Variational Formulation & the Galerkin Method

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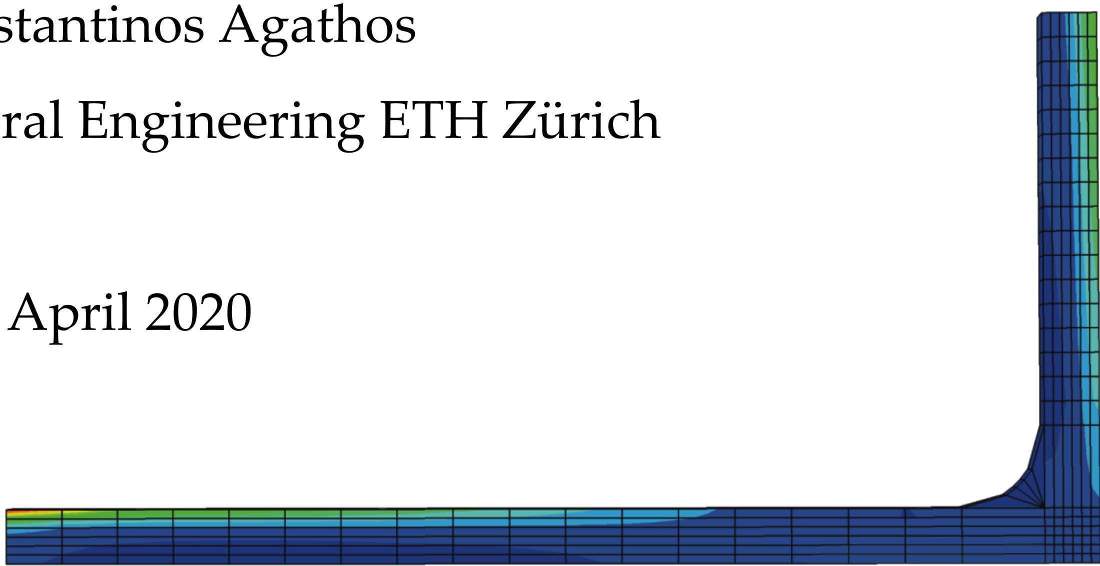
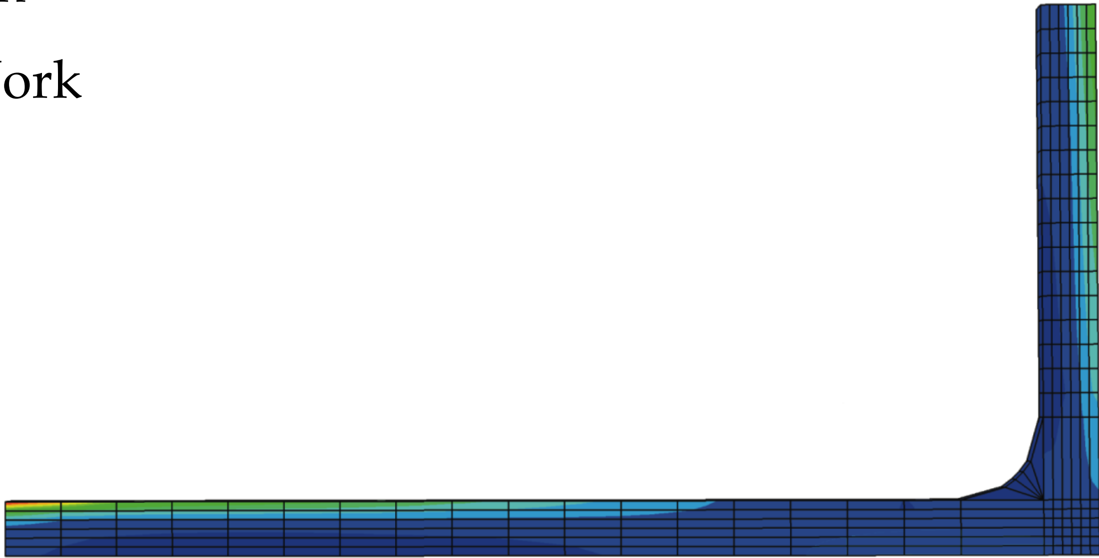


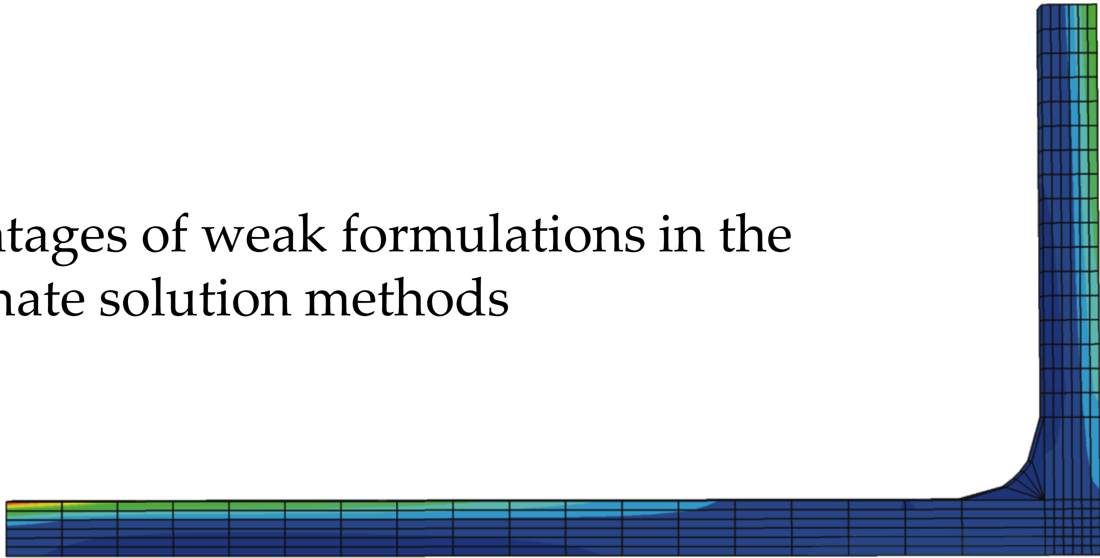
Today’s Lecture Contents:

- Introduction
- Strong form
  - Strong form of a 1D bar
  - Strong form solution for a 1D bar
- Weak form
  - Potential minimization
  - Principle of Virtual Work
  - Galerkin Method
  - Weak form solution



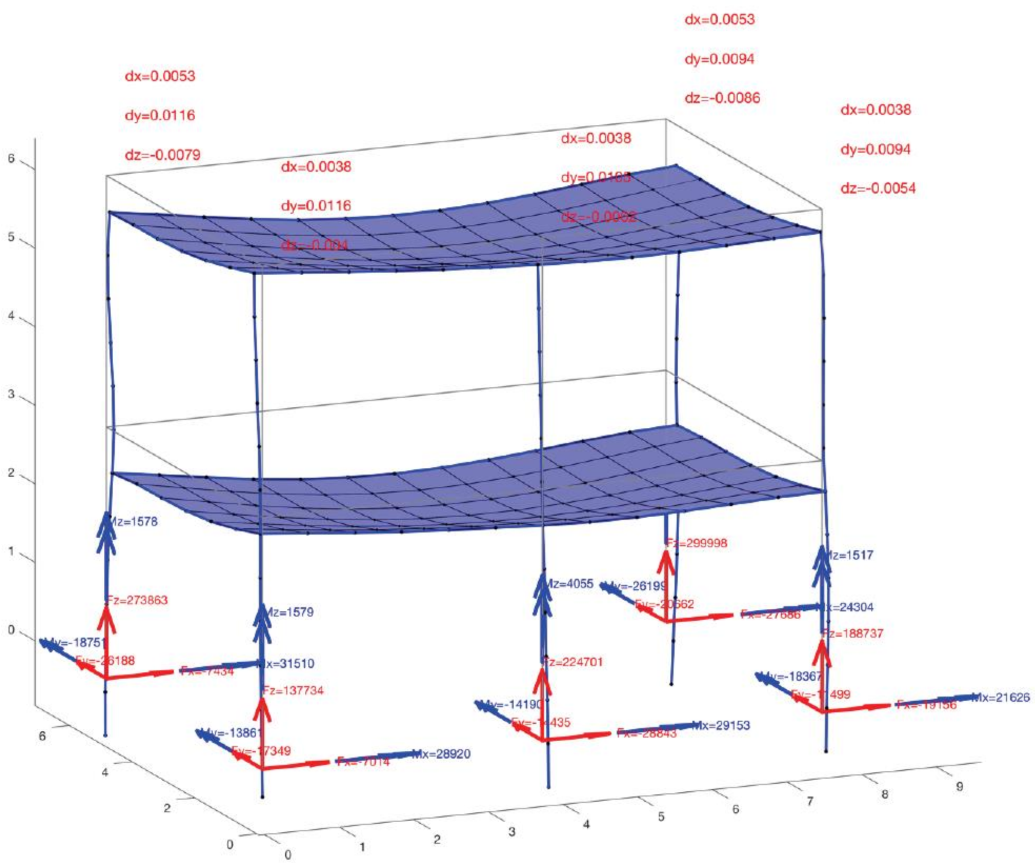
Learning goals:

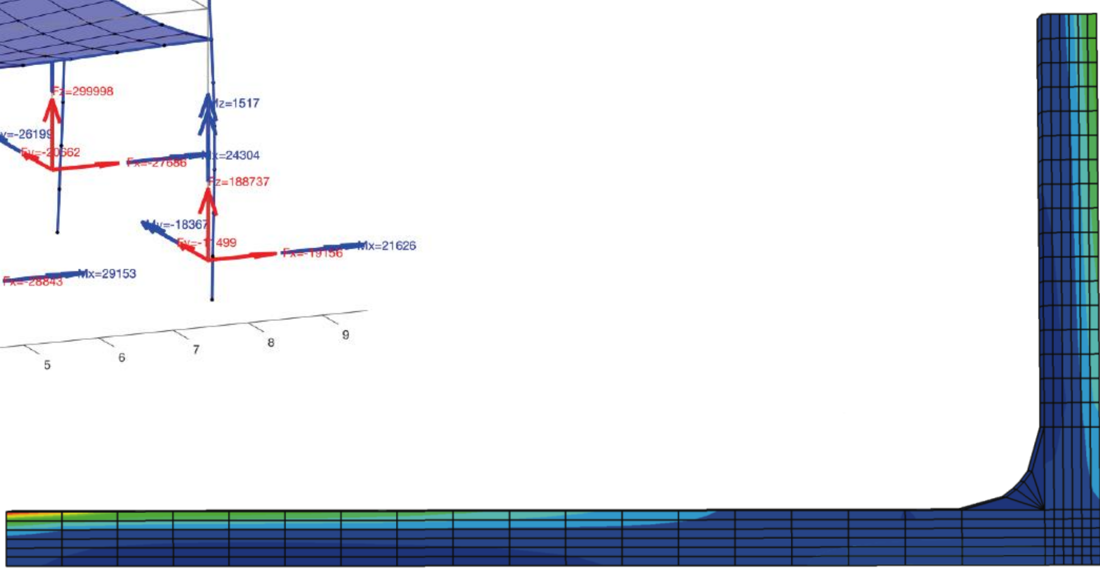
- Understanding strong and weak forms through a simple example
- Demonstrating the equivalence between the two as well as the differences
- Demonstrating the equivalence between alternative weak formulations
- Understanding the advantages of weak formulations in the development of approximate solution methods



FEM applications:

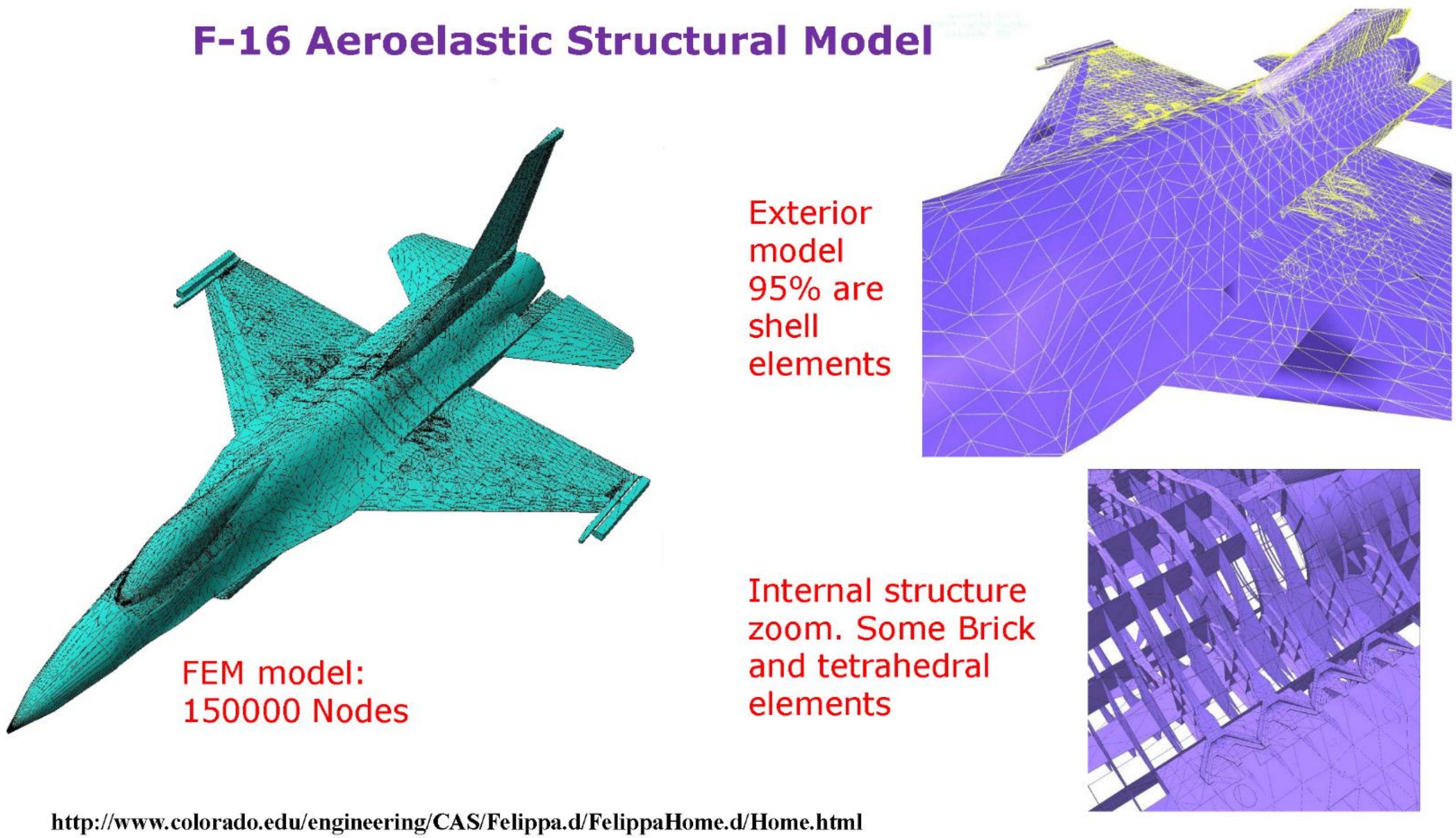
- Simple structural mechanics





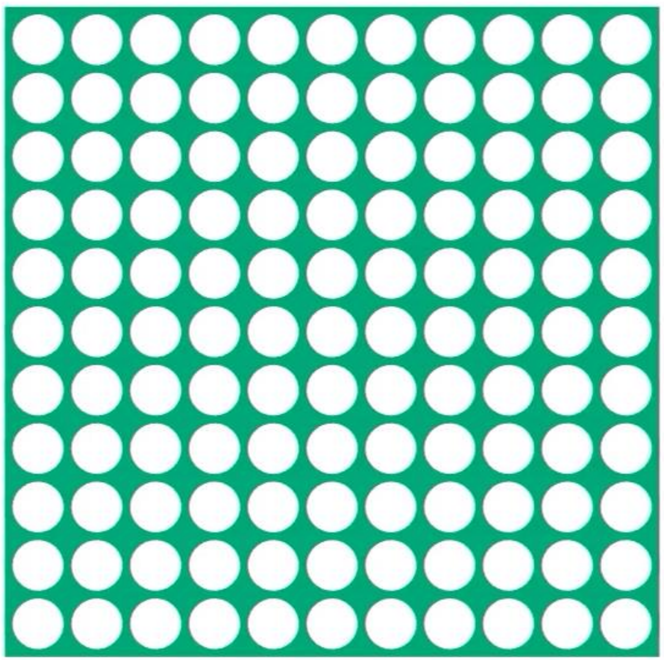
FEM applications:

- Not so simple structural mechanics

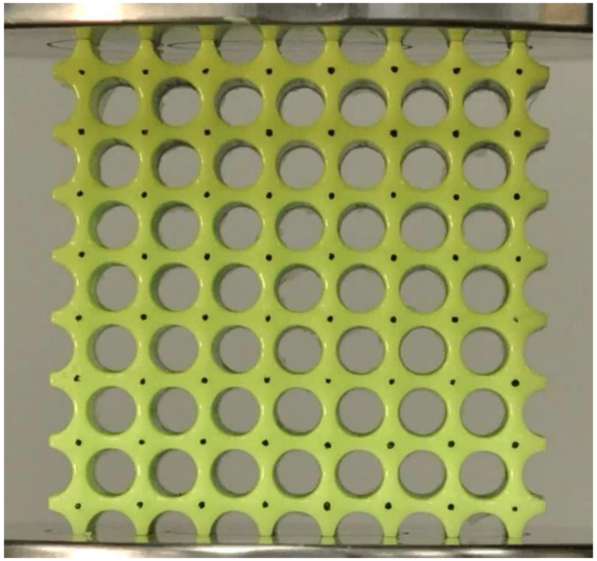


FEM applications:

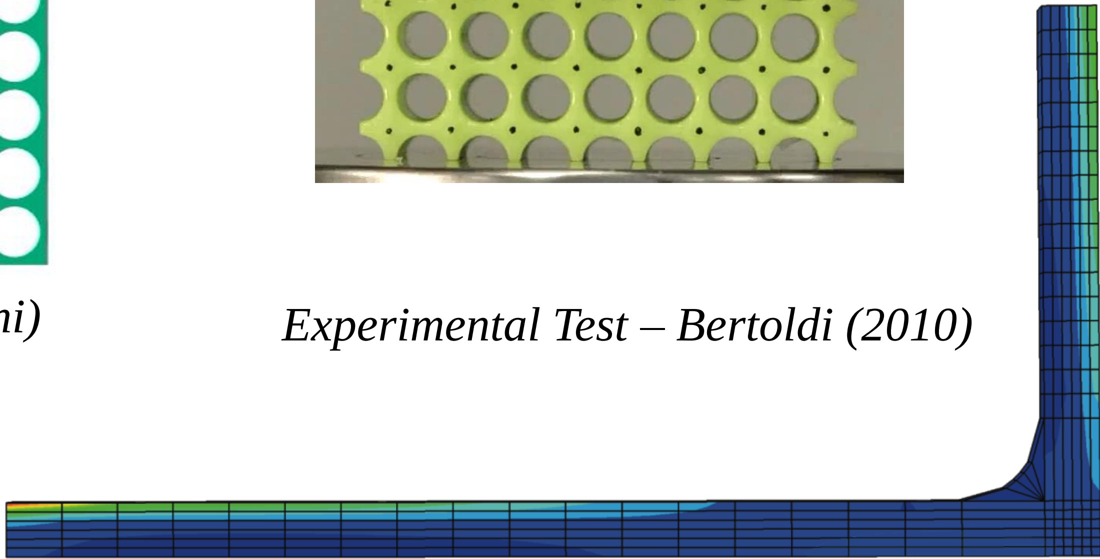
- Solid mechanics – Geometrical nonlinearities



Simulation (Aguzzi & Zaccherini)

