v \theta / T_0 = -H_{i,i} / T_0 - \beta_{ij} e_{ij}.$$

Equation (2) gives the Euler equations and the boundary conditions:

- (3a) $(C_{ijkl}e_{kl} - \beta_{ij}\theta)_{,j} + \rho b_i = \tau_{ij,j} + \rho b_i = \rho \ddot{u}_i$ and $\theta_{,i} + \lambda_{ij}\dot{H}_j = 0$ in V ,
(3b) $(C_{ijkl}e_{kl} - \beta_{ij}\theta)\nu_j = \tau_{ij}\nu_j = \bar{T}_i$ on B_σ , $\delta u_i = 0$ on $B - B_\sigma$,
 $\theta - \bar{\theta} = 0$ on B_θ , $\delta H_i \nu_i = 0$ on $B - B_\theta$,

where $\bar{\theta}$ is the boundary value of θ , ν_j are the direction cosines of the outward normal to the surface B (see Sec. 12.13), and \mathcal{V} is the thermoelastic potential of Biot defined by

$$(4) \quad \mathcal{V} = \int_V (C_{ijkl}e_{ij}e_{kl} + \rho C_v \theta^2 / T_0) dv / 2.$$

Note that if Eq. (2) is integrated with respect to time between an arbitrary interval (t_0, t_1) , under the stipulation that $\delta u_i = 0$ at $t = t_0, t_1$, the term involving the inertia force $-\rho \ddot{u}_i$ can be expressed as $-\delta\mathcal{K}$, where \mathcal{K} is the kinetic energy. Then a formula similar to Hamilton's principle, now including a dissipation function, is obtained.

The special treatment of the dissipation function \mathcal{D} in the variational equations should be noted. The variation $\delta\mathcal{D}$ was defined directly. When the equations of motion are written in Lagrangian form, \mathcal{D} can be identified as a functional which equals to $T_0/2$ times the rate of entropy production of the entire system. From 5th equation of Eq. (12.7:11), the entropy production per unit volume is $-h_j T_{,j} / T_0^2$, where h_j is the heat flux and $T_{,j}$ is the temperature gradient. On introducing Fourier's law of heat conduction in the form $T_{,j} = -\lambda_{jl} h_l$, the rate or entropy production per unit volume is $\lambda_{jl} h_j h_l / T_0^2$. Hence,

$$(5) \quad \mathcal{D} = \int_V \lambda_{jl} h_j h_l dv / (2T_0).$$

In Sec. 12.13, for the boundary condition over $B - B_1$, where the wall temperature is unspecified, it is required that the heat flow be specified exactly. For a variational approach, this boundary condition can be relaxed by including in the dissipation function a term corresponding to entropy production at the boundary. Let there be a heat flux per unit area $h_j \nu_j$ leaving the body at a boundary, T_a the temperature outside the heat transfer layer at the wall and T that of the body inside. (For example, T_a represents the adiabatic wall temperature in the boundary-layer flow in aerodynamic heating). If $T_a \neq T$, the entropy production per unit wall area is $-h_j \nu_j (T^{-1} - T_a^{-1})$. A phenomeno-logical law may be posed as

$$(6) \quad h_n = h_j \nu_j = -K(1/T - 1/T_a).$$

Therefore, the contribution of heat transfer at the boundary to the dissipation function is

$$(7) \quad \mathcal{D}_B = T_0 \int_B K(1/T - 1/T_a)^2 dS / 2 = T_0 \int_B h_n^2 / K dS / 2.$$

If \mathcal{D}_B is added to \mathcal{D} from (5), we have

$$(8) \quad \mathcal{D} = \int_V \lambda_{jl} h_j h_l dv / (2T_0) + T_0 \int_B h_n^2 / K dS / 2,$$

which must be used if the heat flow at the wall is to be considered as a natural boundary condition. When the

expression (8) is used, the variation δH_i can be regarded as arbitrary over the entire boundary.

Problem 12.22. Using Lagrangian multiplier as in Sec. 12.15, show that

$$\begin{aligned}\delta\mathcal{V} + \delta\mathcal{D} &= \delta \int_V [C_{ijkl}e_{kl}e_{ij}/2 - (T_0\beta_{ij}e_{ij} + \rho C_v\theta/2 + H_{i,i})\theta/T_0]dv \\ &+ \int_V \lambda_{ij}\dot{H}_j\delta H_idv/T_0 = \int_V (\rho b_i - \rho\ddot{u}_i)\delta u_idv + \int_{B_\sigma} \bar{T}_i\delta u_idS - \int_{B_\theta} \bar{\theta}\nu_i\delta H_idS/T_0.\end{aligned}$$

is a variational principle of three fields u_i , H_i and θ that gives, in addition to Eq. (3) and (4), the heat conduction equation (1) as an Euler equation in V .

12.15. LAGRANGIAN EQUATIONS FOR HEAT CONDUCTION AND THERMOELASTICITY

When n generalized coordinates q_1, \dots, q_n are introduced to represent the displacement and heat flow vectors by expressions

$$(1) \quad u_i = u_i(q_1, q_2 \dots q_n, \mathbf{x}), \quad H_i = H_i(q_1, q_2 \dots q_n, \mathbf{x}), \quad i = 1, 2, 3,$$

the variational principles of the preceding section may be written as

$$(2) \quad \partial\mathcal{V}/\partial q_j + \partial\mathcal{D}/\partial \dot{q}_j = Q_j,$$

where \mathcal{V} and \mathcal{D} are given by Eqs. (12.14:4) and (12.14:5), respectively, and

$$(3) \quad Q_j = \int_V (\rho b_i - \rho\ddot{u}_i)(\partial u_i/\partial q_j)dv + \int_B [\bar{T}_i \partial u_i/\partial q_j - \theta_0(t)(\partial H_n/\partial q_j)/T_0]dS,$$

H_n is the normal component of H_i on B . The applications of these Lagrangian equations are well-known for the stress problems (Chapters 10 and 11). In what follows we shall give illustration to heat conduction problems.

Example 1. Prescribed Wall Temperature. Consider a semi-infinite homogeneous, isotropic solid, with constant parameters k and C_v , initially at a uniform temperature $\theta = 0$ (Fig. 12.15:1). The boundary at $x = 0$ is heated to a variable temperature $\theta_0(t)$. Ignoring the inertia force and thermoelastic coupling, we shall find the temperature distribution in the half-space.

We shall solve this problem approximately. Following Biot (1957), we assume the temperature distribution to be parabolic and represented by

$$(4) \quad \theta(x, t) = \theta_0[1 - x/q_1(t)]^2 \quad \text{for } x < q_1; \quad \text{and} \quad \theta(x, t) = 0 \quad \text{for } x > q_1.$$

The function $q_1(t)$ can be interpreted physically as a “penetration depth,” and will be taken as our generalized coordinate (see Fig. 12.15:1).

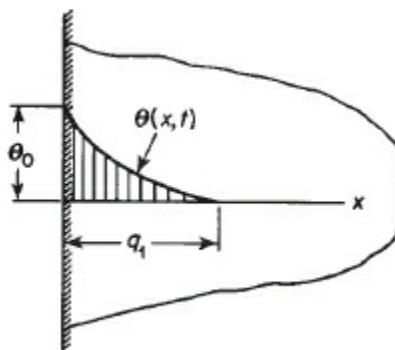


Fig. 12.15:1. Penetration of heat in one-dimensional flow.

The heat flow vector H_i is defined by Eqs. (12.13:17 – 19)

$$(5) \quad \dot{H}_i = h_i, \quad -H_{i,i} = \rho C_v\theta.$$

In the one-dimensional problem posed here, we may assume that only H_1 is different from zero and is a function of x

alone. Hence, from Eq. (5),

$$(6) \quad \partial H_1 / \partial x = -\rho C_v \theta.$$

For $x > q_1$, thermal equilibrium is undisturbed so that $H_1 = 0$. Hence, Eq. (6) and (4) can be integrated to give

$$(7) \quad H_1(x, t) = \rho C_v \int_x^{q_1(t)} \theta(\xi, t) d\xi = \rho C_v \theta_0 [q_1/3 - x + x^2/q_1 - x^3/(3q_1^2)].$$

To evaluate \mathcal{V} and \mathcal{D} , it is sufficient to consider a semi-infinite cylinder of unit cross section with axis parallel to x . Hence,

$$(8) \quad \mathcal{V} = \rho C_v \int_0^{q_1} \theta^2 dx / (2T_0) = \rho C_v \theta_0^2 q_1 / (10T_0),$$

$$(9) \quad \mathcal{D} = \int_0^{q_1} (\partial H_1 / \partial t)^2 dx / (2kT_0),$$

which, from Eq. (7), is

$$\begin{aligned} \mathcal{D} &= \frac{\rho^2 C_v^2}{2kT_0} \int_0^{q_1} \left[\dot{\theta}_0 \left(\frac{q_1}{3} - x + \frac{x^2}{q_1} - \frac{x^3}{3q_1^2} \right) + \theta_0 \dot{q}_1 \left(\frac{1}{3} - \frac{x^2}{q_1^2} + \frac{2x^3}{3q_1^3} \right) \right]^2 dx \\ &= \rho^2 C_v^2 q_1 (13\dot{q}_1^2 \theta_0^2 / 315 + \theta_0 q_1 \dot{\theta}_0 \dot{q}_1 / 21 + \dot{\theta}_0^2 q_1^2 / 63) / (2kT_0). \end{aligned}$$

The generalized force Q_1 is, according to Eq. (3),

$$(10) \quad Q_1(t) = \theta_0 (\partial H_1 / \partial q_1)_{x=0} / T_0 = \rho C_v \theta_0^2 / (3T_0).$$

Hence, the Lagrange equation for heat conduction (2) is

$$(11) \quad \theta_0^2 / (10T_0) + \rho C_v q_1 (26\dot{q}_1 \theta_0^2 / 315 + q_1 \theta_0 \dot{\theta}_0 / 21) / (2kT_0) = \theta_0^2 / (3T_0),$$

which, with $z = q_1^2$, reduces to

$$(12) \quad \dot{z} + 15\dot{\theta}_0 / (13\theta_0) z = 147k / (13\rho C_v).$$

This is a standard differential equation with an integration factor

$$\exp \left[\int 15\dot{\theta}_0 / (13\theta_0) dt \right] = [\theta_0(t)]^{15/13}$$

or

$$d\{z[\theta_0(t)]^{15/13}\} / dt = 147k[\theta_0(t)]^{15/13} / (13\rho C_v).$$

Hence, using the initial condition $q_1 = z = 0$ at $t = 0$, we obtain

$$(13) \quad z = 147k[\theta_0(t)]^{-15/13} \int_0^t [\theta_0(\tau)]^{15/13} d\tau / (13\rho C_v).$$

If the wall temperature follows a power law,

$$(14) \quad \theta_0(t) = \alpha t^n, \quad n \geq 0,$$

we obtain

$$(15) \quad q_1^2 = z = 147kt / [\rho C_v(15n + 13)].$$

This shows that the penetration depth q_1 varies with \sqrt{t} . It is independent of α and depends on the exponent n only through a constant factor. The case, where $\theta_0 = \text{const.}$ corresponds to $n = 0$, yields

$$(16) \quad q_1 = 3.36(kt / \rho C_v)^{1/2}.$$

Comparison of this simple solution of only one generalized coordinate with the exact solution of the problem was

made by Biot (1957) who shows that the approximation (4) is valid if the temperature increases or decreases monotonically. Biot also points out that if this is not the case, one may split up the time interval into segments for which θ_0 varies monotonically, and then apply Eq. (12) to each segment, using the principle of superposition. One can also use the power law solution (14) by dividing the time history of the temperature into segments, each of which may be approximated by a power law, or by an additive combination of such terms including the constant value.

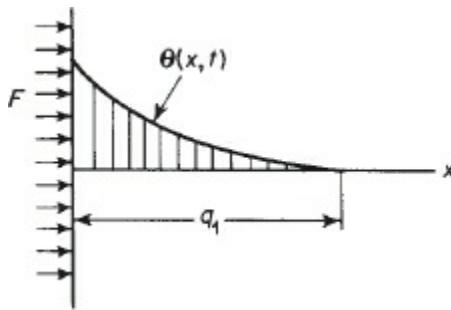


Fig. 12.15:2. Heating of a half-space with prescribed heat flux.

Example 2. Prescribed Heat Flux. The semi-infinite solid described in Example 1 is heated at the wall $x = 0$ with a uniform, constant heat flux $h_x = F$ at the wall (Fig. 12.15:2) (Lardner, T. J., AIAA J. 1, 1, 1963, pp. 196– 206). The problem is one-dimensional with $\mathbf{H} = H$. We have

$$(17) \quad \partial H / \partial x = -\rho C_v \theta.$$

Again, we assume a parabolic temperature distribution

$$(18) \quad \theta(x, t) = A(t)[1 - x/q_1(t)]^2$$

with q_1 being the “penetration depth.” The function $A(t)$ must be so chosen to satisfy the boundary condition at $x = 0$. An integration of Eq. (17) gives

$$(19) \quad H = A\rho C_v q_1 (1 - x/q_1)^3 / 3,$$

whence

$$(20) \quad h_x = \dot{H} = \rho C_v (\dot{A} q_1 + A \dot{q}_1) (1 - x/q_1)^3 / 3 + A \rho C_v \dot{q}_1 x (1 - x/q_1)^2 / q_1.$$

At $x = 0$, the boundary condition $h_x = F$ requires that

$$(21) \quad \rho C_v (\dot{A} q_1 + A \dot{q}_1) (1 - x/q_1)^3 / 3 = F \quad \text{or} \quad A = 3Ft / \rho C_v q_1.$$

Substituting A into Eq. (18) and (19) and then computing the thermal potential ψ and dissipation function \mathcal{D} according to Eqs. (12.14:4) and (12.14:8), we obtain

$$\frac{\partial \psi}{\partial q_1} = -\frac{9F^2 t^2}{10\rho C_v q_1^2}, \quad \frac{\partial \mathcal{D}}{\partial \dot{q}_1} = -\frac{F^2 t}{k} \left(\frac{3\dot{q}_1 t}{35q_1} + \frac{3}{42} \right), \quad Q_1 = \frac{9}{10} t.$$

The governing differential equation (2) becomes

$$(22) \quad 3kq_1(\dot{q}_1 t / 35 + q_1 / 42) / (\rho C_v) = 9t / 10.$$

The solutions for the penetration depth and the surface temperature are

$$(23) \quad q_1 = 2.81 [kt / (\rho C_v)]^{1/2},$$

$$(24) \quad \theta_0 = 3Ft / (\rho C_v q_1) = 1.065F[t / (\rho C_v k)]^{1/2}.$$

The exact solution is known to be

$$(25) \quad \theta_0 = 1.128F[t / (\rho C_v k)]^{1/2}.$$

PROBLEMS

12.23. Consider a slab of thickness b , heated on the surface $x = 0$. Initial condition at $t = 0$ is $\theta = 0$ throughout the plate. The boundary condition at the surface $x = 0$ is $\theta = \theta_0(t)$ for $t > 0$ (same as in Example 1), and that at $x = b$ is $hx = \partial\theta / \partial x = 0$.

$\partial x = 0$. At certain time t_0 , the penetration depth q_1 will equal to b . For time $t > t_0$, the temperature profile will be as shown in Fig. P12.23. Obtain an approximate solution by taking q_3 as the generalized coordinate.

12.24. Consider the same problem as in Problem 12.23 except that the boundary condition at $x = 0$ is, as shown in Fig. P12.24, $hx = F$, a const. (same as in Example 2).

12.25. A long, circular, cylindrical, solid-fuel rocket, upon curing, releases its heat of polymerization throughout its mass at a steady rate, say Q_0 . The external surfaces is held at constant temperature T_0 . Assuming that the cylinder is so long that over the body of the cylinder the heat flow is essentially two-dimensional, determine approximately the temperature distribution in the cross section of the cylinder.

Consider other cylindrical rockets whose cross-sectional shapes are (1) an isosceles right triangle, (2) an equilateral triangle, (3) a square. Let all of these rockets be made of the same material and have the same cross-sectional area. Compare the steady-state temperatures at the centroids of these rockets.

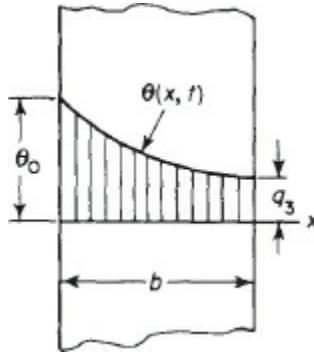


Fig. P12.23

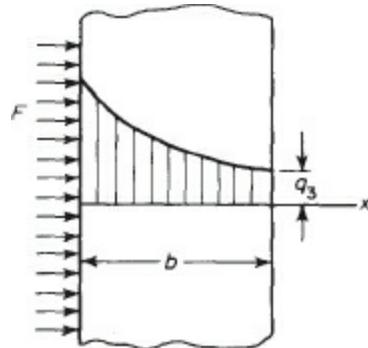


Fig. P12.24

¹We limit our attention to infinitesimal deformations in this chapter. The e_{ij} is the infinitesimal strain tensor. For finite deformations, see Chapter 13, especially Sec. 13.11.

13

LARGE DEFORMATION

In [Chapter 4](#), we discussed briefly the different strain measures for large deformation. In this chapter we shall examine further the consequences of the finiteness of strain fields. The generalization from infinitesimal strains to the nonlinear theory of finite strains opens up a tremendous vista of the theory. The nonlinear theory associated with finite strain is difficult and extensive. In this book, we cover only the theory needed for the development of numerical solutions of three-dimensional elastic solids including plates and shells. In this chapter, the strain measures are developed further, the stresses conjugate to various finite strain measures are defined, the equations of motion are derived, some constitutive equations are discussed, and the variational principles for large deformation are presented.

13.1. COORDINATE SYSTEMS AND TENSOR NOTATION

In this chapter, *rectangular Cartesian coordinates in 3-dimension* will be employed unless otherwise specified. Occasionally we use curvilinear orthogonal coordinates such as cylindrical and spherical coordinates in examples. In that case, it will be clearly specified. We shall use a combination of the Gibbs dyadic, matrix and indicial notations. Normally the Roman indices have a range from 1 to 3 and the Greek indices from 1 to 2. Bold face characters represent vectors, tensors or matrices. To simplify the discussion, a lower case bold face character denotes vector, a tensor of rank 1, and an upper case for tensor of rank 2, unless specifically noted otherwise. *Vectors and second rank tensors* may be expressed in component form:

$$(1) \quad \mathbf{u} = u_I \boldsymbol{\varepsilon}_I, \quad \mathbf{A} = A_{IJ} \boldsymbol{\varepsilon}_I \boldsymbol{\varepsilon}_J.$$

Here the physical and tensorial components are the same. If the reference frame is transformed to another Cartesian system with base vectors \mathbf{e}_i related by

$$(2) \quad \boldsymbol{\varepsilon}_I = R_{Ij} \mathbf{e}_j,$$

with the determinant of the matrix $\det(\mathbf{R}) = \det(R_{ij}) = 1$, then

$$(3) \quad \mathbf{u} = u_I R_{Ij} \mathbf{e}_j = \bar{u}_j \mathbf{e}_j = \bar{\mathbf{u}}, \quad \mathbf{A} = A_{IJ} R_{Ip} R_{Jq} \mathbf{e}_p \mathbf{e}_q = \bar{A}_{pq} \mathbf{e}_p \mathbf{e}_q = \bar{\mathbf{A}},$$

with $\bar{u}_j = u_I R_{Ij}$ and $\bar{A}_{pq} = A_{IJ} R_{Ip} R_{Jq}$. The tensor product of two vectors $\mathbf{e}_i \mathbf{e}_j$ is a tensor of rank 2 and in general is not commutative, i.e., $\mathbf{e}_i \mathbf{e}_j \neq \mathbf{e}_j \mathbf{e}_i$ for $i \neq j$.

In the *cylindrical coordinates* (r, θ, z), a vector \mathbf{v} can be referred to a set of orthogonal unit base vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ in the form

$$(4) \quad \mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z.$$

Here (v_r, v_θ, v_z) are the *physical, not tensorial*, components of \mathbf{v} (see [Sec. 2.14](#)), because $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ are unit vectors related to $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of Cartesian frame of reference through

$$\mathbf{e}_r = \mathbf{e}_1 \cos \theta + \mathbf{e}_2 \sin \theta, \quad \mathbf{e}_\theta = -\mathbf{e}_1 \sin \theta + \mathbf{e}_2 \cos \theta, \quad \mathbf{e}_z = \mathbf{e}_3.$$

According to the notation of Eq. (2), we have

$$\begin{aligned} R_{r1} &= R_{\theta 2} = \cos \theta, & R_{r2} &= -R_{\theta 1} = \sin \theta, \\ R_{r3} &= R_{\theta 3} = R_{z1} = R_{z2} = 0, & R_{z3} &= 1. \end{aligned}$$

The coordinates of the two systems are related by

$$x_1 = x = r \cos \theta, \quad x_2 = y = r \sin \theta, \quad x_3 = z.$$

The unit base vectors of the *spherical coordinates* $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ are related to those of Cartesian frame by:

$$\begin{aligned}\mathbf{e}_r &= (\mathbf{e}_1 \cos \theta + \mathbf{e}_2 \sin \theta) \sin \phi + \mathbf{e}_3 \cos \phi, \\ \mathbf{e}_\phi &= (\mathbf{e}_1 \cos \theta + \mathbf{e}_2 \sin \theta) \cos \phi - \mathbf{e}_3 \sin \phi, \\ \mathbf{e}_\theta &= -\mathbf{e}_1 \sin \theta + \mathbf{e}_2 \cos \theta.\end{aligned}$$

The coordinates (r, ϕ, θ) and (x_1, x_2, x_3) of the two frames are related by

$$x_1 = x = r \sin \phi \cos \theta, \quad x_2 = y = r \sin \phi \sin \theta, \quad x_3 = z = r \cos \phi.$$

If the indices of \mathbf{A} are referred to different coordinate systems, e.g., $\mathbf{A} = A_{ij} \mathbf{e}_i \mathbf{e}_j$, \mathbf{A} is called a *two-point tensor of rank 2* as opposed to a *uniform tensor* for which all indices refer to the same coordinate frame. One may similarly define *mixed tensors of higher rank*, for which each index of a tensor associates with a different coordinate system.

We summarize the basic vector and tensor operations as follows:

- If $\mathbf{B} = B_{ij} \mathbf{e}_i \mathbf{e}_j$, its transpose is $\mathbf{B}^T = B_{ji} \mathbf{e}_j \mathbf{e}_i$. \mathbf{B} is symmetric if $B_{ij} = B_{ji}$, then $\mathbf{B}^T = \mathbf{B}$. If $\mathbf{B} (= B_{ij} \mathbf{e}_i \mathbf{e}_j)$ is a two-point tensor, then $\mathbf{B}^T (= B_{ij} \mathbf{e}_j \mathbf{e}_i)$.
- $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ is the dot product of two unit orthogonal vectors, where δ_{ij} is the Kronecker delta. Let $\mathbf{u} = u_I \mathbf{e}_I$, $\mathbf{v} = v_I \mathbf{e}_I$ and $\varepsilon_I = R_{lj} \mathbf{e}_j$, the *dot product* $\mathbf{u} \cdot \mathbf{v} = u_I v_J (\varepsilon_I \cdot \mathbf{e}_J) = u_I R_{lj} v_j = \mathbf{v} \cdot \mathbf{u}$. If $\varepsilon_i = \mathbf{e}_i$, then $\mathbf{u} \cdot \mathbf{v} = u_i v_i$. Then $\mathbf{e}_i \cdot \mathbf{v} = v_i$ is the *projection or component* of \mathbf{v} along \mathbf{e}_i .
- If $\mathbf{B} = B_{ij} \mathbf{e}_i \mathbf{e}_j$ and $\mathbf{v} = v_I \mathbf{e}_I$, then the *dot products* are defined as $\mathbf{B} \cdot \mathbf{v} = B_{ik} v_k \mathbf{e}_i$ and $\mathbf{v} \cdot \mathbf{B} = v_k B_{ki} \mathbf{e}_i$. In general, $B_{ik} v_k \neq v_k B_{ki}$, i.e. $\mathbf{B} \cdot \mathbf{v} \neq \mathbf{v} \cdot \mathbf{B}$. One can similarly obtain the dot products of vectors and tensors whose components refer to different coordinate systems.
- If $\mathbf{A} = A_{IJ} \varepsilon_I \varepsilon_J$, $\mathbf{B} = B_{ij} \mathbf{e}_i \mathbf{e}_j$ and $\varepsilon_I = R_{lj} \mathbf{e}_j$, then $\mathbf{A} \cdot \mathbf{B} = A_{IJ} B_{ij} \varepsilon_I (\varepsilon_J \cdot \mathbf{e}_i) \mathbf{e}_j = A_{IJ} R_{jl} B_{ij} \varepsilon_I \mathbf{e}_j$ is a two-point tensor of rank 2. In general $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$. One can generalize the dot product for two tensors of arbitrary ranks r_A and r_B that $\mathbf{A} \cdot \mathbf{B}$ involves the contraction of the last index of \mathbf{A} and the first index of \mathbf{B} . The resulting product is a tensor of rank $r_A + r_B - 2$. A vector is a tensor of rank 1 and a scalar is a zero rank tensor. $\mathbf{A}^2 = \mathbf{A} \cdot \mathbf{A}$.
- The *trace of a uniform tensor* \mathbf{A} of rank 2 is a scalar defined as $\text{tr}(\mathbf{A}) = A_{ii}$. If $\mathbf{A} (= A_{ij} \varepsilon_I \mathbf{e}_i)$ is a two-point tensor, $\text{tr}(\mathbf{A}) = A_{jj} \varepsilon_J \cdot \mathbf{e}_i = A_{jj} R_{ji}$.
- The *inner product of two tensors* \mathbf{A} , \mathbf{B} of rank 2 is a scalar defined as $\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A} \cdot \mathbf{B}^T)$. If $\mathbf{A} = A_{KL} \varepsilon_K \varepsilon_L$, $\mathbf{B} = B_{ij} \mathbf{e}_i \mathbf{e}_j$, then $\mathbf{A} : \mathbf{B} = A_{KL} B_{ij} (\varepsilon_K \cdot \mathbf{e}_i) (\varepsilon_L \cdot \mathbf{e}_j) = A_{KL} R_{Ki} R_{Lj} B_{ij}$. If $\mathbf{A} = A_{ij} \mathbf{e}_i \mathbf{e}_j$ and $\mathbf{B} = B_{ij} \mathbf{e}_i \mathbf{e}_j$, then $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$. Since $\text{tr}(\mathbf{A} \cdot \mathbf{B}^T) = \text{tr}(\mathbf{A}^T \cdot \mathbf{B})$, one has $\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A}$ for second rank tensors. If \mathbf{A} and \mathbf{B} have different rank, generally, $\mathbf{A} : \mathbf{B} \neq \mathbf{B} : \mathbf{A}$.
- If $\mathbf{C} = C_{Klij} \varepsilon_K \varepsilon_L \mathbf{e}_i \mathbf{e}_j$ is a *mixed tensor* of rank 4, $\mathbf{A} (= A_{KL} \varepsilon_K \varepsilon_L)$ and $\mathbf{B} (= B_{ij} \mathbf{e}_i \mathbf{e}_j)$ are tensors of rank 2, then the *inner product* $\mathbf{A} : \mathbf{C} : \mathbf{B} = A_{KL} C_{Klij} B_{ij}$ is a scalar.
- For base vectors ε_I and \mathbf{e}_i of two Cartesian frames, the *cross product* $\mathbf{e}_j \times \mathbf{e}_k = e_{ijk} \mathbf{e}_i$ and $\varepsilon_L \times \mathbf{e}_k = R_{lj} \mathbf{e}_j \times \mathbf{e}_k = R_{lj} e_{ijk} \mathbf{e}_i$ are vectors, where e_{ijk} is the *permutation tensor* of rank 3. If $\varepsilon_I = \mathbf{e}_i$, then $\mathbf{u} \times \mathbf{v} = e_{ijk} u_j v_k \mathbf{e}_i$. Note that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{r} = (\mathbf{r} \cdot \mathbf{u})\mathbf{v} - (\mathbf{r} \cdot \mathbf{v})\mathbf{u}$.
- The outer product of two vectors $\mathbf{a} \otimes \mathbf{b} = (a_I \mathbf{e}_I) \otimes (b_J \mathbf{e}_J) = a_I b_J \mathbf{e}_I \mathbf{e}_J$.
- If $\mathbf{u} = u_I \mathbf{e}_I$, $\mathbf{v} = v_I \mathbf{e}_I$, then $\partial \mathbf{u} / \partial \mathbf{v} = (\partial u_I / \partial v_J) \mathbf{e}_I \mathbf{e}_J$ is a 2nd rank tensor.
- The *gradient* and the right-operator ∇ of a tensor \mathbf{A} of rank n with respect to a vector \mathbf{x} is a tensor of rank $n+1$:

$$\begin{aligned}\partial \mathbf{A} / \partial \mathbf{x} &= \text{grad}(\mathbf{A}) = \nabla \mathbf{A} = \partial \mathbf{A} / \partial x_k \otimes \mathbf{e}_k = [\partial(A_{IJ\dots} \varepsilon_I \varepsilon_J \dots) / \partial x_k] \otimes \mathbf{e}_k \\ &= (\partial \mathbf{A} / \partial x) \mathbf{e}_x + (\partial \mathbf{A} / \partial y) \mathbf{e}_y + (\partial \mathbf{A} / \partial z) \mathbf{e}_z \quad (\text{Cartesian coords.}) \\ &= (\partial \mathbf{A} / \partial r) \mathbf{e}_r + (\partial \mathbf{A} / \partial \theta) \mathbf{e}_\theta / r + (\partial \mathbf{A} / \partial z) \mathbf{e}_z \quad (\text{cylindrical coords.}) \\ &= (\partial \mathbf{A} / \partial r) \mathbf{e}_r + [(\partial \mathbf{A} / \partial \phi) \mathbf{e}_\phi + (\partial \mathbf{A} / \partial \theta) \mathbf{e}_\theta / \sin \phi] / r \quad (\text{spherical coords.}).\end{aligned}$$

If \mathbf{x} is a tensor of rank n larger than 1,

$$\partial \mathbf{A} / \partial \mathbf{x} = \partial(A_{IJ\dots} \varepsilon_I \varepsilon_J \dots) / \partial x_{kl\dots} \otimes \mathbf{e}_k \mathbf{e}_l \dots$$

In computing the partial derivatives, one must take into account that the base vectors of the reference frame can

be functions of the spatial coordinates, e.g., let \mathbf{A} be a vector \mathbf{u} in cylindrical coordinates,

$$\begin{aligned}\mathbf{u} &= u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z, \\ \mathbf{e}_r &= \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2, \quad \mathbf{e}_\theta = -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2, \\ \partial \mathbf{u} / \partial \theta &= (\partial u_r / \partial \theta - u_\theta) \mathbf{e}_r + (\partial u_\theta / \partial \theta + u_r) \mathbf{e}_\theta + (\partial u_z / \partial \theta) \mathbf{e}_z.\end{aligned}$$

- The divergence of a tensor \mathbf{A} of rank n is $\nabla \cdot \mathbf{A} = \mathbf{e}_i \cdot (\partial \mathbf{A} / \partial x_i)$, a tensor of rank $n - 1$. The dot operation by the base vectors must be executed from the left as shown above. Note that $(\partial \mathbf{A} / \partial x_i) \cdot \mathbf{e}_i \neq \mathbf{e}_i \cdot (\partial \mathbf{A} / \partial x_i)$. If \mathbf{A} is a vector, then it does not matter. For a vector \mathbf{u} , we have

$$\begin{aligned}\nabla \cdot \mathbf{u} &= \mathbf{e}_i \cdot (\partial \mathbf{u} / \partial x_i) = \partial u_i / \partial x_i && \text{(Cartesian coords.)}, \\ &= \mathbf{e}_r \cdot \partial \mathbf{u} / \partial r + \mathbf{e}_\theta \cdot (\partial \mathbf{u} / \partial \theta) / r + \mathbf{e}_z \cdot \partial \mathbf{u} / \partial z \\ &= (\partial r u_r / \partial r + \partial u_\theta / \partial \theta) / r + \partial u_z / \partial z && \text{(cylindrical coords.)} \\ &= \mathbf{e}_r \cdot \partial \mathbf{u} / \partial r + [\mathbf{e}_\theta \cdot (\partial \mathbf{u} / \partial \theta) / \sin \phi + \mathbf{e}_\phi \cdot \partial \mathbf{u} / \partial \phi] / r \\ &= \frac{1}{r^2} \frac{\partial r^2 u_r}{\partial r} + \frac{1}{r \sin \phi} \left(\frac{\partial u_\theta}{\partial \theta} + \frac{\partial \sin \phi u_\phi}{\partial \phi} \right) && \text{(spherical coords.)}.\end{aligned}$$

- The *curl* of a tensor is defined as

$$\begin{aligned}\nabla \times \mathbf{A} &= \mathbf{e}_i \times (\partial \mathbf{A} / \partial x_i) && \text{(Cartesian coords.)} \\ &= \mathbf{e}_r \times (\partial \mathbf{A} / \partial r) + \mathbf{e}_\theta \times (\partial \mathbf{A} / \partial \theta) / r + \mathbf{e}_z \times (\partial \mathbf{A} / \partial z) && \text{(cylindrical coords.)} \\ &= \mathbf{e}_r \times \frac{\partial \mathbf{A}}{\partial r} + \mathbf{e}_\theta \times \frac{1}{r \sin \phi} \frac{\partial \mathbf{A}}{\partial \theta} + \mathbf{e}_\phi \times \frac{1}{r} \frac{\partial \mathbf{A}}{\partial \phi} && \text{(spherical coords.)}.\end{aligned}$$

The cross product of two base vectors is defined as $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk} \mathbf{e}_k$.

The *Laplace* operation is defined as $\nabla^2 \mathbf{v} = \nabla \cdot (\nabla \mathbf{v})^T = \mathbf{e}_i \cdot (\partial \nabla \mathbf{v} / \partial x_i)^T$. In Cartesian coordinates, $\nabla^2 \mathbf{v} = \partial^2 v_i / (\partial x_i \partial x_j) \mathbf{e}_i$. One can similarly define the Laplace operation in other coordinates.

We shall close the section by identifying some frequently used relations of the inner product and properties of matrix operations. The inner product of strain rate and stress is the rate of work done by the forces acting on the surface of a unit cube as the body deforms. It is related to the change rate of the strain energy, that is, the internal energy under isentropic conditions or the Helmholtz free energy under isothermal conditions of the material based on the first law of thermodynamics. Hence the inner product defined earlier is a frequently used operation. We have

$$\begin{aligned}(5) \quad \mathbf{A} : \mathbf{B} &= \mathbf{B} : \mathbf{A} = \mathbf{A}^T : \mathbf{B}^T \\ (\mathbf{A} \cdot \mathbf{B}) : \mathbf{C} &= \mathbf{C} : (\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \mathbf{C}^T) : \mathbf{A}^T = (\mathbf{C} \cdot \mathbf{B}^T) : \mathbf{A} \\ &= (\mathbf{A}^T \cdot \mathbf{C}) : \mathbf{B} = (\mathbf{C}^T \cdot \mathbf{A}) : \mathbf{B}^T = \mathbf{B} : (\mathbf{A}^T \cdot \mathbf{C}) = \mathbf{A} : (\mathbf{C} \cdot \mathbf{B}^T),\end{aligned}$$

for any second rank tensors \mathbf{A} , \mathbf{B} and \mathbf{C} . If \mathbf{S} is a symmetric tensor, then

$$\mathbf{S} : \mathbf{B} = \mathbf{S} : \mathbf{B}^T = \mathbf{S} : (\mathbf{B} + \mathbf{B}^T)/2 = \mathbf{S} : (\mathbf{B})_s, \quad \mathbf{S} : (\mathbf{B})_a = 0,$$

where $(\mathbf{B})_s = (\mathbf{B} + \mathbf{B}^T)/2$ and $(\mathbf{B})_a = (\mathbf{B} - \mathbf{B}^T)/2$ are the symmetric and skew-symmetric parts of \mathbf{B} , respectively. If \mathbf{A} is skew-symmetric, then

$$\mathbf{A} : \mathbf{B} = -\mathbf{A}^T : \mathbf{B} = -\mathbf{A} : \mathbf{B}^T = \mathbf{A} : (\mathbf{B} - \mathbf{B}^T)/2 = \mathbf{A} : (\mathbf{B})_a, \quad \mathbf{A} : (\mathbf{B})_s = 0.$$

From time to time, we will use matrix notations and follow the rules of matrix operations for scalar, vectors, and tensors of rank 2. In this case, the vectors and tensors are referred to the same orthogonal coordinate system. A vector (*column matrix*) \mathbf{u} and a tensor \mathbf{A} can be written in matrix notation as

$$(6) \quad \underset{3 \times 1}{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = [u_i], \quad \underset{3 \times 4}{\mathbf{A}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix} = [A_{ij}].$$

The vector \mathbf{u} is a *column vector*. The underscore of \mathbf{A} denotes the ranges of the first and second indices. In the case above, \mathbf{A} is called a 3×4 matrix, a matrix of 3 rows and 4 columns. Implicitly we assume the ranges of indices are known

and drop the underscore unless needed for clarity. A column vector has the index ranging from 1 to p , called a ' $p \times 1$ ' matrix, and a row vector of one row and p column is called ' $1 \times p$ ' matrix. The transpose of a column vector \mathbf{u} is a row vector $\mathbf{u}^T (= [u_1 \ u_2 \ u_3])$. A ' $p \times p$ ' matrix is called a *square matrix*. If $A_{ij} = A_{ji}$, $\mathbf{A}(= \mathbf{A}^T)$ is a square symmetric matrix.

The products \mathbf{u} with a matrix \mathbf{A} are defined as

$$\mathbf{A}\mathbf{u} = [A_{ij}u_j], \quad \mathbf{u}^T\mathbf{A} = [u_iA_{ij}],$$

where repeated indices denote summation over the appropriate ranges. In the first case the number of columns of \mathbf{A} must be the same as the rows of \mathbf{u} . In the second case the number of rows of \mathbf{A} must be the same as the columns of \mathbf{u} . The components of $\mathbf{A}\mathbf{u}$ (matrix multiplication) equal those of $\mathbf{A} \cdot \mathbf{u}$ (inner product) and those of $\mathbf{u}^T\mathbf{A}$ equal those of $\mathbf{u} \cdot \mathbf{A}$ in the tensor operation. (In tensor operation the transpose of a vector is the vector itself.) If \mathbf{A}, \mathbf{B} are non-square matrices, their product is

$$\underset{m \times p}{\mathbf{C}} = \underset{m \times n}{\mathbf{A}} \underset{n \times p}{\mathbf{B}} = [A_{ik}B_{kj}],$$

which is the same as $\mathbf{A} \cdot \mathbf{B}$ in tensor operation. Finally, if \mathbf{A} is a square matrix with its inverse \mathbf{A}^{-1} , then

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where \mathbf{I} is an identity matrix with $I_{ij} = 1$ if $i = j$, while $I_{ij} = 0$ if $i \neq j$.

13.2. DEFORMATION GRADIENT

Consider a body which deforms from a volume V in the original or undeformed configuration R_0 into a volume v in the current or deformed configuration R . Let \mathbf{x} denote the position of a point in R associated with a material point originally located at \mathbf{X} in R_0 . Then we have

$$\mathbf{x} = x_i \mathbf{e}_i, \quad \mathbf{X} = X_I \varepsilon_I,$$

where \mathbf{e}_i and ε_I are the unit base-vectors in R and R_0 , respectively. The former, \mathbf{e}_i , is called *Eulerian base-vector* and the latter, ε_I , *Lagrangian base-vector*. The base vectors may translate in Cartesian frame. As a rule, we use capital indices referring to the undeformed coordinates and the lower case indices for the deformed coordinates. We assume the mechanics to be Newtonian. The motion of the body is described by the relation

$$(7) \quad \mathbf{x} = \mathbf{x}(\mathbf{X}, t),$$

where t is time and \mathbf{x} is continuously differentiable with respect to \mathbf{X} and t . To find the transformation between line elements in R and R_0 , we consider a material line element $d\mathbf{X}$ at \mathbf{X} , which becomes $d\mathbf{x}$ after deformation. The relation between the two elements is then

$$(8) \quad d\mathbf{x} = (\partial \mathbf{x} / \partial \mathbf{X}) \cdot d\mathbf{X} = (\nabla_{\mathbf{X}} \mathbf{x}) \cdot d\mathbf{X} = \mathbf{F} \cdot d\mathbf{X},$$

where $\nabla_{\mathbf{X}}$ denotes the gradient operation with respect to the \mathbf{X} coordinates, and \mathbf{F} is called the *deformation gradient tensor* defined as

$$(3) \quad \mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X} = F_{iI} \mathbf{e}_i \varepsilon_I, \quad \text{or} \quad F_{iI} = \partial x_i / \partial X_I,$$

The first index of F_{iI} refers to an *Eulerian* and the second index to a *Lagrangian* basis. Thus \mathbf{F} is a two-point second-rank tensor related to the *Eulerian* and *Lagrangian* coordinates. Equation (2) can be inverted to give

$$(4) \quad d\mathbf{X} = \mathbf{F}^{-1} \cdot d\mathbf{x}$$

provided that the *Jacobian* $J = \det[\mathbf{F}(\mathbf{X}, t)]$ is nonsingular, i.e., $J \neq 0$. The *Jacobian* is denoted also commonly in alternative forms

$$(5) \quad J = \det[\mathbf{F}] = \det \left[\frac{\partial x_i}{\partial X_I} \right] = e_{ijk} \frac{\partial x_i}{\partial X_1} \frac{\partial x_j}{\partial X_2} \frac{\partial x_k}{\partial X_3}.$$

The inverse of \mathbf{F} can be written as

$$(6) \quad \mathbf{F}^{-1} = \partial \mathbf{X} / \partial \mathbf{x}, \quad \text{or} \quad (F_{iI})^{-1} = \partial X_I / \partial x_i.$$

Stretches of Line, Area and Volume. We shall show how line, area and volume elements in R are related to their

counterparts in R_0 through \mathbf{F} . First, define the Cauchy and Almansi tensors \mathbf{C} and \mathbf{B}^{-1} , respectively, as

$$(7) \quad \mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}, \quad \text{or} \quad C_{IJ} = (\partial x_i / \partial X_I)(\partial x_i / \partial X_J),$$

$$(8) \quad \mathbf{B}^{-1} = (\mathbf{F}^{-1})^T \cdot \mathbf{F}^{-1}, \quad \text{or} \quad B_{ij}^{-1} = (\partial X_K / \partial x_i)(\partial X_K / \partial x_j),$$

They both are symmetric and positive definite. Clearly \mathbf{C} is a Lagrangian based tensor because the Eulerian indices i have been contracted; whereas \mathbf{B}^{-1} is an Eulerian based tensor as a result of the contraction of the Lagrangian indices K .

We shall establish the stretching of line elements under deformation. Consider an infinitesimal line element $d\mathbf{x} (= sds)$ at (\mathbf{x}, t) in R and its corresponding line element $d\mathbf{X} (= \mathbf{S}dS)$ at (\mathbf{X}, t) before deformation, where ds and dS are the lengths of the line element before and after deformation, and \mathbf{s} and \mathbf{S} are unit vectors along the directions of $d\mathbf{x}$ and $d\mathbf{X}$. The *stretch ratio* of the element is ds/dS . Denoting as λ_s for $ds \rightarrow 0$, we have

$$(9) \quad \lambda_s = ds/dS = \sqrt{d\mathbf{X} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\mathbf{X}} / |d\mathbf{X}| = \sqrt{\mathbf{S} \cdot \mathbf{C} \cdot \mathbf{S}}, \quad \text{or}$$

$$(10) \quad \lambda_s = ds/dS = |d\mathbf{x}| / \sqrt{d\mathbf{x} \cdot \mathbf{B}^{-1} \cdot d\mathbf{x}} = 1 / \sqrt{\mathbf{s} \cdot \mathbf{B}^{-1} \cdot \mathbf{s}}.$$

Let $d\mathbf{x}(1)$ and $d\mathbf{x}(2)$ be two infinitesimal noncollinear line elements lying in the deformed surface and $d\mathbf{X}(1)$ and $d\mathbf{X}(2)$ be the corresponding line segments in the undeformed configuration. The cross product $d\mathbf{x}(1) \times d\mathbf{x}(2) / |d\mathbf{x}(1) \times d\mathbf{x}(2)|$ is a unit vector normal to $d\mathbf{x}(1)$ and $d\mathbf{x}(2)$. We define

$$d\mathbf{a} = d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} = \mathbf{n}da \quad \text{or} \quad n_i da = e_{ijk} dx_j^{(1)} \times dx_k^{(2)}.$$

Then $d\mathbf{a}$ is the area of an element in the deformed configuration with \mathbf{n} being a unit normal to the plane of $d\mathbf{x}(1)$ and $d\mathbf{x}(2)$. The corresponding area element in the undeformed configuration is

$$(11) \quad d\mathbf{A} = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} = \mathbf{N}da \quad \text{or} \quad N_I da = e_{IJK} dX_J^{(1)} \times dX_K^{(2)},$$

where \mathbf{N} is a unit normal to $d\mathbf{X}(1)$ and $d\mathbf{X}(2)$. Then

$$N_I da = e_{IJK} dX_J^{(1)} dX_K^{(2)} = e_{IJK} (\partial X_J^{(1)} / \partial x_j) (\partial X_K^{(2)} / \partial x_k) dx_j^{(1)} dx_k^{(2)}.$$

Multiplying both sides of Eq. (11) by $J\mathbf{F}^{-1}$ (or $J\partial X_I / \partial x_i$) and simplifying the result by making use of Eq. (5) and the definition of determinant

$$e_{ijk} \det |\partial X_L / \partial x_m| = e_{IJK} (\partial X_I / \partial x_i) (\partial X_J / \partial x_j) (\partial X_K / \partial x_k),$$

we obtain

$$(12) \quad J\mathbf{F}^{-T} \cdot d\mathbf{A} = da \quad \text{or} \\ J N_M \frac{\partial X_M}{\partial x_i} dA = J e_{LMN} \frac{\partial X_L}{\partial x_i} \frac{\partial X_M}{\partial x_j} \frac{\partial X_N}{\partial x_k} dx_j^{(1)} dx_k^{(2)} = n_i da.$$

The area stretch ratio is then

$$(13) \quad \Lambda_N = |da| / |d\mathbf{A}| = J \sqrt{d\mathbf{A} \cdot \mathbf{C}^{-1} \cdot d\mathbf{A}} / |d\mathbf{A}| = J \sqrt{\mathbf{N} \cdot \mathbf{C}^{-1} \cdot \mathbf{N}}.$$

$$(14) \quad \Lambda_n = |da| / |d\mathbf{A}| = J |da| / \sqrt{da \cdot \mathbf{B} \cdot da} = J / \sqrt{\mathbf{n} \cdot \mathbf{B} \cdot \mathbf{n}}.$$

Note that $\Lambda_N = \Lambda_n$. From Eq. (12)–(14), one obtains

$$\mathbf{n} = (J\mathbf{F}^{-T} \cdot d\mathbf{A}) / |da| = \mathbf{F}^{-T} \cdot \mathbf{N} \cdot \sqrt{\mathbf{n} \cdot \mathbf{B} \cdot \mathbf{n}} = \mathbf{F}^{-T} \cdot \mathbf{N} / \sqrt{\mathbf{N} \cdot \mathbf{C}^{-1} \cdot \mathbf{N}},$$

which is the relation between the normal to $d\mathbf{a}$ and that to $d\mathbf{A}$.

Finally, we shall determine the relation between the deformed and undeformed volumes. The volume of the element tetrahedron dv formed by infinitesimal line elements $d\mathbf{x}(1)$, $d\mathbf{x}(2)$ and $d\mathbf{x}(3)$ in R is

$$(15) \quad dv = (d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}) \cdot d\mathbf{x}^{(3)} / 6.$$

It is related to its corresponding undeformed volume dV by

$$(16) \quad dv/dV = J = \det(\mathbf{F}).$$

Details of the proof are left as an exercise. Let ρ and ρ_0 be the mass density per unit deformed and undeformed volumes, respectively. From conservation of mass, we have

$$(17) \quad J = \rho_0/\rho.$$

Problem 13.1. Consider an area element vector $d\mathbf{a} = d\mathbf{x}(1) \times d\mathbf{x}(2)$ in the deformed configuration R . The corresponding area vector in the undeformed configuration is $d\mathbf{A} = d\mathbf{X}(1) \times d\mathbf{X}(2)$. Show that

$$(18) \quad d\mathbf{a} = J d\mathbf{A} \cdot \mathbf{F}^{-1} = J \mathbf{F}^{-T} \cdot d\mathbf{A}.$$

Problem 13.2. Show that Eq. (16) holds for the volume ratio of the deformed to the undeformed configuration. Hint: To determine the relationship between area elements and between volume elements, one needs to use the permutation symbols and identities defined in [Chapter 2](#).

$$\begin{aligned} e_{ijk} &= (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k, & e_{IJK} &= (\mathbf{e}_I \times \mathbf{e}_J) \cdot \mathbf{e}_K, & \mathbf{e}_i \times \mathbf{e}_j &= e_{ijm} \mathbf{e}_m, \\ e_{KLN} J &= e_{KLN} \det(\mathbf{F}) = e_{ijm} F_{iK} F_{jL} F_{mN}, \\ [e_{IJK} dX_J^{(1)} dX_K^{(2)}] (F_{iI})^{-1} \mathbf{e}_i &= (d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)}) \cdot \mathbf{F}^{-1} \end{aligned}$$

13.3. STRAINS

In this section, we shall discuss various strain measures and the relations among them. Let $ds^2 (= d\mathbf{x} \cdot d\mathbf{x})$ be the length square of the line element $d\mathbf{x}$ in R and $dS^2 (= d\mathbf{X} \cdot d\mathbf{X})$ be that of the corresponding undeformed element $d\mathbf{X}$ in R_0 . They can be expressed in terms of the deformation gradient:

$$(1) \quad ds^2 = d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\mathbf{X} = d\mathbf{X} \cdot \mathbf{C} \cdot d\mathbf{X} = C_{IJ} dX_I dX_J,$$

$$(2) \quad dS^2 = d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{x} \cdot (\mathbf{F}^{-1})^T \cdot \mathbf{F}^{-1} \cdot d\mathbf{x} = d\mathbf{x} \cdot \mathbf{B}^{-1} \cdot d\mathbf{x} = B_{ij}^{-1} dx_i dx_j.$$

If the Lagrangian coordinates are used to describe positions of material particles in the undeformed body, the metric tensor \mathbf{C} transforms the undeformed body to the deformed one. On the other hand, if the Eulerian coordinates are used in R , \mathbf{B}^{-1} , the metric tensor turns the deformed body to the undeformed one. A necessary condition (also locally sufficient) for a symmetric positive definite metric \mathbf{C} to be derivable from a configuration \mathbf{x} is the vanishing of a fourth-rank tensor field called the *Riemann–Christoffel tensor* [see Probs. 2.24, 2.31 and 4.9; and Green and Zerna (1968), Eringen (1962)]. The condition is called the compatibility condition, (see [Sec. 4.6](#)).

The *Green strain tensor* \mathbf{E} and the *Almansi strain tensor* \mathbf{e} are defined by

$$(3) \quad \Delta \quad ds^2 - dS^2 = 2d\mathbf{X} \cdot \mathbf{E} \cdot d\mathbf{X} = 2E_{IJ} dX_I dX_J,$$

$$(4) \quad \Delta \quad ds^2 - dS^2 = 2d\mathbf{x} \cdot \mathbf{e} \cdot d\mathbf{x} = 2e_{ij} dx_i dx_j,$$

where like \mathbf{C} ,

$$(5) \quad \Delta \quad \mathbf{E} = (\mathbf{C} - \mathbf{I})/2, \quad E_{IJ} = (C_{IJ} - \delta_{IJ})/2,$$

is a Lagrangian tensor referring to the undeformed coordinates; whereas

$$(6) \quad \Delta \quad \mathbf{e} = (\mathbf{I} - \mathbf{B}^{-1})/2, \quad e_{ij} = (\delta_{ij} - B_{ij}^{-1})/2,$$

like \mathbf{B} , is an Eulerian tensor referring to the deformed coordinates.

We shall examine the characteristics of the various deformation measures. We first consider \mathbf{C} of Eq. (13.2:7). Since \mathbf{C} is positive definite, its eigenvalues can be written in the form λ_i^2 , $i = 1, 2, 3$. We call the directions of the eigenvectors the *principal directions*. Consider a point $\mathbf{X} + d\mathbf{X}$ in R_0 which is displaced to $\mathbf{x}(\mathbf{X} + d\mathbf{X}, t) [= \mathbf{x}(\mathbf{X}, t) + d\mathbf{x}]$ in R . Using Eqs. (13.2:2) and (13.2:7), we obtain

$$d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X} \cdot \mathbf{C} \cdot d\mathbf{X} + O(|d\mathbf{X}|^2).$$

If $\mathbf{X} + d\mathbf{X}$ is a point on a small ellipsoid centered at \mathbf{X} with the semi-axes r/λ_i oriented along the principal directions of \mathbf{C} . Then

$$d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X} \cdot \mathbf{C} \cdot d\mathbf{X} = r^2$$

which indicates that the image of an ellipsoid in R_0 is approximately a sphere of radius r in R . Expressing $d\mathbf{X}$ in terms

of $d\mathbf{x}$ leads to

$$d\mathbf{X} \cdot d\mathbf{X} = d\mathbf{x} \cdot \mathbf{B}^{-1} \cdot d\mathbf{x}.$$

If $d\mathbf{X} \cdot d\mathbf{X} = r^2$, the equation above is an ellipsoid of the principal semi-axes $r\lambda_i$ for $d\mathbf{x}$ as the eigenvalues of \mathbf{B}^{-1} are $1/\lambda_i^2$. Thus, the image of a sphere of radius r in R_0 is approximately an ellipsoid of the principal semi-axes $r\lambda_i$ in R . For line element $d\mathbf{X}$ along the principal direction associated with eigenvalue λ_i , we have

$$d\mathbf{x} = \lambda_i d\mathbf{X}$$

Therefore, λ_i , $i = 1, 2, 3$, are called the *principal stretches* and the ellipsoid in R is called the *strain ellipsoid*.

13.4. RIGHT AND LEFT STRETCH AND ROTATION TENSORS

Another useful strain measure is \mathbf{U} defined as

$$(1) \quad \mathbf{U} \equiv \mathbf{C}^{1/2}.$$

If \mathbf{N}_i is an eigenvector of \mathbf{C} associated with the eigenvalue λ_i^2 , it is also the eigenvector of \mathbf{U} associated with the eigenvalue λ_i . Hence \mathbf{C} and \mathbf{U} are co-axial. Clearly, like \mathbf{C} , \mathbf{U} is also symmetric and positive definite that

$$(2) \quad \mathbf{U} \cdot \mathbf{U} = \mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}.$$

Let $\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1}$. We can easily show that

$$\mathbf{R}^T \cdot \mathbf{R} = (\mathbf{U}^{-1})^T \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \mathbf{U}^{-1} = (\mathbf{U}^{-1}) \cdot \mathbf{U} \cdot \mathbf{U} \cdot \mathbf{U}^{-1} = \mathbf{I},$$

therefore, \mathbf{R} is a *rotation tensor*. Equation (3) gives

$$(3) \quad \mathbf{F} = \mathbf{R} \cdot \mathbf{U},$$

which is called the *right polar decomposition* of \mathbf{F} , and \mathbf{U} is called the *right stretch tensor*.

From Eq. (3), we can interpret the deformation from $d\mathbf{X}$ to $d\mathbf{x}$ as a two-stage transformation and write Eq. (13.2:2) as

$$(4) \quad d\mathbf{X}^* = \mathbf{U} \cdot d\mathbf{X}, \quad d\mathbf{x} = \mathbf{R} \cdot d\mathbf{X}^*.$$

In other words, the line element $d\mathbf{X}$ is first stretched and rotated by \mathbf{U} to become $d\mathbf{X}^*$. (For line elements along the direction of an eigenvector of \mathbf{U} , the deformation involves stretching only.) Then the line element $d\mathbf{X}^*$ is rotated by \mathbf{R} to become $d\mathbf{x}$. The second stage deformation involves no stretching. Such a description of the deformation of $d\mathbf{X}$ is unique only to a rigid body translation.

The deformation gradient \mathbf{F} can also be decomposed into

$$(5) \quad \mathbf{F} = \mathbf{V} \cdot \mathbf{R},$$

where \mathbf{V} is a symmetric tensor and satisfies

$$(6) \quad \mathbf{V} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{R} \cdot \mathbf{R}^T \cdot \mathbf{V}^T = \mathbf{F} \cdot \mathbf{F}^T \equiv \mathbf{B}.$$

Equation (5) is called the *left polar decomposition* and \mathbf{V} is called the *left stretch tensor*. Note that \mathbf{B} and \mathbf{V} have the same eigenvectors, and therefore are coaxial. Clearly \mathbf{V} is positive definite and is linked to \mathbf{U} of the right polar decomposition by

$$(7) \quad \mathbf{V} = \mathbf{F} \cdot \mathbf{R}^T = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T.$$

Writing Eq. (13.2:2) in the form

$$(8) \quad d\mathbf{x}' = \mathbf{R} \cdot d\mathbf{X}, \quad d\mathbf{x} = \mathbf{V} \cdot d\mathbf{x}',$$

we can interpret that the deformation first rotates $d\mathbf{X}$ as a rigid body by \mathbf{R} to form $d\mathbf{x}'$, then stretches and rotates it by \mathbf{V} to form $d\mathbf{x}$. If $d\mathbf{x}'$ is along a principal direction of \mathbf{V} , it is only stretched to become $d\mathbf{x}$.

If λ_i is an *eigenvalue* of \mathbf{U} associated with the *eigenvector* \mathbf{N}_i , then

$$(9) \quad \mathbf{U} \cdot \mathbf{N}^i = \lambda_i \mathbf{N}^i \quad (i \text{ not summed}) \quad \text{and} \\ \lambda_i \mathbf{R} \cdot \mathbf{N}^i \cdot \mathbf{R}^T = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{N}^i \cdot \mathbf{R}^T = \mathbf{V} \cdot \mathbf{R} \cdot \mathbf{N}^i \cdot \mathbf{R}^T, \quad (i \text{ not summed}).$$

Note that λ_i is also an *eigenvalue* of \mathbf{V} associated with the *eigenvector*

$$(10) \quad \mathbf{n}^i = \mathbf{R} \cdot \mathbf{N}^i \cdot \mathbf{R}^T.$$

In other words, \mathbf{V} and \mathbf{U} have same eigenvalues, and their eigenvectors are linked by Eq. (10). Note that \mathbf{N}^i and \mathbf{n}^i are, respectively, also the eigenvectors of \mathbf{C} and \mathbf{B} associated with the eigenvalue λ_i^2 .

13.5. STRAIN RATES

We shall now define the time rate of the strain measures discussed in the previous section. The Eulerian descriptions of the *velocity field* $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ and the *velocity gradient* \mathbf{L} at time t are defined as

$$(1) \quad \mathbf{v}(\mathbf{x}, t) = \dot{\mathbf{x}}(\mathbf{X}, t) = \partial \mathbf{x} / \partial t,$$

$$(2) \quad \mathbf{L} \equiv \partial \mathbf{v} / \partial \mathbf{x}, \quad \text{or} \quad L_{ij} = \partial v_i / \partial x_j.$$

Then the *material time derivative* of deformation gradient is

$$(3) \quad \Delta \quad \dot{\mathbf{F}} = D\mathbf{F}/Dt = \partial \dot{\mathbf{x}} / \partial \mathbf{X} = \partial \dot{\mathbf{x}} / \partial \mathbf{x} \cdot \partial \mathbf{x} / \partial \mathbf{X} = \mathbf{L} \cdot \mathbf{F}, \quad \text{or} \\ \dot{F}_{ij} = D(\partial x_i / \partial X_j) / Dt = \partial \dot{x}_i / \partial X_j = (\partial v_i / \partial x_j)(\partial x_j / \partial X_j) = L_{ij} F_{jj}.$$

Again $\dot{\mathbf{F}}$ is a two-point tensor. Using $\mathbf{F} \cdot \mathbf{F}^{-1} = \mathbf{I}$ and $\dot{\mathbf{F}} \cdot \mathbf{F}^{-1} + \mathbf{F} \cdot \dot{\mathbf{F}}^{-1} = 0$, we establish that

$$(3a) \quad D\mathbf{F}^{-1} / Dt = -\mathbf{F}^{-1} \cdot \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = -\mathbf{F}^{-1} \cdot \mathbf{L}.$$

In general, $(\dot{\mathbf{F}})^{-1} (= \mathbf{F}^{-1} \cdot \mathbf{L}^{-1}) \neq D\mathbf{F}^{-1} / Dt$. From Eq. (13.2:7), we find

$$(4) \quad \dot{\mathbf{C}} = DC/Dt = \mathbf{F}^T \cdot \dot{\mathbf{F}} + \dot{\mathbf{F}}^T \cdot \mathbf{F} = 2\mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F},$$

where

$$(5) \quad \Delta \quad \mathbf{D} \equiv (\mathbf{L} + \mathbf{L}^T)/2$$

is the symmetric part of \mathbf{L} called the (*Eulerian*) *deformation* or *stretching rate tensor*. Another useful rate tensor derived from \mathbf{L} is

$$(6) \quad \Delta \quad \Omega = (\mathbf{L} - \mathbf{L}^T)/2,$$

which is skew-symmetric and called the *spin tensor*.

From Eq. (5) and (13.3:5), we find the *Green's strain rate*

$$(7) \quad \Delta \quad \dot{\mathbf{E}} = \mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F}.$$

We derive the rate of the right stretch tensor from Eq. (7) and (13.4:2) as

$$(8) \quad \dot{\mathbf{U}} \cdot \mathbf{U} + \mathbf{U} \cdot \dot{\mathbf{U}} = 2\dot{\mathbf{E}} = 2\mathbf{R}^T \cdot \mathbf{D} \cdot \mathbf{F}.$$

We then obtain

$$(9) \quad DB^{-1} / Dt = -(\mathbf{L}^T \cdot \mathbf{B}^{-1} + \mathbf{B}^{-1} \cdot \mathbf{L}),$$

$$(9a) \quad \Delta \quad \dot{\mathbf{B}} = -\mathbf{B} \cdot (DB^{-1} / Dt) \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{L}^T + \mathbf{L} \cdot \mathbf{B}.$$

Again $DB^{-1} / Dt \neq (\dot{\mathbf{B}})^{-1}$. The time derivative of Eq. (13.3:6) gives the rate of the Almansi strain tensor

$$(10) \quad \Delta \quad \dot{\mathbf{e}} = \mathbf{D} - (\mathbf{e} \cdot \mathbf{L} + \mathbf{L}^T \cdot \mathbf{e}).$$

From Eq. (3) and (8), $\dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \mathbf{R}^T = 0$ and $\dot{\mathbf{F}} = \dot{\mathbf{R}} \cdot \mathbf{U} + \mathbf{R} \cdot \dot{\mathbf{U}}$ we have

$$(11) \quad \dot{\mathbf{R}} = \omega \cdot \mathbf{R}, \quad \dot{\mathbf{U}} = \mathbf{R}^T \cdot (\mathbf{L} - \omega) \cdot \mathbf{F},$$

where ω is a skew-symmetric tensor.

13.6. MATERIAL DERIVATIVES OF LINE, AREA AND VOLUME ELEMENTS

In Sec. 13.2, it was shown that $d\mathbf{x}$, $d\mathbf{a}$, and dV in R are related to their undeformed counterparts $d\mathbf{X}$, $d\mathbf{A}$ and dV in R_0 through \mathbf{F} . In the following, we shall show that the stretching rate tensor \mathbf{D} defined in Eq. (13.5:5) relates the material rate derivatives of $d\mathbf{x}$, $d\mathbf{a}$, dV to those of $d\mathbf{X}$, $d\mathbf{A}$, dV . The time rate of a line element $d\mathbf{x}$ is

$$\dot{d\mathbf{x}} \equiv Dd\mathbf{x}/Dt = \dot{\mathbf{F}} \cdot d\mathbf{X} = \mathbf{L} \cdot d\mathbf{x},$$

in which the 2nd and 3rd equalities are from the fact that $d\mathbf{X}$ is independent of t and the result in Eq. (13.5:2). The stretch rate of the element is

$$(1) \quad l_s \equiv \frac{1}{|d\mathbf{x}|} \frac{D|d\mathbf{x}|}{Dt} = \frac{\dot{d\mathbf{x}} \cdot d\mathbf{x} + d\mathbf{x} \cdot \dot{d\mathbf{x}}}{2|d\mathbf{x}|^2} = \frac{d\mathbf{x} \cdot \mathbf{D} \cdot d\mathbf{x}}{|d\mathbf{x}|^2} = \bar{s} \cdot \mathbf{D} \cdot \bar{s},$$

where $\bar{s} = d\mathbf{x}/|d\mathbf{x}|$ is the unit vector along the direction of $d\mathbf{x}$. Since $ds = \sqrt{d\mathbf{x} \cdot d\mathbf{x}} = |d\mathbf{x}|$ and $ds/dS = \lambda_s$ [Eq. (13.2:10)], it follows that

$$(2) \quad l_s = D \ln ds/Dt = D \ln (ds/dS)/Dt = D \ln \lambda_s/Dt,$$

where dS is the undeformed length of ds and is independent of t . The equation above gives

$$(3) \quad \dot{\lambda}_s = \lambda_s l_s = \lambda_s \bar{s} \cdot \mathbf{D} \cdot \bar{s} = \lambda_s \bar{s} \cdot \mathbf{L} \cdot \bar{s} = \lambda_s \bar{s} \cdot \mathbf{L}^T \cdot \bar{s}.$$

Using Eq. (1) we can show that

$$(4) \quad \dot{\bar{s}} = \dot{d\mathbf{x}}/|d\mathbf{x}| - (d\mathbf{x} \cdot \dot{d\mathbf{x}})d\mathbf{x}/|d\mathbf{x}|^3 = \mathbf{L} \cdot \bar{s} - (\bar{s} \cdot \mathbf{D} \cdot \bar{s})\bar{s} = (\mathbf{L} - l_s \mathbf{I}) \cdot \bar{s}.$$

The quantity $\ln \lambda_s$ is often called the *natural* or *logarithmic strain*. Equation (2) establishes that the stretch rate of a deformed element is equal to the rate of logarithmic strain. In Eq. (1), taking $\bar{s} = \mathbf{e}_i$ leads to $l_s = D_{ii}$ (i not summed). Thus, with respect to Cartesian coordinates, the diagonal components D_{11} , D_{22} and D_{33} are the logarithmic strain rates along the directions of base vectors \mathbf{e}_1 , \mathbf{e}_2 and \mathbf{e}_3 , respectively.

To determine the meaning of the off-diagonal components of \mathbf{D} , we consider two infinitesimal line elements $d\mathbf{x}(1)$ and $d\mathbf{x}(2)$ along the directions $\bar{s}^{(1)}$ and $\bar{s}^{(2)}$, respectively. The dot product of the two line elements gives

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = |d\mathbf{x}^{(1)}| |d\mathbf{x}^{(2)}| \cos \theta,$$

where θ is the angle between the two vector differentials. Then

$$\begin{aligned} D[d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}]/Dt &= 2d\mathbf{x}^{(1)} \cdot \mathbf{D} \cdot d\mathbf{x}^{(2)} \\ &= \cos \theta D[|d\mathbf{x}^{(1)}||d\mathbf{x}^{(2)}|]/Dt - |d\mathbf{x}^{(1)}||d\mathbf{x}^{(2)}| \sin \theta (D\theta/Dt). \end{aligned}$$

If $d\mathbf{x}(1)$ is perpendicular to $d\mathbf{x}(2)$, $\cos \theta = 0$ and $\sin \theta = 1$. Equating the two equations above gives

$$(5) \quad -D\theta/Dt = 2d\mathbf{x}^{(1)} \cdot \dot{\mathbf{D}} \cdot d\mathbf{x}^{(2)} / (|d\mathbf{x}^{(1)}||d\mathbf{x}^{(2)}|) = 2\bar{s}^{(1)} \cdot \mathbf{D} \cdot \bar{s}^{(2)}.$$

But $-D\theta/Dt$ is the decreasing rate of a right angle, which is precisely the shearing rate. If $\bar{s}^{(1)}$ and $\bar{s}^{(2)}$ are in the directions of two base-vectors \mathbf{e}_i and \mathbf{e}_j where $i \neq j$, then $\bar{s}^{(2)}$

The rate of the volume ratio is determined as follows. Note that

$$\partial J/\partial F_{iI} = \text{cofactor } (F_{iI}) = J(F_{iI}^T)^{-1} \quad \text{or} \quad \partial J/\partial \mathbf{F} = J(\mathbf{F}^T)^{-1}.$$

It is straightforward to find

$$\begin{aligned} (6) \quad D(dv/dV)/Dt &= DJ/Dt = \partial J/\partial \mathbf{F} : \dot{\mathbf{F}} \\ &= J(\mathbf{F}^T)^{-1} : (\mathbf{L} \cdot \mathbf{F}) = J X_{K,i} L_{ij} x_{j,K} = \text{tr}(\mathbf{L})J = \text{tr}(\mathbf{D})J. \end{aligned}$$

Then

$$(7) \quad \Delta \quad D \ln(J)/Dt = \text{tr}(\mathbf{D}) = \partial v_i/\partial x_i = \nabla \cdot \mathbf{v},$$

where $\nabla \cdot (\cdot) = \mathbf{e}_i[\partial(\cdot)/\partial x_i]$ is the Eulerian divergent operator. The divergence of the velocity vector is the logarithmic rate of the volume ratio.

To compute the time rate of the area element $d\mathbf{a}$, whose magnitude is da , we derive from Eqs. (13.2:12), (13.5:3a) and (6) that

$$(8) \quad d\dot{\mathbf{a}} = \left\{ \frac{DJ}{Dt}(\mathbf{F}^{-1})^T + J \frac{D}{Dt}[(\mathbf{F}^{-1})^T] \right\} \cdot d\mathbf{A} = [\text{tr}(\mathbf{L})\mathbf{I} - \mathbf{L}^T] \cdot d\mathbf{a}.$$

The use of Eq. (13.2:14) and $|d\mathbf{a}|^2 = d\mathbf{a} \cdot d\mathbf{a}$ leads to

$$\dot{\Lambda}_n \equiv D(|d\mathbf{a}|/|d\mathbf{A}|)/Dt = (|d\mathbf{a}|/|d\mathbf{A}|)(D|da|/Dt)/|da| = \Lambda_n da \cdot d\dot{\mathbf{a}}/|da|^2.$$

From Eq. (1) and (8), the equation above becomes

$$(9) \quad \dot{\Lambda}_n = \Lambda_n [\text{tr}(\mathbf{D}) - \mathbf{n} \cdot \mathbf{L}^T \cdot \mathbf{n}] = \Lambda_n [\text{tr}(\mathbf{D}) - l_s],$$

where \mathbf{n} is the unit vector normal to the surface and l_s is the logarithmic stretch rate of the line element along the normal direction. Then

$$(10) \quad D \ln \Lambda_n / Dt = \text{tr}(\mathbf{D}) - l_s.$$

Introducing the *Jacobian* $J \equiv \det[\mathbf{F}(\mathbf{X}, t)]$, and using Eq. (2), (6) and (10), we obtain

$$D \ln J / Dt = D \ln \Lambda_n / Dt + D \ln \lambda_s / Dt = D \ln(\lambda_s \Lambda_n) / Dt,$$

which implies

$$(11) \quad J = \lambda_s \Lambda_n.$$

Equation (11) indicates that the volume ratio equals the product of the stretch ratio and the area ratio provided that the direction of the stretch is normal to the area element.

In summary, the rates of the logarithmic line, area and volume ratios of finite deformation are

$$\frac{D}{Dt} \ln(\lambda_s) = \bar{s} \cdot \mathbf{D} \cdot \bar{s}, \quad \frac{D}{Dt} \ln(\Lambda_n) = \text{tr}(\mathbf{D}) - \mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n}, \quad \frac{D}{Dt} \ln(J) = \text{tr}(\mathbf{D}).$$

These results reduce to those of the infinitesimal strain in [Chapter 4](#).

Problem 13.3. Show that $\partial J/\partial \mathbf{F} = J(\mathbf{F}^T)^{-1}$, and $\partial J/\partial \mathbf{E} = J \mathbf{C}^{-1}$, where $J = \det(\mathbf{F})$.

13.7. STRESSES

The stress tensor introduced in [Chapter 3](#) is valid under large deformation, provided that all quantities are Eulerian referred to the current configuration. This stress tensor is called the *Cauchy stress* and denoted by τ , which is an Eulerian tensor with components τ_{ij} . The rate of work done per unit volume by the Cauchy stress acting on the surface of an infinitesimal volume element is $\tau_{ij}D_{ij}$, which is the inner product of the stress tensor and the deformation rate tensor. Let ρ be the density of the material at the deformed state. The rate of work done per unit mass by the stress is $W = \tau_{ij}D_{ij}/\rho$. Then

$$(1) \quad \rho \dot{W} = \tau : \mathbf{D} = \tau_{ij}D_{ij},$$

where \mathbf{D} is given in Eq. (13.5:5). We define that a stress is *conjugate to* (or the *work conjugate variable*) a strain rate, and vice versa, if their inner product is the work rate per unit deformed or undeformed volume. Thus the Cauchy stress τ is the stress measure conjugate to, or the work conjugate variable of, the deformation rate tensor \mathbf{D} when the work rate is referred to the deformed volume.

Since the mass does not change during deformation, the work rate referred to a unit volume of the material in the undeformed state is

$$(2) \quad \rho_0 \dot{W} = (\rho_0/\rho)\rho \dot{W} = J \tau : \mathbf{D} = J \tau_{ij}D_{ij},$$

where $J [= \det(\mathbf{F})]$ is the *Jacobian* of the deformation gradient tensor \mathbf{F} .

[Chapter 4](#) and [Secs. 13.3 and 13.4](#) shows that there are other strain measures, each has a claim to advantage in

certain circumstances. Corresponding to each strain measure is a stress measure whose inner product with the time rate of the strain measure gives the rate of work done per unit deformed or undeformed volume. Then the strain rate and the corresponding stress measure are said to be *conjugate* to each other. The stress measures are defined as follows.

Kirchhoff Stress. Kirchhoff defined

$$(3) \quad \sigma \equiv J\tau,$$

now known as the *Kirchhoff stress tensor*. Obviously σ is symmetric and $\sigma : \mathbf{D}$ equals to $\rho_0 \dot{W}$, the time rate of strain energy density per unit undeformed volume. Therefore, σ is conjugate to \mathbf{D} when the work rate is referred to the undeformed configuration.

First Piola Kirchhoff Stress. The *first Piola–Kirchhoff stress tensor*, also called the *Lagrangian stress tensor*, is defined as

$$(4) \quad \mathbf{T} \equiv \mathbf{F}^{-1} \cdot \sigma = J\mathbf{F}^{-1} \cdot \tau,$$

which is a nonsymmetric two-point tensor with \mathbf{F}^{-1} based on the undeformed state and τ referred to the deformed state. Since σ is symmetric, using Eq. (13.5:3), one can show that

$$(5) \quad \mathbf{T} : \dot{\mathbf{F}}^T = (\mathbf{F}^{-1} \cdot \sigma) : \dot{\mathbf{F}}^T = (\dot{\mathbf{F}} \cdot \mathbf{F}^{-1})^T : \sigma = \sigma : \mathbf{L} = \sigma : \mathbf{D}.$$

Thus $\mathbf{T} : \dot{\mathbf{F}}^T$ equals to $\rho_0 \dot{W}$ and, therefore, \mathbf{T} is conjugate to $\dot{\mathbf{F}}^T$.

Second Piola Kirchhoff Stress. A widely used stress measure is

$$(6) \quad \mathbf{S} \equiv \mathbf{T} \cdot (\mathbf{F}^{-1})^T = \mathbf{F}^{-1} \cdot \sigma \cdot (\mathbf{F}^{-1})^T = J\mathbf{F}^{-1} \cdot \tau \cdot (\mathbf{F}^{-1})^T,$$

known as the *second Piola–Kirchhoff stress tensor*. Clearly \mathbf{S} is symmetric and is defined in Lagrangian description. It can be shown that

$$(7) \quad \begin{aligned} \mathbf{S} : \dot{\mathbf{E}} &= [\mathbf{F}^{-1} \cdot \sigma \cdot (\mathbf{F}^{-1})^T] : \dot{\mathbf{E}} = (\mathbf{F}^{-1} \cdot \sigma) : (\dot{\mathbf{E}} \cdot \mathbf{F}^{-1}) \\ &= \sigma : [(\mathbf{F}^{-1})^T \cdot \dot{\mathbf{E}} \cdot \mathbf{F}^{-1}] = \sigma : \mathbf{D} = \rho_0 \dot{W}, \end{aligned}$$

which establishes \mathbf{S} being conjugate to Green's strain rate \mathbf{E} . Eq. (13.5:7), defined in

Biot Luré Stress. Another stress measure is the *Biot–Luré stress tensor*

which is defined as

$$(8) \quad \mathbf{r}^* = \mathbf{F}^{-1} \cdot \sigma \cdot \mathbf{R} = \mathbf{T} \cdot \mathbf{R} = \mathbf{S} \cdot \mathbf{U}.$$

One can show that

$$(8a) \quad \mathbf{S} : \dot{\mathbf{E}} = \mathbf{S} \cdot \mathbf{U} : \dot{\mathbf{U}} = \mathbf{r}^* : \dot{\mathbf{U}} = \mathbf{r} : \dot{\mathbf{U}},$$

where $\mathbf{r} = (\mathbf{r}^* + \mathbf{r}^{*T})/2$ is the symmetric part of \mathbf{r}^* . Then the conjugate stress measure of $\dot{\mathbf{U}}$ is \mathbf{r}^* since $\mathbf{r}^* : \dot{\mathbf{U}} = \rho_0 \dot{W}$.

Corotational Cauchy Stress. Green and Naghdi (1965) introduced

$$(9) \quad \sigma_r = \mathbf{R}^T \cdot \tau \cdot \mathbf{R}J = \mathbf{U} \cdot \mathbf{S} \cdot \mathbf{U}$$

as the symmetric corotational Cauchy stress. This stress is obtained by rotating the Cauchy stress tensor back to the undeformed body. The conjugate strain measure is $\mathbf{R}^T \cdot \mathbf{D} \cdot \mathbf{R}$ that the strain energy per unit undeformed volume is

$$(9a) \quad \sigma_r : (\mathbf{R}^T \cdot \mathbf{D} \cdot \mathbf{R}) = \sigma : \mathbf{D} = \mathbf{S} : \dot{\mathbf{E}}.$$

A summary of the relations among the different stress measures is

$$(10) \quad \sigma = J\tau = \mathbf{F} \cdot \mathbf{T} = \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T = \mathbf{F} \cdot \mathbf{r}^* \cdot \mathbf{R}^T = \mathbf{R} \cdot \sigma_r \cdot \mathbf{R}^T.$$

The hydrostatic pressure is usually defined as the mean Cauchy stress

$$p = -(\tau : \mathbf{I})/3.$$

It follows from Eq. (10)

$$(11) \blacktriangle -p = \frac{\tau : \mathbf{I}}{3} = \frac{\sigma_r : \mathbf{I}}{3J} = \frac{\sigma : \mathbf{I}}{3J} = \frac{\mathbf{T} : \mathbf{F}^T}{3J} = \frac{\mathbf{S} : \mathbf{C}}{3J} = \frac{\mathbf{r}^* : \mathbf{U}}{3J} = \frac{\mathbf{r} : \mathbf{U}}{3J},$$

which gives the consistent definition of hydrostatic pressure based on different stress measures. Similarly one can consistently define the deviatoric stresses for different stress measures, namely

$$(12) \begin{aligned} \tau'' &= \tau + p\mathbf{I} = \tau - (\tau : \mathbf{I})\mathbf{I}/3, \quad \sigma'' = \sigma + Jp\mathbf{I} = \sigma - (\sigma : \mathbf{I})\mathbf{I}/3, \\ \mathbf{r}'' &= \mathbf{r} + Jp\mathbf{U}^{-1} = \mathbf{r} - (\mathbf{r}^* : \mathbf{U})\mathbf{U}^{-1}/3, \quad \sigma_r'' = \sigma_r + Jp\mathbf{I} = \sigma_r - (\sigma_r : \mathbf{I})\mathbf{I}/3, \\ \mathbf{S}'' &= \mathbf{S} + Jp\mathbf{C}^{-1} = \mathbf{S} - \frac{(\mathbf{S} : \mathbf{C})\mathbf{C}^{-1}}{3}, \quad \mathbf{r}^{*''} = \mathbf{r}^* + Jp\mathbf{U}^{-1} = \mathbf{r}^* - \frac{(\mathbf{r}^* : \mathbf{U})\mathbf{U}^{-1}}{3}, \\ \mathbf{T}'' &= \mathbf{T} + Jp(\mathbf{F}^{-1})^T = \mathbf{T} - (\mathbf{T} : \mathbf{F}^T)(\mathbf{F}^{-1})^T/3. \end{aligned}$$

Note that $\mathbf{F}^{-1} : \mathbf{F}^T = 3$. One can show that

$$(13) \quad \tau'' : \mathbf{I} = \sigma'' : \mathbf{I} = \sigma_r'' : \mathbf{I} = \mathbf{r}'' : \mathbf{U} = \mathbf{S}'' : \mathbf{C} = \mathbf{r}^{*''} : \mathbf{U} = \mathbf{T}'' : \mathbf{F}^T = 0.$$

We have established in Eq. (5), (7), (8a) and (9a) that the work conjugate variables of σ , \mathbf{T} , \mathbf{S} , \mathbf{r}^* and σ_r are $\dot{\mathbf{D}}$, $\dot{\mathbf{F}}^T$, $\dot{\mathbf{E}}$, $\dot{\mathbf{U}}$ and $\mathbf{R} \cdot \mathbf{D} \cdot \mathbf{R}^T$ respectively. Let us define

$$P_d \equiv \int_{R_0} \rho_0 \dot{W} dV = \int_R \rho \dot{W} dv$$

as the *deformation power*, where R_0 is the undeformed body. We can express P_d in alternative forms based on different stress measures, that is,

$$(14) \blacktriangle P_d = \int_R \tau : \mathbf{D} dv = \int_{R_0} \sigma : \mathbf{D} dV = \int_{R_0} \mathbf{T} : \dot{\mathbf{F}}^T dV = \int_{R_0} \mathbf{S} : \dot{\mathbf{E}} dV \\ = \int_{R_0} \mathbf{r}^* : \dot{\mathbf{U}} dV = \int_{R_0} \mathbf{r} : \dot{\mathbf{U}} dV = \int_{R_0} \sigma_r : (\mathbf{R}^T \cdot \mathbf{D} \cdot \mathbf{R}) dV.$$

Using Eq. (10) and (13.2:12), one can derive the following relations among the stress measures:

$$\tau \cdot da = \mathbf{T}^T \cdot dA = \mathbf{F} \cdot \mathbf{S} \cdot dA = \mathbf{R} \cdot \mathbf{r}^{*T} \cdot dA = \mathbf{R} \cdot \sigma_r \cdot \mathbf{U}^{-1} \cdot dA = \mathbf{R} \cdot \sigma_r \cdot \mathbf{R}^T \cdot da/J,$$

which means that the transformation of the force vector $\tau \cdot da$ by \mathbf{I} , \mathbf{F}^{-1} , \mathbf{R}^T gives the force vectors $\mathbf{T}T \cdot dA$, $\mathbf{S} \cdot dA$, $\mathbf{r}^{*T} \cdot dA$, respectively, on the undeformed surface. We also have the rotated force vector $\mathbf{R}T \cdot (\sigma \cdot da) \cdot \mathbf{R} = \sigma_r \cdot da^*$ where $da^* = \mathbf{R}T \cdot da \cdot \mathbf{R}$ is a rotated surface of da . The geometric interpretation of the different force vectors is explained in more details below.

Graphic Description of Stress Measures. Because of the importance of the stress tensors, we would like to offer a graphic description of the Lagrangian, second Piola–Kirchhoff and Biot–Luré stresses below. This is to supplement the graphic details presented in [Chapter 3](#) for the Cauchy stress, and to do the derivation once more in the indicial notation, which was used in [Chapters 3–12](#). Consider an element of a strained solid as shown on the right-hand side of [Fig. 13.7:1](#). Assume that in the original (undeformed) state this element has the configuration as shown on left side of [Fig. 13.7:1](#). A force vector $d\overset{\nu}{\mathbf{T}}$ on the surface $PQRS$. A corresponding force vector $d\overset{\nu}{\mathbf{T}}_0$ acts on the surface $P_0Q_0R_0S_0$. If we assign a rule of correspondence between $d\overset{\nu}{\mathbf{T}}$ and $d\overset{\nu}{\mathbf{T}}_0$ for every corresponding pair of surfaces, and define stress vectors in each case as the limiting ratios $d\overset{\nu}{\mathbf{T}}/da$, $d\overset{\nu}{\mathbf{T}}_0/dA$, where da and dA are the areas of $PQRS$, $P_0Q_0R_0S_0$, respectively, then by the method of [Chapter 3](#) we can define stress tensors in both configurations. The assignment of a correspondence rule between $d\overset{\nu}{\mathbf{T}}$ and $d\overset{\nu}{\mathbf{T}}_0$ is based on the equations derived above.

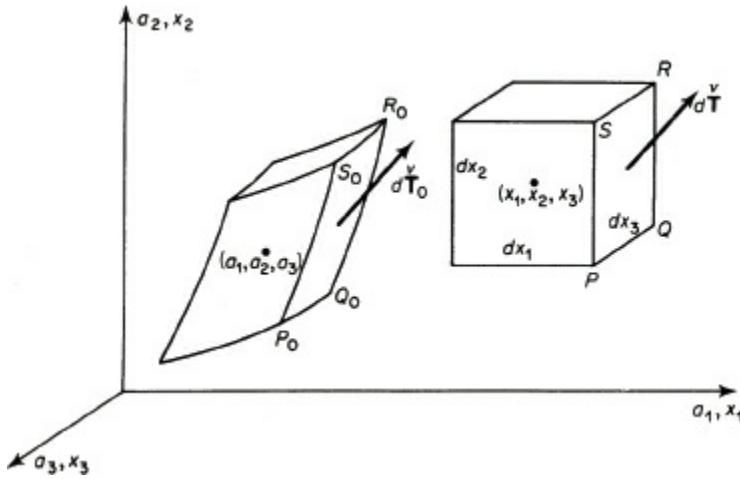


Fig. 13.7:1. The corresponding tractions in the original and deformed state of body.

The following alternative rules are known as the Lagrangian, the second Piola–Kirchhoff and the Biot–Luré rules, respectively (see Fig. 13.7:2)

$$(15) \quad d\overset{\nu}{T}_{0i}^{(L)} = d\overset{\nu}{T}_i \quad \text{or} \quad d\overset{\nu}{T}_0^{(L)} = d\overset{\nu}{T} \quad (\text{Lagrangian rule}),$$

$$(16) \quad d\overset{\nu}{T}_{0i}^{(K)} = \frac{\partial X_i}{\partial x_j} d\overset{\nu}{T}_j \quad \text{or} \quad d\overset{\nu}{T}_0^{(K)} = \mathbf{F}^{-1} \cdot d\overset{\nu}{T} \quad (\text{Piola–Kirchhoff rule}),$$

$$(17) \quad d\overset{\nu}{T}_{0i}^{(B)} = R_{ij} d\overset{\nu}{T}_j \quad \text{or} \quad d\overset{\nu}{T}_0^{(B)} = \mathbf{R} \cdot d\overset{\nu}{T} \quad (\text{Biot–Luré rule}).$$

In other words, the Lagrangian rule assigns the force vector $d\overset{\nu}{T}_0^{(L)}$ acting on the surface element dA in the undeformed configuration to be equal to the force vector $d\overset{\nu}{T}$ acting on the corresponding surface element da in the current configuration; the second Piola–Kirchhoff rule specifies $d\overset{\nu}{T}_0^{(K)}$, the force vector acting on the surface element dA , to be related to $d\overset{\nu}{T}$ by the same rule as the transformation $dX_i = (\partial X_i / \partial x_j) dx_j$; whereas the second Biot–Luré rule specifies $d\overset{\nu}{T}_0^{(B)}$ to be $d\overset{\nu}{T}$ rotated by \mathbf{R} . Note that, in the present case, both the deformed and undeformed configurations have the same base vectors. There is no need to use capital subscripts to distinguish indices referred to the undeformed configuration.

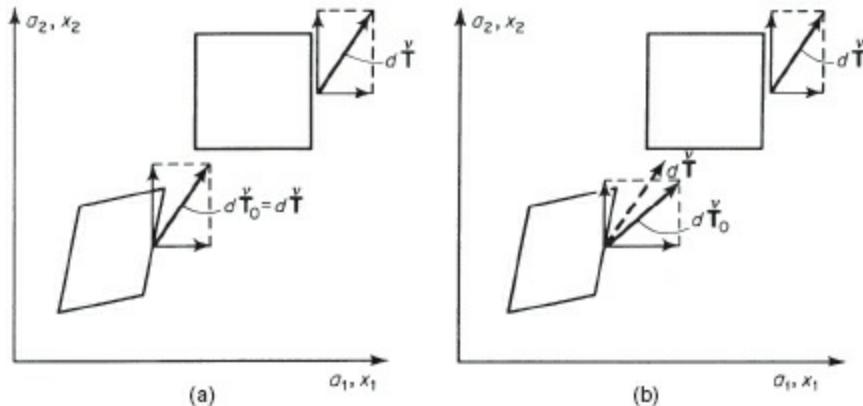


Fig. 13.7:2. The correspondence of force vectors in defining (a) Lagrange's and (b) the second Piola–Kirchhoff stress, illustrated in a two-dimensional case.

Let \mathbf{n} and \mathbf{N} be the unit vectors normal to da and dA , respectively. If τ_{ij} is the Cauchy stress tensor referred to the strained state, from *Cauchy's formula*, the vector $d\overset{\nu}{T}$ with components $d\overset{\nu}{T}_i$, can be written as

$$(18) \quad d\overset{\nu}{T}_i = \tau_{ij} n_j da, \quad \text{or} \quad d\overset{\nu}{T} = \boldsymbol{\tau} \cdot da.$$

We now define stress components referred to the original state by a law similar to Eq. (18). If Eq. (15) is used, we write

$$(19) \quad d\overset{\nu}{T}_{0i}^{(L)} = T_{ji} N_j dA = d\overset{\nu}{T}_i, \quad d\overset{\nu}{T}_0 = \mathbf{T}^T \cdot d\mathbf{A} = d\overset{\nu}{T},$$

according to the Lagrangian rule. Note: $T_{ij} \neq T_{ji}$. Equation (16) gives

$$(20) \quad dT_{0i}^{\nu(K)} = S_{ji}N_j dA = \frac{\partial X_i}{\partial x_j} dT_j^{\nu}, \quad d\mathbf{T}_0^{\nu(K)} = \mathbf{S} \cdot d\mathbf{A} = \mathbf{F}^{-1} \cdot d\mathbf{T}^{\nu},$$

according to the second Piola–Kirchhoff rule. Note that $S_{ij} = S_{ji}$ and that τ_{ij} , T_{ij} , S_{ij} are the Cauchy, the Lagrangian, and the second Piola–Kirchhoff stress tensors, respectively.

The relationship among τ_{ij} , T_{ij} and S_{ij} can now be derived. Using that between $n_i da$ and $N_i dA$ given in Eq. (13.2:12), from Eq. (18) and (19), we obtain the relationship among τ_{ij} , T_{ij} and S_{ij}

$$\begin{aligned} T_{ji}N_j dA &= \tau_{ji}n_j da = J\tau_{mi}(\partial X_j/\partial x_m)N_j dA, \text{ or} \\ \mathbf{N} \cdot \mathbf{T} dA &= \boldsymbol{\tau} \cdot \mathbf{n} da = J\boldsymbol{\tau} \cdot (\partial \mathbf{X}/\partial \mathbf{x}) \cdot \mathbf{N} da \end{aligned}$$

which implies

$$(21) \quad T_{ji} = J(\partial X_j/\partial x_m)\tau_{mi} = (\partial X_j/\partial x_m)\sigma_{mi}, \quad \text{or} \quad \mathbf{T} = J\mathbf{F}^{-1} \cdot \boldsymbol{\tau} = \mathbf{F}^{-1} \cdot \boldsymbol{\sigma},$$

which is the same as that defined in Eq. (4), where J is the Jacobian of the transformation from the undeformed to the deformed configuration defined in Eqs. (13.2:5), (13.2:16), or (13.2:17). Similarly, from Eq. (18) and (20),

$$\begin{aligned} (22) \quad S_{ji} &= J(\partial X_i/\partial x_m)(\partial X_j/\partial x_n)\tau_{nm} = (\partial X_i/\partial x_m)(\partial X_j/\partial x_n)\sigma_{nm}, \quad \text{or} \\ \mathbf{S} &= J\mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot (\mathbf{F}^{-1})^T = \mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot (\mathbf{F}^{-1})^T \end{aligned}$$

as given in Eq. (6). Hence, from the two equations above, we find the relation between \mathbf{S} and \mathbf{T}

$$S_{ji} = T_{in}(\partial X_j/\partial x_n), \quad \text{or} \quad \mathbf{S} = \mathbf{T} \cdot (\mathbf{F}^{-1})^T.$$

Similarly, we can show the relation between $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}_r$ using Eq. (17).

The Eulerian (Cauchy) stress tensor τ_{ij} is symmetric as shown in Chapter 3. From Eq. (21), we see that, in general, the Lagrangian stress tensor T_{ji} is not symmetric, whereas from Eq. (22), the second Piola–Kirchhoff stress tensor S_{ji} is. The Lagrangian tensor is inconvenient to use in a stress-strain law in which the strain tensor is always symmetric, but is often convenient in laboratory work. The second Piola–Kirchhoff stress tensor is more suitable for the purpose.

From the identities

$$\delta_{ij} = (\partial X_i/\partial x_m)(\partial x_m/\partial X_j), \quad \delta_{ij} = (\partial x_i/\partial X_m)(\partial X_m/\partial x_j),$$

we find at once the relations

$$(23) \quad \begin{aligned} \tau_{ji} &= (\partial x_i/\partial X_m)T_{mj}/J = (\partial x_i/\partial X_m)(\partial x_j/\partial X_n)S_{nm}/J, \\ \text{or} \quad \boldsymbol{\tau} &= \mathbf{F} \cdot \mathbf{T}/J = \boldsymbol{\sigma}/J, \end{aligned}$$

$$(24) \quad T_{ij} = S_{im}(\partial x_j/\partial X_m), \quad \text{or} \quad \mathbf{T} = \mathbf{S} \cdot \mathbf{F}^T.$$

13.8. EXAMPLE: COMBINED TENSION AND TORSION LOADS

Consider a thin-walled circular cylinder subjected to combined tension and torsion loads (Khan and Huang^{6,2} 1995). The geometry and deformation are described in terms of polar coordinates. The geometry of the tube is

$$R_m - h/2 \leq R \leq R_m + h/2, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq Z \leq L,$$

where h/R_m , $h/L \ll 1$. The point at (R, Θ, Z) before deformation moves to (r, θ, z) . In polar coordinates, the deformation can be written as

$$r = \alpha R, \quad \theta = \Theta + \omega Z, \quad z = \lambda Z.$$

Here ω and λ are the angle of twist ratio and the axial stretch ratio per unit undeformed length, respectively. The base-vectors for the deformed and undeformed coordinates are, respectively, $(\mathbf{e}_r, \mathbf{e}_{\theta}, \mathbf{e}_z)$ and $(\mathbf{e}_R, \mathbf{e}_{\Theta}, \mathbf{e}_Z)$. They are related to the fixed Cartesian base vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ by

$$(1) \quad \mathbf{e}_r = \cos \theta \mathbf{e}_X + \sin \theta \mathbf{e}_Y = \cos(\Theta + \omega Z) \mathbf{e}_X + \sin(\Theta + \omega Z) \mathbf{e}_Y,$$

$$(2) \quad \begin{aligned} \mathbf{e}_\theta &= -\sin\theta\mathbf{e}_X + \cos\theta\mathbf{e}_Y = -\sin(\Theta + \omega Z)\mathbf{e}_X + \cos(\Theta + \omega Z)\mathbf{e}_Y, \\ \mathbf{e}_z &= \mathbf{e}_Z, \quad \mathbf{e}_R = \cos\Theta\mathbf{e}_X + \sin\Theta\mathbf{e}_Y, \quad \mathbf{e}_\Theta = -\sin\Theta\mathbf{e}_X + \cos\Theta\mathbf{e}_Y. \end{aligned}$$

The base vectors $\mathbf{e}_r, \mathbf{e}_\theta$ are functions of time and the undeformed coordinates. The deformed position and the velocity vector are, respectively,

$$(3) \quad \mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z = R\alpha\mathbf{e}_r + Z\lambda\mathbf{e}_z, \quad \dot{\mathbf{x}} = R\dot{\alpha}\mathbf{e}_r + RZ\alpha\dot{\omega}\mathbf{e}_\theta + Z\dot{\lambda}\mathbf{e}_z.$$

The second term on the right hand side of the last equation arises from the time derivative of \mathbf{e}_r . The deformation gradient tensor \mathbf{F} and its material time derivative $\dot{\mathbf{F}}$ can be derived by taking the spatial derivatives of Eq. (3) with respect to the undeformed coordinates. In doing so, one must account for the fact also that both \mathbf{e}_r and \mathbf{e}_θ are functions of Θ and Z .

As an example, consider the derivation of $\dot{\mathbf{F}}$ from $\dot{\mathbf{x}}$ through the spatial derivative with respect to the undeformed coordinates:

$$\dot{\mathbf{F}} = (\partial\dot{\mathbf{x}}/\partial R)\mathbf{e}_R + (\partial\dot{\mathbf{x}}/\partial\Theta)\mathbf{e}_\Theta/R + (\partial\dot{\mathbf{x}}/\partial Z)\mathbf{e}_Z,$$

where

$$(4) \quad \begin{aligned} \partial\dot{\mathbf{x}}/\partial R &= \mathbf{e}_r\dot{F}_{rR} + \mathbf{e}_\theta\dot{F}_{\theta R} + \mathbf{e}_z\dot{F}_{zR} = \mathbf{e}_r\dot{\alpha} + \mathbf{e}_\theta Z\alpha\dot{\omega}, \\ (\partial\dot{\mathbf{x}}/\partial\Theta)/R &= \mathbf{e}_r\dot{F}_{r\Theta} + \mathbf{e}_\theta\dot{F}_{\theta\Theta} + \mathbf{e}_z\dot{F}_{z\Theta} = -\mathbf{e}_r Z\alpha\dot{\omega} + \mathbf{e}_\theta\dot{\alpha}, \\ \partial\dot{\mathbf{x}}/\partial Z &= \mathbf{e}_r\dot{F}_{rZ} + \mathbf{e}_\theta\dot{F}_{\theta Z} + \mathbf{e}_z\dot{F}_{zZ} = -\mathbf{e}_r RZ\alpha\omega\dot{\omega} + \mathbf{e}_\theta R(\alpha\dot{\omega} + \dot{\alpha}\omega) + \dot{\lambda}\mathbf{e}_z, \end{aligned}$$

where $\dot{F}_{rR}, \dot{F}_{r\Theta}, \dots$ are the components of the two-point tensor $\dot{\mathbf{F}}$. The lowercase subscripts denote the association with the current coordinate frame and the upper-case subscripts denote the association with the reference frame. A comparison of the components on both sides of Eq. (4) yields $\dot{F}_{rR}, \dot{F}_{r\Theta}, \dots$, etc. Following this procedure, one obtains

$$(5) \quad \mathbf{F} = \begin{bmatrix} \partial r/\partial R & (\partial r/\partial\Theta)/R & \partial r/\partial Z \\ r(\partial\theta/\partial R) & r(\partial\theta/\partial\Theta)/R & r(\partial\theta/\partial Z) \\ \partial z/\partial R & (\partial z/\partial\Theta)/R & \partial z/\partial Z \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & \omega\alpha R_m \\ 0 & 0 & \lambda \end{bmatrix},$$

$$(6) \quad \dot{\mathbf{F}} = \begin{bmatrix} \dot{\alpha} & -Z\alpha\dot{\omega} & -R_m Z\alpha\omega\dot{\omega} \\ Z\alpha\dot{\omega} & \dot{\alpha} & R_m(\alpha\dot{\omega} + \dot{\alpha}\omega) \\ 0 & 0 & \dot{\lambda} \end{bmatrix},$$

in which the mean undeformed radius R_m has been used to replace R . From Eq. (5), it follows that

$$\mathbf{F}^{-1} = \begin{bmatrix} 1/\alpha & 0 & 0 \\ 0 & 1/\alpha & -\omega R_m/\lambda \\ 0 & 0 & 1/\lambda \end{bmatrix}, \quad J = \det(\mathbf{F}) = \alpha^2\lambda.$$

To obtain \mathbf{R} , \mathbf{U} and \mathbf{V} in the polar decomposition, one writes the rotational tensor in the form

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}.$$

The solution of the equations $\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$ gives the right and left stretch tensors

$$\mathbf{U} = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & \sin \phi & \omega R_m \sin \phi + \lambda \cos \phi / \alpha \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha \omega R_m \sin \phi + \lambda \cos \phi & \lambda \sin \phi \\ 0 & \lambda \sin \phi & \lambda \cos \phi \end{bmatrix},$$

where

$$\cos \phi = (\lambda + \alpha) / d, \quad \sin \phi = \alpha \omega R_m / d, \quad d = \sqrt{(\lambda + \alpha)^2 + (\alpha \omega R_m)^2}.$$

The other quantities can be determined in a straightforward manner:

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & \omega \alpha^2 R_m \\ 0 & \omega \alpha^2 R_m & \lambda^2 + \omega^2 \alpha^2 R_m^2 \end{bmatrix},$$

$$\mathbf{B}^{-1} = (\mathbf{F}^{-1})^T \cdot \mathbf{F}^{-1} = \begin{bmatrix} 1/\alpha^2 & 0 & 0 \\ 0 & 1/\alpha^2 & -\omega R_m / (\alpha \lambda) \\ 0 & -\omega R_m / (\alpha \lambda) & (1 + \omega^2 R_m^2) / \lambda^2 \end{bmatrix}.$$

The Lagrangian and the Eulerian strains can be obtained by using $\mathbf{E} = (\mathbf{C} - \mathbf{I})/2$ and $\mathbf{e} = (\mathbf{I} - \mathbf{B}^{-1})/2$. The velocity gradient tensor is simply

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \begin{bmatrix} \dot{\alpha}/\alpha & -\dot{\omega}Z & 0 \\ \dot{\omega}Z & \dot{\alpha}/\alpha & \dot{\omega}\alpha R_m / \lambda \\ 0 & 0 & \dot{\lambda}/\lambda \end{bmatrix}.$$

Hence the stretch and spin rates are

$$\mathbf{D} = \begin{bmatrix} \dot{\alpha}/\alpha & 0 & 0 \\ 0 & \dot{\alpha}/\alpha & \dot{\omega}\alpha R_m / (2\lambda) \\ 0 & \dot{\omega}\alpha R_m / (2\lambda) & \dot{\lambda}/\lambda \end{bmatrix},$$

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & -\dot{\omega}Z & 0 \\ \dot{\omega}Z & 0 & \dot{\omega}\alpha R_m / (2\lambda) \\ 0 & -\dot{\omega}\alpha R_m / (2\lambda) & 0 \end{bmatrix}.$$

One can obtain \mathbf{L} directly by expressing $\dot{\mathbf{x}}$ in terms of the base vectors of the current configuration

$$\dot{\mathbf{x}} = r \dot{\alpha} \mathbf{e}_r / \alpha + r z \dot{\omega} \mathbf{e}_\theta / \lambda + z \dot{\lambda} \mathbf{e}_z / \lambda,$$

and taking the spatial derivatives with respect to the deformed coordinates, i.e., $\mathbf{L} = \partial \dot{\mathbf{x}} / \partial \mathbf{x}$.

The Cauchy, the first Piola–Kirchhoff and second Piola–Kirchhoff stresses are given by:

$$\boldsymbol{\tau} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau \\ 0 & \tau & \sigma \end{bmatrix},$$

$$\mathbf{T} = J\mathbf{F}^{-1} \cdot \boldsymbol{\tau} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\tau\alpha^2\omega R_m & \tau\alpha\lambda - \sigma\alpha^2\omega R_m \\ 0 & \tau\alpha^2 & \sigma\alpha^2 \end{bmatrix},$$

$$\mathbf{S} = \mathbf{T} \cdot (\mathbf{F}^{-1})^T = \frac{\alpha}{\lambda} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\tau\omega R_m\lambda + \sigma\alpha\omega^2 R_m^2 & \tau\lambda - \sigma\alpha\omega R_m \\ 0 & \tau\lambda - \sigma\alpha\omega R_m & \sigma\alpha \end{bmatrix}.$$

Here τ is deduced from the applied axial and torsion loads. The constitutive law is needed to determine the relation between τ , σ and λ , ω .

Problem 13.4. Show that $\mathbf{E} = \mathbf{F}^T \cdot \mathbf{e} \cdot \mathbf{F}$ where \mathbf{E} is the Green strain and \mathbf{e} is the Almansi strain.

Problem 13.5. In a simple shear experiment, the motion is described by

$$x = X + \gamma Y, \quad y = Y, \quad z = Z.$$

Find \mathbf{F} , \mathbf{C} , \mathbf{B} , \mathbf{R} , \mathbf{U} , \mathbf{V} , the principal stretches and the eigenvectors of \mathbf{C} and \mathbf{B} .

13.9. OBJECTIVITY

Truesdell and Noll (1965) discussed the concept of *frame indifference* of world events and the significance of the concept in the formulation of mechanics. Eringen (1962) introduced the term *axiom of objectivity*. Simply stated, the constitutive equation must be invariant under changes of reference frame. A quantity or an equation is *frame indifference* or *objective* if it is invariant under changes of reference frame. We give below the mathematical definition of the term and examine what quantities are objective and what are not. To appreciate the beauty and significance of these concepts, one must read the original papers.

Consider two motions $\mathbf{x}(\mathbf{X}, t)$ and $\bar{\mathbf{x}}(\mathbf{X}, t)$ of a body that differ by a rigidbody motion, that is,

$$(1) \quad \bar{\mathbf{x}}(\mathbf{X}, t) = \mathbf{Q}(t) \cdot \mathbf{x}(\mathbf{X}, t) + \mathbf{c}(t),$$

where $\mathbf{Q}(t)$ is a rotational tensor and $\mathbf{c}(t)$ is a vector representing a rigid body translation. All quantities associated with the two deformations will have different values in general. In the subsequent discussion, we shall use the notations $(\bar{\cdot})$ and (\cdot) to denote the same relevant quantity in the barred ($\bar{\mathbf{x}}$) and the unbarred (\mathbf{x}) configurations. For instance, \mathbf{F} and $\bar{\mathbf{F}}$ will denote the deformation gradients in the respective configurations. Truesdell and Noll, and Eringen defined that a vector or a tensor of rank 2 is *frame indifferent* or *objective* if its corresponding quantities in the two configurations are related by the following transformation rules:

$$(2) \quad \mathbf{v} = \mathbf{Q} \cdot \mathbf{v},$$

if \mathbf{v} is a vector, and

$$(3) \quad \bar{\mathbf{G}} = \mathbf{Q} \cdot \mathbf{G} \cdot \mathbf{Q}^T,$$

if \mathbf{G} is a tensor of rank 2.

To examine the objectivity of various quantities we first consider a line element $d\bar{\mathbf{x}}$ in the barred configuration and its corresponding line element $d\mathbf{x}$ in the unbarred configuration. From Eq. (1), one finds

$$(4) \quad d\bar{\mathbf{x}} = \mathbf{Q} \cdot d\mathbf{x},$$

which satisfies the transformation rule Eq. (2) for a vector. Hence the line element $d\mathbf{x}$ is an objective vector. One can also show that all area elements and all volume elements are objective. In fact a volume element is always objective because it is a scalar, which is always so.

Now consider the deformation gradients $\bar{\mathbf{F}}$, \mathbf{F} in the two configurations:

$$\bar{\mathbf{F}} \cdot d\mathbf{X} = d\bar{\mathbf{x}} = \mathbf{Q} \cdot d\mathbf{x} = \mathbf{Q} \cdot \mathbf{F} \cdot d\mathbf{X},$$

it implies

$$(5) \quad \bar{\mathbf{F}} = \mathbf{Q} \cdot \mathbf{F}.$$

This shows that \mathbf{F} is not objective because \mathbf{F} , being a tensor of rank 2, does not follow the transformation rule Eq. (3) for tensor of rank 2. Note that \mathbf{F} is a two-point tensor. The components associated with the deformed coordinates change with the reference frame while those associated with the undeformed coordinates do not. This is because the undeformed coordinates are fixed whereas the reference frame for the deformed coordinates changes. Following Eq. (5), one finds

$$(6) \quad \bar{\mathbf{C}} = \bar{\mathbf{F}}^T \cdot \bar{\mathbf{F}} = \bar{\mathbf{F}}^T \cdot \mathbf{Q} \cdot \mathbf{Q}^T \cdot \bar{\mathbf{F}} = \mathbf{F}^T \cdot \mathbf{F} = \mathbf{C},$$

which indicates that \mathbf{C} is also not objective because the transformation does not follow the rule given in Eq. (3). We should expect \mathbf{C} not to change as shown in Eq. (6) for different reference deformed coordinates because \mathbf{C} is a Lagrangian tensor referred to the original frame, which does not depend on the current reference frame. In the mean time,

$$\bar{\mathbf{B}} = \bar{\mathbf{F}} \cdot \bar{\mathbf{F}}^T = \mathbf{Q} \cdot \mathbf{B} \cdot \mathbf{Q}^T,$$

which transforms like Eq. (3). Therefore, \mathbf{B} is an objective tensor. This is also expected because \mathbf{B} is an Eulerian tensor referred to the reference frame. Similarly, the Lagrangian strain tensor \mathbf{E} is not objective, because

$$\bar{\mathbf{E}} = \mathbf{E},$$

which does not satisfy Eq. (3). On the other hand the Almansi strain tensor \mathbf{e} , an Eulerian tensor, transforms as

$$\bar{\mathbf{e}} = \mathbf{Q} \cdot \mathbf{e} \cdot \mathbf{Q}^T,$$

and is, therefore, objective. Using the polar decomposition and Eq. (5), one derives

$$\bar{\mathbf{F}} = \bar{\mathbf{R}} \cdot \bar{\mathbf{U}} = \mathbf{Q} \cdot \mathbf{F} = \mathbf{Q} \cdot \mathbf{R} \cdot \mathbf{U}.$$

Since $\mathbf{Q} \cdot \mathbf{R}$ is still a rotational tensor, and $\bar{\mathbf{U}}$ and \mathbf{U} are symmetric, and since the polar decomposition is unique, the equation above implies

$$(7) \quad \bar{\mathbf{R}} = \mathbf{Q} \cdot \mathbf{R}, \quad \bar{\mathbf{U}} = \mathbf{U},$$

which shows that neither \mathbf{R} nor \mathbf{U} is objective. However, for the left polar decomposition,

$$(8) \quad \bar{\mathbf{F}} = \bar{\mathbf{V}} \cdot \bar{\mathbf{R}} = \mathbf{Q} \cdot \mathbf{V} \cdot \mathbf{R},$$

one obtains

$$(9) \quad \bar{\mathbf{V}} = \mathbf{Q} \cdot \mathbf{V} \cdot \mathbf{R} \cdot \bar{\mathbf{R}}^T = \mathbf{Q} \cdot \mathbf{V} \cdot \mathbf{Q}^T,$$

which means the left stretch tensor \mathbf{V} is objective. Thus \mathbf{B} , \mathbf{e} , and \mathbf{V} , all Eulerian tensors, are objective while \mathbf{C} , \mathbf{E} , and \mathbf{U} , all Lagrangian tensors, are not.

Vectors and tensors related to the velocity field are in general not objective. To show this result, we first establish the properties of the rotation rate tensor. Let us define $\Psi \equiv \mathbf{Q}^T \cdot \dot{\mathbf{Q}}$. Since $\mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{I}$, it follows that

$$D(\mathbf{Q}^T \cdot \mathbf{Q})/Dt = \dot{\mathbf{Q}}^T \cdot \mathbf{Q} + \mathbf{Q}^T \cdot \dot{\mathbf{Q}} = \Psi^T + \Psi = 0,$$

which indicates that $\Psi (= -\Psi^T)$ is antisymmetric and that

$$\dot{\mathbf{Q}} = -\mathbf{Q} \cdot \dot{\mathbf{Q}}^T \cdot \mathbf{Q}.$$

For an objective vector \mathbf{v} , i.e., $\mathbf{v}^\perp = \mathbf{Q} \cdot \mathbf{v}$, the following is true

$$(10) \quad \dot{\mathbf{v}} = (\dot{\mathbf{Q}} \cdot \mathbf{v}) = \mathbf{Q} \cdot (\dot{\mathbf{v}} + \Psi \cdot \mathbf{v}),$$

which means that $\bar{\mathbf{v}} = \mathbf{Q} \cdot \mathbf{v}$, is in general not objective. Similarly, for an objective tensor \mathbf{G} of rank 2, i.e., $\dot{\mathbf{v}}$ one can show that

$$(11) \quad \dot{\mathbf{G}} = \mathbf{Q} \cdot (\dot{\mathbf{G}} + \Psi \cdot \mathbf{G} - \mathbf{G} \cdot \Psi) \cdot \mathbf{Q}^T.$$

Now consider the deformation rate. It can be shown that

$$\dot{\mathbf{F}} = \mathbf{Q} \cdot (\dot{\mathbf{F}} + \Psi \cdot \mathbf{F}) = \bar{\mathbf{L}} \cdot \bar{\mathbf{F}},$$

which leads to

$$(12) \quad \bar{\mathbf{L}} = \mathbf{Q} \cdot (\dot{\mathbf{F}} + \Psi \cdot \mathbf{F}) \cdot \mathbf{F}^{-1} \cdot \mathbf{Q}^T = \mathbf{Q} \cdot (\mathbf{L} + \Psi) \cdot \mathbf{Q}^T = \mathbf{Q} \cdot (\mathbf{D} + \Omega + \Psi) \cdot \mathbf{Q}^T,$$

where $\mathbf{L} = \mathbf{D} + \Omega$, and \mathbf{D} and Ω are the deformation and spin rate tensors. Splitting $\mathbf{A}\mathbf{u}_n^+$ into the symmetric part $\bar{\mathbf{D}}$ and the antisymmetric parts $\bar{\Omega}$, i.e., $\dot{\mathbf{v}}$ we obtain from Eq. (12)

$$(13) \quad \bar{\mathbf{D}} = \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^T,$$

$$(14) \quad \bar{\Omega} = \mathbf{Q} \cdot (\Omega + \Psi) \cdot \mathbf{Q}^T = \mathbf{Q} \cdot \Omega \cdot \mathbf{Q}^T + \dot{\mathbf{Q}} \cdot \mathbf{Q}^T.$$

One sees that only \mathbf{D} satisfies the transformation law of objectivity while both the velocity gradient tensor \mathbf{L} and the spin tensor Ω do not.

We shall now examine the objectivity of stress measures. The objectivity of the surface traction $\frac{\nu}{\mathbf{T}}$ is based on physical grounds. Then

$$\bar{\tau} \cdot \bar{\mathbf{n}} = \bar{\mathbf{T}} = \mathbf{Q} \cdot \frac{\nu}{\mathbf{T}} = \mathbf{Q} \cdot \tau \cdot \mathbf{Q}^T \cdot \mathbf{Q} \cdot \mathbf{n} = \mathbf{Q} \cdot \tau \cdot \mathbf{Q}^T \cdot \bar{\mathbf{n}},$$

where \mathbf{n} is the unit normal to the element. Therefore

$$(15) \quad \bar{\tau} = \mathbf{Q} \cdot \tau \cdot \mathbf{Q}^T,$$

which indicates that the Cauchy stress is objective. This is expected because, similar to the case of strain measures, the Cauchy stress is an Eulerian tensor and, therefore, is objective. For the Lagrangian stress \mathbf{T} and the second Piola-Kirchhoff stress \mathbf{S} , it is easy to show that

$$(16) \quad \bar{\mathbf{T}} = \mathbf{T} \cdot \mathbf{Q}^T, \quad \bar{\mathbf{S}} = \mathbf{S}, \quad \bar{\mathbf{r}}^* = \mathbf{r}^*, \quad \bar{\sigma}_r = \sigma_r,$$

all of which are not objective. This can be expected because \mathbf{T} and \mathbf{r}^* are two-point tensor and \mathbf{S} and σ_r are Lagrangian tensor.

Finally, we shall discuss the transformation of stress rates. Stress rates are especially needed in plasticity theory where the constitutive equations are often expressed in terms of stress increments (rates). They are also needed in the incremental approach for solving large deformation problems. From Eq. (13.7:10), using Eqs. (13.5:3a) and (13.6:6) for the rates of deformation, one find the rates of various stress measures:

$$(17) \quad \dot{\mathbf{T}} = J\mathbf{F}^{-1} \cdot [\dot{\tau} + \text{tr}(\mathbf{D})\tau - \mathbf{L} \cdot \tau],$$

$$(18) \quad \dot{\mathbf{S}} = J\mathbf{F}^{-1} \cdot [\dot{\tau} + \text{tr}(\mathbf{D})\tau - \mathbf{L} \cdot \tau - \tau \cdot \mathbf{L}^T] \cdot (\mathbf{F}^{-1})^T,$$

$$(19) \quad \dot{\mathbf{r}}^* = J\mathbf{F}^{-1} \cdot [\dot{\tau} + \text{tr}(\mathbf{D})\tau - \mathbf{L} \cdot \tau + \tau \cdot \Omega] \cdot \mathbf{R},$$

$$(20) \quad \dot{\sigma}_r = \mathbf{R}^T \cdot [\dot{\tau} - \Omega \cdot \tau + \tau \cdot \Omega + \text{tr}(\mathbf{D})\tau] \cdot \mathbf{R}.$$

Using Eq. (14) that $\dot{\mathbf{Q}} = \bar{\Omega} \cdot \mathbf{Q} - \mathbf{Q} \cdot \Omega$ and differentiating Eq. (15) and (16), one can establish that

$$(21) \quad \dot{\tau} - \bar{\Omega} \cdot \bar{\tau} + \bar{\tau} \cdot \bar{\Omega} = \mathbf{Q} \cdot (\dot{\tau} - \Omega \cdot \tau + \tau \cdot \Omega) \cdot \mathbf{Q}^T,$$

$$(22) \quad \dot{\bar{\mathbf{T}}} + \bar{\mathbf{T}} \cdot \bar{\Omega} = (\dot{\mathbf{T}} + \mathbf{T} \cdot \Omega) \cdot \mathbf{Q}^T,$$

$$(23) \quad \dot{\bar{\mathbf{S}}} = \dot{\bar{\mathbf{S}}},$$

$$(24) \quad \dot{\bar{\mathbf{r}}^*} = \dot{\mathbf{r}}^*,$$

$$(25) \quad \dot{\bar{\sigma}}_r = \dot{\sigma}_r.$$

None of the material rates $\dot{\tau}$, $\dot{\mathbf{T}}$, $\dot{\mathbf{S}}$, $\dot{\mathbf{r}}^*$ and $\dot{\sigma}_r$ are objective. However, from Eq. (21) we see that

$$(26) \quad \tau^\theta = \dot{\tau} - \Omega \cdot \tau + \tau \cdot \Omega$$

is objective and called the *Jaumann (corotational) stress rate*.

In general, for an objective tensor \mathbf{G} , rates such as

$$\mathbf{G}^\theta = \dot{\mathbf{G}} - \boldsymbol{\Omega} \cdot \mathbf{G} + \mathbf{G} \cdot \boldsymbol{\Omega}, \quad (\text{Jaumann rate})$$

$$\mathbf{G}^\nabla = \dot{\mathbf{G}} - \mathbf{L} \cdot \mathbf{G} - \mathbf{G} \cdot \mathbf{L}^T, \quad (\text{Oldroyd rate})$$

$$\mathbf{G}^\diamond = \dot{\mathbf{G}} + \mathbf{L}^T \cdot \mathbf{G} + \mathbf{G} \cdot \mathbf{L} \quad (\text{Cotter-Rivlin rate})$$

are all objective. They are respectively, called the *Jaumann*, *Oldroyd*, and *Cotter-Rivlin rates*. It can also be shown that the rate $\dot{\mathbf{G}} + \text{tr}(\mathbf{D})\mathbf{G} + \mathbf{L}^T \cdot \mathbf{G} + \mathbf{G} \cdot \mathbf{L}$ introduced by Truesdell is objective if \mathbf{G} is so.

To conclude this section, we mention two useful properties of rates: if \mathbf{G} is a symmetric tensor, then

$$D(\mathbf{G} : \mathbf{G})/Dt = 2\mathbf{G} : \mathbf{G}^\theta;$$

and if \mathbf{G} is a deviatoric tensor, so is \mathbf{G}^θ .

Problem 13.6. Let \mathbf{G}^* be a Lagrangian tensor and \mathbf{G} an Eulerian tensor. If $\mathbf{G} = \mathbf{F}^{-T} \cdot \mathbf{G}^* \cdot \mathbf{F}^{-1}$, then $\dot{\mathbf{G}}^*$ is linearly related to the Cotter-Rivlin rate of \mathbf{G} ; if $\mathbf{G} = \mathbf{F}^{-T} \cdot \mathbf{G}^* \cdot \mathbf{F}^{-1}/J$, then $\dot{\mathbf{G}}^*$ is linearly related to the Truesdell rate $\mathbf{G}^\Delta = \dot{\mathbf{G}} + \text{tr}(\mathbf{D})\mathbf{G} + \mathbf{L}^T \cdot \mathbf{G} + \mathbf{G} \cdot \mathbf{L}$ of \mathbf{G} ; if $\mathbf{G} = \mathbf{F} \cdot \mathbf{G}^* \cdot \mathbf{F}^{-1}$, then $\dot{\mathbf{G}}^*$ is linearly related to the rate $\mathbf{G}^\perp = \dot{\mathbf{G}} - \mathbf{L} \cdot \mathbf{G} + \mathbf{G} \cdot \mathbf{L}$; and if $\mathbf{G} = \mathbf{F} \cdot \mathbf{G}^* \cdot \mathbf{F}^{-1}/J$, then the rate $\dot{\mathbf{G}}^\parallel = \dot{\mathbf{G}} + \text{tr}(\mathbf{D})\mathbf{G} - \mathbf{L} \cdot \mathbf{G} + \mathbf{G} \cdot \mathbf{L}$ is linearly related to $\dot{\mathbf{G}}^*$.

13.10. EQUATION OF MOTION

Consider the body of a continuum occupying a region v with a boundary surface ∂v in the deformed state. The corresponding region in the original (natural or zero-stress) state is V with a boundary surface ∂V . The body is subjected to external loads: the inertia force, the body force \mathbf{b} per unit mass acting on a volume element dv , and the surface traction $\overset{\nu}{\mathbf{T}}$ acting on a surface element $d\mathbf{a}$ whose unit outer normal is \mathbf{n} . All these variables are in Eulerian description. The corresponding body force \mathbf{b}_0 in dV and surface traction \mathbf{T}_0 on $d\mathbf{A}$ of outward unit normal \mathbf{N} are in Lagrangian description. The original density is ρ_0 and that in the deformed state is ρ . The notations to be used are given in the table below.

	In Deformed Configuration	In Undeformed Configuration
Region	v	V
Boundary surface	∂v	∂V
Volume element	$dv = dx_1 dx_2 dx_3$	$dV = dX_1 dX_2 dX_3$
Surface element	$d\mathbf{a}$	$d\mathbf{A}$
Unit outer normal	$\mathbf{n}, (n_1, n_2, n_3)$	$\mathbf{N}, (N_1, N_2, N_3)$
Particle coordinates	$\mathbf{x}, (x_1, x_2, x_3)$	$\mathbf{X}, (X_1, X_2, X_3)$
Body force per unit mass	$\mathbf{b}, (b_1, b_2, b_3)$	$\mathbf{b}_0, (b_{01}, b_{02}, b_{03})$
Surface traction	$\overset{\nu}{\mathbf{T}}, (\overset{\nu}{T}_1, \overset{\nu}{T}_2, \overset{\nu}{T}_3)$	$\overset{\nu}{\mathbf{T}}_0, (\overset{\nu}{T}_{01}, \overset{\nu}{T}_{02}, \overset{\nu}{T}_{03})$
Density	ρ	ρ_0
Stresses	$\tau, (\tau_{ij})$ (Cauchy)	$\mathbf{T}, (T_{ji})$ (1 st Piola-Kirchhoff)

Note that the traction $\overset{\nu}{\mathbf{T}}$ is a vector (tensor of rank 1) and the first Piola–Kirchhoff stress \mathbf{T} is a tensor of rank 2.

Consider the motion of the body. Conservation of mass requires that $\rho dv = \rho_0 dV$ and $\mathbf{b} = \mathbf{b}_0$, i.e., the mass of a volume element dV and the body force per unit mass do not change as the body deforms. The conservation of momentum gives the Eulerian equation of motion of the continuum

$$(1) \quad \nabla_{\mathbf{x}} \cdot \boldsymbol{\tau} + \rho \mathbf{b} = \rho D\mathbf{v}/Dt,$$

[Eq. (5.5:7) in the present notation] in the current configuration. We can write the equation in a different reference frame and express the variables in terms of coordinates in that configuration. One way of doing this is to write the equation in the integral form as follows: the sum of the total body force and inertia force acting on the region $\nu(V)$ is

$$(2) \quad \begin{aligned} \int_v \left(\mathbf{b} - \frac{D\mathbf{v}}{Dt} \right) \rho dv &= \int_V \left(\mathbf{b}_0 - \frac{D\mathbf{v}}{Dt} \right) \rho_0 dV, \quad \text{or} \\ \int_v \left(b_i - \frac{Dv_i}{Dt} \right) \rho dv &= \int_V \left(b_{0i} - \frac{Dv_i}{Dt} \right) \rho_0 dV. \end{aligned}$$

The resultant of the surface traction $\overset{\nu}{\mathbf{T}}$ acting on the surface ∂v (or ∂V) is

$$(3) \quad \begin{aligned} \int_{\partial v} \overset{\nu}{\mathbf{T}} da &= \int_{\partial v} \mathbf{n} \cdot \boldsymbol{\tau} da = \int_{\partial V} \mathbf{N} \cdot \mathbf{T} dA, \quad \text{or} \\ \int_{\partial v} \overset{\nu}{T}_i da &= \int_{\partial v} \tau_j n_j da = \int_{\partial V} N_I T_{Ii} dA, \end{aligned}$$

in which Eq. (13.2:12) has been used and \mathbf{T} is the *Lagrangian* or the *first Piola–Kirchhoff stress tensor*. The use of Gauss' theorem leads to

$$(4) \quad \int_{\partial v} \overset{\nu}{\mathbf{T}} da = \int_V \nabla_{\mathbf{x}} \cdot \mathbf{T} dV, \quad \text{or} \quad \int_{\partial v} \overset{\nu}{T}_i da = \int_V \frac{\partial T_{Ii}}{\partial X_I} dV,$$

where $\nabla_{\mathbf{x}} \cdot$ is the divergence operator with respect to the \mathbf{X} -coordinates. Hence, by Eq. (2) and (4), the equation of motion can be written as

$$(5) \quad \int_V \left(\rho_0 \mathbf{b}_0 - \rho_0 \frac{D\mathbf{v}}{Dt} + \nabla_{\mathbf{x}} \cdot \mathbf{T} \right) dV = 0.$$

Since this equation must be valid for an arbitrary region V , the integrand must vanish. Hence

$$(6) \quad \nabla_{\mathbf{x}} \cdot \mathbf{T} + \rho_0 \mathbf{b}_0 = \rho_0 D\mathbf{v}/Dt, \quad T_{K_i, K} + \rho_0 b_{0i} = \rho_0 Dv_i/Dt$$

is the equation of motion in terms of the *first Piola–Kirchhoff stress*. The divergent operation is with respect to the undeformed coordinates. A substitution of Eq. (13.7:6) into Eq. (6) yields the equation of motion in terms of the *second Piola–Kirchhoff stress*:

$$(7) \quad \nabla_{\mathbf{x}} \cdot (\mathbf{S} \cdot \mathbf{F}^T) + \rho_0 \mathbf{b}_0 = \rho_0 \frac{D\mathbf{v}}{Dt}, \quad \frac{\partial}{\partial X_J} \left(S_{JK} \frac{\partial x_i}{\partial X_K} \right) + \rho_0 b_{0i} = \rho_0 \frac{Dv_i}{Dt}.$$

Since $\mathbf{x} = \mathbf{X} + \mathbf{u}$, where \mathbf{u} is the displacement vector, Eq. (7) can be written as

$$(8) \quad \nabla_{\mathbf{x}} \cdot \{ \mathbf{S} \cdot [\mathbf{I} + (\partial \mathbf{u} / \partial \mathbf{X})^T] \} + \rho_0 \mathbf{b}_0 = \rho_0 D\mathbf{v}/Dt, \quad \text{or}$$

$$(8a) \quad \partial [S_{KJ} (\delta_{IJ} + \partial u_I / \partial X_J)] / \partial X_K + \rho_0 b_{0i} = \rho_0 Dv_i/Dt,$$

if the deformed and undeformed configurations have the same base vectors. In this case, there is no distinction between the upper and lower case indices. Equation (8) exhibits the geometric effect of finite deformation through the displacement gradient $\partial u_I / \partial X_J$.

13.11. CONSTITUTIVE EQUATIONS OF THERMOELASTIC BODIES

Equations that describe the properties of materials are called the *constitutive equations*. Materials may be classified according to their constitutive equations. The constitutive equations do have to be restricted by the laws of thermodynamics. The case of small deformations is discussed in [Chapter 12](#). Large deformation does make the mathematical theory more complex. In this section, the thermo-mechanically-simple case will be discussed.

The constitutive equations must be restricted by the laws of thermodynamics. The *first law of thermodynamics* or *principle of energy balance* requires that the rate of increase of the total energy of the system is equal to the sum of the work rate and the heat flow, as shown in the global energy balance equation (12.2:5) or the local energy balance equation (12.2:7). By introducing the Helmholtz free energy per unit mass (12.6:5), the local energy balance can be expressed by Eq. (12.6:6). The *second law of thermodynamics* gives the entropy production inequality (12.6:8).

As a special class of thermomechanically simple materials, hyper-thermoelasticity is a type of coupled constitutive model for which the responses can be derived from an energy density function and a dissipation potential, which depend on up to the first-order gradients of displacement and temperature fields. The Helmholtz or Gibbs free energy density function may be used alternatively.

In the reference configuration, the entropy production inequality (12.6:7) can be rewritten in terms of the Helmholtz free energy as

$$(1) \quad \rho_0 T D_i \hat{s} / D t = \mathbf{S} : \dot{\mathbf{C}} / 2 - \rho_0 \hat{s} \dot{T} - \rho_0 \dot{\hat{h}} - T \nabla_R \cdot (\mathbf{H} / T - \mathbf{J}_s) - \mathbf{H} \cdot \mathbf{G} / T \geq 0,$$

where $\mathbf{S} = J \mathbf{F}^{-1} \cdot \tau \cdot \mathbf{F} - T$ is the second Piola–Kirchhoff stress tensor (13.7:6), $\mathbf{C} = \mathbf{F} \mathbf{T} \cdot \mathbf{F}$ is the right Cauchy–Green deformation tensor (13.2:7), $J = \rho_0 / \rho = \det[\mathbf{F}]$ is the Jacobian (13.2:5), $\mathbf{H} = J \mathbf{F}^{-1} \cdot \mathbf{h}$ is the heat flux, $\mathbf{J}_s = J \mathbf{F}^{-1} \cdot \mathbf{j}_s$ is the entropy flux and $\mathbf{G} = \nabla_R T$ is the temperature gradient in the reference configuration.

Alternatively, in the reference configuration, the entropy production inequality can be rewritten in terms of the Gibbs free energy as

$$(2) \quad \rho_0 T D_i \hat{s} / D t = -\dot{\mathbf{S}} : \mathbf{E} - \rho_0 \hat{s} \dot{T} - \rho_0 \dot{\hat{g}} - T \nabla_R \cdot (\mathbf{H} / T - \mathbf{J}_s) - \mathbf{H} \cdot \mathbf{G} / T \geq 0,$$

where $\mathbf{E} = (\mathbf{C} - \mathbf{I}) / 2$ is the Green strain tensor (13.3:5).

The Helmholtz free energy is taken to be a function of deformation, temperature, and temperature gradient in the reference configuration, that is,

$$(3) \quad \hat{h} = \hat{h}(\mathbf{C}, T, \mathbf{G}; \mathbf{X}).$$

In order that the entropy production inequality (1) is always valid, it is necessary and sufficient that the state equations fulfill the following thermodynamically admissible conditions:

$$(4) \quad \begin{aligned} \partial \hat{h} / \partial T_{,K} &= 0, & S_{KL} &= 2 \rho_0 \partial \hat{h} / \partial C_{KL}, & \hat{s} &= -\partial \hat{h} / \partial T, \\ \mathbf{H} / T &= \mathbf{J}_s, & D_i \hat{s} / D t &= (\mathbf{H} / T) \cdot (-\mathbf{G} / T) / \rho_0 & \geq 0. \end{aligned}$$

From the 1st equation of (4), the Helmholtz free energy is independent of temperature gradient, and so are the derived variables like stress and entropy. From the last equation of (4) the entropy production of hyper-thermoelastic materials originates from heat conduction, which is equal to the thermodynamic flux multiplied by the thermodynamic force. Thus the the transport law of heat conduction may be derived from the Fourier dissipation potential, a quadratic scalar function of the thermodynamic force $-\mathbf{G} / T$ as,

$$(5) \quad \mathcal{D}_{th} = (-\mathbf{G} / T) \cdot \mathbf{K} \cdot (-\mathbf{G} / T) / 2,$$

where the thermal conductivity tensor \mathbf{K} is symmetric that $\mathbf{K}^T = \mathbf{K}$ and is positive definite.

Hence, the thermodynamic flux for heat conduction depends linearly on the corresponding thermodynamic force with satisfaction of the Onsager principle, which is consistent with the discussion in [Sec. 12.5](#), that is,

$$(6) \quad \mathbf{H} / T = \partial \mathcal{D}_{th} / \partial (-\mathbf{G} / T) = \mathbf{K} \cdot (-\mathbf{G} / T).$$

A substitution of the 3rd equation of (4) into Eq. (12.6:6) using the 4th equation of (4) and Eq. (6) yields the heat transfer equation in terms of the Helmholtz free energy:

$$(7) \quad \rho_0 T D(-\partial \hat{h} / \partial T) / D t = \nabla_R \cdot (\mathbf{K} \cdot \nabla_R T).$$

The Gibbs free energy is a function of stress, temperature, and temperature gradient in the reference configuration, that is,

$$(8) \quad \hat{g} = \hat{g}(\mathbf{S}, T, \nabla_R T; \mathbf{X}) .$$

In order that the entropy production inequality (2) is always valid, it is necessary and sufficient that state equations fulfill the following thermodynamically admissible conditions:

$$(9) \quad \begin{aligned} \partial \hat{g} / \partial T_{,K} &= 0, \quad E_{KL} = -\rho_0 \partial \hat{g} / \partial S_{KL}, \quad \hat{s} = -\partial \hat{g} / \partial T, \\ \mathbf{H}/T &= \mathbf{J}_s, \quad D_i \hat{s} / D t = (\mathbf{H}/T) \cdot (-\mathbf{G}/T) / \rho_0 \geq 0. \end{aligned}$$

Based on the 1st equation of (9), the Gibbs free energy is independent of the temperature gradient, and so the derived variables like strain and entropy are also independent of the temperature gradient. It can be seen that the last equation of (9) is identical to the last equation of (4). Thus the transport law of heat conduction may be derived from the Fourier dissipation potential in the same way.

The heat transfer equation is obtained in terms of the Gibbs free energy,

$$(10) \quad \rho_0 T D(-\partial \hat{g} / \partial T) / D t = \nabla_R \cdot (\mathbf{K} \cdot \nabla_R T) .$$

Based on the fundamental principles of thermodynamics, a theory of finite thermoelasticity for anisotropic materials may be formulated from the free energy density as a function of finite strain and temperature. Expansion of the Helmholtz free energy density function with respect to the Green strain and the temperature deviation up to the second-order terms gives

$$(11) \quad \Delta \quad \rho_0 \hat{h} = \rho_0 \hat{h}_0 - \rho_0 s_0 \theta + S_{IJ}^0 E_{IJ} + \frac{1}{2} C_{IJKL} E_{IJ} E_{KL} - \frac{1}{2T_0} \rho_0 C_v \theta^2 - \beta_{IJ} E_{IJ} \theta,$$

where the Green strain tensor $\mathbf{E} = (\mathbf{C} - \mathbf{I})/2$ and the temperature deviation $\theta = T - T_0$ are used, C_{IJKL} , C_v and β_{IJ} are material properties, $C_{IJKL} = CKLJ = CJIKL = CLKIJ$, $\beta_{IJ} = \beta_{JI}$.

Substitution of Eq. (11) into the 2nd and 3rd equations of (4) yields the following anisotropic thermoelastic constitutive equations:

$$(12) \quad \Delta \quad S_{IJ} = S_{IJ}^0 + C_{IJKL} E_{KL} - \beta_{IJ} \theta,$$

$$(13) \quad \rho_0 \hat{s} = \rho_0 \hat{s}_0 + \beta_{IJ} E_{IJ} + \rho_0 C_v \theta / T_0 .$$

Substituting Eq. (13) into Eq. (7) yields the following coupled heat transfer equation:

$$(14) \quad \Delta \quad D(T_0 \beta_{IJ} E_{IJ} + \rho_0 C_v \theta) / D t = \nabla_R \cdot (\mathbf{K} \cdot \nabla_R \theta) .$$

Similarly, expansion of the Gibbs free energy density function with respect to the second Piola–Kirchhoff stress and the temperature deviation up to the second-order terms gives

$$(15) \quad \Delta \quad \rho_0 \hat{g} = \rho_0 \hat{g}_0 - \rho_0 s_0 \theta - S_{IJ} E_{IJ}^0 - (C_{IJKL}^{-1} S_{IJ} S_{KL} + \rho_0 C_p \theta^2 / T_0) / 2 - \alpha_{IJ} S_{IJ} \theta ,$$

Substitution of Eq. (15) into the 2nd and 3rd equations of (9) yields the following anisotropic thermoelastic constitutive equations:

$$(16) \quad \Delta \quad E_{IJ} = E_{IJ}^0 + C_{IJKL}^{-1} S_{KL} + \alpha_{IJ} \theta ,$$

$$(17) \quad \rho_0 \hat{s} = \rho_0 \hat{s}_0 + \alpha_{IJ} S_{IJ} + \rho_0 C_p \theta / T_0 .$$

Substituting Eq. (17) into Eq. (10) yields the following coupled heat transfer equation:

$$(18) \quad \Delta \quad D(T_0 \alpha_{IJ} S_{IJ} + \rho_0 C_p \theta) / D t = \nabla_R \cdot (\mathbf{K} \cdot \nabla_R \theta) .$$

These coupled thermoelastic constitutive equations are generalization of the Duhamel–Neumann law given in Sec. 12.8 to finite strain. In these equations, all the indices range over 1, 2, 3; C_{IJKL} is the elastic modulus tensor; β_{IJ} is the thermal modulus tensor; $\rho_0 C_v$ is the specific heat at constant volume; $\alpha_{IJ} = C_{IJKL}^{-1} \beta_{KL}$ is the thermal expansion tensor; C_{IJKL}^{-1} is the elastic compliance tensor; and $\rho_0 C_p = T_0 \beta_{IJ} C_{IJKL}^{-1} \beta_{KL} + \rho_0 C_v$ is specific heat at constant pressure.

For isotropic materials, these equations are reduced to

$$(19) \quad \Delta \quad S_{IJ} = S_{IJ}^0 + \lambda \delta_{IJ} E_{KK} + 2G E_{IJ} - \beta \delta_{IJ} \theta ,$$

$$(20) \quad \rho_0 \hat{s} = \rho_0 \hat{s}_0 + \beta E_{KK} + \rho_0 C_v \theta / T_0 ,$$

$$(21) \blacktrianglequad D(T_0\beta E_{KK} + \rho_0 C_v \theta)/Dt = \nabla_R \cdot (k \nabla_R \theta),$$

or

$$(22) \blacktrianglequad E_{IJ} = E_{IJ}^0 + (1 + \nu) S_{IJ}/E - \nu S_{KK} \delta_{IJ}/E + \alpha \delta_{IJ} \theta,$$

$$(23) \qquad \rho_0 \hat{s} = \rho_0 \hat{s}_0 + \alpha S_{KK} + \rho_0 C_p \theta / T_0,$$

$$(24) \blacktrianglequad D(T_0 \alpha S_{KK} + \rho_0 C_p \theta)/Dt = \nabla_R \cdot (k \nabla_R \theta).$$

In isothermal processes, the Helmholtz free energy per unit undeformed volume can serve as the strain energy density function W_0 , so that

$$(25) \qquad \mathbf{S} = \partial W_0 / \partial \mathbf{E}.$$

Hence, from Eq. (13.7:10) the Cauchy stress can be expressed as

$$(26) \qquad \boldsymbol{\tau} = \mathbf{F} \cdot (\partial W_0 / \partial \mathbf{E}) \cdot \mathbf{F}^T / J.$$

If the material is isotropic, W_0 must be a function of invariant of \mathbf{E} or \mathbf{C} ($= \mathbf{I} + 2\mathbf{E}$). A set of such invariant is

$$I_1 = \text{tr}(\mathbf{C}) = 3 + 2E_{KK}, \quad I_2 = (I_1^2 - \mathbf{C} : \mathbf{C})/2, \quad I_3 = \det(\mathbf{C}) = J^2.$$

One can then write W_0 in the form $W_0(T, I_1, I_2, I_3)$. It can be shown that

$$\partial I_1 / \partial \mathbf{C} = \mathbf{I}, \quad \partial I_2 / \partial \mathbf{C} = I_1 \mathbf{I} - \mathbf{C}, \quad \partial I_3 / \partial \mathbf{C} = I_3 \mathbf{C}^{-1}.$$

Then the constitutive law becomes

$$(27) \blacktrianglequad \mathbf{S} = 2\partial W_0 / \partial \mathbf{C} = C_1 \mathbf{I} + C_2(I_1 \mathbf{I} - \mathbf{C}) + C_3 I_3 \mathbf{C}^{-1},$$

where

$$(28) \qquad C_1 = 2\partial W_0 / \partial I_1, \quad C_2 = 2\partial W_0 / \partial I_2, \quad C_3 = 2\partial W_0 / \partial I_3.$$

Since $\mathbf{S} = 0$ for zero deformation, we have

$$(29) \qquad C_1 + 2C_2 + C_3 = 0 \quad \text{or} \quad \partial W_0 / \partial I_1 + 2\partial W_0 / \partial I_2 + \partial W_0 / \partial I_3 = 0,$$

as $I_1 = 3$, $I_2 = 3$, $I_3 = 1$ and $\mathbf{C} = \mathbf{I}$. From Eq. (27) and (13.7:10), one obtains the constitutive law for the Cauchy stress

$$(30) \blacktrianglequad \boldsymbol{\tau} = [C_1 \mathbf{B} + C_2(I_1 \mathbf{B} - \mathbf{B}^2) + C_3 I_3 \mathbf{I}] / J.$$

From Eq. (13.7:11), $p = -(\boldsymbol{\tau} : \mathbf{I})/3 = -(\mathbf{S} : \mathbf{C})/(3J)$, where p is the hydrostatic pressure, it follows

$$(31) \qquad p = -2[(\partial W_0 / \partial I_1) I_1 + 2(\partial W_0 / \partial I_2) I_2 + 3(\partial W_0 / \partial I_3) I_3] / (3J) \\ = -(C_1 I_1 + 2C_2 I_2 + 3C_3 I_3) / (3J).$$

One can split as follows the stresses into a distortional part, which involves the hydrostatic pressure p , and a deviatoric part, which has a zero mean as defined in Eq. (13.7:13):

$$(32) \qquad \mathbf{S} = -p J \mathbf{C}^{-1} + C_1(\mathbf{I} - I_1 \mathbf{C}^{-1}/3) + C_2(I_1 \mathbf{I} - \mathbf{C} - 2I_2 \mathbf{C}^{-1}/3),$$

$$(33) \qquad \boldsymbol{\tau} = -p \mathbf{I} + C_1(\mathbf{B} - I_1 \mathbf{I}/3) / J + C_2(I_1 \mathbf{B} - \mathbf{B}^2 - 2I_2 \mathbf{I}/3) / J.$$

A special form of the energy density is

$$(34a) \qquad W_0 = [C_1(I_1 - 3) + C_2(I_2 - 3) + b_3(I_3 - 1) + b_4(I_3 - 1)^2]/2,$$

with C_1 , C_2 , b_3 , b_4 being constant and $C_3 = b_3 + 2b_4(I_3 - 1)$. Equation (34a) represents an expansion of W_0 in a power series of $I_1 - 3$, $I_2 - 3$, and $I_3 - 1$. If the last two terms of Eq. (34a) are dropped, it reduces to the Mooney–Rivlin model for certain incompressible rubber-like materials.

For infinitesimal deformation, we can approximate

$$I_2 = 3 + 4E_{KK}, \quad I_3 = 1 + 2E_{KK}, \quad J = 1 + E_{KK}, \quad \mathbf{C}^{-1} = \mathbf{I} - 2\mathbf{E},$$

where E 's are strains. The energy density becomes

$$W_0 = (C_1 + 2C_2 + b_3)E_{KK} - (b_3 + C_2)E_{IJ}E_{IJ} + (b_3 + 2b_4 + C_2)(E_{KK})^2.$$

To assure that W_0 is consistent with Hook's law, we require

$$C_1 + 2C_2 + b_3 = 0, \quad -b_3 - C_2 = G, \quad 2b_3 + 4b_4 + 2C_2 = \lambda = 2G\nu/(1 - 2\nu),$$

i.e.,

$$C_2 = G - C_1, \quad b_3 = C_1 - 2G, \quad b_4 = (1 - \nu)G/[2(1 - 2\nu)],$$

where G is the shear modulus, λ the Lamé constant and ν Poisson's ratio. From Eq. (27) and (30) or Eq. (32) and (33), we obtain Hooke's law

$$\tau = S = 2GE + \lambda E_{KK}\mathbf{I} = 2G(E - E_{KK}\mathbf{I}/3) - p\mathbf{I},$$

Another commonly used energy density is

$$(34b) \quad W_0 = [C_1(I_1 - 3) + C_2(I_2 - 3) + b_3(1/I_3^2 - 1) + b_4(I_3 - 1)^2]/2.$$

For infinitesimal deformation, the energy density becomes

$$W_0 = (C_1 + 2C_2 - 2b_3)E_{KK} + (2b_3 - C_2)E_{IJ}E_{IJ} + (4b_3 + 2b_4 + C_2)(E_{KK})^2,$$

with

$$C_1 + 2C_2 - 2b_3 = 0, \quad 2b_3 - C_2 = G, \quad 8b_3 + 4b_4 + 2C_2 = \lambda = 2G\nu/(1 - 2\nu).$$

For *incompressible materials*, $J = 1$, Eq. (25) and (26) no longer hold and so are Eq. (27) and (30). Since the rate of volume change is zero,

$$(35) \quad \Delta \quad DJ/Dt = (DJ/DE) : \dot{\mathbf{E}} = J\mathbf{C}^{-1} : \dot{\mathbf{E}} = 0,$$

the incompressibility condition represents an internal constraint to \mathbf{E} . We can modify Eq. (25) to reflect the constraint of Eq. (35) and obtain

$$(36) \quad \Delta \quad S = -p'\mathbf{C}^{-1} + \partial W_0 / \partial \mathbf{E},$$

where p' is a Lagrangian multiplier relating to the hydrostatic pressure. The expression Eq. (36) is not unique for incompressible materials when $J = 1$ is taken into account. For example, if W_0 is replaced by $W_0 + f(\mathbf{C})(J - 1)$, the value of W_0 remains the same, but the derivative of W_0 with respect to \mathbf{E} changes. The constitutive equation becomes

$$S = -p'\mathbf{C}^{-1} + \partial W_0 / \partial \mathbf{E} + f(\mathbf{C})J\mathbf{C}^{-1}.$$

However no ambiguity arises in the determination of the deviatoric stress as defined in Eq. (13.7:12)

$$(37) \quad S'' = S - (S : \mathbf{C})\mathbf{C}^{-1}/3 = \partial W_0 / \partial \mathbf{E} - (\partial W_0 / \partial \mathbf{E} : \mathbf{C})\mathbf{C}^{-1}/3.$$

The additional term in S gives no contribution to the deviatoric stress. We can then write the constitutive equation in the form

$$(38) \quad S = -p\mathbf{C}^{-1} + \partial W_0 / \partial \mathbf{E} - (\partial W_0 / \partial \mathbf{E} : \mathbf{C})\mathbf{C}^{-1}/3,$$

where

$$(39) \quad p = p' - (\partial W_0 / \partial \mathbf{E} : \mathbf{C})/3$$

is the hydrostatic pressure as defined in Eq. (16.7:11) and has to be determined directly from the field solution. Accordingly the constitutive equation for the Cauchy stress becomes

$$(40) \quad \Delta \quad \tau = -p\mathbf{I} + [\mathbf{F} \cdot (\partial W_0 / \partial \mathbf{E}) \cdot \mathbf{F}^T - (\partial W_0 / \partial \mathbf{E} : \mathbf{C})\mathbf{I}/3]/J.$$

If the material is isotropic and incompressible, one can write Eq. (38) and (40) in the form of Eq. (32) and (33). As the constitutive laws Eq. (32) and (33) are valid for incompressible materials, except that the hydrostatic pressure must be determined directly from the field equations instead of Eq. (31). For infinitesimal deformation, Eq. (32) and (33) reduce to

$$\tau = S = -p\mathbf{I} + 2G(\mathbf{E} - E_{KK}\mathbf{I}/3).$$

We can establish a formulation valid for compressible as well as incompressible materials. Following the division of the stresses into the deviatoric stress and hydrostatic pressure, we split the deformation into a dilatational part, representing the volume change of a differential element of the body, and a distortion part, accounting for its shape change at constant volume. The procedure is known as *kinematic split*, which involves a multiplicative decomposition in the case of finite strain. This formulation is particularly useful for treating nearly incompressible materials. Let us define

$$(41) \quad \mathbf{U}' = \mathbf{U}J^{-1/3}, \quad \det(\mathbf{U}') = \det(\mathbf{U})J^{-1} = 1.$$

We thus obtain the multiplicative decomposition with \mathbf{U}' representing the distortion of the right stretch tensor \mathbf{U} , while $J^{1/3}\mathbf{I}$ is the dilatation. As we shall see later the decomposition of the deformation rate becomes additive and is in the same form as that in the infinitesimal strain case.

Accordingly, one can write the strain energy density in the form

$$(42) \quad W_0^*(\mathbf{C}', J) = W_0(\mathbf{C})$$

such that

$$(43) \quad \mathbf{S}' = \partial W_0^*(\mathbf{C}', J) / \partial \mathbf{C}', \quad p = -\partial W_0^*(\mathbf{C}', J) / \partial J,$$

where $\mathbf{C}' = \mathbf{C}^{J-2/3}$. The relationship among the second Piola–Kirchhoff stress \mathbf{S} , \mathbf{S}' and p [Eq. (43)] can be derived as follows:

$$(44) \quad \mathbf{S} : \delta \mathbf{E} = \delta W_0 = [\partial W_0^*(\mathbf{C}', J) / \partial \mathbf{C}'] : \delta \mathbf{C}' + [\partial W_0^*(\mathbf{C}', J) / \partial J] \delta J \\ = \mathbf{S}' : \delta \mathbf{C}' - p \delta J.$$

From the property of determinant [$J = \sqrt{\det(\mathbf{C})}$], it can be shown that

$$\delta J = (\partial J / \partial \mathbf{C}) : \delta \mathbf{C} = J \mathbf{C}^{-1} : \delta \mathbf{C} / 2.$$

Using the result above, one derives

$$\delta \mathbf{C}' = J^{-2/3} \delta \mathbf{C} - 2J^{-5/3} \mathbf{C} \delta J / 3 = J^{-2/3} \delta \mathbf{C} - J^{-2/3} \mathbf{C} (\mathbf{C}^{-1} : \delta \mathbf{C}) / 3.$$

A substitution of the last two equations into Eq. (44) leads to

$$(45) \quad \mathbf{S} = 2J^{-2/3} [\mathbf{S}' - (\mathbf{S}' : \mathbf{C}) \mathbf{C}^{-1} / 3] - p J \mathbf{C}^{-1}.$$

Taking the inner product of the equation above with \mathbf{C} , one can show that

$$(46) \quad p = -\mathbf{S} : \mathbf{C} / (3J),$$

in which the result $\mathbf{C}^{-1} : \mathbf{C} = 3$ has been used. Equation (46) establishes that p as defined in Eq. (43) is indeed the hydrostatic pressure, the mean of the Cauchy stress. Note that \mathbf{S}' is not deviatoric, since $\mathbf{S}' : \mathbf{C} \neq 0$.

Problem 13.7. Let I_1 , I_2 and I_3 be the principal invariance of \mathbf{C} . Show that

$$I_2 = (I_1^2 - \mathbf{C} : \mathbf{C}) / 2 = I_3 \text{tr}(\mathbf{C}^{-1}).$$

For incompressible materials $I_3 = 1$, then $I_2 = \text{tr}(\mathbf{C}^{-1})$. (*Hint:* Express the invariance in terms of the eigenvalues of \mathbf{C} .)

Problem 13.8. Show that $\mathbf{C} = I_1 \mathbf{I} - I_2 \mathbf{C}^{-1} + I_3 \mathbf{C}^{-2}$, without using the Cayley–Hamilton theorem. (*Hint:* Use the approach as that for Problem 13.7 or differentiate the result of Problem 13.7 with respect to \mathbf{C} .)

Problem 13.9. Using $\mathbf{C} = I_1 \mathbf{I} - I_2 \mathbf{C}^{-1} + I_3 \mathbf{C}^{-2}$ and $I_2 = I_3 \text{tr}(\mathbf{C}^{-1})$, show that

$$\begin{aligned} \mathbf{B}^2 &= I_1 \mathbf{B} - I_2 \mathbf{I} + I_3 \mathbf{B}^{-1}, \\ \mathbf{S} &= -p J \mathbf{C}^{-1} + C_1 (\mathbf{I} - I_1 \mathbf{C}^{-1} / 3) + C_2 (I_2 \mathbf{C}^{-1} / 3 - J^2 \mathbf{C}^{-2}), \\ \boldsymbol{\tau} &= -p \mathbf{I} + C_1 (\mathbf{B} - I_1 \mathbf{I} / 3) / J + C_2 [I_2 \mathbf{I} / (3J) - J \mathbf{B}^{-1}]. \end{aligned}$$

MORE EXAMPLES

Uniform Tension. We consider a unit cube with sides parallel to the Cartesian axes (X , Y , Z) and is deformed into a

cuboid of dimensions $\lambda_1, \lambda_2, \lambda_3$ parallel to the X, Y, Z axes, respectively. The material is compressible and the energy density is in the form as Eq. (34a). The deformed coordinates are referred to the Cartesian axes x, y, z , such that

$$(47) \quad x = \lambda_1 X, \quad y = \lambda_2 Y, \quad z = \lambda_3 Z.$$

The components of the deformation gradient \mathbf{F} , the metric tensors \mathbf{C} and \mathbf{B} , and the Green strain tensor \mathbf{E} are

$$\begin{aligned} \mathbf{B}^2 &= I_1 \mathbf{B} - I_2 \mathbf{I} + I_3 \mathbf{B}^{-1}, \\ \mathbf{S} &= -p \mathbf{J} \mathbf{C}^{-1} + C_1 (\mathbf{I} - I_1 \mathbf{C}^{-1}/3) + C_2 (I_2 \mathbf{C}^{-1}/3 - J^2 \mathbf{C}^{-2}), \\ \tau &= -p \mathbf{I} + C_1 (\mathbf{B} - I_1 \mathbf{I}/3)/J + C_2 [I_2 \mathbf{I}/(3J) - J \mathbf{B}^{-1}]. \end{aligned}$$

in which only \mathbf{B} is objective and \mathbf{F}, \mathbf{C} and \mathbf{E} are not. The strain invariant

$$\begin{aligned} \mathbf{F} &= \text{dia}[\lambda_1 \quad \lambda_2 \quad \lambda_3], \quad \mathbf{C} = \mathbf{B} = \text{dia}[\lambda_1^2 \quad \lambda_2^2 \quad \lambda_3^2], \\ \mathbf{E} &= \text{dia}[\lambda_1^2 - 1 \quad \lambda_2^2 - 1 \quad \lambda_3^2 - 1]/2 \end{aligned}$$

The inverse of \mathbf{C} is simply

$$\begin{aligned} I_1 &= \text{tr}(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= (I_1^2 - \mathbf{C} : \mathbf{C})/2 = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2, \\ I_3 &= \det(\mathbf{C}) = J^2 = \lambda_1^2 \lambda_2^2 \lambda_3^2. \end{aligned}$$

From Eq. (27) and (30), it follows

$$\begin{aligned} \mathbf{S} &= C_1 \mathbf{I} + C_2 \text{dia}[\lambda_2^2 + \lambda_3^2 \quad \lambda_3^2 + \lambda_1^2 \quad \lambda_1^2 + \lambda_2^2] \\ &\quad + C_3 \lambda_1^2 \lambda_2^2 \lambda_3^2 \text{dia}[1/\lambda_1^2 \quad 1/\lambda_2^2 \quad 1/\lambda_3^2], \\ \tau &= \{C_1 \text{dia}[\lambda_1^2 \quad \lambda_2^2 \quad \lambda_3^2] + C_3 \lambda_1^2 \lambda_2^2 \lambda_3^2 \mathbf{I} \\ &\quad + C_2 \text{dia}[\lambda_1^2(\lambda_2^2 + \lambda_3^2) \quad \lambda_2^2(\lambda_3^2 + \lambda_1^2) \quad \lambda_3^2(\lambda_1^2 + \lambda_2^2)]\}/J, \end{aligned}$$

where, from Eqs. (34a)

$$C_3 = b_3 + 2b_4(\lambda_1^2 \lambda_2^2 \lambda_3^2 - 1).$$

Note that τ is objective and \mathbf{S} is not. All stress components are constant and therefore the equilibrium equations are satisfied. Since the deformation involves uniform stretching only, all shear stresses are zero and the components of \mathbf{S} are proportional to those of τ , i.e., $\tau_{ii} = S_{ii} \lambda_i^2/J$, $i = 1, 2, 3$ and are not summed. These results are also true locally for nonuniform deformation if the referenced coordinates coincide with the principal stress directions where λ_i is the stretch in the i th principal direction.

