Chap 3. Virtual Work Formulation (3-d formulation)

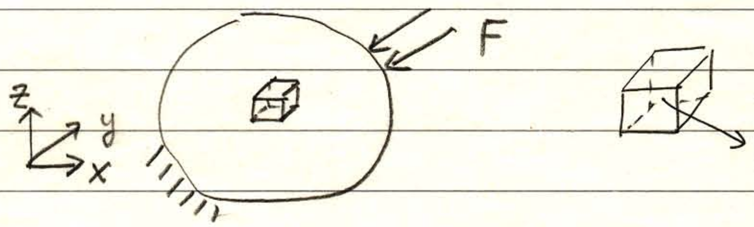
3.1 Basic Equation of Elasticity

(1) Equation of Equilibrium (응력평형식)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + X_B = 0$$

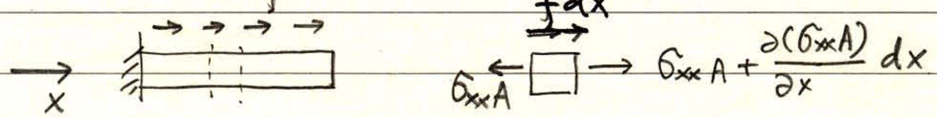
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + Y_B = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z_B = 0$$



$$\vec{F}_B = X_B\vec{i} + Y_B\vec{j} + Z_B\vec{k}$$
$$= \text{body force/volume}$$

(NOTE) 1-d formulation



$$\frac{\partial(\sigma_{xx}A)}{\partial x} + f = 0$$

(2) strain - Displacement relation (Linear)

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \varepsilon_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

(3) stress - strain relation.

$$\underset{\sim}{\sigma} = \underset{\sim}{C}\,(\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}^{\circ})$$

$\underset{\sim}{C}$ = 6×6 elastic constant matrix

$\underset{\sim}{\varepsilon}^{\circ}$ = initial strain (예) thermal strain.

thermally isotropic material 의 경우

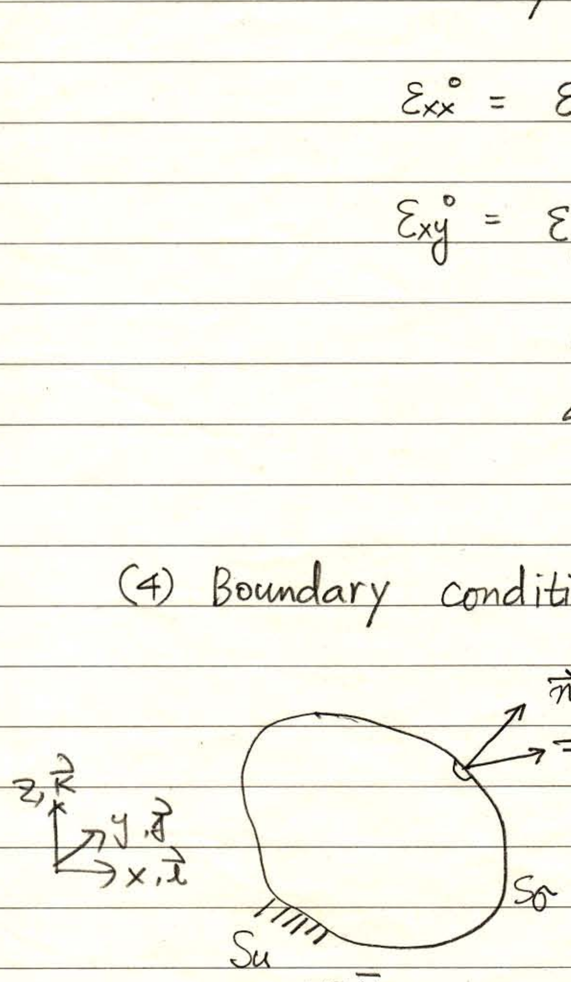
$$\varepsilon_{xx}^{\circ} = \varepsilon_{yy}^{\circ} = \varepsilon_{zz}^{\circ} = \alpha \Delta T$$

$$\varepsilon_{xy}^{\circ} = \varepsilon_{yz}^{\circ} = \varepsilon_{zx}^{\circ} = 0$$

α = 열팽창 계수

ΔT = 기준온도에서 측정한 온도증가.