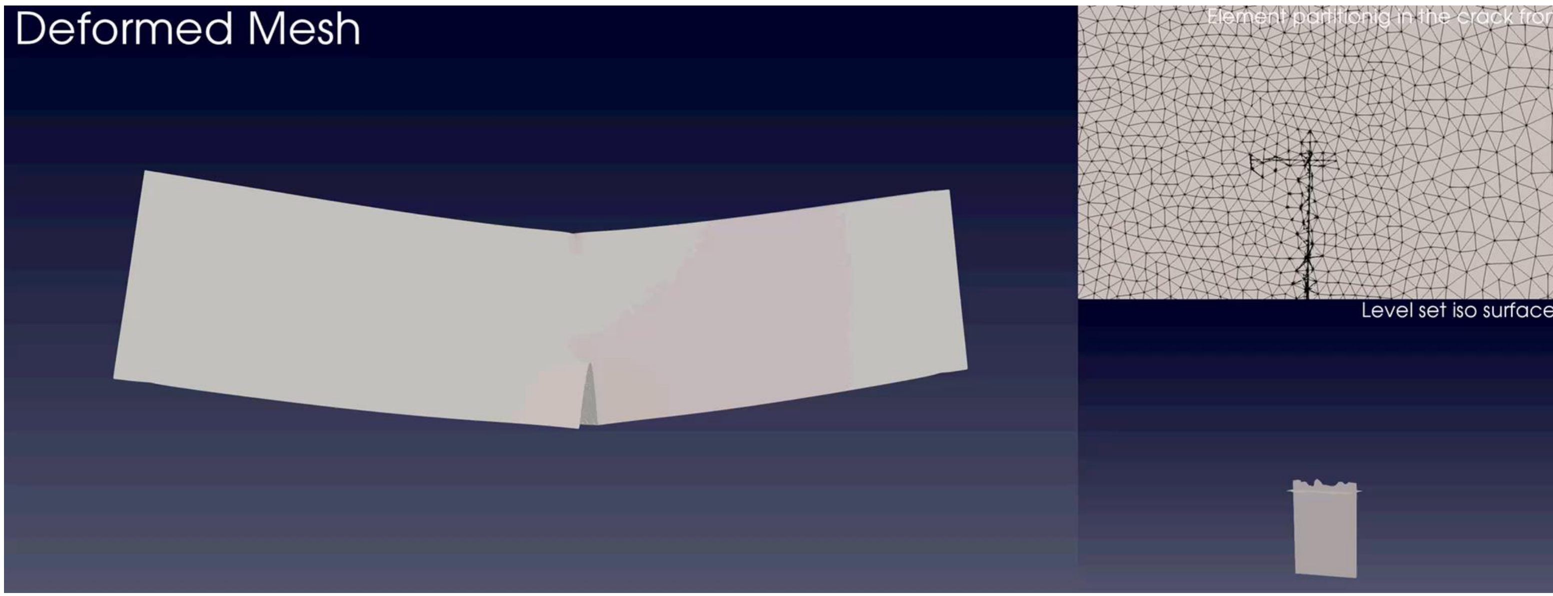


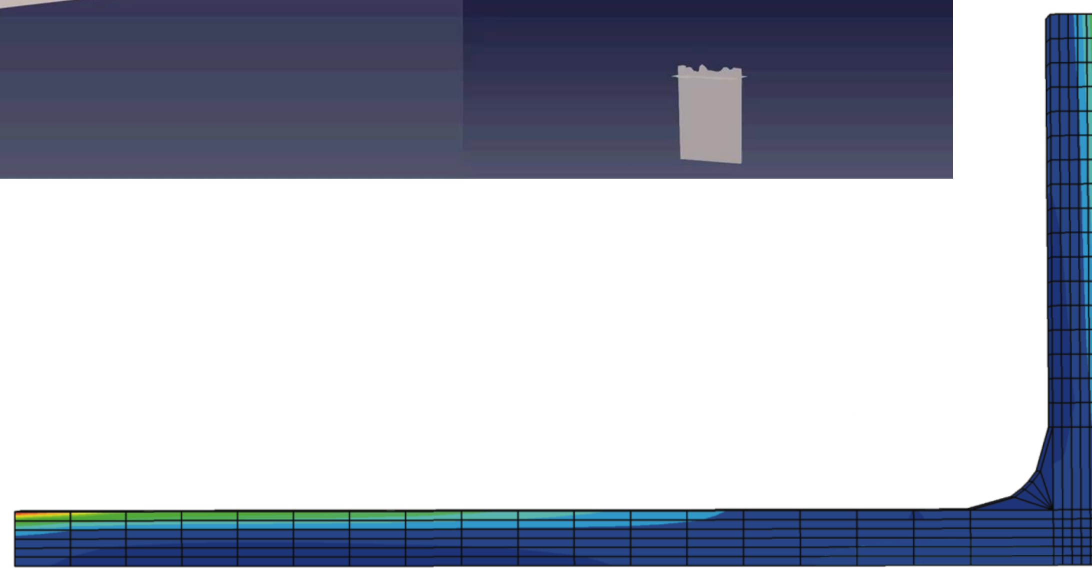
Experimental Test – Bertoldi (2010)



FEM applications:

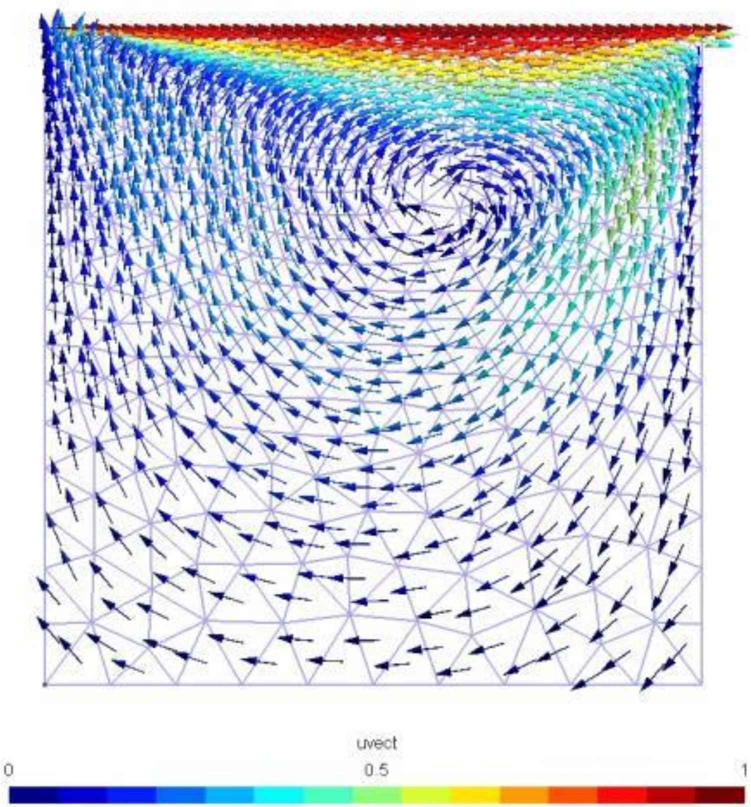
- Solid mechanics - Fracture

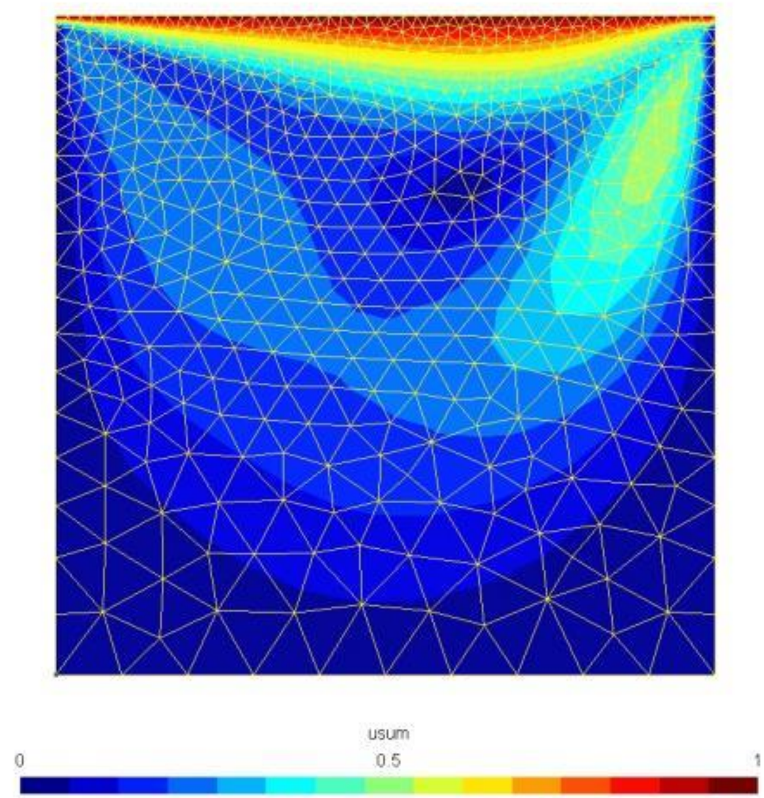




FEM applications:

- Fluid mechanics





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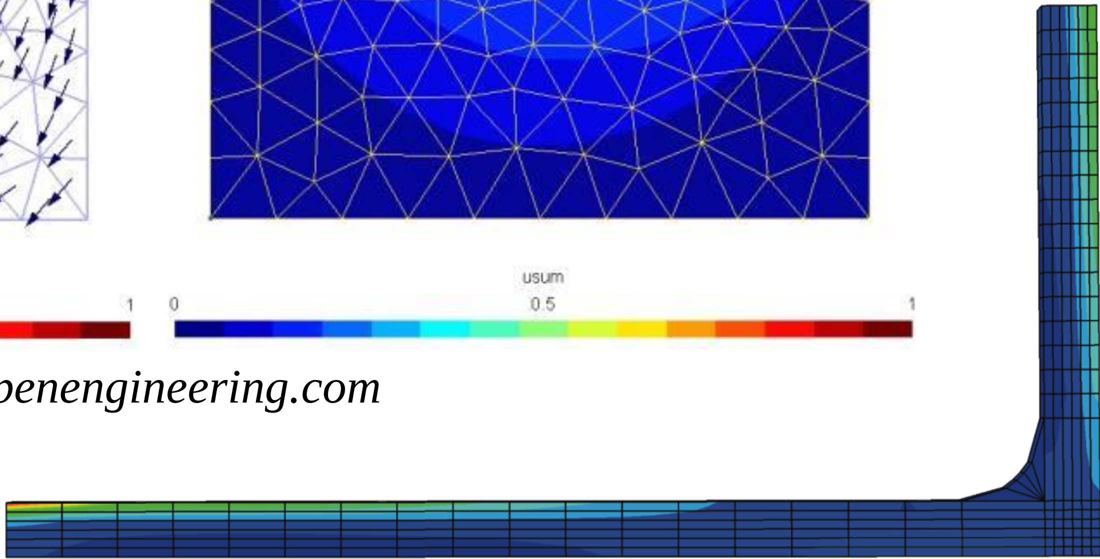
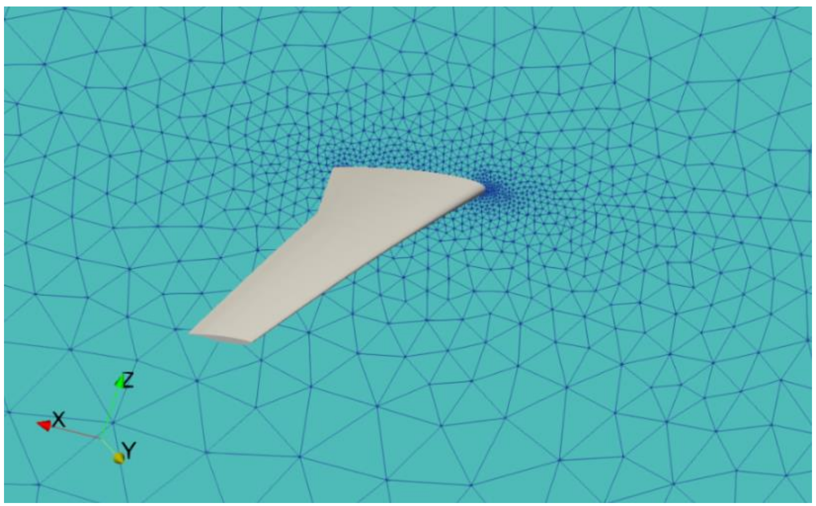
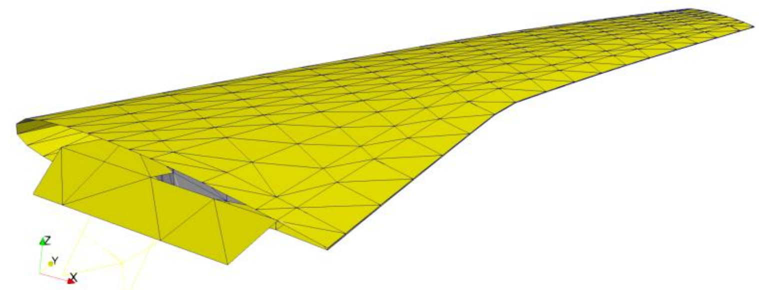


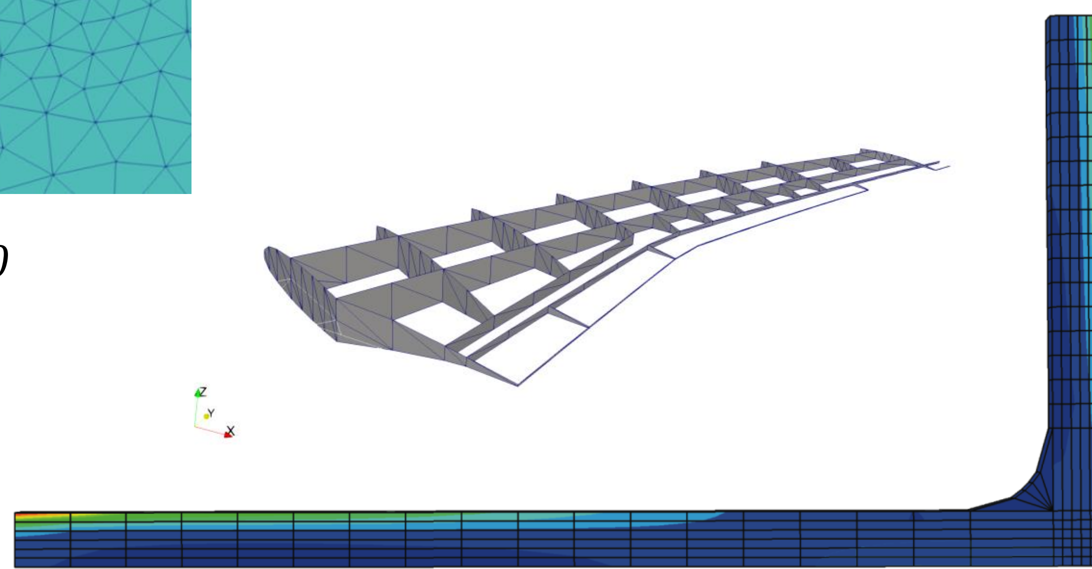
FEM applications:

- Multiphysics - fluid structure interaction



Boncoraglio et al. 2020





FEM for PDEs

The wide range of applicability is attributed to the fact that FEM is essentially a tool for solving partial differential equations (PDEs), for instance:

Bernoulli-Euler beam equilibrium equations:

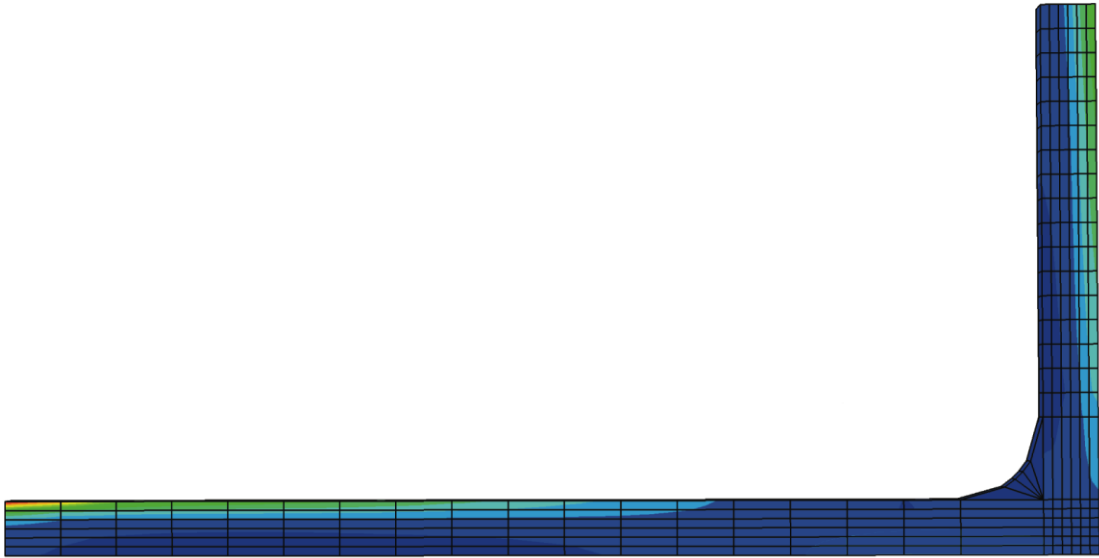
$$EI_z \frac{d^4 w}{dx^4} = f_y(x)$$

2D elasticity (Navier equations):

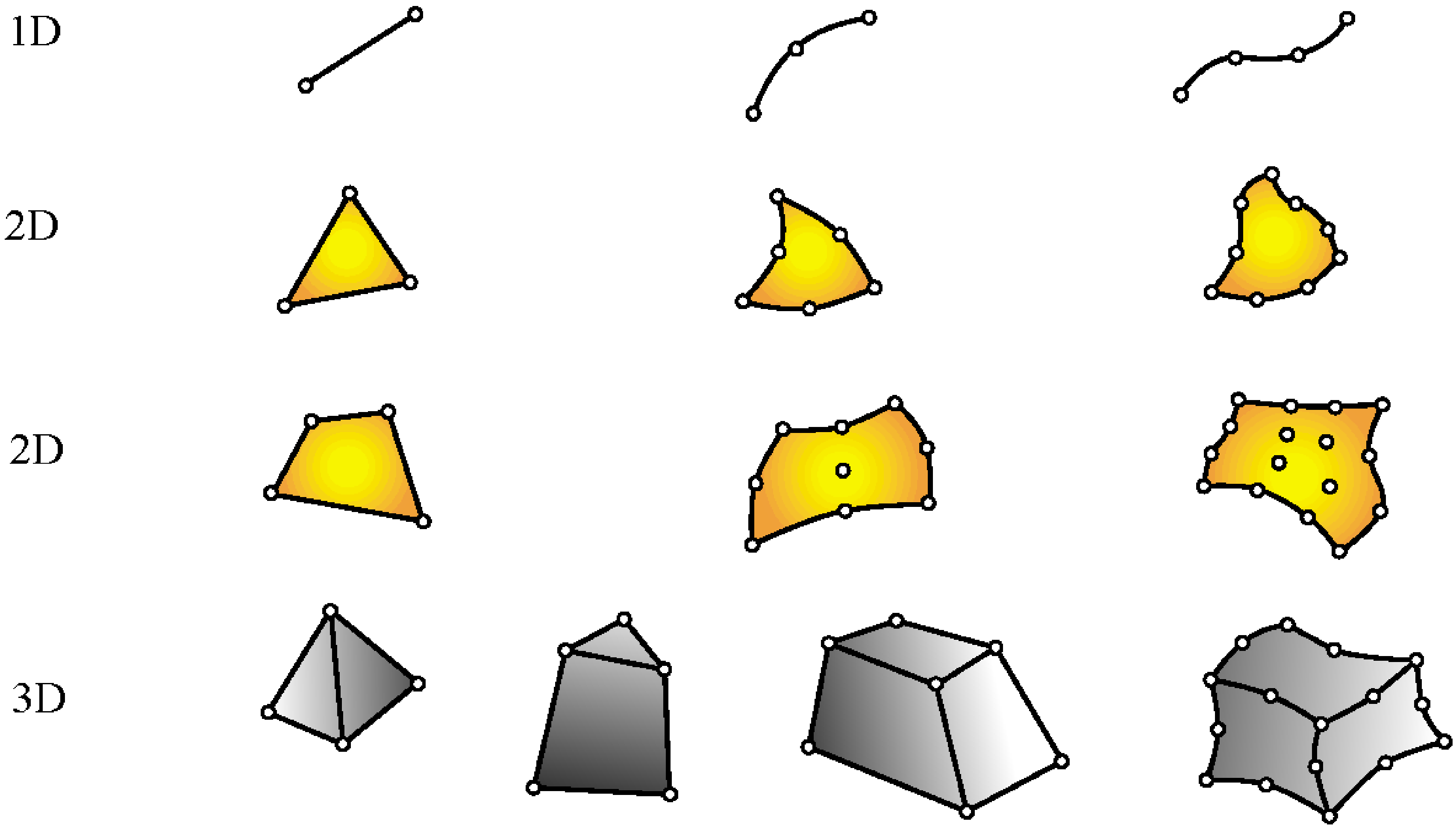
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0$$

Laplace equation:

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = 0$$



FEM across dimensions



FEM for PDEs

A series of steps is typically followed by all FE methods:

Strong form → Elasticity, heat conduction, Navier stokes, etc.

