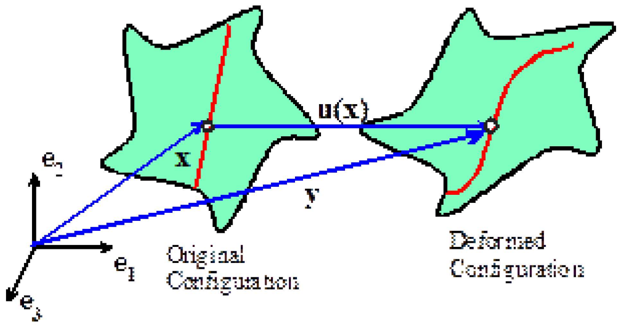
목록

3. Kinematics

3.1 Basic Assumptions

Continuum mechanics is a combination of mathematics and physical laws that approximate the large-scale behavior of matter that is subjected to mechanical loading. It is a generalization of Newtonian particle dynamics, and starts with the same physical assumptions inherent to Newtonian mechanics; and adds further assumptions that describe the structure of matter. Specifically:



- **The Newtonian reference frame:** In classical continuum mechanics, the world is idealized as a three dimensional Euclidean space (a vector space consisting of all triads of real numbers (x_1, x_2, x_3)). A *point* in space is identified by a unique set of three real numbers. A Euclidean space is endowed with a *metric*, which defines the distance between points: $d = \sqrt{x_i x_i}$. Vectors can be expressed as components in a ***basis*** - $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of mutually perpendicular unit vectors. Physical quantities such as force, velocity, acceleration are expressed as vectors in this space. A *Cartesian Coordinate Frame* is a fixed point O together with a basis. A *Newtonian reference frame* is a particular choice of Cartesian coordinate frame in which Newton's laws of motion hold.

- **The Continuum:** Matter is idealized as a *continuum*, which has two properties: (i) it is infinitely divisible (you can subdivide some region of the solid as many times as you wish); and (ii) it is locally homogeneous – in other words if you subdivide it sufficiently many times, all sub-divisions have identical properties (eg mass density). A continuum can be thought of as an infinite set of vanishingly small particles, connected together.

Both the existence of a Newtonian reference frame, and the concept of a continuum, are mathematical idealizations. Experimental evidence suggest that the laws of motion based on these assumptions accurately approximate the behavior of most solid and fluid materials at length scales of order mm-km or so in engineering applications. In some cases continuum models can also approximate behavior at much shorter length scales (for volumes of material containing a few 1000 atoms), but models at these length scales often require different relations between internal forces deformation measures in the solid to those used to model larger volumes.

3.2 Reference and deformed configuration of a solid

The *configuration* of a solid is a region of space occupied (filled) by the solid. When we describe motion, we normally choose some convenient configuration of the solid to use as *reference* - this is often the initial, undeformed solid, but it can be any convenient region that could be occupied by the solid. The material changes its shape under the action of external loads, and at some time *t* occupies a new region which is called the *deformed* or *current* configuration of the solid.

For some applications (fluids, problems with growth or evolving microstructures) a fixed reference configuration can't be identified – in this case we usually use the deformed material as the reference configuration.

Mathematically, we describe a *deformation* as a 1:1 mapping which transforms points from the reference configuration of a solid to the deformed configuration. Specifically, let ξ_i be three numbers specifying the position of some point in the undeformed solid (these could be the three components of position vector in a Cartesian coordinate system, or they could be a more general ccoordinate system, such as polar coordinates). As the solid deforms, each the values of the coordinates change to different numbers. We can write this in general form as $\eta_i = f_i(\xi_k, t)$. This is called a *deformation mapping*.

To be a physically admissible deformation

(i) The coordinates must specify positions in a Newtonian reference frame. This means that it must be possible to find some coordinate transformation $x_i(\xi_k)$, such that x_i are components in an orthogonal basis, which is taken to be 'stationary' in the sense of Newtonian dynamics.

(ii) The functions $f_i(\xi_k)$ must be 1:1 on the full set of real numbers; and f_i must be invertible

(iii) f_i must be continuous and continuously differentiable (we occasionally relax these two assumptions, but this has to be dealt with on a case-by-case basis)

(iv) The mapping must satisfy $\det(\partial \eta_i / \partial \xi_j) > 0$.

To begin with, we will describe all motions and deformations by expressing positions of points in both undeformed and deformed solids as components in a Cartesian reference frame (which is also taken to be an inertial frame). Thus x_i will denote components of the position vector of a materal particle before deformation, and $y_i(x_k)$ will be components of its position vector after deformation.

3.3 The Displacement and Velocity Fields

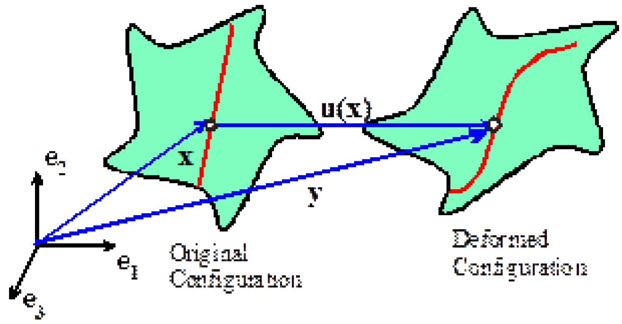
The displacement vector $\mathbf{u}(\mathbf{x},t)$ describes the motion of each point in the solid. To make this precise, visualize a solid deforming under external loads. Every point in the solid moves as the load is applied: for example, a point at position $\mathbf{x}$ in the undeformed solid might move to a new position $\mathbf{y}$ at time t. The displacement vector is defined as

$$\mathbf{y} = \mathbf{x} + \mathbf{u}(\mathbf{x}, t)$$

We could also express this formula using *index notation*, as

$$y_i = x_i + u_i(x_1, x_2, x_3, t)$$

Here, the subscript i has values 1,2, or 3, and (for example) y_i represents the three Cartesian components of the vector $\mathbf{y}$.