(4) Boundary condition.



$$\vec{n} = l\vec{i} + m\vec{j} + n\vec{k}$$

$$\vec{T} = \bar{T}_x\vec{i} + \bar{T}_y\vec{j} + \bar{T}_z\vec{k}$$

$\vec{n}$ = 표면에 수직 단위 vector. $= l\vec{i} + m\vec{j} + n\vec{k}$

$S = S_\sigma + S_u$

$u = \bar{u}$
$v = \bar{v}$
$w = \bar{w}$

$\vec{T}$ = traction force

= force / unit surface area

$\bar{u}, \bar{v}, \bar{w}$ = prescribed displacement

(a) S_σ 에서

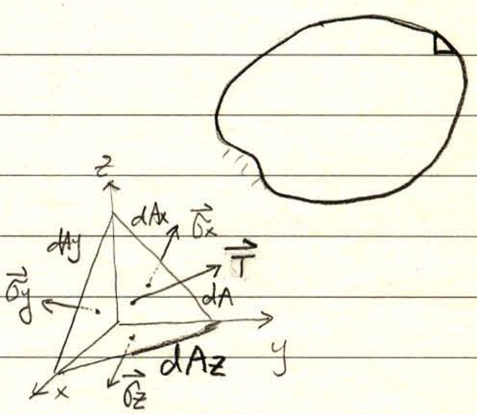
$$T_x = \sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n = \bar{T}_x$$

$$T_y = \sigma_{yx} l + \sigma_{yy} m + \sigma_{yz} n = \bar{T}_y$$

$$T_z = \sigma_{zx} l + \sigma_{zy} m + \sigma_{zz} n = \bar{T}_z$$

(b) S_u 에서

$$u = \bar{u}, \quad v = \bar{v}, \quad w = \bar{w}$$



(NOTE) 사면체 힘평형

$$\vec{T}\, dA = \vec{\sigma}_x\, dA_x + \vec{\sigma}_y\, dA_y + \vec{\sigma}_z\, dA_z$$

$$\vec{T} = \vec{\sigma}_x \frac{dA_x}{dA} + \vec{\sigma}_y \frac{dA_y}{dA} + \vec{\sigma}_z \frac{dA_z}{dz}$$

$$= \vec{\sigma}_x l + \vec{\sigma}_y m + \vec{\sigma}_z n$$

$$= (\sigma_{xx} \vec{i} + \sigma_{xy} \vec{j} + \sigma_{xz} \vec{k})\, l$$
$$+ (\sigma_{yx} \vec{i} + \sigma_{yy} \vec{j} + \sigma_{yz} \vec{k})\, m$$
$$+ (\sigma_{zx} \vec{i} + \sigma_{zy} \vec{j} + \sigma_{zz} \vec{k})\, n$$

한편 $\vec{T} = T_x \vec{i} + T_y \vec{j} + T_z \vec{k}$

4.2 Virtual displacement & virtual work

응력 평형식에 virtual disp δu, δv, δw 도입

$$-\int_V \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + X_B\right) \delta u \, dV$$

$$-\int_V \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + Y_B\right) \delta v \, dV$$

$$-\int_V \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z_B\right) \delta w \, dV = 0 \quad (1)$$

Divergence Theorem

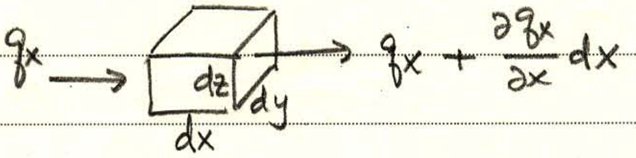
$$\int_V \nabla \cdot \vec{q} \, dV = \int_S \vec{q} \cdot \vec{n} \, ds$$

or $$\int_V \left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right) dV = \int_S (L\ell + Mm + Nn) \, ds$$

여기서 $\vec{q} = L\vec{i} + M\vec{j} + N\vec{k}$

$\vec{n} = \ell\vec{i} + m\vec{j} + n\vec{k}$

(NOTE) divergence theorem 유도



if $\vec{q}$ = mass flow rate vector / unit area ,

volume $dV = dx\,dy\,dz$ 에서 흘러나오는 mass flow rate dF 는

$$dF = \left[\left(q_x + \frac{\partial q_x}{\partial x}dx\right) - q_x\right] dy\,dz$$

$$+ \left[\left(q_y + \frac{\partial q_y}{\partial y}dy\right) - q_y\right] dx\,dz$$

$$+ \left[\left(q_z + \frac{\partial q_z}{\partial z}dz\right) - q_z\right] dx\,dy$$

$$= \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) dx\,dy\,dz$$

$$= \nabla \cdot \vec{q}\, dV$$

= mass flow rate / unit volume.