For the particular case of simple extension under a force parallel to the x -axis, $\lambda_2 = \lambda_3$, $\tau_{22} = \tau_{33} = 0$, hence

$$\lambda_1 \tau_{22} = C_1 + C_2(\lambda_1^2 + \lambda_2^2) + [b_3 + 2b_4(\lambda_1^2 \lambda_2^4 - 1)]\lambda_1^2 \lambda_2^2 = 0.$$

One can solve for λ_2^2 in terms of λ_1^2 . The above equation gives only one real solution for λ_2^2 . Replacing λ_2 with $1 + 2E_{KK}$, ($K = 1, 2$, not summed) and solving for E_{22} , one finds

$$E_{22}/E_{11} = -\nu(E_{11}),$$

in which $\nu(E_{11})$ is a complicated function of E_{11} . For infinitesimal deformation, it reduces to the well-known Poisson's ratio

$$\nu(E_{11}) = (C_2 + b_3 + 2b_4)/(C_2 + b_3 + 4b_4) = \lambda/[2(\lambda + G)].$$

For incompressible materials, $I_3 = 1$, the Cauchy stress for the Mooney–Rivlin material becomes

$$\begin{aligned} \tau &= -(p + C_1 I_1/3 + 2C_2 I_2/3) \mathbf{I} + C_1 \text{dia}[\lambda_1^2 \quad \lambda_2^2 \quad \lambda_3^2] \\ &\quad + C_2 \text{dia}[\lambda_1^2(\lambda_2^2 + \lambda_3^2) \quad \lambda_2^2(\lambda_3^2 + \lambda_1^2) \quad \lambda_3^2(\lambda_1^2 + \lambda_2^2)], \end{aligned}$$

Problem 13.10. Find the Cauchy stress for the uniform extension problem with the strain energy density

$$W_0 = [b_1(I_1 - 3) + b_2(I_2 - 3) + b_3(J - 1) + b_4(J - 1)^2]/2,$$

where b 's are constants. Show that (a) we must have

$$2b_1 + 4b_2 + b_3 = 0,$$

to assure that the stresses are zero for zero deformation; (b) for simple extension in the X -direction

$$\begin{aligned} E_{22} &= -\nu(E_{11})E_{11} \\ &= [(b_1 + 2b_2 + b_4)(\sqrt{1+2E_{11}} - 1) - 2(b_2 + b_4)E_{11}]/\{4[b_2 + b_4(1+2E_{11})]\}. \end{aligned}$$

Simple Shear. Consider the simple shear deformation

$$(48) \quad x = X + \gamma Y, \quad y = Y, \quad z = Z,$$

where (x, y, z) and (X, Y, Z) are, respectively, the deformed and undeformed Cartesian coordinates. The matrices for deformation and deformation rates are

$$\begin{aligned} \mathbf{F} &= \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{U} &= \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ \sin \phi & (1 + \sin^2 \phi)/\cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{C} &= \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 + \gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C}^{-1} = \begin{bmatrix} 1 + \gamma^2 & -\gamma & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{B} &= \begin{bmatrix} 1 + \gamma^2 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \frac{1}{2} \begin{bmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{D} &= \frac{1}{2} \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\Omega} = \frac{1}{2} \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ -\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

where $\phi = \tan^{-1}(\gamma/2)$. The strain invariants are

$$I_1 = I_2 = 3 + \gamma^2, \quad I_3 = J^2 = 1.$$

Note that the simple shear deformation does not involve any dilatation. It follows from Eq. (27) and (30) that

$$\begin{aligned} \mathbf{S} &= C_1 \mathbf{I} + C_2 \begin{bmatrix} 2 + \gamma^2 & -\gamma & 0 \\ -\gamma & 2 & 0 \\ 0 & 0 & 2 + \gamma^2 \end{bmatrix} + C_3 \begin{bmatrix} 1 + \gamma^2 & -\gamma & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \boldsymbol{\tau} &= C_1 \begin{bmatrix} 1 + \gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + C_2 \begin{bmatrix} 2 + \gamma^2 & \gamma & 0 \\ \gamma & 2 & 0 \\ 0 & 0 & 2 + \gamma^2 \end{bmatrix} + C_3 \mathbf{I}. \end{aligned}$$

The stresses are uniform and therefore satisfy the equilibrium equations automatically. It is interesting to note that $\tau_{33} = S_{33} = 0$ unless $\gamma = 0$. In other words, if the body is compressible, we cannot have a traction free plane for simple shear deformation.

This would not be true if the body is incompressible. In this case, the Cauchy stresses are in the form

$$\boldsymbol{\tau} = -p \mathbf{I} + C_1 \begin{bmatrix} 2\gamma^2/3 & \gamma & 0 \\ \gamma & -\gamma^2/3 & 0 \\ 0 & 0 & -\gamma^2/3 \end{bmatrix} + C_2 \begin{bmatrix} \gamma^2/3 & \gamma & 0 \\ \gamma & -2\gamma^2/3 & 0 \\ 0 & 0 & \gamma^2/3 \end{bmatrix},$$

where p is to be evaluated from the field equations. If there is no stress ($\tau_{i3} = S_{i3} = 0$, $i, I = 1, 2, 3$) across the planes where $z = \text{constant}$, it requires that $p = (C_2 - C_1)\gamma^2/3$. Then the stresses become

$$\boldsymbol{\tau} = \begin{bmatrix} C_1\gamma^2 & (C_1 + C_2)\gamma & 0 \\ (C_1 + C_2)\gamma & -C_2\gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can easily compute the surface forces on any surface. For example, let us consider a surface originally at $X = \text{constant}$. The unit vector normal to the surface at the deformed state [Eq. (13.2:12)] is

$$\mathbf{n}^T = [1 \quad -\gamma \quad 0]/\sqrt{1 + \gamma^2}.$$

The surface traction acting over this surface is given by the vector

$$\overset{\nu}{\mathbf{T}}^T = \mathbf{n} \cdot \boldsymbol{\tau} = [-C_2\gamma^2 \quad (C_1 + C_2)\gamma + C_2\gamma^3 \quad 0]/\sqrt{1 + \gamma^2}.$$

The normal and tangential components of \mathbf{T} are, respectively,

$$[-C_2\gamma^4 - (C_1 + 2C_2)\gamma^2]/(1 + \gamma^2), \quad (C_1 + C_2)\gamma/(1 + \gamma^2).$$

For the surface originally at $Y = \text{constant}$, the unit normal to its deformed surface is still in the Y -direction. The surface traction is

$$\overset{\nu}{\mathbf{T}}^T = [(C_1 + C_2)\gamma \quad -C_2\gamma^2 \quad 0].$$

The normal and tangential components of the surface traction are, respectively, $[-C_2\gamma^2, -(C_1 + C_2)\gamma]$. The normal component is proportional to γ^2 as compared to γ^4 for the formal case.

More examples including simultaneous extension, inflation and torsion of a cylindrical tube, flexure of a circular cylinder, zero stress state of artery can be found in Green and Zerna (1968), Green and Adkins (1960), and Fung^{1,3} (1990).

Anisotropic Materials. Isotropic materials are a special group in nature. Almost all biological tissues and most crystalline materials are anisotropic. An account of the constitutive equations of biological tissues is given in Fung (1993). An extensive discussion of the various symmetries in crystals and their constitutive equations is given by Green and Adkins (1960). Current literature can be found in Applied Mechanics Reviews.

13.12. VARIATIONAL PRINCIPLES FOR NONLINEAR ELASTICITY: COMPRESSIBLE MATERIALS

In this section, we shall discuss variational principles for finite strain analysis of hyperelastic materials. An early work is by Koiter (1973) on the complementary energy principle. A general variational principle can involve displacements, and the different strain and stress measures defined in Secs. 13.2–13.7 as independent fields. The following are frequently used relations from Sec. 13.11. Denoting the *strain energy per unit undeformed volume* as $W_0(\mathbf{C})$, one obtains the following constitutive equation

$$(1) \quad \partial W_0 / \partial \mathbf{C} = \partial W_0 / (\partial \mathbf{F}^T \cdot \mathbf{F}) = \mathbf{S}/2 = S_{IJ}\varepsilon_I\varepsilon_J/2,$$

where \mathbf{F} is the *deformation gradient tensor*, $\mathbf{C} (= \mathbf{F}^T \cdot \mathbf{F})$ is the *metric tensor of the deformed body* referred to the undeformed configuration [See Eq. (13.3:1)], and \mathbf{S} is the *second Piola-Kirchhoff stress tensor*. These quantities are defined in Eqs. (13.2:3), (13.2:7) and (13.7:6). Since

$$(2) \quad \frac{\partial(\mathbf{F}^T \cdot \mathbf{F})}{\partial \mathbf{F}} = \frac{\partial(F_{iI}F_{jJ}\varepsilon_I\varepsilon_J)}{\partial F_{iK}} \mathbf{e}_i\varepsilon_K = (F_{iI}\delta_{JK} + F_{iJ}\delta_{IK})\varepsilon_I\varepsilon_J \mathbf{e}_i\varepsilon_K,$$

one can show that

$$(3) \quad \begin{aligned} \partial W_0 / \partial \mathbf{F} &= [\partial W_0 / \partial(\mathbf{F}^T \cdot \mathbf{F})] : [\partial(\mathbf{F}^T \cdot \mathbf{F}) / \partial \mathbf{F}] \\ &= S_{IJ}(F_{iI}\delta_{JK} + F_{iJ}\delta_{IK})\mathbf{e}_i\varepsilon_K/2 = F_{iJ}S_{JK}\mathbf{e}_i\varepsilon_K = \mathbf{F} \cdot \mathbf{S} = \mathbf{T}^T, \end{aligned}$$

which is the constitutive equation for \mathbf{T} , the *Lagrangian or first Piola-Kirchhoff stress tensor*. In Eq. (1) and (2), the base vectors must be in the order consistent with that of the subscripts of F 's as shown.

Let Q_0 be a function of the *right stretch tensor* \mathbf{U} defined in Eq. (13.4:4), as the strain energy per unit undeformed volume. To distinguish it from W_0 , the strain energy density in terms of \mathbf{F} or \mathbf{C} , we define

$$Q_0(\mathbf{U}) \equiv W_0(\mathbf{C})$$

for the same deformation. Then, it can be shown that

$$(4) \quad \partial Q_0 / \partial \mathbf{U} = (\mathbf{T} \cdot \mathbf{R})_s = \mathbf{r},$$

where the subscript $(.)_s$ denotes the symmetric part of the relevant quantity and \mathbf{R} is the *rotation tensor*. Equation (4) is an alternative expression of the constitutive equation for \mathbf{T} .

Following Atluri and Cazzani (1995), we can derive a *mixed four-field variational principle*, which uses the displacement vector $\mathbf{u}(= \mathbf{x} - \mathbf{X})$, the rotational tensor \mathbf{R} , the right stretch tensor \mathbf{U} , and the Lagrangian stress tensor \mathbf{T} as independent fields. The functional is in the form

$$(5) \quad \Pi_1(\mathbf{u}, \mathbf{R}, \mathbf{U}, \mathbf{T}) = \int_{V_0} [Q_0(\mathbf{U}) + \mathbf{T}^T : (\mathbf{I} + \nabla_0 \mathbf{u} - \mathbf{R} \cdot \mathbf{U}) \\ - \rho_0 \mathbf{b} \cdot \mathbf{u}] dV - \int_{S_{u0}} \mathbf{N} \cdot \mathbf{T} \cdot (\mathbf{u} - \bar{\mathbf{u}}) dA - \int_{S_{\sigma0}} \bar{\mathbf{T}} \cdot \mathbf{u} dA,$$

where \mathbf{b} is the body force per unit mass, V_0 the undeformed volume, \mathbf{N} a unit outward normal to the boundary surface $\partial V_0 (= S_0)$, and $S_{\sigma0} + S_{u0} = v \partial V_0 = S_0$ with $S_{\sigma0}$ and S_{u0} being the surfaces where the traction $\bar{\mathbf{T}}$ and the displacement $\bar{\mathbf{u}}$ are prescribed, respectively. The *admissible requirements* for the four field variables are: \mathbf{u} must be continuous, \mathbf{R} orthogonal, \mathbf{U} symmetric and \mathbf{T} a general tensor of rank 2. In Eq. (5), the gradient operator refers to the undeformed coordinates, i.e., $\nabla_0 = \partial(./\partial \mathbf{X}_I) \varepsilon_I$.

The vanish of $\delta\Pi_1$ for arbitrary continuous $\delta\mathbf{u}$, antisymmetric $\delta\mathbf{R} \cdot \mathbf{R}T$, symmetric $\delta\mathbf{U}$ and general tensor $\delta\mathbf{T}$ gives

$$(6) \quad \partial Q_0 / \partial \mathbf{U} = (\mathbf{T} \cdot \mathbf{R})_s \quad (\text{Constitutive law in } V_0),$$

$$(7) \quad \mathbf{I} + \nabla_0 \mathbf{u} = \mathbf{R} \cdot \mathbf{U} \quad (\text{Compatibility condition in } V_0),$$

$$(8) \quad (\mathbf{T}^T \cdot \mathbf{U} \cdot \mathbf{R}^T)_a = (\mathbf{R} \cdot \mathbf{U} \cdot \mathbf{T})_a = 0 \quad (\text{Angular momentum balance in } V_0),$$

$$(9) \quad \nabla_0 \cdot \mathbf{T} + \rho_0 \mathbf{b} = 0 \quad (\text{Equilibrium equation in } V_0),$$

$$(10) \quad \mathbf{N} \cdot \mathbf{T} = \bar{\mathbf{T}} \quad (\text{Traction boundary condition on } S_{\sigma0}),$$

$$(11) \quad \mathbf{u} = \bar{\mathbf{u}} \quad (\text{Displacement boundary condition on } S_{u0}).$$

This variational principle is valid for general elastic materials. In this formulation both the displacement and traction conditions are *natural boundary conditions* (see Sec. 10.6). Note that the angular momentum equation requires $\mathbf{R} \cdot \mathbf{U} \cdot \mathbf{T}$ to be symmetric. As a matter of fact, $\mathbf{R} \cdot \mathbf{U} \cdot \mathbf{T}(= \sigma)$ is the *Kirchhoff stress tensor*, which of course should be symmetric.

One can obtain other variational principles by selectively enforcing Eq. (6)–(11) *a priori*. For example, setting

$$\mathbf{U} = [\mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u})]_s$$

in Eq. (5) will eliminate \mathbf{U} from the functional and result in a variational principle with only three dependent variables \mathbf{u} , \mathbf{R} , and \mathbf{T} .

One can also establish a variational principle *involving only the displacements as the field variable* by requiring *a priori*

$$\mathbf{I} + \nabla_0 \mathbf{u} = \mathbf{R} \cdot \mathbf{U} \quad \text{in } V_0,$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } S_{u0}.$$

Equation (5) then becomes

$$(12) \quad \Pi_3(\mathbf{u}) = \int_{V_0} \{W_0[\mathbf{F}^T \cdot \mathbf{F}] - \rho_0 \mathbf{b} \cdot \mathbf{u}\} dV - \int_{S_{\sigma0}} \bar{\mathbf{T}} \cdot \mathbf{u} dA,$$

wherein

$$\mathbf{F} = \mathbf{I} + \nabla_0 \mathbf{u}.$$

Equation (12) is the functional for the commonly known *principle of minimum potential energy*, which involves only one field variable \mathbf{u} . We have the associated field equations

$$\nabla_0 \cdot (\partial W_0 / \partial \mathbf{F}^T) + \rho_0 \mathbf{b} = \nabla_0 \cdot (\mathbf{S} \cdot \mathbf{F}^T) + \rho_0 \mathbf{b} = 0 \quad \text{in } V_0,$$

and the traction condition (10) on $S_{\sigma 0}$ as a *natural boundary condition*. The displacement boundary condition (11) is now a *rigid boundary condition* (Sec. 10.6). One enforces the rigid condition in the integral formulation and uses $\delta \mathbf{u} = 0$ on S_{u0} in deriving the stationarity condition for Π_3 .

Atluri includes \mathbf{R} as an additional field variable by modifying Π_3

$$(13) \quad \begin{aligned} \Pi_3^*(\mathbf{u}, \mathbf{R}) = & \int_{V_0} \left\{ W_0(\mathbf{F}^T \cdot \mathbf{F}) + \frac{\gamma}{4}(\mathbf{R}^T \cdot \mathbf{F})_a^2 - \rho_0 \mathbf{b} \cdot \mathbf{u} \right\} dV \\ & - \int_{S_{\sigma 0}} \bar{\mathbf{T}} \cdot \mathbf{u} dA, \end{aligned}$$

wherein

$$(\mathbf{R}^T \cdot \mathbf{F})_a^2 = (\mathbf{R}^T \cdot \mathbf{F} - \mathbf{F}^T \cdot \mathbf{R}) : (\mathbf{R}^T \cdot \mathbf{F} - \mathbf{F}^T \cdot \mathbf{R}),$$

γ is a constant, and $(.)_a$ denotes the *skew symmetric part* of the corresponding tensor of rank 2. The constant is of the magnitude of the elastic modulus of the material. One can show that the variation of $(\mathbf{R}^T \cdot \mathbf{F})_a^2$ with respect to \mathbf{R} and \mathbf{u} gives

$$\begin{aligned} \delta(\mathbf{R}^T \cdot \mathbf{F})_a^2 = & 4(\mathbf{F} - \mathbf{R} \cdot \mathbf{F}^T \cdot \mathbf{R}) : \nabla_0 \delta \mathbf{u} \\ & + 4[(\mathbf{R}^T \cdot \mathbf{F} - \mathbf{F}^T \cdot \mathbf{R}) \cdot \mathbf{F}^T \cdot \mathbf{R}] : (\delta \mathbf{R} \cdot \mathbf{R}) \end{aligned}$$

by using the properties of (\cdot) operation described in Sec. 13.1. Applying Eq. (3), one can derive the field equations

$$(14) \quad \nabla_0 \cdot [\mathbf{S} \cdot \mathbf{F}^T + \gamma(\mathbf{F} - \mathbf{R} \cdot \mathbf{F}^T \cdot \mathbf{R})] + \rho_0 \mathbf{b} = 0,$$

$$(15) \quad (\mathbf{F}^T \cdot \mathbf{R})^2 = (\mathbf{R}^T \cdot \mathbf{F})^2, \quad \text{in } V_0 \text{ and}$$

the natural boundary condition (10) on $S_{\sigma 0}$. The boundary condition (11) is a rigid condition. Equation (15) implies that $\mathbf{R}^T \cdot \mathbf{F}$ is symmetric, i.e.,

$$(16) \quad \mathbf{R}^T \cdot \mathbf{F} = \mathbf{F}^T \cdot \mathbf{R} \quad \text{or} \quad \mathbf{F} = \mathbf{R} \cdot \mathbf{F}^T \cdot \mathbf{R}.$$

One sees that γ is the constant factor in the penalty function approach. Atluri and Cazzani (1995) showed that the finite element solution based on Π_3^* provided better accuracy than that based on Π_3 .

One can also establish different three-field variational principles through contact transformation, e.g., using the following contact transformation

$$(17) \quad -Q_c(\mathbf{r}) = Q_0(\mathbf{U}) - (\mathbf{T} \cdot \mathbf{R})_s : \mathbf{U},$$

where $\mathbf{r} = (\mathbf{T} \cdot \mathbf{R})_s$ is the *symmetrized Biot–Luré stress tensor* and Q_c is called the *complementary energy density* in terms of \mathbf{r} (see Sec. 10.9). One can show that Q_c does not explicitly depend on \mathbf{U} , i.e.,

$$-\partial Q_c / \partial \mathbf{U} = \partial Q_0 / \partial \mathbf{U} - \mathbf{r} = 0.$$

A substitution of Eq. (17) into Eq. (5) yields

$$(18) \quad \begin{aligned} \Pi_4(\mathbf{u}; \mathbf{R}; \mathbf{T}) = & \int_{V_0} \{-Q_c[(\mathbf{T} \cdot \mathbf{R})_s] + \mathbf{T}^T : (\mathbf{I} + \nabla_0 \mathbf{u}) - \rho_0 \mathbf{b} \cdot \mathbf{u}\} dV \\ & - \int_{S_{u0}} \mathbf{N} \cdot \mathbf{R} \cdot (\mathbf{u} - \bar{\mathbf{u}}) dA - \int_{S_{\sigma 0}} \bar{\mathbf{T}} \cdot \mathbf{u} dA, \end{aligned}$$

The admissible requirements for \mathbf{u} , \mathbf{R} , and \mathbf{T} are the same as those of Eq. (5). The corresponding field equations of the functional are

$$(19) \quad \partial Q_c / \partial \mathbf{r} = \mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u}) = \mathbf{U} \quad \text{in } V_0,$$

and Eq. (8)–(11). Apparently, Eq. (19) is a compatibility condition for the new functional replacing Eq. (6) and (7).

One can establish variational principles using \mathbf{u} , \mathbf{R} , \mathbf{U} , \mathbf{r}^* as independent fields, where \mathbf{r}^* is the *Biot–Luré stress tensor* defined in Eq. (13.7:8). For example, by substituting \mathbf{T} by $\mathbf{r}^* \cdot \mathbf{R}^T$ into Eq. (5), one obtains

$$(20) \quad \Pi_5(\mathbf{u}, \mathbf{R}, \mathbf{U}, \mathbf{r}^*) = \int_{V_0} \{Q_0(\mathbf{U}) + \mathbf{r}^{*T} : [\mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u}) - \mathbf{U}] \\ - \rho_0 \mathbf{b} \cdot \mathbf{u}\} dV - \int_{S_{u0}} \mathbf{N} \cdot \mathbf{r}^* \cdot \mathbf{R}^T \cdot (\mathbf{u} - \bar{\mathbf{u}}) dA - \int_{S_{\sigma0}} \frac{\nu}{\mathbf{T}} \cdot \mathbf{u} dA.$$

One can derive other variational principles by imposing different conditions *a priori* as before.

PROBLEMS

13.11. Show Eq. (8) using the indicial notation.

13.12. Using the properties of the $(:)$ operation, show that

$$\mathbf{T}^T : (\delta \mathbf{R} \cdot \mathbf{U}) = (\mathbf{T}^T \cdot \mathbf{U} \cdot \mathbf{R}^T) : (\delta \mathbf{R} \cdot \mathbf{R}^T) = (\mathbf{T}^T \cdot \mathbf{U} \cdot \mathbf{R}^T)_a : (\delta \mathbf{R} \cdot \mathbf{R}^T).$$

Hint: $\mathbf{R} \cdot \mathbf{R} = I$ and $\delta \mathbf{R} \cdot \mathbf{R}$ is anti-symmetric.

?**13.13.** Using the properties of the $(:)$ operation, show that

$$\mathbf{T}^T : (\delta \mathbf{R} \cdot \mathbf{U}) = (\mathbf{T}^T \cdot \mathbf{U} \cdot \mathbf{R}^T) : (\delta \mathbf{R} \cdot \mathbf{R}^T) = (\mathbf{T}^T \cdot \mathbf{U} \cdot \mathbf{R}^T)_a : (\delta \mathbf{R} \cdot \mathbf{R}^T).$$

13.14. Derive the field equations including the natural boundary conditions for the functional given in Eq. (5) with $\mathbf{U} = [\mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u})]$ satisfied *a priori*. Identify the admissible conditions for the independent field variables.

13.15. Find that the field equations and boundary conditions for the functional

$$(18) \quad \Pi_4(\mathbf{u}; \mathbf{R}; \mathbf{T}) = \int_{V_0} \{-Q_c[(\mathbf{T} \cdot \mathbf{R})_s] + \mathbf{T}^T : (\mathbf{I} + \nabla_0 \mathbf{u}) - \rho_0 \mathbf{b} \cdot \mathbf{u}\} dV \\ - \int_{S_{u0}} \mathbf{N} \cdot \mathbf{R} \cdot (\mathbf{u} - \bar{\mathbf{u}}) dA - \int_{S_{\sigma0}} \frac{\nu}{\mathbf{T}} \cdot \mathbf{u} dA,$$

of Sec. 13.12.

13.13. VARIATIONAL PRINCIPLES FOR NONLINEAR ELASTICITY: NEARLY INCOMPRESSIBLE OR INCOMPRESSIBLE MATERIALS

For nearly incompressible materials, the previously defined energy density functions W_0 and Q_0 are ill defined and not suitable for evaluating the mean stress. If the material is incompressible, one must use the constitutive law as defined in Eq. (13.11:36) or Eqs. (13.11:38) and (13.11:40). In this section, we shall discuss functionals valid for compressible and incompressible materials using the kinematic split formulation discussed in Sec. 13.11. We multiplicatively decompose the finite strain into a dilatational part and a distortion one and divide the stress into the deviatoric stress and the hydrostatic pressure. As an illustration we shall derive the variational principles involving the *Biot–Luré stress tensor* \mathbf{r}^* .

Following Eq. (13.11:41), we split the right stretch tensor as $\mathbf{U} = \mathbf{U}' J^{1/3}$ to give $\det(\mathbf{U}') = 1$, where J is the Jacobian, \mathbf{U}' represents the *distortion of the right stretch tensor* \mathbf{U} and $J^{1/3}$ the *dilatation*. If the strain is small, this decomposition becomes additive as that for the infinitesimal strain case. Similar to the derivation in Sec. 13.11, we define

$$\mathbf{r}'_s^* = \partial Q_0^*(\mathbf{U}', J) / \partial \mathbf{U}', \quad -p = \partial Q_0^*(\mathbf{U}', J) / \partial J,$$

where $Q_0^*(\mathbf{U}', J) = Q_0(\mathbf{U})$ is the strain energy density function in terms of \mathbf{U} . The relations among \mathbf{r}^* , \mathbf{r}'_s^* , and the hydrostatic pressure p are

$$(1) \quad \mathbf{r}_s^* = J^{-1/3} \mathbf{r}'_s^* - J^{-1/3} (\mathbf{r}'_s^* : \mathbf{U}) \mathbf{U}^{-1}/3 - p J \mathbf{U}^{-1},$$

$$(2) \quad -p = (\mathbf{r}_s^* : \mathbf{U}) / (3J) = (\boldsymbol{\tau} : \mathbf{I}) / 3.$$

Note that \mathbf{r}'_s^* and p are the stresses related to the *distortion* and *dilatation* strains, respectively.

We can establish variational principles having \mathbf{u} , \mathbf{R} , \mathbf{U}' , J , \mathbf{r}'^* , p , $\overset{\nu}{\mathbf{T}}$ as independent field variables where $\overset{\nu}{\mathbf{T}}$ is a traction vector defined only on S_{u0} (Atluri and Cazzani 1995). For example,

$$(3) \quad \Pi(\mathbf{u}, \mathbf{R}, \mathbf{U}', J, \mathbf{r}'^*, p, \overset{\nu}{\mathbf{T}}) = \int_{V_0} \{Q_0^*(\mathbf{U}', J) + \mathbf{r}'^{*T} : [\mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u}) J_u^{-1/3} - \mathbf{U}'] \\ - (J_u - J)p - \rho_0 \mathbf{b} \cdot \mathbf{u}\} dV - \int_{S_{\sigma 0}} \overset{\nu}{\mathbf{T}} \cdot \mathbf{u} dA - \int_{S_{u0}} \overset{\nu}{\mathbf{T}} \cdot (\mathbf{u} - \bar{\mathbf{u}}) dA$$

is the functional for a variational principle of the seven field variables, in which

$$(4) \quad J_u \equiv \det(\hat{\mathbf{U}}), \quad \hat{\mathbf{U}} = [\mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u})]_s.$$

The admissible requirements are: \mathbf{u} is continuous, \mathbf{R} orthogonal, \mathbf{U}' symmetric, J positive, \mathbf{r}'^* a second rank tensor, p piecewise continuous and $\overset{\nu}{\mathbf{T}}$ piecewise continuous vector.

To derive the field equations associated with the functional given in Eq. (3), we need to take into account that

$$\begin{aligned} \mathbf{r}'^{*T} : [\delta \mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u})] &= [\mathbf{r}'^{*T} \cdot (\mathbf{I} + \nabla_0 \mathbf{u})^T] : \delta \mathbf{R}^T = [(\mathbf{I} + \nabla_0 \mathbf{u}) \cdot \mathbf{r}'^*] : \delta \mathbf{R} \\ &= \{(\mathbf{R} \cdot \mathbf{R}^T) \cdot [(\mathbf{I} + \nabla_0 \mathbf{u}) \cdot \mathbf{r}'^*]\} : \delta \mathbf{R} = [\mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u}) \cdot \mathbf{r}'^*] : (\mathbf{R}^T \cdot \delta \mathbf{R}), \\ \mathbf{r}'^{*T} : (\mathbf{R}^T \cdot \nabla_0 \delta \mathbf{u}) &= (\mathbf{R} \cdot \mathbf{r}'^{*T}) : \nabla_0 \delta \mathbf{u} = (\mathbf{r}'^* \cdot \mathbf{R}^T) : (\nabla_0 \delta \mathbf{u})^T, \\ \delta J_u = (\partial J_u / \partial \hat{\mathbf{U}}) : \delta \hat{\mathbf{U}} &= J_u \hat{\mathbf{U}}^{-1} : \delta \hat{\mathbf{U}} = J_u \hat{\mathbf{U}}^{-1} : [\delta \mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u}) + \mathbf{R}^T \cdot \nabla_0 \delta \mathbf{u}] \\ &= J_u \{[\mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u}) \cdot \hat{\mathbf{U}}^{-1}] : (\mathbf{R}^T \cdot \delta \mathbf{R}) + (\hat{\mathbf{U}}^{-1} \cdot \mathbf{R}^T) : (\nabla_0 \delta \mathbf{u})^T\}. \end{aligned}$$

The stationary condition of Π for arbitrary variation of the admissible field variables gives the following field equations:

$$(5) \quad \mathbf{r}'^* = \partial Q_0^*(\mathbf{U}', J) / \partial \mathbf{U}', \quad -p = \partial Q_0^*(\mathbf{U}', J) / \partial J \quad (\text{constitutive laws});$$

$$(6) \quad \mathbf{U}' = J_u^{-1/3} \mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u}), \quad J = J_u \quad (\text{compatibility conditions});$$

$$(7) \quad \{\mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u}) \cdot [3\mathbf{r}'^* - \langle \mathbf{r}'^* : (\mathbf{I} + \nabla_0 \mathbf{u})^T \cdot \mathbf{R} + 3pJ_u^{4/3} \rangle \hat{\mathbf{U}}^{-1}]\}_a = 0 \\ (\text{angular momentum balance condition});$$

$$(8) \quad \nabla_0 \cdot \{J_u^{-1/3} [\mathbf{r}'^* - \langle \mathbf{r}'^* : (\mathbf{I} + \nabla_0 \mathbf{u})^T \cdot \mathbf{R} / 3 + pJ_u^{4/3} \rangle \hat{\mathbf{U}}^{-1}]\} \cdot \mathbf{R}^T + \rho_0 \mathbf{b} = 0 \\ (\text{linear momentum equation}) \text{ in } V_0;$$

$$(9) \quad J_u^{-1/3} \mathbf{N} \cdot [\mathbf{r}'^* - \langle \mathbf{r}'^* : (\mathbf{I} + \nabla_0 \mathbf{u})^T \cdot \mathbf{R} / 3 + pJ_u^{4/3} \rangle \hat{\mathbf{U}}^{-1}] \cdot \mathbf{R}^T = \overset{\nu}{\mathbf{T}},$$

the traction condition on $S_{\sigma 0}$; and

$$(10) \quad J_u^{-1/3} \mathbf{N} \cdot [\mathbf{r}'^* - \langle \mathbf{r}'^* : (\mathbf{I} + \nabla_0 \mathbf{u})^T \cdot \mathbf{R} / 3 + pJ_u^{4/3} \rangle \hat{\mathbf{U}}^{-1}] \cdot \mathbf{R}^T = \overset{\nu}{\mathbf{T}},$$

$$(11) \quad \mathbf{u} = \bar{\mathbf{u}},$$

the displacement and traction compatibility conditions on S_{u0} .

One can obtain a *five-field variational principle through the contact transformation*

$$(12) \quad -Q_c^*[\mathbf{r}'^*, p] = Q_0^*(\mathbf{U}', J) - \mathbf{U}' : \mathbf{r}'^* + pJ$$

A substitution of Eq. (12) into Eq. (3) yields

$$(13) \quad \Pi_8(\mathbf{u}, \mathbf{R}, \mathbf{r}'^*, p, \overset{\nu}{\mathbf{T}}) = \int_{V_0} \left\{ -Q_c^*[\mathbf{r}'^*, p] + \frac{\mathbf{r}'^{*T} : \mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u})}{J_u^{1/3}} \right. \\ \left. - J_u p - \rho_0 \mathbf{b} \cdot \mathbf{u} \right\} dV - \int_{S_{\sigma 0}} \overset{\nu}{\mathbf{T}} \cdot \mathbf{u} dA - \int_{S_{u0}} \overset{\nu}{\mathbf{T}} \cdot (\mathbf{u} - \bar{\mathbf{u}}) dA,$$

which is a functional of five fields. The admissible requirements for the independent fields \mathbf{u} , \mathbf{R} , \mathbf{r}'^* , p , $\overset{\nu}{\mathbf{T}}$ are the same

as those in Π .

We can similarly establish the variational principles involving the first Piola–Kirchhoff stress \mathbf{T} . Let

$$(14) \quad \mathbf{F}' = \mathbf{F} J^{-1/3}, \quad W_0^*(\mathbf{F}', J) \equiv W_0(\mathbf{F}^T \cdot \mathbf{F}).$$

From the last two equations, we can show that

$$(15) \quad (\mathbf{T}')_s = \partial W_0^*(\mathbf{F}', J) / \partial \mathbf{F}', \quad -p = \partial W_0^*(\mathbf{F}', J) / \partial J,$$

$$(16) \quad \mathbf{T}^T = J^{-1/3} [\mathbf{T}'^T - (\mathbf{T}'^T : \mathbf{F}) \mathbf{F}^{-1}/3] - p J \mathbf{F}^{-1}.$$

One can then establish a variational principle using \mathbf{u} , \mathbf{F}' , J , \mathbf{T}' , p , $\frac{\nu}{J}$ as independent field variables.

One can also derive a *two-field principle* through the contact transformation

$$(17) \quad -W_m(\mathbf{F}', p) = W_0^*(\mathbf{F}', J) + p J,$$

The enforcement of compatibility conditions Eq. (6) *a priori* results in

$$\mathbf{F}' = (\mathbf{I} + \nabla_0 \mathbf{u}) J^{-1/3}, \quad J = \det(\mathbf{I} + \nabla_0 \mathbf{u}).$$

Then in Eq. (3), replacing $Q_0^*(\mathbf{U}', J)$ with $W_0^*(\mathbf{F}', J)$, enforcing the displacement boundary condition Eq. (11) on S_{u0} , and introducing the penalty function $\gamma [\mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u})]_a^2$, one obtains

$$(18) \quad \begin{aligned} \Pi_9(\mathbf{u}, \mathbf{R}, p) = & \int_{V_0} \left\{ -W_m(\mathbf{F}', p) - p \det(\mathbf{I} + \nabla_0 \mathbf{u}) \right. \\ & \left. + \frac{\gamma}{4} [\mathbf{R}^T \cdot (\mathbf{I} + \nabla_0 \mathbf{u})]_a^2 - \rho_0 \mathbf{b} \cdot \mathbf{u} \right\} dV - \int_{S_{u0}} \bar{\mathbf{T}} \cdot \mathbf{u} dA. \end{aligned}$$

The penalty function is introduced to improve the accuracy of the weak solution (Atluri and Cazzani 1995). The functional Π_9 for infinitesimal strain with $\gamma = 0$ was derived by Tong (1969).

Note that the function Π of Eq. (3) cannot be used for incompressible materials because $Q_0^*(\mathbf{U}', J) [\equiv Q_0(\mathbf{U})]$ is not defined when $J = 1$. However, both Π_8 and Π_9 of Eq. (13) and (18) are suitable for incompressible and nearly incompressible materials.

Before leaving this section, we summarize below the definitions of the various energy density functions and the associated constitutive laws developed in the last two sections:

Energy density functions	Constitutive laws
$Q_0(\mathbf{U}) = W_0(\mathbf{C}) = W_0(\mathbf{F}^T \cdot \mathbf{F})$	$\mathbf{S} = 2 \frac{\partial W_0}{\partial \mathbf{C}}, \quad \mathbf{T}^T = \frac{\partial W_0}{\partial \mathbf{F}}, \quad (\mathbf{T} \cdot \mathbf{R})_s = \frac{\partial Q_0}{\partial \mathbf{U}}$
$Q_c(\mathbf{r}) = (\mathbf{T} \cdot \mathbf{R})_s; \quad \mathbf{U} = Q_0(\mathbf{U})$	$\mathbf{U} = \partial Q_c / \partial \mathbf{r}$
$Q_0^*(\mathbf{U}', J) \equiv Q_0(\mathbf{U}' J^{1/3})$	$\mathbf{r}_s'^* = \partial Q_0^*(\mathbf{U}', J) / \partial \mathbf{U}', \quad p = -\frac{\partial Q_0^*(\mathbf{U}', J)}{\partial J}$
$-Q_c^*[(\mathbf{r}_s'^*)_s, p] \equiv Q_0^*(\mathbf{U}', J)$ $-\mathbf{U}' : \mathbf{r}_s'^* + p J$	$\mathbf{U}' = \partial Q_c^* / \partial \mathbf{r}_s'^*, \quad J = -\frac{\partial Q_c^*}{\partial p}$
$W_0^*(\mathbf{F}', J) \equiv W_0(\mathbf{F}'^T \cdot \mathbf{F}' J^{2/3})$	$(\mathbf{T}')_s = \frac{\partial W_0^*(\mathbf{F}', J)}{\partial \mathbf{F}'}, \quad p = -\frac{\partial W_0^*(\mathbf{F}', J)}{\partial J}$

$$-W_m(\mathbf{F}', p) \equiv W_0^*(\mathbf{F}', J) + pJ$$

$$(\mathbf{T}')_s = -\frac{\partial W_m(\mathbf{F}', p)}{\partial \mathbf{F}'}, \quad J = -\frac{\partial W_m(\mathbf{F}', p)}{\partial p}$$

13.14. SMALL DEFLECTION OF THIN PLATES

In the following two sections we shall consider the basic equations of isotropic elastic plates. First we will treat small and then large deflections. For engineering applications, plate theory is one of the most important and interesting topics in the elasticity theory.

A *plate* is a body bounded by two surfaces of small curvature. The distance h between the surfaces, called the *plate thickness*, is small in comparison with the surface dimensions. The *middle surface* is of equidistance to the bounding surfaces. A *flat* plate has a plane middle surface in the undeformed configuration. We shall only consider flat plates of uniform thickness of homogeneous linear elastic materials. The concepts developed in Secs. 13.1 to 13.11 will be applied to derive the basic equations of the plate theory.

In straining a plate the traction on surfaces is small as compared with the maximum bending or stretching stresses in the body, e.g., the aerodynamic pressure on airplane wings in flight is of the order of 1 to 10 lb/in.² (7 to 70 kPa), whereas the bending stress in the wing skin is likely to range from 10,000 to 200,000 lb/in.² (7×10^4 to 1.4×10^6 kPa). In many applications, plates are used to transmit forces and moments acting on their edges, and no load on the faces at all. This practical situation is important in simplifying the theory.

A thin flat plate is depicted in Fig. 13.14:1. The x, y -plane coincides with the middle surface of the plate in its initial unloaded state, and the z -axis is normal to it. The plate thickness h is much smaller than the plate dimensions in the x, y -plane. Under load, the elastic displacements u_i ($|u_i| \ll h$) and the strains are small. The constitutive equation is Hooke's law.

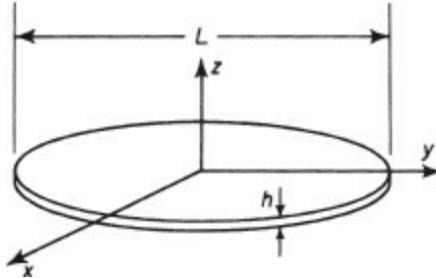


Fig. 13.14:1. A thin plate and the coordinate system.

The plate theory concerns with applications in which the stress components σ_{zx} , σ_{zy} , σ_{zz} are small throughout the plate. Hence, in determining the plate deformation, it is a good approximation to assume that

$$(1) \quad \sigma_{zx} = 0, \quad \sigma_{zy} = 0, \quad \sigma_{zz} = 0, \quad -h/2 \leq z \leq h/2.$$

The plate deformation is mainly stretching and bending, hence one may assume that σ_{xx} and σ_{yy} vary linearly throughout the thickness, i.e.

$$(2) \quad \sigma_{xx} = a_1(x, y) + b_1(x, y)z, \quad \sigma_{yy} = a_2(x, y) + b_2(x, y)z.$$

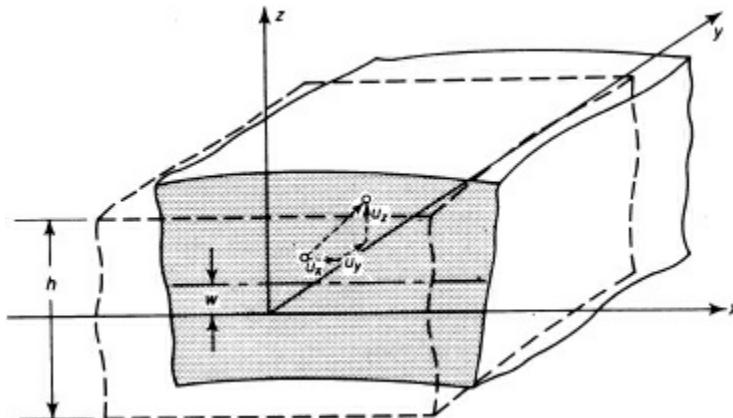


Fig. 13.14:2. Notations for components of displacements.

These assumptions are to be verified *a posteriori* for each boundary-value problem. With the assumptions listed in Eq.

(1) and (2), we have, by Hooke's law,

$$(3) \quad 2e_{zx} = \partial u_x / \partial z + \partial u_z / \partial x = 0, \quad 2e_{zy} = \partial u_y / \partial z + \partial u_z / \partial y = 0.$$

$$(4) \quad e_{zz} = \partial u_z / \partial z = -\nu(\sigma_{xx} + \sigma_{yy})/E.$$

Equations (2) and (4) together yield

$$(5) \quad u_z = w(x, y) - \frac{\nu}{E}[a_1(x, y) + a_2(x, y)]z - \frac{\nu}{E}[b_1(x, y) + b_2(x, y)]\frac{z^2}{2}.$$

The function $w(x, y)$ represents the vertical deflection of the middle surface of the plate. In view of the thinness of the plate, the last two terms in Eq. (5) are in general small and will be neglected in comparison with $w(x, y)$.¹ Substituting $w(x, y)$ for u_z in Eq. (3) and integrating it, we obtain

$$(6) \quad u_x = u(x, y) - z \frac{\partial w(x, y)}{\partial x}, \quad u_y = v(x, y) - z \frac{\partial w(x, y)}{\partial y}, \quad u_z = w(x, y)$$

where $u(x, y)$ and $v(x, y)$ are the in-plane displacements of the midsurface (Fig. 13.14:2).

To simplify the discussion, we shall employ matrix notations and operations in the remaining part of this chapter. From Eq. (6) we deduce the engineering strain matrix:

$$(7) \quad \boldsymbol{\varepsilon} = \mathbf{e} - z\boldsymbol{\kappa},$$

where \mathbf{e} are the in-plane strain column matrix at the mid-surface and $\boldsymbol{\kappa}$ is the curvature column matrix whose components are defined below:

$$(8) \quad \boldsymbol{\varepsilon}^T = [\varepsilon_x \quad \varepsilon_y \quad 2\varepsilon_{xy}], \quad \mathbf{e}^T = [\partial u / \partial x \quad \partial v / \partial y \quad \partial u / \partial y + \partial v / \partial x], \\ \boldsymbol{\kappa}^T = [\partial^2 w / \partial x^2 \quad \partial^2 w / \partial y^2 \quad 2\partial^2 w / (\partial x \partial y)],$$

where the $(\cdot)^T$ denotes the transpose of the relevant quantity. The transpose of a column matrix is a row matrix. In the present definition, $\boldsymbol{\varepsilon}$, \mathbf{e} , and $\boldsymbol{\kappa}$ are no longer tensors. By means of Hooke's law, we find the stress matrix

$$(9) \quad \boldsymbol{\sigma} = [\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy}]^T = \mathbf{D}_e(\mathbf{e} - z\boldsymbol{\kappa}),$$

where

$$(10) \quad \mathbf{D}_e = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

is the elastic coefficient matrix for a homogeneous isotropic material. In this way, the stresses and deformations in the plate are determined on the basis of the assumptions Eq. (1) and (2).

The approximate results embodied in Eq. (6) are called *Kirchhoff's hypothesis*, which states that *every straight line in the plate that was originally perpendicular to the plate middle surface remains straight and perpendicular to the deflected middle surface after the strain*. The theory of plates based on this hypothesis is called the *Kirchhoff theory*.

In most cases the *ad hoc* assumptions Eq. (1) and (2) are approximations. In general, they are not even consistent. Kirchhoff alluded to the theory of simple beams for justification of his hypothesis; but we know from Saint-Venant's beam theory (see Sec. 7.7) that the assumption "plane cross sections remain plane" is incorrect if the resultant shear does not vanish. For these reasons many authors dislike these *ad hoc* assumptions, and have tried other formulations. A systematic scheme of successive approximations has been developed, of which the "first-order" approximation coincides with those of the Kirchhoff theory. These criticisms provide impetus for further developments of plate theory. However, in retrospect, the truth of the matter is that the recognition of Kirchhoff's hypothesis Eq. (6) was the most important discovery in the theory of plates.

The Kirchhoff hypothesis reduces the equations of three-dimensional elastic continuum to those of a two-dimensional non-Euclidean continuum of the middle surface, describing the behavior of the plate in terms of the displacements (and their derivatives) of points on the middle surface.

Let us now consider the equations of equilibrium. Since the approximate stress distribution throughout the thickness is already known [see Eq. (9)], we shall derive the approximate equations of equilibrium by integrating Eqs. (7.1:5) over

the thickness. For small strain, the Cauchy and Kirchhoff stress tensors [see Eq. (13.7:3)] are approximately the same. We shall use σ 's to denote stress in the deformed configuration. We assume that over the faces of the plate ($z = \pm h/2$) there act the normal stress $\sigma_{zz}(\pm h/2)$ and the shear stresses $\sigma_{zx}(\pm h/2)$, $\sigma_{zy}(\pm h/2)$. Figure 13.14:3 shows the directions of stress vectors, if the external loads $\sigma_{zx}(\pm h/2)$, $\sigma_{zy}(\pm h/2)$, $\sigma_{zz}(\pm h/2)$ are positive. By integrating the stresses σ_{xx} , σ_{xy} , σ_{yy} through the thickness, we obtain the stress resultants N_{xx} , N_{xy} , N_{yy} defined by

$$(11) \quad N_x = \int_{-h/2}^{h/2} \sigma_{xx} dz, \quad N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} dz, \quad N_y = \int_{-h/2}^{h/2} \sigma_{yy} dz.$$

Now integrating the first two equations of Eq. (7.1:5) with respect to z from $-h/2$ to $h/2$, we obtain

$$(12) \quad \partial N_x / \partial x + \partial N_{xy} / \partial y + f_x = 0, \quad \partial N_{xy} / \partial x + \partial N_y / \partial y + f_y = 0,$$

where f 's are the *external loads tangent to the plate* (forces per unit area)

$$f_x = \sigma_{zx}(h/2) - \sigma_{zx}(-h/2) + \int_{-h/2}^{h/2} X_x dz,$$

$$f_y = \sigma_{zy}(h/2) - \sigma_{zy}(-h/2) + \int_{-h/2}^{h/2} X_y dz,$$

where X 's are the components of body force per unit volume.

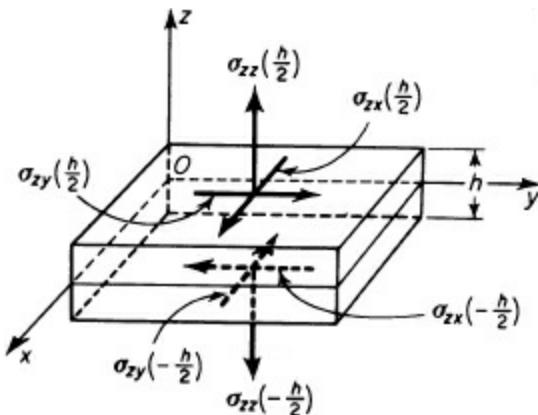


Fig. 13.14:3. External loads acting on the plate surfaces.

Now if we multiply the first two equations of Eq. (7.1:5) by $z dz$ and integrate from $-h/2$ to $h/2$, we will obtain the results

$$(13) \quad \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + m_x = 0, \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + m_y = 0,$$

where

$$(14) \quad M_x = \int_{-h/2}^{h/2} \sigma_{xx} z dz, \quad M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} z dz, \quad M_y = \int_{-h/2}^{h/2} \sigma_{yy} z dz,$$

$$(15) \quad Q_x = \int_{-h/2}^{h/2} \sigma_{zx} dz, \quad Q_y = \int_{-h/2}^{h/2} \sigma_{zy} dz,$$

$$m_x = \frac{h}{2} [\sigma_{zx}(h/2) + \sigma_{zx}(-h/2)] + \int_{-h/2}^{h/2} X_x z dz,$$

$$m_y = \frac{h}{2} [\sigma_{zy}(h/2) + \sigma_{zy}(-h/2)] + \int_{-h/2}^{h/2} X_y z dz.$$

The quantities M_x , M_y are the *bending moments* and M_{xy} the *twisting moment*; m_x , m_y are the *resultant external moments* per unit area about the middle surface; and Q_x , Q_y are the *transverse shears* of dimension of force per unit length. Clearly, the moment arm is $h/2$ for shear on the plate faces. These stress resultants and moments are illustrated in Figs. 13.14:4 and 13.14:5.

If the third equation of Eq. (7.1:5) is multiplied by dz and integrated from $-h/2$ to $h/2$, we obtain

$$(16) \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0$$

where p is the external load per unit area normal to the middle surface defined as

$$p = \sigma_{zz}(h/2) - \sigma_{zz}(-h/2) + \int_{-h/2}^{h/2} X_z dz .$$

Eliminating Q_x , Q_y from Eq. (16), we have the equation of equilibrium in moments,

$$(17) \quad \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = - \frac{\partial m_x}{\partial x} - \frac{\partial m_y}{\partial y} - p .$$

To complete the formulation, we substitute σ_{xx} , σ_{yy} , σ_{xy} from Eq. (9) into Eq. (11) and (14) to obtain

$$(18) \quad \mathbf{N} = [N_x \quad N_y \quad N_{xy}]^T = \mathbf{D}_n \mathbf{e} ,$$

$$(19) \quad \mathbf{M} = [M_x \quad M_y \quad M_{xy}]^T = -\mathbf{D}_b \boldsymbol{\kappa} ,$$

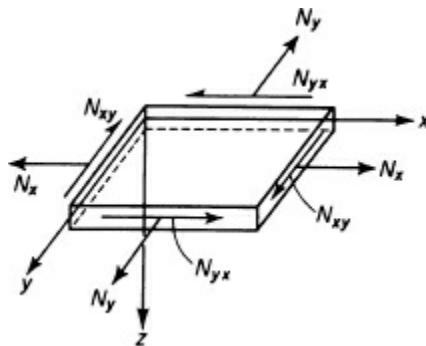
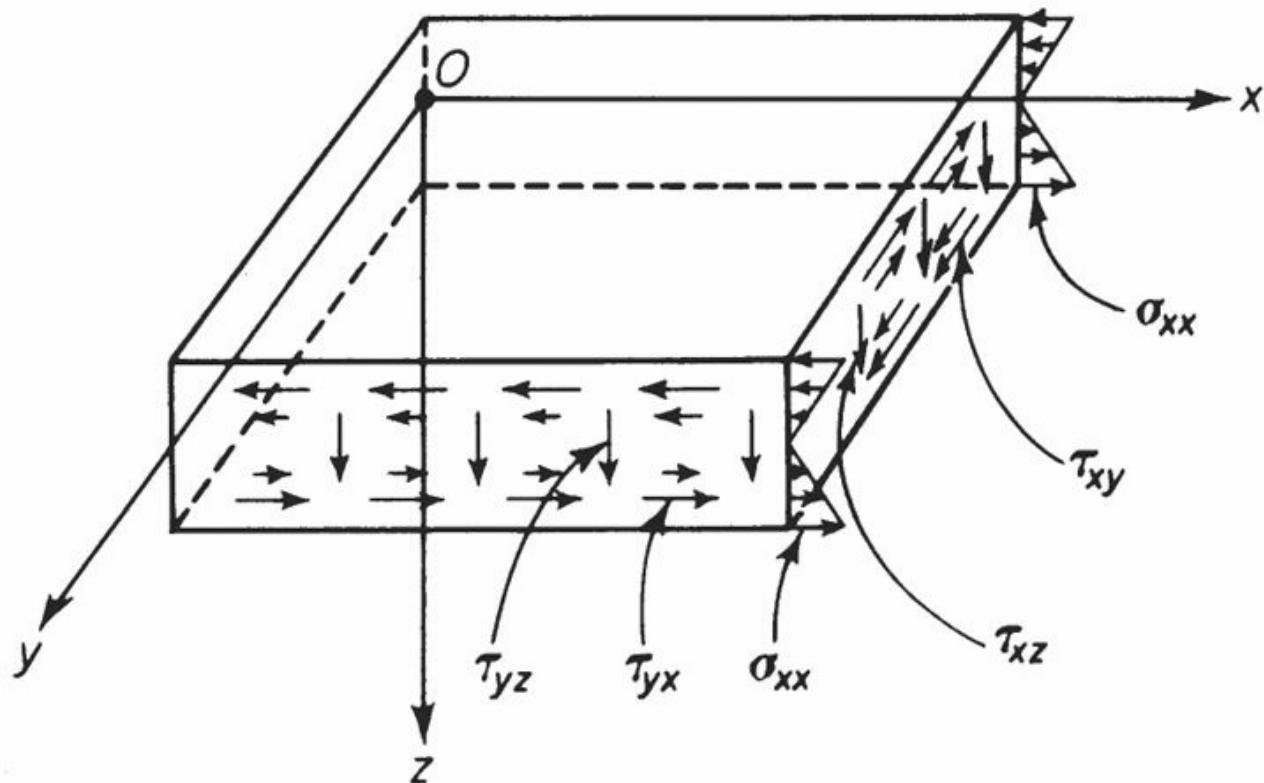
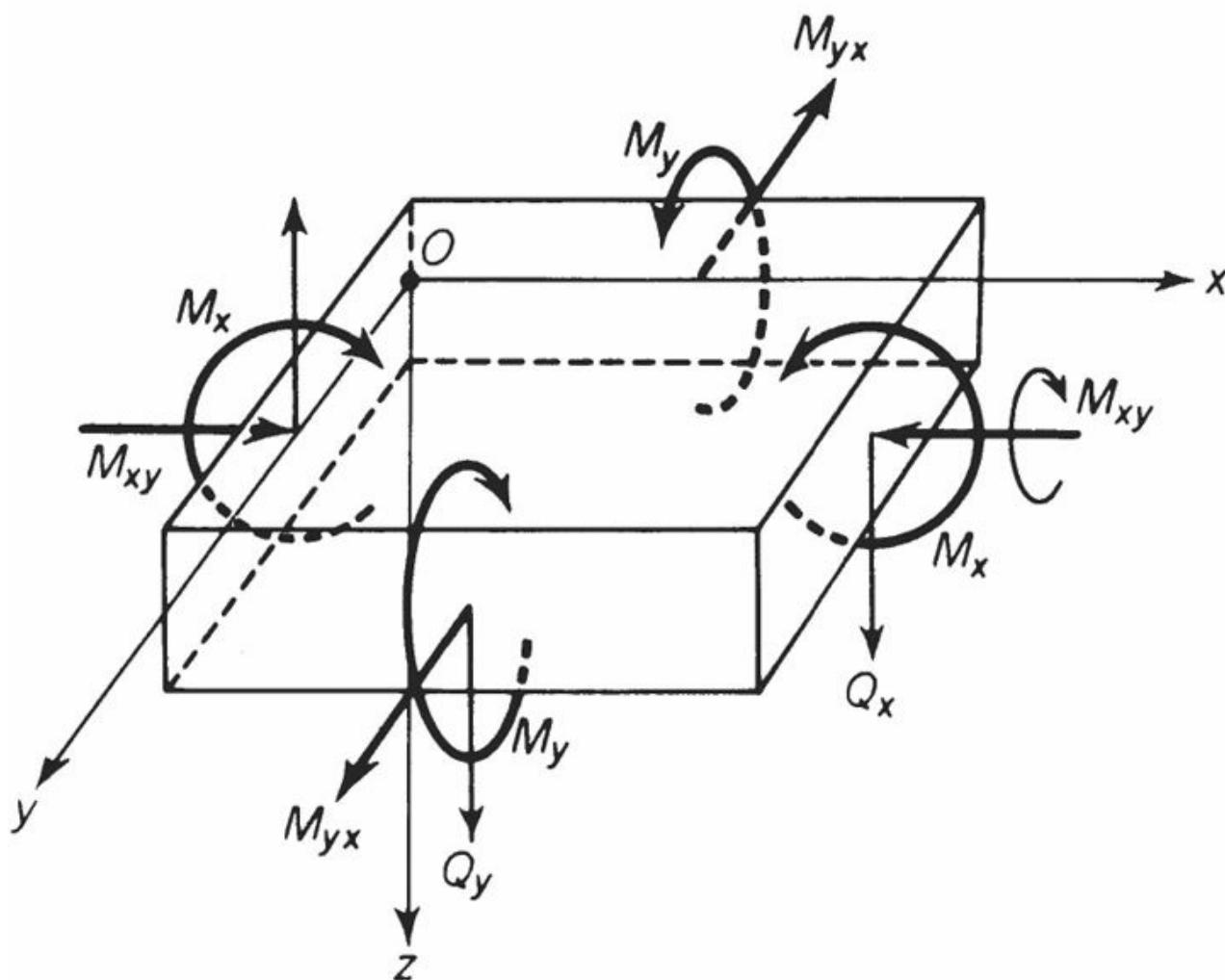


Fig. 13.14:4. Stress resultants.



(a)



(b)

Fig. 13.14:5. Moments.

where

$$(20) \quad D_n = \int_{-h/2}^{h/2} D_e dz,$$

$$D_b = \int_{-h/2}^{h/2} z D_e dz,$$

with D_e given in Eq. (10). For homogeneous isotropic materials,

$$\begin{aligned} D_n &= \frac{Eh}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \\ D_b &= D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \quad D = \frac{Eh^3}{12(1-\nu^2)} \end{aligned}$$

in which D is called the *bending rigidity of the plate*.

Equations (17) and (19) define the *bending problem* for the deflection $w(x, y)$, and Eq. (12) and (18) define the *stretching problem* for the mid-plane displacements u and v . For infinitesimal displacements, there is no coupling between the two responses and the equations are linear. However, for large deflections, they are coupled and nonlinear. A substitution of Eq. (19) into Eq. (17) yields a fourth order equation for w , the fundamental equation of the linear theory of plate bending.

For plates of uniform thickness, the equation becomes

$$(21) \quad \partial^4 w / \partial x^4 + 2\partial^4 w / (\partial x^2 \partial y^2) + \partial^4 w / \partial y^4 = (p + \partial m_x / \partial x + \partial m_y / \partial y) / D.$$

The bending problem is thus reduced to solving a biharmonic equation with appropriate boundary conditions and, therefore, the same as the Airy stress functions in plane elasticity. The plate stretching is identical with plane stress problems and Airy stress functions can also be used.

13.15. LARGE DEFLECTIONS OF PLATES

The theory of small deflections of plates was derived in the previous section under the assumption of infinitesimal displacements. Unfortunately, the results are valid literally only for very small displacements. When the deflections are as large as the thickness of the plate, the results become quite inaccurate. This is in sharp contrast with the theory of beams, for which the linear equation is valid as long as the slope of the deflection curve is small in comparison to unity.

A well-known theory of large deflections of plates is due to von Kármán.

In this theory the following assumptions are made.

(H1) *The plate is thin. The thickness h is much smaller than the characteristic dimension L of the plate in its plane, i.e., $h \ll L$.*

(H2) *The magnitude of the deflection w is of the same order of magnitude as the thickness of the plate, h , but small compared with the typical plate dimension L , i.e.,*

$$|w| = O(h), \quad |w| \ll L.$$

(H3) *The slope is everywhere small, $|\partial w / \partial x| \ll 1$, $|\partial w / \partial y| \ll 1$.*

(H4) *The tangential displacements, u, v are infinitesimal. In the strain-displacement relations, only those nonlinear terms, which depend on $\partial w / \partial x$ and $\partial w / \partial y$ are retained. All other nonlinear terms are neglected.*

(H5) *All strain components are small. Hooke's law holds.*

(H6) Kirchhoff's hypotheses hold ; i.e., the traction on surfaces parallel to the middle surface are small, strains vary linearly with z , the distance from the midplane, within the plate, and normal remains normal.

Thus, von Kármán's theory of plates differs from the linear theory of plates only in retaining certain powers of the derivatives $\partial w/\partial x$ and $\partial w/\partial y$ in the strain-displacement relationship.

The term *large deflection* refers to the fact that w can be of the same order as h , the basic small parameter. As shown in Fig. 13.15:1, the deformed plate differs considerably from the original one. If we retain the rectangular coordinates, with the z -plane fixed on the middle plane of the undeformed plate, then the limits of integration across the plate thickness for a loaded plate can no longer be from $-h/2$ to $h/2$. To resolve the issue we adopt the Lagrangian description for the plate in the plate surfaces are always designated as $z = \pm h/2$.

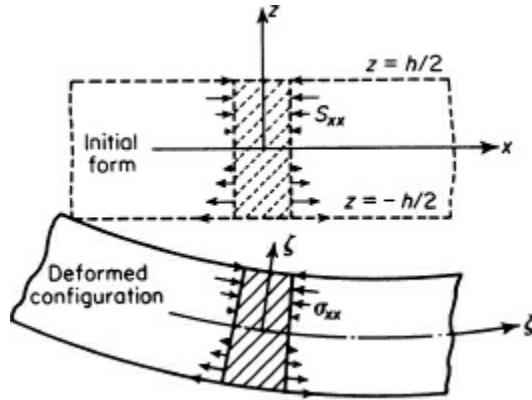


Fig. 13.15:1. Corresponding stresses in the initial and deformed configuration of a plate.

In the following we use *Green's strain tensor* referred to the initial configuration. The strain components [identifying X_1, X_2, X_3 with x, y, z] are given in Eq. (13.3:5) and, according to (H6), the displacement components are given in Eq. (13.14:6). According to (H4) the higher powers of the derivatives of u and v are negligible in comparison with their first powers; and since $|z| \leq h \ll L$, the only nonlinear terms retained in the strain-displacement relations Eqs. (13.14:7) and (13.14:8) are the square and product of $\partial w/\partial x, \partial w/\partial y$. Hence,

$$(1) \quad \mathbf{E} = \mathbf{e} + \mathbf{e}_N/2 - z\kappa,$$

where \mathbf{e}_N are the nonlinear part of the strain at the mid-plane, and

$$(2) \quad \mathbf{e}_N^T = [(\partial w/\partial x)^2 \quad (\partial w/\partial y)^2 \quad 2(\partial w/\partial x)(\partial w/\partial y)].$$

Conjugate to Green's strain tensor is the second Piola-Kirchhoff stress tensor \mathbf{S} [see Eq. (13.7:6)], which can be used in the Lagrangian description. We assume, according to (H5), that \mathbf{S} is related to \mathbf{E} through Hooke's law,

$$\mathbf{S} = [S_{xx} \quad S_{yy} \quad S_{xy}]^T = \mathbf{D}(\mathbf{e} + \mathbf{e}_N/2 - z\kappa),$$

or in terms of the stress resultants and moments

$$(3) \quad \begin{aligned} \mathbf{N}' &= [N'_x \quad N'_y \quad N'_{xy}]^T = \mathbf{D}_n(\mathbf{e} + \mathbf{e}_N/2), \\ \mathbf{M}' &= [M'_x \quad M'_y \quad M'_{xy}]^T = -\mathbf{D}_b\kappa, \end{aligned}$$

where \mathbf{D}_n and \mathbf{D}_b are given in Eq. (13.14:20) and the stress resultants N 's and the moments M 's are defined as

$$(4) \quad N'_x = \int_{-h/2}^{h/2} S_{xx} dz, \quad N'_{xy} = \int_{-h/2}^{h/2} S_{xy} dz, \quad N'_y = \int_{-h/2}^{h/2} S_{yy} dz,$$

$$(5) \quad M'_x = \int_{-h/2}^{h/2} S_{xx} z dz, \quad M'_{xy} = \int_{-h/2}^{h/2} S_{xy} z dz, \quad M'_y = \int_{-h/2}^{h/2} S_{yy} z dz.$$

The prime over N 's and M 's indicates that they are based on the second Piola-Kirchhoff stresses referring to the initial, unloaded configuration of the plate.

The stress tensor \mathbf{S} , the resultants N'_x, N'_y, N'_{xy} , and the moments M'_x, M'_y, M'_{xy} are defined with respect to the original, unloaded configuration of the plate. Since all partial derivatives u, v, w are small in comparison with unity, the values of M'_y, M'_{xy} are in the first order approximation *equal*, respectively, to the Cauchy or Kirchhoff stress tensor σ_{ij} .

the stress resultants N_x , N_y , N_{xy} , and the moments M_x , M_y , M_{xy} defined in the deformed configuration of the plate. See Eq. (13.7:10) which relates the Kirchhoff stress tensor σ_{ij} to the second Piola–Kirchhoff stress tensor S_{ij} . We will drop the prime in the subsequent discussion.

In the Lagrangian description, the condition of equilibrium is given by Eq. (13.10:7)

$$\partial[S_{JK}(\partial x_i/\partial X_K)]/\partial X_J + X_{0i} = 0,$$

in indicial notation, where X_0 's are the body force per unit volume in the original configuration. Identifying X_1 , X_2 , X_3 with x , y , z and identifying x_1 , x_2 , x_3 with $x + u_x$, $y + u_y$, $z + u_z$, we obtain, on retaining only the first-order terms (i.e., neglecting any product terms that include u or v), the equations become

$$(6) \quad \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} + \frac{\partial S_{zx}}{\partial z} + X_{0x} = 0, \quad \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y} + \frac{\partial S_{zy}}{\partial z} + X_{0y} = 0,$$

$$(7) \quad \begin{aligned} \frac{\partial}{\partial x} \left(S_{xx} \frac{\partial w}{\partial x} + S_{xy} \frac{\partial w}{\partial y} + S_{zx} \right) + \frac{\partial}{\partial y} \left(S_{xy} \frac{\partial w}{\partial x} + S_{yy} \frac{\partial w}{\partial y} + S_{zy} \right) \\ + \frac{\partial}{\partial z} \left(S_{zx} \frac{\partial w}{\partial x} + S_{zy} \frac{\partial w}{\partial y} + S_{zz} \right) + X_{0z} = 0. \end{aligned}$$

Note that u_x , u_y , u_z are given in Eq. (13.14:6).

The rest of the development of the large deflection theory parallels closely that of the classical linear theory. Hence, our presentation will be brief, and we refer to the preceding section for many details. If Eq. (6) is multiplied successively by dz and $z dz$ and integrated from $-h/2$ to $h/2$ and if only the first order terms are retained, the governing equations for the membrane stress resultants N 's and the moments M 's are the same as Eqs. (13.14:12) and (13.14:13) with σ_{ij} replaced by S_{ij} in the formulas for the f 's, Q 's, m 's, and M 's.

Finally, an integration of Eq. (7) with respect to z from $-h/2$ to $h/2$ gives

$$(8) \quad \frac{\partial}{\partial x} \left(Q_x + N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left(Q_y + N_{xy} \frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y} \right) + p = 0$$

where p is the lateral load per unit area (of the undeformed middle plane) acting on the plate,

$$(9) \quad p = \left(S_{zx} \frac{\partial w}{\partial x} + S_{zy} \frac{\partial w}{\partial y} + S_{zz} \right)_{-h/2}^{h/2} + \int_{-h/2}^{h/2} X_{0z} dz.$$

The last two terms on the right-hand side of Eq. (9) are the familiar lateral load. The first and second terms represent contributions to the lateral load due to shear acting on the surfaces that are rotated in the deformed position. Equation (8) is substantially different from the corresponding Eq. (13.14:16) of the small-deflection theory. We can eliminate Q_x , Q_y between Eq. (8) and (13.14:13) to obtain

$$\begin{aligned} \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} &= -p - \frac{\partial m_x}{\partial x} - \frac{\partial m_y}{\partial y} \\ &- \frac{\partial}{\partial x} \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) - \frac{\partial}{\partial y} \left(N_{xy} \frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y} \right). \end{aligned}$$

Finally, a substitution of Eq. (3) into the above gives, for a plate of uniform thickness,

$$(10) \quad \Delta \quad \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} \left[p + \frac{\partial m_x}{\partial x} + \frac{\partial m_y}{\partial y} \right. \\ \left. + \frac{\partial}{\partial x} \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_{xy} \frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y} \right) \right].$$

A substitution of Eq. (3) into Eq. (13.14:12) yields immediately the basic equations for u and v . Alternatively, a stress function $F(x; y)$ can be introduced that

$$N_x = \partial^2 F / \partial y^2 - P_x, \quad N_y = \partial^2 F / \partial x^2 - P_y, \quad N_{xy} = -\partial^2 F / (\partial x \partial y)$$

satisfy Eq. (13.14:12) identically, where the P 's are the potentials of the tangential forces given by the indefinite integrals

$$(11) \quad P_x = \int f_x dx, \quad P_y = \int f_y dy.$$

The compatibility condition can be derived by eliminating u and v from Eq. (3). Using Eq. (11), we obtain, for a plate of uniform thickness,

$$(12) \quad \Delta \quad \frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = Eh \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \\ + \frac{\partial^2 (P_y - \nu P_x)}{\partial x^2} + \frac{\partial^2 (P_x - \nu P_y)}{\partial y^2}.$$

Equation (10) and (12) are the famous *von Kármán equations for large deflection of plates*.

In most problems the external loads m 's, and P 's are zero. We then have the well-known pair of equations²

$$(13) \quad \Delta \quad \nabla^4 w = \frac{1}{D} \left(p + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right),$$

$$(14) \quad \Delta \quad \nabla^4 F = Eh \{ [\partial^2 w / (\partial x \partial y)]^2 - (\partial^2 w / \partial x^2)(\partial^2 w / \partial y^2) \},$$

where ∇^4 is the two-dimensional biharmonic operator.

Equations (13) and (14) are coupled and nonlinear. Equation (8) shows that the membrane stresses N_x , N_y , N_{xy} multiplied by the corresponding curvatures and twist, $\partial^2 w / \partial x^2$, $\partial^2 w / \partial y^2$, $\partial^2 w / (\partial x \partial y)$, are equivalent to lateral forces. They represent the feedback of induced membrane stress on bending. The nonlinear term on the right-hand side of Eq. (14) is the Gaussian curvature of the deflected surface $z = w(x, y)$ and vanishes if $z = w(x, y)$ is developable. If a flat plate is deformed into a non-developable surface, its middle surface is stretched in some manner making the curvature nonzero. Thus, the nonlinearity of Eq. (14) is caused by the bending induced stretching that makes the middle surface non-developable. If the nonlinear terms are neglected, Eq. (13) and (14) reduce to the corresponding equations of the small-deflection theory.

There are a large number of papers on the plate theory. Some are devoted to justify the *ad hoc* assumptions (H1)–(H6), or to their removal or generalization; others are concerned with the solution of boundary-value problems. See bibliography 13.13. We shall conclude our discussion here, merely pointing out that the in-plane deformation of plates in small deflection is the same as that of the plane problems of linear elasticity, whereas the large-deflection equations are nonlinear. The theory of large deflection of plates includes many interesting problems of stability, responses, subharmonic resonances, edge layers, etc. The explicit introduction of both Eulerian and Lagrangian descriptions of stress and strain clarifies the foundation of the large-deflection theory of plates.

Problem 13.16. Section 13.9 establishes that the displacement vector \mathbf{u} , the Almansi strain tensor \mathbf{e} , the metric tensor related the deformed body to the undeformed body \mathbf{B}^{-1} , the left stretch tensor \mathbf{V} , the deformation rate tensor \mathbf{D} , and the Cauchy and Kirchhoff stress tensors τ and σ are objective and that the velocity vector \mathbf{v} , the Green strain tensor \mathbf{E} , the metric tensor related the undeformed body to the deformed body C , the deformation gradient \mathbf{F} , the rotational tensor \mathbf{R} , the right stretch tensor \mathbf{U} , the Lagrange stress tensor \mathbf{T} , the second Piola–Kirchhoff stress tensor \mathbf{S} , the Biot stress tensors \mathbf{r}^* , \mathbf{r} , the corotational stress tensor σr , the velocity gradient \mathbf{L} , the spin rate $\mathbf{\Omega}$, and all stress rates are not objective. Work out the details to prove the results.

¹The consistency of this statement has to be verified *a posteriori*. Note, however, that for a successful design using plates as part of a safe and stable structure, the stresses σ_{xx} , σ_{yy} must be bounded by the yielding stress, σ_Y , of the material. Hence, according to Eq. (4), σ_{zz} is bounded by $2\nu\sigma_Y/E$. Thus, the last two terms in Eq. (5) must be smaller than $2\nu(\sigma_Y/E)h$. For most structural materials, (σ_Y/E) is of order 10^{-3} . Hence the last two terms in Eq. (5) are negligible if $w/h > 10^{-3}$.

²These equations were given without proof by von Kármán in 1910. Most books give derivations of these equations without a clear indication as to whether Lagrangian or Eulerian descriptions were used. The explicit use of Lagrangian description introduces a degree of clarity not hitherto achieved.

14

VISCOELASTICITY AND THERMOVISCOELASTICITY

In this chapter we shall consider a class of materials, whose deflection depends not only on the load and temperature but also on time. The present state of deformation of the materials cannot be determined unless the entire history of loading is known. This is particularly true for organic polymers. Thus we must consider other constitutive laws. We shall also generalize the ideas discussed in [Chapters 12](#) and [13](#) to deal with coupled thermal and mechanical processes in thermodynamic nonequilibrium.

14.1. LINEAR SOLIDS WITH MEMORY

[Figure 14.1:1](#) shows three mechanical models of material behavior, namely the Maxwell, Voigt, and Kelvin models, composed of combinations of linear springs with spring constant μ and dashpots (pistons moving in a viscous fluid) with coefficient of viscosity η . The *linear springs*, which produce instantaneously a deformation proportional to the load, are elastic. The *dashpots* produce a velocity proportional to the load at any instant and are viscous. Such hereditary solids are called *viscoelastic* materials.

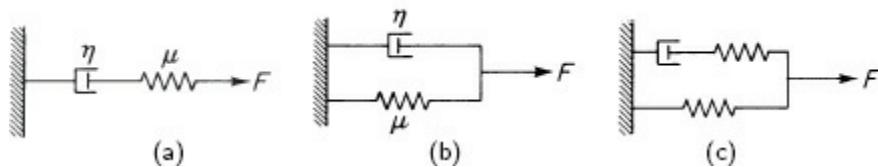


Fig. 14.1:1. Models of linear viscoelasticity: (a) Maxwell, (b) Voigt, (c) Kelvin.

The load-deflection relationships for these models are

- (1) Maxwell model: $\dot{u} = \dot{F}/\mu + F/\eta$, $u(0) = F(0)/\mu$,
- (2) Voigt model: $F = \mu u + \eta \dot{u}$, $u(0) = 0$,
- (3) Kelvin model: $F + \tau_e \dot{F} = E_R(u + \tau_\sigma \dot{u})$, $\tau_e F(0) = E_R \tau_\sigma u(0)$,

where τ_e and τ_σ are constant. To integrate these equations, the initial conditions at $t = 0$ must be prescribed as indicated above.

The *creep functions* $c(t)$, the displacement u in response to a unit-step force $F(t) = \mathbf{1}(t)$ defined in Eq. (7) below, are the solutions of Eqs. (1)–(3):

- (4) Maxwell solid: $c(t) = (1/\mu + t/\eta) \mathbf{1}(t)$,
- (5) Voigt solid: $c(t) = [1 - e^{-(\mu/\eta)t}] \mathbf{1}(t)/\mu$,
- (6) Kelvin solid: $c(t) = [1 - (1 - \tau_e/\tau_\sigma) e^{-t/\tau_\sigma}] \mathbf{1}(t)/E_R$,

where the *unit-step function* $\mathbf{1}(t)$ is defined as

$$(7) \quad \mathbf{1}(t) = \begin{cases} 1 & \text{when } t > 0, \\ 1/2 & \text{when } t = 0, \\ 0 & \text{when } t < 0. \end{cases}$$

A body which obeys a load-deflection relation given by Maxwell's model is called a Maxwell solid. Similarly, Voigt and Kelvin solids are defined.

Interchanging the roles of F and u , we obtain the *relaxation function* as the response $k(t)$ corresponding to an elongation $u(t) = \mathbf{1}(t)$.

$$(8) \text{ Maxwell solid: } k(t) = \mu e^{-(\mu/\eta)t} \mathbf{1}(t),$$

$$(9) \text{ Voigt solid: } k(t) = \eta \delta(t) + \mu \mathbf{1}(t),$$

$$(10) \text{ Kelvin solid: } k(t) = E_R [1 - (1 - \tau_\sigma/\tau_e) e^{-t/\tau_e}] \mathbf{1}(t),$$

where $\delta(t)$ is the *Dirac delta-function*. The functions, $c(t)$ and $k(t)$, are illustrated in Figs. 14.1:2 and 14.1:3, respectively.

For Maxwell solid, a sudden application of load induces an immediate deflection of the elastic spring. The sudden deformation is followed by “creep” of the dashpot, where the stress is relaxed according to an exponential law (8). The factor η/μ , with dimensions of time, called a *relaxation time*, characterizes the force decay rate.

For Voigt solid, a sudden application of force produces no immediate deflection of the spring as it is arranged in parallel with the dashpot, which does not move instantaneously. As shown by Eq. (5) [Fig. 14.1:2(b)] that the deformation gradually builds up, the spring takes a bigger and bigger share of the load and the dashpot displacement relaxes exponentially. The ratio η/μ is a relaxation time characterizing the rate of decay of the deflection.

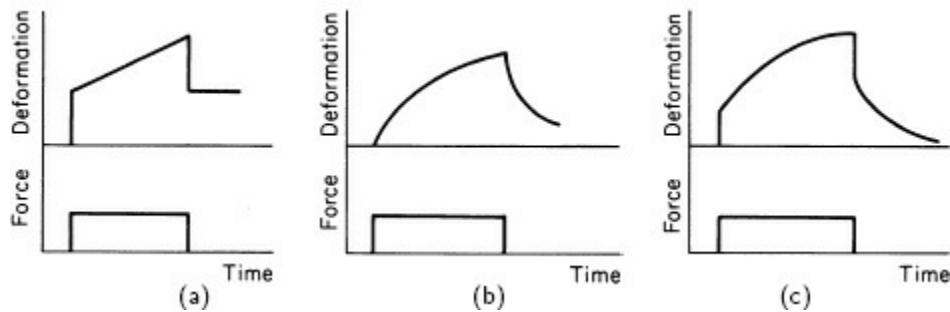


Fig. 14.1:2. Creep function of (a) Maxwell, (b) Voigt, (c) Kelvin solid. A negative phase is superposed at the time of unloading.

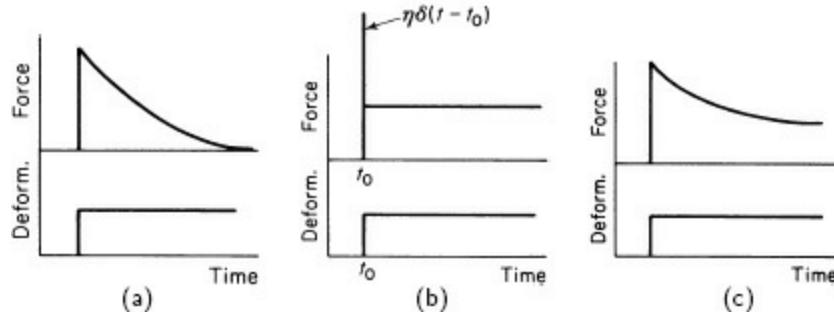


Fig. 14.1:3. Relaxation function, of (a) Maxwell, (b) Voigt, (c) Kelvin solid.

For Kelvin solid, a similar interpretation is applicable: τ_e is the relaxation time of load under a constant deflection [Eq. (10)], whereas τ_σ is the relaxation time of deflection under the condition of constant load [Eq. (4)]. As $t \rightarrow \infty$, the dashpot is completely relaxed, and the load-deflection relation becomes that of the springs, as characterized by the constant E_R in Eqs. (4) and (10). Therefore, E_R is called the *relaxed elastic modulus*.

Load-deflection relations such as (1)–(3) were proposed to extend the classical elasticity theory to include *anelastic* phenomena. Lord Kelvin, on measuring the variation of the dissipation rate of energy with oscillating frequency in various materials, showed the inadequacy of the Maxwell and Voigt equations. A more successful generalization using mechanical models was first made by Poynting and Thomson in their book *Properties of Matter* (London: Griffin and Co., 1902).

14.2. ANISOTROPIC LINEAR VISCOELASTIC MATERIALS

The material obeys a linear hereditary law is said to be *viscoelastic*, although the logic of this terminology is by no means certain. The origin of the

term lies in the simple models such as those of Maxwell, Voigt, or the standard linear solid, built of springs and dashpots. The springs are elastic, the dashpots are viscous; hence the name. Such an etymology is entirely different from that of viscoplasticity, hyperelasticity, hypoelasticity, elastic-plastic materials, etc. However, the terminology of calling a material obeying a linear hereditary law as viscoelastic is popular and well-accepted.

For 3-dimensional continua in a rectangular Cartesian frame, let \mathbf{x} be the position vector of a point. A function of position $f(x_1, x_2, x_3)$ is denoted as $f(\mathbf{x})$. Let σ_{ij} and e_{ij} be the stress and strain tensors at \mathbf{x} and in the time interval $-\infty < t < \infty$. The strain field $e_{ij}(\mathbf{x}, t)$ and the displacement field $u_i(\mathbf{x}, t)$ as well as the velocity field $v_i(\mathbf{x}, t)$ are infinitesimal, and

$$(1) \quad e_{ij} = (u_{i,j} + u_{j,i})/2,$$

where a comma denotes a partial differentiation. Under the assumption of infinitesimal strain and velocity, the partial derivative with respect to time $\partial e_{ij}/\partial t$ equals the material derivative of e_{ij} within the first order. We define a *linear viscoelastic* material to be one for which $\sigma_{ij}(\mathbf{x}, t)$ is related to $e_{ij}(\mathbf{x}, t)$ by a convolution integral

$$(2) \quad \Delta \quad \sigma_{ij}(\mathbf{x}, t) = \int_{-\infty}^t G_{ijkl}(\mathbf{x}, t - \tau) [\partial e_{kl}(\mathbf{x}, \tau)/\partial \tau] d\tau,$$

the stress-strain law of the relaxation type, where G_{ijkl} is a tensor of order 4, called the *tensorial relaxation function* of the material. Its inverse,

$$(3) \quad \Delta \quad e_{ij}(\mathbf{x}, t) = \int_{-\infty}^t J_{ijkl}(\mathbf{x}, t - \tau) [\partial \sigma_{kl}(\mathbf{x}, \tau)/\partial \tau] d\tau,$$

if exists, is called the stress-strain law of the *creep type*. The fourth-order tensor J_{ijkl} is the *tensorial creep function*. Curtin and Sternberg (1962) show that the inverse (3) of Eq. (2) exists if $G_{ijkl}(\mathbf{x}, t)$ is twice differentiable and the initial value of $G_{ijkl}(\mathbf{x}, t)$ at $t = 0$ is not zero.

The lower limits of integration in Eqs. (2) and (3) are taken as $-\infty$, which means that the integration is to be taken before the very beginning of motion. If the motion starts at time $t = 0$, and $e_{ij} = \sigma_{ij} = 0$ for $t < 0$, Eq. (2) reduces to

$$(4) \quad \sigma_{ij}(\mathbf{x}, t) = G_{ijkl}(\mathbf{x}, t) e_{kl}(\mathbf{x}, 0+) + \int_0^t G_{ijkl}(\mathbf{x}, t - \tau) \frac{\partial e_{kl}(\mathbf{x}, \tau)}{\partial \tau} d\tau,$$

where $e_{ij}(\mathbf{x}, 0+)$ is the limiting value of $e_{ij}(\mathbf{x}, t)$ when $t \rightarrow 0$ from the positive side. The first term in Eq. (4) gives the effect of initial disturbance from the jump of $e_{ij}(\mathbf{x}, t)$ at $t = 0$. In fact, it was tacitly assumed that $e_{ij}(\mathbf{x}, t)$ is continuous and differentiable in Eq. (2). Any discontinuity of $e_{ij}(\mathbf{x}, t)$ in the form of a jump will contribute a term similar to the first term in Eq. (4), e.g., if $e_{ij}(\mathbf{x}, t)$ has another jump $\Delta e_{ij}(\mathbf{x}, \xi)$ at $t = \xi$, as shown in Fig. 14.2:1, while G_{ijkl} and $\partial e_{ij}/\partial t$ are continuous elsewhere, then we have

$$(5) \quad \sigma_{ij}(\mathbf{x}, t) = G_{ijkl}(\mathbf{x}, t) e_{kl}(\mathbf{x}, 0) + G_{ijkl}(\mathbf{x}, t) \Delta e_{kl}(\mathbf{x}, \xi) \mathbf{1}(t - \xi) \\ + \int_0^t G_{ijkl}(\mathbf{x}, t - \tau) [\partial e_{kl}(\mathbf{x}, \tau)/\partial \tau] d\tau.$$

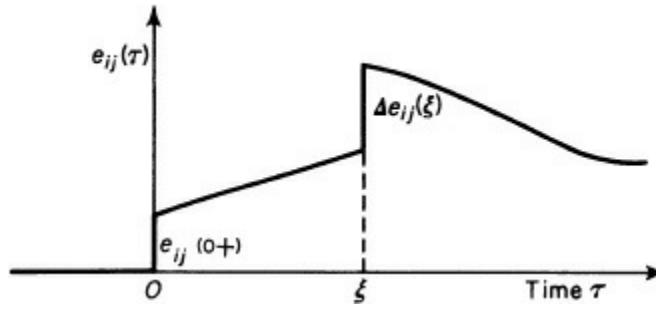


Fig. 14.2:1. Illustration for a loading history with jumps.

In such cases the following forms, which are equivalent to Eq. (4) when $\partial e_{ij}/\partial t$ and $\partial G_{ijkl}/\partial t$ exist and are continuous in $0 \leq t < \infty$, may be used to advantage:

$$(6) \quad \sigma_{ij}(\mathbf{x}, t) = G_{ijkl}(\mathbf{x}, 0) e_{kl}(\mathbf{x}, t) + \int_0^t e_{kl}(\mathbf{x}, t - \tau) [\partial G_{ijkl}(\mathbf{x}, \tau)/\partial \tau] d\tau \\ = \frac{\partial}{\partial t} \int_0^t e_{kl}(\mathbf{x}, t - \tau) G_{ijkl}(\mathbf{x}, \tau) d\tau.$$

These constitutive equations are appropriate for *isothermal conditions*. For temperature influences, see [Bibliography 14.2](#).

A viscoelastic material is defined by specific relaxation or creep functions. In the following discussions we shall regard G_{ijkl} or J_{ijkl} as experimentally determined. Of the variables related by these functions, their current value depends more strongly upon the recent history than the distant past. In other words, G_{ijkl} or J_{ijkl} serve as weighting functions continuously decreasing dependence upon past events. The magnitude of the slope of each component of the relaxation and creep tensors is a continuously decreasing function of time. The relaxation functions approach asymptotically non-zero constants while the creep functions vanish at large time.

In viscoelasticity, we have to write many convolution integrals. We introduce a shorthand notation of *composition products*. Let ϕ and ψ be functions defined on the intervals $0 \leq t < \infty$ and $-\infty < t < \infty$, respectively, and let the integral

$$(7) \quad I(t) = \phi(t)\psi(0) + \int_0^t \phi(t-\tau)[d\psi(\tau)/d\tau]d\tau$$

exist for all t in $(0, \infty)$. Then the function $I(t)$ is called the *convolution of ϕ and ψ* and is denoted by a “composition” product

$$(8) \quad I(t) = \phi * d\psi .$$

The integral in Eq. (7) is in Riemannian or Stieltjes’ sense with the latter being more general and better suited (Gurtin and Sternberg 1962).

The following properties of the convolution of ϕ with ψ and θ (also defined over $-\infty < t < \infty$) can be verified.

$$(9) \quad \phi * d\psi = \psi * d\phi \quad (\text{commutativity}),$$

$$(10) \quad \phi * d(\psi * d\theta) = (\phi * d\psi) * d\theta = \phi * d\psi * d\theta \quad (\text{associativity}),$$

$$(11) \quad \phi * d(\psi + \theta) = \phi * d\psi + \phi * d\theta \quad (\text{distributivity}),$$

$$(12) \quad \phi * d\psi \equiv 0 \text{ implies } \phi \equiv 0 \text{ or } \psi \equiv 0 \quad (\text{Titchmarsh theorem}) .$$

With these notations, Eqs. (2) and (3) may be written as

$$(2a) \quad \sigma_{ij} = G_{ijkl} * de_{kl} = e_{kl} * dG_{ijkl} ,$$

$$(2b) \quad e_{ij} = J_{ijkl} * d\sigma_{kl} = \sigma_{kl} * dJ_{ijkl} .$$

The symmetry of the stress and strain tensors either requires or permits

$$(13) \quad G_{ijkl} = G_{jikl} = G_{ijlk} ,$$

$$(14) \quad J_{ijkl} = J_{jikl} = J_{ijlk} .$$

Furthermore, it is natural to require that the action of a loading at a time t_0 will produce a response only for $t \geq t_0$. Hence, we must have

$$(15) \quad G_{ijkl} = 0, \quad J_{ijkl} = 0, \quad \text{for } -\infty < t < 0 .$$

This requirement is sometimes called *the axiom of nonretroactivity*.

If G_{ijkl} is invariant with respect to rotation of Cartesian coordinates, the material is said to be *isotropic*. A 4th order isotropic tensor with the symmetry properties of Eq. (13) can be written as

$$(16) \quad G_{ijkl} = (G_2 - G_1)\delta_{ij}\delta_{kl}/3 + G_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2 ,$$

where G_1, G_2 are scalar functions satisfying Eq. (15). If Eq. (16) is substituted into Eq. (2a), we obtain the stress-strain laws in the form

$$(17) \blacktrianglequad \sigma'_{ij} = e'_{ij} * dG_1 = G_1 * de'_{ij}, \quad \sigma_{kk} = e_{kk} * dG_2 = G_2 * de_{kk},$$

where σ'_{ij} and e'_{ij} are the components of the stress and strain deviators as

$$(18) \quad \sigma'_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3, \quad e'_{ij} = e_{ij} - \delta_{ij}e_{kk}/3.$$

The functions G_1 and G_2 are the relaxation functions in shear and isotropic compression, respectively.

Let J_1 and J_2 be the creep functions in shear and isotropic compression, respectively. The corresponding stress-strain laws of the creep type for an isotropic material is

$$(19) \blacktrianglequad e'_{ij} = \sigma'_{ij} * dJ_1 = J_1 * d\sigma'_{ij}, \quad e_{kk} = \sigma_{kk} * dJ_2 = J_2 * d\sigma_{kk}.$$

Within a certain range of frequencies, measurements of harmonic responses of many materials are feasible. The bulk of our information about the viscoelastic behavior of high polymers is presented in the form of *frequency responses*.

14.3. STRESS-STRAIN RELATIONS IN DIFFERENTIAL EQUATION FORM

The stress-strain relation may be put in the form of a differential equation as the Maxwell, the Voigt, or the standard linear models discussed in Sec. 14.1. A more general expression may be given as follows. Let D denote the timederivative operator, or *differential operator* defined by

$$(1) \quad Df = \partial f(t)/dt, \quad D^2f = \partial^2 f(t)/dt^2, \quad \text{etc.}$$

where f is a function of time. Let us consider the polynomials

$$(2) \quad P_\alpha(D) = \sum_{k=0}^{n_\alpha} a_{k\alpha} D^k, \quad Q_\alpha(D) = \sum_{k=0}^{m_\alpha} b_{k\alpha} D^k,$$

where $a_{k\alpha}$, $b_{k\alpha}$ ($\neq 0$) are real-valued functions of the spatial coordinates x_i and α ($= 1, 2$) refers to the deviatoric and dilatational functions, respectively. Then the stress-strain relations

$$(3) \blacktrianglequad P_1(D)\sigma'_{ij} = Q_1(D)e'_{ij}, \quad P_2(D)\sigma_{kk} = Q_2(D)e_{kk},$$

specify an isotropic linear viscoelastic material. Here σ_{ij} , e_{ij} , σ'_{ij} , e'_{ij} [see Eq. (14.2:17)] are understood to be functions of \mathbf{x} and t .

Consider the load-deflection relationship of a network of springs and dashpots such as those pictured in Fig. 14.3:1. It can be shown that the load σ and the deflection e are related by an equation of the form of Eq. (3) (see Problem 14.1). For this reason, a viscoelastic body is often represented by a mechanical model. Naturally, not all conceivable polynomials (2) may represent physically realizable systems.

If the stress-strain law of a given material can be expressed in both the integral and differential form, the relationship between the relaxation (or creep) function and the differential operators can be determined. The result is particularly simple if we assume that

$$(4) \quad \sigma_{ij}(t) = e_{ij}(t) = 0 \quad \text{for } t < 0,$$

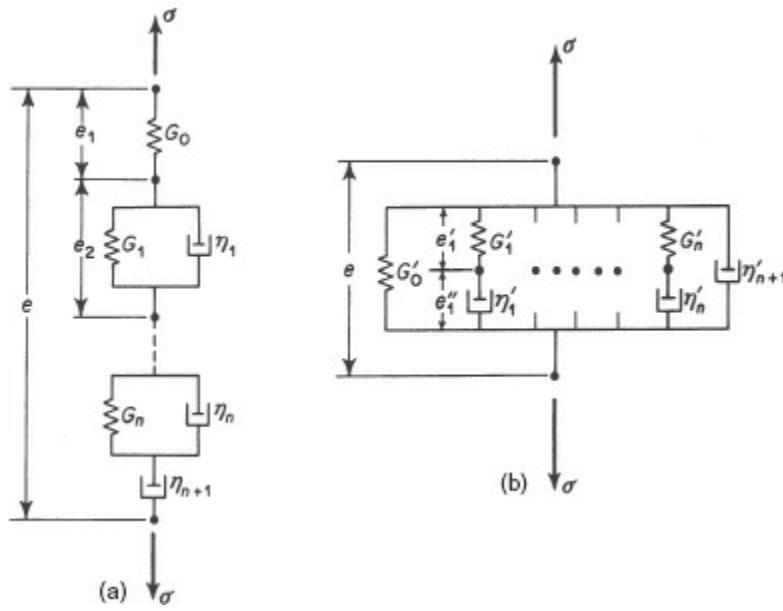


Fig. 14.3:1. Generalized Kelvin and Maxwell models.

and that the Laplace transformations of $\sigma_{ij}(t)$ and $e_{ij}(t)$ exist. Let the transformed function of $f(t)$ be indicated by a bar, then

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt \quad \text{RI } s > s_0,$$

where s_0 is the abscissa of convergence of the Laplace integral.

The Laplace transformations of Eqs. (14.2:17) and (14.2:19) are

$$(5) \quad \bar{\sigma}'_{ij}(s) = s\bar{G}_1(s)\bar{e}'_{ij}(s), \quad \bar{\sigma}_{kk}(s) = s\bar{G}_2(s)\bar{e}_{kk}(s),$$

$$(6) \quad \bar{e}'_{ij}(s) = s\bar{J}_1(s)\bar{\sigma}'_{ij}(s), \quad \bar{e}_{kk}(s) = s\bar{J}_2(s)\bar{\sigma}_{kk}(s),$$

and the transformation of Eq. (3) is

$$(7) \quad \begin{aligned} \bar{P}_\alpha(s)\bar{\sigma}'_{ij}(s) - \sum_{k=1}^{n_\alpha} a_{k\alpha} \sum_{r=1}^k s^{r-1} [\partial^{k-r} \sigma'_{ij}(0)/\partial t^{k-r}] \\ = \bar{Q}_\alpha(s)\bar{e}'_{ij}(s) - \sum_{k=1}^{m_\alpha} b_{k\alpha} \sum_{r=1}^k s^{r-1} [\partial^{k-r} e'_{ij}(0)/\partial t^{k-r}], \end{aligned}$$

where $\alpha = 1$ denoting the deviatoric functions,

$$(8) \quad \bar{P}_\alpha(s) = \sum_{k=0}^{n_\alpha} a_{k\alpha} s^k, \quad \bar{Q}_\alpha(s) = \sum_{k=0}^{m_\alpha} b_{k\alpha} s^k,$$

and $\sigma'_{ij}(0)$, $\partial\sigma'_{ij}/\partial t(0+)$ etc., are the initial values of σ'_{ij} , $\partial\sigma'_{ij}/\partial t$, etc., i.e., the value of $\sigma'_{ij}(x, t)$ with $t \rightarrow 0$ from the positive side. One can find the similar expression for the dilatational functions ($\alpha = 2$).

One identifies Eqs. (7) with the relaxation law (5) if $n_1 \geq m_1$, otherwise with the creep law (6). Then Eq. (7) may be written as¹

$$(9) \quad \begin{aligned} \bar{\sigma}'_{ij}(s) &= [\bar{Q}_1(s)/\bar{P}_1(s)]\bar{e}'_{ij}(s) = s\bar{G}_1(s)\bar{e}'_{ij}(s), \quad \text{if } n_1 \geq m_1, \\ \bar{e}'_{ij}(s) &= [\bar{P}_1(s)/\bar{Q}_1(s)]\bar{\sigma}'_{ij}(s) = s\bar{J}_1(s)\bar{\sigma}_{ij}(s), \quad \text{if } n_1 < m_1, \end{aligned}$$

provided that the following initial condition holds:

$$(10) \quad \sum_{k=1}^{n_1} a_{k1} \sum_{r=1}^k s^{k-r} \partial^{r-1} \sigma'_{ij}(0)/\partial t^{r-1} - \sum_{k=1}^{m_1} b_{k1} \sum_{r=1}^k s^{k-r} \partial^{r-1} e'_{ij}(0)/\partial t^{r-1} = 0.$$

The initial condition (10) is an identity as a polynomial in s ; every coefficient of the polynomial must vanish. Thus

$$(11) \blacktriangle \sum_{r=k}^{N_1} [a_{r1} \partial^{r-k} \sigma'_{ij}(0) / \partial t^{r-k} - b_{r1} \partial^{r-k} e'_{ij}(0) / \partial t^{r-k}] = 0, \quad k = 1, \dots, N_1,$$

where $N_1 = \text{Max}(n_1, m_1)$ is the larger of n_1 and m_1 ; $a_{r1} = 0$, if $n_1 \leq N_1$ and $b_{r1} = 0$, if $m_1 \leq N_1$. The case of $\alpha = 2$ for the dilatational components is to be worked out by readers in Problem 14.3. In both cases, i.e., $\alpha = 1$ and 2, we may identify Eq. (9) with the relaxation law (5), with

$$(12a) \blacktriangle \bar{G}_\alpha(s) = \bar{Q}_\alpha(s) / [s \bar{P}_\alpha(s)], \quad \text{if } n_\alpha \geq m_\alpha,$$

or with the creep law (6),

$$(12b) \blacktriangle \bar{J}_\alpha(s) = \bar{P}_\alpha(s) / [s \bar{Q}_\alpha(s)] = 1 / [s^2 \bar{G}_\alpha(s)], \quad \text{if } n_\alpha \leq m_\alpha.$$

If $n_\alpha = m_\alpha$ then the material may be represented by either the stress-strain laws of relaxation or creep type.

The initial conditions (11) are important. They represent the initial conditions to be imposed when Eqs. (3) are identified as the differential equations for σ'_{ij} with e'_{ij} regarded as forcing functions; or vice versa.

The initial conditions (11) is physically significant that Eq. (14.2:17) in the preceding section is valid for t in $(-\infty, \infty)$, and Eq. (3) of this section for t in $(0, \infty)$ with $\sigma'_{ij} = e'_{ij} = 0$ for t in $(-\infty, 0)$. If

$$(13) \quad e'_{ij} = \partial e'_{ij} / \partial t = \dots = \partial^{m_1-1} e'_{ij} / \partial t^{m_1-1} = 0 \quad \text{at } t = 0,$$

Equations (11) imply

$$(14) \quad \sigma'_{ij} = \partial \sigma'_{ij} / \partial t = \dots = \partial^{n_1-1} \sigma'_{ij} / \partial t^{n_1-1} = 0 \quad \text{at } t = 0.$$

Thus the transition is smooth, as expected. The surprising aspect of Eqs. (11) arises only when $e'_{ij}(t)$ is continuous in $(0, \infty)$ but the limiting values $e'_{ij}(0+)$, $(\partial e'_{ij} / \partial t)(0+)$, etc., are nonzero as $t \rightarrow 0$. In this case there is a jump in $e'_{ij}(t)$ or its derivatives in the neighborhood of $t = 0$. The initial conditions (11) stem from the fact that the differential operators in Eqs. (3) are expected to hold, in a certain sense, during the jump.

One can show this by smoothing off the jump of $e'_{ij}(t, \varepsilon)$ (bounded, continuous, and n_α times differential) over the range $(-\varepsilon, \varepsilon)$ with $e'_{ij}(t, \varepsilon)$ and its derivatives being zero at the left end, $t = -\varepsilon$; whereas assuming the values $e'_{ij}(0+)$, $(\partial e'_{ij} / \partial t)(0+)$, etc., at the right end, $t = \varepsilon$. Such a function serves to describe the jump as we let $\varepsilon \rightarrow 0$. Let $e'_{ij}(t, \varepsilon)$ and $\sigma'_{ij}(t, \varepsilon)$ be connected by the viscoelasticity law (3); i.e.

$$(15) \quad \sum_{k=0}^{N_1} a_{k1} \partial^k \sigma'_{ij}(t, \varepsilon) / \partial t^k = \sum_{k=0}^{N_1} b_{k1} \partial^k e'_{ij}(t, \varepsilon) / \partial t^k,$$

where $a_{k1} = 0$, if $n_1 < k \leq N_1$, and $b_{k1} = 0$, if $m_1 < k \leq N_1$ to eliminate any terms which may be absent from Eq. (3). Repeatedly integrating Eq. (15) with respect to t and taking $\varepsilon \rightarrow 0$ give Eqs. (11).

Problem 14.1. Let σ denote loads and ε displacements. Show that, for the generalized Kelvin model shown in Fig. 14.3:1(a),

$$G_0 e_1 = \sigma, \quad G_0 e_2 + \eta_1 D e_2 = \sigma, \quad \dots, \quad \varepsilon = e_1 + e_2 + \dots + e_n,$$

so that

$$[1/G_0 + 1/(G_1 + \eta_1 D) + \dots + 1/(G_n + \eta_n D) + 1/(\eta_{n+1} D)]\sigma = \varepsilon.$$

For the generalized Maxwell model shown in Fig. 14.3:1(b), we have

$$\begin{aligned} G'_1 e'_1 &= \sigma_1, \quad \eta'_1 D e''_1 = \sigma_1, \quad \varepsilon = e'_1 + e''_1 = [1/G'_1 + 1/(\eta'_1 D)]\sigma_1, \\ \sigma &= \sigma_0 + \sigma_1 + \dots + \sigma_{n+1} \\ &= [G'_0 + G'_1 \eta'_1 D / (G'_1 + \eta'_1 D) + \dots + G'_n \eta'_n D / (G'_n + \eta'_n D) + \eta'_{n+1} D]e. \end{aligned}$$

Reduce these relations to the form of Eq. (3).

Problem 14.2. Let a special case of Eq. (3) be specified by

$$P_1 \sigma_{ij} = a_{21} \ddot{\sigma}_{ij} + a_{11} \dot{\sigma}_{ij} + a_{01} \sigma_{ij}, \quad Q_1 e_{ij} = b_{21} \ddot{e}_{ij} + b_{11} \dot{e}_{ij} + b_{01} e_{ij}$$

where a dot denotes a differentiation with respect to time. Deduce proper initial conditions. Interpret the usual

statement that “the initial response of a viscoelastic body is purely elastic.”

Problem 14.3. Obtain the dilatational stress strain differential operator in the form of Eqs. (7) and (8); and the equivalent relaxation integral form of Eqs. (12), i.e., $\alpha = 2$. Obtain Eqs. (11) by repeated applications of the same process. *Hints:* Replace σ'_{ij}, e'_{ij} with σ_{jj}, e_{jj} in Eqs. (7)–(11).

14.4. STEADY STATE HARMONIC OSCILLATION

Consider the steady state harmonic oscillations in viscoelastic bodies of isotropic material. Consider the stress strain relation

$$(1) \quad \sigma = \int_{-\infty}^t G_\alpha(t - \tau) [d\varepsilon(\tau)/d\tau] d\tau$$

with $\alpha = 1$ or 2 and the strain history being a harmonic function

$$(2) \quad \varepsilon(t) = \varepsilon_0 e^{i\omega t},$$

where ε_0 is the amplitude and ω is the frequency of oscillation. The stress can be written in the form

$$(3) \quad \sigma(t) = \tilde{G}_\alpha(i\omega) \varepsilon_0 e^{i\omega t},$$

where $\tilde{G}_\alpha(i\omega)$ is the *complex modulus* of frequency to be determined. Equation (3) can be written in an alternate form

$$(4) \quad \sigma(t) = |\tilde{G}_\alpha(i\omega)| \varepsilon_0 e^{i(\omega t + \phi_\alpha)},$$

where

$$(5) \quad \tan[\phi_\alpha(\omega)] = G''_\alpha / G'_\alpha$$

is called the loss tangent. Physically, Eq. (4) indicates that, either real or imaginary part of the steady state harmonic stress and strain, the stress leads the strain by a phase angle of ϕ_α .

A substitution of $\sigma(t)$ and $\varepsilon(t)$ into Eq. (1) yields

$$(6) \quad \sigma(t) = \tilde{G}_\alpha(i\omega) \varepsilon_0 e^{i\omega t} = G_\alpha(0) \varepsilon_0 e^{i\omega t} + i\omega \varepsilon_0 \int_{-\infty}^t \tilde{G}_\alpha(t - \tau) e^{i\omega \tau} d\tau,$$

where

$$\tilde{G}_\alpha(t) = G_\alpha(t) - G_\alpha(0), \quad \text{with } \tilde{G}_\alpha(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Changing variable $t - \tau$ to η and removing the common multiplication factor $\varepsilon_0 e^{i\omega t}$, we obtain

$$(7) \quad \tilde{G}_\alpha(i\omega) = [G_\alpha(0) + \omega \int_0^\infty \sin(\omega\eta) \tilde{G}_\alpha(\eta) d\eta + i\omega \int_0^\infty \cos(\omega\eta) \tilde{G}_\alpha(\eta) d\eta].$$

Decomposing $\tilde{G}_\alpha(i\omega)$ into real and imaginary parts, we find

$$(8) \quad \tilde{G}_\alpha(i\omega) = \text{Re}[\tilde{G}_\alpha(i\omega)] + i \text{Im}[\tilde{G}_\alpha(i\omega)] = G'_\alpha(\omega) + iG''_\alpha(\omega),$$

where

$$(9) \quad \begin{aligned} G'_\alpha(\omega) &= G_\alpha(0) + \omega \int_0^\infty \sin(\omega\eta) \tilde{G}_\alpha(\eta) d\eta \\ &= G_\alpha(0) + \tilde{G}_\alpha(0) + \int_0^\infty \cos(\omega\eta) [d\tilde{G}_\alpha(\eta)/d\eta] d\eta, \end{aligned}$$

$$(10) \quad G''_\alpha(\omega) = \omega \int_0^\infty \cos(\omega\eta) \tilde{G}_\alpha(\eta) d\eta = - \int_0^\infty \sin(\omega\eta) [d\tilde{G}_\alpha(\eta)/d\eta] d\eta,$$

which are, respectively, called the storage and loss moduli.

Equations (9) and (10) are Fourier sine and cosine transforms and can be inverted to give the relaxation functions as

$$(11) \quad \widehat{G}_\alpha(t) = \frac{2}{\pi} \int_0^\infty \frac{G'_\alpha(\omega) - G_\alpha(0)}{\omega} \sin(\omega t) d\omega = \frac{2}{\pi} \int_0^\infty \frac{G''_\alpha(\omega)}{\omega} \cos(\omega t) d\omega.$$

The integration by parts of the second term in the first integrand of Eq. (11) further simplifies it to the form

$$(12) \quad G_\alpha(t) = \frac{2}{\pi} \int_0^\infty \frac{G'_\alpha(\omega)}{\omega} \sin(\omega t) d\omega = \frac{2}{\pi} \int_0^\infty \frac{G''_\alpha(\omega)}{\omega} \cos(\omega t) d\omega.$$

At zero frequency, for exponentially decaying type of relaxation function, we have

$$(13) \quad G'_\alpha(0) = G_\alpha(0) = G_\alpha(t)|_{t \rightarrow \infty}, \quad G''_\alpha(0) = 0.$$

At infinite frequency, with the change of variable $\omega\eta = \tau$, one can show that Eqs. (9) and (10) give

$$(14) \quad G'_\alpha(\infty) = G_\alpha(0) + \widehat{G}_\alpha(0) = G_\alpha(t)|_{t \rightarrow 0}, \quad G''_\alpha(\infty) = 0.$$

In other words, as the frequency of excitation becomes large or small, $G''_\alpha(\omega)$ approaches zero. This implies, that undergoing very slow or fast processes, viscoelastic solids behave in an elastic manner.

From Eqs. (9) and (12), eliminating $\widehat{G}_\alpha(t)$, one finds another relationship between the real and imaginary parts of the complex modulus as

$$(15) \quad G'_\alpha(\omega) - G_\alpha(0) = \frac{2}{\pi} \int_0^\infty \frac{G''_\alpha(\lambda)\omega^2}{\lambda(\omega^2 - \lambda^2)} d\lambda.$$

Problem 14.4. Write the Fourier transforms of (14.2:17) as

$$\begin{aligned} \bar{\sigma}'_{ij}(\omega) &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^t G_1(t-\tau) [de_{ij}(\tau)/d\tau] d\tau \right\} e^{-i\omega t} dt, \\ \bar{\sigma}_{kk}(\omega) &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^t G_2(t-\tau) [de_{kk}(\tau)/d\tau] d\tau \right\} e^{-i\omega t} dt. \end{aligned}$$

Use $\tilde{G}_\alpha(i\omega)$ from Eq. (8) to reduce the equations above to

$$\bar{\sigma}'_{ij}(\omega) = \tilde{G}_1(i\omega) \bar{e}'_{ij}(\omega), \quad \bar{\sigma}_{kk}(\omega) = \tilde{G}_2(i\omega) \bar{e}_{kk}(\omega).$$

14.5. BOUNDARY-VALUE PROBLEMS AND INTEGRAL TRANSFORMS

The motion of a viscoelastic body is governed by the laws of conservation of mass and momentum, the stress-strain relations, and the boundary and initial conditions. Let u_i , e_{ij} , σ_{ij} , and X_i be Cartesian components of the displacement, strain, stress, and body force per unit volume, respectively, and ρ be the mass density. If we limit our consideration to infinitesimal motions of homogeneous and isotropic viscoelastic solids, we have

- (1) *Definition of strain*: $e_{ij} = (u_{i,j} + u_{j,i})/2$,
- (2) *Equation of continuity*: $\partial\rho/\partial t + \partial(\rho\partial u_i/\partial t)/\partial x_i = 0$,
- (3) *Equations of motion*: $\sigma_{ij,j} + X_i = \rho\partial^2 u_i/\partial t^2$, $\sigma_{ij} = \sigma_{ji}$,

same as the field equations of elasticity. The stress-strain relation may assume any of the following forms (see Sec. 14.3):

- (4a) *Relaxation laws*: $\sigma'_{ij} = e'_{ij} * dG_1$, $\sigma_{kk} = e_{kk} * dG_2$,
- (4b) *Creep laws*: $e'_{ij} = \sigma'_{ij} * dJ_1$, $e_{kk} = \sigma_{kk} * dJ_2$,
- (4c) *Differential operator laws*: $P_1(D)\sigma'_{ij} = Q_1(D)e'_{ij}$,
 $P_2(D)\sigma_{kk} = Q_2(D)e_{kk}$

where e'_{ij}, σ'_{ij} are the strain and stress deviators defined by

$$(5) \quad \sigma'_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3, \quad e'_{ij} = e_{ij} - e_{kk}\delta_{ij}/3,$$

G_1, G_2 are the relaxation functions, J_1, J_2 are the creep functions, and P_1, P_2, Q_1, Q_2 are polynomials of the time-derivative operator D as discussed in Sec. 14.3. For homogeneous materials, these functions and operators are independent of position.

If the body is originally undisturbed, then the initial conditions are

$$(6) \quad u_i = e_{ij} = \sigma_{ij} = 0, \quad \text{for } -\infty < t < 0.$$

If the differential operator law (4c) is used and there is a jump condition at $t = 0$, then the initial conditions may assume the form of specific assigned values of $e'_{ij}(0+), \partial e'_{ij}/\partial t(0+), \dots, \partial^{N_1} e'_{ij}/\partial t^{N_1}(0+)$, and of $\sigma'_{ij}(0+), \partial \sigma'_{ij}/\partial t(0+), \dots, \partial^{N_1} \sigma'_{ij}/\partial t^{N_1}(0+)$, must be connected by the necessary conditions [Eq. (14.3:11)]:

$$(7) \quad \sum_{r=k}^{N_1} [a_{r1} \partial^{r-k} \sigma'_{ij}/\partial t^{r-k}(0+) - b_{r1} \partial^{r-k} e'_{ij}/\partial t^{r-k}(0+)] = 0, \quad k = 1, \dots, N_1,$$

where N_1 is the larger of the degrees of $P_1(D)$ and $Q_1(D)$. Similar statements hold for initial values of mean stress and mean strain.

The boundary conditions may either

$$(8) \quad \overset{\nu}{T}_i = \sigma_{ij} \nu_j = f_i, \quad \text{over } S_\sigma$$

with unit normal vector ν_i , or

$$(9) \quad u_i = g_i \quad \text{over } S_u,$$

where f_i, g_i are prescribed functions of position and time.

The problems in the theory of linear viscoelasticity usually involve determining u_i, e_{ij} and σ_{ij} for prescribed X_i, f_i, g_i and initial conditions. Except for the stress-strain law, the same equations occur in the linearized theory of elasticity.

In linear elasticity the transformation translates the original dynamic problem involving derivatives and convolution integrals with respect to the time t into an algebraic problem with respect to a parameter s . After solving the algebraic problem, the solution is translated back to the time domain. The last step, of course, is by no means easy.

It appears natural to apply the Laplace transformation to problems in linear viscoelasticity. In fact, the transformed problem, involving the parameter s , can be put into the same form as that in the theory of linear elasticity. If the latter problem can be solved, the viscoelasticity solution can be obtained by an inverse transformation from s back to the time domain. The last step, again, may not be easy.

The Fourier transform may be used in place of the Laplace transform. The selection of the transform depends on the nature of the functions involved. A function which has a Laplace transform may not necessarily have a Fourier transform, and vice versa, e.g., the unit-step function $\mathbf{1}(t)$, or a power t^n , has no Fourier transform in the interval $(0, \infty)$.

The identification of a problem in linear elasticity with one in linear viscoelasticity in the transformed plane, is called the *correspondence principle*. Applications of the correspondence principle will become apparent by examining some examples in Secs. 14.6 and 14.7.

Problem 14.5. Assuming that the functions u_i, G_1, G_2 are continuous and can be differentiated as many times as may be desired, show that the equation of motion may be written (in the form of Navier's equation) as

$$u_{i,jj} * dG_1 + u_{j,ji} * dK + 2X_i = 2\rho \partial^2 u_i / \partial t^2, \quad i = 1, 2, 3$$

where $3K = G_1 + 2G_2$ (Gurtin and Sternberg 1962).

Problem 14.6. From the strain equations of compatibility

$$u_{ij,kk} + u_{kk,ij} - u_{ik,jk} - u_{jk,ik} = 0,$$

the stress-strain relations

$$e_{ij} = \sigma_{ij} * dJ_1 + \delta_{ij} \sigma_{kk} * d(J_2 - J_1)/3,$$

and the equations of equilibrium

$$\sigma_{ik,k} = -X_i,$$

deduce the stress equations of compatibility

$$\sigma_{ij,kk} * dJ_1 + \sigma_{kk,ij} * d\Lambda = \Theta_{ij},$$

where

$$\begin{aligned}\Theta_{ij} &= \delta_{ij} X_{k,k} * d\Omega - (X_{i,j} + X_{j,i}) * dJ_1, \quad 3\Lambda = 2J_1 + J_2, \\ d\Omega &= J_1 * d(J_2 - J_1) * d(J_1 + 2J_2)^{-1}. \quad (\text{Gurtin and Sternberg})\end{aligned}$$

Problem 14.7. Let $\mathbf{u}(u_i)$, $\varepsilon(e_{ij})$ and $\sigma(\sigma_{ij})$ satisfy the equations of viscoelasticity, and $G_1 \neq 0$, $2G_1 + G_2 \neq 0$. Show that, when the body force $X_i = 0$,

$$\begin{aligned}\nabla^2(\nabla \cdot \mathbf{u}) &= 0, \quad \nabla^2(\nabla \times \mathbf{u}) = 0, \quad \nabla^2 e_{kk} = 0, \quad \nabla^2 \sigma_{kk} = 0, \\ \nabla^4 \mathbf{u} &= 0, \quad \nabla^4 \varepsilon = 0, \quad \nabla^4 \sigma = 0. \quad (\text{Gurtin and Sternberg})\end{aligned}$$

14.6. WAVES IN AN INFINITE MEDIUM

We shall consider a viscoelastic medium of infinite extent and look for solutions of Eqs. (14.5:1–6), in which all dependent variables, including body forces, vary sinusoidally with time. Let

$$e_{ij} = \text{Rl}(\bar{e}_{ij} e^{i\omega t}), \quad \sigma_{ij} = \text{Rl}(\bar{\sigma}_{ij} e^{i\omega t}), \quad \text{etc.},$$

where \bar{e}_{ij} , $\bar{\sigma}_{ij}$, etc., are complex-valued functions of the spatial coordinates only. The basic Eqs. (14.5:1–4) now read

$$(1) \quad \bar{e}_{ij} = (\bar{u}_{i,j} + \bar{u}_{j,i})/2, \quad \bar{\sigma}_{ij,j} + \bar{X}_i + \rho\omega^2 \bar{u}_i = 0,$$

$$(2) \quad \bar{\sigma}'_{ij} = i\omega \bar{G}_1(\omega) \bar{e}'_{ij}, \quad \bar{\sigma}_{kk} = i\omega \bar{G}_2(\omega) \bar{e}_{kk},$$

where $\bar{G}_1(\omega)$, $\bar{G}_2(\omega)$ are the deviatoric and the dilatational complex moduli, and are the Fourier transforms of the relaxation functions $G_1(t)$ and $G_2(t)$ respectively. If we write

$$(3) \quad \bar{\lambda}(\omega) = i\omega [\bar{G}_2(\omega) - \bar{G}_1(\omega)]/3, \quad \bar{G}(\omega) = i\omega \bar{G}_1(\omega)/2,$$

and call $\bar{\lambda}(\omega)$ and $\bar{G}(\omega)$ the complex Lamé constants for a viscoelastic material, we see that Eqs. (1)–(3) are identical with those governing linear elasticity theory (Sec. 7.1), except that the Lamé constants are replaced by complex moduli. Hence, without much ado, we can write down the Navier equation [cf. Eq. (7.1:9)]

$$(4) \quad \bar{G}(\omega) \bar{u}_{i,jj} + [\bar{\lambda}(\omega) + \bar{G}(\omega)] \bar{u}_{j,ji} + \bar{X}_i + \rho\omega^2 \bar{u}_i = 0,$$

and the solutions, in case $\bar{X}_i \equiv 0$

$$(5) \quad u_j = \text{Rl} \langle A n_j \exp{i\omega \{t \pm \sqrt{\rho/[3\bar{\lambda}(\omega) + 2\bar{G}(\omega)]} n_k x_k\}} \rangle,$$

$$(6) \quad u_j = \text{Rl} \langle c_j \exp\{i\omega [t \pm \sqrt{\rho/\bar{G}(\omega)} n_k x_k]\} \rangle,$$

where A is a complex constant, n_j is an arbitrary vector, and c_j are components of a vector perpendicular to n_j , i.e., $c_j n_j = 0$.

Equation (5) represents a plane dilatational wave and Eq. (6) a plane shear wave. The exponential factors in Eqs. (5) and (6) are complex. A little reflection shows that the real parts of the factors in front of $n_k x_k$ are the inverse of wave velocities, v_D , v_R ; whereas the imaginary parts are attenuation factors α_D , α_R .

(7) *Dilatational waves:*

$$v_D = \{\text{Rl} \langle \sqrt{\rho/[3\bar{\lambda}(\omega) + 2\bar{G}(\omega)]} \rangle\}^{-1}, \quad \alpha_D = -\omega \text{Im} \{ \sqrt{\rho/[3\bar{\lambda}(\omega) + 2\bar{G}(\omega)]} \},$$

(8) *Rotational waves:*

$$v_R = \{\text{Re}[\sqrt{\rho/G(\omega)}]\}^{-1}, \quad \alpha_R = -\omega \text{Im}[\sqrt{\rho/G(\omega)}].$$

If the material is elastic, $\lambda(\omega)$, and $G(\omega)$ become real numbers, independent of ω , in which case the attenuation factors α_D and α_R vanish and the wave speeds v_D , v_R are independent of frequency.

Any solution of Navier's equation in the theory of linear elasticity, of the form $f = \text{Re}[\bar{f}(x_1, x_2, x_3)e^{i\omega t}]$ offers a corresponding solution for a linear viscoelastic body, if the elastic moduli in $\bar{f}(x_1, x_2, x_3)$ are replaced by the corresponding complex moduli of the material. If the boundary conditions for the elastic and viscoelastic problems are identical, then a solution in viscoelasticity can be obtained by this correspondence principle. However, the operation of separating a complex modulus into its real and imaginary parts has no counterpart in the theory of linear elasticity. Hence, such an operation is excluded from the correspondence principle. For example, if we wish to find the maximum of $|\bar{f}|$ with respect to ω , we must separate the real and imaginary parts, and no direct analogy is available.

Problem 14.8. Work out the details of the derivation of Eqs. (5) and (6) by reference to Sec. 7.8^{14.1}.

Problem 14.9. Find the speed and attenuation factor for Rayleigh surface waves propagating over a viscoelastic half-space (see Sec. 7.9). Reference Bland, *Linear Viscoelasticity*^{14.1} pp. 73–75.

Problem 14.10. Consider a uniform cantilever beam with one end clamped on a wall being forced to oscillate at a constant amplitude w_0 and the other end free (Fig. P14.10). For an elastic beam, its equation of motion is

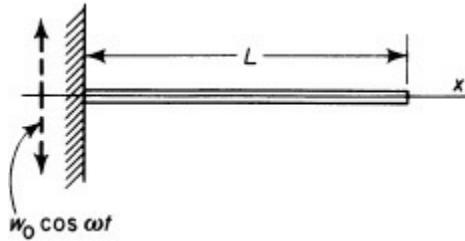


Fig. P14.10

$$EI(\partial^4 w / \partial x^4) + m(\partial^2 w / \partial t^2) = 0,$$

where w is the beam deflection, m the mass per unit length, E Young's modulus, and I the moment of inertia of the cross section. The boundary conditions are:

$$\begin{aligned} w &= w_0 \cos \omega t, \quad \partial w / \partial x = 0 \quad \text{at } x = 0; \quad \text{and} \\ \partial^2 w / \partial x^2 &= \partial^3 w / \partial x^3 = 0 \quad \text{at } x = L. \end{aligned}$$

Determine a solution $w(x, t)$ in the form $\bar{w}(x)e^{i\omega t}$.

If the beam material is viscoelastic, determine (a) the corresponding complex bending rigidity $EI(\omega)$ from the stress-strain relationship of the material and (b) the solution $w(x, t)$.

The amplitude ratio and phase lag of the oscillations at the free and the clamped ends can be used for experimental determination of the complex modulus. Reference Bland and Lee, *J. Appl. Phys.*, 26(1955), p. 1497.

14.7. QUASI-STATIC PROBLEMS

If all functions of concern vanish for $t < 0$ and have Laplace transformations, the transformations of the Eqs. (14.5:1–9) assume exactly the same form as in the linear elasticity theory. In quasi-static problems, the loading is assumed to be so slow that inertia forces may be neglected. In this case, the corresponding elastic problems are static.

Let us denote the Laplace transform of a variable by a bar over it. The basic equations for quasi-static problems of a viscoelastic body occupying a region V with boundary $B = B_\sigma + B_u$ are

$$(1) \quad \bar{e}_{ij} = (\bar{u}_{i,j} + \bar{u}_{j,i})/2 \quad \text{in } V,$$

$$(2) \quad \bar{\sigma}_{ij,j} + \bar{X}_i = 0 \quad \text{in } V,$$

$$(3a) \quad \bar{\sigma}'_{ij} = s\bar{G}_1(s)\bar{e}'_{ij}, \quad \bar{\sigma}_{kk} = s\bar{G}_2(s)\bar{e}_{kk} \quad \text{in } V, \quad \text{or}$$

$$(3b) \quad \bar{e}'_{ij} = s\bar{J}_1(s)\bar{\sigma}'_{ij}, \quad \bar{e}_{kk} = s\bar{J}_2(s)\bar{\sigma}_{kk} \quad \text{in } V, \quad \text{or}$$

$$(3c) \quad P_1(s)\bar{\sigma}'_{ij} = Q_1(s)\bar{e}'_{ij}, \quad P_2(s)\bar{\sigma}_{kk} = Q_2(s)\bar{e}_{kk} \quad \text{in } V,$$

$$(4) \quad \bar{\sigma}_{ij}\nu_j = \bar{f}_i \quad \text{on } B_\sigma,$$

$$(5) \quad \bar{u}_i = \bar{g}_i \quad \text{on } B_u.$$

The initial conditions (14.5:6) and (14.5:7) are satisfied in V and in B . The material properties $\bar{G}_1(s), \bar{G}_2(s)$; or $\bar{J}_1(s), \bar{J}_2(s)$; or $P_\alpha(s), Q_\alpha(s)$ with $\alpha = 1, 2$, are given. If the bars were removed, these would be the same equations that govern the static equilibrium of an elastic body of the same geometry. If the solution of the latter were known, the transformed solution would be obtained, and the solution of the original problem could be obtained by an inversion.

The counterpart of Young's modulus E , the shear modulus G , the bulk modulus K , Poisson's ratio ν , and one of Lamé constants λ (the other Lamé constant is $\mu = G$) can be derived from Eqs. (3a), (3b), and (3c):

$$(6) \quad \bar{E}(s) = \frac{3Q_1(s)Q_2(s)}{Q_1(s)P_2(s) + 2P_1(s)Q_2(s)} = \frac{3s\bar{G}_1(s)\bar{G}_2(s)}{2\bar{G}_2(s) + \bar{G}_1(s)} = \frac{3}{s[2\bar{J}_1(s) + \bar{J}_2(s)]},$$

$$(7) \quad \bar{G}(s) = Q_1(s)/[2P_1(s)] = s\bar{G}_1(s)/2 = 1/[2s\bar{J}_1(s)],$$

$$(8) \quad \bar{K}(s) = Q_2(s)/[3P_2(s)] = s\bar{G}_2(s)/3 = 1/[3s\bar{J}_2(s)],$$

$$(9) \quad \bar{\nu}(s) = \frac{P_1(s)Q_2(s) - P_2(s)Q_1(s)}{Q_1(s)P_2(s) + 2P_1(s)Q_2(s)} = \frac{\bar{G}_2(s) - \bar{G}_1(s)}{2\bar{G}_2(s) + \bar{G}_1(s)} = \frac{\bar{J}_1(s) - \bar{J}_2(s)}{2\bar{J}_1(s) + \bar{J}_2(s)},$$

$$(10) \quad \bar{\lambda}(s) = \bar{K}(s) - 2\bar{G}(s)/3 = s[\bar{G}_2(s) - \bar{G}_1(s)]/3.$$

Example 1. Boussinesq Problem. Consider the problem of a concentrated load Z on a half-space (Fig. 14.7:1). The elastic solution is given in Sec. 8.10 with the stress component σ_{rr} given by Eq. (8.10:5)

$$(11) \quad \sigma_{rr}(r, z) = Z[(1 - 2\nu)R/(R + z) - 3r^2z/R^3]/(2\pi R^2),$$

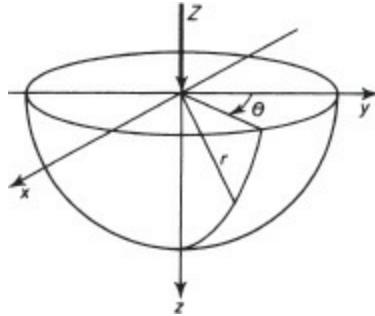


Fig. 14.7:1. Boussinesq problem over a viscoelastic material.

in which $R = \sqrt{r^2 + z^2}$. On applying the correspondence principle, the solution of σ_{rr} in the plane of Laplace transformation is

$$(12) \quad \bar{\sigma}_{rr}(r, z, s) = \bar{Z}(s)\{[1 - 2\bar{\nu}(s)]/(R + z) - 3r^2z/R^4\}/(2\pi R),$$

where $\bar{\nu}(s)$ is given by Eq. (9). The inverse transformation of $\bar{\sigma}_{rr}(r, z, s)$ is,

$$(13) \quad \sigma_{rr}(r, z, t) = [\int_0^t Z(t - \tau)\Phi(\tau)d\tau/(R + z) - 3r^2zZ(t)/R^4]/(2\pi R),$$

in which

$$(14) \quad \Phi(t) = \int_{c-i\infty}^{c+i\infty} [1 - 2\bar{\nu}(s)]e^{st}ds$$

is the inverse Laplace transformation of $1 - 2\bar{\nu}(s)$, the constant c being any number greater than the abscissa of convergence.

Problem 14.11. A concentrated load $Z(t) = Z_0\mathbf{1}(t)$ (Z_0 a constant, $\mathbf{1}(t)$ a unitstep function) acts normal to the free surface of a half-space of Voigt material, for which $\sigma'_{ij} = (2\eta D + 2G)\bar{e}'_{ij}$ and $\sigma_{jj} = 3Ke_{jj}$.

Show that $1 - 2\bar{\nu}(s) = 3(\eta s + G)/(3K + \eta s + G)$ and

$$\sigma_{rr} = Z_0 \left\{ \frac{3G + 9K \exp[-(G + 3K)t/\eta]}{(3K + G)(R+z)} - \frac{3r^2 z}{R^4} \right\} \frac{\mathbf{1}(t)}{2\pi R}.$$

Problem 14.12. Consider a long thick-walled circular cylinder subjected to a uniform internal pressure p_i and uniform external pressure p_0 (Fig. P14.12). For plane strain problem, show that the stress distribution in the cylinder is the same whether the material is linear elastic or linear viscoelastic. Derive the history of deformation u_i if the cylinder is viscoelastic.

In the elasticity theory, the biharmonic equation governing the Airy stress function in plane stress or plane strain problems is independent of the elastic constants if the body force vanishes. However, if the boundary conditions of a 2-dimensional problem involve displacements, then the associated elastic problem will involve elastic moduli and the correspondence principle applies.

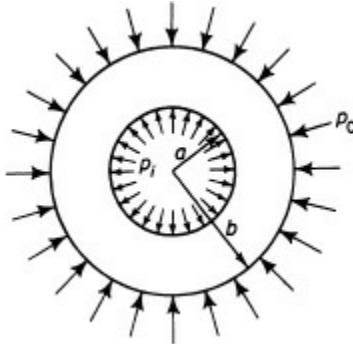


Fig. P14.12

14.8. PROBLEMS OF CONSTANT POISSON'S RATIO

If viscoelastic Poisson's ratio is a real constant, the quasi-static solution can be written in the form of separated variables as

$$(1) \quad u_i(\mathbf{x}, t) = \bar{u}_i(\mathbf{x})g(t), \quad e_{ij}(\mathbf{x}, t) = \bar{e}_{ij}(\mathbf{x})g(t), \quad \sigma_{ij}(\mathbf{x}, t) = \bar{\sigma}_{ij}(\mathbf{x})f(t).$$

This is seen from the stress strain relations and the equilibrium equations

$$(2) \quad \begin{aligned} \sigma_{ij,j} &= \int_{-\infty}^t [\lambda(t-\tau)\bar{e}_{kk,i}(\mathbf{x}) + 2G(t-\tau)\bar{e}_{ij,j}(\mathbf{x})][dg(\tau)/d\tau]d\tau \\ &= \int_{-\infty}^t \{\bar{u}_{k,ki}(\mathbf{x})[\lambda(t-\tau) + G(t-\tau)] + \bar{u}_{i,jj}(\mathbf{x})G(t-\tau)\} \frac{dg(\tau)}{d\tau} d\tau = 0. \end{aligned}$$

The equation above can be satisfied only if

$$\lambda(t) + G(t) = cG(t),$$

where c is a constant. The relations (14.7:9) and (14.7:10) imply

$$\nu(t) = \nu = \text{constant}.$$

This requirement further implies that the ratio of the relaxation modulus in shear and that in volume change, and the corresponding ratio in creep, must be constant, i.e.,

$$(3) \quad G_2(t)/G_1(t) = 3K/(2G) = (1+\nu)/(1-2\nu) = J_1(t)/J_2(t).$$

The special restriction is not needed if $\bar{u}_{i,jj} = 0$ and $\bar{u}_{k,k} = \text{constant}$.

Consider the problem through the separated form with the entire boundary being subject to prescribed stresses

$$(4) \quad \sigma_{ij}(\mathbf{x}, t)\nu_j(\mathbf{x}) = \bar{f}_i(\mathbf{x})f(t) \quad \text{on } B$$

and the prescribed body forces

$$(5) \quad X_i(\mathbf{x}, t) = \bar{X}_i(\mathbf{x})f(t).$$

Using the creep function and the equilibrium equations (14.5:3) for zero inertia forces, we can write the compatibility

equations in terms of stresses

$$(6) \quad \int_0^t J_1(t-\tau) [df(\tau)/d\tau] Y_{ij}(\mathbf{x}) d\tau = 0,$$

where

$$(7) \quad Y_{ij}(\mathbf{x}) = \bar{\sigma}_{ij,kk}(\mathbf{x}) + \bar{\sigma}_{kk,ij}(\mathbf{x})/(1+\nu) + \nu \delta_{ij} \bar{X}_{k,k}(\mathbf{x})/(1-\nu) \\ + \bar{X}_{i,j}(\mathbf{x}) + \bar{X}_{j,i}(\mathbf{x}).$$

To satisfy Eq. (6), it is necessary and sufficient that

$$(8) \quad Y_{ij}(\mathbf{x}) = 0,$$

which are simply the Beltrami–Michell compatibility equations of elasticity theory. One can obtain the solution $\bar{\sigma}_{ij}(\mathbf{x})$ from the equilibrium and the compatibility equations (7) and (8) as that in elasticity.

To determine the displacements, we assume

$$(9) \quad u_i(\mathbf{x}, t) = \bar{u}_i(\mathbf{x})g(t).$$

Then the strains can be written as

$$(10) \quad e_{ij}(\mathbf{x}, t) = \bar{e}_{ij}(\mathbf{x})g(t) = [\bar{u}_{i,j}(\mathbf{x}) + \bar{u}_{j,i}(\mathbf{x})]g(t)/2.$$

Using Eq. (14.2:3), we obtain

$$(11) \quad \bar{e}'_{ij}(\mathbf{x})g(t) = \bar{\sigma}'_{ij}(\mathbf{x}) \int_0^t J_1(t-\tau) [df(\tau)/d\tau] d\tau,$$

$$(12) \quad \bar{e}_{kk}(\mathbf{x})g(t) = \bar{\sigma}_{kk}(\mathbf{x})(1-2\nu) \int_0^t J_1(t-\tau) [df(\tau)/d\tau] d\tau / (1+\nu),$$

where $()'$ denotes the deviatoric component of the corresponding quantity. Taking

$$(13) \quad g(t) = \int_0^t J_1(t-\tau) [df(\tau)/d\tau] d\tau / J_1(0)$$

to be the temporal part of displacement solutions, one get

$$(14) \quad \bar{e}'_{ij}(\mathbf{x}) = J_1(0) \bar{\sigma}'_{ij}(\mathbf{x}),$$

$$(15) \quad \bar{e}_{kk}(\mathbf{x}) = \bar{\sigma}_{kk}(\mathbf{x})J_1(0)(1-2\nu)/(1+\nu) = \bar{\sigma}_{kk}(\mathbf{x})J_2(0).$$

The spatial part $\bar{u}_i(\mathbf{x})$ can be obtained by integrating Eqs. (14) and (15).

Problem 14.13. Consider the problem of zero body and inertia forces subject to the prescribed displacements

$$(P1) \quad u_i(\mathbf{x}, t) = \bar{u}_i(\mathbf{x})g(t)$$

over the entire boundary B . Using the equilibrium equations in terms of displacements, show that the spatial part of the displacements satisfies

$$\bar{u}_{i,kk}(\mathbf{x}) + \bar{u}_{k,ki}(\mathbf{x})/(1-2\nu) = 0,$$

which are simply Navier's equilibrium equations of elasticity theory. Knowing the solution $\bar{u}_i(\mathbf{x})$, subject to boundary conditions (P1), find the stress solution in a separation form in terms of the displacements.

14.9. RECIPROCITY RELATIONS

In the beginning of Sec. 1.1, we derived the Maxwell and Betti–Rayleigh reciprocity relations. These relations can be generalized to elastic or viscoelastic continua and, with proper interpretation, they are valid not only for statics but also for dynamics.

The most general reciprocal theorem in dynamics was asserted by Lamb^{14.3} to be derivable from a remarkable

formula established by Lagrange in the *Mécanique Analytique*(1809) by way of a prelude to Lagrange's theory of the variation of arbitrary constants. Lamb showed how the reciprocal theorems of von Helmholtz, in the theory of least action in acoustics and optics, and of Rayleigh in acoustics, can be derived from Lagrange's formula. Rayleigh (1926) extended the reciprocal theorem to include the action of dissipative forces, and Lamb (1888) showed the complete reciprocity relationships in a moving fluid with reversed-flow conditions.

In the theory of elasticity, a generalization of the reciprocity theorem to dynamic problems was given by Graffi (1939), and certain applications of Graffi's results to the problem of elastic wave propagation were pointed out by DiMaggio and Bleich (1956).

We shall derive the dynamic reciprocity relationship for a viscoelastic solid, which includes the linear elastic solid as a special case. We assume the stressstrain law to be given by one of the equations (14.5:4). The equations of motion and boundary conditions are given by Eqs. (14.5:3) and (14.5:8) or (14.5:9), respectively. Let us consider the problems in which the body force $X_i(\mathbf{x}, t)$, the surface tractions $f_i(\mathbf{x}, t)$ and displacements $g_i(\mathbf{x}, t)$ are given, which start their actions at $t > 0$, under the initial conditions

$$(1) \quad u_i = \partial u_i / \partial t = \dots = \partial^N u_i / \partial t^N = 0, \quad \text{when } t \leq 0,$$

where N is either unity or equal to the highest derivative that occurs in the stress-strain law (14.5:4c). We apply the Laplace transform with respect to the time t to every dependent variable, assuming that the transforms exist. For two problems where the applied body forces, the surface tractions and displacements are specified differently, let the variables involved in these two problems be distinguished by superscripts in parentheses. Then, we have, due to the initial conditions indicated above,

$$(2) \quad \begin{aligned} \bar{\sigma}_{ij}^{(\alpha)} &= \bar{\lambda}(s)\bar{u}_{k,k}^{(\alpha)}\delta_{ij} + \bar{G}(s)(\bar{u}_{i,j}^{(\alpha)} + \bar{u}_{j,i}^{(\alpha)}), \quad s^2\rho\bar{u}_i^{(\alpha)} = \bar{X}_i^{(\alpha)} + \bar{\sigma}_{ij,j}^{(\alpha)} \text{ in } V; \\ \bar{\sigma}_{ij}^{(\alpha)}\nu_j &= \bar{f}_i^{(\alpha)} \text{ on } S_\sigma; \quad \text{and} \quad \bar{u}_i^{(\alpha)} = \bar{g}_i^{(\alpha)} \text{ on } S_u, \end{aligned}$$

where $\alpha = 1, 2$, and $\bar{G}(s), \bar{\lambda}(s)$ are given by Eqs. (14.7:7) and (14.7:10), respectively.

Applying the divergent theorem to get

$$(3) \quad \begin{aligned} L &= \int_S \bar{\sigma}_{ij}^{(1)}\bar{u}_i^{(2)}\nu_j dS + \int_V \bar{X}_i^{(1)}\bar{u}_i^{(2)}dV \\ &= \int_V [\bar{\sigma}_{ij,j}^{(1)}\bar{u}_i^{(2)} + \bar{\sigma}_{ij}^{(1)}\bar{u}_{i,j}^{(2)}]dV + \int_V \bar{X}_i^{(1)}\bar{u}_i^{(2)}dV, \end{aligned}$$

where S_σ and S_u are the parts of S over which the tractions and displacements are specified, respectively. A substitution of the equilibrium equations and the stress strain relations of (2) into Eq. (3) yields

$$L = \int_V [\rho\omega^2\bar{u}_i^{(1)}\bar{u}_i^{(2)} + (\bar{\lambda}\bar{u}_{k,k}^{(1)}\delta_{ij} + \bar{G}\bar{u}_{i,j}^{(1)} + \bar{G}\bar{u}_{j,i}^{(1)})\bar{u}_{i,j}^{(2)}]dV,$$

which is invariant with interchange of the superscripts 1 and 2. We have

$$(4a) \quad L = \int_S \bar{\sigma}_{ij}^{(1)}\bar{u}_i^{(2)}\nu_j dS + \int_V \bar{X}_i^{(1)}\bar{u}_i^{(2)}dV = \int_S \bar{\sigma}_{ij}^{(2)}\bar{u}_i^{(1)}\nu_j dS + \int_V \bar{X}_i^{(2)}\bar{u}_i^{(1)}dV,$$

or

$$(4b) \quad \begin{aligned} &\int_{S_\sigma} \bar{f}_i^{(1)}\bar{u}_i^{(2)}dS + \int_{S_u} \bar{\sigma}_{ij}^{(1)}\bar{g}_i^{(2)}\nu_j dS + \int_V \bar{X}_i^{(1)}\bar{u}_i^{(2)}dV \\ &= \int_{S_\sigma} \bar{f}_i^{(2)}\bar{u}_i^{(1)}dS + \int_{S_u} \bar{\sigma}_{ij}^{(2)}\bar{g}_i^{(1)}\nu_j dS + \int_V \bar{X}_i^{(2)}\bar{u}_i^{(1)}dV. \end{aligned}$$

This is the *general reciprocal relation in the Laplace transformation form*. If we remove the bars and consider the variables to be in the real-time domain, then it becomes Betti's reciprocal relation in elastostatics.

Since the inverse transform of the product of two functions is the convolution of the inverses, we obtain:

$$(5) \int_V \int_0^t X_i^{(1)}(\mathbf{x}, t-\tau) u_i^{(2)}(\mathbf{x}, \tau) d\tau dV + \int_{S_\sigma} \int_0^t f_i^{(1)}(\mathbf{x}, t-\tau) u_i^{(2)}(\mathbf{x}, \tau) d\tau dS \\ + \int_{S_u} \int_0^t \sigma_{ij}^{(1)}(\mathbf{x}, t-\tau) g_i^{(2)}(\mathbf{x}, \tau) \nu_j d\tau dS = \int_V \int_0^t X_i^{(2)}(\mathbf{x}, t-\tau) u_i^{(1)}(\mathbf{x}, \tau) d\tau dV \\ + \int_{S_\sigma} \int_0^t f_i^{(2)}(\mathbf{x}, t-\tau) u_i^{(1)}(\mathbf{x}, \tau) d\tau dS + \int_{S_u} \int_0^t \sigma_{ij}^{(2)}(\mathbf{x}, t-\tau) g_i^{(1)}(\mathbf{x}, \tau) \nu_j d\tau dS.$$

This is the *general reciprocal relation for elasto-kinetics*. This result holds for both viscoelastic and elastic materials. It also holds for inhomogeneous materials of variable density $\rho(\mathbf{x})$.

Generalization to Infinite Region. A generalization of the above result to an infinite or semi-infinite region is possible. Since with a finite wave speed there always exists a finite boundary surface, at any finite $t > 0$, not yet influenced by the loading initiated at $t = 0$. Let S_u be such a surface, then $g_i = 0$ on S_u , and the remainder of the equation holds.

Examples of Applications

(a) *Space-time-separable body forces, surface tractions and displacements.* If

$$X_i^{(\alpha)} = \Xi_i^{(\alpha)}(\mathbf{x}) h(t), \quad f_i^{(\alpha)} = P_i^{(\alpha)}(\mathbf{x}) h(t), \quad g_i^{(\alpha)} = W_i^{(\alpha)}(\mathbf{x}) h(t),$$

where $\alpha = 1, 2$, then Eq. (4b), on cancelling $h(s)$, can be written as

$$\int_{S_\sigma} P_i^{(1)} \bar{u}_i^{(2)} dS + \int_{S_u} \bar{\sigma}_{ij}^{(1)} W_i^{(2)} \nu_j dS + \int_V \Xi_i^{(1)} \bar{u}_i^{(2)} dV \\ = \int_{S_\sigma} P_i^{(2)} \bar{u}_i^{(1)} dS + \int_{S_u} \bar{\sigma}_{ij}^{(2)} W_i^{(1)} \nu_j dS + \int_V \Xi_i^{(2)} \bar{u}_i^{(1)} dV.$$

The inverse transformation gives

$$\int_{S_\sigma} P_i^{(1)} u_i^{(2)}(\mathbf{x}, t) dS + \int_{S_u} \sigma_{ij}^{(1)}(\mathbf{x}, t) W_i^{(2)} \nu_j dS + \int_V \Xi_i^{(1)} u_i^{(2)}(\mathbf{x}, t) dV \\ = \int_{S_\sigma} P_i^{(2)} u_i^{(1)}(\mathbf{x}, t) dS + \int_{S_u} \sigma_{ij}^{(2)}(\mathbf{x}, t) W_i^{(1)} \nu_j dS + \int_V \Xi_i^{(2)} u_i^{(1)}(\mathbf{x}, t) dV.$$

Graffi's well-known formula results if $g_i^{(1)} = g_i^{(2)} = 0$ on S_u .

(b) *Forces applied at different times.* If

$$X_i^{(\alpha)} = \Xi_i^{(\alpha)}(\mathbf{x}) h(t - T_\alpha), \quad f_i^{(\alpha)} = P_i^{(\alpha)}(\mathbf{x}) h(t - T_\alpha), \quad g_i^{(\alpha)} = 0,$$

where $h(t) = 0$ for $t \leq 0$, then on substituting the above into Eq. (4b) and taking the inverse Laplace transform, one obtains

$$\int_{S_\sigma} P_i^{(1)} u_i^{(2)}(\mathbf{x}, t - T_1) dS + \int_V \Xi_i^{(1)} u_i^{(2)}(\mathbf{x}, t - T_1) dV \\ = \int_{S_\sigma} P_i^{(2)} u_i^{(1)}(\mathbf{x}, t - T_2) dS + \int_V \Xi_i^{(2)} u_i^{(1)}(\mathbf{x}, t - T_2) dV.$$

(c) *Concentrated forces.* If the loading consists of concentrated loads $F_i^{(1)}$ and $F_i^{(2)}$ acting at points p_1, p_2 respectively, we may consider Ξ_i or P_i of the Cases (a) and (b) above as suitable delta functions and obtain at once

$$F_i^{(1)}(p_1) u_i^{(2)}(p_1, t - T_1) = F_i^{(2)}(p_2) u_i^{(1)}(p_2, t - T_2),$$

or, if $T_1 = T_2$,

$$F_i^{(1)}(p_1) u_i^{(2)}(p_1, t) = F_i^{(2)}(p_2) u_i^{(1)}(p_2, t).$$

This is the extension of the conventional elastostatic Betti–Rayleigh reciprocal relation to kinetics.

(d) *Impulsive and traveling concentrated forces.* Let an impulsive concentrated force act at a point p_1 ,

$$X_i^{(1)} = F_i^{(1)} \delta(p_1) \delta(t),$$

where $\delta(t)$ is a unit-impulse or delta function, and let a concentrated force $F^{(2)}$ be applied at the origin at $t = 0$, and thereafter moved along the x_1 axis at uniform speed U :

$$X_i^{(2)} = F_i^{(2)} \delta(t - x_1/U) \delta(x_2) \delta(x_3).$$

No other surface loading or displacement is imposed. Then Eq. (5) gives

$$F_i^{(1)} u_i^{(2)}(p_1, t) = F_i^{(2)} \iiint \delta(x_2) \delta(x_3) dx_1 dx_2 dx_3 \int_0^t \delta\left(\tau - \frac{x_1}{U}\right) u_i^{(1)}(\mathbf{x}, t - \tau) d\tau$$

and therefore,

$$F_i^{(1)} u_i^{(2)}(p_1, t) = F_i^{(2)} \int_{-\infty}^{Ut} u_i^{(1)}(x_1, 0, 0, t - x_1/U) dx_1.$$

If $u_i^{(1)}(x_1, 0, 0, t - x_1/U)$ is known, then $u_i^{(2)}(p_1, t)$ can be found from the equation above.

(e) *Suddenly started line load over an elastic half-space.* Ang^{9.2} considered the problem of suddenly started line load acting on the surface of a half-space. Now, according to the reciprocal theorem Ang's problem can be solved by one integration of the solution of Lamb's problem: the impulsive loading at the point (not traveling) inside a two-dimensional half-space. Only the surface displacement due to the point loading needs to be known.

Problem 14.14. The equation of transverse motion of a string stretched between two points is

$$c^2 \partial^2 w / \partial x^2 - \partial^2 w / \partial t^2 = F(x, t), \quad 0 \leq x \leq L.$$

Let $F^{(1)}(x, t)$ and $F^{(2)}(x, t)$ be the dynamic loads for the solutions $w^{(1)}(x, t)$ and $w^{(2)}(x, t)$, respectively. Generalize the dynamic reciprocity relation to the present problem by integrating over the length $0 \leq x \leq L$ as well as over the time $(0, t)$.

Next consider the specific case in which $F^{(2)}(x, t)$ [= $\delta(x)\delta(t)$] is an impulsive concentrated load acting at x , and $F^{(1)}(x, t)$ [= $\delta(x - Vt)$] is a traveling load, where V is a constant. Show that $w^{(1)}(x, t)$ can be derived by the reciprocity relation from the solution $w^{(2)}(x, t)$ corresponding to $F^{(2)}(x, t)$.

14.10. FUNCTIONAL THERMODYNAMICS AND COUPLED CONSTITUTIVE RELATIONS

We shall generalize the constitutive laws to cover thermoviscoelastic materials. In an overview of constitutive equations for nonlinear viscoelastic solids (Schapery 2000), nonequilibrium thermodynamic approaches are essentially of two types: functional thermodynamics and state-variable thermodynamics. In functional thermodynamics, the free energy is expressed as a functional of the histories of deformation (stress), temperature, etc. (see [Biblio. 14.4](#)). The use of “functional” as a mathematical term originates in the calculus of variations, which is concerned with the minimization of a functional with its arguments as functions. Nonequilibrium thermodynamics is taken as an extension of irreversible thermodynamics described in [Chapter 12](#). The problem is formulated based on the energy balance and the entropy production inequality.

14.10.1. Fundamental principles

Based on the first law of thermodynamics, the local balance equations of mass, momentum (linear momentum), moment of momentum (angular momentum), and energy in [Sec. 12.6](#) are repeated here for clarity,

$$(1) \quad D\rho/Dt = -\rho \nabla \cdot \mathbf{v},$$

$$(2) \quad \rho D\mathbf{v}/Dt = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{b},$$

$$(3) \quad \boldsymbol{\tau}^T = \boldsymbol{\tau},$$

$$(4) \quad \rho D\hat{\mathbf{e}}/Dt = \boldsymbol{\tau} : \mathbf{D} - \nabla \cdot \mathbf{h}.$$

The second law of thermodynamics gives the entropy production inequality

$$(5) \quad D_i \hat{s}/Dt \equiv D\hat{s}/Dt + \nabla \cdot \mathbf{j}^s/\rho \geq 0.$$

In the reference configuration V_R , the entropy production inequality can be rewritten as

$$(6) \quad D_i\hat{s}/Dt \equiv D\hat{s}/Dt + \nabla_R \cdot \mathbf{J}^s/\rho_0 \geq 0.$$

14.10.2. Coupled constitutive relations based on Helmholtz free energy functional

Using the Helmholtz free energy $\hat{h} = \hat{e} - T\hat{s}$, we rewrite the local energy balance equation (4) in the reference configuration V_R as (see Sec. 13.11)

$$(7) \quad \rho_0 D\hat{s}/Dt = [\mathbf{S} : \dot{\mathbf{C}}/2 - \rho_0 \hat{s}\dot{T} - \rho_0 D\hat{h}/Dt - \nabla_R \cdot \mathbf{H}]/T.$$

The Green strain tensor (also called the Green–Lagrange strain tensor) and the temperature deviation are

$$(8) \quad \mathbf{E} = (\mathbf{C} - \mathbf{I})/2,$$

$$(9) \quad \theta = T - T_0.$$

The hypothesis of fading memory stipulates that the influence of long past events is weaker than that of recent ones in determining the material response. The Helmholtz free energy is taken to be a functional of the histories of deformation and temperature. Using the Lagrange strain measure and the temperature deviation, for materials with fading memory on intrinsic time scale, we expand the Helmholtz free energy functional up to the second order as

$$\begin{aligned} \rho_0 \hat{h} &= \rho_0 \hat{h}_0 + \int_{-\infty}^{\psi} [L_{IJ}^M(\mathbf{X}, \psi - \psi')[\partial E_{IJ}(\mathbf{X}, \psi')/\partial \psi']] d\psi' \\ &\quad - \int_{-\infty}^{\psi} M^M(\mathbf{X}, \psi - \psi')[\partial \theta(\mathbf{X}, \psi')/\partial \psi'] d\psi' \\ &\quad + \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \{-\beta_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'')[\partial E_{IJ}(\mathbf{X}, \psi')/\partial \psi'][\partial \theta(\mathbf{X}, \psi'')/\partial \psi''] \\ &\quad + G_{IJKL}^M(\mathbf{X}, \psi - \psi', \psi - \psi'')[\partial E_{IJ}(\mathbf{X}, \psi')/\partial \psi'][\partial E_{KL}(\mathbf{X}, \psi'')/\partial \psi'']]/2 \\ &\quad - C_H^M(\mathbf{X}, \psi - \psi', \psi - \psi'')[\partial \theta(\mathbf{X}, \psi')/\partial \psi'][\partial \theta(\mathbf{X}, \psi'')/\partial \psi'']/(2T_0)\} d\psi' d\psi'', \end{aligned} \quad (10)$$

where \hat{h}_0 is the value of the Helmholtz free energy per unit mass in the reference state, $G_{IJKL}^M(\mathbf{X}, \psi', \psi'') = G_{KLIJ}^M(\mathbf{X}, \psi'', \psi')$, $C_H^M(\mathbf{X}, \psi', \psi'') = C_H^M(\mathbf{X}, \psi'', \psi')$, and ψ is the intrinsic time scale governed by

$$(11) \quad D\psi/Dt = \dot{\psi} = a_{t_a, T, \mathbf{E}}(t; t_a^0, T, \mathbf{E}),$$

with t_a^0 as the aging time prior to loading, $t_a = t + t_a^0$ as the total aging time, and $a_{t_a, T, \mathbf{E}}$ as the shift function due to the combined effects of aging, temperature, and strain, which can be determined experimentally.

Substituting Eqs. (7) and (10) into (6) and performing differentiation with respect to time using the Leibnitz rule yield

$$\begin{aligned} D_i\hat{s}/Dt &= \langle S_{IJ} - L_{IJ}^0 - \int_{-\infty}^{\psi} \{G_{IJKL}^M(\mathbf{X}, 0, \psi - \psi')[\partial E_{KL}(\mathbf{X}, \psi')/\partial \psi'] \\ &\quad - \beta_{IJ}^M(\mathbf{X}, 0, \psi - \psi')[\partial \theta(\mathbf{X}, \psi')/\partial \psi']\} d\psi' \rangle \dot{E}_{IJ}/(\rho_0 T) \\ &\quad - \langle \rho_0 \hat{s} - M^0 - \int_{-\infty}^{\psi} \{\beta_{IJ}^M(\mathbf{X}, \psi - \psi', 0)[\partial E_{IJ}(\mathbf{X}, \psi')/\partial \psi'] \\ &\quad + C_H^M(\mathbf{X}, \psi - \psi', 0)[\partial \theta(\mathbf{X}, \psi')/\partial \psi']/T_0\} d\psi' \rangle \dot{T}/(\rho_0 T) \\ &\quad + [\Lambda + \mathbf{H} \cdot (-\nabla_R T)/T + T \nabla_R \cdot (\mathbf{J}^s - \mathbf{H}/T)]/(\rho_0 T) \geq 0, \end{aligned} \quad (12)$$

where the intrinsic dissipation rate is given by

$$\begin{aligned}
\Lambda = & \left\{ \int_{-\infty}^{\psi} \left[\frac{\partial M^M(\mathbf{X}, \psi - \psi')}{\partial \psi} \frac{\partial \theta(\mathbf{X}, \psi')}{\partial \psi'} - \frac{\partial L_{IJ}^M(\mathbf{X}, \psi - \psi')}{\partial \psi} \frac{\partial E_{IJ}(\mathbf{X}, \psi')}{\partial \psi'} \right] d\psi' \right. \\
& - \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \frac{\partial G_{IJKL}^M(\mathbf{X}, \psi - \psi', \psi - \psi'')}{\partial \psi} \frac{\partial E_{IJ}(\mathbf{X}, \psi')}{\partial \psi'} \frac{\partial E_{KL}(\mathbf{X}, \psi'')}{\partial \psi''} \frac{d\psi' d\psi''}{2} \\
& + \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \frac{\partial \beta_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'')}{\partial \psi} \frac{\partial E_{IJ}(\mathbf{X}, \psi')}{\partial \psi'} \frac{\partial \theta(\mathbf{X}, \psi'')}{\partial \psi''} d\psi' d\psi'' \\
& \left. + \frac{1}{2T_0} \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \frac{\partial C_H^M(\mathbf{X}, \psi - \psi', \psi - \psi'')}{\partial \psi} \frac{\partial \theta(\mathbf{X}, \psi')}{\partial \psi'} \frac{\partial \theta(\mathbf{X}, \psi'')}{\partial \psi''} d\psi' d\psi'' \right\} \dot{\psi}. \quad (13)
\end{aligned}$$

Since the entropy production inequality (12) is always valid, the state equations should fulfill the following conditions of thermodynamic admissibility:

$$\begin{aligned}
(14) \quad S_{IJ}(\mathbf{X}, \psi) = & \int_{-\infty}^{\psi} G_{IJKL}^M(\mathbf{X}, 0, \psi - \psi')[\partial E_{KL}(\mathbf{X}, \psi')/\partial \psi'] d\psi' \\
& - \int_{-\infty}^{\psi} \beta_{IJ}^M(\mathbf{X}, 0, \psi - \psi')[\partial \theta(\mathbf{X}, \psi')/\partial \psi'] d\psi' + L_{IJ}^0(\mathbf{X}),
\end{aligned}$$

$$\begin{aligned}
(15) \quad \rho_0 \hat{s}(\mathbf{X}, \psi) = & \int_{-\infty}^{\psi} \beta_{IJ}^M(\mathbf{X}, \psi - \psi', 0)[\partial E_{IJ}(\mathbf{X}, \psi')/\partial \psi'] d\psi' \\
& + \int_{-\infty}^{\psi} C_H^M(\mathbf{X}, \psi - \psi', 0)[\partial \theta(\mathbf{X}, \psi')/\partial \psi'] d\psi'/T_0 + M^0(\mathbf{X}),
\end{aligned}$$

$$(16) \quad \mathbf{J}^s = \mathbf{H}/T,$$

$$(17) \quad D_i \hat{s}/Dt = [\mathbf{H} \cdot (-\nabla_R T)/T + \Lambda]/(\rho_0 T) \geq 0,$$

where G_{IJKL}^M , C_H^M and β_{IJ}^M are thermodynamic property functions. Since E_{IJ} and S_{IJ} are symmetric tensors, we also have $G_{IJKL}^M = G_{JIKL}^M = G_{IJKL}^M$ and $\beta_{IJ}^M = \beta_{JI}^M$. The last terms L_{IJ}^0 and M^0 on the right-hand sides of Eqs. (14) and (15) are the values of S_{IJ} and $\rho_0 \hat{s}$ in the reference state, the first terms the mechanical contribution, and the second terms the thermal contribution. It is shown from (17) that the total dissipation rate is associated with heat conduction and intrinsic dissipative processes. Since the inequality (17) should always be satisfied, the kinetic laws for specific irreversible processes can be determined accordingly.