The displacement field completely specifies the change in shape of the solid. The *velocity field* would describe its motion, as

$$v_i(x_k, t) = \frac{\partial y_i}{\partial t} = \left.\frac{\partial u_i(x_k,t)}{\partial t}\right|_{x_k=\text{const}}$$

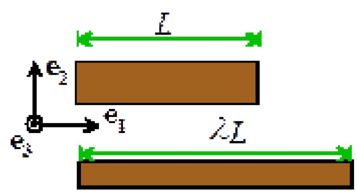
We also define the *acceleration field*

$$a_i(x_k, t) = \frac{\partial^2 y_i}{\partial t^2} = \left.\frac{\partial v_i(x_k,t)}{\partial t}\right|_{x_k=\text{const}}$$

Examples of some simple deformations

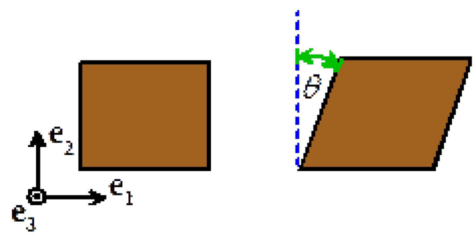
- **Volume preserving uniaxial extension**

$$y_1 = \lambda x_1$$
$$y_2 = x_2/\sqrt{\lambda}$$
$$y_3 = x_3/\sqrt{\lambda}$$



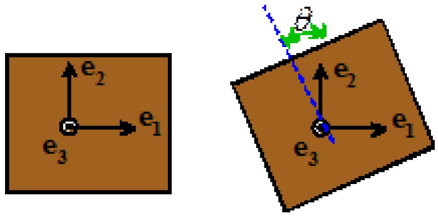
- **Simple shear**

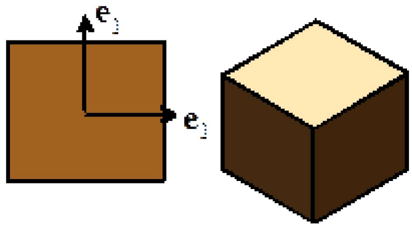
$$y_1 = x_1 + \tan\theta \, x_2$$
$$y_2 = x_2$$
$$y_3 = x_3$$



- **Rigid rotation through angle θ about $\mathbf{e}_3$ axis**

$$y_1 = x_1 \cos\theta - x_2 \sin\theta$$
$$y_2 = x_2 \cos\theta + x_1 \sin\theta$$
$$y_3 = x_3$$





- **General rigid rotation about the origin**

$$\mathbf{y} = \mathbf{R} \cdot \mathbf{x} \quad \text{or} \quad y_i = R_{ij} x_j$$

where **R** must satisfy $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$, det(**R**)>0. (i.e. **R** is proper orthogonal). **I** is the identity tensor with components $\delta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$

Alternatively, a rigid rotation through angle θ (with right hand screw convention) about an axis through the origin that is parallel to a unit vector **n** can be written as

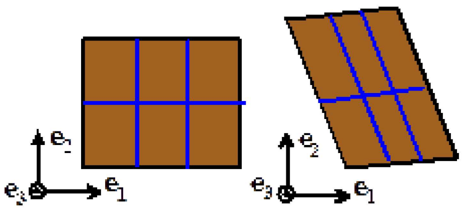
$$\mathbf{y} = \cos\theta \mathbf{x} + (1 - \cos\theta)(\mathbf{n} \cdot \mathbf{x})\mathbf{n} + \sin\theta(\mathbf{n} \times \mathbf{x})$$

The components of **R** are thus

$$R_{ij} = \cos\theta \delta_{ij} + (1 - \cos\theta) n_i n_j + \sin\theta \in_{ikj} n_k$$

where $\in_{ijk}$ is the *permutation symbol*, satisfying

$$\in_{ijk} = \begin{cases} 1 & i,j,k = 1,2,3; \quad 2,3,1 \text{ or } 3,1,2 \\ -1 & i,j,k = 3,2,1; \quad 2,1,3 \text{ or } 1,3,2 \\ 0 & \text{otherwise} \end{cases}$$



- **General homogeneous deformation**

$$y_1 = A_{11} x_1 + A_{12} x_2 + A_{13} x_3 + c_1$$
$$y_2 = A_{21} x_1 + A_{22} x_2 + A_{23} x_3 + c_2$$
$$y_3 = A_{31} x_1 + A_{32} x_2 + A_3 x_3 + c_3$$

or

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x} + \mathbf{c} \quad y_i = A_{ij} x_j + c_i$$

where A_{ij} are constants.

The physical significance of a homogeneous deformation is that all straight lines in the solid remain straight under the deformation. Thus, every point in the solid experiences the same shape change. All the deformations listed above are examples of homogeneous deformations.

3.4 Eulerian and Lagrangian descriptions of motion and deformation.

The displacement and velocity are vector valued functions. In any application, we have a choice of writing the vectors as functions of the position of material particles before deformation x_i

$$y_i = x_i + u_i(x_j, t) \qquad \left.\frac{\partial y_i}{\partial t}\right|_{x_i = \text{const}} = \frac{\partial u_i}{\partial t} = v_i(x_j, t)$$

This is called the *lagrangean* description of motion. It is usually the easiest way to visualize a deformation.

But in some applications (eg fluid flow problems, where it's hard to identify a reference configuration) it is preferable to write the displacement, velocity and acceleration vectors as functions of the *deformed* position of particles.

$$y_i = x_i + u_i(y_j, t) \qquad \left.\frac{\partial y_i}{\partial t}\right|_{x_i = \text{const}} = v_i(y_j, t) \qquad \left.\frac{\partial^2 y_i}{\partial t^2}\right|_{x_i = \text{const}} = a_i(y_j, t)$$

These express displacement, velocity and displacement as functions of a particular point in space (visualize describing air flow, for example). This is called the *Eulerian* description of motion. Of course the functions of x_i and y_i are not the same – we just run out of symbols if we introduce different variables in the Lagrangian and Eulerian descriptions.

The relationships between displacement, velocity, and acceleration are somewhat more complicated in the Eulerian description. In the laws of motion, we normally are interested in the velocity and acceleration of a particular material particle, rather the rate of change of displacement and velocity at a particular point in space. When computing the time derivatives, it is necessary to take into account that y_i is a function of time. Thus, displacement, velocity and acceleration are related by

$$\left(\delta_{ij} - \frac{\partial u_i}{\partial y_k}\right)\frac{\partial y_k}{\partial t}\bigg|_{x_i=\text{const}} = \frac{\partial u_i}{\partial t}\bigg|_{y_i=\text{const}} \qquad \frac{\partial^2 y_i}{\partial t^2}\bigg|_{x_i=\text{const}} = a_i(y_j,t) = \frac{\partial v_i}{\partial t}\bigg|_{y_i=\text{const}} + v_k(y_j,t)\frac{\partial v_i}{\partial y_k}$$

You can derive these results by a simple application of the chain rule.

