LINEAR & NONLINEAR PLATE THEORY

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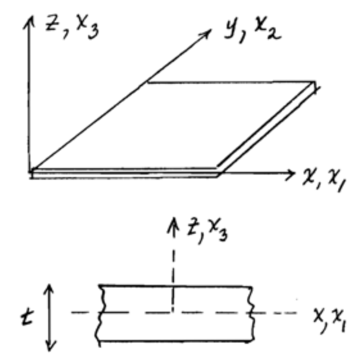
References

Brush and Almroth, *Buckling of bars, plates and shells,* Chp. 3, McGraw-Hill, 1975.

Timoshenko & Woinowsky-Krieger, *Theory of plates and shells,* McGraw-Hill, 1959.

Strain-displacement relations for nonlinear plate theory



The chief characteristic of a thin flat plate is it flexibility for out of plane bending relative to its stiffness with respect to in-plane deformations. The theory we will derive is restricted to *small strains, moderate out-of-plane rotations and small in-plane rotations*. Analogous to the theory derived for curved beams, a 2D theory will be derived based solely on the deformation of the middle surface of the plate. The mid-plane of the undeformed plate is assumed to lie in the (x, y) or (x_1, x_2) plane. Specifically, with reference to the figure, we will assume:

(i) Strains are everywhere small compared to unity.

(ii) Rotations about the z axis (x_3 axis) are small compared to unity, just as in linear elasticity due to in-plane stiffness.

(iii) Rotations about the x and y axes (x_1 and x_2 axes) can be moderately large in the same sense as introduced for curved beams.

(iv) As in beam theory we invoke the Bernoulli-Kirchhoff hypothesis that normals to the middle surface remain normal in the deformed state and that a state of approximate plane stress holds throughout the plate.

The expression for the 3D Lagrangian strain tensor in Cartesian coordinates is

$$\eta_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}u_{k,i}u_{k,j}$$

where $u_i(x_j)$ are the components of the displacement vector with base vectors of the x_i axes for material points initially at x_j. This strain tensor exactly characterizes deformation of a 3D solid for arbitrary large strains and rotations. When the strains are small, η_{ij} may be identified with the stretching strain tensor ε_{ij} without restrictions on the rotations. That is, $\eta_{11} = \varepsilon_{11}$ is the strain of the material line element aligned with the x_1 axis in the undeformed plate, etc. Thus,

$$\varepsilon_{11} = u_{1,1} + \frac{1}{2}\left(u_{1,1}{}^2 + u_{2,1}{}^2 + u_{3,1}{}^2\right), \qquad \varepsilon_{22} = u_{2,2} + \frac{1}{2}\left(u_{1,2}{}^2 + u_{2,2}{}^2 + u_{3,2}{}^2\right)$$

$$\varepsilon_{12} = \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) + \frac{1}{2}\left(u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2}\right)$$

The strain components ε_{33}, ε_{13} and ε_{23} will not enter directly in the theory and we do not need the expression for them in terms of displacement gradients. The rotation about the x_3 axis, assuming it is small (on the order of the strains) is $\omega_3 = u_{2,1} - u_{1,2}$.

Restrictions (i) and (ii) require that the gradients of the in-plane displacements are small, i.e.

$$|u_{1,1}| << 1, \quad |u_{2,2}| << 1, \quad |u_{1,2}| << 1, \quad |u_{2,1}| << 1$$

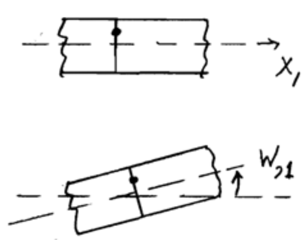
By restricting the deformations to moderately large rotations about the x_1 and x_2 axes, as in (iii), we require

$$u_{3,1}{}^2 << 1, \quad u_{3,2}{}^2 << 1$$

With these requirements the three strain-displacement equations can be approximated by

$$\varepsilon_{11} = u_{1,1} + \frac{1}{2}u_{3,1}{}^2, \quad \varepsilon_{22} = u_{2,2} + \frac{1}{2}u_{3,2}{}^2 \quad \text{and} \quad \varepsilon_{12} = \frac{1}{2}\left(u_{1,2} + u_{2,1}\right) + \frac{1}{2}u_{3,1}\,u_{3,2}$$

The Kirchhoff hypothesis (iv) together with the restrictions to moderate rotations implies that displacements off the mid-surface ($x_3 \neq 0$) can be expressed in terms of mid-surface displacements ($x_3 = 0$) by $u_i(x_1, x_2, x_3) \cong u_i(x_1, x_2, 0) - x_3 u_{3,i}(x_1, x_2, 0), \quad (i = 1, 2)$



This can also be seen by making a Taylor series expansion about the middle surface and using the fact that $\varepsilon_{i3} = (u_{i,3} + u_{3,i})/2 \cong 0$ for $i = 1, 2$ to obtain

$$u_i(x_1, x_2, x_3) \cong u_i(x_1, x_2, 0) + x_3 u_{i,3}(x_1, x_2, 0) \cong u_i(x_1, x_2, 0) - x_3 u_{3,i}(x_1, x_2, 0), \ (i = 1, 2)$$

Next, substitute these expressions in the strain-displacement relations above letting

$$u(x_1,x_2)\equiv u_1(x_1,x_2,0)\,,\quad v(x_1,x_2)\equiv u_2(x_1,x_2,0)\,,\quad w(x_1,x_2)\equiv u_3(x_1,x_2,0)\,,$$

and $z \equiv x_3$ to obtain

$$\varepsilon_{11} = u_{,1} + \frac{1}{2}w_{,1}{}^2 - zw_{,11}\,,\qquad \varepsilon_{22} = v_{,2} + \frac{1}{2}w_{,2}{}^2 - zw_{,22}$$

$$\text{and}\quad \varepsilon_{12} = \frac{1}{2}\left(u_{,2}+v_{,1}\right) + \frac{1}{2}w_{,1}w_{,2} - zw_{,12}$$

These are the strain-displacement relations for a flat plate. The nonlinear terms involve the out-of-plane rotations. The bending strain tensor (the curvature change tensor for the plate mid-surface) is $K_{\alpha\beta} = -w_{,\alpha\beta}$.

From this point on we adopt the following notation. The *displacements of the mid-surface* are denoted by $u_1(x_1,x_2)$, $u_1(x_1,x_2)$, $w(x_1,x_2)$. The *mid-surface strains* are

$$E_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta}+u_{\beta,\alpha}\right) + \frac{1}{2}w_{,\alpha}w_{,\beta} \qquad \text{for}\ \alpha=1,2\ \text{and}\ \beta=1,2$$

The *bending strains* are

$$K_{\alpha\beta} = -w_{,\alpha\beta} \qquad \text{for}\ \alpha=1,2\ \text{and}\ \beta=1,2$$

The *strains at any point in the plate* are

$$\varepsilon_{\alpha\beta} = E_{\alpha\beta} + zK_{\alpha\beta}$$

Stress quantities, Principle of Virtual Work and equilibrium equations

Let δu_α and δw be the virtual displacements of the mid-surface of the plate and let $\delta\varepsilon_{\alpha\beta} = \delta E_{\alpha\beta} + z\delta K_{\alpha\beta}$ be the associated virtual strains. At all points through the thickness of the plate it is assumed that there exists a state of approximate plane stress which means that $\sigma_{33} \cong 0$ and $\varepsilon_{13} \cong \varepsilon_{23} \cong 0$. This is the obvious extension of beam theory to plates. Thus, the strain energy at any point is taken as $\sigma_{\alpha\beta}\varepsilon_{\alpha\beta}/2$ with the convention that a repeated Greek indice is only summed over 1 and 2. The *internal virtual work* of the plate is

$$IVW = \int_S dx_1 dx_2 \int_{-t/2}^{t/2} dz\, \sigma_{\alpha\beta}\delta\varepsilon_{\alpha\beta}$$

where S denotes the area of the middle surface and t is the plate thickness. But using $\delta\varepsilon_{\alpha\beta} = \delta E_{\alpha\beta} + z\delta K_{\alpha\beta}$, one has

$$\int_{-t/2}^{t/2} \sigma_{\alpha\beta}\delta\varepsilon_{\alpha\beta} = \delta E_{\alpha\beta}\int_{-t/2}^{t/2}\sigma_{\alpha\beta}dz + \delta K_{\alpha\beta}\int_{-t/2}^{t/2}\sigma_{\alpha\beta}zdz \equiv N_{\alpha\beta}\delta E_{\alpha\beta} + M_{\alpha\beta}\delta K_{\alpha\beta}$$

where the *resultant membrane stresses* and the *bending moments* are defined by

$$N_{\alpha\beta} = \int_{-t/2}^{t/2}\sigma_{\alpha\beta}dz\ , \quad M_{\alpha\beta} = \int_{-t/2}^{t/2}\sigma_{\alpha\beta}zdz$$

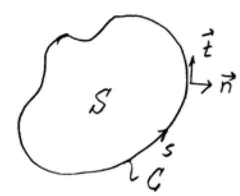
Thus, the internal virtual work can be written as

$$IVW = \int_S \left(N_{\alpha\beta}\delta E_{\alpha\beta} + M_{\alpha\beta}\delta K_{\alpha\beta}\right)dS \qquad (\, dS = dx_1 dx_2 \,)$$

We postulate a *principle of virtual work* and use this principle to derive the equilibrium equations. As in the case of curved beam theory, the principle states that $IVW = EVW$ for all admissible virtual displacements. First, consider the IVW:

$$IVW = \int_S \left(N_{\alpha\beta}\delta u_{\alpha,\beta} + N_{\alpha\beta}w_{,\alpha}\delta w_{,\beta} - M_{\alpha\beta}\delta w_{,\alpha\beta}\right)dS$$

Apply the divergence theorem to the above expression (twice to the third term) to obtain



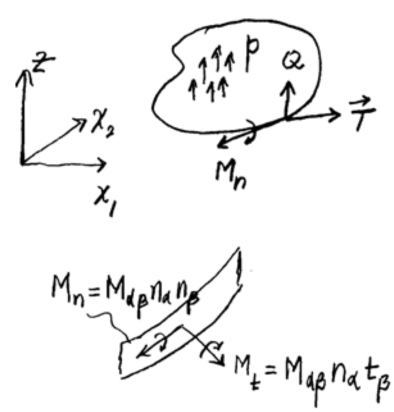
$$IVW = \int_S \left[-N_{\alpha\beta,\beta}\delta u_\alpha - \left(N_{\alpha\beta}w_{,\beta}\right)_{,\beta}\delta w - M_{\alpha\beta,\alpha\beta}\delta w\right]dS +$$
$$\oint_C \left[N_{\alpha\beta}n_\beta\delta u_\alpha + N_{\alpha\beta}w_{,\alpha}n_\beta\delta w - M_{\alpha\beta}n_\beta\delta w_{,\alpha} + M_{\alpha\beta,\beta}n_\alpha\delta w\right]ds$$

Notation is displayed in the figure.