The intrinsic dissipation rate satisfies the inequality:

$$(18) \quad \Lambda \geq 0.$$

As discussed in Secs. 12.7 and 13.11, the transport law of heat conduction may be derived from the Fourier dissipation potential, which is taken as a quadratic scalar function of the thermodynamic force $-\mathbf{G}/T$, [see Eq. (13.11:5)],

$$(19) \quad \mathcal{D}_{th} = (-\mathbf{G}) \cdot \mathbf{K} \cdot (-\mathbf{G})/(2T^2),$$

where the thermal conductivity tensor $\mathbf{K}^T = \mathbf{K}$ is nonnegative-definite, and $\mathbf{G} = \nabla_R T$ is the temperature gradient in the reference configuration.

Hence, the thermodynamic flux for heat conduction depends linearly on the corresponding thermodynamic force with satisfaction of the Onsager principle, consistent with the discussion in Sec. 12.5,

$$(20) \quad \mathbf{H}/T = \partial \mathcal{D}_{th}/\partial(-\mathbf{G}/T) = \mathbf{K} \cdot (-\mathbf{G})/T.$$

Substituting (15) and (20) into (7) yields the following coupled heat transfer equation based on the Helmholtz free energy functional expansion:

$$\begin{aligned}
(21) \quad \frac{D}{Dt} \int_{-\infty}^{\psi} & [\beta_{IJ}^M(\mathbf{X}, \psi - \psi', 0) \frac{\partial E_{IJ}(\mathbf{X}, \psi')}{\partial \psi'} + \frac{C_H^M(\mathbf{X}, \psi - \psi', 0)}{T_0} \frac{\partial \theta(\mathbf{X}, \psi')}{\partial \psi'}] d\psi' \\
& = [\nabla_R \cdot (\mathbf{K} \cdot \nabla_R \theta) + \Lambda]/T_0,
\end{aligned}$$

where the integral involving the strain history gives rise to a coupling between thermal and mechanical effects.

Special forms of integration functions may be used for the Helmholtz free energy functional expansion, e.g.,

$$G_{IJKL}^M(\mathbf{X}, \psi', \psi'') = \tilde{G}_{IJKL}^M(\mathbf{X}, \psi' + \psi''),$$

which is consistent with the relaxation function for the integral form of viscoelastic model described in Sec. 14.2.

14.10.3. Coupled constitutive relations based on Gibbs free energy functional

With the Gibbs free energy $\hat{g} = \hat{e} - T\hat{s} - \mathbf{S} : \mathbf{E}/\rho_0$, the local energy balance equation (4) can be rewritten in the reference configuration V_R as (see Sec. 13.11)

$$(22) \quad \rho_0 D\hat{s}/Dt = [-\dot{\mathbf{S}} : \mathbf{E} - \rho_0 \hat{s}\dot{T} - \rho_0 D\hat{g}/Dt - \nabla_R \cdot \mathbf{H}]/T.$$

The Gibbs free energy is taken to be a functional of the histories of stress and temperature. Using the second Piola–Kirchhoff stress tensor and the temperature deviation, one can expand the Gibbs free energy functional for materials with fading memory on intrinsic time scale up to the second order to obtain

$$(23) \quad -\rho_0 \hat{g} = -\rho_0 \hat{g}_0 + \int_{-\infty}^{\psi} A_{IJ}^M(\mathbf{X}, \psi - \psi')[\partial S_{IJ}(\mathbf{X}, \psi')/\partial \psi']d\psi'$$

$$+ \int_{-\infty}^{\psi} B^M(\mathbf{X}, \psi - \psi')[\partial \theta(\mathbf{X}, \psi')/\partial \psi']d\psi'$$

$$+ \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} J_{IJKL}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') \frac{\partial S_{IJ}(\mathbf{X}, \psi')}{\partial \psi'} \frac{\partial S_{KL}(\mathbf{X}, \psi'')}{\partial \psi''} \frac{d\psi' d\psi''}{2}$$

$$+ \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \alpha_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') \frac{\partial S_{IJ}(\mathbf{X}, \psi')}{\partial \psi'} \frac{\partial \theta(\mathbf{X}, \psi'')}{\partial \psi''} d\psi' d\psi''$$

$$+ \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} C_G^M(\mathbf{X}, \psi - \psi', \psi - \psi'') \frac{\partial \theta(\mathbf{X}, \psi')}{\partial \psi'} \frac{\partial \theta(\mathbf{X}, \psi'')}{\partial \psi''} \frac{d\psi' d\psi''}{2T_0},$$

where \hat{g}_0 is the Gibbs free energy per unit mass in the reference state,

$$J_{IJKL}^M(\mathbf{X}, \psi', \psi'') = J_{KLIJ}^M(\mathbf{X}, \psi'', \psi'), C_G^M(\mathbf{X}, \psi', \psi'') = C_G^M(\mathbf{X}, \psi'', \psi'),$$

and ψ is the intrinsic time scale governed by

$$(24) \quad D\psi/Dt = \dot{\psi} = a_{t_a, T, \mathbf{S}}(t; t_a^0, T, \mathbf{S}),$$

where t_a^0 is the aging time prior to loading, $t_a = t + t_a^0$ the total aging time, and $a_{t_a, T, \mathbf{S}}$ the shift function due to the combined effects of aging, temperature, and stress, which is to be determined experimentally.

Substituting Eqs. (22) and (23) into Eq. (6) and performing differentiation with respect to time using the Leibnitz rule, we obtain

$$(25) \quad D_i \hat{s}/Dt = \left\{ \int_{-\infty}^{\psi} J_{IJKL}^M(\mathbf{X}, 0, \psi - \psi')[\partial S_{KL}(\mathbf{X}, \psi')/\partial \psi']d\psi' \right.$$

$$+ \int_{-\infty}^{\psi} \alpha_{IJ}^M(\mathbf{X}, 0, \psi - \psi')[\partial \theta(\mathbf{X}, \psi')/\partial \psi']d\psi' + A_{IJ}^0 - E_{IJ} \} \dot{S}_{IJ}/\rho_0 T$$

$$- \left\{ \rho_0 \hat{s} - B_0 - \int_{-\infty}^{\psi} \alpha_{IJ}^M(\mathbf{X}, \psi - \psi', 0)[\partial S_{IJ}(\mathbf{X}, \psi')/\partial \psi']d\psi' \right\} \dot{T}/(\rho_0 T)$$

$$- \int_{-\infty}^{\psi} C_G^M(\mathbf{X}, \psi - \psi', 0)[\partial \theta(\mathbf{X}, \psi')/\partial \psi']d\psi'/T_0 \} \dot{T}/(\rho_0 T)$$

$$+ [\Lambda + \mathbf{H} \cdot (-\nabla_R T)/T + T \nabla_R \cdot (\mathbf{J}^s - \mathbf{H}/T)]/(\rho_0 T) \geq 0,$$

where the intrinsic dissipation rate is given by

$$(26) \quad \Lambda = \dot{\psi} \left\{ \int_{-\infty}^{\psi} [\partial A_{IJ}^M(\mathbf{X}, \psi - \psi')/\partial\psi][\partial S_{IJ}(\mathbf{X}, \psi')/\partial\psi']d\psi' \right. \\ + \int_{-\infty}^{\psi} [\partial B^M(\mathbf{X}, \psi - \psi')/\partial\psi][\partial\theta(\mathbf{X}, \psi')/\partial\psi']d\psi' \\ + \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \frac{\partial J_{IJKL}^M(\mathbf{X}, \psi - \psi', \psi - \psi'')}{\partial\psi} \frac{\partial S_{IJ}(\mathbf{X}, \psi')}{\partial\psi'} \frac{\partial S_{KL}(\mathbf{X}, \psi'')}{\partial\psi''} \frac{d\psi' d\psi''}{2} \\ + \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \frac{\partial \alpha_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'')}{\partial\psi} \frac{\partial S_{IJ}(\mathbf{X}, \psi')}{\partial\psi'} \frac{\partial\theta(\mathbf{X}, \psi'')}{\partial\psi''} d\psi' d\psi'' \\ \left. + \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \frac{\partial C_G^M(\mathbf{X}, \psi - \psi', \psi - \psi'')}{\partial\psi} \frac{\partial\theta(\mathbf{X}, \psi')}{\partial\psi'} \frac{\partial\theta(\mathbf{X}, \psi'')}{\partial\psi''} \frac{d\psi' d\psi''}{2T_0} \right\}.$$

Since the entropy production inequality (25) is always valid, the state equations should fulfill the following conditions of thermodynamic admissibility:

$$(27) \quad E_{IJ}(\mathbf{X}, \psi) = \int_{-\infty}^{\psi} J_{IJKL}^M(\mathbf{X}, 0, \psi - \psi')[\partial S_{KL}(\mathbf{X}, \psi')/\partial\psi']d\psi' \\ + \int_{-\infty}^{\psi} \alpha_{IJ}^M(\mathbf{X}, 0, \psi - \psi')[\partial\theta(\mathbf{X}, \psi')/\partial\psi']d\psi' + A_{IJ}^0(\mathbf{X}),$$

$$(28) \quad \rho_0 \hat{s}(\mathbf{X}, \psi) = \int_{-\infty}^{\psi} \alpha_{IJ}^M(\mathbf{X}, \psi - \psi', 0)[\partial S_{IJ}(\mathbf{X}, \psi')/\partial\psi']d\psi' \\ + \int_{-\infty}^{\psi} C_G^M(\mathbf{X}, \psi - \psi', 0)[\partial\theta(\mathbf{X}, \psi')/\partial\psi']d\psi'/T_0 + B^0(\mathbf{X}),$$

$$(29) \quad \mathbf{J}^s = \mathbf{H}/T,$$

$$(30) \quad D_i \hat{s}/Dt = [\mathbf{H} \cdot (-\nabla_R T)/T + \Lambda]/(\rho_0 T) \geq 0,$$

where J_{IJKL}^M , C_G^M , and α_{IJ}^M are thermodynamic property functions. Since E_{IJ} and S_{IJ} are symmetric tensors, we also have $J_{IJKL}^M = J_{JIKL}^M = J_{IJKL}^M$ and $\alpha_{IJ}^M = \alpha_{JI}^M$. The last terms A_{IJ}^0 and B^0 on the right-hand sides of Eqs. (27) and (28) are the values of E_{IJ} and $\rho_0 \hat{s}$ in the reference state, the first terms the mechanical contribution, and the second terms the thermal contribution. It can be seen that the entropy production inequality (30) is identical to (17). Likewise, the kinetic laws for specific irreversible processes may be determined accordingly.

The intrinsic dissipation rate satisfies the inequality:

$$(31) \quad \Lambda \geq 0.$$

The transport law of heat conduction may be derived from the Fourier dissipation potential in the same way, i.e.,

$$(32) \quad \mathbf{H} = \mathbf{K} \cdot (-\nabla_R T).$$

Substituting Eqs. (28) and (32) into Eq. (22) yields the following coupled heat transfer equation based on the Gibbs free energy functional expansion:

$$(33) \quad \frac{D}{Dt} \int_{-\infty}^{\psi} [\alpha_{IJ}^M(\mathbf{X}, \psi - \psi', 0) \frac{\partial S_{IJ}(\mathbf{X}, \psi')}{\partial\psi'} + \frac{C_G^M(\mathbf{X}, \psi - \psi', 0)}{T_0} \frac{\partial\theta(\mathbf{X}, \psi')}{\partial\psi'}]d\psi' \\ = [\nabla_R \cdot (\mathbf{K} \cdot \nabla_R \theta) + \Lambda]/T_0,$$

where the integral involving the stress history gives rise to the coupling between thermal and mechanical effects.

Similarly, special forms of integration functions $J_{IJKL}^M(\mathbf{X}, \psi', \psi'') = \tilde{J}_{IJKL}^M(\mathbf{X}, \psi' + \psi'')$ may be used for the Gibbs free energy functional expansion, which is consistent with the creep function for the integral form of viscoelastic model described in Sec. 14.2.

Problem 14.15. For the generalized Kelvin and Maxwell Models shown in Fig. 14.1:1, express the energy stored in the

springs.

Problem 14.16. Derive the isotropic form of thermoviscoelastic constitutive equations for materials with memory on intrinsic time scale based on the Helmholtz free energy functional, following the procedure in Sec. 14.10.2.

Problem 14.17. Derive the isotropic form of thermoviscoelastic constitutive equations for materials with memory on intrinsic time scale based on the Gibbs free energy functional, following the procedure in Sec. 14.10.3.

14.11. COUPLED THERMOVISCOELASTIC BOUNDARY-INITIAL VALUE PROBLEMS

Coupled thermoviscoelastic boundary-initial value problems can be formulated with the mass balance equation (14.10:1), linear momentum balance equation (14.10:2), angular momentum balance equation (14.10:3), Lagrange strain measure definition (14.10:8), constitutive equations (14.10:14 or 14.10:27), and heat transfer equation (14.10:21 or 14.10:33), when appropriate boundary and initial conditions are specified. Finite element method is often adopted to solve complex nonlinear problems involving thermal and mechanical coupling and dissipative effects.

Initial conditions are taken as

$$(1) \quad \mathbf{u} = \mathbf{0} \quad (t < 0),$$

$$(2) \quad \boldsymbol{\tau} = \mathbf{0} \quad (t < 0),$$

$$(3) \quad \theta = 0 \quad (t < 0).$$

Boundary conditions are given by

$$(4) \quad \mathbf{u} = \mathbf{u}^B \quad \text{on } S_u(t \geq 0),$$

$$(5) \quad \mathbf{n} \cdot \boldsymbol{\tau} = \mathbf{t}^B \quad \text{on } S_\sigma(t \geq 0),$$

$$(6) \quad \theta = \theta^B \quad \text{on } S_\theta(t \geq 0),$$

$$(7) \quad \mathbf{n} \cdot \mathbf{h} = q^B \quad \text{on } S_q(t \geq 0),$$

where the subscript of S denotes the part of the boundary the subscripted quantity is prescribed, e.g., S_u denotes the prescribed displacement boundary. Like thermoelastic problems, other mixed boundary conditions for thermoviscoelastic problems may also be used.

To establish the existence and uniqueness of solutions for viscoelastic and thermoviscoelastic boundary-initial value problems, the reader may refer to the papers by Onat and Breuer (1963) and Gurtin and Sternberg (1962) and the book by Christensen (1982).

14.12. LINEARIZED THEORY AND INTEGRAL TRANSFORMS

The governing equations of thermoviscoelasticity can be reduced to the linearized formulation as follows:

(1) *Equations of motion:* $\rho\dot{\mathbf{v}}_i = \tau_{ij,j} + \rho b_i$, $\tau_{ij} = \tau_{ji}$

(2a) Heat transfer equation based on Helmholtz free energy functional:

$$\blacktriangle \quad \frac{\partial}{\partial t} \int_{-\infty}^t \{ \beta_{ij}(\mathbf{x}, t - \tau, 0) \partial e_{ij}(\mathbf{x}, \tau) / \partial \tau \\ + C_H(\mathbf{x}, t - \tau, 0) [\partial \theta(\mathbf{x}, \tau) / \partial \tau] / T_0 \} d\tau = (k_{ij}\theta_{,j})_{,i} / T_0,$$

(2b) Heat transfer equation based on Gibbs free energy functional:

$$\blacktriangle \quad \frac{\partial}{\partial t} \int_{-\infty}^t \{ \alpha_{ij}(\mathbf{x}, t - \tau, 0) \partial \tau_{ij}(\mathbf{x}, \tau) / \partial \tau \\ + C_G(\mathbf{x}, t - \tau, 0) [\partial \theta(\mathbf{x}, \tau) / \partial \tau] / T_0 \} d\tau = (k_{ij}\theta_{,j})_{,i} / T_0,$$

[k_{ij} is defined in Eq. (12.7:3)]

(3) Infinitesimal strain-displacement relation: $e_{ij} = (u_{i,j} + u_{j,i})/2$,

(4a) *Constitutive relation based on Helmholtz free energy functional:*

$$\Delta \tau_{ij}(x, t) = \int_{-\infty}^t \left[G_{ijkl}(x, 0, t - \tau) \frac{\partial e_{kl}(x, \tau)}{\partial \tau} - \beta_{ij}(x, 0, t - \tau) \frac{\partial \theta(x, \tau)}{\partial \tau} \right] d\tau,$$

(4b) *Constitutive relation based on Gibbs free energy functional:*

$$\Delta e_{ij}(x, t) = \int_{-\infty}^t \left[J_{ijkl}(x, 0, t - \tau) \frac{\partial \tau_{kl}(x, \tau)}{\partial \tau} + \alpha_{ij}(x, 0, t - \tau) \frac{\partial \theta(x, \tau)}{\partial \tau} \right] d\tau.$$

A coupled thermoviscoelastic analysis involves determining the displacement and temperature fields under prescribed boundary and initial conditions. Integral transform methods provide a powerful tool for solving linearized problems. The selection of an appropriate transform depends on the nature of the functions involved. After an integral transform is applied to the basic equations of a thermoviscoelastic problem, the transformed problem can be solved in a manner similar to that of a coupled thermoelastic problem. The final thermoviscoelastic solution is obtained upon the inversion of the transformed solution.

Assuming that the Laplace transform exists, with the initial conditions as specified in Sec. 14.11, the transformed basic equations and boundary conditions are

$$(5) \quad \rho s^2 \bar{u}_i = \bar{\tau}_{ij,j} + \rho \bar{b}_i,$$

$$(6a) \quad (k_{ij}\bar{\theta}_{,j})_i = s^2 \rho \bar{C}_H \bar{\theta} + s^2 T_0 \bar{\beta}_{ij} \bar{e}_{ij},$$

$$(6b) \quad (k_{ij}\bar{\theta}_{,j})_i = s^2 \rho \bar{C}_G \bar{\theta} + s^2 T_0 \bar{\alpha}_{ij} \bar{\tau}_{ij},$$

$$(7) \quad \bar{e}_{ij} = (\bar{u}_{i,j} + \bar{u}_{j,i})/2,$$

$$(8a) \quad \bar{\tau}_{ij} = s \bar{G}_{ijkl} \bar{e}_{kl} - s \bar{\beta}_{ij} \bar{\theta},$$

$$(8b) \quad \bar{e}_{ij} = s \bar{J}_{ijkl} \bar{\tau}_{kl} + s \bar{\alpha}_{ij} \bar{\theta}$$

$$(9) \quad \bar{u}_i = \bar{u}_i^B, \quad \text{on } S_u,$$

$$(10) \quad n_i \bar{\tau}_{ij} = \bar{t}_i^B \quad \text{on } S_\sigma,$$

$$(11) \quad \bar{\theta} = \bar{\theta}^B \quad \text{on } S_\theta,$$

$$(12) \quad -n_i k_{ij} \bar{\theta}_{,j} = \bar{q}^B \quad \text{on } S_q.$$

The correspondence principle of identifying a problem in linear elasticity with one in linear viscoelasticity in the transformed plane in Sec. 14.5 may be extended to identifying a linear thermoelasticity problem with one in linear thermoviscoelasticity.

14.13. REPRESENTATION OF THERMODYNAMIC PROPERTY FUNCTIONS FOR MATERIALS WITH MEMORY ON INTRINSIC TIME SCALE

The thermodynamic formulation of the coupled thermoviscoelastic constitutive relations in Sec. 14.10 allows the dependence of the thermodynamic property functions on aging, temperature, strain or stress, etc. via the introduction of intrinsic time scale. Thus the dependence of the long-term property functions on aging time, temperature, etc. may be determined from shortterm experiments with an accelerated test methodology. Physical aging refers to structural relaxation of the glassy state toward the metastable state, accompanied by changes in almost all physical properties (Hodge 1995). In this section, we will explore the characterization of the thermodynamic property functions for materials with memory on intrinsic time scale.

14.13.1. Thermo-rheological and piezo-rheological simple materials

The time-temperature superposition principle (also called the method of reduced variables) is a well-known procedure to characterize the viscoelastic behavior over a broad range of times or frequencies by shifting data obtained at several temperatures to a reference temperature. The procedure involves the use of temperature-dependent shift factors to obtain a “master curve,”

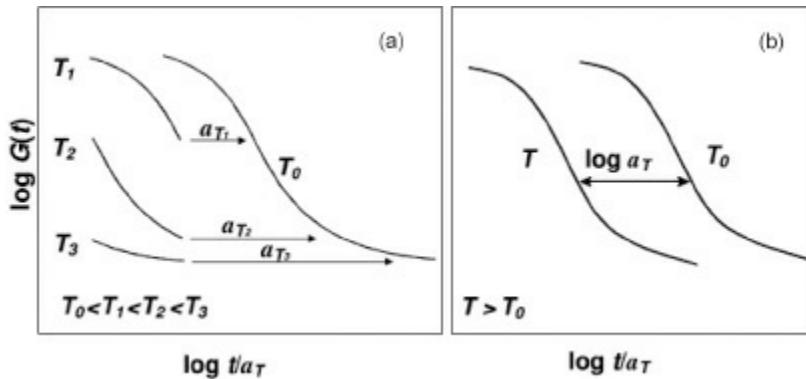


Fig. 14.13:1. (a) Shifting isothermal segments into a master curve; (b) Shift factor $\log(a_T)$ for temperature change from T to T_0 . (From Emri, 2005, with permission from The British Society of Rheology.) leading to development of accelerated testing methods. A schematic illustration is shown in Fig. 14.13:1, thereby extending the time scale beyond the range that could normally be covered in a single experiment.

A special class of materials to which this procedure is applicable is referred to as “thermo-rheological simple materials.” When the viscoelastic property function such as relaxation modulus or creep compliance is shown in double logarithmic plot, data at various constant temperatures can be shifted horizontally by a factor $\log(a_T)$ and vertically by a factor $\log(b_T)$ to generate a master curve at the reference temperature T_0 . Then

$$(1) \quad G^M(\psi) = b_T G(t, T),$$

$$(2) \quad \psi = t/a_T.$$

The Williams–Landel–Ferry (WLF) equation (Williams *et al.* 1955) is widely used to model the temperature dependence of the horizontal shift factor for amorphous polymers in the glass transition region based on the Cohen–Turnbull free volume theory (Cohen and Turnbull 1959, Turnbull and Cohen 1961):

$$(3) \quad \log(a_T) = -C_1^0(T - T_0)/(C_2^0 + T - T_0),$$

where C_1^0 and C_2^0 are empirical parameters depending on the reference temperature T_0 .

By analogy with the reaction rate theory, the Arrhenius equation is often used for describing the temperature dependence of the horizontal shift factor in the plateau and terminal zones:

$$(4) \quad \log(a_T) = E_a(1/T - 1/T_0)/R,$$

where E_a is the activation energy, and R is the universal gas constant.

The molecular theory of viscoelasticity implies that the vertical shift of relaxation modulus has the following temperature dependence:

$$(5) \quad b_T = \rho_0 T_0 / (\rho T).$$

Rather than using Eq. (5), McCrum and Morris (1964) determined the vertical shift directly from the variation of unrelaxed or relaxed compliance or modulus with temperature. A thorough discussion of the time-temperature superposition principle by shifting data horizontally and vertically can be found in the book by Ferry (1980).

A similar procedure, i.e., the time-temperature-pressure superposition principle, may be applied to characterize the effects of temperature and pressure on the properties of “thermo-rheological and/or piezo-rheological simple materials” (see review by Tschoegl *et al.* 2002, Emri 2005). As an extension of the WLF equation, Table 14.13:1 lists the comparison of three models for the logarithmic shift factor $\log(a_{T,P})$, accounting for the effect of pressure in addition to that of temperature. The fraction of the occupied lattice sites, y , may be linked to the fractional free volume f with $y = 1 - f$.

Table 14.13:1. Comparison of the models for time-temperature-pressure superposition. (From Emri, 2005, with permission from The British Society of Rheology.)

Model	$\log a_{T,P}$	Approach
Simha–Somcynsky	$\log a_{T,P} = \frac{B_{SS}}{2.303} \left(\frac{1}{1-y_0} - \frac{1}{1-y} \right)$	Hole theory
Havlíček–Ilvalský–Hrouz	$\log a_{T,P} = \frac{B_{HIH}}{2.303} \left(\frac{1}{TS_c(T, P)} - \frac{1}{T_g S_c(T_g, P_g)} \right)$	Excess entropy
Fillers–Moonan–Tschoegl	$\log a_{T,P} = -\frac{B}{2.303 f_0} \left[\frac{T - T_0 - \theta(P)}{f_0/\alpha_f(P) + T - T_0 - \theta(P)} \right]$	Free volume

14.13.2. Effective time theory for aging materials

It is well known that amorphous polymers do not achieve thermodynamic equilibrium immediately as ambient temperature drops below their glass transition temperature. Physical aging is a thermo-reversible process, during which the segmental mobility of polymer chains changes with the aging time. An effective time theory has been proposed to predict the long-term behavior of amorphous polymers and other materials experiencing physical aging from short-term experiments (Struik 1978). Physical aging that refers to relaxation of the glassy state toward the metastable equilibrium state may have direct implications for long-term behavior of glassy polymers.

The long-term creep curve of polymers and other materials undergoing aging may be fitted well by a stretched exponential function. For illustration, the long-term creep compliance of polymeric materials experiencing physical aging may be expressed as

$$(6) \quad J(t) = J_0 \exp(\psi/\tau)^m,$$

$$(7) \quad \psi = \int_0^t a_{pa}(t') dt' = \int_0^t [t_1/(t_2 + t' + t_a^0)]^\mu dt',$$

where J_0 is instantaneous creep compliance, τ is relaxation time, m is shape factor, $0 < \mu \leq 1$ is shift rate, t_1 and t_2 are reference time parameters.

14.13.3. Time-aging-temperature-strain superposition

In the Helmholtz free energy functional-based formulation for materials with memory on intrinsic time scale, special integration functions with $f_H^M(\psi', \psi'') = \tilde{f}_H^M(\psi' + \psi'')$ may be employed. This implies that when the thermodynamic property functions in the constitutive equations are known, the corresponding integration functions in the Helmholtz free energy functional expansion can be determined.

A long-term property function may be related to a momentary master curve by a superposition principle of time, aging, temperature, and strain as follows:

$$(8) \quad \tilde{f}_H^M(\psi) = b_{t_a, T, E} f_H(t),$$

$$(9) \quad \psi = \int_0^t a_{t_a, T, E}[t'; t_a^0, T(t'), E(t')] dt',$$

where f_H denotes a thermodynamic property function (scalar or tensorial) with real time t as argument, \tilde{f}_H^M is the master function with intrinsic time scale ψ as argument, $a_{t_a, T, E}$ and $b_{t_a, T, E}$ correspond to horizontal and vertical shift functions due to the combined effects of aging, temperature and strain. For simplicity, the argument X of f , a and b are not shown.

The master function \tilde{f}_H^M may be represented in terms of continuous relaxation time spectrum as

$$(10) \quad \tilde{f}_H^M(\psi) = \int_{-\infty}^{\infty} \hat{f}_H(\tau) \phi_H(\psi/\tau) d(\ln \tau),$$

where $\hat{f}_H(\tau)$ is the relaxation time spectrum distribution, and $\phi_H(\psi/\tau)$ is the kernel function.

As the relaxation time spectrum has the form of delta function and the kernel function has the form of exponential function, the Prony series (developed by Gaspard Riche de Prony in 1795) representation can be expressed as

$$(11) \quad \tilde{f}_H^M(\psi) = \hat{f}_{H\infty}^M + \sum_{i=1}^N \hat{f}_{Hi}^M \exp(-\psi/\tau_i),$$

where $\hat{f}_{H\infty}^M$ is the relaxed value, \hat{f}_{Hi}^M are Prony series coefficients, and τ_i are discrete relaxation times.

14.13.4. Time-aging-temperature-stress superposition

In the Gibbs free energy functional-based formulation, special integration functions with $f_G^M(\psi', \psi'') = \tilde{f}_G^M(\psi' + \psi'')$ may be employed. When the thermodynamic property functions in the constitutive equations are known, the corresponding integration functions in the Gibbs free energy functional expansion can be

determined.

A long-term property function may be related to a momentary master curve by a superposition principle of time, aging, temperature, and stress as follows:

$$(12) \blacktriangle \quad \tilde{f}_G^M(\psi) = b_{t_a, T, S} f_G(t),$$

$$(13) \blacktriangle \quad \psi = \int_0^t a_{t_a, T, S}[t'; t_a^0, T(t'), S(t')] dt',$$

where f_G denotes a thermodynamic property function (scalar or tensorial) with real time t as argument, \tilde{f}_G^M is master function with intrinsic time scale ψ as argument, $a_{t_a, T, S}$ and $b_{t_a, T, S}$ correspond to horizontal and vertical shift functions due to the combined effects of aging, temperature and stress. For simplicity, the argument X of f , a and b are not shown.

Similarly, the master function \tilde{f}_G^M may be represented in terms of continuous relaxation time spectrum as

$$(14) \quad \tilde{f}_G^M(\psi) = \int_{-\infty}^{\infty} \hat{f}_G(\tau) \phi_G(\psi/\tau) d(\ln \tau),$$

where $\hat{f}_G(\tau)$ is the relaxation time spectrum distribution, and $\phi_G(\psi/\tau)$ is the kernel function.

In general, the horizontal and vertical shift functions depend on aging, temperature, and stress (strain), etc., and the relationships for a specific material should be characterized through experiments. The coupled theory of thermoviscoelasticity can also be extended to include other effects.

Problem 14.18. Consider an isotropic viscoelastic slab made of thermorheologically simple material subjected to heating. The slab has infinite extent in the (x_1, x_2) plane and finite thickness $2a$, bounded by the planes $x_3 = \pm a$. Formulate the corresponding boundary-initial value problem, and find the displacement field and associated stress field if the temperature field is known as a function of the thickness, that is, $T = T(x_3, t)$ for $-a \leq x_3 \leq a$ (Muki and Sternberg 1961).

Problem 14.19. Consider an isotropic viscoelastic sphere of radius r_0 centered at the origin. The sphere is made of thermorheological simple material subjected to heating. Formulate the corresponding boundary-initial value problem, and find the displacement and stress fields if the temperature is a known function of the radius $T = T(r, t)$ for $r \leq r_0$ (Muki and Sternberg 1961).

¹Note that a polynomial of s has no continuous inverse function. The inverse of s is $\delta(t)$; those of s^2, s^3 are higher order singularities.

15

THERMODYNAMICS WITH INTERNAL STATE VARIABLES AND THERMO-ELASTO-VISCOPLASTICITY

Nonequilibrium thermodynamics provides an effective way of studying the behavior of dissipative materials. There are essentially two types of nonequilibrium thermodynamic approaches to establish the time- or path-dependent constitutive relations: functional thermodynamics and state-variable thermodynamics. In functional thermodynamics, the free energy is expressed as a functional of the histories of strain (stress), temperature, etc. In state-variable thermodynamics, the free energy is expressed as a function of current deformation (stress), temperature, and other variables, including so-called internal state variables (ISVs). Following [Chapter 14](#) on functional thermodynamics, we shall introduce thermodynamics with ISVs and explore its applications to thermo-elasto-viscoplastic problems in this chapter.

15.1. THERMODYNAMICS WITH INTERNAL STATE VARIABLES

A roadmap of the birth of the ISV theory and how different phenomena have been brought into the ISV theory over the years is shown in [Fig. 15.1:1](#). The concept of “hidden” or internal state variables was first used in thermodynamics by Onsager (1931a,b). The convergence of thermodynamics, kinematics, kinetics, and mechanics leads to the formation of the watershed paper by Coleman and Gurtin (1967). Since then, the ISV theory has been employed to study internal structure changes, such as dislocation, creep and damage, for metals, polymers, composites, biomaterials, geo-materials, and other multiphase materials (see [Biblio. 15.1](#)).

The basic idea behind the ISV theory is that, in order to fully characterize the state of a system undergoing irreversible processes, one expands the dimensions of the space of observable state variables (OSVs) commonly used in classical thermodynamics by introducing a set of additional “hidden” variables, also called ISVs, to describe the internal structural changes. The history effects are implicitly included in the evolution of these “hidden” variables or ISVs.

In the ISV theory, for a specified state (\mathbf{X}, t) in the reference configuration V_R , we consider the materials for which the basic variables, such as the Helmholtz free energy per unit mass $\hat{h} = \hat{e} - T\hat{s}$, the entropy per unit mass \hat{s} , the second Piola–Kirchhoff stress tensor $\mathbf{S} = J\mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}^{-T}$, and the heat flux $\mathbf{H} = J\mathbf{F}^{-1} \cdot \mathbf{h}$, can be determined if the current values of the following state variables are known: the deformation gradient \mathbf{F} (or the deformation tensor or the strain tensor $\mathbf{E} = \mathbf{C}/2 - \mathbf{I} = \mathbf{F}\mathbf{T} \cdot \mathbf{F}^T/2 - \mathbf{I}$), the absolute temperature T , the temperature gradient $\mathbf{G} = \nabla_R T$, and a set of “hidden” or internal state variables (scalar, vectorial, or tensorial) $\alpha(m)$ ($m = 1, \dots, N$). The rate of change of each ISV is governed by a general nonlinear evolution equation,

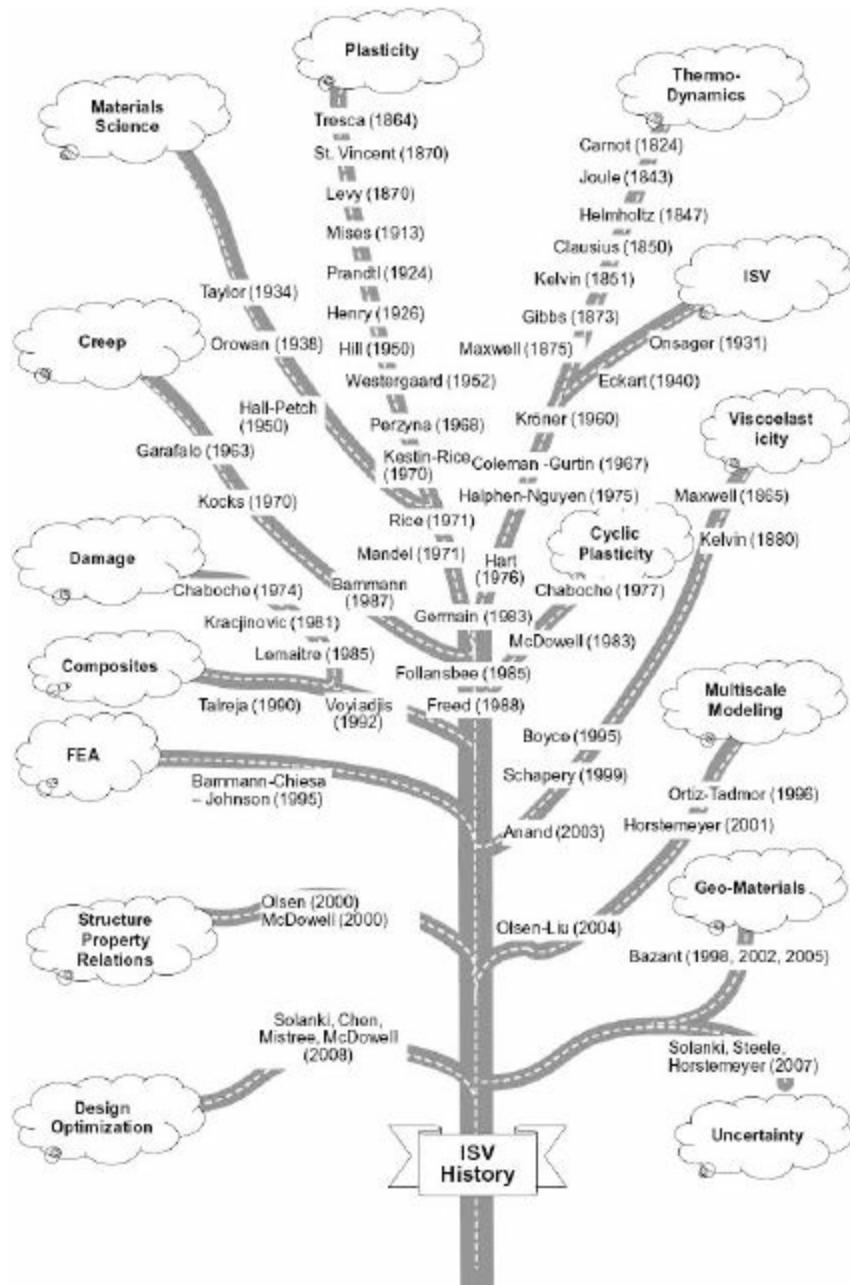


Fig. 15.1:1. Historic summary of different ways for formulating the ISV theory. (From Horstemeyer and Bamann, 2010, with permission from Elsevier.)

$$(1) \quad \dot{\alpha}^{(n)} = f^{(n)}(\mathbf{F}, T, \mathbf{G}; \alpha^{(m)}, \mathbf{X}), \quad m, n = 1, 2, \dots, N,$$

whose solution satisfies the fundamental laws of thermodynamics.

The freedom of choosing the ISVs enables the framework to accommodate a wide range of processes of interests, such as the constituent concentration change in chemical reaction or phase transformation and defect density alternation for dislocation or micro-crack. ISVs can be scalars, vectors or tensors. For specific materials and conditions, one needs to identify the physical meaning and the actual number of the ISVs. The Gibbs free energy may also be adopted to formulate the ISV theory through the Legendre transformation.

The ISV theory is subjected to the general restrictions imposed by the first and second laws of thermodynamics. As described in Chapters 12 and 13, with use of the Helmholtz free energy, the entropy production inequality (12.6:7–8) can be rewritten in the reference configuration as [see Eq. (13.11:1)]

$$(2) \quad \rho_0 T D_i \hat{s} / D t = \mathbf{S} : \dot{\mathbf{C}} / 2 - \rho_0 \dot{\hat{s}} T - \rho_0 \dot{\hat{h}} - T \nabla_R \cdot (\mathbf{H} / T - \mathbf{J}^s) - \mathbf{H} \cdot \mathbf{G} / T \geq 0,$$

leading to the following conditions of thermodynamic admissibility:

$$(3) \quad \partial \hat{h} / \partial G_K = 0,$$

$$(4) \quad S_{KL} = 2 \rho_0 \partial \hat{h} / \partial C_{KL},$$

$$(5) \quad \hat{s} = -\partial \hat{h}/\partial T,$$

$$(6) \quad \mathbf{J}^s = \mathbf{H}/T,$$

$$(7) \quad \rho_0 D_i \hat{s} / D t = (-\mathbf{G}) \cdot \mathbf{H} / T^2 + \sum_m \mathbf{A}^{(m)} \circ \dot{\alpha}^{(m)} / T \geq 0,$$

where ‘ \circ ’ denotes the appropriate product of a thermodynamic force and a thermodynamic flux that are energetically conjugate, and the thermodynamic driving force $\mathbf{A}^{(m)}$ for the corresponding internal structural change is given by

$$(8) \quad \mathbf{A}^{(m)} = -\rho_0 \partial \hat{h} / \partial \alpha^{(m)}.$$

Based on Eq. (3), the Helmholtz free energy is independent of the temperature gradient and so are the derived variables like stress and entropy. It is shown from (7) that the total dissipation rate is the sum of the thermodynamic forces multiplied by the thermodynamic fluxes. It is noted that the role played by the internal dissipation rate in thermodynamics with ISVs, as shown above, is almost identical to that played by the intrinsic dissipation rate in thermodynamics of materials with memory, as described in Sec. 14.10.2. The evolution of irreversible processes must satisfy the non-negative entropy production requirement.

The thermal dissipation rate and the internal dissipation rate are often assumed to be non-negative separately, that is,

$$(9) \quad \phi_{th} = (-\mathbf{G}) \cdot \mathbf{H} / T^2 \geq 0,$$

$$(10) \quad \phi_{in} = \sum_m \mathbf{A}^{(m)} \circ \dot{\alpha}^{(m)} / T \geq 0.$$

15.2. ENERGY–MOMENTUM TENSOR AND INVARIANT INTEGRAL

Eshelby (1951, 1970, 1975) was the first to introduce the concepts of energy–momentum tensor and path-independent integral for studying the force on an imperfection (dislocation, grain boundary, inhomogeneity, etc.). Since the integral of the energy–momentum tensor over any contour enclosing a singularity has the same value, the path-independent integral describes the invariant characteristics of the singularity. Several variations of invariant integrals have been proposed to account for body force, inertia, inelastic, thermal and inhomogeneous effects (see Biblio. 15.2). The following formulation follows Chen (2014).

With the use of Eqs. (15.1:3–6), the global energy balance equation (12.2:5) becomes

$$(1) \quad D(\int_V \mathcal{H} dV) / D t = \int_{\partial V} \tilde{\mathbf{T}} \cdot \dot{\mathbf{u}} dS + \int_V (\rho \mathbf{b} \cdot \dot{\mathbf{u}} - \rho \hat{s} \dot{T} - \sum_m J^{-1} \mathbf{A}^{(m)} \circ \dot{\alpha}^{(m)}) dV,$$

where the *Hamiltonian* density is the sum of the kinetic energy density and the Helmholtz free energy density, that is,

$$(2) \quad \mathcal{H}(\mathbf{v}, \mathbf{F}, T, \mathbf{G}; \alpha^{(m)}, \mathbf{X}) = \rho \hat{k}(\mathbf{v}; \alpha^{(m)}, \mathbf{X}) + \rho \hat{h}(\mathbf{F}, T, \mathbf{G}; \alpha^{(m)}, \mathbf{X}).$$

Consider a 3-D body $\tilde{V}_{\tilde{\Gamma}}$ bounded by a surface $\tilde{\Gamma}$ in a reference frame ($\tilde{\mathbf{X}} = \mathbf{X} - \mathbf{V}^R t$) moving with instantaneous speed \mathbf{V}^R . The global energy balance equation (1) can be rewritten in the reference frame as

$$(3) \quad F(\tilde{\Gamma}) = \int_{\tilde{\Gamma}} \tilde{\mathbf{N}} \cdot [J \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \dot{\mathbf{u}} + (\tilde{\rho} \hat{k} + \tilde{\rho} \hat{h}) \mathbf{V}^R] d\tilde{\Gamma} = \int_{\tilde{V}_{\tilde{\Gamma}}} \tilde{\partial}(\tilde{\rho} \hat{k} + \tilde{\rho} \hat{h}) / \tilde{\partial} t d\tilde{V} + \int_{\tilde{V}_{\tilde{\Gamma}}} (-\tilde{\rho} \mathbf{b} \cdot \dot{\mathbf{u}} + \tilde{\rho} \hat{s} \dot{T} + \sum_m \mathbf{A}^{(m)} \circ \dot{\alpha}^{(m)}) d\tilde{V},$$

where $\tilde{\mathbf{N}}$ is the unit normal vector on the surface $\tilde{\Gamma}$, and $F(\tilde{\Gamma})$ is the total energy flux through $\tilde{\Gamma}$ caused by the work done by the traction and the transport of material with its associated energy density. It is noted that the associated energy density includes the kinetic energy density and the Helmholtz free energy density.

If a field solution is invariant in the reference frame moving at uniform speed \mathbf{V}^R , the field solution depends on t only through the combination $\tilde{\mathbf{X}} = \mathbf{X} - \mathbf{V}^R t$. Then Eq. (3) leads to the following equations:

$$(4) \quad \tilde{J}_K V_K^R = \int_{\tilde{V}_{\tilde{\Gamma}}} [(\tilde{\rho}\hat{k} + \tilde{\rho}\hat{h})_{,K} \Big|_{\text{exp.}} + \tilde{\rho}b_i u_{i,K} - \tilde{\rho}\hat{s}T_{,K} - \sum_m \mathbf{A}^{(m)} \circ \alpha_{,K}^{(m)}] V_K^R d\tilde{V},$$

$$(5) \quad \blacktriangle \quad \tilde{J}_K = \int_{\tilde{\Gamma}} \tilde{N}_J \tilde{T}_{JK} d\tilde{\Gamma},$$

$$(6) \quad \blacktriangle \quad \tilde{T}_{JK} = -J F_{J,l}^{-1} \tau_{li} u_{i,K} + (\tilde{\rho}\hat{k} + \tilde{\rho}\hat{h}) \delta_{JK},$$

where the \tilde{J}_K -integral is expressed by the integral of the normal component of the generalized energy-momentum tensor $\tilde{\mathbf{T}}$ over the surface.

Hence, the divergence of the generalized energy-momentum tensor is related to inhomogeneity, body force, temperature gradient, and internal structural change, i.e.,

$$(7) \quad \blacktriangle \quad \tilde{T}_{JK,J} = (\tilde{\rho}\hat{k} + \tilde{\rho}\hat{h})_{,K} \Big|_{\text{exp.}} + \tilde{\rho}b_i u_{i,K} - \tilde{\rho}\hat{s}T_{,K} - \sum_m \mathbf{A}^{(m)} \circ \alpha_{,K}^{(m)}.$$

The above equations are generally applicable to homogeneous and inhomogeneous materials, including functionally graded materials, in the presence of the effects of inertia, body force, temperature gradient and internal structural change. If there is no inhomogeneity, body force, temperature gradient, internal structural change in a continuous medium, the divergence of the generalized energy-momentum tensor becomes zero and thus the integral of the normal component of the generalized energy-momentum tensor over any closed surface vanishes. For elastic materials under quasi-static and isothermal conditions, the original form of Eshelby stress tensor can be recovered with replacement of $u_{i,K}$ in Eqs. (4), (6) and (7) with F_{iK} based on the linear momentum balance. Further applications can be found in Sec. 22.6.

For steady-state propagation of a crack along the \tilde{X}_1 -direction, a \hat{J} -integral is defined as

$$(8) \quad \blacktriangle \quad \hat{J} = \tilde{J}_1 + \int_{\tilde{V}_{\tilde{\Gamma}}} [-(\tilde{\rho}\hat{k} + \tilde{\rho}\hat{h})_{,1} \Big|_{\text{exp.}} - \tilde{\rho}b_i u_{i,1} + \tilde{\rho}\hat{s}T_{,1} + \sum_m \mathbf{A}^{(m)} \circ \alpha_{,1}^{(m)}] d\tilde{V}.$$

Due to the addition of domain integral terms, the \hat{J} -integral becomes invariant, that is, path-domain independent. The invariant integral- \hat{J} method is not only generally applicable to various material systems and loading conditions but also relatively easy for finite element implementation due to its path-domain independency.

15.3. POTENTIALS OR PSEUDO-POTENTIALS OF DISSIPATION

Potentials or pseudo-potentials of dissipation are usually introduced for specific class of dissipative processes to derive the evolution laws (see Biblio. 15.3) so that the non-negative dissipation requirement can be satisfied, which will be illustrated below.

Thermal dissipation. The transport law of heat conduction may be derived from the Fourier dissipation potential, which is taken as a quadratic scalar function of the thermodynamic force ($-\mathbf{G}/T$), as shown in Secs. 12.7, 13.11 and 14.10,

$$(1) \quad \mathcal{D}_{th} = (-\mathbf{G}) \cdot \mathbf{K} \cdot (-\mathbf{G}) / (2T^2),$$

where the thermal conductivity tensor $\mathbf{K}^T = \mathbf{K}$ is nonnegative-definite.

Hence, the thermodynamic flux for heat conduction depends linearly on the corresponding thermodynamic force with satisfaction of the Onsager principle, consistent with the discussion in Sec. 12.5,

$$(2) \quad \mathbf{H}/T = \partial \mathcal{D}_{th} / \partial (-\mathbf{G}/T) = \mathbf{K} \cdot (-\mathbf{G}) / T.$$

Consequently, the thermal dissipation rate becomes

$$(3) \quad \phi_{th} = 2\mathcal{D}_{th} \geq 0.$$

Rate-independent internal dissipation. For a rate-independent internal dissipative process, the thermodynamic forces are homogeneous functions of degree zero in the thermodynamic fluxes with the internal dissipation being a positively homogeneous function of degree one in thermodynamic fluxes. Convex analysis (Rockafellar 1970) provides a useful tool to study rate-independent internal dissipative effects, as the thermodynamic forces are confined to a closed convex set containing the origin. A set C is said to be convex if, for any x and y in C , $(1 - \lambda)x + \lambda y$ is in C , $\forall \lambda \in [0, 1]$. A real valued function $f: C \rightarrow \mathbb{R}$ defined on a convex set C is called convex if, for any x and y in C ,

$$f[(1-\lambda)x + \lambda y] \leq (1-\lambda)f(x) + \lambda f(y), \quad \forall \lambda \in [0,1].$$

Internal dissipation occurs only if the thermodynamic forces reach a border point of the closed convex set C referred to as the elastic domain, which is mathematically expressed as

$$(4) \quad \Delta \quad \phi_{in}(\dot{\alpha}) = \sup_{\mathbf{A}^* \in C} (\mathbf{A}^* \circ \dot{\alpha}) \geq 0,$$

which is equivalent to the principle of maximum dissipation (see [Sec. 6.9](#)),

$$(5) \quad (\mathbf{A} - \mathbf{A}^*) \circ \dot{\alpha} \geq 0, \quad \forall \mathbf{A}^* \in C.$$

The evolution laws of the internal state variables may be determined by the generalized normality rule,

$$(6) \quad \dot{\alpha} \in N_C(\mathbf{A}),$$

where $N_C(\mathbf{A})$ denotes the cone of the outward normal at the boundary of the convex set C .

The boundary of the convex set C provides the loading surface $f = 0$ in the \mathbf{A} -space. If the loading surface has continuous tangent planes, that is, f is continuously differentiable with respect to \mathbf{A} , then

$$(7) \quad \Delta \quad \dot{\alpha}^{(m)} = \dot{\lambda}(\partial f / \partial \mathbf{A}^{(m)}),$$

where the non-negative scalar factor $\dot{\lambda}$ can be determined by the consistency condition: $\dot{f} = 0$. Hence, the loading function f acts as the pseudo-potential of dissipation.

Rate-dependent internal dissipation. For rate-dependent internal dissipative processes, there may exist a potential of dissipation φ^* so that the evolution law of the internal state variables is derived from

$$(8) \quad \Delta \quad \dot{\alpha}^{(m)} = \partial \varphi^* / \partial \mathbf{A}^{(m)}.$$

If φ^* is a positively homogeneous function of degree n in the thermodynamic forces, with the use of Euler's homogeneous function theorem, we have

$$(9) \quad \Delta \quad \phi_{in} = \sum_m \mathbf{A}^{(m)} \circ \dot{\alpha}^{(m)} / T = \sum_m \mathbf{A}^{(m)} \circ \partial \varphi^* / \partial \mathbf{A}^{(m)} / T = n \varphi^* / T \geq 0.$$

A special case is that φ^* is a positive homogeneous function of degree two in the thermodynamic forces so that the thermodynamic fluxes are homogeneous functions of degree one in the thermodynamic forces.

15.4. ALTERNATIVE FORMULATION OF THEORIES OF PLASTICITY

15.4.1. Thermo-elasto-plasticity

For a rate-independent thermo-elasto-plastic material, the pseudo-potential of dissipation is none other than the plastic potential and the generalized normality rule is the plastic flow rule. The well-known concepts like the elastic domain and the yield surface or loading surface are applicable (see [Biblio. 15.4](#)).

The thermodynamic admissible conditions become

$$(1) \quad \hat{h} = \hat{h}(\mathbf{E}, T; \mathbf{E}^p, \kappa^p, \mathbf{X}),$$

$$(2) \quad \hat{s} = -\partial \hat{h} / \partial T,$$

$$(3) \quad \mathbf{S} = \rho_n \partial \hat{h} / \partial \mathbf{E},$$

$$(4) \quad \mathbf{Y}^p = -\rho_0 \partial \hat{h} / \partial \mathbf{E}^p,$$

$$(5) \quad z^p = -\rho_0 \partial \hat{h} / \partial \kappa^p,$$

$$(6) \quad \dot{\mathbf{E}}^p = \dot{\lambda}(\partial f / \partial \mathbf{Y}^p),$$

$$(7) \quad \dot{\kappa}^p = \dot{\lambda}(\partial f / \partial z^p),$$

where the plastic multiplier $\dot{\lambda} \geq 0$, the plastic strain tensor \mathbf{E}^p and the plastic hardening variable κ^p are the internal state variables, and \mathbf{Y}^p and z^p are the conjugate thermodynamic forces.

If the loading function is taken as

$$(8) \quad \blacktriangle \quad f = \sqrt{3Y_{ij}^{p'} Y_{ij}^{p'}/2} + z^p - \sigma_0 = 0, \quad Y_{ij}^{p'} = Y_{ij}^p - Y_{kk}^p \delta_{ij}/3,$$

then the following plastic flow rule is obtained as

$$(9) \quad \blacktriangle \quad \dot{E}_{ij}^p = 3\dot{\lambda}Y_{ij}^{p'}/(2\sqrt{3Y_{ij}^{p'} Y_{ij}^{p'}/2}),$$

$$(10) \quad \blacktriangle \quad \dot{\kappa}^p = \dot{\lambda} = \sqrt{2\dot{E}_{ij}^p \dot{E}_{ij}^p/3},$$

where σ_0 is the initial yield stress depending on temperature. It can be seen from Eq. (10) that there exists equivalency among the rate of the plastic hardening variable, the plastic multiplier, and the effective plastic strain rate. The consistency condition becomes

$$(11) \quad \blacktriangle \quad (\partial f / \partial Y_{ij}^{p'}) \dot{Y}_{ij}^{p'} + (\partial f / \partial z^p) \dot{z}^p + (\partial f / \partial T) \dot{T} = 0.$$

For small-strain problems, the total strain is assumed to be a sum of the elastic strain and the plastic strain, that is, $\mathbf{e} = \mathbf{e}^e + \mathbf{e}^p$. If the free energy density function can be separated as

$$(12) \quad \rho \hat{h}(\mathbf{e}, T; \mathbf{e}^p, \kappa^p) = \rho \hat{h}^e(\mathbf{e}^e, T) + \rho \hat{h}^p(\mathbf{e}^p, \kappa^p, T),$$

then we have

$$(13) \quad \mathbf{Y}^p = -\rho(\partial \hat{h} / \partial \mathbf{e}^p) = \tau - \rho(\partial \hat{h}^p / \partial \mathbf{e}^p) = \tau - \tau^b,$$

$$(14) \quad z^p = -\rho(\partial \hat{h} / \partial \kappa^p) = -\rho(\partial \hat{h}^p / \partial \kappa^p),$$

$$(15) \quad \dot{\epsilon}^p = \dot{\lambda}(\partial f / \partial \mathbf{Y}^p),$$

$$(16) \quad \dot{\kappa}^p = \dot{\lambda}(\partial f / \partial z^p),$$

where τ^b is the so-called back stress.

Hence, Eq. (8) describes a combination of kinematic and isotropic hardening (i.e., mixed hardening), as discussed in Sec. 6.12. Equation (13) defines the translation of the loading surface with plastic deformation (i.e., kinematic hardening), whereas Eq. (14) defines the expansion of the loading surface with plastic deformation (i.e., isotropic hardening).

15.4.2. Thermo-elasto-viscoplasticity

For rate-dependent thermo-elasto-viscoplastic materials, the thermodynamically admissible conditions become

$$(17) \quad \hat{h} = \hat{h}(\mathbf{E}, T; \mathbf{E}^{vp}, \mathbf{X}), \quad \hat{s} = -\partial \hat{h} / \partial T, \quad \mathbf{S} = \rho_0 \partial \hat{h} / \partial \mathbf{E},$$

$$(18) \quad \mathbf{Y}^{vp} = -\rho_0 \partial \hat{h} / \partial \mathbf{E}^{vp}, \quad \dot{\mathbf{E}}^{vp} = \partial \phi^* / \partial \mathbf{Y}^{vp},$$

where the viscoplastic strain \mathbf{E}^{vp} is the internal state variable, \mathbf{Y}^{vp} is the conjugate thermodynamic force, and ϕ^* is the viscoplastic potential.

Rate-dependent viscoplasticity has often been modeled with the overstress model or its variance (e.g., Perzyna 1971, 1980, 1983, 2001; Chaboche 1989, 1991, 1993, 2008). For small-strain problems, the total strain is assumed to be a sum of the elastic strain and the viscoplastic strain, that is, $\mathbf{e} = \mathbf{e}^e + \mathbf{e}^{vp}$. The viscoplastic strain rate is governed by

$$(19) \quad \dot{\epsilon}^{vp} = \langle \phi(\tilde{f}) \rangle (\partial F / \partial \mathbf{Y}^{vp}) / \eta,$$

where $\tilde{f} = (F - F_s)/\chi$ is the normalized loading function, F_s is the quasi-static yield strength, χ and η are material parameters, $\phi(\tilde{f})$ denotes the overstress function, and $\langle \phi(\tilde{f}) \rangle$ is defined by

$$(20) \quad \langle \phi(\tilde{f}) \rangle = \begin{cases} 0 & \text{as } \tilde{f} \leq 0, \\ \phi(\tilde{f}) & \text{as } \tilde{f} > 0. \end{cases}$$

Thus the effective viscoplastic strain rate can be obtained from Eq. (19) as

$$(21) \quad \dot{\kappa}^{vp} = \sqrt{2\dot{\mathbf{e}}^{vp} : \dot{\mathbf{e}}^{vp}/3} = \phi(\tilde{f}) |\partial F/\partial \mathbf{Y}^{vp}| \sqrt{2/3}/\eta.$$

If the inverse function of φ exists, we have

$$(22) \quad \Delta \quad F = F_s + \chi \phi^{-1} [\eta \dot{\kappa}^{vp} / (\sqrt{2/3} |\partial F/\partial \mathbf{Y}^{vp}|)],$$

which is interpreted as the dynamic yield criterion with inclusion of the strain rate effects.

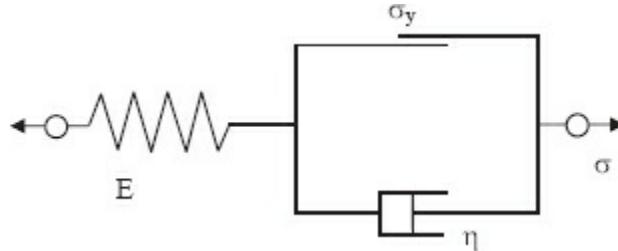


Fig. 15.4:1. Schematic of one-dimensional elasto-viscoplastic model.

As $\eta \rightarrow 0$, the rate-dependent viscoplastic model is reduced to the rate-independent plastic model, that is, the static yield criterion $F = F_s$ is satisfied. For computational advantages, a fictitious material with very low viscosity is sometimes adopted to simulate the rate-independent material behavior.

A widely used expression for the overstress function is in the power law form (see the review papers by Germain *et al.* 1983, Chaboche 2008),

$$(23) \quad \Delta \quad \phi(\tilde{f}) = a_0 \tilde{f}^n,$$

where the coefficient a_0 and the exponent n are to be determined experimentally.

A schematic of the one-dimensional elasto-viscoplastic model of the over-stress type is shown in Fig. 15.4:1. It can be seen that the total strain is equal to the sum of the elastic strain from the spring element and the viscoplastic strain from the plastic and viscous elements acting in parallel. The overstress caused by the difference between the total stress and the yield stress drives the viscoplastic flow.

Schapery (1969, 1997, 1999) introduced ISVs to formulate nonlinear vis-coelastic and viscoplastic constitutive equations based on thermodynamics. Boyce *et al.* (1988) provided rate-dependent constitutive models for describing large inelastic deformation of glassy polymers with the use of the back stress, based on multiplicative decomposition of the deformation gradient into the elastic and plastic parts (Lee 1969). Anand and Gurtin (2003) further developed an ISV theory for modeling the elasto-viscoplastic behavior of amorphous solids undergoing large deformation.

15.5. CONNECTING VISCOPLASTICITY TO VISCOELASTICITY WITH INTRINSIC TIME SCALE

The endochronic theory was first developed by Valanis (1971), who proposed the use of a positive scalar variable, which is not the Newtonian time measured by a clock but in itself is a property of the material at hand, in establishing constitutive relations to model the viscoplastic behavior of metals. It was extended to inelastic problems of other materials such as soil and concrete by Bazant (1978). Lion *et al.* (2010) studied the glass-transition in mechanical and thermal properties of glass-forming materials based on the thermodynamics with ISVs and showed that strain, temperature and ISVs could be expressed as functional of the histories of stress and temperature. This theory is unifying in the sense that many existing theories of viscoplasticity and viscoelasticity can be obtained as special cases by imposing suitable constraints on the material parameters involved. Furthermore, it does not make use of the idea of *a priori* existence of a yield surface. The phenomenon of yield may be deduced as a consequence of a particular definition of the intrinsic time measure in terms of the plastic or viscoplastic strain tensor (see Biblio. 15.5).

As described in the last subsection of Sec. 6.13, the endochronic theory is founded on the underlying principle that the material behavior is governed by constitutive relations with material constitutive invariance on intrinsic time scale. Valanis (1971) defined the intrinsic time measure as a monotonically increasing function of deformation in the following form:

$$(1) \quad d\xi^2 = P_{ijkl} dC_{ij} dC_{kl},$$

where C_{ij} is the right Cauchy–Green deformation tensor, and P_{ijkl} is a positive definite fourth-order tensor which may depend on C_{ij} .

For small-strain problems, Eq. (1) becomes

$$(2) \quad d\xi^2 = P_{ijkl} de_{ij} de_{kl},$$

where e_{ij} is the infinitesimal strain tensor, and P_{ijkl} is a positive definite fourth-order tensor which may depend on e_{ij} .

To represent more closely the dissipative nature (Valanis 1980), the total strain may be replaced by the inelastic strain in the intrinsic time measure defined as,

$$(3) \quad d\xi^2 = P_{ijkl} de_{ij}^{ie} de_{kl}^{ie},$$

where $e_{ij}^{ie} = e_{ij} - e_{ij}^e$ are the infinitesimal inelastic and elastic strain tensors, respectively, and P_{ijkl} is a positive definite fourth-order tensor which may depend on e_{ij}^e . For actual materials, it is logical to generalize the definition of the intrinsic time measure as

$$(4) \quad d\zeta^2 = \alpha^2 d\xi^2 + \beta^2 dt^2,$$

where α and β are scalar material parameters.

Then the intrinsic time scale $Z(\zeta)$ can be chosen such that

$$(5) \quad dZ/d\zeta > 0, \quad 0 < \zeta < \infty.$$

Similar to the treatment of viscoelasticity in [Chapter 14](#), the current value of the stress is expressed by the convolution of the relaxation function and the strain on the intrinsic time scale under isothermal conditions, that is,

$$(6) \quad \tau_{ij} = \int_0^Z G_{ijkl}(Z - Z') [\partial e_{kl}(Z') / \partial Z'] dZ',$$

where $G_{ijkl}(Z - Z')$ is a fourth-order tensor with

$$G_{ijkl}(Z - Z') = G_{jikl}(Z - Z') = G_{ijlk}(Z - Z'),$$

to account for the symmetry of stress and strain tensors.

It can be seen that Eq. (6) has the same form as that for viscoelastic materials formulated based on functional thermodynamics in [Chapter 14](#). As the integration function is an exponential function of the intrinsic time scale, an incremental form of endochronic flow rule exists, which is commonly used in solving plastic and viscoplastic problems (e.g., Valanis and Fan 1983).

For isotropic materials, inelastic constitutive equations may be expressed as the deviatoric and hydrostatic parts:

$$(7) \quad \tau'_{ij} = 2 \int_0^{Z_D} G(Z_D - Z'_D) (\partial e'_{ij} / \partial Z'_D) dZ'_D,$$

$$(8) \quad \tau_{kk} = 3 \int_0^{Z_H} K(Z_H - Z'_H) (\partial e_{kk} / \partial Z'_H) dZ'_H,$$

where the deviatoric and hydrostatic intrinsic time scales and intrinsic time measures are defined by

$$(9) \quad dZ_D^2 = \alpha_{00} d\zeta_D^2 + \alpha_{01} d\zeta_H^2, \quad dZ_H^2 = \alpha_{10} d\zeta_D^2 + \alpha_{11} d\zeta_H^2,$$

$$(10) \quad d\zeta_D = \sqrt{de_{ij}^{ie'} de_{ij}^{ie'}}, \quad d\zeta_H = |de_{kk}^{ie}|,$$

$$(11) \quad de_{ij}^{ie'} = de'_{ij} - de_{ij}^e, \quad de_{kk}^{ie} = de_{kk} - de_{kk}^e,$$

with α_{ij} as a matrix of non-dimensional scalars depending on ζ_D and ζ_H as well as ζ_D and ζ_H . The non-zero coefficients α_{01} and α_{10} represent the coupling between the deviatoric and hydrostatic responses.

The shear and volumetric relaxation functions may be expressed as

$$(12) \quad G(Z_D) = \sum_r G_r e^{-A_r Z_D}, \quad K(Z_H) = \sum_r K_r e^{-B_r Z_H}.$$

As a result, Eqs. (7) and (8) can be rewritten as (Valanis 1978)

$$(13) \quad \tau'_{ij} = S_Y^0 (de_{ij}^{ie'}/dZ_D) + \tau_{ij}^{b'},$$

$$(14) \quad \tau_{kk} = 3\sigma_Y^0 (de_{kk}^{ie}/dZ_H) + \tau_{kk}^b,$$

$$(15) \quad \tau_{ij}^{b'} = 2G_0 \int_0^{Z_D} \rho_1(Z_D - Z'_D) (\partial e_{ij}^{ie'}/\partial Z'_D) dZ'_D,$$

$$(16) \quad \tau_{kk}^b = 3K_0 \int_0^{Z_H} \varphi_1(Z_H - Z'_H) (\partial e_{kk}^{ie}/\partial Z'_H) dZ'_H.$$

Equations (13) and (14) describe the effects of mixed hardening on deviatoric and hydrostatic responses with the first term on the right-hand side representing isotropic hardening and the second term representing kinematic hardening. Equations (15) and (16) define the deviatoric and hydrostatic back stress.

Hence, the loading conditions are expressed by

$$(17) \quad \sqrt{(\tau'_{ij} - \tau_{ij}^{b'})(\tau'_{ij} - \tau_{ij}^{b'})} = S_Y^0 (d\zeta_D/dZ_D),$$

$$(18) \quad |\tau_{kk} - \tau_{kk}^b| = 3\sigma_Y^0 (d\zeta_H/dZ_H).$$

It can be seen that the loading surface is capable of both translation and expansion with the back stress specifying the translation of the loading surface and the derivative of the intrinsic time measure with respect to the intrinsic time scale specifying the expansion of the loading surface. The expansion of the loading surface may depend on the inelastic strain for rate-independent materials or both the inelastic strain and the inelastic strain rate for rate-dependent materials.

The endochronic flow rule is given by

$$(19) \quad de_{ij}^{ie'} = (\tau'_{ij} - \tau_{ij}^{b'})dZ_D/S_Y^0, \quad de_{kk}^{ie} = (\tau_{kk} - \tau_{kk}^b)dZ_H/3\sigma_Y^0.$$

The reader may refer to the papers by Erlicher and Point (2006, 2008) for further information.

P R O B L E M S

5.1. Find the flow rule for a viscoplastic material at small deformation under isothermal condition with the dynamic yield criterion given by $F = (3Y_{ij}^{vp'} Y_{ij}^{vp'}/2)^{1/2} = \sigma_0 + H^i \kappa^{vp} + \eta_0 \dot{\kappa}^{vp}$, where $Y_{ij}^{vp'}$ are the deviatoric components of $\mathbf{Y}^{vp} = \boldsymbol{\tau} - \boldsymbol{\tau}^b$, $\boldsymbol{\tau}^b = \mathbf{H}^k : \boldsymbol{\varepsilon}^{vp}$ is the back stress, \mathbf{H}^k is the kinematic hardening modulus, $\kappa^{vp} = \int_0^t \dot{\kappa}^{vp} dt$ is the accumulated effective viscoplastic strain, H^i is the isotropic hardening modulus, σ_0 is the initial yield strength, and η_0 is viscosity.

15.2. Show the flow rule for an elastoplastic material is recovered as the viscosity η_0 tends to zero in Problem 15.1.

15.3. Find the flow rule for a viscoplastic material at small deformation under isothermal condition with the viscoplastic potential given by $\phi^*(\tilde{f}) = \zeta_0 \chi_0 \tilde{f}^{n+1}/(n+1)$, where ζ_0 and χ_0 are material parameters, and $\tilde{f} = [(3Y_{ij}^{vp'} Y_{ij}^{vp'}/2)^{1/2} - \sigma_0 - H^i \kappa^{vp}]/\chi_0$ is the normalized loading function.

15.4. Derive Eqs. (13)–(16) from Eqs. (7) and (8) using (9)–(12) (Valanis 1978).

16

ELECTRO-THERMO-VISCOELASTICITY/VISCOPLASTICITY

In this chapter, we shall extend the formulation developed in [Chapters 14](#) and [15](#) to coupled electric, thermal and mechanical processes in thermodynamic nonequilibrium.

16.1. INTRODUCTION

Electrodynamics is a branch of physics which studies electric charges in motion, whereas mechanics is the science of force and motion of matter. At the continuum level, there are ten fundamental laws, namely (1) conservation of mass, (2) conservation of linear momentum, (3) conservation of angular momentum, (4) conservation of energy, (5) conservation of electric charges, (6) entropy production inequality, (7) Gauss' law, (8) Faraday's law, (9) Gauss' law for magnetism, and (10) Ampere's law. These general physical laws are insufficient for formulating a deterministic problem. It is necessary to specify the material laws, which rest upon the axioms within the framework of continuum mechanics. Energy can be converted from one form to another due to interactions among magnetic, electric, thermal, and mechanical effects. Because of the increasing applications of magneto- and electrosensitive materials at relatively high load and large deformation, e.g., electroactive polymers used in adaptive control systems and robotic arms (see [Fig. 16.1:1](#)), nonlinear magneto- and electro-thermo-viscoelastic/viscoplastic problems have attracted considerable attention (see [Biblio. 16.1](#)). The effects of hysteresis and aging need to be considered in analyzing the time-dependent responses under combined magnetic, electric, thermal, and mechanical loads. Many aspects of magneto- and electro-thermo-viscoelastic deformation and fracture can be found in the book by Chen and Mai (2012).

16.2. PHYSICAL NOTATIONS

16.2.1. Electromagnetic field quantities

The electromagnetic field can be viewed as a combination of the electric and magnetic fields, represented mathematically as vectors. The electric charge in a body may be positive or negative. The motion of charged particles in a given direction is known as electric current. Since a high velocity close to light speed is not easily achievable in solids, the Galilean approximation is adopted hereafter instead of the relativistic treatment. The reader may refer to the books by Landau and Lifshitz (1960), Truesdell and Toupin (1960) and Eringen (1980) for more detailed descriptions.



Fig. 16.1:1. The icon of the grand challenge for the development of electroactive polymer. (From Bar-Cohen, 2012, with permission from The Korean Society for Aeronautical & Space Sciences.)

In the fixed Galilean frame R_G , also referred to as the laboratory frame, the electromagnetic field variables include the polarization \mathbf{P} , the electric field \mathbf{E} , the electric displacement $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, the magnetization \mathbf{M} , the magnetic induction \mathbf{B} , the magnetic field $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$, and the total electric current \mathbf{j}_e , where ϵ_0 is the vacuum permittivity and μ_0 is the vacuum permeability.

The field quantities in the co-moving frame R_C are given by

$$(1) \quad \mathbf{E} = \mathbf{E} + \mathbf{v} \times \mathbf{B},$$

$$(2) \quad \mathbf{M} = \mathbf{M} + \mathbf{v} \times \mathbf{P},$$

(3)

$$\mathbf{H} = \mathbf{H} - \mathbf{v} \times \mathbf{D},$$

(4)

$$\mathbf{j}_e = \mathbf{j}_e - q_f \mathbf{v},$$

where E is the electromotive intensity, j_e is the conduction current, and q_f is the free electric charge density.

16.2.2. Electromagnetic body force and couple

The force that the electromagnetic field exerts on an electrically charged particle is called the electromagnetic force, which is one of the fundamental forces in nature. The other fundamental forces are the strong interaction, the weak interaction, and the gravitational forces. All other forces are ultimately derived from these four fundamental forces. The Lorentz force law that describes the force acting on a point charge due to the electromagnetic field is used to construct the expressions for the electromagnetic body force and couple in continuous media.

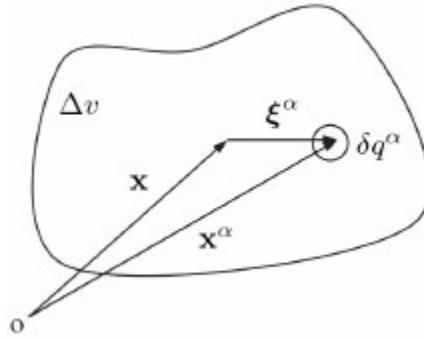


Fig. 16.2:1. Volume element with moving charges.

The Lorentz force acting on a point charge δq^α in a volume element (see Fig. 16.2:1) is

$$(5) \quad \delta \mathbf{f}^\alpha = \delta q^\alpha [\mathbf{e}(\mathbf{x}^\alpha) + \mathbf{v}^\alpha \times \mathbf{b}(\mathbf{x}^\alpha)],$$

where $\mathbf{e}(\mathbf{x}^\alpha)$ and $\mathbf{b}(\mathbf{x}^\alpha)$ are the microscopic electric field and the microscopic magnetic induction at \mathbf{x}^α , respectively.

The electromagnetic force and the electromagnetic couple acting on the volume element are given by, respectively,

$$(6) \quad {}_{em}\mathbf{f} \Delta v = \sum_\alpha \delta q^\alpha \mathbf{e}(\mathbf{x} + \xi^\alpha) + \sum_\alpha \delta q^\alpha (\mathbf{v} + \dot{\xi}^\alpha + \dot{\hat{\xi}}^\alpha) \times \mathbf{b}(\mathbf{x} + \xi^\alpha),$$

$$(7) \quad {}_{em}\mathbf{l} \Delta v = \sum_\alpha \delta q^\alpha (\mathbf{x} + \xi^\alpha) \times \mathbf{e}(\mathbf{x} + \xi^\alpha) \\ + \sum_\alpha \delta q^\alpha (\mathbf{x} + \xi^\alpha) \times [(\mathbf{v} + \dot{\xi}^\alpha + \dot{\hat{\xi}}^\alpha) \times \mathbf{b}(\mathbf{x} + \xi^\alpha)],$$

where $\mathbf{x} + \xi^\alpha$ is the average position of the point charge δq^α and $\dot{\hat{\xi}}^\alpha$ is the fluctuation velocity.

The macroscopic electromagnetic field quantities are defined by

$$(8) \quad \mathbf{E}(\mathbf{x}) = \mathbf{e}(\mathbf{x}),$$

$$(9) \quad \mathbf{B}(\mathbf{x}) = \mathbf{b}(\mathbf{x}),$$

$$(10) \quad q_f \Delta v = \sum_\alpha \delta q^\alpha,$$

$$(11) \quad \mathbf{P} \Delta v = \sum_\alpha \delta q^\alpha \xi^\alpha,$$

$$(12) \quad M \Delta v = \sum_\alpha \delta q^\alpha \xi^\alpha \times \dot{\xi}^\alpha / 2,$$

$$(13) \quad j_e \Delta v = \sum_\alpha \delta q^\alpha \dot{\hat{\xi}}^\alpha.$$

Thus, the expressions for the electromagnetic body force ${}_{em}\mathbf{f}$ and the electro-magnetic body couple ${}_{em}\mathbf{c}$ are

$$(14) \quad {}_{em}\mathbf{f} = q_f \mathbf{E} + (j_e + \dot{\mathbf{P}}) \times \mathbf{B} + (\mathbf{P} \cdot \nabla) \mathbf{E} + (\nabla \mathbf{B}) \cdot \mathbf{M},$$

$$(15) \quad {}_{em}\mathbf{c} = {}_{em}\mathbf{l} - \mathbf{x} \times {}_{em}\mathbf{f} = \mathbf{P} \times \mathbf{E} + \mathbf{M} \times \mathbf{B},$$

where the convective time derivative denoted by an overhead asterisk is defined as

$$(16) \quad \dot{\mathbf{P}} = \dot{\mathbf{P}} - (\mathbf{P} \cdot \nabla) \mathbf{v} + \mathbf{P}(\nabla \cdot \mathbf{v}).$$

Hence, polarization is related to the electric dipole moment, magnetization is related to the magnetic moment, and conduction current is related to the fluctuation velocity of charges. A material is electrically polarized if \mathbf{P} is nonzero, whereas a material is magnetized if \mathbf{M} is nonzero. Materials can also be classified as conducting, semiconducting, or insulating, based on their ability to conduct electric current.

16.2.3. Electromagnetic stress tensor and momentum vector

There exist an electromagnetic stress tensor ${}_{em}\tau_{ij}$ and an electromagnetic momentum vector \mathbf{G}_k (Collet and Maugin 1974) such that

$$(17) \quad {}_{em}f_k = {}_{em}\tau_{ik,i} - \partial G_k / \partial t,$$

$$(18) \quad {}_{em}c_k = \varepsilon_{kij} {}_{em}\tau_{ij}.$$

One solution is

$$(19) \quad {}_{em}\tau = \mathbf{P} \otimes \mathbf{E} - \mathbf{B} \otimes \mathbf{M} + (\mathbf{M} \cdot \mathbf{B})\mathbf{I} + \varepsilon_0 \mathbf{E} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{B}/\mu_0 - {}_{em}u^f \mathbf{I},$$

$$(20) \quad \mathbf{G} = \varepsilon_0 \mathbf{E} \times \mathbf{B},$$

where the symbol \otimes is the dyadic product denoting the outer product of two vectors (see Sec. 13.1), ${}_{em}u^f = \varepsilon_0 \mathbf{E} \cdot \mathbf{E}/2 + \mathbf{B} \cdot \mathbf{B}/(2\mu_0)$ is the energy density of the free electromagnetic field, and \mathbf{I} is the second-order unit tensor.

Hence, with the use of the electromagnetic stress tensor ${}_{em}\tau$ and the electromagnetic momentum vector \mathbf{G} , the local linear momentum and angular momentum balance equations can be expressed as

$$(21) \quad \rho dv_k / dt = \tau_{ik,i} + {}_{em}\tau_{ik,i} + \rho b_k - \partial G_k / \partial t,$$

$$(22) \quad \varepsilon_{kij}(\tau_{ij} + {}_{em}\tau_{ij}) = 0,$$

where v_k is the velocity vector and ε_{kij} the permutation tensor. Thus, the total stress tensor ${}_t\tau = \tau + {}_{em}\tau$ is symmetric, that is, ${}_t\tau_{ij} = {}_t\tau_{ji}$, although the Cauchy stress tensor τ or the electromagnetic stress tensor ${}_{em}\tau$ may not be symmetric.

By introducing the Maxwell stress tensor involving only the free electromagnetic field

$$(23) \quad {}_F\tau = \varepsilon_0 \mathbf{E} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{B}/\mu_0 - {}_{em}u^f \mathbf{I},$$

the electromagnetic stress tensor ${}_{em}\tau$ can be decomposed as

$$(24) \quad {}_{em}\tau = {}_F\tau + {}_{em}\bar{\tau},$$

with

$$(25) \quad {}_{em}\bar{\tau} = \mathbf{P} \otimes \mathbf{E} - \mathbf{B} \otimes \mathbf{M} + (\mathbf{M} \cdot \mathbf{B})\mathbf{I}.$$

The electromagnetic body force and couple can also be expressed as

$$(26) \quad {}_{em}f_k = L\bar{f}_k + {}_{em}\bar{\tau}_{ik,i},$$

$$(27) \quad {}_{em}c_k = \varepsilon_{kij} {}_{em}\bar{\tau}_{ij},$$

where $L\bar{f} = q^{eff}E + j_e^{eff} \times \mathbf{B} = \nabla \cdot {}_F\tau - \partial \mathbf{G} / \partial t$ is the effective Lorentz force, $q^{eff} = q_f - \nabla \cdot \mathbf{P}$ is the effective charge density, and $j_e^{eff} = j_e + \dot{\mathbf{P}} + \nabla \times \mathbf{M}$ is the effective current.

16.2.4. Electromagnetic power

The electromagnetic power is the rate of work done by the electromagnetic forces,

$$(28) \quad {}_{em}w\Delta v = \sum_{\alpha} \delta q^{\alpha}(\mathbf{v} + \dot{\xi}^{\alpha} + \ddot{\xi}^{\alpha}) \cdot \mathbf{e}(\mathbf{x} + \xi^{\alpha}).$$

The useful equivalent expressions for the electromagnetic power density in terms of different time derivatives are listed as follows:

$$(29) \quad {}_{em}w = \mathbf{E} \cdot \partial \mathbf{P} / \partial t - \mathbf{M} \cdot \partial \mathbf{B} / \partial t + \nabla \cdot [\mathbf{v}(\mathbf{E} \cdot \mathbf{P})] + \mathbf{j}_e \cdot \mathbf{E},$$

$$(30) \quad {}_{em}w = {}_{em}\mathbf{f} \cdot \mathbf{v} + \rho \mathbf{E} \cdot \dot{\pi} - \mathbf{M} \cdot \dot{\mathbf{B}} + \mathbf{j}_e \cdot \mathbf{E},$$

$$(31) \quad {}_{em}w = {}_{em}\mathbf{f} \cdot \mathbf{v} + {}_{em}\mathbf{c} \cdot \omega + {}_{em}\bar{\tau} : \mathbf{d} + \mathbf{E} \cdot \mathbf{\hat{P}} - \mathbf{M} \cdot \mathbf{\hat{B}} + \mathbf{j}_e \cdot \mathbf{E},$$

where $\pi = \mathbf{P}/\rho$ is the polarization per unit mass and ρ is the mass density.

The Poynting vector, which represents the flux of the electromagnetic energy, is denoted by $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ in the laboratory frame R_G and by $S = E \times H$ in the co-moving frame R_C .

The Poynting theorem in R_G gives the identity

$$(32) \quad \mathbf{H} \cdot \partial \mathbf{B} / \partial t + \mathbf{E} \cdot \partial \mathbf{D} / \partial t = -\mathbf{j}_e \cdot \mathbf{E} - \nabla \cdot \mathbf{S}.$$

With this identity, the electromagnetic power density can be rewritten in a new form

$$(33) \quad {}_{em}w = -\partial_{em}u^f / \partial t - \nabla \cdot [\mathbf{S} - \mathbf{v}(\mathbf{E} \cdot \mathbf{P})].$$

The Poynting theorem in R_C gives the identity

$$(34) \quad \mathbf{H} \cdot \mathbf{\hat{B}} + \mathbf{E} \cdot \mathbf{\hat{D}} = -\mathbf{j}_e \cdot \mathbf{E} - \nabla \cdot \mathbf{S}.$$

With the use of this identity, the electromagnetic power density can be rewritten in another form

$$(35) \quad {}_{em}w = -\rho[D({}_{em}u^f / \rho) / Dt] + \nabla \cdot [({}_{em}\tau + \mathbf{v} \otimes \mathbf{G}) \cdot \mathbf{v} - S].$$

Hence, the local energy balance equation can be expressed as

$$(36) \quad \rho[D(\hat{k} + \hat{e} + {}_{em}u^f / \rho) / Dt] = -\nabla \cdot \mathbf{h} + \nabla \cdot [({}_t\tau + \mathbf{v} \otimes \mathbf{G}) \cdot \mathbf{v} - S] + \rho \mathbf{b} \cdot \mathbf{v}.$$

16.3. BASIC FIELD EQUATIONS FOR ELECTROSENSITIVE MATERIALS

Under the quasi-electrostatic approximation, i.e., near-absence of timevarying magnetic field, the local balance equations for electrosensitive materials are given in a fixed Galilean frame R_G as follows:

$$(1) \quad \blacktriangle \quad \nabla \cdot \mathbf{D} = q_f,$$

$$(2) \quad \blacktriangle \quad \nabla \times \mathbf{E} = 0,$$

$$(3) \quad \blacktriangle \quad -\partial q_f / \partial t = \nabla \cdot \mathbf{j}_e,$$

$$(4) \quad \blacktriangle \quad D\rho / Dt = -\rho \nabla \cdot \mathbf{v},$$

$$(5) \quad \blacktriangle \quad \rho D\mathbf{v} / Dt = \nabla \cdot {}_t\tau + \rho \mathbf{b},$$

$$(6) \quad \blacktriangle \quad {}_t\tau^T = {}_t\tau,$$

$$(7) \quad \blacktriangle \quad \rho[D(\hat{k} + \hat{e} + {}_em u^f / \rho) / Dt] = -\nabla \cdot \mathbf{h} + \nabla \cdot ({}_t\tau \cdot \mathbf{v} - S) + \rho \mathbf{b} \cdot \mathbf{v},$$

where ${}_t\tau = \tau + {}_e\tau$ is the total stress tensor as a sum of the Cauchy stress tensor τ and the electric stress tensor ${}_e\tau = \mathbf{D} \otimes \mathbf{E} - {}_em u^f \mathbf{I}$, ${}_em u^f = \epsilon_0 \mathbf{E} \cdot \mathbf{E} / 2$ is the energy density of the free electric field, and $S = -\mathbf{E} \times (\mathbf{v} \times \mathbf{D})$ is the Poynting vector in the co-moving frame R_C .

As described in Sec. 12.4, the entropy production inequality is given by

$$(8) \quad D_i \hat{s} / Dt = D \hat{s} / Dt + \nabla \cdot \mathbf{j}^s / \rho \geq 0.$$

16.4. AUGMENTED HELMHOLTZ AND GIBBS FREE ENERGY FUNCTIONALS

16.4.1. Expansion of augmented Helmholtz free energy functional

With the introduction of the augmented Helmholtz free energy including the contribution of the energy of the free electric field,

$$(1) \quad \hat{h} = \hat{e} - T\hat{s} + {}_e u^f / \rho,$$

the local energy balance equation becomes

$$(2) \quad \rho[D(\hat{k} + \tilde{h} + T\hat{s})/Dt] = -\nabla \cdot \mathbf{h} + \nabla \cdot ({}_t\tau \cdot \mathbf{v} - S) + \rho \mathbf{b} \cdot \mathbf{v}.$$

In the reference configuration V_R , the local energy balance equation can be rewritten as

$$(3) \quad \rho_0 D\hat{s}/Dt = -\nabla_R \cdot (\mathbf{H}/T) + \mathbf{H} \cdot \nabla_R(1/T) + {}_t\Sigma : \dot{\mathbf{C}}/(2T) + \hat{\mathbf{E}} \cdot \dot{\hat{\mathbf{D}}}/T + J_e \cdot \hat{\mathbf{E}}/T - \rho_0 \hat{s}\dot{T}/T - \rho_0(D\tilde{h}/Dt)/T,$$

where ${}_t\Sigma = J\mathbf{F}^{-1} \cdot {}_t\tau \cdot \mathbf{F}^{-T}$ is the second Piola-Kirchhoff total stress tensor, $J_e = J\mathbf{F}^{-1} \cdot j_e$, $\hat{\mathbf{D}} = J\mathbf{F}^{-1} \cdot \mathbf{D}$, and $\hat{\mathbf{E}} = \mathbf{E} \cdot \mathbf{F}$. In the same reference configuration V_R , the entropy production inequality is rewritten as

$$(4) \quad D_i\hat{s}/Dt \equiv D\hat{s}/Dt + \nabla_R \cdot \mathbf{J}^s/\rho_0 \geq 0.$$

For electro-sensitive materials, the augmented Helmholtz free energy including the contribution of the energy of the free electric field is taken to be a functional of the histories of deformation, temperature, and electric displacement in V_R on the intrinsic time scale. Using the Lagrange strain measure $\mathbf{E} = (\mathbf{C} - \mathbf{I})/2$, the temperature deviation $\theta = T - T_0$, and the Lagrange electric displacement deviation $\hat{\mathbf{d}} = \hat{\mathbf{D}} - \hat{\mathbf{D}}_0$, and expanding the augmented Helmholtz free energy functional up to the second order yield

$$(5) \quad \begin{aligned} \rho_0 \tilde{h} &= \rho_0 \tilde{h}_0 + \int_{-\infty}^{\psi} \{ L_{IJ}^M(\mathbf{X}, \psi - \psi') [\partial E_{IJ}(\mathbf{X}, \psi') / \partial \psi'] \\ &\quad - M^M(\mathbf{X}, \psi - \psi') [\partial \theta(\mathbf{X}, \psi') / \partial \psi'] + N_I^M(\mathbf{X}, \psi - \psi') [\partial \hat{d}_I(\mathbf{X}, \psi') / \partial \psi'] \} d\psi' \\ &\quad + \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \{ -\beta_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial E_{IJ}(\mathbf{X}, \psi') / \partial \psi'] [\partial \theta(\mathbf{X}, \psi'') / \partial \psi''] \\ &\quad + G_{IJKL}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial E_{IJ}(\mathbf{X}, \psi') / \partial \psi'] [\partial E_{KL}(\mathbf{X}, \psi'') / \partial \psi''] / 2 \\ &\quad - f_{KIJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial E_{IJ}(\mathbf{X}, \psi') / \partial \psi'] [\partial \hat{d}_K(\mathbf{X}, \psi'') / \partial \psi''] \\ &\quad - C_H^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial \theta(\mathbf{X}, \psi') / \partial \psi'] [\partial \theta(\mathbf{X}, \psi'') / \partial \psi''] / (2T_0) \\ &\quad - \gamma_I^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial \hat{d}_I(\mathbf{X}, \psi') / \partial \psi'] [\partial \theta(\mathbf{X}, \psi'') / \partial \psi''] \\ &\quad + \bar{\chi}_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial \hat{d}_I(\mathbf{X}, \psi') / \partial \psi'] [\partial \hat{d}_J(\mathbf{X}, \psi'') / \partial \psi''] / 2 \} d\psi' d\psi'', \end{aligned}$$

where \tilde{h}_0 is the augmented Helmholtz free energy per unit mass in the reference state,

$$\begin{aligned} G_{IJKL}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') &= G_{KLIJ}^M(\mathbf{X}, \psi - \psi'', \psi - \psi'), \\ C_H^M(\mathbf{X}, \psi - \psi', \psi - \psi'') &= C_H^M(\mathbf{X}, \psi - \psi'', \psi - \psi'), \\ \bar{\chi}_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') &= \bar{\chi}_{JI}^M(\mathbf{X}, \psi - \psi'', \psi - \psi'). \end{aligned}$$

The intrinsic time scale can be used to handle electro-thermo-viscoelastic and electro-thermo-viscoplastic problems from a unified point of view. Similar to the endochronic theory as described in Sec. 15.5, the intrinsic time scale ψ may be related to Newtonian time t and intrinsic time measure ξ such that

$$(6) \quad d\psi^2 = \alpha_H^2 d\xi^2 + \beta_H^2 dt^2,$$

where α_H is a material function related to electro-thermo-viscoplastic process, and β_H is that related to electro-thermo-viscoelastic process. If α_H vanishes, the above definition can be taken as an extension of the definition of the intrinsic time scale for thermo-viscoelastic problems in Chapter 14.

16.4.2. Expansion of augmented Gibbs free energy functional

With the introduction of the augmented Gibbs free energy including the contribution of the free electric field, that is,

$$(7) \quad \tilde{g} = \hat{e} - T\hat{s} + {}_e u^f / \rho - {}_t \Sigma : \mathbf{E} / \rho_0 - \hat{\mathbf{E}} \cdot \hat{\mathbf{D}} / \rho_0,$$

the local energy balance equation can be rewritten in the reference configuration as

$$(8) \quad \rho_0 D\hat{s}/Dt = -\nabla_R \cdot (\mathbf{H}/T) - \mathbf{H} \cdot (\nabla_R T)/T^2 - {}_t \dot{\Sigma} : \mathbf{E}/T \\ - \dot{\hat{\mathbf{E}}} \cdot \hat{\mathbf{D}}/T + J^e \cdot \hat{\mathbf{E}}/T - \rho_0 \hat{s}\dot{T}/T - \rho_0 (D\tilde{g}/Dt)/T.$$

For electro-sensitive materials, the augmented Gibbs free energy including the contribution of the free electric field is taken to be a functional of the histories of total stress, temperature, and electric field in the reference configuration V_R on intrinsic time scale. Using the second Piola–Kirchhoff total stress tensor ${}_t \Sigma = J\mathbf{F}^{-1} \cdot {}_t \tau \cdot \mathbf{F}^{-T}$, the temperature deviation $\theta = T - T_0$, and the Lagrange electric field $\hat{\mathbf{E}} = \mathbf{E} \cdot \mathbf{F}$, and expanding the augmented Gibbs free energy functional up to the second order yields

$$(9) \quad -\rho_0 \tilde{g} = -\rho_0 \tilde{g}_0 + \int_{-\infty}^{\psi} \{ A_{IJ}^M(\mathbf{X}, \psi - \psi') [\partial_t \Sigma_{IJ}(\mathbf{X}, \psi') / \partial \psi'] \\ + B^M(\mathbf{X}, \psi - \psi') [\partial \theta(\mathbf{X}, \psi') / \partial \psi'] + C_I^M(\mathbf{X}, \psi - \psi') [\partial \hat{E}_I(\mathbf{X}, \psi') / \partial \psi'] \} d\psi' \\ + \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \{ \alpha_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial_t \Sigma_{IJ}(\mathbf{X}, \psi') / \partial \psi'] [\partial \theta(\mathbf{X}, \psi'') / \partial \psi''] \\ + g_{KIJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial_t \Sigma_{IJ}(\mathbf{X}, \psi') / \partial \psi'] [\partial \hat{E}_K(\mathbf{X}, \psi'') / \partial \psi''] \\ + J_{IJKL}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial_t \Sigma_{IJ}(\mathbf{X}, \psi') / \partial \psi'] [\partial_t \Sigma_{KL}(\mathbf{X}, \psi'') / \partial \psi''] / 2 \\ + C_G^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial \theta(\mathbf{X}, \psi') / \partial \psi'] [\partial \theta(\mathbf{X}, \psi'') / \partial \psi''] / (2T_0) \\ + \eta_I^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial \hat{E}_I(\mathbf{X}, \psi') / \partial \psi'] [\partial \theta(\mathbf{X}, \psi'') / \partial \psi''] \\ + \kappa_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') [\partial \hat{E}_I(\mathbf{X}, \psi') / \partial \psi'] [\partial \hat{E}_J(\mathbf{X}, \psi'') / \partial \psi''] / 2 \} d\psi' d\psi'',$$

where \tilde{g}_0 is the value of the augmented Gibbs free energy per unit mass in the reference state, and

$$J_{IJKL}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') = J_{KLIJ}^M(\mathbf{X}, \psi - \psi'', \psi - \psi'), \\ C_G^M(\mathbf{X}, \psi - \psi', \psi - \psi'') = C_G^M(\mathbf{X}, \psi - \psi'', \psi - \psi'), \\ \kappa_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') = \kappa_{JI}^M(\mathbf{X}, \psi - \psi'', \psi - \psi').$$

The intrinsic time scale can be used to handle electro-thermo-viscoelastic and electro-thermo-viscoplastic problems from a unified point of view. Similar to the endochronic theory as described in Sec. 15.5, the intrinsic time scale ψ may be related to Newtonian time t and intrinsic time measure ξ such that

$$(10) \quad d\psi^2 = \alpha_G^2 d\xi^2 + \beta_G^2 dt^2,$$

where α_G is a material function related to electro-thermo-viscoplastic process, and β_G is a material function related to electro-thermo-viscoelastic process. If α_G vanishes, the above definition can be taken as an extension of the definition of the intrinsic time scale for thermo-viscoelastic problems in Chapter 14.

16.5. FINITE ELECTRO-THERMO-VISCOELASTICITY

16.5.1. Finite electro-thermo-viscoelasticity based on augmented Helmholtz free energy functional

From the entropy production inequality, state equations should satisfy the following conditions of thermodynamic admissibility based on the augmented Helmholtz free energy functional:

$$(1) \quad {}_t \Sigma_{IJ} = L_{IJ}^0 + \int_{-\infty}^{\psi} \{ G_{IJKL}^M(\mathbf{X}, 0, \psi - \psi') [\partial E_{KL}(\mathbf{X}, \psi') / \partial \psi'] d\psi' \\ - \beta_{IJ}^M(\mathbf{X}, 0, \psi - \psi') [\partial \theta(\mathbf{X}, \psi') / \partial \psi'] - f_{KIJ}^M(\mathbf{X}, 0, \psi - \psi') [\partial \hat{d}_K(\mathbf{X}, \psi') / \partial \psi'] \} d\psi',$$

$$(2) \quad \Delta \rho \hat{s} = M^0 + \int_{-\infty}^{\psi} \{ \beta_{IJ}^M(\mathbf{X}, \psi - \psi', 0) [\partial E_{IJ}(\mathbf{X}, \psi') / \partial \psi'] \\ + C_H^M(\mathbf{X}, \psi - \psi', 0) [\partial \theta(\mathbf{X}, \psi') / \partial \psi'] / T_0 + \gamma_I^M(\mathbf{X}, \psi - \psi', 0) [\partial \hat{d}_I(\mathbf{X}, \psi') / \partial \psi'] \} d\psi',$$

$$(3) \quad \Delta \hat{E}_I = N_I^0 + \int_{-\infty}^{\psi} \{ -f_{IJK}^M(\mathbf{X}, \psi - \psi', 0) [\partial E_{JK}(\mathbf{X}, \psi') / \partial \psi'] \\ - \gamma_I^M(\mathbf{X}, 0, \psi - \psi') [\partial \theta(\mathbf{X}, \psi') / \partial \psi'] + \bar{\chi}_{IJ}^M(\mathbf{X}, 0, \psi - \psi') [\partial \hat{d}_J(\mathbf{X}, \psi') / \partial \psi'] \} d\psi',$$

$$(4) \quad \mathbf{J}^s = \mathbf{H}/T,$$

$$(5) \quad \rho_0 D_i \hat{s} / Dt = \mathbf{H} \cdot (-\nabla_R T) / T^2 + \mathbf{J}^e \cdot \hat{\mathbf{E}} / T + \Lambda / T \geq 0,$$

where G_{IJKL}^M , C_H^M , $\bar{\chi}_{IJ}^M$, β_{IJ}^M , f_{KIJ}^M , and γ_I^M are thermodynamic property functions. Since E_{IJ} and $t\Sigma_{IJ}$ are symmetric tensors, we also have $G_{IJKL}^M = G_{JIKL}^M = G_{IJKL}^M$, $\beta_{IJ}^M = \beta_{JI}^M$, $f_{KIJ}^M = f_{KJI}^M$. The intrinsic dissipation rate is given by

$$(6) \quad \Lambda = \dot{\psi} \left\{ \int_{-\infty}^{\psi} \{ -\partial L_J^M(\mathbf{X}, \psi - \psi') / \partial \psi [\partial E_{IJ}(\mathbf{X}, \psi') / \partial \psi'] \right. \\ + [\partial M^M(\mathbf{X}, \psi - \psi') / \partial \psi] [\partial \theta(\mathbf{X}, \psi') / \partial \psi'] \\ - [\partial N_I^M(\mathbf{X}, \psi - \psi') / \partial \psi] [\partial \hat{d}_I(\mathbf{X}, \psi') / \partial \psi'] \} d\psi' \\ + \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \{ [\partial \beta_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') / \partial \psi] [\partial E_{IJ}(\mathbf{X}, \psi') / \partial \psi'] [\partial \theta(\mathbf{X}, \psi'') / \partial \psi''] \\ - [\partial G_{IJKL}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') / \partial \psi] [\partial E_{IJ}(\mathbf{X}, \psi') / \partial \psi'] [\partial E_{KL}(\mathbf{X}, \psi'') / \partial \psi''] / 2 \\ + [\partial \beta_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') / \partial \psi] [\partial E_{IJ}(\mathbf{X}, \psi') / \partial \psi'] [\partial \theta(\mathbf{X}, \psi'') / \partial \psi''] \\ + [\partial f_{KIJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') / \partial \psi] [\partial E_{IJ}(\mathbf{X}, \psi') / \partial \psi'] [\partial \hat{d}_K(\mathbf{X}, \psi'') / \partial \psi''] \\ + [\partial C_H^M(\mathbf{X}, \psi - \psi', \psi - \psi'') / \partial \psi] [\partial \theta(\mathbf{X}, \psi') / \partial \psi'] [\partial \theta(\mathbf{X}, \psi'') / \partial \psi''] / (2T_0) \\ - [\partial \bar{\chi}_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'') / \partial \psi] [\partial \hat{d}_I(\mathbf{X}, \psi') / \partial \psi'] [\partial \hat{d}_J(\mathbf{X}, \psi'') / \partial \psi''] / 2 \\ \left. + [\partial \gamma_I^M(\mathbf{X}, \psi - \psi', \psi - \psi'') / \partial \psi] [\partial \hat{d}_I(\mathbf{X}, \psi') / \partial \psi'] [\partial \theta(\mathbf{X}, \psi'') / \partial \psi''] \right\} d\psi' d\psi''.$$

The terms L_{IJ}^0 , M^0 , and N_I^0 on the right-hand sides of Eq. (1)–(3) are the values of $t\Sigma_{IJ}$, $\rho_0 \hat{s}$, and \hat{E}_I in the reference state; the first terms in the multiple integrals denote the mechanical contribution, the second terms the thermal contribution, and the third terms the electric contribution. It is shown from that the total dissipation rate is associated with heat conduction, electrical conduction and intrinsic dissipative processes. Since the entropy production inequality (5) must always be satisfied, the kinetic laws for specific irreversible processes may be determined accordingly.

The intrinsic dissipation rate satisfies the inequality,

$$(7) \quad \Lambda \geq 0.$$

The transport laws of heat conduction and electrical conduction may be derived from a dissipation potential, which is taken as a quadratic scalar function of the thermodynamic forces $-\nabla_R T/T$ and $\hat{\mathbf{E}}/T$, that is,

$$(8) \quad \mathcal{D} = \frac{(-\nabla_R T)}{2T} \cdot \mathbf{L}^{qq} \cdot \frac{(-\nabla_R T)}{T} - \frac{\nabla_R T}{T} \cdot \mathbf{L}^{qe} \cdot \frac{\hat{\mathbf{E}}}{T} + \frac{\hat{\mathbf{E}}}{2T} \cdot \mathbf{L}^{ee} \cdot \frac{\hat{\mathbf{E}}}{T},$$

where the conduction coefficient matrix

$$(9) \quad \begin{bmatrix} \mathbf{L}^{qq} & \mathbf{L}^{qe} \\ \mathbf{L}^{eq} & \mathbf{L}^{ee} \end{bmatrix}^T = \begin{bmatrix} \mathbf{L}^{qq} & \mathbf{L}^{qe} \\ \mathbf{L}^{eq} & \mathbf{L}^{ee} \end{bmatrix}$$

is nonnegative-definite.

Hence, the thermodynamic fluxes for heat conduction and electrical conduction depend linearly on the corresponding thermodynamic forces with satisfaction of the Onsager principle discussed in Sec. 12.5,

$$(10) \blacktriangle \quad \mathbf{H}/T = \partial \mathcal{D}/\partial(-\nabla_R T/T) = \mathbf{L}^{qq} \cdot (-\nabla_R T)/T + \mathbf{L}^{qe} \cdot \hat{\mathbf{E}}/T,$$

$$(11) \blacktriangle \quad \mathbf{J}_e = \partial \mathcal{D}/\partial(\hat{\mathbf{E}}/T) = \mathbf{L}^{eq} \cdot (-\nabla_R T)/T + \mathbf{L}^{ee} \cdot \hat{\mathbf{E}}/T,$$

which are generalized transport laws extending well-known Fourier's and Ohm's laws to describe the reciprocal phenomena of thermoelectricity originated from the interference of heat conduction and electrical conduction, including the Peltier, Seebeck, and Thomson effects, as discussed in Sec. 12.5.

Substituting Eq. (11) into Eq. (16.3:3) yields the coupled electric charge balance equation:

$$(12) \quad DQ_f/Dt + \nabla_R \cdot [-(\mathbf{L}^{eq} \cdot \nabla_R T)/T + \mathbf{L}^{ee} \cdot \hat{\mathbf{E}}/T] = 0,$$

where $Q_f = Jq_f$. A substitution of Eq. (2), (10) and (11) into Eq. (16.4:3) gives the coupled heat transfer equation based on the augmented Helmholtz free energy functional expansion:

$$\begin{aligned} (13) \quad & \frac{D}{Dt} \int_{-\infty}^{\psi} \{ \beta_{IJ}^M(\mathbf{X}, \psi - \psi', 0) [\partial E_{IJ}(\mathbf{X}, \psi')/\partial \psi'] \\ & + C_H^M(\mathbf{X}, \psi - \psi', 0) [\partial \theta(\mathbf{X}, \psi')/\partial \psi'] / T_0 \\ & + \gamma_I^M(\mathbf{X}, \psi - \psi', \theta) [\partial \hat{d}_I(\mathbf{X}, \psi')/\partial \psi'] \} d\psi' \\ & = [-\nabla_R \cdot (-\mathbf{L}^{qq} \cdot \nabla_R T + \mathbf{L}^{qe} \cdot \hat{\mathbf{E}}) + \hat{\mathbf{E}} \cdot (-\mathbf{L}^{eq} \cdot \nabla_R T + \mathbf{L}^{ee} \cdot \hat{\mathbf{E}})/T + \Lambda]/T, \end{aligned}$$

where the strain history term in the integral gives rise to the coupling between thermal and mechanical effects, while the electric displacement history term gives rise to the coupling between thermal and electric effects.

16.5.2 Finite electro-thermo-viscoelasticity based on augmented Gibbs free energy functional

From the entropy production inequality, the state equations should satisfy the following conditions of thermodynamic admissibility based on the augmented Gibbs free energy functional:

$$\begin{aligned} (14) \blacktriangle \quad & E_{IJ} = A_{IJ}^0 + \int_{-\infty}^{\psi} \{ J_{IJKL}^M(\mathbf{X}, 0, \psi - \psi') [\partial_t \Sigma_{KL}(\mathbf{X}, \psi')/\partial \psi'] \\ & + \alpha_{IJ}^M(\mathbf{X}, 0, \psi - \psi') [\partial \theta(\mathbf{X}, \psi')/\partial \psi'] \\ & + g_{KIJ}^M(\mathbf{X}, 0, \psi - \psi') [\partial \hat{E}_K(\mathbf{X}, \psi')/\partial \psi'] \} d\psi', \end{aligned}$$

$$\begin{aligned} (15) \blacktriangle \quad & \rho_0 \hat{s} = B^0 + \int_{-\infty}^{\psi} \{ \alpha_{IJ}^M(\mathbf{X}, \psi - \psi', 0) [\partial_t \Sigma_{IJ}(\mathbf{X}, \psi')/\partial \psi'] \\ & + C_G^M(\mathbf{X}, \psi - \psi', 0) [\partial \theta(\mathbf{X}, \psi')/\partial \psi'] / T_0 \\ & + \eta_I^M(\mathbf{X}, \psi - \psi', 0) [\partial \hat{E}_I(\mathbf{X}, \psi')/\partial \psi'] \} d\psi', \end{aligned}$$

$$\begin{aligned} (16) \blacktriangle \quad & \hat{D}_I = C_I^0 + \int_{-\infty}^{\psi} \{ g_{IJK}^M(\mathbf{X}, \psi - \psi', 0) [\partial_t \Sigma_{JK}(\mathbf{X}, \psi')/\partial \psi'] \\ & + \eta_I^M(\mathbf{X}, 0, \psi - \psi') [\partial \theta(\mathbf{X}, \psi')/\partial \psi'] \\ & + \kappa_{IJ}^M(\mathbf{X}, 0, \psi - \psi') [\partial \hat{E}_J(\mathbf{X}, \psi')/\partial \psi'] \} d\psi', \end{aligned}$$

$$(17) \quad \mathbf{J}^s = \mathbf{H}/T,$$

$$(18) \quad \rho_0 D_i \hat{s} / Dt = \mathbf{H} \cdot (-\nabla_R T)/T^2 + \mathbf{J}^e \cdot \hat{\mathbf{E}}/T + \Lambda/T \geq 0,$$

where J_{IJKL}^M , C_G^M , κ_{IJ}^M , α_{IJ}^M , g_{KIJ}^M , and η_I^M are thermodynamic property functions. Since E_{IJ} and $\partial_t \Sigma_{IJ}$ are symmetric tensors, we also have $J_{IJKL}^M = J_{JIKL}^M = J_{IJKL}^M$, $\alpha_{IJ}^M = \alpha_{JI}^M$, $g_{KIJ}^M = g_{KJI}^M$. The intrinsic dissipation rate is given by

$$(19) \quad \Lambda = \dot{\psi} \int_{-\infty}^{\psi} \{ [\partial A_{IJ}^M(\mathbf{X}, \psi - \psi')/\partial\psi] [\partial_t \Sigma_{IJ}(\mathbf{X}, \psi')/\partial\psi'] \\ + [\partial B^M(\mathbf{X}, \psi - \psi')/\partial\psi] [\partial\theta(\mathbf{X}, \psi')/\partial\psi'] \\ + [\partial C_I^M(\mathbf{X}, \psi - \psi')/\partial\psi] [\partial \hat{E}_I(\mathbf{X}, \psi')/\partial\psi'] \} d\psi' \\ + \int_{-\infty}^{\psi} \int_{-\infty}^{\psi} \{ [\partial \alpha_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'')/\partial\psi] [\partial_t \Sigma_{IJ}(\mathbf{X}, \psi')/\partial\psi'] [\partial\theta(\mathbf{X}, \psi'')/\partial\psi''] \\ + [\partial J_{IJKL}^M(\mathbf{X}, \psi - \psi', \psi - \psi'')/\partial\psi] [\partial_t \Sigma_{IJ}(\mathbf{X}, \psi')/\partial\psi'] [\partial_t \Sigma_{KL}(\mathbf{X}, \psi'')/\partial\psi''] / 2 \\ + [\partial g_{KIJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'')/\partial\psi] [\partial_t \Sigma_{IJ}(\mathbf{X}, \psi')/\partial\psi'] [\partial \hat{E}_K(\mathbf{X}, \psi'')/\partial\psi''] \\ + [\partial C_G^M(\mathbf{X}, \psi - \psi', \psi - \psi'')/\partial\psi] [\partial\theta(\mathbf{X}, \psi')/\partial\psi'] [\partial\theta(\mathbf{X}, \psi'')/\partial\psi''] / (2T_0) \\ + [\partial \kappa_{IJ}^M(\mathbf{X}, \psi - \psi', \psi - \psi'')/\partial\psi] [\partial \hat{E}_I(\mathbf{X}, \psi')/\partial\psi'] [\partial \hat{E}_J(\mathbf{X}, \psi'')/\partial\psi''] / 2 \\ + [\partial \eta_I^M(\mathbf{X}, \psi - \psi', \psi - \psi'')/\partial\psi] [\partial \hat{E}_I(\mathbf{X}, \psi')/\partial\psi'] [\partial\theta(\mathbf{X}, \psi'')/\partial\psi''] \} d\psi' d\psi''.$$

The terms A_{IJ}^0 , B^0 and C_I^0 on the right-hand sides of Eq. (14)–(16) are the values of E_{IJ} , $\rho_0 \hat{s}$ and \hat{D}_I in the reference state; the first, second and third terms in the multiple integrals denote the mechanical, thermal, and electric contributions, respectively. It is shown from (18) that the total dissipation rate is associated with heat conduction, electrical conduction and intrinsic dissipative processes.

Analogous to the treatments in Sec. 16.5.1, since the entropy production inequality (18) must always be satisfied, the kinetic laws for specific irreversible processes may be determined accordingly. Hence, the intrinsic dissipation rate satisfies the inequality (7), and the transport laws of heat and electrical conductances are described by (10–11). The electric charge balance is governed by Eq. (12).

Substituting Eq. (10), (11) and (15) into Eq. (16.4:8) yields the coupled heat transfer equation based on the augmented Gibbs free energy functional expansion:

$$(20) \quad \frac{D}{Dt} \int_{-\infty}^{\psi} \{ \alpha_{IJ}^M(\mathbf{X}, \psi - \psi', 0) [\partial_t \Sigma_{IJ}(\mathbf{X}, \psi')/\partial\psi'] \\ + C_G^M(\mathbf{X}, \psi - \psi', 0) [\partial\theta(\mathbf{X}, \psi')/\partial\psi'] / T_0 \\ + \eta_I^M(\mathbf{X}, \psi - \psi', 0) [\partial \hat{E}_I(\mathbf{X}, \psi')/\partial\psi'] \} d\psi' \\ = [-\nabla_R \cdot (-\mathbf{L}^{qq} \cdot \nabla_R T + \mathbf{L}^{qe} \cdot \hat{\mathbf{E}}) + \hat{\mathbf{E}} \cdot (-\mathbf{L}^{eq} \cdot \nabla_R T + \mathbf{L}^{ee} \cdot \hat{\mathbf{E}})] / T + \Lambda / T,$$

where the stress history term in the integral gives rise to the coupling between thermal and mechanical effects, while the electric field history term gives rise to the coupling between thermal and electric effects.

If the electric effect is shut off, the equations given in this section reduce to those given in Sec. 14.10.

16.6. BOUNDARY-INITIAL VALUE PROBLEMS FOR ELECTROSENSITIVE MATERIALS

As an extension of Sec. 14.11, the mathematical boundary-initial value problems for electro-sensitive materials are given by Gauss' law (16.3:1), Faraday's law (16.3:2), the mass balance equation (16.3:4), the linear momentum balance equation (16.3:5), the angular momentum balance equation (16.3:6), the electric charge balance equation (16.5:12), the heat transfer equation (16.5:13 or 16.5:20), and the constitutive equations (16.5:1–3) or (16.5:14–16), with specified initial conditions

$$(1) \quad \mathbf{u} = \mathbf{0} \quad (t < 0),$$

$$(2) \quad t\tau = \mathbf{0} \quad (t < 0),$$

$$(3) \quad \theta = 0 \quad (t < 0),$$

$$(4) \quad \mathbf{D} = \mathbf{0} \quad (t < 0),$$

$$(5) \quad \mathbf{E} = \mathbf{0} \quad (t < 0),$$

and jump/boundary conditions

- (6) $\blacktriangle \quad \mathbf{n} \cdot [[\mathbf{D}]] = \varpi_f \quad (t \geq 0),$
- (7) $\blacktriangle \quad \mathbf{n} \times [[\mathbf{E}]] = \mathbf{0} \quad (t \geq 0),$
- (8) $\blacktriangle \quad \mathbf{n} \cdot [[\mathbf{j}^e - q_f \mathbf{v}]] = 0 \quad (t \geq 0),$
- (9) $\blacktriangle \quad \mathbf{u} = \mathbf{u}^B \quad \text{on } S_u \quad (t \geq 0),$
- (10) $\blacktriangle \quad \mathbf{n} \cdot {}_t\tau = \mathbf{t}^B \quad \text{on } S_\sigma \quad (t \geq 0),$
- (11) $\blacktriangle \quad \theta = \theta^B \quad \text{on } S_\theta \quad (t \geq 0),$
- (12) $\blacktriangle \quad \mathbf{n} \cdot \mathbf{h} = q^B \quad \text{on } S_q \quad (t \geq 0),$

where ϖ_f is the free surface charge density, $[[\dots]]$ represents the jump of the field quantity inside the double square brackets across the boundary S , with $S_u \cup S_\sigma = S$, and $S_\theta \cup S_q = S$. Other mixed boundary conditions may also be applied. It is noted that the total stress and the total traction should be considered in the basic equations as well as the boundary and initial conditions.

16.7. LINEARIZED THEORY AND INTEGRAL TRANSFORMS

The linearized electro-thermo-viscoelastic theoretical framework is summarized as follows:

(1) \blacktriangle Equations of motion: $\rho \ddot{\mathbf{v}}_i = {}_t\tau_{ij,j} + \rho b_i, \quad {}_t\tau_{ij} = {}_t\tau_{ji},$

(2) \blacktriangle Electric charge balance equation:

$$\partial D_{i,i}/\partial t + (-l_{ij}^{eq}\theta_{,j}/T_0 + l_{ij}^{ee}E_j/T_0)_{,i} = 0,$$

(3a) \blacktriangle Heat transfer equation based on augmented Helmholtz free energy functional:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{-\infty}^t \{ \beta_{ij}(\mathbf{x}, t-\tau, 0) \frac{\partial e_{ij}(\mathbf{x}, \tau)}{\partial \tau} + \frac{C_H(\mathbf{x}, t-\tau, 0)}{T_0} \frac{\partial \theta(\mathbf{x}, \tau)}{\partial \tau} \\ + \gamma_i(\mathbf{x}, t-\tau, 0) [\partial D_i(\mathbf{x}, \tau)/\partial \tau] \} d\tau = (l_{ij}^{qq}\theta_{,j} - l_{ij}^{qe}E_j)_{,i}/T_0, \end{aligned}$$

(3b) Heat transfer equation based on augmented Gibbs free energy functional:

$$\begin{aligned} {}_t\tau_{ij}(\mathbf{x}, t) &= \int_{-\infty}^t \{ G_{ijkl}(\mathbf{x}, 0, t-\tau) [\partial e_{kl}(\mathbf{x}, \tau)/\partial \tau] \\ &\quad - \beta_{ij}(\mathbf{x}, 0, t-\tau) [\partial \theta(\mathbf{x}, \tau)/\partial \tau] - f_{kij}(\mathbf{x}, 0, t-\tau) [\partial D_k(\mathbf{x}, \tau)/\partial \tau] \} d\tau \\ E_i(\mathbf{x}, t) &= \int_{-\infty}^t \{ -f_{ijk}(\mathbf{x}, t-\tau, 0) [\partial e_{jk}(\mathbf{x}, \tau)/\partial \tau] \\ &\quad - \gamma_i(\mathbf{x}, 0, t-\tau) [\partial \theta(\mathbf{x}, \tau)/\partial \tau] + \bar{\chi}_{ij}(\mathbf{x}, 0, t-\tau) [\partial D_j(\mathbf{x}, \tau)/\partial \tau] \} d\tau, \end{aligned}$$

(4) \blacktriangle Infinitesimal strain-displacement relation: $e_{ij} = (u_{i,j} + u_{j,i})/2,$

(5) \blacktriangle Electric field-electric potential relation: $E_i = -\phi_{,i}$

(6a) \blacktriangle Constitutive relations based on augmented Helmholtz free energy functional:

$$\begin{aligned} {}_t\tau_{ij}(\mathbf{x}, t) &= \int_{-\infty}^t \{ G_{ijkl}(\mathbf{x}, 0, t-\tau) [\partial e_{kl}(\mathbf{x}, \tau)/\partial \tau] \\ &\quad - \beta_{ij}(\mathbf{x}, 0, t-\tau) [\partial \theta(\mathbf{x}, \tau)/\partial \tau] - f_{kij}(\mathbf{x}, 0, t-\tau) [\partial D_k(\mathbf{x}, \tau)/\partial \tau] \} d\tau \\ E_i(\mathbf{x}, t) &= \int_{-\infty}^t \{ -f_{ijk}(\mathbf{x}, t-\tau, 0) [\partial e_{jk}(\mathbf{x}, \tau)/\partial \tau] \\ &\quad - \gamma_i(\mathbf{x}, 0, t-\tau) [\partial \theta(\mathbf{x}, \tau)/\partial \tau] + \bar{\chi}_{ij}(\mathbf{x}, 0, t-\tau) [\partial D_j(\mathbf{x}, \tau)/\partial \tau] \} d\tau, \end{aligned}$$

(6b) \blacktriangle Constitutive relations based on augmented Gibbs free energy functional:

$$e_{ij}(\mathbf{x}, t) = \int_{-\infty}^t \{ J_{ijkl}(\mathbf{x}, 0, t - \tau) [\partial_t \tau_{kl}(\mathbf{x}, \tau) / \partial \tau] \\ + \alpha_{ij}(\mathbf{x}, 0, t - \tau) [\partial \theta(\mathbf{x}, \tau) / \partial \tau] + g_{kij}(\mathbf{x}, 0, t - \tau) [\partial E_k(\mathbf{x}, \tau) / \partial \tau] \} d\tau,$$

$$D_i(\mathbf{x}, t) = \int_{-\infty}^t \{ g_{ijk}(\mathbf{x}, t - \tau, 0) [\partial_t \tau_{jk}(\mathbf{x}, \tau) / \partial \tau] \\ + \eta_i(\mathbf{x}, t - \tau) [\partial \theta(\mathbf{x}, \tau) / \partial \tau] + \kappa_{ij}(\mathbf{x}, 0, t - \tau) [\partial E_j(\mathbf{x}, \tau) / \partial \tau] \} d\tau.$$

A coupled electro-thermo-viscoelastic analysis usually consists of determining electric, thermal and mechanical field quantities under prescribed boundary and initial conditions. Similar to the discussion on solving linearized thermoviscoelastic boundary-initial value problems in Sec. 14.12, integral transform methods provide a powerful tool for solving linearized electro-thermoviscoelastic boundary-initial value problems. For illustration, we will examine the Laplace transform of the foregoing equations based on the augmented Gibbs free energy functional with use of the initial conditions and boundary/jump conditions specified in Sec. 16.6. They are given by

$$(7) \quad \rho s^2 \bar{u}_i = t \bar{\tau}_{ij,j} + \rho \bar{b}_i,$$

$$(8) \quad s \bar{D}_{i,i} + (-l_{ij}^{eq} \bar{\theta}_{,j} / T_0 + l_{ij}^{ee} \bar{E}_j / T_0)_{,i} = 0,$$

$$(9) \quad (l_{ij}^{qq} \bar{\theta}_{,j} - l_{ij}^{qe} \bar{E}_j)_{,i} = s^2 \rho \bar{C}_p \bar{\theta} + s^2 T_0 \bar{\alpha}_{ij} \bar{\sigma}_{ij} + s^2 \bar{\eta}_i \bar{E}_i,$$

$$(10) \quad \bar{e}_{ij} = (\bar{u}_{i,j} + \bar{u}_{j,i}) / 2,$$

$$(11) \quad \bar{E}_i = -\bar{\phi}_{,i},$$

$$(12) \quad \bar{e}_{ij} = s J_{ijkl} \bar{\tau}_{kl} + s \bar{\alpha}_{ij} \bar{\theta} + s \bar{g}_{kij} \bar{E}_k,$$

$$(13) \quad D_i = s \bar{g}_{ijkt} \bar{\tau}_{jk} + s \bar{\eta}_i \bar{\theta} + s \bar{\kappa}_{ij} \bar{E}_j,$$

$$(14) \quad n_i [[\bar{D}_i]] = \bar{\varpi}_f \text{ across } S,$$

$$(15) \quad \varepsilon_{ijk} n_i [[\bar{E}_j]] = 0 \text{ across } S,$$

$$(16) \quad n_i [[-l_{ij}^{eq} \bar{\theta}_{,j} + l_{ij}^{ee} \bar{E}_j]] = 0 \text{ across } S,$$

$$(17) \quad \bar{u}_i = \bar{u}_i^B \text{ on } S_u,$$

$$(18) \quad n_{it} \bar{\tau}_{ij} = \bar{t}_j^B \text{ on } S_\sigma,$$

$$(19) \quad \bar{\theta} = \bar{\theta}^B \text{ on } S_\theta,$$

$$(20) \quad -n_i l_{ij}^{qq} \bar{\theta}_{,j} + n_i l_{ij}^{qe} \bar{E}_j = \bar{q}^B \text{ on } S_q.$$

After an integral transform is applied to the basic equations for a linearized electro-thermo-viscoelastic problem, the transformed boundary value problems can be solved in a manner similar to that for a linearized thermoviscoelastic problem. The final electro-thermo-viscoelastic solution is then obtained upon inversion of the transformed solution. The correspondence principle may be extended to the identification of the problem in linearized electro-thermoelasticity with one in linearized electro-thermo-viscoelasticity in the transformed plane.

16.8. REPRESENTATION OF THERMODYNAMIC PROPERTY FUNCTIONS FOR ELECTROSENSITIVE MATERIALS WITH MEMORY ON INTRINSIC TIME SCALE

The thermodynamic formulation of coupled electro-thermo-viscoelastic constitutive laws allows the dependence of the thermodynamic property functions on aging, temperature, strain (stress), electric displacement (electric field), etc. via the introduction of intrinsic time scale. Thus the dependence of a long-term property function on aging, temperature, stress (strain), electric field (electric displacement) may be determined from short-term experiments with an accelerated test method. From experimental work on the aging behavior of electrosensitive materials, piezoelectric and dielectric properties have been reported to follow the stretched exponential law (see Biblio. 16.2). It appears that electric field (electric displacement) may have influence on structural relaxation of the glassy state toward the metastable state, similar to the effects of stress (strain) and temperature.

16.8.1. Time-aging-temperature-strain-electric displacement superposition

In the augmented Helmholtz free energy functional-based formulation for electrosensitive materials with memory on intrinsic time scale, special forms of integration functions with arguments added may be employed, that is, $f_H^M(\psi - \psi', \psi - \psi'') = \tilde{f}_H^M(2\psi - \psi' - \psi'')$. This implies that when the thermodynamic property functions in the constitutive equations are known, the corresponding integration functions in the augmented Helmholtz free energy functional expansion can be determined.

A long-term property function may be related to a momentary master curve by a superposition principle of time, aging, temperature, strain, and electric displacement as follows:

$$(1) \blacktriangle \quad \tilde{f}_H^M(\psi) = b_{t_a, T, E, \hat{d}} f_H(t),$$

$$(2) \blacktriangle \quad \psi = \int_0^t a_{t_a, T, E, \hat{d}} [t'; t_a^0, T(t'), E(t'), \hat{d}(t')] dt',$$

where f_H denotes a thermodynamic property function (scalar or tensorial) with real time t as argument, \tilde{f}_H^M is master function with intrinsic time scale ψ as argument, t_a^0 is the aging time prior to loading, $t_a = t + t_a^0$ is the total aging time, $a_{t_a, T, E, \hat{d}}$ and $b_{t_a, T, E, \hat{d}}$ correspond to horizontal and vertical shift functions due to the combined effects of aging, temperature, strain, and electric displacement. For simplicity, the argument X of f , a and b are not shown.

As discussed in Sec. 14.13.3, the master function \tilde{f}_H^M may be represented in terms of continuous relaxation time spectrum as

$$(3) \quad \tilde{f}_H^M(\psi) = \int_{-\infty}^{\infty} \hat{f}_H(\tau) \phi_H(\psi/\tau) d(\ln \tau),$$

where $\hat{f}_H(\tau)$ is relaxation time spectrum distribution, and $\phi_H(\psi/\tau)$ is kernel function.

16.8.2. Time-aging-temperature-stress-electric field superposition

In the augmented Gibbs free energy functional-based formulation for electrosensitive materials with memory on intrinsic time scale, special forms of integration functions with arguments added such that

$$f_G^M(\psi - \psi', \psi - \psi'') = \tilde{f}_G^M(2\psi - \psi' - \psi'')$$

may be employed. This implies that when the thermodynamic property functions in the constitutive equations are known, the corresponding integration functions in the augmented Gibbs free energy functional expansion can be determined.

A long-term property function may be related to a momentary master curve by a superposition principle of time, aging, temperature, stress, and electric field as follows:

$$(4) \blacktriangle \quad \tilde{f}_G^M(\psi) = b_{t_a, T, t \Sigma, \hat{E}} f_G(t),$$

$$(5) \blacktriangle \quad \psi = \int_0^t a_{t_a, T, t \Sigma, \hat{E}} [t'; t_a^0, T(t'), t \Sigma(t'), \hat{E}(t')] dt',$$

where f_G denotes a thermodynamic property function (scalar or tensorial) with real time t as argument, \tilde{f}_G^M is master function with intrinsic time scale ψ as argument, $a_{t_a, T, t \Sigma, \hat{E}}$ and $b_{t_a, T, t \Sigma, \hat{E}}$ correspond to horizontal and vertical shift functions due to the combined effects of aging, temperature, stress, and electric field. For simplicity, the argument X of f , a and b are not shown.

Similarly, the master function \tilde{f}_G^M may be represented in terms of continuous relaxation time spectrum as

$$(6) \quad \tilde{f}_G^M(\psi) = \int_{-\infty}^{\infty} \hat{f}_G(\tau) \phi_G(\psi/\tau) d(\ln \tau),$$

where $\hat{f}_G(\tau)$ is relaxation time spectrum distribution, and $\phi_G(\psi/\tau)$ is kernel function.

16.9. REDUCTION TO ELECTRO-THERMO-ELASTICITY

The coupled theories of electro-thermo-viscoelasticity can be reduced to those of electro-thermo-elasticity, as the augmented Helmholtz free energy is taken to be a function of basic variables such as deformation and electric displacement. Alternative formulations which consider stress and electric fields as basic variables can also be established through the Legendre transformation. Equations (16.5:1–3) and (16.5:14–16) are, respectively, reduced to those for finite electro-thermo-elasticity as

$$(1) \Delta \quad {}_t\Sigma_{IJ} = {}_t\Sigma_{IJ}^0 + C_{IJKL}E_{KL} - \beta_{IJ}\theta - f_{KIJ}\hat{d}_K,$$

$$(2) \Delta \quad \rho_0\hat{s} = \rho_0\hat{s}^0 + \beta_{IJ}E_{IJ} + C_H\theta/T_0 + \gamma_I\hat{d}_I,$$

$$(3) \Delta \quad \hat{E}_I = \hat{E}_I^0 - f_{IJK}E_{JK} - \gamma_I\theta + \bar{\chi}_{IJ}\hat{d}_J,$$

$$(4) \Delta \quad E_{IJ} = E_{IJ}^0 + C_{IJKL}^{-1}{}_t\Sigma_{KL} + \alpha_{IJ}\theta + g_{KIJ}\hat{E}_K,$$

$$(5) \Delta \quad \rho_0\hat{s} = \rho_0\hat{s}^0 + \alpha_{IJ}{}_t\Sigma_{IJ} + C_G\theta/T_0 + \eta_I\hat{E}_I,$$

$$(6) \Delta \quad \hat{D}_I = \hat{D}_I^0 + g_{IJK}{}_t\Sigma_{JK} + \eta_I\theta + \kappa_{IJ}\hat{E}_J.$$

For infinitesimal cases, Eqs. (4)–(6) become

$$(7) \Delta \quad \varepsilon_{ij} = \varepsilon_{ij}^0 + C_{ijkl}^{-1}{}_t\tau_{kl} + \alpha_{ij}\theta + g_{kij}E_k,$$

$$(8) \Delta \quad \rho\hat{s} = \rho\hat{s}_0 + \alpha_{ij}{}_t\tau_{ij} + C_G\theta/T_0 + \eta_iE_i,$$

$$(9) \Delta \quad D_i = D_i^0 + g_{ijk}{}_t\tau_{jk} + \eta_i\theta + \kappa_{ij}E_j,$$

where $\varepsilon_{ij} = (u_{ij} + u_{ji})/2$ is the infinitesimal strain.

We can rewrite the above equations with strain, temperature change, and electric field as independent variables:

$$(10) \Delta \quad {}_t\tau_{ij} = {}_t\tau_{ij}^0 + C_{ijkl}\varepsilon_{kl} - \beta_{ij}\theta - e_{kij}E_k,$$

$$(11) \Delta \quad \rho\hat{s} = \rho\hat{s}_0 + \beta_{ij}\varepsilon_{ij} + C_v\theta/T_0 + \omega_iE_i,$$

$$(12) \Delta \quad D_i = D_i^0 + e_{ijk}\varepsilon_{jk} + \omega_i\theta + \kappa_{ij}E_j.$$

It is obvious that the material constants in the equivalent constitutive representations are related, which means that the material constants in one constitutive representation can be transformed to those in another constitutive representation. Materials with nonzero e_{kij} exhibit piezoelectricity, i.e., mechanical load or deformation can produce electric polarization or electric field, a direct effect, and *vice versa*, an inverse effect. Materials with nonzero ω_i exhibit pyroelectricity, i.e., temperature change can produce electric polarization or electric field, a direct effect, and *vice versa*, an inverse effect. The physical meanings for commonly used material constants are listed in [Table 16.9:1](#).

Table 16.9:1. Physical meanings of material constants.

Symbol	Physical Meaning
$C_{ijkl} = C_{klji} = C_{jikl} = C_{ijlk}$	elastic moduli
$\kappa_{ij} = \kappa_{ji}$	dielectric permittivity
$e_{kij} = e_{kji}$	piezoelectric coefficients
ω_i	pyroelectric coefficients
$\beta_{ij} = \beta_{ji}$	thermal moduli
C_v	specific heat at constant volume

New examples of constitutive relations for practical applications may be implemented as user subroutines in commercial finite element analysis software packages such as ABAQUS or ANSYS. The number of material properties required for coupled multifield analysis depends on the material type. The reader may refer to the Bibliography for further information on solving nonlinear electroelastic problems involving large deformation (see [Biblio. 16.3](#)).

PROBLEMS

Problem 16.1. Derive the transversely isotropic form of electro-thermoviscoelastic constitutive equations for electro-sensitive materials with memory on intrinsic time scale based on the augmented Helmholtz free energy functional, following the procedure in [Secs. 16.4.1](#) and [16.5.1](#).

Problem 16.2. Derive the transversely isotropic form of electro-thermoviscoelastic constitutive equations for electro-sensitive materials with memory on intrinsic time scale based on the augmented Gibbs free energy functional, following the procedure in [Secs. 16.4.2](#) and [16.5.2](#).

Problem 16.3. Consider a transversely isotropic electro-viscoelastic slab made of electro-sensitive material with memory on intrinsic time scale subjected to electric field. The slab has infinite extent in the (x_1, x_2) plane and finite thickness $2a$, bounded by the planes $x_3 = \pm a$. The symmetry axis is along the x_3 -axis. Formulate the corresponding boundary-initial value problem, and find the displacement and the associated stress field if the electric field is known as a function of the thickness, $E_i = E_i(x_3, t)$ for $-a \leq x_3 \leq a$, $i = 1, 2, 3$.

Problem 16.4. Consider inflation of a spherical shell made of incompressible isotropic electroelastic material. A pressure P is applied to the inner boundary $r = a$, and no traction on the outer boundary $r = b$. The electric field is along the radial direction. Derive the nonlinear electroelastic constitutive equations with the deformation and electric displacement as basic variables, and solve the corresponding boundary value problem with spherical symmetry (Dorfmann and Ogden 2006).

17

INCREMENTAL APPROACH TO SOLVING SOME NONLINEAR PROBLEMS

The field equations of mechanics of solids subjected to large deformation are nonlinear with respect to displacements and velocities. The constitutive equations are linear or nonlinear depending on the material. The boundary conditions could be nonlinear if the external forces and the displacements of particles on the boundary are coupled nonlinearly. The external loading may depend on the deformation as in aeroelasticity or in some special structural problems. These features can result in a nonlinear boundary or initial value problem, which is difficult to solve. One observes, however, that large deformations are arrived at through many infinitesimal steps, and for each step the linearized equations prevail in some sense. The incremental approach is to solve the linearized equations at each step and track all the infinitesimal increments. For problems such as plasticity, viscoelasticity, and biomechanics, incremental approach is natural because the constitutive laws of the materials can only be described incrementally.

In this chapter, we shall derive the linearized equations for the increments of the deformations and the corresponding variational principles for integral formulation. For simplicity, we refer “updated *Lagrangian description*” as “*updated description*,” the symbol ‘≡’ as definition and ‘≈’ as approximation. We also drop the explicit reference N to the material point \mathbf{X}^N , the *material rate* $\overset{\circ}{\mathbf{A}}^N$ of a quantity \mathbf{A} and the parameter increment Δt_N at the current deformation state N by simply using \mathbf{X} , $\overset{\circ}{\mathbf{A}}$ and Δt , respectively, except occasionally use the reference N to emphasize clarity.

Computational methods for solving the linearized incremental equations will be discussed in the sequent chapters. Finite element modeling of nonlinear elasticity, viscoelasticity, plasticity, viscoplasticity, and creep will be discussed in [Chapter 21](#).

17.1. UPDATED LAGRANGIAN DESCRIPTION

We describe the deformation as a multi-increment process (Atluri and Cazzani 1995) whose evolution is described by a parameter t , which can be actual time as in dynamic or creep problems but often it is the magnitude of a proportional load. Implicit in the assumption is that the solution is independent of the rate of change of the parameter. Let D_0 be the *reference configuration* with region V_0 occupied by the body at $t = t_0$ and D_1, D_2, \dots , be the sequence of configurations occupied by the body at $t = t_1, t_2, \dots$, respectively. We use D_0 to denote the undeformed configuration, D_N the current state, and D_{N+1} the state at the subsequent increment. We seek the incremental solution at $t = t_{N+1}$ with the solution at t_N known. *Bold* characters $\mathbf{A}_I^J(\mathbf{X})$ are used as a tensor or vector and non-bold italic characters such as $\mathbf{A}_I^J(\mathbf{X})$ for scalars at the material point \mathbf{X} , in which the superscript J denotes the quantity at the D_J configuration at $t = t_J$ but refers to the configuration D_I at $t = t_I$. For example, we have $\mathbf{u}_I^J(\mathbf{X})$ describe the relative displacement vector at the material point \mathbf{X} between the deformation state t_J and t_I . As another example, the first Piola–Kirchhoff stress tensor $\mathbf{T} [= \mathbf{T}_0^N(\mathbf{X})]$ in Eq. (13.7:4) is a stress measure in the current configuration, $t = t_N$, referred to the undeformed coordinates. The subscript 0 refers to the undeformed configuration and the superscript N denotes the current one. In subsequent discussion, we shall drop the explicit reference to the material point \mathbf{X} by simply writing \mathbf{A}_I^J, A_I^J for $\mathbf{A}_I^J(\mathbf{X}), A_I^J(\mathbf{X})$, respectively, except occasionally use \mathbf{X} to emphasize clarity.

We shall use the *Lagrangian description* as discussed in [Sec. 5.2](#) to describe the material and spatial changes, and the *updated Lagrangian description* in the incremental approach by referring all latest incremented quantities to the current configuration, e.g., \mathbf{A}_N^{N+1} or $\overset{\circ}{\mathbf{A}}_N^{N+1}$. In computation, an updated configuration at each increment is needed. The *material rate* of a quantity \mathbf{A} at $t = t_N$ is defined as

$$(1) \quad \overset{\circ}{\mathbf{A}}^N(\mathbf{X}) \equiv [\mathbf{A}_N^{N+1}(\mathbf{X}) - \mathbf{A}_N^N(\mathbf{X})]/\Delta t_N \quad \text{or} \quad \mathbf{A}_N^{N+1} = \mathbf{A}_N^N + \overset{\circ}{\mathbf{A}}\Delta t,$$

where \mathbf{A} can be a tensor, a vector or a scalar (for non-boldfaced italic character) for small Δt_N ($= t_{N+1} - t_N$). The material rate in the *total Lagrangian description* is defined as

$$D\mathbf{A}_0^N(\mathbf{X})/Dt \equiv \dot{\mathbf{A}} \cong [\mathbf{A}_0^{N+1}(\mathbf{X}) - \mathbf{A}_0^N(\mathbf{X})]/\Delta t_N.$$

The definition of stress rate in the updated description needs much care. Since stress is defined by a force vector and a surface, one must consider both the rate of change of the force vector and the associated surface with respect to the parameter when one attempts to define a stress rate. In fact, stress rate can be defined in a number of different ways. By

introducing the parameter t , one liberalizes but not necessarily simplifies or settles the matter (see Sec. 17.3).

The incremental approach is to determine $\overset{\circ}{\mathbf{A}}$ approximately by linearizing the governing equations for all material rates at D_N configuration and evaluate the incremental solution according to Eq. (1).

17.2. LINEARIZED RATE OF DEFORMATION

We shall use the notations of Sec. 13.2 and denote \mathbf{u} as a relative displacement vector. Then, \mathbf{u}_N^N is the relative displacement vector in the same deformation state N , and \mathbf{F}_N^N is the deformation gradient tensor associated with \mathbf{u}_N^N . Hence

$$(1) \quad \mathbf{u}_N^N = 0,$$

$$(2) \quad \mathbf{F}_N^N = \mathbf{I}.$$

From Eq. (13.2:5) and the equation below (13.4:2), we find

$$(3) \quad J_N^N = \det(\mathbf{F}_N^N) = \det(\mathbf{I}) = 1,$$

$$(4) \quad \mathbf{U}_N^N = \mathbf{R}_N^N = \mathbf{I}.$$

The approximate rate of change of the displacement \mathbf{u} for small Δt is the *velocity-like vector* denoted by

$$(5) \quad \overset{\circ}{\mathbf{u}} \equiv \mathbf{u}_N^{N+1}/\Delta t \cong \mathbf{v}.$$

There is no definition for acceleration-like vector in the updated description.

In general, since the spatial coordinates of quantities are referred to the current configuration in the updated Lagrangian description, the material rate of a strain measure is equivalent to the corresponding rate in the Eulerian description. Thus, one can determine the rates of deformation in the updated Lagrangian description from the rates given in Secs. 13.5 and 13.6 by replacing the undeformed state with the current one as the reference configuration. From Eq. (2) and (13.5:3), (13.5:5) and (13.5:6), the updated rate of the deformation gradient tensor is

$$(6) \quad \blacktriangle \quad \overset{\circ}{\mathbf{F}} \equiv (\mathbf{F}_N^{N+1} - \mathbf{F}_N^N)/\Delta t = [\partial(\mathbf{X}^N + \mathbf{u}_N^{N+1})/\partial\mathbf{X}^N - \mathbf{I}]/\Delta t \\ = \partial(\mathbf{u}_N^{N+1}/\Delta t)/\partial\mathbf{X}^N \cong \partial\mathbf{v}/\partial\mathbf{X}^N = \mathbf{L} = \mathbf{D} + \boldsymbol{\Omega},$$

where \mathbf{L} is a velocity-like gradient. Since the spatial coordinates of quantities are referred to the current configuration in the updated description, the material rate of a strain measure is equivalent to the corresponding rate in the Eulerian description. One can determine the deformation rates in the updated description from the rates given in Secs. 13.5 and 13.6 by replacing the undeformed state with the current state as the reference configuration. Since $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$, one has

$$(7) \quad \blacktriangle \quad \overset{\circ}{\mathbf{F}} \cong (\mathbf{R} \cdot \dot{\mathbf{U}} + \dot{\mathbf{R}} \cdot \mathbf{U})_N^N = \mathbf{R}_N^N \cdot \overset{\circ}{\mathbf{U}} + \overset{\circ}{\mathbf{R}} \cdot \mathbf{U}_N^N = \overset{\circ}{\mathbf{U}} + \overset{\circ}{\mathbf{R}},$$

Notice that the decomposition of the deformation gradient rate is additive as opposed to the multiplicative polar decomposition for the finite deformation given in Eq. (13.4:3). Since $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$, its deformation rate is zero,

$$\overset{\circ}{\mathbf{R}} \cdot (\mathbf{R}_N^N)^T + \mathbf{R}_N^N \cdot \overset{\circ}{\mathbf{R}}^T = \overset{\circ}{\mathbf{R}} + \overset{\circ}{\mathbf{R}}^T = 0,$$

which implies that $\overset{\circ}{\mathbf{R}}$ is antisymmetric. Since \mathbf{U} is symmetric, then

$$\overset{\circ}{\mathbf{U}} \equiv (\mathbf{U}_N^{N+1} - \mathbf{U}_N^N)/\Delta t$$

is symmetric. We call $\overset{\circ}{\mathbf{U}}$ the *updated Lagrangian strain rate tensor* and $\overset{\circ}{\mathbf{R}}$ the *updated Lagrangian spin tensor*. A comparison of Eq. (6) and (7), and the use of the symmetric and antisymmetric properties of $\overset{\circ}{\mathbf{U}}$ and $\overset{\circ}{\mathbf{R}}$ give

$$(8) \quad \blacktriangle \quad \overset{\circ}{\mathbf{U}} \cong \mathbf{D} = (\mathbf{L})_s, \quad \overset{\circ}{\mathbf{R}} \cong \boldsymbol{\Omega} = (\mathbf{L})_a,$$

where $(\cdot)_s$ and $(\cdot)_a$ denote the symmetric and antisymmetric parts of the relevant quantity. Thus, the *updated Lagrangian strain rate* and the *updated Lagrangian spin rate tensors* are the symmetric and antisymmetric parts of the *velocity-like gradient tensor*. Similarly, from Eq. (13.6:6), one finds

$$(9) \blacktriangle \quad \overset{\circ}{J} \cong \text{tr}(\mathbf{L}) J_N^N = \text{tr}(\mathbf{L}) = \text{tr}(\mathbf{D}).$$

One can also show that the *Cauchy* and *Almansi tensor rates* are

$$(10) \quad \overset{\circ}{\mathbf{C}} = \overset{\circ}{\mathbf{B}} = 2\mathbf{D}.$$

The use of the definitions above gives the updated solution of various quantities

$$(11) \quad \mathbf{F}_N^{N+1} = \mathbf{I} + \overset{\circ}{\mathbf{F}} \Delta t = \mathbf{I} + \mathbf{L} \Delta t,$$

$$(12) \quad \mathbf{U}_N^{N+1} = \mathbf{I} + \overset{\circ}{\mathbf{U}} \Delta t \cong \mathbf{I} + \mathbf{D} \Delta t,$$

$$(13) \quad \mathbf{R}_N^{N+1} = \mathbf{I} + \overset{\circ}{\mathbf{R}} \Delta t \cong \mathbf{I} + \mathbf{\Omega} \Delta t,$$

$$(14) \quad \mathbf{u}_N^{N+1} = \overset{\circ}{\mathbf{u}} \Delta t = \mathbf{v} \Delta t,$$

$$(15) \quad J_N^{N+1} = \det(\mathbf{F}_N^{N+1}) = 1 + \overset{\circ}{J} \Delta t \cong 1 + \text{tr}(\mathbf{D}) \Delta t$$

in terms of their updated Lagrangian material rate. Since $\mathbf{F} = \mathbf{I} + \nabla \mathbf{u}$, one has

$$(16) \quad \mathbf{F}_N^{N+1} = \mathbf{I} + \nabla_N \mathbf{u}_N^{N+1} = \mathbf{I} + \nabla_N \mathbf{v} \Delta t,$$

where ∇_N is the gradient operator defined in the current configuration. Comparing Eq. (16) with Eq. (11), one finds

$$(17) \quad \mathbf{L} = \nabla_N \mathbf{v}.$$

Note that \mathbf{L} , \mathbf{D} , $\mathbf{\Omega}$ and all variables $(\overset{\circ}{\cdot})$ are referred to the current configuration D_N and that $\mathbf{F}(= \mathbf{I})$, $\mathbf{R}(= \mathbf{I})$ and $J(= 1)$ are the current deformation gradient tensor, rotation tensor and Jacobian in the updated description.

With \mathbf{X}^N denoting the material point in D_N , its corresponding point in D_{N+1} is

$$(18) \quad \mathbf{X}^{N+1} = \mathbf{X}^N + \mathbf{u}_N^{N+1},$$

which is the *updated coordinates* for the next increment.

The gradient operation ∇_N is calculated as follows. If the current configuration is characterized by generalized coordinates η^{iN} with corresponding contravariant base vectors \mathbf{g}^{iN} , the gradient operator is in the form

$$\nabla_N(\cdot) = [\partial(\cdot)/\partial \eta^{iN}] \mathbf{g}^{iN} \quad (N \text{ not summed}).$$

Then

$$(19) \quad \nabla_N \mathbf{v} = [\partial \mathbf{v} / \partial \eta^{iN}] \mathbf{g}^{iN} = (\partial v_j / \partial X^{iN}) \mathbf{e}_j \mathbf{e}_i \quad (N \text{ not summed}),$$

where \mathbf{e}_i are the base vectors of Cartesian coordinates. Note that $\mathbf{e}_j \mathbf{e}_i \neq \mathbf{e}_i \mathbf{e}_j$ for $i \neq j$.

One can easily establish the following relations between the update Lagrangian quantities and those referenced to the undeformed state:

$$(20) \quad \mathbf{u}_0^{N+1} = \mathbf{u}_0^N + \mathbf{v} \Delta t,$$

$$(21) \quad \mathbf{F}_0^{N+1} = \partial \mathbf{X}^N / \partial \mathbf{X}^0 + (\partial \mathbf{u}_N^{N+1} / \partial \mathbf{X}^N) \cdot (\partial \mathbf{X}^N / \partial \mathbf{X}^0) = (\mathbf{I} + \mathbf{L} \Delta t) \cdot \mathbf{F}_0^N,$$

$$(22) \quad \mathbf{C}_0^{N+1} = (\mathbf{F}_0^{N+1})^T \cdot \mathbf{F}_0^{N+1} = (\mathbf{F}_0^N)^T \cdot [\mathbf{I} + 2\mathbf{D} \Delta t + \mathbf{L}^T \cdot \mathbf{L} (\Delta t)^2] \cdot \mathbf{F}_0^N,$$

$$(23) \quad \mathbf{R}_0^{N+1} = \mathbf{R}_0^N + (\mathbf{R}_0^{N+1} - \mathbf{R}_0^N) \cong (\mathbf{I} + \mathbf{\Omega} \Delta t) \cdot \mathbf{R}_0^N,$$

$$(24) \quad \mathbf{U}_0^{N+1} = \mathbf{R}_0^{N+1} \cdot \mathbf{F}_0^{N+1} \cong [\mathbf{I} + (\mathbf{R}_0^N)^T \cdot \mathbf{D} \cdot \mathbf{R}_0^N \Delta t] \cdot \mathbf{U}_0^N.$$

In Eq. (22), the second order term is retained to derive the constitutive laws in Sec. 17.5. Equation (20) is by definition and Eq. (21) and (22) follow. Equations (23) and (24) are approximate.

17.3. LINEARIZED RATES OF STRESS MEASURES

We denote the Cauchy stress tensor τ_N^N in the current configuration as τ ; and the current Kirchhoff, first and second Piola–Kirchhoff, and Biot–Luré stress tensors as σ , \mathbf{T} , \mathbf{S} and \mathbf{r}^* , respectively, that

$$\begin{aligned}\sigma_N^N &= J_N^N \tau_N^N = \tau, & \mathbf{T}_N^N &= J_N^N (\mathbf{F}_N^N)^{-1} \cdot \tau_N^N = \tau, \\ \mathbf{S}_N^N &= \mathbf{T}_N^N \cdot [(\mathbf{F}^{-1})_N^N]^T = \tau, & \mathbf{r}_N^{*N} &= J_N^N (\mathbf{F}_N^N)^{-1} \cdot \tau_N^N \cdot \mathbf{R}_N^N = \tau.\end{aligned}$$

In other words, all *updated Lagrangian stress tensors* in the current configuration D_N equal the Cauchy stress tensor. To determine their *material rates* in the *updated description*, we first consider the rate of the first Piola–Kirchhoff stress tensor \mathbf{T}

$$\overset{\circ}{\mathbf{T}} \equiv (\mathbf{T}_N^{N+1} - \mathbf{T}_N^N) / \Delta t,$$

which gives

$$(1) \quad \mathbf{T}_N^{N+1} = \tau + \overset{\circ}{\mathbf{T}} \Delta t.$$

From \mathbf{T}_N^{N+1} , one can evaluate the updated Cauchy stress tensor τ_N^{N+1} , denoted as τ^{N+1} , in D_{N+1} . From Eq. (13.7.21), we have

$$\tau_N^{N+1} \equiv \tau^{N+1} = \mathbf{F}_N^{N+1} \cdot \mathbf{T}_N^{N+1} / J_N^{N+1}.$$

In linearized form, the Cauchy stress in D_{N+1} becomes

$$(2) \quad \tau^{N+1} \cong (\mathbf{I} + \mathbf{L} \Delta t) \cdot (\tau + \overset{\circ}{\mathbf{T}} \Delta t) / [1 + \text{tr}(\mathbf{L} \Delta t)] \cong \tau + [\overset{\circ}{\mathbf{T}} + \mathbf{L} \cdot \tau - \text{tr}(\mathbf{L}) \tau] \Delta t,$$

which shall be used to determine the solution for the next increment. Since $\overset{\circ}{\tau}$ is the limit of the solution of the following equation as $\Delta t \rightarrow 0$,

$$(3) \quad \tau^{N+1} = \tau + \overset{\circ}{\tau} \Delta t,$$

we can equate Eq. (2) and (3) to determine the *Cauchy stress rate* $\overset{\circ}{\tau}$ in the updated description. Thus, we obtain:

$$(4) \quad \overset{\circ}{\tau} \cong \overset{\circ}{\mathbf{T}} + \mathbf{L} \cdot \tau - \text{tr}(\mathbf{L}) \tau, \quad \text{or} \quad \overset{\circ}{\mathbf{T}} \cong \overset{\circ}{\tau} - \mathbf{L} \cdot \tau + \text{tr}(\mathbf{L}) \tau.$$

We can then determine the updated Lagrangian rates of $\sigma (= J\tau)$, $\mathbf{r}^* (= J\mathbf{F}^{-1} \cdot \tau \cdot \mathbf{R})$, $\mathbf{S} [= \mathbf{T} \cdot (\mathbf{F}^{-1})^T]$ and $\sigma_r (= J\mathbf{R}^T \cdot \tau \cdot \mathbf{R})$, the *Kirchhoff*, the *Biot–Luré*, the *second Piola–Kirchhoff* and the *Green–Naghdi stress tensors*. On examining the equations

$$\begin{aligned}\sigma_N^{N+1} &= \tau + \overset{\circ}{\sigma} \Delta t = J_N^{N+1} \tau_N^{N+1} \\ &\cong (1 + \overset{\circ}{J} \Delta t)(\tau + \overset{\circ}{\tau} \Delta t) \cong \tau + (\overset{\circ}{J} \tau + \overset{\circ}{\tau}) \Delta t, \\ \mathbf{r}_N^{*N+1} &= \tau + \overset{\circ}{\mathbf{r}}^* \Delta t = (\mathbf{F}^{-1})_N^{N+1} \cdot \tau^{N+1} \cdot \mathbf{R}_N^{N+1} J_N^{N+1} \\ &\cong (\mathbf{I} - \mathbf{L} \Delta t) \cdot (\tau + \overset{\circ}{\tau} \Delta t) \cdot (\mathbf{I} + \Omega \Delta t) [1 + \text{tr}(\mathbf{D}) \Delta t] \\ &\cong \tau + [\overset{\circ}{\tau} + \tau \cdot \text{tr}(\mathbf{D}) - \mathbf{L} \cdot \tau + \tau \cdot \Omega] \Delta t, \\ \mathbf{S}_N^{N+1} &= \tau + \overset{\circ}{\mathbf{S}} \Delta t = \mathbf{T}_N^{N+1} \cdot [(\mathbf{F}^{-1})_N^{N+1}]^T \\ &\cong (\tau + \overset{\circ}{\mathbf{T}} \Delta t) \cdot (\mathbf{I} - \mathbf{L}^T \Delta t) = \tau + (\overset{\circ}{\mathbf{T}} - \tau \cdot \mathbf{L}^T) \Delta t, \\ \sigma_{rN}^{N+1} &= (\mathbf{I} + \Omega^T \Delta t) \cdot (\tau + \overset{\circ}{\tau} \Delta t) \cdot (\mathbf{I} + \Omega \Delta t) [1 + \text{tr}(\mathbf{D}) \Delta t] \\ &= \tau + (\overset{\circ}{\tau} + \Omega^T \cdot \tau + \tau \cdot \Omega + \text{tr}(\mathbf{D}) \tau) \Delta t,\end{aligned}$$

one can derive the following rates of stress measures:

$$(5) \quad \overset{\circ}{\sigma} \cong \overset{\circ}{\tau} + \overset{\circ}{J} \tau = \overset{\circ}{\tau} + \text{tr}(\mathbf{D}) \tau,$$

$$(6) \quad \overset{\circ}{\mathbf{r}}^* \cong \overset{\circ}{\tau} - \mathbf{L} \cdot \tau + \text{tr}(\mathbf{D}) \tau + \tau \cdot \Omega,$$

$$(7) \quad \overset{\circ}{\mathbf{S}} \cong \overset{\circ}{\mathbf{T}} - \tau \cdot \mathbf{L}^T = \overset{\circ}{\tau} - \mathbf{L} \cdot \tau - \tau \cdot \mathbf{L}^T + \text{tr}(\mathbf{D}) \tau,$$

$$(8) \blacktriangle \quad \overset{\circ}{\sigma}_r \cong \overset{\circ}{\tau} - \boldsymbol{\Omega} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega} + \text{tr}(\mathbf{D})\boldsymbol{\tau}.$$

Note that $\text{tr}(\mathbf{D}) = \text{tr}(\mathbf{L})$. From Eq. (4) and (6), one establishes the relation between the rates of \mathbf{T} and \mathbf{r}^* in the updated description as

$$\overset{\circ}{\mathbf{r}}^* = \overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega}.$$

Also from Eq. (6), since $\overset{\circ}{\tau}$ and $\boldsymbol{\tau}$ are symmetric,

$$(\overset{\circ}{\mathbf{r}}^* + \mathbf{L} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_a = (\overset{\circ}{\mathbf{r}}^* + \mathbf{D} \cdot \boldsymbol{\tau})_a = 0.$$

We evaluate $\boldsymbol{\tau}^{N+1}$ from Eq. (3) and $\overset{\circ}{\tau}$ from Eq. (4)–(8). The choice of the specific expression for $\overset{\circ}{\tau}$ depends on which of the stress rates $\overset{\circ}{\mathbf{T}}$, $\overset{\circ}{\mathbf{r}}^*$, $\overset{\circ}{\mathbf{S}}$ and $\overset{\circ}{\sigma}_r$ is directly available. If one uses $\overset{\circ}{\mathbf{T}}$ or $\overset{\circ}{\mathbf{r}}^*$, one will need the value of the spin rate $\boldsymbol{\Omega}$, the antisymmetric part of \mathbf{L} . The accuracy of $\boldsymbol{\Omega}$ strongly affects that of the overall solution (Iura and Atluri 1992). Details will be discussed later in the numerical implementation sections.