3.5 The Displacement gradient and Deformation gradient tensors

These quantities are defined by

- **Displacement Gradient Tensor:** $\nabla \mathbf{u}$ is a tensor with components $\frac{\partial u_i}{\partial x_k}$
- **Deformation Gradient Tensor:** $\mathbf{F} = \mathbf{I} + \nabla\mathbf{u}$ or in Cartesian components $F_{ik} = \delta_{ik} + \frac{\partial u_i}{\partial x_k}$

where $\mathbf{I}$ is the identity tensor, with components described by the Kronekor delta symbol:

$$\delta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

and ∇ represents the gradient operator. Formally, the gradient of a vector field $\mathbf{u}(\mathbf{x})$ is defined so that

$$[\nabla \mathbf{u}]\cdot\mathbf{n} = \lim_{\alpha\to 0}\frac{\mathbf{u}(\mathbf{x}+\alpha\mathbf{n}) - \mathbf{u}(\mathbf{x})}{\alpha}$$

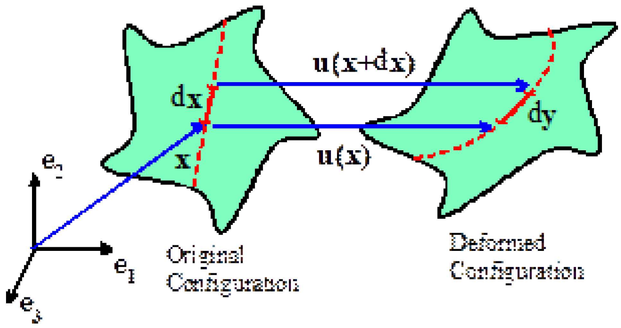
but in practice the component formula $\partial u_i/\partial x_j$ is more useful.

Note also that

$$\nabla\mathbf{y} = \nabla\left(\mathbf{x} + \mathbf{u}(\mathbf{x})\right) = \mathbf{I} + \mathbf{F}$$

$$\text{or} \quad \frac{\partial y_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(x_i + u_i\right) = \delta_{ij} + \frac{\partial u_i}{\partial x_j} = F_{ij}$$

The concepts of displacement gradient and deformation gradient are introduced to quantify the change in shape of infinitesimal line elements in a solid body. To see this, imagine drawing a straight line on the undeformed configuration of a solid, as shown in the figure. The line would be mapped to a smooth curve on the deformed configuration. However, suppose we focus attention on a line segment $\mathbf{dx}$, much shorter than the radius of curvature of this curve, as shown. The segment would be straight in the undeformed configuration, and would also be (almost) straight in the deformed configuration.



Thus, no matter how complex a deformation we impose on a solid, infinitesimal line segments are merely stretched and rotated by a deformation.The infinitesimal line segments $\mathbf{dx}$ and $\mathbf{dy}$ are related by

$$d\mathbf{y} = \mathbf{F}\cdot d\mathbf{x} \quad \text{or} \quad dy_i = F_{ik}dx_k$$

Written out as a matrix equation, we have

$$\begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \end{bmatrix} = \begin{bmatrix} 1+\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1+\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1+\frac{\partial u_3}{\partial x_3} \end{bmatrix}\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

To derive this result, consider an infinitesimal line element $\mathbf{dx}$ in a deforming solid. When the solid is deformed, this line element is stretched and rotated to a deformed line element $\mathbf{dy}$. If we know the displacement field in the solid, we can compute **dy=[x+dx+u(x+dx)]-[x+u(x)]** from the position vectors of its two end points

$$dy_i = x_i + dx_i + u_i(x_k + dx_k) - (x_i + u_i(x_k))$$

Expand $u_i(x_k + dx_k)$ as a Taylor series

$$u_i(x_k + dx_k) \approx u_i(x_k) + \frac{\partial u_i}{\partial x_k}dx_k$$

so that

$$dy_i = dx_i + \frac{\partial u_i}{\partial x_k}dx_k = \left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)dx_k$$

We identify the term in parentheses as the deformation gradient, so

$$dy_i = F_{ik}dx_k$$

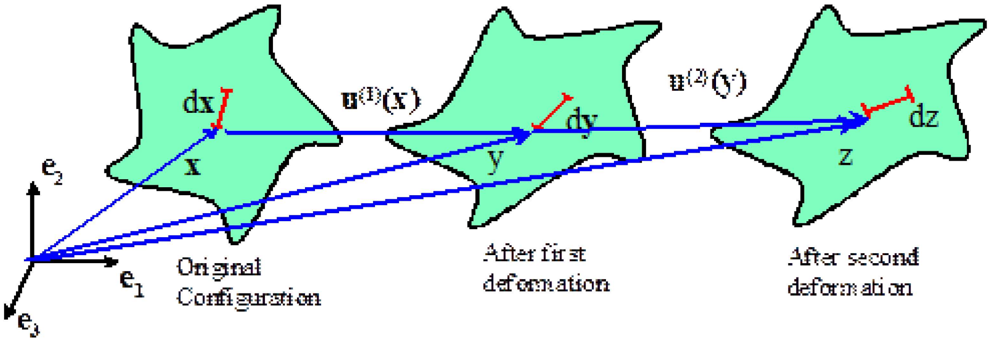
The **inverse** of the deformation gradient $\mathbf{F}^{-1}$ arises in many calculations. It is defined through

$$dx_i = F_{ik}^{-1} dy_k$$

or alternatively

$$F_{ij}^{-1} = \frac{\partial x_i}{\partial y_j}$$

3.6 Deformation gradient resulting from two successive deformations



Suppose that two successive deformations are applied to a solid, as shown. Let

$$d\mathbf{y} = \mathbf{F}^{(1)} \cdot d\mathbf{x} \quad d\mathbf{z} = \mathbf{F}^{(2)} \cdot d\mathbf{y} \quad \text{or} \quad dy_i = F_{ij}^{(1)} dx_j \quad dz_i = F_{ij}^{(2)} dy_j$$

map infinitesimal line elements from the original configuration to the first deformed shape, and from the first deformed shape to the second, respectively, with

$$\mathbf{F}^{(1)} = \mathbf{y} \otimes \nabla_{\mathbf{x}} \quad \mathbf{F}^{(2)} = \mathbf{z} \otimes \nabla_{\mathbf{y}} \quad \text{or} \quad F_{ij}^{(1)} = \frac{\partial y_i}{\partial x_j} \quad F_{ij}^{(2)} = \frac{\partial z_i}{\partial y_j}$$

The deformation gradient that maps infinitesimal line elements from the original configuration directly to the second deformed shape then follows as

$$d\mathbf{z} = \mathbf{F} \cdot d\mathbf{x} \quad \text{with } \mathbf{F} = \mathbf{F}^{(2)} \cdot \mathbf{F}^{(1)} \quad \text{or} \quad dz_i = F_{ij} dx_j \quad F_{ij} = F_{ik}^{(2)} F_{kj}^{(1)}$$

Thus, the cumulative deformation gradient due to two successive deformations follows by multiplying their individual deformation gradients.