Next define the *external virtual work*, EVW:

$$EVW = \int_S p\delta w dS + \oint_C \left[T_\alpha\delta u_\alpha + Q\delta w - M_n\delta w_{,\alpha}n_\alpha\right]ds + \sum P\delta w$$

The *dead load* contributions are as follows (see the figure): the normal pressure distribution, p, the in-plane edge resultant tractions (force/length), T_α, the normal edge force/length, Q, and the component of the edge moment that works through the negative of $\delta w_{,n} \equiv \delta w_{,\alpha}n_\alpha$. The possibility of concentrated loads acting perpendicular to the plate away from the edge is



also noted but it will not be explicitly taken into account below. The reason for the notation will become clearer below. One might be tempted to include a contribution like $M_t \delta w_{,t}$ ($\delta w_{,t} = \delta w_{,\alpha} t_\alpha$)—see figure, but we will see that this does not turn out to be an independent contribution.

We are still not in a position to enforce $IVW = EVW$ because the term $M_{\alpha\beta} n_\beta \delta w_{,\alpha}$ in the line integral for the IVW is not in a form which permits identification of independent variations. Noting that $\delta w_{,\alpha} = (\delta w_{,\beta} n_\beta) n_\alpha + (\delta w_{,\beta} t_\beta) t_\alpha \equiv \delta w_{,n} n_\alpha + \delta w_{,t} t_\alpha$, one see that $\delta w_{,n}$ can be varied independently of δw along C, but $\delta w_{,t}$ cannot because it equals $d\delta w / ds$ along C. Write

$$\oint_C M_{\alpha\beta} n_\beta \delta w_{,\alpha} ds = \oint_C M_{\alpha\beta} n_\beta \left[\delta w_{,n} n_\alpha + \delta w_{,t} t_\alpha \right] ds$$

Let $M_t \equiv M_{\alpha\beta} t_\alpha n_\beta$ and integrate the second term by parts using $(\ )_{,t} = (\ )_{,\alpha} t_\alpha$

$$\oint_C M_t \delta w_{,t} ds = -\oint_C M_{t,t} \delta w ds + M_t \delta w \Big|_{A^-}^{A^+}$$

where the contributions at A are meant to represent points along C such as corners at which $\mathbf{t}$ is discontinuous. Thus, the troublesome term in IVW becomes

$$\oint_C M_{\alpha\beta} n_\beta \delta w_{,\alpha} ds = \oint_C \left[M_{\alpha\beta} n_\alpha n_\beta \delta w_{,n} - M_{t,t} \delta w \right] ds + M_t \delta w \Big|_{\text{corners}}$$

And we may finally write the internal virtual work as

$$IVW = \int_S \left[-N_{\alpha\beta,\beta} \delta u_\alpha - \left(N_{\alpha\beta} w_{,\beta} \right)_{,\beta} \delta w - M_{\alpha\beta,\alpha\beta} \delta w \right] dS +$$
$$\oint_C \left[N_{\alpha\beta} n_\beta \delta u_\alpha + \left(M_{\alpha\beta,\beta} n_\alpha + M_{t,t} + N_{\alpha\beta} w_{,\alpha} n_\beta \right) \delta w - M_{\alpha\beta} n_\alpha n_\beta \delta w_{,n} \right] ds + M_t \delta w \Big|_{\text{corners}}$$

Now it is possible to enforce $IVW = EVW$ by independently varying δu_α and δw with $\delta w_{,n}$ varied independently on C. The following equilibrium equations and boundary conditions are the outcome.

Equilibrium equations in S:

$$N_{\alpha\beta,\beta} = 0 \quad \text{and} \quad M_{\alpha\beta,\alpha\beta} + N_{\alpha\beta} w_{,\alpha\beta} = -p$$

Relation between boundary forcse and moment and internal stress quantities on C:

$$T_\alpha = N_{\alpha\beta} n_\beta, \quad Q = M_{\alpha\beta,\beta} n_\alpha + M_{t,t} + N_{\alpha\beta} w_{,\alpha} n_\beta \quad \text{and} \quad M_n = M_{\alpha\beta} n_\alpha n_\beta$$

Boundary conditions on C. Specify three conditions:

$$u_\alpha \text{ or } T_\alpha, \quad w \text{ or } Q, \quad w_{,\alpha} \text{ or } M_n$$

At a corner, if $M_t \delta w\big|_{A^-}^{A^+}$ is nonzero, there must be a concentrated load at the corner with:

$$P = M_t(A^+) - M_t(A^-)$$

Note that we have not considered distributed in-plane forces nor concentrated loads acting perpendicular to the plate either within S or on a smooth section of C, but these can be included. The moment equilibrium equation is nonlinear, coupling to the in-plane resultant membrane stresses. The relation of the normal force Q on C to the moment and membrane stress quantities is also nonlinear.

The fact that there are three and not four boundary conditions (i.e. it is not possible to apply M_t as a fourth condition) is due to the two dimensional nature of the theory. Understanding this subtlety, which emerges clearly from variational considerations associated with the PVW, is attributed to Kirchhoff.

Linear elastic constitutive relation for plates

For reasons similar to those discussed for curved beams, attention is restricted to materials that respond linearly in the small strain range. Plates formed as composite laminates are often anisotropic and they may display coupling between stretch and bending. The general form of constitutive relation for linear plates have an energy/area given by

$$U(\mathbf{E},\mathbf{K}) = \frac{1}{2}\left(S_{\alpha\beta\kappa\gamma}E_{\alpha\beta}E_{\kappa\gamma} + 2C_{\alpha\beta\kappa\gamma}E_{\alpha\beta}K_{\kappa\gamma} + D_{\alpha\beta\kappa\gamma}K_{\alpha\beta}K_{\kappa\gamma}\right)$$

where all the fourth order tensors share reciprocal symmetry, $C_{\alpha\beta\kappa\gamma} = C_{\kappa\gamma\alpha\beta}$ and U is positive definite. The stress-strain relations are then

$$N_{\alpha\beta} = \partial U / \partial E_{\alpha\beta} = S_{\alpha\beta\kappa\gamma}E_{\kappa\gamma} + C_{\alpha\beta\kappa\gamma}K_{\kappa\gamma} \quad , \quad M_{\alpha\beta} = \partial U / \partial K_{\alpha\beta} = D_{\alpha\beta\kappa\gamma}K_{\kappa\gamma} + C_{\alpha\beta\kappa\gamma}E_{\kappa\gamma}$$

This general constitutive relation can be used with the strain-displacement relations and the equilibrium equations used below, but we will focus on plates of uniform thickness, t, made of uniform, isotropic linearly elastic materials. In plane stress,

$$\varepsilon_{\alpha\beta} = \frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E}\sigma_{\gamma\gamma}\delta_{\alpha\beta} \ , \qquad \sigma_{\alpha\beta} = \frac{E}{1-\nu^2}\left[(1-\nu)\varepsilon_{\alpha\beta} + \nu\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right]$$

Using $\varepsilon_{\alpha\beta} = E_{\alpha\beta} + zK_{\alpha\beta}$, $N_{\alpha\beta} = \int_{-t/2}^{t/2} \sigma_{\alpha\beta} dz$ and $M_{\alpha\beta} = \int_{-t/2}^{t/2} \sigma_{\alpha\beta} z dz$, one finds

$$E_{\alpha\beta} = \frac{1+\nu}{Et} N_{\alpha\beta} - \frac{\nu}{Et} N_{\gamma\gamma}\delta_{\alpha\beta} \ , \qquad N_{\alpha\beta} = \frac{Et}{1-\nu^2}\left[(1-\nu)E_{\alpha\beta} + \nu E_{\gamma\gamma}\delta_{\alpha\beta}\ \right]$$

and

$$K_{\alpha\beta} = \frac{12}{Et^3}\left[(1+\nu)M_{\alpha\beta} - \nu M_{\gamma\gamma}\delta_{\alpha\beta}\right], \qquad M_{\alpha\beta} = D\left[(1-\nu)K_{\alpha\beta} + \nu K_{\gamma\gamma}\delta_{\alpha\beta}\ \right]$$

where the widely used bending stiffness is defined as $D = Et^3 / \left[12(1-\nu^2)\right]$. These are the constitutive relations for an isotropic elastic plate. Note that bending and stretching are decoupled, just as in the case of straight beams.

Homework Problem #7 Stationarity of potential energy of the plate-loading system

Consider a plate with the general energy density $U(\mathbf{E},\mathbf{K})$ introduced above. Suppose the plate boundary is divided into two portions one, C_T, on which the force quantities are prescribed (i.e. $T_\alpha = \bar{T}_\alpha(s)$, $Q = \bar{Q}(s)$ and $M_n = \bar{M}_n(s)$) and the remainder, C_u, on which displacements are prescribed (i.e. $u_\alpha = \bar{u}_\alpha$, $w = \bar{w}(s)$ and $w_{,n} = \bar{w}_{,n}(s)$). The potential energy of the system is

$$PE(\mathbf{u}, w) = \int_S \left[U - pw\right] dS - \int_{C_T} \left(\bar{T}_\alpha u_\alpha + \bar{Q}w - \bar{M}_n w_{,n}\right) ds$$

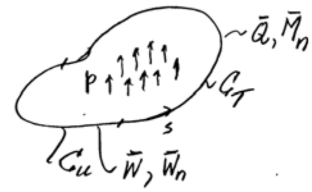
Using the PVW, show that the first variation of the potential energy vanishes ($\delta PE = 0$) for all admissible variations of the displacements for any solution to all the field equations. An admissible displacement must satisfy the displacement boundary conditions on C_u. In linear elasticity (or in linear plate theory) the stationarity point corresponds to a global minimum of the PE. Can you provide an example which illustrates the fact the this cannot generally be the case for the nonlinear theory (no details are required)?