Note that $\boldsymbol{\sigma}_N^N$, \mathbf{T}_N^N , \mathbf{S}_N^N , \mathbf{r}_N^{*N} , $\boldsymbol{\sigma}_{rN}^N$ all equal $\boldsymbol{\tau}_N^N (= \boldsymbol{\tau}^N = \boldsymbol{\tau})$, and differ from $\boldsymbol{\sigma}_0^N$, \mathbf{T}_0^N , \mathbf{S}_0^N , \mathbf{r}_0^{*N} , $\boldsymbol{\sigma}_{r0}^N$, respectively. The latter refers to the undeformed coordinates, while $\boldsymbol{\tau}_N^N = \boldsymbol{\tau}_0^N$ because it is the stress at the deformed state and $N \neq 0$ independent of the original configuration. This distinction is important for the determination of the constitutive relations. The rates of the stress measures in the total Lagrangian description are simply those of $\boldsymbol{\tau}_0^N$, \mathbf{T}_0^N , $\boldsymbol{\sigma}_0^N$, \mathbf{r}_0^{*N} , \mathbf{S}_0^N , $\boldsymbol{\sigma}_{r0}^N$. The rates of $\boldsymbol{\tau}$ in both descriptions are the same ($\overset{\circ}{\tau} = \dot{\tau}$), since $\boldsymbol{\tau}_N^N = \boldsymbol{\tau}_0^N$. From Eq. (13.7:3), the rate of $\boldsymbol{\sigma}_0^N$ is

$$D\boldsymbol{\sigma}/Dt = J(D\boldsymbol{\tau}/Dt) + (DJ/Dt)\boldsymbol{\tau} = J[\dot{\boldsymbol{\tau}} + \text{tr}(\mathbf{D})\boldsymbol{\tau}],$$

in which DJ/Dt is from Eq. (13.6:6) and the superscript ‘ N ’ and the subscript ‘ 0 ’ have been dropped to make the notation consistent with Sec. 13.6. One can also obtain the rates of \mathbf{T}_0^N , $\boldsymbol{\sigma}_0^N$, \mathbf{r}_0^{*N} , \mathbf{S}_0^N , $\boldsymbol{\sigma}_{r0}^{*N}$ by taking the time derivatives of Eqs. (13.7:3), (13.7:4), (13.7:6), (13.7:8) and (13.7:9). The results are given in Eqs. (13.9:17–20). Comparing them with Eq. (4)–(8), we obtain

$$(9) \quad \overset{\circ}{\mathbf{T}} = \mathbf{F} \cdot \frac{D\mathbf{T}}{Dt} \frac{1}{J} = \overset{\circ}{\boldsymbol{\tau}} - \mathbf{L} \cdot \boldsymbol{\tau} + \text{tr}(\mathbf{D})\boldsymbol{\tau},$$

$$(10) \quad \overset{\circ}{\boldsymbol{\sigma}} = \frac{D\boldsymbol{\sigma}}{Dt} \frac{1}{J} = \overset{\circ}{\boldsymbol{\tau}} + \overset{\circ}{J}\boldsymbol{\tau} = \overset{\circ}{\boldsymbol{\tau}} + \text{tr}(\mathbf{D})\boldsymbol{\tau},$$

$$(11) \quad \overset{\circ}{\mathbf{r}}^* = \mathbf{F} \cdot \frac{D\mathbf{r}^*}{Dt} \cdot \mathbf{R}^T \frac{1}{J} = \overset{\circ}{\boldsymbol{\tau}} - \mathbf{L} \cdot \boldsymbol{\tau} + \text{tr}(\mathbf{D})\boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega},$$

$$(12) \quad \overset{\circ}{\mathbf{S}} = \mathbf{F} \cdot \frac{D\mathbf{S}}{Dt} \cdot \mathbf{F}^T \frac{1}{J} = \overset{\circ}{\boldsymbol{\tau}} - \mathbf{L} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{L}^T + \text{tr}(\mathbf{D})\boldsymbol{\tau},$$

$$(13) \quad \overset{\circ}{\boldsymbol{\sigma}}_r = \mathbf{R} \cdot \frac{D\boldsymbol{\sigma}_r}{Dt} \cdot \mathbf{R}^T = \overset{\circ}{\boldsymbol{\tau}} - \boldsymbol{\Omega} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega} + \text{tr}(\mathbf{D})\boldsymbol{\tau},$$

the relation between the time rates in the updated description and those in the total Lagrangian description. The strain energy density functions, for materials (called *hyper-elastic materials*) the functions are well defined, are often given in terms of strain measures referred to the undeformed state. The material rates of stresses in the total Lagrangian description can be expressed in terms of the derivatives of the energy density function with respect to deformation. In turn, which will be shown later, we can determine the corresponding rates in the updated description from such derivatives.

Recall that $\boldsymbol{\tau}$, \mathbf{L} , \mathbf{D} , $\boldsymbol{\Omega}$, \mathbf{F} , \mathbf{R} , J refer to the current configuration D_N and that \mathbf{F} , \mathbf{R} and J are deformation gradient tensor, the rotation tensor and the Jacobian.

Problem 17.1. Prove that $d\mathbf{F}^{-1}/Dt = -\mathbf{F}^{-1} \cdot \mathbf{L}$ where \mathbf{F} is the deformation gradient tensor and \mathbf{L} is the velocity gradient tensor.

17.4. INCREMENTAL EQUATIONS OF MOTION

The equation of motion in terms of the Cauchy stress in D_N is

$$(1) \quad \nabla_N \cdot \boldsymbol{\tau} + \rho_N \mathbf{b}^N = 0,$$

where ∇_N is the divergent operator, ρ_N the density and \mathbf{b}^N the sum of the body force and the inertia force per unit mass referring to the deformation state in D_N . The inertia force is mass time acceleration, and is a function of the rigid body motion and the deformation of the body. The acceleration must be updated in real time. Equation (1) is the equation of equilibrium, which, when expressed in terms of \mathbf{T}_N^{N+1} in D_N , becomes

$$(2) \quad \nabla_N \cdot \mathbf{T}_N^{N+1} + \rho_N \mathbf{b}_N^{N+1} = 0.$$

Using $\mathbf{b}_N^{N+1} = \mathbf{b}^N + \overset{\circ}{\mathbf{b}}\Delta t$ and $\mathbf{T}_N^N = \boldsymbol{\tau}^N$, one gets

$$\nabla_N \cdot (\boldsymbol{\tau}^N + \overset{\circ}{\mathbf{T}}\Delta t) + \rho_N(\mathbf{b}^N + \overset{\circ}{\mathbf{b}}\Delta t) = 0,$$

or

$$(3) \quad \nabla_N \cdot \overset{\circ}{\mathbf{T}} + \rho_N \overset{\circ}{\mathbf{b}} = 0,$$

where $\overset{\circ}{\mathbf{b}}$ is the sum of the incremental rate of the body force and inertia force in the updated description. Equation (3) is linear in $\overset{\circ}{\mathbf{T}}$ and is similar in form as Eq. (2) in terms of \mathbf{T}_N^{N+1} , but $\overset{\circ}{\mathbf{b}}$ can be a nonlinear function of strain, strain rate, displacement and velocities. When the system is in equilibrium so that the inertia force vanishes, then the effect of large deformation appears only implicitly in Eq. (3) through the gradient operator ∇_N . Substituting $\mathbf{T}_N^{N+1} = \mathbf{r}_N^{*N+1} \cdot (\mathbf{R}_N^{N+1})^T$ into Eq. (2) yields

$$\nabla_N \cdot [\mathbf{r}_N^{*N+1} \cdot (\mathbf{R}_N^{N+1})^T] + \rho_N \mathbf{b}_N^{N+1} = 0.$$

Using $\mathbf{r}_N^{*N} = \boldsymbol{\tau}$ and \mathbf{R}_N^{N+1} from Eq. (17.2:13), one finds

$$\nabla_N \cdot [(\boldsymbol{\tau} + \overset{\circ}{\mathbf{r}}^*\Delta t) \cdot (\mathbf{I} + \boldsymbol{\Omega}\Delta t)^T] + \rho_N(\mathbf{b}^N + \overset{\circ}{\mathbf{b}}\Delta t) = 0.$$

Expanding and linearizing the equation above and taking into account that $\boldsymbol{\Omega}^T = -\boldsymbol{\Omega}$, one obtains

$$(4) \quad \nabla_N \cdot (\overset{\circ}{\mathbf{r}}^* - \boldsymbol{\tau} \cdot \boldsymbol{\Omega}) + \rho_N \overset{\circ}{\mathbf{b}} = 0,$$

the equilibrium equation for the Biot–Luré stress rate. Equation (4) shows explicitly the effect of large deformation through the term $\boldsymbol{\tau} \cdot \boldsymbol{\Omega}$. The large deformation also affects the equilibrium equation implicitly through the gradient operator ∇_N . Similarly, the replacement of \mathbf{T} by $\mathbf{S} \cdot \mathbf{F}^T$ in Eq. (2)

$$\nabla_N \cdot [(\boldsymbol{\tau} + \overset{\circ}{\mathbf{S}}\Delta t) \cdot (\mathbf{I} + \mathbf{L}^T\Delta t)] + \rho_N(\mathbf{b}^N + \overset{\circ}{\mathbf{b}}\Delta t) = 0,$$

gives

$$(5) \quad \nabla_N \cdot (\overset{\circ}{\mathbf{S}} + \boldsymbol{\tau} \cdot \mathbf{L}^T) + \rho_N \overset{\circ}{\mathbf{b}} = 0,$$

the equilibrium equation for the second Piola–Kirchhoff stress rate with $\overset{\circ}{\mathbf{b}}$ representing the rate of inertia and body forces. In this case, the large deformation effects are through the term $\boldsymbol{\tau} \cdot \mathbf{L}$ and the gradient operator, and the rate of inertia force $\overset{\circ}{\mathbf{b}}$. The equilibrium equation for the corotational stress rate in the updated Lagrangian approach is

$$(6) \quad \nabla_N \cdot (\overset{\circ}{\sigma}_r - \mathbf{D} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\Omega}) + \rho_N \overset{\circ}{\mathbf{b}} = 0.$$

17.5. CONSTITUTIVE LAWS

A constitutive law of a material should be independent of the frame of reference. Ideally every term in a constitutive law must be objective, i.e., unaffected by translation and rotation of the frame of reference. A stress rate can be brought about not only by the strain rate \mathbf{D} but also by the spin rate $\boldsymbol{\Omega}$ of the material element. However, if the strain rate is zero or the local material motion is characterized by pure-spin, objectivity requires that there is no change in the stress noted by observer who is spinning along with the material (Truesdell and Noll 1965). Therefore, in establishing a consistent rate constitutive law, we must define a stress rate that does not change in the body fixed coordinates when the body undergoes a rigid spin (Atluri 1980, 1984, Reed and Atluri 1983, Rubinstein and Atluri 1983). Such a rate is called *objective rate*. We thus assume that the objective stress rate is a function of the strain rate only (not a function of the spin rate) and postulate their relationship in the spinning coordinate system.

Objective tensors are discussed in Sec. 13.9 and the updated stress rates in Sec. 17.3. Stresses and their rates in the updated description are in general not objective. However, for an objective tensor \mathbf{A} and its updated rate $\overset{\circ}{\mathbf{A}}$, the

following rate tensors are objective:

$$(1) \quad \overset{\partial}{\mathbf{A}} \equiv \overset{\circ}{\mathbf{A}} - \boldsymbol{\Omega} \cdot \mathbf{A} + \mathbf{A} \cdot \boldsymbol{\Omega} \quad (\text{Jaumann rate}),$$

$$(2) \quad \overset{\nabla}{\mathbf{A}} \equiv \overset{\circ}{\mathbf{A}} - \mathbf{L} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{L}^T \quad (\text{Oldroyd rate}).$$

Here $(\overset{\circ}{\cdot})$, $(\overset{\partial}{\cdot})$, and $(\overset{\nabla}{\cdot})$ denote, respectively, the *updated Lagrangian*, *Jaumann*, and *Oldroyd rates*, $\boldsymbol{\Omega}$ the spin tensor and \mathbf{L} the velocity gradient tensor. The Jaumann and Oldroyd rates of an objective tensor differ from one another by terms in \mathbf{L} and $\boldsymbol{\Omega}$. They are among the best known objective rates of objective tensors.

For the Kirchhoff stress, the Jaumann and Oldroyd rates are, respectively,

$$(3) \quad \overset{\partial}{\boldsymbol{\sigma}} = \overset{\circ}{\boldsymbol{\sigma}} - \boldsymbol{\Omega} \cdot \boldsymbol{\sigma}_N^N + \boldsymbol{\sigma}_N^N \cdot \boldsymbol{\Omega} \\ = \overset{\circ}{\boldsymbol{\tau}} + \text{tr}(\mathbf{D})\boldsymbol{\tau} - \boldsymbol{\Omega} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega} = \overset{\partial}{\boldsymbol{\tau}} + \text{tr}(\mathbf{D})\boldsymbol{\tau} \quad (\text{Jaumann rate}),$$

$$(4) \quad \overset{\nabla}{\boldsymbol{\sigma}} = \overset{\circ}{\boldsymbol{\sigma}} - \mathbf{L} \cdot \boldsymbol{\sigma}_N^N - \boldsymbol{\sigma}_N^N \cdot \mathbf{L}^T \\ = \overset{\circ}{\boldsymbol{\tau}} + \text{tr}(\mathbf{D})\boldsymbol{\tau} - \mathbf{L} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{L}^T = \overset{\nabla}{\boldsymbol{\tau}} + \text{tr}(\mathbf{D})\boldsymbol{\tau} \quad (\text{Oldroyd rate}).$$

Clearly the two rates are linked by the following equation

$$(5) \quad \overset{\partial}{\boldsymbol{\sigma}} = \overset{\nabla}{\boldsymbol{\sigma}} + \mathbf{D} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{D}.$$

The relation between the rates of various stress measures and the *Jaumann rate of the Kirchhoff stress tensor* in the updated description can be derived from Eqs. (17.3:4–8). One finds

$$(6) \quad \blacktriangle \quad \overset{\circ}{\mathbf{T}} = \overset{\circ}{\boldsymbol{\tau}} + \text{tr}(\mathbf{D})\boldsymbol{\tau} - \boldsymbol{\Omega} \cdot \boldsymbol{\tau} - \mathbf{D} \cdot \boldsymbol{\tau} = \overset{\partial}{\boldsymbol{\sigma}} - \mathbf{D} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\Omega},$$

$$(7) \quad \blacktriangle \quad \overset{\circ}{\mathbf{r}}^* = \overset{\circ}{\boldsymbol{\tau}} + \text{tr}(\mathbf{D})\boldsymbol{\tau} - \boldsymbol{\Omega} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega} - \mathbf{D} \cdot \boldsymbol{\tau} = \overset{\partial}{\boldsymbol{\sigma}} - \mathbf{D} \cdot \boldsymbol{\tau},$$

$$(8) \quad \blacktriangle \quad \overset{\circ}{\mathbf{S}} = \overset{\circ}{\mathbf{T}} - \boldsymbol{\tau} \cdot \mathbf{L}^T = \overset{\partial}{\boldsymbol{\sigma}} - \mathbf{D} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{D} = \overset{\nabla}{\boldsymbol{\sigma}},$$

$$(8a) \quad \blacktriangle \quad \overset{\circ}{\boldsymbol{\sigma}}_r = \overset{\partial}{\boldsymbol{\sigma}}.$$

Clearly $\overset{\circ}{\boldsymbol{\sigma}}_r$ is objective. From Eq. (5) and (8), one can show that $\overset{\circ}{\mathbf{S}}$ is also objective. Note that $\overset{\circ}{\mathbf{T}}$ and $\overset{\circ}{\mathbf{r}}^*$ are not objective.

Following Hill (1967), one can establish a *rate potential* $\overset{\circ}{V}_J$ in terms of the strain rate tensor \mathbf{D} such that the *constitutive equation for the Jaumann rate of the Kirchhoff stress tensor* in the update description is

$$(9) \quad \overset{\partial}{\boldsymbol{\sigma}} = \partial \overset{\circ}{V}_J(\mathbf{D}) / \partial \mathbf{D}.$$

Let $\overset{\circ}{U}$, $\overset{\circ}{W}$ and $\overset{\circ}{Q}$ denote the *rate potentials* for the first Piola–Kirchhoff stress tensor rate $\overset{\circ}{\mathbf{T}}$, the second Piola–Kirchhoff stress tensor rate $\overset{\circ}{\mathbf{S}}$, and the Biot–Luré stress tensor rate $\overset{\circ}{\mathbf{r}}^*$, respectively. Then

$$(10) \quad \overset{\circ}{\mathbf{T}} = \frac{\partial \overset{\circ}{U}(\mathbf{L})}{\partial \mathbf{L}^T}, \quad \overset{\circ}{\mathbf{S}} = \frac{\partial \overset{\circ}{W}(\mathbf{D})}{\partial \mathbf{D}}, \quad \overset{\circ}{\mathbf{r}} \equiv (\overset{\circ}{\mathbf{r}}^*)_s = \frac{\partial \overset{\circ}{Q}(\mathbf{D})}{\partial \mathbf{D}}, \quad \overset{\circ}{\boldsymbol{\sigma}}_r = \frac{\partial \overset{\circ}{V}_J(\mathbf{D})}{\partial \mathbf{D}}.$$

Then, noting that

$$\overset{\circ}{\mathbf{r}} = \overset{\circ}{\mathbf{S}} + (\boldsymbol{\tau} \cdot \mathbf{D} + \mathbf{D} \cdot \boldsymbol{\tau})/2, \quad \overset{\circ}{\boldsymbol{\sigma}}_r = \overset{\circ}{\mathbf{S}} + \boldsymbol{\tau} \cdot \mathbf{D} + \mathbf{D} \cdot \boldsymbol{\tau}$$

and using Eq. (8)–(10), we can show that the rate potential is

$$(11) \quad \overset{\circ}{W}(\mathbf{D}) = \overset{\circ}{V}_J(\mathbf{D}) - \boldsymbol{\tau} : (\mathbf{D} \cdot \mathbf{D}).$$

From Eq. (7), (9) and (10), we have

$$(12) \quad \overset{\circ}{Q}(\mathbf{D}) = \overset{\circ}{V}_J(\mathbf{D}) - \boldsymbol{\tau} : (\mathbf{D} \cdot \mathbf{D})/2 = \overset{\circ}{W}(\mathbf{D}) + \boldsymbol{\tau} : (\mathbf{D} \cdot \mathbf{D})/2.$$

Finally, from Eq. (6), (9) and (10), we obtain

$$(13) \quad \overset{\circ}{U}(\mathbf{L}) = \overset{\circ}{V}_J(\mathbf{D}) - \boldsymbol{\tau} : (\mathbf{D} \cdot \mathbf{D}) + \boldsymbol{\tau} : (\mathbf{L}^T \cdot \mathbf{L})/2.$$

The relations among the rate potentials can be easily derived

$$(14) \quad \begin{aligned} \overset{\circ}{E}(\mathbf{L}) &= \overset{\circ}{W}(\mathbf{D}) + \boldsymbol{\tau} : (\mathbf{L}^T \cdot \mathbf{L})/2 = \overset{\circ}{Q}(\mathbf{D}) + \boldsymbol{\tau} : (\mathbf{L}^T \cdot \mathbf{L})/2 - \boldsymbol{\tau} : (\mathbf{D} \cdot \mathbf{D})/2 \\ &= \overset{\circ}{V}_0(\mathbf{D}) + \boldsymbol{\tau} : (\mathbf{L}^T \cdot \mathbf{L})/2 = \overset{\circ}{U}(\mathbf{L}). \end{aligned}$$

Depending on applications, one expresses \mathbf{L} and \mathbf{D} in terms of displacement increments, \mathbf{Q} , etc.

The corresponding constitutive law for the *Oldroyd rate of the Kirchhoff stress tensor* σ is:

$$(15) \quad \overset{\nabla}{\sigma} = \partial \overset{\circ}{V}_0(\mathbf{D}) / \partial \mathbf{D},$$

where the *Oldroyd rate potential* $\overset{\circ}{V}_0$ is related to the *Jaumann rate potential* $\overset{\circ}{V}_J$ and the rate potential $\overset{\circ}{W}$ by

$$(16) \quad \overset{\circ}{V}_0(\mathbf{D}) = \overset{\circ}{V}_J(\mathbf{D}) - \boldsymbol{\tau} : (\mathbf{D} \cdot \mathbf{D}) = \overset{\circ}{W}(\mathbf{D}).$$

We can then express the rate potentials $\overset{\circ}{U}$, $\overset{\circ}{W}$ and $\overset{\circ}{Q}$ in terms $\overset{\circ}{V}_0$ by the following equations

$$(17) \quad \blacktriangle \quad \overset{\circ}{W}(\mathbf{D}) = \overset{\circ}{V}_0(\mathbf{D}),$$

$$(18) \quad \blacktriangle \quad \overset{\circ}{Q}(\mathbf{D}) = \overset{\circ}{V}_0(\mathbf{D}) + \boldsymbol{\tau} : (\mathbf{D} \cdot \mathbf{D})/2,$$

$$(19) \quad \blacktriangle \quad \overset{\circ}{U}(\mathbf{L}) = \overset{\circ}{V}_0[(\mathbf{L} + \mathbf{L}^T)/2] + \boldsymbol{\tau} : (\mathbf{L}^T \cdot \mathbf{L})/2.$$

Note that the *Oldroyd rate potential* $\overset{\circ}{V}_0$ is the rate potential $\overset{\circ}{W}$ for the *rate of the second Piola–Kirchhoff stress* in the updated description. The equations above are similar to Eq. (11)–(13). Substituting Eq. (15) into Eq. (6)–(8), one can express the updated Lagrangian stress rates in terms of the Oldroyd rate of the Kirchhoff stress

$$(20) \quad \overset{\circ}{\mathbf{T}} = \overset{\nabla}{\sigma} + \boldsymbol{\tau} \cdot \mathbf{L}^T, \quad \overset{\circ}{\mathbf{S}} = \overset{\nabla}{\sigma}, \quad \overset{\circ}{\mathbf{r}}^* = \overset{\nabla}{\sigma} + \boldsymbol{\tau} \cdot \mathbf{D}, \quad \overset{\circ}{\sigma}_r = \overset{\nabla}{\sigma} + \mathbf{D} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{D}.$$

Thus the Oldroyd rate of the Kirchhoff stress is actually the rate of the second Piola–Kirchhoff stress in the updated Lagrangian description.

For hyperelastic materials, there exists a strain energy function $W_0(\mathbf{C})$ with its second variation defined as $(\partial^2 W_0 / \partial C_{KL} \partial C_{IJ}) \delta C_{KL} \delta C_{IJ}$. The rate potential $\overset{\circ}{W}$ for the second Piola–Kirchhoff stress rate is derivable from the second variation of $W_0(\mathbf{C})$. From Eqs. (13.11:25) and (13.11:27), we have

$$\mathbf{S} = \partial W_0 / \partial \mathbf{E} = 2 \partial W_0 / \partial \mathbf{C} = 2(\partial W_0 / \partial C_{IJ}) \varepsilon_I \varepsilon_J,$$

where \mathbf{S} is the second Piola–Kirchhoff stress referred to the *undeformed coordinates*. Then its material rate in the total Lagrangian description is

$$\frac{D\mathbf{S}}{Dt} = 2 \frac{D}{Dt} \left(\frac{\partial W_0}{\partial \mathbf{C}} \right) = 2 \frac{\partial^2 W_0}{\partial \mathbf{C}^2} : \frac{D\mathbf{C}}{Dt} = 4 \frac{\partial^2 W_0}{\partial \mathbf{C}^2} : (\mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F}),$$

or, in indicial notation

$$D\mathbf{S}/Dt = 4(\partial^2 W_0 / \partial C_{KL} \partial C_{IJ})(\partial x_k / \partial X_K)(\partial x_l / \partial X_L) D_{kl} \varepsilon_I \varepsilon_J$$

where x_i and X_I are the *deformed* and the *undeformed coordinates referring to the D_N and D_0 configurations*, respectively. From Eq. (17.3:12), one has

$$(21) \quad \blacktriangle \quad \overset{\circ}{\mathbf{S}} = \mathbf{F} \cdot (D\mathbf{S}/Dt) \cdot \mathbf{F}^T / J = d_{ijkl} D_{kl} \mathbf{e}_i \mathbf{e}_j, \quad \text{or} \quad \overset{\circ}{S}_{ij} = d_{ijkl} D_{kl},$$

where $\mathbf{d} = d_{ijkl} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l$ is a fourth rank tensor with \mathbf{e}_i being the unit base vectors in the current configuration D_N and

$$(22) \quad d_{ijkl} = \frac{4}{J} \frac{\partial^2 W_0}{\partial C_{KL} \partial C_{IJ}} \frac{\partial x_i}{\partial X_I} \frac{\partial x_j}{\partial X_J} \frac{\partial x_k}{\partial X_K} \frac{\partial x_l}{\partial X_L},$$

which have the same symmetric properties as those of the elastic constants for anisotropic materials discussed in Sec. 6.1. Note that $\overset{\circ}{\mathbf{S}}$ and \mathbf{D} are objective. From Eq. (21), the rate potential can be written as

$$(23) \quad \overset{\circ}{W}(\mathbf{D}) = d_{ijkl} D_{ij} D_{kl} / 2,$$

which is a quadratic function of \mathbf{D} related to the second variation of W_0 (Pian and Tong 1971). With $\boldsymbol{\varepsilon}_I, \mathbf{e}_I$ being unit base vectors, $2(\partial^2 W_0 / \partial C_{KL} \partial C_{IJ})/J$ are the physical components of a fourth rank tensor referred to the undeformed coordinates and d_{ijkl} are the components of the corresponding tensor referred to the deformed coordinates. If $\boldsymbol{\varepsilon}_I, \mathbf{e}_I$ are contravariant base vectors, then d_{ijkl} are the corresponding contravariant components of the tensor $2(\partial^2 W_0 / \partial C_{KL} \partial C_{IJ})/J$ in the deformed coordinates. In this case, we generally denote $\boldsymbol{\varepsilon}_I, \mathbf{e}_I, d_{ijkl}$ by $\mathbf{G}^I, \mathbf{g}^I, d^{ijkl}$, respectively.

If the material is isotropic with $W_0(\mathbf{C}) = W_0(I_1, I_2, I_3)$, we have

$$(\partial^2 W_0 / \partial C_{KL} \partial C_{IJ}) \boldsymbol{\varepsilon}_I \boldsymbol{\varepsilon}_J \boldsymbol{\varepsilon}_K \boldsymbol{\varepsilon}_L = (\partial W_0 / \partial I_2)(\mathbf{II} - \mathbf{H}) + (\partial W_0 / \partial I_3)\mathbf{A} + \mathbf{G},$$

where $\mathbf{H}, \mathbf{A}, \mathbf{G}$ are fourth rank tensors defined as

$$\begin{aligned} \mathbf{H} &= \partial \mathbf{C} / \partial \mathbf{C} = \delta_{IK} \delta_{JL} \boldsymbol{\varepsilon}_I \boldsymbol{\varepsilon}_J \boldsymbol{\varepsilon}_K \boldsymbol{\varepsilon}_L, \\ \mathbf{A} &= \partial(I_3 \mathbf{C}^{-1}) / \partial \mathbf{C} = e_{JKN} e_{ILM} C_{MN} \boldsymbol{\varepsilon}_I \boldsymbol{\varepsilon}_J \boldsymbol{\varepsilon}_K \boldsymbol{\varepsilon}_L, \\ \mathbf{G} &= \frac{\partial^2 W_0}{\partial I_1^2} \mathbf{II} + \frac{\partial^2 W_0}{\partial I_2^2} (I_1 \mathbf{I} - \mathbf{C})(I_1 \mathbf{I} - \mathbf{C}) + \frac{\partial^2 W_0}{\partial I_3^2} I_3^2 \mathbf{C}^{-1} \mathbf{C}^{-1} \\ &\quad + \frac{\partial^2 W_0}{\partial I_1 \partial I_2} (2I_1 \mathbf{II} - \mathbf{IC} - \mathbf{CI}) + \frac{\partial^2 W_0}{\partial I_1 \partial I_3} I_3 (\mathbf{IC}^{-1} + \mathbf{C}^{-1} \mathbf{I}) \\ &\quad + \frac{\partial^2 W_0}{\partial I_2 \partial I_3} I_3 [\mathbf{C}^{-1} (I_1 \mathbf{I} - \mathbf{C}) + (I_1 \mathbf{I} - \mathbf{C}) \mathbf{C}^{-1}], \end{aligned}$$

where e_{JKN} is the permutation tensor. The derivation is left to the readers. Note that $\mathbf{I}(= \delta_{IJ} \boldsymbol{\varepsilon}_I \boldsymbol{\varepsilon}_J)$ is the identity tensor. Both $\mathbf{II}(= \delta_{IJ} \delta_{KL} \boldsymbol{\varepsilon}_I \boldsymbol{\varepsilon}_J \boldsymbol{\varepsilon}_K \boldsymbol{\varepsilon}_L)$ and $\mathbf{IC}[= (\mathbf{CI})^T]$ are tensor of rank 4.

A substitution of Eq. (23) into Eq. (11), (12) and (14) yields the rate potentials for other stress measures and into Eq. (16), (18) and (19) yields the rate potentials for stress measures related to the Oldroyd rate [Eq. (15)]. From Eq. (6)–(8), or (20), the constitutive laws can be written as

$$(24) \quad \begin{aligned} \overset{\circ}{\mathbf{S}} &= \mathbf{d} : \mathbf{D}, & \overset{\circ}{\mathbf{T}} &= \mathbf{d} : \mathbf{D} + \boldsymbol{\tau} \cdot \mathbf{L}^T, \\ \overset{\circ}{\boldsymbol{\sigma}} &= \mathbf{d} : \mathbf{D} + \mathbf{L} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{L}^T, & \overset{\circ}{\boldsymbol{\tau}} &= \mathbf{d} : \mathbf{D} + \mathbf{L} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{L}^T - \text{tr}(\mathbf{L})\boldsymbol{\tau} \\ \overset{\circ}{\mathbf{r}} &= \mathbf{d} : \mathbf{D} + \boldsymbol{\tau} \cdot \mathbf{D}, & \overset{\circ}{\boldsymbol{\sigma}}_r &= \mathbf{d} : \mathbf{D} + \mathbf{D} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{D}. \end{aligned}$$

One sees that the constitutive laws consist of two parts: one part relates to the energy density represented by the first term on the right hand side of the equations above. The second part includes the remaining terms of the equations involving the interaction between the current Cauchy stress tensor $\boldsymbol{\tau}$ and the incremental change of geometry.

Problem 17.2. Show that

$$\partial \mathbf{C}^{-1} / \partial \mathbf{C} = -\mathbf{C}^{-1} \mathbf{C}^{-1} + e_{JKN} e_{ILM} C_{MN} \boldsymbol{\varepsilon}_I \boldsymbol{\varepsilon}_J \boldsymbol{\varepsilon}_K \boldsymbol{\varepsilon}_L / I_3.$$

(Hint: Consider $\mathbf{C}: \mathbf{C}^{-1} = \mathbf{I}$)

17.6. INCREMENTAL VARIATIONAL PRINCIPLES IN TERMS OF $\overset{\circ}{\mathbf{T}}$

We shall construct variational functionals involving stress and deformation rates in such a way that the conditions for the vanishing of the variation of the functional leads to Eulerian equations (the field equations), for the rate variables in the continuum. Thus, the solution of the field equations of the boundary-value problem corresponds to the stationary condition of the functional and vice versa. Pian and Tong (1971) first introduced incremental functional for finite deformation analyses. Different variational principles can be derived by selectively relaxing constraints on the field variables using Lagrange multipliers to increase the number of independent field variables of the functional (Sec. 10.5), or by enforcing selected constraints *a priori* to reduce the independent field variable number. The next two subsections follows mostly the work of Atluri (1979, 1980a, 1995 with Cazzani).

A Four-Field Principle. The following is a variational principle of the displacement rate tensor \mathbf{v} , the spin rate tensor $\boldsymbol{\Omega}$, the stretch rate tensor \mathbf{D} , and the material rate of the second Piola–Kirchhoff stress tensor $\overset{\circ}{\mathbf{T}}$ in the updated description:

$$(1) \quad \ddot{\Pi}_1(\mathbf{v}, \boldsymbol{\Omega}, \mathbf{D}, \overset{\circ}{\mathbf{T}}) = \int_{V_N} \left\{ \dot{Q}(\mathbf{D}) + \frac{1}{2} \boldsymbol{\tau} : (\boldsymbol{\Omega}^T \cdot \boldsymbol{\Omega}) + \boldsymbol{\tau} : (\boldsymbol{\Omega}^T \cdot \mathbf{D}) \right. \\ \left. + \overset{\circ}{\mathbf{T}}^T : (\nabla_N \mathbf{v} - \boldsymbol{\Omega} - \mathbf{D}) - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \right\} dV \\ - \int_{S_{uN}} \mathbf{n} \cdot \overset{\circ}{\mathbf{T}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) dA - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA,$$

where $\overset{\circ}{\mathbf{T}}$, $\bar{\mathbf{v}}$ are respectively the prescribed increment rates of the traction on $S_{\sigma N}$ and the displacement on S_{uN} . V_N the volume of the body in the reference configuration D_N , and \mathbf{n} the unit outward normal to the surface $S_{\sigma N}$ or S_{uN} . The admissible requirements are: \mathbf{v} continuous, $\boldsymbol{\Omega}$ skew-symmetric and \mathbf{D} symmetric. The last two conditions are called the rigid constraint, which must be enforced in constructing a weak form solution. In this case, $\boldsymbol{\Omega}$ has only three independent components and \mathbf{D} has six rather than nine for a tensor of rank 2.

We shall derive the stationary condition of $\overset{\circ}{\Pi}_1$ for arbitrary variation of the four field variables subjected to the rigid constraints to obtain the appropriate field equations and natural boundary conditions. To derive the variation of $\overset{\circ}{\Pi}_1$, one must take into account that

$$\overset{\circ}{\mathbf{T}}^T : \delta \mathbf{D} = (\overset{\circ}{\mathbf{T}}^T)_s : \delta \mathbf{D}, \quad \overset{\circ}{\mathbf{T}}^T : \delta \boldsymbol{\Omega} = -\overset{\circ}{\mathbf{T}}^T : \delta \boldsymbol{\Omega}^T = -\overset{\circ}{\mathbf{T}} : \delta \boldsymbol{\Omega},$$

for symmetric $\delta \mathbf{D}$ and skew-symmetric $\delta \boldsymbol{\Omega}$. One also needs to use the following trace and divergent properties:

$$\begin{aligned} \boldsymbol{\tau} : (\boldsymbol{\Omega}^T \cdot \delta \mathbf{D}) &= \boldsymbol{\tau} : (\delta \mathbf{D} \cdot \boldsymbol{\Omega}) = -\boldsymbol{\tau} : (\delta \mathbf{D} \cdot \boldsymbol{\Omega}^T) = -(\boldsymbol{\tau} \cdot \boldsymbol{\Omega}) : \delta \mathbf{D}, \\ \boldsymbol{\tau} : (\delta \boldsymbol{\Omega}^T \cdot \mathbf{D}) &= (\boldsymbol{\tau} \cdot \mathbf{D}) : \delta \boldsymbol{\Omega}^T = (\mathbf{D} \cdot \boldsymbol{\tau}) : \delta \boldsymbol{\Omega}, \\ \boldsymbol{\tau} : \delta(\boldsymbol{\Omega}^T \cdot \boldsymbol{\Omega}) &= \boldsymbol{\tau} : (\delta \boldsymbol{\Omega}^T \cdot \boldsymbol{\Omega} + \boldsymbol{\Omega}^T \cdot \delta \boldsymbol{\Omega}) \\ &= (\boldsymbol{\tau} \cdot \boldsymbol{\Omega}^T) : \delta \boldsymbol{\Omega}^T + (\boldsymbol{\Omega} \cdot \boldsymbol{\tau}) : \delta \boldsymbol{\Omega} = 2(\boldsymbol{\Omega} \cdot \boldsymbol{\tau}) : \delta \boldsymbol{\Omega}, \\ \int_{V_N} \overset{\circ}{\mathbf{T}} : \nabla_N \delta \mathbf{v} dV &= \int_{S_N} \mathbf{n} \cdot \overset{\circ}{\mathbf{T}} \cdot \delta \mathbf{v} dA - \int_{V_N} (\nabla_N \cdot \overset{\circ}{\mathbf{T}}) \cdot \delta \mathbf{v} dV. \end{aligned}$$

The first variation of $\overset{\circ}{\Pi}_1$ is

$$\begin{aligned} \delta \overset{\circ}{\Pi}_1 &= \int_{V_N} \left\{ \left[\frac{\partial \dot{Q}}{\partial \mathbf{D}} - (\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s \right] : \delta \mathbf{D} + \delta \overset{\circ}{\mathbf{T}}^T : (\nabla_N \mathbf{v} - \boldsymbol{\Omega} - \mathbf{D}) \right. \\ &\quad \left. + (\overset{\circ}{\mathbf{T}} + \mathbf{D} \cdot \boldsymbol{\tau} + \boldsymbol{\Omega} \cdot \boldsymbol{\tau}) : \delta \boldsymbol{\Omega} - (\nabla_N \cdot \overset{\circ}{\mathbf{T}} + \rho_N \overset{\circ}{\mathbf{b}}) \cdot \delta \mathbf{v} \right\} dV \\ &\quad - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} - \mathbf{n} \cdot \overset{\circ}{\mathbf{T}} \cdot \delta \mathbf{v} dA - \int_{S_{uN}} \mathbf{n} \cdot \overset{\circ}{\mathbf{T}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) dA, \end{aligned}$$

with the rigid constraints $(\delta \mathbf{D})_a = 0$ and $(\delta \boldsymbol{\Omega})_s = 0$. The stationary condition of $\overset{\circ}{\Pi}_1$ gives the following incremental field equations,

$$(2) \quad \partial \dot{Q} / \partial \mathbf{D} = (\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s, \quad (\text{constitutive law in } V_N),$$

$$(3) \quad \nabla_N \mathbf{v} = \mathbf{D} + \boldsymbol{\Omega} \quad (\text{compatibility condition in } V_N),$$

$$(4) \quad (\overset{\circ}{\mathbf{T}} + \mathbf{D} \cdot \boldsymbol{\tau} + \boldsymbol{\Omega} \cdot \boldsymbol{\tau})_a = 0 \quad (\text{angular momentum balance in } V_N),$$

$$(5) \quad \nabla_N \cdot \overset{\circ}{\mathbf{T}} + \rho_N \overset{\circ}{\mathbf{b}} = 0 \quad (\text{linear momentum balance in } V_N),$$

and the following natural boundary conditions,

$$(6) \quad \mathbf{n} \cdot \overset{\circ}{\mathbf{T}} = \overset{\circ}{\mathbf{T}} \quad (\text{traction boundary condition on } S_{\sigma N}),$$

$$(7) \quad \mathbf{v} = \bar{\mathbf{v}} \quad (\text{displacement boundary condition on } S_{uN}).$$

In the following subsections, we will discuss several variational principles with *a priori* imposed conditions to reduce the number of field variables in Eq. (1), and with the Lagrange multipliers to introduce new field variables.

A Three-Field Principle. A variational principle with a functional involving three tensors \mathbf{v} , $\boldsymbol{\Omega}$ and $\overset{\circ}{\mathbf{T}}$ can be obtained

by imposing

$$(8) \quad \mathbf{D} = (\nabla_N \mathbf{v})_s ,$$

a priori in Eq. (1) to give

$$(9) \quad \begin{aligned} \overset{\circ}{\Pi}_2(\mathbf{v}, \boldsymbol{\Omega}, \overset{\circ}{\mathbf{T}}) &= \int_{V_N} \{ \overset{\circ}{Q}[(\nabla_N \mathbf{v})_s] + \frac{1}{2} \boldsymbol{\tau} : (\boldsymbol{\Omega}^T \cdot \boldsymbol{\Omega}) + \boldsymbol{\tau} : [\boldsymbol{\Omega}^T \cdot (\nabla_N \mathbf{v})_s] \\ &+ \overset{\circ}{\mathbf{T}}^T \cdot [(\nabla_N \mathbf{v})_a - \boldsymbol{\Omega}] - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \} dV - \int_{S_{uN}} (\mathbf{n} \cdot \overset{\circ}{\mathbf{T}}) \cdot (\mathbf{v} - \bar{\mathbf{v}}) dA - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA . \end{aligned}$$

The admissible requirements for the field variables are: \mathbf{v} is continuous and $\boldsymbol{\Omega}$ is skew-symmetric. The stationary condition of $\overset{\circ}{\Pi}_2$ gives the same set of equations as Eq. (2)–(7) except that \mathbf{D} is replaced by $(\nabla_N \mathbf{v})_s$ in Eq. (1) and (4) and that Eq. (3) becomes

$$(10) \quad (\nabla_N \mathbf{v})_a = \boldsymbol{\Omega} .$$

A Purely Kinematic Principle. A variational principle with a functional involving \mathbf{v} only can be derived by requiring, *a priori*,

$$(11) \quad \nabla_N \mathbf{v} = \mathbf{D} + \boldsymbol{\Omega} = \mathbf{L} , \quad \mathbf{D} = (\nabla_N \mathbf{v})_s , \quad \boldsymbol{\Omega} = (\nabla_N \mathbf{v})_a$$

and $\mathbf{v} = \bar{\mathbf{v}}$ on S_{uN} . Equation (1) reduces to

$$(12) \quad \overset{\circ}{\Pi}_3(\mathbf{v}) = \int_{V_N} \{ \overset{\circ}{E}(\mathbf{v}) - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \} dV - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA ,$$

where $\overset{\circ}{E}$ is given in Eq. (17.5:14) whose expression depends on the choice of rate potential. In Eq. (12), \mathbf{v} must be continuous and equals $\bar{\mathbf{v}}$ on S_{uN} .

If $\overset{\circ}{E}$ is in terms of $\overset{\circ}{W}$, Eq. (12) is the second variation of the functional of Eq. (13.13:12) (Pian and Tong 1971) derived by using Eqs. (17.2:20–22). One can show that

$$W(\mathbf{C}_0^{N+1}) = W(\mathbf{C}_0^N) + \boldsymbol{\tau}^N : \mathbf{D} \Delta t + [\overset{\circ}{W}(\mathbf{D}) + \boldsymbol{\tau}^N : (\mathbf{L}^T \cdot \mathbf{L})/2] \Delta t^2 ,$$

where $\overset{\circ}{W}(\mathbf{D})$ is given in Eq. (17.5:23). Equation (12) then becomes

$$(12a) \quad \begin{aligned} \overset{\circ}{\Pi}_3(\mathbf{v}) &= \int_{V_N} \{ \overset{\circ}{W}[(\nabla_N \mathbf{v})_s] + \frac{1}{2} \boldsymbol{\tau} : [(\nabla_N \mathbf{v})^T \cdot (\nabla_N \mathbf{v})] - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \} dV \\ &- \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA . \end{aligned}$$

One can introduce $\boldsymbol{\Omega}$ as an additional independent field variable by modifying $\overset{\circ}{\Pi}_3$ as

$$(13) \quad \begin{aligned} \overset{\circ}{\Pi}_3^*(\mathbf{v}) &= \int_{V_N} \{ \overset{\circ}{W}[(\nabla_N \mathbf{v})_s] + \frac{1}{2} \boldsymbol{\tau} : [(\nabla_N \mathbf{v})^T \cdot (\nabla_N \mathbf{v})] \\ &+ \frac{c}{2} [(\nabla_N \mathbf{v})_a - \boldsymbol{\Omega}]^2 - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \} dV - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA , \end{aligned}$$

where c is a given positive constant, and

$$[(\nabla_N \mathbf{v})_a - \boldsymbol{\Omega}]^2 = [(\nabla_N \mathbf{v})_a - \boldsymbol{\Omega}] : [(\nabla_N \mathbf{v})_a - \boldsymbol{\Omega}] .$$

The admissibility requirements for the field variables are: \mathbf{v} is continuous and equals $\bar{\mathbf{v}}$ on S_{uN} , and $\boldsymbol{\Omega}$ is skew-symmetric. Equation (13) is equivalent to the *penalty function approach*. One usually chooses c to be of the order of the elastic modulus of the material.

A Complementary Three-Field Principle. We shall derive a *complementary principle* involving \mathbf{v} , $\boldsymbol{\Omega}$ and $\overset{\circ}{\mathbf{T}}$ based on the *complementary strain energy density*. The concepts of complementary work and complementary strain energy are explained in Sec. 10.9. We call the variational principle complementary as it is a function of stress-rate tensor. We introduce the contact transformation

$$(14) \quad -\overset{\circ}{Q}_c[(\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s] = \overset{\circ}{Q}(\mathbf{D}) + \boldsymbol{\tau} : (\boldsymbol{\Omega}^T \cdot \mathbf{D}) - (\overset{\circ}{\mathbf{T}})_s : \mathbf{D},$$

to eliminate the explicit dependence on \mathbf{D} in Eq. (1). Now $\overset{\circ}{Q}_c$ is a function of a stress rate $(\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s$ and is called the *complementary rate potential*. We first show that the gradient of $\overset{\circ}{Q}_c$ with respect to $(\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s$ is the stretch rate tensor \mathbf{D} . Consider the variation of $\overset{\circ}{Q}_c$

$$-\delta\overset{\circ}{Q}_c = [\frac{\partial\overset{\circ}{Q}(\mathbf{D})}{\partial\mathbf{D}} - (\boldsymbol{\tau} \cdot \boldsymbol{\Omega})] : \delta\mathbf{D} - (\boldsymbol{\tau} \cdot \delta\boldsymbol{\Omega} + \delta\boldsymbol{\tau} \cdot \boldsymbol{\Omega}) : \mathbf{D} - \delta\overset{\circ}{\mathbf{T}}_s : \mathbf{D} - \overset{\circ}{\mathbf{T}}_s : \delta\mathbf{D}.$$

Taking into account the constitutive law Eq. (2) and the identities

$$\begin{aligned} \boldsymbol{\tau} : (\delta\boldsymbol{\Omega}^T \cdot \mathbf{D}) &= (\boldsymbol{\tau} \cdot \delta\boldsymbol{\Omega}^T) : \mathbf{D}, \quad \boldsymbol{\tau} : (\boldsymbol{\Omega}^T \cdot \delta\mathbf{D}) = -(\boldsymbol{\tau} \cdot \boldsymbol{\Omega}) : \delta\mathbf{D}, \\ (\boldsymbol{\tau} \cdot \boldsymbol{\Omega})_a : \delta\mathbf{D} &= \delta(\boldsymbol{\tau} \cdot \boldsymbol{\Omega})_a : \mathbf{D} = 0, \end{aligned}$$

for symmetric $\boldsymbol{\tau}$ and \mathbf{D} , and skew-symmetric $\boldsymbol{\Omega}$, one finds

$$\begin{aligned} -\delta\overset{\circ}{Q}_c &= (\boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s : \delta\mathbf{D} - (\boldsymbol{\tau} \cdot \boldsymbol{\Omega}) : \delta\mathbf{D} - \delta(\boldsymbol{\tau} \cdot \boldsymbol{\Omega}) : \mathbf{D} - (\delta\overset{\circ}{\mathbf{T}})_s : \mathbf{D} \\ &= -(\boldsymbol{\tau} \cdot \boldsymbol{\Omega})_a : \delta\mathbf{D} - \delta(\boldsymbol{\tau} \cdot \boldsymbol{\Omega})_a : \mathbf{D} - \delta(\boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s : \mathbf{D} - \delta(\overset{\circ}{\mathbf{T}})_s : \mathbf{D} \\ &= -\delta(\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s : \mathbf{D}. \end{aligned}$$

In other words, $\overset{\circ}{Q}_c$ is a function of $(\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s$ only and

$$(15) \quad \partial\overset{\circ}{Q}_c / \partial(\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s = \mathbf{D}.$$

A substitution of Eq. (14) into Eq. (1) yields

$$(16) \quad \overset{\circ}{\Pi}_4(\mathbf{v}, \boldsymbol{\Omega}, \overset{\circ}{\mathbf{T}}) = \int_{V_N} \left\{ -\overset{\circ}{Q}_c[(\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s] + \frac{\boldsymbol{\tau} : (\boldsymbol{\Omega}^T \cdot \boldsymbol{\Omega})}{2} - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \right. \\ \left. + \overset{\circ}{\mathbf{T}}^T : (\nabla_N \mathbf{v} - \boldsymbol{\Omega}) \right\} dV - \int_{S_{uN}} \mathbf{n} \cdot \overset{\circ}{\mathbf{T}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) dA - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA.$$

The admissibility requirements for the field variables are: \mathbf{v} is continuous, $\overset{\circ}{\mathbf{T}}$ is piecewise continuous and $\boldsymbol{\Omega}$ is skew-symmetric. It can be shown that the field equations associated with $\overset{\circ}{\Pi}_4$ are

$$(17) \quad \nabla_N \mathbf{v} = \partial\overset{\circ}{Q}_c / \partial(\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s + \boldsymbol{\Omega}, \quad (\text{compatibility condition}),$$

$$(18) \quad \left[\overset{\circ}{\mathbf{T}} + \frac{\partial\overset{\circ}{Q}_c}{\partial(\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s} \cdot \boldsymbol{\tau} + \boldsymbol{\Omega} \cdot \boldsymbol{\tau} \right]_a = 0, \quad (\text{angular moment balance}),$$

and the equilibrium equation Eq. (5) in V_N . The natural boundary conditions are Eq. (6) and (7).

One can independently enforce Eq. (18) by adding the term $c[\overset{\circ}{\mathbf{T}} + (\nabla_N \mathbf{v})^T \cdot \boldsymbol{\tau}]_a^2/2$ to the volume integral in Eq. (16). The new functional is

$$(19) \quad \overset{\circ}{\Pi}_4^*(\mathbf{v}, \boldsymbol{\Omega}, \overset{\circ}{\mathbf{T}}) = \int_{V_N} \left\{ -\overset{\circ}{Q}_c[(\overset{\circ}{\mathbf{T}} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega})_s] + \frac{\boldsymbol{\tau} : (\boldsymbol{\Omega}^T \cdot \boldsymbol{\Omega})}{2} + \overset{\circ}{\mathbf{T}} : (\nabla_N \mathbf{v} - \boldsymbol{\Omega}) \right. \\ \left. + \frac{c}{2} (\overset{\circ}{\mathbf{T}} + \nabla_N \mathbf{v} \cdot \boldsymbol{\tau})_a^2 - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \right\} dV - \int_{S_{uN}} \mathbf{n} \cdot \overset{\circ}{\mathbf{T}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) dA - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA,$$

where c is a positive constant of the penalty function approach. In Eq. (19), c is of order $1/\tau^N$ or larger.

17.7. INCREMENTAL VARIATIONAL PRINCIPLES IN TERMS OF $\overset{\circ}{\mathbf{r}}^*$

One can derive incremental variational principles in terms $\overset{\circ}{\mathbf{r}}^*$ simply by substituting

$$\overset{\circ}{\mathbf{T}} = \overset{\circ}{\mathbf{r}}^* - \boldsymbol{\tau} \cdot \boldsymbol{\Omega}$$

into Eq. (17.6:1) and using the following properties

$$\begin{aligned} -(\Omega^T \cdot \tau) : (\nabla_N \mathbf{v} - \Omega - \mathbf{D}) &= (\Omega \cdot \tau) : (\nabla_N \mathbf{v} - \Omega - \mathbf{D}) \\ &= \tau : (\Omega^T \cdot \nabla_N \mathbf{v}) - \tau : (\Omega^T \cdot \Omega) - \tau : (\Omega^T \cdot \mathbf{D}). \end{aligned}$$

One finds

$$(1) \quad \overset{\circ}{\Pi}_1(\mathbf{v}, \Omega, \mathbf{D}, \overset{\circ}{\mathbf{T}}) = \int_{V_N} \left\{ \overset{\circ}{Q}(\mathbf{D}) + \frac{1}{2} \tau : (\Omega^T \cdot \Omega) + \tau : (\Omega^T \cdot \mathbf{D}) \right. \\ \left. + \overset{\circ}{\mathbf{T}}^T : (\nabla_N \mathbf{v} - \Omega - \mathbf{D}) - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \right\} dV \\ - \int_{S_{uN}} \mathbf{n} \cdot \overset{\circ}{\mathbf{T}} \cdot (\mathbf{v} - \bar{\mathbf{v}}) dA - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA,$$

The admissibility requirements for the field variables are: \mathbf{v} is continuous, Ω skew-symmetric, \mathbf{D} symmetric, and $\overset{\circ}{\mathbf{r}}^*$ piecewise continuous.

By requiring $\mathbf{D} = (\nabla_N \mathbf{v})_s$ a priori in Eq. (1), one obtains a three-field principle in terms \mathbf{v} , Ω and $\overset{\circ}{\mathbf{r}}^*$. The functional is

$$(2) \quad \overset{\circ}{\Pi}_6(\mathbf{v}, \Omega, \overset{\circ}{\mathbf{r}}^*) = \int_{V_N} \left\{ \overset{\circ}{Q}[(\nabla_N \mathbf{v})_s] + \tau : [(\nabla_N \mathbf{v} - \frac{\Omega}{2})^T \cdot \Omega] - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \right. \\ \left. + \overset{\circ}{\mathbf{r}}^{*T} : [(\nabla_N \mathbf{v})_a - \Omega] \right\} dV - \int_{S_{uN}} \mathbf{n} \cdot (\overset{\circ}{\mathbf{r}}^{*T} - \tau \cdot \Omega) \cdot (\mathbf{v} - \bar{\mathbf{v}}) dA - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA.$$

The admissibility requirements for the field variables are the same of those for the functional $\overset{\circ}{\Pi}_5$, except that \mathbf{D} does not appear here. If we further require that $\Omega = (\nabla_N \mathbf{v})_a$ and $\mathbf{v} = \bar{\mathbf{v}}$ on S_{uN} are satisfied *a priori*, $\overset{\circ}{\Pi}_6$ reduces to $\overset{\circ}{\Pi}_3$ as given in Eq. (17.6:12).

As in the previous subsection, the use of the contact transformation

$$(3) \quad \overset{\circ}{Q}(\mathbf{D}) - (\overset{\circ}{\mathbf{r}}^*)_s : \mathbf{D} = -\overset{\circ}{Q}_c[(\overset{\circ}{\mathbf{r}}^*)_s]$$

in Eq. (1) leads to an alternate three-field principle with the functional

$$(4) \quad \overset{\circ}{\Pi}_7(\mathbf{v}, \Omega, \overset{\circ}{\mathbf{r}}^*) = \int_{V_N} \left\{ -\overset{\circ}{Q}_c[(\overset{\circ}{\mathbf{r}}^*)_s] + \tau : [(\nabla_N \mathbf{v} - \frac{\Omega}{2})^T \cdot \Omega] - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \right. \\ \left. + \overset{\circ}{\mathbf{r}}^{*T} : [(\nabla_N \mathbf{v})_a - \Omega] \right\} dV - \int_{S_{uN}} \mathbf{n} \cdot (\overset{\circ}{\mathbf{r}}^{*T} - \tau \cdot \Omega) \cdot (\mathbf{v} - \bar{\mathbf{v}}) dA - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA.$$

The admissibility requirements for the field variables are also the same of those for the functional $\overset{\circ}{\Pi}_5$ without the dependence on \mathbf{D} .

Problem 17.3. Show that a four-field principle involving \mathbf{v} , Ω , \mathbf{D} and $\overset{\circ}{\mathbf{S}}$ can be written as

$$\overset{\circ}{\Pi}_1(\mathbf{v}, \Omega, \mathbf{D}, \overset{\circ}{\mathbf{S}}) = \int_{V_N} \left\{ \overset{\circ}{W}(\mathbf{D}) + \overset{\circ}{\mathbf{S}} : (\nabla_N \mathbf{v} - \Omega - \mathbf{D}) - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \right. \\ \left. + \tau : [(\nabla_N \mathbf{v})^T - (\mathbf{D} + \Omega)^T / 2] \cdot (\mathbf{D} + \Omega) \right\} dV \\ - \int_{S_{uN}} \mathbf{n} \cdot [\overset{\circ}{\mathbf{S}} + \tau \cdot (\mathbf{D} + \Omega)^T] \cdot (\mathbf{v} - \bar{\mathbf{v}}) dA - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA.$$

17.8. INCOMPRESSIBLE AND NEARLY INCOMPRESSIBLE MATERIALS

For materials approaching incompressibility, the formulation in Secs. 17.6 and 17.7 breaks down because the equivalent bulk modulus tends to infinity. To circumvent the difficulty, we shall modify the variational principles discussed before. We first introduce the deviatoric variables

$$(1) \quad \mathbf{L}' = \mathbf{L} - \mathbf{I} \overset{\circ}{J} / 3, \quad \mathbf{D}' = \mathbf{D} - \mathbf{I} \overset{\circ}{J} / 3,$$

in which $\overset{\circ}{J}$ is, from Eq. (17.2:9),

$$(2) \quad \overset{\circ}{J} = \text{tr}(\mathbf{L}) = \text{tr}(\mathbf{D}).$$

Denoting the rate potentials given in Sec. 17.5 as follows:

$$(3) \quad \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J}) \equiv \overset{\circ}{Q}(\mathbf{D}), \quad \overset{\circ}{W}^*(\mathbf{D}', \overset{\circ}{J}) \equiv \overset{\circ}{W}(\mathbf{D}), \quad \overset{\circ}{U}^*(\mathbf{L}', \overset{\circ}{J}) \equiv \overset{\circ}{U}(\mathbf{L}),$$

one has the constitutive laws

$$(4) \quad \overset{\circ}{\mathbf{S}}' \equiv \frac{\partial \overset{\circ}{W}^*(\mathbf{D}', \overset{\circ}{J})}{\partial \mathbf{D}'}, \quad (\overset{\circ}{\mathbf{r}'^*})_s \equiv \frac{\partial \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J})}{\partial \mathbf{D}'}, \quad \overset{\circ}{\mathbf{T}}'^T \equiv \frac{\partial \overset{\circ}{U}^*(\mathbf{L}', \overset{\circ}{J})}{\partial \mathbf{L}'},$$

$$(5) \quad -\overset{\circ}{p}_r \equiv \partial \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J}) / \partial \overset{\circ}{J} = [\partial \overset{\circ}{Q}(\mathbf{D}) / \partial \mathbf{D}] : \mathbf{I} / 3 = \overset{\circ}{\mathbf{r}}^* : \mathbf{I} / 3 = \text{tr}(\overset{\circ}{\mathbf{r}}^*) / 3,$$

$$-\overset{\circ}{p}_s \equiv \partial \overset{\circ}{W}^*(\mathbf{D}', \overset{\circ}{J}) / \partial \overset{\circ}{J} = \text{tr}(\overset{\circ}{\mathbf{S}}) / 3, \quad -\overset{\circ}{p}_t \equiv \partial \overset{\circ}{U}^*(\mathbf{L}', \overset{\circ}{J}) / \partial \overset{\circ}{J} = \text{tr}(\overset{\circ}{\mathbf{T}}) / 3,$$

which relate deformation rates to different measures of stress rates. One can show that

$$(6) \quad \begin{aligned} \delta \overset{\circ}{Q} &= (\overset{\circ}{\mathbf{r}}^*)_s : \delta \mathbf{D} = [\partial \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J}) / \partial \mathbf{D}'] : \delta \mathbf{D}' \\ &\quad + [\partial \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J}) / \partial \overset{\circ}{J}] \delta \overset{\circ}{J} = (\overset{\circ}{\mathbf{r}'^*})_s : (\delta \mathbf{D} - \mathbf{I} \delta \overset{\circ}{J} / 3) - \overset{\circ}{p}_r \delta \overset{\circ}{J} \\ &= (\overset{\circ}{\mathbf{r}'^*})_s : \delta \mathbf{D} - [(\overset{\circ}{\mathbf{r}'^*})_s : \mathbf{I}] (\mathbf{I} : \delta \mathbf{D}) / 3 - \overset{\circ}{p}_r \mathbf{I} : \delta \mathbf{D}, \end{aligned}$$

which implies

$$(7) \quad (\overset{\circ}{\mathbf{r}}^*)_s = (\overset{\circ}{\mathbf{r}'^*})_s - [(\overset{\circ}{\mathbf{r}'^*})_s : \mathbf{I}] \mathbf{I} / 3 - \overset{\circ}{p}_r \mathbf{I}.$$

Similarly, one can establish that

$$(8) \quad \overset{\circ}{\mathbf{S}} = \overset{\circ}{\mathbf{S}}' - (\overset{\circ}{\mathbf{S}}' : \mathbf{I}) \mathbf{I} / 3 - \overset{\circ}{p}_s \mathbf{I},$$

$$(9) \quad \overset{\circ}{\mathbf{T}} = \overset{\circ}{\mathbf{T}}' - (\overset{\circ}{\mathbf{T}}' : \mathbf{I}) \mathbf{I} / 3 - \overset{\circ}{p}_t \mathbf{I}.$$

The relations between various updated Lagrangian pressure rates can be obtained by differentiating Eq. (13.7:11) with respect to t with all quantities referring to the current configuration in D_N :

$$(10) \quad \begin{aligned} -\overset{\circ}{p} &= \overset{\circ}{\tau} : \mathbf{I} / 3 = -\overset{\circ}{p}_t + \text{tr}(\mathbf{D}) p + \boldsymbol{\tau} : \mathbf{D} / 3 \\ &= -\overset{\circ}{p}_s + \text{tr}(\mathbf{D}) p + 2\boldsymbol{\tau} : \mathbf{D} / 3 = -\overset{\circ}{p}_r + \text{tr}(\mathbf{D}) p + \boldsymbol{\tau} : \mathbf{D} / 3, \end{aligned}$$

where $\overset{\circ}{p}$ is the hydrostatic pressure rate and $p [= -(\boldsymbol{\tau} : \mathbf{I}) / 3]$ is the hydrostatic pressure. In general, $\overset{\circ}{p}_t = \overset{\circ}{p}_r \neq \overset{\circ}{p}_s \neq \overset{\circ}{p}$.

If one replaces

$$\overset{\circ}{Q} \rightarrow \overset{\circ}{Q}^*, \quad \overset{\circ}{W} \rightarrow \overset{\circ}{W}^*, \quad \overset{\circ}{U} \rightarrow \overset{\circ}{U}^*, \quad \mathbf{D} \rightarrow \mathbf{D}' + \overset{\circ}{J} \mathbf{I}, \quad \mathbf{L} \rightarrow \mathbf{L}' + \overset{\circ}{J} \mathbf{I},$$

by substituting Eq. (1)–(3) and (7)–(9) into the functional defined in Secs. 17.6 and 17.7, one obtains new variational principles in terms of the new field variables. For example, a substitution of Eq. (1)–(3) and (7) into Eq. (17.7:1) yields

$$(11) \quad \begin{aligned} \overset{\circ}{\Pi}(\mathbf{v}, \boldsymbol{\Omega}, \mathbf{D}', \overset{\circ}{J}, \overset{\circ}{\mathbf{r}}'^*, \overset{\circ}{p}_r) &= \int_{V_N} \left\{ \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J}) + \boldsymbol{\tau} : [(\nabla_N \mathbf{v} - \frac{\boldsymbol{\Omega}}{2})^T \cdot \boldsymbol{\Omega}] - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \right. \\ &\quad \left. + \overset{\circ}{\mathbf{r}}'^* T : [\nabla_N \mathbf{v} - \boldsymbol{\Omega} - \mathbf{D}' - \text{tr}(\nabla_N \mathbf{v}) \mathbf{I} / 3] - \overset{\circ}{p}_r [\text{tr}(\nabla_N \mathbf{v}) - \overset{\circ}{J}] \right\} dV \\ &\quad - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA - \int_{S_{u N}} \mathbf{n} \cdot [\overset{\circ}{\mathbf{r}}'^* - \text{tr}(\overset{\circ}{\mathbf{r}}'^*) \mathbf{I} / 3 - \overset{\circ}{p}_r \mathbf{I} - \boldsymbol{\tau} \cdot \boldsymbol{\Omega}] \cdot (\mathbf{v} - \bar{\mathbf{v}}) dA. \end{aligned}$$

The functional (similar to that by Seki and Atluri 1994) holds for all materials for continuous \mathbf{v} , skew symmetric $\boldsymbol{\Omega}$, symmetric deviatoric \mathbf{D}' , positive $\overset{\circ}{J}$, and piecewise continuous deviatoric $\overset{\circ}{\mathbf{r}}'^*$ and $\overset{\circ}{p}_r$. In a similar fashion, one can derive the variational functionals using \mathbf{v} , $\boldsymbol{\Omega}$, \mathbf{L}' , $\overset{\circ}{J}$, $\overset{\circ}{\mathbf{T}}'$, $\overset{\circ}{p}_t$ or \mathbf{v} , $\boldsymbol{\Omega}$, \mathbf{D}' , $\overset{\circ}{J}$, $\overset{\circ}{\mathbf{S}}'$, $\overset{\circ}{p}_s$ as independent field variables. The details are left to the reader.

One can derive different variational principles from Eq. (11) using the contact transformation

$$(12) \quad -\overset{\circ}{Q}_c[(\overset{\circ}{\mathbf{r}}'^*)_s, \overset{\circ}{p}_r] = \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J}) - (\overset{\circ}{\mathbf{r}}'^*)_s : \mathbf{D}' + \overset{\circ}{p}_r \overset{\circ}{J}.$$

One can show that

$$\partial \overset{\circ}{Q}_c[(\overset{\circ}{\mathbf{r}}'^*)_s, \overset{\circ}{p}_r] / \partial (\overset{\circ}{\mathbf{r}}'^*)_s = \mathbf{D}', \quad -\partial \overset{\circ}{Q}_c[(\overset{\circ}{\mathbf{r}}'^*)_s, \overset{\circ}{p}_r] / \partial \overset{\circ}{p}_r = \overset{\circ}{J}.$$

A substitution of Eq. (12) into Eq. (11) yields

$$(13) \quad \overset{\circ}{\Pi}_8(\mathbf{v}, \boldsymbol{\Omega}, \overset{\circ}{\mathbf{r}}'^*, \overset{\circ}{p}_r) = \int_{V_N} \left\{ -\overset{\circ}{Q}_c(\overset{\circ}{\mathbf{r}}'^*, \overset{\circ}{p}_r) + \boldsymbol{\tau} : [\boldsymbol{\Omega}^T \cdot (\nabla_N \mathbf{v} - \boldsymbol{\Omega}/2)] \right. \\ \left. - \overset{\circ}{p}_r \text{tr}(\nabla_N \mathbf{v}) + \overset{\circ}{\mathbf{r}}'^* T : [\nabla_N \mathbf{v} - \boldsymbol{\Omega} - \text{tr}(\nabla_N \mathbf{v}) \mathbf{I}/3] - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \right\} dV \\ - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA - \int_{S_{uN}} \mathbf{n} \cdot [\overset{\circ}{\mathbf{r}}'^* - \text{tr}(\overset{\circ}{\mathbf{r}}'^*) \mathbf{I}/3 - \overset{\circ}{p}_r \mathbf{I} - \boldsymbol{\tau} \cdot \boldsymbol{\Omega}] \cdot (\mathbf{v} - \bar{\mathbf{v}}) dA,$$

which is a functional with the four-field variables $\mathbf{v}, \boldsymbol{\Omega}, \overset{\circ}{\mathbf{r}}'^*, \overset{\circ}{p}_r$. The stationary condition of $\overset{\circ}{\Pi}_8$ under the variation of the field variables subjected to the rigid condition $(\boldsymbol{\Omega})_s = 0$ gives the appropriate field equations and natural boundary conditions.

One can also introduce a mixed contact transformation

$$(14) \quad \overset{\circ}{Q}_m(\mathbf{D}', \overset{\circ}{p}_r) = \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J}) + \overset{\circ}{p}_r \overset{\circ}{J} \\ \partial \overset{\circ}{Q}_m(\mathbf{D}', \overset{\circ}{p}_r) / \partial \mathbf{D}' = (\overset{\circ}{\mathbf{r}}'^*)_s, \quad -\partial \overset{\circ}{Q}_m(\mathbf{D}', \overset{\circ}{p}_r) / \partial \overset{\circ}{p}_r = \overset{\circ}{J}.$$

A substitution of Eq. (14) into Eq. (11) yields

$$(15) \quad \overset{\circ}{\Pi}_9(\mathbf{v}, \boldsymbol{\Omega}, \mathbf{D}', \overset{\circ}{\mathbf{r}}'^*, \overset{\circ}{p}_r) = \int_{V_N} \left\{ \overset{\circ}{Q}_m(\mathbf{D}', \overset{\circ}{p}_r) - \boldsymbol{\tau} : [(\nabla_N \mathbf{v} + \boldsymbol{\Omega}/2)^T \cdot \boldsymbol{\Omega}] \right. \\ \left. - \overset{\circ}{p}_r \text{tr}(\nabla_N \mathbf{v}) + \overset{\circ}{\mathbf{r}}'^* T : [\nabla_N \mathbf{v} - \boldsymbol{\Omega} - \mathbf{D}' - \text{tr}(\nabla_N \mathbf{v}) \mathbf{I}/3] - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \right\} dV \\ - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA - \int_{S_{uN}} \mathbf{n} \cdot [\overset{\circ}{\mathbf{r}}'^* - \text{tr}(\overset{\circ}{\mathbf{r}}'^*) \mathbf{I}/3 - \overset{\circ}{p}_r \mathbf{I} - \boldsymbol{\tau} \cdot \boldsymbol{\Omega}] (\mathbf{v} - \bar{\mathbf{v}}) dA,$$

which is functional with the five-field variables $\mathbf{v}, \boldsymbol{\Omega}, \mathbf{D}', \overset{\circ}{\mathbf{r}}'^*, \overset{\circ}{p}_r$.

In Eq. (14) and (15), $\overset{\circ}{J}$ does not appear explicitly and $\overset{\circ}{Q}_c$ and $\overset{\circ}{Q}_m$ are well behaved as the material approaches incompressibility. Thus both $\overset{\circ}{\Pi}_8$ and $\overset{\circ}{\Pi}_9$ are suitable for incompressible or nearly incompressible materials.

One can derive additional functionals by imposing constraint condition *a priori*. For example, if the compatibility condition are satisfied *a priori*

$$(16) \quad \mathbf{D}' = (\nabla_N \mathbf{v})_s - \mathbf{I} : (\nabla_N \mathbf{v})_s / 3, \quad \text{and} \quad \boldsymbol{\Omega} = (\nabla_N \mathbf{v})_a \quad \text{in } V_N,$$

$$(17) \quad \mathbf{v} = \bar{\mathbf{v}} \quad \text{on } S_{uN},$$

then Eq. (15) is reduced to

$$(18) \quad \overset{\circ}{\Pi}_{10}(\mathbf{v}, \overset{\circ}{p}_r) = \int_{V_N} \left\{ \overset{\circ}{Q}_m[(\nabla_N \mathbf{v})_s - \text{tr}(\nabla_N \mathbf{v})/3, \overset{\circ}{p}_r] - \overset{\circ}{p}_r \text{tr}(\nabla_N \mathbf{v}) \right. \\ \left. + \boldsymbol{\tau} : [(\nabla_N \mathbf{v})_a^T \cdot (\nabla_N \mathbf{v} - (\nabla_N \mathbf{v})_a/2)] - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \right\} dV - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA,$$

which is a functional of the two fields, piecewise continuous $\overset{\circ}{p}_r$ and continuous \mathbf{v} . Tong (1969) first derived $\overset{\circ}{\Pi}_{10}$ for infinitesimal deformation.

When $\boldsymbol{\Omega}$ is re-introduced as part of penalty function, we can extend the two-field functional $\overset{\circ}{\Pi}_{10}$ to the following three-field functional:

$$(19) \quad \begin{aligned} \overset{\circ}{\Pi}_{10}^*(\mathbf{v}, \boldsymbol{\Omega}, \overset{\circ}{p}_r) = & \int_{V_N} \{ \overset{\circ}{Q}_m [(\nabla_N \mathbf{v})_s - \text{tr}(\nabla_N \mathbf{v})/3, \overset{\circ}{p}_r] \\ & + \tau : [(\nabla_N \mathbf{v})_a^T \cdot (\nabla_N \mathbf{v} - (\nabla_N \mathbf{v})_a/2)] + c[(\nabla_N \mathbf{v})_a - \boldsymbol{\Omega}]^2/2 \\ & - \overset{\circ}{p}_r \text{tr}(\nabla_N \mathbf{v}) - \rho_N \overset{\circ}{\mathbf{b}} \cdot \mathbf{v} \} dV - \int_{S_{\sigma N}} \overset{\circ}{\mathbf{T}} \cdot \mathbf{v} dA, \end{aligned}$$

where c is the constant factor of the penalty function. The functional $\overset{\circ}{\Pi}_{10}^*$, with the condition Eq. (16) satisfied *a priori*, is different from $\overset{\circ}{\Pi}_9$ in which $\overset{\circ}{\mathbf{r}}^*$ is an independent field variable.

Having $\boldsymbol{\Omega}$ as an independent field rather than computing it from $(\nabla_N \mathbf{v})_a$ is important from the computational point of view. As demonstrated by Cazzani and Atluri (1992), and Seki and Atluri (1994), the solution accuracy for large deformation problems depends strongly on the accuracy of $\boldsymbol{\Omega}$. The spin rate is needed to compute the updated solution. The formulation of Eq. (19) is generally more accurate as it gives $\boldsymbol{\Omega}$ directly without having to differentiate the numerical solution of \mathbf{v} .

17.9. UPDATED SOLUTION

In each increment, one solves for the stress increment rates $\overset{\circ}{\mathbf{T}}$, $\overset{\circ}{\mathbf{r}}^*$ or $\overset{\circ}{\mathbf{S}}$, the deformation rates \mathbf{D} , $\boldsymbol{\Omega}$ and the displacement rate \mathbf{v} . One evaluates the Cauchy increment rate $\overset{\circ}{\tau}$ according to Eq. (17.3:4) or Eqs. (17.3:5–8). In solving for the rates, we need the Cauchy stress $\tau (= \tau N)$, the increment rate of boundary traction $\overset{\circ}{\mathbf{T}}^{\sigma N}$ on $S_{\sigma N}$, that of boundary displacement $\overset{\circ}{\mathbf{v}}$ on S_{uN} , the deformed geometry of the body in D_N , and the constitutive law that relates the stress rates to the deformation rates. The constitutive law generally depends on the total deformation gradient \mathbf{F}_0^N and τ . If a constitutive law is known in the form of the gradient of a rate-potential in terms of a specific strain rate such as that given in Eq. (17.5:9) or (17.5:15), one can derive other rate-potentials through Eqs. (17.5:11–13) or (17.5:16–19) and determine the corresponding constitutive laws accordingly. Hill (1967) and Atluri (1979, 1980a) have constructed a rate-potential $\overset{\circ}{V}_J(\mathbf{D})$ in terms of the strain rate tensor \mathbf{D} . Pian and Tong (1971) derive $\overset{\circ}{W}(\mathbf{D})$ from the strain energy density $W_0(\mathbf{C})$. Equations (17.5:22) and (17.5:23) show the dependence of $\overset{\circ}{W}(\mathbf{D})$ on the total deformation gradient and related deformation matrices.

The updated Cauchy stress is simply $\tau^{N+1} = \tau + \overset{\circ}{\tau} \Delta t$, which is used to determine the solution of the next increment. The current Cauchy stress is simply the sum of all previous increments, i.e.,

$$(1) \quad \tau^{N+1} = \sum_{i=1}^N \overset{\circ}{\tau}_i \Delta t_i,$$

where $\overset{\circ}{\tau}_i \Delta t_i$ is the Cauchy stress increment of the i^{th} step. The direct summation of the increments of other stress measures such as $\overset{\circ}{\mathbf{T}}$, $\overset{\circ}{\mathbf{r}}^*$ and $\overset{\circ}{\mathbf{S}}$ does not have any physical meaning, because each increment refers to a different configuration.

The updated total displacement vector and the updated position vector of material point after each increment are

$$(2) \quad \mathbf{u}^{N+1} = \mathbf{u}^N + \mathbf{v} \Delta t, \quad \mathbf{X}^{N+1} = \mathbf{X}^N + \mathbf{v} \Delta t = \mathbf{X}^0 + \mathbf{u}^{N+1}.$$

Based on the updated \mathbf{u}^{N+1} , and \mathbf{X}^{N+1} , one can evaluate the updated total deformation gradient and the related deformation matrices. Together with the updated τ , one can update the rate-potential Eq. (17.5:23) and the constitutive equation Eq. (17.5:24) for the next increment.

In Secs. 13.9 and 17.5, we discuss the need of maintaining objectivity in establishing the constitutive laws. There is also a need to maintain objectivity in computation. The rate of change of stress depends not only on the strain rate \mathbf{D} but also on the spin rate $\boldsymbol{\Omega}$. An objective constitutive law gives the relation between the strain rate and the objective stress rate in the coordinate frame rigidly spinning with the material element. The relation between the strain rate, the spin rate, and the other stress rate measures can be obtained by the transformation of the objective stress rate tensor and the strain rate from the spinning coordinate system.

How well objectivity is maintained in computation depends on the accuracy of the explicit or implicit determination of the spin rate and the spinning coordinates. The covariant base vector \mathbf{g}_M of a rigid frame spinning at a rate of $\boldsymbol{\Omega}(t)$ satisfies the equation

$$(3) \quad d\mathbf{g}_M/dt = \boldsymbol{\Omega} \cdot \mathbf{g}_M,$$

at any loading stage. The solution is

$$(4) \quad \mathbf{g}_M(t) = \mathbf{R}(t) \cdot \mathbf{g}_M(t_N)$$

where $\mathbf{g}_M(t_N)$ is the initial value at $t = t_N$, $\mathbf{R}(t)$ is the rotation tensor of the deformation if $\Omega(t)$ is as defined in Eq. (13.5:11). Thus to maintain objectivity, we should assure that the computed $\mathbf{R}(t)$ is a rotation tensor (Hughes and Winget 1981, Rubinstein and Atluri 1983).

It can be shown that the linearized solution of Eq. (13.5:11) is

$$\mathbf{R}(t_N + \Delta t) = [\mathbf{I} + \Omega(t_N)\Delta t] \cdot \mathbf{R}(t_N),$$

where $\Omega(t_N)$ is the spin rate of the deformation in the updated Lagrange description. Then, obviously

$$(5) \quad \begin{aligned} & \mathbf{R}(t_N + \Delta t)^T \cdot \mathbf{R}(t_N + \Delta t) \\ &= \mathbf{R}(t_N)^T \cdot [\mathbf{I} + \Omega(t_N)^T \Delta t] \cdot [\mathbf{I} + \Omega(t_N)\Delta t] \cdot \mathbf{R}(t_N) \\ &= \mathbf{I} + \mathbf{R}(t_N)^T \cdot \Omega(t_N) \cdot \Omega(t_N)^T \cdot \mathbf{R}(t_N) \Delta t^2, \end{aligned}$$

since $\mathbf{R}(t_N)^T \cdot \mathbf{R}(t_N) = \mathbf{I}$ and $\Omega(t_N)^T + \Omega(t_N) = 0$. The error in rotation is of the order Δt^2 . Objectivity is maintained only in the limit of $\Delta t \rightarrow 0$.

In computation, Δt is finite. In order to improve the preservation of objectivity of the incremental solution, we modify the computational algorithm by using an appropriate value $\Omega(t_*)$ (where $t_N \leq t_* \leq t_N + \Delta t$), instead of $\Omega(t_N)$ in the Euler approximation given above. This minimizes the error caused by the finite incremental steps. Hughes and Winget (1981), Rubinstein and Atluri (1983), and Key (1980) show that the midpoint rule with $t_* = t_N + \Delta t/2$ provides an accuracy of the order $(\Delta t)^2$.

The implementation of the mid-point rule in computation is to use the value of $\Omega(t_N + \Delta t/2)$ at each increment. In determining the solution in the D_{N+1} configuration, instead of using the operator ∇_{N+1} , we use

$$(6) \quad \nabla_{N+1/2} = [\partial(\cdot)/\partial X_I^{N+1/2}] \varepsilon_I,$$

where $\mathbf{X}^{N+1/2} = \mathbf{X}^N + \mathbf{v}\Delta t/2$. Then, we define the rotation and the deformation gradient as

$$(7) \quad \mathbf{R}(t_N + \Delta t) = [\mathbf{I} + \Omega(t_N + \Delta t/2)\Delta t] \cdot \mathbf{R}(t_N)$$

$$(8) \quad \mathbf{F} = \mathbf{F}_0^{N+1} = \partial \mathbf{X}^{N+1/2} / \partial \mathbf{X}^0 = \mathbf{F}(t_N + \Delta t/2),$$

where

$$(9) \quad \Omega(t_N + \Delta t/2) = [\nabla_{N+1/2} \mathbf{v} - (\nabla_{N+1/2} \mathbf{v})^T]/2.$$

We may also compute the orthogonal rotation by

$$(10) \quad \mathbf{R}(t_N + \Delta t) = \exp(\Omega \Delta t) \mathbf{R}(t_N),$$

which can further be approximated as

$$(11) \quad \mathbf{R}(t_N + \Delta t) = [\mathbf{I} + \Omega \Delta t + \Omega^T \cdot \Omega(\Delta t)^2/2] \mathbf{R}(t_N).$$

We calculate \mathbf{C}_0^{N+1} , \mathbf{B}_0^{N+1} and \mathbf{U}_0^{N+1} from \mathbf{F}_0^{N+1} and \mathbf{R}_0^{N+1} . Depending on the approach one has the solution in terms of $\overset{\circ}{\mathbf{T}}$, $\overset{\circ}{\mathbf{r}}^*$, $\overset{\circ}{\mathbf{S}}$ or $\overset{\circ}{\boldsymbol{\sigma}}_r$. Then one of the following relations is used to calculate $\overset{\circ}{\tau}$,

$$(12) \quad \overset{\circ}{\tau} = \overset{\circ}{\boldsymbol{\sigma}}_r + \Omega \cdot \tau - \tau \cdot \Omega - \text{tr}(\mathbf{L})\tau,$$

$$(13) \quad \overset{\circ}{\tau} = \overset{\circ}{\mathbf{S}} + \mathbf{L} \cdot \tau + \tau \cdot \mathbf{L}^T - \text{tr}(\mathbf{L})\tau,$$

$$(14) \quad \overset{\circ}{\tau} = \overset{\circ}{\mathbf{r}}^* + \mathbf{L} \cdot \tau - \text{tr}(\mathbf{L})\tau - \tau \cdot \Omega,$$

$$(15) \quad \overset{\circ}{\tau} = \overset{\circ}{\mathbf{T}} + \mathbf{L} \cdot \tau - \text{tr}(\mathbf{L})\tau,$$

where \mathbf{L} and Ω are the half-step values defined in Eq. (6) and (9). Incremental numerical solutions are numerous, e.g., Pian and Tong (1971), Murakawa and Atluri (1978, 1979) and Seki and Atluri (1994).

17.10. INCREMENTAL LOADS

The value of load increment at each increment step depends on the nature of applied loads. Normally the applied loads are either a *deadweight* or *pressure type*. The direction of a deadweight load does not change (e.g., the gravitational force per unit mass) and its magnitude is in terms of unit undeformed volume or surface. On the other hand, a pressure loads is in terms of unit deformed area and both its direction and magnitude can change at each increment.

We are going to determine the applied traction rate $\overset{\circ\nu}{\bar{\mathbf{T}}}$ in the updated Lagrangian description in the N configuration. The rate depends on the nature of the applied loads. From Eqs. (17.6:6) and (17.3:4), we obtain

$$(1) \quad \overset{\circ\nu}{\bar{\mathbf{T}}} = \mathbf{N} \cdot \overset{\circ}{\bar{\mathbf{T}}} = \mathbf{N} \cdot \overset{\circ}{\tau} + \mathbf{N} \cdot [\text{tr}(\mathbf{L})\mathbf{I} - \mathbf{L}] \cdot \boldsymbol{\tau},$$

where \mathbf{N} is a unit normal of $d\mathbf{A}^N$ ($= \mathbf{N}dS^N$), \mathbf{L} the rate of deformation gradient and $\overset{\circ}{\bar{\mathbf{T}}}, \overset{\circ}{\tau}$ are Lagrangian and Cauchy stress rates in the updated description. The first term of the right most equation above reflects the change in the loads. The second term $[\text{tr}(\mathbf{L})\mathbf{I} - \mathbf{L}] \cdot \boldsymbol{\tau}$ results from the deformation of the boundary surface.

For applied *pressure* loads (without shear),

$$\begin{aligned} \bar{p}^{N+1}d\mathbf{A}^{N+1} &= (\bar{p}^N + \overset{\circ}{\bar{p}}\Delta t)[1 + \text{tr}(\mathbf{D})\Delta t](\mathbf{I} - \mathbf{L}^T\Delta t) \cdot d\mathbf{A}^N \\ &= \bar{p}^N d\mathbf{A}^N + \{\overset{\circ}{\bar{p}}\mathbf{I} + [\text{tr}(\mathbf{D})\mathbf{I} - \mathbf{L}^T]\bar{p}^N\}\Delta t \cdot d\mathbf{A}^N, \end{aligned}$$

where $\overset{\circ}{\bar{p}}$ is the *prescribed incremental rate of pressure*. Note that, from Eq. (13.2:18), we have

$$d\mathbf{A}^{N+1} = J_N^{N+1}[(\mathbf{F}^{-1})^T]_N^{N+1}d\mathbf{A}^N.$$

Then, from $\overset{\circ\nu}{\bar{\mathbf{T}}}d\mathbf{A}^N = \bar{p}^{N+1}d\mathbf{A}^{N+1} - \bar{p}^N d\mathbf{A}^N$, the applied load increment is

$$(2) \quad \overset{\circ\nu}{\bar{\mathbf{T}}} = \overset{\circ}{\bar{p}}\mathbf{N} + [\text{tr}(\mathbf{D})\mathbf{N} - \mathbf{L}^T \cdot \mathbf{N}]\bar{p}^N,$$

for the *rate of traction increment* on the *deformed traction-prescribed surfaces* in the updated Lagrangian description.

For a *deadweight type* of applied loads, the forces are normally referred to v the undeformed configuration. Let $d\overset{\nu}{\bar{\mathbf{T}}}^{N+1}$ be the applied force vector acting on an infinitesimal surface $d\mathbf{A}^{N+1}$ on the boundary in the D_{N+1} configuration. We have

$$(3) \quad \begin{aligned} d\overset{\nu}{\bar{\mathbf{T}}}^{N+1} &= \mathbf{T}_0^{N+1} \cdot d\mathbf{A}^0 = (\mathbf{T}_0^N + \Delta\mathbf{T}) \cdot d\mathbf{A}^0 \\ &= \boldsymbol{\tau}^N \cdot d\mathbf{A}^N + \Delta\overset{\nu}{\bar{\mathbf{T}}}dS^0 = \overset{\nu}{\bar{\mathbf{T}}}^N dS^N + \Delta\overset{\nu}{\bar{\mathbf{T}}}dS^0, \end{aligned}$$

where the superscript “0” refers to the *undeformed state*, $\mathbf{T}_0^{N+1}, \overset{\nu}{\bar{\mathbf{T}}}^N$ are the first Piola–Kirchhoff stress at the $N+1$ and N deformation states referred to the undeformed state. Thus $\Delta\overset{\nu}{\bar{\mathbf{T}}}$ is the *given traction increment*. The force vector $d\overset{\nu}{\bar{\mathbf{T}}}^{N+1}$ also relates to the traction $\overset{\nu}{\bar{\mathbf{T}}}^N$ and the surface area $dS^N = |d\mathbf{A}^N|$ in the D_N configuration by the equation

$$(4) \quad d\overset{\nu}{\bar{\mathbf{T}}}^{N+1} = \overset{\nu}{\bar{\mathbf{T}}}^{N+1}dS^N = (\overset{\nu}{\bar{\mathbf{T}}}^N + \overset{\circ\nu}{\bar{\mathbf{T}}}\Delta t)dS^N.$$

Comparing Eq. (3) and (4) gives

$$\overset{\circ\nu}{\bar{\mathbf{T}}} = (\Delta\overset{\nu}{\bar{\mathbf{T}}}/\Delta t)(dS^0/dS^N),$$

Using Eqs. (13.2:14), one finds

$$(5) \quad \overset{\circ\nu}{\bar{\mathbf{T}}} = (\Delta\overset{\nu}{\bar{\mathbf{T}}}/\Delta t)\sqrt{\mathbf{N} \cdot \mathbf{B}_0^N \cdot \mathbf{N}}/J_0^N,$$

for the *rate of traction increment on the deformed traction-prescribed surfaces* in the updated Lagrangian description, where $\mathbf{B}_0^N [= \mathbf{F}_0^N(\mathbf{F}_0^N)^T]$ is the Almansi tensor in terms of the deformation gradient \mathbf{F}_0^N given in Eq. (13.2:8), and J_0^N is the Jacobian referred to the undeformed state. Note, lower case \mathbf{n} is used as the unit normal to the deformed surface $d\mathbf{a}$ in Eq. (13.2:8) and presently they are denoted by \mathbf{N} and $d\mathbf{A}^N$, respectively.

For body forces proportional to the mass per unit undeformed volume, the *rate of body force increment* in the updated Lagrangian description is

$$(6) \quad \overset{\circ}{\mathbf{b}} = (\Delta \mathbf{b} / \Delta t)(dV^0 / dV^N) = (\Delta \mathbf{b} / \Delta t) / J_0^N ,$$

where $\Delta \mathbf{b}$ is the body force increment per unit mass at the N^{th} step of increment.

We summarize the rate potentials and the constitutive laws developed in this chapter as follows:

Rate potential	Constitutive law
$\overset{\circ}{W}(\mathbf{D}) = \overset{\circ}{V}_0(\mathbf{D}) = d_{ijkl} D_{ij} D_{kl} / 2$	$\overset{\circ}{\mathbf{S}} = \partial \overset{\circ}{W}(\mathbf{D}) / \partial \mathbf{D}$
$\overset{\circ}{Q}(\mathbf{D}) = \overset{\circ}{V}_0(\mathbf{D}) + \tau : (\mathbf{D} \cdot \mathbf{D}) / 2$	$\overset{\circ}{\mathbf{r}} \equiv (\overset{\circ}{\mathbf{r}}^*)_s = \partial \overset{\circ}{Q}(\mathbf{D}) / \partial \mathbf{D}$
$\overset{\circ}{U}(\mathbf{L}) = \overset{\circ}{V}_0\left(\frac{\mathbf{L} + \mathbf{L}^T}{2}\right) + \frac{1}{2} \tau : (\mathbf{L}^T \cdot \mathbf{L})$	$\overset{\circ}{\mathbf{T}} = \partial \overset{\circ}{U}(\mathbf{L}) / \partial \mathbf{L}^T = \partial \overset{\circ}{\mathbf{U}} / \partial (\nabla_N \mathbf{v})^T$
$\overset{\circ}{V}_J(\mathbf{D}) = \overset{\circ}{V}_0(\mathbf{D}) + \tau : (\mathbf{D} \cdot \mathbf{D})$	$\overset{\circ}{\sigma}_r = \partial \overset{\circ}{V}_J(\mathbf{D}) / \partial \mathbf{D}$
$\overset{\circ}{Q}_c[(\overset{\circ}{\mathbf{T}} + \tau : \Omega)_s] = -\overset{\circ}{Q}(\mathbf{D})$ $+ \tau : (\Omega \cdot \mathbf{D}) + (\overset{\circ}{\mathbf{T}})_s : \mathbf{D}$	$\mathbf{D} = \partial \overset{\circ}{Q}_c / \partial (\overset{\circ}{\mathbf{T}} + \tau : \Omega)_s$
$\overset{\circ}{W}^*(\mathbf{D}', \overset{\circ}{J}) = \overset{\circ}{W}(\mathbf{D})$	$\overset{\circ}{\mathbf{S}}' = \frac{\partial \overset{\circ}{W}^*(\mathbf{D}', \overset{\circ}{J})}{\partial \mathbf{D}'}, \overset{\circ}{p}_s = -\frac{\partial \overset{\circ}{W}^*(\mathbf{D}', \overset{\circ}{J})}{\partial \overset{\circ}{J}}$
$\overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J}) = \overset{\circ}{Q}(\mathbf{D})$	$(\overset{\circ}{\mathbf{r}}^*)_s = \frac{\partial \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J})}{\partial \mathbf{D}'}, \overset{\circ}{p}_r = -\frac{\partial \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J})}{\partial \overset{\circ}{J}}$
$\overset{\circ}{U}^*(\mathbf{L}', \overset{\circ}{J}) = \overset{\circ}{U}(\mathbf{L}) = \overset{\circ}{E}(\mathbf{L})$	$\overset{\circ}{\mathbf{T}}'^T = \frac{\partial \overset{\circ}{U}^*(\mathbf{L}', \overset{\circ}{J})}{\partial \mathbf{L}'}, \overset{\circ}{p}_t = -\frac{\partial \overset{\circ}{U}^*(\mathbf{L}', \overset{\circ}{J})}{\partial \overset{\circ}{J}}$

Note: The rate potentials are defined in [Secs. 17.5–17.7](#) with $\overset{\circ}{V}_0(\mathbf{D}) = \overset{\circ}{V}_J(\mathbf{D}) - \tau : (\mathbf{D} \cdot \mathbf{D}) = \overset{\circ}{W}(\mathbf{D})$.

17.11. INFINITESIMAL STRAIN THEORY

The incremental variational principles previously formulated can be reduced to the case of the infinitesimal strain theory by simply regarding the current configuration D_N as the initial configuration and assuming no initial stress, i.e., $\tau^N = 0$. In this case, all stress measures become the same

$$\begin{aligned} \tau^{N+1} &= \mathbf{T}^{N+1} = \mathbf{r}^{*N+1} = \mathbf{S}^{N+1} = \overset{\circ}{\tau} \Delta t = \overset{\circ}{\mathbf{T}} \Delta t = \overset{\circ}{\mathbf{r}}^* \Delta t = \overset{\circ}{\mathbf{S}} \Delta t, \\ \overset{\circ}{\mathbf{T}}' &= \overset{\circ}{\mathbf{r}}'^* = \overset{\circ}{\mathbf{S}}' = \overset{\circ}{\tau}' = \overset{\circ}{\tau} - \text{tr}(\overset{\circ}{\tau})/3, \\ \overset{\circ}{p} &= \overset{\circ}{p}_r = \overset{\circ}{p}_t = \overset{\circ}{p}_s. \end{aligned}$$

The following rate potentials are also equal,

$$\overset{\circ}{V}_J(\mathbf{D}) = \overset{\circ}{Q}(\mathbf{D}) = \overset{\circ}{W}(\mathbf{D}) = \overset{\circ}{U}(\mathbf{L}).$$

For linear elastic isotropic material, one has

$$\begin{aligned} \overset{\circ}{Q}(\mathbf{D}) &= (D_{ii})^2 \lambda / 2 + G D_{ij} D_{ij}, \\ \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J}) &= \lambda \overset{\circ}{J}^2 / 2 + G(D'_{ij} + \overset{\circ}{J} \delta_{ij}/3)(D'_{ij} + \overset{\circ}{J} \delta_{ij}/3) \\ &= (\lambda + 2G/3) \overset{\circ}{J}^2 / 2 + G D'_{ij} D'_{ij}, \\ \overset{\circ}{Q}_m(\mathbf{D}', \overset{\circ}{p}) &= \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J}) - \overset{\circ}{p} \overset{\circ}{J} = G D'_{ij} D'_{ij} - \overset{\circ}{p}^2 / (2K), \\ -\overset{\circ}{Q}_c(\tau', \overset{\circ}{p}) &= \overset{\circ}{Q}^*(\mathbf{D}', \overset{\circ}{J}) - \tau'_{ij} D'_{ij} - \overset{\circ}{p} \overset{\circ}{J} = -\tau'_{ij} \tau'_{ij} / (4G) - \overset{\circ}{p}^2 / (2K), \end{aligned}$$

where λ is the Lamé constant, G the shear modulus and K the bulk modulus

$$K = \lambda + 2G/3.$$

For materials approaching incompressibility, λ and K tend to infinity while \hat{J} approaches zero. Therefore \hat{Q} and \hat{Q}^* are not suitable for numerical computation. However, both \hat{Q}_m and \hat{Q}_c are well behaved since \hat{p} is finite.

P R O B L E M S

17.4. Derive a variational principle using $\mathbf{v}, \Omega, \mathbf{L}', \hat{J}, \hat{\mathbf{T}}', \hat{p}_t$ as the field variables where $\hat{\mathbf{T}}', \hat{p}_t$ are defined in Eqs. (17.8:4) and (17.8:5). Also, derive the field equations and the natural boundary conditions associated with the stationary condition of the functional.

17.5. Derive the field equations and the natural boundary conditions for the functionals $\hat{\Pi}_8, \hat{\Pi}_9, \hat{\Pi}_{10}, \hat{\Pi}_{10}^*$ defined in Sec. 17.8.

18

FINITE ELEMENT METHODS

Numerical approach is a ‘must’ to establish satisfactory approximate solutions for most practical problems. There are many numerical methods. A treatise on the entire subject is beyond the scope of this book. In the remaining of this book, we will focus on the *finite element method* (FEM), the meshless method of Atluri, and the method based on the Eshelby energy conservation law. These are versatile and powerful methods for solving linear as well as nonlinear complex problems. We will explore the fundamental concepts and approximations of the methods, construct the numerical equations and solutions, and use incremental approach to solve nonlinear problems.

The FEM was first developed in the 1950’s by engineers as an outgrowth of the so-called matrix method (Argyris 1955, 1965, Argyris and Kelsey 1960) for analyzing complex structures with large number of components. Over the years, the finite element method and the other numerical methods have spread to applications in many fields of engineering, science and medicine.

Foundations of FEMs have been established for continuum mechanics, magneto-hydrodynamics, radiation analysis, heat transfer, etc. The method first establishes approximate solutions in terms of unknown parameters in subregions called “elements” and then deduces an approximate solution for the whole domain by enforcing relations among the solutions of all elements. For structural analysis the procedure relates the displacements and the internal forces at selected nodal points of individual structural components in the form of a system of algebraic equations. The unknowns can be nodal displacements, nodal internal forces, or both and some other general parameters. Depending on the unknowns and dependent variables selected, the method is qualified with word like *displacement, force, hybrid or mixed*. The system of equations is written in matrix notation, and solved by high-speed computers.

We will focus on formulations based on variational principles or principles of virtual work. There are other approaches include the *boundary element method*, the *meshless method*, *adaptive solution techniques*, etc. Up-to-date references can be found periodically in the Journal Applied Mechanics Reviews. Finite-element computer programs including software to ease the burden on input and output are commercially available. (See references in [Biblio. 18.1](#) and published literature.)

To lay down the foundation for all aspects of the finite element method with such extensive applications is not easy; the learning of it will demand great attention. But this is the task we set for ourselves for this chapter.

18.1. BASIC APPROACH

Turner *et al.* (1956) in their celebrated paper first applied the displacement method to plane stress problems. They divided a structure into triangular or rectangular sub-domains, called “elements” and designated their vertices as nodes. The behavior of each element was represented by the nodal displacements, the unknown variables, and an element-stiffness matrix relating the displacements to the forces at the nods of the element. The element-stiffness matrices were assembled to form a system of algebraic equations to assure interelement equilibrium. The equations were solved using high-speed computers. There are two key aspects in the formulation: the determination of the element stiffness matrices and the assembly of element-stiffness matrices to form a system of algebraic equations, called the global system, for the selected unknowns. The former involves the local approximation for establishing the relations between the element nodal forces and displacements.

Equally important, if not more, in the finite element method is its *implementation*. The success of the finite element method lies largely in the development of efficient pre-and post-processors, and algorithms for solving large systems of equations. The pre-processor enables users to describe efficiently and in a relatively error free manner to describe the geometry, material properties, loading conditions, etc., of a complex problem through inputs. The post-processor is essential to make the voluminous outputs in an understandable form through interactive graphic, tables, charts, and summaries. The need for efficient solution algorithms for large system goes without saying, as the problem often involves hundreds of thousands, even millions, of equations. Even with the advent of computers today, efficient algorithms are needed to make it possible to obtain the solution, especially for nonlinear problems, at a reasonable cost and in reasonable time.

The concept of finite element method dates back to Courant (1943) who used triangle meshes to solve a two-dimensional Laplace equation, or earlier when mathematicians formulated differential calculus with piecewise smooth functions. But the method did not catch on until the availability of computers to efficiently solve large number of algebraic equations, and handle the tedious inputs and the voluminous outputs.

Due to scope limitation, we will not address the issues of preand post-processors, but touch upon lightly the solution algorithms for large systems. We will focus on (a) methods for determining element stiffness matrices, and (b) the process to assemble the element matrices into a global system of algebraic equations for numerical solution.

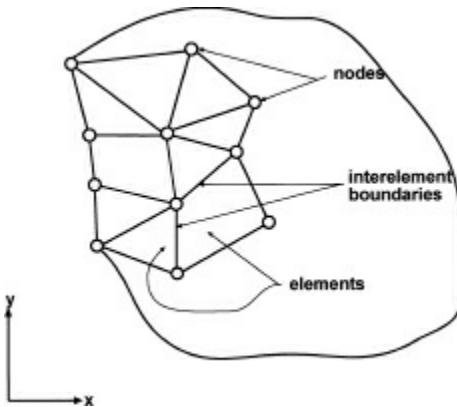


Fig. 18.1:1. Two-dimensional elements in a continuum.

In the finite element method, the continuum is divided into a finite number of sub-regions V_e in 3-dimension or A_e in 2-dimension domain, called *elements* (Fig. 18.1:1). The boundary between two adjacent elements is called the *interelement boundary*, which is a point for a one-, a line or a curve for a 2-, and a plane or a curve surface for a 3-dimensional continuum. One may visualize the two adjacent elements as being connected at a number of discrete points, called *nodes*, on their common boundaries. The nodes are usually the vertices of the element, the mid-point, one-third and two-third points along an element boundary, the centroid or some specific points on the element boundary surfaces. Additional points within an element may also be selected as nodes called interior nodes of the element. *The finite element method seeks to approximate the field variables within an element by interpolating their values at the nodes by shape functions.* From the interpolated or shape functions, one then determines the approximate behavior of the element. The nodal values or the magnitudes of the shape functions are parameters, called *degrees-of-freedom* or *generalized coordinates* to be determined. These two terms are used interchangeably in this chapter. One may also include the nodal values or the magnitudes of the spatial derivatives of the field variables for this purpose, especially when dealing with higher order differential equations. *The number of degrees-of-freedom at each node and that within each element are the number of degrees-of-freedom per node and per element, respectively.* The element behavior is characterized in terms of these parameters (such as the nodal values of the field variables) by element matrices such as the *element stiffness matrix* and the *element applied force vector*. The local interpolated or shape functions must satisfy certain continuity requirements along the common boundaries of adjacent elements to assure that the finite-element solution will converge to the exact solution as the size of elements reduces to zero. One approach is to *construct the element matrices from the functional associated with a variational principle or from an integrally weighted-average of the governing differential equations.* Then, one *derives the equations relating the parameters of all elements based on the zero condition of a variational functional, or the weighted-average of the governing differential equations.* This process is called an *integral formulation*. In the method, the interpolation (or shape) functions, the assembling process, and the solution methods for a large system of equations are generic. For efficient implementation, a *local system to label quantities associated with elements and a global system to describe quantities of the entire system* are employed.

In this chapter, we will deal almost exclusively with matrix operation as discussed (Sec. 16.2). Lower case bold letters denote column vectors for generalized coordinates. Specifically, bold σ and e represent the stress and strain column matrices (not tensors), and bold T denotes the traction column matrix with components subscripted T_j unless otherwise specified.

In the next section, for clarity, we shall write out explicitly most of the components of matrices. In the later sections, we shall resort to matrix notations and operations to simplify the expressions of field variables and the derivation of equations.

18.2. ONE-DIMENSIONAL PROBLEMS GOVERNED BY SECOND ORDER DIFFERENTIAL EQUATIONS

In this section, we shall discuss the construction of interpolation functions and element matrices for one-dimensional problems. In the process, we will point out features that are important to the numerical implementation of the finite element method.

We divide a domain $0 \leq X \leq L$ into a finite number of non-overlapping segments, called *elements* (Fig. 18.2:1). The *inter-element boundary* of two adjacent elements is a point, designated as a *node*. Additional points within the element may also be selected as nodes. Each node of the whole domain is assigned with a unique number, called the *global nodal number*. For instance, for the element over the region $X_2 \leq X \leq X_3$ with nodes at X_2 and X_3 [Fig. 18.2:1(a)], the subscripts 2

and 3 are the *global nodal labels* denoting the 2nd and 3rd nodes of the system.

Consider a field variable $u(X)$ over $0 \leq X \leq L$. We approximate $u(X)$ within an element by the interpolation of the selected parameters associated with the function at the nodes. Let the values of $u(X)$ be u_2, u_3 at X_2, X_3 , respectively. Each of these values, called a *degree-of-freedom* (DOF) or *generalized coordinate*, is generally not known *a priori*. In other words, u_2, u_3 are called the 2nd and 3rd degrees-of-freedom of the system with the subscripts denoting the *global labels of the DOFs*. In this case there is only one DOF at each node, the global DOF label and the global nodal label are the same. If there are more than one field-variable or parameter at the nodes, the global DOF label at a node will be different from the global nodal label of the node. Proper labeling for all nodes of the entire domain and all DOFs are essential to the success of the finite element analysis, as we usually deal with large number of DOFs.

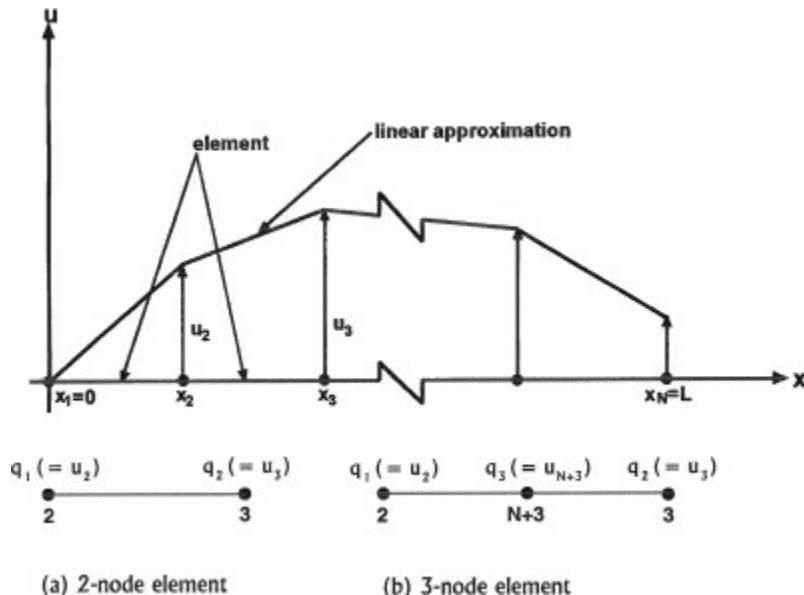


Fig. 18.2:1. Subdivision of a one-dimensional domain.

C^p Function. A function is said to have C^p continuity if its derivatives up to order p are continuous. Thus, a C^0 function is continuous and a C^1 function not only is continuous but also has continuous first derivatives.

Linear Interpolation (C^0 function). We approximate a function u by the first two terms of its Taylor series or by interpolating its nodal values at X_2 and X_3 in the form

$$(1) \quad u(X) \cong q_1 + q_2(X - X_2) = [u_2(X_3 - X) + u_3(X - X_2)]/(X_3 - X_2).$$

If u is represented in the same manner for all elements, we obtain a continuous piecewise linear interpolated function to approximate u over the whole domain. Here the subscripts 2 and 3 are the global label.

We shall write u in a form more suitable for the finite element process. We introduce a new coordinate ξ , called the *normalized local coordinate*, or *natural coordinate of the element*, that

$$(2) \quad \Delta \quad \xi = (2X - X_2 - X_3)/\varepsilon$$

with its origin at the mid-point of the element and the range between ± 1 , where $\varepsilon = X_3 - X_2$ is the *size of the element*. We can write Eq. (1) as

$$(3) \quad \Delta \quad u(\xi) = q_1 h_1(\xi) + q_2 h_2(\xi),$$

where

$$(4) \quad \Delta \quad h_1(\xi) = (1 - \xi)/2, \quad h_2(\xi) = (1 + \xi)/2, \quad u_2 = q_1, \quad u_3 = q_2,$$

while q_1 and q_2 are the values of u at the first and second nodes of the element, called the *local generalized coordinates*, while u_2 and u_3 are the values of u at the second and third nodes of the whole domain, called the *global generalized coordinates*. Thus the subscripts of q 's are the *local nodal labels* denoting the first and second nodes of the element, respectively, while those of u 's are the *global labels* denoting the second and third nodes of the system. The h 's, called *shape* or *interpolation functions*, satisfy

$$(5) \quad h_1(-1) = h_2(1) = 1 \quad h_1(1) = h_2(-1) = 0.$$

Obviously u of Eq. (3) is *continuous* over the whole domain. The function h_i , which equals to 1 at node i of an element and zero at all other nodes, is the Lagrange interpolation.

Equation (1) is the first two terms of the Taylor series expansion of the function. The error in approximating a smooth function over an element of size ε is of the order $O(\varepsilon^2)$, which is small if ε is so. *Equation (1) is a typical finite element representation of u by linear interpolation function within an element, whereas Eq. (3) is a generic expression useful in numerical implementation of the finite element method.* This last expression has the following properties:

- (1) The range of the *natural coordinate* ξ is always from -1 to 1 for all elements. Equation (2) is the transformation between the global coordinates (the coordinate for the entire system) and the natural coordinates (the coordinate conveniently describes the element). The use of local coordinate is not essential here but is often necessary for more complex problems.
- (2) When the interpolation function h_i is expressed in terms of the *natural coordinate* ξ , the geometric characteristics of an element, represented by the element size ε here, do not appear at all.
- (3) Here q_i denotes the value of u at the i^{th} node of the element. The subscript i ($= 1, 2$) is the local label of the element.

The representation of a field variable in the generic form of Eq. (3) does not depend on the specific location of the element, or the global labels of the DOFs of the element. One can represent the field variable in other elements in exactly the same way. For example, one can write u in the element $X_n \leq X \leq X_{n+1}$ in the form of Eq. (3) with

$$(6) \quad \xi = (2X - X_n - X_{n+1})/\varepsilon, \quad \varepsilon = X_{n+1} - X_n, \quad u_n = q_1, \quad u_{n+1} = q_2.$$

The *generic representation allows us to express a field variable in an element without knowing the specifics of the variable itself*. In other words, we can approximate any continuous function in a small neighborhood in the form of Eq. (3) with Eq. (4) as the shape functions. This property forms the basis of general and efficient algorithms for constructing element matrices.

The relation between the subscripts of q 's and those of u 's given in Eq. (6) is the relation between the local labels of the DOFs of an element and those of global system of the whole domain. The array that lists the relationship is called the **assembly list**, which is used for assembling the element matrices into a system of algebraic equations, and will be further examined later.

Using the natural coordinate ξ rather than the global spatial coordinate X provides advantages more than just representing a function by the generic interpolation functions. The details will be given when we discuss two-dimensional problems.

Quadratic Interpolation (C^0 function). One can improve the approximation by shrinking the element, i.e., refining the subdivision, or by increasing the order of the Taylor expansion. The latter approach means constructing *higher order elements*. For example, we represent u in the element $X_2 \leq X \leq X_3$ as

$$(7) \quad \blacktriangle u \cong \alpha_1 + \alpha_2(X - X_2) + \alpha_3(X - X_2)^2 = q_1 h_1(\xi) + q_2 h_2(\xi) + q_3 h_3(\xi),$$

where q 's are the nodal values at the element ends $\xi = \pm 1$ and in the element interior at the mid-point $\xi = 0$ [Fig. 18.2:1(b)], and

$$(8) \quad \blacktriangle h_1(\xi) = -(1 - \xi)\xi/2, \quad h_2(\xi) = \xi(1 + \xi)/2, \quad h_3(\xi) = (1 - \xi)(1 + \xi).$$

The function h_i are the Lagrange interpolation, which equal to 1 at node i and 0 at other nodes [e.g., $h_1(-1) = 1$ and $h_1(0) = h_1(1) = 0$]. Nodes 1 and 2 are boundary nodes while node 3 is an interior node. The truncation error of Eq. (7) is of the order $O(\varepsilon^3)$. Again, the generic character of Eq. (7) does not depend on any element specifics. The relationship between the local and global DOFs is

$$(9) \quad u_2 = q_1, \quad u_3 = q_2, \quad u_{N+3} = q_3.$$

The subscripts 2, 3 and $N+3$ of u are the global nodal labels of the DOFs at the 1st, 2nd and 3rd nodes of the element [Fig. 18.2:1(b)]. The representation Eq. (7) in terms of the interpolation functions in the form of Eq. (8) can be used for any element. Similarly, one can establish representations of higher orders (Tong and Rossettos 1977).

Problem 18.1. Derive the Lagrange interpolation, which gives unity at $\xi = \xi k$ and equals zero at other m points.

Element Matrices. The element matrices relate q 's to other field quantities. We construct element matrices using *variational functional*. Consider a rod subjected to a distributed load $P(X)$ along its axis with boundary conditions

$$(10) \quad u = \bar{u} \quad \text{at } X = 0 \text{ (rigid condition)}$$

$$(11) \quad EA(du/dX) = \bar{T} \quad \text{at } X = L \text{ (natural condition).}$$

The longitudinal deformation satisfies the second-order differential equation

$$(12) \quad d[EA(X)(du/dX)]/dX = P(X).$$

We divide the domain into elements. We introduce the natural coordinate $\xi = (2X - X_n - X_{n+1})/\varepsilon$, $\varepsilon = X_{n+1} - X_n$ for the element with $X_n \leq X \leq X_{n+1}$ and express the elastic modulus EA and the load P in terms of the local coordinate ξ ,

$$ea(\xi) = EA(X), \quad p(\xi) = P(X).$$

Following the derivation of Sec. 10.1, the functional for the *principle of minimum potential energy* is

$$(13) \quad \Pi = \int_0^L [EA(X)(du/dX)(du/dX)/2 + P(X)u] dX - u(L)\bar{T} = \sum_{n=1}^N \Pi_n,$$

where Π_n is in terms of the local coordinates that

$$\Pi_n = \int_{-1}^1 \varepsilon [2ea(\xi)(du/d\xi)(du/d\xi)/\varepsilon^2 + p(\xi)u] d\xi/2 - \alpha q_2 \bar{T},$$

in which $\alpha = 0$ for $n < N$ and $\alpha = 1$ for $n = N$ with the last node $N + 1$ at $X = L$. Thus $q_2 [= u(L)]$ is the unknown parameter associated with the last node in the last element. The **admissibility condition** requires that u be C^0 continuous for $0 \leq X \leq L$ and satisfies the rigid boundary condition Eq. (10) $u = \bar{u}$ at $X = 0$ (Sec. 10.6). The admissibility condition assures *the sum of the functional of all elements equal to the functional of the whole domain*, a feature of great importance to finite element method. We call Π a *functional* and u the *field or field variable* of the functional.

In the variational form, it is required that

$$(14) \quad \delta\Pi = \int_0^L [EA(X) \frac{du}{dX} \frac{d\delta u}{dX} + P(X)\delta u] dX - \bar{T}\delta u(L) = \sum_{n=1}^N \delta\Pi_n = 0,$$

with α , shown below, equals zero except in the last element that

$$(15) \quad \delta\Pi_n = \frac{\varepsilon}{2} \int_{-1}^1 \left[\frac{4ea(\xi)}{\varepsilon^2} \frac{du}{d\xi} \frac{d\delta u}{d\xi} + p(\xi)\delta u \right] d\xi - \alpha \bar{T}\delta u(L).$$

The *admissible conditions* require that both u and δu be C^0 continuous for $0 \leq X \leq L$, and that $u = \bar{u}$ and $\delta u = 0$ at $X = 0$. The determination of the solution based on Eq. (14) is called the *integral formulation*.

The finite element method constructs an approximate solution using selected admissible functions of the *integral formulation*. Both the linear and quadratic interpolation functions discussed before are admissible to the present problem. The rigid condition is to require $u_1 [= u(0)] = \bar{u}$ and $\delta u_1 [= \delta u(0)] = 0$, i.e., to impose constraint(s) on the system of algebraic equations to be discussed later. The boundary condition (11) is accounted for by the term $u(L)\bar{T}$ in Π . There is no restriction on $\delta u(L)$, since it is a *natural boundary condition* (Sec. 10.6). The boundary with natural boundary condition is called a *force boundary*.

Using $u = h_j(\xi)q_j$ and $\delta u = h_j(\xi)\delta q_j$ and substituting Eqs. (3) and (4) or Eqs. (7) and (8) into Eq. (15) give

$$(16) \quad \delta\Pi_n = \sum_{i,j} \delta q_i \left\{ \int_{-1}^1 \frac{2ea(\xi)}{\varepsilon} \frac{dh_i}{d\xi} \frac{dh_j}{d\xi} d\xi q_j + \int_{-1}^1 \frac{\varepsilon p(\xi)h_i(\xi)}{2} d\xi \right\} - \alpha \bar{T}\delta q_2,$$

where the range of summation is from 1 to 2 for linear interpolation and from 1 to 3 for quadratic interpolation. The rigid condition at $X = 0$ is satisfied by requiring $q_1 = \bar{u}$ and $\delta q_1 = 0$ in the first element.

Equation (16) can be written in the matrix form

$$(17) \quad \delta\Pi_n \equiv \delta\mathbf{q}^T \mathbf{Q} = \delta\mathbf{q}^T (\mathbf{k}\mathbf{q} - \mathbf{f}),$$

where \mathbf{q} and \mathbf{Q} are the *generalized element nodal displacement and force vectors* i.e.,

$$(18) \quad \mathbf{q}^T = [q_1 \quad q_2 \quad \dots], \quad \mathbf{Q}^T = (\mathbf{k}\mathbf{q} - \mathbf{f})^T = [Q_1 \quad Q_2 \quad \dots],$$

\mathbf{k} is the *element stiffness matrix* and \mathbf{f} is the *element external or applied force matrix* within the element and on the

boundary of the last element. The terminology of stiffness and force is borrowed from structural mechanics. In general \mathbf{q} and \mathbf{Q} may not have the dimension of displacement and force, respectively. The components of \mathbf{k} and \mathbf{f} are

$$(19) \quad k_{ij} = \frac{2}{\varepsilon} \int_{-1}^1 ea(\xi) \frac{dh_i}{d\xi} \frac{dh_j}{d\xi} d\xi, \quad f_i = -\frac{\varepsilon}{2} \int_{-1}^1 p(\xi) h_i(\xi) d\xi + \delta_{i2} \alpha \bar{T},$$

in which the range of i, j is from one to the number of degrees-of-freedom of the element, and δ_{i2} is the Kronecker delta. These matrices are *independent of the boundary conditions* except for the element with the force boundary. The element stiffness and external force matrices relate the element nodal displacement \mathbf{q} to the element nodal force \mathbf{Q} . Clearly \mathbf{k} is symmetric and positive semi-definite provided that $ea(\xi) > 0$. The zero eigenvalue(s) and zero eigenvector(s) of \mathbf{k} associate with rigid body motion of the element, which gives zero strain energy.

Linear Element. For linear interpolation function, there are only two DOFs per element

$$\mathbf{q}^T = [q_1 \quad q_2], \quad \delta\mathbf{q}^T = [\delta q_1 \quad \delta q_2].$$

If ea and p are approximated as constants within the element, the integration can be carried out analytically to give

$$(20) \quad \mathbf{k} = \frac{ea}{\varepsilon} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = -\frac{p\varepsilon}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ T \end{bmatrix}.$$

In this case, \mathbf{k} has a zero eigenvector $q_1 = q_2$. The finite element process lumps half of the applied distributed load over the element to each node. Since the shape functions are linear in ξ , we call the element a **linear element**.

Quadratic Element. We use higher order approximations for the field variable(s) to derive higher order elements. Let us use the quadratic interpolation functions of Eq. (8) and the linear approximations for ea and p , i.e.,

$$ea(\xi) = ea_0 + ea_1 \varepsilon \xi / 2, \quad p(\xi) = p_0 + p_1 \varepsilon \xi / 2,$$

where ea_0, ea_1, p_0, p_1 are constants, we obtain from Eq. (19)

$$(21) \quad \mathbf{k} = \frac{ea_0}{3\varepsilon} \begin{bmatrix} 7 & & \text{sym} \\ 1 & 7 & \\ -8 & -8 & 16 \end{bmatrix} + \frac{2ea_1}{3} \begin{bmatrix} -1 & & \text{sym} \\ 0 & 1 & \\ 1 & -1 & 0 \end{bmatrix},$$

$$(22) \quad \mathbf{f}^T = -p_0 \varepsilon [1 \quad 1 \quad 4]/6 - p_1 \varepsilon^2 [-1 \quad 1 \quad 0]/12 + \alpha [0 \quad \bar{T} \quad 0]$$

If $p_1 = 0$, the finite element process lumps $1/6$ of the distributed load over the element to each of the end nodes and $2/3$ to the midpoint node. There are three degrees-of-freedom for the element. The rank of \mathbf{k} is two as $ea(\xi)$ must be non-negative. Since the interpolation functions involve all quadratic functions of ξ , we call the element a *quadratic element*.

Similarly even higher order elements can be constructed (Tong and Rossettos 1977). Analytical integration of Eq. (19) is not essential because one can always evaluate the integrals numerically. In certain practical applications, one sometime purposely lowers the order approximation in integration to reduce the rigidity of the stiffness matrix. Also, in practice, ea or p may be available only in numerical form that Eq. (19) can only be integrated numerically. The formation of the system of equations for the whole domain from the element matrices and solving the algebraic equations will be discussed in later sections.

Problem 18.2. Find the limit of ea_0 and ea_1 for the rank of \mathbf{k} in Eq. (21) to be one and explain why such a condition is unphysical.

18.3. SHAPE FUNCTIONS AND ELEMENT MATRICES FOR HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

The interpolation functions and element matrices derived up to this point may not be suitable for problems governed by differential equations of order higher than 2. Consider the bending of a clamped cantilever beam

$$(1) \quad d^2[EI(X)(d^2u/dX^2)]/dX^2 - P(X) = 0$$

over $0 \leq X \leq L$, with boundary conditions

$$(2) \quad u = \bar{u}, \quad du/dX = \bar{u}', \quad \text{at } X = 0 \text{ (rigid condition)},$$

$$(3) \quad EI \frac{d^2u}{dX^2} = \bar{M}, \quad \frac{d}{dX} \left(EI \frac{d^2u}{dX^2} \right) = \bar{Q} \quad \text{at } X = L \text{ (natural condition)},$$

where overhead bar denotes prescribed quantity. The principle of minimum potential energy of the beam is defined in Eq. (10.8:3). The first variation of the functional of an element is

$$(4) \quad \delta\Pi_n = \frac{\varepsilon}{2} \int_{-1}^1 \left[\frac{16ei(\xi)}{\varepsilon^4} \frac{d^2u}{d\xi^2} \frac{d^2\delta u}{d\xi^2} - p(\xi)\delta u \right] d\xi - \alpha \left(\frac{2\bar{M}}{\varepsilon} \frac{d\delta u}{d\xi} - \bar{Q}\delta u \right) \Big|_{\substack{\xi=1 \\ (X=L)}}$$

where $\alpha = 0$ for all elements except the one containing the node $X = L$, then $\alpha = 1$. The admissibility condition requires that u be C^1 continuous, i.e., u and du/dx are continuous over $0 \leq X \leq L$ to assure that the sum of the functional of all elements equals that of the entire domain, and that u satisfies the rigid conditions Eq. (2). The boundary conditions at $X = L$ are natural associated with applied moment and shear force. The point $X = L$ is a *force boundary*.

Obviously, u defined in Eqs. (18.2:3) and (18.2:7) is not C^1 continuous. We need to construct new interpolation functions making u be so. First we assume

$$(5) \quad u = q_1 h_1(\xi) + q_2 h_2(\xi) + q_3 h_3(\xi) + q_4 h_4(\xi)$$

for the element between X_n and X_{n+1} , where ξ is the local coordinate with range from -1 to 1 as defined before and the interpolation functions h_i and $dh_i/d\xi$ are zero at $\xi = \pm 1$ except that,

$$h_1(-1) = h_3(1) = 1, \quad dh_2/d\xi(-1) = dh_4/d\xi(1) = \varepsilon/2.$$

The functions as defined are called the *Hermite interpolation* that

$$(6) \quad \begin{aligned} h_1(\xi) &= (2 + \xi)(\xi - 1)^2/4, & h_2(\xi) &= \varepsilon(\xi + 1)(\xi - 1)^2/8, \\ h_3(\xi) &= (2 - \xi)(\xi + 1)^2/4, & h_4(\xi) &= \varepsilon(\xi - 1)(\xi + 1)^2/8. \end{aligned}$$

In this case q_1 and q_3 are the nodal values of u and that q_2 and q_4 are the values of du/dX at $\xi = \pm 1$, respectively, i.e.,

$$(7) \quad \begin{aligned} u_{2n-1} &\equiv u(X_{n-1}) = q_1, & u_{2n} &\equiv du/dX(X_{n-1}) = q_2, \\ u_{2n+1} &\equiv u(X_n) = q_3, & u_{2n+2} &\equiv du/dX(X_n) = q_4. \end{aligned}$$

Obviously, u is C^1 continuous over $0 \leq X \leq L$. We call the q 's and u 's the *generalized coordinates* or *DOFs*. Now we have two DOFs at each node. They have different physical dimension.

Substituting Eq. (5) into (4) yields, in the same form as Eq. (18.2:17),

$$\delta\Pi_n = \delta\mathbf{q}^T (\mathbf{k}\mathbf{q} - \mathbf{f})$$

for the element where the components of \mathbf{k} and \mathbf{f} are

$$(8) \quad k_{ij} = (8/\varepsilon^3) \int_{-1}^1 ei(\xi) (d^2 h_i/d\xi^2) (d^2 h_j/d\xi^2) d\xi,$$

The matrix \mathbf{k} is symmetric and positive semi-definite if $ei(\xi) > 0$. It has two zero eigenvectors associated with rigid body translation and rotation. Thus, if Eq. (8) is integrated exactly, the rank of \mathbf{k} equals the number of q 's minus 2. One often integrates Eq. (8) approximately by numerical means. The approximation may lower the rank of \mathbf{k} and introduce spurious deformation mode(s), i.e., nonrigid body motion with no contribution to energy.

If ei and p are treated as constant within the element, we have

$$(10) \quad \mathbf{k} = \frac{ei}{\varepsilon^3} \begin{bmatrix} 12 & & & \text{sym} \\ 6\varepsilon & 4\varepsilon^2 & & \\ -12 & -6\varepsilon & 12 & \\ 6\varepsilon & 2\varepsilon^2 & -6\varepsilon & 4\varepsilon^2 \end{bmatrix}, \quad \mathbf{f} = \frac{p\varepsilon}{12} \begin{bmatrix} 6 \\ \varepsilon \\ 6 \\ -\varepsilon \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 0 \\ -\bar{Q} \\ \bar{M} \end{bmatrix}.$$

Each element has 4 DOFs and each node has two. For an element with its 1st and 2nd nodes being the I_1^{th} and I_2^{th} global nodes, the relationship between the global and the local labels is given in the table below. The array I_r is used in the assembly process to form the global \mathbf{K} and \mathbf{F} of the system.

Table 18.3:1. Global and local degrees and nodal labels.

Global degree label	Global nodal label	Local degree label	Local nodal label
$2I_1 - 1$	I_1	1	1
$2I_1$	I_1	2	1
$2I_2 - 1$	I_2	3	2
$2I_2$	I_2	4	2

As before we see, from Eq. (8), that \mathbf{k} is independent of loads and boundary conditions. Equation (9) shows that the interior load contributes to the element applied force matrix \mathbf{f} . The boundary loads contribute to \mathbf{f} of the elements on the force boundary only.

One can use Eqs. (8) and (9) to construct element matrices of higher order elements with higher order shape functions. For example, consider an element of m nodes at $\xi_j, j = 1, \dots, m$. We write the function u in the form

$$u = \sum_{j=1}^{2m} q_j h_j(\xi),$$

where

$$(11) \quad \begin{aligned} q_{2j-1} &= u(\xi_j), & q_{2j} &= du/dX(\xi_j), \\ h_{2j-1} &= [1 + \frac{d\psi_j(\xi_j)}{d\xi} \frac{\xi_j - \xi}{\psi_j(\xi_j)}] \frac{\psi_j(\xi)}{\psi_j(\xi_j)}, & h_{2j} &= \frac{\varepsilon(\xi - \xi_j)\psi_j(\xi)}{2\psi_j(\xi_j)}, \\ \psi_j(\xi) &= \prod_{i=1}^m (\xi - \xi_i)^2 / (\xi - \xi_j)^2, & (j \text{ not summed}). \end{aligned}$$

Obviously, the interpolation functions given in Eq. (6) or (11) are applicable to the differential equation such as

$$(12) \quad \frac{d^2}{dX^2} \left[EI(X) \frac{d^2u}{dX^2} \right] - \frac{d}{dX} \left[N(X) \frac{du}{dX} \right] + K(X)u - P(X) = 0$$

with the first variation of the potential energy in the form

$$(13) \quad \delta\Pi_n = \frac{\varepsilon}{2} \int_{-1}^1 \left[\frac{16ei}{\varepsilon^4} \frac{d^2u}{d\xi^2} \frac{d^2\delta u}{d\xi^2} + \frac{4n}{\varepsilon^2} \frac{du}{d\xi} \frac{d\delta u}{d\xi} + ku\delta u - p\delta u \right] d\xi + B,$$

where $ei = EI(X)$, $n = N(X)$, $k = K(X)$, and $p = P(X)$ within the element, and B involves terms related to natural boundary conditions for element with force boundary. The element matrices can be similarly computed and assembled into a system of algebraic equations for numerical solution. **The finite element process reduces the task of solving differential equations to numerically solving algebraic equations. The complexity of constructing analytical solutions is replaced by the routine integration, which can be performed numerically, in evaluating the element matrices.**

Problem 18.3. Show that the rank of \mathbf{k} given in Eq. (10) is two.

Problem 18.4. Using the interpolation functions defined in Eqs. (18.2:4) and (6), determine the element matrices associated with $\int_{-1}^1 k(\xi)u\delta u d\xi$ for $k = k_0 + k_1\varepsilon\xi$.

Problem 18.5. Consider the bending of a beam as defined in Eq. (1). The prescribed boundary conditions are

$$\begin{aligned} EI(X)(d^2u/dX^2) &= \bar{M}, & d(EI(X)[d^2u/dX^2])/dX &= \bar{Q} \quad \text{at } X = 0, \\ u &= \bar{u}, & du/dX &= \bar{u}' \quad \text{at } X = L. \end{aligned}$$

(a) Show that the first variation of the potential energy of an element is

$$\delta\Pi_n = \frac{\varepsilon}{2} \int_{-1}^1 \left[\frac{16ei(\xi)}{\varepsilon^4} \frac{d^2u}{d\xi^2} \frac{d^2\delta u}{d\xi^2} - p(\xi)\delta u \right] d\xi + \alpha \left(\frac{2\bar{M}}{\varepsilon} \frac{d\delta u}{d\xi} - \bar{Q}\delta u \right) \Big|_{\substack{\xi=-1 \\ (X=0)}},$$

where $\alpha = 0$ for all elements except for the one containing the node $X = 0$, then $\alpha = 1$. The equation above has a same form as Eq. (4) except for the last term with a sign difference. This shows clearly that \mathbf{k} does not depend on boundary conditions.

- (b) Explain the reason for the sign difference.
- (c) Derive the element force vector for the element with the node $X = 0$.

18.4. ASSEMBLING AND CONSTRAINING GLOBAL MATRICES

We need to obtain the global matrices \mathbf{K} and \mathbf{F} to solve for the unknowns u 's. It is \mathbf{K} and \mathbf{F} that relate the *nodal generalized coordinates* of one node to those of all other nodes. The finite element method divides the solution process in three steps:

- (1) Assemble the element matrices into the global matrices based on Eq. (18.2:14) with $\delta\Pi_n$ in the form of Eq. (18.2:17). $\delta\Pi_n$ is given in Eq. (18.2:16) for rods and Eq. (18.3:4) for beams.
- (2) Impose rigid constraints.
- (3) Solve the system of algebraic equations by computers.

Assembling the element matrices to form \mathbf{K} and \mathbf{F} can be symbolized as

$$\mathbf{K} = \sum_m \mathbf{k}_m, \quad \mathbf{F} = \sum_m \mathbf{f}_m,$$

where the subscript m denotes the m^{th} element. Assembling is actually a numerical summing process to carry out the summation of Eq. (18.2:14) for all elements. Starting with null matrices \mathbf{K} and \mathbf{F} , we add to them the contributions from \mathbf{k} and \mathbf{f} of each element. When the last element has been added, the global \mathbf{K} and \mathbf{F} are formed. We implicitly assume that all elements have the same definition of generalized coordinates at their common node to allow the direct addition of the components of \mathbf{k} and \mathbf{f} from different elements. If the generalized coordinates of an element are different from those of the global system, a transformation of \mathbf{k} and \mathbf{f} (to be discussed later) is needed before addition.

In finite element application, most \mathbf{k} and \mathbf{K} are symmetric. We shall illustrate the details of assembling of the lower triangle of \mathbf{k} and \mathbf{K} only. While assembling we shall impose rigid constraints at the same time.

Consider the assembling of element m . Let I_r be the global DOF label of the r^{th} DOF of the element and p be the number of DOFs. The array I_r is called the assembly list. Then the assembly is simply

$$(1) \quad \text{add } (k_{rs})_m \text{ to } K_{rs}, \quad \text{add } (f_r)_m \text{ to } F_{I_r}, \quad \text{for } r, s = 1, 2, \dots, p$$

where $R = \text{Max}(I_r, I_s)$ and $S = \text{Min}(I_r, I_s)$. If any of the DOF is prescribed, say $u_r = \bar{u}$ being a rigid condition, we multiply the components $(k_{ir})_m$ of r^{th} column of \mathbf{k}_m by \bar{u} and subtract them from F_{I_r} , i.e., replace F_{I_r} by $F_{I_r} - (k_{ir})_m \bar{u}$. In the meantime, skip the addition r^{th} column of \mathbf{k}_m to \mathbf{K} , and set $F_{I_r} = \bar{u}$ and $K_{I_r I_r} = 1$. After we proceed through all the elements, we have the final global matrices \mathbf{K} and \mathbf{F} . The resulting equation

$$(2) \quad \mathbf{K}\mathbf{u} = \mathbf{F}$$

automatically gives the solution satisfying the rigid constraints at the constrained nodes and the natural conditions in the integral sense.

To illustrate assembling, let us assemble the quadratic element, say r shown in Fig. 18.2:1(b). The second node is constrained with prescribed value \bar{u} and its global label is 3. The assembly list is

Global label (Assembly list I_r)	Local label
2	1
3	2
$N+3$	3

To assemble the element, we

add $(k_{11})_r$ to K_{22}	set $K_{3,2} = 0$	set $K_{33} = 1$
add $(k_{31})_r$ to $K_{N+3,2}$	set $K_{N+3,3} = 0$	add $(k_{33})_r$ to $K_{N+3,N+3}$
add $f_1 - (k_{21})_r \bar{u}$ to F_2	set $F_3 = \bar{u}$	add $f_3 - (k_{32})_r \bar{u}$ to F_{N+3}

in which the subscript $(\cdot)_r$ signifies the quantity of the r^{th} element.

Consider the case of the 3-element division as shown in Fig. 18.4:1(a) with node 1 of the first element being a constrained node. The assembly list for the elements is a 2-dimensional array given below:

Element 1	1	2	-
Element 2	2	3	-
Element 3	3	5	4

Note that for element 3 the global label of its 2nd DOF is 5 and that of its 3rd DOF (associated with the interior node of the element) is 4. The first two elements have two DOFs each with element matrices given in Eq. (18.2:20). Element 3 has three DOFs with its element matrices given in Eqs. (18.2:21) and (18.2:22).

For constant ea and p within each element, the constrained assembled matrices are

$$\mathbf{K} = \begin{bmatrix} 1 & & & & & \text{sym} \\ 0 & \left(\frac{ea}{\varepsilon}\right)_1 + \left(\frac{ea}{\varepsilon}\right)_2 & & & & \\ 0 & -\left(\frac{ea}{\varepsilon}\right)_2 & \left(\frac{ea}{\varepsilon}\right)_2 + \left(\frac{7ea}{3\varepsilon}\right)_3 & & & \\ 0 & 0 & -[8ea/(3\varepsilon)]_3 & [16ea/(3\varepsilon)]_3 & & \\ 0 & 0 & [ea/(3\varepsilon)]_3 & -[8ea/(3\varepsilon)]_3 & [7ea/(3\varepsilon)]_3 & \end{bmatrix},$$

$$\mathbf{F}^T = - \begin{bmatrix} -\bar{u} & (p\varepsilon/2)_1 + (p\varepsilon/2)_2 & \left(\frac{p\varepsilon}{2}\right)_2 + \left(\frac{p\varepsilon}{6}\right)_3 & \left(\frac{p\varepsilon}{3}\right)_3 & \left(\frac{p\varepsilon}{6}\right)_3 - (\bar{T})_3 \end{bmatrix}.$$

The solution of $\mathbf{Ku} = \mathbf{F}$ gives $u_1 = \bar{u}$. No special effort is required to enforce the natural boundary condition.

The natural conditions are accounted for by the integral formulation and reflected in the generalized force vector of elements with prescribed natural boundary. The conditions are satisfied automatically in an approximate sense, which makes meeting natural boundary conditions easy. In the present case, $EA(L)du/dX = \bar{T}$ at $X = L$ is a natural boundary condition. The term \bar{T} in \mathbf{f} of the last element accounts for this natural condition.

The ease of handling boundary conditions, rigid and natural, for any boundary is an important advantage of the finite element method over other approximated methods such as the finite difference method. In the finite element method, \mathbf{K} is generally *banded and sparse*. The semi-bandwidth of a row of a matrix is the number of components, named also as *entries*, between the first nonzero entry of the row to the diagonal. For symmetric matches, only semi-bandwidth is essential. The semi-bandwidth depends strongly on how the global DOFs are labeled. For example, for the label scheme shown in Fig. 18.4:1(c), the semi-bandwidths are 1, 2, 2, 2, 3, while the corresponding semi-bandwidths are 1, 2, 2, 4, 4 for Fig. 18.4:1(d).

The semi-bandwidth strongly affects the computer storage for the assembled matrices and the computer time for solving the system of equations. For the *variable bandwidth solution algorithm* (Tong and Rossettos 1977), the optimum-labeling scheme is approximately the one that gives a minimum average semi-bandwidth. For the simple problem considered, the labeling scheme shown in Fig. 18.4:1(d) does not give a minimum semi-bandwidth. However, for more complicated 2- or 3-dimensional problems, determining the labeling scheme to minimize the average semi-bandwidth of the global stiffness matrix is non-trivial (see George and Liu 1981).

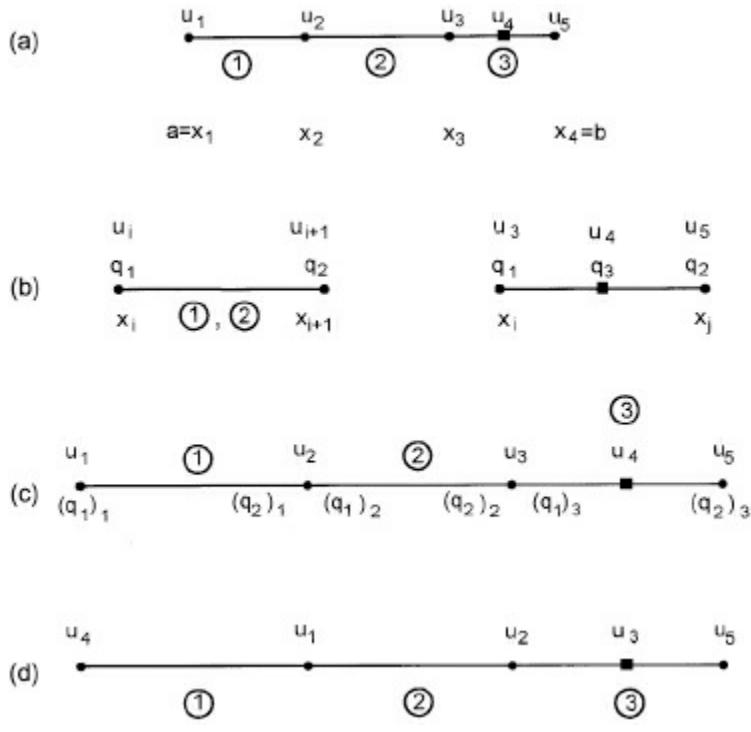


Fig. 18.4:1. Local and global labels in three-finite-element division of a domain. (c) and (d) are two alternate global schemes.

One can improve the accuracy of the finite element analysis using more and/or higher order elements. For the same number of elements, the system associated with higher order elements has more DOFs and higher semi-bandwidth for \mathbf{K} . The higher order system generally gives better approximation.

Problem 18.6. Construct the 4-node (two-end nodes and two equally spaced interior nodes) Lagrange interpolation over $0 \leq x \leq \varepsilon$.

Problem 18.7. Consider the problem defined by Eqs. (18.2:10–12) with constant EA . (a) Show the finite element solution using linear elements always gives the exact solution at the nodes for any distributed load $p(X)$ and any number of elements of any size. (*Hint:* See Tong 1970a.) (b) Do you expect the same is true if quadratic elements are used? Prove your result.

18.5. EQUATION SOLVING

The finite element analysis inevitably ends up solving a large set of algebraic equations. The feasibility of finite element analysis hinges on the ability to solve large system of algebraic equations with acceptable accuracy and reasonable time. The first point associates with round-off error, which can render the numerical solution meaningless. The second depends on the operation counts involved in the solution algorithm. Usually \mathbf{K} is *symmetric* and *sparse* with relatively small semi-bandwidth. Sparseness is defined as having small number of nonzero entries. Both the sparseness and bandwidth of each row affect strongly the operation counts of the solution algorithm.

One can use either a direct or iterative method to solve algebraic equations. A direct method would lead to the “exact” solution of the algebraic system after a finite number of operations if there is no round-off error. On the other hand, an iterative method would converge to a solution close to the exact one after a finite number of iterations. Detailed discussion of solution methods is beyond the scope of this book. We will just outline briefly the concept of two direct methods, namely the Gauss elimination and triple factoring, and identify a number of iterative schemes. Readers are referred to literature for details.

Gaussian Elimination. The idea underlying Gaussian elimination is simple. For a system of n algebraic equations, one uses the first equation to eliminate the first variable u_1 from the last $n-1$ equations, then uses the second equation to eliminate u_2 from the updated last $n-2$ equations, and so on. After $n-1$ such eliminations, the resulting matrix equation is triangular. This process is termed *forward elimination*. At this stage the last equation involves only one unknown u_n , which can be determined. Then one determines u_{n-1} from the $(n-1)^{\text{th}}$ equation, and so on until all unknowns are obtained. The later procedure is called *backward substitution*.

There are many time saving techniques to reduce the operation counts. The main one is to recognize the zero entries of \mathbf{K} in the elimination procedure. When u_i is being eliminated from the j^{th} equation, if the coefficient of u_i in the equation is zero, the operation to eliminate it can be skipped. For a sparse or narrow semi-bandwidth matrix, this provides great timesaving. Recognizing zero entries of \mathbf{K} efficiently and skipping the unnecessary operations are key to

the success of implementing finite element analysis in practical applications.

The essential information enabling us to recognize the unnecessary operation in the forward elimination and backward substitution processes is the column number of the first nonzero entry of each row, denoted by $C(i)$ for the i^{th} row of \mathbf{K} . Then the semi-bandwidth of the row is $i - C(i) + 1$. All operations to eliminate the first through $[C(i) - 1]^{\text{th}}$ variable from the i^{th} row can be skipped in forward elimination. In addition, there is no need to store K_{ij} for $j < C(i)$. This reduces the data storage requirement, as the semi-bandwidth of K can be small in practice. Therefore, it is important to label the nodes in such way to minimize average semi-bandwidth.

For symmetric \mathbf{K} , the first zero row of the i^{th} column of the resulting matrix after forward elimination is also $C(i)$. This allows us to use the same computer storage for the lower triangle of \mathbf{K} before forward elimination to store the resulting upper triangle after the process. The determination of $C(i)$ will be discussed later (Tong and Rossettos 1977).

Triple Factoring Method. This is also a commonly used direct method. The basic idea is to express a symmetric matrix \mathbf{K} in the form

$$\mathbf{K} = \mathbf{LDL}^T,$$

in which \mathbf{D} is a diagonal matrix and \mathbf{L} is a lower triangular matrix with all diagonal terms being unity and the off diagonal entries of the upper triangle being zero. The process of determining \mathbf{L} and \mathbf{D} is named *factorization*. Then we solve the equation

$$\mathbf{Lu}^* = \mathbf{F}$$

for \mathbf{u}^* , a process is called *forward substitution*. Finally we determine \mathbf{u} from

$$\mathbf{L}^T \mathbf{u} = \mathbf{D}^{-1} \mathbf{u}^*$$

by *backward substitution*. One can show that \mathbf{L} and \mathbf{K} have same semibandwidth $C(i)$. Thus one can use the same storage location for \mathbf{L} and \mathbf{D} . One can also use the same storage location for \mathbf{F} , \mathbf{u}^* , and \mathbf{u} as they have the same size and are generated sequentially.

Factorization in the triple factoring method and forward elimination in Gaussian elimination method are the most time consuming parts of the solution procedures. For many problems, they consume 85 to 95 percent of the total computational effort. The multiplication counts for both methods are of the order n^3 for a fully populated matrix of order n . The counts are reduced to the order nB^2 for a banded matrix of average semi-bandwidth B . The counts are further reduced if the variability of the bandwidth is accounted for. During factorization or forward elimination of a highly sparse matrix, only some of the zero entries between the first nonzero entry and the diagonal of a row are *filled-in* (become nonzero), while most of zero entries remain. Recognizing those unaltered entries and avoiding the unnecessary operations to update them can reduce the operation counts further. Therefore, it is important to recognize the '*filling-ins*' to reduce both memory requirements and the number of arithmetic operations. Algorithms and software have been developed to reorder \mathbf{K} for this purpose. Unfortunately, the algorithms best to minimize the average semibandwidth are in general not the best for minimizing '*filling-ins*'.

Many advances in the direct method are in parallel computation, i.e., to have many central process units (CPUs) working together to solve the equations. The computation process can be very involved. Since the direct method is inherently a sequential process, many operations require information from the results of earlier steps, which may be computed by different CPUs. Also, often the assembled global stiffness matrix is so huge, that no single CPU can store all the information. When a large number of CPUs is involved, communication between different CPUs, and synchronizing the processes¹ can consume both CPU time and clock time. The speed up of computer time is not linearly proportional to the number of CPUs used. Balancing the workloads among the CPUs and minimize communications among them are essential for efficient parallel computation. Readers are referred to literature for various efficient equation solvers for sparse matrices and parallel computation.

Iterative Schemes. There are numerous iterative schemes. The methods proceed from an initial guessed solution $\mathbf{u}^{(0)}$ and define a sequence of successive approximation $\mathbf{u}^{(r)}$ until it converges to a solution close to the exact one. The procedure can be written in the form

$$\mathbf{Nu}^{(r)} = \mathbf{Pu}^{(r-1)} + \mathbf{F},$$

where \mathbf{N} and \mathbf{P} are matrices derived from \mathbf{K} . The matrix \mathbf{N} is usually diagonal or triangular (e.g., the upper or lower triangular part of \mathbf{K} while \mathbf{P} the remaining part) so that $\mathbf{u}^{(r)}$ can be readily solved. In each step, extensive operations involve the multiplication of \mathbf{P} and $\mathbf{u}^{(r-1)}$. One advantage of some of the iterative methods is that the product can be computed at the element level. This makes it unnecessary to physically form the global \mathbf{P} to save computer storage and easily adaptable for parallel processing. Commonly used iterative methods include Jacob's iteration, the Gauss-Seidel iteration, the conjugate gradient method, the variable metric methods, etc. (see Isaacson and Keller 1966).

The accuracy of the solution by the direct method and the number of iterations required by the iterative methods to reach a converged solution depend strongly on the *condition number*, the ratio of the highest to the lowest eigenvalue of the constrained \mathbf{K} . The condition number of the finite element equations can be estimated from the element stiffness matrices (Tong 1970b, Hughes 1987). Reconditioning and reordering (including pivoting) \mathbf{K} are often necessary in order to obtain numerical solution of acceptable accuracy (George and Liu 1981, Chen *et al.* 1996) for very large systems.

Within a tolerable error, there is no clear cut answer as to which method is best. Generally, iterative methods are better for highly sparse systems of over one hundred thousand equations. It is easier to implement for parallel computation if the scheme does not require assembled global matrices. The direct methods are often faster, if they can be used at all. The practical considerations of adopting a specific method often depends on user's familiarity to a particular computer program and its availability.

The finite element processes of assembling, constraining, and solving algebraic equations are very general and applicable to multi-fields and multidimensional problems. In assembling an element, one needs to know the assembly list, which identifies the global labels of the generalized coordinates of elements. In constraining the assembled matrices, one needs to know the DOFs to be constrained and their constrained values. Constraining can be performed as the elements are being assembled. In solving an algebraic equation by a direct method, one needs to know the semi-bandwidths and sparseness of the assembled stiffness matrix for efficient computation. This is important for saving computer time and the computer storage for the global stiffness matrix.

18.6. TWO-DIMENSIONAL PROBLEMS BY ONE-DIMENSIONAL ELEMENTS

Plane truss is a class of two-dimensional problems can be modeled by one-dimensional elements. Consider a two-member plane truss as shown in Fig. 18.6:1. Each member can be approximated by one or more one-dimensional elements of Eq. (18.2:20) or (18.2:21) and (18.2:22). Only one element is sufficient for a truss member of uniform cross-section. We use the nodal displacements (u , v) in the x , y -directions as the generalized coordinates. Let q 's be the axial nodal displacements of elements. To assemble the element matrices in plane truss applications, we need first to transform the element matrices from those associated with q 's to those with $(u$, v). We then assemble the transformed element matrices to form the global matrices.

For a truss of uniform cross-section with nodal displacements q_1 and q_2 the element matrices are given in Eq. (18.2:20). Let (\bar{q}_1, \bar{q}_2) and (\bar{q}_3, \bar{q}_4) be the nodal displacements (u, v) at the end nodes. One has

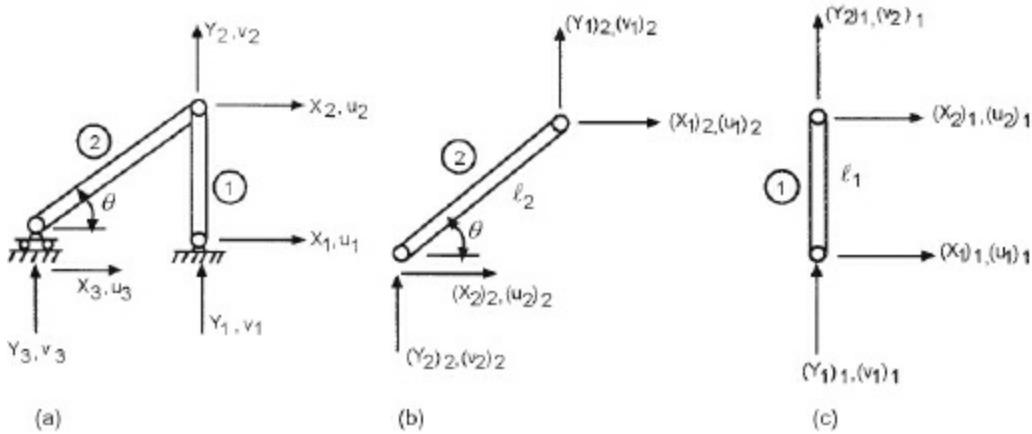


Fig. 18.6:1. Two-member plane truss assembly.

$$\mathbf{q} = \mathbf{R}\bar{\mathbf{q}}$$

where

$$\bar{\mathbf{q}}^T = [\bar{q}_1 \quad \bar{q}_2 \quad \bar{q}_3 \quad \bar{q}_4], \quad \mathbf{R} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} c\bar{q}_1 + s\bar{q}_2 \\ c\bar{q}_3 + s\bar{q}_4 \end{bmatrix},$$

in which $c = \cos \theta$, $s = \sin \theta$ with θ denoting the orientation of the element with respect to the x -axis and \mathbf{R} is the rotation matrix.

The element matrices associated with $\bar{\mathbf{q}}$ can be derived based on the first variation of the functional for the element

$$\delta\bar{\mathbf{q}}^T (\bar{\mathbf{k}}\bar{\mathbf{q}} - \bar{\mathbf{f}}) = \delta\mathbf{q}^T (\mathbf{k}\mathbf{q} - \mathbf{f}) = \delta\bar{\mathbf{q}}^T (\mathbf{R}^T \mathbf{k} \mathbf{R} \bar{\mathbf{q}} - \mathbf{R}^T \mathbf{f}).$$

The element matrices associated $\bar{\mathbf{q}}$ are thus

$$\bar{\mathbf{k}}_{4 \times 4} = \underset{4 \times 2}{\mathbf{R}}^T \underset{2 \times 2}{\mathbf{k}} \underset{2 \times 4}{\mathbf{R}}, \quad \bar{\mathbf{f}}_{4 \times 1} = \underset{4 \times 2}{\mathbf{R}}^T \underset{2 \times 1}{\mathbf{f}},$$

which are to be assembled according to Eq. (18.4:1).

Similarly one can generate 6-degree-of-freedom 2-node 3-dimensional elements from 2-node 1-dimensional elements by expressing the nodal axial displacement in terms of a 3-dimensional vector (u, v, w) .

18.7. GENERAL FINITE ELEMENT FORMULATION

We have demonstrated the construction of finite element equations for rods and beams governed by 2nd and 4th order differential equation, respectively. The process includes: *dividing the domain into a finite number of discrete elements, evaluating the element matrices based on a functional in integral formulation with admissible functions and selected generalized coordinates, assembling the element matrices to form global matrices, imposing constraints to enforce the rigid boundary conditions, and finally solving the constrained algebraic equations numerically by high-speed computer.*

In constructing the element stiffness matrices, one represents the field variable \mathbf{u} over an element in terms of *nodal parameters (generalized coordinates)* and admissible shape functions. The problem can have one or more field variables in multi-dimensional domain. Consider a problem

$$(1) \quad \mathbf{A}(\mathbf{u})^T \equiv [A_1(\mathbf{u}) \quad A_2(\mathbf{u}) \quad \dots] = 0 \quad \text{in } V,$$

$$(2) \quad \mathbf{B}(\mathbf{u})^T \equiv [B_1(\mathbf{u}) \quad B_2(\mathbf{u}) \quad \dots] = 0 \quad \text{on } \partial V$$

where A 's and B 's are, respectively, differential operators defined in V and on

$$(3) \quad \delta\Pi = \int_V W(\mathbf{u}, \delta\mathbf{u}) dV + \int_{\partial V} w(\mathbf{u}, \delta\mathbf{u}) dS = 0,$$

for all admissible \mathbf{u} and $\delta\mathbf{u}$, where $W(\mathbf{u}, \delta\mathbf{u})$ and $w(\mathbf{u}, \delta\mathbf{u})$ are the first variation of the integrands of a functional. The admissible requirements are: \mathbf{u} and $\delta\mathbf{u}$ are sufficiently smooth to allow $\delta\Pi$ be mathematically defined, and satisfy the rigid conditions (see Sec. 10.6). Both $W(\mathbf{u}, \delta\mathbf{u})$ and $w(\mathbf{u}, \delta\mathbf{u})$ can be function of \mathbf{u} and its derivative. For example, for the bending problem considered in Sec. 18.3, we use both w and dw/dx as variables.

In the finite element method, the domain is divided into elements. Equation (3) can be written in the form

$$(4) \quad \Delta \quad \delta\Pi = \sum_{\text{all elements}} \delta\Pi_e = \sum_{\text{all elements}} \left[\int_{V_e} W(\mathbf{u}, \delta\mathbf{u}) dV + \int_{\partial V_e} w(\mathbf{u}, \delta\mathbf{u}) dS \right] = 0,$$

where V_e is the volume of an element and $\partial V_{e\sigma}$ is the portion of ∂V_e with prescribed natural boundary condition. Generally the admissibility condition requires C^0 continuity for \mathbf{u} and $\delta\mathbf{u}$ if Eqs. (1) and (2) involve second order differential operator(s), and C^1 continuity if fourth order differential operator(s) is involved. The continuity requirement at interelement boundaries is called the *compatibility condition*. The expression $\delta\Pi_e$ of rod requires C^0 continuity for u and δu as in Eq. (18.2:15).

The specific form of Eq. (3) depends on the nature of the problem. For a rod discussed in Sec. 18.2, we have,

$$A_1(\mathbf{u}) = d[EA(X)(du/dX)]/dX - P(X) = 0,$$

over $0 \leq X \leq L$, and

$$B_1(\mathbf{u}) = u - \bar{u} = 0 \quad \text{as the rigid boundary condition at } X = 0,$$

$$B_2(\mathbf{u}) = EA(L) \frac{du}{dX} - \bar{T} = 0 \quad \text{as the natural boundary condition at } X = L.$$

The functional for the minimum of potential energy principle is given in Eq. (18.2:13) with its first variation given in Eq. (18.2:14)

$$\delta\Pi = \int_0^L [EA(X)(du/dX)(d\delta u/dX) + P(X)\delta u] dX - \bar{T}\delta u(L) = 0.$$

The admissibility requirements are: u and δu are C^0 continuous over $0 \leq X \leq L$, and $u = \bar{u}$ and $\delta u = 0$ at $X = 0$. For a single element, we have

$$\int_{V_e} W(\mathbf{u}, \delta\mathbf{u}) dV = \frac{\varepsilon}{2} \int_{-1}^1 \left[\frac{4ea(\xi)}{\varepsilon^2} \frac{du}{d\xi} \frac{d\delta u}{d\xi} + p(\xi) \delta u \right] d\xi,$$

$$\int_{\partial V_e} w(\mathbf{u}, \delta\mathbf{u}) dS = -\alpha \bar{T} \delta u(L),$$

as given in Eq. (18.2:15).

Establishing the solution based on Eq. (3) or (4) is termed integral formulation. It is called the ‘weak’ formulation meaning that the requirements of satisfying the governing differential equations in V and the natural boundary conditions on ∂V_σ are weakened.

The field variable \mathbf{u} within an element is often written in the form

$$(5) \quad \mathbf{u} = \mathbf{h}(\mathbf{X})\mathbf{q},$$

where the entries of $\mathbf{h}(\mathbf{X})$ are *shape or interpolation* functions so chosen to make \mathbf{u} *admissible* and *complete*² (to be discussed in the next section) and \mathbf{q} is a unknown *parameter or generalized coordinate vector* associated with nodal values of \mathbf{u} and possibly its spatial derivatives too. The completeness and smoothness of \mathbf{u} (*admissibility requirements*) is to assure its convergence to the exact solution. Some components or transformed components of \mathbf{u} must equal its prescribed values on the rigid boundaries. For example, in Problem 18.5, the rigid constraints are $u = \bar{u}$ and $du/dX = \bar{u}'$ at $X = L$.

One usually use the same spatial variation for \mathbf{u} and $\delta\mathbf{u}$, i.e.,

$$(6) \quad \delta\mathbf{u} = \mathbf{h}(\mathbf{X})\delta\mathbf{q}.$$

The rigid constraint is to require those components of $\delta\mathbf{q}$ associated with the rigid condition equal zero.

Again, for the beam example in Sec. 18.3, we have

$$\mathbf{u} = \mathbf{h}(\mathbf{X})\mathbf{q} = [h_1(\xi) \ h_2(\xi) \ h_3(\xi) \ h_4(\xi)][q_1 \ q_2 \ q_3 \ q_4]^T.$$

The rigid constraints are $\bar{q}_1 = \bar{u}$, $\bar{q}_2 = \bar{u}'$ and $\delta u = \delta q_1 = d\delta u/dX = \delta q_2 = 0$ at $X = 0$ of the first element. The weak form (3) is the first variation of the potential energy or the principle of virtual work.