To see this, write the cumulative mapping as $z_i\left(y_j(x_k)\right)$ and apply the chain rule

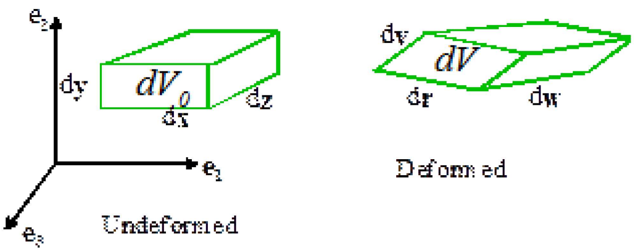
$$dz_i = \frac{\partial z_i}{\partial y_j} \frac{\partial y_j}{\partial x_k} dx_k$$

3.7 The Jacobian of the deformation gradient – change of volume

The Jacobian is defined as

$$J = \det(\mathbf{F}) = \det\left(\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right)$$

It is a measure of the volume change produced by a deformation. To see this, consider the infinitessimal volume element shown with sides $d\mathbf{x}$, $d\mathbf{y}$, and $d\mathbf{z}$ in the figure above. The original volume of the element is



$$dV_0 = d\mathbf{z} \cdot (d\mathbf{x} \times d\mathbf{y}) = \in_{ijk} dz_i dx_j dy_k$$

Here, $\in_{ijk}$ is the permutation symbol. The element is mapped to a paralellepiped with sides $d\mathbf{r}$, $d\mathbf{v}$, and $d\mathbf{w}$ with volume given by

$$dV = \in_{ijk} dw_i dr_j dv_k$$

Recall that

$$dr_i = F_{il} dx_l, \quad dv_j = F_{jm} dy_m, \quad dw_k = F_{kn} dz_n$$

so that

$$dV = \in_{ijk} F_{il} dx_l F_{jm} dy_m F_{kn} dz_n = \in_{ijk} F_{il} F_{jm} F_{kn} dx_l dy_m dz_n$$

Recall that

$$\in_{ijk} A_{il} A_{jm} A_{kn} = \in_{lmn} \det(\mathbf{A})$$

so that

$$dV = \det(\mathbf{F}) \in_{lmn} dx_l dy_m dz_n = \det(\mathbf{F}) dV_0$$

Hence

$$\frac{dV}{dV_0} = \det(\mathbf{F}) = J$$

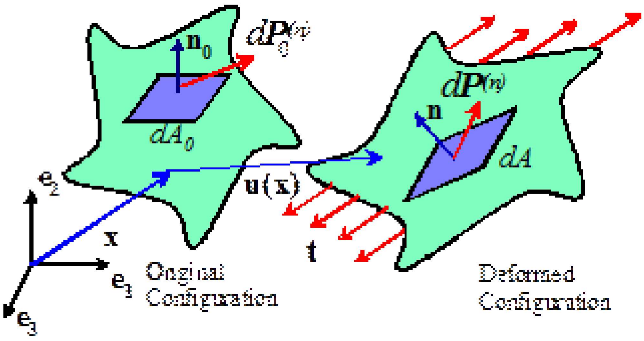
Observe that

- For any physically admissible deformation, the volume of the deformed element must be positive (no matter how much you deform a solid, you can't make material disappear). Therefore, all physically admissible displacement fields must satisfy *J>0*
- If a material is *incompressible*, its volume remains constant. This requires *J=1*.
- If the *mass density* of the material at a point in the undeformed solid is ρ_0, its mass density in the deformed solid is $\rho = \rho_0 / J$

Derivatives of J. When working with constitutive equations, it is occasionally necessary to evaluate derivatives of *J* with respect to the components of **F**. The following result (which can be proved e.g. by expanding the Jacobian using index notation – see HW1, problem 7, eg) is extremely useful

$$\frac{\partial J}{\partial F_{ij}} = J F^{-1}_{ji}$$

3.8 Transformation of internal surface area elements



When we deal with internal forces in a solid, we need to work with forces acting on internal surfaces in a solid. An important question arises in this treatment: if we identify an element of area dA_0 with normal $\mathbf{n}_0$ in the reference configuration, and then what are the area of dA and normal $\mathbf{n}$ of this area element in the deformed solid?

The two are related through

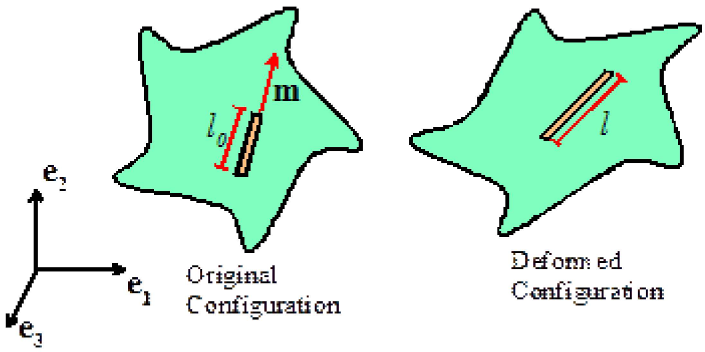
$$dA\mathbf{n} = J\mathbf{F}^{-T} \cdot dA_0 \mathbf{n}_0 \qquad dAn_i = JF^{-1}_{ki} n^0_k dA_0$$

To see this,

1. let dv^0_i, dw^0_j be two infinitesimal material fibers with different orientations at some point in the reference configuration. These fibers bound a parallelapiped with area and normal
$$dA_0 n^0_i = \in_{ijk} dw^0_j dv^0_k$$
2. The vectors map to $dv_j = F_{jq} dv^0_q$, $dw_k = F_{kn} dw^0_n$ in the deformed solid
3. In the deformed solid the area element is thus
$$dAn_i = \in_{ijk} F_{jq} dv^0_q F_{kn} dw^0_n = F_{ml} F^{-1}_{li} \in_{mjk} F_{jq} F_{kn} dv^0_q dw^0_n$$
4. Recall the identity $\in_{lmn} \det(\mathbf{A}) = \in_{ijk} A_{il} A_{jm} A_{kn}$ - so $dAn_i = F^{-1}_{li} \in_{lqn} J dv^0_q dw^0_n = JF^{-1}_{ki} n^0_k dA_0$

3.8 The Lagrange strain tensor

The Lagrange strain tensor is defined as



$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) \quad \text{or} \quad E_{ij} = \frac{1}{2}(F_{ki} F_{kj} - \delta_{ij})$$

The components of Lagrange strain can also be expressed in terms of the displacement gradient as

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_j} \frac{\partial u_k}{\partial x_i} \right)$$

The Lagrange strain tensor quantifies the changes in length of a material fiber, and angles between pairs of fibers in a deformable solid. It is used in calculations where large shape changes are expected.