Homework Problem #8 Minimum PE principle for linear bending theory of plates

A special case of the above potential energy principle is that for the linear bending theory of plates. Now consider a positive definite energy density, $U(\mathbf{K}) = \frac{1}{2} D_{\alpha\beta\kappa\gamma} K_{\alpha\beta} K_{\kappa\gamma}$, with $K_{\alpha\beta} = -w_{,\alpha\beta}$ and the functional

$$PE(w) = \int_S [U - pw] dS - \int_{C_T} \left(\bar{Q} w - \bar{M}_n w_{,n} \right) ds$$

where Q and M_n are prescribed on C_T and $\bar{w}$ and $\bar{w}_{,n}$ are prescribed on C_u. Prove that PE is stationary and minimized by the (unique) solution.



Reduction to von Karman plate equations in terms of w and a stress function F

We have not introduced in-plane distributed forces and therefore one notes immediately that an Airy stress function $F(x_1, x_2)$ can be used to satisfy the in-plane equilibrium equations, $N_{\alpha\beta,\beta} = 0$:

$$N_{11} = F_{,22} \quad , \quad N_{22} = F_{,11} \quad , \quad N_{12} = -F_{,12}$$

Compatibility of the in-plane displacements requires (derive this):

$$E_{11,22} + E_{22,11} - 2E_{12,12} = w_{,12}{}^2 - w_{,11} w_{,22}$$

(In tensor notation with $\varepsilon_{\alpha\beta}$ as the 2D permutation tensor (and NOT the strain) these two equations are: $N_{\alpha\beta} = \varepsilon_{\alpha\kappa}\varepsilon_{\beta\gamma} F_{,\kappa\gamma}$ and $\varepsilon_{\alpha\beta}\varepsilon_{\gamma\mu}\left[E_{\alpha\gamma,\beta\mu} + \frac{1}{2} w_{,\alpha\gamma} w_{,\beta\mu} \right] = 0$.) To obtain the final form of *the von Karman plate equations*, first use the constitutive equation and the stress function to express the compatibility equation in terms of F

$$\frac{1}{Et}\nabla^4 F = w_{,12}{}^2 - w_{,11} w_{,22} \quad (= \tfrac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\mu} w_{,\alpha\gamma} w_{,\beta\mu})$$

with

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^4}{\partial^4 x_1} + 2\frac{\partial^4}{\partial^2 x_1 \partial^2 x_2} + \frac{\partial^4}{\partial^4 x_2} \right)$$

Lastly, again using the constitutive relation, express the moment equilibrium equation in terms of w and F as

$$D\nabla^4 w = F_{,22} w_{,11} + F_{,11} w_{,22} - 2F_{,12} w_{,12} + p \qquad (= \varepsilon_{\alpha\beta}\varepsilon_{\gamma\mu} F_{,\alpha\gamma} w_{,\beta\mu} + p)$$

The two fourth order equations are coupled through the nonlinear terms. If one linearizes the equations one obtains two sets of uncoupled equations:

$$\frac{1}{Et}\nabla^4 F = 0\,, \quad \text{(the Airy equation for plane stress)}$$

$$D\nabla^4 w = p\,, \quad \text{(the linear plate bending equation)}$$

Homework Problem #9 Equations for axisymmetric deformations of circular plates

For 3D axisymmetric deformations restricted to have small strains and moderate rotations the two nonzero strains in the radial and circumferential directions are

$$\varepsilon_r = u_{r,r} + \tfrac{1}{2}u_{3,r}{}^2\,, \quad \varepsilon_\theta = r^{-1}u_r$$

where $u_r(r,z)$ and $u_3(r,z)$ are the radial and vertical displacements. Follow the steps performed for the 2D plates laid out above and derive the strain-displacement relations for axisymmetric deformations of circular plates:

$$E_r = u' + \tfrac{1}{2}w'^2\,, \quad E_\theta = r^{-1}u\,, \quad K_r = -w''\,, \quad K_r = -r^{-1}w'$$

where $u(r)$ and $w(r)$ are the radial and normal displacements of the middle surface and $(\ )' = d(\ )/dr$. Define the stress quantities, $N_r, N_\theta, M_r, M_\theta$, and the internal virtual work. Postulate the PVW and derive the equilibrium equations. For a uniform plate of linear isotropic material, derive the equations governing nonlinear axisymmetric deformations of circular plates for axisymmetric pressure distributions $p(r)$.

Several illustrative linear plate bending problems

(i) Circular plate subject to uniform pressure p: clamped or simply supported

This is an axisymmetric problem for $w(r)$. With $(\ )' = d(\ )/dr$

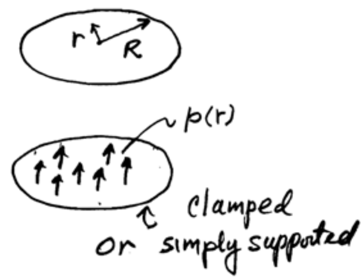
$$D\nabla^4 w = p\,, \quad \nabla^2(\ ) = (\ )'' + r^{-1}(\ )' = r^{-1}(r(\ )')'$$

The general solution to this 4th order ode is

$$w = c_1 + c_2 r^2 + c_3 \ln r + c_4 r^2 \ln r + \frac{pr^4}{64D}$$

For both sets of boundary conditions, the solution *w* is bounded at $r = 0$, and thus $c_3 = 0$ and $c_4 = 0$. For *clamped boundary conditions* at $r = R$, $w = w' = 0$ which requires $c_1 = pR^4/(64D)$ and $c_2 = -pR^2/(32D)$ and

$$w = \frac{pR^4}{64D}\left(1 - 2\left(\frac{r}{R}\right)^2 + \left(\frac{r}{R}\right)^4\right), \quad w(0) = \frac{pR^4}{64D}$$



For *simply supported boundary conditions* at $r = R$, $w = 0$ and $M_r = 0$. From Homework Problem #9, $M_r = D\left[K_r + \nu K_\theta\right] = -D\left[w_{,rr} + \nu r^{-1} w_{,r}\right]$, thus $M_r = 0$ requires $w_{,rr} + \nu r^{-1} w_{,r} = 0$. These conditions give

$c_1 = pR^4(5+\nu)/(64(1+\nu)D)$,

$c_2 = -pR^2(3+\nu)/(32(1+\nu)D)$, and

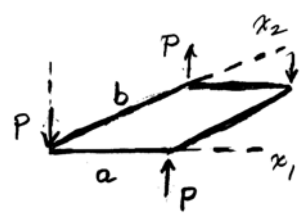
$$w = \frac{pR^4}{64(1+\nu)D}\left(5 + \nu - 2(3+\nu)\left(\frac{r}{R}\right)^2 + (1+\nu)\left(\frac{r}{R}\right)^4\right), \quad w(0) = \frac{pR^4}{64D}\frac{(5+\nu)}{(1+\nu)}$$

The ratio of the center deflections for the two support conditions is

$$\frac{w(0)_{\text{clamped}}}{w(0)_{\text{simply supported}}} = \frac{1+\nu}{5+\nu}$$

(ii) Homework Problem #10: Rectangular plate with concentrated forces at it corners

Here is a teaser. What problem does $w = Axy$ satisfy? Note that $\nabla^4 w = 0$ and the only non-zero curvature is $K_{12} = -A$. Verify that $Q = 0$ and $M_n = 0$ on all four sides of the rectangular plate. Noting the corner condition obtained in setting up the PVW, determine A in terms of the concentrated load with magnitude P acting at each corner as depicted in the figure.



(iii) Simply supported rectangular plate subject to normal pressure distribution

Two useful preliminary observations. Consider a simply supported edge parallel to the x_2 axis. The conditions along this edge are $w = 0$ and

$$M_n = M_{11} = 0 \Rightarrow D\left[K_{11} + \nu K_{22}\right] = -D\left[w_{,11} + \nu w_{,22}\right] = 0 \Rightarrow w_{,11} = 0$$

The last step follows because $w = 0$ along $x_1 = const$ and, thus, $w_{,22} = 0$. Therefore along any simply supported edge parallel to the x_2 axis, the conditions $w = 0$ and $w_{,11} = 0$ can be used. Similarly, along any simply supported edge parallel to the x_1 axis, $w = 0$ and $w_{,22} = 0$ can be used.

Two theorems related to differentiation of sine and cosine series. These are due to Stokes and will be proved in class.

Conditions for differentiating infinite sine series. Suppose

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{where} \quad a_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)dx \ .$$

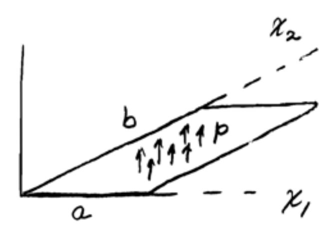
The series can be differentiated term by term to give $df(x)/dx = \sum_{n=1}^{\infty} a_n \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)$ if $f(x)$ is continuous on $(0, L)$ and if $f(0) = f(L) = 0$.

Conditions for differentiating infinite cosine series. Suppose

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x)dx \quad (n \geq 1)$$

The series can be differentiated term by term to give $df(x)/dx = \sum_{n=1}^{\infty} -a_n \frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)$ if $f(x)$ is continuous on $(0, L)$. There are no conditions on *f* at the ends of the interval.