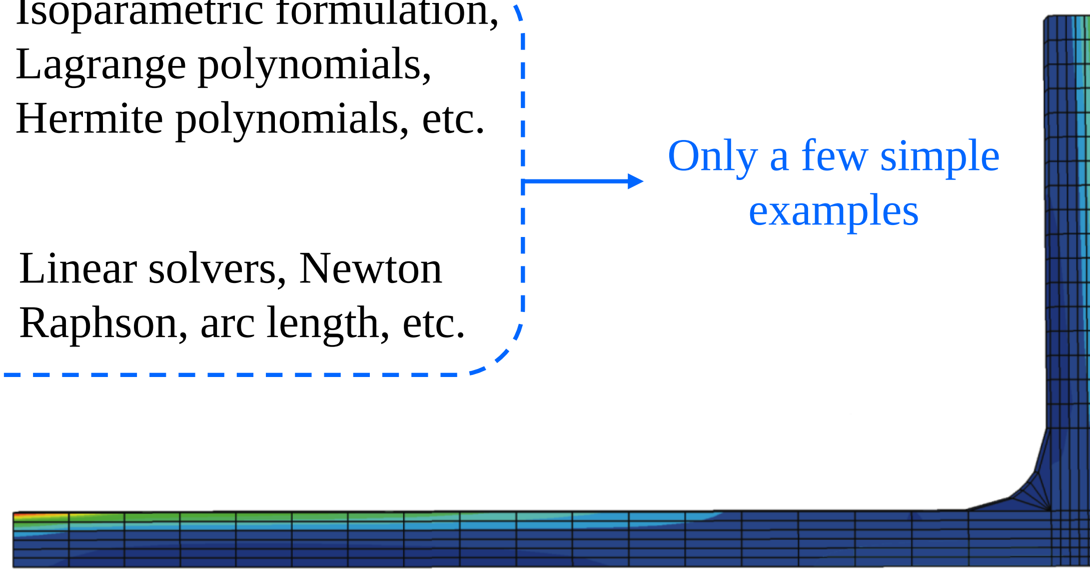
Weak form → Potential minimization, principle of virtual work, Galerkin method, etc.

Today's main focus!

Discretization → Isoparametric formulation, Lagrange polynomials, Hermite polynomials, etc.

Solution → Linear solvers, Newton Raphson, arc length, etc.

Only a few simple examples



Strong form

General form of 2D second order partial differential equations (PDEs)

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = \phi\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

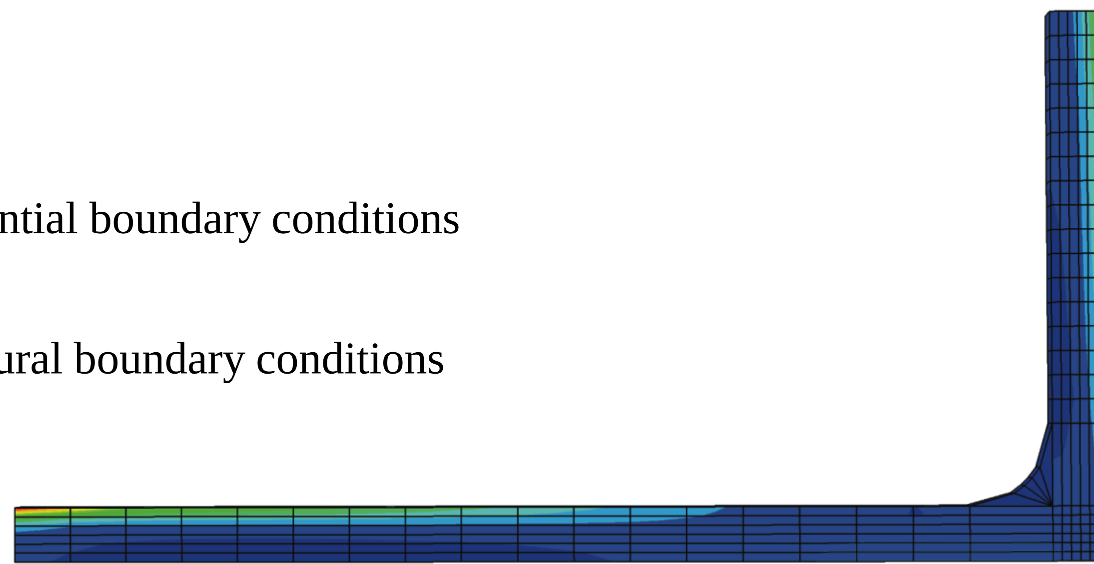
Categorization:

$$B^2 - AC = \begin{cases} < 0 \rightarrow \text{elliptic} \\ = 0 \rightarrow \text{parabolic} \\ > 0 \rightarrow \text{hyperbolic} \end{cases}$$

Boundary conditions (BCs):

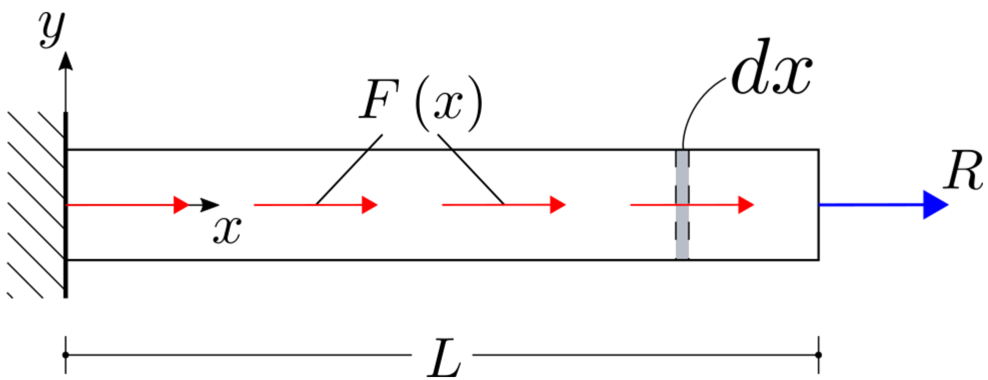
$u(x_0, y_0) = u_0$ Dirichlet or essential boundary conditions

$\dfrac{\partial^{m+n} u}{\partial x^m \partial y^n} = \bar{u}(x,u)$ Neumann of natural boundary conditions



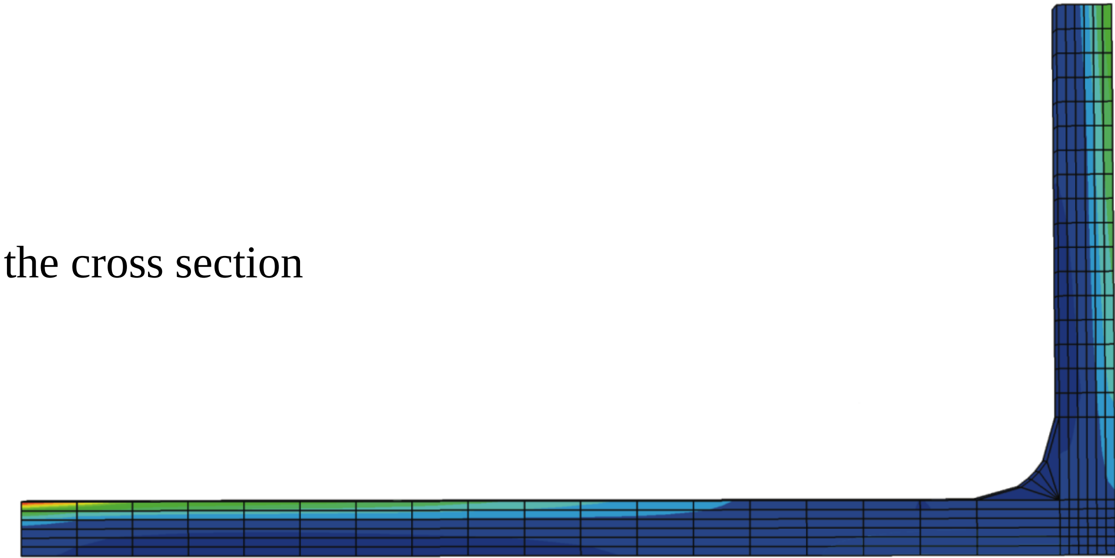
Strong form – 1D bar

Illustrative example – 1D bar with a distributed and an end load

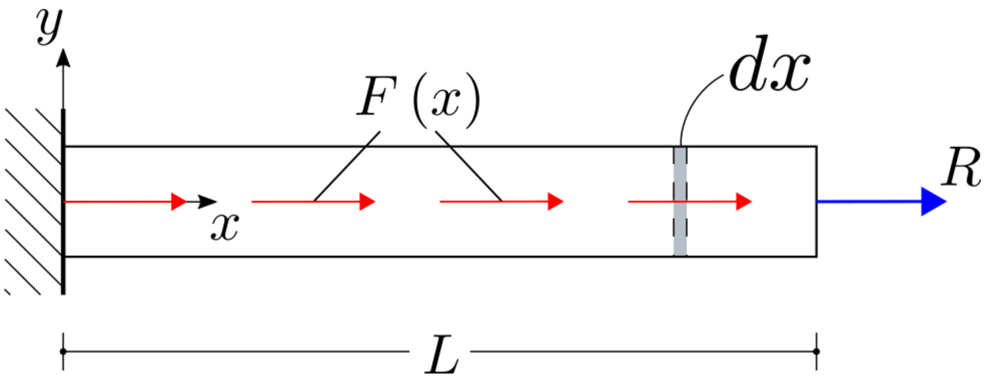


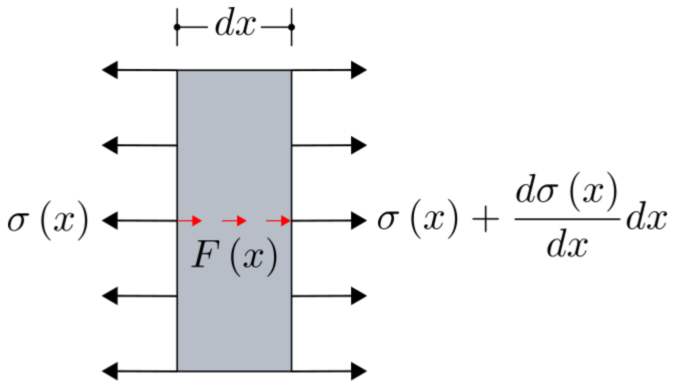
Assumptions:

- Constant cross section
- Linear elastic material
- Loads applied at the centroid of the cross section
- Arbitrary distributed load



Strong form



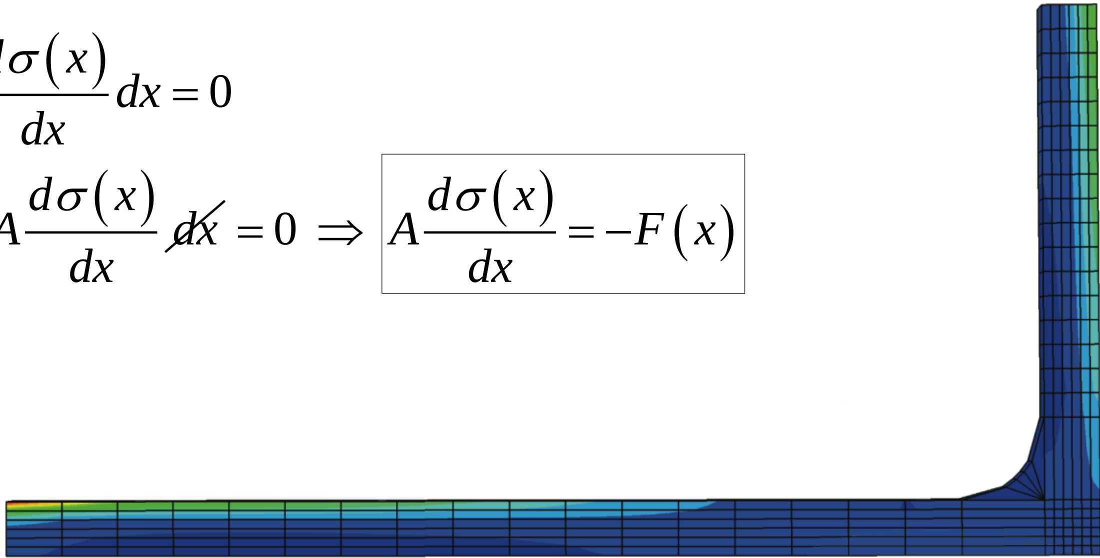


Equilibrium of forces along *x* direction

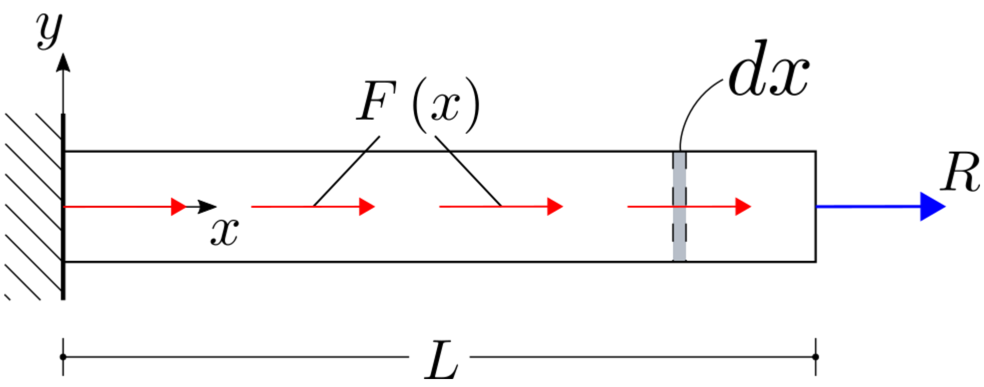
$$A\sigma(x) - F(x)dx - A\sigma(x+dx) = 0$$

$$\Rightarrow A\sigma(x) - F(x)dx - A\sigma(x) - A\frac{d\sigma(x)}{dx}dx = 0$$

$$\Rightarrow \cancel{A\sigma(x)} - F(x)\cancel{dx} - \cancel{A\sigma(x)} - A\frac{d\sigma(x)}{dx}\cancel{dx} = 0 \Rightarrow \boxed{A\frac{d\sigma(x)}{dx} = -F(x)}$$



Strong form



Kinematic equation (strain definition)

$$\varepsilon = \frac{du}{dx}$$

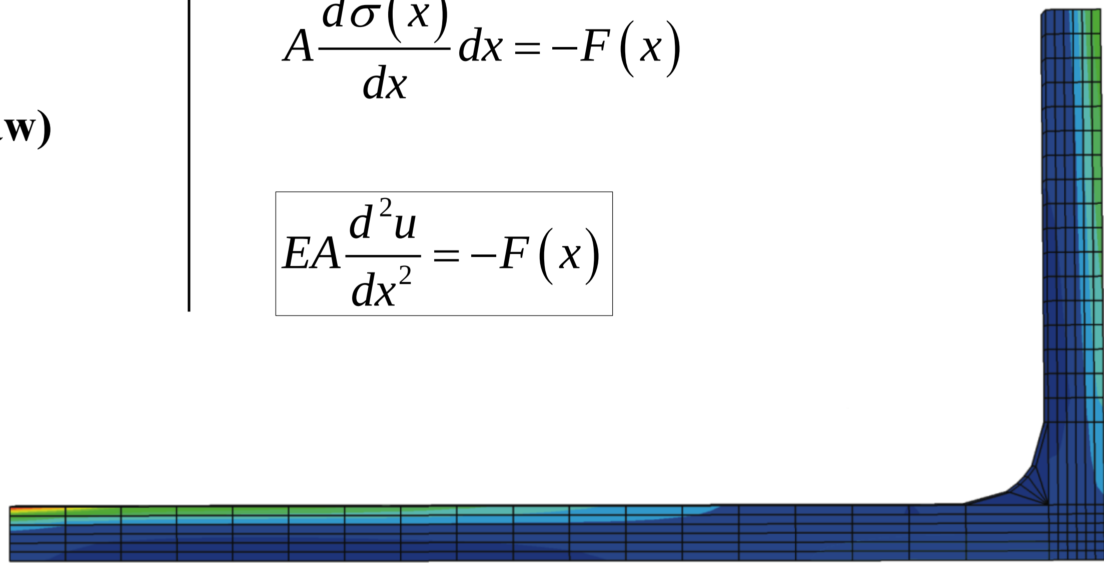
Constitutive Equation (Hooke's law)

$$\sigma = E\varepsilon = E\frac{du}{dx}$$

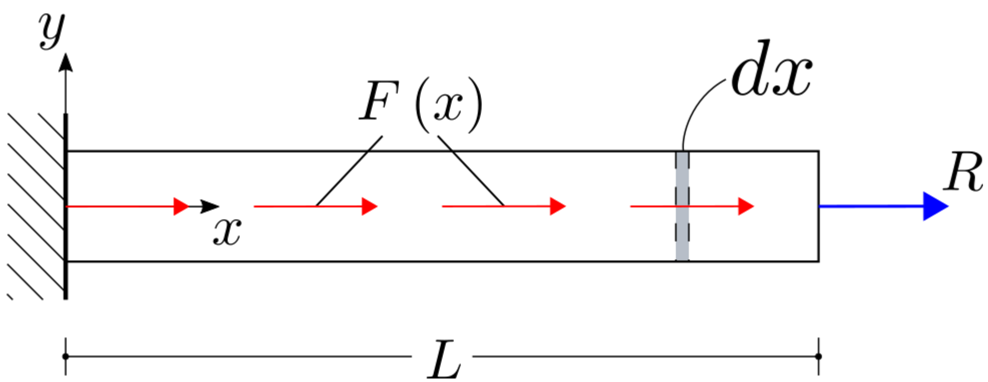
Equilibrium Equation

$$A\frac{d\sigma(x)}{dx}dx = -F(x)$$

$$\boxed{EA\frac{d^2u}{dx^2} = -F(x)}$$



Strong form



Equilibrium Equation

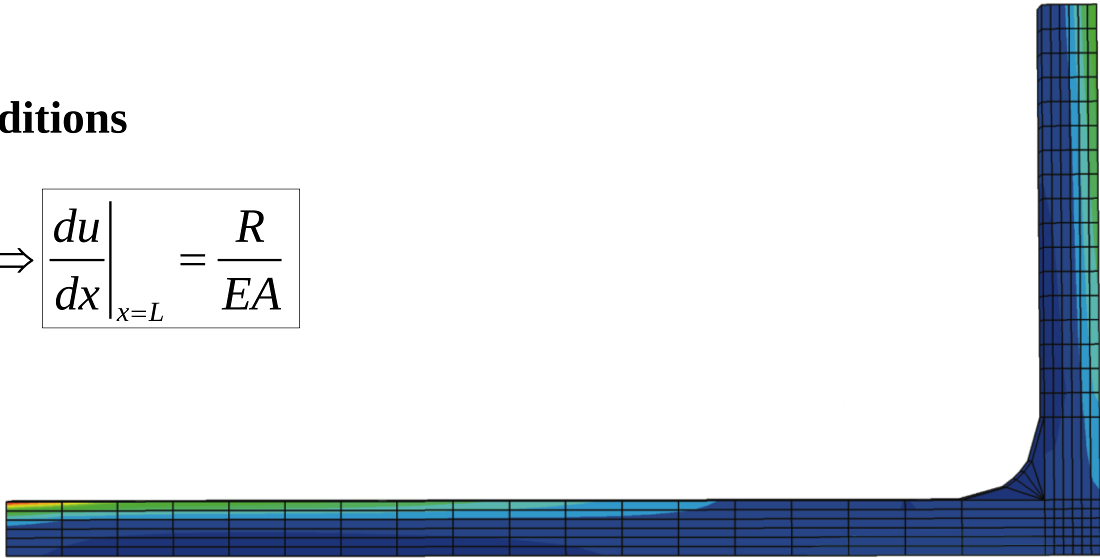
$$\boxed{EA\frac{d^2u}{dx^2} = -F(x)}$$

Dirichlet (essential) boundary conditions

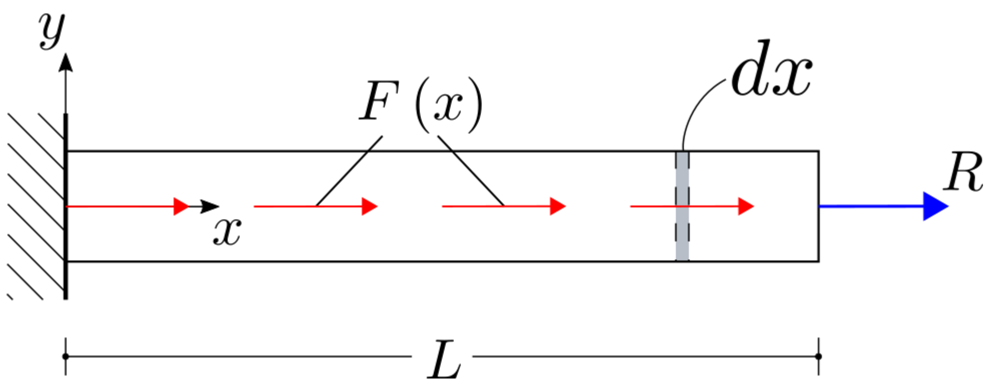
$$\boxed{u(x=0)=0}$$

Neumann (natural) boundary conditions

$$A\sigma(x=L)=R \Rightarrow AE\left.\frac{du}{dx}\right|_{x=L} = R \Rightarrow \boxed{\left.\frac{du}{dx}\right|_{x=L} = \frac{R}{EA}}$$