Equation (4) is used to derive the system of algebraic equations for the approximate solution. We write the algebraic equations in the form

$$(7) \quad \mathbf{K}\mathbf{r} = \mathbf{F}$$

where \mathbf{K} and \mathbf{F} may depend on \mathbf{q} if the problem is nonlinear. \mathbf{q} is the *generalized coordinate* of the elements, whereas \mathbf{r} is that of the whole system. The well-known Rayleigh–Ritz process uses the similar approach in which the shape function matrix $\mathbf{h}(\mathbf{X})$ in Eq. (5) is expressed in terms of functions valid throughout the whole domain and \mathbf{q} is just a parameter vector, which may or may not associate with specific nodal values of \mathbf{u} . In this case there is no distinction between \mathbf{q} and \mathbf{r} .

The *basic premise of the finite element method is to use elements that are small and/or simple in shape*. Within the small region, we can represent with good accuracy any field variable by polynomials (shape functions), e.g., Eqs. (18.2:1) and (18.2:7), or special functions that interpolate the values of a field variable at selected points. However, having q_i as chosen in Sec. 18.2 makes it easy to meet the compatible requirements along the inter-element boundaries, which we shall soon see.

In the finite element method, meeting the continuity requirement at the interelement boundaries generally is not difficult. This would not be the case for problems requiring C^1 continuity in 2- and 3-dimensional problems. The *interelement boundary* for 2-D problems is a line or curve and that for 3-D problems is a plane or curved surface. An element boundary is defined by a number of discrete *nodes*. Figure 18.1:1 shows the interelement lines between two nodes. The nodal values of \mathbf{u} and its spatial derivatives [for problems governed by 4th or higher order differential equations such as bending of beams (Sec. 18.3), plates and shells] are chosen to be the generalized coordinates. They have the same values at their common nodes between elements. The difficulty is to find shape functions to assure that \mathbf{u} is smooth within the element and C^1 continuous over the entire interelement boundary. Also the shape functions must be chosen to uniquely determine the value of the integral form or the functional of the element defined in Eq. (4) as a function of \mathbf{q} except for possible rigid body motions. (This is to avoid the *zero energy modes* to be discussed later.) These chosen shape functions will assure the convergence of the finite element solution and allow simple assembling of element matrices.

Element meeting the admissibility requirements is named *compatible element*. Choosing interpolation functions to assure \mathbf{u} being C^1 continuous can be difficult for 2- and 3-dimensional problems. Thus the *admissibility condition is sometime violated* and the element is identified as an *incompatible element*.

The key difference between the conventional Ritz method and the finite element method is the manner in which the field variable(s) is represented. The Ritz process uses “*assumed functions*” that is valid for the entire domain leading generally to a fully populated \mathbf{K} in Eq. (7). In the finite element process, interpolation functions are assumed for each element and each nodal parameter associates only with elements connected to the node. One can use the latter property to determine the semi-bandwidth of a particular DOF. This characteristics also leads to a sparse and usually banded \mathbf{K} . By its nature the conventional Ritz method is limited to problems over regions of relatively simple shape since the assumed functions must be valid for the entire domain and satisfy all rigid boundary conditions. For finite element method, interpolation functions can be constructed more easily because elements generally have simple shapes. An assembly of simple shape elements can represent complex configurations.

Equation (4) or (18.2:13) and (18.2:14), which represents the functional of the whole domain as the sum of the functional of individual element or subdomain, is a very powerful and versatile concept. It permits the use of different approximations for each element. Of course, we must make sure that elements are compatible so that the sum of the functional of all elements is equal to that of the whole system, e.g., in Eqs. (18.2:13) and (18.2:14) \mathbf{u} must be C^1 continuous over the entire domain. Allowing different approximations for different elements is particularly useful in dealing with problems with singularity or steep gradient. For instance the hybrid singular element (Tong *et al.* 1973, Tong 1977, 1984) is used for domain involving cracks. The flexibility also allows easy handling of problems such as solid-fluid interactions in which part of domain is solid and part is liquid, and such as structures involving solid, beams and shells with part of the domain being governed by second order and part by fourth order differential equations.

Therefore, the finite element method is inherently an *element base method* in which local approximate solutions in terms of unknown parameters are constructed based on the integral form for each element or sub-domain. The local solutions must compatible and complete to assure convergence. The approximate solution for the whole domain is determined based on the integral form of Eq. (4).

The finite element method involves selection of interpolation functions, construction of element matrices, and assembly of the element matrices into a system of simultaneous algebraic equations for numerical solution. The process is generic that it can be applied to almost any type of problems. The element stiffness matrix depends on the problem type such as beams, plane problems, 3-dimensional solids, etc., rather than on the specifics of the problem such as dimensions, boundary conditions, and load distributions. The element applied force matrix depends on the distributed loads, but the process of determining it is again generic. We can use the same type of element stiffness matrix to solve problems of a plane with a square hole or with a circular hole. The nature of the problem is reflected in the rigid and natural constraints on the boundaries, and the distributed load. These inherent generic characteristics make it possible to apply the finite element method to almost any problems with ease.

18.8. CONVERGENCE

We can examine the convergence of finite-element solution to the exact solution from two points of view. One is to increase the number of DOFs per element with the element size fixed. If the shape functions are p^{th} degree complete polynomials within the element, the procedure leads to the so-called *p-convergence*. The other is to subdivide the domain into smaller and smaller elements with the number of DOFs per element is fixed. Both cases improve the accuracy of the representation within the element. This former procedure is not commonly used in practice because increasing the number of DOFs of an element means reformulating the local representation with higher order interpolation functions, which can be tedious and increase the bandwidth of \mathbf{K} . The latter procedure leads to the so-called *h-convergence* where h denotes the size of the element. In the latter procedure, we only have to continuously sub-divide the domain being considered, which is relatively straightforward.

The *p-convergence*, i.e., increasing the order of polynomials as the approximate function, can be proved from the convergence of the *Taylor series expansion*. The *h-convergence*, i.e., reducing the element size, can be established as follows: The finite element method constructs the solution of the entire domain from the local approximate solutions based on an *integral formulation*. If the solution is to converge to the exact solution, the *functional* or the *integral form* itself must converge as the size of the element diminishes. In proving convergence, then one only has to show that the convergence of the functional or the integral form implies the convergence of the function itself.

In practice, there are two requirements to assure convergence:

- (1) The functional must be mathematically defined over the entire domain, i.e., the assumed functions for the independent fields must meet the admissibility requirements of the functional: C^0 and C^1 continuity for the independent fields involving first and second order differentials, respectively, and the enforcement of rigid boundary conditions.

(2) The generalized coordinates for the highest differentials of the independent fields must be able to represent any constant within an element as the element size approaches zero.

Interested readers are referred to the literature (Tong and Pian 1967, Strang and Fix 1973, Babuska 1971, Oden 1969). In the subsequent sections, we shall discuss the shape functions that meet the requirements set forth above and the associated element matrices.

18.9. TWO-DIMENSIONAL SHAPE FUNCTIONS

Consider a two-dimensional domain (Fig. 18.1:1), which is divided into a finite number of *triangular and quadrilateral elements*. The *inter-element boundaries* are line segments with two or more nodes. A node can be a vertex, an interior point of the element or a point on the element boundaries. We will use the nodal values of the field variables as parameters, also called *DOFs* or *generalized coordinates*, to characterize the element. For simplicity, in subsequent discussion, we will use the local spatial coordinates to define the elements.

Consider a family of triangular elements with progressively increasing number of nodes, indicated by dots, on the element boundaries (Fig. 18.9:1). We seek to represent a 2-dimensional smooth function in an element by interpolating the values of the function at the nodes with polynomials. Each nodal value is a *DOF*. The number of polynomial terms needed to interpolate the function equals the number of element nodes.

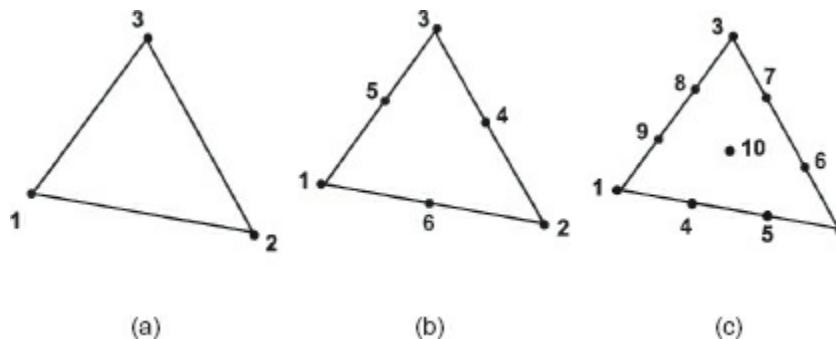


Fig. 18.9:1. (a) Linear, (b) quadratic, (c) cubic triangular elements. Node 10 is an interior node at the centroid.

The polynomial representation is a *truncated Taylor series* expansion. The order of approximation equals one plus the highest order of complete polynomial in the expansion. Therefore, for a given number of DOFs in an element, it is desirable to have an expansion with the highest order of complete polynomials. This will give the highest order of approximation in the element. The more the DOFs, the higher the order of complete polynomials can be achieved. Thus elements with more nodes and higher order approximation are called *higher order elements*.³

Shape Function Variation on Element Boundaries. Other than choosing polynomial terms to give the highest order of completeness for the shape functions, we need to select the polynomials to ensure full inter-element compatibility, i.e., meeting the admissibility requirements of the integral formulation. This can limit the choice of certain higher order polynomial terms and thus limit the order of complete polynomial. For C^0 continuity, an interpolated function is smooth within the element and equals to that of the adjacent elements at their common nodes. However, we must assure that the continuity at the common nodes of two adjacent elements implies continuity along the entire common boundary. This limits the polynomial terms of the shape function at the element boundary. For a plane element with j nodes on a line boundary, the polynomials must be of the order no more than $j - 1$ to assure compatibility along the entire boundary. Thus, the shape function can only be linear on the edge with two nodes [Fig. 18.9:1(a)]. A six-node triangle [Fig. 18.9:1(b)] has three nodes on each side. Full compatibility is assured along the boundary, if the shape functions are only linear or quadratic on the side, and so on.

In summary, the number of DOFs for a field variable is the same as the number of nodes in the element if the nodal values of the variable are used as the generalized coordinates. It is desirable that the shape functions have the highest order of complete polynomials for a given number of DOFs. This is to get the highest order of approximation. However, the polynomial terms can be restricted by the compatibility requirements, and thus limiting the allowable order of complete polynomials. For two- and three-dimensional problems, the inter-element boundaries are lines and planes, respectively. The finite element method enforces the compatibility condition at the inter-element nodes. The interpolation function must be so chosen that *compatibility at the common nodes of an inter-element boundary automatically guarantees compatibility over the entire boundary*. See examples below:

Shape Functions for 3-Node Triangle. Over the area of a 3-node triangle [Fig. 18.9:1(a)] with the nodal coordinates (x_i, y_i) , $i = 1, 2, 3$, we represent a field variable ϕ by the first 3 terms of the Taylor expansion as

$$(1) \quad \phi = \alpha_1 + \alpha_2 x + \alpha_3 y .$$

The error in the approximation within the triangle is of the order ε^2 where ε is the maximum linear dimension of the triangle. The approximated ϕ can be written in the form

$$(2) \quad \phi(x, y) = \sum_{j=1}^3 q_j h_j(x, y) = \mathbf{h}\mathbf{q},$$

where \mathbf{h} is the *shape function matrix* and \mathbf{q} the *element generalized coordinate matrix*. The components q 's are the nodal values of ϕ at the vertices. Solving Eq. (1) for α 's in terms of q 's, we find

$$(3) \quad \mathbf{h}^T = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \phi(x_1, y_1) \\ \phi(x_2, y_2) \\ \phi(x_3, y_3) \end{bmatrix},$$

$$(4) \quad \Delta = (x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1)/2,$$

$$(5) \quad a_1 = x_2y_3 - x_3y_2, \quad b_1 = y_2 - y_3, \quad c_1 = x_3 - x_2,$$

whereas a_2, b_2, c_2 and a_3, b_3, c_3 are obtainable from Eq. (5) by cyclic permutation of the subscripts 1, 2, 3, and Δ is the area of the triangular element. Note that $\zeta_i(x_j, y_j) = \delta_{ij}$, i.e., $\zeta_i = 1$ at node i and zero at the other two nodes. Thus $\zeta_i(x, y)$ are *Lagrange interpolations* and are commonly referred to as the *triangular coordinates*. One can show that

$$(6) \quad \zeta_1 = \Delta_{P23}/\Delta, \quad \zeta_2 = \Delta_{P31}/\Delta, \quad \zeta_3 = \Delta_{P12}/\Delta,$$

where Δ_{Pjk} is the area of the triangle with nodes P, j, k as shown in Fig. 18.9:2 and P is a point within the triangle. The ζ 's are not independent that they satisfy the equation

$$\zeta_1 + \zeta_2 + \zeta_3 = 1.$$

One can show that $\zeta_j = 1$ is a linear function of x, y .

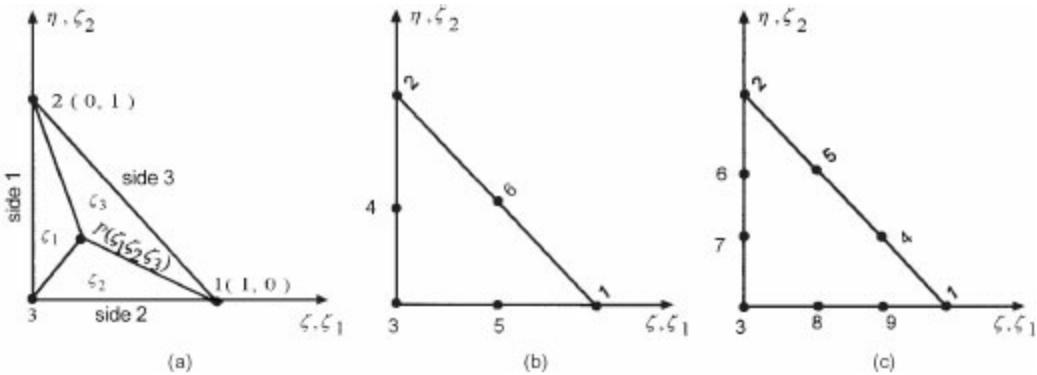


Fig. 18.9:2. Triangular coordinates.

If ϕ is as defined in Eq. (2) for all triangles over a domain and continuous at all inter-connecting nodes, then ϕ is piecewise linear continuous over all interelement boundaries and thus the whole domain. From Eq. (1), one sees that h_i is a complete linear polynomial. In Eq. (2), the subscript of q_j refers to the local label, meaning that q_j is the j^{th} generalized coordinate of the element. To assemble the element matrices, we need the assembly list, the global nodal label I_j of the j^{th} node of the element.

Shape Functions for 6-Node Triangle. Consider a six-node triangle [Fig. 18.9:1(b)] with nodes 1, 2, 3 being the vertices and nodes 4, 5, 6 the midside nodes. The nodal coordinates are $(x_j, y_j), j = 1, 2, \dots, 6$. The function

$$(7) \quad \phi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 = \sum_{j=1}^6 q_j h_j(x, y) = \mathbf{h}\mathbf{q}$$

gives an error in approximation of the order of ε^2 . The q 's are the values of ϕ at the nodes, and h 's are the interpolation functions that

$$(8) \quad h_i(x, y) = \zeta_i(2\zeta_i - 1), \quad h_{i+3}(x, y) = 4\zeta_j\zeta_k, \quad i = 1, 2, 3$$

where i, j, k are the cyclic permutations of 1, 2, 3. The shape functions satisfy

$$(9) \quad h_i(x_j, y_j) = \delta_{ij} \quad i, j = 1, \dots, 6.$$

Thus, h_i is the Lagrange interpolation, which equals 1 at node i and zero at the other five nodes. Equation (7) is in the same form as Eq. (2) except now \mathbf{h} and \mathbf{q} have six components.

The shape functions can be systematically derived by the use of the triangular coordinates (Zienkiewicz 1971, Gallagher 1974, 1975, Tong and Rossettos 1977). The shape function h_j is a product of two-line functions. All nodes except node j are passed by at least one of the lines and each line passes through at least one node other than node j . The product is normalized to unity at node j . For example $h_1(x, y)[= \zeta_1(2\zeta_1 - 1)]$ is the product of two line functions $\zeta_1 = 0$ and $2\zeta_1 = 1$ normalized to unity at node 1 ($\zeta_1 = 1$). The line function $\zeta_1 = 0$ passes nodes 2, 4, 3 on the side of the element opposite to node 1 [Fig. 18.9:1(b)] and $2\zeta_1 = 1$ passes the midside nodes 5 and 6. The shape function $h_4(x, y)[= 4\zeta_1\zeta_2]$ is the product of two line-functions $\zeta_1 = \zeta_2 = 0$ passing nodes 1, 5, 3 and nodes 1, 6, 2, respectively. The factor 4 makes $h_4 = 1$ at node 4 ($\zeta_2 = \zeta_3 = 1/2$). Note that the product of the two lines is a quadratic shape functions in x, y .

In this case the shape functions are quadratic along the side of the element, the compatibility at the three nodes on a side guarantees compatibility along the entire side. The six quadratic functions given Eq. (8) are all linearly independent, thus h 's are quadratic complete polynomials. The function ϕ as defined in Eq. (7) is piecewise quadratic over the whole domain. The array $I_i, i = 1, \dots, 6$, specifying the global nodal label of the i^{th} node of the element is needed for assembly.

A cubic element has four nodes on each side for a total of 9 boundary nodes. A complete cube polynomial has 10 terms and needs an extra node in the interior of the triangle. The shape function for the tenth node is $27\zeta_1\zeta_2\zeta_3$, the product of 3-line functions $\zeta_1 = \zeta_2 = \zeta_3 = 0$ of the sides of the triangle normalized to unity at the centroid of the triangle ($\zeta_1 = \zeta_2 = \zeta_3 = 1/3$). The function is called a *bulb function* because it vanishes on all sides. The other 9 shape functions are products of three line functions, of which one passes the centroid. For example, in $h_1(x, y) = \zeta_1(3\zeta_1 - 1)(3\zeta_1 - 2)/2$ and $h_4(x, y) = 2\zeta_1\zeta_2(3\zeta_1 - 1)/9$, the line function $3\zeta_1 - 1 = 0$ passes the centroid. Thus all shape functions associated with boundary nodes are zero at the centroid.

Using the triangular coordinates, one can derive families of higher order interpolation functions for triangular elements with ease. It can be shown (Problem 18.10) that triangular elements allow complete polynomials of all orders in Cartesian coordinates. If the number of boundary nodes needed to assure compatibility is less or equal to the number of the complete polynomial terms, one can derive an element with shape functions being complete polynomials. In Fig. 18.9:3, the complete polynomial terms needed for triangular elements are shown. The needed terms are arranged in the shape of a triangle called the *truncated Pascal triangle*. Terms through the second row are needed for linear elements, those through the third row for quadratic elements, and so on.

Pascal Triangle	Polynomial degree, p	Number of terms, n	Triangular element (number of nodes = number of terms)
1	0 (constant)	1	
$x \quad y$	1 (linear)	3	$n = \frac{(p+1)(p+2)}{2}$
$x^2 \quad xy \quad y^2$	2 (quadratic)	6	
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3 (cubic)	10	
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$	4 (quartic)	15	

Fig. 18.9:3. Relation between type of triangular elements and polynomial terms used for interpolation.

The representation of a field variable in the form of Eq. (2) or (7), derived without knowing the specifics of the variable, is generic. The use of natural coordinates makes it easy to construct C^0 continuous shape functions. The generic representation can be used to determine the finite element solutions for any problems that require C^0 continuity.

Problem 18.8. Use the triangular coordinates to derive the shape functions of C^0 continuity for a 10-node triangular element. The 10 nodes are the three vertices, the $1/3$ - and $2/3$ -points of each edge, and the centroid of the triangle.

Problem 18.9. Show that it is possible to construct the shape functions of complete polynomial of any order higher one for triangular elements. [Hint: Show that the minimum number of boundary nodes required for complete p^{th} order polynomial expansion is $3p$ while the number of complete polynomial terms is $(p + 1)(p + 2)/2$.]

18.10. ELEMENT MATRICES FOR SECOND-ORDER ELLIPTICAL EQUATIONS

To illustrate the derivation of element matrices, we consider a second-order elliptical equation

$$(1) \quad \partial[D_{\alpha\beta}(x)(\partial\phi/\partial x_\beta)]\partial x_\alpha - g(x)\phi = b(x) \quad \text{on } A,$$

where $D_{12} = D_{21}$ and the range of α, β is from 1 to 2 with $x_1 = x, x_2 = y$. For Eq. (1) to be elliptical, the quadratic form is positive definite, i.e., $D_{\alpha\beta} x_\alpha x_\beta > 0$ for all x 's not equal zero, where repeated indices denote summation. The boundary conditions are

$$(2) \quad \phi = \bar{\phi} \quad \text{on } \partial A_\phi,$$

$$(3) \quad n_\alpha D_{\alpha\beta} \partial\phi/\partial x_\beta = \bar{\phi}_\sigma \quad \text{on } \partial A_\sigma,$$

where ∂A_ϕ and ∂A_σ are the boundaries over which ϕ and $n_\alpha D_{\alpha\beta} \partial\phi/\partial x_\beta$ are specified to be $\bar{\phi}$ and $\bar{\phi}_\sigma$, respectively, and n (n_α) is the unit outward normal to the boundary. The total boundary of A is $\partial A (= \partial A_\phi + \partial A_\sigma)$.

The integral formulation of Eq. (1) can be expressed as the first variation of the potential energy Eq. (18.7:4). The variation of the functional for an element with $\phi = \mathbf{h}\mathbf{q}$ and $\delta\phi = \mathbf{h}\delta\mathbf{q}$ can be written in the matrix form

$$(4) \quad \delta\Pi_e = \delta\mathbf{q}^T (\mathbf{k}\mathbf{q} - \mathbf{f}),$$

where

$$\delta\mathbf{q}^T \mathbf{k}\mathbf{q} = \int_{A_e} \left(D_{\alpha\beta} \frac{\partial\phi}{\partial x_\alpha} \frac{\partial\delta\phi}{\partial x_\beta} + g\phi\delta\phi \right) dA, \quad \delta\mathbf{q}^T \mathbf{f} = - \int_{A_e} b\delta\phi dA + \int_{\partial A_{\sigma e}} \bar{\phi}_\sigma \delta\phi ds,$$

in which A_e is the element area and $\partial A_{\sigma e}$ is the portion of the boundary ∂A_σ . The admissible requirement is that ϕ is C^0 continuous and satisfies the rigid boundary condition (2). The natural condition Eq. (3) is accounted for automatically by the integration term over $\partial A_{\sigma e}$, while the rigid condition must be enforced. The components of \mathbf{k} and \mathbf{f} are

$$(5) \quad k_{ij} = \int_{A_e} \left(D_{\alpha\beta} \frac{\partial h_i}{\partial x_\alpha} \frac{\partial h_j}{\partial x_\beta} + g h_i h_j \right) dA, \quad f_i = - \int_{A_e} b h_i dA + \int_{\partial A_{\sigma e}} \bar{\phi}_\sigma h_i ds.$$

The range of i, j is from 1 to 3 for a three-node triangle and from 1 to 6 for a 6-node triangle. One can also express \mathbf{k} and \mathbf{f} in the matrix form

$$(6) \quad \mathbf{k} = \int_{A_e} (\mathbf{B}^T \mathbf{D}_e \mathbf{B} + g \mathbf{h}^T \mathbf{h}) dA, \quad \mathbf{f} = - \int_{A_e} b \mathbf{h}^T dA + \int_{\partial A_{\sigma e}} \bar{\phi}_\sigma \mathbf{h}^T ds,$$

where

$$(7) \quad \mathbf{h} = [h_1 \ h_2 \ \dots \ h_p], \quad \mathbf{B} = \mathbf{d}\mathbf{h} = \begin{bmatrix} \partial\mathbf{h}/\partial x \\ \partial\mathbf{h}/\partial y \end{bmatrix}, \quad \mathbf{D}_e = \begin{bmatrix} D_{11} & D_{12} \\ D_{12} & D_{22} \end{bmatrix},$$

in which \mathbf{D}_e is the coefficient matrix for elliptical equations or the elastic modulus matrix in elasticity and \mathbf{h} is an *interpolation matrix*. If \mathbf{h} is in terms of the triangular coordinates, its differentials with respect to x, y are

$$\partial\mathbf{h}/\partial x = \sum_{i=1}^3 (b_i/2\Delta) \partial\mathbf{h}/\partial\zeta_i, \quad \partial\mathbf{h}/\partial y = \sum_{i=1}^3 (c_i/2\Delta) \partial\mathbf{h}/\partial\zeta_i,$$

where b 's, c 's and Δ are defined in Eqs. (18.9:4) and (18.9:5).

One can in principle integrate Eq. (5) analytically by approximating D 's, g , b , $\bar{\phi}_\sigma$ as polynomials. In practice, numerical integration is frequently used, which will be discussed later in the chapter.

Clearly \mathbf{k} is a symmetric $p \times p$ matrix. The rank of the matrix is p if $g > 0$ in the element. This is because the first term of the integrand of Eq. (5) is non-negative for an elliptical equation and the second term is always positive. If $g = 0$, the rank of the matrix is at most $p - 1$ because \mathbf{k} has at least one zero-eigenvector, which corresponds $\phi = \text{constant}$, i.e., all components of \mathbf{q} are the same. This will result in $\partial\phi/\partial x_\alpha = 0$. As pointed out before, the rank of \mathbf{k} can be lower if one uses integration approximation for Eq. (5).

Three-Node Triangular Element. For a three-node triangular element, the shape functions are given in Eq. (18.9:3). For the special case that D_{ij} and g are constant, we find

$$\mathbf{k} = \frac{D_{11}}{4\Delta} \begin{bmatrix} b_1^2 & & \text{sym} \\ b_2 b_1 & b_2^2 & \\ b_3 b_1 & b_3 b_2 & b_3^2 \end{bmatrix} + \frac{D_{22}}{4\Delta} \begin{bmatrix} c_1^2 & & \text{sym} \\ c_2 c_1 & c_2^2 & \\ c_3 c_1 & c_3 c_2 & c_3^2 \end{bmatrix} + \frac{D_{12}}{2\Delta} \begin{bmatrix} b_1 c_1 & & \text{sym} \\ (b_2 c_1 + b_1 c_2)/2 & b_2 c_2 & \\ (b_3 c_1 + b_1 c_3)/2 & (b_3 c_2 + b_2 c_3)/2 & b_3 c_3 \end{bmatrix} + \frac{g\Delta}{12} \begin{bmatrix} 2 & & \text{sym} \\ 1 & 2 & \\ 1 & 1 & 2 \end{bmatrix}.$$

If b is a constant and no element boundary is on $\partial A_{\sigma e}$, then

$$\mathbf{f}^T = -b\Delta[1 \ 1 \ 1]/3.$$

The finite element process lumps 1/3 of the applied load over the element to each of the vertices.

Six-Node Triangular Element. For a six-node triangular element (Fig. 18.9:1), the interpolation functions are given in Eq. (18.9:8). In this case, \mathbf{k} is a 6×6 and \mathbf{f} is a 6×1 matrix. The components of \mathbf{k} and \mathbf{f} can be evaluated from Eq. (5) or (6). If D_{ij} , g and b are all constant, we find

$$\mathbf{k} = [D_{11}\bar{\mathbf{k}}(\mathbf{b}, \mathbf{b}) + 2D_{12}\bar{\mathbf{k}}(\mathbf{b}, \mathbf{c}) + D_{22}\bar{\mathbf{k}}(\mathbf{c}, \mathbf{c})]/\Delta + g\mathbf{k}_2\Delta.$$

Both $\bar{\mathbf{k}}$ and \mathbf{k}_2 are symmetric. The components of $\bar{\mathbf{k}}(\mathbf{b}, \mathbf{c})$ are

$$\begin{aligned} \bar{k}_{ii} &= b_i c_i / 4, \quad i = 1, 2, 3 \quad (i \text{ not summed}) \\ \bar{k}_{ij} &= -(b_i c_j + b_j c_i) / 24, \quad i \neq j, \quad i, j = 1, 2, 3, \\ \bar{k}_{61} &= \bar{k}_{62} = (b_2 c_1 + b_1 c_2) / 6, \\ \bar{k}_{42} &= \bar{k}_{43} = (b_3 c_2 + b_2 c_3) / 6, \\ \bar{k}_{53} &= \bar{k}_{51} = (b_1 c_3 + b_3 c_1) / 6, \\ \bar{k}_{63} &= \bar{k}_{52} = \bar{k}_{41} = 0, \\ \bar{k}_{66} &= [(b_1 + 2b_2)c_2 + (2b_1 + b_2)c_1] / 3, \\ \bar{k}_{44} &= [(b_2 + 2b_3)c_3 + (2b_2 + b_3)c_2] / 3, \\ \bar{k}_{55} &= [(b_3 + 2b_1)c_1 + (2b_3 + b_1)c_3] / 3, \\ \bar{k}_{64} &= [(b_2 + 2b_3)c_1 + (b_1 + 2b_2 + b_3)c_2 + (2b_1 + b_2)c_3] / 6, \\ \bar{k}_{54} &= [(b_3 + 2b_1)c_2 + (b_2 + 2b_3 + b_1)c_3 + (2b_2 + b_3)c_1] / 6, \\ \bar{k}_{65} &= [(b_1 + 2b_2)c_3 + (b_3 + 2b_1 + b_2)c_1 + (2b_3 + b_1)c_2] / 6. \end{aligned}$$

$$\mathbf{k}_2 = \frac{1}{180} \begin{bmatrix} 6 & & & & & \\ -1 & 6 & & & & \text{sym} \\ -1 & -1 & 6 & & & \\ -4 & 0 & 0 & 32 & & \\ 0 & -4 & 0 & 16 & 32 & \\ 0 & 0 & -4 & 16 & 16 & 32 \end{bmatrix}, \quad \mathbf{f} = -\frac{b\Delta}{3} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\alpha \bar{\phi}_\sigma \varepsilon}{6} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$$

where $\alpha = 0$ for elements without $\partial A_{\sigma e}$ and equals 1 for elements with $\partial A_{\sigma e}$ of size ε and nodes 1, 2 and 6 on $\partial A_{\sigma e}$. We see that the finite element process lumps 1/3 of the applied load over the element to each of the midside nodes and nothing to the vertices. This is different from the linear triangle, which lumps 1/3 of the applied load to each of the vertices.

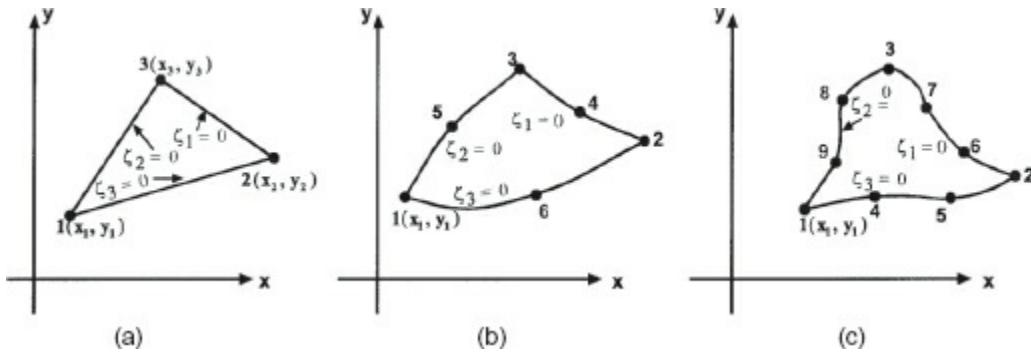


Fig. 18.10:1. Triangular elements with curved sides: (a) linear, (b) quadratic, and (c) cubic.

For higher order elements, the process of deriving element matrices is identical to that for the lower order elements except the use of appropriate higher order interpolation functions. The evaluation involves integration of quantities defined in terms of triangular coordinates. The following exact integration formula can be useful

$$(8) \quad \int_A \zeta_1^\alpha \zeta_2^\beta \zeta_3^\gamma dA = 2\Delta \alpha! \beta! \gamma! / (\alpha + \beta + \gamma + 2)!.$$

18.11. COORDINATE TRANSFORMATION

In principle, triangular elements can approximate a domain of any shape. For a domain with sharp curved boundary, many small elements of straight edges are needed to approximate the boundary. One can improve the representation using elements with curved sides. Then one has to find interpolation functions to fulfill the compatibility requirements. To do so, one maps the curve elements to elements with straight edges and makes the previous shape functions usable. In the mapped domain there should be *no gap between elements* and the *compatibility at the nodes of two adjacent elements implies compatibility along their whole common boundary*.

For the transformation $[x, y] = [x(\xi, \eta), y(\xi, \eta)]$, we have

$$(1) \quad \mathbf{d}_n = \begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{bmatrix} = \begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta \end{bmatrix} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \mathbf{J} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix},$$

$$(2) \quad \mathbf{d} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \begin{bmatrix} \partial\xi/\partial x & \partial\eta/\partial x \\ \partial\xi/\partial y & \partial\eta/\partial y \end{bmatrix} \begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \end{bmatrix},$$

Where

$$(3) \quad \mathbf{J} = \begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta \end{bmatrix}$$

$$(4) \quad \mathbf{J}^{-1} = \begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta \end{bmatrix}^{-1} = \begin{bmatrix} \partial\xi/\partial x & \partial\eta/\partial x \\ \partial\xi/\partial y & \partial\eta/\partial y \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \partial y/\partial\eta & -\partial y/\partial\xi \\ -\partial x/\partial\eta & \partial x/\partial\xi \end{bmatrix},$$

$$(5) \quad J = \det(\mathbf{J}) = (\partial x/\partial\xi)(\partial y/\partial\eta) - (\partial y/\partial\xi)(\partial x/\partial\eta)$$

where J is the Jacobian of the transformation. For physical reasons, we must limit the transformation to $J > 0$. The integration is related as follows:

$$(6) \quad \int_{A_e} u(x, y) dx dy = \int_{A_e} u[x(\xi, \eta), y(\xi, \eta)] J d\xi d\eta.$$

Recall that, in Chapters 2–4, the components of coordinates are defined as the contravariant components of a tensor, and those of deformation gradient or strains are the covariant components of a tensor. The transformation of components between coordinate systems follows the tensor transformation laws. If the coordinate system is orthogonal and if the physical components for those quantities are used, there is no need to distinguish the covariant and contravariant transformations. The natural coordinates for triangle and those for general quadrilaterals used in the finite element methods are not orthogonal. The transformation described in this section is the covariant transformation of gradient operators in matrix form.

18.12. TRIANGULAR ELEMENTS WITH CURVED SIDES

A family of such elements is shown in Fig. 18.10:1. In terms of the ξ, η -coordinates

$$\xi = \zeta_1, \quad \eta = \zeta_2, \quad \zeta_3 = 1 - \zeta_1 - \zeta_2 = 1 - \xi - \eta,$$

we have

$$\mathbf{d} = \frac{\partial}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \frac{\partial y}{\partial \zeta_2} - \frac{\partial y}{\partial \zeta_3} & \frac{\partial y}{\partial \zeta_3} - \frac{\partial y}{\partial \zeta_1} \\ \frac{\partial x}{\partial \zeta_3} - \frac{\partial x}{\partial \zeta_2} & \frac{\partial x}{\partial \zeta_1} - \frac{\partial x}{\partial \zeta_3} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \zeta_1} - \frac{\partial}{\partial \zeta_3} \\ \frac{\partial}{\partial \zeta_2} - \frac{\partial}{\partial \zeta_3} \end{bmatrix}.$$

The element in the ξ, η -plane is an isosceles right triangles in the first quadrant $0 \leq \xi, \eta \leq 1$ (Fig. 18.9:2) of area equal $1/2$.

Consider the transformation of an element with p boundary nodes

$$(1) \quad x = \sum_{j=1}^p x_j h_j(\zeta_1, \zeta_2, \zeta_3), \quad y = \sum_{j=1}^p y_j h_j(\zeta_1, \zeta_2, \zeta_3),$$

where h 's are the shape functions as defined earlier, and (x_i, y_i) are the co-ordinates of the nodes of the triangle. The elements so transformed is called an *isoparametric element* (Taig 1961, Irons 1966), in which one uses the same interpolation functions for field variables and spatial coordinates.

The element matrices are in the same form as given in Eq. (18.10:6) except that the integrand is now in terms of ζ 's and

$$(2) \quad \mathbf{B} = \mathbf{d}\mathbf{h} = \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial \mathbf{h}}{\partial \xi} \\ \frac{\partial \mathbf{h}}{\partial \eta} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \frac{\partial y}{\partial \zeta_2} - \frac{\partial y}{\partial \zeta_3} & \frac{\partial y}{\partial \zeta_3} - \frac{\partial y}{\partial \zeta_1} \\ \frac{\partial x}{\partial \zeta_3} - \frac{\partial x}{\partial \zeta_2} & \frac{\partial x}{\partial \zeta_1} - \frac{\partial x}{\partial \zeta_3} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{h}}{\partial \zeta_1} - \frac{\partial \mathbf{h}}{\partial \zeta_3} \\ \frac{\partial \mathbf{h}}{\partial \zeta_2} - \frac{\partial \mathbf{h}}{\partial \zeta_3} \end{bmatrix}.$$

The boundary integral in Eq. (18.10:6) becomes an integral along a straight line. Suppose we have to integrate from node i to node j , where $i, j = 1, 2$, or 3 ($i \neq j$). Let node k be the third vertex. Along the side, we have $\zeta_i = 1 - \zeta_j$ and

$$(3) \quad \int_{\partial A_e} \mathbf{h}^T \bar{\phi}_\sigma ds = \int_0^1 \mathbf{h}^T \bar{\phi}_\sigma \sqrt{\left(\frac{\partial x}{\partial \zeta_j} - \frac{\partial x}{\partial \zeta_i} \right)^2 + \left(\frac{\partial y}{\partial \zeta_j} - \frac{\partial y}{\partial \zeta_i} \right)^2} \Big|_{\zeta_k=0}^{1-\zeta_j} d\zeta_j$$

(j not summed). The integration is now with respect to ζ_j ranging from 0 to 1.

One can express the area integrals in Eq. (18.10:6) in the form

$$\int_{A_e} (\dots) dx dy = \int_0^1 \int_0^{1-\zeta_2} (\dots) J d\zeta_1 d\zeta_2$$

with $\zeta_3 = 1 - \zeta_1 - \zeta_2$ in the integrand. In practice, we carry out the integration numerically rather than according to the formula Eq. (18.10:8).

It can be seen that under the transformation, the interpolated function is continuous over all interelement boundaries. The mapping of the common boundary of two adjacent elements is the same, because from Eq. (1) the mapping of a side depends only the coordinates of the nodes on the side, as the shape functions associated with other nodes vanish. Now, we have the same coordinates along the common boundary of the two adjacent elements before and after transformation, which implies that there is no gap between adjacent elements after mapping. Compatibility is no different from that without transformation that compatibility at all interelement nodes guarantees compatibility along the entire curved interelement boundary.

The element matrices are in the same form as those in Eq. (18.10:6). Assembling, constraining and equation solving are the same as described in Secs. 18.4 and 18.5.

18.13. QUADRILATERAL ELEMENTS

Although triangles have the advantage of approximating domains of any shape, the linear triangular elements are very stiff that fine finite element meshes are often needed to achieve an acceptable level of accuracy in applications. To gain better accuracy, one has the option of using higher order triangular elements or elements of other shapes. Rectangles or quadrilaterals are often used as they are generally more accurate than triangular elements. We shall derive the shape functions and construct the element matrices for rectangles and quadrilaterals.

Rectangular Element. Consider the *natural or intrinsic* coordinates

$$(1) \quad \xi = 2(x - x_1)/\varepsilon_x - 1, \quad \eta = 2(y - y_1)/\varepsilon_y - 1,$$

where x_1, y_1 are the coordinates of node 1, $\varepsilon_x \times \varepsilon_y$ are the element size in the x, y plane and ξ, η are defined over the domain $-1 \leq \xi, \eta \leq 1$ (Fig. 18.13:1). We express a field variable in terms of shape functions as

$$\phi(x, y) = \sum_{j=1}^4 q_j h_j(\xi, \eta),$$

where q_j is the value of ϕ at node j at (x_j, y_j) (j not summed), and

$$h_1(\xi, \eta) = (1 - \xi)(1 - \eta)/4, \quad h_2(\xi, \eta) = (1 + \xi)(1 - \eta)/4, \\ h_3(\xi, \eta) = (1 + \xi)(1 + \eta)/4, \quad h_4(\xi, \eta) = (1 - \xi)(1 + \eta)/4.$$

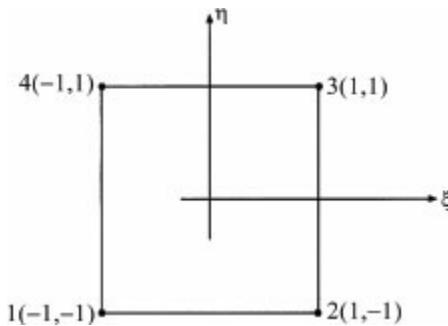


Fig. 18.13:1. A four-node square element.

These are bilinear functions of ξ, η , i.e., linear in ξ as well as in η . The shape functions can be written in a more compact form:

$$(2) \quad h_j(\xi, \eta) = (1 + \xi_j \xi)(1 + \eta_j \eta)/4 \quad j = 1, \dots, 4 \quad (\text{not summed})$$

where $\xi_1 = \xi_4 = \eta_1 = \eta_2 = -1$ and $\xi_2 = \xi_3 = \eta_3 = \eta_4 = 1$. Thus $h_j = 1$ at node j and 0 at all other nodes.

Since ϕ varies linearly along all sides and is compatible at the two corners, it is compatible along the entire inter-element boundary. Thus, ϕ is C^0 continuous over the whole domain. If g, p and $D_{\alpha\beta}$ are all constant, the element matrices defined by Eqs. (18.10:1–3) are

$$k_{ij} = \left(\frac{D_{11}}{4} \frac{\varepsilon_y}{\varepsilon_x} + \frac{D_{22}}{4} \frac{\varepsilon_x}{\varepsilon_y} \right) \xi_i \xi_j \left(1 + \frac{\eta_i \eta_j}{3} \right) + \frac{D_{12}}{4} (\xi_i \eta_j + \xi_j \eta_i) + g \varepsilon_x \varepsilon_y (k_2)_{ij} \\ (k_2)_{ij} = \varepsilon_x \varepsilon_y \left(1 + \frac{\eta_i \eta_j}{3} \right) \left(1 + \frac{\xi_i \xi_j}{3} \right), \quad \mathbf{f}^T = -\frac{b \varepsilon_x \varepsilon_y}{4} [1 \ 1 \ 1 \ 1] + \frac{\alpha \bar{\phi}_\sigma \varepsilon_1}{2} [1 \ 1 \ 0 \ 0].$$

where $\alpha = 0$ for elements without ∂A_{oe} and equals 1 for elements with ∂A_{oe} of size ε_1 and nodes 1, 2 on ∂A_{oe} . The finite element process accounts for the distributed load over the element and on the element boundaries by lumping the distributed load equally to the corner nodes. Note that $\xi_i \xi_j (1 + \eta_i \eta_j / 3) = \eta_i \eta_j (1 + \xi_i \xi_j / 3)$ (i, j are not summed).

Figure 18.13:2 shows a family of rectangular elements, called *Lagrange elements*⁴ where progressively increasing number of nodes are placed on the element boundaries and interior. Heavy dots denote boundary nodes, whereas open circles indicate interior ones. To ensure continuity on the edges the shape functions must be linear, quadratic, cubic, etc. for the increasingly higher order elements. One can generate higher order shape functions for corner nodes by subtracting the midside and bubble functions from low order shape functions. For a quadratic element (Fig. 18.13:2), h_1 (shape function associated with node 1) can be obtained by subtracting from the bilinear function $(1 - \xi)(1 - \eta)/4$, halves of the two adjacent midside shape functions (h_5 and h_8) and a quarter of the bubble function h_9 of the interior node. The factors of these functions are to make h_1 zero at nodes 5, 8 and 9. The midside shape function, say h_5 , can be obtained by subtracting the bubble function from the product of the line functions of the three sides without node 5 (Table 18.3:1). For a cubic element, we generate the corner shape function by subtracting the adjacent midside and bubble functions

from the bilinear function. In this case, there are four adjacent midside and four interior nodes.

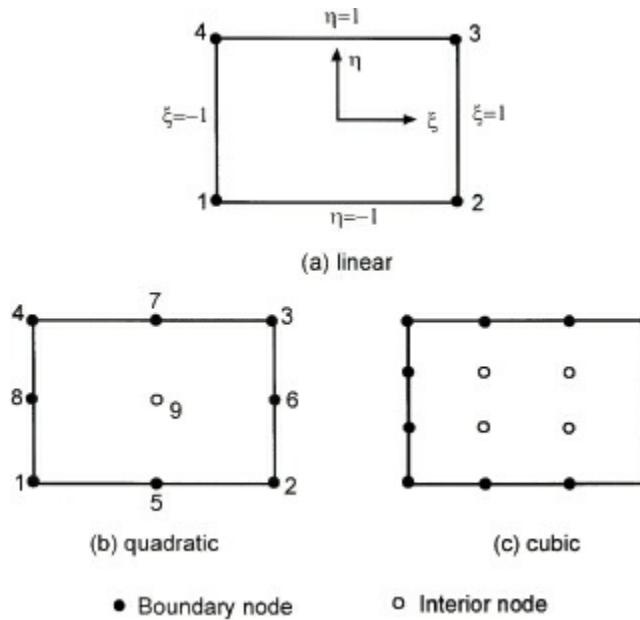


Fig. 18.13:2. A family of rectangular Lagrange elements: (a) linear, (b) quadratic, (c) cubic.

Table 18.13:1. Shape functions of a rectangle.

Shape function	Include only if node j is present in the element				
	$j = 5$	$j = 6$	$j = 7$	$j = 8$	$j = 9$
$h_1 = (1 - \xi)(1 - \eta)/4$	$-h_5/2$			$-h_8/2$	$-h_9/4$
$h_2 = (1 + \xi)(1 - \eta)/4$	$-h_5/2$	$-h_6/2$			$-h_9/4$
$h_3 = (1 + \xi)(1 + \eta)/4$		$-h_6/2$	$-h_7/2$		$-h_9/4$
$h_4 = (1 - \xi)(1 + \eta)/4$			$-h_7/2$	$-h_8/2$	$-h_9/4$
$h_5 = (1 - \xi^2)(1 - \eta)/2$					$-h_9/2$
$h_6 = (1 + \xi)(1 - \eta^2)/2$					$-h_9/2$
$h_7 = (1 - \xi^2)(1 + \eta)/2$					$-h_9/2$
$h_8 = (1 - \xi)(1 - \eta^2)/2$					$-h_9/2$
$h_9 = (1 - \xi^2)(1 - \eta^2)$					

Table 18.13:1 gives the shape functions for the 9-node quadrilateral Lagrange element. One can obtain the shape functions for elements with missing midside node(s) by removing the shape functions of the missing nodes from Table 18.13:1. For example, for an element without node 5 as shown in Fig. 18.13:3, we remove h_5 from h_1 and h_2 to give

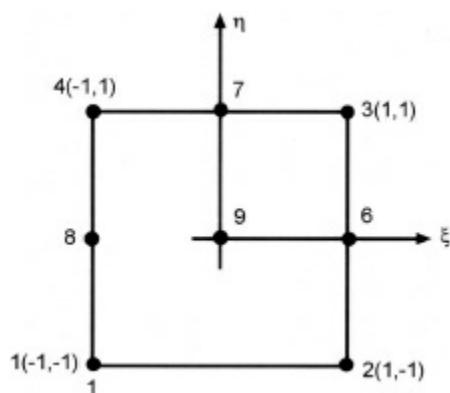


Fig. 18.13:3. Rectangular element with no mid-side node between nodes 1 and 2.

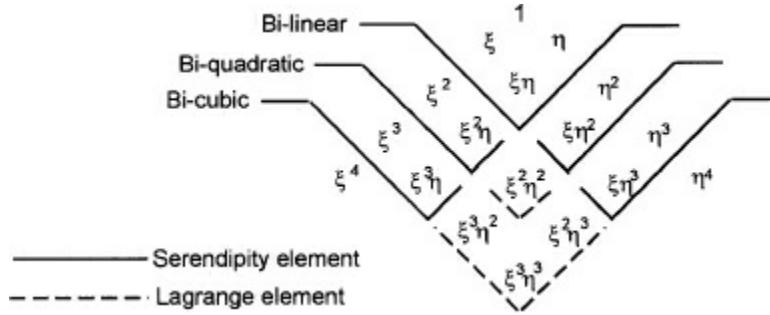


Fig. 18.13:4. Polynomial terms in plane serendipity and Lagrange elements.

$$h_1 = (1 - \xi)(1 - \eta)/4 - h_8/2 - h_9/4, \quad h_2 = (1 + \xi)(1 - \eta)/4 - h_6/2 - h_9/4.$$

The shape functions do not involve h_5 . On the side with missing midside node, the field variable varies linearly. The side can be connected to a linear element while the other sides to quadratic elements. This arrangement is useful for transition from linear elements to quadratic ones in a domain.

It is instructional to examine the polynomial terms that are missed from the shape functions. In Fig. 18.13:4 polynomial terms above the solid lines are those included in the expansions of shape functions of the corresponding Serendipity elements while those above dotted/solid lines are included in the corresponding Lagrange elements. The linear element includes one quadratic term $\xi\eta$ and the highest order of complete polynomial is only linear. For the quadratic and cubic elements, the highest complete polynomials are quadratic and cubic, respectively. For the quartic Serendipity element without interior node, the expansion of the shape functions associated with the 12 boundary nodes misses the $\xi^2\eta^2$ term. The error in approximation is of the order ε^4 . The higher order elements contain some very high order terms while omitting some lower order ones. For example, a cubic Lagrange element has 16 nodes. It contains the term $\xi^3\eta^3$ while omits terms like ξ^4 and η^4 . The accuracy of higher order elements is limited by the omitted terms.

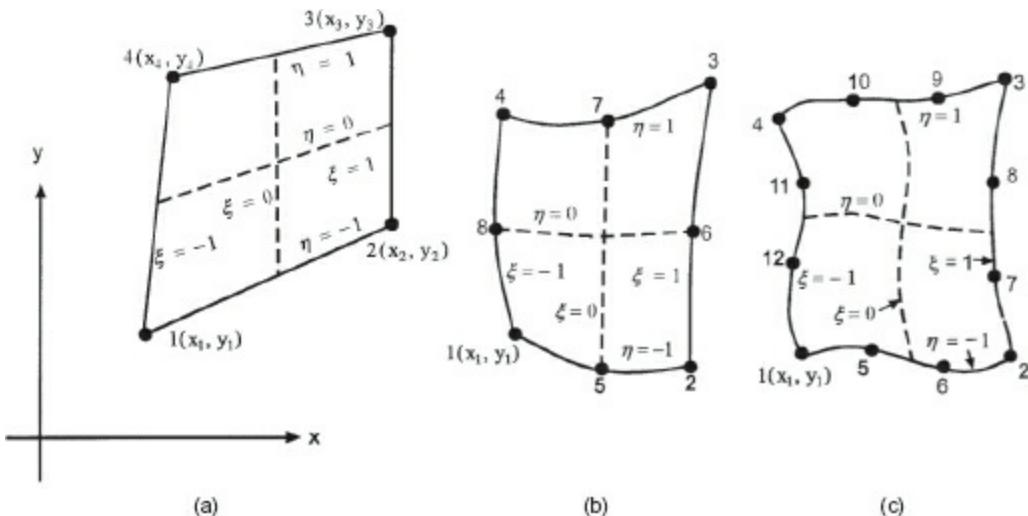


Fig. 18.13:5. Curved quadrilateral elements.

Isoparametric Quadrilateral Elements. Rectangular elements are generally more accurate than the triangular elements of the same order. However, rectangular element suffers the shortcomings of not being able to fit in irregular domains. We introduce curved quadrilateral elements (Fig. 18.13:5) to approximate the domains. Following the approach for curved triangles, we use the shape functions to map the element into a square and construct the element matrices in the transformed coordinates. The mapping is in the same form as Eq. (18.12:1)

$$(3) \quad x = \sum_{j=1}^p x_j h_j(\xi, \eta), \quad y = \sum_{j=1}^p y_j h_j(\xi, \eta).$$

For 4- and 8-node quadrilaterals, h 's are given in Eq. (2) and in Table 18.13:1, respectively. The mapped element is a square over $-1 \leq \xi, \eta \leq 1$ in the ξ, η -plane (Fig. 18.13:1).

The element matrices given in Eq. (18.10:6) can be now written as

$$(4) \quad \mathbf{k} = \int_{-1}^1 \int_{-1}^1 (\mathbf{B}^T \mathbf{D}_e \mathbf{B} + \mathbf{g} \mathbf{h}^T \mathbf{h}) J d\xi d\eta, \quad \mathbf{f} = - \int_{-1}^1 \int_{-1}^1 \mathbf{b} \mathbf{h}^T J d\xi d\eta + \int_{\partial A_{\sigma e}} \bar{\phi}_{\sigma} \mathbf{h}^T ds,$$

where, similar to Eq. (18.12:2),

$$(5) \quad \mathbf{B} = \mathbf{d}\mathbf{h} = \mathbf{J}^{-1}\mathbf{d}_n\mathbf{h} = \mathbf{J}^{-1} \begin{bmatrix} \partial\mathbf{h}/\partial\xi \\ \partial\mathbf{h}/\partial\eta \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \partial y/\partial\eta & -\partial y/\partial\xi \\ -\partial x/\partial\eta & \partial x/\partial\xi \end{bmatrix} \begin{bmatrix} \partial\mathbf{h}/\partial\xi \\ \partial\mathbf{h}/\partial\eta \end{bmatrix},$$

in which \mathbf{d} and \mathbf{d}_n are defined in Eqs. (18.11:1) and (18.11:2). The resulting element is called an *isoparametric* element as the shape functions are used for coordinate transformation and for variable interpolation.

The boundary integral in Eq. (4) involves integration along the side for ξ or $\eta = \pm 1$ if $\delta A_{\sigma e}$ is not null. If the integration is along the side $\xi = \alpha$ where $\alpha = 1$ or -1 , the boundary integral can be written as

$$(6) \quad \int_{\partial A_e} \mathbf{h}^T \bar{\phi}_\sigma ds = \int_0^1 \mathbf{h}^T \bar{\phi}_\sigma \sqrt{(\partial x/\partial\eta)^2 + (\partial y/\partial\eta)^2} \Big|_{\xi=\alpha} d\eta.$$

If the integration is along the side $\eta = \alpha$, the boundary integral is the same except the interchange of ξ and η in the right integral.

The integrating Eq. (4) is complicated, because of the presence of J in the denominator in \mathbf{B} . Even for the simple 4-node isoparametric element with constant \mathbf{D}_e analytical integration, though possible, is not practical. Numerical integration is an easy way to evaluate the element matrices.

Static Condensation. There are interior nodes for cubic triangles, Lagrange quadrilaterals and elements of higher orders. The shape functions associated with the interior nodes are bubble functions. Their generalized coordinates do not appear outside the element and can be eliminated by the stationary condition of the functional, called *static condensation*.

We partition \mathbf{q} , \mathbf{k} and \mathbf{f} of Eq. (18.10:4) in the forms

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_b \\ \mathbf{q}_a \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} \mathbf{k}_{bb} & \mathbf{k}_{ab} \\ \mathbf{k}_{ab}^T & \mathbf{k}_{aa} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}_b \\ \mathbf{f}_a \end{bmatrix},$$

where subscripts a and b denote quantities associated with the interior and boundary nodes, respectively. There is one interior node for quadratic elements, and four for cubic elements (Fig. 18.13:2). Since \mathbf{q}_a does not appear in other elements, the vanish of the first variation of the functional with respect to \mathbf{q}_a gives

$$\mathbf{k}_{ab}\mathbf{q}_b + \mathbf{k}_{aa}\mathbf{q}_a - \mathbf{f}_a = 0 \quad \text{or} \quad \mathbf{q}_a = -\mathbf{k}_{aa}^{-1}\mathbf{k}_{ab}\mathbf{q}_b + \mathbf{k}_{aa}^{-1}\mathbf{f}_a.$$

A substitution of \mathbf{q}_a into Eq. (18.10:4) yields

$$\delta\Pi_e = \delta\mathbf{q}_b^T [(\mathbf{k}_{bb} - \mathbf{k}_{ab}^T \mathbf{k}_{aa}^{-1} \mathbf{k}_{ab}) \mathbf{q}_b - (\mathbf{f} + \mathbf{k}_{ab}^T \mathbf{k}_{aa}^{-1} \mathbf{f}_a)] = \delta\mathbf{q}_b^T (\mathbf{k}^* \mathbf{q}_b - \mathbf{f}^*),$$

where \mathbf{k}^* and \mathbf{f}^* , called the reduced element matrices, are to be assembled to form the global matrices.

18.14. PLANE ELASTICITY

Up to now, we have considered problems with one field variable. Generalizing the finite element process to problems of multiple fields is straightforward. Consider a problem in plane elasticity with

$$(1) \quad \mathbf{u} = [u \ v]^T, \quad \boldsymbol{\sigma} = [\sigma_x \ \sigma_y \ \sigma_{xy}]^T, \\ \mathbf{e} = [e_x \ e_y \ \gamma_{xy}]^T = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}^T \mathbf{u} = \mathbf{d}_e \mathbf{u},$$

where \mathbf{u} , $\boldsymbol{\sigma}$ and \mathbf{e} are the displacement, the stress and the engineering strain vectors, **not tensors**. Their components are different in different coordinate systems. Consider two coordinate systems x_i and \tilde{x}_i , which are related by $\tilde{x}_i = R_{ij}x_j$. Then we have, in the matrix form,

$$\tilde{\mathbf{u}} = \mathbf{R}\mathbf{u}.$$

The **tensor transformation and matrix rule** give, respectively,

$$(2) \quad \begin{bmatrix} \tilde{\sigma}^{11} & \tilde{\sigma}^{12} \\ \tilde{\sigma}^{21} & \tilde{\sigma}^{22} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{21} \\ R_{12} & R_{22} \end{bmatrix} \begin{bmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{21} & \sigma^{22} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

$$\tilde{\sigma} = \begin{bmatrix} \tilde{\sigma}^{11} \\ \tilde{\sigma}^{22} \\ \tilde{\sigma}^{12} \end{bmatrix} = \begin{bmatrix} R_{11}^2 & R_{12}^2 & 2R_{11}R_{12} \\ R_{21}^2 & R_{22}^2 & 2R_{22}R_{21} \\ R_{11}R_{21} & R_{22}R_{12} & R_{11}R_{22} + R_{12}R_{21} \end{bmatrix} \begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{12} \end{bmatrix}.$$

In general, $\mathbf{u} \neq \bar{\mathbf{u}}$, $\boldsymbol{\sigma} \neq \tilde{\boldsymbol{\sigma}}$ and $\mathbf{e} \neq \tilde{\mathbf{e}}$ based on the matrix definition. Based on the tensor definition, $\mathbf{u} = \bar{\mathbf{u}}$, $\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}$ and $\mathbf{e} = \tilde{\mathbf{e}}$, but their components are different in different coordinates. Note that $\boldsymbol{\sigma}^T \mathbf{e} = \mathbf{e}^T \boldsymbol{\sigma}$ in the matrix definition and equal $\boldsymbol{\sigma}^T : \mathbf{e}$ ($= \mathbf{e}^T : \boldsymbol{\sigma}$) if $\boldsymbol{\sigma}$ and \mathbf{e} are the tensors. The use of engineering shear strain γ_{xy} in Eq. (1) rather than the tensorial shear strain e_{xy} [$= (\partial u / \partial y + \partial v / \partial x) / 2$] [Eq. (4.2:7)] is to make the *elastic constant or modulus matrix* \mathbf{D}_e (defined below) symmetric.

The generalized Hooke's law can be written in the matrix form

$$(3) \quad \boldsymbol{\sigma} = \mathbf{D}_e \mathbf{e} = \mathbf{D}_e \mathbf{d}_e \mathbf{u} = \begin{bmatrix} D_{11} & & \text{sym} \\ D_{12} & D_{22} & \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \mathbf{d}_e \mathbf{u},$$

where \mathbf{d}_e is the differential operator defined in Eq. (1). **The matrix \mathbf{D}_e is also not a tensor but its components are related to the elastic modulus tensor C^{ijkl} of Eq. (6.1:1).**

Consider the problem over a domain A with boundary conditions

$$(4) \quad \mathbf{u} = [u \ v]^T = [u_x \ u_y]^T = \bar{\mathbf{u}} \text{ on } \partial A_u,$$

$$(5) \quad \mathbf{T} = [T_x \ T_y]^T = [n_x \sigma_{xx} + n_y \sigma_{xy} \ n_x \sigma_{xy} + n_y \sigma_{yy}]^T = \bar{\mathbf{T}} \text{ on } \partial A_\sigma,$$

$$(6) \quad u_n = \bar{u}_n, \ T_t = \bar{T}_t \text{ on } \partial A'_\sigma \text{ and } u_t = \bar{u}_t, \ T_n = \bar{T}_n \text{ on } \partial A''_\sigma,$$

where \mathbf{T} is the boundary traction and \mathbf{n} ($= [n_x \ n_y]$) is a unit normal on ∂A_σ with n and t denoting the normal and tangential components of quantities on ∂A_e . In subsequent discussions, we shall *use capital \mathbf{T} to denote traction rather than Lagrangian stress used in previous chapters*.

In integral formulation, the potential energy functional of an element is

$$(7) \quad \Pi_e = \int_{A_e} \left[\frac{1}{2} (\mathbf{d}_e \mathbf{u})^T \mathbf{D}_e (\mathbf{d}_e \mathbf{u}) - \mathbf{u}^T \mathbf{b} \right] dA - \int_{\partial A_{\sigma e}} \mathbf{u}^T \bar{\mathbf{T}} ds$$

$$- \int_{\partial A'_{\sigma e}} u_t \bar{T}_t ds - \int_{\partial A''_{\sigma e}} u_n \bar{T}_n ds,$$

where \mathbf{b} is the distributed body force. We require *the first variation of the functional for the whole domain to vanish with respect to the admissible \mathbf{u}* . The prescribed displacement conditions in Eqs. (4) and (6), rigid conditions, are enforced while the traction conditions in Eqs. (5) and (6), natural conditions, are accounted for by the integrals over $\partial A_{\sigma e}$, $\partial A'_{\sigma e}$ and $\partial A''_{\sigma e}$.

To establish the element matrices for a p -node element with nodal coordinate x_j, y_j and the nodal values of \mathbf{u} are

$$q_{2j-1} = u(x_j, y_j), \quad q_{2j} = v(x_j, y_j),$$

where q 's are the generalized coordinates with their subscripts being the local label. Each node now has two DOFs as opposed to one considered before. Utilizing the shape functions derived earlier, we can interpolate the field variables in the form

$$(8) \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \sum_{j=1}^p \begin{bmatrix} h_j(x, y) q_{2j-1} \\ h_j(x, y) q_{2j} \end{bmatrix} = \mathbf{h} \mathbf{q}, \quad \mathbf{e} = \mathbf{d}_e \mathbf{u} = \mathbf{d}_e \mathbf{h} \mathbf{q} = \mathbf{B} \mathbf{q},$$

where \mathbf{d}_e is a differential operator matrix defined in Eq. (1), \mathbf{B} a *strain-displacement* matrix, \mathbf{h} a $2 \times 2p$ *interpolation matrix* [as opposed to a row matrix for problems of one field variable in Eq. (18.10:7)]

$$(9) \quad \mathbf{h}_{2 \times 2p} = \begin{bmatrix} h_1 & 0 & \dots & h_p & 0 \\ 0 & h_1 & \dots & 0 & h_p \end{bmatrix},$$

and \mathbf{q} a nodal displacement matrix

$$\mathbf{q}_{1 \times 2p}^T = [q_1 \ q_2 \ \dots \ q_{2p-1} \ q_{2p}].$$

A substitution of Eq. (8) into Eq. (7) yields the equation for the element

$$(10) \quad \Pi_e = \mathbf{q}^T \mathbf{k} \mathbf{q} / 2 - \mathbf{q}^T \mathbf{f},$$

where

$$(11) \quad \mathbf{k} = \int_{A_e} \mathbf{B}^T \mathbf{D}_e \mathbf{B} dA,$$

$$\mathbf{f} = \int_{A_e} \mathbf{h}^T \mathbf{b} dA + \int_{\partial A_{\sigma e}} \mathbf{h}^T \bar{\mathbf{T}} ds + \int_{\partial A'_{\sigma e}} u_t \bar{T}_t ds + \int_{\partial A''_{\sigma e}} u_n \bar{T}_n ds.$$

These equations are similar in form as Eq. (18.10:6) except \mathbf{D}_e , \mathbf{B} are defined in Eqs. (3) and (8), respectively.

The stiffness matrix \mathbf{k} is symmetric of rank $2p - 3$, because there are 3 rigid body modes for plane elasticity. However, the rank can be higher if the shape functions are incapable of representing certain rigid body modes, or lower if the integration in Eq. (11) is carried out approximately, as it can introduce spurious deformation.

For a three-node triangle ($p = 3$), h_j are linear functions of x, y [or ζ, ζ_2, ζ_3 , Eq. (18.9:3)] and the strain is constant. Therefore, a three node triangular element is also called *constant strain triangle*. A 6-node triangular element uses quadratic shape functions Eq. (18.9:8) and has linear strains. It is called *linear strain triangle*.

For elements with curved sides, we have

$$(12) \quad \mathbf{k} = \int_{A_e} \mathbf{B}^T \mathbf{D}_e \mathbf{B} J d\xi d\eta,$$

$$\mathbf{f} = \int_{A_e} \mathbf{h}^T \mathbf{b} J d\xi d\eta + \int_{\partial A_{\sigma e}} \mathbf{h}^T \bar{\mathbf{T}} ds + \int_{\partial A'_{\sigma e}} u_t \bar{T}_t ds + \int_{\partial A''_{\sigma e}} u_n \bar{T}_n ds$$

where ξ, η are the transformed coordinates, A_e is a right isosceles triangle of area 1/2 for triangular elements and a square of area 4 for quadrilateral elements, and the integrals along $\partial A_{\sigma e}$, $\partial A'_{\sigma e}$ and $\partial A''_{\sigma e}$ are in the form of Eq. (18.12:3) for triangular elements and Eq. (18.13:6) for quadrilateral elements with $\mathbf{h}^T \bar{\phi}_{\sigma}$ replaced by $\mathbf{h}^T \bar{\mathbf{T}}$, and

$$(13) \quad \mathbf{B} = \mathbf{d}_e \mathbf{h} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \mathbf{h} = \begin{bmatrix} \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} & 0 \\ 0 & \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} \\ \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} & \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} \end{bmatrix} \mathbf{h}.$$

The evaluation of the matrix \mathbf{B} is tedious. According to Eqs. (18.11:3–5) and (18.13:3),

$$(14) \quad \mathbf{J} = \begin{bmatrix} \partial x / \partial \xi & \partial y / \partial \xi \\ \partial x / \partial \eta & \partial y / \partial \eta \end{bmatrix} = \sum_{j=1}^p \begin{bmatrix} x_j \partial h_j / \partial \xi & y_j \partial h_j / \partial \xi \\ x_j \partial h_j / \partial \eta & y_j \partial h_j / \partial \eta \end{bmatrix},$$

$$(15) \quad \mathbf{J}^{-1} = \frac{1}{J} \begin{bmatrix} \partial y / \partial \eta & -\partial y / \partial \xi \\ -\partial x / \partial \eta & \partial x / \partial \xi \end{bmatrix} = \frac{1}{J} \sum_{j=1}^p \begin{bmatrix} y_j \partial h_j / \partial \eta & -y_j \partial h_j / \partial \xi \\ -x_j \partial h_j / \partial \eta & x_j \partial h_j / \partial \xi \end{bmatrix},$$

$$(16) \quad J = (\partial x / \partial \xi)(\partial y / \partial \eta) - (\partial x / \partial \eta)(\partial y / \partial \xi).$$

Then from Eq. (8), we obtain

$$(17) \quad \mathbf{e} = \mathbf{B}\mathbf{q} = [\partial u/\partial x \quad \partial v/\partial y \quad \partial u/\partial y + \partial v/\partial x]^T$$

$$= \frac{1}{J} \sum_{i=1}^p \begin{bmatrix} [(\partial h_i/\partial \xi)(\partial y/\partial \eta) - (\partial h_i/\partial \eta)(\partial y/\partial \xi)]q_{2i-1} \\ [(\partial h_i/\partial \eta)(\partial x/\partial \xi) - (\partial h_i/\partial \xi)(\partial x/\partial \eta)]q_{2i} \\ \left(\frac{\partial h_i}{\partial \eta} \frac{\partial x}{\partial \xi} - \frac{\partial h_i}{\partial \xi} \frac{\partial x}{\partial \eta} \right) q_{2i-1} + \left(\frac{\partial h_i}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial h_i}{\partial \eta} \frac{\partial y}{\partial \xi} \right) q_{2i} \end{bmatrix}.$$

Note that we have

$$(18) \quad B_{1,2i-1} = \partial h_i/\partial x = [(\partial h_i/\partial \xi)(\partial y/\partial \eta) - (\partial h_i/\partial \eta)(\partial y/\partial \xi)]/J,$$

$$B_{2,2i} = \partial h_i/\partial y = [(\partial h_i/\partial \eta)(\partial x/\partial \xi) - (\partial h_i/\partial \xi)(\partial x/\partial \eta)]/J,$$

$$B_{3,2i-1} = B_{2,2i}, \quad B_{3,2i} = B_{1,2i-1}, \quad i = 1, \dots, p.$$

All other B 's are zero. The geometry of the element affects the element matrices through the values J and \mathbf{J}^{-1} .

For a 4-node quadrilateral element, from Eq. (18.13:2), we have

$$\partial h_i/\partial \xi = \xi_i(1 + \eta_i \xi)/4, \quad \partial h_i/\partial \eta = \eta_i(1 + \xi_i \eta)/4 \quad (i \text{ not summed})$$

$$\mathbf{J}^{-1} = \frac{1}{4J} \sum_{i=1}^4 \begin{bmatrix} y_i \eta_i(1 + \xi_i \xi) & -y_i \xi_i(1 + \eta_i \eta) \\ -x_i \eta_i(1 + \xi_i \xi) & x_i \xi_i(1 + \eta_i \eta) \end{bmatrix},$$

$$J = \sum_{i,j=1}^4 (x_i y_j - x_j y_i) \xi_i \eta_j (1 + \eta_i \eta) (1 + \xi_j \xi) / 16.$$

The Jacobian J is not a constant for elements other than rectangles and parallelograms. Thus, it is best to integrate Eq. (12) numerically.

Plane Stress. For plane stress problems of orthotropic materials,

$$(19) \quad \mathbf{D}_e = \frac{1}{1 - \nu_{xy}\nu_{yx}} \begin{bmatrix} E_x & & \text{sym} \\ E_x \nu_{xy} & E_y & \\ 0 & 0 & G_{xy}(1 - \nu_{xy}\nu_{yx}) \end{bmatrix},$$

in which $E_x \nu_{xy} = E_y \nu_{yx}$. If the material is *isotropic*, $E_x = E_y = E$, $\nu_{xy} = \nu_{yx} = \nu$ and $G_{xy} = G = E/[2(1 + \nu)]$, where E is Young's modulus and ν is Poisson's ratio.

Plane Strain. For plane strain problems, if the material is isotropic

$$(20) \quad \mathbf{D}_e = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & & \text{sym} \\ \nu & 1 - \nu & \\ 0 & 0 & (1 - 2\nu)/2 \end{bmatrix}.$$

Axisymmetric Problems. For axisymmetric problems, there are four components of strain

$$e_r = \partial u/\partial r, \quad e_\theta = u/r, \quad e_z = \partial w/\partial z, \quad e_{rz} = \partial u/\partial z + \partial w/\partial r,$$

which can be written as

$$\mathbf{e} = \mathbf{d}_a \mathbf{u} = \mathbf{d}_a \begin{bmatrix} u \\ w \end{bmatrix} = \mathbf{d}_a \mathbf{h} \mathbf{q} = \begin{bmatrix} \partial/\partial r & 1/r & 0 & \partial/\partial z \\ 0 & 0 & \partial/\partial z & \partial/\partial r \end{bmatrix}^T \mathbf{h} \mathbf{q} = \mathbf{B}_a \mathbf{q}.$$

Expressing u, w in the form of Eq. (8), we obtain the element matrices

$$\mathbf{k} = \int_{A_e} \mathbf{B}_a^T \mathbf{D}_a \mathbf{B}_a r dr dz ,$$

$$\mathbf{f} = \int_{A_e} \mathbf{h}^T \mathbf{b} r dr dz + \int_{\partial A_{\sigma e}} \mathbf{h}^T \bar{\mathbf{T}} ds + \int_{\partial A'_{\sigma e}} u_t \bar{T}_t ds + \int_{\partial A''_{\sigma e}} u_n \bar{T}_n ds ,$$

which have the similar form as Eq. (11) except that \mathbf{D}_a is a 4×4 *elastic coefficient or modulus matrix*. For isotropic materials

$$(21) \quad \mathbf{D}_a = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & & & \text{sym} \\ \nu & 1-\nu & & \\ \nu & \nu & 1-\nu & \\ 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} .$$

Problem 18.10. Determine the nominal rank of \mathbf{k} for axial symmetric deformation.

Problem 18.11. Consider the coordinate transform

$$r = r(\xi, \eta), \quad z = z(\xi, \eta) .$$

Find $\mathbf{d}\mathbf{a}$ in terms of differentials $\partial/\partial\xi$ and $\partial/\partial\eta$.

Incompatible Element. The accuracy of triangular and rectangular elements, more so for triangles, deteriorate when the *aspect ratio* (the ratio of one of the element dimension to the other) becomes large. In particular, when the material is approaching incompressible (ν near 1/2), the element will be overly stiff. The effect is especially pronounced in simulating bending of slender beams using plane elements. If only one element is used through the thickness, the element is very stiff due to spurious shear. Wilson *et al.* (1973) introduced quadratic terms in the deformation that

$$(22) \quad \mathbf{u} = \mathbf{h}\mathbf{q} + (1 - \xi^2)\mathbf{a}_5 + (1 - \eta^2)\mathbf{a}_6 ,$$

where \mathbf{h} is the same as that for a 4-node rectangle given in Eq. (9) and \mathbf{a} 's are unknown parameters. The additional terms are zero at the 4 nodes and, therefore, do not affect \mathbf{q} . But these terms result in discontinuous displacements between two elements along the interelement boundaries. Thus, the element is called incompatible element.

The strain matrix can be written as

$$(23) \quad \mathbf{e} = \mathbf{B}\mathbf{q} + 2\xi\mathbf{B}_5\mathbf{a}_5/J + 2\eta\mathbf{B}_6\mathbf{a}_6/J ,$$

where

$$\mathbf{B}_5 = \begin{bmatrix} -\partial y / \partial \eta & 0 & \partial x / \partial \eta \\ 0 & \partial x / \partial \eta & -\partial y / \partial \eta \end{bmatrix}^T, \quad \mathbf{B}_6 = \begin{bmatrix} \partial y / \partial \xi & 0 & -\partial x / \partial \xi \\ 0 & -\partial x / \partial \xi & \partial y / \partial \xi \end{bmatrix}^T ,$$

in which \mathbf{B} is the same as that of Eq. (10), J is as before. Note that both \mathbf{B}_5 and \mathbf{B}_6 are constant for rectangle and parallelogram. One can derive the element matrices accordingly. Since \mathbf{a} 's do not appear in other elements, they can be eliminated from the element by *static condensation*.

The incompatible element improves the bending performance. However, its behavior is erratic when the element is not a rectangle or parallelogram. Taylor *et al.* (1976) modified the incompatible modes by treating \mathbf{B}_5 and \mathbf{B}_6 as constants of their values at $\xi = \eta = 0$. This removes the spurious shear induced by the isoparametric transformation when the element is not a rectangle or parallelogram. Approximating \mathbf{B}_5 and \mathbf{B}_6 as constant is a form of *reduced integration*, a technique often used to reduce the stiffness of elements. Taylor's element satisfies the *patch test*, a condition required for convergence of incompatible elements. The details of reduced integration and patch test will be discussed later.

Rotation. In order to use the assembly procedures described in Sec. 18.4, the generalized coordinates of different elements must have the same physical dimension at their common nodes. For example, in plane elasticity, the displacement vectors of different elements at the common node must refer to the same coordinate system. In generating the element matrices, using a local coordinate system, which can be different from the global one, may be more convenient. Modeling a two-dimensional truss by one-dimensional rods discussed in Sec. 18.6 is such an example. For orthotropic materials, it may be desirable to have the local coordinates aligned with the material coordinates of the element in deriving the element matrices. Let the prime and un-prime denote quantities in the global and local systems, respectively (Fig. 18.14:1). The two quantities are related by

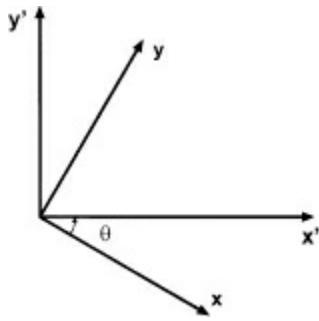


Fig. 18.14:1. Coordinate rotation.

$$\mathbf{x}_j = \beta_j \mathbf{x}'_j = \begin{bmatrix} \cos \theta_j & -\sin \theta_j \\ \sin \theta_j & \cos \theta_j \end{bmatrix} \mathbf{x}'_j, \quad [q_{2j-1} \quad q_{2j}]^T = \beta_j [q'_{2j-1} \quad q'_{2j}]^T$$

where x_j is the local coordinate of, β_j is the rotation matrix of, and q 's are the nodal values of \mathbf{u} at node j (j not summed). For a p-node element, we have

$$(24) \quad \underset{2p \times 1}{\mathbf{q}} = \underset{2p \times 2p}{\text{dia}} \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_p \end{bmatrix}_{2 \times 2} \underset{2p \times 1}{\mathbf{q}'} = \underset{2p \times 2p}{\mathbf{R}} \underset{2p \times 1}{\mathbf{q}'},$$

with \mathbf{R} being the *rotation matrix* for the element. We derive the element matrices associated with \mathbf{q}' by substituting Eq. (24) into Eq. (10),

$$(25) \quad \delta\Pi_e = \delta\mathbf{q}^T(\mathbf{k}\mathbf{q} - \mathbf{f}) = \delta\mathbf{q}'^T(\mathbf{R}^T\mathbf{k}\mathbf{R}\mathbf{q}' - \mathbf{R}^T\mathbf{f}) = \delta\mathbf{q}'^T(\mathbf{k}'\mathbf{q}' - \mathbf{f}').$$

The element matrices \mathbf{k}' and \mathbf{f}' are to be used for assembly.

There are other circumstances that one must rotate the coordinates. In mixed-boundary problems, for instance, the components of the prescribed displacement can be in the directions differ from the global coordinates. This will require a transformation of the coordinates of those boundary nodes to the directions of prescribed displacements.

In summary, the finite element method derives an approximate solution based on the weak form formulations, the principles of virtual work or variational calculus. All field variables are expressed in terms of shape functions and generalized coordinates. For most cases, the shape functions are polynomials independent of the type of problems. Thus the method allows generating element matrices in a generic way. The element matrices are assembled systematically into a system of algebraic equations with constraints imposed to enforce the rigid boundary conditions. The finite element process assures that the first variation of the functional of the entire domain equals that of the sum of those of all elements that

$$\sum_{\text{all element}} \delta\Pi_e = \sum_{\text{all element}} \delta\mathbf{q}^T(\mathbf{k}\mathbf{q} - \mathbf{f}) = 0.$$

We denote *symbolically* the process as

$$(26) \quad \mathbf{Ku} - \mathbf{F} = \int_A \mathbf{B}^T \mathbf{D}_e \mathbf{B} u dA - \int_A \mathbf{h}^T \mathbf{b} dA - \int_{\partial A_\sigma} \mathbf{h}^T \bar{\mathbf{T}} ds - \int_{\partial A'_\sigma} \mathbf{h}_t^T \bar{\mathbf{T}}_t ds - \int_{\partial A''_\sigma} \mathbf{h}_n^T \bar{\mathbf{T}}_n ds = 0.$$

Here \mathbf{u} represents the nodal unknown matrix for the whole system, and n and t refer to the normal and tangent directions on the boundary. Equation (26) is to be solved using high speed computers. Thus in the finite element method, *solving a set of differential equations with prescribed boundary conditions becomes the task of discretizing the domain into elements, constructing the element matrices, labeling the nodes, assembling the element matrices to form a set of algebraic equations, imposing constraints to enforce the rigid boundary conditions, and finally solving the algebraic equations*.

Problem 18.12. Show that rotation does not change the rank of \mathbf{k} .

Problem 18.13. Let the constitutive law be in the form

$$\boldsymbol{\sigma} = \mathbf{D}_t [\varepsilon_x \quad \varepsilon_y \quad \varepsilon_{xy}]^T = \mathbf{D}_t [\partial u / \partial x \quad \partial v / \partial y \quad (\partial u / \partial y + \partial v / \partial x) / 2]^T,$$

where ε 's are components of the strain tensor. Show that \mathbf{D}_t is in general not symmetric.

18.15. THREE-DIMENSIONAL SHAPE FUNCTIONS

Generalizing the finite element process to three-dimensional continua is conceptually straightforward. It is just a matter of discretizing the three-dimensional body into a finite number of *solid elements*, deriving the appropriate *3-dimensional shape functions*, and generating the corresponding element matrices based on appropriate integral formulation. The procedures of assembling, constraining, and solving algebraic equations for one-, two- or three-dimensional problems are the same. The complication of three-dimensional problems is in the implementation such as describing the geometry, dealing with many algebraic equations of high semi-bandwidths, etc. Discretizing complex three-dimensional body into small solids, especially hexahedrals, is not a trivial matter. Optimum labeling the nodes to give better average semibandwidth is very involved.

We shall limit ourselves to the derivation of three-dimensional shape functions and element matrices. In the process, we represent a field variable ϕ by its nodal values q_j and the shape functions h_j over the element as

$$(1) \quad \phi = \sum_{j=1}^p h_j(x, y, z) q_j,$$

where p is the number of nodes of the element. The element boundaries are surfaces, full admissibility means that admissibility at the nodes on an interelement surface and its edges implies admissibility over the entire side.

Shape Functions for Tetrahedrons. For tetrahedral elements (Fig. 18.15:1), it is most convenient to construct the shape functions in terms of the *volume or natural coordinates*,

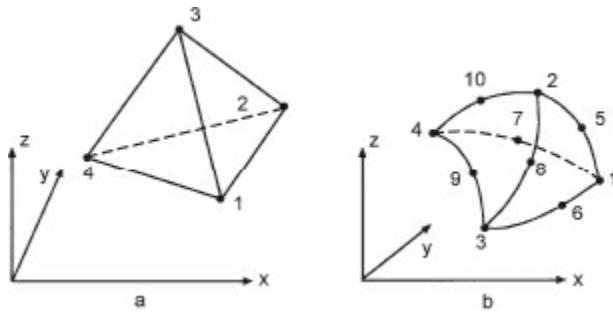


Fig. 18.15:1. Tetrahedral elements.

$$(2) \quad \zeta_j = (a_j + b_j x + c_j y + d_j z) / (6\Delta_v) \quad j = 1, \dots, 4,$$

where Δ_v is the volume of the tetrahedral element

$$(3) \quad \Delta_v = \frac{1}{6} \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}, \quad a_1 = \begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix},$$

$$b_1 = - \begin{vmatrix} 1 & 1 & 1 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}, \quad c_1 = \begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ z_2 & z_3 & z_4 \end{vmatrix}, \quad d_1 = - \begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \end{vmatrix},$$

with a_2, b_2, c_2 , etc. obtainable through the cyclic permutation of the subscripts. The natural coordinates are, similar to the triangular coordinates, the normalized volumes of tetrahedrons,

$$(4) \quad \zeta_1 = \Delta_{P234} / \Delta_v,$$

where Δ_{P234} is the volume of the tetrahedron with nodes $P, 2, 3, 4$ in which P is a point within the tetrahedron, and $\zeta_2, \zeta_3, \zeta_4$ are obtained by the cyclic permutations of the subscripts 1 to 4. Thus $\zeta_i = 1$ at node i and = 0 at all other three nodes, is a 3-D Lagrange function. The volume coordinates satisfy

$$\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 = 1.$$

One can use the formula to evaluate the volume integral for element matrices:

$$(5) \quad \int_{\Delta_v} \zeta_1^\alpha \zeta_2^\beta \zeta_3^\gamma \zeta_4^\lambda dV = 6\Delta_v \alpha! \beta! \gamma! \lambda! / (\alpha + \beta + \gamma + \lambda + 3)!.$$

However, numerical integration is usually more practical.

Four-Node Tetrahedron. A four-node tetrahedral is the simplest solid element. The shape functions are simply

$$(6) \quad h_j = \zeta_j ,$$

which are linear in x, y, z in the element and on each of its side. Each side has 3 nodes and thus full compatibility is assured.

Ten-Node Tetrahedron. The shape function h_j are the product of two parallel plane functions,

$$(7) \quad \begin{aligned} h_j &= \zeta_j(2\zeta_j - 1), \quad j = 1, 2, 3, 4, \\ h_5 &= 4\zeta_1\zeta_2, \quad h_6 = 4\zeta_1\zeta_3, \quad h_7 = 4\zeta_1\zeta_4, \\ h_8 &= 4\zeta_2\zeta_3, \quad h_9 = 4\zeta_2\zeta_4, \quad h_{10} = 4\zeta_3\zeta_4, \end{aligned}$$

which can represent a complete quadratic polynomial. The element has 6 nodes on each side, and full compatibility is assured.

Higher order shape functions can be derived from the products of plane functions and the addition of side node shape functions.

Shape Functions for Hexahedron. The shape functions for hexahedra can be derived easily like those of quadrilateral elements using the natural coordinates ξ, η, ζ . The shape functions for an 8-node cube with sides bounded by the plane functions $\xi = \pm 1, \eta = \pm 1$, and $\zeta = \pm 1$ are simply products of these side functions:

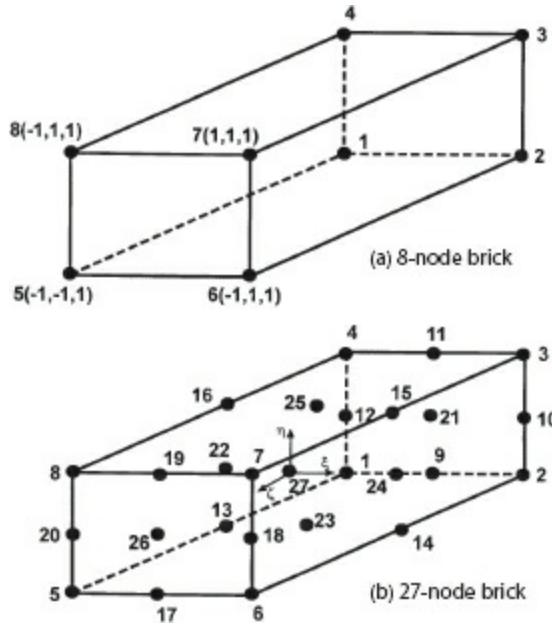


Fig. 18.15:2. Hexahedral elements.

$$(8) \quad h_j(\xi, \eta) = (1 + \xi_j \xi)(1 + \eta_j \eta)(1 + \zeta_j \zeta)/8 \quad (j \text{ not summed}),$$

a trilinear function with

$$\begin{aligned} \xi_1 &= \xi_4 = \xi_5 = \xi_8 = \eta_1 = \eta_2 = \eta_5 = \eta_6 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = -1, \\ \xi_2 &= \xi_3 = \xi_6 = \xi_7 = \eta_3 = \eta_4 = \eta_7 = \eta_8 = \zeta_5 = \zeta_6 = \zeta_7 = \zeta_8 = 1. \end{aligned}$$

The h_j are the products of three plane functions $1 + \xi_j \xi = 0, 1 + \eta_j \eta = 0$, and $1 + \zeta_j \zeta = 0$. These shape functions are bilinear on an element side and have 4 independent terms. Since each boundary side has four nodes and thus full compatibility is assured.

The shape functions for a 27-node cube can be derived also by combinations of different shape functions. The one associated the centroid node 27 is a bulb function

$$h_{27} = (1 - \xi^2)(1 - \eta^2)(1 - \zeta^2).$$

Those associated nodes 21, 22 and 23, the midside nodes adjacent to node 1 at $\zeta = -1, \xi = -1$ and $\eta = -1$ are, respectively,

$$h_{21} = (1 - \xi^2)(1 - \eta^2)(1 - \zeta)/2 - h_{27}/2$$

$$h_{22} = (1 - \eta^2)(1 - \zeta^2)(1 - \xi)/2 - h_{27}/2$$

$$h_{23} = (1 - \zeta^2)(1 - \xi^2)(1 - \eta)/2 - h_{27}/2.$$

The function h_{21} is the product of five plane functions not connected to node 21. The function is normalized to unity at $\xi = 0, \eta = \zeta = -1$ and nulled at $\xi = \eta = \zeta = 0$ by minus $h_{27}/2$. The shape functions associated nodes 9, 12 and 13, the midedge nodes adjacent to node 1, are

$$h_9 = (1 - \xi^2)(1 - \eta)(1 - \zeta)/4 - (h_{21} + h_{23})/2 - h_{27}/4,$$

$$h_{12} = (1 - \xi)(1 - \eta^2)(1 - \zeta)/4 - (h_{22} + h_{21})/2 - h_{27}/4,$$

$$h_{13} = (1 - \xi)(1 - \eta)(1 - \zeta^2)/4 - (h_{23} + h_{22})/2 - h_{27}/4.$$

The subtraction of functions in the equations above is to make h_9, h_{12} and h_{13} equal zero at all nodes other than nodes 9, 12 and 13, respectively. The function h_1 is

$$h_1 = \frac{(1 - \xi)(1 - \eta)(1 - \zeta)}{8} - \frac{(h_9 + h_{12} + h_{13})}{2} - \frac{(h_{21} + h_{22} + h_{23})}{4} - \frac{h_{27}}{8}.$$

On the element sides, the shape functions are bi-quadratic with 9 independent terms. The 27-node hexahedron has 9 nodes on each side and thus full compatibility is assured. The shape functions for a 20-node element, an element without midside and central nodes, can be obtained simply from those of the 27-node element by removing h_{21}, \dots, h_{27} .

Problem 18.14. Derive the shape functions for triangular prisms as shown. The heavy dots are nodes of the element. The prism has two triangular and three rectangular surfaces.

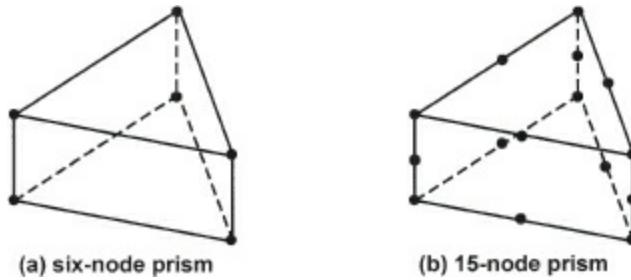


Fig. P18.14. Triangular prism: (a) 6-node prism and (b) 15-node prism.

Problem 18.15. Derive the shape functions for a quartic tetrahedral element. Show that an interior node inside the tetrahedron is needed to achieve complete quartic polynomials for the shape functions.

18.16. THREE-DIMENSIONAL ELASTICITY

We express the field variables of 3-dimensional elasticity in matrix form

$$(1) \quad \mathbf{u}^T = [u \ v \ w],$$

$$(2) \quad \boldsymbol{\sigma}^T = [\sigma_x \ \sigma_y \ \sigma_z \ \sigma_{xy} \ \sigma_{yz} \ \sigma_{zx}],$$

$$(3) \quad \mathbf{e}^T = [e_x \ e_y \ e_z \ \gamma_{xy} \ \gamma_{yz} \ \gamma_{zx}],$$

where \mathbf{u} , $\boldsymbol{\sigma}$, and \mathbf{e} are the displacement, the stress, and the strain vectors (**not tensor**), respectively. Note that $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ are the *engineering shear strains*. The stress and strain vectors are related by the constitutive law

$$(4) \quad \boldsymbol{\sigma} = \mathbf{D}_e \mathbf{e} = \begin{bmatrix} D_{11} & & & & & \text{sym} \\ D_{12} & D_{22} & & & & \\ \dots & \dots & \dots & & & \\ D_{16} & D_{26} & \dots & D_{66} & & \end{bmatrix} \mathbf{e},$$

where \mathbf{D}_e is the elastic constant matrix, whose components are related to the coefficients of Hooke's law ($\sigma_{ij} = d_{ijkl}e_{kl}$) by the following equations:

$$(5) \quad D_{ij} = D_{ji} = d_{iijj}, \quad D_{i,j+3} = D_{j+3,i} = d_{iijm(j)}, \\ D_{i+3,j+3} = D_{j+3,i+3} = d_{im(i)jm(j)},$$

where $i, j = 1, 2, 3$, and $m(j) = j + 1$ for $j = 1, 2$, $m(j) = 1$ for $j = 3$. The engineering shear strain for \mathbf{e} make \mathbf{D}_e symmetric. For isotropic materials,

$$D_{11} = D_{22} = D_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad D_{44} = D_{55} = D_{66} = \frac{E}{2(1+\nu)}, \\ D_{12} = D_{21} = D_{13} = D_{31} = D_{32} = D_{23} = E\nu/[(1+\nu)(1-2\nu)]$$

while the unlisted D 's are zero. Let \mathbf{d}_e , a differential operator matrix, be defined as below, we can express the strain \mathbf{e} in terms of the displacements as

$$(6) \quad \mathbf{e} = \mathbf{d}_e \mathbf{u} = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial y & 0 & \partial/\partial z \\ 0 & \partial/\partial y & 0 & \partial/\partial x & \partial/\partial z & 0 \\ 0 & 0 & \partial/\partial z & 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}^T \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Consider a problem over V with distributed body force vector \mathbf{b} and prescribed boundary conditions

$$(7) \quad \mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \partial V_u,$$

$$(8) \quad \mathbf{T} = \bar{\mathbf{T}} \quad \text{on } \partial V_\sigma,$$

$$(9) \quad u_n = \bar{u}_n, \quad T_{t_1} = \bar{T}_{t_1}, \quad T_{t_2} = \bar{T}_{t_2} \quad \text{on } \partial V'_\sigma,$$

where $\partial V_u + \partial V_\sigma + \partial V'_\sigma = \partial V$. Equation (9) is a mixed condition with the subscripts n denoting the normal, and t_1 and t_2 the tangential components of the corresponding vector on $\partial V'_\sigma$. The potential energy for an element is

$$(10) \quad \Pi_e = \int_{V_e} (\mathbf{e}^T \mathbf{D}_e \mathbf{e} / 2 - \mathbf{u}^T \mathbf{b}) dV - \int_{\partial V_{\sigma e}} \mathbf{u}^T \bar{\mathbf{T}} dS - \int_{\partial V'_{\sigma e}} (u_{t_1} \bar{T}_{t_1} + u_{t_2} \bar{T}_{t_2}) dS,$$

where subscript e denotes association with elements. If the mixed condition involves a combination of different prescribed components of displacement and traction, the last two terms of Eq. (9) must be modified accordingly.

One can derive the element matrices routinely from Eq. (10) and the finite element equations from the first variation

$$\sum_{\text{all element}} \delta \Pi_e = 0$$

with respect to admissible \mathbf{u} . The admissible conditions are: \mathbf{u} and $\partial \mathbf{u}$ be C^0 continuous over V , $\mathbf{u} = \bar{\mathbf{u}}$, and $\delta \mathbf{u} = 0$ on ∂V_u , and $u = \bar{u}_n$, $\delta u_n = 0$ on $\partial V'_\sigma$. The prescribed tractions in Eq. (8) and (9) are natural conditions, which are accounted for by the integrals over $\partial V_{\sigma e}$ and $\partial V'_{\sigma e}$.

The procedure for establishing the finite element equations is the same as that for plane elasticity. For an element with p nodes at x_j, y_j, z_j , let $\mathbf{u}_j^T = \mathbf{q}_j^T = [u_j, v_j, w_j]$ and $\mathbf{q} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_p]^T = [q_1 \ q_2 \dots \ q_{3p-1} \ q_{3p}]^T$, where q 's are the generalized coordinates. Each node now has *three DOFs* as opposed to one or two considered earlier. We write

$$(11) \quad \mathbf{u} = \begin{matrix} \mathbf{h} \\ 3 \times 3p \end{matrix} \begin{matrix} \mathbf{q} \\ 3p \times 1 \end{matrix}, \quad \begin{matrix} \mathbf{h} \\ 3 \times 3p \end{matrix} = [h_1 \mathbf{I}_{3 \times 3} \ \dots \ h_p \mathbf{I}_{3 \times 3}], \\ \mathbf{e} = \mathbf{d}_e \mathbf{u} = \mathbf{d}_e \mathbf{h} \mathbf{q} = \mathbf{B} \mathbf{q}, \quad \mathbf{B} = \mathbf{d}_e \mathbf{h},$$

where h_i are defined in Sec. 18.15.

The element matrices can be written in the standard form

$$(12) \quad \mathbf{k} = \int_{V_e} \mathbf{B}^T \mathbf{D}_e \mathbf{B} dV,$$

$$(13) \quad \mathbf{f} = \int_{V_e} \mathbf{h}^T \mathbf{b} dV + \int_{\partial V_{\sigma e}} \mathbf{h}^T \bar{\mathbf{T}} dS + \int_{\partial V'_{\sigma e}} (\mathbf{h}_{t_1}^T \bar{T}_{t_1} + \mathbf{h}_{t_2}^T \bar{T}_{t_2}) dS,$$

where \mathbf{h}_{t_1} and \mathbf{h}_{t_2} are so defined that $u_{t_1} = \mathbf{h}_{t_1} \mathbf{q}$ and $u_{t_2} = \mathbf{h}_{t_2} \mathbf{q}$ on $\partial V'_{\sigma e}$, respectively. For four-node tetrahedrons, the

strain $\mathbf{e} (= \mathbf{B}\mathbf{q})$ is constant and the element is called *constant strain tetrahedron*. A 10-node tetrahedron has quadratic displacements and linear strains and is, therefore, called *linear strain tetrahedron*. The rank of \mathbf{k} is $3p - 6$ if the shape functions are capable of representing all 6 rigid body modes. Approximate integration of Eq. (12) can lower the rank of \mathbf{k} and introduce spurious deformation mode(s).

For elements with p -node curved surfaces, we introduce the transformation

$$(14) \quad [x \ y \ z] = \sum_{j=1}^p h_j(\xi, \eta, \zeta) [x_j \ y_j \ z_j]$$

to map a tetrahedral element V_e into a right angle tetrahedron in the first quadrant or a hexahedral element into a cube of $-1 \leq \xi, \eta, \zeta \leq 1$. The element matrices can be written in the similar form as Eq. (18.14:12)

$$(15) \quad \mathbf{k} = \int_{V_e} \mathbf{B}^T \mathbf{D}_e \mathbf{B} J d\xi d\zeta d\eta,$$

$$(16) \quad \mathbf{f} = \int_{V_e} \mathbf{h}^T \mathbf{b} J d\xi d\zeta d\eta + \int_{\partial V_{\sigma e}} \mathbf{h}^T \bar{\mathbf{T}} dS + \int_{\partial V'_{\sigma e}} (\mathbf{h}_{t_1}^T \bar{T}_{t_1} + \mathbf{h}_{t_2}^T \bar{T}_{t_2}) dS,$$

where $\mathbf{B} = \mathbf{d}_e \mathbf{h}$ and \mathbf{d}_e is defined in Eq. (18.16:6). The inverse transformations $\xi = \xi(x, y, z)$, $\eta = \eta(x, y, z)$ and $\zeta = \zeta(x, y, z)$ are generally not available, we evaluate $\partial\xi/\partial x, \dots$ using the following relations:

$$\mathbf{d} = [\partial/\partial x \ \partial/\partial y \ \partial/\partial z]^T = \mathbf{J}^{-1} [\partial/\partial \xi \ \partial/\partial \eta \ \partial/\partial \zeta]^T,$$

$$\mathbf{J}^{-1} = \begin{bmatrix} \partial\xi/\partial x & \partial\eta/\partial x & \partial\zeta/\partial x \\ \partial\xi/\partial y & \partial\eta/\partial y & \partial\zeta/\partial y \\ \partial\xi/\partial z & \partial\eta/\partial z & \partial\zeta/\partial z \end{bmatrix} = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi & \partial z/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta & \partial z/\partial \eta \\ \partial x/\partial \zeta & \partial y/\partial \zeta & \partial z/\partial \zeta \end{bmatrix}^{-1}.$$

The integration over $\partial V_{\sigma e}$ and $\partial V'_{\sigma e}$ becomes the integration over a right isosceles triangle of area of 1/2 for tetrahedral elements in the transformed coordinates; or a square of area of 4 for cubic elements. If the surface coordinates of the boundary are ξ_1, ξ_2 , which are two of the three natural coordinates ξ, η, ζ . Then

$$dS = \sqrt{(\partial x_k / \partial \xi_1)^2 + (\partial x_k / \partial \xi_2)^2 - (\partial x_k / \partial \xi_1)(\partial x_k / \partial \xi_2)} d\xi_1 d\xi_2,$$

where $x_1 = x, y_1 = y$ and $z_1 = z$, and repeated indices denote summation from 1 to 3. For a boundary surface with the normal in the ζ -direction, ξ_1, ξ_2 are ξ, η and so on.

For tetrahedral elements, the shape functions are actually in terms of the volume coordinates $\zeta_1, \zeta_2, \zeta_3, \zeta_4$. We have

$$\xi = \zeta_1, \quad \eta = \zeta_2, \quad \zeta = \zeta_3, \quad \zeta_4 = 1 - \zeta_1 - \zeta_2 - \zeta_3.$$

The differentiations with respect to the ξ, η, ζ -coordinates relate to those with respect to the volume coordinates by

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \zeta_1} - \frac{\partial}{\partial \zeta_4}, \quad \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \zeta_2} - \frac{\partial}{\partial \zeta_4}, \quad \frac{\partial}{\partial \zeta} = \frac{\partial}{\partial \zeta_3} - \frac{\partial}{\partial \zeta_4}.$$

The finite element method permits a systematic process for selecting shape functions, constructing element matrices, assembling them into a system of algebraic equations with imposed rigid constraints, and solving the equations by high speed computer. The process can be applied to almost any types of problem. The element stiffness matrix depends on problem type rather than the specifics of the problem. The applied force matrix depends on the distributed loads, but the process to determine it is generic. We can use same type of element matrices to solve a problem with a square hole or a circular one. The nature of the problem is reflected in the element size, shape and distribution, the boundary conditions, and the distributed load. The generic characteristics make the finite element method applicable to almost any problems.

18.17. DYNAMIC PROBLEMS OF ELASTIC SOLIDS

The finite-element method for dynamic problems results in a set of simultaneous equations in the form

$$(1) \quad \mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F},$$

where \mathbf{u} is the matrix of generalized coordinates or unknown parameters for the entire domain, a dot(s) over a character denotes differentiation with respect to time t , \mathbf{M} and \mathbf{C} are the assembled mass and damping matrices, respectively, and \mathbf{F} is the force vector. \mathbf{M} is symmetric and positive definite, and \mathbf{C} is also symmetric if linear viscous

damping prevails. We now have an initial-valued problem for $\mathbf{u}(t)$ with initial conditions:

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0.$$

We use Hamilton's principle to derive the mass and the damping matrices,

$$(2) \quad I = \int_{t_1}^{t_2} (U - W - K) dt$$

where U is the strain energy, K the kinetic energy, and W the work of the applied loads. Hamilton's principle requires

$$(3) \quad \delta I = 0,$$

for all time for displacements that satisfy the compatibility and rigid boundary conditions, and the conditions at time t_1 and t_2 .

To illustrate these ideas, consider the axial displacement $u(x, t)$ of a rod with prescribed loads \bar{F}_1 and \bar{F}_2 at the ends. The governing equation is

$$(4) \quad \partial[AE(x, t)(\partial u / \partial x)] / \partial x + b(x, t) = \rho A(x, t) \ddot{u}$$

where ρA is the mass per unit length, b the applied distributed axial load, AE the rod axial stiffness, and $u(x, t_1)$ and $u(x, t_2)$ are prescribed. Let the rod be divided into N elements. For the n^{th} element between points x_n and x_{n+1} , the nodal values at these points are denoted by $q_n = [u(x_n, t)]$. A weak form $\delta I = 0$ for Eq. (4) can be written as

$$(5) \quad \delta I = \int_{t_1}^{t_2} \left[\sum_{j=1}^N \int_{x_j}^{x_{j+1}} (AE(\partial u / \partial x)(\partial \delta u / \partial x) - b \delta u - \rho A u \delta \ddot{u}) dx \right. \\ \left. + \bar{F}_1(t) \delta u(x_0, t) - \bar{F}_2(t) \delta u(x_{N+1}, t) \right] dt = 0.$$

We assume u in the j^{th} element in the form

$$(6) \quad u(x, t) = \mathbf{h}(\xi) \mathbf{q}_j(t), \quad \delta u(x, t) = \mathbf{h}(\xi) \delta \mathbf{q}_j(t),$$

where $\mathbf{h}(\xi)$ is the shape function and $\mathbf{q}_j(t)$ is the generalized-coordinate vector as defined in Sec. 18.2. Assuming the field variable as the product of the space-like $\mathbf{h}(\xi)$ and the time-like $\mathbf{q}_j(t)$ is **most fundamental** in the finite element approach to dynamic problems. Integrating $\delta \dot{\mathbf{q}}_j(t)$ by parts, with $\delta \mathbf{q}_j(t_1) = \delta \mathbf{q}_j(t_2) = 0$, we can write Eq. (5) as

$$(7) \quad \delta I = \int_{t_1}^{t_2} \sum_{j=1}^N \delta \mathbf{q}_j^T [\mathbf{k}_j \mathbf{q}_j - \mathbf{f}_j + \partial(\mathbf{m}_j \dot{\mathbf{q}}_j) / \partial t] dt = 0$$

where

$$(8) \quad \begin{aligned} \mathbf{k}_j &= (2/\varepsilon_j) \int_{-1}^1 AE(\partial \mathbf{h}^T / \partial \xi)(\partial \mathbf{h} / \partial \xi) d\xi, \\ \mathbf{f}_j &= (\varepsilon_j/2) \int_{-1}^1 b \mathbf{h}^T d\xi - [\alpha \bar{F}_1 \quad 0 \quad \dots]^T + [\dots \quad 0 \quad \beta \bar{F}_2]^T, \\ \mathbf{m}_j &= (\varepsilon_j/2) \int_{-1}^1 \rho A \mathbf{h}^T \mathbf{h} d\xi, \end{aligned}$$

where \mathbf{k}_j , \mathbf{f}_j and \mathbf{m}_j are the j^{th} element stiffness, force and mass matrices, respectively, $\alpha = 1$ at the first node of the first element and 0 elsewhere, and $\beta = 1$ at the last node of the last element and 0 elsewhere. In most structural applications, ρA and AE are independent of time. This is not true in general for biological or living materials, where the time variation of the material plays an essential role influencing the behavior of the system.

After all elements are assembled and constraints imposed, we have

$$(9) \quad \delta I = \int_{t_1}^{t_2} \delta \mathbf{u}^T [\mathbf{Ku} - \mathbf{F} + \partial(\mathbf{Mu} / \partial t)] dt = 0,$$

where \mathbf{u} is the nodal parameters of u . The Euler equation is

$$(10) \quad \partial(\mathbf{M}\ddot{\mathbf{u}})/\partial t + \mathbf{K}\mathbf{u} = \mathbf{F},$$

for the dynamic system. The matrices \mathbf{K} , \mathbf{F} and \mathbf{M} are from the strain energy, the external work, and the kinetic energy, respectively.

In Eq. (8), if the same shape functions are used for \mathbf{k} and \mathbf{m} , the mass matrix is referred to as a *consistent mass matrix*, which is usually fully populated. In practice, *diagonal or lumped mass matrices* are often employed due to their general economy especially in the explicit time-integration schemes. Each of these approaches has its advantages and disadvantages with regard to computer implementation.

The lumped mass formulation assumes that a certain amount of structural mass surrounding a given node is *lumped* at that node. There are many *ad hoc* ways to construct lumped mass matrices (Tong *et al.* 1971, Hughes *et al.* 1976, Fried and Malkus 1976, Hinton *et al.* 1976). One approach sums the quantities of each row of the consistent matrix to the diagonal if all generalized coordinates have the same physical dimension. No general theory of obtaining higher-order accurate mass matrices has yet been established.

In Eq. (8), for constant ρA and

$$u = q_1(1 - \xi)/2 + q_2(1 + \xi)/2,$$

the consistent and lumped element mass matrices are, respectively

$$(11) \quad \mathbf{m}_j = \frac{\rho A \varepsilon_j}{3} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}, \quad (\mathbf{m}_j)_L = \frac{\rho A \varepsilon_j}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Eqs. (18.3:5) and (18.3:6), we obtain the consistent and lumped element mass matrices of a beam with uniform mass distribution are

$$(12) \quad \mathbf{m}_j = \frac{\rho A \varepsilon_j}{420} \begin{bmatrix} 156 & & & & \text{sym} \\ 22\varepsilon_j & 4\varepsilon_j^2 & & & \\ 54 & 13\varepsilon_j & 156 & & \\ -13\varepsilon_j & -3\varepsilon_j^2 & -22\varepsilon_j & 4\varepsilon_j^2 & \end{bmatrix},$$

$$(13) \quad (\mathbf{m}_j)_L = \rho A \varepsilon_j \text{dia}[1 \quad \varepsilon_j^2/12 \quad 1 \quad \varepsilon_j^2/12]/2.$$

Lumping the rotary inertia at a node as $\rho A \varepsilon_j^3/24$ is *ad hoc* at best. At times one even neglects it completely. In this case, the mass matrix becomes singular. This will make numerical integration of Eq. (1) by an explicit integration scheme more cumbersome.

For the triangular elements in plane elasticity, using the interpolation of Sec. 18.7, we find the consistent element mass matrix

$$(14) \quad \mathbf{m}_j = \frac{\rho A_j d}{12} \begin{bmatrix} 2 & & & & & \\ 0 & 2 & & & & \text{sym} \\ 1 & 0 & 2 & & & \\ 0 & 1 & 0 & 2 & & \\ 1 & 0 & 1 & 0 & 2 & \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix},$$

where d is thickness, A_j area, and ρ mass per unit volume of the element.

$$(15) \quad (\mathbf{m}_j)_L = \rho A_j d \text{dia}[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]/12$$

is the lumped element mass equally divided the mass among the nodes.

In Eq. (14) and (15), the mass matrix is based on the total mass of the element. In Eq. (18.14:11), the element stiffness and loading matrices are values per unit thickness. For dynamic study, one must multiply those matrices by the element thickness. If the thickness varies over the element, we multiply \mathbf{D}_e , \mathbf{b} , and $\bar{\mathbf{T}}$ in Eq. (18.14:11) or (18.14:12) by the thickness and modify Eq. (14) and (15) accordingly to evaluate the corresponding matrices. In plane strain, we deal with problems of unit thickness. In this case, we simply set d to 1 in Eq. (14) and (15).

To determine the damping matrix \mathbf{C} , we consider the virtual work done by the damping force. For example, consider

the case where the damping force is proportional to velocity. The virtual work done is

$$\int_{A_j} \delta \dot{\mathbf{u}}^T \gamma \dot{\mathbf{u}} dA = \delta \dot{\mathbf{q}}_j^T (\int_{A_j} \gamma \mathbf{h}^T \mathbf{h} dA) \dot{\mathbf{q}}_j = \delta \dot{\mathbf{q}}_j^T \mathbf{C}_j \dot{\mathbf{q}}_j,$$

in which \mathbf{C}_j is similar to the element mass matrix. An often-used form of \mathbf{C} is the *Rayleigh damping matrix*

$$\mathbf{C}_j = a\mathbf{M}_j + b\mathbf{K}_j,$$

where a and b are constant parameters.

All element matrices can be assembled as before. Equation (1) is to be solved by time integration or modal analysis. The eigenvalue problems (free vibration) corresponding to the homogeneous system of Eq. (1) can be treated by well-established techniques (Clough and Bathe 1972).

With respect to computation and accuracy, here are the pros and cons of employing consistent and lumped mass matrices. Lumped mass matrix is often diagonal. This requires less computer storage and time to generate than those of consistent mass matrix. The diagonal form also facilitates calculation in explicit time-integration scheme, where new vectors to be computed are known from previous steps (see [Biblio. 18.7](#)). An example of $\mathbf{C} = 0$ by the central difference scheme is

$$\mathbf{M}\mathbf{u}_{n+1} = \mathbf{M}(2\mathbf{u}_n - \mathbf{u}_{n-1}) - \mathbf{K}\mathbf{u}_n(\Delta t)^2 + \mathbf{F}(t_n), \quad \mathbf{u}(0) = \mathbf{u}_0,$$

where \mathbf{u}_n denotes the solution for \mathbf{u} at time $t = t_n$ and Δt is the time step of integration. If \mathbf{M} is a diagonal matrix, \mathbf{u}_{n+1} can be evaluated rapidly, especially for very large scale problems using parallel computation. Otherwise, solving for \mathbf{u}_{n+1} requires considerably computer time.

A useful feature of consistent mass matrix is that, if the element is compatible and all element matrices are carried out with full integration, the lowest natural frequency obtained is an upper bound. The use of lumped mass matrix tends to lower the frequencies. There may be instances where more accurate frequencies are obtained by the lumped mass approach. In regards to the rate of convergence of mode shapes and frequencies using consistent and lumped mass formulations, it has been shown (Tong *et al.* 1971) that, for the lowestorder elements, such as the constant-strain triangle or four-node quadrilateral elements, lumped and consistent mass matrices provide the same order of accuracy for second-order differential equations. No systematic procedure has been found to lump the mass that guarantees better convergence for higher order elements or for problems of higher order differential equations. Consistent mass matrix often gives higher-order approximation. In general, it is recommended to use lumped mass matrix in conjunction with low-order elements as it provides the same order of approximation.

For real world problems, such as vehicles, buildings, and bridges, the mass distribution is often concentrated in certain areas. One often uses a gross approximation for the stiffness description. In these cases, one should use a lumped-mass approach. An example where this concept has been used to advantage concerns the structural deformation of vehicles in a crash (Ros-settos and Weinstock 1974, Tong and Rossettos 1974). The vehicle can be conveniently divided into individual modules, each with its own stiffness and mass characteristics. The modules are connected by the standard finite-element methodology. Mass lumping is a clear choice for such problems (e.g., engine mass, etc.). One can solve Eq. (1) by modal analysis (see [Biblio. 18.7](#)).

We will discuss time integration methods. The accuracy is measured in terms of the truncation error. An integration scheme is said to be m th order accurate if the truncation error is of the order Δt^{m+1} .

Newmark β -Method (Newmark 1959). This is one of the most widely used integration methods for 2nd order differential equations. The Newmark β -method rewrites Eq. (1) in the approximate form:

$$(16) \quad \begin{aligned} \mathbf{M}_{n+1}\mathbf{a}_{n+1} + \mathbf{C}_{n+1}\mathbf{v}_{n+1} + \mathbf{K}_{n+1}\mathbf{q}_{n+1} &= \mathbf{F}_{n+1}, \\ \mathbf{q}_{n+1} &= \mathbf{q}_n + \Delta t \mathbf{v}_n + \Delta t^2 [(1 - 2\beta)\mathbf{a}_n + 2\beta\mathbf{a}_{n+1}] / 2, \\ \mathbf{v}_{n+1} &= \mathbf{v}_n + \Delta t [(1 - \gamma)\mathbf{a}_n + \gamma\mathbf{a}_{n+1}], \end{aligned}$$

where $(.)_n$ denotes the quantity at $t = t_n$ with \mathbf{q}_n , \mathbf{v}_n and \mathbf{a}_n being \mathbf{q}_n , $\dot{\mathbf{q}}_n$ and $\ddot{\mathbf{q}}_n$, respectively. In most structural problems, \mathbf{M} and \mathbf{C} are independent of time. This is not in the case of biological systems.

The following recursive formula for \mathbf{a}_{n+1} can be derived from Eq. (16)

$$(17) \quad (\mathbf{M} + \gamma \Delta t \mathbf{C} + \beta \Delta t^2 \mathbf{K})_{n+1} \mathbf{a}_{n+1} = \mathbf{F}_{n+1} - \mathbf{C}_{n+1} \hat{\mathbf{v}}_{n+1} - \mathbf{K}_{n+1} \hat{\mathbf{q}}_{n+1},$$

$$(18) \quad \mathbf{q}_{n+1} = \hat{\mathbf{q}}_{n+1} + \beta \Delta t^2 \mathbf{a}_{n+1} / 2, \quad \mathbf{v}_{n+1} = \hat{\mathbf{v}}_{n+1} + \gamma \Delta t \mathbf{a}_{n+1},$$

$$(19) \quad \hat{\mathbf{q}}_{n+1} = \mathbf{q}_n + \Delta t \mathbf{v}_n + \Delta t^2 (1 - 2\beta) \mathbf{a}_n / 2, \quad \hat{\mathbf{v}}_{n+1} = \mathbf{v}_n + \Delta t (1 - \gamma) \mathbf{a}_n.$$

To start the integration process, from the initial condition $\mathbf{q} = \mathbf{q}_0$ and $\mathbf{v} = \mathbf{v}_0$, one first calculates \mathbf{a}_0 from the first equation of Eq. (16)

$$\mathbf{M}_0 \mathbf{a}_0 = \mathbf{F}_0 - \mathbf{C}_0 \mathbf{v}_0 - \mathbf{K}_0 \mathbf{q}_0$$

and evaluates $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{v}}_1$ from Eq. (19). One then solves for \mathbf{a}_1 from Eq. (17), determines $\mathbf{q}_1, \mathbf{v}_1$ from Eq. (18), proceeds to determine $\mathbf{a}_2, \mathbf{q}_2$ and \mathbf{v}_2 of the next time increment.

An integration scheme is called *explicit* if \mathbf{a}_{n+1} can be obtained without solving the algebraic equation (17), otherwise the scheme is called *implicit*. Obviously if \mathbf{M} and \mathbf{C} are diagonal and $\beta = 0$ (\mathbf{K} is usually not diagonal), Eq. (17) is an explicit scheme. In practice one calls the scheme explicit as long as $\beta = 0$ even \mathbf{M} and \mathbf{C} are not diagonal. Many well-known integration schemes are special cases of the Newmark method. For example $\beta = 1/4$ and $\gamma = 1/2$ gives the unconditionally stable implicit trapezoidal rule; $\beta = 1/6$ and $\gamma = 1/4$ gives the linear implicit acceleration method; and $\beta = 0$ and $\gamma = 1/2$ is the explicit central difference method. The accuracy of the last two cases is of the order Δt^2 , said to be *second order accurate*.

An integration algorithm, always stable regardless the size of Δt , is said to be unconditionally stable. For some β, γ and Δt , the integrated solution grows exponentially, called *numerical instability*. These parameters also affect the accuracy of the solution. In practice one chooses proper β, γ and Δt to control both the accuracy and the stability of the algorithm. The stability conditions (Goudreau and Taylor 1972, Krieg and Key 1973) for the Newmark method with $\mathbf{C} = a\mathbf{M} + b\mathbf{K}$ are

$$\text{Unconditionally stable: } 2\beta \geq \gamma \geq 1/2$$

$$\text{Conditionally stable: } 2\beta < \gamma, \quad \omega \Delta t \leq \Omega_{crit},$$

where

$$\Omega_{crit} = \frac{\xi(\gamma - 1/2) + [\gamma/2 - \beta + \xi^2(\gamma - 1/2)^2]^{1/2}}{\gamma/2 - \beta}, \quad \xi = \frac{1}{2} \left(\frac{a}{\omega} + b\omega \right),$$

ω is the maximum undamped frequency of the system and ξ is the corresponding damping ratio. For $\gamma > 1/2$, one can use $\Omega_{crit} = (\gamma/2 - \beta)^{-1/2}$ as a conservative estimate when a realistic damping coefficient is not available. The maximum frequency of the system can be estimated from the maximum eigenvalues of individual elements (Tong 1970b). Hughes (1987) listed the maximum frequencies for a number of elements.

For explicit schemes, the time increment Δt required for stability is inversely proportional to the maximum frequency (Tong *et al.* 1971, Clough and Bathe 1972, Hughes 1987). One can use larger time step in the lumped mass approach since it usually has lower maximum frequency than that of the consistent mass formulation. This advantage is no longer clear when an implicit integration scheme is employed.

Multi-Step Method for First-Order Equations (Gear 1971). The method is used to integrate first order equations of the form

$$(20) \quad \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t).$$

Equation (1) can be converted into a first order equation as below:

$$\dot{\mathbf{y}} = \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{C} & \mathbf{K} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{y} \right) = \mathbf{f}(\mathbf{y}, t).$$

The integration is defined as

$$(21) \quad \sum_{i=0}^k [\alpha_i \mathbf{y}_{n+1-i} + \beta_i \Delta t \mathbf{f}(\mathbf{y}_{n+1-i}, t_{n+1-i})] = \mathbf{0},$$

where α 's and β 's are constant parameters. The method is explicit if $\beta_0 = 0$, otherwise implicit.

Many algorithms of practical interest take the form of Eq. (21). For example, with $k = 1$, $\alpha_0 = -\alpha_1 = 1$, $\beta_0 = \alpha$ and $\beta_1 = 1 - \alpha$, Eq. (21) becomes a one-step *trapezoidal method* in the form

$$(22) \quad \mathbf{y}_{n+1} = \mathbf{y}_n + \Delta t [\alpha \mathbf{f}(\mathbf{y}_{n+1}, t_{n+1}) + (1 - \alpha) \mathbf{f}(\mathbf{y}_n, t_n)].$$

The forward and backward difference methods correspond to $\alpha = 0$ (an explicit method) and $\alpha = 1$ (an implicit

method), respectively.

For a linear system with $\mathbf{f}(\mathbf{y}, t) = \mathbf{H}\mathbf{y} + \mathbf{F}$. The stability of the multi-step method for constant \mathbf{H} can be determined as follows: let

$$(23) \quad \mathbf{y}_j = \eta^j \mathbf{a}_\lambda,$$

where η is a parameter and \mathbf{a}_λ is the eigenvector of \mathbf{H} associated with the eigenvalue λ . Setting $\mathbf{F} = \mathbf{0}$. A substitution of Eq. (23) into Eq. (22) (the particular solution has no effect on stability) yields the stability equation

$$(24) \quad \sum_{i=1}^k (\alpha_i + \Delta t \lambda \beta_i) \eta^{n+1-i} = 0.$$

If $|\eta|_{\max}$ of all roots of Eq. (24) is less than one for all λ , the integration scheme is stable. Dahlquist (1963) showed that

- No explicit unconditionally stable multi-step method exists.
- No 3rd order accurate unconditionally stable multi-step method exists.
- The trapezoidal rule has the smallest error constant among the unconditionally stable multi-step methods of 2nd order accurate.

For one-step method, the stability equation gives

$$\eta = [1 + (1 - \alpha)\lambda\Delta t] / (1 - \alpha\lambda\Delta t).$$

We see that the backward difference method ($\alpha = 1$) is unconditionally stable if $\text{RI}(\lambda) < 0$ for all λ of \mathbf{H} , while the forward difference method ($\alpha = 0$) is stable only if $|1 + \lambda\Delta t| \leq 1$ for all eigenvalues.

Park (1975) proposed a second-order accurate 3-step implicit method, which is unconditionally stable with good accuracy in the low frequencies and strong dissipation in high frequencies. The Park method is defined by

$$\begin{aligned} k = 3, \quad \alpha_0 &= -1, \quad \alpha_1 = 1.5, \quad \alpha_2 = -0.6, \\ \alpha_3 &= 0.1, \quad \beta_0 = 0.6, \quad \beta_1 = \beta_2 = \beta_3 = 0. \end{aligned}$$

Multi-Step Method for Second-Order Equations. Consider a system

$$(25) \quad \ddot{\mathbf{y}} = \mathbf{f}_1(\mathbf{y}, t)\dot{\mathbf{y}} + \mathbf{f}_0(\mathbf{y}, t).$$

Most of structural dynamics equations can be put in this form. The multi-step method is defined by (Geradin 1974)

$$(26) \quad \sum_{i=0}^k [\alpha_i \mathbf{y}_{n+1-i} + \beta_i \Delta t \mathbf{f}_1(\mathbf{y}_{n+1-i}, t_{n+1-i}) \mathbf{y}_{n+1-i} + \gamma_i \Delta t^2 \mathbf{f}_0(\mathbf{y}_{n+1-i}, t_{n+1-i})] = 0.$$

One sees that the Newmark method is a two-step method. The stability properties of Eq. (26) can be examined in a similar fashion (Krieg 1973).

Generally, conditionally stable schemes require time steps of the order of the shortest vibration period of the structure. Thus one often is forced to use time steps much smaller than those needed for accuracy. Therefore, unconditionally stable algorithms may be preferable, especially if only the low-frequency response is of interest. In this case, it is desirable if the algorithm can effectively damp out the high-frequency modes. Assessment of numerical dissipation can be found in Hughes (1983).

There are many integration schemes such as the Wilson θ -method (Wilson 1968), Runge–Kutta predictor-corrector method and the α -method (Hilber *et al.* 1977). See [Biblio. 18.7](#) (e.g., Hughes 1987).

18.18. NUMERICAL INTEGRATION

Derivation of element matrices involves line, area, or volume integrals. With exception of simple elements exact integration are complicated. Numerical integration is essential in almost all practical applications. We shall discuss the principles of numerical integration and give the tables of numerical coefficients or weighting factors for commonly used schemes.

One-Dimensional Integration. If we approximate a function $u(x) \approx \sum_{j=1}^m h_j(x)q_j$ over $-1 < x < 1$, where q 's are the values of u at m selected locations and h 's are interpolation (shape) functions, then

$$\int_{-1}^1 u(x)dx \cong \sum_{j=1}^m W_j q_j ,$$

where ± 1 are the normalized integration limits and

$$W_j = \int_{-1}^1 h_j(x)dx .$$

There are many choices of m and the point locations. The *Gauss quadrature*, defined as follows, stands out as quite accurate for a fixed m . We approximate u as the m^{th} order *Hermite interpolation*,

$$I = \int_{-1}^1 u(\xi)d\xi = \sum_{j=1}^m W_j u(\xi_j) + \sum_{j=1}^m W'_j du(\xi_j)/d\xi ,$$

where

$$W_j = \int_{-1}^1 h_{2j-1}(\xi)d\xi , \quad W'_j = \int_{-1}^1 h_{2j}(\xi)d\xi ,$$

in which h 's are defined in Eq. (18.3:11) and W 's are called *weighting factor*. If the stations $\xi_1, \xi_2, \dots, \xi_m$ are chosen such that all W'_j vanish,

$$I = \int_{-1}^1 u(\xi)d\xi = \sum_{j=1}^m W_j u(\xi_j)$$

and the resulting formula is called the m^{th} order Gauss quadrature, which integrates a complete polynomial of degree $2m - 1$ exactly. The *accuracy* of the quadrature is $2m - 1$ degree. The error is of the order ε^{2m} . [Table 18.18:1](#) gives the stations and the weighting factors of the Gauss quadrature of different order. The integration of a one-dimensional function is now approximated as a summation of the products of the function's value at selected stations and their corresponding weighting factors.

Table 18.18:1. Stations and weighting factors of Gauss quadrature.

No. of stations	ξ_i (Station locations)	Weighting factors (W_j)	Order	Error
1	0	2	Linear	$O(\varepsilon^2)$
2	$\pm 1/\sqrt{3}$	1	Cubic	$O(\varepsilon^4)$
3	0 $\pm \sqrt{3/5}$	8/9 5/9	Quintic	$O(\varepsilon^6)$

Two- and Three-Dimensional Integrations. Multi-dimensional Gauss integration rules can be formed by successive application of the one-dimensional Gauss rule. In a 2-dimensional square, we have

$$I = \int_{-1}^1 \int_{-1}^1 u(\xi, \eta)d\xi d\eta \cong \sum_i \int_{-1}^1 W_i u(\xi_i, \eta)d\eta \cong \sum_i \sum_j W_i W_j u(\xi_i, \eta_j) ,$$

where W 's are those of one-dimensional integration. A m^{th} order Gauss quadrature can correctly integrate any term $\xi_i \eta_j$, $i, j \leq 2m - 1$. For Eq. (18.14:12), the integrand $\mathbf{B}^T \mathbf{D}_e \mathbf{B}^J$ has $p(1 + p)/2$ distinguished entries of functions of ξ and η for an element of p DOFs to be integrated separately. For an 8 DOFs element, it would require 36 separate integrations to determine \mathbf{k} .

In three-dimension cases, the integration rule has the similar form

$$I = \int_{-1}^1 \int_{-1}^1 u(\xi, \eta)d\xi d\eta \cong \sum_i \int_{-1}^1 W_i u(\xi_i, \eta)d\eta \cong \sum_i \sum_j W_i W_j u(\xi_i, \eta_j) ,$$

A m^{th} order Gauss quadrature can correctly integrate any term $\xi^i \eta^j \zeta^k$, $i, j, k \leq 2m - 1$. For Eq. (18.16:15), an 8-node hexahedron for 3-dimensional elasticity has 24 DOFs and 300 distinguished entries of $\mathbf{B}^T \mathbf{D}_e \mathbf{B}^J$. A 20-node hexahedron has 60 DOFs with 1891 distinguished entries. The required number of integrations increases rapidly for high order 3-dimensional elements. In practice, it is not necessary to use the same weighting stations in all directions of integration. Just choose enough Gauss stations according to need.

Iron (1971) designed a more efficient integration rule to integrate complete polynomials (of order $2m - 1$):

$$(1) \quad I = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 u(\xi, \eta, \zeta) d\xi d\eta d\zeta \cong Au(0, 0, 0) \text{ (1 term)}$$

$$+ B[u(-b, 0, 0) + u(b, 0, 0) + u(0, -b, 0) + \dots] \text{ (6 terms)}$$

$$+ C[u(-c, -c, -c) + u(c, -c, -c) + u(c, c, -c) + \dots] \text{ (8 terms)}$$

$$+ D[u(-d, -d, 0) + u(d, -d, 0) + u(d, 0, -d) + u(-d, 0, -d) + \dots] \text{ (12 terms).}$$

The weighting factors A, B, C, D and the corresponding coordinates are listed in [Table 18.18:2](#). From the truncation errors, we see that, to correctly integrate a given complete order polynomial, Irons formulae require less number of integration points than that of the Gauss quadrature.

Table 18.18:2. Weighting factors and coordinates for Eq. (18.18:1).

Number integration points	Coordinates	Weighting factors*	Order	Error
1		$A = 8$	Linear	$O(\varepsilon^2)$
6	$b = 1$	$B = 8/6$	Cubic	$O(\varepsilon^4)$
14	$b = 0.795822426$ $c = 0.758786911$	$B = 0.886426593$ $C = 0.335180055$	Quintic	$O(\varepsilon^6)$
27	$b = 0.848418001$ $c = 0.652816472$ $d = 1.106412899$	$A = 0.788073483$ $B = 0.499362002$ $C = 0.478508449$ $D = 0.032303742$	Hept	$O(\varepsilon^8)$

*unlisted weighting factors are zero

Problem 18.16. Derive the equivalence of the Irons formulae for square: Determine (a) the number of points needed for integration; (b) the weight factors: and (c) the corresponding coordinates of the integration points.

Numerical Integration for Triangles or Tetrahedrons. A triangle can be treated as a degenerated quadrilateral. Hence, the integration can be carried out using the Gauss quadrature. However, it is not advisable to do so because there is a singularity in mapping a triangle to a square, which causes lost in accuracy. An alternative is to apply the 1-dimensional approach directly to a triangular area

$$I = \int_0^1 \int_0^{1-\eta} u(\xi, \eta) d\xi d\eta \cong \sum_j W_j u(\xi_j, \eta_j).$$

The integration limits now involve an independent variable itself. The integration stations, weights and error orders of different formulae (Hammer *et al.* 1956, Felippa 1966, Cowper 1973) are given in [Table 18.18:3](#). Similarly one can derive the formulae for tetrahedrons given in [Table 18.18:4](#) (Hammer *et al.* 1956).

Table 18.18:3. Numerical integration stations and weights for triangles.

Figure	Integ. Points	Triangular coordinates	Weights	Order	Error
	a	1/3,1/3,1/3	1	Linear	$O(\varepsilon^2)$
	a b c	1/2, 1/2, 0 0, 1/2, 1/2 1/2, 0, 1/2	1/3 1/3 1/3	Quadratic	$O(\varepsilon^3)$
	a b c d	1/3, 1/3, 1/3 0.6, 0.2, 0.2 0.2, 0.6, 0.2 0.2, 0.2, 0.6	-27/48 25/48	Cubic	$O(\varepsilon^4)$
	a b c d e f g	1/3, 1/3, 1/3 $a_1, b_1, b_1 \}$ $b_1, a_1, b_1 \}$ $b_1, b_1, a_1 \}$ $a_2, b_2, b_2 \}$ $b_2, a_2, b_2 \}$ $b_2, b_2, a_2 \}$ $a_1 = 0.0597158717$ $b_1 = 0.4701420641$ $a_2 = 0.7974269853$ $b_2 = 0.1012865073$	0.250000000 0.1323941527 0.1259391805	Quintic	$O(\varepsilon^6)$

Required Order of Numerical Integration. Numerical integration consumes a significant amount of computer time, it is of interest to determine the minimum integration points required for a specific level of convergence. The finite element method approximates the functional to the order $n = 2(p - m)$, $n = 0, 2, 4, \dots$, where p is the order of complete polynomials of the shape functions and m the order of differentials in the functional. The minimum number of integration points needed to achieve such convergence depends on the numerical scheme used. Thus, an integration of accuracy order higher than the approximation of the shape functions will not change the order of convergence of the overall solution. In fact, using integration of higher order sometime makes the element stiffer and adversely affects the accuracy of the finite element solution.

Table 18.18:4. Numerical integration stations and weights for tetrahedrons.

Figure	Points	Tetrahedral coordinates	Weights	Order	Error
	a	1/4,1/4,1/4,1/4	1	Linear	$O(\varepsilon^2)$
	a b c d	$\alpha, \beta, \beta, \beta$ $\beta, \alpha, \beta, \beta$ $\beta, \beta, \alpha, \beta$ $\beta, \beta, \beta, \alpha$ $\alpha = 0.58541021$ $\beta = 0.13819660$	1/4 1/4 1/4 1/4	Quadratic	$O(\varepsilon^3)$
	a b c d e	1/4,1/4,1/4,1/4 1/3,1/6,1/6,1/6 1/6,1/3,1/6,1/6 1/6,1/6,1/3,1/6 1/6,1/6,1/6,1/3	-4/5 9/20 9/20 9/20 9/20	Cubic	$O(\varepsilon^4)$

18.19. PATCH TEST

We have presented elements with incompatible displacements. The general theory no longer assures the convergence of the solution. Irons (1966, 1966a) introduced the concept of *patch test* to examine the correctness of such elements. The test has been generalized to examine whether or not an element with nonstandard features such as *reduced integration* and *incompatible interpolation* converges. Even though compatible displacement, hybrid and mixed elements pass patch tests, the assumed variables may contain undesirable deformation modes. One can use the test to check the correctness of all element formulations and their implementation as Irons originally intended.

A *patch* is a small group of contiguous irregularly shaped elements. Figure 18.19:1 shows two typical 2-dimensional patches frequently used for bench-mark test (Robinson and Blackham 1979, Hughes 1987). An element passes the patch test if its shape functions can represent the linear deformations for a 2nd order system. For plane elasticity, a 2-dimensional 2nd order system, the linear deformations are simply

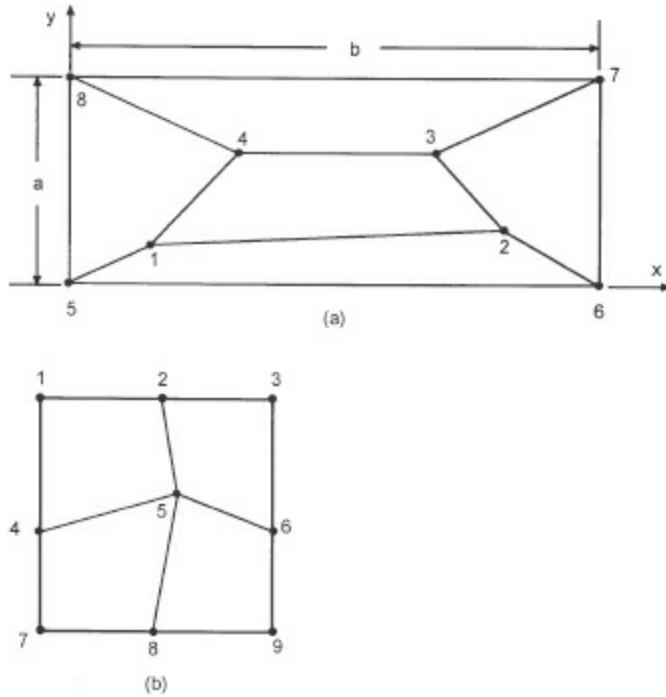


Fig. 18.19:1. A two-dimensional patch of element.

$$(1) \quad u = a_1 + a_2x + a_3y, \quad v = a_4 + a_5x + a_6y.$$

One version of the patch tests specifies at all exterior nodes (nodes 5–8 in Fig. 18.19:1(a) or nodes 1–4 and 6–9 in Fig. 18.19:1(b) for arbitrary a 's. One solves the six displacement boundary value problems to examine the deformations of the six linear states. The patch test is passed if, in the limit of the element size approaching zero, (a) the solutions give the collect values u, v at all interior nodes and (b) the strains within each element give $\partial u / \partial x = a_2, \partial u / \partial y = a_3, \partial v / \partial x = a_5$, and $\partial v / \partial y = a_6$.

For the strain to be exact, the interior nodal forces must be zero,

$$(2) \quad \sum_{\text{all patch elem.}} \int_{A_e} \mathbf{q}^T (\mathbf{d}_e \mathbf{h})^T \boldsymbol{\sigma} dA = 0 \quad \text{or} \quad \boldsymbol{\sigma}^T \left\{ \sum_{\text{all patch elem.}} \left(\int_{A_e} \mathbf{d}_e \mathbf{h} dA \right) \mathbf{q} \right\} = 0,$$

for any arbitrary q 's associated with the interior nodes of the patch and constant $\boldsymbol{\sigma}^T = [\sigma_{xx} \sigma_{yy} \sigma_{xy}]$. This is equivalent to requiring self-equilibrium at all interior nodes. If q_j and their associated shape functions h_j are defined within a single element only, from Eq. (2), we have

$$(3) \quad \int_{A_e} \mathbf{d}_e h_j dA = 0$$

for the element. This is a necessary condition for any arbitrary \mathbf{q} and their associated shape functions to pass the constant strain patch test.

Version 2 involves testing of rigid body motion and constant strain. For a patch in plane elasticity as shown in Fig. 18.19:2, first we specify

$$u = a_1 + a_2 y, \quad v = a_3 - a_2 x,$$

at all nodes. Equation (2) must be satisfied at all nodes for arbitrary a 's. We then impose enough nodal constraints to restrain the rigid body motion, and apply appropriate loads to the unconstrained exterior nodes to produce a constant stress state over the patch. The patch test passes if the finite element solution gives the correct constant stresses in all elements. For plane elasticity, the nodal forces are

$$\sum_{\text{all patch elem.}} \int_{\partial A_{\sigma e}} \mathbf{h}^T \begin{bmatrix} n_x \sigma_{xx} + n_y \sigma_{xy} \\ n_x \sigma_{xy} + n_y \sigma_{yy} \end{bmatrix} ds,$$

where σ 's are the stresses associated with the linear deformation

$$u = e_{xx}x + \alpha e_{xy}y, \quad v = (1 - \alpha)e_{xy}x + e_{yy}y,$$

where $0 < \alpha < 1$ is a constant, e strain, \mathbf{n} a unit normal to the exterior boundaries, and $\partial A_{\sigma e}$ the portion of ∂A_e coinciding with the exterior boundaries of the patch. For elements with straight edges, $n_x \sigma_{xx} + n_y \sigma_{xy}$ and $n_x \sigma_{xy} + n_y \sigma_{yy}$ are constant. The integration above is straightforward.

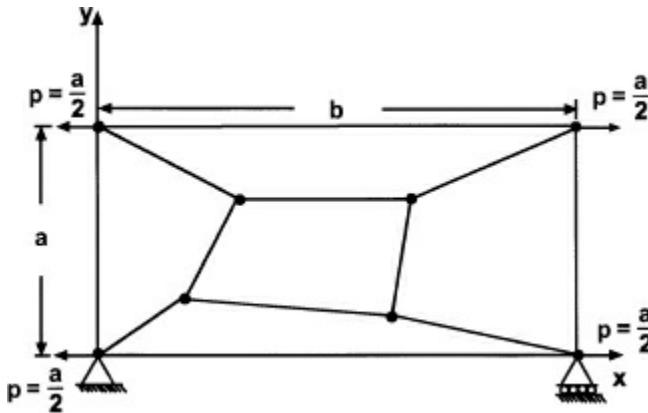


Fig. 18.19:2. Patch test with applied loads.

A patch test can be applied to a single element to determine whether the strains within the element are correctly evaluated from nodal displacements or forces. But it cannot determine whether the element is compatible with its adjacent elements. Single element patch test is less useful than multi-element patch test. However, a single element test is good to detect the existence of spurious zero energy modes, which are usually induced by approximate integration of element matrices to be discussed in more detail later.

We can similarly establish higher order patch tests, e.g., requiring that the shape functions exactly represent quadratic deformations. This is needed for fourth order equations such as bending of thin plates and shells, in which the curvature and twist are proportional to second derivatives of the out-ofplane displacement. Quadratic deformation represents only constant curvature and twist. One generally has to solve many more boundary-value problems for higher order patch test.

If the material is homogeneous, the patch test will be independent of element size for plane elasticity.

18.20. LOCKING-FREE ELEMENTS

The incorrect interpolation of displacements and their derivatives can cause the failure of patch tests. There are two sources of errors. Approximation for higher order functions occurs naturally since the finite element method uses mostly polynomials of limited order. Errors are also introduced through the nonlinearity of isoparametric coordinate transformation (Tong and Rossettos 1977, MacNeal 1994). For example, for an 8-node quadrilateral element, the shape functions contain only complete quadratic polynomials in ξ, η . However, the transformation

$$x = \sum_{j=1}^8 x_j h_j(\xi, \eta), \text{ etc.,}$$

given in Eq. (18.13:3) includes quadratic and higher order terms of ξ, η . Then the shape functions cannot exactly represent x, y, x^2, \dots

Interpolation failure has other consequences. A condition is excessive stiffness called locking, which often occurs in bending of incompressible or nearly incompressible materials due to spurious shear and dilatation (see [Biblio. 18.10](#)). Babuska (1990) identified the parameters characterizing locking and its effects on convergent rates.

Increasing the order of shape functions generally reduces the disorder due to interpolation errors and thus reduces the seriousness of locking or rids it completely. Unfortunately, this usually fails for elements involving nonlinear coordinate transformation. For example, the isoparametric transformation introduces quadratic interpolation errors for the 8-node quadrilateral element (Tong and Rossettos 1977). Barlow (1989) gave a comprehensive analysis of errors due to shape distortions of 8-node quadrilaterals and 20-node bricks.

Locking is the result of incorrect representation of some strain components. However, there are locations within the element where components are correct. Thus, locking can be eliminated by evaluating those strains only at the locations with correct values. This procedure is called *selectively-reduced integration*. If one reduces the number of integration points for all strains, the procedure is called *uniformly reduced integration*, which may introduce *spurious zero energy modes* from nonrigid body motion. Selectively-reduced integration can retain sufficient number of evaluations of strains to prevent spurious modes. (See [Biblio. 18.10](#)). Hybrid and mixed formulations are alternatives to circumvent locking. Their full potential has yet to be fully explored.

We shall illustrate shear and dilatation lockings by examples below.

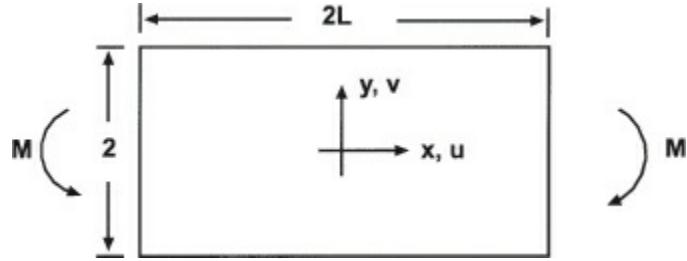


Fig. 18.20:1. Four-node rectangular element subject to in-plane bending.

Shear Locking. Consider a 4-node rectangle ($-L \leq x \leq L$, $-1 \leq y \leq 1$) subjected to bending as shown in [Fig. 18.20:1](#). The displacements are

$$(1) \quad u = xy, \quad v = -x^2/2,$$

which give a constant curvature along the x -axis. The corresponding strains, stresses, and strain energy for a *plane stress* element are

$$\begin{aligned} e_x &= u_{,x} = y, \quad e_y = v_{,y} = 0, \quad \gamma_{xy} = u_{,y} + v_{,x} = 0, \\ \sigma_x &= E_x y / (1 - \nu_{xy} \nu_{yx}), \quad \sigma_y = E_x \nu_{xy} y / (1 - \nu_{xy} \nu_{yx}), \quad \sigma_{xy} = 0, \\ W &= \int_{-1}^1 \int_{-L}^L \sigma^T \epsilon dxdy / 2 = 2E_x L / [3(1 - \nu_{xy} \nu_{yx})], \end{aligned}$$

where E_x is Young's modulus, and ν_{xy} and ν_{yx} are Poisson's ratios. The 4-node element representation of the same nodal displacements are

$$\begin{aligned} u &= xy, \quad v = -L^2/2, \\ e_x &= u_{,x} = y, \quad e_y = v_{,y} = 0, \quad \gamma_{xy} = u_{,y} + v_{,x} = x, \\ \sigma_x &= E_x y / (1 - \nu_{xy} \nu_{yx}), \quad \sigma_y = E_x \nu_{xy} y / (1 - \nu_{xy} \nu_{yx}), \quad \sigma_{xy} = G_{xy} x, \\ W &= \int_{-1}^1 \int_{-L}^L \sigma^T \epsilon dxdy / 2 = 2L [E_x / (1 - \nu_{xy} \nu_{yx}) + G_{xy} L^2] / 3, \end{aligned}$$

where G_{xy} is the shear modulus. Obviously, the shear strain, the shear stress and the strain energy are not correct. The ratio of the strain energy of the finite element solution to the exact one is $1 + G_{xy} (1 - \nu_{xy} \nu_{yx}) L^2 / E_x$, which represents the bending stiffness ratio of finite element model to the exact solution. The ratio is large even for a moderately slender element (moderately large L) and/or if G_{xy} / E_x is large. The high element stiffness is mainly from the incorrect shear strain. The phenomenon is called *shear locking*, which precludes the use of regular 3-D displacement elements to represent thin plates or shells by reducing the thickness of the element. This is due to the characteristics that the maximum displacement ratio v/u is the same order as the slender ratio of the element. The situation is further exacerbated if G_{xy} / E_x is large.

An effective remedy to shear locking is to use the value of γ_{xy} at $x = y = 0$ for all integration points (Doherty *et al.* 1969) to give the correct shear. This technique is called selectively-reduced-integration because, in effect, we employ a single point to evaluate the shear strain energy and the 2×2 or higher order Gauss rule to evaluate e_x and e_y . If γ_{xy} is

coupled to e_x and e_y , in evaluating $(D_{13}e_x + D_{23}e_y)\gamma_{xy}$, one would just use the value of γ_{xy} at the center. Another remedy is to use nonconforming element by adding a deformation proportional to x^2 (Bazeley *et al.* 1966, Taylor *et al.* 1976) in the y -direction or use the hybrid formulation to avoid the spurious shear strain (Razzaque 1973).

The 8-node rectangle also locks, but to a lesser degree. The bending deformation is

$$u = x^2y, \quad v = -x^3/3, \quad e_x = 2xy, \quad e_y = \gamma_{xy} = 0,$$

and that of the finite element model is

$$u = x^2y, \quad v = -L^2x/3, \quad e_x = 2xy, \quad e_y = 0, \quad \gamma_{xy} = x^2 - L^2/3.$$

The finite element representation gives correct shear only at the integration stations ($\pm 1/\sqrt{3}$) of the 2×2 Gauss rule, thus locking can be removed by such integration (Zienkiewicz *et al.* 1971) for all strains. The reduced integration introduces rank deficiency. The situation can be corrected by using the 2×2 integration rule for the shear and the 3×3 rule for bending. If the shear couples with the tensile strains, one has to extrapolate the shear strain from the 2×2 stations to the 3×3 ones ($\xi_i = 0, \pm\sqrt{0.6}$). Unfortunately the extrapolation makes the element fail the patch test. However, it still provides good results in many practical applications.

An 8-node brick element locks due to incorrect shear when the dimension in the bending direction is much less than the others. It can be avoided by using one-point integration for the shear strain energy. The constant strain triangle experiences interpolation error for all quadratic or higher order displacements. Locking in in-plane bending making the element useless for such case. A 6-node triangular element does not lock for any quadratic deformation because it correctly interpolates such displacements. However, for elements with curve sides, locking occurs due to errors in the nonlinear isoparametric transformation.

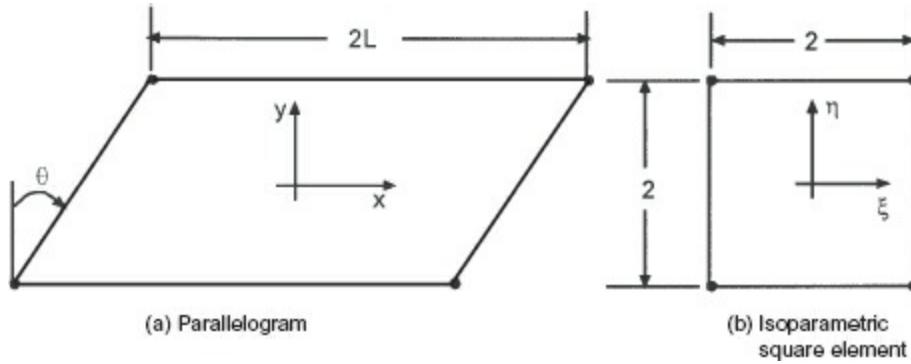


Fig. 18.20:2. Parallelogram element.

Reduced integration generally fails to fix the locking of distorted elements. For example, consider the parallelogram shown in Fig. 18.20:2 with the displacement field (MacNeal 1994)

$$u = xy, \quad v = -x^2/2, \quad e_x = y, \quad e_y = \gamma_{xy} = 0.$$

The transformation to the natural coordinates gives

$$(2) \quad \begin{aligned} x &= L\xi + \eta \tan \theta, \quad y = \eta \\ u &= L\xi\eta + \eta^2 \tan \theta, \quad v = -(L\xi + \eta \tan \theta)^2/2, \\ e_x &= \eta, \quad e_y = \tan \theta(\eta \tan \theta - L\xi), \quad \gamma_{xy} = L\xi - 2\eta \tan \theta. \end{aligned}$$

Clearly γ_{xy} and e_y are in error and will lead to locking if L and/or $L \tan \theta$ are large compare to one. If we evaluate both γ_{xy} and e_y at the origin to eliminate the error, the element would have no stiffness in in-plane bending. Similar results can be shown for a 4-node trapezoid.

Mixed Formulation. An alternative to reduced integration is to use mixed formulation. We separate the functional of Eq. (18.14:6) into the distortion energy Π_{de} , the shear energy Π_{se} , and the external work Π_{fe} as

$$(3) \quad \Pi_e = \Pi_{de} + \Pi_{se} - \Pi_{fe},$$

where

$$\Pi_{de} = \int_{A_e} E(u_x^2 + v_y^2 + 2\nu u_x v_y) dx dy / [2(1-\nu)]^2,$$

$$(4) \quad \begin{aligned} \Pi_{de} &= \int_{A_e} E(u_x^2 + v_y^2 + 2\nu u_{,x} v_{,y}) dx dy / [2(1-\nu)]^2, \\ \Pi_{se} &= \int_{A_e} G(u_{,y} + v_{,x})^2 dx dy / 2, \\ \Pi_{fe} &= \int_{A_e} (ub_x + vb_y) dx dy + \int_{\partial A_{\sigma e}} (u\bar{T}_x + v\bar{T}_y) ds. \end{aligned}$$

Consider a $2l \times 2h$ 4-node rectangular element and write the displacements and γ_{xy} in terms of natural coordinates ($x = l\xi$, $y = h\eta$) as

$$(5) \quad \begin{aligned} u &= a_1 + a_2\xi + a_3\eta + a_4\xi\eta, & v &= b_1 + b_2\xi + b_3\eta + b_4\xi\eta, \\ \gamma_{xy} &= a_3/h + b_2/l + a_4\xi/h + b_4\eta/l, \\ a_1 &= (u_1 + u_2 + u_3 + u_4)/4, & a_2 &= (-u_1 + u_2 + u_3 - u_4)/4, \\ a_3 &= (-u_1 - u_2 + u_3 + u_4)/4, & a_4 &= (u_1 - u_2 + u_3 - u_4)/4, \end{aligned}$$

in which u 's are the nodal values of u . Similarly one can express b 's in terms of v 's. The shear energy of the element is

$$(6) \quad \Pi_{se} = 2Ghl[(a_3/h + b_2/l)^2 + (a_4/h)^2/3 + (b_4/l)^2/6].$$

In the limit $l/h \rightarrow \infty$ (high aspect ratio), the shear energy approaches zero ($\Pi_{se} \rightarrow 0$). This is equivalent to forcing

$$(7) \quad a_3/h + b_2/l \rightarrow 0,$$

$$(8) \quad a_4 \rightarrow 0, \quad b_4 \rightarrow 0.$$

These three conditions together over constrain the system and lead to shear locking. Prathap (1993) argued that a constraint involving all related field variables, such as a_3 and b_2 from u and v in Eq. (7), can be satisfied in the limit and is said to be *field-consistent*. The constraints in Eq. (8) involving only one field variable incorrectly constrains the deformation and causes locking, and is called *field-inconsistent representation*.

We can remove the spurious constraints using the mixed or hybrid formulation. Replace Π_{se} by the Reissner functional [Eq. (10.10:9)]

$$(9) \quad \Pi_{se} = \int_{A_e} G[\tilde{\gamma}_{xy}(u_{,y} + v_{,x}) - \tilde{\gamma}_{xy}^2/2] dx dy,$$

in which $\tilde{\gamma}_{xy}$ is a new independent field variable for the element to be determined from the stationary condition of Π_{se} with respect to $\tilde{\gamma}_{xy}$,

$$(10) \quad \delta\Pi_{se} = \int_{A_e} G(u_{,y} + v_{,x} - \tilde{\gamma}_{xy}) \delta\tilde{\gamma}_{xy} dx dy = 0.$$

Judicious selection of the shear strain $\tilde{\gamma}_{xy}$ can remove the spurious constraint(s) and alleviate locking. An element with field-consistent $\tilde{\gamma}_{xy}$ is said to be *fieldconsistent*.

4-Node Field-Consistent Rectangular Element. For a 4-node rectangle, assuming constant $\tilde{\gamma}_{xy}$ within the element, from Eq. (10) and (5) one obtain

$$(11) \quad \tilde{\gamma}_{xy} = a_3/h + b_2/l = [(u_3 + u_4 - u_1 - u_2)/h + (v_3 - v_4 - v_1 + v_2)/l]/4.$$

A substitution of $\tilde{\gamma}_{xy}$ into Eq. (9) yields

$$(12) \quad \Pi_{se} = Ghl[(-u_1 - u_2 + u_3 + u_4)/h + (-v_1 + v_2 + v_3 - v_4)/l]^2/8.$$

Using Π_{se} above, together with Π_{de} and Π_{fe} of Eq. (4), one can drive the element matrices. Equation (12) is equivalent to evaluating Π_{se} by one point. The modified element performs well for $l/h \gg 1$ in modeling bending of thin beams. One should use *field-consistent* $\tilde{\gamma}_{xy}$ to calculate the stress since γ_{xy} derived from the assumed displacements [Eq. (5)] often shows rapid linear variation within the element. Wilson's 4-node incompatible rectangular element discussed in Sec. 18.14 is field-consistent.

Generally field-inconsistent quadrilateral elements lock badly in modeling flexure of thin beams. Optimum solution has yet to be found. Wilson's modified 4-node incompatible element ([Sec. 18.14](#)) provides satisfactory solution. *Field-consistency does not guarantee locking-free* as the shear strains of constant triangle and tetrahedron are field-consistent and yet lock in thin beams, plate and shell applications due to insufficient deformation modes.

8-Node Field-Consistent Rectangular Element. For 8-node elements, the displacements and the shear strain are

$$(13) \quad \begin{aligned} u &= a_1 + a_2\xi + a_3\eta + a_4\xi\eta + a_5\xi^2 + a_6\eta^2 + a_7\xi^2\eta + a_8\xi\eta^2, \\ v &= b_1 + b_2\xi + b_3\eta + b_4\xi\eta + b_5\xi^2 + b_6\eta^2 + b_7\xi^2\eta + b_8\xi\eta^2, \\ \gamma_{xy} &= a_3/h + b_2/l + (a_7/h + b_8/l)/3 + (a_4/h + 2b_5/l)\xi + (2a_6/h + b_4/l)\eta \\ &\quad + 2(a_8/h + b_7/l)\xi\eta + a_7(\xi^2 - 1/3)/h + b_8(\eta^2 - 1/3)/l, \end{aligned}$$

where a 's and b 's are in terms of the nodal values of u and v , and γ_{xy} is in terms of Legendre polynomials. The spurious constraints are

$$(14) \quad a_7 \rightarrow 0, \quad b_8 \rightarrow 0.$$

Using the orthogonal properties of Legendre polynomials, one can show that the field-consistent shear strain is

$$(15) \quad \begin{aligned} \tilde{\gamma}_{xy} &= a_3/h + b_2/l + (a_7/h + b_8/l)/3 + (a_4/h + 2b_5/l)\xi \\ &\quad + (2a_6/h + b_4/l)\eta + 2(a_8/h + b_7/l)\xi\eta, \end{aligned}$$

which is also the field-consistent shear of the 9-node rectangular element. The shear stress calculated from $\tilde{\gamma}_{xy}$ is usually smoother than that from γ_{xy} in flexure of thin beams (Prathap 1993).