Now consider a rectangular plate ($0 \leq x_1 \leq a$, $0 \leq x_2 \leq b$) simply supported along all four edges and loaded by an arbitrary pressure distribution, $p(x_1, x_2)$ such that the governing equation and boundary conditions are



$$\nabla^4 w = p/D, \quad w = w_{,11} = 0 \text{ on } x_1 = 0, a \quad \text{and} \quad w = w_{,22} = 0 \text{ on } x_2 = 0, b$$

Look for a solution of the form

$$w(x_1, x_2) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\left(\frac{n\pi x_1}{a}\right)\sin\left(\frac{m\pi x_2}{b}\right)$$

Consider the Stokes conditions for differentiating this series. For example, it follows that

$$\frac{\partial^4 w}{\partial^4 x_1} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \left(\frac{n\pi}{a}\right)^4 \sin\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{b}\right)$$

The first differentiation is valid because w vanishes at the ends of the intervals, the second differentiation is valid because the series in x_1 is a cosine series, the third is valid because $w_{,xx}$ vanishes at the ends of the intervals, and, finally, the four differentiation is valid because the series is a cosine series. Applying this reasoning for all the differentiations one obtains

$$\nabla^4 w = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right)^2 \sin\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{b}\right) =$$

$$= \frac{p}{D} = \frac{1}{D} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} p_{nm} \sin\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_2}{b}\right)$$

where

$$p_{nm} = \frac{4}{ab} \int_0^a \int_0^b \sin\left(\frac{n\pi x_1}{a}\right) \sin\left(\frac{m\pi x_1}{b}\right) p(x_1, x_2) dx_1 dx_2$$

Orthogonally implies

$$a_{nm} = \left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]^{-2} \frac{p_{nm}}{D}$$

For *uniform pressure*,

$$p_{11} = \frac{16p}{\pi^2},\ p_{12} = p_{21} = 0,\ p_{13} = p_{31} = \frac{16p}{3\pi^2},\ p_{22} = 0,\ p_{23} = p_{32} = 0,\ \ldots$$

such that

$$w = \frac{16p}{\pi^2 D} \left[\frac{1}{(\pi/a)^2 + (\pi/b)^2}\right]^2 \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{b}\right) +$$

$$+ \frac{16p}{3\pi^2 D} \left[\frac{1}{(3\pi/a)^2 + (\pi/b)^2}\right]^2 \sin\left(\frac{3\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{b}\right)$$

$$+ \frac{16p}{3\pi^2 D} \left[\frac{1}{(\pi/a)^2 + (3\pi/b)^2}\right]^2 \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{3\pi x_2}{b}\right) + \ldots$$

For a square plate with $a = b$, the deflection at the center of the plate is

$$w(a/2, b/2) = \frac{16pa^4}{\pi^6 D} \left\{\frac{1}{4} - \frac{1}{300} - \frac{1}{300} + \ldots\right\}$$

It is seen clearly that first term associated with a_{11} is of dominant importance; for the case of uniform pressure the first term alone gives a result for the deflection that is accurate to within a few percent. Note that if $p = p_{11}\sin(\pi x_1/a)\sin(\pi x_2/b)$, the one-term solution is exact.

(iv) Rectangular plate clamped on two opposite edges and simply supported on the others

We will use the following example to illustrate the direct method of the calculus of variations. Prior to the availability of finite element methods to solve plate problems, the method illustrated below was very effective in generating approximate solutions to plate problems for which exact solutions cannot be expected to be found. The book by L.V. Kantorivich and V.I. Krylov (*Approximate methods of higher analysis*, published in English in 1958) contains many more illustrations.

Preliminary to the application we derive a useful simplification of the energy functional for all clamped plates and for rectangular plates with $w = 0$ on the boundaries. Recall that bending solutions for the clamped circular plate and rectangular clamped and simply-supported plates do not depend on Poisson's ratio ν. We will show that for special cases ν can be eliminated from the energy functional. The bending energy is

$$\begin{aligned}\tfrac{1}{2}\int_S M_{\alpha\beta}K_{\alpha\beta}dS &= \tfrac{1}{2}D\int_S\left[(1-\nu)w_{,\alpha\beta}w_{,\alpha\beta} + \nu w_{,\gamma\gamma}{}^2\right]dS \\ &= \tfrac{1}{2}D\int_S\left[\left(\nabla^2 w\right)^2 + 2(1-\nu)\left(w_{,12}{}^2 - w_{,11}w_{,22}\right)\right]dS\end{aligned}$$

In the following we will derive conditions such that the terms multiplying $(1-\nu)$ vanish:

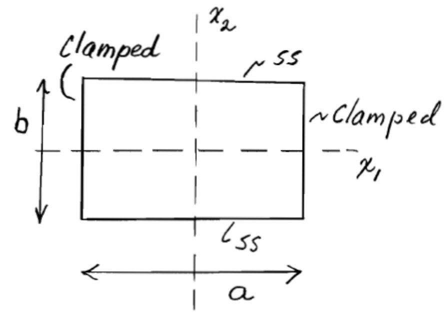
$$\begin{aligned}\int_S\left(w_{,12}{}^2 - w_{,11}w_{,22}\right)dS &= \tfrac{1}{2}\int_S \varepsilon_{\alpha\mu}\varepsilon_{\beta\gamma}w_{,\alpha\beta}w_{,\mu\gamma}dS \\ &= \tfrac{1}{2}\int_S\left[\varepsilon_{\alpha\mu}\varepsilon_{\beta\gamma}(w_{,\alpha}w_{,\mu\gamma})_{,\beta} - \varepsilon_{\alpha\mu}\varepsilon_{\beta\gamma}w_{,\alpha}w_{,\mu\gamma\beta}\right]dS \\ &= \tfrac{1}{2}\int_C \varepsilon_{\alpha\mu}\varepsilon_{\beta\gamma}w_{,\alpha}w_{,\mu\gamma}n_\beta ds = -\tfrac{1}{2}\int_C \varepsilon_{\alpha\mu}w_{,\alpha}w_{,\mu\gamma}t_\gamma ds\end{aligned}$$

In the second line, $\varepsilon_{\beta\gamma}w_{,\mu\gamma\beta} = 0$ is used and in the last line $t_\gamma = -\varepsilon_{\beta\gamma}n_\beta$ is used. At any point on the boundary C, pick the coordinate system such that $\mathbf{t} = (1,0)$; then, $\varepsilon_{\alpha\mu}w_{,\alpha}w_{,\mu\gamma}t_\gamma = w_{,1}w_{,12} - w_{,2}w_{,11}$. For any clamped plate, this vanishes on C because both $w_{,1}$ and $w_{,2}$ vanish. For rectangular plates, both $w_{,1}$ and $w_{,11}$ vanish if $w = 0$. Thus, for a clamped plate of any shape or for rectilinear plates having edges with $w=0$ on all the edges (e.g., either clamped or simply supported) the bending energy can be expressed as

$$\tfrac{1}{2}\int_S M_{\alpha\beta}K_{\alpha\beta}dS = \tfrac{1}{2}D\int_S \left(\nabla^2 w\right)^2 dS$$

Now consider a *rectangular plate* $(-a/2 \le x_1 \le a/2,\ -b/2 \le x_2 \le b/2)$ that is *simply supported on the top and bottom edges and clamped on the left and right edge*s and subject to uniform pressure p. The potential energy functional for the system is

$$PE(w) = \int_S \left[\tfrac{1}{2}D\left(\nabla^2 w\right)^2 - pw\right]dS$$



Noting symmetry about $x_2=0$, represent the solution as

$$w(x_1,x_2) = \sum_{n=1,3,5}^{\infty} f_n(x_1)\cos\left(\frac{n\pi x_2}{b}\right)$$

The simply support conditions along the top and bottom edges $(w = w_{,22} = 0)$ are satisfied by each term in this series. Substitute this into $PE(w)$ and integrate with respect to x_2, making use of orthogonally and the Stokes rules, to obtain

$$PE(w) = PE(f) = \frac{b}{4}\sum_{n=1,3,5}^{\infty}\int_{-a/2}^{a/2}\left\{D\left[f_n'' - \left(\frac{n\pi}{b}\right)^2 f_n\right]^2 - (-1)^{\frac{n-1}{2}}\frac{8}{\pi n}pf_n\right\}dx_1$$

Now use the calculus of variations to minimize PE with respect to each $f_n(x_1)$ noting that the clamped conditions on the left and right edges require

$$f_n(\pm a/2) = 0, \quad f_n'(\pm a/2) = 0$$

In addition to the boundary conditions, $\delta PE = 0$ requires

$$f_n'''' - 2\left(\frac{n\pi}{b}\right)^2 f_n'' + \left(\frac{n\pi}{b}\right)^4 f_n = (-1)^{\frac{n-1}{2}}\frac{4}{\pi n}\frac{p}{D}, \qquad n = 1,3,5,\ .....$$

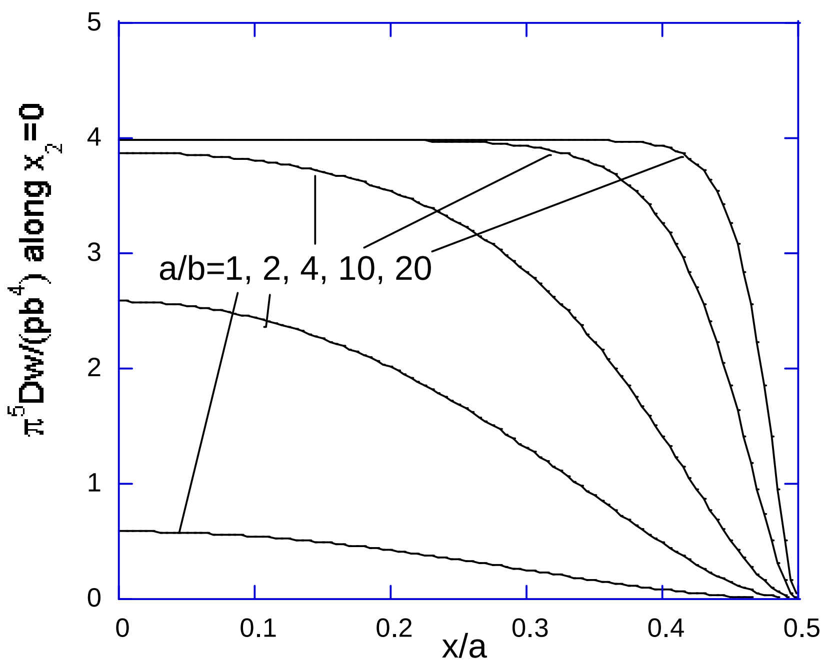
The general solution to these ode's is (noting symmetry about $x_1 = 0$):

$$f_n(x_1) = c_n\cosh\left(\frac{n\pi x_1}{b}\right) + d_n x_1\sinh\left(\frac{n\pi x_1}{b}\right) + (-1)^{\frac{n-1}{2}}\frac{4pb^4}{\pi^5 n^5 D}$$

Enforcing the boundary conditions gives

$$c_n \cosh\left(\frac{n\pi a}{2b}\right) + d_n \frac{a}{2}\sinh\left(\frac{n\pi a}{2b}\right) = -(-1)^{\frac{n-1}{2}} \frac{4pb^4}{\pi^5 n^5 D}$$
$$c_n \sinh\left(\frac{n\pi a}{2b}\right) + d_n \left[\frac{a}{2}\cosh\left(\frac{n\pi a}{2b}\right) + \frac{b}{n\pi}\sinh\left(\frac{n\pi a}{2b}\right)\right] = 0$$

The solution is complete, although elementary numerical work is needed to compute the deflection or moment distributions. The figure plots the deflection along $x_2 = 0$. The normalized defection is defined as $\pi^5 Dw/(pb^4)$.