To visualize the physical significance of **E**, suppose we mark out an imaginary tensile specimen with (very short) length l_0 on our deforming solid, as shown in the picture. The orientation of the specimen is arbitrary, and is specified by a unit vector $\mathbf{m}$, with components m_i. Upon deformation, the specimen increases in length to $l = l_0 + \delta l$. Define the strain of the specimen as

$$\varepsilon_L(m_i) = \frac{l^2 - l_0{}^2}{2l_0{}^2} = \frac{\delta l}{l_0} + \frac{(\delta l)^2}{2l_0^2}$$

Note that this definition of strain is similar to the definition $\varepsilon = \delta l / l_0$ you are familiar with, but contains an additional term. The additional term is negligible for small δl. Given the Lagrange strain components E_{ij}, the strain of the specimen may be computed from

$$\varepsilon_L(\mathbf{m}) = \mathbf{m} \cdot \mathbf{E} \cdot \mathbf{m} \quad \text{or} \quad \varepsilon_L(m_i) = E_{ij} m_i m_j$$

We proceed to derive this result. Note that

$$dx_i = l_0 m_i$$

is an infinitesimal vector with length and orientation of our undeformed specimen. From the preceding section, this vector is stretched and rotated to

$$dy_k = \left(\delta_{kj} + \frac{\partial u_k}{\partial x_j}\right) dx_j = \left(\delta_{kj} + \frac{\partial u_k}{\partial x_j}\right) l_0 m_j$$

The length of the deformed specimen is equal to the length of $\mathbf{dy}$, so we see that

$$\begin{aligned} l^2 = dy_k dy_k &= \left(\delta_{kj} + \frac{\partial u_k}{\partial x_j}\right) l_0 m_j \left(\delta_{ki} + \frac{\partial u_k}{\partial x_i}\right) l_0 m_i \\ &= \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) l_0{}^2 m_j m_i \end{aligned}$$

Hence, the strain for our line element is

$$\varepsilon_L(m_i) = \frac{l^2 - l_0{}^2}{2l_0{}^2} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial x_i}\right) m_i m_j$$

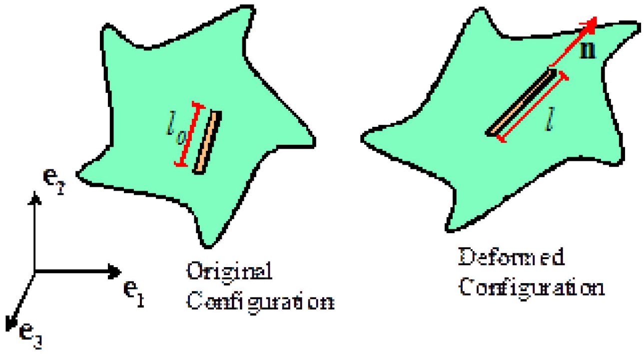
giving the results stated.

3.9 The Eulerian strain tensor

The Eulerian strain tensor is defined as

$$\mathbf{E}^* = \tfrac{1}{2}(\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}) \quad \text{or} \quad E^*_{ij} = \tfrac{1}{2}(\delta_{ij} - F^{-1}_{ki} F^{-1}_{kj})$$

Its physical significance is similar to the Lagrange strain tensor, except that it enables you to compute the strain of an infinitesimal line element from its orientation *after* deformation.



Specifically, suppose that $\mathbf{n}$ denotes a unit vector parallel to the deformed material fiber, as shown in the picture. Then

$$\varepsilon_E(\mathbf{n}) = \frac{l^2 - l_0{}^2}{2l^2} = \mathbf{n} \cdot \mathbf{E}^* \cdot \mathbf{n} \quad \text{or} \quad \varepsilon_E(n_i) = E^*_{ij} n_i n_j$$

The proof is left as an exercise.

3.10 The Infinitesimal strain tensor

The infinitesimal strain tensor is defined as

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left(\mathbf{u}\nabla + (\mathbf{u}\nabla)^T\right) \quad \text{or} \quad \varepsilon_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

where $\mathbf{u}$ is the displacement vector. Written out in full

$$\varepsilon_{ij} \equiv \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_1} + \dfrac{\partial u_1}{\partial x_2}\right) & \dfrac{\partial u_2}{\partial x_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_1} + \dfrac{\partial u_1}{\partial x_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_2} + \dfrac{\partial u_2}{\partial x_3}\right) & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix}$$

The infinitesimal strain tensor is an *approximate* deformation measure, which is only valid for small shape changes. It is more convenient than the Lagrange or Eulerian strain, because it is linear.

Specifically, suppose the deformation gradients are small, so that all $\partial u_i / \partial x_j << 1$. Then the Lagrange strain tensor is

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial x_i}\right) \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \approx \varepsilon_{ij}$$

so the infinitesimal strain approximates the Lagrange strain. You can show that it also approximates the Eulerian strain with the same accuracy.

Properties of the infinitesimal strain tensor

- For small strains, the engineering strain of an infinitesimal fiber aligned with a unit vector **m** can be estimated as

$$\varepsilon_e(\mathbf{m}) = \frac{l-l_0}{l_0} \approx \varepsilon_{ij} m_i m_j$$

- Note that

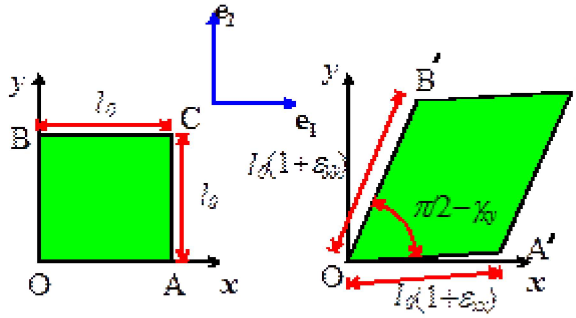
$$\text{trace}\,(\boldsymbol{\varepsilon}) \equiv \varepsilon_{kk} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{dV - dV_0}{dV_0}$$

(see below for more details)

- The infinitesimal strain tensor is closely related to the strain matrix introduced in elementary strength of materials courses. For example, the physical significance of the (2 dimensional) strain matrix

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} \\ \gamma_{yx} & \varepsilon_{yy} \end{bmatrix}$$

is illustrated in the figure.



