Linear Algebra

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Triangularization

Closely related to the problem of diagonalization is the problem of triangularization. We shall use this concept as a stepping stone toward the solution of diagonalization problem.

Definition

Two $n \times n$ matrices A and B are said to be congruent if there exists a unitary matrix C such that $C^*AC = B$.

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Obviously all diagonalizable matrices are triangularizable. The following result says that triangularizability causes least problem:

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$A_1 \mathbf{e}_1 = C_1^{-1} A C_1 \mathbf{e}_1 = C_1^{-1} A \mathbf{v}_1 = \mu C_1^{-1}(\mathbf{v}_1) = \mu \mathbf{e}_1$.

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$$A_1 = \left[\begin{array}{ll} \mu & \star \\ 0_{n-1} & B \end{array}\right]$$

where 0_{n-1} is the column of size $n-1$ consisting of zeros.

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Then M_1 is unitary and hence $C = C_1 M_1$ is also unitary. Clearly $C^{-1}AC = M_1^{-1}C_1^{-1}AC_1M_1 = M_1^{-1}A_1M_1$ which is of the form

$$\begin{bmatrix} 1 & 0^t_{n-1} \\ 0_{n-1} & M^{-1} \end{bmatrix} \begin{bmatrix} \mu & * \\ 0_{n-1} & B \end{bmatrix} \begin{bmatrix} 1 & 0^t_{n-1} \\ 0_{n-1} & M \end{bmatrix} = \begin{bmatrix} \mu & * \\ 0_{n-1} & M^{-1}BM \end{bmatrix}$$

and hence is upper triangular. ♠

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For a real square matrix A with all its eigenvalues real, there exists an orthogonal matrix C such that $C^t AC$ is upper triangular.

Normal Matrices

Definition

A square matrix A is called normal if $A^*A = AA^*$.

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(iv) Certainly Hermitian matrices are normal. Of course, there are normal matrices which are not Hermitian. For example, take $A = \begin{bmatrix} \imath & 0 \\ 0 & -\imath \end{bmatrix}$.

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Proof: $\|A\mathbf{x}\|^2 = (A\mathbf{x})^*(A\mathbf{x}) = \mathbf{x}^*A^*A\mathbf{x} = \mathbf{x}^*AA^*\mathbf{x} = (A^*\mathbf{x})^*(A^*\mathbf{x}) = \|A^*\mathbf{x}\|^2$.

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Taking $\mathbf{x} = \mathbf{e}_i$ we get that norms of i^{th}-column of A is equal to the norm of i^{th}-column of A^*= norm of the i^{th}-row of $\bar{A}$=norm of the i^{th}-row of A. ♠

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Proof: Observe that if A is normal then $A - \mu I$ is also normal. Now $(A - \mu I)(\mathbf{v}) = 0$ iff $\|(A - \mu I)(\mathbf{v})\| = 0$ iff $\|(A - \mu I)^*\mathbf{v}\| = 0$ iff $(A^* - \overline{\mu} I)(\mathbf{v}) = 0$.
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(b) A real symmetric matrix is real-congruent to a diagonal matrix.

Proof: (a) We simply observe that a Hermitian matrix is normal and apply the above theorem.

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Proof: (a) We simply observe that a Hermitian matrix is normal and apply the above theorem.
(b) We first recall that for a real symmetric matrix, all eigenvalues are real.

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$$Q_A(\mathbf{x}) := Q_A((x_1, x_2, \ldots, x_n)^t) := \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

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If $A = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ then $Q(\mathbf{x}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$ is called a **diagonal form**.