The figure was computed with terms $n = 1, 3$ but the plot computed with just one term ($n = 1$) is indistinguishable. Two features are highlighted by the figure. As a/b becomes large the deflection over most of the plate, except near the ends, is independent of x_1. This is because the effect of the ends is only felt over distances of order b from the ends. In addition, for large a/b, the deflection in the central region becomes independent of a/b. The behavior of the plate in the central regions is what would be predicted for the 1D problem of an infinitely long plate in the x_1 direction that is simply supported along the top and bottom edges. For moderate values of a/b, e.g. a/b =1,2 in the figure, the fact that the left and right edges are clamped significantly lowers the deflection of the plate even at the center.

(v) Homework Problem#11: An approximate solution for a square clamped plate subject to uniform pressure

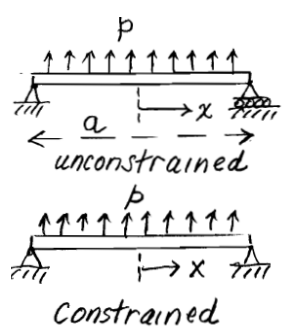
To illustrate another version of the direct method of the calculus of variations, consider a square plate $(-a/2 \le x_1 \le a/2,\ -a/2 \le x_2 \le a/2)$ that is clamped on all four edges and subject to a uniform pressure p. Consider the following approximation to the deflection:

$$w(x_1,x_2) = w_0\left(1-\frac{2x_1}{a}\right)^2\left(1+\frac{2x_1}{a}\right)^2\left(1-\frac{2x_2}{a}\right)^2\left(1+\frac{2x_2}{a}\right)^2$$
$$= w_0\left(1-\left(\frac{2x_1}{a}\right)^2\right)^2\left(1-\left(\frac{2x_2}{a}\right)^2\right)^2$$

Verify that the clamped boundary conditions are met and find w_0, noting that it is the deflection at the center of the plate. Compare your result to the result obtained earlier in the notes for the maximum deflection of a clamped circular plate of radius $a/2$.

Nonlinear plate behavior—coupled bending and stretching

As an introduction to the coupling between bending and stretching in plates we first consider the coupling for *an initially straight, simply supported beam* within the context of small strain/moderate rotation theory. Two in-plane support conditions will be considered—unconstrained and constrained, as illustrated in the accompanying figure. The governing equations obtained from the equations for plates subject to 1D deformations (or specialized from the notes on curved beams) are:



$$\varepsilon = u' + \tfrac{1}{2}w'^2\ , \quad K = -w'' \quad \text{(strain-displacement)}$$

$$N' = 0\,, \quad M'' + Nw'' = -p \quad \text{(equilibrium)}$$

$$N = S\varepsilon\ , \quad M = DK \quad \text{(stress-strain)}$$

Both problems have $w(\pm a/2)=0$, $w''(\pm a/2)=0$ and $u(-a/2)=0$. The unconstrained case has $N(a/2)=0$ while the constrained case has $u(a/2)=0$. For a wide plate, $S=Et/(1-\nu^2)$ and $D=Et^3/12(1-\nu^2)$ while for a 1D beam, $S=EA$ and $D=EI$.

Beam with unconstrained in-plane support at right end.

The condition $N(a)=0$ along with $N'=0$ implies that $N(x)=0$. The moment equilibrium equation then gives $Dw''''=p$ whose solution satisfying the boundary conditions on w is

$$w(x)=\frac{pa^4}{D}\left(\frac{5}{384}-\frac{1}{16}\left(\frac{x}{a}\right)^2+\frac{1}{24}\left(\frac{x}{a}\right)^4\right)\quad,\quad \delta\equiv w(0)=\frac{5pa^4}{384D}$$

The unconstrained in-plane support results in a *linear* out-of-plane bending response of the beam; w is precisely that for linear beam theory. No in-plane stretching develops.

Beam with constrained in-plane support at right end

Now we will have to enforce $u(\pm a/2)=0$. In-plane equilibrium implies that N is independent of x; it is unknown at this point. From the strain displacement relation:

$$\varepsilon=N/S=u'+\frac{1}{2}w'^2$$

Integrate both sides of this equation from $-a/2$ to $a/2$ using $u(\pm a/2)=0$ to obtain

$$N=\frac{S}{2a}\int_{-a/2}^{a/2}w'^2dx$$

Next, the moment equilibrium equation requires

$$Dw''''-Nw''=p$$

The general solution to this equation is (note that $N>0$)

$$w=c_1+c_2x+c_3\cosh(\mu x)+c_4\sinh(\mu x)-\frac{p}{2N}x^2$$

where $\mu=\sqrt{N/D}$. Enforcing the boundary conditions on w (noting symmetry with respect to $x=0$ to make life easier) gives $c_2=c_4=0$ and

$$c_3=\frac{p}{N\mu^2\cosh(\mu a/2)}\ ,\quad c_1=\frac{p}{N}\left(\frac{a^2}{8}-\frac{1}{\mu^2}\right)$$

To obtain N, we need $w' = c_3\mu\sinh\left(\mu x\right) - px/N$. Then, with $\bar{\mu} = \mu a = \sqrt{N/Da}$,

$$N = \frac{S}{2a}\int_{-a/2}^{a/2} w'^2 dx \Rightarrow \frac{Na}{S} = \frac{p^2a^3}{N^2}\int_0^{1/2}\left(\xi - \frac{1}{\bar{\mu}}\frac{\sinh(\bar{\mu}\xi)}{\cosh(\bar{\mu}/2)}\right)^2 d\xi$$

The integral can be evaluated in closed form:

$$\frac{Na}{S} = \frac{p^2a^3}{N^2} f(\bar{\mu})$$

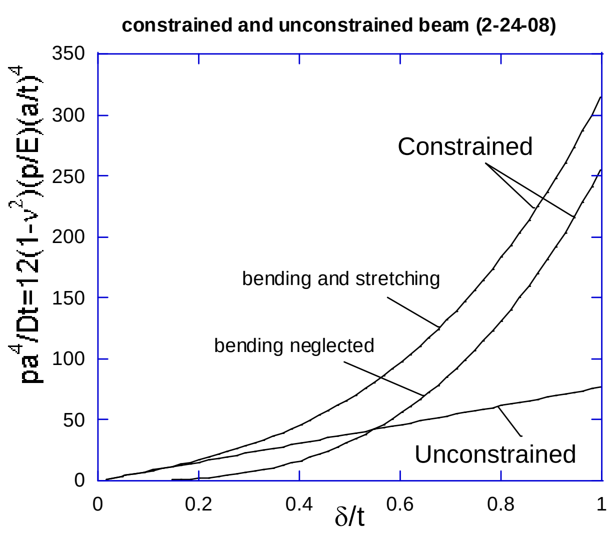
$$\equiv \frac{p^2a^3}{N^2}\left\{\frac{1}{24} - \frac{1}{\bar{\mu}^2}\left(1 - \frac{2\tanh(\bar{\mu}/2)}{\bar{\mu}}\right) - \frac{1}{4\bar{\mu}^2\cosh^2(\bar{\mu}/2)}\left(1 - \frac{\sinh(\bar{\mu})}{\bar{\mu}}\right)\right\}$$

Noting $D = St^2/12$ with t as the thickness of a beam having rectangular cross-section, this can be rewritten as

$$\left(\frac{pa^4}{Dt}\right)^2 = \frac{\bar{\mu}^6}{12 f(\bar{\mu})}$$

In addition, with $\delta = w(0)$ as the deflection in the center of the beam, one finds

$$\frac{\delta}{t} = \left(\frac{pa^4}{Dt}\right)\frac{1}{\bar{\mu}^2}\left[\frac{1}{8} - \frac{1}{\bar{\mu}^2}\left(1 - \frac{1}{\cosh(\bar{\mu}/2)}\right)\right]$$



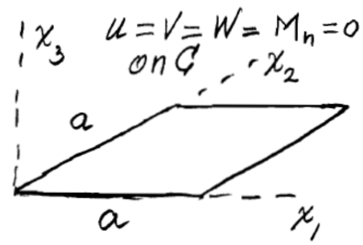
This is a highly nonlinear result. The curve in the figure is plotted by specifying values of $\bar{\mu}$ and then using the formulas to compute the normalized p and δ (not untypical for nonlinear problems). Note that $pa^4/Dt = 12(p/\bar{E})(a/t)^4$. The result for the unconstrained case is included in the figure. The two problems are close to one another only for $\delta / t < 0.3$—this is the regime in which stretching is not important. For larger deflections, stretching dominates the behavior of the constrained beam and the behavior is highly nonlinear. The solution which neglects bending (nonlinear membrane theory solution) which is set in Homework Problem #12 below is also included in the figure. The complete solution, accounting for both bending and stretching, transitions from the bending solution to the membrane solution.

Homework Problem 12: Nonlinear membrane solution for a beam with zero bending stiffness

Repeat the above analysis for the constrained case by neglecting the bending stiffness (i.e. set D to zero in the moment equilibrium equation). This analysis is much easier! You should obtain $\frac{\delta}{t} = \frac{1}{4}\left(\frac{3pa^4}{St^3}\right)^{1/3} = \frac{1}{4}\left(\frac{pa^4}{4Dt}\right)^{1/3}$, where the last expression obtained using $D \equiv St^2/12$ allows one to compare the this result from nonlinear membrane theory with the result in the above figure which accounts for bending stiffness.

Simply supported square plate with constrained in-plane edge displacements

Consider the square plate $(0 \le x_1 \le a,\ 0 \le x_2 \le a)$ with $w = v = u = M_n = 0$ along all its edges and subject to a uniform normal pressure p, as depicted in the figure. The condition of free rotation, $M_n = 0$, together with $w = 0$ is equivalent to $w_{,11} = 0$ along edges parallel to the x_2 axis and to $w_{,22} = 0$ along the x_1 axis. We will develop an approximate solution using the potential energy functional for nonlinear plate theory:



$$PE(u,v,w) = \int_S \left[\tfrac{1}{2} D\left(\nabla^2 w\right)^2 + \tfrac{1}{2} N_{\alpha\beta} E_{\alpha\beta} - pw \right] dS$$

This is the sum of the bending energy, the stretching energy and the potential energy of the loads. The *geometric boundary conditions* associated with stationarity of PE are $u = v = w = 0$ on the boundary.

The following fields, which satisfy the geometric boundary conditions (and $M_n = 0$ on the edges), will be used to obtain an approximate solution:

$$w = \xi \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)$$
$$u = \mu \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)$$
$$v = \mu \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)$$

These displacements are consistent with symmetry of deformations about the center of the plate. The free amplitude variables are ξ and μ; they will be obtained by rendering PE stationary. The amplitudes of u and v are the same due to symmetry.