To relate this to the infintesimal strain tensor, let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis, with $\mathbf{e}_1$ parallel to x and $\mathbf{e}_2$ parallel to y as shown. Let ε_{ij} denote the components of the infinitesimal strain tensor in this basis. Then

$$\varepsilon_{11} = \varepsilon_{xx}$$
$$\varepsilon_{22} = \varepsilon_{yy}$$
$$\varepsilon_{12} = \varepsilon_{21} = \gamma_{xy}/2 = \gamma_{yx}/2$$

3.11 Engineering shear strains

For a general strain tensor (which could be any of $\mathbf{E}$, $\mathbf{E}^*$ or $\boldsymbol{\varepsilon}$, among others), the diagonal strain components $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ are known as `direct' strains, while the off diagonal terms $\varepsilon_{12} = \varepsilon_{21}$ $\varepsilon_{13} = \varepsilon_{31}$ $\varepsilon_{23} = \varepsilon_{32}$ are known as 'shear strains'

The shear strains are sometimes reported as 'Engineering Shear Strains' which are related to the formal definition by a factor of 2 i.e.

$$\gamma_{12} = 2\varepsilon_{12} \qquad \gamma_{13} = 2\varepsilon_{13} \qquad \gamma_{23} = 2\varepsilon_{23}$$

This factor of 2 is an endless source of confusion. Whenever someone reports shear strain to you, be sure to check which definition they are using. In particular, many commercial finite element codes output engineering shear strains.

3.12 Decomposition of infinitesimal strain into volumetric and deviatoric parts

The **volumetric infinitesimal strain** is defined as trace$(\boldsymbol{\varepsilon})$ $\equiv \varepsilon_{kk}$
The **deviatoric infinitesimal strain** is defined as $\mathbf{e} = \boldsymbol{\varepsilon} - \frac{1}{3}\mathbf{I}\,\text{trace}(\boldsymbol{\varepsilon}) \quad \equiv \quad e_{ij} = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk}$

The volumetric strain is a measure of volume changes, and for small strains is related to the Jacobian of the deformation gradient by $\varepsilon_{kk} \approx J - 1$. To see this, recall that

$$J = \det \begin{bmatrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1 + \frac{\partial u_3}{\partial x_3} \end{bmatrix} \approx \left(1 + \frac{\partial u_1}{\partial x_1}\right)\left(1 + \frac{\partial u_2}{\partial x_2}\right)\left(1 + \frac{\partial u_3}{\partial x_3}\right) \approx 1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

The deviatioric strain is a measure of shear deformation (shear deformation involves no volume change).

3.13 The Infinitesimal rotation tensor

The infinitesimal rotation tensor is defined as

$$\mathbf{w} = \frac{1}{2}\left(\mathbf{u}\nabla - (\mathbf{u}\nabla)^T\right) \quad \text{or} \quad w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

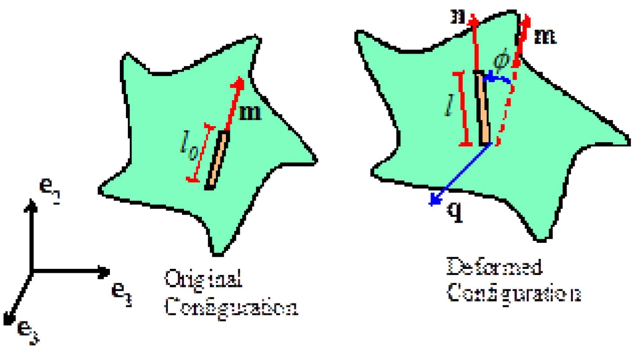
Written out as a matrix, the components of w_{ij} are

$$w_{ij} \equiv \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{bmatrix}$$

Observe that w_{ij} is *skew symmetric*: $w_{ij} = -w_{ji}$.

A skew tensor represents a rotation through a small angle. Specifically, the operation $dy_i = (\delta_{ij} + w_{ij})\, dx_j$ rotates the infinitesimal line element dx_j through a small angle $\theta = \sqrt{w_{ij}w_{ij}/2}$ about an axis parallel to the unit vector $n_i = \in_{ijk} w_{kj}/(2\theta)$. (A skew tensor also sometimes represents an angular velocity).

To visualize the significance of w_{ij}, consider the behavior of an imaginary, infinitesimal, tensile specimen embedded in a deforming solid. The specimen is stretched, and then rotated through an angle ϕ about some axis $\mathbf{q}$. If the displacement gradients are small, then $\phi << 1$.



The rotation of the specimen depends on its original orientation, represented by the unit vector $\mathbf{m}$. One can show (although one would rather not do all the algebra) that w_{ij} represents the *average* rotation, over all possible orientations of $\mathbf{m}$, of material fibers passing through a point.

As a final remark, we note that a general deformation can always be decomposed into an infinitesimal strain and rotation

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ij} + w_{ij}$$

Physically, this sum of ε_{ij} and w_{ij} can be regarded as representing two successive deformations – a small strain, followed by a rotation, in the sense that

$$dy_i = (\delta_{ik} + w_{ik})(\delta_{kj} + \varepsilon_{kj})\, dx_j \approx dx_i + (\varepsilon_{ij} + w_{ij})\, dx_j$$

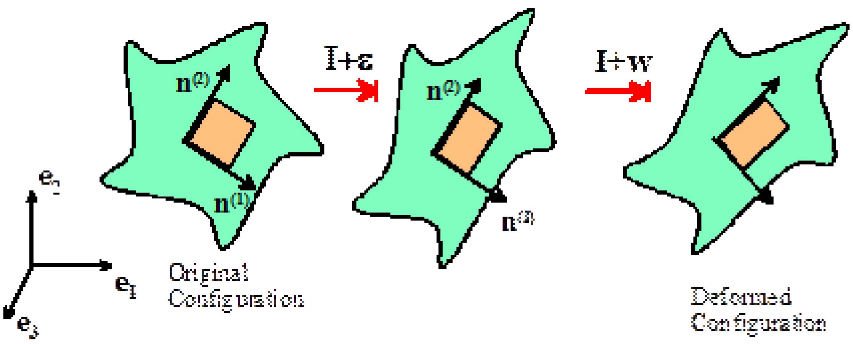
first stretches the infinitesimal line element, then rotates it.

