

# Tensor products

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## 1 The definition

Let  $V, W, X$  be three vector spaces. A *bilinear map* from  $V \times W$  to  $X$  is a function  $H : V \times W \rightarrow X$  such that

$$\begin{aligned} H(av_1 + v_2, w) &= aH(v_1, w) + H(v_2, w) \text{ for } v_1, v_2 \in V, w \in W, a \in \mathbb{F} \\ H(v, aw_1 + w_2) &= aH(v, w_1) + H(v, w_2) \text{ for } v \in V, w_1, w_2 \in W, a \in \mathbb{F} \end{aligned}$$

Let  $V$  and  $W$  be vector spaces. A *tensor product* of  $V$  and  $W$  is a vector space  $V \otimes W$  along with a bilinear map  $\phi : V \times W \rightarrow V \otimes W$ , such that for every vector space  $X$  and every bilinear map  $H : V \times W \rightarrow X$ , there exists a unique linear map  $T : V \otimes W \rightarrow X$  such that  $H = T \circ \phi$ .

In other words, giving a linear map from  $V \otimes W$  to  $X$  is the same thing as giving a bilinear map from  $V \times W$  to  $X$ .

If  $V \otimes W$  is a tensor product, then we write  $v \otimes w := \phi(v, w)$ . Note that there are two pieces of data in a tensor product: a vector space  $V \otimes W$  and a bilinear map  $\phi : V \times W \rightarrow V \otimes W$ .

Here are the main results about tensor products summarized in one theorem.

**Theorem 1.1.** (i) Any two tensor products of  $V, W$  are isomorphic.

(ii)  $V, W$  has a tensor product.

(iii) If  $v_1, \dots, v_n$  is a basis for  $V$  and  $w_1, \dots, w_m$  is a basis for  $W$ , then

$$\{v_i \otimes w_j\}_{1 \leq i \leq n, 1 \leq j \leq m}$$

is a basis for  $V \otimes W$ .

In particular, the last part of the theorem shows we can think of elements of  $V \otimes W$  as  $n \times m$  matrices with entries in  $\mathbb{F}$ .

## 2 Existence

We will start by proving that the tensor product exists. To do so, we will construct an explicit tensor product. This construction only works if  $V, W$  are finite-dimensional.

Let  $B(V^*, W^*; \mathbb{F})$  be the vector space of bilinear maps  $H : V^* \times W^* \rightarrow \mathbb{F}$ . If  $v \in V$  and  $w \in W$ , then we can define a bilinear map  $v \otimes w$  by

$$(v \otimes w)(\alpha, \beta) = \alpha(v)\beta(w).$$

Just as we saw before, we have the following result.

**Theorem 2.1.** *Let  $v_1, \dots, v_n$  be a basis for  $V$  and let  $w_1, \dots, w_m$  be a basis for  $W$ . Then  $\{v_i \otimes w_j\}_{1 \leq i \leq n, 1 \leq j \leq m}$  is a basis for  $B(V^*, W^*; \mathbb{F})$*

Now, we define a map  $\phi : V \times W \rightarrow B(V^*, W^*; \mathbb{F})$  by  $\phi(v, w) = v \otimes w$ .

**Theorem 2.2.**  *$B(V^*, W^*; \mathbb{F})$  along with  $\phi$  is a tensor product for  $V, W$ .*

Note that this proves parts (ii) and (iii) of our main theorem.

*Proof.* Fix bases  $v_1, \dots, v_n$  for  $V$  and  $w_1, \dots, w_m$  for  $W$ .

Let  $X$  be a vector space and let  $H : V \times W \rightarrow X$  be a bilinear map. We define a linear map  $T : V \otimes W \rightarrow X$  by defining it on our basis as  $T(v_i \otimes w_j) = H(v_i, w_j)$ . Then  $T \circ \phi$  and  $H$  are two bilinear maps from  $V \times W$  to  $X$  which agree on basis vectors, hence they are equal. (Note that it is easy to show that for any  $(v, w) \in V \times W$ ,  $T(v \otimes w) = H(v, w)$ .)

Finally, note that  $T$  is the unique linear map with this property, since it is determined on the basis for  $B(V^*, W^*, \mathbb{F})$ .  $\square$

Using the same ideas, it is easy to see that  $L(V^*, W)$ ,  $L(W^*, V)$ , and  $B(V, W; \mathbb{F})^*$  are all also tensor products of  $V, W$ .

### 3 Uniqueness

Now we prove uniqueness. Here is the precise statement.

**Theorem 3.1.** *Let  $(V \otimes W)_1, \phi_1$  and  $(V \otimes W)_2, \phi_2$  be two tensor products of  $V, W$ . Then there exists a unique isomorphism  $T : (V \otimes W)_1 \rightarrow (V \otimes W)_2$  such that  $\phi_2 = T \circ \phi_1$ .*

*Proof.* Let us apply the definition of tensor product to  $(V \otimes W)_1, \phi_1$  with the role of  $X, H$  taken by  $(V \otimes W)_2, \phi_2$ . By the definition, we obtain a (unique) linear map  $T : (V \otimes W)_1 \rightarrow (V \otimes W)_2$  such that  $\phi_2 = T \circ \phi_1$ .

Reversing the roles of  $(V \otimes W)_1, \phi_1$  and  $(V \otimes W)_2, \phi_2$ , we find a linear map  $S : (V \otimes W)_2 \rightarrow (V \otimes W)_1$  such that  $\phi_1 = S \circ \phi_2$ .

We claim that  $T \circ S = I_{(V \otimes W)_2}$  and  $S \circ T = I_{(V \otimes W)_1}$  and hence  $T$  is an isomorphism. We will now prove  $S \circ T = I_{(V \otimes W)_1}$ .

Note that  $(S \circ T) \circ \phi_1 = S \circ \phi_2 = \phi_1$  by the above equations. Now, apply the definition of tensor product to  $(V \otimes W)_1, \phi_1$  with the role of  $X, H$  taken by  $(V \otimes W)_1, \phi_1$ . Then both  $S \circ T$  and  $I_{(V \otimes W)_1}$  can play the role of  $T$ . So by the uniqueness of “ $T$ ” in the definition, we conclude that  $S \circ T = I_{(V \otimes W)_1}$  as desired.

Finally to see that the  $T$  that appears in the statement of the theorem is unique, we just note from the first paragraph of this proof, it follows that there is only one linear map  $T$  such that  $\phi_2 = T \circ \phi_1$ .  $\square$