Strong form solution



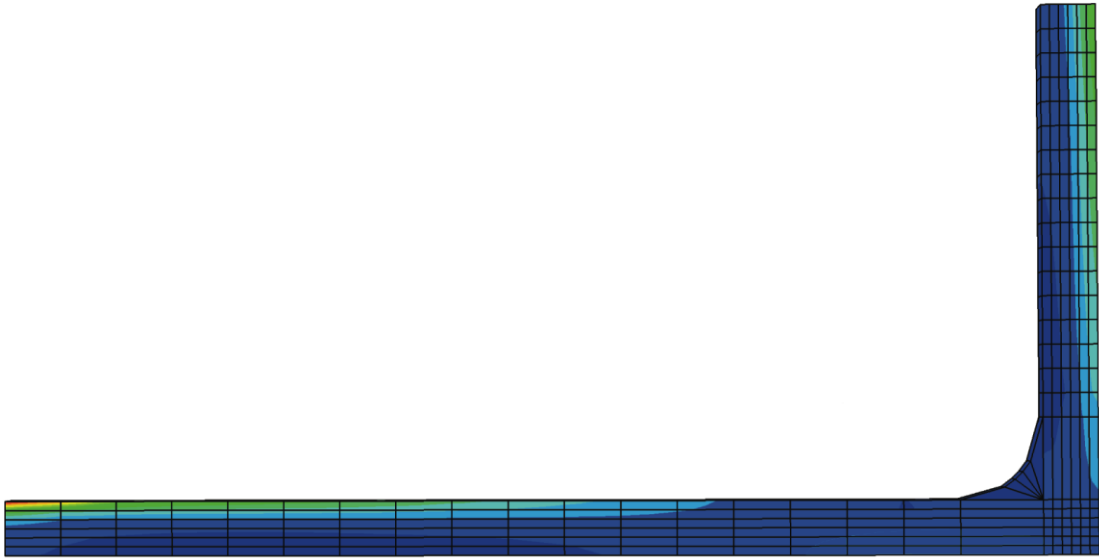
Assuming no distributed load:

$$F(x) = 0$$

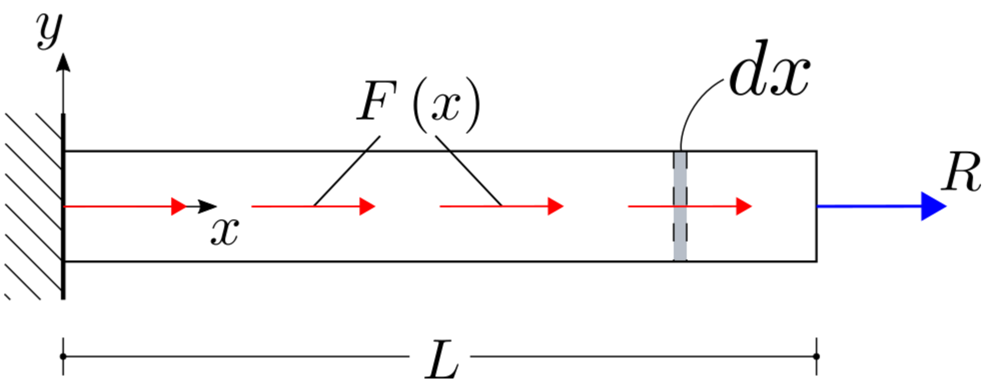
$$EA\frac{d^2u}{dx^2} = 0$$

$$u(x=0) = 0$$

$$\left.\frac{du}{dx}\right|_{x=L} = \frac{R}{EA}$$



Strong form solution



Assuming no distributed load:

$$F(x) = 0$$

$$EA\frac{d^2u}{dx^2} = 0$$

$$u(x=0) = 0$$

$$\left.\frac{du}{dx}\right|_{x=L} = \frac{R}{EA}$$

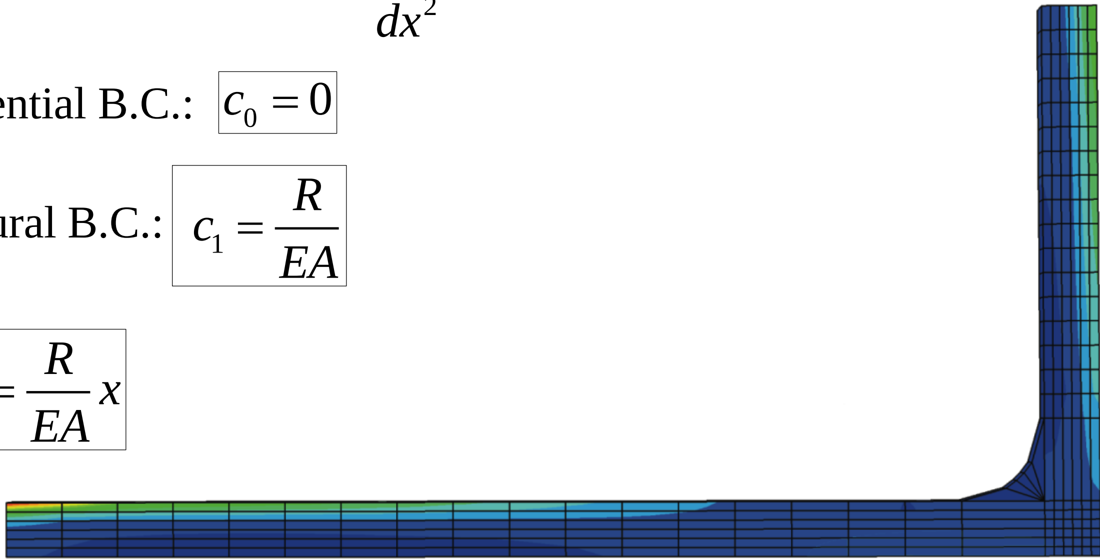
The solution should be of the form: $u(x) = c_0 + c_1 x$

Equilibrium is satisfied: $EA\frac{d^2(c_0 + c_1 x)}{dx^2} = 0$

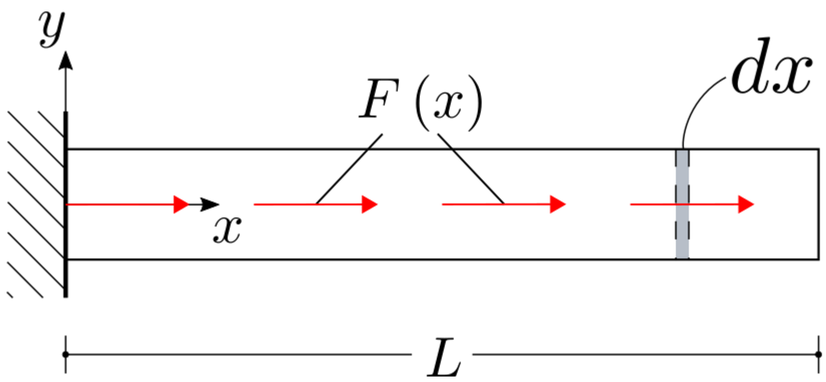
From the essential B.C.: $\boxed{c_0 = 0}$

From the natural B.C.: $\boxed{c_1 = \frac{R}{EA}}$

Final form of the solution: $\boxed{u(x) = \frac{R}{EA}x}$



Strong form solution



Similarly, assuming a linear distributed load and no end load:

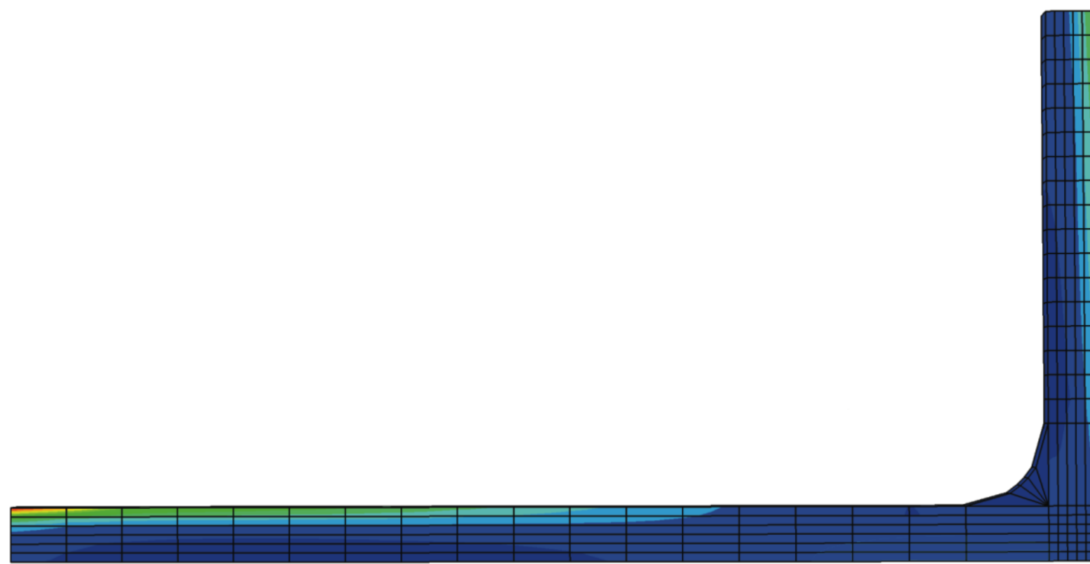
$$F(x) = ax$$

$$R = 0$$

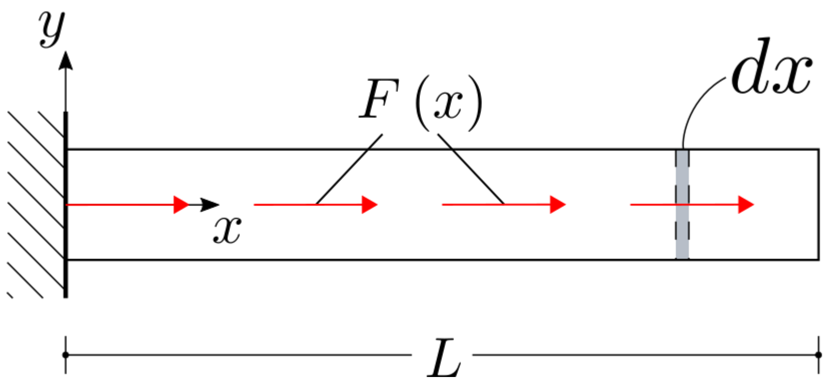
$$EA\frac{d^2u}{dx^2} = -ax$$

$$u(x=0) = 0$$

$$\left.\frac{du}{dx}\right|_{x=L} = 0$$



Strong form solution



Similarly, assuming a linear distributed load and no end load:

$$F(x) = ax$$

$$R = 0$$

$$EA\frac{d^2u}{dx^2} = -ax$$

$$u(x=0) = 0$$

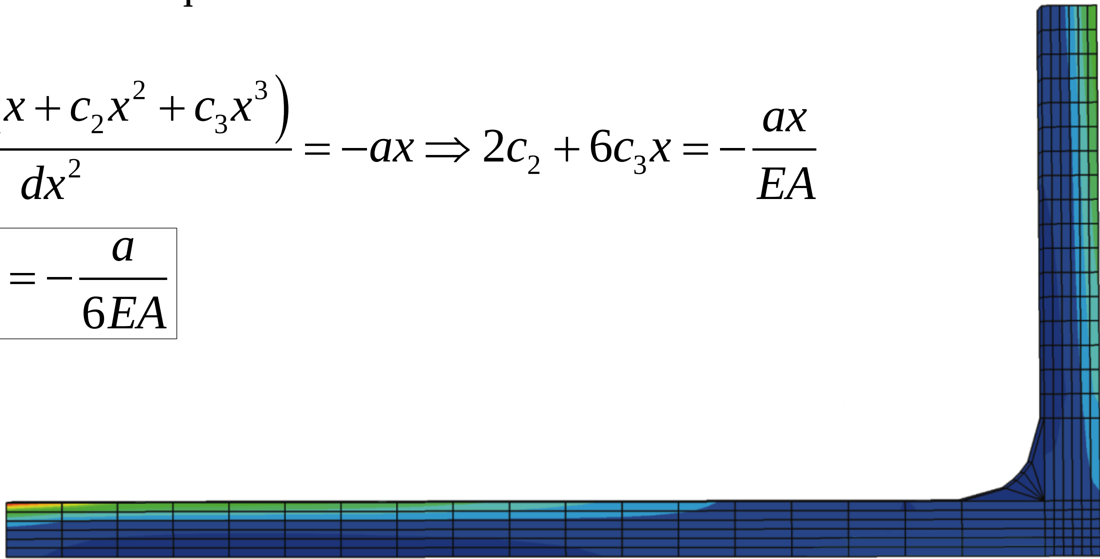
$$\left.\frac{du}{dx}\right|_{x=L} = 0$$

The solution should be of the form: $u(x) = c_0 + c_1x + c_2x^2 + c_3x^3$

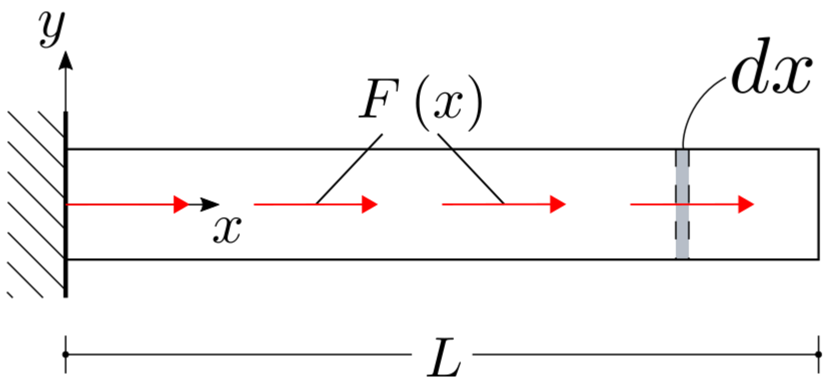
From the equilibrium equation:

$$EA\frac{d^2\left(c_0 + c_1x + c_2x^2 + c_3x^3\right)}{dx^2} = -ax \Rightarrow 2c_2 + 6c_3x = -\frac{ax}{EA}$$

$$\Rightarrow \boxed{c_2 = 0, \quad c_3 = -\frac{a}{6EA}}$$



Strong form solution



Similarly, assuming a linear distributed load and no end load:

$$F(x) = ax$$

$$R = 0$$

$$\boxed{\begin{aligned} &EA\frac{d^2u}{dx^2} = -ax \\ &u(x=0) = 0 \\ &\left.\frac{du}{dx}\right|_{x=L} = 0 \end{aligned}}$$

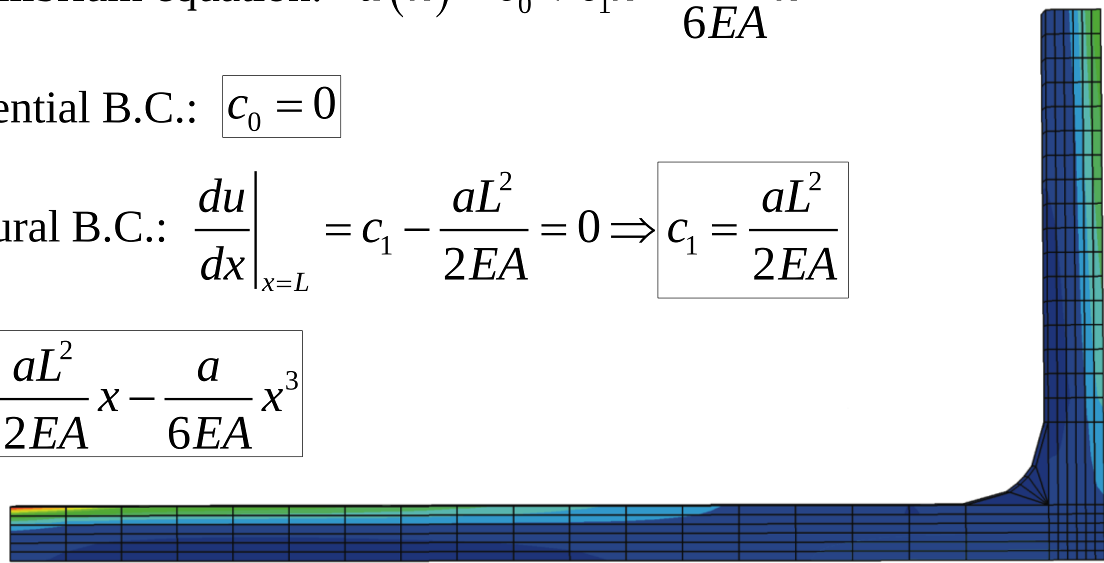
The solution should be of the form: $u(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

From the equilibrium equation: $u(x) = c_0 + c_1 x - \dfrac{a}{6EA}x^3$

From the essential B.C.: $\boxed{c_0 = 0}$

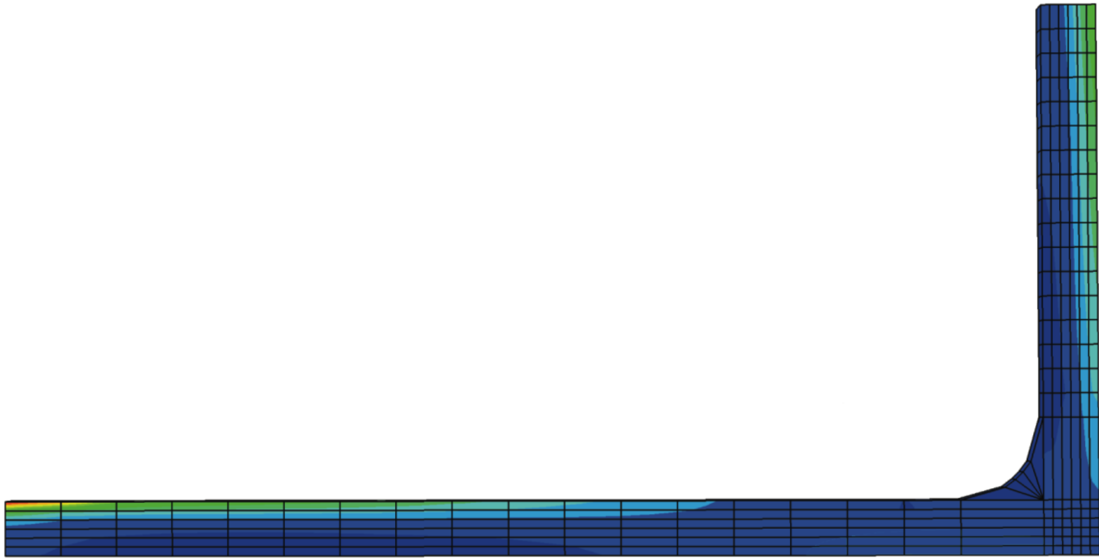
From the natural B.C.: $\left.\dfrac{du}{dx}\right|_{x=L} = c_1 - \dfrac{aL^2}{2EA} = 0 \Rightarrow \boxed{c_1 = \dfrac{aL^2}{2EA}}$