3.14 Principal values and directions of the infinitesimal strain tensor

The three principal values e_i and directions $\mathbf{n}^{(i)}$ of the infinitesimal strain tensor satisfy

$$\boldsymbol{\varepsilon} \cdot \mathbf{n}^{(i)} = e_i \mathbf{n}^{(i)}$$

$$\text{or} \quad \varepsilon_{kl} n_l^{(i)} = e_i n_l^{(i)}$$



Clearly, e_i and $\mathbf{n}^{(i)}$ are the eigenvalues and eigenvectors of $\boldsymbol{\varepsilon}$. There are three principal strains and three principal directions, which are always mutually perpendicular.

Their significance can be visualized as follows.

1. Note that the decomposition $\frac{\partial u_i}{\partial x_j} = \varepsilon_{ij} + w_{ij}$ can be visualized as a small strain, followed by a small rigid rotation, as shown in the picture.
2. The formula $\boldsymbol{\varepsilon} \cdot \mathbf{n}^{(i)} = e_i \mathbf{n}^{(i)}$ indicates that a vector $\mathbf{n}$ is mapped to another, parallel vector by the strain.
3. Thus, if you draw a small cube with its faces perpendicular to $\mathbf{n}^{(i)}$ on the undeformed solid, this cube will be stretched perpendicular to each face, with a fractional increase in length $e_i = \delta l_i / l_0$. The faces remain perpendicular to $\mathbf{n}^{(i)}$ after deformation.
4. Finally, $\mathbf{w}$ rotates the small cube through a small angle onto its configuration in the deformed solid.

3.15 Strain Equations of Compatibility for infinitesimal strains

It is sometimes necessary to *invert* the relations between strain and displacement – that is to say, given the strain field, to compute the displacements. In this section, we outline how this is done, for the special case of *infinitesimal deformations*.

For infinitesimal motions the relation between strain and displacement is

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Given the six strain components ε_{ij} (six, since $\varepsilon_{ij} = \varepsilon_{ji}$) we wish to determine the three displacement components u_i. First, note that you can never completely recover the displacement field that gives rise to a particular strain field. Any rigid motion produces no strain, so the displacements can only be completely determined if there is some additional information (besides the strain) that will tell you how much the solid has rotated and translated. However, integrating the strain field can tell you the displacement field to within an arbitrary rigid motion.

Second, we need to be sure that the strain-displacement relations can be integrated at all. The strain is a symmetric second order tensor field, but not all symmetric second order tensor fields can be strain fields. The strain-displacement relations amount to a system of six scalar differential equations for the three displacement components u_i.

To be integrable, the strains must satisfy the **compatibility conditions**, which may be expressed as

$$\in_{ipm} \in_{jqn} \frac{\partial^2 \varepsilon_{mn}}{\partial x_p \partial x_q} = 0$$

Or, equivalently

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_l} = 0$$

Or, once more equivalently

$$\begin{aligned}
&\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0 \qquad &&\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} - \frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) = 0 \\
&\frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = 0 \qquad &&\frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} - \frac{\partial}{\partial x_2}\left(-\frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1}\right) = 0 \\
&\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} - 2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0 \qquad &&\frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} - \frac{\partial}{\partial x_3}\left(-\frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2}\right) = 0
\end{aligned}$$

It is easy to show that all strain fields must satisfy these conditions - you simply need to substitute for the strains in terms of displacements and show that the appropriate equation is satisfied. For example,

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^4 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^4 u_2}{\partial x_2 \partial x_1^2} - 2\frac{\partial^2}{\partial x_1 \partial x_2}\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$

and similarly for the other expressions.

Not that for planar problems for which $\varepsilon_{13} = \varepsilon_{23} = 0$ and $\frac{d\varepsilon_{ij}}{dx_3} = 0$, all of these compatibility equations are satisfied trivially, with the exception of the first: $\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0$

It can be shown that

(i) If the strains do not satisfy the equations of compatibility, then a displacement vector can not be integrated from the strains.

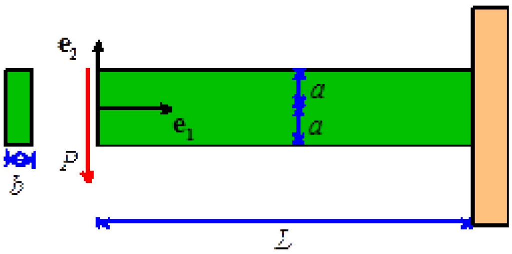
(ii) If the strains satisfy the compatibility equations, and the solid *simply connected* (i.e. it contains no holes that go all the way through its thickness), then a displacement vector can be integrated from the strains.

(iii) If the solid is not simply connected, a displacement vector can be calculated, but it may not be *single valued* – i.e. you may get different solutions depending on how the path of integration encircles the holes.

Now, let us return to the question posed at the beginning of this section. Given the strains, how do we compute the displacements?

2D strain fields

For 2D (plane stress or plane strain) the procedure is quite simple and is best illustrated by working through a specific case



As a representative example, we will use the strain field in a 2D (plane stress) cantilever beam with Young's modulus E and Poisson's ratio ν loaded at one end by a force P. The beam has a rectangular cross-section with height $2a$ and out-of-plane width b. We will show later (Sect 5.2.4) that the strain field in the beam is

$$\varepsilon_{11} = 2Cx_1x_2 \quad \varepsilon_{22} = -2\nu Cx_1x_2 \quad \varepsilon_{12} = (1+\nu)\,C\left(a^2 - x_2^2\right), \quad C = \frac{3P}{4Ea^3b}$$

We first check that the strain is compatible. For 2D problems this requires

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0$$

which is clearly satisfied in this case.

For a 2D problem we only need to determine $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ such that

$$\frac{\partial u_1}{\partial x_1} = \varepsilon_{11}, \frac{\partial u_2}{\partial x_2} = \varepsilon_{22} \text{ and } \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 2\varepsilon_{12} \; .$$

The first two of these give

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 2Cx_1x_2 \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = -2\nu C x_1 x_2$$

We can integrate the first equation with respect to x_1 and the second equation with respect to x_2 to get

$$u_1 = Cx_1^2 x_2 + f_1(x_2) \qquad u_2 = -\nu C x_1 x_2^2 + f_2(x_1)$$

where $f_1(x_2)$ and $f_2(x_1)$ are two functions of x_2 and x_1, respectively, which are yet to be determined. We can find these functions by substituting the formulas for $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ into the expression for shear strain

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = (1+\nu)\, C\left(a^2 - x_2^2\right)$$

$$\frac{1}{2}\left(Cx_1^2 - \nu C x_2^2 + \frac{df_1}{dx_2} + \frac{df_2}{dx_1}\right) = (1+\nu)\, C\left(a^2 - x_2^2\right)$$