The first and last contributions to PE are readily computed:

$$\frac{1}{2}D\int_S \left(\nabla^2 w\right)^2 dS = \frac{1}{2}\frac{\pi^4 D}{a^2}\xi^2 \ , \qquad \int_S pwdS = \frac{4a^2}{\pi^2}p\xi$$

The middle term in PE can be computed in a straightforward, but more lengthy, manner:

$$\frac{1}{2}N_{\alpha\beta}E_{\alpha\beta} = \frac{Et}{2(1-\nu^2)}\left[E_{11}{}^2 + E_{11}{}^2 + 2\nu E_{11}E_{22} + 2(1-\nu)E_{12}{}^2\right]$$

$$E_{11} = \frac{2\pi\mu}{a}\cos\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right) + \frac{1}{2}\left(\frac{\pi\xi}{a}\right)^2 \cos^2\left(\frac{\pi x_1}{a}\right)\sin^2\left(\frac{\pi x_2}{a}\right)$$

$$E_{22} = \frac{2\pi\mu}{a}\sin\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{2\pi x_2}{a}\right) + \frac{1}{2}\left(\frac{\pi\xi}{a}\right)^2 \sin^2\left(\frac{\pi x_1}{a}\right)\cos^2\left(\frac{\pi x_2}{a}\right)$$

$$E_{12} = \frac{\pi\mu}{2a}\left[\sin\left(\frac{2\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right) + \cos\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)\right]$$
$$+\frac{1}{2}\left(\frac{\pi\xi}{a}\right)^2 \sin\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\cos\left(\frac{\pi x_2}{a}\right)$$

The integrations in the stretching contribution are straight forward (but a bit tedious):

$$\tfrac{1}{2}\int_S N_{\alpha\beta}E_{\alpha\beta}dS = \frac{Et}{2(1-\nu^2)}\left\{H_1\mu^2 + H_2\frac{\mu\xi^2}{a} + H_3\frac{\xi^4}{a^2}\right\}$$

where

$$H_1 = 2\pi^2 + \frac{32}{9}\nu + (1-\nu)\left(\frac{\pi^2}{4} + \frac{64}{9}\right), \quad H_2 = \frac{\pi^2}{3}\left(5-\nu\right), \quad H_3 = \frac{5\pi^4}{64}$$

Combining all the terms in the potential energy gives

$$PE(\xi,\mu) =$$
$$\frac{Dt^2}{2a^2}\left\{\pi^4\left(\frac{\xi}{t}\right)^2 + 12H_1\left(\frac{a\mu}{t^2}\right)^2 + 12H_2\left(\frac{a\mu}{t^2}\right)\left(\frac{\xi}{t}\right)^2 + 12H_3\left(\frac{\xi}{t}\right)^4 - \frac{8pa^4}{\pi^2 Dt}\left(\frac{\xi}{t}\right)\right\}$$

Let $\bar{\xi} = \xi / t$ and $\bar{\mu} = a\mu / t^2$, and then render $PE(\xi,\mu)$ stationary with respect to $\bar{\xi}$ and $\bar{\mu}$ to obtain the two equations:

$$\bar{\xi} + \frac{12}{\pi^4}H_2\bar{\mu}\bar{\xi} + \frac{24}{\pi^4}H_3\bar{\xi}^3 = \bar{p} \equiv \frac{4pa^4}{\pi^6 Dt}$$

$$24H_1\bar{\mu} + 12H_2\bar{\xi}^2 = 0$$

Next eliminate $\bar{\mu}$ in the first equation using the second equation:

$$\bar{\xi} + c\bar{\xi}^3 = \bar{p}\ , \qquad \text{where} \quad c = \frac{6}{\pi^4}\left(4H_3 - \frac{H_2^{\ 2}}{H_1}\right)$$

This is the nonlinear equation relating the normalized amplitude of the deflection to the pressure. Three cases are of interest. The linearized equation for *linear bending with no stretching* is $\bar{\xi} = \bar{p}$, and this solution is exactly what we obtained from the one-term solution to the linear problem analyzed earlier. If you trace back in the analysis and neglect the bending energy you will see that you obtain $c\bar{\xi}^3 = \bar{p}$, which is the same as $c\xi^3 = \tilde{p}$ with $\tilde{p} = 48(1-\nu^2)pa^2/(\pi^6 Et)$ involving only the in-plane stretching stiffness. This is the *nonlinear membrane solution* which neglects any effect of bending. Fully *coupled bending and stretching* is governed by $\bar{\xi} + c\bar{\xi}^3 = \bar{p}$, which is plotted in the accompanying figure along with the other two limiting cases. The dimensionless results have been plotted with $\nu = 1/3$, for which $H_1 = 27.31$, $H_2 = 15.35$, $H_3 = 7.61$ and $c = 1.34$.

Important points emerge from these results.

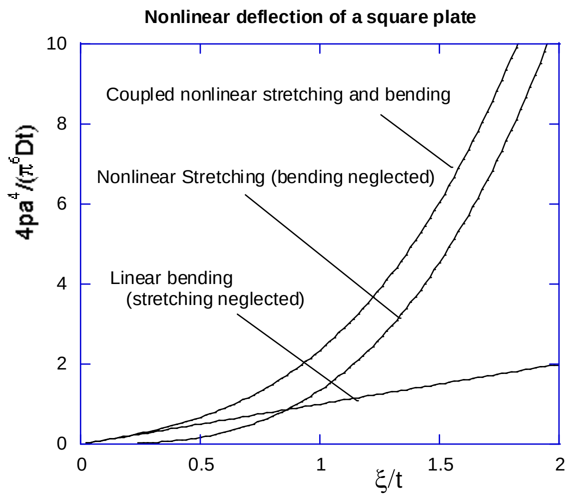
(i) Nonlinear effects (stretching) already become important after normal deflections of less than about $t/3$. This is typical for plate problems, as the earlier beam problem emphasizes. For even larger deflections, stretching becomes dominant and membrane theory becomes a better and better approximation. At the center of the plate, for example,

$$E_{11} = E_{22} = -2\pi(t/a)^2 \bar{\mu} = 2.38(t/a)^2 \bar{\xi}^2$$

and thus the strains can still be small when $\xi / t >> 1$ if t/a is sufficiently small.

(ii) It is readily verified that the rotations about the x_3 axis, $\omega = (u_{2,1} - u_{1,2})$, are of the same order of $E_{\alpha\beta}$, justifying this assumption made in deriving the nonlinear plate equations.

(iii) The in-plane displacements are much smaller than the normal displacement:



$\bar{\mu} = -(H_2 / 2H_1)\bar{\xi}^2$. In dimensional terms, this is $\mu / t = -(H_2 / 2H_1)(t/a)(\xi / t)^2$.

Vibrations of plates

An excellent reference to the subject is *Vibrations of Plates* by A. W. Leissa (a NASA Report SP-160, 1969). We consider out-of-plane motion, $w(\mathbf{x}, t)$, governed by the linearized Karman plate equations (i.e. the linear bending equation with p identified with $-\rho w_{,tt}$ where ρ is the mass/area of the plate):

$$D\nabla^4 w = -\rho w_{,tt}$$

together with relevant homogeneous boundary conditions on C. In the standard investigation of the vibration frequencies and the associated vibration modes, one looks for solutions of the form

$$w(\mathbf{x}, t) = W(\mathbf{x})\{\sin \omega t \text{ or } \cos \omega t\} \quad \Rightarrow \quad D\nabla^4 W = \rho\omega^2 W \text{ plus homogenerous BCs}$$

This is an eigenvalue problem. The eigenvalues are the vibration frequencies denoted by ω_n and usually ordered according to $\omega_1 < \omega_2 < \omega_3$, etc., and the associated eigenmodes are the vibration modes labeled as W_n. It is not uncommon that a given vibration frequency might have multiple modes associated with it and thus the numbering system has to take that into account.

Homework Problem 13: Vibration frequencies and modes of a simply supported rectangular plate

Determine the vibration frequencies and modes of a simply supported rectangular plate ($-a/2 \le x_1 \le a/2$, $-b/2 \le x_2 \le b/2$).

Vibration frequencies and modes of a clamped circular plate of radius a

First consider general solutions to $D\nabla^4 W = \rho\omega^2 W$ where in circumferential coordinates $\nabla^2 W = r^{-1}(rW_{,r})_{,r} + r^{-2}W_{,\theta\theta}$. Introduce the dimensionless radial coordinate, $\eta = r/a$, with $0 \le \eta \le 1$. The pde admits separated solutions of the form

$$W(\eta,\theta) = f(\eta)\sin n\theta \;, \quad \text{or} \quad W(\eta,\theta) = f(\eta)\cos n\theta$$

where $f(\eta)$ satisfies the 4th order ode

$$L_n\left(L_n(f)\right) - \lambda^4 f = 0 \;\text{ with }\; \lambda^4 \equiv \frac{\rho\omega^2 a^4}{D} \;\text{ and }\; L_n(f) = \eta^{-1}(\eta f')' - n^2\eta^{-2} f$$

It is very useful to note that the 4th order ode can be "split" as

$$\left(L_n(\;) + \lambda^2\right)\left(L_n(\;) - \lambda^2\right) f = 0$$

The general solution is the two set of two linearly independent solutions to each of the 2nd order equations. Using standard notation for Bessel functions, the solution is

$$f(\eta) = c_1 J_n(\lambda\eta) + c_2 Y_n(\lambda\eta) + c_3 I_n(\lambda\eta) + c_4 K_n(\lambda\eta)$$

Recall that Y_n and K_n are unbounded at $\eta = 0$ and thus we must take $c_2 = c_4 = 0$.