8-Node Field-Consistent Brick Element ([Fig. 18.15:2](#)). In 3-dimensional elasticity, the shear strain functional are

$$(16) \quad \Pi_{se} = \int_{V_e} \frac{G}{2} [(u_{,y} + v_{,x})^2 + (v_{,z} + w_{,y})^2 + (w_{,x} + u_{,z})^2] dx dy dz \quad (\text{disp. model}),$$

$$(17) \quad \Pi_{se} = \int_{V_e} G [\tilde{\gamma}_{xy}(u_{,y} + v_{,x}) + \tilde{\gamma}_{yz}(v_{,z} + w_{,y}) + \tilde{\gamma}_{zx}(w_{,x} + u_{,z}) - (\tilde{\gamma}_{xy}^2 + \tilde{\gamma}_{yz}^2 + \tilde{\gamma}_{zx}^2)/2] dx dy dz \quad (\text{mixed model}).$$

For an 8-node $2l \times 2h \times 2d$ brick, the displacement and the shear strain are:

$$(18) \quad \begin{aligned} u &= a_1 + a_2\xi + a_3\eta + a_4\zeta + a_5\xi\eta + a_6\eta\zeta + a_7\zeta\xi + a_8\xi\eta\zeta, \\ v &= b_1 + b_2\xi + b_3\eta + b_4\zeta + \dots, \quad w = c_1 + c_2\xi + c_3\eta + c_4\zeta + \dots, \\ \gamma_{xy} &= (a_3 + a_5\xi + a_6\zeta + a_8\xi\zeta)/h + (b_2 + b_5\eta + b_7\zeta + b_8\eta\zeta)/l, \\ \gamma_{yz} &= (b_4 + b_6\eta + b_7\zeta + b_8\eta\zeta)/d + (c_3 + c_6\zeta + c_5\xi + c_8\xi\zeta)/h, \\ \gamma_{zx} &= (c_2 + c_7\zeta + c_5\eta + c_8\zeta\eta)/l + (a_4 + a_7\xi + a_6\eta + a_8\xi\eta)/d, \end{aligned}$$

where a 's, b 's and c 's are related to the nodal values of u , v and w , respectively. In the limit of $\gamma_{xy} \rightarrow 0$ in bending of slender beams, all polynomial coefficients vanish. The constraints

$$a_5 \rightarrow 0, \quad a_8 \rightarrow 0, \quad b_5 \rightarrow 0, \quad b_8 \rightarrow 0,$$

are spurious and can cause locking. Similarly, by considering γ_{yz} or γ_{zx} , one obtains the spurious constraints,

$$c_5 \rightarrow 0, \quad c_8 \rightarrow 0.$$

It can be shown that the field-consistent shear strains are

$$(19) \quad \begin{aligned} \tilde{\gamma}_{xy} &= (a_3 + a_6\zeta)/h + (b_2 + b_7\zeta)/l, \\ \tilde{\gamma}_{yz} &= (b_4 + b_7\xi)/d + (c_3 + c_5\xi)/h, \\ \tilde{\gamma}_{zx} &= (c_2 + c_5\eta)/l + (a_4 + a_6\eta)/d. \end{aligned}$$

An alternative to reduce locking is to use incompatible elements:

$$(20) \quad u = a_1 + a_2\xi + a_3\eta + a_4\zeta + a_5\xi\eta + a_6\eta\zeta + a_7\zeta\xi + a_8\xi\eta\zeta + a_9(1 - \eta^2) + a_{10}(1 - \zeta^2) + a_{11}(1 - \xi^2),$$

and similar for v and w . The last three terms are added for mitigating dilatation locking. The displacement in Eq. (20) is incompatible but gives field-consistent shear and dilatation strains.

27-Node Brick Element. The 20-node brick element performs poorly in slender beam, thin plate and thin shell applications. It is difficult to find the field-consistent strains that will make the element robust in those applications. For a 27-node brick as shown in Fig. 18.15:2, the displacements are triquadratic functions of the natural coordinates ξ , η , ζ and so are the derived shear strains. It can be shown that the field-consistent shear strains are bilinear in ξ , η and quadratic in ζ for $\tilde{\gamma}_{xy}$, bilinear in η , ζ and quadratic in ξ for $\tilde{\gamma}_{yz}$, and so on. Similar to Eq. (13), if γ_{xy} is expanded in terms of the Legendre polynomials of ξ , η , from the orthogonal properties of the polynomials, $\tilde{\gamma}_{xy}$ is in the form similar to Eq. (15), which linear in ξ , η . Similarly one can derive $\tilde{\gamma}_{yz}$ and $\tilde{\gamma}_{zx}$. The proof is left for the readers. The field-consistent $\tilde{\gamma}$'s in Eq. (17) is equivalent to evaluating the shear strain energy in Eq. (16) with the $2 \times 2 \times 3$ integration rule for γ_{yz} , the $2 \times 3 \times 2$ rule for γ_{yz} and so on. Again stresses calculated from the field-consistent $\tilde{\gamma}$'s are smoother.

If the edge and the surface nodes of the brick element (nodes 9 to 26 in Fig. 18.15:2) are not located at the middle of the edges or the center of the sides, the isoparametric transformation of the coordinates involves nonlinear mapping and $\tilde{\gamma}$'s are no longer field-consistent. Establishing proper $\tilde{\gamma}$'s becomes formidable. Using the $2 \times 2 \times 2$ integration rule for all γ 's in Eq. (16) is simple and practical even at the risk of introducing zero energy modes, which in some rare occasions leads to poor results. Stresses should be evaluated at the integration points and extrapolated to the nodes.

Since locking is a result of interpolation error, higher order elements usually do not lock or lock to a lesser extent. Selectively-reduced integration does wonder in removing shear locking for rectangular elements but is ineffective for general quadrilaterals as isoparametric mapping introduces spurious strains (MacNeal 1990). One often resorts to the field-consistent or incompatible approach to reduce locking. The limit of the efficacy of reduced integration to rectangles may not be serious in practice. Normally it is the spurious out-of-plane shear that causes locking in bending. In practice the sides normal to the midplane are generally rectangular or nearly rectangular for which shear locking can be effectively removed by selectively-reduced integration. We shall consider the general distorted brick elements of thin shells in the next chapter.

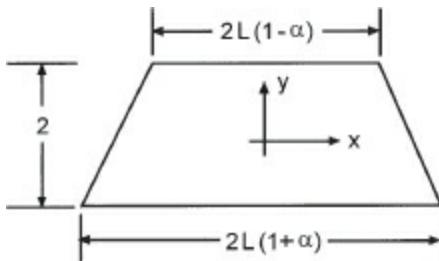


Fig. P18.17. Trapezoid.

Problem 18.17. Consider a trapezoid as shown in the figure with the displacements given in Eq. (18.20:1). The transformation of Cartesian coordinates to the natural coordinates is

$$x = L\xi(1 - \alpha\eta), \quad y = \eta, \quad \text{where } 0 \leq \alpha \leq 1.$$

(a) Find the displacements and strains in terms of the natural coordinates. Show that locking cannot be eliminated by reduced integration. (b) Find the ratio of the strain energy of the finite element model to the correct energy of the element.

Dilatation Locking (MacNeal 1994). The problem is commonly associated with incompressible or nearly incompressible materials. Consider the bending of a rectangle in *plane strain* with

$$(21) \quad \begin{aligned} u &= xy, \quad v = -x^2/2 - \nu y^2/[2(1 - \nu)], \quad e_x = y, \quad e_y = -\nu y/(1 - \nu), \\ \gamma_{xy} &= 0, \quad \sigma_x = E y / (1 - \nu^2), \quad \sigma_y = \sigma_{xy} = 0, \\ e &= e_x + e_y + e_z = (1 - 2\nu)y/(1 - \nu), \end{aligned}$$

$$W = \int_{-1}^1 \int_{-L}^L \boldsymbol{\sigma}^T \mathbf{e} dx dy / 2 = 2EL/[3(1 - \nu^2)].$$

The dilatation e tends to zero as $\nu \rightarrow 1/2$.

For a 4-node rectangular element, the deformation is represented by

$$\begin{aligned} u &= xy, \quad v = -L^2/2 - \nu/[2(1-\nu)], \quad e_x = y, \quad e_y = 0, \quad \gamma_{xy} = x, \\ \sigma_x &= \frac{E(1-\nu)y}{(1+\nu)(1-2\nu)}, \quad \sigma_y = \frac{E\nu y}{(1+\nu)(1-2\nu)}, \quad \sigma_{xy} = \frac{Ex}{2(1+\nu)}, \\ e &= e_x + e_y + e_z = y, \\ W_e &= \int_{-1}^1 \int_{-L}^L \boldsymbol{\sigma}^T \mathbf{e} dx dy / 2 = 2EL/[(1-\nu)/(1-2\nu) + L^2/2]/[3(1+\nu)], \end{aligned}$$

in which the dilatation is independent of Poisson's ratio. The error magnifies the strain energy by a factor proportional to $1/(1-2\nu)$ as $\nu \rightarrow 1/2$. The error in shear also contributes to the high stiffness if L is large.

Nagtegaal *et al.* (1974) introduced a strain-projection scheme to treat the nearly incompressible case. Write the strain-displacement matrix \mathbf{B} as defined in Eq. (18.16:11) in the form

$$\mathbf{B} = [\mathbf{B}_1^{dev} + \mathbf{B}_1^{dil} \quad \mathbf{B}_2^{dev} + \mathbf{B}_2^{dil} \quad \dots \quad \mathbf{B}_p^{dev} + \mathbf{B}_p^{dil}],$$

where p is the number of nodes of the element, and B^{dev} , B^{dil} denote the deviation and dilatation parts of \mathbf{B} 's, respectively, associated with node i :

$$(23) \quad \mathbf{B}_i^{dev} = \frac{1}{3} \begin{bmatrix} 2B_{xi} & -B_{yi} & -B_{zi} \\ -B_{xi} & 2B_{yi} & -B_{zi} \\ -B_{xi} & -B_{yi} & 2B_{zi} \\ B_{yi} & B_{xi} & 0 \\ 0 & B_{zi} & B_{yi} \\ B_{zi} & 0 & B_{xi} \end{bmatrix}, \quad \mathbf{B}_i^{dil} = \frac{1}{3} \begin{bmatrix} B_{xi} & B_{yi} & B_{zi} \\ B_{xi} & B_{yi} & B_{zi} \\ B_{xi} & B_{yi} & B_{zi} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_{\alpha i} = \partial h_i / \partial \alpha, \quad \alpha = x, y, z.$$

For nearly incompressible materials, one approximates \mathbf{B}_i^{dil} by its value at the center of or its average value over the element. One may also approximate \mathbf{B}_i^{dil} by interpolating the values of \mathbf{B}_i^{dil} at the Gauss points of a reduced integration rule. Unfortunately, these techniques cannot eliminate dilatation locking completely.

An alternative to remove dilatation locking is to use the mixed or hybrid formulation. Separate the strain energy Π_{de} of Eq. (4) into

$$\Pi_{de} = \Pi_{ve} + \Pi_{ae},$$

where Π_{ae} and Π_{ve} are the strain energies associated with the deviatoric strain and the dilatation $e (= u_{,x} + v_{,y} + w_{,z})$, respectively. We have

$$(24a) \quad \Pi_{ae} = \frac{1}{2} \int_{V_e} \frac{E}{1+\nu} [(u_{,x} - \frac{e}{3})^2 + (v_{,y} - \frac{e}{3})^2 + (w_{,z} - \frac{e}{3})^2] dx dy dz,$$

$$(24b) \quad \Pi_{ve} = \int_{V_e} E e^2 / [6(1-2\nu)] dx dy dz \quad (\text{displacement model}),$$

$$(24c) \quad \Pi_{ve} = \int_{V_e} E(\tilde{e}e - \tilde{e}^2/2) / [3(1-2\nu)] dx dy dz \quad (\text{mixed model}),$$

in which \tilde{e} is new independent field variable. We can determine \tilde{e} from the stationary condition of Eq. (24c) with respect to \tilde{e} ,

$$(25) \quad \delta \Pi_{ve} = \int_{V_e} E(e - \tilde{e}) \delta \tilde{e} / [3(1-2\nu)] dx dy dz = 0.$$

8-Node Brick. For an 8-node element, from Eq. (18), we obtain

$$e = a_2/l + b_3/h + c_4/d + (b_5/h + c_7/d)\xi + (a_5/l + c_6/d)\eta + (b_6/h + a_7/l)\zeta + a_8\eta\zeta/l + b_8\zeta\xi/h + c_8\xi\eta/d.$$

As the material approaches incompressible, all coefficients of the equation above tend to zero. Following the field-consistent argument, we conclude that, of all the constraints, only

$$a_2/l + b_3/h + c_4/d \rightarrow 0$$

is genuine while others are spurious. Assuming constant \tilde{e} , from Eq. (25), we find the field-consistent dilatation,

$$\tilde{e} = a_2/l + b_3/h + c_4/d,$$

which is in terms of the nodal values of u , v and w . A substitution of \tilde{e} into Eq. (24c) yields

$$\Pi_{ve} = 4Elhd(a_2/l + b_3/h + c_4/d)^2/[3(1 - 2\nu)].$$

This is equivalent to evaluating the dilatation energy in Eq. (24b) with one-point integration. Using Π_{ve} above, together with Π_{ae} of Eq. (24a) and Π_{se} of Eq. (16) or (17), we can establish the element stiffness matrix in terms of the nodal values of u , v and w .

This element is able to deform in nearly incompressible manner. However, this simple dilatation is not sufficient to represent flexure action, which involves linear strain variation. A remedy is to introduce in the incompatible displacements with bubble functions such as the last three terms in Eq. (20). The dilatation derived from the assumed displacements is field-consistent and can be used directly in Eq. (24b) to construct the element stiffness matrix. Such element is free from dilatation locking and can represent bending.

27-Node Brick. The dilatation of a 27-node brick is a triquadratic function of the natural coordinates. It can be shown that the constraints associated with ξ^2 , η^2 and ζ^2 are spurious. Thus the field-consistent dilatation is simply a trilinear function of ξ , η , ζ . Expanding e in terms of Legendre polynomials and dropping all terms proportional to $1 - \xi^2/3$, $1 - \eta^2/3$ and $1 - \zeta^2/3$ will give the field-consistent \tilde{e} .

More on incompressible and nearly incompressible materials will be discussed in [Sec. 19.6](#). For distorted bricks, field-consistent dilatation is yet to be determined.

Problem 18.18. Use the displacement field in Eq. (18.20:21) to examine the dilatation locking of a 4-node rectangular element in plane stress.

Problem 18.19. For 27-node $2l \times 2h \times 2d$ brick elements, the displacements can be written in the form

$$u = \sum_{i,j,k=0}^2 a_{ijk}\xi^i\eta^j\zeta^k, \quad v = \sum_{i,j,k=0}^2 b_{ijk}\xi^i\eta^j\zeta^k, \quad w = \sum_{i,j,k=0}^2 c_{ijk}\xi^i\eta^j\zeta^k,$$

where a' , b' , c 's are constants in terms of the nodal values of u , v and w . Derive the field-consistent shear strains. Hint:

$$\begin{aligned} \gamma_{xy} &= \sum_{i,j,k=0}^2 (a_{ijk}j\xi^i\eta^{j-1}/h + b_{ijk}i\xi^{i-1}\eta^j/l)\zeta^k \\ &= \sum_{k=0}^2 \sum_{j=1}^2 j\zeta^k\eta^{j-1} [a_{0jk} + a_{2jk}/3 + a_{1jk}\xi + a_{2jk}(\xi^2 - 1/3)]/h + \dots \end{aligned}$$

18.21. SPURIOUS MODES IN REDUCED INTEGRATION

MacNeal (1994) defined a working definition of reduced integration. If p is the degree of complete polynomials of the shape functions, the Gauss quadrature of $p+1$ or more points in each direction is full and the that with p points or less in a direction is reduced. Reduced order of integration has many benefits. It reduces computing time as the number of multiplications is proportional to the number of integration points. In the last section, we showed that reduced integration is a potential remedy for many locking problems resulting from interpolation failure. Barlow (1968, 1976) noticed that strains computed at the reduced-Gauss points in rectangular elements were more accurate than those at other points. Unfortunately, reduced integration has its shortcomings. For one, it is less accurate. More seriously, it may introduce *spurious zero energy modes*, which can destroy the accuracy of the finite element solution.

The spurious modes are the eigenvectors of the stiffness matrix associated with zero eigenvalue but nonzero strains. They are distinguished from rigid body motions. Strains are evaluated at integration points. A non-zero strain gives zero strain energy if the strain is zero at the integration point. Thus, the number of strain evaluations at all integration points represents the number of constraints for the occurrence of spurious modes. Let S_e be the number of strain evaluations,

which equal the number of strain components times the total number of integration points, and S_s be the number of independent strain states provided by the nodal displacements. Spurious modes occur if $S_e < S_s$. The difference $|S_s - S_e|$ is the maximum number of spurious modes that can exist in the element. For plane elasticity, there are 3 strains. Thus $S_e = 3p^2$ for the $p \times p$ integration and $S_e = 2(p+1)^2 + p^2$ for $p \times p$ integration of shear and $(p+1) \times (p+1)$ integration of direct strains. For 3-dimensional elasticity, there are 6 strain components and $S_e = 6p^3$ for $p \times p \times p$ integration.

In the meantime, S_s equals to the total number of DOFs minus the number of rigid body modes. The 9-node element in plane elasticity has 18 DOFs and 3 rigid body modes. Therefore, $S_s = 15$. If we use the 2×2 reduced integration rule, $S_e = 12$. Then, there can be 3 spurious zero energy modes.

Spurious Modes in Quadrilateral Elements. For a 4-node plane quadrilateral, $S_s = 5 (= 8 - 3)$. The 2×2 Gauss quadrature has 12 strain evaluations so that there is no spurious mode. On the other hand, $S_e = 3$ for one-point integration, the element has two spurious modes, $u = \xi\eta$, $v = 0$ and $u = 0$, $v = \xi\eta$. Similarly, for an 8-node brick with one integration point, there are 12 spurious modes of which one of the displacements u , v or w equals $\xi\eta$, $\eta\xi$, $\zeta\xi$, or $\xi\eta\zeta$ while others are zero.

We can find the spurious modes for higher order Lagrange elements under the $p \times p$ Gauss quadrature rule. Consider a 2-dimensional displacement field in the form

$$(1) \quad u = f_p(\xi)f_p(\eta), \quad v = 0$$

for a $(p+1)^2$ -node quadrilateral, where f_p are Legendre polynomials of degree p vanishing at the p Gauss quadrature points. Thus u , v and the strains vanish at all the $p \times p$ Gauss points.

$$\begin{aligned} e_x &= u_{,x} = f'_p(\xi)f_p(\eta)\xi_{,x} + f_p(\xi)f'_p(\eta)\eta_{,x}, \quad e_y = v_{,y} = 0, \\ \gamma_{xy} &= u_{,y} + v_{,x} = f'_p(\xi)f_p(\eta)\xi_{,y} + f_p(\xi)f'_p(\eta)\eta_{,y}. \end{aligned}$$

The displacement field given in Eq. (1) is a spurious mode. The other spurious mode is given by interchanging u and v of Eq. (1).

Another type of spurious mode is given by

$$(2) \quad \begin{aligned} u &= f'_p(\xi)f_p(\eta)\xi_{,x} - f_p(\xi)f'_p(\eta)\eta_{,x}, \\ v &= f'_p(\xi)f_p(\eta)\xi_{,y} - f_p(\xi)f'_p(\eta)\eta_{,y}, \end{aligned}$$

provided that the isoparametric mapping is linear, i.e., ξ_x , ξ_y , η_x and η_y are constant, the case of rectangle or parallelogram. For example, for an 8-node rectangle ($p = 2$), $\xi_y = \eta_x = 0$ and ξ_x , η_y are constant. For the Gauss quadrature at $\xi, \eta = \pm 1/\sqrt{3}$, the spurious mode is

$$u = \xi(\eta^2 - 1/3)\xi_{,x}, \quad v = -\eta(\xi^2 - 1/3)\eta_{,y}.$$

Spurious Modes in Hexahedral Elements. For $(p+1)^3$ -node Lagrange hexahedral elements in 3-dimensional elasticity, one type of spurious modes of $p \times p \times p$ reduced integration is in the form

$$\begin{aligned} u &= f_p(\xi)f_p(\eta)(a_1 + \dots + a_{p+1}\zeta^p) + f_p(\eta)f_p(\zeta)(a_{p+2} + \dots + a_{2p+1}\xi^{p-1}) \\ &\quad + f_p(\zeta)f_p(\xi)(a_{2p+2} + \dots + a_{3p+1}\eta^{p-1}) \\ v &= w = 0 \end{aligned}$$

with the linear isoparametric mapping and constant a 's. The other types of spurious mode are given by

$$\begin{aligned} u &= [f'_p(\xi)f_p(\eta)\xi_{,x} - f_p(\xi)f'_p(\eta)\eta_{,x}](a_{3p+2} + \dots + a_{4p+1}\zeta^{p-1}), \\ v &= [f'_p(\xi)f_p(\eta)\xi_{,y} - f_p(\xi)f'_p(\eta)\eta_{,y}](a_{3p+2} + \dots + a_{4p+1}\zeta^{p-1}), \\ w &= 0. \end{aligned}$$

One can obtain other spurious modes by interchanging u , ξ with v , η , respectively, u , ξ with w , ζ and so on. For $p = 2$, the 27-node brick element has 21 spurious modes of the first type and 6 of the second type.

The determination of spurious zero energy modes resulting from reduced integration for triangular elements and elements involving nonlinear isoparametric mapping (the Jacobian of the mapping is not constant) is not trivial. One can always determine the spurious modes from the zero eigenvectors of the element stiffness matrix numerically.

If a spurious mode of a single element exists and in the assembly of elements, it is called a *communicating spurious mode*. If the same type of a spurious mode exists over the entire domain, then it is a *global spurious mode*. On the other hand if the same mode occurs over only part of the domain, it is called the *local mode*. If the spurious mode exists only in a single element, the mode is *non-communicating*. Communicating spurious mode is a serious flaw in practical application. If it exists after boundary constraints are imposed, the global stiffness matrix is singular and the finite element solution is very sensitive to small changes in loading or boundary condition. It can render the solution useless (Irons and Hellen 1978, Bićcanić and Hinton 1979). MacNeal (1994) showed that many of type 1 spurious modes are communicating, global in plane strain or plane stress, while local in 3-dimension. On the other hand, type 2 spurious modes are generally non-communicating. There is at least one exception — a beam modeled by reduced integration of 20-node brick elements. Depending on end conditions, one has as many spurious modes as the number of elements plus or minus one.

Spurious modes can also cause errors when there is a mixture of stiff and soft materials even when, strictly speaking, a global spurious mode does not exist. Consider the example of a stiff block on the top of a soft foundation as shown in Fig. 18.21:1 (Zienkiewicz and Taylor, volume 1, 1989). The use of 2×2 quadrature throughout leads to the answer shown in Fig. 18.21:1(a) while the result of 3×3 quadrature in Fig. 18.21:1(b) is appropriate. Even though no zero-energy mode exists for the assembled system, the low resistance of the material in the foundation allows the penetration of spurious deformation into the soft elements.

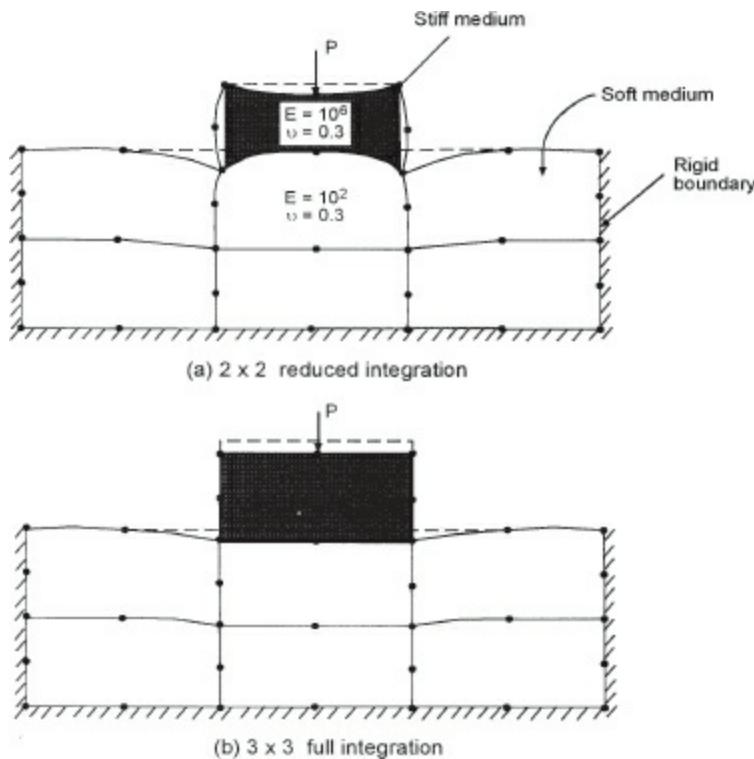


Fig. 18.21:1. Propagation of spurious mode from reduced-integration elements.

We have covered primarily the displacement model of the finite element method for problems governed by linear second order differential equations. In the remaining chapters, we will examine mixed and hybrid models, plates and shells (governed by fourth order differential equations), and non-linear problems.

18.22. PERSPECTIVE

Since its introduction in the early 1960's, the finite element method has become one of the most powerful numerical tools for analysis and design. The method has been widely applied to many fields in engineering, science and medicine. The mathematical foundation of the method has been well established and understood. Three-node triangular and 4-node quadrilateral membrane elements were first introduced because of their simplicity. The problem of locking in applications to bending of thin plates and shells and to incompressible or nearly incompressible materials was soon discovered. Reduced integration was introduced to alleviate the problem. Unfortunately such technique also brings with it spurious zero energy deformation mode(s), which often requires specialized schemes to stabilize the stiffness matrix. Mixed, hybrid and field-consistent approaches, originally developed to circumvent the difficulty of meeting the C^1 continuity requirements for higher order differential equations, are alternatives for mitigating locking.

In spite of the advent of computer technology, computational efficiency is still a consideration in engineering applications, e.g., adaptive remeshing for the analysis of strain localization. Simple and efficient elements are still essential to computational efficiency and accuracy in thin plate and shell applications. The quest for such perfect elements continues.

The greatest future of the finite element method lies in its applications. Metal forming, bioengineering, nonlinear material behavior and large deformation are the essence. The material and thin-film technologies in microelectronics necessitate computational modeling at the atomic level. Tailoring materials for optimum strength, stiffness, fracture toughness and acoustic characteristics in meso- and micro-mechanics of composite materials will make the finite element method indispensable in material development. Applications of smartmaterials to sensors and actuators increasingly require precise knowledge of material performance and reliability as miniaturization of these devices continues. Material modeling to examine slip planes and effects of grain boundaries of multiple crystal systems would be required. Material failure often associates with local material instability or strain localization including material separation or void growth. Genomic biomechanics linking gene expression, protein function, and cell behavior to tissue, organ and continuum function will be new frontiers in finite element applications. Finite element method will be indispensable in simulation and prediction of these phenomena.

¹Often the needed information is generated by a different CPU. The task to generate the information may not have been completed when it is needed by another CPU.

²A polynomial or a function represented by polynomials is p^{th} order complete if it includes all terms of order less of equal p , e.g., u in Eqs. (18.2:2) and (18.2:7) are linear and quadratic complete, respectively.

³Elements with more DOFs do not necessary give high order approximation if lower order polynomial terms are left out in the shape functions.

⁴A family of elements, called Serendipity family, which differ from the Lagrange one in the number of interior node(s). We can obtain the 8-node *quadrilateral serendipity element* by simply setting h_9 zero in [Table 18.13:1](#). Zienkiewicz coined the name after Prince of Serendipity noted for this chance of discoveries.

19

MIXED AND HYBRID FORMULATIONS

So far we have considered mostly the displacement finite element model. There are other types of formulation that demand less stringent admissible requirements on continuity. Among them are the mixed and hybrid formulations, which provide more flexibility to deal with problems involving singularities, infinite domain, and higher order differential equations. The displacement formulation is based on Eq. (18.14:7) or (18.16:10) and is an *irreducible formulation*, whereas the mixed and hybrid finite element formulations are based on the mixed and hybrid variational principles, which use multiple fields as dependent variables (see Sec. 10.3). An irreducible formulation is one for which none of the field variables can be eliminated without solving the differential equations or the weak form of equations. The hybrid formulation allows some field variables defined only along the element boundaries to be different from those within the element. Thus the hybrid method is very versatile. Gallagher proposed to restrict the definition of the hybrid method as one based on variational principles of multi-fields, yet the resulting matrix equations consist only nodal values of displacements as unknown. Pian (1995, 1998) surveyed and assessed the advances in hybrid/mixed finite element method.

19.1. MIXED FORMULATIONS

We can establish the finite element equations using the *Reissner–Hellinger principle* as given in Eq. (10.10:8) that employs both stress and displacement as independent fields. We shall illustrate the derivation by considering problems of plane elasticity. For plane stress or plane strain, the functional Π_{Re} of an element can be written in the form:

$$(1) \quad \Delta \quad \Pi_{Re} = \int_{A_e} (\sigma^T e - \sigma^T C_e \sigma / 2 - u^T b) dA - \int_{\partial A_{\sigma e}} u^T \bar{T} ds,$$

in which e is the engineering *strain* in terms of the displacement u , σ the *stress*, b the *body force per unit volume*, \bar{T} the *prescribed traction* over the boundary $\partial A_{\sigma e}$ of the element, C_e the elastic flexibility matrix, the inverse of the elastic coefficient matrix D_e defined in Eq. (18.14:3), and A_e the element area:

$$\begin{aligned} e &= [e_x \quad e_y \quad \gamma_{xy}]^T = [\partial u / \partial x \quad \partial v / \partial y \quad \partial u / \partial y + \partial v / \partial x]^T, \\ u &= [u \quad v]^T, \quad \sigma = [\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy}]^T, \quad \bar{T} = [\bar{T}_x \quad \bar{T}_y]^T \end{aligned}$$

for plane elasticity. The functional for 3-dimensional elasticity is similar except that the integrals over A_e and $\partial A_{\sigma e}$ in Eq. (1) are replaced by the integrals over V_e and $\partial V_{\sigma e}$, respectively. For isotropic materials, we have

$$\begin{aligned} C_e &= \frac{1}{E} \begin{bmatrix} 1 & & & \text{sym} \\ -\nu & 1 & & \\ 0 & 0 & 2(1+\nu) & \end{bmatrix} && \text{(for plane stress),} \\ C_e &= \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & & & \text{sym} \\ -\nu & 1-\nu & & \\ 0 & 0 & 2 & \end{bmatrix} && \text{(for plane strain),} \\ C_e &= \frac{1}{E} \begin{bmatrix} 1 & & & \text{sym} \\ -\nu & 1 & & \\ -\nu & -\nu & 1 & \\ 0_{3 \times 3} & & 2(1+\nu)_{3 \times 3} I_3 & \end{bmatrix} && \text{(3-D elasticity).} \end{aligned}$$

The displacement boundary condition given in Eq. (18.14:4) is a rigid condition whereas the traction condition given in Eq. (18.14:5) is natural. The stationary condition of the variational functional of the entire domain with respect to σ and u gives the equilibrium equations in stresses, the stress-strain relations, and the traction conditions on $\partial A_{\sigma e}$ as those given in Eqs. (10.10:3)–(10.10:5) in index notation.

In finite element formulation, both the displacements and the stresses are approximated in the usual manner by

interpolating their nodal parameters with shape functions. The admissibility requirement is that either the traction or the displacements are C^0 continuous and the latter satisfy the rigid condition(s). The use of C^0 continuous stresses may result in higher accuracy than possible with the displacement formulation discussed previously (Zienkiewicz *et al.* 1985, 1985a). Poor choice of shape functions may result in poor results. Babuska (1971) and Brezzi (1974) formulated a criterion for the choice of dependent variables, known as the LBB condition. Tong and Pian (1969) proposed to assume the stresses and the derived strains from the assumed displacements to have the same order of approximation. Fraeijs de Veubeke (1964, 1965) showed that if the approximate stresses are capable of reproducing the same variation as that from the displacement model, then the two formulations give identical answer. In many cases, the mixed method provides superior results for stresses by requiring continuity of traction between elements.

Assuming $\mathbf{u} = \mathbf{h}\mathbf{q}$ in the usual manner, we derive $\mathbf{e} = \mathbf{d}_e \mathbf{h}\mathbf{q}$ as given by Eq. (18.14:8) and approximate the stress independently as $\boldsymbol{\sigma} = \mathbf{h}_\sigma \hat{\boldsymbol{\sigma}}$ where $\hat{\boldsymbol{\sigma}}$ is the unknown parameter matrix and \mathbf{h}_σ the shape function matrix for the stresses, where \mathbf{h}_σ and \mathbf{h} are in general different. In mixed formulation, we allow \mathbf{u} to be discontinuous between elements, if the assumed traction derived from σ is C^0 continuous. In terms of \mathbf{q} and $\hat{\boldsymbol{\sigma}}$, Π_{Re} becomes

$$(2) \quad \Delta \quad \Pi_{Re} = \hat{\boldsymbol{\sigma}}^T \mathbf{k}_{\sigma q} \mathbf{q} - \hat{\boldsymbol{\sigma}}^T \mathbf{k}_\sigma \hat{\boldsymbol{\sigma}} / 2 - \mathbf{q}^T \mathbf{f},$$

where

$$(3) \quad \Delta \quad \mathbf{k}_\sigma = \int_{A_e} \mathbf{h}_\sigma^T \mathbf{C}_e \mathbf{h}_\sigma dA, \quad \mathbf{k}_{\sigma q} = \int_{A_e} \mathbf{h}_\sigma^T \mathbf{d}_e \mathbf{h} dA, \quad \mathbf{f} = \int_{A_e} \mathbf{h}^T \mathbf{b} dA + \int_{\partial A_{\sigma e}} \mathbf{h}^T \bar{\mathbf{T}} dA.$$

Note that \mathbf{k}_σ is symmetric and positive definite and \mathbf{f} is in the same form as Eq. (18.14:11). One can assemble \mathbf{k}_σ , $\mathbf{k}_{\sigma q}$, \mathbf{f} and enforce the displacement constraints based on the stationary condition of all Π_{Re} [Eq. (10.11:6)] to obtain the finite element equations in the form

$$\mathbf{K}_{Re} \begin{bmatrix} \sum \hat{\boldsymbol{\sigma}} \\ \sum \mathbf{q} \end{bmatrix} = \begin{bmatrix} -\sum \mathbf{k}_\sigma & \sum \mathbf{k}_{\sigma q} \\ \sum \mathbf{k}_{\sigma q}^T & 0 \end{bmatrix} \begin{bmatrix} \sum \hat{\boldsymbol{\sigma}} \\ \sum \mathbf{q} \end{bmatrix} = \begin{bmatrix} 0 \\ \sum \mathbf{f} \end{bmatrix},$$

where the summation denotes assembling of all elements. We borrow the earlier terminology and call \mathbf{K}_{Re} the assembled constrained stiffness matrix.

With the additional dependent variable σ , generally there will be more unknowns. The assembled matrix is still symmetric but no longer positive definite as all diagonal terms associated with q 's are zero. These terms become nonzero in the process of factorization and Gauss elimination if the unknowns are properly ordered and if the constrained stiffness matrix is nonsingular. If σ is chosen independently for each element, $\hat{\boldsymbol{\sigma}}$ can be eliminated at the element level from the stationary condition of Π_{Re} with respect to $\hat{\boldsymbol{\sigma}}$,

$$\mathbf{k}_\sigma \hat{\boldsymbol{\sigma}} - \mathbf{k}_{\sigma q} \mathbf{q} = 0, \quad \text{or} \quad \hat{\boldsymbol{\sigma}} = \mathbf{k}_\sigma^{-1} \mathbf{k}_{\sigma q} \mathbf{q}.$$

A substitution of $\hat{\boldsymbol{\sigma}}$ above into Eq. (2) leads to

$$(4) \quad \Pi_{Re} = \mathbf{q}^T \mathbf{k}_{\sigma q}^T \mathbf{k}_\sigma^{-1} \mathbf{k}_{\sigma q} \mathbf{q} / 2 - \mathbf{q}^T \mathbf{f} = \mathbf{q}^T \mathbf{k} \mathbf{q} / 2 - \mathbf{q}^T \mathbf{f}$$

with \mathbf{k} being the deduced element stiffness matrix. One can then assemble the element matrices in the usual manner.

Obviously \mathbf{k} is symmetric. The rank of \mathbf{k} depends on the form and number of $\hat{\boldsymbol{\sigma}}$ and u modes. Let n_σ be the number of $\hat{\boldsymbol{\sigma}}$'s, n_d that of u 's, and r that of rigid body modes associated with the shape functions h 's. Then

$$(5) \quad \text{rank of } \mathbf{k} \leq \min(n_\sigma, n_d - r).$$

There will be a rank deficiency for \mathbf{k} if $n_\sigma < n_d - r$ leading to spurious deformation mode(s).

19.2. HYBRID FORMULATIONS

A special mixed formulation is the *hybrid finite element method*, which uses different field variables in different parts of the domain. The hybrid concept was initiated by Pian (1964). Tong and Pian (1969) rationalized the approach using the variational formulation with the functional defined in Eq. (10.10:8). The concept was later generalized into different finite element models (Tong 1970, Pian and Tong 1969, 1972) and used in numerous applications (see [Biblios. 19.2](#) and [19.5](#)).

In plane elasticity, a four-field hybrid principle derived from (10.10:7) is

$$(1) \quad \Pi_H = \sum_{\text{all elements}} \Pi_{He}(\sigma, \mathbf{u}, \hat{\mathbf{u}}, \mathbf{T}),$$

in which

$$\begin{aligned} \Pi_H &= \sum_{\text{all element}} \Pi_H(\sigma, \mathbf{u}, \hat{\mathbf{u}}, \hat{\mathbf{T}}) \\ \Pi_{He} &= \int_{\partial A_e} (\hat{\mathbf{u}} - \mathbf{u}) \hat{\mathbf{T}} ds + \int_{A_e} (\sigma^T \mathbf{e} - \sigma^T \mathbf{C}_e \sigma / 2 - \mathbf{u}^T \mathbf{b}) dA - \int_{\partial A_{\sigma e}} \hat{\mathbf{u}}^T \mathbf{T} ds, \end{aligned}$$

where $\hat{\mathbf{T}}$ is the boundary traction of the element and $\hat{\mathbf{u}}$ is the interelement boundary displacement. In other words, $\hat{\mathbf{T}}$ of different elements are independent of each other while $\hat{\mathbf{u}}$ is the same for adjacent elements along their common boundaries and satisfies the prescribed displacement condition

$$(2) \quad \hat{\mathbf{u}} = \bar{\mathbf{u}} \quad \text{on } \partial A_u.$$

The hybrid functional Π_H given in Eq. (1) for the entire domain involves multiple fields, namely \mathbf{u} and σ within each element (the strain is in terms of \mathbf{u}) as well as $\hat{\mathbf{T}}$ and $\hat{\mathbf{u}}$ on the element boundaries. The admissible requirements are: piecewise continuity of $\hat{\mathbf{u}}$ along inter-element boundaries, $\hat{\mathbf{u}} = \bar{\mathbf{u}}, \delta \hat{\mathbf{u}} = 0$ on ∂A_{ue} , and smooth \mathbf{u} and σ within the element.

The stationary condition of Π_H with respect to the admissible fields gives the Euler equations

$$(3) \quad \sigma_{ix} + \mathbf{b} = 0 \quad (\text{equilibrium equation in each element } A_e),$$

$$(4) \quad \mathbf{e} = \mathbf{C}_e \sigma \quad (\text{constitutive law in } A_e),$$

$$(5) \quad \hat{\mathbf{u}} = \mathbf{u} \quad (\text{displacement matching condition on } \partial A_e),$$

$$(6) \quad \hat{\mathbf{T}} = \mathbf{T} \quad (\text{element traction condition on } \partial A_e),$$

$$(7) \quad \hat{\mathbf{T}} = \bar{\mathbf{T}} \quad (\text{traction boundary condition on } \partial A_{\sigma e}),$$

$$(8) \quad \sum_{\text{all elements}} \int_{\partial A_e} \delta \hat{\mathbf{u}}^T \hat{\mathbf{T}} ds = 0 \quad (\text{inter-element equilibrium}),$$

where $\mathbf{T} (= \mathbf{n} \cdot \sigma = n_j \sigma_{ji})$ is the traction and \mathbf{n} the unit normal outward to ∂A_e . There is no summation in Eqs. (3)–(7) because $\delta \hat{\mathbf{T}}, \delta \hat{\mathbf{u}}$ from different elements are independent. On the other hand, $\delta \hat{\mathbf{u}}$ are common to the adjacent elements and Eq. (8) implies equilibrium between two adjacent elements I and II that

$$(9) \quad \hat{\mathbf{T}}_I + \hat{\mathbf{T}}_{II} = 0.$$

In the formulation above, inter-element compatibility of \mathbf{u} is satisfied in an average sense through Eq. (5) and inter-element equilibrium of σ is achieved in average through Eq. (6) and (8). The derivation of the element matrices based on Eq. (1) by interpolating the field variables in terms of their unknown parameters is straightforward. Since the inter-element admissibility requirement involves only $\hat{\mathbf{u}}$ on the element boundaries while no restriction on \mathbf{u} and σ , one has great flexibility to select their shape functions. This alleviates the difficulties of C^1 encountered in problems governed by 4th order differential equations and provides easy means to treat problems with steep stress gradients and singularities (Tong and Pian 1972). It also permits better stress description to improve the stress approximation.

Through selectively enforcing Eqs. (3)–(9) *a priori*, the number of independent field variables of Eq. (1) can be reduced, which leads to different hybrid models to be discussed below.

Hybrid Stress Model. Pian (1964) originally proposed this widely used model. Choosing an equilibrium stress σ that satisfies also the element traction condition Eq. (6) *a priori*, we can simplify Π_{He} and denoting it as

$$(10) \quad \Delta \quad \Pi_{HS e} = \int_{\partial A_e} \hat{\mathbf{u}}^T \mathbf{T} ds - \int_{A_e} \sigma^T \mathbf{C}_e \sigma dA / 2 - \int_{\partial A_{\sigma e}} \hat{\mathbf{u}}^T \bar{\mathbf{T}} ds$$

for the hybrid stress principle Eq. (1). To derive the element matrices, we shall consider only the case of zero body force, i.e., $\mathbf{b} = 0$. We assume σ in the form

$$(11) \quad \sigma = \mathbf{h}_{\beta} \boldsymbol{\beta} = \mathbf{h}_{\beta} [\beta_1 \quad \beta_2 \quad \dots \quad \beta_p]^T,$$

where

$$(12) \quad \mathbf{h}_\beta = \begin{bmatrix} 1 & 0 & 0 & y & 0 & x & 0 & \dots \\ 0 & 1 & 0 & 0 & x & 0 & y & \dots \\ 0 & 0 & 1 & 0 & 0 & -y & -x & \dots \end{bmatrix}_{3 \times p}.$$

The matrix \mathbf{h}_β is so chosen that σ satisfies the equilibrium equations for arbitrary β . We assume $\hat{\mathbf{u}}$ in terms of its nodal values along the element boundaries. An example of quadrilateral element is shown in Fig. 19.2:1 with q_{2i-1} and q_{2i} , $i = 1, 2, 3, 4$ being the nodal values of \hat{u}, \hat{v} , respectively. Then $\hat{\mathbf{u}}$ can be written as

$$(13) \quad \hat{\mathbf{u}} = [\hat{u} \quad \hat{v}]^T = \mathbf{h}\mathbf{q},$$

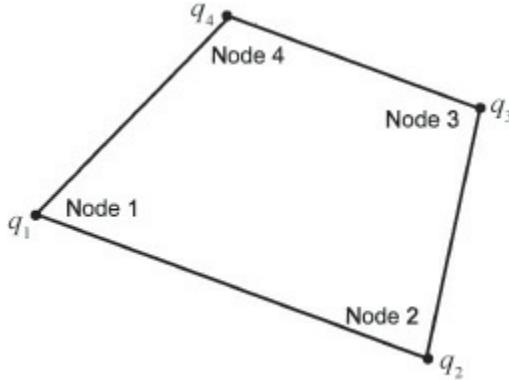


Fig. 19.2:1. Quadrilateral element.

where

$$\mathbf{h} = \begin{bmatrix} 1 - s/s_{1,2} & 0 & s/s_{1,2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 - s/s_{1,2} & 0 & s/s_{1,2} & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{between nodes 1 and 2}),$$

$$\mathbf{h} = \begin{bmatrix} 0 & 0 & 1 - s/s_{2,3} & 0 & s/s_{2,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 - s/s_{2,3} & 0 & s/s_{2,3} & 0 & 0 \end{bmatrix} \quad (\text{between nodes 2 and 3}),$$

...

$$\mathbf{h} = \begin{bmatrix} s/s_{4,1} & 0 & 0 & 0 & 0 & 1 - s/s_{4,1} & 0 \\ 0 & s/s_{4,1} & 0 & 0 & 0 & 0 & 1 - s/s_{4,1} \end{bmatrix} \quad (\text{between nodes 4 and 1}).$$

Here s is the distance along the element boundary and $s_{i,j}$ is the distance between nodes i and j . Note that $\hat{\mathbf{u}}$ has the same displacement along the element boundaries as that of a 4-node quadrilateral displacement element.

At the element boundaries, one has

$$\mathbf{T} = \mathbf{n}^T \boldsymbol{\sigma} = \begin{bmatrix} n_x \sigma_{xx} + n_y \sigma_{xy} \\ n_x \sigma_{xy} + n_y \sigma_{yy} \end{bmatrix} = \mathbf{n}^T \mathbf{h}_\beta \boldsymbol{\beta}, \quad \mathbf{n}^T = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix},$$

where \mathbf{n} is a unit outward normal to the boundary in matrix notation. Recall that in Chapters 14 and 15, traction is defined as $\mathbf{T} = \mathbf{n} \cdot \boldsymbol{\sigma}$ in tensor notation, where $\boldsymbol{\sigma}$ is a second rank tensor and \mathbf{n} a unit normal vector. Presently $\boldsymbol{\sigma}$ is defined in Eq. (18.14:1) as a column matrix.

A substitution of Eq. (11) and (13) into Eq. (10) yields

$$(14) \quad \Pi_{HSe} = \boldsymbol{\beta}^T \mathbf{k}_{\beta q} \mathbf{q} - \boldsymbol{\beta}^T \mathbf{k}_{\beta \beta} \boldsymbol{\beta} / 2 - \mathbf{q}^T \mathbf{f},$$

where

$$(15) \blacktriangleleft \quad \mathbf{k}_{\beta\beta} = \int_{A_e} \mathbf{h}_\beta^T \mathbf{C}_e \mathbf{h}_\beta dA, \quad \mathbf{k}_{\beta q} = \int_{\partial A_e} \mathbf{h}_\beta^T \mathbf{n} h dA, \quad \mathbf{f} = \int_{\partial A_{\sigma e}} \mathbf{h}^T \bar{\mathbf{T}} dA.$$

If $\mathbf{b} \neq 0$, we must include the particular solution of Eq. (3) in Eq. (11), e.g., for constant $\mathbf{b} = [b_x \ b_y]^T$, we assume

$$\boldsymbol{\sigma} = \mathbf{h}_\beta \boldsymbol{\beta} + \boldsymbol{\sigma}_p = \mathbf{h}_\beta \boldsymbol{\beta} + [b_x x \ b_y y]^T.$$

There will be additional terms in \mathbf{f} through $\boldsymbol{\sigma}_p$ (Tong and Rossettos 1977).

In Eq. (15) β of different elements is independent of each other and can be eliminated from the stationary condition of Π_{HSe} with respect to β .

$$(16) \quad \mathbf{k}_{\beta\beta} \boldsymbol{\beta} - \mathbf{k}_{\beta q} \mathbf{q} = 0, \quad \text{or} \quad \boldsymbol{\beta} = \mathbf{k}_{\beta\beta}^{-1} \mathbf{k}_{\beta q} \mathbf{q}.$$

Substituting Eq. (16) into Eq. (14), we obtain again for the element

$$(17) \quad \Pi_{HSe} = \mathbf{q}^T \mathbf{k}_{\beta q}^T \mathbf{k}_{\beta\beta}^{-1} \mathbf{k}_{\beta q} \mathbf{q} / 2 - \mathbf{q}^T \mathbf{f} = \mathbf{q}^T \mathbf{k} \mathbf{q} / 2 - \mathbf{q}^T \mathbf{f}.$$

The element matrices can be assembled and constrained in the usual manner to obtain the finite element equations for the entire system.

Similar to Eq. (19.1:5), we have

$$\text{rank of } \mathbf{k} \leq \min(n_\beta, n_q - n_r),$$

in which n_β , n_q , n_r are, respectively, the number of β , \mathbf{q} and rigid body modes. In other words, the rank of \mathbf{k} is no larger than the number of β 's. For a quadrilateral element, there are 8 q 's and 3 rigid body motions. If the number of β 's is less than 5, the rank of \mathbf{k} is deficient and spurious deformation mode(s) with zero energy exists. The five- β hybrid stress element is commonly used. In this case, Eq. (12) becomes

$$\mathbf{h}_\beta = \begin{bmatrix} 1 & 0 & 0 & y & 0 \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{3 \times 5}, \quad \boldsymbol{\beta}^T = [\beta_1 \ \beta_2 \ \dots \ \beta_5].$$

For distorted elements, instead of assuming the stresses to satisfy equilibrium equation exactly as in Eq. (12), Pian and Sumihara (1984) proposed to express the stresses in terms of the element's nature coordinates to have the assumed stresses satisfy the equilibrium equation only in an integral or average sense.

Hybrid Displacement Model (Tong 1970). If the constitutive law $\boldsymbol{\sigma} = \mathbf{D}_e \mathbf{e} = \mathbf{D}_e \mathbf{d}_e \mathbf{u}$ is satisfied *a priori* in A_e and the traction $\bar{\mathbf{T}}$ equals to $\mathbf{T}(= \mathbf{n}^T \boldsymbol{\sigma})$ on ∂A_e , the functional Eq. (19.1:1) becomes the two-field $\hat{\mathbf{u}}$ and \mathbf{u} hybrid displacement principle of Eq. (10.10:10),

$$(18) \blacktriangleleft \quad \Pi_{HD_e} = \int_{\partial A_e} (\hat{\mathbf{u}} - \mathbf{u})^T \mathbf{T} ds + \int_{A_e} \left[\frac{(\mathbf{d}_e \mathbf{u})^T \mathbf{D}_e \mathbf{d}_e \mathbf{u}}{2} - \mathbf{u}^T \mathbf{b} \right] dA - \int_{\partial A_{\sigma e}} \hat{\mathbf{u}}^T \mathbf{T} ds,$$

where the elastic modulus matrix \mathbf{D}_e , and the differential operator \mathbf{d}_e are defined in Sec. 18.14. Assuming $\hat{\mathbf{u}}$ as given in Eq. (13) and \mathbf{u} and \mathbf{T} in the form

$$\mathbf{u} = \mathbf{h}_\alpha \boldsymbol{\alpha} = \mathbf{h}_\alpha [\alpha_1 \ \alpha_2 \ \dots \ \alpha_p]^T, \\ \mathbf{T} = \mathbf{n}^T \boldsymbol{\sigma} = \begin{bmatrix} n_x \sigma_{xx} + n_y \sigma_{xy} \\ n_x \sigma_{xy} + n_y \sigma_{yy} \end{bmatrix} = \mathbf{n}^T \mathbf{D}_e \mathbf{d}_e \mathbf{h}_\alpha \boldsymbol{\alpha},$$

where \mathbf{n} is a unit outward normal on ∂A_e , Eq. (18) can be written as

$$(19) \quad \Pi_{HD_e} = \boldsymbol{\alpha}^T \mathbf{k}_{\alpha q} \mathbf{q} - \boldsymbol{\alpha}^T \mathbf{k}_{\alpha\alpha} \boldsymbol{\alpha} / 2 - \mathbf{q}^T \mathbf{f}.$$

The element matrices are given by

$$\begin{aligned} \mathbf{k}_{\alpha q} &= \int_{\partial A_e} (\mathbf{d}_e \mathbf{h}_\alpha)^T \mathbf{D}_e \mathbf{n} h dA, \quad \mathbf{k}_{\alpha \sigma} = \int_{\partial A_e} (\mathbf{n}^T \mathbf{D}_e \mathbf{d}_e \mathbf{h}_\alpha)^T \mathbf{h}_\sigma dA, \\ \mathbf{k}_{\alpha \alpha} &= \mathbf{k}_{\alpha \sigma} + \mathbf{k}_{\alpha \sigma}^T - \int_{A_e} (\mathbf{d}_e \mathbf{h}_\alpha)^T \mathbf{D}_e \mathbf{d}_e \mathbf{h}_\alpha dA, \quad \mathbf{f} = \int_{\partial A_{\sigma e}} \mathbf{h}^T \bar{\mathbf{T}} dA. \end{aligned}$$

In Eq. (19) α 's of different elements are independent of each other. From the stationary condition of Π_{HSe} with respect to α , we have

$$\mathbf{k}_{\alpha \alpha} \alpha - \mathbf{k}_{\alpha q} \mathbf{q} = 0, \quad \text{or} \quad \alpha = \mathbf{k}_{\alpha \alpha}^{-1} \mathbf{k}_{\alpha q} \mathbf{q}.$$

A substitution of the equation above into Eq. (19) yields

$$(20) \quad \Pi_{HDe} = \mathbf{q}^T \mathbf{k}_{\alpha q}^T \mathbf{k}_{\alpha \alpha}^{-1} \mathbf{k}_{\alpha q} \mathbf{q} / 2 - \mathbf{q}^T \mathbf{f} = \mathbf{q}^T \mathbf{k} \mathbf{q} / 2 - \mathbf{q}^T \mathbf{f}.$$

If we further require that \mathbf{u} satisfies the equilibrium equations, Eq. (18) reduces to a functional involving only integrals along element boundaries

$$(21) \quad \Pi_{HDe} = \int_{\partial A_e} (\hat{\mathbf{u}} - \mathbf{u}/2)^T \mathbf{T} ds - \int_{\partial A_{\sigma e}} \hat{\mathbf{u}}^T \bar{\mathbf{T}} ds,$$

which is a special case of the hybrid stress model with compatible stresses.

Problem 19.1. Show that the element stiffness matrix of the hybrid stress model is independent of the body force.

Problem 19.2. Derive the element force matrix of the hybrid stress model for plane elasticity with constant body force.

Problem 19.3. Show that the hybrid stress stiffness matrix for triangular element is the same as that of the constant strain triangle of the displacement model regardless of the number of β 's in the stress mode.

Problem 19.4. Derive the hybrid stress element stiffness matrix for an 8-node hexahedral. Determine the rank of \mathbf{k} for different number of stress modes.

Problem 19.5. Derive the element stiffness matrix of the hybrid displacement model:

- (a) Use all three variables \mathbf{u} , $\hat{\mathbf{u}}$ and $\bar{\mathbf{T}}$ as field variables.
- (b) Discuss how the rank of \mathbf{k} is affected by the number of unknown \mathbf{u} 's and $\bar{\mathbf{T}}$'s.

Problem 19.6. Derive the element force matrix of the hybrid displacement model corresponding to the field variable $\hat{\mathbf{u}}$.

Problem 19.7. Derive the element force matrix for the mixed boundary problem,

$$u_1 = \bar{u}_1, \quad (\mathbf{n}^T \boldsymbol{\sigma})_2 = T_2, \quad \text{on} \quad \partial A_{ue},$$

where \mathbf{n} is a unit vector on ∂A_{ue} as defined in Subsec. Hybrid Stress Model.

19.3. HYBRID SINGULAR ELEMENTS (SUPER-ELEMENTS)

For smooth solutions, the order of finite element approximation is proportional to the order of the complete polynomial of the shape functions. When a problem involves singularity, the order of approximation is dominated by the order of singularity (Key 1966, Tong and Pian 1972). For example, the asymptotic solution near a crack tip is (Sih and Liebowitz 1968)

$$(1) \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{K_I \cos \frac{\theta}{2}}{\sqrt{2\pi r}} \begin{bmatrix} 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \end{bmatrix} + \frac{K_{II}}{\sqrt{2\pi r}} \begin{bmatrix} -\sin \frac{\theta}{2} (2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}) \\ \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \\ \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) \end{bmatrix} + \dots,$$

$$(2) \quad \begin{bmatrix} u \\ v \end{bmatrix} = \frac{K_I(\kappa - \cos \theta)}{2G} \sqrt{\frac{r}{2\pi}} \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} + \frac{K_{II}}{2G} \sqrt{\frac{r}{2\pi}} \begin{bmatrix} (2 + \kappa + \cos \theta) \sin \frac{\theta}{2} \\ (2 - \kappa - \cos \theta) \cos \frac{\theta}{2} \end{bmatrix} + \dots,$$

where G is the shear modulus, r the distance from the crack tip, θ the angle measuring from the crack plane, $\kappa = (3-v)/(1+v)$ (for plane stress), $\kappa = 3-4v$ (for plane strain) with v being Poisson's ratio, and K_I and K_{II} are constants commonly referred to as the *stress intensity factors associated with symmetric (Mode I) and antisymmetric (Mode II) deformations*. The stresses and strains are proportional to $1/\sqrt{r}$ and the displacements are proportional to \sqrt{r} at the crack tip.

Near the singularity, the polynomial-base finite element method performs badly. Henshell and Shaw (1975) and Barsoum (1976, 1977) mapped isoparametrically the mid-points of two adjacent sides of the 6-node rectangle to quarter nodes 5 and 8 as shown in Fig. 19.3:1. The transformation has a singularity near node 1 with singular strain of $1/\sqrt{r}$ along the edge of the shifted nodes. The transformation is

$$(3) \quad \begin{aligned} x &= \sum_{i=1}^5 x_i h_i + x_8 h_8 = a\xi + x_5 h_5 = a[\xi - (1 - \xi^2)(1 - \eta)/4], \\ y &= \sum_{i=1}^5 y_i h_i + y_8 h_8 = b\eta + y_8 h_8 = b[\eta - (1 - \eta^2)(1 - \xi)/4], \end{aligned}$$

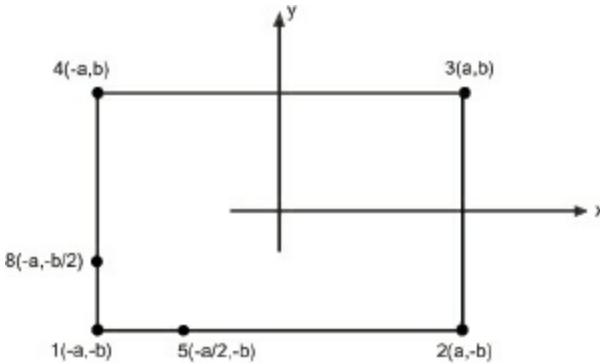


Fig. 19.3:1. A quarter point rectangular element.

which maps the midside points $(0, -1)$ and $(-1, 0)$ in the ξ, η plane to the quarter points $(-a/2, -b)$ and $(-a, -b/2)$ in the x, y plane with h 's defined in Table 18.13:1. The inverse of the Jacobian

$$\begin{aligned} \mathbf{J}^{-1} &= \begin{bmatrix} \partial\xi/\partial x & \partial\eta/\partial x \\ \partial\xi/\partial y & \partial\eta/\partial y \end{bmatrix} \\ &= \begin{bmatrix} (2 + \eta - \xi\eta)/a & (\eta^2 - 1)/(2a) \\ (\xi^2 - 1)/(2b) & (2 + \xi - \xi\eta)/b \end{bmatrix} \frac{8}{(3\xi\eta - \xi - \eta - 5)(\xi\eta - \xi - \eta - 3)} \end{aligned}$$

is singular at $\xi = \eta = -1$ corresponding to $1/\sqrt{r}$ singularity at node 1, $x = -a$ and $y = -b$. The strains e_{xx} , e_{yy} and e_{xy} have the same singularity.

The hybrid approach allows the incorporation of appropriate singularity in the shape functions of field variables (see Biblio. 19.3). Consider the domain with a crack as shown in Fig. 19.3:2. We use such an element, called *super-element*, to surround the crack tip and regular polynomial-based elements discussed before around the crack element. The assumed functions for the crack element and those for the surrounding elements are, in general, not continuous at the interelement boundaries. The hybrid model enforces the continuity on average.

The shape of crack element is generally a polygon. The number of nodes and locations along the element boundaries are selected to match those of the adjacent elements (Fig. 19.3:2). Within the element the crack tip can be anywhere. This feature makes the element ideal for crack growth simulation. As the crack advances within the element, one simply changes the crack tip location to reflect the crack growth, calculates new crack element stiffness matrix and analyzes the new configuration without remeshing. When the crack tip gets too close to the element boundary, the solution can deteriorate because the adjacent element is normally polynomial base. One may then enlarge the crack element by including those close to the crack tip and divide the area of the crack element far from the crack tip into regular elements. This process will involve only changes in the assembly list, which defines the relation between the local and global labels of each element, for the altered elements without having to remesh the entire domain.

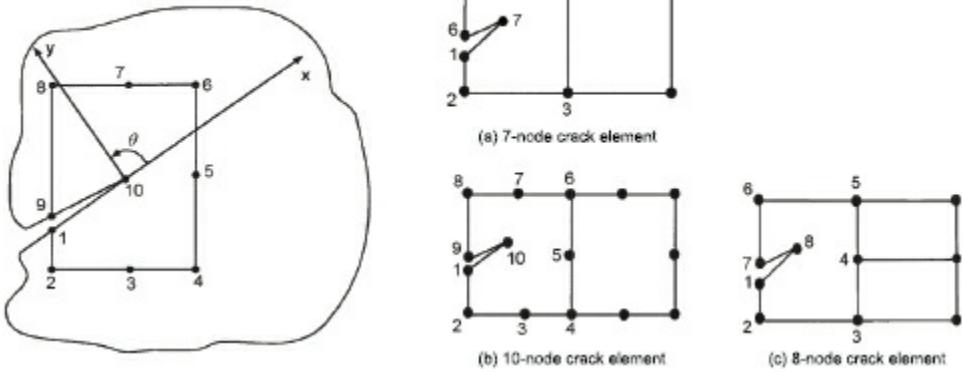


Fig. 19.3:2. Crack super-element connects to: (a) and (c) 4-node quadrilaterals and (b) 8-node quadrilaterals.

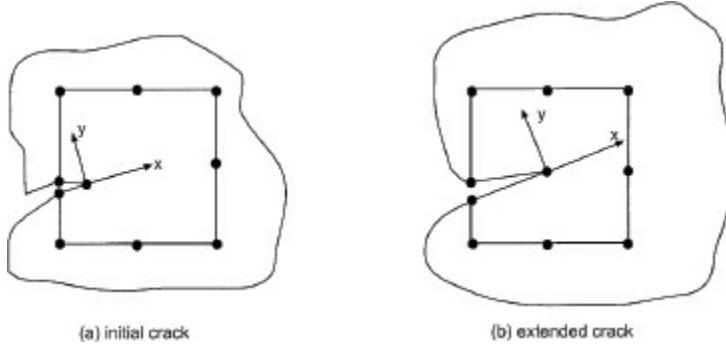


Fig. 19.3:3. Crack growth.

Based on Eq. (19.2:18), in constructing the element matrices, we use the asymptotic solution (1) as the shape functions for the field variables within the element and along the crack surfaces, and assume \hat{u} in the same form as that of the adjacent elements on the element boundary.

We illustrate the concept using a 10-node crack element as shown in Fig. 19.3:2(b). The crack surfaces are between nodes 9 and 10 and between nodes 10 and 1 and nodes 3, 5 and 7 are mid-side node. $\hat{u} = h\hat{q}$ is the displacement along the element boundary from node 2 to 8, where \mathbf{h} is a 2×14 matrix and $[q_{2i-1} \ q_{2i}]$ is the value of \hat{u} at node i . Let $s_{i,j}$ be the distance between nodes i and j , and s the distance from node i . The nonzero components of \mathbf{h} 's are

$$(4) \quad \begin{aligned} h_{1,1} &= h_{2,2} = 1 - s/s_{1,2}, \quad h_{1,3} = h_{2,4} = s/s_{1,2} \quad (\text{between nodes 1 and 2}), \\ h_{1,15} &= h_{2,16} = 1 - s/s_{8,9}, \quad h_{1,17} = h_{2,18} = s/s_{8,9} \quad (\text{between nodes 8 and 9}), \\ h_{1,2i-1} &= h_{2,2i} = (1 - s/s_{2i,2i+2})(1 - 2s/s_{2i,2i+2}), \\ h_{1,2i+1} &= h_{1,2i+2} = 4(1 - s/s_{2i,2i+2})s/s_{2i,2i+2}, \\ h_{1,2i+3} &= h_{2,2i+4} = (2s/s_{2i,2i+2} - 1)s/s_{2i,2i+2}, \end{aligned}$$

(between nodes $2i$ and $2i + 2$), where $i = 1, 2, 3$. A comma is added to the subscripts of h 's to distinguish the two indices with the first subscript of 1 denoting association with u and 2 with v .

An Alternative Hybrid Formulation. As an alternative we can express the shape functions in the crack element in terms of the complex potentials as functions of $z = x + iy$, in which $i = \sqrt{-1}$ is the imaginary number¹. Consider the case of zero body force. For isotropic materials, \mathbf{u} and σ can be derived from the complex potentials ψ, χ as given in Eqs. (9.6:2–4) repeated here for convenience

$$(5) \quad \begin{aligned} \sigma_{xx} + \sigma_{yy} &= 2[\psi'(z) + \overline{\psi'(z)}], \quad \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\bar{z}\psi''(z) + \chi''(z)], \\ 2G(u + iv) &= \kappa\psi(z) - z\overline{\psi'(z)} - \overline{\chi'(z)}, \end{aligned}$$

where $\overline{(.)}$ denotes the complex conjugate of the associated function, a prime denotes differentiation with respect to z , and $\kappa = (3-v)/(1+v)$ for plane stress and $(3-4v)$ for plane strain. The complex potentials are analytic functions.

To construct the element matrices, we assume the complex potentials in terms of unknown parameters and derive the stresses and the displacements according to Eq. (5). The derived stresses and displacements satisfy both the compatibility and equilibrium equations within the element. We shall choose ψ, χ' containing the appropriate singularity at the crack tip. This can be done easily by introducing the transformation

$$z = \zeta^2 = (\xi + i\eta)^2 \quad \text{or} \quad x = \xi^2 - \eta^2, \quad y = 2\xi\eta \quad \text{or}$$

$$\xi = \sqrt{(r+x)/2}, \quad \eta = (y/|y|)\sqrt{(r-x)/2}, \quad r = \sqrt{x^2 + y^2},$$

which is known as the conforming mapping (Muskhelishvili 1953) that maps the x, y -plane to the right half ξ, η plane. The upper ($y = 0^+$) and lower ($y = 0^-$) crack surfaces on the negative x -axis are mapped, respectively, to the positive and negative parts of the η -axis. If we assume ψ, χ' as polynomials of ζ , the derived stresses and strains will contain the appropriate square root singularity at the crack tip.

The derivatives with respect to z and ζ are related by the equation $2d(\cdot)/dz = (1/\zeta)d(\cdot)/d\zeta$. From Eq. (5), one can write the displacements and stresses as functions of ξ, η

$$(6) \quad \begin{aligned} \sigma_{xx} + \sigma_{yy} &= \frac{1}{\zeta} \frac{d\psi(\zeta)}{d\zeta} + \frac{1}{\bar{\zeta}} \frac{\overline{d\psi(\zeta)}}{d\zeta}, \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= \frac{\bar{\zeta}^2}{2\zeta} \frac{d}{d\zeta} \left[\frac{1}{\zeta} \frac{d\psi(\zeta)}{d\zeta} \right] + \frac{1}{\zeta} \frac{d\chi'(z)}{d\zeta}, \\ 2G(u + iv) &= \kappa\psi(\zeta) - \frac{\zeta^2}{2\bar{\zeta}} \frac{\overline{d\psi(\zeta)}}{d\zeta} - \overline{\chi'(z)}. \end{aligned}$$

Clearly, the displacements and stresses are proportional to $\zeta, 1/\zeta$ (i.e., $\sqrt{r}, 1/\sqrt{r}$), respectively, at the crack tip.

From Eq. (9.6:7), the traction free condition can be written as

$$(7) \quad \begin{aligned} \overline{\psi(z)} + \bar{z}\psi'(z) + \chi'(z) &= 0, \quad \text{or} \\ \overline{\psi(\zeta)} + \bar{\zeta}^2 [d\psi(\zeta)/d\zeta]/(2\zeta) + \chi'(z) &= 0 \end{aligned}$$

on the crack surfaces $z = x (< 0)$ in the z -plane, i.e., on the imaginary axis η in the ζ -plane. We obtain $\chi'(z)$ in terms of $\psi(\zeta)$ by analytic continuation (Tong 1984, Muskhelishvili 1953), i.e., replace $\bar{\zeta}$ by $-\zeta$ in Eq. (7) (because $\bar{\zeta} = -\zeta$ on the imaginary axis) to make $\chi'(\zeta)$ analytic. Thus

$$(8) \quad \chi'(z) = -\bar{\psi}(-\zeta) - \zeta [d\psi(\zeta)/d\zeta]/2.$$

The short bar on ψ denotes the complex conjugate of the function while ζ remains in the form as shown in the argument. For example,

$$(9) \quad \psi(\zeta) = a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + \dots, \quad \bar{\psi}(-\zeta) = -\bar{a}_1\zeta + \bar{a}_2\zeta^2 - \bar{a}_3\zeta^3 + \bar{a}_4\zeta^4 - \dots,$$

where a 's are complex unknowns. We have

$$(10) \quad \chi'(z) = (\bar{a}_1 - a_1/2)\zeta - (\bar{a}_2 + a_2)\zeta^2 + (\bar{a}_3 - 3a_3/2)\zeta^3 + \dots$$

As mentioned before, the assumed complex potentials of Eq. (9) and (10) give an $1/\sqrt{r}$ stress singularity at the crack tip. The coefficient of the singular term at the crack tip is a measure of the intensity of the singularity. Thus, we have [see the last two equations of Sec. 9.6 with the origin of the coordinates at the crack tip]

$$(11) \quad K_I - iK_{II} = 2\sqrt{2\pi} \lim_{z \rightarrow 0} (\sqrt{z} d\psi/dz) = \sqrt{2\pi} (d\psi/d\zeta)|_{\zeta=0} = \sqrt{2\pi} a_1,$$

where K_I, K_{II} are real constants referred to as the *stress intensity factors of modes I* and *II*, respectively (Sih and Liebowitz 1968). Mode I is a crack opening deformation and mode II corresponds to shear along the crack plane at the crack tip. Using the hybrid approach, we can evaluate the stress intensity factors directly from a_1 . Stress intensity factors are important parameters in fracture mechanics for structural integrity assessment.

The following identity is used for the integration in Eq. (19.2:21)

$$(12) \quad \begin{aligned} T_c = T_x - iT_y &= (\sigma_{xx} + \sigma_{yy})e^{-i\theta}/2 - (\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy})e^{i\theta}/2 \\ &= \left[\frac{1}{\zeta} \frac{d\psi(\zeta)}{d\zeta} + \frac{1}{\bar{\zeta}} \frac{\overline{d\psi(\zeta)}}{d\zeta} \right] e^{-i\theta} - \left\{ \frac{\bar{\zeta}^2}{2\zeta} \frac{d}{d\zeta} \left[\frac{1}{\zeta} \frac{d\psi(\zeta)}{d\zeta} \right] + \frac{1}{\zeta} \frac{d\chi'(z)}{d\zeta} \right\} e^{i\theta}, \end{aligned}$$

where θ is the angle measuring counter clockwise from the x -axis to the boundary normal of the element. For a straight side, θ is a constant along the side. Using ψ, χ' of Eq. (9) and (10), we obtain

$$(13) \quad T_c = T_x - iT_y = \begin{matrix} \mathbf{G}_\alpha \\ 1 \times (2r-1) \end{matrix} \begin{matrix} \alpha \\ (2r-1) \times 1 \end{matrix}, \quad u_c = u + iv = \begin{matrix} \mathbf{h}_\alpha \\ 1 \times (2r-1) \end{matrix} \begin{matrix} \alpha \\ (2r-1) \times 1 \end{matrix},$$

where $\begin{matrix} \mathbf{G}_\alpha \\ 1 \times (2r-1) \end{matrix} = [G_1 \ G_2 \ \dots]$ is from Eq. (12), $\begin{matrix} \mathbf{h}_\alpha \\ 1 \times (2r-1) \end{matrix} = [h_{\alpha 1} \ h_{\alpha 2} \ \dots]$ from Eq. (6), r is the number of polynomial terms in Eq. (9), and α 's are real unknowns related to the coefficients a 's of ψ that, for $j = 3, 4, \dots, r$,

$$\begin{aligned} a_1 &= \alpha_1 + i\alpha_2, \quad a_2 = \alpha_3, \quad a_j = \alpha_{2j-2} + i\alpha_{2j-1}, \\ G_1 &= (1/\zeta + 1/\bar{\zeta})e^{-i\theta} + (\bar{\zeta}^2 - \zeta^2)e^{i\theta}/(2\zeta^3) \\ G_2 &= i[(1/\zeta - 1/\bar{\zeta})e^{-i\theta} + (\bar{\zeta}^2 + 3\zeta^2)e^{i\theta}/(2\zeta^3)] \\ G_3 &= 8\cos\theta, \dots, \\ G_{2j-2} &= j(\zeta^{j-2} + \bar{\zeta}^{j-2})e^{-i\theta} - j\zeta^{j-2}[(j-2)\bar{\zeta}^2/\zeta^2] \\ G_{2j-1} &= ij[(\zeta^{j-2} - \bar{\zeta}^{j-2})e^{-i\theta} - \zeta^{j-2}[(j-2)\bar{\zeta}^2/\zeta^2]] \\ h_{\alpha 1} &= (2\kappa\zeta - \bar{\zeta} - \zeta^2/\bar{\zeta})/(4G), \quad h_{\alpha 2} = i(2\kappa\zeta - 3\bar{\zeta}) \\ h_{\alpha 3} &= (\kappa\zeta^2 + 2\bar{\zeta}^2 - \zeta^2)/(2G), \dots, \\ (h_\alpha)_{2j-2} &= \{2\kappa\zeta^j - j\zeta^2\bar{\zeta}^{j-2} - [2(-)^{j+1} - j]\bar{\zeta}^j\}/(\\ (h_\alpha)_{2j-1} &= i\{2\kappa\zeta^j + j\zeta^2\bar{\zeta}^{j-2} - [2(-)^{j+1} + j]\bar{\zeta}^j\}/ \end{aligned}$$

We have set $\text{Im}(\alpha_2)$ to zero because it does not contribute to stresses. Using Eq. (12), we can express

$$(14) \quad \begin{aligned} \mathbf{u}^T \mathbf{T} &= \text{Rl}(u_c T_c) = \alpha^T \text{Rl}(\mathbf{h}_\alpha^T \mathbf{G}_\alpha + \mathbf{G}_\alpha^T \mathbf{h}_\alpha) \alpha / 2 \quad \text{on } \partial A_e, \\ \mathbf{u}^T \mathbf{T} &= \alpha^T [\text{Rl}(\mathbf{h}_\alpha^T) T_x - \text{Im}(\mathbf{h}_\alpha^T) T_y] \quad \text{on } \partial A_{\sigma e}. \end{aligned}$$

The displacement $\hat{\mathbf{u}} = \mathbf{h}\mathbf{q}$ along the element boundary is the same as before with \mathbf{h} being real and linear [Eq. (19.2:3)] on a 2-node side, while real and quadratic [Eq. (19.3:4)] on 3-node side. Let us write $\hat{\mathbf{u}}$ in the form

$$(15) \quad \hat{\mathbf{u}} = [\hat{u} \ \hat{v}]^T = \begin{matrix} \mathbf{h} \\ 2 \times 2p \ 2p \times 1 \end{matrix} \mathbf{q} = \left[\begin{matrix} \mathbf{h}_u & \mathbf{q} \\ 1 \times 2p & 2p \times 1 \end{matrix} \right]^T \quad \text{on } \partial A_e,$$

where \mathbf{h}_u and \mathbf{h}_v are associated with u and v , respectively, and p is the total number of nodes of the element. Then we have $h_{uj} = h_{1,j}$, $h_{vj} = h_{2,j}$, and

$$(16) \quad \hat{\mathbf{u}}^T \mathbf{T} = \alpha^T [\text{Rl}(\begin{matrix} \mathbf{G}_\alpha \\ (2r-1) \times 1 \end{matrix})^T \begin{matrix} \mathbf{h}_u \\ 1 \times 2p \end{matrix} - \text{Im}(\begin{matrix} \mathbf{G}_\alpha \\ (2r-1) \times 1 \end{matrix})^T \begin{matrix} \mathbf{h}_v \\ 1 \times 2p \end{matrix}] \mathbf{q}.$$

Generally we choose $r \geq p$ to avoid mode deficiency.

A substitution of these complex functions into Eq. (19.2:21) yields

$$(17) \quad \Pi_{HDe} = \alpha^T \mathbf{k}_{\alpha q} \mathbf{q} - \alpha^T \mathbf{k}_{\alpha \alpha} \alpha / 2 - \mathbf{q}^T \mathbf{f}_q,$$

which is in the same form as Eq. (19.2:20) except that

$$(18) \quad \begin{aligned} \mathbf{k}_{\alpha q} &= \int_{\partial A_e} [\text{Rl}(\mathbf{G}_\alpha^T) \mathbf{h}_u - \text{Im}(\mathbf{G}_\alpha^T) \mathbf{h}_v] ds, \\ \mathbf{k}_{\alpha \alpha} &= \int_{\partial A_e} \text{Rl}(\mathbf{h}_\alpha^T \mathbf{G}_\alpha + \mathbf{G}_\alpha^T \mathbf{h}_\alpha) ds / 2, \quad \mathbf{f}_q = \int_{\partial A_{\sigma e}} \mathbf{h}^T \bar{\mathbf{T}} ds. \end{aligned}$$

One can derive the equivalent element stiffness matrices accordingly. It can be shown that $\mathbf{k}_{\alpha \alpha}$ is positive definite.

The integration along ∂A_e with the crack surfaces excluded can be performed numerically in the z -plane where ξ, η are related to x, y by Eq. (6). The integration along the crack surfaces can be performed analytically in the ζ -plane. One can also set $\hat{\mathbf{u}} = \mathbf{u}$ on the crack surfaces to simplify the computation.

If the crack element is too small or when the crack tip is too close to an element boundary, the crack element will lose its effectiveness (Orringer *et al.* 1977) because we approximate $\hat{\mathbf{u}}$ and the displacements of the adjacent elements as polynomials. To improve the accuracy of crack analysis, one could increase the number of boundary nodes of the crack

element and/or the order of the polynomials for the complex potential ψ . One may also move the element boundaries farther away from the crack tip if possible.

Problem 19.8. Derive the following expression from Eq. (6)

$$2(\sigma_{yy} + i\sigma_{xy}) = \frac{1}{\zeta} \frac{d\psi(\zeta)}{d\zeta} + \frac{1}{\bar{\zeta}} \frac{d\bar{\psi}(\zeta)}{d\zeta} + \frac{\zeta^2}{2\zeta} \frac{d}{d\zeta} \left[\frac{1}{\zeta} \frac{d\psi(\zeta)}{d\zeta} \right] + \frac{1}{\zeta} \frac{d\chi'(z)}{d\zeta}.$$

Find a particular solution ψ_p, χ_p that satisfies the traction condition

$$\mathbf{T} = \pm[(\sigma_{xy})_n x^n \quad (\sigma_{yy})_n x^n]^T \text{ on the crack surfaces } (y = 0^\mp, x < 0),$$

where n is an integer and $(\sigma_{xy})_n, (\sigma_{yy})_n$ are constants.

Problem 19.9. Show Eq. (12).

19.4. ELEMENTS FOR HETEROGENEOUS MATERIALS

Tong and Mei (1992) applied super-element concept to the treated composites made of periodic fibers imbedded in a matrix material. They used the method of homogenization modeling an individual fiber and its surrounding material as a basic cell in the form of a *polygon*. Accorsi (1988), Ghosh and his associates (1991, 1994), Zhang and Katsume (1994) modeled the inclusions of a heterogeneous material as two-dimensional elliptical particles of different sizes, aspect ratios, and orientations scattered in the parent material. They used polygonal elements, called *Voronoi cell elements*, which may contain inclusions of various shapes and sizes. The hybrid approach is adopted to determine the stiffness matrices of such elements (Ghosh and Moorthy 1995, Ghosh and Liu 1995, Dong and Atluri 2011). Such approach can be applied to elastic-plastic and thermo-elastic problems.

19.5. ELEMENTS FOR INFINITE DOMAIN

For problems in infinite domain with rapidly decaying solution at far field, one may approximate domain with finite boundaries far from the region of interest and solve the problem by finite element method. However, for certain problems (such as wave propagation), the disturbance is transmitted to infinity. Any truncation to a finite domain will produce false result.

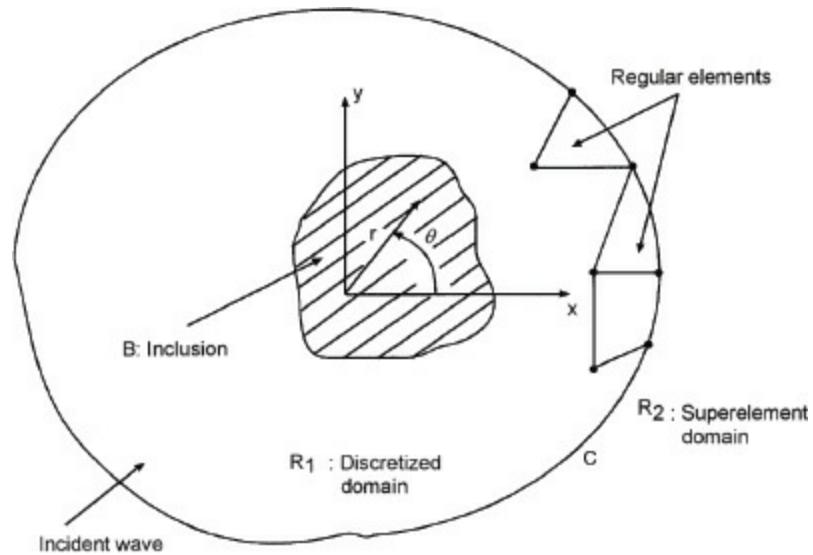


Fig. 19.5:1. A finite-domain element.

Hybrid techniques can be employed to resolve the problem. One separates an infinite domain into R_1 and R_2 by a curve C (Fig. 19.5:1) with R_1 sufficiently large to cover all area of interest and R_2 the remaining part. We approximate R_1 by elements as discussed before and treat R_2 as a single element with asymptotic solutions of infinite domain. A hybrid technique is used to match the two solutions. We shall consider two examples for illustration.

Plane Elasticity. Consider a plane elasticity problem with no body force and decaying deformation at infinity. If neither case is so, we can represent the solution by a particular solution so that the problem is reduced to one with zero body force and decaying deformation at far field. A variational functional for the entire domain is written as

$$\Pi = \Pi_{R_1} + \Pi_{R_2},$$

where Π_{R_1} is the functional of the finite domain R_1 , and Π_{R_2} is the hybrid functional from Eq. (19.2:18) for R_2 . We have

$$(1) \quad \Pi_{R_2} = \int_C (\hat{\mathbf{u}}^T - \mathbf{u}^T/2) \mathbf{T} ds,$$

in which \mathbf{u} is in terms of the decaying asymptotic solutions of the problem, \mathbf{T} is derived from \mathbf{u} , and $\hat{\mathbf{u}}$ is the displacement vector along C between R_1 and R_2 in the form of Eq. (19.2:13) or (19.3:4). The integration along C is carried out one segment at a time. The integral over the circle of radius $R \rightarrow \infty$ vanishes due to the decaying of the asymptotic solutions. Consider the variation $\delta\mathbf{u}$ and the derived $\delta\mathbf{T}$, the inter-region compatibility is ensured in the sense of

$$(2) \quad \int_C (\hat{\mathbf{u}}^T - \mathbf{u}^T) \delta \mathbf{T} ds = 0.$$

We express the asymptotic solution in terms of complex potentials

$$(3) \quad \psi(z) = a_1/z + a_2/z^2 + \cdots + a_r/z^r,$$

$$(4) \quad \chi'(z) = a_{r+1}/z + a_{r+2}/z^2 + \cdots + a_{2r}/z^r,$$

where r is an integer and $a_j = \alpha_{2j-1} + i\alpha_{2j}$ with α 's being real unknowns of a total $4r$ in number. The derived displacements and stresses satisfy the equilibrium and compatibility conditions and decay to zero at infinity. From Eqs. (19.3:5) and (19.3:12), we find \mathbf{u} and \mathbf{T} in the form of Eq. (19.3:13) with G_α 's and h_α 's being

$$\begin{aligned} G_{2j-1} &= -j(1/\bar{z}^{j+1} + 1/z^{j+1})e^{-i\theta} - j(j+1)\bar{z}e^{i\theta}/z^{j+2}, \\ G_{2j} &= i[j(1/\bar{z}^{j+1} - 1/z^{j+1})e^{-i\theta} - j(j+1)\bar{z}e^{i\theta}/z^{j+2}], \\ G_{2r+2j-1} &= je^{i\theta}/z^{j+1}, \quad G_{2r+2j} = -ije^{i\theta}/z^{j+1}, \\ (h_\alpha)_{2j-1} &= (\kappa/z^j + jz/\bar{z}^{j+1})/(2G), \\ (h_\alpha)_{2j} &= i(\kappa/z^j - jz/\bar{z}^{j+1})/(2G), \\ (h_\alpha)_{2r+2j-1} &= -1/(2G\bar{z}^j), \quad (h_\alpha)_{2r+2j} = i/(2G\bar{z}^j), \end{aligned}$$

for $j = 1, 2, \dots, r$. One can write Eq. (1) in the form

$$(5) \quad \Pi_{R_2} = \alpha^T \mathbf{k}_{\alpha q} \mathbf{q} - \alpha^T \mathbf{k}_{\alpha\alpha} \alpha / 2,$$

where

$$\begin{aligned} \mathbf{k}_{\alpha q} &= \int_C [\text{Rl}(\mathbf{G}_\alpha^T \mathbf{h}_u) - \text{Im}(\mathbf{G}_\alpha^T \mathbf{h}_v)] ds, \\ \mathbf{k}_{\alpha\alpha} &= \int_C \text{Rl}(\mathbf{h}_\alpha^T \mathbf{G}_\alpha + \mathbf{G}_\alpha^T \mathbf{h}_\alpha) ds / 2, \end{aligned}$$

which are similar to the element matrices defined in Eq. (19.3:14). Note that \mathbf{G}_α^T and \mathbf{h}_α^T are a $4r \times 1$ column matrix. From the zero condition of the first variation of Π_{R_2} with respect to α , we can eliminate α from Eq. (5) to give

$$\Pi_{R_2} = \mathbf{q}_{\alpha q}^T \mathbf{k}_{\alpha q}^T \mathbf{k}_{\alpha\alpha}^{-1} \mathbf{k}_{\alpha q} \mathbf{q} / 2 = \mathbf{q}^T \mathbf{k} \mathbf{q} / 2.$$

The equivalent element stiffness matrix of the infinite element can be assembled together with the element matrices from R_1 to form the finite element equation.

Long Wave in Potential Flow. Figure 19.5:1 shows an incident wave being scattered by a solid B in the x, y -plane. Under the long wave theory for potential flow (Stoker 1957, Chen and Mei 1974), the velocity potential $\phi \exp(-i\omega t)$ is governed by

$$(6) \quad \nabla \cdot (f \nabla \phi) + \omega^2 \phi / g = 0 \quad \text{in } R_1 + R_2,$$

where $f(x, y)$ is the water depth, and g is the gravitational acceleration. The boundary condition on ∂B is zero normal flow ($\partial\phi/\partial n = 0$). For constant water depth at far field, an incident wave can be written in the form

$$(7) \quad \phi_I = -igc_0 \exp[ikr \cos(\theta - \alpha)] / \omega,$$

where r, θ are the cylindrical coordinates, $k = \omega/\sqrt{gf}$, c_0 a constant and α the angle of the incident wave direction. Interested readers can prove that $\varphi_I \exp(-i\omega t)$ is a plane wave satisfying Eq. (6). The scattered wave potential, defined as $\varphi_s = \phi - \phi_I$, must satisfy the far field conditions

$$(8) \quad \lim_{r \rightarrow \infty} [\sqrt{r}(\partial/\partial r - ik)\phi_s] = 0, \quad \text{and} \quad \lim_{r \rightarrow \infty} (\sqrt{r}\phi_s) \text{ is bounded,}$$

which represents the behavior of an outgoing wave and is called the radiation condition. As before, the hybrid functional can be written as

$$(9) \quad \Pi(\phi, \hat{\phi}) = \Pi_{R_1}(\hat{\phi}) + \Pi_{R_2}(\phi, \hat{\phi}),$$

where

$$\begin{aligned} \Pi_{R_1}(\hat{\phi}) &= \int_{R_1} [f(\nabla \hat{\phi})^2 - \omega^2 \hat{\phi}^2/g] dA/2, \\ \Pi_{R_2}(\phi, \hat{\phi}) &= \int_C \{f(\hat{\phi} - \phi)[\partial(\phi - \phi_I)/\partial n] + h\hat{\phi}[\partial\phi_I/\partial n]\} ds \\ &\quad + \frac{1}{2} \int_{R_2} \{f[\nabla(\hat{\phi} - \phi_I)]^2 - \frac{\omega^2}{g}(\hat{\phi} - \phi_I)^2\} dA - \frac{i}{2} \int_{R \rightarrow \infty} kh(\phi - \phi_I)^2 ds, \end{aligned}$$

in which $\partial\phi/\partial n$ is the outward normal derivative along $\partial R_2 (= C)$. In the equations above, Π_{R_1} is a regular functional over the finite domain surrounding the body B and Π_{R_2} involves integration over an infinite domain. One can show that the vanishing of the first variation of Π with respect to $\hat{\phi}$ and ϕ gives Eq. (6) as the Euler equations for the two field variables in R_1 and R_2 , Eq. (8) as the boundary condition for $\phi_s (= \phi - \phi_I)$, and the matching conditions

$$\hat{\phi} = \phi, \quad \partial\hat{\phi}/\partial n = \partial\phi/\partial n \quad \text{on } C.$$

Let us choose $\phi - \phi_I$ satisfying Eq. (6) and (8) in R_2 *a priori*. We can use the Gauss theorem to eliminate the area integration of Π_{R_2} such that

$$(10) \quad \Pi_{R_2}(\phi, \hat{\phi}) = \int_C (\hat{\phi} - \frac{\phi_s}{2}) \frac{\partial\phi_s}{\partial n} f ds - \int_C \phi_I \frac{\partial\phi_s}{\partial n} f ds + \int_C \hat{\phi} \frac{\partial\phi_I}{\partial n} f ds,$$

which involves only line integral along curve C a finite region. The superelement matrices of R_2 are to be evaluated from Eq. (10).

To construct the finite element equations, we divide R_1 and C into elements as shown in Fig. 19.5:1 and regard R_2 as a superelement. Consider the case of constant f in R_2 , the scattered-wave potential ϕ_s , which satisfies the Helmholtz equation (6) and the radiation condition (8), can be written in the form

$$(11) \quad \begin{aligned} \phi_s(r, \theta) &= \alpha_1 H_0(kr) + \sum_{j=1}^m [\alpha_{2j} \cos(j\theta) + \alpha_{2j+1} \sin(j\theta)] H_j(kr) \\ &= \frac{G(r, \theta)}{\alpha}, \end{aligned}$$

where m is an integer, H 's are the Hankel functions of the first kind and α 's are real unknowns of a total of $2m + 1$ in number.

Let us divide C into N segments with the inter-segment points labeled as nodes 1 through N , and define $\text{mod}(j) = j$ for $j \leq N$ and $\text{mod}(j) = 1$ for $j > N$. Along C , we assume $\hat{\phi}$ of R_2 to be the same as that of R_1 . For linear interpolation between nodes j and $\text{mod}(j + 1)$ segment, $\hat{\phi}(s)$ is written in the form

$$(12) \quad \hat{\phi}(s) = h_j(s)q_j + h_{\text{mod}(j+1)}(s)q_{\text{mod}(j+1)},$$

where s is the distance measuring from node j that

$$h_j(s) = 1 - s/s_{j,\text{mod}(j+1)}, \quad h_{\text{mod}(j+1)}(s) = s/s_{j,\text{mod}(j+1)},$$

and all unlisted entries of \mathbf{h} are zero. For quadratic interpolation between nodes j and $\text{mod}(j + 1)$, we have

$$(13) \quad \hat{\phi}(s) = h_j(s)q_j + h_{\text{mod}(j+1)}(s)q_{\text{mod}(j+1)} + h_{j+N}(s)q_{j+N},$$

where

$$\begin{aligned} h_j(s) &= (1 - s/s_{j,\text{mod}(j+1)})(1 - 2s/s_{j,\text{mod}(j+1)}) , \\ h_{\text{mod}(j+1)}(s) &= (2s/s_{j,\text{mod}(j+1)} - 1)s/s_{j,\text{mod}(j+1)} , \\ h_{j+N}(s) &= 4(1 - s/s_{j,\text{mod}(j+1)})s/s_{j,\text{mod}(j+1)} , \end{aligned}$$

in which node $j + N$ is the mid-point nodal of the element. All unlisted entries of \mathbf{h} are zero.

For the case of M elements with mid-point nodes, substituting Eq. (11)–(13) into Eq. (10) yields

$$(14) \quad \Pi_{R2} = \alpha^T \mathbf{k}_{\alpha q} \mathbf{q} - \alpha^T \mathbf{k}_{\alpha \alpha} \alpha / 2 - \alpha^T \mathbf{f}_\alpha - \mathbf{q}^T \mathbf{f}_q ,$$

where

$$\begin{aligned} \mathbf{k}_{\alpha \alpha}^{(2m+1) \times (2m+1)} &= \int_C \{\mathbf{G}^T(r, \theta) [\partial \mathbf{G}(r, \theta) / \partial n] + [\partial \mathbf{G}^T(r, \theta) / \partial n] \mathbf{G}(r, \theta)\} f ds \\ \mathbf{k}_{\alpha q}^{(2m+1) \times (N+M)} &= \sum_{j=1}^N \int_{\text{node } j}^{\text{node mod}(j+1)} [\partial \mathbf{G}^T(r, \theta) / \partial n] \mathbf{h}(s) f ds , \\ (15) \quad \mathbf{f}_\alpha^{(2m+1) \times 1} &= \int_C [\partial \mathbf{G}^T(r, \theta) / \partial n] \phi_I(r, \theta) f ds , \\ \mathbf{f}_q^{(N+M) \times 1} &= - \sum_{j=1}^N \int_{\text{node } j}^{\text{node mod}(j+1)} \mathbf{h}^T(s) [\partial \phi_I(r, \theta) / \partial n] f ds . \end{aligned}$$

From $\delta \Pi_{R2} = 0$ of Eq. (11) with respect to α , we can eliminate α to obtain

$$(16) \quad \Pi_{R2} = \mathbf{q}^T \mathbf{k} \mathbf{q} / 2 - \mathbf{q}^T \mathbf{f} + \mathbf{f}_\alpha^T \mathbf{k}_{\alpha \alpha}^{-1} \mathbf{f}_\alpha ,$$

in which

$$\alpha = \mathbf{k}_{\alpha \alpha}^{-1} \mathbf{k}_{\alpha q} \mathbf{q} - \mathbf{k}_{\alpha \alpha}^{-1} \mathbf{f}_\alpha , \quad \mathbf{k} = \mathbf{k}_{\alpha q}^T \mathbf{k}_{\alpha \alpha}^{-1} \mathbf{k}_{\alpha q} , \quad \mathbf{f} = \mathbf{k}_{\alpha q}^T \mathbf{k}_{\alpha \alpha}^{-1} \mathbf{f}_\alpha + \mathbf{f}_q .$$

These element matrices are to be assembled, like the regular element matrices, to form the algebraic equation for \mathbf{q} . The size of $\mathbf{k}_{\alpha \alpha}$ can be quite large since there can be many nodes along the entire curve C . One may assemble the matrices of the infinite element with the element matrices of R_1 without first eliminating α and solve for \mathbf{q} and α simultaneously.

Using the orthogonal property of the Hankel function, one can show that $\mathbf{k}_{\alpha \alpha}$ becomes diagonal, if one chooses C as a circle of radius R , that is,

$$(17) \quad \alpha^T \mathbf{k}_{\alpha \alpha} \alpha = \pi R k f [2\alpha_1^2 H_0(kR) H'_0(kR) + \sum_{j=1}^m (\alpha_{2j}^2 + \alpha_{2j+1}^2) H_j(kR) H'_j(kR)]$$

where the prime denotes differentiation with respect to r and

$$\begin{aligned} \mathbf{f}_\alpha &= \int_0^{2\pi} \partial \mathbf{G}^T(r, \theta) / \partial r \Big|_{r=R} \phi_I(R, \theta) f R d\theta , \\ \mathbf{f}_\alpha &= \int_0^{2\pi} \partial \mathbf{G}^T(r, \theta) / \partial r \Big|_{r=R} \phi_I(R, \theta) f R d\theta , \\ (18) \quad \alpha^T \mathbf{f}_\alpha &= \alpha^T \int_0^{2\pi} \partial \mathbf{G}^T(r, \theta) / \partial r \Big|_{r=R} \phi_I(R, \theta) f R d\theta \\ &= - \int_0^{2\pi} \frac{ig c_0 k f}{\omega} \{ \alpha_1 H'_0(kR) + \sum_{j=1}^m [\alpha_{2j} \cos(j\theta) \\ &\quad + \alpha_{2j+1} \sin(j\theta)] H'_j(kR) \} \exp[ikR \cos(\theta - \alpha)] R d\theta . \end{aligned}$$

Recall that f is the water depth at inter-boundaries of regions R_1 and R_2 .

A more general discussion of water-wave problems and their finite-element solutions can be found in Chen and Mei (1974) and Mei and Chen (1975).

19.6. INCOMPRESSIBLE OR NEARLY INCOMPRESSIBLE ELASTICITY

The volume of many materials are essentially preserved in deformation. Rubber is often modeled as such a material. For incompressible or nearly incompressible isotropic Hookian materials, Poisson's ratio approaches 1/2 and

$$K = \lambda - 2G/3 = 2G(1 + \nu)/[3(1 - 2\nu)] \rightarrow \infty,$$

where K is the bulk modulus and λ the Lamé constant [Eq. (6.2:9)]. The limit creates problems since the constitutive equations Eq. (6.2:7) are

$$\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + G(u_{i,j} + u_{j,i}) = \lambda e_{kk} \delta_{ij} + G\gamma_{ij}.$$

An alternative formulation is to include the *hydrostatic pressure* p as a dependent field and the penalty function, respectively,

$$p/K = -e = -u_{k,k}, \quad f = K(u_{k,k})^2 + p u_{k,k} + p^2/K,$$

in which e is the *dilatation*. A penalty type of variational functional (Sec. 10.10) for an element can be written as

$$(1) \quad \Pi_e = \int_{V_e} \{ \mathbf{e}^T \mathbf{D}_I \mathbf{e} / 2 - \alpha [p^2 / (2K) + p u_{k,k}] - \mathbf{u}^T \mathbf{b} \} dV - \int_{\partial V_{\sigma_e}} \mathbf{u}^T \bar{\mathbf{T}} dS,$$

where

$$(2) \quad \mathbf{D}_I = \mathbf{D}_e - \alpha K \tilde{\mathbf{D}}, \quad \tilde{\mathbf{D}}_{6 \times 6} = \begin{bmatrix} 1 & & & & & \text{sym} \\ 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ & \mathbf{0} & & \mathbf{0} & & \\ & 1 \times 3 & & & 3 \times 3 & \end{bmatrix},$$

$$\mathbf{e}^T = [e_{11} \quad e_{22} \quad e_{33} \quad \gamma_{12} \quad \gamma_{23} \quad \gamma_{31}],$$

\mathbf{D}_e is the elastic modulus matrix (see Sec. 18.14 for plane and Sec. 18.16 for 3-dimensional elasticity), $\mathbf{0}$ a zero matrix, K the bulk modulus and α a constant of the magnitude of order 1 chosen to ensure that \mathbf{D}_I is bounded and positive definite.

For isotropic materials, we choose

$$\alpha = (\lambda - 2G\beta)/K,$$

where β is also selected a constant of order 1. Then

$$(3) \quad \mathbf{D}_I = \mathbf{D}'_e + 2\beta G \tilde{\mathbf{D}}, \quad \mathbf{D}'_e = G \text{dia}(2 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1).$$

For $\beta \geq 0$, \mathbf{D}_I is positive definite. If $\beta = 0$, Eq. (1) reduces to the conventional mixed functional (Herrmann 1965, Tong 1969) involving the two fields \mathbf{u} and p .

$$(4) \quad \Pi_e = \int_{V_e} [\mathbf{e}'^T \mathbf{D}'_e \mathbf{e}' / 2 - p^2 / (2K) - p u_{k,k} - \mathbf{u}^T \mathbf{b}] dV - \int_{\partial V_{\sigma_e}} \mathbf{u}^T \bar{\mathbf{T}} dS,$$

where $\mathbf{e}'^T = [e'_{xx} \quad e'_{yy} \quad e'_{zz} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx}]$ is the *deviatoric engineering strain matrix*, and p plays the role of the Lagrange multiplier to enforce the pressure-dilatation relation. In the limiting case of incompressible materials, it enforces the condition of zero dilatation.

The functional above involves both strain and pressure and, therefore, is a mixed model. Admissibility simply requires that \mathbf{u} is C^0 continuous and satisfies the prescribed displacement conditions. Discontinuous p is permitted. The convergence of mixed methods can be found in Tong (1969), Babuska (1971), Brezzi and Pitkaranta (1974), Oden and Carey (1984), and Brezzi (1984).

In finite element formulation, we express

$$(5) \quad \mathbf{u} = \mathbf{h}\mathbf{q}, \quad p = \mathbf{g}\mathbf{r}, \quad e = \text{div}(\mathbf{u}) = \text{dia}(\partial/\partial x \quad \partial/\partial y \quad \partial/\partial z)\mathbf{h}\mathbf{q} = \mathbf{B}_d\mathbf{q},$$

where, as usual, q 's are the nodal values of \mathbf{u} , r 's are those of p , usually of different nodes, and \mathbf{h} and \mathbf{g} are interpolation matrices. The h 's is in the form of Eq. (18.14:9) for plane or axisymmetric problems and Eq. (18.16:11) for three-dimensional elasticity. In general, we use low order shape functions for \mathbf{g} because p has no continuity requirement.

A substitution of Eq. (5) into (1) yields

$$(6) \quad \Pi_e = \mathbf{q}^T \mathbf{k}_{qq} \mathbf{q} / 2 - \mathbf{r}^T \mathbf{k}_{rq} \mathbf{q} - \mathbf{r}^T \mathbf{k}_{rr} \mathbf{r} / 2 - \mathbf{q}^T \mathbf{f},$$

where

$$(7) \quad \begin{aligned} \mathbf{k}_{qq} &= \int_{V_e} \mathbf{B}^T \mathbf{D}_I \mathbf{B} dV, & \mathbf{k}_{rq} &= \alpha \int_{V_e} \mathbf{g}^T \mathbf{B}_d dV, \\ \mathbf{k}_{rr} &= \alpha \int_{V_e} \mathbf{g}^T \mathbf{g} dV / K, & \mathbf{f} &= \int_{V_e} \mathbf{h}^T \mathbf{b} dV + \int_{\partial V_{se}} \mathbf{h}^T \bar{\mathbf{T}} dS, \end{aligned}$$

with \mathbf{B} being defined in Eq. (18.16:11) and \mathbf{f} the same as that in Eq. (18.16:13). We then can derive the global stiffness and force matrices by assembling the element matrices in the usual manner.

One may assume r 's separately for each element to be eliminated before assembling. The procedure fails because \mathbf{k}_{rr} becomes zero for incompressible materials.

Arbitrary combinations of displacement and pressure interpolations may not be effective. Each pressure DOF is equivalent to a constraint to the deformation mode. Too few pressure parameters can lead to poor approximation and/or spurious deformation with zero strain energy mode(s), which can make the finite element equations unsolvable. For example, pure dilatation with zero pressure ($\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = p = 0$, and $e_{xx} = e_{yy} = e_{zz}$) give zero strain energy in Eq. (4).

Too many pressure modes will overly constrain the deformation and lead to locking. Consider the constant strain triangle with one pressure unknown. For the element mesh shown in Fig. 19.6:1, the constant volume (area) condition for element 2 precludes the horizontal motion of node 1, while the same condition for element 1 restrains node 1 to move only horizontally. As a result, the displacements at node 1 can only be zero. With displacements at node 1 being zero, it infers that node 2 has zero displacements and so on. This leads zero displacements for every node in the entire mesh, no matter how many elements are in the mesh.

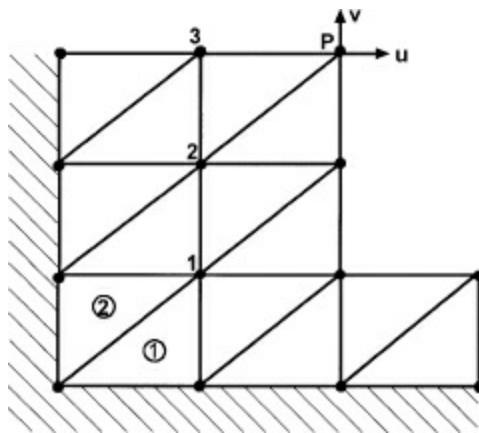


Fig. 19.6:1 Mesh for which incompressibility dictates zero displacement.

Hughes (1987) introduced a heuristic approach to predict the performance of an element in incompressible and nearly incompressible applications. Let n_{eq} be the total number of displacement equations after boundary constraints imposed and n_c be that of incompressibility constraints. For linear problems, n_c is the total number of pressure unknowns. Hughes defined the *constraint ratio* as

$$r = n_{eq} / n_c$$

and conjectured that r should mimic the space dimension of the problem. So in two-dimension, r should be 2. If $r \leq 1$, there are more constraints on \mathbf{q} than the number of q 's available. Severe locking can be resulted. If r is greater than 2, it indicates a lack of incompressibility constraint which can lead to poor solution. The results are illustrated in Fig. 19.6:2 (Hughes 1987).

Another approach to handle the nearly incompressible issue is to split the elastic coefficient matrix \mathbf{D}_e into the G -part (shear) and λ -part (Lamé constant) (Hughes 1987)

$$\mathbf{D}_e = \hat{\mathbf{D}} + \lambda \tilde{\mathbf{D}},$$

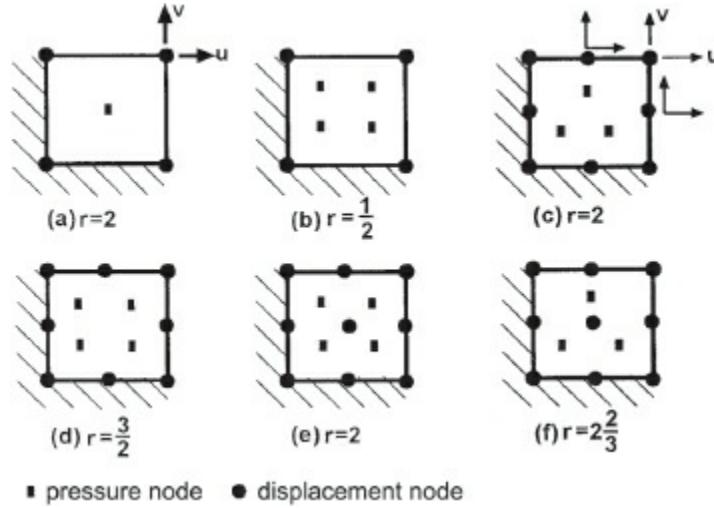


Fig. 19.6.2. Rectangular elements with discontinuous pressure field.

where $\hat{\mathbf{D}} = \mathbf{D}'$ is given in Eq. (3) and $\tilde{\mathbf{D}}$ is given in Eq. (2) for isotropic materials. For plane strain,

$$\hat{\mathbf{D}} = G \text{dia} (2 \quad 2 \quad 1), \quad \tilde{\mathbf{D}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using the displacement formulation, from Eq. (18.16:12), the element stiffness matrix can be written in the form

$$\mathbf{k} = \hat{\mathbf{k}} + \tilde{\mathbf{k}},$$

where

$$\hat{\mathbf{k}} = \int_{V_e} \mathbf{B}^T \hat{\mathbf{D}} \mathbf{B} dV, \quad \tilde{\mathbf{k}} = \lambda \int_{V_e} \mathbf{B}^T \tilde{\mathbf{D}} \mathbf{B} dV.$$

If λ/G is very large, the numerical values of $\tilde{\mathbf{k}}$ can much larger than that of $\hat{\mathbf{k}}$. One simple and important practical way to lower the stiffness of $\tilde{\mathbf{k}}$ is to reduce the order of numerical quadrature by 1 below that ‘normally’ used in evaluating $\hat{\mathbf{k}}$. For instance, for the Gauss quadrature, if we use $(p+1)$ -quadrature to evaluate $\hat{\mathbf{k}}$, we will use only p -point to evaluate $\tilde{\mathbf{k}}$. This is a form of *selectively-reduced integration*, a remedy to eliminate shear locking discussed before. Malkus and Hughes (1978) showed that many elements of mixed formulation are equivalent to elements of selectively-reduced integration.

¹We shall use i as $\sqrt{-1}$ throughout the remaining chapter.

20

FINITE ELEMENT METHODS FOR PLATES AND SHELLS

Plates and shells are particular forms of a three-dimensional solid in which the thickness is small as compared with other dimensions. Conceptually, plates and shells do not present difficulty for the finite element method (FEM). However, the 3-D elements for thin plates and shells have very large aspect ratio (in-plane dimension to thickness), which makes the elements overly stiff in bending, known as '*locking*' (Sec. 18.20). To obtain a solution of acceptable accuracy, it often requires elements of size of the order of thickness. This could be costly in computation.

In this chapter, we shall discuss FEM modeling of plates and shells based on the Kirchhoff (Secs. 13.14 and 13.15) and Reissner–Mindlin theories. The latter is for moderately thick plates. These theories use linear stress through the thickness to reduce the problem to two-dimension. For functionally graded or thick-section laminated composites, an alternative element model using linear strains and over-integration in the thickness direction will be discussed (Atluri *et al.* 2014a,b and an extension by Tong in an unpublished note). We will also discuss reduced integration and mixed formulations to avoid locking. The new elements work well for thick as well as thin beams, plates, and shells, and can be combined with other 3-D C^0 elements. Throughout the chapter, t will be used as the plate and shell thicknesses.

20.1. LINEARIZED BENDING THEORY OF THIN PLATES¹

Let the x, y plane of Cartesian coordinates be the midplane of the plate. The *Kirchhoff theory* assumes that (1) the deformation in the midplane is like that of the plane stress discussed in Sec. 9.1, (2) strains vary linearly along lines normal to the midplane, and (3) the normal lines remain normal to the deformed midsurface after deformation. Under the Kirchhoff assumptions, the displacements at any point of the plate are given by

$$u = u_0 - z \frac{\partial w}{\partial x}, \quad v = v_0 - z \frac{\partial w}{\partial y}, \quad w = w,$$

where z is the distance from the midplane. u_0 and v_0 are the in-plane displacements associated with midplane stretching and shear, w relates to bending, and $\partial w / \partial y$ and $-\partial w / \partial x$ represent the rotations along the x , y -axes, respectively. In linearized theory midplane stretching and bending are decoupled. In Secs. 1 through 4, since only plate bending is considered, we shall drop terms involving u_0 and v_0 . When shells are considered, we will include the mid-surface in-plane displacements to account for stretching and bending coupling.

The bending strains associated with the off-midplane deflection are proportional to the change of curvatures of the midplane, denoted as

$$(1) \quad \kappa = [\kappa_x \quad \kappa_y \quad \kappa_{xy}]^T = [\partial^2 / \partial^2 x \quad \partial^2 / \partial^2 y \quad 2\partial^2 / \partial x \partial y]^T w = \mathbf{d}_b w,$$

where \mathbf{d}_b is a 3×1 by matrix. Let \mathbf{M} be the *bending moments* and \mathbf{Q} be *transverse shear forces* per unit length on the mid-surface in the form

$$(2) \quad \mathbf{M} = [M_x \quad M_y \quad M_{xy}]^T, \quad \mathbf{Q} = [Q_x \quad Q_y]^T.$$

The *constitutive law* (13.14:19) gives

$$(3) \quad \mathbf{M} = -\mathbf{D}_b \kappa = -\mathbf{D}_b \mathbf{d}_b w,$$

where \mathbf{D}_b is the *bending rigidity matrix*. For isotropic elastic material,

$$(4) \quad \mathbf{D}_b = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix},$$

with t being the thickness of the plate (see Sec. 6.2). The shear \mathbf{Q} is related to \mathbf{M} through the moment equilibrium equation (13.14:13).

Consider a domain A with the following two sets of boundary conditions:

$$(5) \quad w = \bar{w} \quad \text{on } \partial A_w,$$

$$(6) \quad n_x Q_x + n_y Q_y - dM_t/ds = \overline{Q - dM_t/ds}, \quad \text{on } \partial A_Q,$$

and

$$(7) \quad \partial w / \partial n = \overline{\partial w / \partial n}, \quad \text{on } \partial A_{w_n},$$

$$(8) \quad M_n = n_x^2 M_x + 2n_x n_y M_{xy} + n_y^2 M_y = \overline{M_n}, \quad \text{on } \partial A_{m_n}.$$

Note that $M_t = n_x n_y (M_y - M_x) + (n_x^2 - n_y^2) M_{xy}$. The overbar denotes prescribed quantities, M_n , M_t are the normal and twist moments, n_x , n_y are the direction cosines of the outward normal to the boundary and $\partial/\partial n$ denotes the outward normal derivative along the boundary, and M 's and Q 's are in terms of ω through the constitutive law. At any given boundary location, only one condition from Eq. (5) or (6) and one from Eq. (7) or (8) are specified. Equations (5a) and (6a) are *rigid* and (5b) and (6b) are *natural boundary conditions*.

The functional for the *principle of minimum potential energy theorem* is

$$\Pi = \int_A [(\mathbf{d}_b w)^T \mathbf{D}_b \mathbf{d}_b w / 2 - pw] dA + \int_{\partial A_{mn}} \bar{M}_n \partial w / \partial n ds - \int_{\partial A_Q} \bar{Q} w ds,$$

where $p = p(x, y)$ is the lateral pressure. Provided that ω is C^1 continuous, the functional can be written as the sum of the functional of all elements:

$$(9) \quad \Pi = \sum_{\text{all elements}} (\Pi_{be} - \Pi_{bfe})$$

where

$$\Pi_{be} = \int_{A_e} (\mathbf{d}_b w)^T \mathbf{D}_b \mathbf{d}_b w dA / 2,$$

$$\Pi_{bfe} = \int_{A_e} pw dA - \int_{\partial A_{mne}} \bar{M}_n \partial w / \partial n ds + \int_{\partial A_{Qe}} \bar{Q} w ds,$$

in which the subscript e denotes element level quantities, Π_{be} is the element strain energy, and Π_{bfe} the external work. The admissibility condition requires that ω is C^1 continuous and satisfies the rigid conditions. For *convergence*, the shape functions must be able to approximate the *state of constant curvature* within an element, i.e., have *complete quadratic terms*.

Constructing multi-dimensional C^1 -interpolations created difficulties in the early development of the finite element method, especially for triangular elements. Elements with only C^0 shape functions are non-conforming or incompatible. Various conforming C^1 plate elements of the displacement model have subsequently been developed. The compatible shape functions are generally complicated. The elements are stiff and converge slowly. Hybrid and mixed formulations are alternatives to circumvent the difficulty.

Conforming elements, especially the lower order ones, may yield inferior results as compared to non-conforming elements, which are widely used in practice. For many structural applications, abrupt change in thickness or discrete stiffening or members meeting at angles are part of the design. At such locations, C^1 continuity should *not* be enforced.

Higher order conforming bending elements can improve the accuracy. Unfortunately they are generally complicated. If one weighs the expenses against the incremental increase in accuracy, simpler non-compatible elements could be a good choice. However, in specific applications where accurate stress (bending moments, which are proportional to the second derivatives of w) prediction is important, especially in regions of high stress gradients, higher-order elements may be necessary.

In the following, two non-compatible elements based the variational functional given in Eq. (9) will be developed.

Incompatible Rectangular Element. Consider the rectangular plate element in the x, y -plane shown Fig. 20.1:1 and take the nodal values of ω , ω_x , ω_y at the four corners as the generalized coordinates that

$$\mathbf{q}_b^T = [w(\xi_1, \eta_1) \quad w_{,x}(\xi_1, \eta_1) \quad w_{,y}(\xi_1, \eta_1) \quad w(\xi_2, \eta_1) \quad w_{,x}(\xi_2, \eta_1) \quad \dots],$$

where $\xi_1 = \eta_1 = -1$, $\xi_2 = \eta_2 = 1$, and $\xi = x/a$, $\eta = y/b$ are the *natural coordinates*. We approximate ω within the element in the form

$$(10) \quad w(x, y) = \sum_{i=1}^2 \sum_{j=1}^2 [h_{ij}w(\xi_i, \eta_j) + h_{xij}w_{,x}(\xi_i, \eta_j) + h_{yij}w_{,y}(\xi_i, \eta_j)] = \mathbf{h}\mathbf{q}_b,$$

where

$$(11) \quad \begin{aligned} \mathbf{h} &= [h_{11} \quad h_{x11} \quad h_{y11} \quad h_{21} \quad h_{x21} \quad h_{y21} \quad \dots \quad h_{y12}], \\ h_{ij} &= \xi_i \eta_j (2 + \xi \xi_i + \eta \eta_j - \xi^2 - \eta^2) (\xi + \xi_i) (\eta + \eta_j) / 8, \\ h_{xij} &= \eta_j (\xi^2 - 1) (\xi + \xi_i) (\eta + \eta_j) a / 8, \\ h_{yij} &= \xi_i (\eta^2 - 1) (\xi + \xi_i) (\eta + \eta_j) b / 8, \quad (i, j \text{ not summed}). \end{aligned}$$

Note that the polynomials in Eq. (10) are bicubic, continuity of ω and its tangential derivative along the boundary with the same type of adjacent element are assured. However, since both ω_x and ω_y are cubic, the *normal derivative* to the boundary is cubic also. The normal derivative of ω is matched only at the two nodes. Its *normal derivative* (or *normal slope*) along the interelement boundary, discontinuity in of ω is expected. Thus, the interpolation function given in Eq. (10) is not compatible.

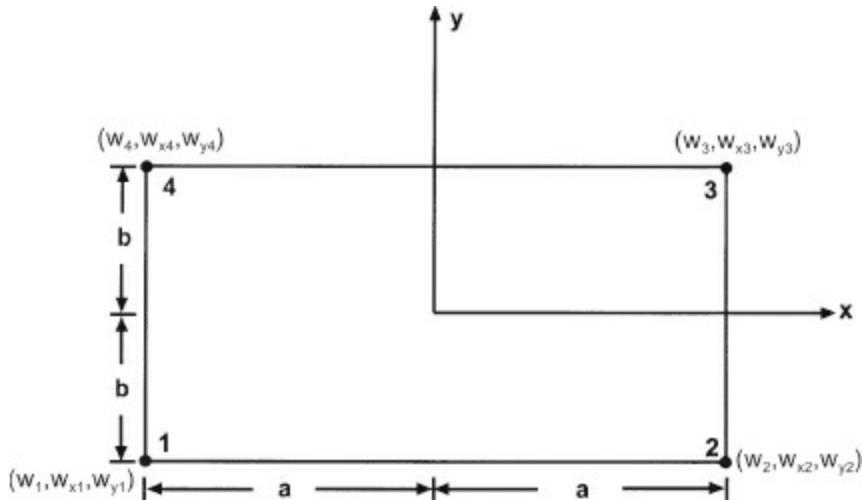


Fig. 20.1:1. A rectangular element.

Using more DOFs can resolve the problem. For example, if one adds the twist ω_{xy} as an additional DOF at each node, the compatibility of normal slope along the interelement boundaries can be achieved. The four terms added to the interpolating polynomials are

$$\alpha_{13}x^2y^2 + \alpha_{14}x^3y^2 + \alpha_{15}x^2y^3 + \alpha_{16}x^3y^3.$$

One has to modify Eq. (10) accordingly.

A substitution of Eq. (10) into Eq. (9) yields the element matrices

$$\Pi_{be} - \Pi_{bfe} = \mathbf{q}_b^T \mathbf{k}_b \mathbf{q}_b / 2 - \mathbf{q}_b^T \mathbf{f}_b,$$

where

$$(12) \quad \mathbf{k}_b = \int_{A_e} (\mathbf{d}_b \mathbf{h})^T \mathbf{D}_b \mathbf{d}_b \mathbf{h} dA,$$

$$(13) \quad \mathbf{f}_b = \int_{A_e} p(x, y) \mathbf{h}^T dA - \int_{\partial A_{mne}} \bar{M}_n \partial \mathbf{h}^T / \partial n ds + \int_{\partial A_{Qe}} \bar{Q} \mathbf{h}^T ds,$$

with

$$(\mathbf{d}_b \mathbf{h})^T = \left[\frac{1}{a^2} \frac{\partial^2 \mathbf{h}^T}{\partial \xi^2} \quad \frac{1}{b^2} \frac{\partial^2 \mathbf{h}^T}{\partial \eta^2} \quad \frac{2}{ab} \frac{\partial^2 \mathbf{h}^T}{\partial \xi \partial \eta} \right].$$

Equations (12) and (13) can be integrated analytically if \mathbf{D}_b and p are constant, or numerically otherwise.

Distorted quadrilateral elements based on the isoparametric transformation of Eq. (18.13:3) perform badly. In this

case, the interpolated function cannot represent exactly the deformation of constant curvature except for parallelogram (Dawe 1966, Argyris 1966).

Incompatible Triangular Elements. Considerable difficulty was encountered in constructing the C^1 shape functions directly in terms of x, y in early development of triangular elements. For an element with three degrees-of-freedom per node, as shown in Fig. 20.1:2, there are only nine parameters whereas a complete cubic polynomial has ten terms. Several possibilities to limit the number of unknowns to nine have been investigated. Various problems arose, which include the lack of symmetric appearance of the ten terms, the non-uniqueness of shape functions, etc.

Specht (1988) who assumed ω in the form $w = \frac{\mathbf{P}}{1 \times 9 \times 1} \alpha$, where

$$P_i = \zeta_i, \quad P_{i+3} = \zeta_j \zeta_k, \quad P_{i+6} = \zeta_i^2 \zeta_j + \zeta_1 \zeta_2 \zeta_3 [3(1 - \mu_k) \zeta_i + (1 + 3\mu_k)(\zeta_k - \zeta_j)],$$

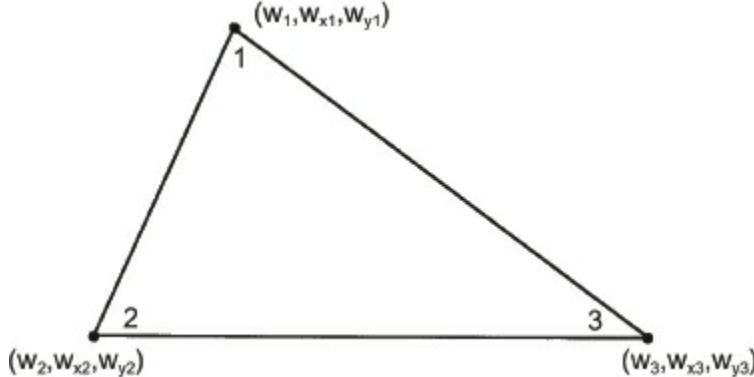


Fig. 20.1:2. Triangular element with three degrees of freedom per node.

with $i = 1, 2, 3$, $\mu_i = (l_k^2 - l_j^2)/l_i^2$, i, j, k (not summed) the cyclic permutation of 1, 2, 3, and l_k the length of side l of the triangle. The shape functions associated with the nodal unknowns $\omega, \omega_{xi}, \omega_{yi}$ at node i are

$$\begin{bmatrix} h_{3(i-1)+1} \\ h_{3(i-1)+2} \\ h_{3(i-1)+3} \end{bmatrix} = \begin{bmatrix} \zeta_i(1 - \zeta_j + \zeta_k) + 2(P_{i+6} - P_{k+6}) \\ -b_j(P_{k+6} - \zeta_i \zeta_k) - b_k P_{i+6} \\ -c_j(P_{k+6} - \zeta_i \zeta_k) - c_k P_{i+6} \end{bmatrix}, \quad i = 1, 2, 3,$$

where b 's and c 's are defined in Eq. (18.9:5). The shape functions are C^0 continuous and satisfy the patch test. The parameter μ_i relates to the normal derivative of side i by

$$\partial/\partial n = l_i [\partial/\partial \zeta_j + \partial/\partial \zeta_k - 2\partial/\partial \zeta_i + \mu_i (\partial/\partial \zeta_k - \partial/\partial \zeta_j)]/(4\Delta).$$

The element matrices are given by Eqs. (12) and (13) with $A_e = \Delta$ being the element area and \mathbf{d}_b the differential operator defined in Eq. (20.1:1):

$$(14) \quad \mathbf{d}_b = \sum_{i,j=1}^3 [b_i b_j \quad c_i c_j \quad 2b_i c_j]^T [\partial^2/(\partial \zeta_i \partial \zeta_j)]/(4\Delta^2).$$

The integrals in Eqs. (12) and (13) can be evaluated analytically using Eq. (18.10:8). However, the explicit integration is lengthy and error prone (Cheung *et al.* 1968). It is more practical to use numerical integration.

Simpler elements of C^0 continuity can be constructed. Morley (1971) proposed an element with ω at the triangle vertices and its normal slope at the element midsides as nodal unknowns. The shape functions form the complete quadratic polynomials and can represent constant curvatures. For homogenous plate, the element is same as the hybrid element of the same displacement at the element boundaries because the constant curvatures satisfy the moment equilibrium equations. With constant moments, the element also satisfies inter-element equilibrium conditions (Hinton and Huang 1986, Zienkiewicz *et al.* 1990). For incompatible elements, we have to rely on the *patch test* to ensure the convergence of the approximate solution. Constructing C^1 shape functions is much harder than constructing C^0 functions.

The differential equation for the out-plane displacement ω is fourth order. The *condition number* (see Sec. 18.5 and Tong 1970c) of the resulting stiffness matrix grows as $1/\epsilon^4$ where ϵ is the minimum dimension of all elements. The condition number grows rapidly as the size of elements decreases, which can lead to round-off problem in the finite element solution of fine meshes.

The discussion up to this point has been focused on resolving compatibility difficulties of displacement models.

Alternative is to use the hybrid method (Pian 1964, Tong 1970c, Atluri *et al.* 2014a,b) to be discussed later. The success of the hybrid approach lies in the flexibility in formulation, whereby stress and displacement fields are assumed over different parts of the element. Hybrid plate elements have yielded excellent results for stresses, deflections and vibration (Mau *et al.* 1973, Rossettos *et al.* 1972, Spilker and Munir 1980).

20.2. REISSNER–MINDLIN PLATES

For moderately thick plates (Fig. 13.14:5) the transverse shears across the thickness of the plate become significant. The *Reissner–Mindlin plate theory* (Reissner 1946) accounts for the shears by allowing the rotations of the fibers normal to the midplane about the x -, y -axes be different from the slopes w_y and $-w_x$. Like thin plates, the in-plane deformations still decouple from bending and can be considered separately. The theory also assumes zero normal stress σ_{zz} and the off-midplane displacements in the form

$$(1) \quad u = -z\theta_y, \quad v = -z\theta_x, \quad w = w(x, y),$$

where θ_y and $-\theta_x$ are the rotations of the cross sectional plane about the x - and y -axes, respectively. The transverse shears are then $w_x - \theta_x$ and $w_y - \theta_y$. Strictly speaking the assumption $\sigma_{zz} = 0$ contradicts the assumption $\omega = w(x, y)$. However, the inconsistency does not affect the usefulness of the theory in applications.

Following the procedure for the Kirchhoff plate theory, the Reissner–Mindlin plate functional can be written as

$$(2) \quad \Pi_R = \sum_{\text{all elements}} \Pi_{Re} = \sum_{\text{all elements}} (\Pi_{be} + \Pi_{\gamma e} - \Pi_{fe}),$$

where

$$\Pi_{be} = \int_{A_e} (\mathbf{d}_e \boldsymbol{\theta})^T \mathbf{D}_b \mathbf{d}_e \boldsymbol{\theta} dA/2,$$

$$(3) \quad \begin{aligned} \Pi_{\gamma e} &= \int_{A_e} (\mathbf{d}\boldsymbol{\gamma})^T \mathbf{D}_s \mathbf{d}\boldsymbol{\gamma} dA/2, \\ \Pi_{fe} &= \int_{A_e} p w dA - \int_{\partial A_{mne}} \bar{M}_n \theta_n ds - \int_{\partial A_{mte}} \bar{M}_t \theta_t ds + \int_{\partial A_{Qe}} \bar{Q} w ds, \end{aligned}$$

$$(4) \quad \boldsymbol{\theta}^T = [\theta_x \ \theta_y], \quad \boldsymbol{\kappa} = [\kappa_x \ \kappa_y \ \kappa_{xy}]^T = -\mathbf{d}_e \boldsymbol{\theta}, \quad \boldsymbol{\gamma} = [\gamma_x \ \gamma_y]^T = \mathbf{d}w - \boldsymbol{\theta},$$

$$\mathbf{d}_e^T = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}, \quad \mathbf{d} = \partial/\partial \mathbf{x} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix},$$

in which overbars denote prescribed quantities on the boundaries, $\boldsymbol{\theta}$, $\boldsymbol{\kappa}$ and $\boldsymbol{\gamma}$ are the rotation, and the bending and the transverse shear matrices, respectively, with the differential operator matrices \mathbf{d}_e and \mathbf{d} as defined in Eqs. (18.11:2) and (18.13:1). The constitutive law is

$$(5) \quad \mathbf{M} = -\mathbf{D}_b \boldsymbol{\kappa} = \mathbf{D}_b \mathbf{d}_e \boldsymbol{\theta}, \quad \mathbf{Q} = \mathbf{D}_s (\mathbf{d}w - \boldsymbol{\theta}),$$

where \mathbf{D}_b (3×3) and \mathbf{D}_s (2×2) are symmetric and positive definite. For isotropic materials, \mathbf{D}_b is given in Eq. (20.1:4) and

$$(6) \quad \mathbf{D}_s = \kappa t G \mathbf{I},$$

where κ ($\approx 5/6$) is a constant to account for the nonuniform shear through the thickness. The boundary conditions can be divided into three sets:

$$(7a) \quad w = \bar{w} \quad \text{on } \partial A_{we},$$

$$(7b) \quad n_x Q_x + n_y Q_y = \bar{Q} \quad \text{on } \partial A_{Qe};$$

$$(8a) \quad n_x \theta_x + n_y \theta_y = \bar{\theta}_n \quad \text{on } \partial A_{\theta ne},$$

$$(8b) \quad n_x^2 M_x + 2n_x n_y M_{xy} + n_y^2 M_y = \bar{M}_n \quad \text{on } \partial A_{Mne};$$

$$(9a) \quad -n_y \theta_x + n_x \theta_y = \bar{\theta}_t \quad \text{on } \partial A_{\theta te},$$

$$(9b) \quad n_x n_y (M_y - M_x) + (n_x^2 - n_y^2) M_{xy} = \bar{M}_t \quad \text{on } \partial A_{Mte},$$

where $\bar{\theta}_t, \bar{\theta}_n$ are the prescribed rotations about the normal and the tangent to the boundary. At a boundary location, one and only one condition from each of the three sets (7), (8) and (9) above is specified. Equations (7a), (8a) and (9a) are *rigid* while Eqs. (7b), (8b) and (9b) are *natural conditions*. There are three boundary conditions for the *Reissner* theory vs. two for *Kirchhoff's* theory. The so-called simply supported condition, prescribed w, M_n and M_t refers as *soft support* (a more realistic condition) and the specification of w, M_n and θ_t is *hard support*. A boundary subjects to both rigid and natural conditions is called a *mixed boundary*.

The functional in Eq. (2) involves the first order differentials of θ_x, θ_y , and w . The admissibility requirements are C^0 continuity for θ 's and ω and the satisfaction of Eqs. (7a), (8a) and (9a) if prescribed. Thus the element formulation of the bending is similar to that of plane elasticity in [Sec. 18.14](#).

To evaluate the strain energy associated with the transverse shears, if the same type of shape functions for ω and θ is used, the latter will contain higher degree terms not present in w_x and w_y causing shear locking in thin plate applications. For example, for pure bending, the change of curvature is constant and the transverse shears are zero. This implies that θ is linear and ω quadratic. Unless the shape functions for ω can correctly represent the quadratic field, locking will occur. Using the approximation of ω one order higher than that of θ can eliminate or minimize locking.

An alternative is to integrate the bending and the shear with same lower-order rule; or integrate the bending with the normal, whereas the transverse shear with lower-order rule. Unfortunately, reduced integration introduces spurious zero energy mode(s), which can cause serious errors in applications. A summary of reduced integration rules for parallelograms and their associated zero-energy modes are given in [Table 20.2:1](#).

Problem 20.1. Let $f_n(\xi)$ be the *Legendre polynomial* of degree n ,

$$(n+1)f_{n+1}(\xi) = (2n+1)\xi f_n(\xi) - nf_{n-1}(\xi).$$

Show that

$$(a) \quad f_0(\xi) = 1, \quad f_1(\xi) = \xi, \quad f_2(\xi) = (3\xi^2 - 1)/2, \quad f_3(\xi) = \xi(5\xi^2 - 3)/2, \dots$$

(b) For parallelogram elements, the displacement and the rotations

$$\begin{aligned} w &= 0, \quad \theta_x = \xi_{,x} f'_n(\xi) f_n(\eta) - \eta_{,x} f'_n(\eta) f_n(\xi), \\ \theta_y &= \xi_{,y} f'_n(\xi) f_n(\eta) - \eta_{,y} f'_n(\eta) f_n(\xi), \end{aligned}$$

give zero bending and transverse strains at the integration points of the $n \times n$ Gauss quadrature, where prime denotes the derivative of the function. Note, $f_n(\xi), f_n(\eta)$ equal zero at the corresponding Gauss points.

Locking Modes of 8-Node Elements. An 8-node serendipity quadrilateral element of shape other than parallelogram cannot interpolate $\omega = x^2$ correctly, because the shape functions include only the terms 1, ξ , η , ξ^2 , $\xi\eta$, η^2 , $\xi^2\eta$, $\xi\eta^2$, $x^2 \{= [\sum_{i=1}^8 h_i(\xi, \eta)x_i]^2\}$ while involves $\xi^2\eta^2$. The incorrect interpolation not only introduces spurious shear, causing locking, but also makes the element fail the patch test. The 2×2 reduced integration (Zienkiewicz *et al.* 1971) behaves poorly in the thin plate limit (Pugh *et al.* 1978). One may use the nine-node Lagrange element ([Fig. 18.13:2](#), MacNeal and Harder 1992) to include the term $\xi^2\eta^2$ in the base functions. Unfortunately, the nine-node element introduces an additional spurious communicating zero energy mode, which if not suppressed can destroy the accuracy of analysis. Hughes and Cohen (1978) introduced the *Heterosis* element, which uses just 8 nodes for ω while 9 nodes for θ . They also used the 2×2 integration rule for transverse shear and the 3×3 rule for curvature.

Table 20.2:1. Reduced integration rules and associated zero-energy modes for parallelograms.

Element	4-node	8-node (serendipity)	9-node (Lagrange)
Shape function for w, θ_x, θ_y	Bilinear	Incomplete biquadratic	Biquadratic
Selectively reduced integration	1×1 shear 2×2 bending (S1)	2×2 shear 3×3 bending (S2)	2×2 shear 3×3 bending (S2)
Zero-energy modes	1. $w = \xi\eta$, (G) $\theta_x = \theta_y = 0$ 2. $w = 0$, $\theta_x = y$ $\theta_y = -x$ (G)		1. $w = \xi^2\eta^2 - \frac{\xi^2 + \eta^2}{3}$ $\theta_x = \theta_y = 0$ (G)
Uniformly reduced integration	1×1 (U1) (Gauss station $\xi = \eta = 0$)	2×2 (U2) (Gauss stations $\xi = \eta = \pm 1/\sqrt{3}$)	2×2 (U2)
Zero-energy modes	(In addition to those of S1) 3. $w = \theta_y = 0$ $\theta_x = \xi\eta$ (G) 4. $w = \theta_x = 0$ $\theta_y = \xi\eta$ (G)	1. $w = 0$, (N) $\theta_x = \xi(\eta^2 - 1/3)\xi_{,x}$ $- \eta(\xi^2 - 1/3)\eta_{,x}$ 2. $w = \theta_y = 0$, (N) $\theta_y = \xi(\eta^2 - 1/3)\xi_{,y}$ $- \eta(\xi^2 - 1/3)\eta_{,y}$ ($\xi_{,x}, \xi_{,y}, \eta_{,x}, \eta_{,y}$ are constant)	(In addition to that of S2) 2. $w = \theta_x = 0$, (G) $\theta_y = (\xi^2 - 1/3)(\eta^2 - 1/3)$ 3. $w = \theta_y = 0$, (G) $\theta_x = (\xi^2 - 1/3)(\eta^2 - 1/3)$ 4. $w = 0$, (N) $\theta_x = \xi(\eta^2 - 1/3)\xi_{,x}$ $- \eta(\xi^2 - 1/3)\eta_{,x}$ $\theta_y = \xi(\eta^2 - 1/3)\xi_{,y}$ $- \eta(\xi^2 - 1/3)\eta_{,y}$

S: Selective-reduced integration,

U: Uniform-reduced integration,

G: Global communicating mode,

N: Non-communicating mode.

To illustrate locking, consider the deformation

$$\theta_x = x^2y, \quad \theta_y = x^3/3, \quad w = x^3y/3,$$

with the curvature and transverse shear strains

$$(10) \quad \kappa_{xx} = 2xy, \quad \kappa_{xy} = 2x^2, \quad \kappa_{yy} = \gamma_x = \gamma_y = 0.$$

The bi-quadratic shape functions of an 8-node ($2a \times 2b$) rectangular element cannot correctly interpolate x^3 . For example, the finite element representation of such deformation is

$$(11) \quad \begin{aligned} \theta_x &= x^2y, \quad \theta_y = xa^2/3, \quad w = a^2xy/3, \\ \kappa_{xx} &= 2xy, \quad \kappa_{xy} = x^2 + a^2/3, \quad \kappa_{yy} = \gamma_y = 0, \quad \gamma_x = y(x^2 - a^2/3), \end{aligned}$$

with γ_x being spurious and κ_{xy} approximate. The ratio of spurious energy, based on Eq. (11) above, to the correct energy from Eq. (10) is

$$\frac{(\Pi_{be} + \Pi_{\gamma e})_{finite\ element}}{(\Pi_{be} + \Pi_{\gamma e})_{exact}} = 1 + \left(\frac{4a^2}{9t^2}\kappa - \frac{a^2}{3b^2} \right) / \left[\frac{a^2}{b^2} + \frac{10}{9(1-\nu)} \right],$$

which is proportional to the square of the element's size-to-thickness ratio (a^2/t^2). The ratio can be very large, which means locking. For 2×2 reduced integration, the shears are evaluated at the Gauss points $x = \pm a/\sqrt{3}$, where the shears are correct and thus locking is avoided. The finite element representation of the twist curvature κ_{xy} is incorrect. However, its influence is not serious as the incorrect twist curvature is not the lowest elastic mode required for convergence, thus the effect is less as the element size progressively reduces.

Problem 20.2. Consider a beam along the x -axis subjected to a concentrated load at $x = 0$, show that the deflection near the origin can be expressed in the form $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ with $a_3 \neq 0$.

Four-Node Element. The shape functions of a four-node quadrilateral cannot correctly represent quadratic displacements. However, one can obtain the correct strains by using the values at points where the first derivatives of a quadratic field are correct. For example, the spurious shear strains $\gamma_x = x$ and $\gamma_y = y$ cause locking for pure bending ($\theta_x = x, \theta_y = 0, \omega = x^2/2$, or $\theta_x = 0, \theta_y = y, \omega = y^2/2$, see Sec. 18.20). But, their values at $x, y = 0$ are correct. One obtains correct shears by using only the values of γ_x, γ_y at $x = y = 0$ (Hughes *et al.* 1977). MacNeal (1978) used the values of γ_x at $\xi = 0, \eta = \pm 1/\sqrt{3}$ and the values of γ_y at $\xi = \pm 1/\sqrt{3}, \eta = 0$. Both approaches work well for rectangular elements but fail to eliminate shear locking for elements of other shapes.

As a/t becomes large, the results of the 8-node serendipity element (Fig. 18.13:2 without the 9th node) with full integration deteriorate rapidly. Reduced integration improves the result drastically for the simply supported case, but not for clamped support, when a/t is large. The 9-node Lagrange element with reduced integration performs well in most circumstances. The rectangular elements subjected to a concentrated load at the center have been extensively evaluated (see Biblio. 20.2).

Elements with Discrete Constraints. Reduced integration generally fails to eliminate locking for general quadrilateral elements. An alternative is to impose discrete constraints on shears. In evaluating Π_{be} of Eq. (3),

$$(12) \quad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_x \\ \gamma_y \end{bmatrix} = \mathbf{P}(\xi, \eta)\boldsymbol{\alpha}, \quad \boldsymbol{\theta} = \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = \mathbf{h}_\theta(\xi, \eta)\mathbf{q}_\theta, \quad \mathbf{w} = \mathbf{h}_w(\xi, \eta)\mathbf{q}_w,$$

where \mathbf{P} is a polynomial matrix of the natural coordinates, $\boldsymbol{\alpha}$ is a column matrix of unknowns, and \mathbf{h}_θ and \mathbf{h}_w are interpolation matrices associated with nodal unknowns $\mathbf{q}_\theta, \mathbf{q}_w$ of $\boldsymbol{\theta}$ and \mathbf{w} , respectively. Rather than evaluating the transverse shear γ according to the strain/displacement relations (4), one uses the approximation of Eq. (12) for γ in Eq. (3) and obtains

$$(13) \quad \Pi_{\gamma e} = \boldsymbol{\alpha}^T \left[\int_{A_e} \mathbf{P}^T \mathbf{D}_s \mathbf{P} dA / 2 \right] \boldsymbol{\alpha},$$

by collocating the component of γ parallel to side nearest to the collocating point with $\tilde{\boldsymbol{\gamma}}$ derived from the assumed ω and θ of Eq. (4):

$$-m_x \gamma_x + m_y \gamma_y = -m_x \tilde{\gamma}_x + m_y \tilde{\gamma}_y.$$

This gives $\boldsymbol{\alpha}$'s in terms of the nodal values \mathbf{q}_ω of ω and \mathbf{q}_θ of θ :

$$\boldsymbol{\alpha} = \mathbf{A}_w \mathbf{q}_w + \mathbf{A}_\theta \mathbf{q}_\theta.$$

Then Eq. (13) can be written as

$$\Pi_{\gamma e} = [\mathbf{q}_w^T \quad \mathbf{q}_\theta^T] [\mathbf{A}_w \quad \mathbf{A}_\theta]^T \left[\int_{A_e} \mathbf{P}^T \mathbf{D}_s \mathbf{P} dA \right] [\mathbf{A}_w \quad \mathbf{A}_\theta] [\mathbf{q}_w^T \quad \mathbf{q}_\theta^T]^T / 2.$$

The other element matrices are evaluated according to Eq. (3). Hughes and Tezduyar (1981), and MacNeal (1982) assume

$$\boldsymbol{\gamma} = \sum_{i=1}^3 \zeta_i [\alpha_i \quad \alpha_{i+1}]^T,$$

for triangles and

$$\boldsymbol{\gamma} = \mathbf{J}^{-1} \tilde{\boldsymbol{\gamma}} = \mathbf{J}^{-1} \begin{bmatrix} 1 & \eta & 0 & 0 \\ 0 & 0 & 1 & \xi \end{bmatrix} \boldsymbol{\alpha},$$

for quadrilaterals, where $\boldsymbol{\gamma}$ is the strain and \mathbf{J}^{-1} is the inverse of the coordinate transformation matrix

$$\mathbf{J}^{-1} = \frac{1}{x, \xi y, \eta - x, \eta y, \xi} \begin{bmatrix} y, \eta & -y, \xi \\ -x, \eta & x, \xi \end{bmatrix}.$$

The collocating points are the midpoint of each side of the element. Another option is to employ the stations of the reduced integration. Discrete constraints have achieved varied degrees of success in reducing locking.

Problem 20.3. (a) Derive the element matrices of a quadrilateral element with 8 DOFs for θ and 8 for w . (b) Show that the element no longer locks for

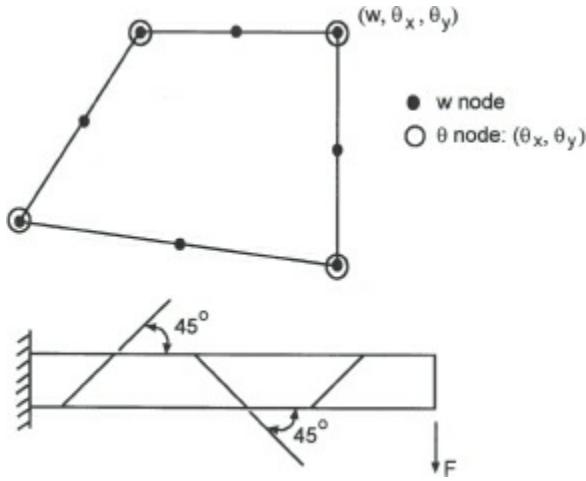


Fig. P20.3

$$\theta_x = x, \quad \theta_y = 0, \quad w = x^2/2,$$

and

$$\theta_x = 0, \quad \theta_y = y, \quad w = y^2/2.$$

(c) Program the new element and solve the bending of a cantilever plate using a rectangular mesh and the mesh as shown in Fig. P20.3. Compare the results to the quadrilateral elements with 8 DOFs for θ and 4 DOFs at vertices for w .

Problem 20.4. Repeat Problem 20.3 with triangular elements of 6 DOFs for θ and 6 DOFs (3 at the vertices and 3 at the midsides) for ω and compare with the results of triangles of 6 DOFs for θ and 3 DOFs at the vertices for w . The mesh is the same as before with each quadrilateral replaced by two triangles.

20.3. MIXED FUNCTIONAL OF REISSNER PLATE THEORY

We can modify Π_{ye} of Eq. (20.2:3) in the form (see [Biblios. 18.1](#) and [20.2](#))

$$(1) \quad \Pi_{ye} = \int_{A_e} \tilde{\gamma}^T D_s (dw - \theta) - \tilde{\gamma}^T D_s \tilde{\gamma}/2) dA$$

with $\tilde{\gamma}$ being a new transverse shear field. Judicious selection of $\tilde{\gamma}$ provides flexibility to avoid shear lock. The admissibility requirement is C^0 continuity for θ and ω or $\tilde{\gamma}_n$, the normal component of $\tilde{\gamma}$ to the element boundaries. Following the standard approach, we assume θ and ω in the form of Eq. (20.2:12) and

$$(2) \quad \tilde{\gamma} = \begin{bmatrix} 1 & 0 & y & 0 & x & 0 & \dots \\ 0 & 1 & 0 & x & 0 & y & \dots \end{bmatrix} \alpha,$$

and derive the element matrices from Eq. (20.2:2).

Tong and Pian (1969), Babuska (1971) and Brezzi (1974) examined the stability associated with the assumed functions for the multiple-field functional. One issue is the existence of spurious modes in the finite element solution. To avoid the spurious modes, the following *stability criterion*

$$(3) \quad n_{\tilde{\gamma}} \geq n_w - \max(3, n_c)$$

should be satisfied, for problems with 3 rigid body modes, n_c is the number of constraints on the parameters for ω , $n_{\tilde{\gamma}}$ are the number of parameters for $\tilde{\gamma}$ and n_{ω} for ω . This stability criterion can be applied to an assembly of elements if all elements are similar. However, the criterion is not applicable if some elements of the assembly fail while some pass.

The stationary condition of Eq. (1) is

$$(4) \quad \delta \Pi_{ye} = \int_{A_e} \delta \tilde{\gamma}^T D_s (dw - \theta - \tilde{\gamma}) dA = 0,$$

and the second variation of Π_{ye} with respect to its field variables is

$$(5) \quad \delta^2 \Pi_{\gamma e} = \int_{A_e} (\mathbf{d}_e \delta \theta)^T [\mathbf{D}_b (\mathbf{d}_e \delta \theta) - (\delta \tilde{\gamma})^T \mathbf{D}_s (\delta \tilde{\gamma})] dA / 2.$$

Thus, the modified mixed functional is a minimum-maximum principle (Tong and Pian 1969). In other words, for fixed $\theta(q_\theta)$'s and $\omega(q_\omega)$'s, the functional is a maximum principle with respect to $\tilde{\gamma}$ (α 's). With $\tilde{\gamma}$ fixed, the functional is a minimum principle with respect to θ and ω . One can also show that the mixed functional is bounded from above by the functional for the principle of minimum potential energy defined in Eq. (20.2:2). In finite element formulation, increasing the number of $\alpha(n_\gamma)$ tends to increase the value of the maximum for given θ and ω and thus, increase the minimum of the maxima. Physically, increasing the number of α makes the model stiffer. In developing a new element, if one starts with the minimum number of n_γ needed by Eq. (3) and if the element is too flexible, then increase n_γ ; on the other hand, if the element is already too stiff, one reduces the number of α 's or increase the order of θ and/or ω at the risk of introducing spurious mode(s).

Field-Consistent Element. We can examine locking relieve from another point of view. For isotropic materials, Eq. (20.2:3) and (1) become

$$(6a) \quad \Pi_{be} = \int_{A_e} \frac{Et^3}{12} \left[\frac{\theta_{x,x}^2 + \theta_{y,y}^2 + 2\nu\theta_{x,x}\theta_{y,y}}{1-\nu^2} + \frac{(\theta_{x,y} + \theta_{y,x})^2}{2(1+\nu)} \right] dA,$$

$$(6b) \quad \Pi_{\gamma e} = \int_{A_e} \frac{Et\kappa}{2(1+\nu)} (\mathbf{d}w - \boldsymbol{\theta})^T (\mathbf{d}w - \boldsymbol{\theta}) dA \quad (\text{displacement model}),$$

$$(6c) \quad \Pi_{\gamma e} = \int_{A_e} \frac{Et\kappa}{2(1+\nu)} [\tilde{\gamma}^T (\mathbf{d}w - \boldsymbol{\theta}) - \frac{1}{2} \tilde{\gamma}^T \tilde{\gamma}] dA \quad (\text{mixed model}).$$

4-Node Field-Consistent Rectangle. Consider a 4-node $2l \times 2h$ rectangular element with $x = l\xi$ and $y = h\eta$. We assume

$$(7) \quad \begin{aligned} w &= a_1 + a_2\xi + a_3\eta + a_4\xi\eta, & \theta_x &= b_1 + b_2\xi + b_3\eta + b_4\xi\eta, \\ \theta_y &= c_1 + c_2\xi + c_3\eta + c_4\xi\eta, \end{aligned}$$

where a 's are related to the nodal values of ω [see Eq. (18.20:5)],

$$(8) \quad \begin{aligned} a_1 &= (w_1 + w_2 + w_3 + w_4)/4, & a_2 &= (-w_1 + w_2 + w_3 - w_4)/4, \\ a_3 &= (-w_1 - w_2 + w_3 + w_4)/4, & a_4 &= (w_1 - w_2 + w_3 - w_4)/4, \end{aligned}$$

similarly b 's and c 's are related to those of θ_x and θ_y , respectively. In displacement formulation, the shear strains γ_x and γ_y derived from the assumed ω , θ_x and θ_y are

$$(9) \quad \begin{aligned} \gamma_x &= a_2/l - b_1 + (a_4/l - b_3)\eta - b_2\xi - b_4\xi\eta, \\ \gamma_y &= a_3/h - c_1 + (a_4/h - c_2)\xi - c_3\eta - c_4\xi\eta. \end{aligned}$$

As t approaches zero, the shear strains of the finite element solution approaches zero. This is equivalent to forcing

$$(10a) \quad a_2/l - b_1 \rightarrow 0, \quad a_4/l - b_3 \rightarrow 0,$$

$$(10b) \quad b_2 \rightarrow 0, \quad b_4 \rightarrow 0.$$

Equations (10b) involve the coefficients of one field variable only and thus are considered to be spurious constraints that can lead to locking in the limit of thin plate. There are similar spurious shear constraints for c

In the mixed formulation, we assume

$$(11) \quad \tilde{\gamma}_x = \beta_1 + \beta_2\eta, \quad \tilde{\gamma}_y = \beta_3 + \beta_4\xi.$$

Using Eqs. (4) and (7), one can show that

$$(12) \quad \begin{aligned} \beta_1 &= a_2/l - b_1 = [(-w_1 + w_2 + w_3 - w_4)/l - (\theta_{x1} + \theta_{x2} + \theta_{x3} + \theta_{x4})]/4, \\ \beta_2 &= a_4/l - b_3 = [(w_1 - w_2 + w_3 - w_4)/l - (-\theta_{x1} - \theta_{x2} + \theta_{x3} + \theta_{x4})]/4, \\ \beta_3 &= a_3/h - c_1 = [(-w_1 - w_2 + w_3 + w_4)/h - (\theta_{y1} + \theta_{y2} + \theta_{y3} + \theta_{y4})]/4, \\ \beta_4 &= a_4/h - c_2 = [(w_1 - w_2 + w_3 - w_4)/h - (-\theta_{y1} + \theta_{y2} + \theta_{y3} - \theta_{y4})]/4. \end{aligned}$$

From Eq. (6c), one has

$$(13) \quad \Pi_{\gamma e} = \kappa E t a b (\beta_1^2 + \beta_2^2/3 + \beta_3^2 + \beta_4^2/3)/(1 + \nu),$$

which can be expressed in terms of the nodal values w 's and θ 's as given in Eq. (12). One can then derive the element matrices in a routine matter.

4-Node Field-Consistent Quadrilateral. For general quadrilaterals, it is difficult to obtain the appropriate γ precisely. We shall determine an approximate γ using the covariant rotations and shear strains defined as

$$(14) \quad \begin{bmatrix} \theta_\xi \\ \theta_\eta \end{bmatrix} = \mathbf{J} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix}, \quad \begin{bmatrix} \gamma_\xi \\ \gamma_\eta \end{bmatrix} = \begin{bmatrix} w, \xi - \theta_\xi \\ w, \eta - \theta_\eta \end{bmatrix} = \mathbf{J} \begin{bmatrix} \gamma_x \\ \gamma_y \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} x, \xi & y, \xi \\ x, \eta & y, \eta \end{bmatrix}.$$

Instead of following Eq. (4), we assume $\omega, \theta_\xi, \theta_\eta$ in the form of Eq. (7) and $\tilde{\gamma}_\xi$ and $\tilde{\gamma}_\eta$ in the form of Eq. (11). Using the stationary condition of Eq. (1)

$$(15a) \quad \delta \Pi_{\gamma e} = \int_{Ae} \delta \tilde{\gamma}^T \mathbf{D}_s (\gamma - \tilde{\gamma}) dA = 0 \quad (\text{for anisotropic materials}),$$

$$(15b) \quad \delta \Pi_{\gamma e} = \int_{Ae} \frac{E t \kappa}{1 + \nu} \delta \tilde{\gamma}^T (\gamma - \tilde{\gamma}) dA = 0 \quad (\text{for isotropic materials}),$$

and if the element is a rectangle,

$$(15c) \quad \delta \Pi_{\gamma e} = \int_{-1}^1 \int_{-1}^1 \frac{E t \kappa l h}{1 + \nu} \left[\frac{\delta \tilde{\gamma}_\xi (\gamma_\xi - \tilde{\gamma}_\xi)}{l^2} + \frac{\delta \tilde{\gamma}_\eta (\gamma_\eta - \tilde{\gamma}_\eta)}{h^2} \right] d\xi d\eta = 0,$$

we find that the β 's of $\tilde{\gamma}_\xi$ and $\tilde{\gamma}_\eta$ are given by Eq. (12) with θ_x replaced by θ_ξ and θ_y by θ_η . The nodal values of θ_ξ and θ_η can be transformed to those of θ_x and θ_y by Eq. (14). We then defined

$$(16) \quad \begin{bmatrix} \tilde{\gamma}_x \\ \tilde{\gamma}_y \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \tilde{\gamma}_\xi \\ \tilde{\gamma}_\eta \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \beta_1 & \beta_2 \eta & 0 & 0 \\ 0 & 0 & \beta_3 & \beta_4 \xi \end{bmatrix},$$

with β 's in terms of the nodal values of θ_x, θ_y and ω . We also approximate $\Pi_{\gamma e}$ of Eq. (6c) as

$$(17) \quad \Pi_{\gamma e} = \frac{1}{2} \int_{Ae} \frac{E t \kappa}{1 + \nu} \tilde{\gamma}^T \tilde{\gamma} dA = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 \frac{E t \kappa l h}{1 + \nu} \left(\frac{\tilde{\gamma}_\xi^T \tilde{\gamma}_\xi}{l^2} + \frac{\tilde{\gamma}_\eta^T \tilde{\gamma}_\eta}{h^2} \right) d\xi d\eta.$$

Using Eq. (6a) and (17), we can derive the element stiffness matrix. This element passes the patch test.

9-Node Field-Consistent Quadrilateral. For a 9-node quadrilateral, $\omega, \theta_\xi, \theta_\eta$ are biquadratic functions of natural coordinates

$$(18) \quad w = \sum_{i,j=0}^2 a_{ij} \xi^i \eta^j, \quad \theta_\xi = \sum_{i,j=0}^2 b_{ij} \xi^i \eta^j, \quad \theta_\eta = \sum_{i,j=0}^2 c_{ij} \xi^i \eta^j,$$

where a' , b' and c' s are related to the nodal values of ω, θ_ξ and θ_η . The transverse shear strains can be expressed in terms of Legendre polynomials of ξ, η as follows:

$$(19) \quad \begin{aligned} \gamma_\xi &= \sum_{j=0}^2 [a_{1j} - b_{0j} - b_{2j}/3 + (2a_{2j} - b_{1j})\xi - b_{2j}(\xi^2 - 1/3)] \eta^j, \\ \gamma_\eta &= \sum_{j=0}^2 [a_{j1} - c_{j0} - c_{j2}/3 + (2a_{j2} - c_{j1})\eta - c_{j2}(\eta^2 - 1/3)] \xi^j. \end{aligned}$$

Since the coefficients b_{2j} and c_{2j} depend on the parameters of one of the rotations only, therefore, are field-inconsistent. The element derived from such field-inconsistent strains converges slowly and can have strong quadratic oscillation in the shear stresses within an element.

Using the orthogonal properties of the Legendre polynomial, from the stationary condition (15), we find the field-consistent shear strains

$$(20) \quad \begin{aligned} \tilde{\gamma}_\xi &= \sum_{j=0}^2 [a_{1j} - b_{0j} - b_{2j}/3 + (2a_{2j} - b_{1j})\xi] \eta^j, \\ \tilde{\gamma}_\eta &= \sum_{j=0}^2 [a_{j1} - c_{j0} - c_{j2}/3 + (2a_{j2} - c_{j1})\eta] \xi^j. \end{aligned}$$

Using Eqs. (6a), (16) and (17), we can derive the element stiffness matrix. This element is locking-free and the stresses from the field-consistent strains are smooth. It is more difficult to establish field-consistent strains for 8-node quadrilateral due to incomplete terms to represent strains.

Approximate field-consistent approach is a rational means to construct locking-free elements, even though exact field-consistent shear is yet to be established.