Final form of the solution: $\boxed{u(x) = \dfrac{aL^2}{2EA}x - \dfrac{a}{6EA}x^3}$



Strong form solution

- Analytical solutions satisfy the PDE at every point of the domain, thus the PDE is called the “strong” form of the problem
- It is not possible to derive such solutions for complex combinations of PDEs, geometries and BCs
- Typically, approximate, numerical solutions are sought for
- In what follows, some tools to systematically derive such solutions will be presented



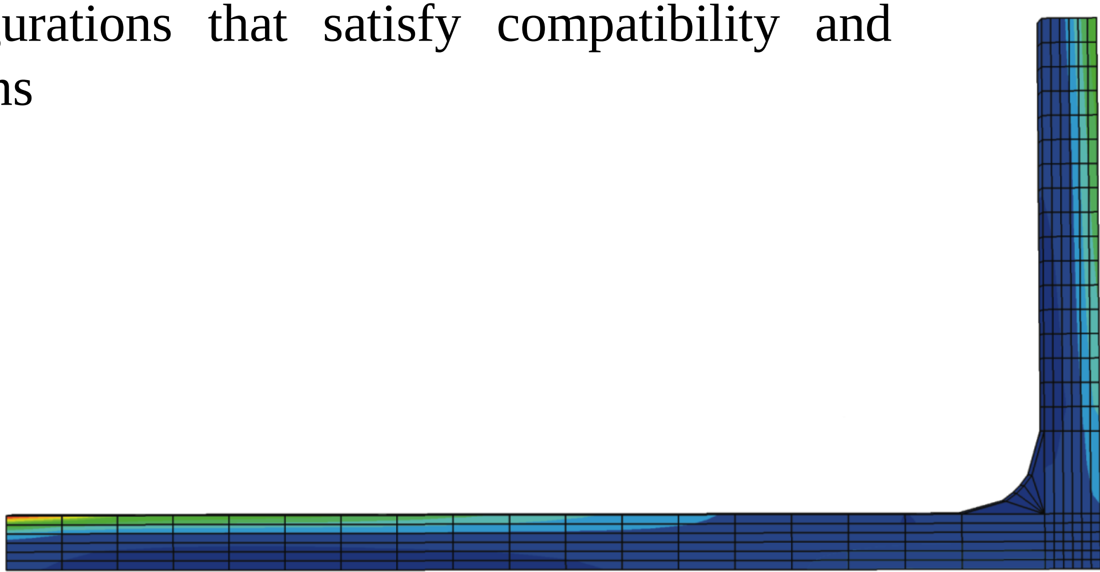
Potential minimization

Principle of stationary potential energy

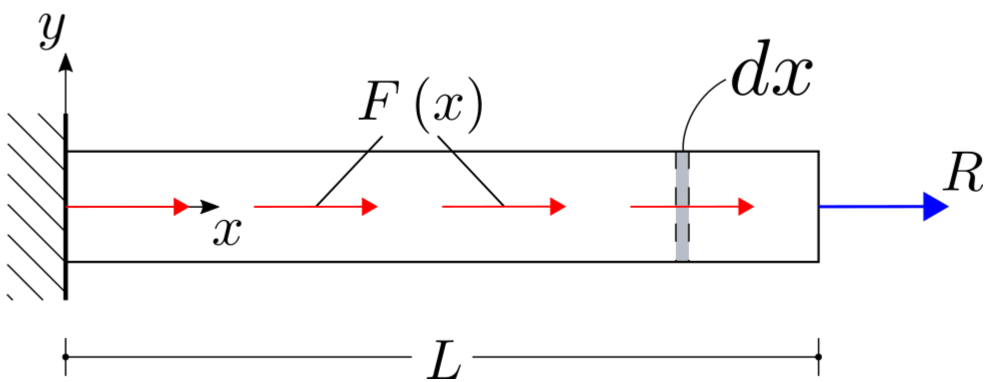
"Among all admissible configurations of a conservative system, those that satisfy the equations of equilibrium, make the potential energy stationary with respect to small admissible variations of the displacements."

where:

- Admissible are all configurations that satisfy compatibility and essential boundary conditions



Potential minimization



For the problem of the bar, potential energy has two components:

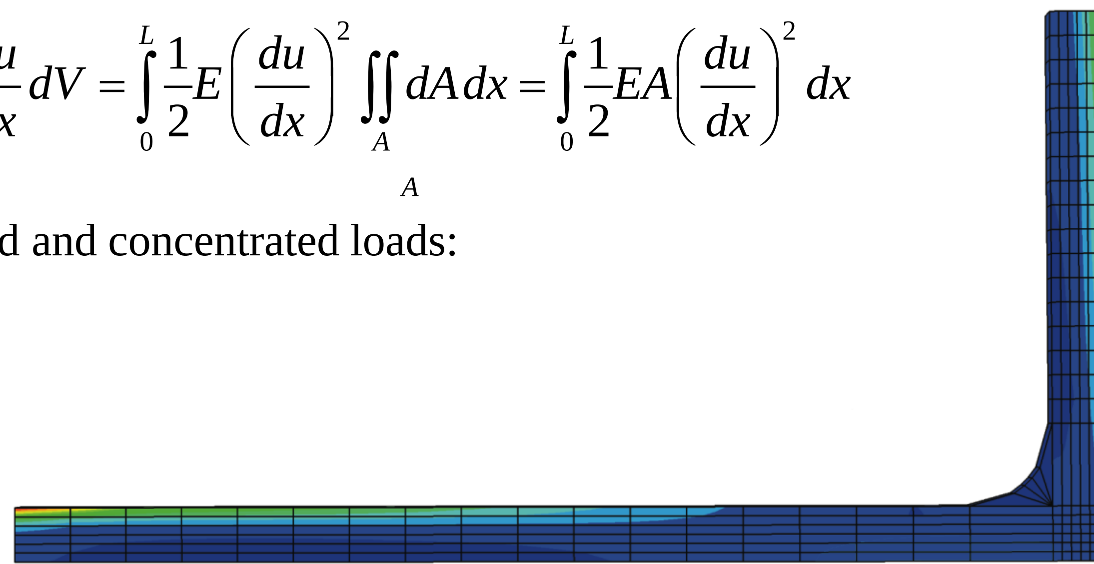
- Internal strain energy

$$U = \iiint_V \frac{1}{2}\sigma\,\varepsilon\, dV = \iiint_V \frac{1}{2}E\,\frac{du}{dx}\frac{du}{dx}\,dV = \int_0^L \frac{1}{2}E\left(\frac{du}{dx}\right)^2 \iint_A dA\,dx = \int_0^L \frac{1}{2}EA\left(\frac{du}{dx}\right)^2 dx$$

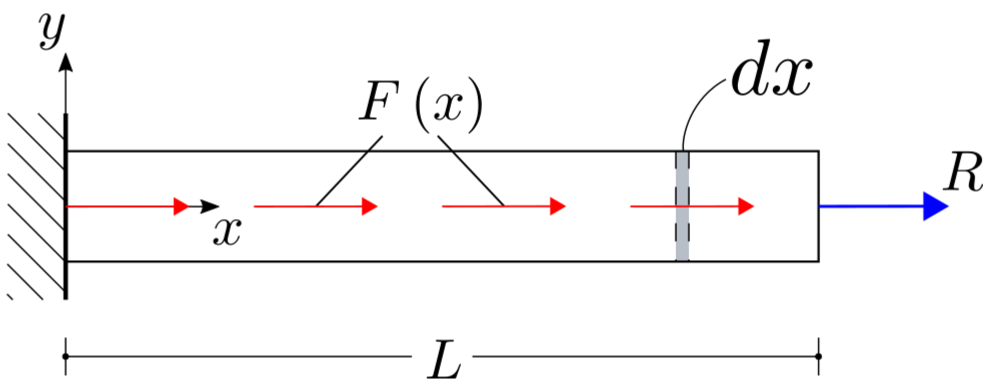
A

- Work produced by the distributed and concentrated loads:

$$W = -\int_0^L F(x)\,u(x)\,dx - Ru(L)$$



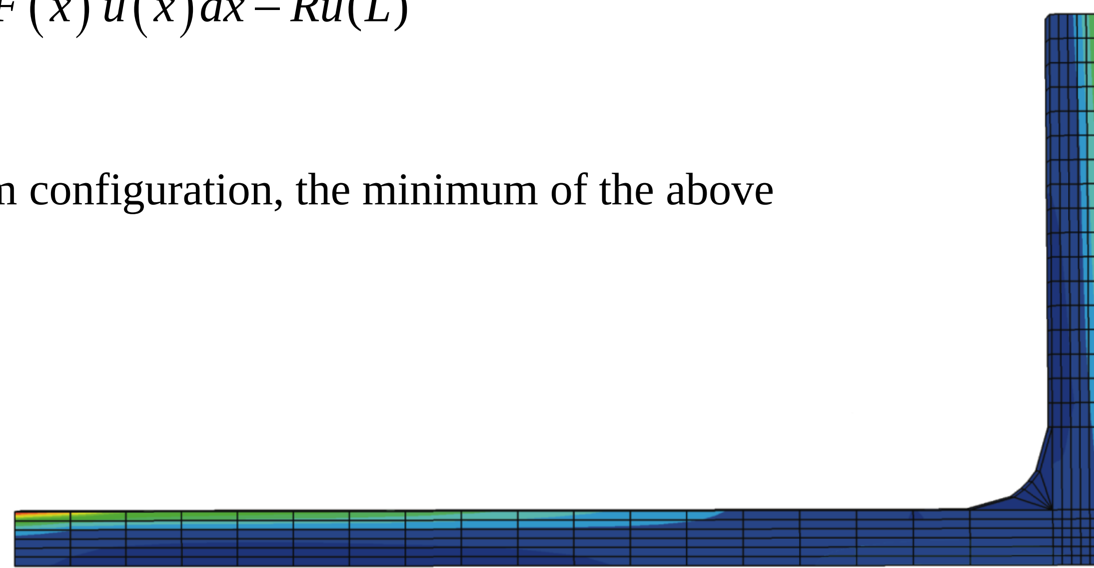
Potential minimization



Total potential energy:

$$\Pi = U + W = \int_0^L \frac{1}{2} EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L F\left(x\right) u\left(x\right) dx - Ru(L)$$

- In order to obtain the equilibrium configuration, the minimum of the above expression should be sought



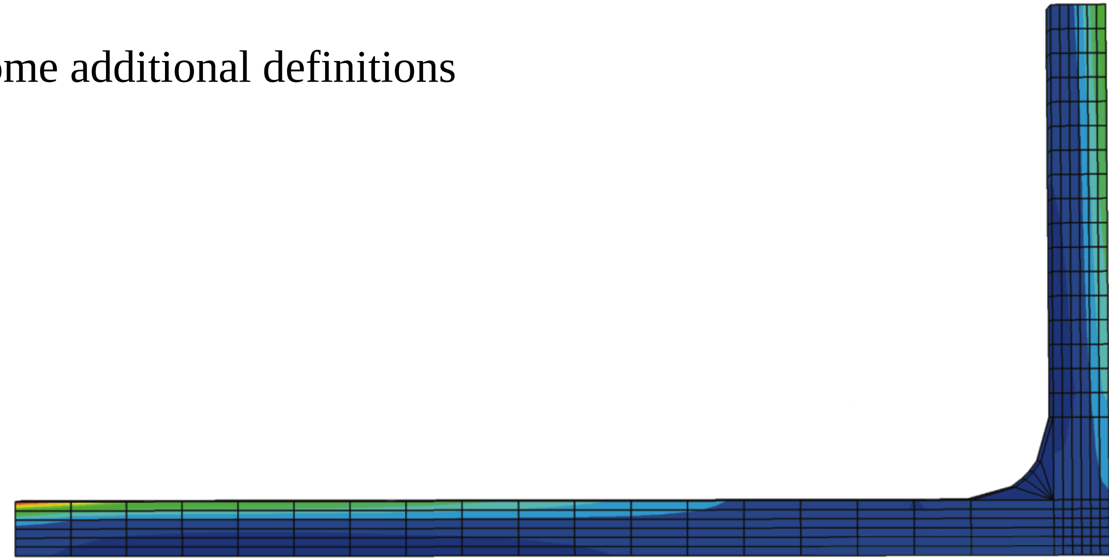
Potential minimization

The expression derived for the total potential energy is a functional, i.e. a mapping from a function space to the real numbers.

In simpler words:

- Functions take numbers as input and return numbers as output, i.e. they map numbers to numbers
- Functionals take functions as input and return numbers as output, i.e. they map functions to numbers

To minimize a functional, we need some additional definitions



Potential minimization

Variation of a function

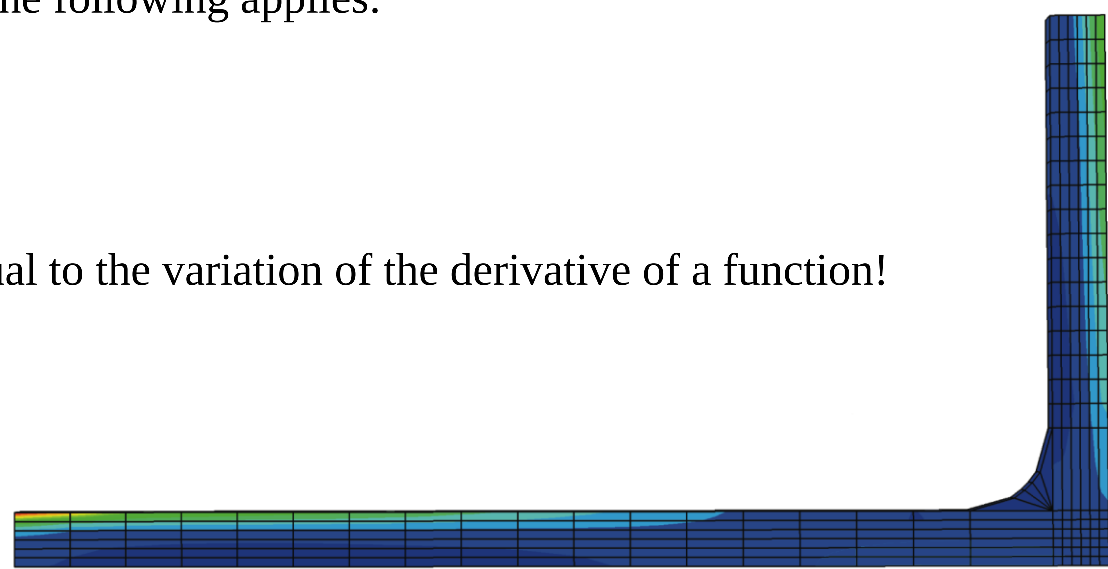
The variation of a function $u(x)$ is defined as an arbitrary and sufficiently smooth function $u(x)$ that vanishes at the points where boundary conditions are applied:

$$\delta u = \eta$$

For the derivatives of the variation, the following applies:

$$\frac{d^n \eta}{dx^n} = \frac{d^n \delta u}{dx^n} = \delta\left(\frac{d^n u}{dx^n}\right)$$

The derivative of the variation is equal to the variation of the derivative of a function!



Potential minimization

Variation of a functional

The variation of a functional F of a function u and its derivatives $\left(u',u'',\ldots,u^n\right)$ is defined as:

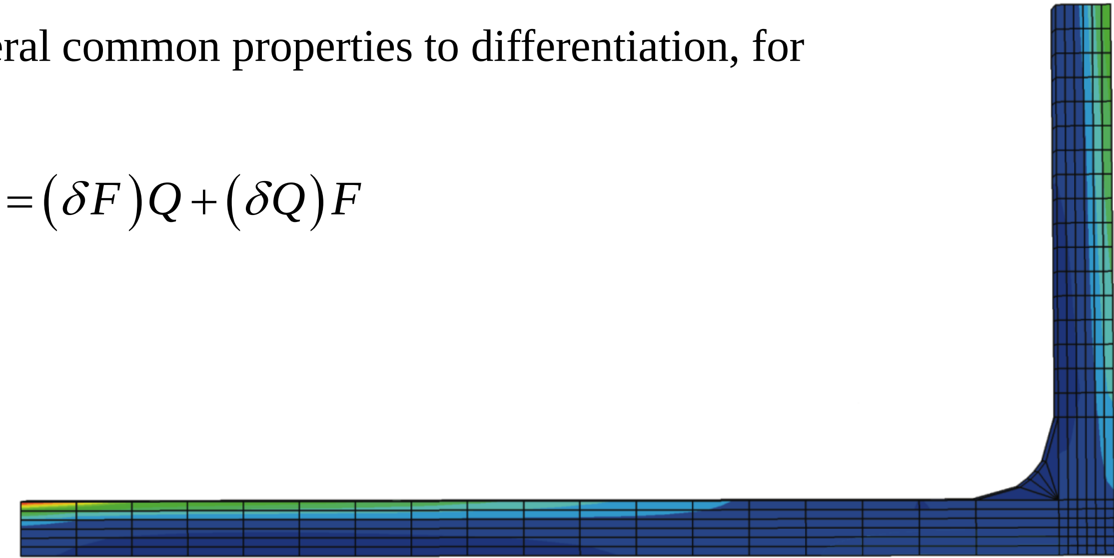
$$\delta F=\lim_{\varepsilon\to 0}\frac{F\left[u+\varepsilon\eta,\left(u+\varepsilon\eta\right)',\left(u+\varepsilon\eta\right)'',\ldots,\left(u+\varepsilon\eta\right)^n\right]-F\left[u,u',u'',\ldots,u^n\right]}{\varepsilon}$$

- Similar to functions, the variations of functionals vanish at stationary points.
- The variational operator has several common properties to differentiation, for instance:

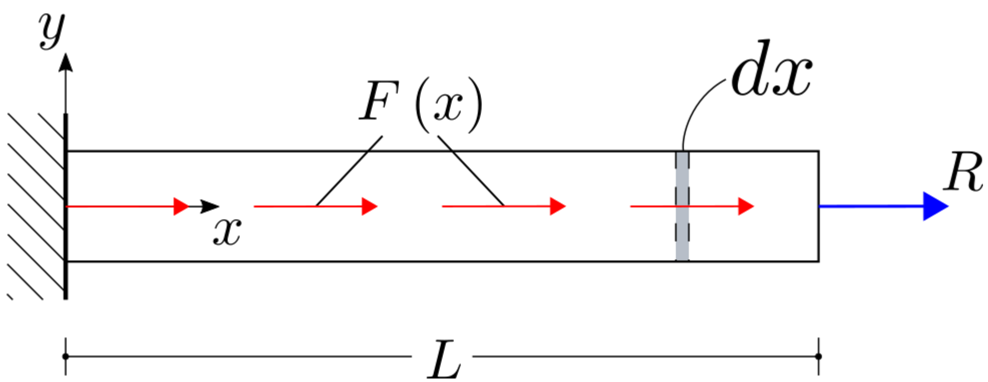
$$\delta\left(F+Q\right)=\delta F+\delta Q,\quad \delta\left(FQ\right)=\left(\delta F\right)Q+\left(\delta Q\right)F$$