$\vec{q} \cdot \vec{n}$ = mass flow rate/unit area of close surf

부분 적분을 이용하여 (1) 식을 다시 쓰면.

$$-\int_V \Big[\frac{\partial}{\partial x}\big(\underbrace{\sigma_{xx}\delta u + \sigma_{xy}\delta v + \sigma_{xz}\delta w}_{= L}\big)$$

$$+ \frac{\partial}{\partial y}\big(\underbrace{\sigma_{xy}\delta u + \sigma_{yy}\delta v + \sigma_{yz}\delta w}_{= M}\big)$$

$$+ \frac{\partial}{\partial z}\big(\underbrace{\sigma_{xz}\delta u + \sigma_{yz}\delta v + \sigma_{zz}\delta w}_{= N}\big) \Big] dV$$

$$+\int_V \left[\sigma_{xx}\frac{\partial \delta u}{\partial x} + \sigma_{xy}\left(\frac{\partial \delta v}{\partial x} + \frac{\partial \delta u}{\partial y}\right) + \sigma_{xz}\left(\frac{\partial \delta w}{\partial x} + \frac{\partial \delta u}{\partial z}\right) + \sigma_{yy}\frac{\partial \delta v}{\partial y} + \sigma_{yz}\left(\frac{\partial \delta w}{\partial y} + \frac{\partial \delta v}{\partial z}\right) + \sigma_{zz}\frac{\partial \delta w}{\partial z} \right] dV$$

$$-\int_V (X_B\,\delta u + Y_B\,\delta v + Z_B\,\delta w)\, dV = 0 \qquad (2)$$

(2)식의 첫항에 divergence theorem 을 적용하여 정리하면

$$-\int_S \Big[(\underbrace{\sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n}_{=T_x})\delta u + (\underbrace{\sigma_{xy} l + \sigma_{yy} m + \sigma_{yz} n}_{=T_y})\delta v + (\underbrace{\sigma_{xz} l + \sigma_{yz} m + \sigma_{zz} n}_{=T_z})\delta w \Big] dS$$

$$+\int_V \left[\sigma_{xx}\frac{\partial \delta u}{\partial x} + \sigma_{xy}\left(\frac{\partial \delta v}{\partial x} + \frac{\partial \delta u}{\partial y}\right) + \cdots \right] dV$$

$$-\int_V (X_B\,\delta u + Y_B\,\delta v + Z_B\,\delta w)\, dV = 0 \qquad (3)$$

(NOTE) $S = S_\sigma + S_u$

$\delta u = \delta v = \delta w = 0$ on S_u (geometric B.C)

$T_x = \overline{T_x}$, $T_y = \overline{T_y}$, $T_z = \overline{T_z}$ on S_σ (force B.C.)

Then 식 (3)은.

$$\delta\Pi = \int_V \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz}\right.$$

$$\left. + \sigma_{xy}\delta\varepsilon_{xy} + \sigma_{yz}\delta\varepsilon_{yz} + \sigma_{zx}\delta\varepsilon_{zx}\right) dV$$

$$- \int_{S_\sigma} \left(\bar{T}_x \delta u + \bar{T}_y \delta v + \bar{T}_z \delta w\right) dS$$

$$- \int_V \left(X_B \delta u + Y_B \delta v + Z_B \delta w\right) dV$$

$$= \delta U - \delta W = 0$$

Matrix form 으로 표시하면.

$$\delta\Pi = \int_V \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, dV - \int_V \delta\underset{\sim}{u}^T \underset{\sim}{F_B}\, dV - \int_{S_\sigma} \delta\underset{\sim}{u}^T \underset{\sim}{\bar{T}}\, ds = 0$$

여기서

$$\delta\underset{\sim}{\varepsilon} = [\delta\varepsilon_{xx}\ \delta\varepsilon_{yy}\ \delta\varepsilon_{zz}\ \delta\varepsilon_{xy}\ \delta\varepsilon_{yz}\ \delta\varepsilon_{zx}]^T$$

$$\underset{\sim}{\sigma} = [\sigma_{xx}\ \sigma_{yy}\ \sigma_{zz}\ \sigma_{xy}\ \sigma_{yz}\ \sigma_{zx}]^T$$

$$\delta\underset{\sim}{u} = [u\ v\ w]^T$$

$$\underset{\sim}{F_B} = [X_B\ Y_B\ Z_B]^T$$

$$\underset{\sim}{\bar{T}} = [\bar{T}_x\ \bar{T}_y\ \bar{T}_z]^T$$

$\underset{\sim}{\sigma} = \underset{\sim}{C} (\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}^{\circ})$ 도입

$$\delta \Pi = \underbrace{\int_V \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{C} \underset{\sim}{\varepsilon} \, dV}_{\text{lead to stiffness matrix}} \; \boxed{- \int_V \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{C} \underset{\sim}{\varepsilon}^{\circ} dV - \int_V \delta \underset{\sim}{u}^T \underset{\sim}{F_B} \, dV - \int_{S_\sigma} \delta \underset{\sim}{u}^T \underset{\sim}{T} \, dS} = 0$$

lead to force vector.