Problem 20.5. Show that one can express a 's of Eq. (18) in terms of the nodal values of ω [Fig. 18.13:2(b)] as

$$\begin{aligned} a_{00} &= w_9, \quad a_{01} = (w_7 - w_5)/2, \quad a_{02} = (w_7 - 2w_9 + w_5)/2, \quad a_{10} = (w_6 - w_8)/2, \\ a_{11} &= (w_1 - w_2 + w_3 - w_4)/4, \quad a_{12} = (-w_1 + w_2 + w_3 - w_4)/4 + (w_8 - w_6)/2 \\ a_{20} &= (w_8 - 2w_9 + w_6)/2, \quad a_{21} = (-w_1 - w_2 + w_3 + w_4)/4 + (w_5 - w_7)/2 \\ a_{22} &= (w_1 + w_2 + w_3 + w_4)/4 - (w_5 + w_6 + w_7 + w_8)/4 + w_9. \end{aligned}$$

20.4. HYBRID FORMULATION FOR PLATES

The hybrid finite element method (Pian 1964, Tong and Pian 1969) is an excellent alternative to overcome the compatibility difficulty for thin plate elements. The method is also good for avoiding locking of thick plates.

Assumed Stress Formulation. The functional in Eq. (20.3:1) becomes the hybrid stress model if we express the transverse shear strain γ in terms of the transverse shear \mathbf{Q} and have \mathbf{Q} satisfy *a priori* the equilibrium equation (13.14:16). Then Π_{ye} and Π_{bfe} of Eqs. (20.2:3) and (20.3:1) becomes

$$(1) \quad \Pi_{ye} = \int_{A_e} (-\mathbf{Q}^T \boldsymbol{\theta} - \mathbf{Q}^T \mathbf{D}_s^{-1} \mathbf{Q}/2) dA + \int_{\partial A_e} w Q_n ds,$$

$$(2) \quad \Pi_{bfe} = - \int_{\partial A_{mne}} \bar{M}_n \theta_n ds - \int_{\partial A_{mte}} \bar{M}_t \theta_t ds + \int_{\partial A_{Qe}} \bar{Q} w ds,$$

where Q_n is normal components of \mathbf{Q} along the element boundaries. The hybrid stress functional, like the mixed one, involves the three fields $\boldsymbol{\theta}$, \mathbf{Q} and ω . However, since ω is now needed only along the element boundaries, we have more freedom in the selection of interpolation polynomials for ω . Similar to the case of the mixed formulation, we assume

$$(3) \quad \boldsymbol{\theta} = \mathbf{h}_\theta \mathbf{q}_\theta, \quad \mathbf{Q} = \mathbf{Q}_h + \mathbf{Q}_p = \mathbf{h}_Q \boldsymbol{\beta} + \mathbf{Q}_p, \quad w = \mathbf{h}_w \mathbf{q}_w,$$

where \mathbf{h}_θ and \mathbf{h}_ω are the interpolation matrices associated with the nodal values of $\boldsymbol{\theta}$ and ω except that \mathbf{h}_ω is needed only along ∂A_e , \mathbf{Q}_h and \mathbf{Q}_p are the homogeneous and particular solutions of Eq. (13.14:16), and

$$(4) \quad \mathbf{h}_Q = \begin{bmatrix} 1 & 0 & x & y & 0 & \dots \\ 0 & 1 & -y & 0 & x & \dots \end{bmatrix}.$$

Within an element, one can approximate $p(x, y)$ as a polynomial and determine \mathbf{Q}_p accordingly. The expression of \mathbf{Q}_p is not unique. The higher order of approximation of \mathbf{Q}_p than that of \mathbf{Q}_h has no effect on the order of accuracy of the finite element solution as the accuracy order is determined by \mathbf{Q}_h . Substituting Eq. (3) into Eqs. (20.2:2), (1) and (2) gives Π_{ye} in terms of \mathbf{q}_ω , \mathbf{q}_θ and $\boldsymbol{\beta}$ (Tong and Rossettos 1977). If $\boldsymbol{\beta}$ is independently assumed for each element, it can be eliminated at the element level based on the stationary condition of Π_{be} . One can then obtain the element matrices in terms of \mathbf{q}_θ , \mathbf{q}_ω .

General Hybrid Formulation. Introducing Lagrange multiplier \hat{w} and setting $\mathbf{Q} = \mathbf{D}_s \boldsymbol{\gamma}$, $\bar{Q}_n = \mathbf{n}^T \mathbf{D}_s \boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma} = \mathbf{d}w - \boldsymbol{\theta}$, one can write Π_{ye} of Eq. (20.3:1) in the form (Pian and Tong 1969, Tong 1970),

$$\Pi_{ye} = \int_{\partial A_e} (\hat{w} - w) \mathbf{n}^T \mathbf{D}_s \boldsymbol{\gamma} ds + \int_{A_e} \boldsymbol{\gamma}^T \mathbf{D}_s \boldsymbol{\gamma} dA/2,$$

called the *hybrid displacement model*. The new functional has three dependent fields, θ , ω , \hat{w} of which \hat{w} is defined

along the element boundary only. The admissibility requirements are that θ and \hat{w} are C^0 continuity at inter-element boundaries and satisfy the prescribed rigid constraint(s) at element boundary $\partial A_{\omega e}$, $\partial A_{\theta ne}$ and $\partial A_{\theta te}$. The finite element equations can be established by assuming

$$\theta = \mathbf{h}_\theta \mathbf{q}_\theta, \quad w = \mathbf{h}_w \mathbf{q}_w, \quad \hat{w} = \mathbf{h}_{\hat{w}} \mathbf{q}_{\hat{w}},$$

in which $\mathbf{h}_{\hat{w}}$ (see equations below Eq. (19.2:13) is assumed along the element boundaries and there is no restriction on the selection of \mathbf{h}_ω . The functional is also called the *hybrid assumed strain model* (Tang *et al.* 1983).

Problem 20.6. Repeat Problem 20.3 by deriving the corresponding hybrid displacement elements.

Remarks. One often judges a *thick* plate element based on its performance in *thin* plate applications. The 9-node Lagrangian element and the heterosis element (9-node for θ and 8-node for w) with rectangular mesh and selectively reduced integration perform well in clamped and simply supported *square thin plates* (Zienkiewics and Taylor 1989, Hughes 1984). The results deteriorate rapidly when the mesh is distorted. Mixed (field-consistent) and discrete constraint methods are alternatives to eliminate locking of general triangular and quadrilateral elements. The field-consistent and the discretely-constrained elements discussed previously and the mixed triangular elements (see [Biblio. 20.3](#)) all perform reasonably well. Locking can be removed also using hybrid formulation (Pian 1993).

The thick plate theory requires three boundary conditions while the thin plate theory needs only two. The two theories give different deformations for different boundary conditions when the plate is thick (l/t is of order 10, where l is dimension of the plate). Generally the thick plate theory gives considerably larger deflections. With l/t of the order 100 and more, the thick elements generally converge to the Kirchhoff plate for *hard support conditions* (prescribed ω , M_n and θ_t), but to slightly higher values for *soft support conditions* (prescribed of ω , M_n and M_t). Atluri *et al.* (2014) introduced a 4-node plane and an 8-node solid elements (to be discussed at the end of the chapter) with selective collocations and over-integration. The elements perform well for thick and thin plates and shells. Tong introduced a corresponding 6-node plane and 16-node solid elements to improve the geometric approximation for curved beams and shells.

20.5. GENERAL SHELL ELEMENTS

Shell, like plate, is a particular form of three-dimensional solid whose thickness is small as compared with other dimensions. The early use of general 3-dimensional solid elements in thin shell applications was fraught with ‘locking’ (see [Sec. 18.20](#)) unless the element sizes are of the order of thickness.

In shells the stress resultants resist the pressure normal to the middle surface through the curvature, a very efficient load carrying mechanism in engineering applications. The combination of stretching and bending plus the complexities of the curved geometry make the precise derivation of 2-dimensional governing equations for shells difficult and lead to various shell theories. Modeling the different shell theories results in distinguish shell elements and is a subject of continued research.

To avoid the complication induced by the curvatures of the shell surface in early finite element development, one approximates a shell as an assembly of flat plates. In this model, through the discrete angle at the junction of neighboring elements, the midplane stress resultants of an element is transmitted to its immediate neighbors as transverse shear forces to resist the lateral load. As the size of subdivision decreases, the geometry converges to the shell. Intuitively, one expects that the flat-element assembly behaves like the shell in the limit.

One advantage of such approximation is the ease of coupling with edge beams and stiffeners in practical applications. However, flat elements are not compatible in shell applications if the order of polynomials used for ‘in-plane’ displacements differs from that for out-plane. Also for meshes involving elements of more than three nodes, the elements may not be flat. The number of surfaces totally covered by orthogonal curvilinear coordinates is known to be 8 only, such as cylinders or spheroids. Only those shells are representable by flat quadrilaterals. The analysis of cylindrical and spherical shells using flat quadrilateral elements has met early success. For more general surfaces, especially due to the poor bending performance of the early triangular plate elements, analysis was not satisfactory.

Since plate has no representation of the rotation about the axis normal to the midplane, such a degree-of-freedom (DOF), say θ_z , which involves no strain energy, must be added. A problem arises if all the elements meeting a node are co-planar, e.g., at straight boundaries along the cylindrical axis. Due to zero stiffness associated with θ_z , the assembled constraint stiffness matrix is singular. Even though the formulation is still valid because the generalized force corresponding to this DOF is also zero. The singularity of the matrix poses problem in numerical computation. One often adds stiffness to those DOFs and call them the *drilling DOF* (see [Biblio. 20.5](#)). Unfortunately, the drilling freedom often induces spurious modes. The advent of modeling shells with solid elements (to be discussed at the end of this chapter) has made approximation shells by plate assembly less attractive.

Conceptually, shells can be modeled by curved 3-dimensional solids. Unfortunately, simple reduction of the thickness dimension leads to overly stiff elements in thin shell applications unless the in-plane dimensions of the element is of the

order of the thickness. To avoid such difficulty, like plates, finite element formulations were introduced to directly model the Kirchhoff thin shells and the Reissner–Mindlin moderately thick shells. Reviews of the subject are given by Gilewski and Radwanska (1991) and Ibrahimbegovic (1997).

In general finite element modeling of thin shells is a difficult subject. Due to bending and stretching coupling, it is harder than plate applications to satisfy the compatibility requirements along the interelement boundaries. The coupling causes severe locking, especially for lower order elements. Thus a general shell element is either complicated or simple but accurate only for limited shell configurations.

Even though locking is a problem, direct discretization of 3-dimensional solids in terms of nodal variables on the midsurface provides great attraction of simplicity and does not depend on the various forms of shell theories. With attentions to the thickness deformation and transverse shears, one can effectively remove locking in many thin shell applications. Elements so derived are called *degenerated solid shell elements*. In subsequent discussion, we shall develop the degeneration process.

Shell Geometry and Coordinates. We shall use four coordinate systems to account for the double curvatures, plane stress conditions, and in-extension of fiber through the shell thickness. The four systems are: (1) the local coordinates with Cartesian unit base vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ for the element geometry; (2) the natural coordinates with covariant base vectors $\mathbf{g}_\xi, \mathbf{g}_\eta, \mathbf{g}_\zeta$ for strain-displacement relations; (3) the normal coordinates with unit base vectors $\bar{\mathbf{e}}_\zeta, \bar{\mathbf{e}}_\xi, \bar{\mathbf{e}}_\eta$ for defining stress strain relations and enforcing plane stress condition; and (4) the global coordinates with Cartesian unit base vectors $\tilde{\mathbf{e}}_x, \tilde{\mathbf{e}}_y, \tilde{\mathbf{e}}_z$ for assembling of elements and imposing rigid constraints. The first three systems are local. The second is usually non-orthogonal curvilinear for general shells while the third is orthogonal for orthotropic materials. Following the tensor rules (Chapter 2), one can easily transform quantities from one system to another.

Let \mathbf{x} be a point in the shell (Fig. 20.5:1) that

$$(1) \quad \mathbf{x} = \mathbf{x}_0(\xi, \eta) + \zeta \mathbf{g}_\zeta(\xi, \eta) = x_0 \mathbf{e}_x + y_0 \mathbf{e}_y + z_0 \mathbf{e}_z + \zeta \mathbf{g}_\zeta(\xi, \eta),$$

in which the subscript 0 denotes the value at the reference surface, and \mathbf{g}_ζ is the covariant base vector in the direction of a fiber through the point at ξ, η on the reference surface. In finite element modeling, \mathbf{x}_0 is related to the natural coordinates ξ, η in the form of shape functions h_i (Chapter 18):

$$(2) \quad \mathbf{x}_0(\xi, \eta) = \sum_i h_i(\xi, \eta) (\mathbf{x}_0)_i, \quad \mathbf{g}_\zeta(\xi, \eta) = \sum_i h_i(\xi, \eta) (ts)_i / 2,$$

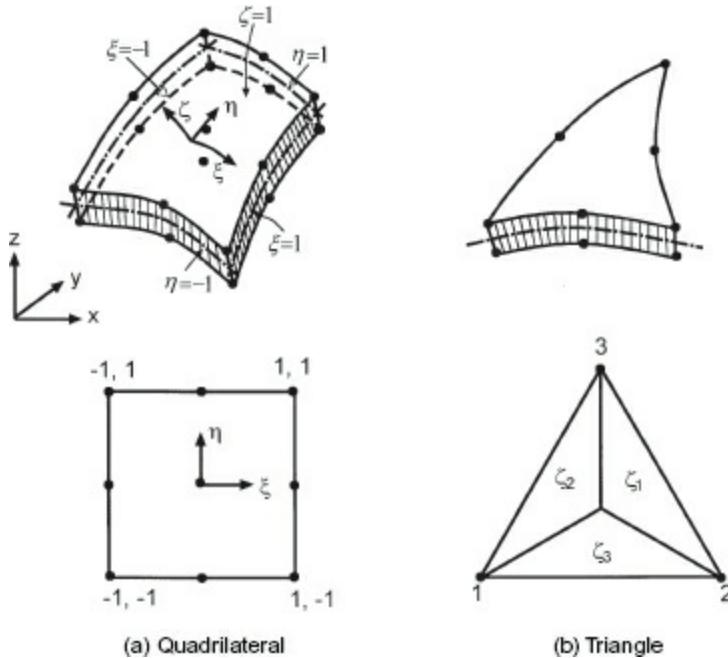


Fig. 20.5:1. Curved shell element.

in which i sums over all the nodes defining the element reference surface, the subscript i indicates association with node i , $(\mathbf{x}_0)_i$ is the position vector on the reference surface, and t_i is the thickness and s_i the unit vector in the fiber direction defined as

$$(3) \quad t_i = |\mathbf{x}_i^+ - \mathbf{x}_i^-|, \quad s_i = (\mathbf{x}_i^+ - \mathbf{x}_i^-) / |\mathbf{x}_i^+ - \mathbf{x}_i^-|,$$

where \mathbf{x}_i^+ , \mathbf{x}_i^- are the position vectors of node i at the shell's top and bottom surfaces. The reference surface is the neutral surface of the shell with the ranges of the natural coordinates ξ, η, ζ being ± 1 .

From Eq. (1), we can derive the covariant base vectors [Eq. (2.14:7)] of the natural coordinates of the shell:

$$(4) \quad \mathbf{g}_\xi = \mathbf{x}_{,\xi}, \quad \mathbf{g}_\eta = \mathbf{x}_{,\eta}, \quad \mathbf{g}_\zeta = \mathbf{x}_{,\zeta},$$

These base vectors in general are not orthogonal for general shells.

The unit base vectors are:

$$(5) \quad \mathbf{e}_\zeta = \mathbf{g}_\zeta / |\mathbf{g}_\zeta|, \quad \mathbf{e}_\eta = \mathbf{g}_\eta / |\mathbf{g}_\eta|, \quad \mathbf{e}_\xi = \mathbf{g}_\xi / |\mathbf{g}_\xi|,$$

in which \mathbf{e}_ζ is in the direction of a fiber and equals \mathbf{s}_i at node i . The contravariant base vectors of the natural coordinates (Sec. 2.14) are:

$$(6) \quad \mathbf{g}^\zeta = \mathbf{g}_\xi \times \mathbf{g}_\eta, \quad \mathbf{g}^\xi = \mathbf{g}_\eta \times \mathbf{g}_\zeta, \quad \mathbf{g}^\eta = \mathbf{g}_\zeta \times \mathbf{g}_\xi.$$

The *normal base coordinate* system has the ζ -axis normal to the ξ, η -plane at $\zeta = 0$. The other two base vectors are chosen to coincide with the material coordinates.

For isotropic materials, the latter two base vectors can be arbitrarily chosen. One choice is

$$(7) \quad \bar{\mathbf{e}}_\zeta = \mathbf{g}_\xi \times \mathbf{g}_\eta / |\mathbf{g}_\xi \times \mathbf{g}_\eta|, \quad \bar{\mathbf{e}}_\xi = \mathbf{g}_\xi / |\mathbf{g}_\xi|, \quad \bar{\mathbf{e}}_\eta = \mathbf{g}_\eta / |\mathbf{g}_\eta|.$$

If $\mathbf{g}_\xi, \mathbf{g}_\eta$ are orthogonal, then $\bar{\mathbf{e}}_\xi, \bar{\mathbf{e}}_\eta$ and $\mathbf{e}_\zeta, \mathbf{e}_\eta$ coincide.

Covariant Strains in Natural Coordinates. We shall derive the covariant strains in natural coordinates. Let $\theta_\eta, -\theta_\xi$ represent the rotation of a fiber in the ξ, η -directions, respectively. The displacement vector can be written as

$$(8) \quad \mathbf{u} = \mathbf{u}_0(\xi, \eta) - t\zeta(\theta_\xi \mathbf{e}_\xi + \theta_\eta \mathbf{e}_\eta)/2,$$

where \mathbf{u}_0 is the displacement of the midsurface. The second term denotes the fiber rotation. One sees that the fiber can rotate but not stretch. In finite element modeling, we represent the displacement by

$$(9) \quad \mathbf{u} = \sum_i h_i(\xi, \eta)(\mathbf{u}_0)_i - \zeta \sum_i h_i(\xi, \eta)[(\theta_\xi \mathbf{e}_\xi + \theta_\eta \mathbf{e}_\eta)t]_i/2 = \mathbf{h}_a \mathbf{q},$$

where the shape functions h 's are different from those in Eq. (2) if the nodes for interpolating displacements differ from those for defining the curved shell surface. The matrix \mathbf{h}_a is so defined to make the components of \mathbf{q} be those of \mathbf{u} in the natural coordinates. The interpolation functions of a 9-node element are given in Table 18.13:1. The first sum in Eq. (9) is the *interpolated displacement* of the midsurface. The second sum is that *induced by fiber rotations*. Alternatively one can represent the displacement in terms of $\bar{\mathbf{e}}_\xi, \bar{\mathbf{e}}_\eta$:

$$(10) \quad \mathbf{u} = \sum_i h_i(\xi, \eta)(\mathbf{u}_0)_i - \zeta \sum_i h_i(\xi, \eta)[(\theta_\xi \bar{\mathbf{e}}_\xi + \theta_\eta \bar{\mathbf{e}}_\eta)t]_i/2 = \bar{\mathbf{h}}_a \bar{\mathbf{q}},$$

where $\bar{\mathbf{h}}_a$ is so defined to make the components of $\bar{\mathbf{q}}$ be those of \mathbf{u} in the normal coordinates.

From Eq. (9), we find the deformation gradients:

$$(11) \quad \begin{aligned} \partial \mathbf{u} / \partial \xi &= \sum_i h_{i,\xi} [\mathbf{u}_0 - t\zeta(\theta_\xi \mathbf{e}_\xi + \theta_\eta \mathbf{e}_\eta)/2]_i, \\ \partial \mathbf{u} / \partial \eta &= \sum_i h_{i,\eta} [\mathbf{u}_0 - t\zeta(\theta_\xi \mathbf{e}_\xi + \theta_\eta \mathbf{e}_\eta)/2]_i, \\ \partial \mathbf{u} / \partial \zeta &= - \sum_i t(\theta_\xi \mathbf{e}_\xi + \theta_\eta \mathbf{e}_\eta)_i h_i/2, \end{aligned}$$

The covariant in-plane and transverse shear strains can be written as

$$(12) \quad e_{\xi\xi} = \frac{\partial \mathbf{u}}{\partial \xi} \cdot \mathbf{g}_\xi, \quad e_{\eta\eta} = \frac{\partial \mathbf{u}}{\partial \eta} \cdot \mathbf{g}_\eta, \quad e_{\xi\eta} = \frac{1}{2} \left(\frac{\partial \mathbf{u}}{\partial \xi} \cdot \mathbf{g}_\eta + \frac{\partial \mathbf{u}}{\partial \eta} \cdot \mathbf{g}_\xi \right);$$

$$(13) \quad e_{\xi\zeta} = \frac{1}{2} \left(\frac{\partial \mathbf{u}}{\partial \xi} \cdot \mathbf{g}_\zeta + \frac{\partial \mathbf{u}}{\partial \zeta} \cdot \mathbf{g}_\xi \right), \quad e_{\eta\zeta} = \frac{1}{2} \left(\frac{\partial \mathbf{u}}{\partial \eta} \cdot \mathbf{g}_\zeta + \frac{\partial \mathbf{u}}{\partial \zeta} \cdot \mathbf{g}_\eta \right).$$

For example, from Eqs. (9)–(13),

$$e_{\xi\zeta} = \sum_i \{h_{i,\xi} (\mathbf{u}_0)_i \cdot \mathbf{g}_\zeta / 2 - [t(\theta_\xi \mathbf{e}_\xi + \theta_\eta \mathbf{e}_\eta)]_i \cdot (h_i \mathbf{g}_\xi + \zeta_i h_{i,\xi} \mathbf{g}_\zeta) / 4\}.$$

The term involving $\zeta \mathbf{g}_\zeta$ represents the contribution of curvature changes to the transverse shear and is zero for flat

elements. A similar expression can be obtained for using Eq. (10).

The engineering shear strains are simply 2 times the physical components of the shear strains. Since \mathbf{g}_ξ and \mathbf{g}_η are linear in ζ , $e_{\xi\xi}$, $e_{\eta\eta}$, $e_{\xi\eta}$ are quadratic and $e_{\xi\zeta}$, $e_{\eta\zeta}$ are linear in ζ . Note that $e_{\zeta\zeta} = (\partial \mathbf{u} / \partial \zeta) \cdot \mathbf{g}_\zeta$ is only approximately equal to zero due to the approximation of Eq. (9) or (10). One often neglects $e_{\zeta\zeta}$ in practical computation. The use of the representation of Eq. (9) or (10) avoids the explicit calculation of the *Eulerian Christoffel symbols* [Eq. (2.15:4)] in deriving the deformation gradient.

Example-Ring Element. Consider a 2-node ring element of thickness t and radius r_0 over the angle $-\varphi_0 \leq \varphi \leq \varphi_0$. The nodes are at $r = r_0$, $\varphi_{1,2} = \pm\varphi_0$. The unit base vectors \mathbf{e}_r , \mathbf{e}_φ of the fiber base coordinates coincide with those of the cylindrical coordinates at the nodes. We have

$$\mathbf{x} = \mathbf{x}_0 + \frac{t\zeta}{2}\mathbf{e}_r = (r_0 + \frac{t\zeta}{2}) \sum_{i=1}^2 (\mathbf{e}_r)_i h_i(\xi), \quad \mathbf{u} = \sum_{i=1}^2 h_i(\xi) (\mathbf{u}_0 + \frac{t\zeta\theta_\xi}{2}\mathbf{e}_\varphi)_i,$$

where

$$\zeta = 2(r - r_0)/t, \quad \xi = \varphi/\varphi_0, \quad h_1 = (1 - \xi)/2, \quad h_2 = (1 + \xi)/2.$$

The covariant base vectors and the deformation gradient are

$$\begin{aligned} \mathbf{g}_\xi &= (r_0 + t\zeta/2)[(\mathbf{e}_r)_2 - (\mathbf{e}_r)_1]/2, \quad \mathbf{g}_\zeta = t[(\mathbf{e}_r)_1 h_1(\xi) + (\mathbf{e}_r)_2 h_2(\xi)]/2, \\ \frac{\partial \mathbf{u}}{\partial \xi} &= \frac{(\mathbf{u}_0)_2 - (\mathbf{u}_0)_1}{2} + \frac{(\theta_\xi \mathbf{e}_\varphi)_2 - (\theta_\xi \mathbf{e}_\varphi)_1}{4} t\zeta, \quad \frac{\partial \mathbf{u}}{\partial \zeta} = \frac{t}{2} \sum_{i=1}^2 h_i(\xi) (\theta_\xi \mathbf{e}_\varphi)_i. \end{aligned}$$

The covariant strains of the natural coordinates are

$$\begin{aligned} \varepsilon_{\xi\xi} &= (r_0 + t\zeta/2)\{(\mathbf{u}_0)_2 - (\mathbf{u}_0)_1 + t\zeta[(\theta_\xi \mathbf{e}_\varphi)_2 - (\theta_\xi \mathbf{e}_\varphi)_1]/2\}[(\mathbf{e}_r)_2 - (\mathbf{e}_r)_1]/4, \\ \varepsilon_{\xi\zeta} &= t\langle (r_0 + t\zeta/2)[(\mathbf{e}_r)_2 - (\mathbf{e}_r)_1] \cdot \sum_{i=1}^2 h_i(\xi) (\theta_\xi \mathbf{e}_\varphi)_i \\ &\quad + \{(\mathbf{u}_0)_2 - (\mathbf{u}_0)_1 + [(\theta_\xi \mathbf{e}_\varphi)_2 - (\theta_\xi \mathbf{e}_\varphi)_1]t\zeta/2\} \cdot \sum_{i=1}^2 h_i(\xi) (\mathbf{e}_r)_i \rangle /8. \end{aligned}$$

The physical components of the strains [Eq. (4.10:11)] are

$$e_{\xi\xi} = \varepsilon_{\xi\xi} / [\mathbf{g}_\xi \cdot \mathbf{g}_\xi], \quad e_{\xi\zeta} = \varepsilon_{\xi\zeta} / |\mathbf{g}_\xi| |\mathbf{g}_\zeta|.$$

Constitutive Equation. A general plane stress constitutive equation can be written in the form with respect to the normal base coordinates

$$(14) \quad \bar{\sigma} = \bar{\mathbf{D}}_{3e} \bar{\varepsilon},$$

where $\bar{\sigma}$, $\bar{\varepsilon}$ are, respectively, the in-plane stress and strain matrices

$$(15) \quad \bar{\sigma} = [\bar{\sigma}_{\xi\xi} \quad \bar{\sigma}_{\eta\eta} \quad \bar{\sigma}_{\xi\eta} \quad \bar{\sigma}_{\eta\xi}]^T, \quad \bar{\varepsilon} = [\bar{\varepsilon}_{\xi\xi} \quad \bar{\varepsilon}_{\eta\eta} \quad 2\bar{\varepsilon}_{\xi\eta} \quad 2\bar{\varepsilon}_{\eta\xi} \quad 2\bar{\varepsilon}_{\xi\xi}]^T,$$

$\bar{\mathbf{D}}_{3e}$ is a 5×5 symmetric elastic modulus matrix for plane stress derived from the 5×5 modulus matrix for 3-dimensional elasticity [Eq. (18.16:4)] by enforcing zero normal stress to the shell surface ($\bar{\sigma}_{\zeta\zeta} = 0$). The subscript 3 is used to distinct $\bar{\mathbf{D}}_{3e}$ from the 2-dimensional case. For materials with transverse shear strains decoupled from the rest of the strain components, $\bar{\mathbf{D}}_{3e}$ can be written as

$$(16) \quad \bar{\mathbf{D}}_{3e} = \text{dia}(\bar{\mathbf{D}}_e \quad \bar{\mathbf{D}}_s),$$

where $\bar{\mathbf{D}}_e$ is a 3×3 elastic modulus matrix for plane stress [see Eq. (18.14:19) for orthotropic materials], $\bar{\mathbf{D}}_s$ is a 2×2 transverse shear modulus matrix. To attain results consistent with classical bending theory, a shear factor needs to be introduced, which usually amounts to multiplying the transverse shearing moduli by $\kappa = 5/6$. For isotropic materials, $\bar{\mathbf{D}}_s$ is a diagonal matrix given in Eq. (20.2:6).

Strain Transformation. Strain transformation between coordinate systems follows the tensor transformation law (see Sec. 2.5). The physical components of the strain tensors in the normal base coordinates can be expressed in terms of those in the natural coordinates as

$$(17) \quad \begin{bmatrix} \bar{\varepsilon}_{\xi\xi} & \bar{\varepsilon}_{\xi\eta} & \bar{\varepsilon}_{\xi\zeta} \\ \bar{\varepsilon}_{\xi\eta} & \bar{\varepsilon}_{\eta\eta} & \bar{\varepsilon}_{\eta\zeta} \\ \bar{\varepsilon}_{\xi\zeta} & \bar{\varepsilon}_{\eta\zeta} & \bar{\varepsilon}_{\zeta\zeta} \end{bmatrix} = \bar{\mathbf{R}}^T \begin{bmatrix} \varepsilon_{\xi\xi} & \varepsilon_{\xi\eta} & \varepsilon_{\xi\zeta} \\ \varepsilon_{\xi\eta} & \varepsilon_{\eta\eta} & \varepsilon_{\eta\zeta} \\ \varepsilon_{\xi\zeta} & \varepsilon_{\eta\zeta} & \varepsilon_{\zeta\zeta} \end{bmatrix} \bar{\mathbf{R}},$$

$$(18) \quad \bar{\mathbf{R}} = \begin{bmatrix} \mathbf{g}^\xi \cdot \bar{\mathbf{e}}_\xi & \mathbf{g}^\xi \cdot \bar{\mathbf{e}}_\eta & \mathbf{g}^\xi \cdot \bar{\mathbf{e}}_\zeta \\ \mathbf{g}^\eta \cdot \bar{\mathbf{e}}_\xi & \mathbf{g}^\eta \cdot \bar{\mathbf{e}}_\eta & \mathbf{g}^\eta \cdot \bar{\mathbf{e}}_\zeta \\ \mathbf{g}^\zeta \cdot \bar{\mathbf{e}}_\xi & \mathbf{g}^\zeta \cdot \bar{\mathbf{e}}_\eta & \mathbf{g}^\zeta \cdot \bar{\mathbf{e}}_\zeta \end{bmatrix}.$$

From Eq. (17) and (18) together with Eq. (12) and (13), we can obtain $\bar{\varepsilon}$'s in terms of shape functions and nodal values of \mathbf{u}_0 , θ_ξ , θ_η . To be consistent with the shell theory, one can neglect $\varepsilon_{\zeta\zeta}$ in Eq. (17). Then we can write the strain-displacement relation in the matrix form

$$(19) \quad \bar{\varepsilon} = \bar{\mathbf{B}}\mathbf{q} = \sum_{i=1}^m \bar{\mathbf{B}}_i \mathbf{q}_i,$$

where $\bar{\mathbf{B}}$ is the strain displacement matrix, m is the number of nodes of the element, the subscript i denotes association with node i and

$$(20) \quad \mathbf{q}_i = [(\mathbf{u}_o)_i^T \quad (\theta_\xi)_i \quad (\theta_\eta)_i]^T, \quad \mathbf{q} = [\mathbf{q}_1^T \quad \mathbf{q}_2^T \quad \dots \quad \mathbf{q}_p^T]^T.$$

The calculation of $\bar{\mathbf{B}}$ from Eqs. (1)–(6), (9)–(13), (17) and (18) is tedious but straightforward.

Element Matrices. The shell element shown in Fig. 20.5:1 is a solid whose top and bottom surfaces are curved and whose sides are generated by straight lines (fibers). We shall derive the matrices for the shell element from the potential energy given in Eq. (18.16:10),

$$(21) \quad \Pi_e = \int_{A_e} \int_{-1}^1 (\bar{\varepsilon}^T \bar{\mathbf{D}}_{3e} \bar{\varepsilon} / 2 - \mathbf{u}^T \mathbf{b}) t J_A d\zeta dA - \int_{\partial A_{\sigma e}} \int_{-1}^1 \mathbf{u}^T \bar{\mathbf{T}} t J_s d\zeta ds \\ + \int_{A_e} p \mathbf{u}_0(\xi, \eta) \cdot \bar{\mathbf{e}}_{\zeta 0} d\zeta dA,$$

with zero shear and the prescribed normal traction \bar{T}_n^\pm on the upper and lower shell surfaces such that

$$(22) \quad p = \bar{T}_n^- J_A^- - \bar{T}_n^+ J_A^+ \cong \bar{T}_n^- - \bar{T}_n^+,$$

in which J_A , J_s are, respectively, the *Jacobians* for the volume and the surface integrals normalized by their corresponding values at the midsurface

$$(23) \quad J_A = \mathbf{g}_\zeta \cdot (\mathbf{g}_\xi \times \mathbf{g}_\eta) / |\mathbf{g}_{\zeta 0} \cdot (\mathbf{g}_{\xi 0} \times \mathbf{g}_{\eta 0})|, \quad J_s = |\mathbf{g}_\zeta \times \mathbf{g}_\alpha| / |\mathbf{g}_{\zeta 0} \times \mathbf{g}_{\alpha 0}|,$$

where $\alpha = \xi$ if the boundary is along the curve $\eta = \text{constant}$, and $\alpha = \eta$ if the boundary is along the curve $\xi = \text{constant}$. Both \mathbf{g}_ξ and \mathbf{g}_η are functions of ξ , η , ζ and so are J_A and J_s . If the boundary of the shell element is part of cylindrical surface, J_s is 1. In the case of a plate, J_A and J_s are 1. In practice, one may approximate J_A and J_s as unity if the shell is thin.

Using Eq. (10) and (19), we can derive the element matrices from Eq. (21) in the form

$$\Pi_e = \frac{1}{2} \mathbf{q}^T \mathbf{k} \mathbf{q} - \mathbf{q}^T \mathbf{f}$$

with

$$(24) \quad \mathbf{k} = \int_{A_e} \int_{-1}^1 t \bar{\mathbf{B}}^T \bar{\mathbf{D}}_{3e} \bar{\mathbf{B}} J_A d\zeta dA, \\ \mathbf{f} = \int_{A_e} \int_{-1}^1 (\bar{\mathbf{h}}_a^T \mathbf{b}) t J_A d\zeta dA - \int_{A_e} p (\bar{\mathbf{h}}_a^T)_0 \cdot \bar{\mathbf{e}}_{\zeta 0} d\zeta dA + \int_{\partial A_{\sigma e}} \int_{-1}^1 \bar{\mathbf{h}}_a^T \bar{\mathbf{T}} t J_s d\zeta ds.$$

A two-point Gauss rule in integration is adequate to sense the bending and membrane actions of a single homogeneous layer shell. For layered laminates through the thickness, the one- or two-point Gauss rule has to be applied to each layer.

Global Coordinates. We must transform \mathbf{q} of the element to the global coordinates before assembling. Let

$\tilde{\mathbf{e}}_x, \tilde{\mathbf{e}}_y, \tilde{\mathbf{e}}_z$ be the unit base vectors of the global system. The displacement and rotation matrices in the global coordinates are written as

$$(25) \quad \tilde{\mathbf{q}} = [\tilde{\mathbf{q}}_1^T \ \tilde{\mathbf{q}}_2^T \ \dots \ \tilde{\mathbf{q}}_p^T]^T, \quad \tilde{\mathbf{q}}_i = [\tilde{\mathbf{u}}_0^T \ \tilde{\boldsymbol{\theta}}^T]_i^T, \quad \tilde{\boldsymbol{\theta}}_i = [\theta_x \ \theta_y \ \theta_z]_i^T.$$

Note that $\tilde{\mathbf{q}}_i$ is a column matrix of six components and that $\tilde{\theta}_y, -\tilde{\theta}_x$ denote the rotations about the x, y -axes. Then, in the global coordinates,

$$(26) \quad (\mathbf{u}_0)_i = \begin{bmatrix} \mathbf{e}_x \cdot \tilde{\mathbf{e}}_x & \mathbf{e}_x \cdot \tilde{\mathbf{e}}_y & \mathbf{e}_x \cdot \tilde{\mathbf{e}}_z \\ \mathbf{e}_y \cdot \tilde{\mathbf{e}}_x & \mathbf{e}_y \cdot \tilde{\mathbf{e}}_y & \mathbf{e}_y \cdot \tilde{\mathbf{e}}_z \\ \mathbf{e}_z \cdot \tilde{\mathbf{e}}_x & \mathbf{e}_z \cdot \tilde{\mathbf{e}}_y & \mathbf{e}_z \cdot \tilde{\mathbf{e}}_z \end{bmatrix}_i (\tilde{\mathbf{u}}_0)_i,$$

$$(27) \quad \begin{bmatrix} \theta_\xi \\ \theta_\eta \end{bmatrix}_i = \begin{bmatrix} \hat{\mathbf{e}}_\eta \cdot \tilde{\mathbf{e}}_\eta & -\hat{\mathbf{e}}_\eta \cdot \tilde{\mathbf{e}}_\xi & \hat{\mathbf{e}}_\eta \cdot \tilde{\mathbf{e}}_\zeta \\ -\hat{\mathbf{e}}_\xi \cdot \tilde{\mathbf{e}}_\eta & \hat{\mathbf{e}}_\xi \cdot \tilde{\mathbf{e}}_\xi & \hat{\mathbf{e}}_\xi \cdot \tilde{\mathbf{e}}_\zeta \end{bmatrix}_i \tilde{\boldsymbol{\theta}}_i.$$

Using the transformation above we obtain the element matrices associated with displacements and rotations in global coordinates suitable for assembly and enforcing rigid constraints. Note that if the displacement by Eq. (9) is used, we simply replace $\tilde{\mathbf{e}}$'s with e 's.

Drilling Degree-of-Freedom and Coordinate Transformation. In Eq. (27), two local rotations at each node are transformed to three in global coordinates. Like the plate case, the assembled stiffness matrix will be singular, if all elements meet at a node have the same fiber direction at the common node, which is very likely. To avoid this potential singularity, one can modify the element formulation by making the rotations about the nodal fibers contribute to strains using drilling DOFs. One defines

$$\Pi_d = G \int_{V_e} \{ \alpha_x[\theta_x - (w_{,y} - v_{,z})^2/2] + \alpha_y[\theta_y - (u_{,z} - v_{,x})^2/2] + \alpha_z[\theta_z - (v_{,x} - u_{,y})^2/2] \} dV/2,$$

where $\alpha_x, \alpha_y, \alpha_z$ are selected constants. An alternative is to identify a global fiber base coordinates for each node. In this case, there will be only two rotations (or five DOFs total) per node. Another alternative is to use the normal base coordinates for rotations given in Eq. (10). Due to the approximation of shell geometry by Eq. (1), the normal vectors of different elements at the common node are usually different. Thus singularity may be avoided.

Remarks. A comment on strain derivation is in order. In literature, the normal base coordinates are often called *the running local Cartesian coordinates*. The strains referred to the normal coordinates are defined as

$$(28) \quad \bar{\varepsilon}_{\bar{\xi}\bar{\xi}} = \partial \bar{u}_{\bar{\xi}} / \partial \bar{\xi}, \quad \bar{\varepsilon}_{\bar{\xi}\bar{\eta}} = (\partial \bar{u}_{\bar{\xi}} / \partial \bar{\eta} + \partial \bar{u}_{\bar{\eta}} / \partial \bar{\xi})/2, \dots, \text{etc.}$$

One then expresses \bar{u} 's and $\bar{\xi}$'s in terms of their corresponding components in the local fixed Cartesian coordinates based on the coordinate transformation rule and proceeds with the partial differentiation. The transformation does not change the fact that the strains in Eq. (28) are defined based only on partial differentiation rather than on covariant differentiation (Sec. 2.12). Even though the normal coordinates are orthogonal, the *orientation of the running local coordinates changes with location within the element for general shells*. This change contributes to strains in the terms of the *Euclidean Christoffel symbols*. In other words, *strains should be derived from the deformation gradients, $\partial \mathbf{u} / \partial \bar{\xi}$, $\partial \mathbf{u} / \partial \bar{\eta}$ etc. in order to account for the covariant or contravariant differentiation, rather than just from the partial derivatives of the components of the displacement*.

In the exercises below, we use \mathbf{e}_r and \mathbf{e}_θ to denote the unit base vectors of the cylindrical coordinates and α the rotation about the z -axis.

Problem 20.7. Consider a ring element of thickness t and radius r_0 over $-\theta_0 \leq \theta \leq \theta_0$. Let

$$\mathbf{x} = \mathbf{x}_0 + t\zeta \mathbf{e}_r / 2 = (r_0 + t\zeta/2) \mathbf{e}_r,$$

where

$$\mathbf{e}_r = \mathbf{e}_x \cos \theta + \mathbf{e}_y \sin \theta, \quad \zeta = 2(r - r_0)/t, \quad \xi = \theta/\theta_0.$$

The element has three nodes at $\theta_{1,2} = \pm \theta_0, \theta_3 = 0$. Assuming the displacement

$$\mathbf{u} = \sum_{i=1}^3 h_i(\xi) (\mathbf{u}_0 + t\zeta \alpha \mathbf{e}_\theta / 2)_i, \quad \mathbf{e}_\theta = -\mathbf{e}_x \sin \theta + \mathbf{e}_y \cos \theta, \\ h_1(\xi) = -\xi(1-\xi)/2, \quad h_2(\xi) = \xi(1+\xi)/2, \quad h_3(\xi) = 1 - \xi^2,$$

show that the covariant base vectors and the metric tensor are

(a) $\mathbf{g}_\xi = (r_0 + t\zeta/2)\theta_0 \mathbf{e}_\theta, \mathbf{g}_\zeta = t\mathbf{e}_r/2,$

(b) $(g_{\xi\xi})^{1/2} = (r_0 + t\zeta/2)\theta_0, (g_{\zeta\zeta})^{1/2} = t/2, g_{\xi\zeta} = 0,$ and

(c) the physical component of the strain in the radial direction is

$$4e_{\zeta\zeta}/t^2 = 4(g_\zeta \cdot \partial \mathbf{u} / \partial \zeta) / t^2 = \{(t\alpha)_1 h_1(\xi) \sin[\theta_0(\xi+1)] \\ + (t\alpha)_2 h_2(\xi) \sin[\theta_0(\xi-1)]/t + (t\alpha)_3 h_3(\xi) \sin(\theta_0 \xi)\} / t,$$

which is not zero whereas its representation in the shell theory is zero. This quantity is of the order θ_0^2 for small element (i.e., small θ_0). [Hint: Use the following relations:

$$\mathbf{e}_{\theta i} \cdot \mathbf{e}_\theta = \mathbf{e}_{ri} \cdot \mathbf{e}_r = \cos(\theta - \theta_i), \quad \mathbf{e}_{\theta i} \cdot \mathbf{e}_r = -\mathbf{e}_{ri} \cdot \mathbf{e}_\theta = \sin(\theta - \theta_i), \\ \mathbf{u}_{0i} = u_{x0i} \mathbf{e}_x + u_{y0i} \mathbf{e}_y = u_{r0i} \mathbf{e}_r + u_{\theta0i} \mathbf{e}_\theta, \quad \theta_2 = -\theta_1 = \theta_0, \theta_3 = 0.]$$

Problem 20.8. Repeat Problem 20.7 using

$$\mathbf{x} = \sum_{i=1}^3 h_i(\xi) (\mathbf{x}_0 + \zeta \mathbf{e}_\theta / 2)_i, \quad (\mathbf{x}_0)_i = r_0 (\mathbf{e}_x \cos \theta_i + \mathbf{e}_y \sin \theta_i)$$

Problem 20.9. Find the difference between the strains of the quadrilateral and those of the ring element (Fig. P20.9).



Fig. P20.9. Shell of flat surfaces.

Problem 20.10. Let the displacements of the ring discussed above be

$$\mathbf{u} = u_r(\theta) \mathbf{e}_r + [u_\theta(\theta) + \alpha(\theta)(r - r_0)] \mathbf{e}_\theta,$$

(a) Derive the physical component of the strains $e_{rr}, e_{\theta\theta}, e_{r\theta}$; (b) let

$$u_r(\theta) \mathbf{e}_r + u_\theta(\theta) \mathbf{e}_\theta = \sum_{i=1}^3 (\mathbf{u}_0)_i h_i(\xi), \quad \alpha(\theta) = \sum_{i=1}^3 \alpha_i h_i(\xi).$$

Find the differences between the strains derived from the displacements above and those from Problem 20.8. Show the difference is of the order θ_0^2 for small θ_0 .

Problem 20.11. Using tensor transformation rule, determine the components of $\bar{\mathbf{D}}_e, \bar{\mathbf{D}}_s$ in the fiber base coordinates in terms of those of $\mathbf{D}_e, \mathbf{D}_s$ in the normal base coordinates. [Hint: $\bar{\boldsymbol{\varepsilon}}^T \bar{\mathbf{D}}_{3e} \bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}^T \mathbf{D}_{3e} \boldsymbol{\varepsilon}$,]

Problem 20.12. Use Eq. (15) and (16) and $\bar{\boldsymbol{\varepsilon}}^T \bar{\mathbf{D}}_{3e} \bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}^T \mathbf{D}_{3e} \boldsymbol{\varepsilon}$, express the components of \mathbf{D}_{3e} in terms of those of $\bar{\mathbf{D}}_{3e}$.

20.6. LOCKING AND STABILIZATION IN SHELL APPLICATIONS

Doubly curved shell elements can lock in thin shell applications. To illustrate the potential of locking, let us examine the characteristics of $\varepsilon_{\xi\xi}$ and $\varepsilon_{\xi\eta}$ of a 4-node quadrilateral element of constant thickness. Let

$$\mathbf{x} = \mathbf{a}_1 + \mathbf{a}_2 \xi + \mathbf{a}_3 \eta + \mathbf{a}_4 \xi \eta + \zeta \mathbf{g}_\zeta(\xi, \eta) = \mathbf{x}_0 + \zeta \mathbf{g}_\zeta(\xi, \eta),$$

$$\mathbf{u} = \mathbf{b}_1 + \mathbf{b}_2 \xi + \mathbf{b}_3 \eta + \mathbf{b}_4 \xi \eta - t\zeta(\theta_\xi \mathbf{e}_\xi + \theta_\eta \mathbf{e}_\eta) = \mathbf{u}_0 - t\zeta(\theta_\xi \mathbf{e}_\xi + \theta_\eta \mathbf{e}_\eta).$$

From Eqs. (20.5:4) and (20.5:11–13), we have

- (1) $g_{\xi 0} = \partial x_0 / \partial \xi = a_2 + a_4 \eta, \quad g_\zeta = t \sum_{i=1}^4 h_i(\xi, \eta) s_i / 2$
- (2) $\partial u_0 / \partial \xi = b_2 + b_4 \eta, \quad \partial u / \partial \zeta = -t \sum_{i=1}^4 [\theta_\xi e_\xi + \theta_\eta e_\eta]_i h_i(\xi, \eta) / 2,$
- (3) $\varepsilon_{\xi \xi 0} = (a_2 + a_4 \eta) \cdot (b_2 + b_4 \eta),$
- (4) $\varepsilon_{\xi \zeta 0} = t[(b_2 + b_4 \eta) \cdot \sum_{i=1}^4 h_i s_i - (a_2 + a_4 \eta) \cdot \sum_{i=1}^4 (\theta_\xi e_\xi + \theta_\eta e_\eta)_i h_i] / 4,$

where s_i is a unit vector in the fiber direction at node i and

- (5) $a_2 = (-x_{01} + x_{02} + x_{03} - x_{04}) / 4, \quad a_4 = (x_{01} - x_{02} + x_{03} - x_{04}) / 4,$
- (6) $b_2 = (-u_1 + u_2 + u_3 - u_4) / 4, \quad b_4 = (u_1 - u_2 + u_3 - u_4) / 4.$

In the limit of inextensional bending and zero transverse shear for thin shells, the coefficients of the polynomial for $\varepsilon_{\xi \xi 0}$ and $\varepsilon_{\xi \zeta 0}$ vanish. This represents constraints on the parameters appearing in the coefficients. Since $\varepsilon_{\xi \xi 0}$ is quadratic in η and independent of ξ while $\varepsilon_{\xi \zeta 0}$ is quadratic in η but linear in ξ , there are 9 possible constraints (3 from $\varepsilon_{\xi \xi 0}$ and 6 from $\varepsilon_{\xi \zeta 0}$) for the parameters. For general shells, it is difficult to identify which of these constraints is field-consistent (Sec. 20.3) and which is not. For flat parallelograms $a_4 = 0, s_1 = s_2 = s_3 = s_4$, one can show that only the transverse shear $\varepsilon_{\xi \zeta 0}$ has spurious terms which are from the coefficients of ξ and $\xi \eta$ in the last term of Eq. (4). We define

$$\sum_{i=1}^4 [\theta_\xi e_\xi + \theta_\eta e_\eta]_i h_i = \sum_{i=1}^4 \Psi_i h_i = \Theta_1 + \Theta_2 \xi + \Theta_3 \eta + \Theta_4 \xi \eta,$$

where

$$\begin{aligned} \Psi_i &= (\theta_\xi e_\xi + \theta_\eta e_\eta)_i \\ \Theta_1 &= (\Psi_1 + \Psi_2 + \Psi_3 + \Psi_4) / 4, \quad \Theta_2 = (-\Psi_1 + \Psi_2 + \Psi_3 - \Psi_4) / 4, \\ \Theta_3 &= (-\Psi_1 - \Psi_2 + \Psi_3 + \Psi_4) / 4, \quad \Theta_4 = (\Psi_1 - \Psi_2 + \Psi_3 - \Psi_4) / 4. \end{aligned}$$

The spurious constraints are then,

$$(7) \quad \Theta_2 \cdot a_2 \rightarrow 0, \quad \Theta_4 \cdot a_2 \rightarrow 0.$$

Similarly we can determine the spurious constraints for $\varepsilon_{\xi \zeta 0}$ as

$$\Theta_3 \cdot b_3 \rightarrow 0, \quad \Theta_4 \cdot b_3 \rightarrow 0.$$

Then the field-consistent transverse shear strains are (note: $\sum_{i=1}^4 h_i = 1$)

$$(8) \quad \begin{aligned} \varepsilon_{\xi \zeta} &= t[(b_2 + b_4 \eta) \cdot \sum_{i=1}^4 s_i / 4 - a_2 \cdot (\Theta_1 + \Theta_3 \eta)] / 4, \\ \varepsilon_{\eta \zeta} &= t[(a_3 + a_4 \xi) \cdot \sum_{i=1}^4 s_i / 4 - b_3 \cdot (\Theta_1 + \Theta_2 \xi)] / 4, \end{aligned}$$

which are used to calculate the shear strain energy to remove locking and to avoid the oscillations in evaluating the transversal shear stresses within the element. This equivalent to evaluating $\varepsilon_{\xi \zeta}$ at $\xi = 0$ and $\varepsilon_{\eta \zeta}$ at $\eta = 0$.

Literature is full of locking removal techniques, such as uniformly reduced integration, selectively reduced integration, drilling DOFs, etc. No single technique can completely rid of locking for general shell elements. The uniformly one-point integration for 4-node and 2×2 reduced integration for 9-node quadrilaterals are most used in practice. Reduced integration offers substantial savings in computation. Further computer savings can be achieved by taking advantages of the symmetric and anti-symmetric properties of $\partial h_i / \partial \xi$ and $\partial h_i / \partial \eta$ at integration points. The choice of elements in applications must balance the computational efficiency against the desired accuracy of the results.

Stabilization. Uniformly reduced integration always and selectively reduced integration usually introduce spurious zero energy modes (kinematic modes). These modes occasionally give spurious solution in static analysis and often give spatially oscillatory solutions in dynamic simulation. The oscillations are called the *hourglass modes*. Schemes of using stabilization matrix have been devised to eliminate the kinematic modes (see Biblio. 20.7) for 4-node quadrilateral plate bending element with one point integration. Consider

$$(9) \quad \mathbf{q} = [w^T \quad \theta_x^T \quad \theta_y^T]^T$$

of the nodal values of the out-plane displacement w 's and the rotations θ 's that

$$(10) \quad \mathbf{w} = [w_1 \ w_2 \ w_3 \ w_4]^T, \quad \boldsymbol{\theta}_x = [\theta_{x1} \dots \theta_{x4}]^T, \quad \boldsymbol{\theta}_y = [\theta_{y1} \dots \theta_{y4}]^T,$$

$$(11) \quad \mathbf{r} = [1 \ 1 \ 1 \ 1]^T, \quad \mathbf{g} = [1 \ -1 \ 1 \ -1]^T, \quad \mathbf{x} = [x_1 \ \dots \ x_4]^T, \quad \mathbf{y} = [y_1 \ \dots \ y_4]^T,$$

$$(12) \quad \mathbf{b}_1 = [y_2 - y_4 \ y_3 - y_1 \ y_4 - y_2 \ y_1 - y_3]^T/2,$$

$$(13) \quad \mathbf{b}_2 = -[x_2 - x_4 \ x_3 - x_1 \ x_4 - x_2 \ x_1 - x_3]^T/2.$$

The rigid body modes are

$$(14) \quad \mathbf{q} = [\mathbf{r}^T \ 0 \ 0]^T, \quad [\mathbf{x}^T \ \mathbf{r}^T \ 0]^T, \quad [0 \ \mathbf{y}^T \ \mathbf{r}^T]^T.$$

The kinematic modes are

$$(15) \quad \mathbf{q} = [\mathbf{g}^T \ 0 \ 0]^T, \quad [0 \ \mathbf{g}^T \ 0]^T, \quad [0 \ 0 \ \mathbf{g}^T]^T, \quad [a_1 \mathbf{x}^T + a_2 \mathbf{y}^T \ \mathbf{x}^T \ \mathbf{y}^T]^T,$$

where

$$a_1 = \mathbf{r}^T \mathbf{y} / 4, \quad a_2 = \mathbf{r}^T \mathbf{x} / 4.$$

The stabilization matrix is in the form

$$\mathbf{k}_H = \text{dia} [c_1 \ c_2 \ c_2] \boldsymbol{\alpha} \boldsymbol{\alpha}^T,$$

where

$$\boldsymbol{\alpha} = \mathbf{g} - [(\mathbf{g}^T \mathbf{x}) \mathbf{b}_1 + (\mathbf{g}^T \mathbf{y}) \mathbf{b}_2] / (\mathbf{x}^T \mathbf{b}_1),$$

is chosen to be orthogonal to the rigid body motion and to assure that the stabilization does not affect the genuine zero strain energy modes. Note that

$$\mathbf{r}^T \mathbf{g} = \mathbf{r}^T \mathbf{b}_1 = \mathbf{r}^T \mathbf{b}_2 = 0, \quad \mathbf{g}^T \mathbf{b}_1 = \mathbf{g}^T \mathbf{b}_2 = \mathbf{y}^T \mathbf{b}_1 = \mathbf{x}^T \mathbf{b}_2 = 0, \quad \mathbf{x}^T \mathbf{b}_1 = \mathbf{y}^T \mathbf{b}_2.$$

The constants c 's are constants to be selected judiciously. If they are too large, the element will lock just like the fully integrated element whereas too small will fail to control oscillations in the solution. Note that $\boldsymbol{\alpha}$ does not control the last kinematic mode of Eq. (15) because $\boldsymbol{\alpha}$ is orthogonal to the mode. In practice, it does not matter because the last kinematic mode cannot exist in a mesh of two or more elements.

Belytschko and Tsay (1983) proposed to define c 's in the form

$$c_1 = c_w \kappa G t^3 (\mathbf{b}_1^T \mathbf{b}_1 + \mathbf{b}_2^T \mathbf{b}_2) / [12(\mathbf{x}^T \mathbf{b}_1)^2]$$

$$c_2 = c_\theta E t^3 (\mathbf{b}_1^T \mathbf{b}_1 + \mathbf{b}_2^T \mathbf{b}_2) / [192(1 - \nu^2) \mathbf{x}^T \mathbf{b}_1]$$

where c_w, c_θ are universal constant independent of shape and material properties of the element. The range of c_w is between 0.03 to 0.1. The solution is generally insensitive to c_θ . It can have a value between 0.001 to 10. The first kinematic mode ($\mathbf{w} = \mathbf{g}$), controlled by c_1 , is most sensitive.

20.7. ELEMENTS FOR HETEROGENEOUS MATERIALS

The complexity of shell elements is still a roadblock to progress of robust shell element development, especially for elements with varying material properties, such as functionally graded or thick-section laminated composites. The linear stress through the thickness is generally not adequate if a coarse mesh is used. Atluri *et al.* (2014a,b) assumed linear strains within the element and employed over-integration in the thickness direction to model tension, bending, and shear of beams, plates, and shells. This is to capture the basic kinematics of the element, even if only a few elements are used.

To illustrate the concept, we first construct a 4-node isoparametric element, called CEQ4 (Atluri *et al.* 2014a), by assuming

$$(1) \quad \mathbf{e} = \begin{bmatrix} e_{xx} \\ e_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & y & 0 \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_5 \end{bmatrix} = \mathbf{h}_\alpha(\xi, \eta) \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_5 \end{bmatrix}_{3 \times 5 \times 1},$$

$$\tilde{\mathbf{e}} = [\tilde{e}_{xx} \quad \tilde{e}_{yy} \quad \tilde{\gamma}_{xy}]^T = \mathbf{d}_e \mathbf{h}(\xi, \eta) \mathbf{q} = \mathbf{B}(\xi, \eta) \mathbf{q},$$

where \mathbf{h}_α is defined in Eq. (1), \mathbf{h} in Eq. (18.14:9) and \mathbf{d}_e in Eq. (18.14:1), (x_i, y_i) are the Cartesian nodal coordinates, α 's are unknowns, q 's are the nodal displacements of the element, and $\tilde{\mathbf{e}}$ is the strain derived from the interpolated displacement.

To capture the basic kinematics of the element, we collocate the extension ratios of strains $\tilde{\mathbf{e}}$ and \mathbf{e} at points 1 and 2 at ($(\xi = 0, \eta = \mp 1/\sqrt{3})$) along the ξ -axis, and at points 3 and 4 at ($(\xi = \mp 1/\sqrt{3}, \eta = 0)$) along the η -axis. Let \mathbf{n} denotes the unit vector along the ξ -axis for points 1 and 2 in Cartesian frame and along the η -axis for points 3 and 4. The collocations give, in tensor notation,

$$(2) \quad \mathbf{n} \cdot \tilde{\mathbf{e}}(\xi, \eta) \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{e}(\xi, \eta) \cdot \mathbf{n}$$

or in matrix notation,

$$\tilde{e}_{xx} n_x^2 + \tilde{e}_{yy} n_y^2 + 2\tilde{\gamma}_{xy} n_x n_y = e_{xx} n_x^2 + e_{yy} n_y^2 + \gamma_{xy} n_x n_y.$$

At point 5 ($\xi = \eta = 0$), we equate the changes in angle, derived from \mathbf{e} and ($\xi = 0, \eta = \mp 1/\sqrt{3}$) between the unit fibers \mathbf{n}_b and \mathbf{n}_c unit vectors of the ξ, η -axes in Cartesian frame. We have,

$$(3) \quad (\mathbf{n}_b - \mathbf{n}_c) \cdot \tilde{\mathbf{e}}(\xi, \eta) \cdot (\mathbf{n}_b - \mathbf{n}_c) = (\mathbf{n}_b - \mathbf{n}_c) \cdot \mathbf{e}(\xi, \eta) \cdot (\mathbf{n}_b - \mathbf{n}_c)$$

$$\tilde{e}_{xx}(n_{bx} n_{by} - n_{cx} n_{cy}) + \tilde{e}_{yy}(n_{cx} n_{cy} - n_{bx} n_{by}) + \tilde{\gamma}_{xy}(n_{cx}^2 - n_{bx}^2)$$

$$= e_{xx}(n_{bx} n_{by} - n_{cx} n_{cy}) + e_{yy}(n_{cx} n_{cy} - n_{bx} n_{by}) + \tilde{\gamma}_{xy}(n_{cx}^2 - n_{bx}^2).$$

From Eq. (2) and (3), one can solve for α 's in terms of q 's, say $\alpha = \mathbf{C}\mathbf{q}$, in which \mathbf{C} is a constant matrix. A substitution of α into Eq. (1) yields

$$\mathbf{e} = \mathbf{h}_\alpha \mathbf{C} \mathbf{q} = \mathbf{B}_\alpha(\xi, \eta) \mathbf{q}$$

From the potential energy functional, we obtain element matrices

$$(4) \quad \Pi_e = \mathbf{q}^T \mathbf{D} \mathbf{q} / 2 - \mathbf{q}^T \mathbf{f},$$

where \mathbf{D} is a 3×3 material matrix for plane strain or stress, and

$$\mathbf{k} = \int_{A_e} \mathbf{B}_\alpha^T \mathbf{D} \mathbf{B}_\alpha dA, \quad \mathbf{f} = \int_{A_e} \mathbf{h}^T \mathbf{b} dA + \int_{\partial A_{\theta_e}} \mathbf{h}^T \bar{\mathbf{T}} ds.$$

For linearly-varying field, a 2×2 Gauss quadrature is sufficient to model homogeneous materials. However, for inhomogeneous materials, such as functionally-graded or thick-section laminated composites, we use 3×3 Gauss quadrature to compute the stiffness matrix. However, for very-thick laminates, it is better to use either a layer-wise 2×2 Gauss quadrature, or a simple trapezoidal rule in the thickness direction to evaluate the element stiffness matrix. The sampling points depends on the number of plies in the thickness.

In applying CEQ4 to model thick-beam section, one obtains better transverse normal and shear stresses by using the bending stress σ_{xx} computed from Eq. (1) to evaluate numerically the transverse stresses:

$$\sigma_{xy} = \int_{y_0}^y [b_x - (\partial \sigma_{xx} / \partial x)] dy \quad \text{and} \quad \sigma_{yy} = \int_{y_0}^y [b_y - (\partial \sigma_{xy} / \partial x)] dy$$

where y_0 denotes the lower edge of the beam.

For 3-dimensional problems, we assume

$$(5) \quad \begin{aligned} \mathbf{e}_{6 \times 1} &= [e_{xx} \ e_{yy} \ e_{zz} \ \gamma_{yz} \ \gamma_{zx} \ \gamma_{xy}]^T = \mathbf{h}_\alpha(\xi, \eta)_{6 \times 18} \alpha_{18 \times 1} \\ &= \left[\begin{array}{ccccccccc} y & z & yz & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z & x & zx & 0 & 0 & 0 \\ \mathbf{I}_{6 \times 6} & 0 & 0 & 0 & 0 & 0 & 0 & x & y \\ & & & & & & & xy & \\ & & & & & & & \mathbf{0}_{3 \times 9} & \\ & & & & & & & & \mathbf{dia}(\mathbf{x})_{3 \times 3} \end{array} \right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{18} \end{bmatrix}. \end{aligned}$$

The collocation points for the extension ratios of fibers along the ξ , η , ζ -directions are $(\xi = 0, \eta = \pm 1/\sqrt{3}, \zeta = \pm 1/\sqrt{3})$, $(\xi = \pm 1/\sqrt{3}, \eta = 0, \zeta = \pm 1/\sqrt{3})$, and $(\xi = \pm 1/\sqrt{3}, \eta = \pm 1/\sqrt{3}, \zeta = 0)$, respectively, and those for the angle change between the fibers along the ξ , η , the η , ζ , and ζ , ξ -axes are at $(\xi = 0, \eta = 0, \zeta = \pm 1/\sqrt{3})$ at $(\xi = \pm 1/\sqrt{3}, \eta = 0, \zeta = 0)$, and at $(\xi = 0, \eta = \pm 1/\sqrt{3}, \zeta = 0)$, respectively.

As before, to improve the stress approximation for inhomogeneous materials, we use a 3-point, a layer-wise 2-point Gauss quadrature, or a simple trapezoidal rule in the thickness direction to evaluate the element stiffness matrix. One also uses the equilibrium equations of 3-D linear elasticity to compute the transverse normal and shear stresses. The derivation is left to the readers.

The elements work well for thick as well as thin beams, plates, and shells, and can be combined with other 3-D C^0 elements such as 20- and 27-node bricks. These locking free solid elements for functionally-graded and laminated plates and shells save the trouble of developing specific theories for such structures, and thus provide a “fit all” procedure for universal modeling of bulk solids and engineering structures.

One can improve the geometry approximation of plane curved beams by introducing the midside nodes 5 and 6 at $\xi = 0, \eta = \pm 1$. We assume

$$(6) \quad \tilde{\mathbf{e}} = \begin{bmatrix} \tilde{e}_x \\ \tilde{e}_y \\ \tilde{\gamma}_{xy} \end{bmatrix} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_9 \end{bmatrix}^T.$$

We collocate the extension ratios along the ξ - and the η -axes at $(\xi = \mp 1/\sqrt{3}, \eta = \mp 1/\sqrt{3})$ (4 in number) and at $(\xi = 0, \eta = 0, \mp \sqrt{3}/5)$ (3 in number), respectively; and collocate the change of the angle between the ξ - and η -axes at $(\xi = \mp 1/\sqrt{3}, \eta = 0)$ (2 in number).

Similarly we can modify the 8-node brick by adding midside nodes on the top and bottom surfaces $\zeta = \pm 1$ [Fig. 20.5:1(a)]. The assumed strains are

$$(7) \quad \begin{aligned} \tilde{e}_x &= \alpha_1 + \alpha_2\eta + \alpha_3\zeta + \alpha_4\eta\zeta + \xi(\alpha_5 + \alpha_6\eta + \alpha_7\zeta + \alpha_8\eta\zeta), \\ \tilde{e}_y &= \alpha_9 + \alpha_{10}\xi + \alpha_{11}\zeta + \alpha_{12}\xi\zeta + \eta(\alpha_{13} + \alpha_{14}\xi + \alpha_{15}\zeta + \alpha_{16}\xi\zeta), \\ \tilde{e}_z &= \alpha_{17} + \alpha_{18}\xi + \alpha_{19}\eta + \alpha_{20}\xi\eta + (\alpha_{21}\xi + \alpha_{22}\eta)\xi\eta + \alpha_{23}\xi^2 + \alpha_{24}\eta^2, \\ \tilde{\gamma}_{yz} &= \alpha_{25} + \alpha_{26}\xi + \eta(\alpha_{27} + \alpha_{28}\xi) + \alpha_{29}\xi^2 + \alpha_{30}\eta^2, \\ \tilde{\gamma}_{xz} &= \alpha_{31} + \alpha_{32}\eta + \xi(\alpha_{33} + \alpha_{34}\eta) + \alpha_{35}\eta^2 + \alpha_{36}\xi^2, \\ \tilde{\gamma}_{xy} &= \alpha_{37} + \alpha_{38}\zeta + \xi(\alpha_{39} + \alpha_{40}\zeta) + \eta(\alpha_{41} + \alpha_{42}\zeta), \end{aligned}$$

of a total of 42 parameters. The collocation points for the stretch ratios of fibers along the ξ , η , ζ -directions are the 3×3 Gauss quadrature stations with the origin excluded on the η , ζ , the ζ , ξ , and the ξ , η -planes (a total of 24), respectively. We also collocate the angle changes between the ξ , η -, the η , ζ , and the ζ , ξ -axes at the 3×2 Gaussian stations of the corresponding surfaces.

We can also construct a triangular prism element with 6 nodes each on the top and bottom surfaces [Fig. 20.5:1(b)] by assuming

$$\begin{aligned}
\tilde{e}_x &= \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\zeta + \alpha_5\xi\eta + \alpha_6\eta\zeta, \\
\tilde{e}_y &= \alpha_7 + \alpha_8\xi + \alpha_9\eta + \alpha_{10}\zeta + \alpha_{11}\xi\eta + \alpha_{12}\zeta\xi, \\
(8) \quad \tilde{e}_z &= \alpha_{13} + \alpha_{14}\xi + \alpha_{15}\eta + \alpha_{16}\xi\eta + \alpha_{17}\xi^2 + \alpha_{18}\eta^2, \\
\tilde{\gamma}_{yz} &= \alpha_{19} + \alpha_{20}\xi + \eta(\alpha_{21} + \alpha_{22}\xi), \\
\tilde{\gamma}_{xz} &= \alpha_{23} + \alpha_{24}\eta + \xi(\alpha_{25} + \alpha_{26}\eta), \\
\tilde{\gamma}_{xy} &= \alpha_{27} + \alpha_{28}\xi + \alpha_{29}\eta + \alpha_{30}\zeta,
\end{aligned}$$

of a total of 30 parameters. The collocation points for the stretch ratio of the fiber along the ξ -directions are the 6 stations on the η, ζ -plane shown the 4th figure in [Table 18.18:3](#) with the origin excluded; that along the η, ζ -directions are the 3×2 Gauss stations on the ζ, ξ -, and the ξ, η -planes, respectively. We also collocate the angle changes between the η, ζ -axes at the 4 points shown in the 3rd figure in [Table 18.18:3](#), and those between the ζ, ξ , and the ξ, η -axes are the 2×2 Gauss stations on the corresponding planes.

As before one solves for α 's in terms of nodal q 's and construct the element matrices. The strains as given in Eqs. (7) and (8) are symmetric with respect to the coordinates. The optimum selection of strain components and collocation locations require further study.

Problem 20.13. Determine numerically the transverse normal and shear stresses using the in-plane stresses derived from Eq. (5) and the equilibrium equations of 3-D linear elasticity.

Problem 20.14. Evaluate numerically the distribution of transverse stresses of $\sigma_{r\theta}, \sigma_{rz}$ and σ_{rr} of a cylindrical shell by solving the equilibrium equations of 3-D linear elasticity with $\sigma_{\theta\theta}, \sigma_{\theta z}$ and σ_{zz} derived from Eq. (5).

Remarks. Development of shell elements is conceptually straightforward but practically complicated. Locking is among the most perplexing problems in thin shell applications. It is of a lesser problem for higher order elements. General techniques for locking mitigation include reduced integration, mixed and hybrid formulations, addition of drilling freedoms, etc. Reduced integration is very popular because it is simple to implement and saves computation time from reduced integration operations. Locking removal introduces rank deficiency (zero energy mode), which requires stabilization, especially for dynamic simulations. Shown in the last section, the elements deduced from solid directly for shells, thin as well as thick, seem to be simple and robust (does not lock, passes the patch test, has no hourglass mode, does not require extensive multiplications, etc.). The advent has made modeling of complex shell structures a routine matter.

In practical applications, the inclination is to use simple elements, especially for dynamic simulations involving large deformation such as metal forming and crashworthiness studies. Choice of element must balance the computational efforts against the desired accuracy of the results. The new robust elements are among the populists.

¹The linearized small deflection theory of plates ([Sec. 13.14](#)) and shells will be discussed in this chapter. To solve the nonlinear problem, an incremental approach will be discussed in [Chapter 21](#). In incremental approach, we linearize the problem at every step. Hence the materials discussed herein are important.

21

FINITE ELEMENT MODELING OF NONLINEAR ELASTICITY, VISCOELASTICITY, PLASTICITY, VISCOPLASTICITY, AND CREEP

Most of the accomplishments of the classical mechanics are based on the principle of superposition, which is rooted in the linearity of the governing equations. Every branch of mechanics is now examining beyond the linear world the effects of nonlinearity. Nonlinear dynamics of the simple Duffing's equation describing the motion of a single particle opened our eyes to the richness of the field. The theory of turbulence in fluid mechanics tells us how deep, controversial, and refractory to research efforts the subject can be. In solid mechanics, the topics awaiting for attention include: large deformation, nonlinear constitutive laws, nonlinear boundary conditions, mechanical properties of living tissues, new materials of complex composition, materials in micromachines at nanoscales, living cells and tissues that cannot be identified as continua in the mathematical sense, heat transfer with large heat flux that occurs in computer and electronic communication hardware, and so on. The axioms of current mechanics may have to be changed in dealing with genomics, photonics, DNA, proteins, and enzymes. Every new change of an axiom leads to new scientific fields. Obviously, nonlinear mechanics is the mechanics of the future. Computational methods will have an important role to play.

In [Chapter 17](#), the use of the incremental approach to deal with nonlinear problems in solid mechanics was discussed. In [Chapters 18–20](#), the finite element method was applied to linear problems. In this chapter, we will discuss nonlinear problems including large deformation, viscoelasticity, plasticity, viscoplasticity, creep, buckling of thin shells, and self-equilibrating residual stresses and strains, and examine the computational aspects of nonlinearity. Oden (1967) gave early finite element solutions for large deformation of incompressible materials. Pian and Tong (1971)[17.1](#) discussed incremental finite element solution.

In incremental approach, we linearize the governing equations from a known state with small incremental step. One of the distinct characteristics of nonlinear problems is that the solution may not be unique, which lies much of nature's secret.

21.1. UPDATED LAGRANGIAN SOLUTION FOR LARGE DEFORMATION

We illustrated in [Chapter 17](#) how the updated Lagrangian approach for large deformation problems led to a set of linear equations for the stress increment rate $\overset{\circ}{\mathbf{T}}, \overset{\circ}{\mathbf{r}}^*, \overset{\circ}{\mathbf{S}}, \overset{\circ}{\tau}$, the deformation rates \mathbf{D}, Ω and the displacement rate \mathbf{v} . In this section, and this section only, following the derivation and notations of [Chapter 17](#), we use boldfaced letters as tensors rather than matrices and the tensor operation rules, unless specifically stated otherwise. We use lower case sub or superscript n to denote the current value. In solving for the rates, we need the current Cauchy stress $\tau (= \tau_n)$, the increment rates of the boundary traction $\overset{\circ}{\mathbf{T}}$ on $S_{\sigma n}$ and the boundary displacement $\overset{\circ}{\mathbf{v}}$ on S_{un} , the deformed geometry in V_n , and the constitutive laws relating the desired stress rates to various deformation rates and the deformation gradient \mathbf{F}^n .

The finite element equations can be derived based on the integral formulations outlined in [Chapter 18](#). Consider the variation of Eq. (17.6:12) in terms of the rate-potential $\overset{\circ}{W}$ defined in Eq. (17.5:23):

$$(1) \quad \overset{\circ}{\Pi}_3 = \sum_{\text{all elements}} \overset{\circ}{\Pi}_{3e},$$

where

$$(2) \quad \overset{\circ}{\Pi}_{3e} = \int_{V_{en}} [\overset{\circ}{W}(\Delta \mathbf{e}) + \frac{\tau : (\Delta \mathbf{L} \cdot \Delta \mathbf{L}^T)}{2} - \rho_n \Delta \mathbf{b} \cdot \Delta \mathbf{u}] dV - \int_{S_{\sigma en}} \Delta \bar{\mathbf{T}} \cdot \Delta \mathbf{u} dA$$

in which

$$\Delta \mathbf{u} = \mathbf{v} \Delta t, \quad \Delta \mathbf{e} = (\nabla_n \Delta \mathbf{u})_s, \quad \Delta \mathbf{L} = \nabla_n \Delta \mathbf{u}, \quad \Delta \mathbf{b} = \overset{\circ}{\mathbf{b}} \Delta t, \quad \Delta \bar{\mathbf{T}} = \overset{\circ}{\mathbf{T}} \Delta t,$$

with $\Delta \mathbf{u}$ being the displacement increment and the subscript n refers to the current configuration. The prescribed

pressure and dead-weight type of boundary traction increment rate $\overset{\circ}{\mathbf{T}}$ are given in Eqs. (17.10:2) and (17.10:5), respectively. The prescribed body force increment is given in Eq. (17.10:6). The compatibility requirements are: $\Delta\mathbf{u}$ is continuous and equal to $\Delta\bar{\mathbf{u}}$ on S_{un} (rigid constraint). We assume $\Delta\mathbf{u}$ in the form $\Delta\mathbf{u} = \mathbf{h}\Delta\mathbf{q}$, where \mathbf{h} and $\Delta\mathbf{q}$ are the interpolation and the parameter matrices, respectively. The incremental finite element equations are obtained from the stationary condition of Π_3 with respect to $\Delta\mathbf{q}$, i.e.,

$$(3) \quad \Delta \overset{\circ}{\Pi}_3 = \sum_{\text{all elements}} \delta \overset{\circ}{\Pi}_{3e} = \sum_{\text{all elements}} \left\{ \int_{V_{en}} [\overset{\circ}{W}(\Delta\mathbf{e}) + \tau : (\Delta\mathbf{L} \cdot \delta\Delta\mathbf{L}^T) - \rho_n \Delta\mathbf{b} \cdot \delta\Delta\mathbf{u}] dV - \int_{S_{\sigma en}} \Delta\bar{\mathbf{T}} \cdot \delta\Delta\mathbf{u} dA \right\} = 0$$

subjected to the rigid constraint $\delta\Delta\mathbf{q} = 0$ over the surface where displacements are prescribed.

From the finite element solution, we can calculate the 2nd Piola–Kirchhoff stress increment in the updated Lagrange description [Eq. (17.5:10)]

$$\Delta\mathbf{S} = \partial\overset{\circ}{W}/\partial\Delta\mathbf{e},$$

and the Cauchy stress increment [Eq. (17.9:13)]

$$(4) \quad \Delta\tau = \Delta\mathbf{S} + \Delta\mathbf{L} \cdot \tau + \tau \cdot \Delta\mathbf{L}^T - \text{tr}(\Delta\mathbf{e})\tau.$$

Note that $\Delta\mathbf{S}$ equals the Truesdell rate of the Cauchy stress. The updated displacements, coordinates and Cauchy stresses, are

$$(5) \quad \mathbf{u}_{n+1} = \mathbf{u}_n + \Delta\mathbf{u}, \quad \mathbf{X}_{n+1} = \mathbf{X}_n + \Delta\mathbf{u} = \mathbf{X}_0 + \mathbf{u}_{n+1}, \quad \tau_{n+1} = \tau_n + \Delta\tau.$$

Here \mathbf{u}_n is the total displacement vector at the n^{th} state.

One may include $\Delta\Omega(t)$ as an independent field in Eq. (2) [Eq. (17.6:13)],

$$(6) \quad \overset{\circ}{\Pi}_{3e} = \int_{V_{en}} [\overset{\circ}{W}(\Delta\mathbf{e}) + \tau : (\Delta\mathbf{L} \cdot \Delta\mathbf{L}^T)/2 + c[(\nabla_n \Delta\mathbf{u})_a - \Delta\Omega]^2/2 - \rho_n \Delta\mathbf{b} \cdot \Delta\mathbf{u}] dV - \int_{S_{\sigma n}} \Delta\bar{\mathbf{T}} \cdot \Delta\mathbf{u} dA,$$

where c is a positive constant of the order of the elastic modulus, and

$$(\nabla_n \Delta\mathbf{u})_a = [\nabla_n \Delta\mathbf{u} - (\nabla_n \Delta\mathbf{u})^T]/2, \\ [(\nabla_n \Delta\mathbf{u})_a - \Delta\Omega]^2 = [(\nabla_n \Delta\mathbf{u})_a - \Delta\Omega] : [(\nabla_n \Delta\mathbf{u})_a - \Delta\Omega].$$

Equation (7) is equivalent to the *penalty function approach* in enforcing

$$(7) \quad \Delta\Omega = (\nabla_n \Delta\mathbf{u})_a.$$

If we express the functional in the form

$$(8) \quad \overset{\circ}{\Pi}_{3e} = \int_{V_{en}} [\overset{\circ}{P}(\Delta\mathbf{e}) + \tau : (\Delta\mathbf{L} \cdot \Delta\mathbf{L}^T)/2 - \rho_n \Delta\mathbf{b} \cdot \Delta\mathbf{u} - \Delta\mathbf{e} : \Delta\mathbf{e}] dV - \int_{S_{\sigma en}} \Delta\bar{\mathbf{T}} \cdot \Delta\mathbf{u} dA,$$

then since [Eq. (17.5:10)]

$$(9) \quad \Delta\sigma_r = \partial\overset{\circ}{P}/\partial\Delta\mathbf{e}$$

gives the increment of the corotational stress, which is the Jaumann rate of the Kirchhoff stress. From Eq. (17.9:12), we have

$$(10) \quad \Delta\tau = \Delta\sigma_r + \Delta\Omega \cdot \tau - \tau \cdot \Delta\Omega - \text{tr}(\mathbf{D})\tau$$

for updating the Cauchy stress.

To improve objectivity in computation, we use the midpoint rule Eqs. (17.9:6) and (17.9:9) that

$$\nabla_{n+1/2} = \varepsilon_I \partial / \partial X_I^{n+1/2}, \quad \Delta \mathbf{L} = \nabla_{n+1/2}(\Delta \mathbf{u}), \\ \Delta \mathbf{e} = [\nabla_{n+1/2}(\Delta \mathbf{u}) + (\nabla_{n+1/2} \Delta \mathbf{u})^T]/2, \quad \Delta \Omega = [\nabla_{n+1/2}(\Delta \mathbf{u}) - (\nabla_{n+1/2} \Delta \mathbf{u})^T]/2.$$

One also updates the total deformation gradient, the related deformation matrices, and the rate-potential using the midpoint rule, e.g., Eq. (17.9:8) for \mathbf{F} and Eq. (17.9:7) or (17.9:11) for \mathbf{R} . For the formulation of Eq. (6), we use $\Delta \Omega$ directly from the finite element solution in Eq. (10).

21.2. INCREMENTAL SOLUTION

Assume that the body force and the prescribed surface traction are in the form¹

$$\mathbf{b} = p \mathbf{b}_0, \quad \bar{\mathbf{T}} = p \bar{\mathbf{T}}_0,$$

where p is a loading parameter. Let \mathbf{q}_n be the current solution corresponding to p_n and $\mathbf{q}_{n+1} = \mathbf{q}_n + \Delta \mathbf{q}$. From Eq. (21.1:3) we obtain

$$(1) \quad \Delta \mathbf{q} = \Delta \mathbf{F} = (p_{n+1} - p_n) \mathbf{F}_0 = \Delta p \mathbf{F}_0,$$

where \mathbf{K} and $\Delta \mathbf{F}$, which can be functions of \mathbf{q}_n , are the assembled constraint stiffness and the load increment matrices, respectively. Equation (1) is a first order ordinary differential equation of p , and can be integrated using the *predictor-corrector method* (Pian and Tong 1971, Ralston 1965).

At certain solution stage, say $p = p_c$, \mathbf{K} may become singular, the numerical integration with respect to p cannot proceed further. At that stage a small increment in p can give a large increment in \mathbf{q} . It is then better to integrate the equation with respect to a deformation parameter near $p = p_c$ (see [Biblio. 19.1](#)). Let q_r be a component of \mathbf{q} , which changes rapidly with respect to p . We partition Eq. (1) in the form

$$(2) \quad \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{21}^T \\ \mathbf{K}_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{q}' \\ \Delta q_r \end{bmatrix} = \Delta p \begin{bmatrix} \mathbf{F}' \\ F_r \end{bmatrix} \text{ or } \begin{bmatrix} \mathbf{K}_{11} & -\mathbf{F}' \\ \mathbf{K}_{21} & -F_r \end{bmatrix} \begin{bmatrix} \Delta \mathbf{q}' / \Delta q_r \\ \Delta p / \Delta q_r \end{bmatrix} = \begin{bmatrix} -\mathbf{K}_{21}^T \\ -K_{22} \end{bmatrix},$$

where \mathbf{K}_{11} is nonsingular and symmetric. We can integrate Eq. (2) with respect to the parameter q_r to determine $\Delta \mathbf{q}'$ and Δp .

Let $q_r = (q_r)_c$ at $p = p_c$. If $\Delta p / \Delta q_r$ changes sign at $(q_r)_c$, no solution for p exists beyond p_c in the vicinity of $(q_r)_c$ while solution(s) below p_c exists. In other words an infinitesimal increment in p beyond p_c can lead to a finite increment in q_r or the collapse of the structure, a nonuniqueness phenomenon due to nonlinearity. The structure is said to be *buckled* (Fung and Sechler 1960, Timoshenko and Gere 1961). On the other hand, if $\Delta p / \Delta q_r$ is a monotonically increasing function of q_r , the structure is stable and can sustain load beyond p_c . This is called *stable buckling* but q_r changes rapidly in the vicinity of p_c .

The coefficient matrix in second equation of Eq. (2) is in general asymmetric. We can, however, avoid dealing with asymmetric matrix by solving the equation as follows: Let

$$(3) \quad \Delta \mathbf{q}' / \Delta q_r = \Delta \mathbf{q}'_1 / \Delta q_r + (\Delta p / \Delta q_r)(\Delta \mathbf{q}'_2 / \Delta q_r),$$

in which $\Delta \mathbf{q}'_1 / \Delta q_r$ and $\Delta \mathbf{q}'_2 / \Delta q_r$ are the solutions of

$$(4) \quad \mathbf{K}_{11}(\Delta \mathbf{q}'_1 / \Delta q_r) = -\mathbf{K}_{21}^T, \quad \mathbf{K}_{11}(\Delta \mathbf{q}'_2 / \Delta q_r) = \mathbf{F}'.$$

A substitution of Eq. (3) into Eq. (2) with $\Delta \mathbf{q}'_1$ and $\Delta \mathbf{q}'_2$ from Eq. (4) yields

$$\Delta p / \Delta q_r = [K_{22} + \mathbf{K}_{21}(\Delta \mathbf{q}'_1 / \Delta q_r)] / (F_r - \mathbf{K}_{21}(\Delta \mathbf{q}'_2 / \Delta q_r)).$$

21.3. DYNAMIC SOLUTION

For dynamic simulation, if explicit scheme is used, one can obtain $\Delta \mathbf{u}$ or $\Delta \mathbf{q}$ without having to solve the algebraic equations. The equations of motion are derived from the principle of virtual work ([Sec. 18.17](#)),

$$(1) \quad \delta\Pi = \int_{V_{n+1}} [\delta\mathbf{u}^T \ddot{\mathbf{u}} \rho_{n+1} + (\delta\mathbf{e})^T \boldsymbol{\tau}_{n+1} - \rho_{n+1} \delta\mathbf{u}^T \mathbf{b}_{n+1}] dV \\ - \int_{(\partial V_\sigma)_{n+1}} \delta\mathbf{u}^T \bar{\mathbf{T}}_{n+1} dS = 0,$$

in which

$$(2) \quad \boldsymbol{\tau}_{n+1} = [\tau_{11} \ \tau_{22} \ \cdots \ \tau_{31}]_{n+1}^T, \quad \delta\mathbf{u} = \mathbf{h}\delta\mathbf{q}, \quad \delta\mathbf{e} = \mathbf{B}\delta\mathbf{q},$$

where $\boldsymbol{\tau}_{n+1}$ is the updated Cauchy stress given in Eq. (21.1:5), $\delta\mathbf{e}$ is a strain variation derived from Eq. (18.16:6), and \mathbf{h} and \mathbf{B} are interpolation matrices relating the parameter $\delta\mathbf{q}$ to the displacement and strain variations as defined in Eq. (18.16:11). The boundary force is calculated from the surface integral in Eq. (1). For pressure surface load

$$(3) \quad \bar{\mathbf{T}}_{n+1} = -\bar{p}_{n+1} \mathbf{N}_{n+1},$$

and for dead-weight load

$$(4) \quad \bar{\mathbf{T}}_{n+1} = (\bar{\mathbf{T}}_0)_{n+1} dS_0 / dS_{n+1} = (\bar{\mathbf{T}}_0)_{n+1} \sqrt{[(\mathbf{N}^T \mathbf{F})(\mathbf{F}^T \mathbf{N})]_{n+1}} / J_{n+1},$$

where \bar{p}_{n+1} is the prescribed pressure referred to the deformed surface in D_{n+1} configuration, and $(\bar{\mathbf{T}}_0)_{n+1}$ is prescribed traction at the $(n+1)^{\text{th}}$ stage referred to the undeformed surface. The element mass matrix is derived from the first term in the first integral that

$$(5) \quad \mathbf{m}_{n+1} \ddot{\mathbf{q}} = \left[\int_{(V_e)_{n+1}} \rho_{n+1} \mathbf{h}^T \mathbf{h} dV \right] \ddot{\mathbf{q}},$$

which is in the same form as that for small deformation discussed in Sec. 18.17 except that the integration in Eq. (5) is over the deformed element volume. In practice, one often uses the lumped mass matrix by summing all terms of each row to the diagonal.

From Eq. (1), one can calculate the acceleration at stage $n + 1$. Using an explicit integration scheme, one determines $\Delta\mathbf{q}$ (i.e., $\Delta\mathbf{u}$) at the $(n+1)^{\text{th}}$ stage, then the increment $\Delta\tau$ from Eq. (21.1:5) or (21.1:10) and the updated solution from Eq. (21.1:5), and then proceed to the next stage.

21.4. NEWTON–RAPHSON ITERATION METHOD

The Newton–Raphson method (Rafston 1965) is one of the most commonly used methods for solving nonlinear algebraic equations. The method can be used for the nonlinear finite element equations of plasticity, viscoplasticity and creep. Almost all solution methods for nonlinear problem involve some sort of iterations. It is no exception for the Newton–Raphson method. The process is illustrated in Fig. 21.4:1. Consider a set of nonlinear algebraic equations in the form

$$(1) \quad \mathbf{R}(\mathbf{q}) = \gamma(\mathbf{q}) - \mathbf{F} = 0,$$

where γ is a nonlinear matrix function of \mathbf{q} and \mathbf{F} is a loading matrix normally independent of \mathbf{q} . Assuming that the solution \mathbf{q}_n at $\mathbf{F} = \mathbf{F}_n$ is known, one seeks the solution \mathbf{q}_{n+1} at $\mathbf{F} = \mathbf{F}_{n+1} = \mathbf{F}_n + \Delta\mathbf{F}_n$. Let \mathbf{q}_{n+1}^i be a known approximate solution for \mathbf{q}_{n+1} at the i^{th} iteration, one improves the approximation in the form $\mathbf{q}_{n+1}^{i+1} = \mathbf{q}_{n+1}^i + d\mathbf{q}^i$. To the first order approximation, Eq. (1) can be written as

$$(2) \quad \mathbf{K}_T^i d\mathbf{q}^i + \mathbf{R}_n^i = 0,$$

where the tangent stiffness matrix \mathbf{K}_T^i and the residual matrix \mathbf{R}_n^i are defined

$$\mathbf{K}_T^i = \partial\gamma/\partial\mathbf{q} \Big|_{\mathbf{q}=\mathbf{q}_{n+1}^i}, \quad \mathbf{R}_n^i = \gamma(\mathbf{q}_{n+1}^i) - \gamma(\mathbf{q}_n) - \Delta\mathbf{F}_n = \gamma(\mathbf{q}_{n+1}^i) - \mathbf{F}_n - \Delta\mathbf{F}_n.$$

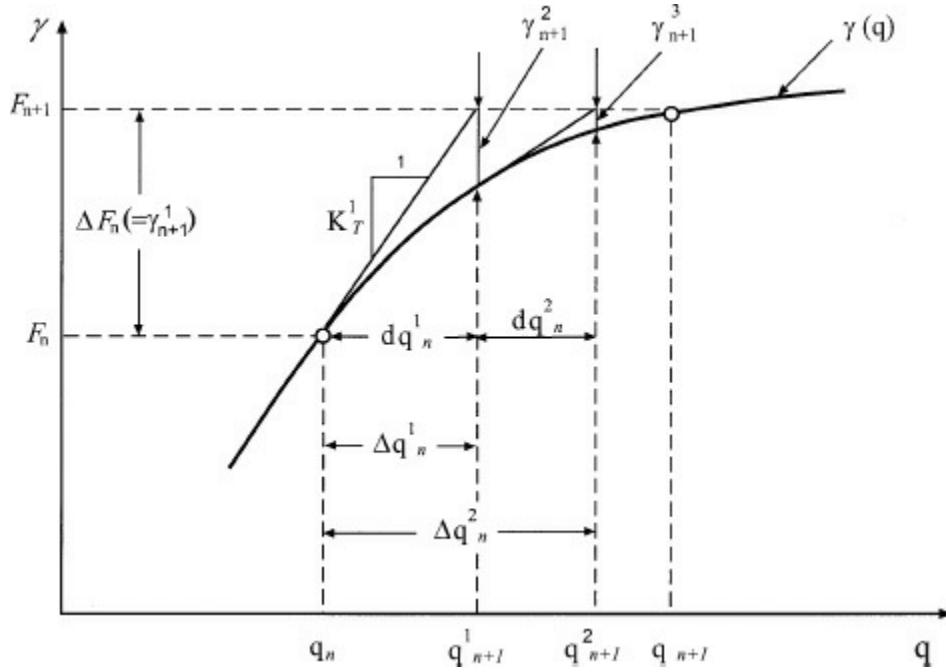


Fig. 21.4:1. Newton–Raphson method.

One repeats the process until

$$(\mathbf{R}_n^i)^T \mathbf{R}_n^i < e_2 (\Delta \mathbf{F}_n)^T \Delta \mathbf{F}_n, \quad (i, n \text{ not summed})$$

where e_2 is a chosen small constant to control the accuracy of the solution for \mathbf{q}_{n+1} and $(\mathbf{R}_n^i)^T (\mathbf{R}_n^i)$ is usually called the square norm of the residual.

There are variants to the Newton–Raphson method. The most common one is the *modified Newton–Raphson method*, which uses a fixed \mathbf{K}_T^i for several iterations without update. Solving for $d\mathbf{q}^i$ in Eq. (2) requires refactorization of the matrix \mathbf{K}_T^i (see Sec. 18.5), if \mathbf{K}_T^i is updated. Factorization is a time consuming process. Updating \mathbf{K}_T^i less frequently saves computing time. One can even use \mathbf{K}_T^0 . In this case, \mathbf{K}_T^0 throughout the entire iteration process of determining \mathbf{q}_{n+1} . In this case, \mathbf{K}_T^0 is factored only once. Such procedure is likely to converge at a slower rate but the computer time required to reach a converged solution may actually be reduced.

A *linear search procedure* can be used to accelerate the convergence of the Newton–Raphson method. The schematics of the procedure is shown in Fig. 21.4:2. Instead of using $\mathbf{q}_{n+1}^i + d\mathbf{q}^i$, we define

$$\mathbf{q}_{n+1}^{i+1} = \mathbf{q}_{n+1}^i + (1 + \eta) d\mathbf{q}^i$$

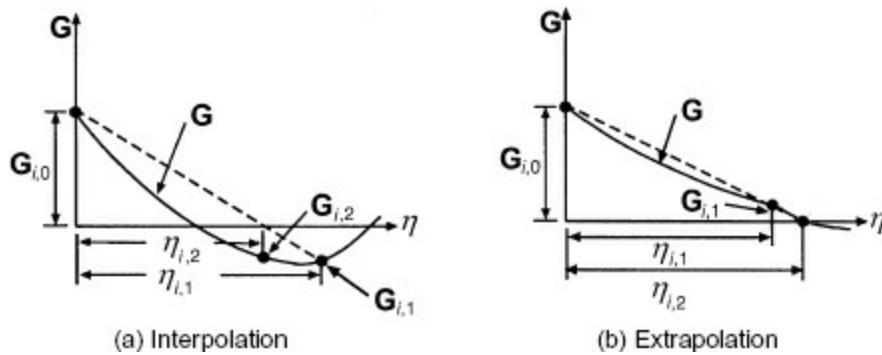


Fig. 21.4:2. Linear search schematic.

as an updated solution. The quantity η is determined by requiring the projection of the residual on the search direction $d\mathbf{q}^i$ to be zero, i.e.,

$$\begin{aligned} G_i(\eta) &= (d\mathbf{q}^i)^T \mathbf{R}[\mathbf{q}_{n+1}^i + (1 + \eta)d\mathbf{q}^i] \\ &= (d\mathbf{q}^i)^T \{\gamma[\mathbf{q}_{n+1}^i + (1 + \eta)d\mathbf{q}^i] - \mathbf{F}_{n+1}\} = 0 \quad (i \text{ not summed}). \end{aligned}$$

The solution for η can be obtained by iteration with $\eta = 0$ as the initial approximation. The linear search needs

additional evaluations of the residual \mathbf{R} . One normally undertakes the search only if

$$|G(\eta)| > e_3 |(d\mathbf{q}^i)^T [\gamma(\mathbf{q}_{n+1}^i) - \mathbf{F}_{n+1}]|,$$

where e_3 is a number close to 0.5 (Matthies and Strang 1979).

In many problems, for a loading in the form

$$\mathbf{F} = p\mathbf{F}_0, \quad \Delta\mathbf{F}_n = (p_{n+1} - p_n)\mathbf{F}_0 = \Delta p_n \mathbf{F}_0,$$

no solution exists above a certain maximum value of p . When this occurs, one can recast the formulation to the increment of a single displacement component as shown in Sec. 21.2.

21.5. VISCOELASTICITY

Viscoelasticity is characterized by a constitutive law that the current state of stress depends not only on the current strains, but also on the full history of their development as described in Chapter 14. In the present study, we assume the constitutive equation in the form

$$(1) \quad \sigma = \mathbf{D}_r(t)\mathbf{e}(0^+) + \int_0^t \mathbf{D}_r(t-\tau) \frac{\partial \mathbf{e}}{\partial \tau} d\tau,$$

as given in Eq. (14.2:6), where σ , \mathbf{e} , $\mathbf{D}_r(t)$ are the stress, strain, relaxation matrices, respectively. If the material is inhomogeneous, $\mathbf{D}_r(t)$ will be a function of the spatial coordinates also. Furthermore, in some problems, $\mathbf{D}_r(t)$ may depend on σ and/or \mathbf{e} . In finite element analysis, we write Eq. (1) in the incremental form

$$(2) \quad \sigma_{n+1} - \sigma_n = \dot{\sigma}_n \Delta t = \mathbf{D}_r(0) \Delta \mathbf{e} + [\dot{\mathbf{D}}_r(t_n) \mathbf{e}(0^+) + \mathbf{Q}_n] \Delta t,$$

where σ_{n+1} , σ_n denote the stresses at $t = t_{n+1}$, t_n , respectively,

$$(3) \quad \Delta \mathbf{e} = \mathbf{e}_{n+1} - \mathbf{e}_n = \mathbf{B} \Delta \mathbf{q}, \quad \mathbf{Q}_n = \int_0^{t_n} \dot{\mathbf{D}}_r(t_n - \tau) (\partial \mathbf{e} / \partial \tau) d\tau,$$

in which \mathbf{B} is the strain-displacement matrix relating the generalized coordinates associated with displacements to strains. In Eq. (2), the first term on right most hand side is the elastic solution with $\mathbf{D}_r(0)$ equivalent to the elastic modulus matrix and the second term is the viscoelastic contribution.

The finite element equation of motion can be written in the form [Sec. (18.16)]

$$(4) \quad \mathbf{R} = \int_V \mathbf{B}^T \sigma dV - \mathbf{F} = 0,$$

where \mathbf{F} is the loading matrix. Substituting Eq. (2) into Eq. (4) and using the strain displacement relation $\Delta \mathbf{e} = \mathbf{B} \Delta \mathbf{q}$, we find

$$(5) \quad \mathbf{K}^0 \Delta \mathbf{q}_n = \mathbf{R}_n,$$

where \mathbf{K}^0 is the standard elastic stiffness matrix that

$$(6) \quad \mathbf{K}^0 = \int_V \mathbf{B}^T \mathbf{D}_r(0) \mathbf{B} dV$$

and \mathbf{R}_n is the equilibrium residual defined as

$$(7) \quad \mathbf{R}_n = \mathbf{F}_{n+1} - \mathbf{F}_n - \Delta t \int_V \mathbf{B}^T [\dot{\mathbf{D}}_r(t_n) \mathbf{e}(0^+) + \mathbf{Q}_n] dV.$$

One can integrate Eq. (5) with initial condition $\mathbf{e}(0^+)$ given. The practical limitation in evaluating Eq. (5) is the requirement of the full history of \mathbf{e} as shown in the integration of \mathbf{Q} in Eq. (3). When t becomes large, this will require extensive computer storage and computation. In practice, often \mathbf{D}_r is an exponentially decaying function of the form

$$\mathbf{D}_r(t) \approx \exp(-t/\tau_0),$$

where τ_0 is a relaxation constant. Equation (3) can be approximated as

$$(8) \quad Q_n \approx \int_{t_n - a\tau_0}^{t_n} \dot{D}_r(t_n - \tau) \frac{\partial \mathbf{e}}{\partial \tau} d\tau,$$

where a is a constant of order 1. Then it is only necessary to store the recent history of \mathbf{e} from $t_n - a\tau_0$ to t_n .

For linear problems, one can proceed to the next time step after $\Delta \mathbf{q}_n$ is determined from Eq. (5). For nonlinear viscoelastic problems, \mathbf{B} can be a function of \mathbf{q} . One may have to use the Newton–Raphson iteration to obtain the updated $\Delta \mathbf{q}_n$.

21.6. PLASTICITY

The linear *incremental plasticity* theory for infinitesimal deformation described in [Chapter 6](#) are summarized below in both matrix and indicial notations:

$$(1) \quad \mathbf{e} = \mathbf{e}^e + \mathbf{e}^p, \quad e_{ij} = e_{ij}^e + e_{ij}^p \quad (\text{elastic and plastic strain-splits}),$$

$$(2) \quad \boldsymbol{\tau} = \mathbf{D}_e \mathbf{e}^e = \mathbf{D}_e(\mathbf{e} - \mathbf{e}^p), \quad \tau_{ij} = D_{ijkl}^e e_{kl}^e = D_{ijkl}^e(e_{kl} - e_{kl}^p) \quad (\text{elastic constitutive equation}),$$

$$(3) \quad \dot{\mathbf{e}}^p = \dot{\Lambda} \partial h / \partial \boldsymbol{\tau}, \quad \dot{e}_{ij}^p = \dot{\Lambda} \partial h / \partial \tau_{ij} \quad (\text{normality condition flow rule}),^2$$

$$(4) \quad \eta = \boldsymbol{\tau} - \boldsymbol{\alpha}, \quad \eta_{ij} = \tau_{ij} - \alpha_{ij},$$

$$(5) \quad \eta' = \boldsymbol{\tau}' - \boldsymbol{\alpha}', \quad \eta'_{ij} = \tau'_{ij} - \alpha'_{ij} = \eta_{ij} - \delta_{ij} \eta_{kk} / 3 \quad (\text{deviatoric components}),$$

$$(6) \quad \dot{\boldsymbol{\alpha}} = \dot{\Lambda} \mathbf{g}, \quad \dot{\alpha}_{ij} = \dot{\Lambda} g_{ij} \quad (\text{back stresses}),$$

$$(7) \quad \dot{e}^p = \sqrt{2/3} |\dot{e}_{ij}^p| = \dot{\Lambda} \sqrt{2/3} |\partial h / \partial \tau_{ij}| \quad (\text{equivalent plastic-strain rate}),$$

$$(8) \quad h = h(\eta_{ij}, \xi_i) \quad (\text{yield potential}),$$

$$(9) \quad f(\eta_{ij}, \xi_i) = 0 \quad (\text{yield surface}),$$

$$(10) \quad \dot{\Lambda} = 0 \quad \text{if} \quad f < 0 \quad (\text{elastic deformation}),$$

$$\dot{\Lambda} > 0 \quad \text{if} \quad f = 0 \quad (\text{plastic deformation}),$$

$$f > 0 \quad (\text{not allowed}),$$

$$(11) \quad \dot{e}_{kk}^p = \partial h / \partial \tau_{kk} = 0,$$

where τ_{ij} , α_{ij} , e_{ij} are the stress, back stress,³ and strain tensors, a primed quantity, $(-)_I$, denotes the deviatoric component of the corresponding quantity, and ξ_i are internal variables. The *yield potential* h , the *yield surface* f and the back stress g_{ij} [defined in Eq. (6)] are functions of η_{ij} and *internal variables* ξ_i . The flow rule is *associated* if $h = f$. The internal variables can be the back stresses, plastic strains and temperature. Equation (10) can be written as

$$\dot{\Lambda} \geq 0, \quad \dot{\Lambda} f = 0,$$

is known as the Kuhn–Tucker form for loading or unloading. In elastic-plastic analysis, one seeks the solution in terms of displacements and stresses that satisfy the equations of motion and the boundary conditions with Eq. (1)–(11) as constraints.

The plasticity equations are highly nonlinear and can be solved in most cases only by numerical means. There is a very large literature on the subject (see [Biblio. 21.3](#)). To construct the incremental solution, consider the interval (t_n, t_{n+1}) , where t is time for dynamic problems and a loading parameter that characterizes the magnitude of the applied loads or prescribed displacements for static problems. For simplicity we will just call it time in the subsequent discussion. At t_n we assume that the total plastic and equivalent plastic strains, the stress and the back stress, \mathbf{e}_n , \mathbf{e}_n^p , ε_n^p , $\boldsymbol{\tau}_n$, $\boldsymbol{\alpha}_n$, are known. We first construct the updated fields \mathbf{e}_{n+1} , \mathbf{e}_{n+1}^p , ε_{n+1}^p , at t_{n+1} in terms of the increment displacement $\Delta \mathbf{u}$ in the manner consistent with the constraints of Eq. (3)–(10), and then proceed to the next increment.

The updating process is to determine \mathbf{e}_{n+1}^p , ε_{n+1}^p and then derives other updated fields based on \mathbf{e}_{n+1}^p , ε_{n+1}^p . In principle, \mathbf{e}_{n+1}^p , ε_{n+1}^p and $\boldsymbol{\alpha}_{n+1}$ can be determined by integrating the flow and hardening rules over the step (t_n, t_{n+1}) .

for a given yield potential h and yield surface f . To illustrate the update process, we consider an associated von Mises material with only internal variable ε^p . If the material is isotropic and satisfies a linear kinematic and nonlinear isotropic hardening law, then

$$(12) \quad \dot{\alpha} = 2c\dot{\Lambda}\eta/3,$$

$$(13) \quad h(\eta, \varepsilon^p) = f(\eta, \varepsilon^p) = \eta^T \mathbf{P} \eta/2 - \sigma_Y^2(\varepsilon^p)/3 = 0,$$

where σ_Y is the yield stress in unidirectional test, and

$$(14a) \quad \varepsilon^p = \int_0^t \dot{\varepsilon}^p dt,$$

$$(14b) \quad \mathbf{P} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad (\text{plane stress}),$$

$$(14c) \quad \mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{bmatrix}, \quad \mathbf{P}_1 = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

$$\mathbf{P}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (\text{3-D, plane strain}).$$

Note that \mathbf{P} is the *projection matrix* that makes $\mathbf{P}\eta$ deviatoric.

Following the approach described in [Sec. 6.12](#) and assuming that

$$f(\tau_n + \mathbf{D}_e \Delta \mathbf{e}, \varepsilon_n^p) > 0,$$

i.e., yielding will occur, from Eq. (7), we find

$$(15) \quad \dot{\varepsilon}^p = \dot{\Lambda}(2\eta^T \mathbf{P} \eta/3)^{1/2} = 2\dot{\Lambda}\sigma_Y/3.$$

Differentiating Eq. (13) with respect to t gives the consistent equation

$$(16) \quad \eta^T \mathbf{P}(\dot{\tau} - 2c\dot{\Lambda}\eta/3) - 2\sigma_Y(\partial\sigma_Y/\partial\varepsilon^p)\dot{\varepsilon}^p/3 = 0.$$

Replacing (\cdot) quantities with $\Delta(-)$ in Eq. (15) and (16), solving for $\Delta\Lambda$ and substituting it into Eq. (3) and into $\Delta\tau = \mathbf{D}_e(\Delta\mathbf{e} - \Delta\mathbf{e}^p)$, we obtain

$$(17) \quad \mathbf{D}^{ep} = \{\mathbf{I} + \mathbf{D}_e \mathbf{P} \eta (\mathbf{P} \eta)^T / [4\sigma_Y^2(c + \partial\sigma_Y/\partial\varepsilon^p)/9]\}^{-1} \mathbf{D}_e,$$

$$(18) \quad \Delta\tau = \mathbf{D}^{ep} \Delta\mathbf{e},$$

$$(19) \quad \Delta\Lambda = \eta^T \mathbf{P} \Delta\tau / [4\sigma_Y^2(c + \partial\sigma_Y/\partial\varepsilon^p)/9],$$

$$(20) \quad \Delta\mathbf{e}^p = \mathbf{P} \eta \Delta\Lambda,$$

$$(21) \quad \Delta\alpha = 2c\eta\Delta\Lambda/3,$$

where $\eta = \eta_n$ $\sigma_Y = \sigma_Y(\varepsilon_n^p)$. The updated fields are simply

$$(22) \quad \tau_{n+1} = \tau_n + \Delta\tau, \quad \mathbf{e}_{n+1}^p = \mathbf{e}_{n+1}^p + \Delta\mathbf{e}^p \dots$$

This process is called the *tangent modulus approach*.

For ideal plasticity, i.e., $c = \partial\sigma_Y/\partial\varepsilon^p = 0$, we derive \mathbf{D}^{ep} , $\Delta\Lambda$ according to [Sec. 6.10](#) [see Eqs. (6.10:8–13)],

$$(22a) \quad \begin{aligned} \mathbf{D}^{ep} &= \mathbf{D}_e - \mathbf{D}_e \mathbf{P} \tau \tau^T \mathbf{P} \mathbf{D}_e / (\tau^T \mathbf{P} \mathbf{D}_e \mathbf{P} \tau), \\ \Delta\Lambda &= \tau^T \mathbf{P} \mathbf{D}_e \Delta\mathbf{e} / (\tau^T \mathbf{P} \mathbf{D}_e \mathbf{P} \tau). \end{aligned}$$

The other equations are in the same form as before.

For finite $\Delta\mathbf{u}$, the updated fields satisfy the yield condition with an error of the order $(\Delta\mathbf{u})^2$, i.e., $f(\eta_{n+1}, \varepsilon_{n+1}^p) = O(\Delta\mathbf{u})^2$. We seek improved updated fields in the form (Simo and Taylor 1986):

$$(23) \quad \Delta\mathbf{e} = \mathbf{d}\Delta\mathbf{u} = \mathbf{B}\Delta\mathbf{q},$$

$$(24) \quad \Delta\mathbf{e}_{n+a}^p = \Gamma\mathbf{P}\eta_{n+a},$$

$$(25) \quad \Delta\alpha_{n+a} = 2c\Gamma\eta_{n+a}/3,$$

$$(26) \quad \Delta\varepsilon_{n+a}^p = \sqrt{2/3}\Gamma(\eta_{n+a}^T\mathbf{P}\eta_{n+a})^{1/2},$$

$$(27) \quad \mathbf{e}_{n+1} = \mathbf{e}_n + \Delta\mathbf{e} = \mathbf{e}_n + \mathbf{B}\Delta\mathbf{q}, \quad \mathbf{e}_{n+a} = \mathbf{e}_n + a\Delta\mathbf{e},$$

$$(28) \quad \tau_{n+1} = \tau_n + \mathbf{D}_e(\Delta\mathbf{e} - \Delta\mathbf{e}_{n+a}^p), \quad \tau_{n+a} = \tau_n + a\mathbf{D}_e(\Delta\mathbf{e} - \Delta\mathbf{e}_{n+a}^p),$$

$$(29) \quad \alpha_{n+1} = \alpha_n + \Delta\alpha_{n+a}, \quad \alpha_{n+a} = \alpha_n + a\Delta\alpha_{n+a},$$

$$(30) \quad \mathbf{e}_{n+1}^p = \mathbf{e}_n^p + \Delta\mathbf{e}_{n+a}^p = \mathbf{e}_n^p + \Gamma\mathbf{P}\eta_{n+a}, \quad \mathbf{e}_{n+a}^p = \mathbf{e}_n^p + a\Delta\mathbf{e}_{n+a}^p,$$

$$(31) \quad \varepsilon_{n+1}^p = \varepsilon_n^p + \Delta\varepsilon_{n+a}^p = \varepsilon_n^p + \Gamma(2\eta_{n+a}^T\mathbf{P}\eta_{n+a}/3)^{1/2},$$

$$\varepsilon_{n+a}^p = \varepsilon_n^p + a\Delta\varepsilon_{n+a}^p,$$

where a is a constant with value commonly $1/2 \leq a \leq 1$, chosen to improve the accuracy of Eq. (15)–(17) and Γ is a positive constant to be determined from the yield condition Eq. (13)

$$(32) \quad f(\eta_{n+a}, \varepsilon_{n+a}^p) = \eta_{n+a}^T\mathbf{P}\eta_{n+a}/2 - \sigma_Y^2(\varepsilon_{n+a}^p)/3 = 0.$$

In Eq. (32), η_{n+a} and ε_{n+a}^p are defined as

$$(33) \quad \eta_{n+a} = \tau_n + a\mathbf{D}_e\Delta\mathbf{e} - a\Gamma\mathbf{D}_e\mathbf{P}\eta_{n+a} - \alpha_n - 2ac\Gamma\eta_{n+a}/3,$$

$$(34) \quad \varepsilon_{n+a}^p = \varepsilon_n^p + a\Gamma\sqrt{2\eta_{n+a}^T\mathbf{P}\eta_{n+a}/3}.$$

Solving for η_{n+a} from Eq. (33) gives

$$(35) \quad \eta_{n+a} = [(1 + 2ac\Gamma/3)\mathbf{I} + a\Gamma\mathbf{D}_e\mathbf{P}]^{-1}(\eta_n + a\mathbf{D}_e\Delta\mathbf{e}).$$

A substitution of Eq. (34) and (35) into Eq. (32) furnishes a nonlinear algebraic equation for Γ to be solved by iteration. One can then update all fields according to Eq. (27)–(31), which are *second order accurate*, i.e.,

$$f(\eta_{n+1}, \varepsilon_{n+1}^p) = O(\Delta\mathbf{u})^3.$$

For other yield surfaces, one may not be able to establish a single algebraic equation for Γ from the plastic constrained equations. In this case, one has to solve for Γ , η_{n+a} , \mathbf{e}_{n+a}^p , and ε_{n+a}^p in terms of $\Delta\mathbf{u}$ simultaneously.

For plane strain, $\Delta e_{31} = \Delta e_{32} = \Delta e_{33} = 0$ and $\Delta\mathbf{e}$ is a function of x and y only. For plane stress, $\eta_{31} = \eta_{32} = \eta_{33} = 0$, and \mathbf{P} and \mathbf{D}_e are given in Eqs. (21.6:14b) and (18.14:19). The solution is also a function of x and y only. To assure $(\tau_{33})_{n+1} = 0$, we update e_{33} as

$$(e_{33})_{n+1} = -\nu[(\tau_{11})_{n+1} + (\tau_{22})_{n+1}]/E - [(e_{11}^p)_{n+1} + (e_{22}^p)_{n+1}],$$

for isotropic materials, and according to the appropriate strain-stress relation for anisotropic materials.

Isotropic Materials. It can be shown that, for isotropic materials,

$$(35a) \quad \mathbf{D}_e\mathbf{P} = \mathbf{P}\mathbf{D}_e, \quad \mathbf{D}_e\mathbf{P}\Delta = 2G\mathbf{P}\eta, \quad \eta^T\mathbf{P}\mathbf{D}_e\mathbf{P}\eta = 2G\eta^T\mathbf{P}\eta,$$

if the problem is plane strain or 3-dimensional and,

$$(35b) \quad \mathbf{D}_e \mathbf{P} \boldsymbol{\eta} = \frac{E}{3(1-\nu^2)} \begin{bmatrix} (2-\nu)\eta_{11} - (1-2\nu)\eta_{22} \\ (2-\nu)\eta_{22} - (1-2\nu)\eta_{11} \\ 3(1-\nu)\eta_{12} \end{bmatrix},$$

$$\boldsymbol{\eta}^T \mathbf{P} \mathbf{D}_e \mathbf{P} \boldsymbol{\eta} \neq 2G\boldsymbol{\eta}^T \mathbf{P} \boldsymbol{\eta},$$

if the problem is plane stress. Then,

$$(36) \quad \mathbf{P} = \mathbf{Q} \Lambda_p \mathbf{Q}^T,$$

$$(37) \quad \mathbf{D}_e = \mathbf{Q} \Lambda_d \mathbf{Q}^T,$$

where

$$(38) \quad \mathbf{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix},$$

$$(39) \quad \Lambda_p = \text{dia}[1/3 \ 1 \ 2],$$

$$(40) \quad \Lambda_d = E \text{ dia}[1 + \nu \ 1 - \nu \ (1 - \nu)/2]/(1 - \nu^2),$$

$$(41) \quad \Lambda_d \Lambda_p = E \text{ dia}[(1 + \nu)/3 \ 1 - \nu \ 1 - \nu]/(1 - \nu^2),$$

for plane stress, and

$$(42) \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & 0 \\ 0 & \mathbf{I} \end{bmatrix}, \quad \mathbf{Q}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & -1 & \sqrt{2} \\ -\sqrt{3} & -1 & \sqrt{2} \\ 0 & 2 & \sqrt{2} \end{bmatrix},$$

$$(43) \quad \Lambda_p = \text{dia}[1 \ 1 \ 0 \ 2 \ 2 \ 2],$$

$$(44) \quad \Lambda_d = \frac{E}{1 + \nu} \text{ dia} \left[1 \ 1 \ \frac{1 + \nu}{1 - 2\nu} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \right],$$

$$(45) \quad \Lambda_d \Lambda_p = E \text{ dia}[1 \ 1 \ 0 \ 1 \ 1 \ 1]/(1 + \nu),$$

for 3-dimensional and plane strain problems, in which dia [...] denotes a diagonal matrix with the diagonal terms given in the bracket and \mathbf{I} is a 3×3 identity matrix. A substitution of Eq. (36)–(45) into Eq. (35) yields

$$(46) \quad \boldsymbol{\eta}_{n+a} = \mathbf{Q}[(1 + 2ac\Gamma/3)\mathbf{I} + a\Gamma\Lambda_d\Lambda_p]^{-1}\mathbf{Q}^T(\boldsymbol{\eta}_n + a\mathbf{D}_e\Delta\mathbf{e}),$$

in which $\Lambda_d\Lambda_p$ is a diagonal matrix. For pure kinematic hardening materials, i.e., σ_Y is constant, Eq. (32) reduces to a quartic equation in Γ for plane stress. The quartic equation has only one positive root for Γ and can be solved in closed form (Simo and Govindjee 1988).

If the material is also ideal plastics, Eq. (17) and (19) become

$$(46a) \quad \mathbf{D}^{ep} = \mathbf{D}_e - 3GP\tau(\mathbf{P}\tau)^T/\sigma_Y^2, \quad \Delta\Lambda = 3\tau^T \mathbf{P} \Delta\mathbf{e}/(2\sigma_Y^2),$$

for plane strain and 3-dimensional cases. Other equations can be derived accordingly. For plane stress, $\mathbf{D}_e \mathbf{P} \boldsymbol{\eta}$ in Eq. (22a) is given by Eq. (35b).

For clarity, readers are reminded of variables and constants: Total, elastic and plastic strains (\mathbf{e} , \mathbf{e}^e , \mathbf{e}^p), Cauchy stress (τ), elastic modulus matrix (\mathbf{D}^e), normality flow rule ($\mathbf{e}^p = \dot{\Lambda} \partial h / \partial \tau$), flow rule proportional rate factor ($\dot{\Lambda}$), deviatoric stress ($\boldsymbol{\eta}$), back stress rate ($\dot{\alpha} = \dot{\Lambda} \mathbf{g}$), equivalent plastic strain rate ($\dot{\varepsilon}^p = \sqrt{2/3}|\dot{\epsilon}_{ij}^p| = \dot{\Lambda} \sqrt{2/3}|h_\sigma|$), yield potential (h), and yield surface ($f = 0$) in Eq. (1) to (9) on page 711; linear kinematic and nonlinear isotropic hardening ($\dot{\alpha} = 2c\dot{\Lambda}\eta/3$), associated von Mises flow rule [$f = \boldsymbol{\eta}^T \mathbf{P} \boldsymbol{\eta}/2 - \sigma_Y^2(\varepsilon^p)/3 = h = 0$] in Eq. (12) and (13) on page 712; projection matrix (\mathbf{P}), plastic modulus (\mathbf{D}^{ep}), the incremental, intermediate and updated Cauchy stresses ($\Delta\tau$, $\tau_{n+\alpha}$, τ_{n+1}), strains ($\Delta\mathbf{e}_n$, $\mathbf{e}_{n+\alpha}$, \mathbf{e}_{n+1}), equivalent plastic strains ($\Delta\varepsilon_n^p$, $\varepsilon_{n+\alpha}^p$, ε_{n+1}^p), and back stresses ($\Delta\alpha_n$, $\alpha_{n+\alpha}$, α_{n+1}),

respectively, and flow rule proportional factor increment ($\Delta\Lambda$) in Eq. (14)–(35) on pages 712 to 714.

Sublayer Hardening Model. An alternative to model isotropic hardening is to use the overlay technique (Pian 1987). In this approach, two or more nonhardening materials are subjected to the same strains. Figure 21.6:1 illustrates a two-nonhardening-material model with the equivalent stress-strain relation in uni-axial tension being

$$\sigma = \begin{cases} E\varepsilon & \text{for } \varepsilon \leq \varepsilon_1 \\ a_2E(\varepsilon - \varepsilon_1) + \sigma_1 & \text{for } \varepsilon_1 < \varepsilon \leq \varepsilon_2 \\ a_1\sigma_1 + a_2\sigma_2 & \text{for } \varepsilon_2 < \varepsilon \end{cases}$$

in which $a_1, a_2 (= 1 - a_1)$ are the fractions of the two materials, σ_1, σ_2 are their yield stresses and $\varepsilon_1, \varepsilon_2$ are the corresponding strains at the onset of yielding.

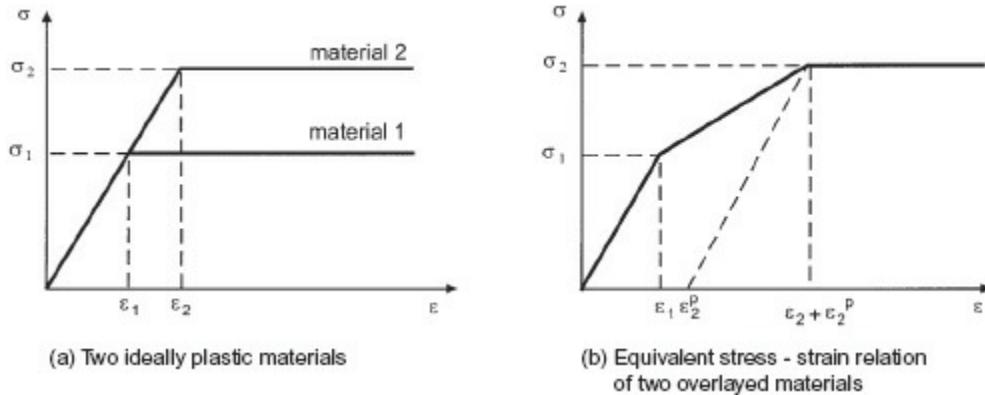


Fig. 21.6:1. Isotropic hardening simulated by two non-hardening materials.

The yield strain is

$$\varepsilon_1^p = \begin{cases} 0 & \text{for } \varepsilon \leq \varepsilon_1 \\ a_1(\varepsilon - \varepsilon_2) & \text{for } \varepsilon_1 < \varepsilon \end{cases} \quad \text{for material 1,}$$

$$\varepsilon_2^p = \begin{cases} 0 & \text{for } \varepsilon \leq \varepsilon_2 \\ a_2(\varepsilon - \varepsilon_1) & \text{for } \varepsilon_2 < \varepsilon \end{cases} \quad \text{for material 2,}$$

Dynamic Analysis. In dynamic analysis, $\Delta\mathbf{u} = \mathbf{u}_{n+1} - \mathbf{u}_n$ is calculated from acceleration in explicit integration. The acceleration at t_{n+1} is derived from the equation of motion based on the principle of virtual work,

$$(47) \quad \Delta \delta\Pi = \left[\int_V (\delta\mathbf{u}^T \ddot{\mathbf{u}}\rho + \delta\mathbf{e}^T \boldsymbol{\tau} - \rho\delta\mathbf{u}^T \mathbf{b}) dV - \int_{\partial V_\sigma} \delta\mathbf{u}^T \bar{\mathbf{T}} dS \right]_{n+1} = 0$$

with $\boldsymbol{\tau}_{n+1}$ from Eq. (28) in terms of $\Delta\mathbf{e}$, $\Delta\mathbf{e}_{n+a}^p$ from Eq. (23) and (24). From the acceleration, one determines the new $\Delta\mathbf{u}$ and proceeds to the next time increment.

Static Analysis. For static analyses, $\Delta\mathbf{u}$ is not known *a priori* at each increment. It must satisfy the incremental equation of motion, which can be derived from Eq. (47) in the form

$$(48) \quad \delta\Delta\Pi = \int_V [\delta\mathbf{e}^T \mathbf{D}_e (\Delta\mathbf{e} - \Delta\mathbf{e}_{n+a}^p) - \rho_n \delta\mathbf{u}^T \Delta\mathbf{b}_n] dV - \int_{\partial V_\sigma} \delta\mathbf{u}^T \Delta\bar{\mathbf{T}}_n dS = 0,$$

or

$$(49) \quad \mathbf{K}\Delta\mathbf{q} = \mathbf{F}_p(\Delta\mathbf{e}_{n+a}^p) + \Delta\mathbf{F},$$

where \mathbf{F}_p is derived from the volume integration of $\delta\mathbf{e}^T \mathbf{D}_e \Delta\mathbf{e}_{n+a}^p$, and $\Delta\mathbf{F}$ from integrals involving the body force increment $\Delta\mathbf{b}_n$, the prescribed traction increment $\Delta\bar{\mathbf{T}}_n$ and the prescribed boundary displacement increment through enforcing rigid constraints. Equation (49) differs from the elasticity equation only by the term $\mathbf{F}_p(\Delta\mathbf{e}_{n+a}^p)$, which itself is a function of $\Delta\mathbf{q}$. Recall that $\Delta\mathbf{u} = \mathbf{h}\Delta\mathbf{q}$ and $\Delta\mathbf{e} = \mathbf{B}\Delta\mathbf{q}$.

Equation (49) is to be solved by iterative process together with the plastic constraints of Eq. (23)–(31), a set of highly nonlinear coupled equations. One approach is to start the iteration by assuming $\Delta\mathbf{u}$ equal to that of the previous step as

a predictor, calculating the updated fields according to Eq. (23)–(31) subjected to the constraint of Eq. (32) for locations where yielding takes place, evaluating $\mathbf{F}_p(\Delta\mathbf{e}_{n+a}^p)$ based on the predicted $\Delta\mathbf{u}$ at the yielded locations, then solving Eq. (49) for the corrected $\Delta\mathbf{u}$. One repeats the process until the estimated error for Eq. (32) is within a specified limit. In this approach, \mathbf{K} is the same as that of the elastic solution and remains constant throughout the entire solution process. All plasticity effects contain in $\mathbf{F}_p(\Delta\mathbf{e}_{n+a}^p)$. This iteration process is sometime called the *initial strain approach*, i.e., an initial strain (in terms of $\Delta\mathbf{u}$) is first assumed and corrections are then sought. Various Newton–Raphson schemes have also used to determine $\Delta\mathbf{u}$.

Another approach is to modify Eq. (48) to the form

$$(50) \quad \int_V [\delta\mathbf{e}^T \mathbf{D}_e (\Delta\mathbf{e} - \Delta\mathbf{e}_n^p)] dV = \int_V \delta\mathbf{e}^T \mathbf{D}_n^{ep} \Delta\mathbf{e} dV = \int_V \rho \delta\mathbf{u}^T \Delta\mathbf{b}_n dV \\ + \int_V [\delta\mathbf{e}^T \mathbf{D}_e (\Delta\mathbf{e}_{n+a}^p - \Delta\mathbf{e}_n^p)] dV + \int_{\partial V_\sigma} \delta\mathbf{u}^T \Delta\bar{\mathbf{T}}_n dS$$

or

$$(51) \quad \mathbf{K}_n^{ep} \Delta\mathbf{q} = \mathbf{F}'_p(\Delta\mathbf{e}_{n+a}^p - \Delta\mathbf{e}_n^p) + \Delta\mathbf{F},$$

where \mathbf{D}_n^{ep} and $\Delta\mathbf{e}_n^p$ are defined in Eq. (17) and (20) and $\mathbf{F}'_p(\Delta\mathbf{e}_{n+a}^p - \Delta\mathbf{e}_n^p)$ is derived from the 1st integral on the right most side in Eq. (50). Thus \mathbf{e}_n^p is derived using \mathbf{D}_n^{ep} rather than \mathbf{D}_e . For general materials, \mathbf{D}_n^{ep} and \mathbf{e}_n^p are derivable from formulae given in Secs. 6.10–6.12. One can use the same iterative method described earlier to obtain the solution for $\Delta\mathbf{q}$. One starts the iteration with assumed $\Delta\mathbf{q}$ to calculate $\mathbf{F}'_p(\Delta\mathbf{e}_{n+a}^p - \Delta\mathbf{e}_n^p)$ and solves Eq. (51) for the corrected $\Delta\mathbf{q}$. The method is called the *modified tangent modulus approach*. The *conventional tangent modulus approach* assumes \mathbf{K}_n^{ep} to be zero. Since \mathbf{K}_n^{ep} depends on the field values, one must refactor \mathbf{K}_n^{ep} in solving for $\Delta\mathbf{q}$, which is time consuming. However, the approach often improves the rate of convergence.

One can also update \mathbf{K}_n^{ep} using the current iterated field values during the iteration for a given load increment. This is done in particular if many locations change from loading to unloading or vice versa in the step. Such changes greatly affect the value of \mathbf{D}_{ep} in the evaluation of \mathbf{K}_n^{ep} .

Large Deformation. To avoid confusion between strain/stress matrices and strain/stress tensors, we shall use Cartesian coordinates and index notation in the remaining and the next sections. For 3-dimensional problems, a strain matrix is a column matrix of 6 components [Eq. (18.16:3)],

$$\mathbf{e}^T = [e_{11} \ e_{22} \ e_{33} \ 2e_{12} \ 2e_{23} \ 2e_{31}],$$

whereas a strain tensor has 9 components e_{ij} . If one assumes that the plasticity constraints Eq. (24)–(26) hold for the incremental plasticity fields, i.e.,

$$(52) \quad (\Delta e_{ij}^p)_{n+a} = \Gamma(\partial f / \partial \tau_{ij})_{n+a}, \quad (\Delta \alpha_{ij})_{n+a} = 2c\Gamma(\partial f / \partial \tau_{ij})_{n+a}/3, \\ (\Delta e^p)_{n+a} = \sqrt{2/3} |\Delta e_{ij}^p|_{n+a},$$

where $(\Delta e_{ij}^p)_{n+a}$ denotes the value at t_{n+a} , and the strain increments are calculated based on the current configuration,⁴ i.e.,

$$(53) \quad \Delta e_{ij} = \Delta D_{ij} = \frac{1}{2} \left[\frac{\partial \Delta u_i}{\partial X_j^n} + \frac{\partial \Delta u_j}{\partial X_i^n} \right], \quad \Delta \Omega_{ij} = \frac{1}{2} \left[\frac{\partial \Delta u_i}{\partial X_j^n} - \frac{\partial \Delta u_j}{\partial X_i^n} \right],$$

the elastic-plastic analysis carries over for large formation by properly defining the stress measure. For example if the stress increment is based on the Jaumann rate of the Cauchy stress [Eq. (13.9:26) or (17.5:3)], i.e.,

$$(54) \quad (\Delta \tau_{ij})_{n+1} = D_{ijkl}^e [\Delta e_{kl} - (\Delta e_{kl}^p)_{n+a}], \\ (\Delta \tau_{ij})_{n+a} = a D_{ijkl}^e [\Delta e_{kl} - (\Delta e_{kl}^p)_{n+a}],$$

where D_{ijkl}^e is the incremental elastic modulus matrix which can be a function of the deformation gradient and the Cauchy stress at t_n , then the updated Cauchy and back stresses are

$$(55) \quad (\tau_{ij})_{n+1} = (\tau_{ij} + \Delta \Omega_{ik} \tau_{kj} - \tau_{ik} \Delta \Omega_{kj})_n + (\Delta \tau_{ij})_{n+1},$$

$$(56) \quad (\alpha_{ij})_{n+1} = (\alpha_{ij} + \Delta\Omega_{ik}\alpha_{kj} - \alpha_{ik}\Delta\Omega_{kj})_n + (\Delta\alpha_{ij})_{n+1}.$$

If the increment of the 2nd Piola–Kirchhoff stress in the updated Lagrange description is used in the incremental constitutive equation Eq. (54), we then calculate the Cauchy and back stress increments according to Eqs. (21.6:18) and (21.6:21).

For finite deformation, solutions obtained by different objective rates can be substantially different. Some may be even physically unreasonable. For example (Truesdell 1955), consider a linear hypoelastic material subjected to a finite simple shear. When the Jaumann rate of the Cauchy stress is used in the linear incremental constitutive equation, the shear stress is oscillatory.

For dynamic analysis, the equation of motion has exactly the same form as Eq. (21.3:1), which is reproduced here in index notation

$$(57) \quad \delta\Pi = \left[\int_V (\delta u_i \ddot{u}_i \rho + \delta e_{ij} \tau_{ij} - \rho \delta u_i b_i) dV - \int_{\partial V_\sigma} \delta u_i \bar{T}_i dS \right]_{n+1} = 0,$$

with the updated Cauchy stress and the Jaumann rate defined in Eq. (55) and (54), respectively. The plasticity effects are contained in the incremental stress-strain relations in terms of $(\Delta e_{kl}^p)_{n+a}$ as given in Eq. (54). Thus once $(\Delta\tau_{ij})_{n+1}$ and $(\Delta\alpha_{ij})_{n+a}$ are determined from the known Δu_i , one uses Eq. (57) to determine the acceleration at t_{n+1} and predict Δu_i for the next increment.

For static analysis, Δu_i is not known *a priori* and must be solved for from the equation of motion together with the plasticity constraints. The equation of motion is the same as Eq. (48) with additional terms to account for large deformation shown below:

$$(58) \quad \delta\Delta\Pi = \int_{V_n} \{ \delta e_{ij} [\Delta\Omega_{ik}\tau_{kj} - \tau_{ik}\Delta\Omega_{kj} + (\Delta\tau_{ij})_{n+1}] - \delta u_i [(\rho b_i)_{n+1} - (\rho b_i)_n] \} dV - \int_{\partial V_\sigma} \delta u_i [(T_i)_{n+1} - (T_i)_n] dS = 0.$$

It goes without saying that the integrals in Eq. (58) are over the current configuration. The iteration process for small deformation applies to large deformation as well.

The integration can be unstable if the incremental step is too large. Readers are referred to literature for stability criterion (Cormeau^{21.3} 1975, Ortiz and Popov^{21.3} 1985). A practical criterion for dynamic analysis is to limit the time step below the minimum of the characteristic dimension of all elements divided by the adiabatic sound speed (Hallquist^{21.3} 1998). The characteristic dimension is usually defined as the element volume divided by the largest surface area of the sides of the element.

21.7. VISCOPLASTICITY

For viscoplasticity materials, assuming linear viscosity and small deformation for simplicity, we can write the constitutive relations in the form

$$(1) \quad \dot{e}_{ij}^p = \frac{H(f)}{\eta} \frac{\partial f(\tau_{ij}, \xi_i)}{\partial \tau_{ij}}, \quad \dot{\xi}_i = \frac{H(f)}{\eta} U_i(\tau_{ij}, \xi_i), \quad \dot{\tau}_{ij} = D_{ijkl}^e (\dot{e}_{ij} - \dot{e}_{ij}^p),$$

where η is a known “viscosity” parameter, f is the yield function (only associated material are considered here), U_i is a known function characterizing the internal variable ξ_i , and H is the Heaviside function defined as

$$(2) \quad H(f) = 0 \quad \text{if } f < 0, \\ = f \quad \text{if } f \geq 0.$$

The internal variable can be the equivalent plastic strain and/or the back stress. The finite element equation of motion is derived from

$$\int_V \delta e_{ij} \dot{\tau}_{ij} dV - \dot{F}_i = \int_V \delta e_{ij} D_{ijkl}^e (\dot{e}_{kl} - \dot{e}_{kl}^p) dV - \dot{F}_i = 0,$$

which leads the finite element equation

$$(3) \quad \mathbf{K}_T^0 \dot{\mathbf{q}} = \mathbf{F}_p(e_{ij}^p) + \dot{\mathbf{F}},$$

where \mathbf{K}_T^0 is the stiffness matrix which is the same as elasticity derived from the terms associated with $\dot{\epsilon}_{kl}$, $\mathbf{F}_p(\dot{\epsilon}_{ij}^p)$ is from the terms associated with $\dot{\epsilon}_{ij}^p$ which contains all the viscoplasticity effects, and $\dot{\mathbf{F}}$ is the rate of applied load, and is equivalent to $\Delta\mathbf{F}$ of Eq. (21.6:49). Equations (1)–(3) are a set of first order nonlinear differential equations of $\dot{q}_i, \dot{\epsilon}_{ij}^p, \dot{\xi}_i$, which can be integrated with the initial conditions or solved by iteration. Note that, as the stress associated with $[-D_{ijkl}^e \dot{\epsilon}_{kl}^p = -H(f) D_{ijkl}^e (\partial f / \partial \tau_{kl}) / \eta]$ always points toward the surface of the yield potential $\dot{\mathbf{F}}$ when yielding occurs, from Eq. (1), the stress state lying outside ($f > 0$) moves toward the yield surface. The solution approaches the yield surface, i.e., $f(\tau_{ij}, \xi_i) \rightarrow 0$ at the steady state.

The main differences between the viscoplasticity and the classical plasticity theories are that the proportional factor $H(f)/\eta$ for the rates of variables in viscoplasticity is specified explicitly in terms of the yield function, and that the time is a physical parameter.

21.8. CREEP

Creep is a time-dependent phenomenon even under a constant load. In infinitesimal deformation, the total strain is split into the instantaneous strain e_{ij}^e and the creep strain e_{ij}^c as

$$(1) \quad e_{ij} = e_{ij}^e + e_{ij}^c.$$

The constitutive law is defined in the form

$$(2) \quad \dot{\tau}_{ij} = D_{ijkl}^e (\dot{e}_{ij} - \dot{e}_{ij}^c)$$

for the instantaneous strain rate and

$$(3) \quad \dot{e}_{ij}^c = \chi_{ij}(\tau),$$

for the creep rate, where $\chi_{ij}(\tau)$ are known functions of the current state of stress. The finite element equation of motion is the same as Eq. (21.7:3) with \mathbf{F}_p derived from terms associated \dot{e}_{ij}^c . The creep equations can be solved by iterative methods (see [Biblio. 21.3](#)).

In this chapter, we have only touched upon briefly the application of finite element method to various nonlinear problems. Readers are referred to literature, which is voluminous, for details. Computational mechanics can provide us tools to explore the enormous richness of nonlinear mechanics: large deformation, composite structures, microand nanotechnologies, living cells, DNA, proteins, enzymes and beyond.

Problem 21.1. Write the incremental energy function Eq. (17.5:25) in matrix notation

$$\overset{\circ}{W}(\mathbf{D}) = \bar{\mathbf{D}}^T \bar{\mathbf{D}}_e \bar{\mathbf{D}} / 2 = \mathbf{D}^T \mathbf{D}_e \mathbf{D} / 2,$$

where \mathbf{D} is the symmetric velocity-like gradient referred to the current configuration \mathbf{x} that $\mathbf{D}^T = [D_{11} \ D_{22} \ D_{33} \ 2D_{12} \ 2D_{23} \ 2D_{31}]$, and similarly for $\bar{\mathbf{D}}$ with D 's replaced by \bar{D} 's. They are related by the deformation gradient matrix \mathbf{F} that

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ \text{sym} & & \bar{D}_{33} \end{bmatrix} = \mathbf{F} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ \text{sym} & & D_{33} \end{bmatrix} \mathbf{F}^T, \quad \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \bar{\mathbf{X}}}.$$

The quantities $\mathbf{D}_e, \bar{\mathbf{D}}_e$ are incremental elastic modulus matrices. Show that

$$\bar{\mathbf{D}}_e = 4[(\partial W_0 / \partial I_2)(\mathbf{II} - \mathbf{H}) + (\partial W_0 / \partial I_3)\mathbf{A} + \mathbf{G}] / J,$$

$$\mathbf{II} - \mathbf{H} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix}, \quad \mathbf{Q}_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q}_2 = -\frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} 0 & C_{33} & C_{22} & 0 & -C_{23} & 0 \\ & 0 & C_{11} & 0 & 0 & -C_{31} \\ & & 0 & -C_{12} & 0 & 0 \\ & & & -C_{33}/2 & C_{31}/2 & C_{23}/2 \\ \text{sym} & & & & -C_{11}/2 & C_{31}/2 \\ & & & & & -C_{22}/2 \end{bmatrix},$$

where $J [= \det(\mathbf{F})]$, C 's are the components of the matrix $\mathbf{C}(= \mathbf{F}^T \mathbf{F})$ and, for the energy density function W_0 defined in Eq. (13.11:34a),

$$\mathbf{G} = (\partial^2 W_0 / \partial^2 I_3) I_3 \hat{\mathbf{C}} \hat{\mathbf{C}}^T = b_4 I_3 \hat{\mathbf{C}} \hat{\mathbf{C}}^T, \quad \hat{\mathbf{C}}^T = [\hat{C}_{11} \ \hat{C}_{22} \ \hat{C}_{33} \ \hat{C}_{12} \ \hat{C}_{23} \ \hat{C}_{31}],$$

where \hat{C} 's are the components of the inverse of \mathbf{C} . Expressing $\bar{\mathbf{D}}$ in terms of \mathbf{D} in the form $\bar{\mathbf{D}} = \mathbf{R} \mathbf{D} \mathbf{R}$, one has $\mathbf{D}_e = \mathbf{R}^T \bar{\mathbf{D}}_e \mathbf{R}$. Find \mathbf{R} in terms of the components of \mathbf{F} .

21.9. RETURN MAPPING FORMULATION WITH VON MISES YIELD SURFACE

For finite incremental step, iterative method is most commonly used for solution of nonlinear problems. We illustrate the approach using a *return mapping formulation* (Wilkins 1964, Krieg and Key 1976, Krieg and Krieg 1977). The accuracy of such formulation can be of the second order with respect to the size of the increment (Ortiz and Popov 1985). Another commonly used formulation is called the *secant method* (Rice and Tracey 1973). Interested readers are referred to literature for details.

We shall consider materials obeying the *associative von Mises yield criterion with isotropic and kinematic hardening* and have the solution satisfy both equilibrium and yielding conditions at plastic Gauss points, whereas only equilibrium condition at elastic Gauss points.

We first focus on the treatment of plastic Gauss points. At such a point, we introduce an intermediate solution signified by $(_)_{n+a}$ with $0 < a \leq 1$. One commonly assigns $a = 1/2$ or 1. In the latter case, $(_)_{n+a} = (_)_{n+1}$, the intermediate solution is a solution after the n^{th} increment. In general, the solution at the $(n+1)^{\text{th}}$ increment is assumed to be

$$(1) \quad \mathbf{e}_{n+1} = \mathbf{e}_n + \mathbf{B} \Delta \mathbf{q}, \quad \mathbf{e}_{n+1}^p = \mathbf{e}_n^p + \Gamma \mathbf{P} \boldsymbol{\eta}_{n+a},$$

$$\mathbf{e}_{n+1}^p = \mathbf{e}_n^p + \sqrt{2/3} \Gamma (\boldsymbol{\eta}_{n+a}^T \mathbf{P} \boldsymbol{\eta}_{n+a})^{1/2},$$

$$(2) \quad \alpha_{n+1} = \alpha_n + 2c\Gamma \boldsymbol{\eta}_{n+a}/3,$$

$$\tau_{n+1} = \mathbf{D}_e(\mathbf{e}_{n+1} - \mathbf{e}_{n+1}^p) = \tau_n + \mathbf{D}_e(\mathbf{B} \Delta \mathbf{q} - \Gamma \mathbf{P} \boldsymbol{\eta}_{n+a}),$$

in which $\Gamma(= \dot{\Lambda})$ is a nonnegative parameter and $(_)_{n+a}$'s are defined as,

$$(3) \quad \mathbf{e}_{n+a} = a\mathbf{e}_{n+1} + (1-a)\mathbf{e}_n = \mathbf{e}_n + a\mathbf{B} \Delta \mathbf{q},$$

$$(4) \quad \mathbf{e}_{n+a}^p = a\mathbf{e}_{n+1}^p + (1-a)\mathbf{e}_n^p = \mathbf{e}_n^p + a\Gamma \mathbf{P} \boldsymbol{\eta}_{n+a},$$

$$(5) \quad \boldsymbol{\varepsilon}_{n+a}^p = a\boldsymbol{\varepsilon}_{n+1}^p + (1-a)\boldsymbol{\varepsilon}_n^p = \boldsymbol{\varepsilon}_n^p + \sqrt{2/3} a \Gamma (\boldsymbol{\eta}_{n+a}^T \mathbf{P} \boldsymbol{\eta}_{n+a})^{1/2},$$

$$(6) \quad \alpha_{n+a} = a\alpha_{n+1} + (1-a)\alpha_n = \alpha_n + 2ac\Gamma \boldsymbol{\eta}_{n+a}/3,$$

$$(7) \quad \tau_{n+a} = \mathbf{D}_e(\mathbf{e}_{n+a} - \mathbf{e}_{n+a}^p) = a\tau_{n+1} + (1-a)\tau_n = \tau_n + a\mathbf{D}_e(\mathbf{B} \Delta \mathbf{q} - \Gamma \mathbf{P} \boldsymbol{\eta}_{n+a}),$$

$$\boldsymbol{\eta}_{n+a} = \tau_{n+a} - \alpha_{n+a} = \tau_n - \alpha_n + a[\mathbf{D}_e(\mathbf{B} \Delta \mathbf{q} - \Gamma \mathbf{P} \boldsymbol{\eta}_{n+a}) - 2c\Gamma \boldsymbol{\eta}_{n+a}/3],$$

or

$$(8) \quad \boldsymbol{\eta}_{n+a} = [(1 + 2ac\Gamma/3)\mathbf{D}_e^{-1} + a\Gamma \mathbf{P}]^{-1}(\mathbf{D}_e^{-1}\boldsymbol{\eta}_n + a\mathbf{B} \Delta \mathbf{q}).$$

The yield criterion is

$$(9) \quad f = \boldsymbol{\eta}_{n+a}^T \mathbf{P} \boldsymbol{\eta}_{n+a}/2 - \kappa_{n+a}^2/3 = 0, \quad \Gamma > 0 \quad \text{for loading,}$$

$$f < 0 \quad \Gamma = 0 \quad \text{for unloading,}$$

where $\kappa_{n+a} = \kappa(\varepsilon_{n+a}^p)$ and κ is a known function of ε^p with $\partial\kappa/\partial\varepsilon^p \geq 0$. The finite element equations of motion are

$$(10) \quad \begin{aligned} \mathbf{R}_n &= \int_V \mathbf{B}^T (\boldsymbol{\tau}_{n+1} - \boldsymbol{\tau}_n) dV - \Delta \mathbf{F}_n \\ &= \int_V \mathbf{B}^T \mathbf{D}_e (\mathbf{B} \Delta \mathbf{q} - \Gamma \mathbf{P} \boldsymbol{\eta}_{n+a}) dV - \Delta \mathbf{F}_n = 0. \end{aligned}$$

Equations (3)–(10) are a set of nonlinear equations of $(\cdot)_{n+a}$ and $\Delta \mathbf{q}$, which can be solved only by iterative means. Then the solution at the $(n + 1)^{\text{th}}$ step is computed from Eq. (1) and (2). Since the integration in Eq. (10) is carried out using the Gaussian quadrature, only the solution of Eq. (3)–(9) at the plastic Gauss points of all elements will be needed.

The Newton–Raphson iterative method is commonly used to solve the nonlinear equations. Discussion below is a two-step procedure for a fixed load increment $\Delta \mathbf{F}_n$. First, for any increment $\Delta \mathbf{q}^i$, we obtain the corresponding elastic-plastic deformation $\mathbf{e}_{n+a}^p, \varepsilon_{n+a}^p, \alpha_{n+a}, \boldsymbol{\tau}_{n+a}, \boldsymbol{\eta}_{n+a}, \Gamma$ at every Gauss point. We then determine $d\mathbf{q}^i$, a correction to $\Delta \mathbf{q}^i$, to make sure that $\Delta \mathbf{q}^{i+1} = \Delta \mathbf{q}^i + d\mathbf{q}^i$ satisfies the finite element equations of equilibrium Eq. (10). We call the first step the *yield and equilibrium checks*. The process is repeated until both Eq. (10) and all yield equations Eq. (3)–(9) are satisfied.

Yield Check. One starts the iteration with the elastic increment $\Delta \mathbf{q}^o$ from

$$(11) \quad \left(\int_V \mathbf{B}^T \mathbf{D}_e \mathbf{B} dV \right) \Delta \mathbf{q}^o - \Delta \mathbf{F}_n = 0,$$

and proceeds to find the elastic plastic deformation associated with $\Delta \mathbf{q}^o$ in an iteration process. One first determines the updated deformation

$$\Gamma^o = 0, \boldsymbol{\eta}^o = \boldsymbol{\eta}_n + a \mathbf{D}_e \mathbf{B} \Delta \mathbf{q}^o, (\mathbf{e}^p)^o = \mathbf{e}_n^p, (\varepsilon^p)^o = \varepsilon_n^p, \boldsymbol{\tau}^o = \boldsymbol{\tau}_n + a \mathbf{D}_e \mathbf{B} \Delta \mathbf{q}^o,$$

associated with $\Delta \mathbf{q}^o$ and then checks for yielding at all Gauss points. If the yield condition is violated, one will modify $\Delta \mathbf{q}^o$ by iteration.

Let the superscript i be the counter signifying the i^{th} iteration. For a given $\Delta \mathbf{q}^i$, one checks for yielding at all Gauss points by first evaluating $f(\boldsymbol{\eta}_n + a \mathbf{D}_e \mathbf{B} \Delta \mathbf{q}^i, \kappa_n)$ if $f \leq 0$, the incremental deformation is elastic at the point, which will be identified as an elastic Gauss point. Then

$$(12) \quad \Gamma^i = 0, \alpha_{n+1}^i = \alpha_n, (\mathbf{e}^p)_{n+1}^i = \mathbf{e}_n, (\varepsilon^p)^i = \varepsilon_n^p, \boldsymbol{\tau}_{n+1}^i = \boldsymbol{\tau}_n + \mathbf{D}_e \mathbf{B} \Delta \mathbf{q}^i,$$

and the plastic increments for all quantities are zero. One proceeds to the next Gauss point. If $f > 0$, the point is identified as a plastic Gauss point. One first examines the yield condition using the current values of $\varepsilon_{n+a}^p, \boldsymbol{\eta}_{n+a}, \Gamma$. If

$$f(\boldsymbol{\eta}_{n+a}, \kappa_{n+a}) > e_1 |f[0, \kappa(0)]|,$$

where e_1 is a chosen small positive constant. This means the solution has deviated from the yield surface by an amount considered significant, the updated plastic solution corresponding to $\Delta \mathbf{q}^i$ must be established. The process involves substituting $\varepsilon_{n+a}^p, \boldsymbol{\eta}_{n+a}$ of Eq. (5), (8) into Eq. (9) and then solving for Γ (Simo and Taylor 1986).

Let $(\varepsilon^p)^i, \boldsymbol{\eta}^i, \Gamma^i$ denote the current iterated values of $\varepsilon_{n+a}^p, \boldsymbol{\eta}_{n+a}$, and let $(\varepsilon^p)^i + d(\varepsilon^p)^i, \boldsymbol{\eta}^i + d\boldsymbol{\eta}^i, \Gamma^i + d\Gamma^i$ be their updates. The subscript $n + a$ has been dropped for simplicity. Substituting the updated variables into Eq. (8) and (9) and linearizing them around $\varepsilon^i, \boldsymbol{\eta}^i, \Gamma^i$ gives

$$\begin{aligned} r \mathbf{Z}^{-1} d\boldsymbol{\eta}^i + a(2c \mathbf{D}^{-1}/3 + \mathbf{P}) \boldsymbol{\eta}^i d\Gamma^i &= r \mathbf{Z}^{-1} d\boldsymbol{\eta}^i + a(2c \mathbf{Z}^{-1}/3 + \mathbf{P}) \boldsymbol{\eta}^i d\Gamma^i = 0, \\ (\boldsymbol{\eta}^i)^T \mathbf{P} d\boldsymbol{\eta}^i - \frac{2a}{3} \sqrt{\frac{2}{3}} \frac{\kappa^i}{\phi} \left(\frac{\partial \kappa}{\partial \varepsilon^p} \right)^i [\Gamma^i \boldsymbol{\eta}^i \mathbf{P} d\boldsymbol{\eta}^i + \phi^2 d\Gamma^i] + f(\boldsymbol{\eta}^i, \kappa^i) &= 0, \end{aligned}$$

where $\phi = \sqrt{(\boldsymbol{\eta}^i)^T \mathbf{P} d\boldsymbol{\eta}^i}$. The solutions for $d\boldsymbol{\eta}^i, d\Gamma^i$ are

$$(13) \quad d\boldsymbol{\eta}^i = -a(2c/3 + \mathbf{Z} \mathbf{P} / r) \boldsymbol{\eta}^i d\Gamma^i / r, \quad d\Gamma^i = r f^i / (\alpha \beta),$$

where $f^i = f(\boldsymbol{\eta}^i, \kappa^i)$ is the yield surface residual, $\mathbf{Z} = [\mathbf{D}_e^{-1} + a \Gamma^i \mathbf{P} / r]^{-1}$, $r = 1 + 2ac\Gamma^i / 3$, and

$$(14) \quad \beta = [1 - \frac{2}{3} \sqrt{\frac{2}{3}} \frac{a\kappa^i \Gamma^i}{\phi} (\frac{\partial \kappa}{\partial \varepsilon^p})^i] \frac{\eta^{iT} \mathbf{P} \mathbf{Z} \mathbf{P} \eta^i}{r} + \frac{2}{3} \sqrt{\frac{2}{3}} \frac{a\kappa^i \Gamma^i}{\phi} (\frac{\partial \kappa}{\partial \varepsilon^p})^i + \frac{2}{3} c\phi^2.$$

The updated Γ is $\Gamma^i + d\Gamma^i$. One can then calculate the updated η_{n+a} from Eq. (8), followed by $e_{n+a}^p, \varepsilon_{n+a}^p$ from Eq. (4) and (5), and lastly $\kappa(\varepsilon_{n+a}^p)$ from the known function. The iteration for the given $\Delta \mathbf{q}^i$ continued until

$$(15) \quad |f(\eta_{n+a}, \kappa_{n+a})| < e_1 |f[0, \kappa(0)]|.$$

The process is repeated for all Gauss points. One can define other convergent criteria, if so desired. The converged $\eta_{n+a}, e_{n+a}^p, \varepsilon_{n+a}^p$, and κ_{n+a} are the *updated elastic-plastic solution* corresponding the load increment $\Delta \mathbf{F}_n$.

Equilibrium Check. After one has checked all Gauss points for yielding and determine the yield deformation associated with the displacement $\Delta \mathbf{q}^i$, if any, one then checks whether $\Delta \mathbf{q}^i$ and its associated elastic-plastic deformation satisfy the equation of equilibrium. One first obtains τ_{n+1}^i from Eq. (2) and perform the equilibrium check. Denoting $\Gamma^i, \eta^i, (e^p)^i, (\varepsilon^p)^i, \tau_{n+1}^i$ as the converged elastic-plastic solution of $\Gamma, \eta_{n+a}, e_{n+a}^p, \varepsilon_{n+a}^p, \tau_{n+1}$, respectively, for the fixed $\Delta \mathbf{q}^i$, we then evaluate the *equilibrium residual* using Eq. (10)

$$(16) \quad \begin{aligned} \mathbf{R}_n^i &= \int_V \mathbf{B}^T (\tau_{n+1}^i - \tau_n) dV - \Delta \mathbf{F}_n \\ &= \int_V \mathbf{B}^T \mathbf{D}_e (\mathbf{B} \Delta \mathbf{q}^i - \Gamma^i \mathbf{P} \eta^i) dV - \Delta \mathbf{F}_n. \end{aligned}$$

Convergence is again defined as

$$(17) \quad (\mathbf{R}_n^i)^T \mathbf{R}_n^i \leq e_2 (\Delta \mathbf{F}_n)^T \Delta \mathbf{F}_n,$$

where e_2 is a chosen small positive constant.

If condition (17) fails, we modify $\Delta \mathbf{q}^i$ using Eq. (10). Assume the updated variables in the form

$$(18) \quad \Delta \mathbf{q}^{i+1} = \Delta \mathbf{q}^i + d\mathbf{q}^i, \quad \Gamma^{i+1} = 0, \quad \eta^{i+1} = \eta^i + a\mathbf{D}_e \mathbf{B} d\mathbf{q}^i, \quad (\varepsilon^p)^{i+1} = \varepsilon_n^p,$$

at the elastic Gauss points and

$$(19) \quad \begin{aligned} \Delta \mathbf{q}^{i+1} &= \Delta \mathbf{q}^i + d\mathbf{q}^i, \quad \Gamma^{i+1} = \Gamma^i + d\Gamma^i, \quad \eta^{i+1} = \eta^i + d\eta^i, \\ (\varepsilon^p)^{i+1} &= (\varepsilon^p)^i + (d\varepsilon^p)^i, \end{aligned}$$

at the plastic Gauss points. For the plastic Gauss points, substituting Eq. (19) into Eq. (8), (9) and linearizing them with respect to $d\mathbf{q}^i, d\Gamma^i, d\eta^i, (d\varepsilon^p)^i$ lead to

$$\begin{aligned} r\mathbf{Z}^{-1} d\eta^i + a(2c\mathbf{Z}^{-1}/3 + \mathbf{P}) \eta^i d\Gamma^i &= a\mathbf{B} d\mathbf{q}^i, \\ (\eta^T)^i \mathbf{P} d\eta^i - (2/3)^{3/2} a\kappa^i (\Gamma^i \eta^{iT} \mathbf{P} d\eta^i / \phi + \phi d\Gamma^i) (\partial \kappa^i / \partial \varepsilon^p) &= 0. \end{aligned}$$

The equations above give

$$(20) \quad d\eta^i = a[\mathbf{Z} \mathbf{B} d\mathbf{q}^i - (2c/3 + \mathbf{Z} \mathbf{P} / r) \eta^i d\Gamma^i] / r,$$

$$(21) \quad d\Gamma^i = [1 - (2/3)^{3/2} a\kappa^i \Gamma^i (\partial \kappa / \partial \varepsilon^p)^i / \phi] (\eta^T)^i \mathbf{P} \mathbf{Z} \mathbf{B} d\mathbf{q}^i / \beta,$$

where $\mathbf{Z}, r, \phi, \beta$ are defined in Eq. (14). Note that Eq. (20) and (21) do not involve any residuals because the current state variables are supposed to satisfy the elastic-plastic equations.

We can now derive the equations for $d\mathbf{q}^i$. For the plastic Gauss points, the differentiation of Eq. (6) and (7) gives

$$\begin{aligned}
d\alpha^i &= 2ac[\eta^i d\Gamma^i + \Gamma^i(d\tau_{n+a}^i - d\alpha^i)]/3 = 2ac(\eta^i d\Gamma^i + \Gamma^i d\tau_{n+a}^i)/(3r) \\
d\eta^i &= d\tau_{n+a}^i - d\alpha^i = d\tau_{n+a}^i - 2ac(\eta^i d\Gamma^i + \Gamma^i d\tau_{n+a}^i)/(3r) \\
&= (d\tau_{n+a}^i - 2ac\eta^i d\Gamma^i/3)/r \\
d\tau_{n+a}^i &= aD_e(Bdq^i - P\eta^i d\Gamma^i - \Gamma^i Pd\eta^i) \\
&= aD_e[Bdq^i - P\eta^i d\Gamma^i - \Gamma^i P(d\tau_{n+a}^i - 2ac\eta^i d\Gamma^i/3)/r].
\end{aligned}$$

The subscript $n + a$ for $d\tau_{n+a}$ is retained to emphasize that $d\tau_{n+a}$ is the increment of τ_{n+a} . The last equation above gives

$$(22) \quad d\tau_{n+a}^i = aZ(Bdq^i - P\eta^i d\Gamma^i/r).$$

Eliminating $d\Gamma^i$ from Eq. (22) by the use of Eq. (21) results in

$$(23) \quad d\tau_{n+a}^i = aD^{ep}Bdq^i,$$

where

$$(24) \quad D^{ep} = Z - [1 - (2/3)^{3/2}a\kappa^i\Gamma^i(\partial\kappa/\partial\varepsilon^p)^i/\phi]ZP\eta^i\eta^{iT}PZ/(r\beta),$$

which was first derived by Simo and Taylor (1986) for plane stress elastoplasticity. Note that $D^{ep} = D_e$ at the elastic Gauss points. Replacing τ_{n+1} by $\tau_{n+1}^i + d\tau_{n+1}^i$ in Eq. (10) and noting that, from Eq. (6), $d\tau_{n+1} = d\tau_{n+a}/a$, we can write the equation in the form

$$(25) \quad K_T dq^i = R_n^i,$$

where

$$(26) \quad K_T = \int_V B^T D^{ep} B dV$$

is the equivalent tangent modulus matrix. We then use Δq^{i+1} , $(\varepsilon^p)^{i+1}$, η^{i+1} , defined in Eq. (19) as the current values of the state variables and repeat the iteration for elastic-plastic solutions starting at the step shown in Eq. (12).