- Also:

$$\delta\int F\left(x\right)dx=\int\delta F\left(x\right)dx$$



Potential minimization



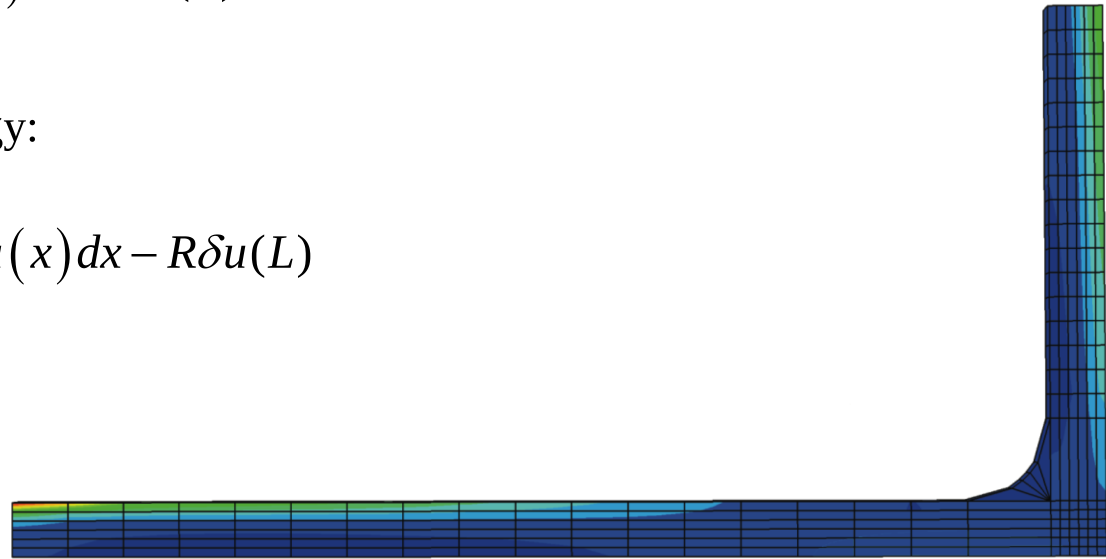
In order to minimize the potential energy functional, its variation should be computed and set to zero

Total potential energy:

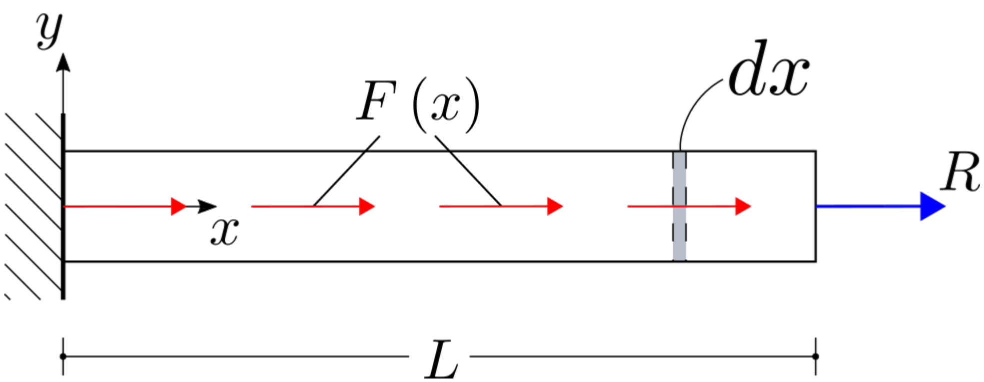
$$\Pi = \int_0^L \frac{1}{2} EA \left(\frac{du}{dx} \right)^2 dx - \int_0^L F(x)\, u(x) dx - Ru(L)$$

Variation of the total potential energy:

$$\delta\Pi = \int_0^L EA \frac{du}{dx} \delta \frac{du}{dx} dx - \int_0^L F(x)\, \delta u(x) dx - R\delta u(L)$$



Potential minimization



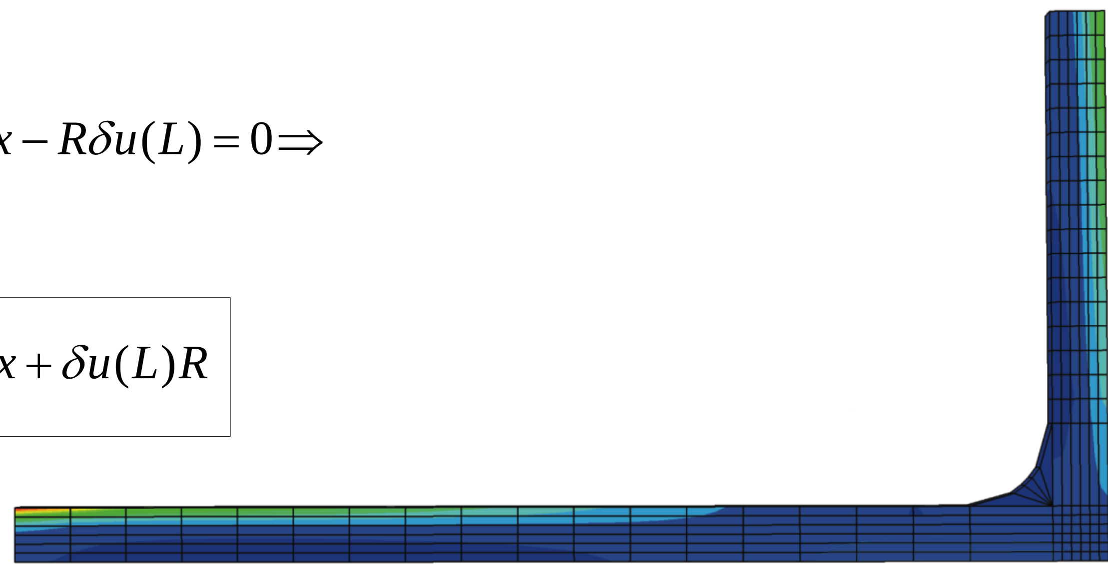
In order to minimize the potential energy functional, its variation should be computed and set to zero

Stationarity condition:

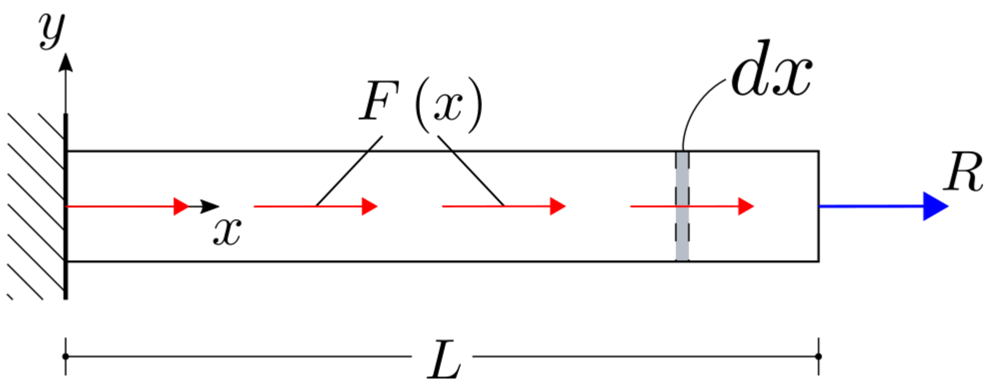
$$\delta\Pi = 0 \Rightarrow$$

$$\int_0^L EA\frac{du}{dx}\delta\frac{du}{dx}dx - \int_0^L F(x)\,\delta u(x)dx - R\delta u(L) = 0 \Rightarrow$$

$$\boxed{\int_0^L EA\frac{du}{dx}\delta\frac{du}{dx}dx = \int_0^L \delta u(x)F(x)\,dx + \delta u(L)R}$$



Principle of Virtual Work



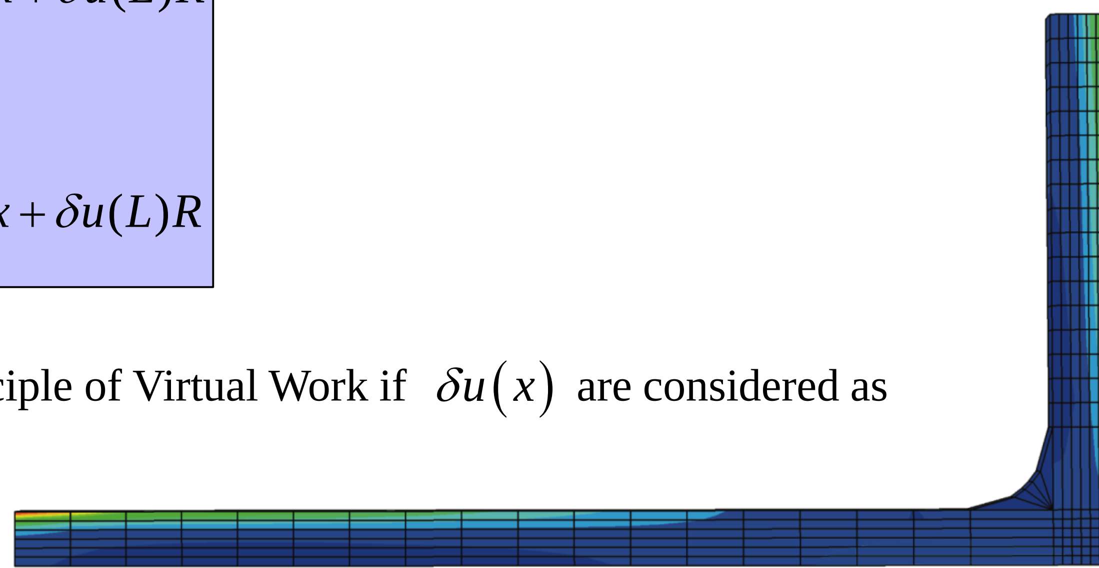
Principle of stationary potential energy:

$$\int_0^L EA\frac{du}{dx}\delta\frac{du}{dx}dx = \int_0^L \delta u(x)F(x)\,dx + \delta u(L)R$$

or

$$\int_0^L EA\frac{du}{dx}\frac{d\delta u}{dx}dx = \int_0^L \delta u(x)F(x)\,dx + \delta u(L)R$$

The above is equivalent to the Principle of Virtual Work if $\delta u(x)$ are considered as virtual displacements



Potential minimization

The principle of stationary potential energy is equivalent to the equilibrium equations and natural BCs, to show that integration by parts is required.

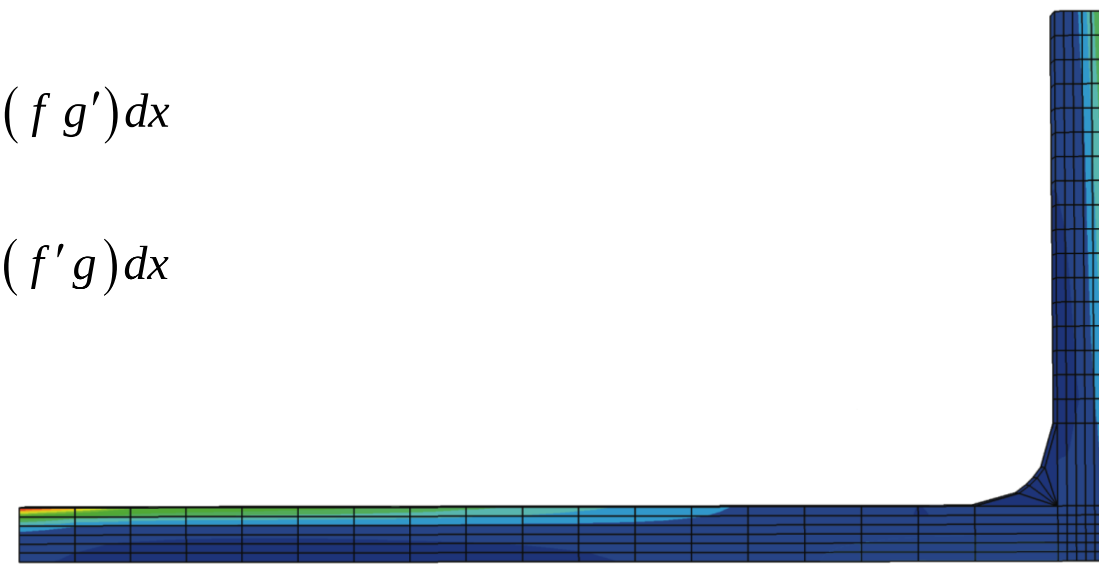
Reminder: Integration by parts

$$(f\,g)' = f'g + f\,g'$$

$$\Rightarrow \int_a^b (f\,g)'\,dx = \int_a^b (f'g + f\,g')\,dx$$

$$\Rightarrow f\,g\big|_{x=b} - f\,g\big|_{x=a} = \int_a^b (f'g)\,dx + \int_a^b (f\,g')\,dx$$

$$\Rightarrow \int_a^b (f\,g')\,dx = f\,g\big|_{x=b} - f\,g\big|_{x=a} - \int_a^b (f'g)\,dx$$



Potential minimization

Principle of stationary potential energy:

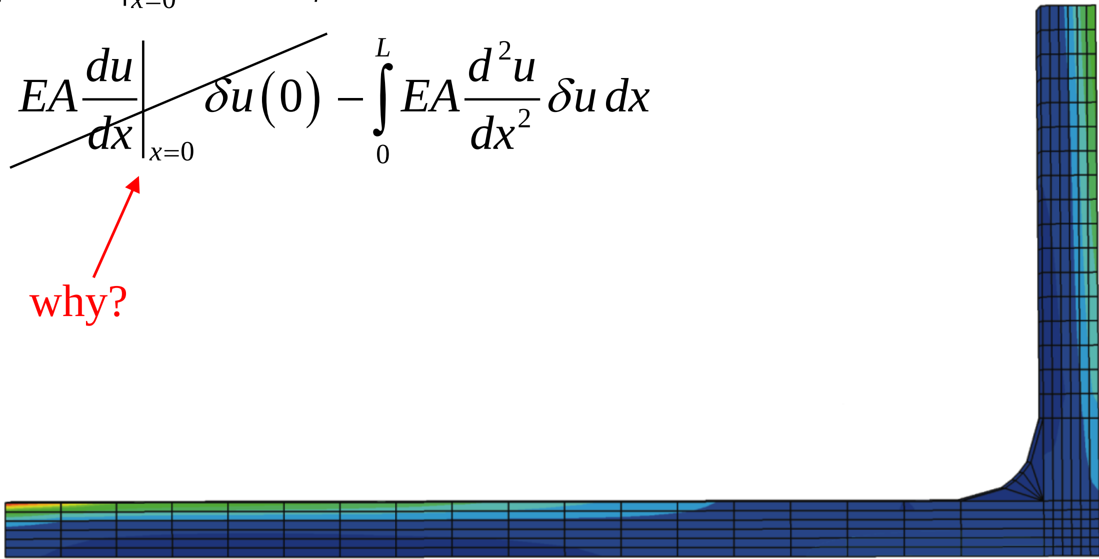
$$\boxed{\int_0^L EA\frac{du}{dx}\delta\frac{du}{dx}\,dx - \int_0^L \delta u(x)F(x)\,dx - \delta u(L)R = 0}$$

Integration by parts for the first term:

$$\int_0^L \underbrace{EA\frac{du}{dx}}_{f}\underbrace{\delta\frac{du}{dx}}_{g'}dx = \underbrace{EA\frac{du}{dx}}_{f}\underbrace{\delta u}_{g}\Bigg|_{x=L} - \underbrace{EA\frac{du}{dx}}_{f}\underbrace{\delta u}_{g}\Bigg|_{x=0} - \int_0^L \underbrace{EA\frac{d^2u}{dx^2}}_{f'}\underbrace{\delta u}_{g}\,dx =$$

$$= EA\frac{du}{dx}\bigg|_{x=L}\delta u(L) - \cancel{EA\frac{du}{dx}\bigg|_{x=0}\delta u(0)} - \int_0^L EA\frac{d^2u}{dx^2}\delta u\,dx$$

why?



Potential minimization

Combining the two parts:

$$EA\left.\frac{du}{dx}\right|_{x=L}\delta u(L)-\int_0^L EA\frac{d^2u}{dx^2}\delta u\,dx-\int_0^L \delta u(x)F(x)\,dx-\delta u(L)R=0 \Leftrightarrow$$

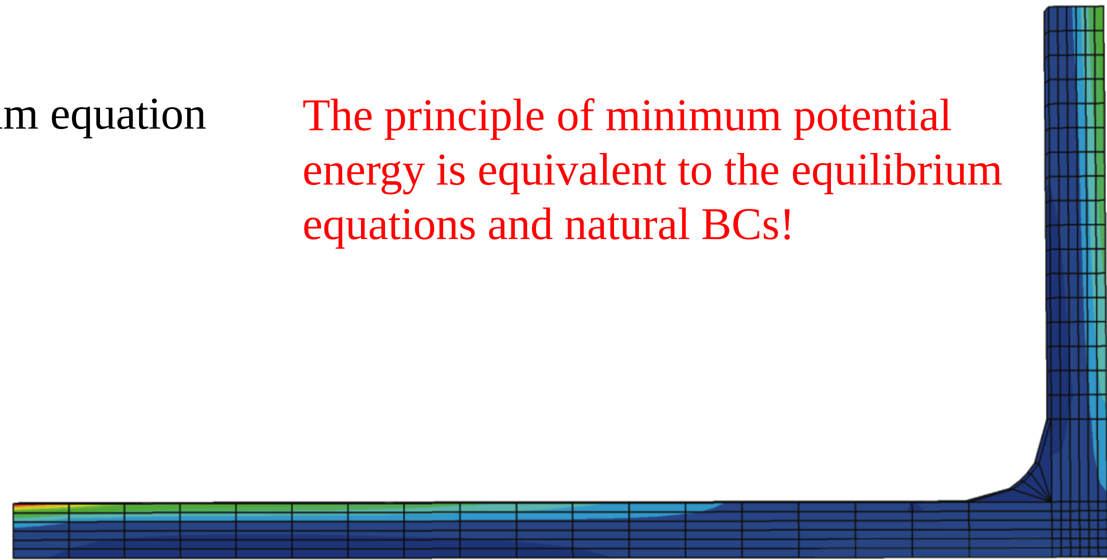
$$-\int_0^L\left(EA\frac{d^2u}{dx^2}+F(x)\right)\delta u\,dx+\left(EA\left.\frac{du}{dx}\right|_{x=L}-R\right)\delta u(L)=0$$

Since both δu and $\delta u(L)$ are arbitrary, the equation only holds if:

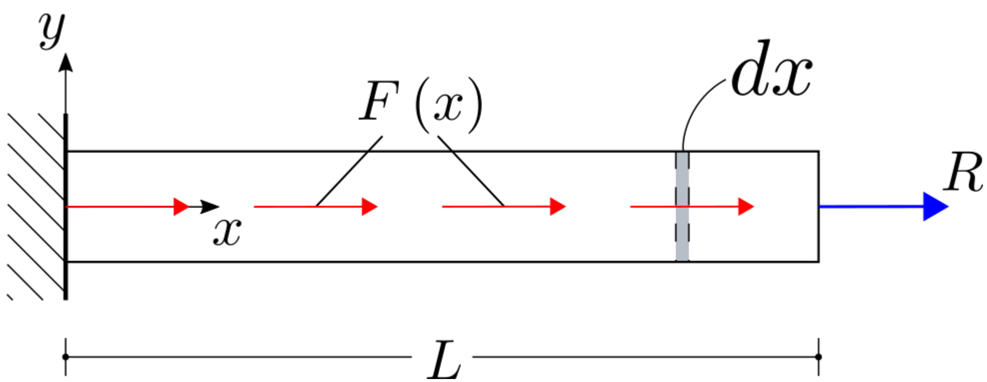
$$EA\frac{d^2u}{dx^2}+F(x)=0 \longrightarrow \text{Equilibrium equation}$$

$$EA\left.\frac{du}{dx}\right|_{x=L}-R=0 \longrightarrow \text{Natural BC}$$

The principle of minimum potential energy is equivalent to the equilibrium equations and natural BCs!