Quadratic forms

Proposition

$$Q(\mathbf{x}) = [x_1, x_2, \ldots, x_n]A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{x}^t A\mathbf{x} \text{ where } \mathbf{x} = (x_1, x_2, \ldots, x_n)^t.$$

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$$= \sum_{j=1}^{n} a_{1j}x_jx_1 + \cdots + \sum_{j=1}^{n} a_{nj}x_jx_n$$

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$$Q(\mathbf{x}) = [x_1, x_2, \ldots, x_n] A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{x}^t A \mathbf{x} \text{ where } \mathbf{x} = (x_1, x_2, \ldots, x_n)^t.$$

Proof:

$$[x_1, x_2, \ldots, x_n] A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1, x_2, \ldots, x_n] \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix}$$

$$= \sum_{j=1}^{n} a_{1j} x_j x_1 + \cdots + \sum_{j=1}^{n} a_{nj} x_j x_n = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j = Q(x).$$

♠

Quadratic forms

Example

(1) $A = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$. *Then*

$$X^t A X = [x, y] \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [x, y] \begin{bmatrix} x + y \\ 3x + 5y \end{bmatrix} = x_1^2 + 4xy + 5y^2.$$

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Notice that A *and* B *give rise to same* $Q(\mathbf{x})$ *and* $B = \frac{1}{2}(A + A^t)$ *is a symmetric matrix.*

Proposition

For any $n \times n$ matrix A and the column vector $\mathbf{x} = (x_1, x_2, \ldots x_n)^t$

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$$\mathbf{x}^t A\mathbf{x} = \frac{1}{2}\mathbf{x}^t A\mathbf{x} + \frac{1}{2}\mathbf{x}^t A^t\mathbf{x} = \mathbf{x}^t \frac{1}{2}(A + A^t)\mathbf{x} = \mathbf{x}^t B\mathbf{x}.$$

$\square$

Quadratic forms and their diagonalization

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$$\mathbf{x}^t A\mathbf{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2,$$

where $\mathbf{y} = (y_1, \ldots, y_n)$ are define by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = U \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = U\mathbf{y}.$$

Quadratic forms and their diagonalization

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Since $U^t AU = \text{ diag } (\lambda_1, \lambda_2, \ldots, \lambda_n)$, we get

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Check that $U^t AU =$ diag $(1, 6)$.

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to diagonalize the quadratic form $Q(x, y)$ to the diagonal form $\lambda_1 u^2 + \lambda_2 v_2^2$. The orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ determines a new set of coordinate axes with respect to which the locus of the equation $[x, y]A[x, y]^T + B[x, y]^T + f = 0$ with $B = [d, e]$ is same as the locus of the equation

$$\begin{aligned} 0 &= [u, v] \text{ diag } (\lambda_1, \lambda_2)[u, v]^T + (BU)[u, v]^T + f \\ &= \lambda_1 u^2 + \lambda_2 v^2 + [d, e][\mathbf{v}_1, \mathbf{v}_2][u, v]^T + f. \end{aligned} \tag{12}$$

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i.e., $x = t(2u + v)$ *and* $y = t(-u + 2v)$. *Substitute these into the original equation to get*

$$u^2 + 6v^2 - \sqrt{5}u + 6\sqrt{5}v - \frac{1}{4} = 0.$$

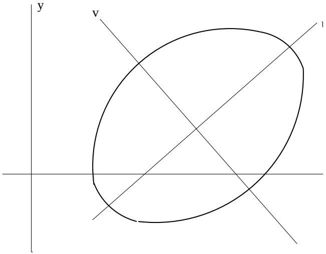
Complete the square to write this as

$$(u - \frac{1}{2}\sqrt{5})^2 + 6(v + \frac{1}{2}\sqrt{5})^2 = 9.$$

This is an equation of ellipse with center $(\frac{1}{2}\sqrt{5}, -\frac{1}{2}\sqrt{5})$ *in the uv-plane.*

Conic Sections: Examples

The u-axis and v-axis are determined by the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ as indicated in the following figure :



Conic Sections: Examples

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The transformed equation becomes

$$3u^2 - 2v^2 - 4t(2u + v) + 10t(-u + 2v) - 13 = 0$$

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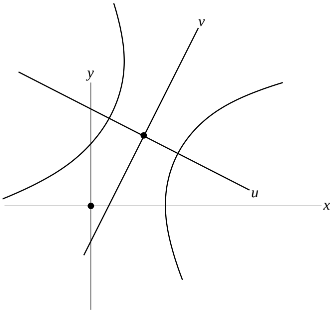

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The equation in uv-plane is $u^2 + v = 0$. *This is an equation of parabola with its vertex at the origin.*