In principle, one should examine all Gauss points for the elastic-plastic solution for a fixed Δq^i . In practice, one checks only the points where yielding is anticipated or occurred during the previous load increment.

There are variants to the iteration procedures. For instance, instead of starting the iteration for a load increment with the elastic solution Δq° given by Eq. (11), we determine Δq° from

$$\left(\int_V B^T D^{ep} B dV \right) \Delta q^\circ - \Delta F_n = 0,$$

where D^{ep} is the last elastic-plastic modulus matrix of the n^{th} load increment. We also defines

$$\tau_{n+a} = \tau_n + aD^{ep}B\Delta q^\circ, \quad \alpha_{n+a} = \alpha_n + 2ac\Gamma\eta_n, \quad \varepsilon_{n+1}^p = \varepsilon_n^p + a\Gamma(\eta_n^T P\eta_n)^{1/2}.$$

The value of Γ depends on location. At the elastic Gauss point of the n^{th} load increment, $\Gamma = 0$. At the plastic Gauss points, for ideally-plastic materials,

$$\begin{aligned}
f(\tau) &= 0, \quad f_{,\tau}^T \dot{\tau} = 0, \quad \dot{\tau} = D_e(\dot{e} - \dot{e}^p) = D_e(\dot{e} - \Gamma h_{,\tau}), \\
\Gamma &= f_{,\tau}^T D_e \dot{e} / (f_{,\tau}^T D_e h_{,\tau}) = f_{,\tau}^T D_e B \Delta q^\circ / (f_{,\tau}^T D_e h_{,\tau});
\end{aligned}$$

and for hardening materials, $f(\tau - \alpha, \kappa) = 0$ with $\dot{\alpha} = 2\dot{\Lambda}c\eta/3$, from the *consistent equation*

$$(\partial f / \partial \tau_{ij})(\dot{\tau}_{ij} - \dot{\alpha}_{ij}) + (\partial f / \partial \kappa)(\partial \kappa / \partial \varepsilon^p)\dot{\varepsilon}^p = f_{,\tau}^T \dot{\tau} - \Gamma K_p = 0,$$

where the capital subscript(s) of a function denotes the partial derivatives of the function with respect to the subscript variable(s), e.g., $f_{,\tau}^T = (\partial f / \partial \tau)^T = [\partial f / \partial \tau_{11} \quad \partial f / \partial \tau_{22} \quad \dots]$. We have

$$\Gamma = f_{,\tau}^T \dot{\tau} / K_p = f_{,\tau}^T D^{ep} B \Delta q^\circ / [2c f_{,\tau}^T \eta / 3 - (\partial f / \partial \kappa)(\partial \kappa / \partial \varepsilon^p)|h_{,\tau}|],$$

in the stress space formulation or Eq. (6.14:16) in the strain space formulation, where $f_{,\tau}$, $h_{,\tau}$ are evaluated at η_n , ε_n^p . If Λ is negative, it is set to be zero.

Isotropic Materials. The iteration calculation can be simplified if the material is isotropic. In this case, Eq. (8) becomes

$$\Gamma = f_{,\tau}^T \dot{\tau} / K^p = f_{,\tau}^T D^{ep} B \Delta q^o / [2c f_{,\tau}^T \eta / 3 - (\partial f / \partial \kappa)(\partial \kappa / \partial \varepsilon^p) |h_{,\tau}|],$$

where G is the shear modulus, η'_e , η'_{n+a} are the deviatoric components of η_e , η_{n+a} and

$$\eta'_{n+a} = \eta'_e / [1 + 2a\Gamma(G + c/3)],$$

In other words, η'_{n+a} is linearly proportional to η'_e . Thus this formulation is also called the *radial return model*. The yielding condition Eq. (9) is in the simple form

$$(27) \quad f = \eta_e^T P \eta_e / [1 + 2a\Gamma(G + c/3)]^2 / 2 - \kappa_{n+a}^2 / 3 = 0.$$

For *non-hardening* materials, κ is constant. The yielding condition gives

$$\begin{aligned} \eta'_{n+a} &= \sqrt{2/3} \kappa \eta'_e / |\eta'_e| = \sqrt{2/3} \kappa \eta'_e / (\eta_e^T P \eta_e)^{1/2}, \\ \Gamma &= (\sqrt{3/2} |\eta'_e| - \kappa) / [2a(G + c/3)\kappa], \end{aligned}$$

in the stress space formulation. The deformation is elastic if $\Gamma \leq 0$. One simply sets $\Gamma = 0$.

For *isotropic hardening materials*, κ is a function of ε_{n+a}^p , which can now be written as

$$\varepsilon_{n+a}^p = \varepsilon_n^p + a\Gamma(\eta_e^T P \eta_e)^{1/2} / [1 + 2a\Gamma(G + c/3)].$$

Equation (13) becomes

$$d\Gamma^i = f^i \frac{[1 + 2a\Gamma(G + c/3)]^2}{2a(\eta_e^T P \eta_e)} \left[\frac{1}{(\eta_e^T P \eta_e)^{1/2}} \frac{\kappa^i \partial \kappa^i}{3} + \frac{G + c/3}{1 + 2a\Gamma(G + c/3)} \right]^{-1}.$$

For equilibrium check, we will use new variables to put the results in a more compacted form. It can be shown that

$$(28) \quad P = Q \Lambda_p Q^T,$$

where

$$(29) \quad Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \quad (\text{for plane strain and stress}),$$

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1/\sqrt{3} & \sqrt{2/3} & & \\ -1 & -1/\sqrt{3} & \sqrt{2/3} & & 0 \\ 0 & 2/\sqrt{3} & \sqrt{2/3} & & \\ & 0 & & \sqrt{2} \text{ dia}_{3 \times 3} [1 \ 1 \ 1] & \end{bmatrix}_{3 \times 3} \quad (\text{for 3-dimension}),$$

$$(30) \quad \Lambda_p = \begin{cases} \text{dia}[1/3 \ 1 \ 2] & (\text{for plane strain}) \\ \text{dia}[(1 - 2\nu)^2/3 \ 1 \ 2] & (\text{for plane strain, isotropic}) \\ \text{dia}[1 \ 1 \ 0 \ 2 \ 2 \ 2] & (\text{for three-dimension}). \end{cases}$$

One can also show that, for isotropic materials, the elastic coefficient matrix can be written in the form

$$(31) \quad D_e = Q \Lambda_d Q^T,$$

where, for plane stress,

$$(32) \quad D_e = \frac{1}{1-\nu^2} \begin{bmatrix} 1 & & \text{sym} \\ -\nu & 1 & \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \quad \Lambda_d = \frac{E}{1+\nu} \text{dia} \left[\frac{1+\nu}{1-\nu} \ 1 \ \frac{1}{2} \right];$$

for plane strain

$$(33) \quad D_e = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & & \text{sym} \\ \nu & 1-\nu & \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}, \quad \Lambda_d = \frac{E}{1+\nu} \text{dia} \left[\frac{1}{1-2\nu} \ 1 \ \frac{1}{2} \right]$$

and for three-dimensional problems

$$(34) \quad D_e = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & & \mathbf{0}_{3 \times 3} \\ \nu & \nu & 1-\nu & & & \\ \mathbf{0}_{3 \times 3} & & & (1-2\nu) \text{dia}_{3 \times 3} [1 \ 1 \ 1]/2 & & \end{bmatrix},$$

$$\Lambda_p = E \text{dia}_{6 \times 6} [1 \ 1 \ (1+\nu)(1-2\nu) \ 1/2 \ 1/2 \ 1/2] / (1+\nu).$$

Let $\zeta = \mathbf{Q}\eta$. Equation (13) becomes

$$d\zeta^i = -a(2c/3 + \mathbf{Z}'\Lambda_p/r)\zeta^i d\Gamma^i/r,$$

and Eq. (20), (21), (24) become

$$\begin{aligned} d\zeta^i &= a[\mathbf{Z}'\mathbf{Q}\mathbf{B}d\mathbf{q}^i - (2c/3 + \mathbf{Z}'\Lambda_p/r)\zeta^i d\Gamma^i]/r, \\ d\Gamma^i &= [1 - (2/3)^{3/2}a\kappa^i\Gamma^i(\partial\kappa/\partial\varepsilon^p)^i/\phi](\zeta^i)^T \Lambda_p \mathbf{Z}' \mathbf{Q} \mathbf{B} d\mathbf{q}^i / \beta, \\ \mathbf{D}^i &= \mathbf{Q}^T \{ \mathbf{Z}' - \frac{1}{r\beta} [1 - \frac{2}{3} \sqrt{\frac{2}{3}} \frac{a\kappa^i\Gamma^i}{\phi} (\frac{\partial\kappa}{\partial\varepsilon^p})^i] \mathbf{Z}' \Lambda_p \zeta^i \zeta^{iT} \Lambda_p \mathbf{Z}' \} \mathbf{Q}, \end{aligned}$$

in which

$$\mathbf{Z}' = [\Lambda_d^{-1} + a\Gamma^i \Lambda_p / r]^{-1}, \quad \phi = \sqrt{\zeta^{iT} \Lambda_p \zeta^i},$$

where \mathbf{Z}' , Λ_d , Λ_p are diagonal matrices and Λ_d , Λ_p , \mathbf{Q} are given in Eq. (29)–(34).

Solving for $d\mathbf{q}^i$ with updated \mathbf{K}_T can be time consuming, because one has to refactor \mathbf{K}_T every time when it is updated. Instead, one updates \mathbf{K}_T every few iterations or even uses the initial \mathbf{K}_T^0 , the elastic stiffness matrix, throughout the entire iteration process. The modified procedures converge generally at a slower rate but may take less total computer time to reach a converged solution by avoiding or minimizing the refactoring of \mathbf{K}_T .

21.10. IMPLICIT SCHEME FOR GENERAL YIELD SURFACES

For more complex yield surfaces (see Chapter 6), the iteration procedure is the same but solving the elastic-plastic equations for a given $\Delta\mathbf{q}^i$ is more involved. The elastic-plastic solutions are governed by

$$(1) \quad \mathbf{e}_{n+1} = \mathbf{e}_n + \mathbf{B}\Delta\mathbf{q}, \quad \mathbf{e}_{n+1}^p = \mathbf{e}_n^p + \Gamma h_{,\tau}(\tau_{n+a}, \xi_{n+a}),$$

$$(2) \quad \tau_{n+1} = \tau_n + \mathbf{D}_e[\mathbf{B}\Delta\mathbf{q} - \Gamma h_{,\tau}(\tau_{n+a}, \xi_{n+a})], \quad \xi_{n+1} = \xi_n + \Gamma \mathbf{U}(\tau_{n+a}, \xi_{n+a}),$$

in which ξ_k are the internal variables, $h_{,\tau}$ is a known *flow rule* and \mathbf{U} is a vector function characterizing the internal variables. The finite element equations are the same as Eq. (21.9:10). The intermediate elastic-plastic equations are

$$(3) \quad \mathbf{f}_s = \tau_{n+a} - \tau_n - a\mathbf{D}_e[\mathbf{B}\Delta\mathbf{q} - \Gamma h_{,\tau}(\tau_{n+a}, \xi_{n+a})],$$

$$(4) \quad \mathbf{f}_{ep} = \mathbf{e}_{n+a}^p - \mathbf{e}_n^p - a\Gamma h_{,\tau}(\tau_{n+a}, \xi_{n+a}),$$

$$(5) \quad f_{in} = \xi_{n+a} - \xi_n - a\Gamma\mathbf{U}(\tau_{n+a}, \xi_{n+a}),$$

$$(6) \quad f = f(\tau_{n+a}, \xi_{n+a}) = 0.$$

Yield Check. The iteration process for the elastic-plastic solution at the $n+1^{\text{th}}$ load increment $\Delta\mathbf{F}_n$ is similar to that described before, which involves both the yield and equilibrium checks. For *yield check*, one determines $(\mathbf{e}^p)_{n+a}$, τ_{n+a} , ξ_{n+a} , Γ corresponding to a known $\Delta\mathbf{q}^i$ by iteration. At a given Gauss point, if yielding occurs, let $(\mathbf{e}^p)^i$, τ^i , ξ^i and Γ^i denote the current iterated values of \mathbf{e}^p , τ , ξ and Γ . (The subscript $n+a$ at the intermediate stage has been dropped for simplicity.) The *residuals of the elastic-plastic equations* \mathbf{f}_s^i , \mathbf{f}_{ep}^i , \mathbf{f}_{in}^i and f^i are defined in Eq. (3)–(6),

$$(7) \quad \mathbf{f}_s^i = \mathbf{f}_s(\tau^i, \xi^i), \quad \mathbf{f}_{ep}^i = \mathbf{f}_{ep}(\tau^i, \xi^i), \quad \mathbf{f}_{in}^i = \mathbf{f}_{in}(\tau^i, \xi^i), \quad f^i = f(\tau^i, \xi^i).$$

The convergence of the elastic-plastic equations is defined as

$$(8) \quad \gamma_1 \mathbf{f}_s^{iT} \mathbf{f}_s^i + \gamma_2 \mathbf{f}_{ep}^{iT} \mathbf{f}_{ep}^i + \gamma_3 \mathbf{f}_{in}^{iT} \mathbf{f}_{in}^i + \gamma_4 f^{i+1} f^{i+1} < e_1 |f[0, \kappa(0)]|,$$

where γ 's, e_1 are selected positive constants.

If Eq. (8) is not satisfied, we seek an updated elastic-plastic solution in the form of $(\mathbf{e}^p)^i + d(\mathbf{e}^p)^i$, $\tau^i + d\tau^i$, $\xi^i + d\xi^i$, $\Gamma^i + d\Gamma^i$. Linearizing Eq. (3)–(6) around the current values of the state variables gives

$$(9) \quad d\tau^i = a\mathbf{D}_e(r_e \mathbf{B}d\mathbf{q}^i - h_{,\tau} d\Gamma - \Gamma h_{,\tau\tau} d\tau^i - \Gamma h_{,\tau\xi} d\xi^i) - r_p \mathbf{f}_s^i,$$

$$(10) \quad (de^p)^i = a\mathbf{D}_e(h_{,\tau} d\Gamma + \Gamma h_{,\tau\tau} d\tau^i + \Gamma h_{,\tau\xi} d\xi^i) - r_p \mathbf{f}_{ep}^i,$$

$$(11) \quad d\xi^i = a[\mathbf{U}^i d\Gamma^i + \Gamma^i (\mathbf{U}_{,\tau}^i d\tau^i + \mathbf{U}_{,\xi}^i d\xi^i)] - r_p \mathbf{f}_{in}^i \quad (i \text{ not summed}),$$

$$(12) \quad f_{,\tau}^i d\tau^i + (f_{,\xi}^i) d\xi^i = -r_p f^i \quad (i \text{ not summed}),$$

where $(_)^i$ denotes quantities evaluated at τ_{n+a}^i and ξ_{n+a}^i , the gradients are

$$f_{,\sigma} = (\partial f / \partial \sigma)^T = [\partial f / \partial \sigma_{11} \quad \partial f / \partial \sigma_{22} \quad \partial f / \partial \sigma_{12}] \quad (\text{for plane}),$$

$$= [\partial f / \partial \sigma_{11} \quad \partial f / \partial \sigma_{22} \quad \partial f / \partial \sigma_{33} \quad \partial f / \partial \sigma_{12} \quad \partial f / \partial \sigma_{23} \quad \partial f / \partial \sigma_{31}] \quad (\text{for 3-D}),$$

$$f_{,\xi} = (\partial f / \partial \xi)^T = [\partial f / \partial \xi_1 \quad \partial f / \partial \xi_2 \quad \dots]$$

and $r_e = 0$ and $r_p = 1$. These constants are introduced for later convenience. One then updates $(\mathbf{e}^p)_{n+a}$, τ_{n+a} , ξ_{n+a} , Γ and examines Eq. (8). The process is repeated until the solution converges.

Equilibrium Check. After the updated elastic-plastic solution has been obtained at all plastic Gauss points, *equilibrium check* is performed. We calculate τ_{n+1}^i from Eq. (2) using the updated τ_{n+a} , ξ_{n+a} etc., and evaluate the *equilibrium residual*

$$(13) \quad \begin{aligned} \mathbf{R}_n^i &= \int_V \mathbf{B}^T (\tau_{n+1}^i - \tau_n) dV - \Delta\mathbf{F}_n \\ &= \int_V \mathbf{B}^T \mathbf{D}_e [\mathbf{B} \Delta\mathbf{q}^i - \Gamma^i h_{,\tau}(\tau^i, \xi^i)] dV - \Delta\mathbf{F}_n. \end{aligned}$$

The convergent criterion is given by Eq. (21.7:17) requiring \mathbf{R}_n^i to be sufficiently small. If the equilibrium check fails, we introduce new updates in the form

$$\Delta\mathbf{q}^{i+1} = \Delta\mathbf{q}^i + d\mathbf{q}^i, \quad (\mathbf{e}^p)^{i+1} = \mathbf{e}_n^p, \quad \tau^{i+1} = \tau^i + \mathbf{D}_e \mathbf{B} d\mathbf{q}^i, \quad \xi^{i+1} = \xi_n, \quad \Gamma^{i+1} =$$

at elastic Gauss points, and

$$\begin{aligned} \Delta\mathbf{q}^{i+1} &= \Delta\mathbf{q}^i + d\mathbf{q}^i, \quad (\mathbf{e}^p)^{i+1} = (\mathbf{e}^p)^i + (de^p)^i, \quad \tau^{i+1} = \tau^i + d\tau^i, \\ \xi^{i+1} &= \xi^i + d\xi^i, \quad \Gamma^{i+1} = \Gamma^i + d\Gamma^i, \end{aligned}$$

at plastic Gauss points. Substituting the updated state variables into Eq. (3)–(6) and (21.7:10), and linearizing them around the current state of the variables, we obtain the exact the same set of equations as Eq. (9) and (12) with now the

constants $r_e = 1$ and $r_p = 0$. This is due to the introduction of $d\mathbf{q}^i$ and the fact that the current state variables satisfy the elastic-plastic equations. From Eq. (13), we obtain the equations of motion

$$(14) \quad \int_V \mathbf{B}^T d\tau_{n+1}^i dV + \mathbf{R}_n^i = \int_V \mathbf{B}^T d\tau_{n+a}^i dV/a + \mathbf{R}_n^i = 0.$$

Solving Eq. (9)–(12) for τ_{n+1}^i in terms of $d\mathbf{q}^i$ and substituting it into Eq. (14), we obtain the equation for $d\mathbf{q}^i$ in terms of \mathbf{R}_n^i . We then repeat the process to determine the elastic-plastic solution for the increment $\Delta\mathbf{q}^{i+1} = \Delta\mathbf{q}^i + d\mathbf{q}^i$.

In this chapter, we have only touched upon briefly the application of finite element method to various nonlinear problems. Readers are referred to the voluminous literature for details. Computational mechanics provide us tools to explore the enormous richness of nonlinear mechanics: large deformation, composite structures, microand nanotechnologies, living cells, DNA, proteins, enzymes, genomics and beyond.

Problem 21.2. Derive the two-stage algorithm as that in Sec. 21.5 for solving viscoplasticity problems. The two-stage approach is outlined as follows: For a given $\Delta\mathbf{q}^i$, find the corresponding elastic-plastic solution $\Delta\mathbf{q}^{i+1} = \Delta\mathbf{q}^i + d\mathbf{q}^i$, and then uses the equations of motion to update $\Delta\mathbf{q}^i$. The process is repeated until the equilibrium solution is obtained. [Solution: For fixed $\Delta\mathbf{q}$, the governing equations for elasticplastic deformation are

$$(P1) \quad \tau = -\mathbf{D}_e \dot{\mathbf{e}}^p = -H(f) \mathbf{D}_e h_{,\tau} / \eta,$$

$$(P2) \quad df/dt = (f_{,\tau})^T \dot{\tau} + (f_{,\xi}) \dot{\xi} = -H(f)/\bar{\tau},$$

where

$$(P3) \quad \bar{\tau} = \eta/\bar{\tau} = \eta/[(f_{,\tau})^T \mathbf{D}_e h_{,\tau} - (f_{,\xi}) \mathbf{U}]$$

is the *instantaneous relaxation time*, which is always positive. Equations (21.7:1) and (P2) are a set of first order nonlinear differential equations, which can be integrated with the initial conditions

$$\tau_{n+1}^i = \tau_{n+1}^{i-1} + \mathbf{D}_e \mathbf{B} \Delta\mathbf{q}^i, \quad \xi_{n+1}^i = \xi_{n+1}^{i-1}, \quad (\mathbf{e}^p)_{n+1}^i = (\mathbf{e}^p)_{n+1}^{i-1}.$$

Converged $\tau_{n+1}^i, \xi_{n+1}^i, (\mathbf{e}^p)_{n+1}^i$ correspond to $f(\tau_{n+1}^i, \xi_{n+1}^i) \rightarrow 0$. Physically (P2) defines the return path to the yield surface $f=0$.

The equilibrium check is simply defined as Eq. (21.5:17)

$$(\mathbf{R}_n^i)^T \mathbf{R}_n^i \leq e_2(\Delta\mathbf{F}_n)^T \Delta\mathbf{F}_n,$$

where

$$\mathbf{R}_n^i = \int_V \mathbf{B}^T \mathbf{D}_e [\mathbf{B} \Delta\mathbf{q}^i - (\mathbf{e}^p)_{n+1}^i + \mathbf{e}_n^p] dV - \Delta\mathbf{F}_n.$$

If the equilibrium check fails, we update $\Delta\mathbf{q}^i$. For simplicity, only the explicit form of equilibrium update is considered. A substitution of $\Delta\mathbf{q}^i + d\mathbf{q}^i, (\mathbf{e}^p)_{n+1}^i$ into Eq. (21.7:6) gives

$$\mathbf{K}_T^o d\mathbf{q}^i + \mathbf{R}_n^i = 0.$$

One then solves for $d\mathbf{q}^i$ and uses $\Delta\mathbf{q}^{i+1} = \Delta\mathbf{q}^i + d\mathbf{q}^i$ as the update for the next iteration.]

¹A more general form of external load is

$$\mathbf{b} = pc_1 \mathbf{b}_0 + \mathbf{b}', \quad \bar{\mathbf{T}} = pc_2 \bar{\mathbf{T}}_0 + \bar{\mathbf{T}}',$$

where c_1, c_2 are given constants and are given preload. This allows the body force and the surface traction to increase at different ratio.

²In Eq. (21.6:3), h is considered to be a function of 6 stress components, i.e., for $i \neq j$, τ_{ij}, τ_{ji} are the same variable leading to $2\dot{\mathbf{e}}_{ij}^p = \Lambda \partial h / \partial \tau_{ij}$ in matrix notation, while h is considered to be a function of 9 stress components, i.e., for $i \neq j$, τ_{ij}, τ_{ji} are the different variables leading to $\dot{\mathbf{e}}_{ij}^p = \Lambda \partial h / \partial \tau_{ij}$ in index notation. Note that: $\tau_t: \mathbf{e}_t = \tau_m^T \mathbf{e}_m$ where the subscript t and m denote the tensor and matrix quantities, respectively.

³Back stress tensor is used to model kinematic hardening as defined in Secs. 6.12 and 6.13.

⁴For dynamic analysis, $\Delta\mathbf{u}$ is calculated from acceleration at each increment. One often defines

$$\Delta e_{ij} = \frac{1}{2} \left[\frac{\partial \Delta u_i}{\partial X_j^{n+1/2}} + \frac{\partial \Delta u_j}{\partial X_i^{n+1/2}} \right], \quad \Delta \Omega_{ij} = \frac{1}{2} \left[\frac{\partial \Delta u_i}{\partial X_j^{n+1/2}} - \frac{\partial \Delta u_j}{\partial X_i^{n+1/2}} \right],$$

where $X_i^{n+1/2} = X_i^n + \frac{1}{2}\Delta u_i$.

22

MESHLESS LOCAL PETROV–GALERKIN AND ESHELBY–ATLURI METHODS

Engineering problems often involve complex geometrical configurations and nonlinearities. Numerical computation is a main tool to tackle these problems in the past five decades and more. The finite element method (FEM) is among the most useful tools. Recently in a radical departure from the popular FEMs, Han and Atluri (2014a,b) introduced a fundamentally different approach blending the energy–momentum conservation laws of Noether (1918) and Eshelby (1951, 1975) and the Meshless Local Petrov–Galerkin methods (MLPGs) of Atluri (1998, 2004), designated as the Eshelby–Atluri method (EAM) for simulations of problems in solids.

The FEM suffers the drawback of tedious meshing, especially in large scale 3-dimensional multi-physics simulations, and in problems involving discontinuities, moving boundaries such as crack propagation, strain localization in shear band formation, mesh distortion in large deformation, etc. The meshless approach is a way to circumvent some of these drawbacks. In 1992, Nayroles *et al.* introduced the diffuse element method. Other publications followed: element free Galerkin method by Belytschko *et al.* (1994), reproducing kernel particle method by Liu *et al.* (1996), hp-cloud method by Duarte and Oden (1996), and many more. These methods do not require meshes for interpolation of solution variables, but use shadow elements (“background meshes”) for integration of the weak forms. Atluri and Zhu (1998a,b) and Atluri *et al.* (1999) introduced the Meshless Local Petrov–Galerkin method (MLPG) that eliminates the need for background mesh for integration with remarkable successes (see [Biblio. 22](#)). Many aspects of the meshless method can be found in Atluri’s book (2004).

This chapter follows Atluri and his colleagues’ works on MLPG and EAM. These methods can be extended to boundary integral equations, and problems with defects and of multi-physics. We first discuss the MLPGs.

22.1. WEAK FORMS

Consider a problem domain Ω . The MLPG uses a local weak form over sub-domains Ω_s inside Ω . The global stiffness matrix is constructed over Ω_s , which may overlap other sub-domains and can have arbitrary shapes. The sub-domains do not form a contiguous mesh globally.

To illustrate the concept, we consider the 2-dimensional linear Poisson’s equation over Ω as sketched in [Fig. 18.1:1](#).

$$(1) \quad \nabla^2\phi(\mathbf{x}) = p(\mathbf{x}) \quad \mathbf{x} \in \Omega,$$

where \mathbf{x} ($= x_\alpha \mathbf{e}_\alpha$) denotes the position of a point in Ω and $p(\mathbf{x})$ the external force. The boundary conditions over the boundaries Γ_ϕ and Γ_σ ($\partial\Omega = \Gamma_\phi + \Gamma_\sigma$) are

$$(2) \quad \phi = \bar{\phi} \quad \text{on} \quad \Gamma_\phi,$$

$$(3) \quad \phi_{,n} = \partial\phi/\partial n = \bar{\phi}_n \quad \text{on} \quad \Gamma_\sigma,$$

where n is the outward normal to $\partial\Omega$, Eq. (2) is the rigid and Eq. (3) the natural boundary condition, and the overbar denotes prescribed quantities. A local weak form penalty formulation of the problem can be written as

$$(4) \quad \int_{\Omega} (\nabla^2\phi - p)\psi d\Omega - \int_{\Gamma_\sigma} (\phi_{,n} - \bar{\phi}_n)\psi d\Gamma - \gamma \int_{\Gamma_\phi} (\phi - \bar{\phi})\psi_{,n} d\Gamma = 0,$$

where γ is the penalty parameter with $\gamma \gg 1$ to enforce the rigid condition (2) on Γ_ϕ . Note that $\psi_{,n}$ can be independently chosen. For simplicity, we shall choose ψ and $\psi_{,n}$ from the same shape function space. We call ϕ and ψ the trial and test functions, respectively. The trial function ϕ is an approximate solution of Eqs. (1) to (3), which satisfies Eq. (4) for a set of selected test functions ψ and $\psi_{,n}$. If one chooses $\psi(\mathbf{x}) = \psi_{,n} = \delta(\mathbf{x} - \mathbf{x}_i)$, the Dirac delta function as the test function for all nodal points \mathbf{x}_i , Eq. (4) reduces to the collocation approximation.

The trial and test functions must satisfy certain requirements, called *admissible conditions*. For example, in Eq. (4), ϕ is required to be twice differentiable, i.e., C^1 continuous, and there is no requirement on the test functions $\psi_{,n}$. In this case, the requirements on ϕ and ψ are different and Eq. (4) is called an *asymmetric* weak form.

Employing the divergent theorem in Eq. (4), we obtain

$$(5) \quad \int_{\Omega} (\phi_{,\alpha} \psi_{,\alpha} + p\psi) d\Omega - \int_{\Gamma_{\phi}} \phi_{,n} \psi d\Gamma + \gamma \int_{\Gamma_{\phi}} (\phi - \bar{\phi}) \psi_{,n} d\Gamma = \int_{\Gamma_{\sigma}} \bar{\phi}_{,n} \psi d\Gamma.$$

In this case, the trial and test functions ϕ and ψ have the same requirement of C^0 continuity. We call Eq. (5) a *symmetric* weak form.

If the weak form in Eq. (4) or (5) is satisfied for arbitrary admissible ψ and ϕ , Eq. (1) in Ω and the boundary conditions Eq. (2) and (3) on Γ_{ϕ} and Γ_{σ} , are satisfied *a posteriori*. If γ is zero, the trial functions are required to satisfy Eq. (2) and the test functions to vanish on Γ_{ϕ} where the rigid condition is specified. For Eq. (5), both the trial and test functions can be chosen from the same space. One achieves localization through the local support of interpolation functions.

The weak forms provide great flexibility for deriving alternate meshless methods on a rational basis. More weak forms with different choices of the trial and test functions can be found in the book by Atluri (2004).

22.2. INTERPOLATION WITH A LOCAL SUPPORT

To determine a meshless weak form solution, we first construct the trial functions. The approach uses the local interpolation or approximation of the unknown variables at selected nodes within the sub-domains as trial functions. The interpolation satisfies the following requirements:

- (i) locality,
- (ii) continuity,
- (iii) consistency (or completeness).

The first requirement minimizes the bandwidth and non-zero entry of the stiffness matrix and makes the computation more efficient. The second ensures the weak form definable. The third relates to the convergence of the interpolantbased weak form method. Also the interpolation scheme in each interpolation domain must not be too sensitive to allow the flexibility of adding, moving or removing nodes.

We shall discuss below a variety of schemes to interpolate the data at scattered points within a domain.

22.2.1. Moving least-square interpolation

We shall find a function $\phi^h(\mathbf{x})$ that best approximates $\phi(\mathbf{x})$ based on the data associated with a set of randomly distributed nodes $\mathbf{x}_j, j = 1, 2, \dots, N$, within Ω_s and on its boundary $\partial\Omega_s$. Let $\phi^h(\mathbf{x})$ be written in the form

$$(1) \quad \phi^h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) \mathbf{a}(\mathbf{x}) \quad \mathbf{x} \in \Omega_s \cup \partial\Omega_s \quad (\text{union of } \Omega_s \text{ and } \partial\Omega_s),$$

where $\mathbf{p}(\mathbf{x}) = [p_1(\mathbf{x}) \ p_2(\mathbf{x}) \ \dots \ p_m(\mathbf{x})]^T$ is a $m \times 1$ column matrix of a complete monomial basis and $\mathbf{a}(\mathbf{x}) = [a_1(\mathbf{x}) \ a_2(\mathbf{x}) \ \dots \ a_m(\mathbf{x})]^T$ is a coefficient column matrix. If the basis includes the complete polynomials of order t , then for l -dimensional problems, $m = (t+1)(t+2) \dots (t+l)/2$. In the 2-dimension case,

$$(2) \quad \mathbf{p}^T(\mathbf{x}) = [1 \ x \ y], \quad m = 3 \text{ for a linear basis } (t = 1);$$

$$(3) \quad \mathbf{p}^T(\mathbf{x}) = [1 \ x \ y \ x^2 \ xy \ y^2], \quad m = 6 \text{ for a quadratic basis } (t = 2).$$

We define a weighted discrete L_2 norm as

$$(4) \quad L_2(\mathbf{a}) = \sum_{J=1}^N w_J(\mathbf{x}) [\mathbf{p}^T(\mathbf{x}_J) \mathbf{a}(\mathbf{x}) - \tilde{\phi}_J]^2 = [\mathbf{P}\mathbf{a}(\mathbf{x}) - \tilde{\phi}]^T \mathbf{W}(\mathbf{x}) [\mathbf{P}\mathbf{a}(\mathbf{x}) - \tilde{\phi}],$$

where $\tilde{\phi}^T = [\tilde{\phi}_1 \ \tilde{\phi}_2 \ \dots \ \tilde{\phi}_N]$ is the data vector with $\tilde{\phi}_J$ being the datum associated with the node at x_J , $w_J(\mathbf{x})$ is a positive weight function in the support Ω_J (the domain outside which the function vanishes), N is the number of nodes in $\Omega \cup \partial\Omega$ with $w_J(\mathbf{x}) > 0$ and can have different value for different nodes. To simplify future references, we refer Ω_J as the support of node J , synonymous with the support of $w_J(\mathbf{x})$. One often chooses the support as an l -dimensional sphere, ellipsoid or parallelepiped centered at \mathbf{x}_J . The sizes/shapes of supports of different nodes can be different.

The matrices \mathbf{P} and \mathbf{W} are defined as

$$(5) \quad \mathbf{P} = [\mathbf{p}(\mathbf{x}_1), \mathbf{p}(\mathbf{x}_2), \dots, \mathbf{p}(\mathbf{x}_N)]^T \quad (N \times m \text{ matrix}),$$

$$(6) \quad \mathbf{W}(\mathbf{x}) = \text{dia}[w_1(\mathbf{x}) \ w_2(\mathbf{x}) \ \dots \ w_N(\mathbf{x})] \quad (\text{diagonal } N \times N \text{ matrix}).$$

Minimization of $L_2(\mathbf{a})$ with respect to a 's gives the coefficient vector as

$$(7) \quad \underset{(m \times 1)}{\mathbf{a}(\mathbf{x})} = \underset{(m \times m)}{\mathbf{U}(\mathbf{x})^{-1}} \underset{(m \times N)}{\mathbf{V}(\mathbf{x})} \underset{(N \times 1)}{\tilde{\phi}^T},$$

where

$$(8) \quad \underset{(m \times m)}{\mathbf{U}(\mathbf{x})} = \underset{(m \times N)(N \times N)(N \times m)}{\mathbf{P}^T \mathbf{W}(\mathbf{x}) \mathbf{P}} = \sum_{J=1}^N w_J(\mathbf{x}) \underset{(m \times N)}{\mathbf{p}(\mathbf{x}_J)} \underset{(N \times m)}{\mathbf{p}^T(\mathbf{x}_J)},$$

$$(9) \quad \underset{(m \times N)}{\mathbf{V}(\mathbf{x})} = \underset{(m \times N)(N \times N)}{\mathbf{P}^T \mathbf{W}(\mathbf{x})} = [w_1(\mathbf{x}) \underset{(m \times 1)}{\mathbf{p}(\mathbf{x}_1)} \ w_2(\mathbf{x}) \underset{(m \times 1)}{\mathbf{p}(\mathbf{x}_2)} \ \dots \ w_N(\mathbf{x}) \underset{(m \times 1)}{\mathbf{p}(\mathbf{x}_N)}].$$

A substitution of Eq. (7) into (1) yields

$$(10) \quad \phi^h(\mathbf{x}) = \underset{(1 \times N)}{\Phi^T(\mathbf{x})} \underset{(N \times 1)}{\tilde{\phi}} = \sum_{J=1}^N \Phi_J(\mathbf{x}) \tilde{\phi}_J,$$

in which

$$(11) \quad \Phi^T(\mathbf{x}) = [\Phi_1 \ \Phi_2 \ \dots \ \Phi_N] = \underset{(1 \times m)}{\mathbf{p}^T(\mathbf{x})} \underset{(m \times m)}{\mathbf{U}(\mathbf{x})^{-1}} \underset{(m \times N)}{\mathbf{V}(\mathbf{x})},$$

$$(12) \quad \Phi_J(\mathbf{x}) = \sum_{j=1}^m p_j(\mathbf{x}) [\mathbf{U}(\mathbf{x})^{-1} \mathbf{V}(\mathbf{x})]_{jJ},$$

$\Phi_J(\mathbf{x})$ is the shape function associated with node J , the node at \mathbf{x}_J . Clearly ϕ^h at \mathbf{x} is affected by $\tilde{\phi}_J$, $J = 1, 2, \dots, N$, whose weight functions do not vanish at \mathbf{x} . In general the vector

$$\tilde{\phi}^T = [\tilde{\phi}_1 \ \tilde{\phi}_2 \ \dots \ \tilde{\phi}_N] \neq [\phi^h(\mathbf{x}_1) \ \phi^h(\mathbf{x}_2) \ \dots \ \phi^h(\mathbf{x}_N)],$$

i.e., $\tilde{\phi}_J$ is not the value of the function $\phi^h(\mathbf{x})$ at \mathbf{x}_J . We thus call $\tilde{\phi}_J$ a fictitious nodal value and $\tilde{\phi}^T$ a fictitious nodal vector. Also usually $\Phi_J(\mathbf{x})$ does not have a unity value at $\mathbf{x} = \mathbf{x}_J$, nor vanishes at other nodes within the support of $w_J(\mathbf{x})$. This is in contrast to the Kronecker delta property of the shape functions of finite element methods, i.e., equal to 1 at node J and zero at all other nodes.

The shape function $\Phi_J(\mathbf{x})$ has a complex structure. From Eq. (11), the J^{th} column of $\mathbf{V}(\mathbf{x})$ is zero if $w_J(\mathbf{x}) = 0$ and hence $\Phi_J(\mathbf{x}) = 0$. Thus the support of $\Phi_J(\mathbf{x})$ is no bigger than that of $w_J(\mathbf{x})$ and can be denoted as Ω_J . The value of the trial function ϕ^h at \mathbf{x} is affected by all the nodal weight functions with positive values at the point. In contrast, in the FEMs the trial function at any point \mathbf{x} is affected by nodes of the element in which \mathbf{x} lies. Thus in evaluating a trial function at a point, one has to identify all nodes, in whose supports the point lies. One then includes the weight functions of all these nodes to evaluate the shape functions of these nodes. Consider an example of a domain Ω as shown in Fig. 22.2:1. The solid circular dots are nodes. Some of the nodes have their nodal number indicated. The patterned areas are supports of nodes 1, 3, 7 and 8 are marked by $\Omega_1, \Omega_3, \Omega_7$ and Ω_8 , respectively. Node 2 is close to the domain boundary $\partial\Omega$, while node 9 is actually on it. The point at \mathbf{x}_0 lies in and only in the intersection of supports of nodes 3 and 7. The values of $\Phi_3(\mathbf{x}_0)$ and $\Phi_7(\mathbf{x}_0)$ will depend only on the weight functions $w_3(\mathbf{x}_0)$ and $w_7(\mathbf{x}_0)$ and not on any others.

Since $\Phi(\mathbf{x})$ is obtained by the minimization of a discrete L_2 norm over a sub-domain Ω_x containing all nodes within Ω or on $\partial\Omega$ with weight functions positive at the point \mathbf{x} , we call the scheme the moving least square interpolation and Ω_x “the domain of definition of \mathbf{x} .” For example in Fig. 22.2:1, the intersection of Ω_3 and Ω_7 ($\Omega_3 \cup \Omega_7$) is the domain of definition of \mathbf{x}_0 . Under special circumstances, one may purposely skip some nodes in the domain of definition to evaluate the trial function.

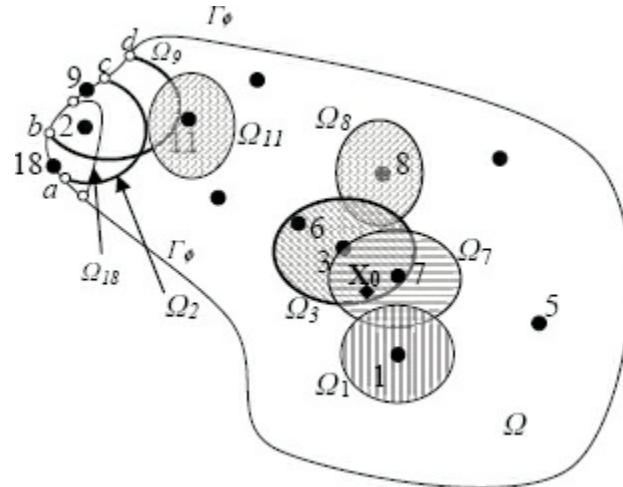


Fig. 22.2:1. Schematics of nodes and supports.

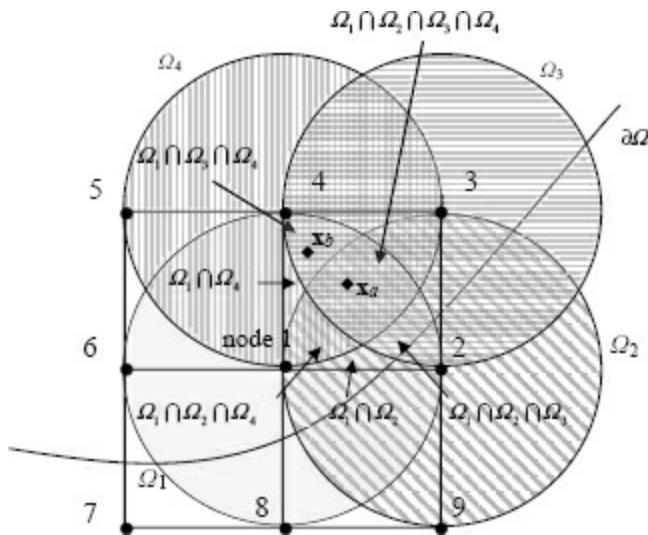


Fig. 22.2:2. Domain of definition.

Figure 22.2:2 further illustrates how the shape functions depend on the domains of definition of different points. In the figure, nodes are evenly distributed. The supports of nodes 1, 2, 3 and 4 are the circular area centered at the respective nodes. Patterned areas pointed by arrows are the intersections of supports. The supports involves in the intersection are identified at the end of the respective arrow. The shape functions of nodes whose supports are not part of an intersection are zero within the intersection. Thus only the weight functions with supports indicated at the end of the arrow contribute to the trial function $\phi^h(\mathbf{x})$ in the intersection. For locations in different intersections, the weight functions that affect the shape functions and so the trial function are different. Consider two locations $\mathbf{x}_a \in \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4$ and $\mathbf{x}_b \in \Omega_1 \cap \Omega_3 \cap \Omega_4$ shown in Fig. 22.2:2. The trial function $\phi^h(\mathbf{x}_a)$ involves only the shape functions $\Phi_1(\mathbf{x}_a), \dots, \Phi_4(\mathbf{x}_a)$ and, in turn, the weight functions $w_1(\mathbf{x}_a), \dots, w_4(\mathbf{x}_a)$. On the other hand, $\phi^h(\mathbf{x}_b)$ involves $\Phi_1(\mathbf{x}_b), \Phi_3(\mathbf{x}_b)$ and $\Phi_4(\mathbf{x}_b)$ and thus depends only on the weight functions $w_1(\mathbf{x}_b), w_3(\mathbf{x}_b)$ and $w_4(\mathbf{x}_b)$. For points near the boundary of Ω , we exclude the part of an intersection outside Ω . Also no nodes outside Ω should be included in defining the domain of definition and any intersection.

Problem 22.1. In Fig. 22.2:2, if the boundary $\partial\Omega$ divides nodes 1, 3, 4, 5, 6 and nodes 2, 7, 8, 9 such that nodes 1, 3, 4, 5, 6 are inside the domain Ω while the other nodes are outside. For radial support of nodal distance, identify the admissible intersections (i.e., the parts of intersections of supports of the interior nodes within Ω) involving the supports of these nodes.

The order of completeness of the function ϕ^h as defined in Eq. (1) is the same as that of the basis of $\mathbf{p}(\mathbf{x})$. The consistency condition implies that,

$$(13) \quad \sum_{J=1}^N \Phi_J(\mathbf{x}) = 1, \quad \sum_{J=1}^N \Phi_J(\mathbf{x}) x_J = x, \quad \sum_{J=1}^N \Phi_J(\mathbf{x}) y_J = y,$$

for a linear basis over the support of the function. In addition, we have

$$(14) \quad \sum_{J=1}^N \Phi_J(\mathbf{x}) x_J^2 = x^2, \quad \sum_{J=1}^N \Phi_J(\mathbf{x}) x_J y_J = xy, \quad \sum_{J=1}^N \Phi_J(\mathbf{x}) y_J^2 = y^2,$$

for a quadratic basis. From Eq. (13), we have

$$(15) \quad \sum_{J=1}^N [\Phi_J(\mathbf{x})]_{,i} = 0, \quad i = 1, 2, \dots, l,$$

where l is the spatial dimension of the problem.

The shape functions of a linear basis in 2-D can be written out explicitly:

$$\begin{aligned} \mathbf{U}(\mathbf{x}) &= \begin{bmatrix} \sum_{I=1}^N w_I(\mathbf{x}) & & \text{symmetric} \\ \sum_{I=1}^N w_I(\mathbf{x})x_I & \sum_{I=1}^N w_I(\mathbf{x})x_I^2 & \\ \sum_{I=1}^N w_I(\mathbf{x})y_I & \sum_{I=1}^N w_I(\mathbf{x})x_Iy_I & \sum_{I=1}^N w_I(\mathbf{x})y_I^2 \end{bmatrix} \\ &= \begin{bmatrix} u_{11}(\mathbf{x}) & u_{12}(\mathbf{x}) & u_{13}(\mathbf{x}) \\ u_{12}(\mathbf{x}) & u_{22}(\mathbf{x}) & u_{23}(\mathbf{x}) \\ u_{13}(\mathbf{x}) & u_{23}(\mathbf{x}) & u_{33}(\mathbf{x}) \end{bmatrix}, \\ \mathbf{V}(\mathbf{x}) &= \begin{bmatrix} w_1(\mathbf{x}) & w_2(\mathbf{x}) & \cdots & w_N(\mathbf{x}) \\ x_1w_1(\mathbf{x}) & x_2w_2(\mathbf{x}) & \cdots & x_Nw_N(\mathbf{x}) \\ y_1w_1(\mathbf{x}) & y_2w_2(\mathbf{x}) & \cdots & y_Nw_N(\mathbf{x}) \end{bmatrix}. \end{aligned}$$

The inverse of \mathbf{U} is simply

$$\begin{aligned} \mathbf{U}^{-1}(\mathbf{x}) &= \frac{1}{\det(\mathbf{U})} \begin{bmatrix} u_{22}u_{33} - u_{23}^2 & & \text{symmetric} \\ u_{13}u_{23} - u_{12}u_{33} & u_{11}u_{33} - u_{13}^2 & \\ u_{12}u_{23} - u_{13}u_{22} & u_{12}u_{13} - u_{11}u_{23} & u_{11}u_{22} - u_{12}^2 \end{bmatrix} \\ &= \begin{bmatrix} c_{11}(\mathbf{x}) & c_{12}(\mathbf{x}) & c_{13}(\mathbf{x}) \\ c_{12}(\mathbf{x}) & c_{22}(\mathbf{x}) & c_{23}(\mathbf{x}) \\ c_{13}(\mathbf{x}) & c_{23}(\mathbf{x}) & c_{33}(\mathbf{x}) \end{bmatrix}, \end{aligned}$$

with

$$\det(\mathbf{U}) = u_{11}u_{22}u_{33} + 2u_{12}u_{13}u_{23} - u_{11}u_{23}^2 - u_{22}u_{13}^2 - u_{33}u_{12}^2$$

being the determinant of \mathbf{U} . Thus $c_{ij}(\mathbf{x})$ are rational functions and so is

$$\Phi_J(\mathbf{x}) = w_J(\mathbf{x})[c_{11} + x_Jc_{12} + c_{13}y_J + (c_{12} + x_Jc_{22} + c_{23}y_J)x + (c_{13} + x_Jc_{23} + c_{33}y_J)y]$$

if all $w_J(\mathbf{x}), J = 1, \dots, N$ are. It is straightforward to find $\Phi_J(\mathbf{x})$ for quadratic basis and for 3-dimensional cases. Details are left for the readers.

Atluri and Zhu (1998) suggested using of Gaussian or Spline functions with compact support as the weight functions in Eq. (4). The Gaussian weight function is defined as

$$(16) \quad w_J(\mathbf{x}) = \begin{cases} \frac{\exp[-(d_J/c_J)^{2k}] - \exp[-(r_J/c_J)^{2k}]}{1 - \exp[-(r_J/c_J)^{2k}]} & 0 \leq d_J \leq r_J \\ 0, & d_J > r_J \end{cases}$$

where $d_J = |\mathbf{x} - \mathbf{x}_J|$, k is a positive integer and c_J a constant controlling the relative weight of $w_J(\mathbf{x})$. The support of $w_J(\mathbf{x})$ is an l -dimensional sphere of radius r_J centered at \mathbf{x}_J . In practice one often chooses $k = 1$. Clearly the Gaussian type of weight function is only C^0 continuous over Ω . The shape and the trial functions are thus C^0 only, even though $\mathbf{p}(\mathbf{x})$ is C^∞ . The constants c_J affect the value of the shape functions. Belytschko *et al.* (1994) recommended a method of choosing c_J . However, the optimum choice of c_J still needs to be explored.

The Spline type of weight function can be written in the form

$$(17) \quad w_J(\mathbf{x}) = \begin{cases} 1 + (d_J/r_J)^{t_0} \sum_{i=1}^{t_1+1} a_i (d_J/r_J)^i, & 0 \leq d_J \leq r_J \\ 0, & d_J > r_J \end{cases}$$

in which $p = t_0 + t_1 + 1$ is the order of Spline function, t_0 and t_1 are the order of continuity of $w_J(\mathbf{x})$ for $d_J/r_J = 0$ and 1, respectively, and a_i are constants chosen to satisfy

$$(18) \quad [d^i w_J(\mathbf{x})/d(d_J)^i] \Big|_{d_J=r_J} = 0, \quad i = 1, 2, \dots, t_1.$$

The weight function $\omega_J(\mathbf{x})$ as defined in Eq. (17) is C^{t_0} continuous at $d_J = 0$ and C^{t_1} continuous at $d_J = r_J$.

Since the Spline function is a polynomial, the corresponding nodal shape functions in Eq. (12) as well as the trial functions in Eq. (11) are rational. Spline functions of the first 4 orders are given below:

$$\begin{aligned} a_1 &= -1, & \text{for } p = 1 \quad (t_0 = 0, t_1 = 0) \\ a_1 &= -2, \quad a_2 = 1, & \text{for } p = 2 \quad (t_0 = 0, t_1 = 1) \\ a_1 &= -3, \quad a_2 = 2, & \text{for } p = 3 \quad (t_0 = 1, t_1 = 1) \\ a_1 &= -6, \quad a_2 = 8, \quad a_3 = -3 & \text{for } p = 4 \quad (t_0 = 1, t_1 = 2) \end{aligned}$$

According to Atluri *et al.* (1999), the 4th order Spline weight function is preferred for problems in linear elasticity governed by 2nd order partial differential equations. Numerical examples showed that the 4th order function gives smoother first derivatives of the shape functions. This is important as the first derivatives are used to calculate energy. The discontinuities or kinks in the derivatives also make it difficult to accurately integrate the weak form numerically. The 4th order Spline function is C^1 continuous over Ω and thus are the corresponding shape and trial functions.

Even though the shape function $\Phi_J(\mathbf{x})$ associated with Spline weight functions is a rational function, its exact form depends on the location of \mathbf{x} . This is because, as pointed out before, different weight functions are involved in $\Phi_J(\mathbf{x})$ in different parts of Ω_J . For instance, in Fig. 22.2:2, if $\mathbf{x} \in \Omega_1 \cap \Omega_2 \cap \Omega_4$ and is not in any supports of other nodes, only the weight functions $w_1(\mathbf{x})$, $w_2(\mathbf{x})$ and $w_4(\mathbf{x})$ contribute to $\Phi_1(\mathbf{x})$, $\Phi_2(\mathbf{x})$ and $\Phi_4(\mathbf{x})$. In other words, at any \mathbf{x} , only the weight function $\omega_J(\mathbf{x})$ with $\Omega_J \cap \Omega \neq \emptyset$ contributes to $\Phi_J(\mathbf{x})$ for $\mathbf{x}_I, \mathbf{x}_J \in \Omega \cup \partial\Omega$.

The sizes of supports of the weight functions strongly affect the shape functions as the sizes determine the domain of definition of points in Ω . The weight function supports should be large enough to make the domain of definition of any point cover enough number of nodes ($N \geq m$) whose weight functions do not vanish at the point. This is to ensure that the rank of \mathbf{U} equals m . Otherwise \mathbf{U} will be singular and cannot be inverted. The shape function will not be definable. On the other hand, the domain of definition should be small enough to maintain the local character of the moving least square approximation. However, a very small weight function support may result in a relatively large numerical error in calculating the entries of the system matrix by Gauss numerical quadrature.

In the moving least square approximation, the order of continuity of the shape and trial functions equals the minimum of that of the basis function $\mathbf{p}(\mathbf{x})$ and those of the weight functions involved. Since $\mathbf{p}(\mathbf{x})$ is often C^∞ , the smoothness of the shape and the trial functions is usually the same as that of the least smooth weight function involved. Thus one can achieve a higher order of continuity for the shape and the trial functions by using smoother weight functions.

One can generalize the moving least square interpolation to include in the L_2 norm in Eq. (4) the information on the derivatives and the values of the function at scattered points for better approximation (Atluri *et al.* 1999).

Problem 22.2. In evaluating the weak forms in Eq. (4) and (5), it is necessary to compute the partial derivatives of the trial functions and hence those of \mathbf{U}^{-1} . Show that

$$(\mathbf{U}^{-1})_{,k} = -\mathbf{U}^{-1}\mathbf{U}_{,k}\mathbf{U}^{-1}, \quad (\mathbf{U}^{-1})_{,kj} = \mathbf{U}^{-1}[\mathbf{U}_{,j}\mathbf{U}^{-1}\mathbf{U}_{,k} + \mathbf{U}_{,k}\mathbf{U}^{-1}\mathbf{U}_{,j} - \mathbf{U}_{,kj}]\mathbf{U}^{-1}.$$

(Hint: $\mathbf{U}^{-1}\mathbf{U} = \mathbf{I}$, where \mathbf{I} is the identity matrix.)

Problem 22.3. Derive the 5th through 7th Spline functions with $t_0 = 2$ and 3 [the order of continuity at $d_J = 0$, see Eq. (2.17)].

22.2.2. Shepard functions

Shepard (1968) used inverse distance weighting to fit $\tilde{\phi}_J$ in Ω_s in the form

$$(19) \quad \phi^h(\mathbf{x}) = \sum_{J=1}^N \chi_J(\mathbf{x}) \tilde{\phi}_J \quad \mathbf{x} \in \Omega_s,$$

where N is the number of nodes in the sub-domain Ω_s and $\chi_J(\mathbf{x})$ is the shape function associated with node J defined as

$$(20) \quad \chi_J(\mathbf{x}) = w_J(\mathbf{x}) / \sum_{J=1}^N w_J(\mathbf{x})$$

with $\omega_J(\mathbf{x})$ being a weight function associated with node J in the form

$$w_J(\mathbf{x}) = d_J^{-\mu} = |\mathbf{x} - \mathbf{x}_J|^{-\mu}.$$

Typically $\mu = 2$, $\chi_J(\mathbf{x}_J) = \delta_{JJ}$ and thus $\phi^h(\mathbf{x}_J) = \phi_J$.

The Shepard function of Eq. (19) with the weight function defined above is global ($\neq 0$ anywhere in Ω). Franke (1982) introduced the weight function

$$(21) \quad w_J(\mathbf{x}) = \begin{cases} (r_J - d_J)^2 / (r_J d_J)^2, & 0 \leq d_J \leq r_J \\ 0, & d_J > r_J \end{cases}$$

to make the shape function local with the l -dimensional spherical support of radius r_J centered at \mathbf{x}_J . The sub-domain Ω_s is then the domain of definition of the point at \mathbf{x} and denoted by Ω_x . The weight function as defined is C^0 continuous at $d_J = 0$ and C^1 at $d_J = r_J$. The Shepard shape function has a simpler structure than that of the moving least square ones but is only C^0 as opposed to C^1 for the shape function discussed before.

From Eq. (19), obviously

$$(22) \quad \sum_{J=1}^N \chi_J(\mathbf{x}) = 1 \quad \text{and} \quad \sum_{J=1}^N [\chi_J(\mathbf{x})]_{,i} = 0, \quad i = 1, 2, \dots, l.$$

Thus the Shepard function is only complete to the zeroth order, i.e., only able to represent exactly a constant, while the order of completeness of $\Phi_J(\mathbf{x})$ of the moving least square method is the same as that of the basis of the function $\mathbf{p}(\mathbf{x})$ and is usually 1 or higher [see Eqs. (2) and (3)].

There are other approaches to construct the shape functions, such as the partition of unity methods (Babuska and Melenk 1997), the reproducing kernel particle interpolation (Liu *et al.* 1996), compactly supported radial basis function (Wendland 1995), etc. Readers are referred to literature for details.

22.2.3. Interpolation errors

Various interpolation approaches produce small oscillations or indentations and unwanted waves in the first derivative of the interpolated functions (Atluri *et al.* 1999). The derivatives are important, because they appear in the weak form. Smaller supports produce larger oscillation. Oscillation makes accurate numerical integration difficult.

The rates of convergence of the interpolations depend upon the nodal distances and the support sizes of the trial functions. Let us define the error norm ε_1 and the L^2 -based H_k Sobolev norm as

$$(23) \quad \varepsilon_1 = \|\phi^h - \phi\|_1 / \|\phi\|_1,$$

$$(24) \quad \|\phi\|_k = (\sum_{\alpha=0}^k \int_{\Omega} |\partial^{\alpha} \phi / \partial \mathbf{x}^{\alpha}|^2 d\Omega)^{1/2}.$$

The H_1 -norm is associated with energy and called the energy norm. The interpolation errors converge more or less linearly in logarithmic scale as the nodal distance h decreases or the nodal support r increases (Fig. 22.2-3). One usually obtains better solution with larger support domain.

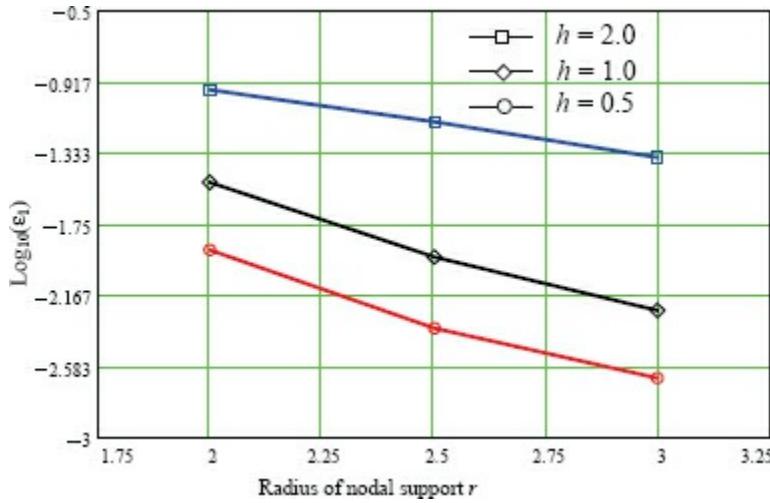


Fig. 22.2:3. Convergence of interpolation error (energy norm) vs. the size of nodal support for different nodal distances.

Problem 22.4. Examine numerically the errors [Eq. (24)] of the linear basis moving least square method with the 4th order Spline weight function and the Shepard method with Franke's weight function to represent the function

$$\phi(x, y) = x^3 y^3 \sin[3\pi(x - 0.1)] \cos[3\pi(y - 0.1)]$$

over the rectangular domain of $-4 \leq x, y \leq 4$. Use evenly distributed nodes of spacing h and consider two radial sizes of $r_j = h$ and $1.5h$ for the circular supports of all nodes.

Insights can be gained by examining the cluster of nodes around node 1 with uniform nodal distance of h as shown in Fig. 22.2:4. The size of nodal supports influences the values of the shape functions. Consider 3 circular subdomains, Ω_{1a} , Ω_{1b} and Ω_{1c} of radii h , $1.25h$ and $1.5h$, respectively. In the case the radius of support of all nodes is 1, the domain of influence for any point in Ω_{1a} involves the weight function associated with nodes 1 through 9, which contribute to the values of the shape functions 1 through 9. For example, at x_a , lies in the intersection of the supports of nodes 1, 4, 5 and 8. Only the weight functions of these 4 nodes affect the values of the corresponding shape functions. If the radii of the nodal supports are $1.25h$ such as Ω_{1b} for node 1, the size of the intersection involves supports of nodes 1, 4, 5 and 8 will increase. The values of the shape functions at points in the intersection outside the original one changes. Supports of additional nodes overlap Ω_{1b} also affect the values of the shape functions of nodes involved in the new intersection. For example, if the radius of the support of node 12 is also $1.25h$, we have Ω_{12} and Ω_{1b} overlap ($\Omega_{12} \cap \Omega_{1b} \neq \emptyset$). At a point x in $\Omega_{12} \cap \Omega_{1b}$, the weight function $w_{12}(x)$ will affect the value of $\Phi_1(x)$ and vice versa. In contrast to the previous case, whereas the radii of the supports of node 1 and 12 are 1, $w_{12}(x)$ does not affect $\Phi_1(x)$ at all and vice versa. We will see later in more details that this will increase the non-zero entries of the global stiffness matrix. If the nodal supports are $1.5h$ for all the nodes or some of the nodes, such as Ω_{1c} for node 1, there will be additional changes in the support intersections and even more new nodal support intersections. More nodal weight functions will be involved in the shape functions of nodes. In other words, many nodes will interact with larger number of nodes. This increases more non-zero entries of the global stiffness matrix and reduces the local characteristics of the shape functions.

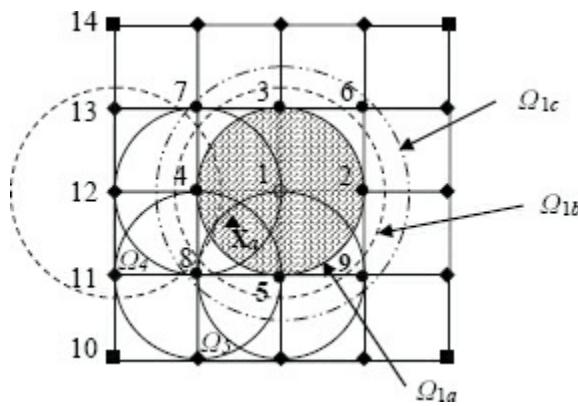


Fig. 22.2:4. Cluster of nodes with different sub-domains Ω_s .

22.2.4. Summary

For most meshless methods there is no need for any specific mesh or fixed relation among nodes and trial functions. The only requirement is that all nodes are in the closed domain Ω . Interpolation errors reduce with nodal distance. The

nodal supports should be large enough to reduce oscillatory errors but small enough to retain the local characteristics of shape functions. A higher order continuous trial function can be constructed easily using weight functions of higher continuity order. This does not increase the non-zero entries of the global stiffness matrix as the number of nodes involved with a particular trial function remains the same as the continuity of weight functions changes. However, change in the size of nodal support affects the bandwidths of the global stiffness matrix. Trial functions with higher order continuity are needed for higher order differential equations. In elasticity, higher order continuity in the trial functions can provide better (smooth) approximation for stresses and strains. But high order of continuity can be a disadvantage for problems with discontinuities.

Nodal shape functions are generally in a rational form and highly complex. This makes integration of the weak form difficult. Without the Kronecker delta property in the shape functions of the moving least square method makes harder to satisfy the rigid boundary conditions exactly. One way to overcome the difficulty is to use penalty function as given in Eqs. (4) to (6). Other ways of dealing with rigid boundary conditions in an approximate manner will be discussed in a later section.

22.3. DOMAIN DISCRETIZATION

In meshless methods, it is not necessary to choose the trial and the test functions from the same function-space or have the same supports. Different choices of these functions lead to different meshless methods. One advantage of the meshless methods over the conventional FEMs is that there is no assembly process to form the global stiffness and force matrices.

Accurate numerical integration of the weak forms is essential, but often difficult as the trial functions are often complex and rapidly varying (especially the derivatives). To make the integration simpler, one chooses the supports of regular-shape such as l -dimensional spheres, ellipsoids or parallelepipeds, and chooses local test functions to minimize domain integral in the weak form. Atluri (2004) suggested a variety of choices of test functions in a given Ω_s :

- (i) an element from the space of weight functions;
- (ii) the Dirac delta function;
- (iii) the Heaviside step function over a local sub-domain Ω_{te} , and
- (iv) a modified fundamental solution to the differential equation.

We shall solve Poisson's equation (22.1:1) with boundary conditions (22.1:2) and (22.1:3) to illustrate the features of different test functions.

22.3.1. Weight functions as test functions

From the weak form in Eq. (22.1:5), we obtain the discrete equations for unknowns $\tilde{\phi}_s$ using the 4th order Spline weight function of Eq. (22.2:17) as the local test function in the moving least square method and the weight function of Eq. (22.2:21) of Shepard approach. Equation (22.1:5) reduces to

$$(1) \int_{\Omega_I^{te} \cap \Omega} \phi(x), \alpha w_{I,\alpha}^{te}(x) d\Omega - \int_{\Gamma_{I\phi}} \phi, n(x) w_I^{te}(x) d\Gamma + \int_{\Gamma_{I\phi}} \gamma \phi(x) w_{I,n}^{te}(x) d\Gamma \\ = \int_{\Gamma_{I\sigma}} \bar{\phi}_n(x) w_I^{te}(x) d\Gamma + \int_{\Gamma_{I\phi}} \gamma \bar{\phi}(x) w_{I,n}^{te}(x) d\Gamma - \int_{\Omega_I^{te} \cap \Omega} p(x) w_I^{te}(x) d\Omega,$$

where $w_{I,n}^{te} = \partial w_I^{te} / \partial n$, $I = 1, \dots, M$ with M being the total number of nodes in the closed domain Ω , Ω_I^{te} is the support of $\omega_I(x)$ for the test function, and $\Gamma_{I\phi} [= \Gamma_\phi \cap (\Omega_I^{te} \cup \partial\Omega_I^{te})]$ and $\Gamma_{I\sigma} [= \Gamma_\sigma \cap (\Omega_I^{te} \cup \partial\Omega_I^{te})]$ lie inside Ω_I^{te} or on the boundary $\partial\Omega_I^{te}$. The weight functions for test functions may be different from those for trial functions and $\omega_{I,n}$ can differ from ω_I . The superscripts "te" and "tr" denote quantities associated with test and trial functions, respectively. In practice, one chooses the same type of weight functions for the test and the trial functions but with different support sizes. When the weigh functions and their supports for test and trial are the same, we will drop the superscripts.

We use ϕ^h of Eq. (22.2:10) as the trial function of the moving least square scheme. We substitute Eq. (22.2:10) into Eq. (1) and yield the following discretized system of linear algebraic equations:

$$(2) \quad \mathbf{K}\tilde{\phi} = (K_{IJ}\tilde{\phi}_J) = \mathbf{f} = (f_I),$$

in which $\tilde{\phi}^T = [\tilde{\phi}_1 \quad \tilde{\phi}_2 \quad \dots \quad \tilde{\phi}_M]$ is the fictitious nodal vector, \mathbf{K} is the global stiffness matrix and \mathbf{f} is the force vector with

$$(3) \quad K_{IJ} = \int_{\Omega_{IJ}} [\Phi_J^{tr}(\mathbf{x})]_{,\alpha} [w_I^{te}(\mathbf{x})]_{,\alpha} d\Omega - \int_{\Gamma_{IJ\phi}} [\partial\Phi_J^{tr}(\mathbf{x})/\partial n] w_I^{te}(\mathbf{x}) d\Gamma \\ + \gamma \int_{\Gamma_{IJ\phi}} \Phi_J^{tr}(\mathbf{x}) w_{I,n}^{te}(\mathbf{x}) d\Gamma,$$

$$(4) \quad f_I = \int_{\Gamma_{I\sigma}} \bar{\phi}_n(\mathbf{x}) w_I^{te}(\mathbf{x}) d\Gamma - \int_{\Omega \cap \Omega_I^{te}} p(\mathbf{x}) w_I^{te}(\mathbf{x}) d\Omega + \gamma \int_{\Gamma_{I\phi}} \bar{\phi}(\mathbf{x}) w_{I,n}^{te}(\mathbf{x}) d\Gamma,$$

in which $\Omega_{IJ} = \Omega \cap \Omega_I^{te} \cap \Omega_J^{tr}$, $\Gamma_{IJ\phi} = \Gamma_\phi \cap (\Omega_I^{te} \cup \partial\Omega_I^{te}) \cap (\Omega_J^{tr} \cup \partial\Omega_J^{tr})$, $\Gamma_{IJ\sigma} = \Gamma_\sigma \cap (\Omega_I^{te} \cup \partial\Omega_I^{te}) \cap (\Omega_J^{tr} \cup \partial\Omega_J^{tr})$, $I = 1, 2, \dots, M$. The reason for limiting the integration to the domains and boundaries shown above is because $\Phi_J^{tr}(\mathbf{x})$ and $w_I^{te}(\mathbf{x})$ are zero outside their respective supports. Obviously, if Ω_I^{te} or Ω_J^{tr} completely inside Ω , there will be no boundary integrals as indicated.

We define two number sets of sizes N_I and $N_{I\phi}$, respectively, for later use:

$$S_I = \{\text{the nodal numbers of all nodes whose supports overlap with } \Omega \cap \Omega_I^{te}\},$$

$$S_{I\phi} = \{\text{the nodal numbers of all nodes whose supports overlap with } \Gamma_{I\phi} = \Gamma_\phi \cap (\Omega_I^{te} \cup \partial\Omega_I^{te})\}.$$

The elements of the sets are denoted by $S_I(i)$, $i = 1, \dots, N_I$ and $S_{I\phi}(i)$, $i = 1, \dots, N_{I\phi}$.

Let us consider an example as shown in Fig. 22.2:1 with the test and trial functions from the same space. The small open circles are locations of intersections of nodal supports and $\partial\Omega$. For node 3, support Ω_3 overlaps supports of nodes 6, 7 and 8, and the union of all these supports is within Ω . Then

$$S_3 = \{3, 6, 7, 8\}, \quad S_{3\phi} = \Gamma_{3\phi} = \Gamma_{3J\phi} = 0, \quad \text{for all } J.$$

There are 4 nodal numbers in S_3 and zero in $S_{3\phi}$, thus $N_3 = 4$ and $N_{3\phi} = 0$. The elements of $S_I(i)$ takes the values 3, 6, 7 and 8, $i = 1, 2, 3, 4$.

For node 2, Ω_2 overlaps the supports of nodes 9 and 18. We have $S_2 = \{2, 9, 18\}$. The support Ω_2 intersects Γ_ϕ at points a and c , thus $\Gamma_{2\phi} = \widehat{ac}$. The supports Ω_9 intersects Γ_ϕ at b and d . The overlap between $\Gamma_{2\phi}$ and \widehat{bd} is \widehat{bc} , thus $\Gamma_{2,9\phi} = \widehat{bc}$. The arc \widehat{bc} intersects with the supports of nodes 2, 9 and 18, thus $S_{2\phi} = \{2, 9, 18\}$. In this case, we have $S_2 = S_{2\phi}$. One can show that $S_{I\phi} \subseteq S_I$ and $\Gamma_{IJ\phi} = \Gamma_{JI\phi}$.

Problem 22.5. Determine S_9 , $S_{9\phi}$, $\Gamma_{9\phi}$, $\Gamma_{9,2\phi}$ in Fig. 22.2:1. Show that $S_{I\phi} \subseteq S_I$ and $\Gamma_{IJ\phi} = \Gamma_{JI\phi}$ (Ans. $S_9 = \{2, 9, 11, 18\}$, $S_{2\phi} = \{2, 9, 18\}$, $\Gamma_{9\phi} = \widehat{bd}$, $\Gamma_{9,2\phi} = \widehat{bc}$).

Normally many of entries of \mathbf{K} are zero. The schematic of the domain of integration of K_{IJ} is shown in Fig. 22.3:1. K_{IJ} is zero if $\Omega_{IJ} (= \Omega \cap \Omega_I^{te} \cap \Omega_J^{tr}) = 0$. Thus Eq. (2) can be written in the index notation as

$$(5) \quad \sum_{i=1}^{N_I} K_{I,I(i)} \tilde{\phi}_{I(i)} = f_I, \quad I = 1, 2, \dots, M$$

where N_I is the size of the set S_I , which contains the nodal numbers of all nodes whose trial function supports overlap with Ω_I^{te} and $I(i) \in S_I$.

Usually there are not too many nodes in both number sets, i.e., $N_I \ll M$ for all $I = 1, 2, \dots, M$. Thus, the majority of the entries K_{IJ} are zero, if none of the nodal supports cover many nodes. This makes it possible to efficiently solve for $\tilde{\phi}$ numerically. Since usually $K_{IJ} \neq 0$, if $\Omega_I^{te} \cap \Omega_J^{tr} \neq 0$. As the sizes of nodal supports increase, there will be more such node pairs and thus more non-zero K_{IJ} . After one has determined the fictitious vector $\tilde{\phi}$ from Eq. (2), one can use Eq. (22.2:10) to evaluate $\phi(\mathbf{x})$.

In FEMs, the bandwidth of the rows associated with the DOFs of a node, say node I , in the global stiffness matrix equals the number of DOFs per node times the number of nodes in the contiguous elements of node I ; while those in the meshless methods using ϕ^h of Eq. (22.2:10) as the trial function equals the number of DOFs per node times N_h , the

number of nodes whose supports overlap with that of node I .

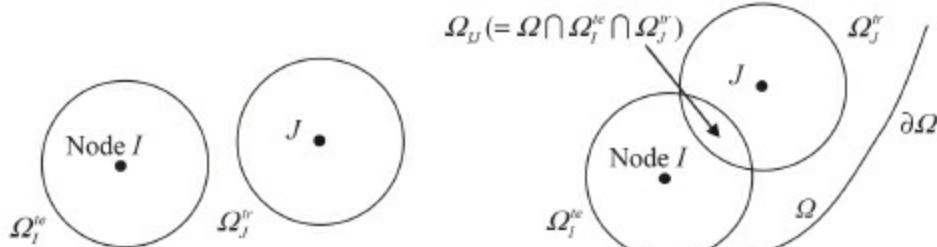
In the Shepard approach, Eq. (22.2:19) is used as the trial function. A substitution into Eq. (22.1:5) leads to the matrix equations in the same form as Eqs. (2) or (5). The vector $\tilde{\phi}$ is the actual nodal vector

$$(6) \quad \tilde{\phi}^T = [\phi^h(x_1) \ \phi^h(x_2) \ \dots \ \phi^h(x_M)].$$

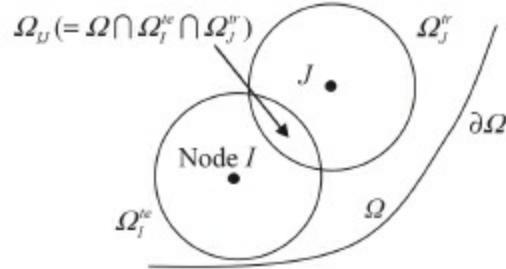
Similar in form as Eq. (3), the entries of stiffness matrix \mathbf{K} now involve the shape functions $\chi_{S_I(i)}(x)$ of Eq. (22.2:20) and the weight functions of Eq. (22.2:21),

$$(7) \quad K_{IJ} = \int_{\Omega \cap \Omega_I^{te} \cap \Omega_J^{tr}} \chi_{I(i),\alpha}^{tr}(x) w_{I,\alpha}^{te}(x) d\Omega - \int_{\Gamma_{I\phi}} \chi_{I(i),n}^{tr}(x) w_I^{te}(x) d\Gamma \\ + \gamma \int_{\Gamma_{I\phi}} \chi_{I(i)}^{tr}(x) w_{I,n}^{te}(x) d\Gamma.$$

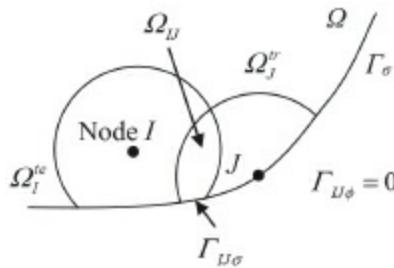
The components of the force vector are in the same form as that in Eq. (4) with the weight function $\omega_J(x)$ defined in Eq. (22.2:21). One can use the same shape functions for $\omega_{J,n}$ as those of ω_J . In both cases, one obtains the entries of the global stiffness matrix directly from Eq. (3) or (7) and those of the force vector from Eq. (4) without an assembly process. Since $\Phi_J^{tr}(x)$ and $\chi_J^{tr}(x)$ differ from the corresponding weight functions $w_J^{tr}(x)$ for the moving least square and Shepard methods, the stiffness matrix \mathbf{K} is asymmetric.



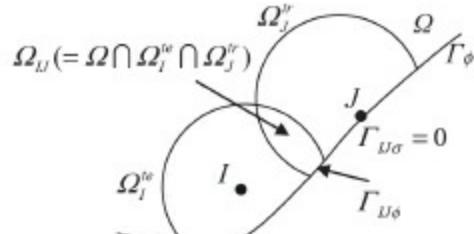
$$(a) \Omega_U = \Omega \cap \Omega_I^{te} \cap \Omega_J^{tr} = 0 \\ K_{IJ} = 0$$



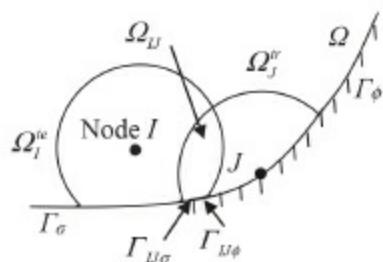
$$(b) \Gamma_{IJ\phi} = \Gamma_{IJ\sigma} = S_{I\phi} = 0, \\ K_{IJ} = \int_{\Omega_U} [\Phi_J(x)]_\alpha [w_I(x)]_\alpha d\Omega, \\ (\Omega_{IJ} = 0, K_{IJ} = 0, \text{ if } J \notin S_I)$$



$$(c) K_{IJ} = \int_{\Omega_U} [\Phi_J(x)]_\alpha [w_I(x)]_\alpha d\Omega \\ \Gamma_{U\phi} = S_{I\phi} = 0, \quad J \in S_I$$



$$(d) \Gamma_{U\phi} \neq 0, S_{I\phi} \neq 0 \\ K_{IJ} \text{ is that in Eq. (3.3) for, } J \in S_I \cup S_{I\phi}$$



$$(e) K_{IJ} \text{ is that in Eq. (3.3),} \\ \Gamma_{IJ\phi} \neq 0, \Gamma_{IJ\sigma} \neq 0, S_{I\phi} \neq 0, \quad J \in S_I \cup S_{I\phi}$$

Fig. 22.3:1. K_{IJ} and its domains of integration.

In the implementation of the present method,

- (i) choose a finite number of M nodes over the closed domain Ω of the given physical problem;
- (ii) choose the basis and the weight functions (their type, order and support radius) to define the shape and test functions (for the Shepard trial function, there is no need to choose the basis function);
- (iii) loop over node I ($I = 1, 2, \dots, M$) inside Ω and on $\partial\Omega$ to carry out the necessary numerical integrations associated with node I ;
 - 1. evaluate the integrals in Eq. (4) using Gaussian quadrature,
 - 2. define S_I as the number set containing the nodal numbers of all nodes with supports overlap that of node I ,
 - 3. loop over node J at \mathbf{x}_J for $J \in S_I$
 - a. loop over the Gaussian quadrature points $\mathbf{x}_q \in \Omega_I^{te} \cap \Omega_J^{tr}$,
 - determine $\Phi_J^{tr}(\mathbf{x}_q)$ [or $\chi_J^{tr}(\mathbf{x}_q)$] and $\omega_I(\mathbf{x}_q)$ and their first partial derivatives at \mathbf{x}_q
 - evaluate numerically the integrals in Eqs. (3) or (7)
 - b. end of quadrature loop
 - 4. end of loop over J ;
- (iv) end of loop over node I ;
- (v) solve the linear equation (2); and
- (vi) calculate the values of the unknown variables ϕ and its derivatives using the trial function Eq. (22.2:10) or (22.2:19).

Problem 22.6. Show that, even though K as defined is not a symmetric matrix, if $\Omega_I^{te} = \Omega_I^{tr}$ and $\Omega_J^{te} = \Omega_J^{tr}$ and if $K_{IJ} = 0$, generally $K_{JI} = 0$. How can one use this information to numerically solve Eq. (5) more efficiently? (Hint: Examine the domains of integration for K_{IJ} and K_{JI} and see the direct methods for numerical solution in Sec. 18.5.)

Problem 22.7. Show that, in general, $\Omega_{IJ} \neq \Omega_{JI}$ if $\Omega_I^{tr} \neq \Omega_I^{te}$ and/or $\Omega_J^{tr} \neq \Omega_J^{te}$.

22.3.2. Dirac delta function as test function

If the Dirac delta function is used as the test function

$$\psi = \psi_{,n} \delta(\mathbf{x} - \mathbf{x}_I), \quad I = 1, 2, \dots, M,$$

where \mathbf{x}_I is the location of node I , Eq. (22.1:4) can be written as

$$(8) \quad \sum_{i=1}^{N_I} [\Phi_{I(i)}^{tr}(\mathbf{x}_I)]_{,kk} \tilde{\phi}_{I(i)} = p(\mathbf{x}_I) \text{ for } \mathbf{x}_I \in \Omega \text{ (} I \text{ not summed),}$$

in which $I(i) \in S_I$ and N_I are defined in the previous subsection with

$$(9) \quad \gamma \sum_{i=1}^{N_{I\phi}} \Phi_{I\phi(i)}^{tr}(\mathbf{x}_I) \tilde{\phi}_{I\phi(i)} = \gamma \bar{\phi}(\mathbf{x}_I) \text{ for } \mathbf{x}_I \in \Gamma_{I\phi} \text{ (} I \text{ not summed),}$$

$$(10) \quad \sum_{i=1}^{N_{I\sigma}} [\partial \Phi_{I\sigma(i)}^{tr}(\mathbf{x}_I) / \partial n] \tilde{\phi}_{I\sigma(i)} = \bar{q}(\mathbf{x}_I) \text{ for } \mathbf{x}_I \in \Gamma_{I\sigma} \text{ (} I \text{ not summed),}$$

where $I_\phi(i)$ and $I_\sigma(i)$ are numbers in sets $S_{I\phi}$ and $S_{I\sigma}$ respectively. Set $S_{I\phi}$ contains the nodal numbers of all nodes whose supports associated with the trial functions overlap $\Gamma_{I\phi}$, i.e., $\Gamma_\phi \cap (\Omega_I^{te} \cup \partial\Omega_I^{te}) \cap (\Omega_{I(i)}^{tr} \cup \partial\Omega_{I(i)}^{tr}) \neq 0$ and $N_{I\phi}$ is the size of $S_{I\phi}$ and $S_{I\sigma}$ and $N_{I\sigma}$ are similarly defined based on the boundary $\Gamma_\sigma \cap (\Omega_I^{te} \cup \partial\Omega_I^{te}) \cap (\Omega_{I(i)}^{tr} \cup \partial\Omega_{I(i)}^{tr}) \neq 0$ (I not summed).

Equations (8) to (10) can be written in the same form as Eq. (5) with

$$(11) \quad K_{I,I(i)} = [\Phi_{I(i)}^{tr}(\mathbf{x}_I)]_{,kk}, \quad f_I = p(\mathbf{x}_I), \quad I(i) \in S_I, \quad \mathbf{x}_I \in \Omega_{I,I(i)},$$

$$(12) \quad K_{I,I\phi(i)} = \gamma \Phi_{I\phi(i)}^{tr}(\mathbf{x}_I), \quad f_I = \gamma \bar{\phi}_n(\mathbf{x}_I), \quad I\phi(i) \in S_{I\phi}, \quad \mathbf{x}_I \in \Gamma_{I\phi},$$

$$(13) \quad K_{I,I\sigma(i)} = \Phi_{I\sigma(i)}^{tr}(\mathbf{x}_I), \quad f_I = \bar{\phi}_n(\mathbf{x}_I), \quad I\sigma(i) \in S_{I\sigma}, \quad \mathbf{x}_I \in \Gamma_{I\sigma} \text{ (} I \text{ not summed).}$$

In the Shepard approach [Eq. (22.2:19)], one simply replaces $\Phi_J(\mathbf{x})$ in Eq. (8) to (13) with $\chi_J(\mathbf{x})$ of Eq. (22.2:20) and replaces $\tilde{\phi}_I$ in Eqs. (2) and (5) with $\phi^h(\mathbf{x})$. No numerical integration is needed in this approach. Since γ can be eliminated in Eq. (2), the penalty parameter plays no role in this approach. However, the calculation of the second derivatives of the shape functions can be time consuming.

22.3.3. Heaviside step function as test function in Ω_I^{te}

We can use the Heaviside function as the test function over the supports of nodes. From Eq. (22.1:4), which admits discontinuous test functions, we obtain

$$(14) \quad \int_{\Gamma_{I\sigma}} \phi_{,n} d\Gamma - \int_{\Omega_I^{te}} \nabla^2 \phi d\Omega + \int_{\Gamma_{I\phi}} \gamma \phi d\Gamma = \int_{\Gamma_{I\sigma}} \bar{\phi}_n d\Gamma - \int_{\Omega \cap \Omega_I^{te}} p d\Omega + \int_{\Gamma_{I\phi}} \gamma \bar{\phi} d\Gamma,$$

where $I = 1, 2, \dots, M$. Integrating the domain integral by part, we obtain

$$(15) \quad - \int_{\partial\Omega - \Gamma_{I\sigma}} \phi_{,n} d\Gamma + \gamma \int_{\Gamma_{I\phi}} \phi d\Gamma = \int_{\Gamma_{I\sigma}} \bar{\phi}_n d\Gamma - \int_{\Omega \cap \Omega_I^{te}} p d\Omega + \gamma \int_{\Gamma_{I\phi}} \bar{\phi} d\Gamma.$$

The change of the integration domain of p is to limit the integration inside Ω . Substituting Eq. (22.2:10) into Eq. (15) yields Eq. (2) with

$$(16) \quad K_{IJ} = - \int_{\partial\Omega - \Gamma_{I\sigma}} [\partial \Phi_J^{tr}(\mathbf{x}) / \partial n] d\Gamma + \gamma \int_{\Gamma_{IJ\phi}} \Phi_J^{tr}(\mathbf{x}) d\Gamma,$$

$$(17) \quad f_I = \int_{\Gamma_{I\sigma}} \bar{\phi}_n(\mathbf{x}) d\Gamma - \int_{\Omega \cap \Omega_I^{te}} p(\mathbf{x}) d\Omega + \gamma \int_{\Gamma_{I\phi}} \bar{\phi}(\mathbf{x}) d\Gamma,$$

where

$$\int_{\Gamma_{IJ\phi}} \Phi_J^{tr}(\mathbf{x}) d\Gamma = 0, \quad \text{if } J \notin \{\text{nodal numbers of nodes whose support overlaps } \Gamma_{I\phi}\},$$

$$K_{IJ} = 0, \quad \text{if } J \notin \{\text{nodal numbers of nodes whose support overlaps } \partial\Omega - \Gamma_{I\sigma}\}.$$

The domain integral is altogether avoided in Eq. (16). This will take less time to compute K_{IJ} . In the Shepard approach one simply replace the shape function Φ by χ .

22.3.4. Shape function as test function

Using $\Phi_J^{te}(\mathbf{x})$ of Eq. (22.2:12) as test functions, we can construct the discretized equations based on Eq. (22.1:5) as $\Phi_J^{te}(\mathbf{x})$ is at least C^0 continuous. One may use shape functions from different spaces. Often one uses the same type of shape functions as trial functions but of different support. For simplicity, we choose $\Phi_I^{te}(\mathbf{x})$ as the shape function for test function $w_{I,n}^{te}(\mathbf{x})$ in the remaining section. The weak form becomes

$$(18) \quad \begin{aligned} & \int_{\Omega \cap \Omega_I^{te}} \phi_{,\alpha} [\Phi_I^{te}(\mathbf{x})]_{,\alpha} d\Omega - \int_{\Gamma_{I\phi}} \phi_{,n} \Phi_I^{te}(\mathbf{x}) d\Gamma + \gamma \int_{\Gamma_{I\phi}} \phi \Phi_I^{te}(\mathbf{x}) d\Gamma \\ &= \int_{\Gamma_{I\sigma}} \bar{\phi}_n \Phi_I^{te}(\mathbf{x}) d\Gamma - \int_{\Omega \cap \Omega_I^{te}} p \Phi_I^{te}(\mathbf{x}) d\Omega + \gamma \int_{\Gamma_{I\phi}} \bar{\phi} \Phi_I^{te}(\mathbf{x}) d\Gamma. \end{aligned}$$

A substitution of Eq. (22.2:10) into Eq. (18) yields Eq. (2) to (5) with $w_I(\mathbf{x})$ and $\omega_{I,n}(\mathbf{x})$ replaced by $\Phi_I^{te}(\mathbf{x})$. Equations (3) and (4) become

$$(19) \quad K_{IJ} = \int_{\Omega_{IJ}} [\Phi_J^{tr}(\mathbf{x})]_{,\alpha} [\Phi_I^{te}(\mathbf{x})]_{,\alpha} d\Omega - \int_{\Gamma_{IJ\phi}} [\partial\Phi_J^{tr}(\mathbf{x})/\partial n] \Phi_I^{te}(\mathbf{x}) d\Gamma \\ + \gamma \int_{\Gamma_{IJ\phi}} \Phi_J^{tr}(\mathbf{x}) \Phi_I^{te}(\mathbf{x}) d\Gamma,$$

$$(20) \quad f_I = \int_{\Gamma_{I\sigma}} \bar{\phi}_n(\mathbf{x}) \Phi_I^{te}(\mathbf{x}) d\Gamma - \int_{\Omega \cap \Omega_I^{te}} p(\mathbf{x}) \Phi_I^{te}(\mathbf{x}) d\Omega + \int_{\Gamma_{I\phi}} \gamma \bar{\phi}(\mathbf{x}) \Phi_I^{te}(\mathbf{x}) d\Gamma.$$

In Eq. (19), if $\Gamma_{IJ\phi} [= \Gamma_\phi \cap (\Omega_I^{te} \cup \partial\Omega_I^{te}) \cap (\Omega_J^{tr} \cup \partial\Omega_J^{tr})] = 0$, $\Omega_I^{te} = \Omega_I^{tr}$ and $\Omega_J^{te} = \Omega_J^{tr}$, we have $K_{IJ} = K_{JI}$. This is usually for nodes away from Γ_ϕ , which is often the case for the majority of the nodes. Thus \mathbf{K} is almost symmetric.

The domain integration for Eq. (19) and (20) is more time consuming than that for Eq. (3) and (4) as the shape functions and their derivatives are more complex than those of the weight functions. However, this may not be a severe disadvantage as, in either case, one must evaluate the shape functions and their derivatives in evaluating \mathbf{K} and \mathbf{f} . Since \mathbf{K} is almost symmetric, one can achieve extensive savings in solving the algebraic equation (2).

In the Shepard approach, one simply replaces Φ 's by the corresponding χ 's in Eq. (19) and (20), and uses Eq. (6) as the unknown vector.

22.3.5. Summary

Using Eq. (22.1:6) as the weak form and the solution of Eq. (22.1:1) as the test function leads to the integral equations for ϕ . One can establish the meshless equation over $\partial\Omega$ (Zhu *et al.* 1998). Different choices of trial and test functions lead to alternate meshless methods. Only a number of cases have been tested partially or extensively and reported in literature (Atluri 2004). We have examined four different types of test functions. The collocation method of using the Dirac delta function in Sec. 22.3.2 does not involve any integration to generate the stiffness matrices and force vectors, thus is simplest to implement. However, the approximate solution is sensitive to the selection of collocation points. The use of shape functions in Secs. 22.3.1 and 22.3.4 for the test (weight) function requires domain integrals. Generally domain integral is tedious due to the complexity of the integrand. The Heaviside step function method in Sec. 22.3.3 involves only regular boundary integrals and thus is an attractive approach among the meshless methods.

Weak form formulations provide a rational basis for the meshless methods with great flexibility. All methods presented satisfy the patch test and thus convergence is assured (Atluri 2004). Due to the many advantages, different meshless methods have been developed for a variety of applications. Interested readers are referred to literature.

22.4. APPROXIMATION IN RIGID BOUNDARY CONDITION

The rigid boundary condition Eq. (22.1:2) can be imposed approximately by collocation without invoking the penalty parameter γ . This can be done in a straightforward manner if the trial functions possess the Kronecker delta properties, like the Shepard function approach. In this case, we simply set

$$(1) \quad \phi^h(\mathbf{x}_L) = \bar{\phi}(\mathbf{x}_L)$$

in Eq. (7) for $\mathbf{x}_L \in \Gamma_\phi$ and drop the terms involving γ in the stiffness matrix and the force vector. Equations (8) and (5) become, respectively,

$$(2) \quad K_{IJ} = \int_{\Omega_{IJ}} [\chi_J^{tr}(\mathbf{x})]_{,\alpha} [w_I^{te}(\mathbf{x})]_{,\alpha} d\Omega - \int_{\Gamma_{I\phi}} [\partial\chi_J^{tr}(\mathbf{x})/\partial n] w_I^{te}(\mathbf{x}) d\Gamma,$$

$$(3) \quad f_I = \int_{\Gamma_{I\sigma}} \bar{\phi}_n(\mathbf{x}) w_I^{te}(\mathbf{x}) d\Gamma - \int_{\Omega \cap \Omega_I^{te}} p(\mathbf{x}) w_I^{te}(\mathbf{x}) d\Omega.$$

Let us define two number sets:

$$S = \{\text{nodal numbers of all nodes over } \Omega \cup \partial\Omega\} = \{1, 2, \dots, M\}$$

$$S_\phi = \{\text{nodal numbers of all nodes on } \Gamma_\phi\}$$

where M_ϕ is the total number of nodes in S_ϕ . Assume that we have labeled the nodes in S_ϕ . We can partition \mathbf{K} , \mathbf{f} and $\tilde{\phi}$ in the form

$$(4) \quad \begin{bmatrix} \mathbf{K}_0 & \mathbf{K}_\phi \\ (M-M_\phi) \times (M-M_\phi) & (M-M_\phi) \times M_\phi \\ \mathbf{K}_1 & \mathbf{K}_2 \\ M_\phi \times (M-M_\phi) & M_\phi \times M_\phi \end{bmatrix} \begin{bmatrix} \tilde{\phi}_0 \\ (M-M_\phi) \times 1 \\ \tilde{\phi}_\phi \\ M_\phi \times 1 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_0 \\ (M-M_\phi) \times 1 \\ \mathbf{f}_\phi \\ M_\phi \times 1 \end{bmatrix},$$

where

$$(5) \quad \begin{bmatrix} \tilde{\phi}_0^T \\ [1 \times (M-M_\phi)] \end{bmatrix} = [\phi^h(\mathbf{x}_1) \quad \phi^h(\mathbf{x}_2) \quad \dots \quad \phi^h(\mathbf{x}_{M-M_\phi})]$$

is the unknown nodal vector of the nodes in $\Omega \cup \Gamma_\sigma$, and

$$(6) \quad \begin{aligned} \begin{bmatrix} \tilde{\phi}_\phi^T \\ (1 \times M_\phi) \end{bmatrix} &= [\phi^h(\mathbf{x}_{M-M_\phi+1}) \quad \phi^h(\mathbf{x}_{M-M_\phi+2}) \quad \dots \quad \phi^h(\mathbf{x}_M)] \\ &= [\bar{\phi}(\mathbf{x}_{M-M_\phi+1}) \quad \bar{\phi}(\mathbf{x}_{M-M_\phi+2}) \quad \dots \quad \bar{\phi}(\mathbf{x}_M)] = \bar{\phi}_\phi^T \end{aligned}$$

is the known nodal vector of nodes on Γ_ϕ . The first $M - M_\phi$ equations of Eq. (4) can be rewritten in the form

$$(7) \quad \mathbf{K}_0 \tilde{\phi}_0 = \mathbf{f}_0^*, \quad \mathbf{f}_0^* = \mathbf{f}_0 - \mathbf{K}_\phi \bar{\phi}_\phi \quad (\phi \text{ not summed}).$$

One obtains the unknown vector $\tilde{\phi}$ by solving Eq. (7). The last M_ϕ equations can be simply ignored.

We have thus avoided evaluating terms involving the penalty parameter γ in Eqs. (22.3:3) and (22.3:4) and reduced the number of equations in Eq. (7). The non-zero entries of \mathbf{K}_0 are fewer too, as K_{IJ} for $J \in S_\phi$, have been moved to the right hand side. This also makes the numerical solution more efficient.

In practice, one can obtain Eq. (7) directly without having to physically reorder the labels of nodes and partition the matrices. The present method of imposing rigid boundary conditions is the same as that in the FEM described in Sec. 18.4 in generating the modified stiffness matrices \mathbf{K}^* and \mathbf{f}^* directly. As K_{IJ} and f_I are being computed, for $I \notin S_\phi$, we first computes $f_I^* = f_I - K_{IJ} \bar{\phi}(\mathbf{x}_J)$ (J not summed), then set $K_{IJ} = 0$ for $J \neq I$ and $f_I^* = K_{II} \bar{\phi}(\mathbf{x}_I)$ (I not summed) for $I \in S_\phi$. Note that \mathbf{K}^* and \mathbf{f}^* are $M \times M$ and $M \times 1$ matrices, respectively and \mathbf{f}^* defers from \mathbf{f}_0^* . Defining \mathbf{K}^* and \mathbf{f}^* in this way assures that, automatically, the solution of

$$(8) \quad \mathbf{K}^* \tilde{\phi} = \mathbf{f}^*$$

satisfies the rigid boundary condition at nodes on Γ_ϕ (Sec. 18.4).

The approximate solution of $\phi(\mathbf{x})$, represented by $\phi^h(\mathbf{x})$ in the form of Eq. (22.2:12) with $\tilde{\phi}$ being the solution of Eq. (8) satisfies the rigid condition only nodes on Γ_ϕ , but may not equal to $\bar{\phi}(\mathbf{x})$ exactly at other locations. As a matter of fact, this is the same as the FEM, in which the prescribed rigid condition is satisfied in the interpolated sense.

If the trial functions are not the Dirac delta functions, the collocation procedure is more complicated. We can partition \mathbf{K} , \mathbf{f} and $\tilde{\phi}$ in the form of Eq. (4) as before. In the present case, \mathbf{K} and \mathbf{f} are based on trial and weight functions from space other than that of the Shepard approach and $\tilde{\phi}_\phi$ is simply

$$\tilde{\phi}_\phi^T = [\phi^h(\mathbf{x}_{M-M_\phi+1}) \quad \phi^h(\mathbf{x}_{M-M_\phi+2}) \quad \dots \quad \phi^h(\mathbf{x}_M)]$$

and is no long equal to known vector $\bar{\phi}_\phi$ as Eq. (6).

Instead of having Eq. (1), for the moving least square scheme, we set

$$(9) \quad \phi^h(\mathbf{x}_L) = \sum_{J=1}^{N_L} \Phi_J^{tr}(\mathbf{x}_L) \tilde{\phi}_J = \bar{\phi}(\mathbf{x}_L), \quad \text{for } \mathbf{x}_L \in \Gamma_\phi,$$

where N_L is the number of nodes in the domain of definition of \mathbf{x}_L . Equation (9) can be written in matrix form as

$$(10) \quad \tilde{\phi}_\phi = \mathbf{A} \tilde{\phi}_0 + \mathbf{B} \tilde{\phi}_\phi$$

where

$$\begin{aligned} A_{IJ} &= \Phi_J^{tr}(\mathbf{x}_I), \quad I = 1, 2, \dots, M_\phi; \quad J = 1, 2, \dots, M - M_\phi, \\ B_{IJ} &= \Phi_J^{tr}(\mathbf{x}_I), \quad I, J - (M - M_\phi) = 1, 2, \dots, M_\phi; \quad \text{i.e., } J \in S_\phi. \end{aligned}$$

Clearly $A_{IJ} = 0$, if $J \notin S_I$, and $B_{IJ} = 0$ if $J \notin S_{I\phi}$, in which S_I is the nodal number set containing all nodes in the domain of definition of node I and $S_{I\phi}$ is the set containing all nodes in the intersection of the closed support of node I and Γ_ϕ . Thus \mathbf{B} is a narrow band matrix and \mathbf{A} is sparse.

From Eq. (10), one obtain

$$(11) \quad \tilde{\phi}_\phi = \mathbf{B}^{-1}(\bar{\phi}_\phi - \mathbf{A}\phi_0).$$

Substituting in Eq. (4) yields

$$(12) \quad (\mathbf{K}_0 - \mathbf{K}_\phi \mathbf{B}^{-1} \mathbf{A})\tilde{\phi}_0 = \mathbf{f}_0^*, \quad \mathbf{f}_0^* = \mathbf{f}_0 - \mathbf{K}_\phi \mathbf{B}^{-1} \bar{\phi}_\phi.$$

In practice, it does not require a major effort to determine $\tilde{\phi}_\phi$ in Eq. (11) since the bandwidth of \mathbf{B} is small. In fact, one can space the nodes on Γ_ϕ in such a way that no supports of two nodes on Γ_ϕ intersect to make \mathbf{B} diagonal. Same as the case that the trial function possess the Dirac delta properties, $\tilde{\phi}_0$ from Eq. (12) satisfies the rigid condition only in the sense of interpolation.

22.5. NUMERICAL INTEGRATION OF THE WEAK FORMS

It is sometimes difficult to integrate the weak forms accurately in meshless methods due to the complexity of the trial and test functions. To obtain the full rates of convergence, the energy in the weak form formulation needs to be computed accurately. We shall present a variety of approaches to improve the accuracy of the numerical integration (Atluri 2004).

The weak forms of using weight function, shape function and Heaviside step function as test functions discussed in Sec. 22.3 require domain and/or boundary integrations. The domain of integration is the intersection of the supports of interior nodes in, near or on the boundary of Ω (Fig. 22.3:1). Figure 22.5:1 shows an example of integration over $\Omega_{IJ} (= \Omega \cap \Omega_I^{te} \cap \Omega_J^{tr})$. A 2×3 Gauss quadrature is used over a rectangle R_{IJ} sufficiently large to enclose Ω_{IJ} ($\Omega_{IJ} \subseteq R_{IJ}$). The solid dots are the integration stations. The integrand is zero outside Ω_{IJ} .

One can improve the integration accuracy using higher order quadrature. But the approach is usually not very effective as Gauss quadrature is designed to integrate relatively smooth functions represented by polynomials. The present integrand usually has a highly complex structure and varies rapidly over the integration domain. A better approach is to subdivide R_{IJ} into smaller sub-domains such as the smaller rectangles marked by 1, 2, 3, ..., etc. in Fig. 22.5:1 and then carry out the numerical integration separately over each of the sub-domains using low order quadratures. One can drop the subdomain entirely if none of the Gauss stations over the sub-domain is in Ω_{IJ} . For example, the sub-domain marked by 3 is likely in this category.

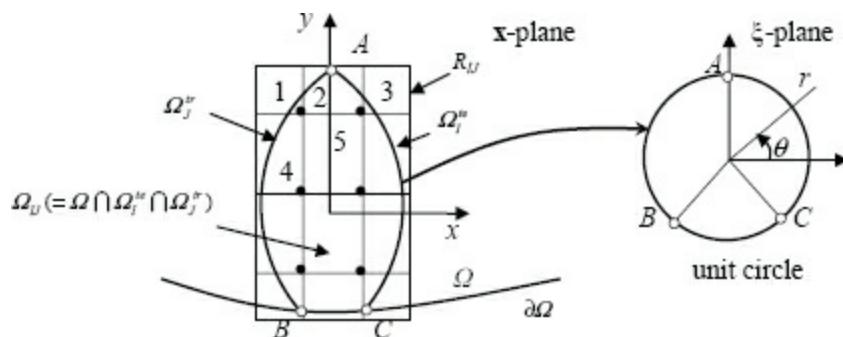


Fig. 22.5:1. Typical domain of integration.

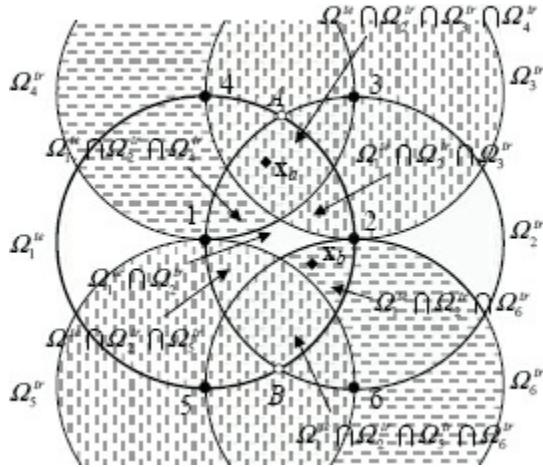


Fig. 22.5.2. Sub-domain within intersection of Ω_1 and Ω_2 .

Atluri (2004) suggested to transform the domain Ω_{IJ} to a unit circle for 2-D and a unit sphere for 3-D problems, then divide the circle or sphere into small regular sub-domains and carry out the numerical integration separately over each of the sub-domains in the mapped space. The transformation is given by

$$\xi = (\mathbf{x} - \mathbf{x}_0)/|\mathbf{x}_1 - \mathbf{x}_0|$$

where ξ is the point in the unit domain corresponding to \mathbf{x} , \mathbf{x}_0 is a nodal position or the center of the intersection, and \mathbf{x}_1 is a point on $\partial\Omega_{IJ}$ (Fig. 22.5.1).

Figure 22.5.2 illustrates the complexity of domain integral over $\Omega_{12} = \widehat{A1B2A} = \Omega_1^{te} \cap \Omega_2^{tr}$, the intersection of Ω_1^{te} and Ω_2^{tr} . The integrand is zero outside the intersection. Within it, there are six sub-domains. In each of them, the integrand involves weight functions associated different nodal supports indicated at the ends of the arrows. For example, at \mathbf{x}_a , which is in the sub domain $\Omega_1^{te} \cap \Omega_2^{tr} \cap \Omega_3^{tr} \cap \Omega_4^{tr}$, the shape function for the integrand involves the weight functions associated with nodes 1, 2, 3 and 4; while at \mathbf{x}_b , the shape function involves the weight functions associated with nodes 1, 2 and 6. Thus at different points, different weight and shape functions and their derivatives are needed to evaluate the integrand.

The followings are two typical integrands in the evaluation of the symmetric weak form in Ω_I^{te} with the shape function as the test function:

$$F_1 = [\partial\Phi_J^{tr}(\mathbf{x})/\partial x][\partial\Phi_I^{te}(\mathbf{x})/\partial x] + [\partial\Phi_J^{tr}(\mathbf{x})/\partial y][\partial\Phi_I^{te}(\mathbf{x})/\partial y],$$

$$F_2 = [\partial\Phi_J^{tr}(\mathbf{x})/\partial x][\partial\Phi_I^{te}(\mathbf{x})/\partial y] + [\partial\Phi_J^{tr}(\mathbf{x})/\partial y][\partial\Phi_I^{te}(\mathbf{x})/\partial x].$$

Problem 22.8. Assume that Ω_{IJ} shown in Fig. 22.5.1 is the intersection of three circles. The two circles of radius 1 are centered at $(\pm 0.7, 0)$ and the circle of radius 10.5 is centered at $(0, 10)$. The vertices of the rectangle R_{IJ} are $(\pm 0.3, 0.714)$ and $(\pm 0.3, -0.5)$. Consider the integral

$$A = \int_{\Omega_{IJ}} F(x, y) d\Omega,$$

where

$$F(x, y) = \begin{cases} (d_I - 1)(d_J - 1)xy \sin[3\pi(x - 0.1)] \cos[3\pi(y - 0.1)] & \text{for } (x, y) \in \Omega_{IJ}, \\ 0 & \text{for } (x, y) \in \Omega_{IJ}, \end{cases}$$

$$d_I = \sqrt{(x + 0.7)^2 + y^2}, \quad d_J = \sqrt{(x - 0.7)^2 + y^2}.$$

Determine A by numerical integration:

- use the 10×10 Gauss quadrature over R_{IJ} ;
- subdivide R_{IJ} into 25 uniform rectangles and use the 2×2 Gauss quadrature over each of the small rectangles;
- transform R_{IJ} to a unit circle by $\xi = \mathbf{x}/|\mathbf{x}_1|$ (in complex variable notation $\xi = re^{i\theta}$);
- determine the Jacobian of the coordinate transformation from (x, y) to (r, θ) of the unit circle;

ii. divide the unit circle into sub-domains of 3 equal divisions in the radial direction and 8 equal ones in the θ -direction; then

iii. use the 2×2 Gauss quadrature over each of the sub-domains; and

d. refine one of the approaches above by employing higher order quadrature and/or smaller sub-domains to assess the accuracy and efficiency of the integration approaches.

Atluri and colleagues (2004) demonstrated the convergence of meshless methods numerically by solving the Laplace equation [Eq. (22.1:1) with $p = 0$]. They found that the MLPGs converge better than the FEMs of the same mesh size. They showed that the exponent R of the convergent rates hR is larger than 2 except for the collocation approach, for which $R = 1.5$, where h is the nodal distance. The FEMs with triangular or rectangular elements have $R = 2$. Generally the moving least square methods converge faster than the Shepard function approach (not shown). The Heaviside function as the test function gives the best results. They also found that numerical integration must be sufficiently accurate to assure convergence and that the optimum support size for the test function is about the nodal distance.

Analysis costs, human and computer, are an important consideration for assessing the utility of a methodology. A major human cost is preparing inputs for analysis, for which the MLPG holds advantage. The major computational costs are generating global stiffness matrix and solving the resulting algebraic equations. The computational time for a single entry of the global stiffness matrix of the MLPGs is higher than that of FEMs, since the shape and test functions of the MLPGs are generally more complex. However, due to the higher rate of convergence, the meshless methods (Atluri 2004) needs much fewer nodes to achieve the same level of accuracy of the FEMs. This provides huge computational savings in solving the resulting algebraic equations, especially for problems of higher order differential equations.

Higher order of continuity can be achieved in the MLPGs by adjusting the order of continuity of the weight functions without increasing the DOF per node, while that for elements of the FEMs will increase rapidly and hence the total number of unknowns even the number of nodes remains constant. The bandwidth of the stiffness matrix will also increase, both of which will increase the computational cost of solving for the unknowns.

22.6. ESHELBY–ATLURI METHODS (EAMs)

This section presents the EAM for finite deformations of solids. The method blends the energy–momentum conservation laws of Noether/Eshelby and the MLPGs of Atluri (Han and Atluri 2014a,b). We shall use the notations of Chapter 13 except that now \mathbf{T} denotes the Eshelby stress tensor and \mathbf{P} the first Piola–Kirchhoff stress tensor, called Lagrangian stress tensor \mathbf{T} before. Using the energy–momentum tensor, Eshelby (1975) defined

$$(1) \quad \mathbf{T} = W\mathbf{I} - \mathbf{P} \cdot \mathbf{F}, \quad T_{IJ} = W\delta_{IJ} - P_{Ik}F_{kJ} = W\delta_{IJ} - P_{Ik}(\partial u_k / \partial X_J + \delta_{kI}),$$

where \mathbf{I} is an identity tensor, \mathbf{F} the deformation gradient, and W the strain energy density per unit initial volume. Both \mathbf{F} and \mathbf{P} are two-point tensors.

22.6.1. Balance laws of Eshelby stress tensor \mathbf{T}

We may write, equivalently, the Eshelby stress tensor as

$$(2) \quad \mathbf{T} = W\mathbf{I} - \mathbf{P} \cdot \mathbf{F} = W\mathbf{I} - \mathbf{S} \cdot \mathbf{F}^T \cdot \mathbf{F} = W\mathbf{I} - \mathbf{S} \cdot \mathbf{C} = W\mathbf{I} - J\mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F},$$

which is in general asymmetric, unless \mathbf{S} and \mathbf{C} are co-axial as the case of isotropic materials. Like $\mathbf{S} \cdot \mathbf{F}^T \cdot \mathbf{F}$, \mathbf{T} is a tensor in the undeformed configuration while $\boldsymbol{\tau}$ and \mathbf{S} are the Cauchy and the second Piola–Kirchhoff stress tensors, respectively. Even for small deformation, \mathbf{T} is quadratic in \mathbf{F} . This poses some difficulties in computation, if \mathbf{T} is used directly as variable. The relations among the stress tensors can be expressed in an alternative form

$$\mathbf{P} = \mathbf{S} \cdot \mathbf{F}^T = J\mathbf{F}^{-1} \cdot \boldsymbol{\tau} = (W\mathbf{I} - \mathbf{T}) \cdot \mathbf{F}^{-1} = \partial W / \partial \mathbf{F}^T.$$

From Eq. (2), with $W_{,J} = W_{,J}|_{\text{exp.}} + P_{Ii}F_{ij,b}$, we have,

$$(3) \quad T_{IJ,I} = W_{,J}|_{\text{exp.}} + P_{Ii}F_{iJ,I} - P_{Ii,IF_{iJ}} - P_{Ii}F_{iJ,I} = W_{,J}|_{\text{exp.}} - P_{Ii,IF_{iJ}} \quad \text{or}$$

$$(4) \quad \partial T_{IJ} / \partial X_I - \rho_0 b_J = -(\partial P_{Ii} / \partial X_I + \rho_0 f_i)F_{iJ} = 0 \quad \text{in } \Omega,$$

where $b_J = W_{,J}|_{\text{exp.}}/\rho_0 + f_iF_{ij}$ is ‘a distributed force’. Equation (4) is the energy–momentum balance law for \mathbf{T} which implies that \mathbf{T} inherently involves the weak-form of the balance law for \mathbf{P} , multiplied by the ‘test function’ \mathbf{F} . We see that for finite deformations of non-isotropic and non-homogeneous hyperelastic solids, the balance laws for \mathbf{T} are a set of linear partial differential equations in the undeformed coordinates \mathbf{X} .

For comparison, recall the balance laws in Newtonian mechanics,

$$\begin{aligned}\partial\tau_{ij}/\partial x_i + \rho f_j &= 0, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^T, \\ \partial P_{Ij}/\partial X_I + \rho_0 f_j &= 0, \quad \mathbf{P} = \partial W/\partial \mathbf{F}^T, \quad \mathbf{F} \cdot \mathbf{P} = \mathbf{P}^T \cdot \mathbf{F}^T, \\ \partial(S_{IK}F_{jI})/\partial X_K + \rho_0 f_j &= 0, \quad \mathbf{S} = 2\partial W/\partial(\mathbf{F}^T \cdot \mathbf{F}), \quad \mathbf{S} = \mathbf{S}^T, \\ \partial P_{Ij}/\partial X_I &= J\partial\tau_{ij}/\partial x_i.\end{aligned}$$

A “traction vector” \mathbf{t}^* at a boundary with a unit normal \mathbf{N} in the initial configuration can be written as:

$$(5) \quad \mathbf{t}^* = \mathbf{N} \cdot \mathbf{T} = WN - \mathbf{N} \cdot \mathbf{P} \cdot \mathbf{F} \quad \text{or} \quad t_J^* = N_K T_{KJ} = WN_J - N_K P_{Ki} F_{iJ}.$$

The integral of \mathbf{t}^* over a small volume Ω_ϵ , which encloses a crack-tip in 2-D, or a crack-segment in 3-D (Fig. 22.6:1), is

$$\begin{aligned}(6) \quad \mathbf{T}^* &= \int_{\partial\Omega_\epsilon} \mathbf{t}^* dS = \int_{\partial\Omega_\epsilon} \mathbf{N} \cdot \mathbf{T} dS = \int_{\partial\Omega} \mathbf{N} \cdot \mathbf{T} dS - \int_{\Omega - \Omega_\epsilon} \nabla_{\mathbf{X}} \cdot \mathbf{T} d\Omega \quad \text{or} \\ T_K^* &= \int_{\partial\Omega_\epsilon} N_I T_{IK} dS = \int_{\partial\Omega} N_I T_{IK} dS - \int_{\Omega - \Omega_\epsilon} T_{IK,I} d\Omega \\ &= \int_{\partial\Omega} (WN_K - N_I P_{Ij} F_{jK}) dS - \int_{\Omega - \Omega_\epsilon} (W\delta_{IK} - P_{Ij} F_{jK})_{,I} d\Omega,\end{aligned}$$

which is a vector of ‘force on the defect’ (Eshelby 1957, 1975) quantifying the singular nature of the stress field in Ω and $\Omega - \Omega_\epsilon$ is free from defects and singularities. We have, from the last term of the above equation,

$$\begin{aligned}(7) \quad \int_{\Omega - \Omega_\epsilon} (W\delta_{IK} - P_{Ij} F_{jK})_{,I} d\Omega &= \int_{\Omega - \Omega_\epsilon} [W_{,K} |_{\text{exp.}} + P_{Ij} F_{jK,I} \\ &\quad - (P_{Ij,I} F_{jK} + P_{Ij} F_{jK,I})] d\Omega = \int_{\Omega - \Omega_\epsilon} (W_{,K} |_{\text{exp.}} + \rho_0 f_j F_{jK}) d\Omega,\end{aligned}$$

in which the following equations have been used:

$$(8) \quad \partial P_{Ik}/\partial X_I + \rho_0 f_k = 0, \quad F_{jK,I} = F_{jI,K}.$$

For an elastostatic defective homogenous solid with no body forces or crack face tractions, we obtain a “path-independent” integral

$$(9) \quad T_K^* = \int_{\partial\Omega_\epsilon} N_I T_{IK} dS = \int_{\partial\Omega_\epsilon} (WN_K - N_I P_{Ij} F_{jK}) dS = \int_{\partial\Omega} (WN_K - N_I P_{Ij} F_{jK}) dS.$$

In this case, Eq. (4) is the energy-momentum balance equation in $\Omega - \Omega_\epsilon$, a domain free from defects, and \mathbf{b} is “the distributed force on the defect”.

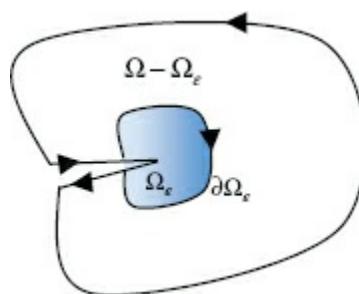


Fig. 22.6:1. A close volume Ω includes a crack tip within Ω_ϵ .

For finite-deformation, hyperelasticity problems of a defective homogenous anisotropic solids, with no body forces or crack face tractions, we can define a “generalized weak-form” of Eq. (4) as

$$(10) \quad \int_{\Omega - \Omega_\epsilon} \partial T_{IJ}/\partial X_I w_{Jd} d\Omega = \int_{\Omega - \Omega_\epsilon} \rho_0 b_J w_{Jd} d\Omega,$$

where w_J are test functions. Since $\Omega - \Omega_\epsilon$ is free of any singularities, an application of the divergence theorem leads

to:

$$(10a) \quad \int_{\partial(\Omega - \Omega_e)} t_J^* w_{Jd} dS - \int_{\Omega - \Omega_e} (T_{IJ} w_{J,I} + \rho_0 b_J w_J) d\Omega = 0.$$

Using different test functions in Eq. (10), one obtains a variety of weak-forms and ‘paths-independent’ representations \mathbf{T}^* (the vectors of ‘concentrated force on the defect’) as defined in Eq. (9). These non-hyper-singular integral equations lead to convenient algorithms for traction boundary value problems in 3-dimensional fracture and fatigue (Han and Atluri 2002, 2014a).

22.6.2. Stress tensors $\tilde{\tau}, \tilde{\mathbf{P}}, \tilde{\mathbf{S}}, \mathbf{T}$

We shall define a set of corresponding stress tensors $\tilde{\tau}, \tilde{\mathbf{P}}, \tilde{\mathbf{S}}, \mathbf{T}$ refer to the deformed configuration. Let $\tilde{\mathbf{u}}$ be ‘the displacement of the deformed body’ with $\mathbf{X} = \mathbf{x} + \tilde{\mathbf{u}}(\mathbf{x})$. Then

$$\tilde{\mathbf{u}}(\mathbf{x}) = -\mathbf{u}(\mathbf{x}) = -\mathbf{u}[\mathbf{X} + \mathbf{u}(\mathbf{X})],$$

where \mathbf{u} is the relative displacement in the undeformed body. We have the deformation gradient of the inverse deformation

$$(11) \quad F_{Ij}^{-1} = \partial X_I / \partial x_j \equiv X_{Ij} = \tilde{u}_{I,j} + \delta_{Ij}$$

as oppose to $F_{jl} (= \partial x_j / \partial X_l \equiv x_{jl} = u_{j,l} + \delta_{jl})$ of the undeformed body.

We define, corresponding to the first Piola–Kirchhoff stress tensor $\mathbf{P} = \partial W / \partial \mathbf{F}^T$ (denoted as \mathbf{T} in Sec. 13.13) in the undeformed configuration,

$$(12) \quad \tilde{\mathbf{P}} = \partial \tilde{W} / \partial \mathbf{F}^{-T}$$

as that in the inverse deformation with

$$\tilde{W}(\mathbf{x}) = W(\mathbf{X})/J$$

being the strain energy per unit *deformed configuration*. Since

$$(13) \quad \frac{\partial W}{\partial F_{Mn}^{-1}} = \frac{\partial J \tilde{W}}{\partial F_{Mn}^{-1}} = J \frac{\partial \tilde{W}}{\partial F_{Mn}^{-1}} + \tilde{W} \frac{\partial J}{\partial F_{Mn}^{-1}} = J \frac{\partial \tilde{W}}{\partial F_{Mn}^{-1}} - \tilde{W} J F_{nM} = J \frac{\partial \tilde{W}}{\partial F_{Mn}^{-1}} - W F_{nM},$$

from Eq. (12) and (13),¹ one obtains the constitutive relation

$$(14) \quad \tilde{\mathbf{P}} = \frac{1}{J} (W \mathbf{F} - \mathbf{F} \cdot \frac{\partial W}{\partial \mathbf{F}^T} \cdot \mathbf{F}) = \frac{1}{J} (W \mathbf{F} - \mathbf{F} \cdot \mathbf{P} \cdot \mathbf{F}) = \frac{1}{J} \mathbf{F} \cdot \mathbf{T}.$$

Note that $\tilde{\mathbf{P}}$ is a two-point tensor (with the first leg in the current and the second in the undeformed configuration).

Since $\mathbf{F}^{-1}\mathbf{F} = \mathbf{I}$, we have

$$(\partial F_{Mm}^{-1} / \partial F_{jI}) F_{mI} + F_{Mj}^{-1} \delta_{JI} = 0, \quad \partial F_{Mn}^{-1} / \partial F_{jI} = -F_{Mj}^{-1} F_{In}^{-1}.$$

From Eq. (13.12:3), it can be shown that

$$(15) \quad P_{Ij} = \partial W / \partial F_{jI} = (\partial W / \partial F_{Mn}^{-1}) (\partial F_{Mn}^{-1} / \partial F_{jI}), \quad \text{or} \\ \mathbf{P} = \partial W / \partial \mathbf{F}^T = -\mathbf{F}^{-1} \cdot (\partial W / \partial \mathbf{F}^{-T}) \cdot \mathbf{F}^{-1}.$$

¹Note that $\tilde{P}_{nM} = \frac{\partial \tilde{W}}{\partial F_{Mn}^{-1}} = \frac{1}{J} (WF_{nM} + \frac{\partial W}{\partial F_{Mn}^{-1}}) = \frac{1}{J} (WF_{nM} - F_{nI} \cdot \frac{\partial W}{\partial F_{jI}} \cdot F_{jM}).$

Then, from Eq. (15), (12) and (14), we have

$$(16) \quad \tilde{\mathbf{P}} \cdot \mathbf{F}^{-1} = (W \mathbf{I} - \mathbf{F} \cdot \mathbf{P}) / J = \tilde{W} \mathbf{I} - \boldsymbol{\tau} = \mathbf{F} \cdot \mathbf{T} \cdot \mathbf{F}^{-1} / J \equiv \tilde{\mathbf{S}},$$

in which $\tilde{\mathbf{S}}$ is an Eulerian symmetric tensor (and also called “chemical potential tensor”) in the current configuration.

With $\mathbf{F} \cdot \delta\mathbf{X} = \delta\mathbf{x}$ and $\mathbf{J}\mathbf{N}dS = \mathbf{n} \cdot \mathbf{F}ds$ the weak-forms of the path-independent integrals for the Eshelby stress tensor may be written in terms of $\tilde{\mathbf{S}}$ in the current configuration

$$(17) \quad \int_{\delta\Omega} \mathbf{N} \cdot \mathbf{T} \cdot \delta\mathbf{X} dS = \int_{\delta\Omega} \frac{\mathbf{n}}{J} \cdot \mathbf{F} \cdot \mathbf{T} \cdot \mathbf{F}^{-1} \delta\mathbf{x} ds = \int_{\partial\tilde{\Omega}} \mathbf{n} \cdot \tilde{\mathbf{S}} \cdot \delta\mathbf{x} ds.$$

Let $\tilde{\mathbf{B}}$ be the left Cauchy–Green deformation tensor of the inverse deformation defined as

$$(18) \quad \tilde{\mathbf{B}} = \mathbf{F}^{-1} \cdot \mathbf{F}^{-T} = (F_{Im}^{-1} F_{Jm}^{-T}) = \mathbf{C}^{-1}, \text{ then}$$

$$(19) \quad \partial\tilde{\mathbf{B}}/\partial\mathbf{F}^{-T} = \partial(F_{Im}^{-1} F_{Jm}^{-T})/\partial F_{Kj}^{-1} = \delta_{IK} F_{Jj}^{-1} + \delta_{JK} F_{Ij}^{-1}.$$

Equation (12) may be rewritten as

$$\tilde{\mathbf{P}} = \partial\tilde{W}/\partial\mathbf{F}^{-T} = (\partial\tilde{W}/\partial\tilde{\mathbf{B}})(\partial\tilde{\mathbf{B}}/\partial\mathbf{F}^{-T}) = 2\mathbf{F}^{-1} \cdot (\partial\tilde{W}/\partial\tilde{\mathbf{B}}).$$

Multiplying the equation above with $\mathbf{I} (= \mathbf{FT} \cdot \mathbf{F}^{-T})$ gives

$$(20) \quad \tilde{\mathbf{P}} = \partial\tilde{W}/\partial\mathbf{F}^{-T} = 2\mathbf{F}^{-1} \cdot (\partial\tilde{W}/\partial\tilde{\mathbf{B}}) \cdot \mathbf{F}^T \cdot \mathbf{F}^{-T} = \tilde{\tau} \cdot \mathbf{F}^{-T},$$

where $\tilde{\tau}$ is a function of $\tilde{\mathbf{B}}$ in the current configuration defined as

$$(21) \quad \tilde{\tau} = 2\mathbf{F}^{-1} \cdot (\partial\tilde{W}/\partial\tilde{\mathbf{B}}) \cdot \mathbf{F}^T = \tilde{\mathbf{P}} \cdot \mathbf{F}^T.$$

From Eq. (16) and (20), one has,

$$(22) \quad \tilde{\tau} = \tilde{\mathbf{S}} \cdot (\mathbf{F} \cdot \mathbf{F}^T) = \tilde{\mathbf{S}} \cdot \mathbf{B} \quad \text{or} \quad \tilde{\mathbf{S}} = \tilde{\tau} \cdot (\mathbf{F}^{-T} \cdot \mathbf{F}^{-1}),$$

where $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T$ is the left Cauchy–Green deformation tensor. The stress tensor $\tilde{\tau}$ is not symmetric for general anisotropic materials.

Let $j(\mathbf{x})$ be the Jacobian determinant of the inverse deformation tensor,

$$(23) \quad j(\mathbf{x}) = |\mathbf{F}^{-1}(\mathbf{x})| = 1/J(\mathbf{X}).$$

By their definitions, we have the relations between the various stress tensors:

$$(24a) \quad \tilde{\mathbf{P}} = j(\mathbf{x})\mathbf{F} \cdot \mathbf{T} = \tilde{\tau} \cdot \mathbf{F}^{-T} = \tilde{\mathbf{S}} \cdot \mathbf{F} = \partial\tilde{W}/\partial\mathbf{F}^{-T} = W\mathbf{I} - \mathbf{P} \cdot \mathbf{F},$$

$$(24b) \quad \tilde{\tau} = j(\mathbf{x})\mathbf{F} \cdot \mathbf{T} \cdot \mathbf{F}^T = \tilde{\mathbf{P}} \cdot \mathbf{F}^T = \tilde{\mathbf{S}} \cdot \mathbf{F} \cdot \mathbf{F}^T = (\partial\tilde{W}/\partial\mathbf{F}^{-T}) \cdot \mathbf{F}^T,$$

$$(24c) \quad \mathbf{T} = J(\mathbf{X})\mathbf{F}^{-1} \cdot \tilde{\mathbf{P}} = J\mathbf{F}^{-1} \cdot \tilde{\tau} \cdot \mathbf{F}^{-T} = J\mathbf{F}^{-1} \cdot \tilde{\mathbf{S}} \cdot \mathbf{F},$$

which give the duality between the stress tensors \mathbf{S} , \mathbf{P} , τ and $\tilde{\tau}$, $\tilde{\mathbf{P}}$, \mathbf{T} . The stress tensors \mathbf{S} , \mathbf{P} , τ are commonly used in Newtonian Momentum Balance Laws, while the stress tensors $\tilde{\tau}$, $\tilde{\mathbf{P}}$, \mathbf{T} are used to write the Noether/Eshelby energy–momentum conservation laws. The new stress tensor $\tilde{\tau}$ is analogous to the second Piola–Kirchhoff stress tensor \mathbf{S} .

For isotropic materials, $\tilde{\tau}$ becomes symmetric

$$(25) \quad \tilde{\tau} = 2\partial\tilde{W}/\partial\tilde{\mathbf{C}},$$

and

$$(26) \quad \tilde{\mathbf{P}} = \partial\tilde{W}/\partial\mathbf{F}^{-T} = (\partial\tilde{W}/\partial\tilde{\mathbf{C}})\partial\tilde{\mathbf{C}}/\partial\mathbf{F}^{-T} = 2(\partial\tilde{W}/\partial\tilde{\mathbf{C}}) \cdot \mathbf{F}^{-T}.$$

22.6.3. Noether/Eshelby energy–momentum conservation laws in terms of $\tilde{\tau}$, $\tilde{\mathbf{P}}$, $\tilde{\mathbf{S}}$, \mathbf{T}

We now consider the “configuration invariance of the energy–momentum” type of conservation laws based on Noether’s (1918) Theorem of configuration invariance energy. The strong form balance laws for \mathbf{T} , $\tilde{\mathbf{P}}$, $\tilde{\mathbf{S}}$ and $\tilde{\tau}$ may be derived, following the procedures given in Han and Atluri (2014b), as

$$(27a) \quad T_{IJ,I} - \rho_0 b_J = -(P_{Ik,I} + \rho_0 f_k)F_{kJ} = 0,$$

$$(27b) \quad \tilde{P}_{iJ,i} + \rho b_J = 0,$$

$$(27c) \quad (JF_{Ik}^{-1}\tilde{S}_{kl}F_{lJ}),_I - \rho_0 b_J = 0,$$

$$(27d) \quad (\tilde{\tau}_{ik}F_{jk}^{-1}),_i - \rho b_J = 0,$$

where

$$b_J = W_{,J}|_{\text{exp.}}/\rho_0 + f_k F_{kJ}.$$

Again, the equivalence of Eqs. (27a) and (27b) needs to be guaranteed by the geometric identity for any finite deformation, that:

$$(28) \quad \partial[j(x)\partial x_i/\partial X_K]/\partial x_i = 0.$$

The symmetry of the Eshelby stress tensor with respect to the strain tensor \mathbf{C} is mandatory, as

$$(29) \quad \mathbf{C} \cdot \mathbf{T} = \mathbf{T}^T \cdot \mathbf{C}, \quad \mathbf{F}^T \cdot \tilde{\mathbf{P}} = \tilde{\mathbf{P}}^T \cdot \mathbf{F}, \quad \mathbf{F}^T \cdot \tilde{\boldsymbol{\tau}} \cdot \mathbf{F}^{-T} = \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}, \quad \tilde{\mathbf{S}} = \tilde{\mathbf{S}}^T.$$

22.6.4. Tangential material stiffness coefficients of Noether/Eshelby energy-momentum conservation laws

We now consider the “incremental” variables suitable for computational analyses of quasi-static finite deformations of solids. Let $\Delta\mathbf{u}$ be the incremental displacement of a material particle from the current configuration $C^{(N)}$ to the next configuration $C^{(N+1)}$ corresponding to the configurational change $\tilde{\mathbf{v}}(\mathbf{X})$ in the undeformed configuration $C^{(0)}$ of the solid. Then $\Delta\mathbf{u}$ is related to $\tilde{\mathbf{v}}(\mathbf{X})$ by

$$(30) \quad \Delta\mathbf{u}(\mathbf{x}) = \mathbf{F} \cdot \tilde{\mathbf{v}}(\mathbf{X}) \quad \text{or} \quad \Delta u_i = F_{ij}\tilde{v}_j.$$

Then

$$(31) \quad \Delta L_{ij} = \frac{\partial \Delta u_i}{\partial x_j} = F_{ik}\frac{\partial \tilde{v}_k}{\partial x_j} = F_{ik}\frac{\partial \tilde{v}_k}{\partial X_L}F_{Lj}^{-1} = F_{ik}F_{Lj}^{-1}\Delta L_{KL}^* \quad \text{or}$$

$$\Delta \mathbf{L} = \mathbf{F} \cdot \Delta \mathbf{L}^* \cdot \mathbf{F}^{-1},$$

$$(32) \quad 2\Delta D_{ij} = F_{ik}F_{Lj}^{-1}\Delta L_{KL}^* + F_{jk}F_{Li}^{-1}\Delta L_{KL}^* = (F_{ik}F_{Lj}^{-1} + F_{Li}^{-1}F_{jk})\Delta L_{KL}^*,$$

where $\Delta \mathbf{L}^*$ is the gradient of $\tilde{\mathbf{v}}(\mathbf{X})$

$$(33) \quad \Delta \mathbf{L}^* = \partial \tilde{\mathbf{v}} / \partial \mathbf{X} \quad \text{or} \quad \Delta L_{KL}^* = \partial \tilde{v}_K / \partial X_L.$$

The constitutive relation for the stress increment $\Delta \mathbf{S}$ in the undeformed configuration $C^{(0)}$ is [see Sec. (17.5) and use Eq. (32)]

$$(34) \quad \Delta S_{IJ} = [\partial^2 W / (\partial E_{IJ} \partial E_{KL})] F_{kk} F_{ll} \Delta D_{kl} \equiv C_{IJKL}^* \Delta L_{KL}^*,$$

where

$$(35) \quad C_{IJKL}^* \equiv [\partial^2 W / (\partial E_{IJ} \partial E_{MN})] F_{km} F_{ln} (F_{kk} F_{ll}^{-1} + F_{lk}^{-1} F_{kl}) / 2$$

is the 4th order tangential material stiffness tensor. We designate $\Delta \mathbf{S}$ and $\Delta \mathbf{L}^*$ as a conjugate pair postulating a piecewise linear constitutive relation in the Noether/Eshelby energy-momentum conservation laws.

22.6.5. MLPG weak-forms of energy-momentum conservation laws

We shall employ the configurational change of $C^{(0)}\delta\mathbf{X}$ as the local trial function, and that of $C(N)\delta\mathbf{x}$ as the local test function (Atluri 1998, 2004).

Meshless Trial Functions. We use the moving least square approximation (MLS)² to construct the trial functions $\tilde{\mathbf{v}}(\mathbf{X})$ based on the fictitious nodal value $\tilde{\mathbf{v}}^{(K)}$ [Atluri (2004)]

$$(36) \quad \tilde{\mathbf{v}}^{MLS}(\mathbf{X}) = \sum_{K=1}^n \Phi^{(K)}(\mathbf{X}) \tilde{\mathbf{v}}^{(K)}, \quad \Delta \mathbf{L}^{*MLS}(\mathbf{X}) = \frac{\partial \tilde{\mathbf{v}}^{MLS}(\mathbf{X})}{\partial \mathbf{X}},$$

for $\forall \mathbf{X} \in \Omega(I)$, where the gradient $\Delta \mathbf{L}^*(\mathbf{X})$ is derived from $\tilde{\mathbf{v}}(\mathbf{X})$ through direct differentiation. The continuity of the trial functions $\tilde{\mathbf{v}}^{MLS}$ depends on that of the weight functions $w^{(k)}$ in the moving least square (MLS) interpolation. With the fourth-order spline function as the weight function, the trial function is continuous in the local domain.

Meshless Test Functions. In EAM, the test functions are the configurational changes of the deformed body. They can be chosen independently from the trial functions to make it more suitable for numerical implementation. We consider two types of test functions. First, Heaviside functions are chosen within each local test sub-domain

$$(37) \quad \delta \mathbf{x}^{FVM}(\mathbf{X}) = \delta \mathbf{x}^{(I)} = \text{const.} \quad \text{for } \forall \mathbf{X} \in \Omega^{(I)},$$

in which $\delta \mathbf{x}^{(I)}$ are the variations of the vertex nodes. The constant test functions make the method computationally efficient, because all domain integrals vanish for piecewise linear materials. We call the method the finite volume method.

Alternatively linear shape functions are adopted as the test functions for 2-D triangles or 3-D tetrahedrons,

$$(38) \quad \delta \mathbf{x}^{FEM}(\zeta) = \sum_M \mathbf{N}^{(M)}(\zeta) \delta \mathbf{x}^{(M)}.$$

We call the resultant method the linear test function method.

Local Weak Forms of Energy–Momentum Conservation for Solution Predictor. Following Han and Atluri (2014b) to develop piecewise predictor solutions, we linearize the local weak forms of the balance laws (27) using the tangential material stiffness in an incremental form. We refer the solutions, displacements, deformation gradients, and stresses, in the configuration $C^{(N+1)}$ to the immediately preceding state $C^{(N)}$. The initial configuration $C^{(0)}$ is the reference configuration. The solution variables in $C^{(N)}$ may be the trial solutions of M^{th} iteration, which may not satisfy the balance laws exactly.

Let \mathbf{X} and \mathbf{x} be the coordinates of a particle in the initial and the deformed configurations $C^{(0)}$ and $C^{(N)}$, respectively, $\tilde{\mathbf{v}}(\mathbf{X})$ the trial function of the configurational change of C^0 , and $\Delta \mathbf{u}(\mathbf{x})$ the corresponding displacement change of $C^{(N)}$ as given in Eq. (30). We now solve for the stress increment from $C^{(N)}$ to $C^{(N+1)}$. Taking $\delta \mathbf{X}$ as the test functions, for the variables in $C^{(N+1)}$, we can write the local weak forms of the balance laws Eq. (27) for each local sub-domain $\Omega(I)$ in $C^{(0)}$ as,

$$(39a) \quad \int_{\Omega^{(I)}} T_{IJ,I}^{(N+1)} \delta X_J d\Omega = \int_{\Omega^{(I)}} \rho_0 b_J \delta X_J d\Omega.$$

If all trial variables are continuous within each local sub-domain $\Omega(I)$, an alternate weak form may be written for each local sub-domain from Eq. (27b) as,

$$(39b) \quad \int_{\Omega^{(I)}} P_{Ij,I}^{(N+1)} F_{jK}^{(N+1)} \delta X_K d\Omega + \int_{\Omega^{(I)}} \rho_0 f_j^{(N+1)} F_{jK}^{(N+1)} \delta X_K d\Omega = 0.$$

As the test functions may be assumed independently, one may replace $\delta \mathbf{X}$ with some other test functions, denoted as $\delta \mathbf{x}$, within each sub-domain $\Omega^{(I)}$ as

$$(40) \quad \delta x_j = F_{jK}^{(N+1)} \delta X_K,$$

where $\mathbf{F}^{(N+1)}$ is continuous. Thus the weak form in Eq. (39b) may be re-written as,

$$(39c) \quad \int_{\Omega^{(I)}} P_{Ij,I}^{(N+1)} \delta x_j d\Omega + \int_{\Omega^{(I)}} \rho_0 f_j^{(N+1)} \delta x_j d\Omega = 0.$$

Since $\mathbf{P}(\mathbf{X}) = J \mathbf{F}^{-1} \cdot \boldsymbol{\sigma}$, the above weak form may be also written in terms of the Cauchy stress in $C^{(N+1)}$ that

$$(39d) \quad \int_{\Omega_{N+1}^{(I)}} \tau_{ij,i}^{(N+1)} \delta x_j d\Omega + \int_{\Omega_{N+1}^{(I)}} \rho_N f_j^{(N+1)} \delta x_j d\Omega = 0.$$

As the configuration $C^{(N+1)}$ is not yet known, the solution variables may be linearized from their values in the immediately preceding state $C^{(N)}$ as

$$(39e) \quad \int_{\Omega_N^{(I)}} \{[(S_{(N)}^{(N+1)})_{IJ} (F_{(N)}^{(N+1)})_{jJ}]_{,I} + \rho_N (f_j^{(N)} + \Delta f_j)\} \delta x_j d\Omega = 0,$$

where $(\cdot)_{(N)}^{(N+1)}$ denotes the variable in $C^{(N+1)}$ with $C^{(N)}$ as the reference configuration. The first integrand in Eq. (39e) may be linearized in the incremental form as

$$(41) \quad [(S_{(N)}^{(N+1)})_{IJ}(F_{(N)}^{(N+1)})_{jJ}],_I = \{[\tau_{IJ}^{(N)} + (\Delta S_{(N)}^{(N+1)})_{IJ}][\delta_{jJ} + (\Delta F_{(N)}^{(N+1)})_{jJ}]\},_I \\ = \{[\tau_{IJ}^{(N)} + (\Delta S_{(N)}^{(N+1)})_{IJ}]\delta_{jJ} + \tau_{IJ}^{(N)}(\Delta F_{(N)}^{(N+1)})_{jJ}\},_I .$$

Notice that

$$(42) \quad \tau^{(N+1)} = (\mathbf{F}_{(0)}^{(N+1)}) \cdot (\mathbf{S}_{(0)}^{(N+1)}) \cdot (\mathbf{F}_{(0)}^{(N+1)})^T / J^{(N+1)} \\ = J^{(N)} (\mathbf{F}_{(N)}^{(N+1)}) \cdot (\mathbf{S}_{(N)}^{(N+1)}) \cdot (\mathbf{F}_{(N)}^{(N+1)})^T / J^{(N+1)},$$

$$(43) \quad \mathbf{S}_{(N)}^{(N+1)} = (\mathbf{F}_{(0)}^{(N+1)}) \cdot (\mathbf{S}_{(0)}^{(N+1)}) \cdot (\mathbf{F}_{(0)}^{(N+1)})^T / J^{(N)} .$$

By definition, we have

$$(44) \quad \mathbf{S}_{(N)}^{(N+1)} = \tau^{(N)} + \Delta \mathbf{S}_{(N)}^{(N+1)}, \quad \mathbf{S}_{(0)}^{(N+1)} = \mathbf{S}_{(0)}^{(N)} + \Delta \mathbf{S}_{(0)}, \\ J^{(N)} \tau^{(N)} = \mathbf{F}_{(0)}^{(N)} \cdot \mathbf{S}_{(0)}^{(N)} \cdot \mathbf{F}_{(0)}^{(N)T} .$$

A substitution of Eq. (44) into Eq. (43) yields the tangential material stiffness

$$(45) \quad \Delta \mathbf{S}_{(N)}^{(N+1)} = \mathbf{F}_{(0)}^{(N)} \cdot \Delta \mathbf{S}_{(0)} \cdot \mathbf{F}_{(0)}^{(N)T} / J^{(N)} .$$

With the tangential material stiffness for $\Delta \mathbf{S}(0)$ in C^0 as in Eq. (34), one obtains the spatial tangential material stiffness for $\Delta \mathbf{S}_{(N)}^{(N+1)}$ in $C^{(N)}$ as

$$(46) \quad (\Delta S_{(N)}^{(N+1)})_{ij} = (F_{(0)}^{(N)})_{iM} (F_{(0)}^{(N)})_{jN} C_{MNKL}^* \Delta L_{KL}^{(N)} / J^{(N)} \equiv c_{ijKL}^* \Delta L_{KL}^* .$$

The incremental stress in Eq. (41) can be linearized with respect to $\Delta \mathbf{L}^*$ as

$$(47) \quad (\Delta S_{(N)}^{(N+1)})_{IJ} + \tau_{IM}^{(N)} (\Delta F_{(N)}^{(N+1)})_{JM} = (c_{ijKL}^* + \tau_{IM}^{(N)} F_{jK} F_{LM}^{-1}) \Delta L_{KL}^* \\ \equiv c_{ijKL}^* \Delta L_{KL}^*,$$

with the use of Eqs. (31) and (46). Substituting Eqs. (41) and (47) into Eq. (39e) and applying the divergence theorem to the first integrand of Eq. (39e), one obtains the local weak forms of the energy-momentum conservation laws:

$$(48a) \quad \left(\int_{\partial \Omega_N^{(I)}} n_i c_{ijKL}^* \Delta L_{KL}^* \delta x_j dS - \int_{\Omega_N^{(I)}} c_{ijKL}^* \Delta L_{KL}^* \delta x_{j,i} d\Omega \right) + \int_{\Omega_N^{(I)}} \rho_N \Delta f_j \delta x_j d\Omega \\ + \left(\int_{\partial \Omega_N^{(I)}} n_i \tau_{ij}^{(N)} \delta x_j dS - \int_{\Omega_N^{(I)}} \tau_{ij}^{(N)} \delta x_{j,i} d\Omega + \int_{\Omega_N^{(I)}} \rho_N f_j^{(N)} \delta x_j d\Omega \right) = 0,$$

in which the first term corresponds to the tangential stiffness matrix of the local solid domain, the second term to the load increments, and the last term to the residual forces. All variables are defined in $C^{(N)}$ within a local sub-domain.

If the configurational change $\tilde{\mathbf{v}}$ (local trial functions) in $C^{(0)}$ and the position change $\delta \mathbf{x}$ (local test functions) in $C^{(N)}$ are both continuous between the non-overlapping subdomains, Eq. (48a) is equivalent to the weak form of \mathbf{T} in Eq. (39a) provided that the geometric identity $\partial[j(\mathbf{x})x_{i,K}]_i = 0$ is satisfied. However, it is hard to construct $\tilde{\mathbf{v}}$ and the corresponding $\Delta \mathbf{u}$ that are continuous and satisfy the geometric identity globally. We shall just construct continuous $\tilde{\mathbf{v}}$ that satisfies the geometric identify within each local sub-domain only, and allow it to be discontinuous between the non-overlapping sub-domains. One may also introduce an additional term $\mathbf{A}(\zeta)$ in the integral along the share boundaries of the non-overlapping subdomains in Eq. (48a). The additional term is linear and in the direction normal to the boundary,

$$(49) \quad \mathbf{A}(\zeta) = (1 - \zeta) \mathbf{A}^{(I)} + \zeta \mathbf{A}^{(J)},$$

where $\partial\Omega^{(I)}$ is the boundary-segment between the non-overlapping local subdomains $\Omega^{(I)}$ and $\Omega^{(J)}$ with the parameter ζ . One may write the additional terms separately for the preceding solution $\mathbf{u}(\mathbf{X})$ in $C^{(N)}$ and the solution $\tilde{\mathbf{v}}(\mathbf{X})$ for the configurational change in C^0 as

$$(48) \quad \begin{aligned} & \left\{ \int_{\partial\Omega_N^{(IJ)}} n_i [(c_{ijKL}^{\text{tangent}} \Delta L_{KL}^* \delta x_j)^{(J)} - (c_{ijKL}^{\text{tangent}} \Delta L_{KL}^* \delta x_j)^{(I)}] dS \right. \\ & - \int_{\partial\Omega_N^{(IJ)}} n_i [(c_{ijKL}^{\text{tangent}} \Delta L_{KL}^*)^{(J)} + (c_{ijKL}^{\text{tangent}} \Delta L_{KL}^*)^{(I)}] (\delta x_j^{(J)} - \delta x_j^{(I)}) dS/2 \} \\ & + \left\{ \int_{\partial\Omega_N^{(IJ)}} n_i [(\tau_{ij}^{(N)} \delta x_j)^{(J)} - (\tau_{ij}^{(N)} \delta x_j)^{(I)}] dS \right. \\ & - \left. \int_{\partial\Omega_N^{(IJ)}} n_i [(\tau_{ij}^{(N)})^{(J)} + (\tau_{ij}^{(N)})^{(I)}] (\delta x_j^{(J)} - \delta x_j^{(I)}) dS/2 \} \end{aligned}$$

to be added to Eq. (48a) to form the global stiffness equations if discontinuous trial and test functions are used. Thus the local weak forms of the energy-momentum conservation laws can be written as,

$$(48b) \quad \begin{aligned} & \int_{\Omega_N^{(I)}} c_{ijKL}^{\text{tangent}} \Delta L_{KL}^* \delta x_{j,i} d\Omega + \frac{1}{2} \int_{\partial\Omega_N^{(IJ)}} n_i [(c_{ijKL}^{\text{tangent}} \Delta L_{KL}^*)^{(J)} \\ & + (c_{ijKL}^{\text{tangent}} \Delta L_{KL}^*)^{(I)}] (\delta x_j^{(J)} - \delta x_j^{(I)}) dS = \int_{\Omega_N^{(I)}} \rho_N \Delta f_j \delta x_j d\Omega + \left\{ \int_{\Omega_N^{(I)}} \rho_N f_j \delta x_j d\Omega \right. \\ & \left. - \int_{\Omega_N^{(I)}} \tau_{ij}^{(N)} \delta x_{j,i} d\Omega - \int_{\partial\Omega_N^{(IJ)}} n_i [(\tau_{ij}^{(N)})^{(J)} + (\tau_{ij}^{(N)})^{(I)}] (\delta x_j^{(J)} - \delta x_j^{(I)}) dS/2 \right\}. \end{aligned}$$

With a constant test function $\delta\mathbf{x}$ over each local sub-domain, the domain integrals vanish and the local weak forms of the energy-momentum conservation laws may be further simplified as,

$$(48c) \quad \begin{aligned} & \int_{\partial\Omega_N^{(IJ)}} n_i [(c_{ijKL}^{\text{tangent}} \Delta L_{KL}^*)^{(J)} + (c_{ijKL}^{\text{tangent}} \Delta L_{KL}^*)^{(I)}] (\delta x_j^{(J)} - \delta x_j^{(I)}) dS/2 \\ & = \int_{\Omega_N^{(I)}} \rho_N \Delta f_j \delta x_j d\Omega + \left\{ \int_{\Omega_N^{(I)}} \rho_N f_j \delta x_j d\Omega \right. \\ & \left. - \int_{\partial\Omega_N^{(IJ)}} n_i [(\tau_{ij}^{(N)})^{(J)} + (\tau_{ij}^{(N)})^{(I)}] (\delta x_j^{(J)} - \delta x_j^{(I)}) dS/2 \right\}. \end{aligned}$$

This formulation is computationally efficient as i) no domain integrals are involved except when the body forces exist; ii) all derivatives are evaluated at nodal points only; and iii) one point integral per one shared boundary segment may be used.

Once the trial functions $\tilde{\mathbf{v}}(\mathbf{X})$, configurational changes in $C^{(0)}$, are solved through Eq. (48) due to either the unbalanced force or the incremental loading, $\Delta\mathbf{u} = \mathbf{F} \cdot \tilde{\mathbf{v}}$ can be obtained through Eq. (30). It is clear that Eq. (48) can be applied to both the total and updated Lagrangian formulations by setting the displacements $\mathbf{u}(\mathbf{X})$ in $C^{(N)}$ accordingly.

Corrector Iterations based on Local Weak Forms in Current Configuration. Equations (48) represent the linearized weak forms of the Noether/Eshelby energy-momentum conservation laws based on the configurational changes in $C^{(0)}$ for the predictor solution. An iteration process is necessary to correct the predicted trial solution in $C^{(N+1)}$, if it does not satisfy the solution exactly. The last bracketed terms of Eq. (48) are residual forces and have the physical meaning as the rate of work done by the unbalanced forces of the $C^{(N+1)}$ trial solutions in $C^{(N)}$. It has the form

$$(51) \quad [\delta e_{(N)}^{(N+1)}]^{\text{resid}(I)} = \int_{\Omega_N^{(I)}} (\tau_{ij,j}^{(N)} + \rho_N f_i) \delta x_i d\Omega.$$

This term is used to enforce Newtonian momentum balance laws in terms of the Cauchy stress in the deformed configuration. Thus the present method enforces the momentum balance laws through the predictor, and the energy-momentum conservation laws through the corrector iteration.

The predictor allows the discontinuity in the trial function based on the known configuration $C^{(N)}$, which may not be valid if “defects” are developed during the current increment of deformation. Thus, corrector iterations may also be performed based on the path-independent integrals for Eulerian Eshelby stress tensor $\tilde{\mathbf{S}}$ in $C^{(N)}$. From Eq. (16), one has

$$(52) \quad \tilde{\mathbf{S}}^{(N+1)} = \tilde{W}^{(N+1)} \mathbf{I} - \tau^{(N+1)}.$$

The corresponding corrector iterations are performed, based on the local weak forms of the Noether/Eshelby energy–momentum conservation laws in $C^{(N+1)}$,

$$(53) \quad \begin{aligned} \delta e_{(0)}^{(N+1)} &= \int_{\Omega} (J F_{Ik}^{-1} \tilde{S}_{kl}^{(N+1)} F_{lJ})_{,I} \delta X_J d\Omega - \int_{\Omega} \rho_0 b_J \delta X_J d\Omega \\ &= \int_{\partial\Omega^{(N+1)}} n_i \tilde{S}_{ij}^{(N+1)} \delta x_j dS - \int_{\Omega^{(N+1)}} \tilde{S}_{ij}^{(N+1)} \delta x_{j,i} dS - \int_{\Omega^{(N+1)}} \rho_N b_j \delta x_j d\Omega \text{ for } \forall \Omega_{N+1}, \end{aligned}$$

with the use of Eq. (27c). One may choose constant test functions $\delta \mathbf{x}$ over the subdomain and re-write Eq. (53) as,

$$(54) \quad \delta e_{(0)}^{(N+1)} = \int_{\partial\Omega^{(N+1)}} n_i (\tilde{W}^{(N+1)} \delta_{ij} - \tau_{ij}^{(N+1)}) \delta x_j dS - \int_{\Omega^{(N+1)}} \rho_N b_j \delta x_j d\Omega = 0$$

for $\forall \Omega_{N+1}$, which is in general the weak form of the Noether/Eshelby energy–momentum conservation laws in $C^{(N)}$ (Han and Atluri 2014b).

Both Newtonian momentum balance laws and the Noether/Eshelby energy–momentum conservation laws need to be satisfied to ensure that the trial functions are the true solutions. In EAM, Newtonian momentum balance laws are enforced directly through the iteration process, regardless of the continuity of the trial functions. For elastic materials, it can be verified that Eq. (53) and (54) are always satisfied within each local sub-domain if the trial functions are so chosen that the strong form of the momentum equilibrium is satisfied, i.e., no energy is lost within a local sub-domain. Thus, the energy–momentum conservation laws are enforced across the inter-subdomain boundaries and assure no-rupture of the predictor of Eq. (48) during deformation at these boundaries. On other hand, one may predefine an enclosed path and track the energy portion of the weak form in Eq. (54) as

$$(55) \quad \int_{\partial\Omega^{(N+1)}} \tilde{W}^{(N+1)} n_i \delta x_i dS$$

while the stress portion may be evaluated based on the Cauchy stress in the deformed configuration. If a crack develops, the integral becomes the criterion to initiate and control the damage processes. The details are under studies.

Atluri and Han's Remarks (2014b): i) There are no continuity requirements across local subdomains. These functions can be constructed within a local sub-domain to satisfy the geometric identities exactly. This improves the solution accuracy and the rate of convergence. ii) Because no inter-subdomain continuity is required, discontinuities may be naturally introduced based on the energy–momentum conservation criteria to allow the formation of cracks. iii) Iteration based on the energy–momentum conservation laws provides additional corrections to other non-elastic processes if multi-physics are involved; iv) Higher order continuity requirements are achieved by adjusting the order of continuity of the weight functions without increasing the degree of freedom per node in plate and shell applications.

One uses the weak forms of the energy–momentum conservation laws to construct the invertible mapping between the variables in all configurations. The mapping should be compatible and satisfy the geometric identities. It is not possible to define such mappings between the configurations in the global domain. In contrast, the local linearized invertible and compatible relationships among configurations can be constructed in terms of closed form functions of constant coefficients (Atluri 1998, 2004).

When the configurational changes $\tilde{\mathbf{v}}$ of C^0 are employed as the trial functions of the weak forms, $\tilde{\mathbf{v}}$ need to be admissible within the local neighboring domains because $\Delta \mathbf{u}$ must satisfy the boundary conditions. The test functions $\delta \mathbf{x}$ are configurational changes in C^N . It is convenient to choose linear $\delta \mathbf{x}$ within each test domain (Han and Atluri 2014a).

22.7. SUMMARY

In this chapter, we present the basic concept of MLPGs and EAMs in solid mechanics. We have considered two types of trial functions: those derived by the moving least square scheme and those of the Shepard functions; and four types of test functions: the weight, Dirac delta, shape and Heaviside step functions. The shape functions are in general in a rational form and are more complicated than the polynomial based shape functions of FEMs. The complexity makes accurate numerical integration of the weak forms difficult. High order Gauss quadrature or smaller sub-domains are required to integrate the weak forms accurately. Thus it is good to use the Heaviside step function as test functions to avoid domain integration. Larger supports for the weight functions generally make the shape functions more complicated and non-local. Hence a smaller support is desired. However, small supports for weight functions introduces unwanted waviness in the derivatives of the trial functions, which is not desirable.

Numerical examples show good rates of convergence for the state variables and their derivatives. The trial functions

are generally smooth enough allowing the direct calculation of derivatives without post smooth processing. The combined use of Noether/Eshelby energy-momentum conservation laws and Atluri's MLPGs in EAMs is a radical departure from the popular FEMs. Numerical solutions (Biblio. of this chapter) show that the EAMs converge faster and are accurate. The computation and human-labor costs are generally lower than those of FEMs. The MLPGs and EAMs hold great promise in engineering applications.

$$^1 \text{Note that } \tilde{P}_{nM} = \frac{\partial \tilde{W}}{\partial F_{Mn}^{-1}} = \frac{1}{J}(WF_{nM} + \frac{\partial W}{\partial F_{Mn}^{-1}}) = \frac{1}{J}(WF_{nM} - F_{nI} \cdot \frac{\partial W}{\partial F_{JI}} \cdot F_{jM}).$$

²Among the many meshless approximation schemes, the MLS is generally considered to be the best to interpolate random data with reasonable accuracy, because of its locality, completeness, robustness and continuity.

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