The Galerkin Method

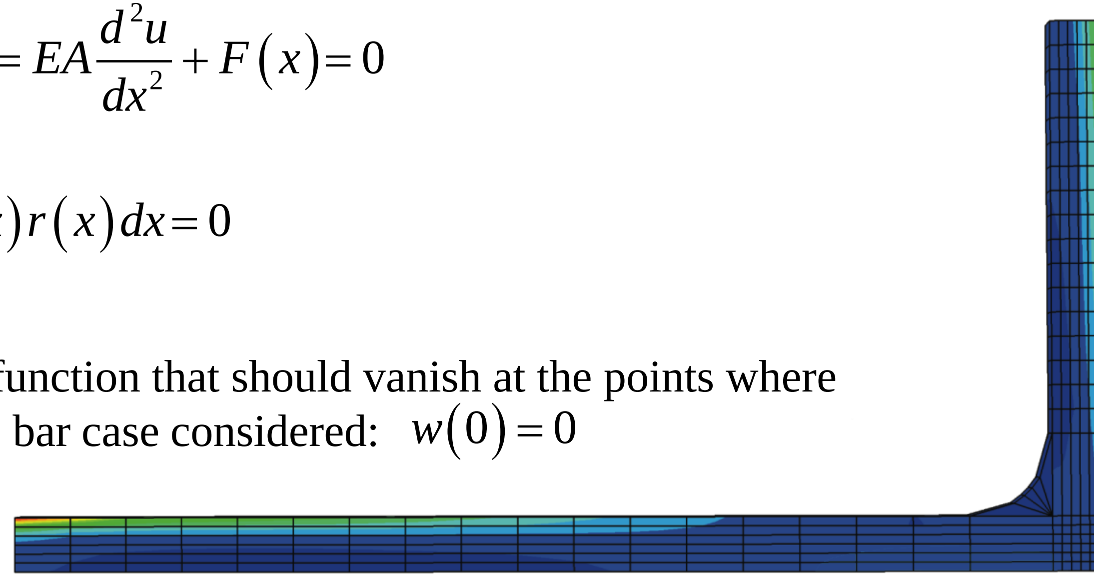


Differential equation $\rightarrow$ $$EA\frac{d^2u}{dx^2} = -F(x)$$

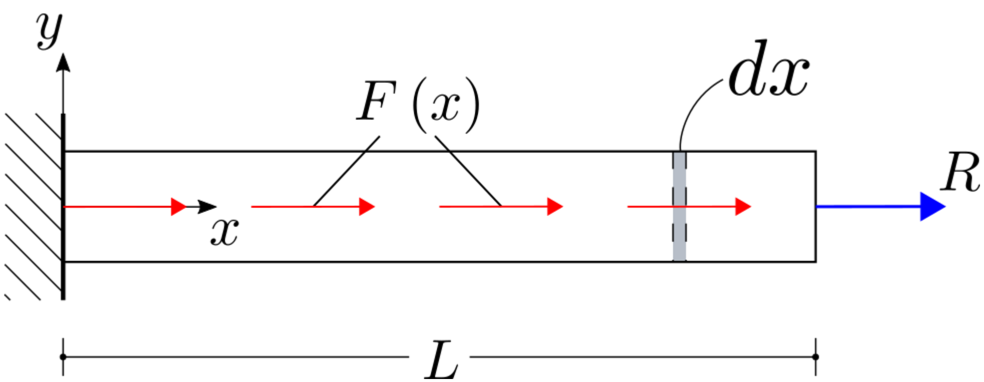
Residual $\rightarrow$ $$r(x) = EA\frac{d^2u}{dx^2} + F(x) = 0$$

Weighted Residual $\rightarrow$ $$\int_0^L w(x)r(x)dx = 0$$

Where $w(x)$ is an arbitrary weight function that should vanish at the points where essential BCs are applied, for the 1D bar case considered: $w(0) = 0$



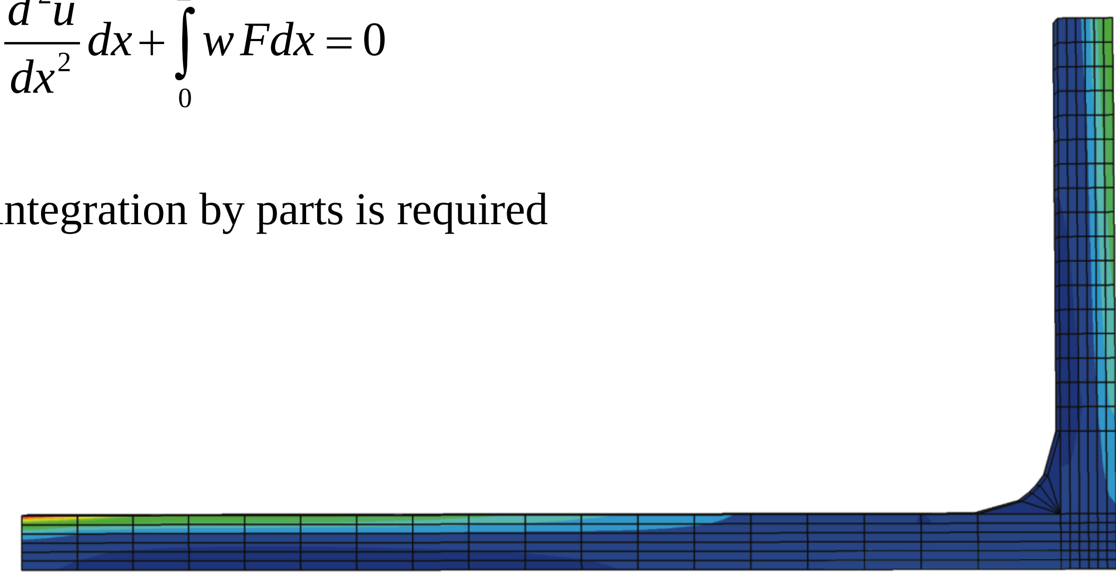
The Galerkin Method



Weighted Residual

$$\int_0^L w\left(EA\frac{d^2u}{dx^2}+F\right)dx=0 \Rightarrow \int_0^L wEA\frac{d^2u}{dx^2}dx+\int_0^L wFdx=0$$

To further process this expression, integration by parts is required



The Galerkin Method

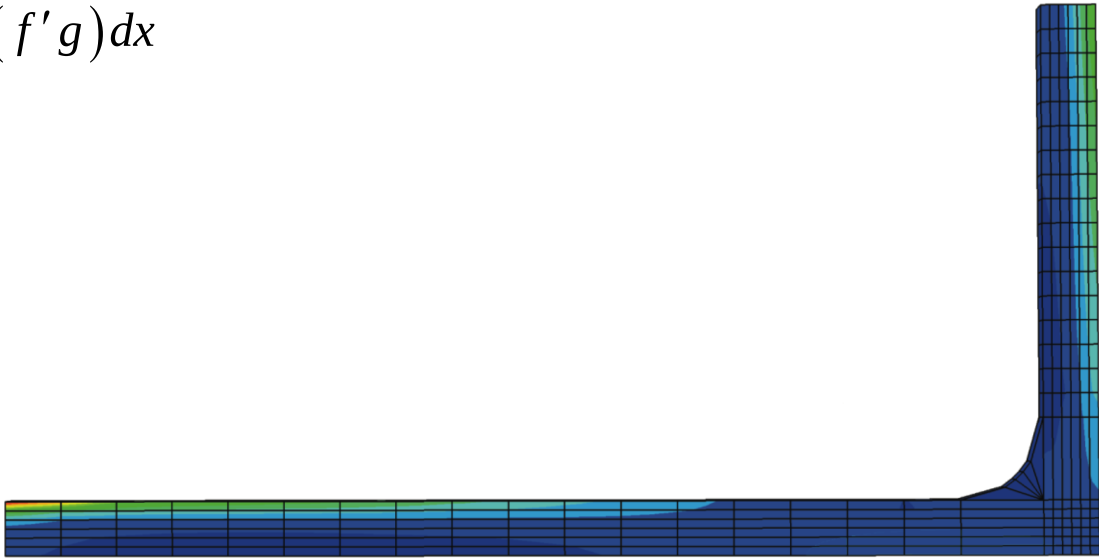
Reminder (again!): Integration by parts

$$(f\,g)' = f'g + f\,g'$$

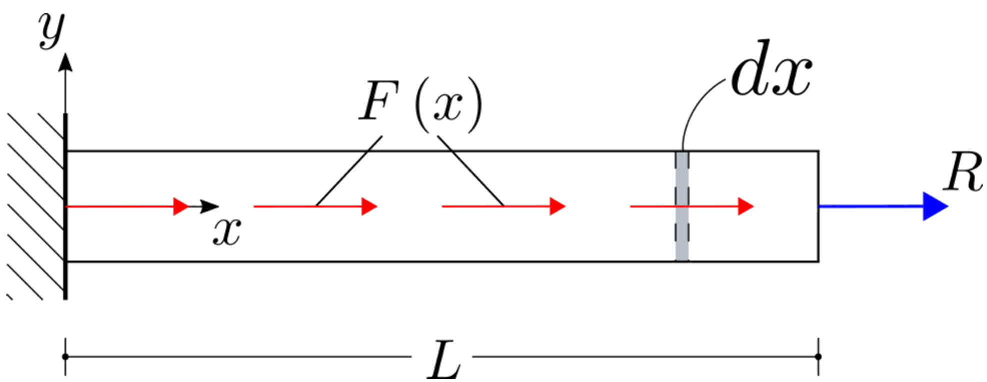
$$\Rightarrow \int_a^b (f\,g)'\,dx = \int_a^b (f'g + f\,g')\,dx$$

$$\Rightarrow f\,g\big|_{x=b} - f\,g\big|_{x=a} = \int_a^b (f'g)\,dx + \int_a^b (f\,g')\,dx$$

$$\Rightarrow \int_a^b (f\,g')\,dx = f\,g\big|_{x=b} - f\,g\big|_{x=a} - \int_a^b (f'g)\,dx$$



The Galerkin Method



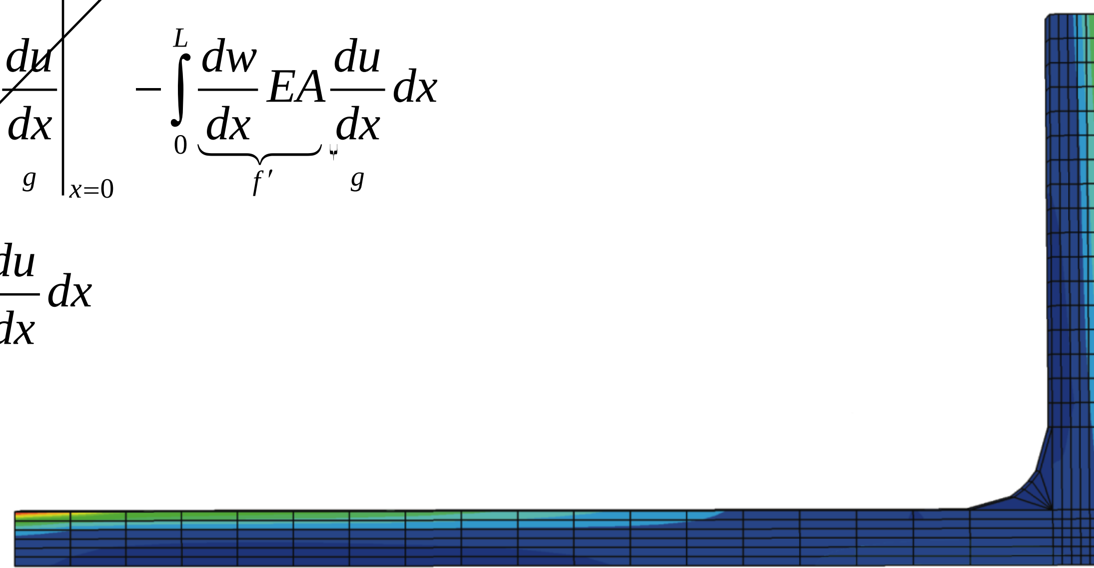
Essential BC: $\left. EA\frac{du}{dx}\right|_{x=L} = R$

Weight: $w(0)=0$

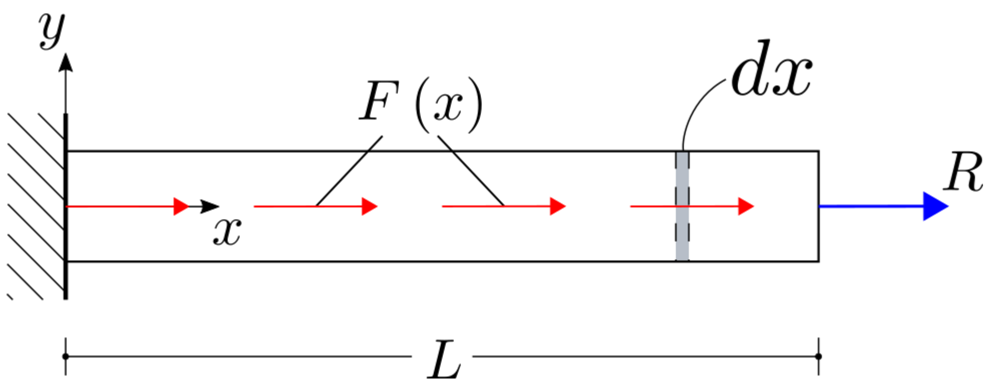
Integration by parts of the first term of the weighted residual

$$\int_0^L \underbrace{w}_{f} EA \underbrace{\frac{d^2u}{dx^2}}_{g'} dx = \left. \underbrace{w}_{f} EA \underbrace{\frac{du}{dx}}_{g} \right|_{x=L} - \cancel{\left. \underbrace{w}_{f} EA \underbrace{\frac{du}{dx}}_{g} \right|_{x=0}} - \int_0^L \underbrace{\frac{dw}{dx}}_{f'} EA \underbrace{\frac{du}{dx}}_{g} dx$$

$$= w(L)R - \int_0^L \frac{dw}{dx} EA \frac{du}{dx} dx$$



The Galerkin Method

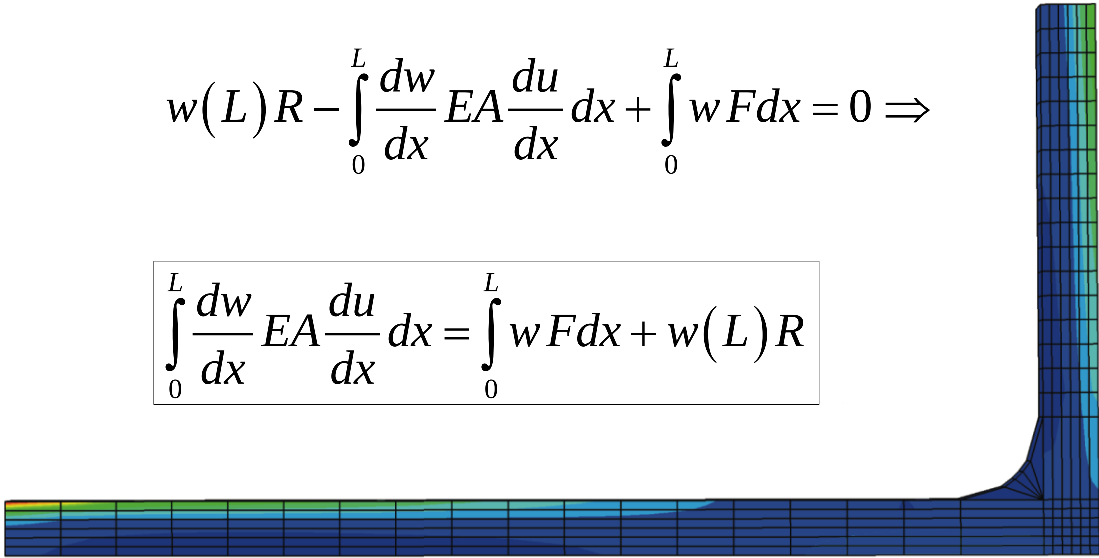


Substituting in the initial expression:

$$\int_0^L wEA\frac{d^2u}{dx^2}dx+\int_0^L wFdx=0\Rightarrow$$

$$w(L)R-\int_0^L \frac{dw}{dx}EA\frac{du}{dx}dx+\int_0^L wFdx=0\Rightarrow$$

$$\boxed{\int_0^L \frac{dw}{dx}EA\frac{du}{dx}dx=\int_0^L wFdx+w(L)R}$$



Weak form comparison

All weak formulations presented are equivalent:

Principle of stationary potential energy:

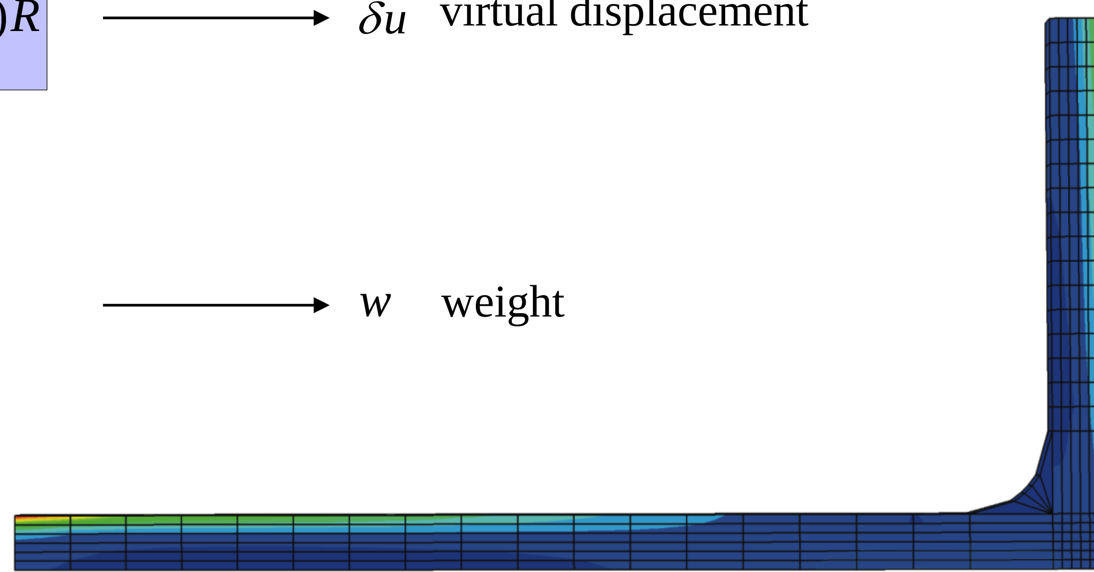
$$\int_0^L EA\frac{du}{dx}\delta\frac{du}{dx}dx = \int_0^L \delta u\,F\,dx + \delta u(L)R \longrightarrow \delta u \quad \text{displacement variation}$$

Principle of Virtual Work:

$$\int_0^L EA\frac{du}{dx}\frac{d\delta u}{dx}dx = \int_0^L \delta u\,F\,dx + \delta u(L)R \longrightarrow \delta u \quad \text{virtual displacement}$$

Galerkin method:

$$\int_0^L \frac{dw}{dx}EA\frac{du}{dx}dx = \int_0^L w\,F dx + w(L)R \longrightarrow w \quad \text{weight}$$



Weak vs Strong form

Weak form:

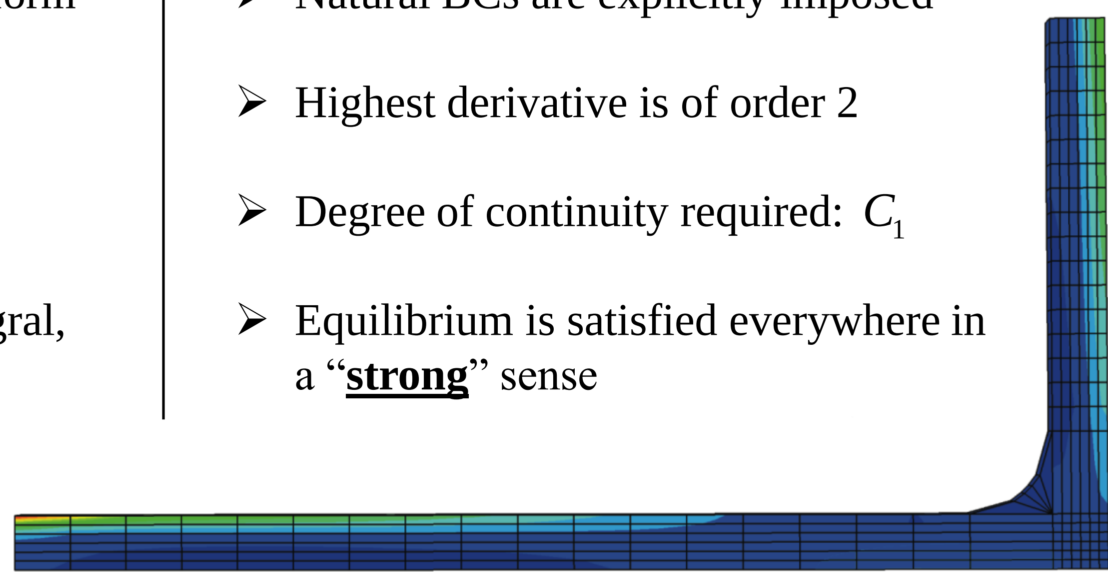
$$\int_0^L \frac{dw}{dx} EA \frac{du}{dx} dx = \int_0^L w F dx + w(L) R$$
$$u(0) = 0$$

- Natural BCs are part of the weak form
- Highest derivative is of order 1
- Degree of continuity required: C_0
- Equilibrium is satisfied in an integral, "**<u>weak</u>**" sense