For a *clamped plate*, $f(1) = f'(1) = 0$, requiring

$$c_1 J_n(\lambda) + c_3 I_n(\lambda) = 0 \;\;\&\;\; c_1 J_n'(\lambda) + c_3 I_n'(\lambda) = 0$$

The identities for the derivatives, $\lambda J_n' = nJ_n - \lambda J_{n+1}$ and $\lambda I_n' = nI_n + \lambda I_{n+1}$ allow us to obtain the eigenvalue equation for the vibration frequencies

$$J_n(\lambda_{nm}) I_{n+1}(\lambda_{nm}) + I_n(\lambda_{nm}) J_{n+1}(\lambda_{nm}) = 0$$

The notation, $\lambda_{nm} = \left(\rho\omega_{nm}{}^{2}a^{4}/D\right)^{1/4}$, is used because for each $n = 0,\infty$ there are infinitely many ($m = 1,\infty$) roots to the eigenvalue equation. The roots have to be obtained numerically, but the particular equation above arises in various contexts so it has been tabulated (e.g. in Abramowitz and Stegun)—see also Leissa. The lowest frequency is associated with $n = 0$ (an axisymmetric mode) and is given by

$$\omega = 10.2158\,a^{2}\sqrt{D/\rho} \quad (n = 0)$$

The next lowest frequency is

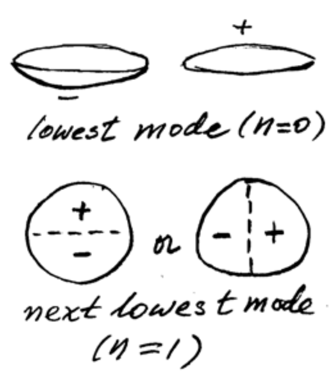
$$\omega = 21.26\,a^{2}\sqrt{D/\rho} \quad (n = 1)$$

The mode shapes are sketched in the figure below.

TABLE 2.1.—*Values of $\lambda^2 = \omega a^2\sqrt{\rho/D}$ for a Clamped Circular Plate*

s	λ^2 for values of n of—														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	10.2158	21.26	34.88	51.04	69.6659	90.7390	114.2126	140.0561	168.2445	198.7561	231.5732	266.6790	304.0601	343.7038	385.5996
1	39.771	60.82	84.58	111.01	140.1079	171.8029	206.0706	242.8782	282.1977	324.0036	368.2734	--------	--------	--------	--------
2	89.104	120.08	153.81	190.30	229.5186	271.4283	316.0015	363.2097	--------	--------	--------	--------	--------	--------	--------
3	158.183	199.06	242.71	289.17	338.4113	390.3896	--------	--------	--------	--------	--------	--------	--------	--------	--------
4	247.005	297.77	351.38	407.72	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------
5	355.568	416.20	479.65	545.97	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------
6	483.872	554.37	627.75	703.95	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------
7	631.914	712.30	795.52	881.67	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------
8	799.702	889.95	983.07	1079.0	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------
9	987.216	1087.4	1190.4	1296.2	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------	--------

The whole range of frequencies and modes are given by Leissa (see above), along with frequencies and modes of other boundary conditions and plate shapes. One picture from Leissa of Chladni patterns using fine power to highlight the nodes of the vibration modes is shown for a triangular plate which is unsupported (free) on all its edges.



Mode shapes for two lowest modes of a clamped circular plate.

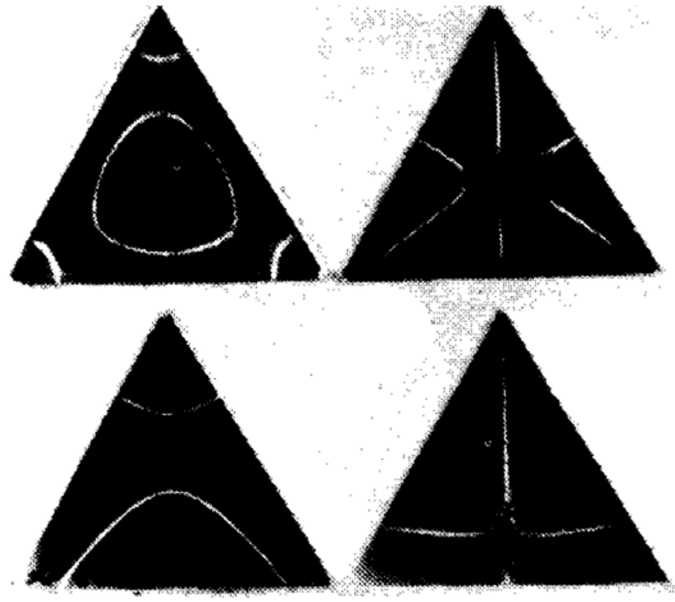


FIGURE 7.44.—Nodal patterns for a F–F–F equilateral triangular plate; material, brass. (From ref. 7.38)

Classical plate buckling

Classical plate buckling problems are characterized by an in-plane prebuckling state of stress, $N^0_{\alpha\beta}$, which is assumed to be an exact solution to the plane stress equations (i.e. to the von Karman plate equations with $w = 0$). A perturbation expansion is used to identify the lowest critical value of $N^0_{\alpha\beta}$ such that an out-of-plane displacement w develops beginning as a bifurcation from the in-plane state. In this section of the notes, we will only obtain the critical value (the lowest eigenvalue) and the associated mode—i.e. the classical analysis analogous to the buckling analysis of beams. Continuing the perturbation expansion gives information on the post-buckling behavior.

With reference to the coupled nonlinear von Karman plate equations, the prebuckling solution is

$$w^{(0)} = 0,\ \ F^{(0)} = \tfrac{1}{2} N^0_{11} x_2{}^2 + \tfrac{1}{2} N^0_{22} x_1{}^2 - N^0_{12} x_1 x_2$$

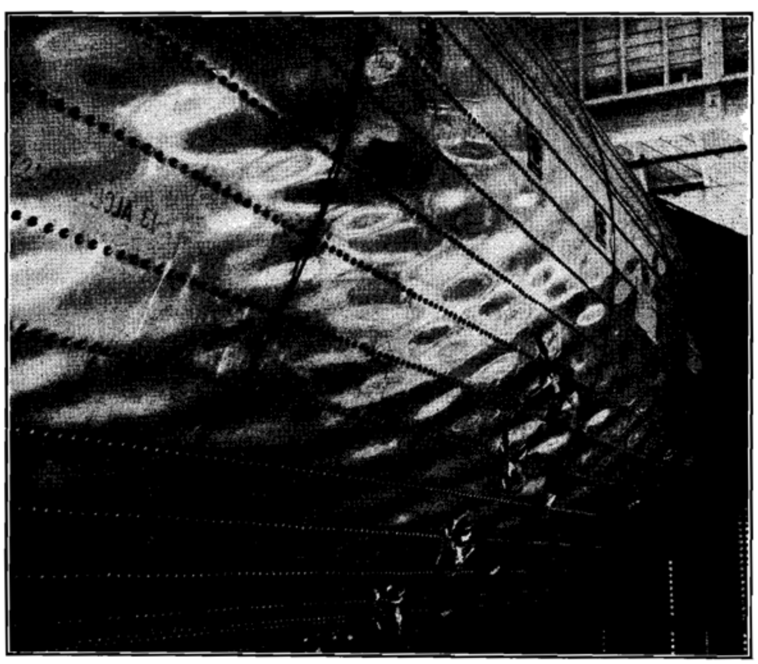
where the in-plane stresses $N^0_{\alpha\beta}$ are assumed to be related to the intensity of the loading system applied to the plate. Investigate the possibility of a bifurcation from the in-plane state by looking for a solution of the form

$$w = w^{(0)} + \xi w^{(1)} + ..., \quad F = F^{(0)} + \xi F^{(1)}$$

with ξ as a perturbation parameter. Substitute into the von Karman plate equations and linearize with respect to ξ to obtain

$$D\nabla^4 w^{(1)} - N^0_{\alpha\beta} w^{(1)}{}_{,\alpha\beta} = 0 \ , \qquad (Et)^{-1}\nabla^4 F^{(1)} = 0$$

The first equation is the classical buckling equation. It depends only on $w^{(1)}$ and is decoupled from $F^{(1)}$. Consider boundary conditions such as clamped, free or simply supported; these lead to homogeneous conditions on $w^{(1)}$. For example, for clamped conditions, $w^{(1)} = w^{(1)}_{,n} = 0$. Thus, the buckling problem is an *eigenvalue problem*. The prebuckling stress, $N^0_{\alpha\beta}$, is the eigenvalue and $w^{(1)}$ is the eigenmode. Note that in a typical problem, the relative proportions of the components $N^0_{\alpha\beta}$ are fixed and there is a single amplitude factor. In almost all buckling problems, it is only the lowest eigenvalue that is of interest—it is called the buckling stress. An example of shear buckling of nominally flat plates between stringers of a fuselage is shown below. The fuselage has been loaded to induce shear in the skin panels.



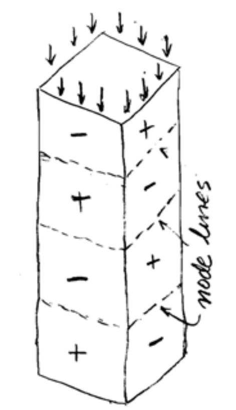
(b)

Fig. 25.12. **Buckling of flat plates in shear. Photograph shows test of an airplane-fuselage structure. (*Lockheed Aircraft Corporation.*)**

The picture has been copied from *Strength of Materials* by F.R. Stanley.

Buckling of a simply supported rectangular plate under uniaxial compression

Consider a simply-supported plate subject to an uniaxial in-plane state of uniaxial compression, $N_{11}^0 = -N^0$, $N_{12}^0 = N_{22}^0 = 0$. These boundary conditions are relevant to multiple plate panels stiffened by stringers spaced a distance b apart when the torsional stiffness of the stringer is not large and to a long thin-walled square tube of width b where symmetry dictates that the buckles alternate in sign from plate to plate as one traverses the tube as sketched in the accompanying figure. The buckling equation and boundary conditions are:



$$D\nabla^4 w^{(1)} + N^0 w_{,11}^{(1)} = 0 \; ; \quad w^{(1)} = w_{,11}^{(1)} = 0 \text{ on } x_1 = 0, a \; ; \; w^{(1)} = w_{,22}^{(1)} = 0 \text{ on } x_2 = 0, b$$

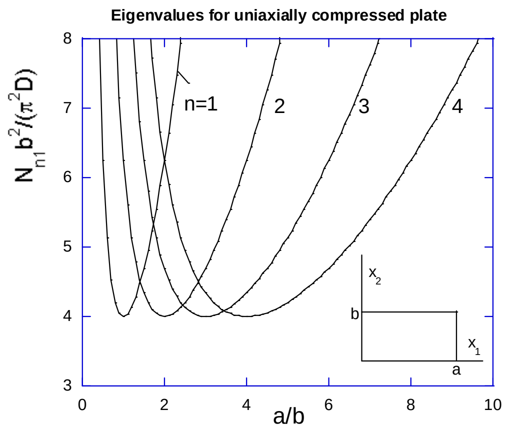
The equation and boundary conditions admit exact solutions of the form (with n and m as any integers): $w^{(1)} = \sin(n\pi x_1 / a)\sin(m\pi x_2 / b)$. Moreover, a double series representation comprised of these solutions is complete so exploring solutions for all n and m will provide the desired solution. With $w^{(1)} = \sin(n\pi x_1 / a)\sin(m\pi x_2 / b)$, the eigenvalue equation becomes

$$N_{nm}^0 = \pi^2 D \left[\frac{\left((n/a)^2 + (m/b)^2 \right)^2}{(n/a)^2} \right]$$

Now we determine the lowest eigenvalue. Clearly, the lowest value will be associated with $m = 1$ which can be written as

$$N_{n1}^0 = \frac{\pi^2 D}{b^2} \left[\frac{\left(n^2 (b/a) + (a/b) \right)^2}{n^2} \right]$$

The integer value of n giving the lowest eigenvalue depends on a/b. The following plot is easily constructed.