Strong form:

$$EA \frac{d^2u}{dx^2} = -F$$
$$\left.\frac{du}{dx}\right|_{x=L} = \frac{R}{EA}$$
$$u(x=0) = 0$$

- Natural BCs are explicitly imposed
- Highest derivative is of order 2
- Degree of continuity required: C_1
- Equilibrium is satisfied everywhere in a "**<u>strong</u>**" sense



Weak form solution

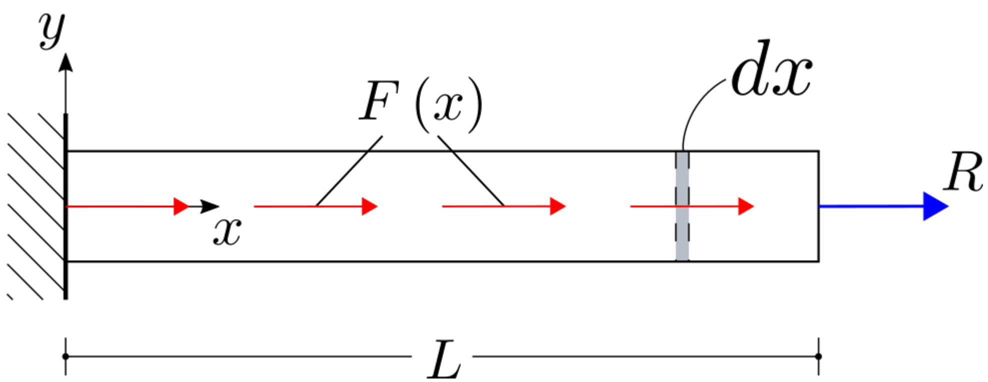
A general process:

- ➢ Assume a general form for the solution
- ➢ Plug into weak form
- ➢ Obtain unknown coefficients

In this process:

- ➢ The problem formulation poses restrictions with respect to possible forms of the solution
- ➢ The weak form is much less restrictive than the corresponding strong form

Weak form solution - Exact



Linear distributed load and no end load:

$$F(x) = ax, \;\; R = 0$$

Analytical solution:

$$\boxed{u(x) = \frac{aL^2}{2EA}x - \frac{a}{6EA}x^3}$$

Galerkin weak form**:**

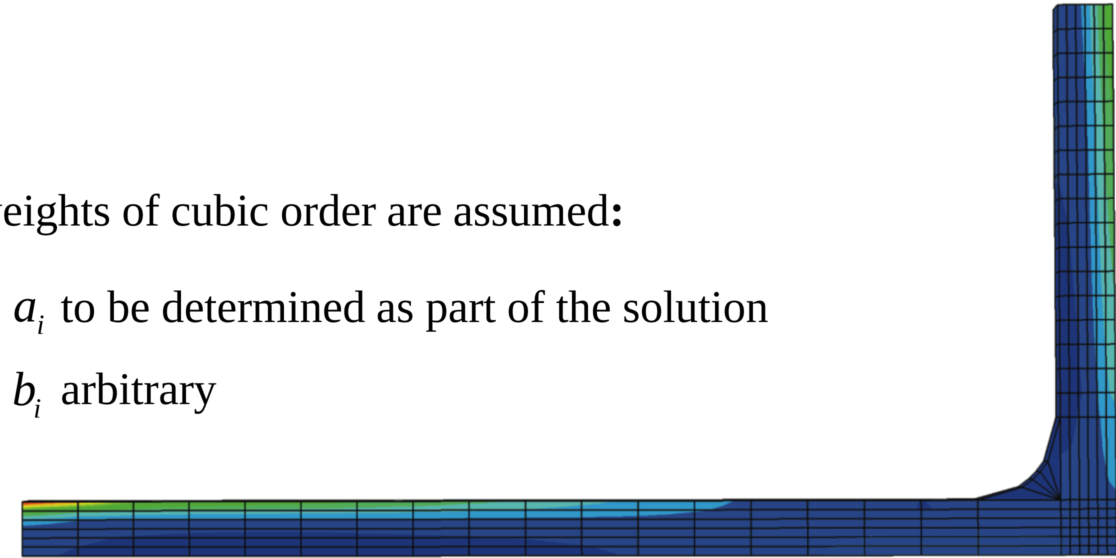
$$\int_0^L \frac{dw}{dx} EA \frac{du}{dx} dx = \int_0^L w F dx$$

$$u(0) = 0, w(0) = 0$$

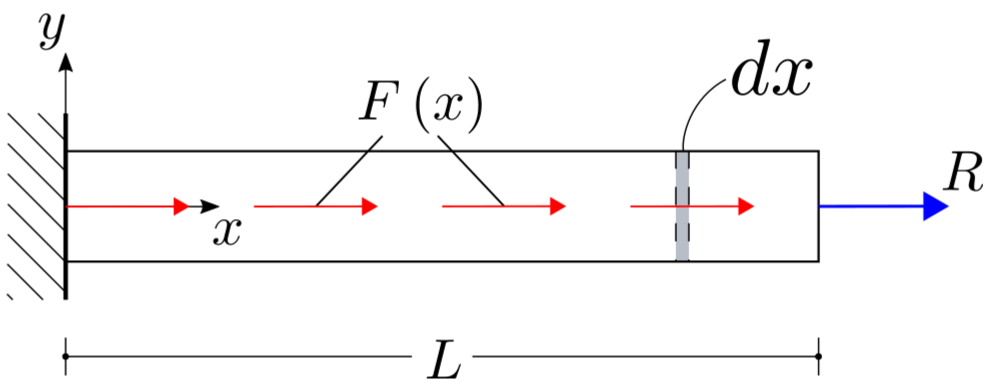
Polynomial displacements and weights of cubic order are assumed**:**

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \rightarrow a_i \text{ to be determined as part of the solution}$$

$$w(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 \rightarrow b_i \text{ arbitrary}$$



Weak form solution - Exact



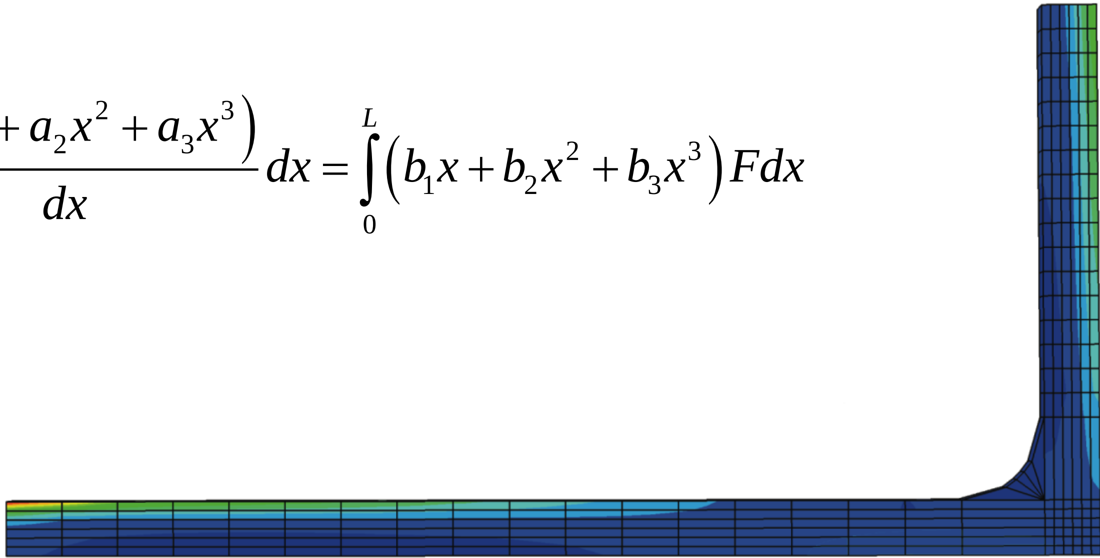
Analytical solution:

$$\boxed{u(x) = \frac{aL^2}{2EA}x - \frac{a}{6EA}x^3}$$

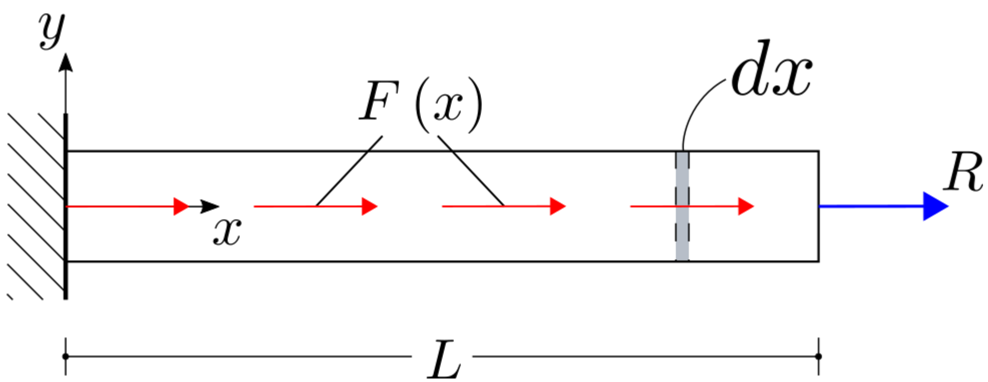
From essential BCs: $u(0) = 0, w(0) = 0 \Rightarrow a_0 = b_0 = 0$

From weak form:

$$\int_0^L \frac{d\left(b_1x + b_2x^2 + b_3x^3\right)}{dx} EA \frac{d\left(a_1x + a_2x^2 + a_3x^3\right)}{dx} dx = \int_0^L \left(b_1x + b_2x^2 + b_3x^3\right) F dx$$



Weak form solution - Exact



Analytical solution:

$$u(x)=\frac{aL^2}{2EA}x-\frac{a}{6EA}x^3$$

After a few rearrangements… $\boxed{b_1 I_1 + b_2 I_2 + b_3 I_3 = 0}$

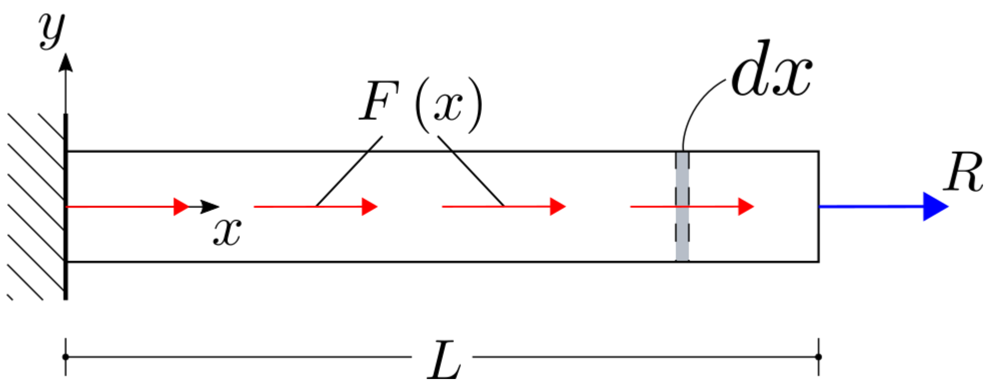
with:

$$I_1=\int_0^L\left(a_1+2a_2x+3a_3x^2-\frac{ax^2}{EA}\right)dx = La_1+L^2a_2+L^3a_3-\frac{aL^3}{3EA}$$

$$I_2=\int_0^L\left(2a_1x+4a_2x^2+6a_3x^3-\frac{ax^3}{EA}\right)dx = L^2a_1+\frac{4L^3}{3}a_2+\frac{3L^4}{2}a_3-\frac{aL^4}{4EA}$$

$$I_3=\int_0^L\left(3a_1x^2+6a_2x^3+9a_3x^5-\frac{ax^4}{EA}\right)dx = L^3a_1+\frac{3L^4}{2}a_2+\frac{9L^5}{5}a_3-\frac{aL^5}{5EA}$$

Weak form solution - Exact



Analytical solution:

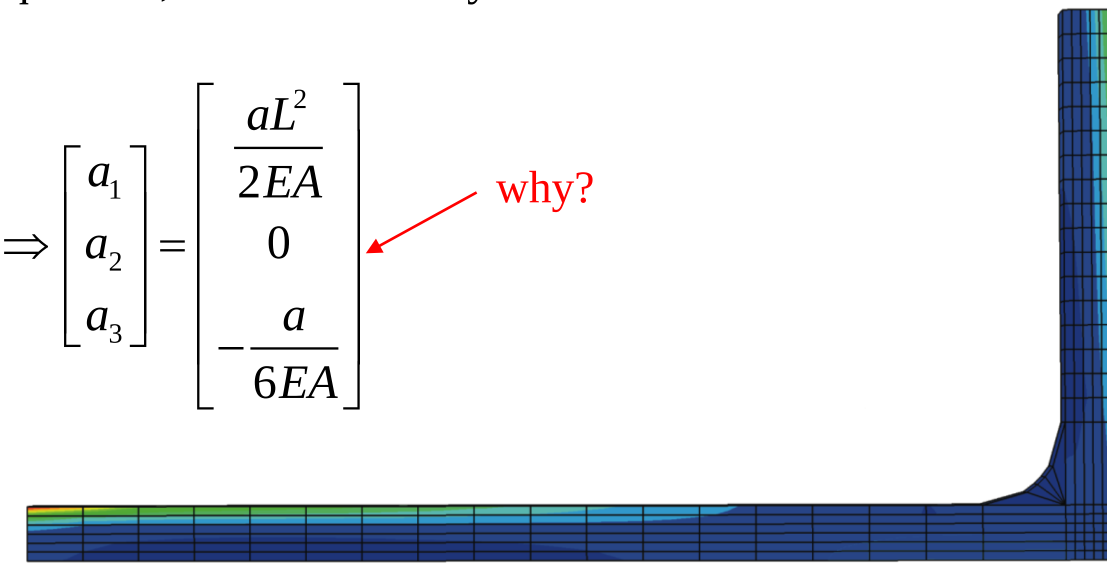
$$u(x)=\frac{aL^2}{2EA}x-\frac{a}{6EA}x^3$$

$\boxed{b_1 I_1 + b_2 I_2 + b_3 I_3 = 0}$ Since b_i are arbitrary, the equation can only hold if $I_1 = I_2 = I_3 = 0$

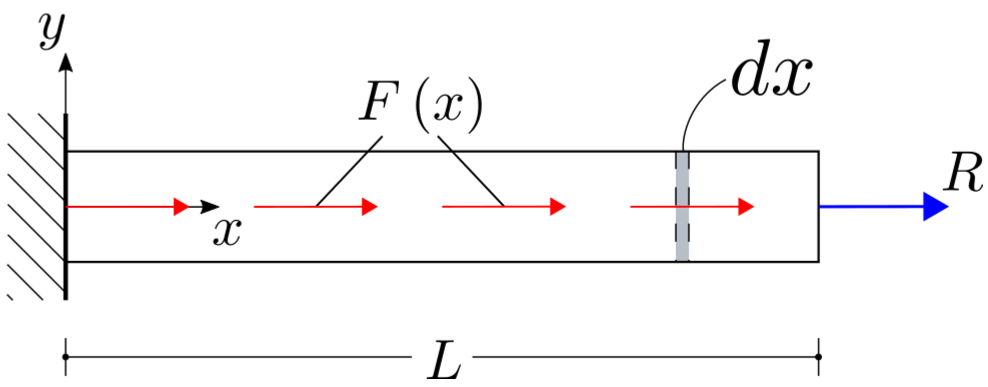
This results in a linear system of equations, whose solution yields the exact coefficients:

$$\begin{bmatrix} L & L^2 & L^3 \\ L^2 & \frac{4L^3}{3} & \frac{3L^4}{2} \\ L^3 & \frac{3L^4}{2} & \frac{9L^5}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{aL^3}{3EA} \\ \frac{aL^4}{4EA} \\ \frac{aL^5}{5EA} \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{aL^2}{2EA} \\ 0 \\ -\frac{a}{6EA} \end{bmatrix}$$

why?



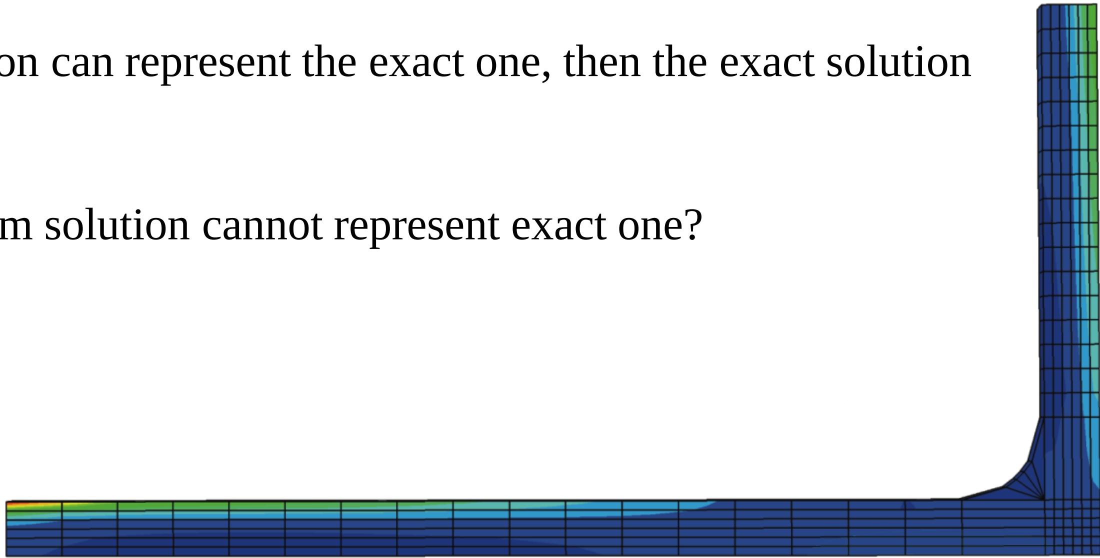
Weak form solution - Exact



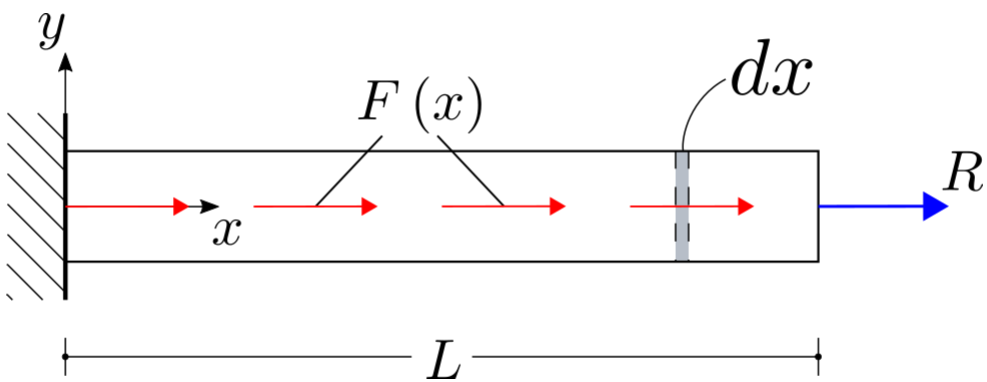
Analytical solution:

$$u(x) = \frac{aL^2}{2EA}x - \frac{a}{6EA}x^3$$

- The weak form is equivalent to the differential equation
- If the assumed form of the solution can represent the exact one, then the exact solution will be obtained
- What happens if the assumed form solution cannot represent exact one?



Weak form solution - Approximate



Analytical solution:

$$\boxed{u(x)=\frac{aL^2}{2EA}x-\frac{a}{6EA}x^3}$$

Linear displacements and weights are assumed**:**

$$u(x)=a_0+a_1x$$

$$w(x)=b_0+b_1x$$

From essential BCs**:** $u(0)=0, w(0)=0 \Rightarrow a_0=b_0=0$

From weak form:

$$\int_0^L \frac{d(b_1x)}{dx}EA\frac{d(a_1x)}{dx}dx=\int_0^L (b_1x)Fdx \Rightarrow \cancel{b_1}EALa_1=\cancel{b_1}\frac{aL^3}{3} \Rightarrow a_1=\frac{aL^2}{3EA}$$

Weak form solution - Approximate

Exact solution:

$$u(x) = \frac{aL^2}{2EA}x - \frac{a}{6EA}x^3$$

Approximate solution:

$$\bar{u}(x) = \frac{aL^2}{3EA}x$$