For $a/b=1$, the critical (lowest) eigenvalue is $N_{CR}b^2/(\pi^2 D)=4$ with $n=1$. Note that as a/b becomes large, $N_{CR}b^2/(\pi^2 D)$ also approaches 4. To obtain this limiting result analytically, assume $a/b >> 1$, treat n as a continuous variable, and minimize $N_{CR}b^2/(\pi^2 D)$ with respect to n. You will obtain $N_{CR}b^2/(\pi^2 D)=4$ and $n=a/b$. When a/b is large but not an integer, there exists an integer nearly equal to a/b which gives an eigenvalue only slightly larger than $N_{CR}b^2/(\pi^2 D)=4$, as evidenced in the figure above. The mode shape with $n=a/b$ and $m=1$ is $w^{(1)}=\sin(\pi x_1/b)\sin(\pi x_2/b)$ corresponding to a pattern with equal half-wavelength in the two directions.

Variational statement of the buckling eigenvalue problem

For rectangular plates with $w=0$ on the edges or for arbitrary shaped plates with clamped conditions, the following functional, $\Phi(w)$, provides a variational principle for the buckling eigenvalue problem:

$$\Phi(w)=\frac{1}{2}\int_S \left[D\left(\nabla^2 w\right)^2 + N^0_{\alpha\beta} w_{,\alpha} w_{,\beta} \right] dS \qquad (w \equiv w^{(1)})$$

Note that the full bending energy expression must be used in place of $D\left(\nabla^2 w\right)^2$ for more general boundary conditions (e.g. free) or other shapes.

Homework Problem #14: Variational principle for the buckling eigenvalue problem

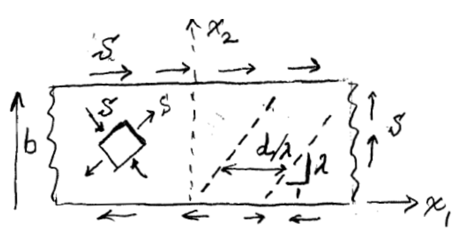
As an exercise, show that the buckling equation above follows from rendering $\delta\Phi = 0$ for all admissible δw (assume the plate is either simply-supported or clamped on C). Moreover, for any eigenmode, w, associated with an eigenvalue, $N^0_{\alpha\beta}$, one can readily show that (just set $\delta w = w$ in the variational equation):

$$\Phi(w) = 0 \Rightarrow \int_S \left[N^0_{\alpha\beta} w_{,\alpha} w_{,\beta} \right] dS = -\int_S \left[D\left(\nabla^2 w\right)^2 \right] dS$$

This result generalizes for arbitrary shapes and boundary conditions but then one has to use the full expression for the bending energy.

One immediately sees that buckling requires $N^0_{\alpha\beta} w_{,\alpha} w_{,\beta}$ <0 over at least part of S. Thus, for example, if $N^0_{22} \neq 0$ and $N^0_{12} = N^0_{11} = 0$, $N^0_{\alpha\beta} w_{,\alpha} w_{,\beta} = N^0_{22} w_{,2}{}^2$ proving that buckling cannot occur for $N^0_{22} > 0$.

An interesting and important example is buckling of a *long simply supported rectangular plate in shear* ($N^0_{12} \equiv S > 0$ and $N^0_{11} = N^0_{22} = 0$). For a wide plate stiffened by stringers spaced a distance b apart, as in the picture of the buckled fuselage, the assumption of simple support at the stiffeners is often a reasonable approximation because a stiffener is effective at suppressing normal deflection but not rotation due to its low torsional stiffness. Thus, consider an infinitely long simply supported plate in shear with the lower edge along $x_2 = 0$ and the upper edge along $x_2 = b$. (These conditions are the same as those for a mode that is required to be periodic in the x_2 direction with half-period b and vanishing at $x_2 = 0, b, 2b, 3b, \ldots$, as illustrated by the fuselage). No simple solutions to the shear buckling problem exist, but a solution based on a doubly infinite sine series is given by Timoshenko and Gere (*Theory of Elastic Stability*). In what follows, we will make a judicious choice of mode in $\Phi(w)$ to generate an approximation to the lowest buckling mode, along the lines also given by Timoshenko and Gere. Although we will not prove it here, the approach used leads to an upper bound to the lowest eigenvalue.



With $N_{12}^0 \equiv S > 0$, note as illustrated in the figure that the in-plane shear stress creates compression, $-S$, across lines parallel to $x_2 = x_1$ and tension, S, across lines parallel to $x_2 = -x_1$. We can anticipate that the buckles will be oriented perpendicular to the compressive direction. Let $\xi = (x_2 + x_1)/\sqrt{2}$ and $\eta = (x_2 - x_1)/\sqrt{2}$ be coordinates oriented at 45 degrees to the plate axes as shown in the figure. Then note that for any function $w(x_1, x_2) = f(\eta)$, $N_{\alpha\beta}^0 w_{,\alpha} w_{,\beta} = 2Sw_{,1}w_{,2} = -S(f'(\eta))^2$ which will be negative as required by the identity above.

For the admissible function w in $\Phi(w)$, take

$$w = \sin\left(\frac{\pi}{d}(x_2 - \lambda x_1)\right)\sin\left(\frac{\pi}{b}x_2\right)$$

The two free parameters, d and λ, determine the slant of the nodal lines and the spacing, d/λ, between the nodal lines in the x_1-direction. Note that the function satisfies the admissibility requirement, $w = 0$, on the edges (plus periodicity in the x_1-direction). It does not satisfy the natural boundary condition, $M_n = 0 \Rightarrow w_{,22} = 0$, on the edges, but this is not necessary in applying the principle.

Substitute w into $\Phi(w)$ and perform the integrations to obtain

$$\Phi = \frac{bd}{4}\left\{\pi^4 D\left[\frac{(\lambda^2+1)^2}{d^4} + \frac{1}{b^4} + \frac{2(\lambda^2+6)}{b^2 d^2}\right] - 2\pi^2 S\frac{\lambda}{d^2}\right\}$$

where the integration is taken over one full wavelength, $2d$, in the x_1-direction. Now set $\Phi = 0$, to obtain the (upper bound) estimate to the lowest eigenvalue, S,

$$\frac{Sb^2}{\pi^2 D} = \frac{1}{2\lambda}\left\{\frac{(\lambda^2+1)^2}{\xi^2} + \xi^2 + 2(\lambda^2+3)\right\}, \quad \text{with} \quad \xi \equiv \frac{d}{b}$$

It is straightforward to minimize S with respect to the two free parameters to obtain

$$\xi = \sqrt{\lambda^2+1} = \sqrt{3}\,, \quad \lambda = \sqrt{2} \quad \text{and} \quad \frac{S_{CR}b^2}{\pi^2 D} = 4\sqrt{2} = 5.66$$

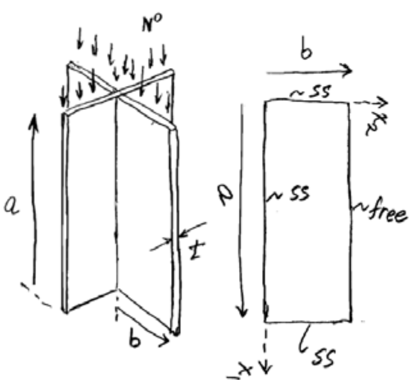
The exact solution obtained by the infinite series method gives the dimensionless shear stress at buckling as 5.35 (Timoshenko and Gere).

Homework Problem #15: Buckling of an infinitely long clamped plate

Consider an infinitely long plate clamped along its top and bottom edges, $x_2 = \pm a/2$, and compressed uniformly parallel to the edges, $N_{11}^0 = -N^0$. Look for solutions of the form $w = \sin(\alpha \pi x_1 / a)\, f(x_2)$ where α is a free parameter which determines the wavelength of the mode in the x_1-direction. Determine f either by rendering Φ stationary or by working directly with the pde. Then compute the lowest eigenvalue associated with the critical buckling stress, N_{CR}^0 (modest numerical work will be necessary). Is the solution exact?

Homework Problem #16: Buckling of a rectangular plate with one free edge

The cruciform column shown in the figure is one "popular" configuration that has been used to carry out buckling experiments in both the elastic and plastic ranges. The cruciform undergoes torsional bucking about the common central axis. Consider any one of the plate webs and take the in-plane coordinates such that $0 \le x_1 \le a$ and $0 \le x_2 \le b$ as shown. The edge along $x_2 = 0$ can be regarded as simply supported because under torsion of the whole column the edge remains straight and there is no resistance to rotation. The edge along $x_2 = b$ is free. Model the ends at $x_1 = 0$ and $x_1 = a$ as being simply supported. The in-plane prebuckling compression is $N_{11}^0 = -N^0$. Determine an estimate of the critical buckling stress, N_{CR}^0, using the following method.



Obtain an (upper bound) estimate of N_{CR}^0 by using the field $w = x_2 \sin(n\pi x_1 / a)$ in the functional $\Phi(w)$. State clearly why this field is admissible and be sure to note that the simplified expression for the bending energy density cannot be used. Note that the minimum estimate is given by $n = 1$. Moreover, for $a/b >> 1$, $N_{CR}^0 \to 6(1-\nu)D/b^2 = Gt(t/b)^2$ where $G = E/[2(1+\nu)]$ is the shear modulus and t is the plate thickness. (This, in fact, is the exact limiting result for N_{CR}^0 when $a/b >> 1$).

The exact solution to the above problem for general a/b can be obtained by assuming $w = f(x_2)\sin(n\pi x_1/a)$. Can you obtain the fourth order ode for f together with the boundary conditions? While this eigenvalue problem can be solved with modest numerical work, you are not being asked to do so.