We can re-write this as

$$\left(\frac{df_2}{dx_1} + Cx_1^2\right) + \left(\frac{df_1}{dx_2} - \nu C x_2^2 - 2\,(1+\nu)\, C\left(a^2 - x_2^2\right)\right) = 0$$

The two terms in parentheses are functions of x_1 and x_2, respectively. Since the left hand side must vanish for all values of x_1 and x_2, this means that

$$\left(\frac{df_2}{dx_1} + Cx_1^2\right) = \omega$$

$$\left(\frac{df_1}{dx_2} - \nu C x_2^2 - 2\,(1+\nu)\, C\left(a^2 - x_2^2\right)\right) = -\omega$$

where ω is an arbitrary constant. We can now integrate these expressions to see that

$$f_1 = \left(2(1+\nu)Ca^2 - \omega\right) x_2 - \frac{C}{3}(2+\nu)x_2^3 + c$$

$$f_2 = \omega x_1 - \frac{C}{3}x_1^3 + d$$

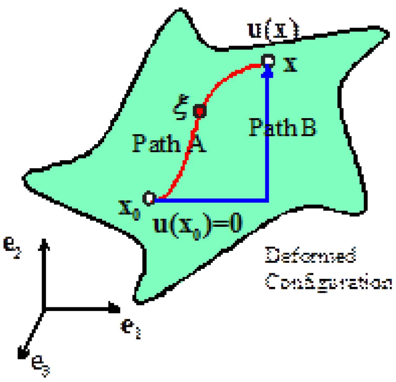
where *c* and *d* are two more arbitrary constants. Finally, the displacement field follows as

$$u_1 = Cx_1^2 x_2 - \frac{C}{3}(2+\nu)x_2^3 + 2(1+\nu)Ca^2 x_2 - \omega x_2 + c$$

$$u_2 = -\nu C x_1 x_2^2 - \frac{C}{3}x_1^3 + \omega x_1 + d$$

The three arbitrary constants ω, *c* and *d* can be seen to represent a small rigid rotation through angle ω about the x_3 axis, together with a displacement (*c*,*d*) parallel to (x_1, x_2) axes, respectively.

3D strain fields



For a general, three dimensional field a more formal procedure is required. Since the strains are the derivatives of the displacement field, so you might guess that we compute the displacements by integrating the strains. This is more or less correct. The general procedure is outlined below.

We first pick a point $\mathbf{x}_0$ in the solid, and arbitrarily say that the displacement at $\mathbf{x}_0$ is zero, and also take the rotation of the solid at $\mathbf{x}_0$ to be zero. Then, we can compute the displacements at any other point $\mathbf{x}$ in the solid, by integrating the strains along any convenient path. In a simply connected solid, it doesn't matter what path you pick.

Actually, you don't exactly integrate the strains – instead, you must evaluate the following integral

$$u_i(\mathbf{x}) = \int_{\mathbf{x}0}^{\mathbf{x}} U_{ij}(\mathbf{x}, \xi)\, d\xi_j$$

where

$$U_{ij}(\mathbf{x}, \xi) = \varepsilon_{ij}(\xi) + (x_k - \xi_k)\left[\frac{\partial \varepsilon_{ij}(\xi)}{\partial \xi_k} - \frac{\partial \varepsilon_{kj}(\xi)}{\partial \xi_i}\right]$$

Here, x_k are the components of the position vector at the point where we are computing the displacements, and ξ_j are the components of the position vector ξ of a point somewhere along the path of integration. The fact that the integral is path-independent (in a simply connected solid) is guaranteed by the compatibility condition. Evaluating this integral in practice can be quite painful, but fortunately almost all cases where we need to integrate strains to get displacement turn out to be two-dimensional.

3.16 Cauchy-Green Deformation Tensors

There are two Cauchy-Green deformation tensors – defined through

- **The Right Cauchy Green Deformation Tensor** $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} \qquad C_{ij} = F_{ki}F_{kj}$
- **The Left Cauchy Green Deformation Tensor** $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T \qquad B_{ij} = F_{ik}F_{jk}$

They are called `left' and `right' tensors because of their relation to the `left' and 'right' stretch tensors defined below. They can be regarded as quantifying the squared length of infinitesimal fibers in the deformed configuration, by noting that if a material fiber $d\mathbf{x} = l_0\mathbf{m}$ in the undeformed solid is stretched and rotated to $d\mathbf{y} = l\mathbf{n}$ in the deformed solid, then

$$\frac{l^2}{l_0^2} = \mathbf{m} \cdot \mathbf{C} \cdot \mathbf{m} \qquad \frac{l_0^2}{l^2} = \mathbf{n} \cdot \mathbf{B}^{-1} \cdot \mathbf{n}$$

3.17 Rotation tensor, and Left and Right Stretch Tensors

The definitions of these quantities are

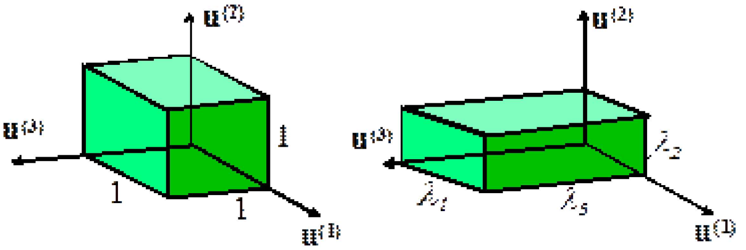
- **The Right Stretch Tensor** $\mathbf{U} = \mathbf{C}^{1/2} = \left(\mathbf{F}^T \cdot \mathbf{F}\right)^{1/2} \qquad U_{ij} = C_{ij}^{1/2}$
- **The Left Stretch Tensor** $\mathbf{V} = \mathbf{B}^{1/2} \qquad V_{ij} = B_{ij}^{1/2}$
- **The Rotation Tensor** $\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1} = \mathbf{V}^{-1} \cdot \mathbf{F} \qquad R_{ij} = F_{ik}U_{kj}^{-1} = V_{ik}^{-1}F_{kj}$

To calculate these quantities you need to remember how to calculate the square root of a matrix. For example, to calculate the square root of $\mathbf{C}$, you must

1. Calculate the eigenvalues of $\mathbf{C}$ – we will call these λ_n^2, with n=1,2,3. Since $\mathbf{C}$ and $\mathbf{B}$ are both symmetric and positive definite, the eigenvalues λ_n^2 are all positive real numbers, and therefore their square roots λ_n are also positive real numbers.
2. Calculate the eigenvectors of $\mathbf{C}$ and *normalize them so they have unit magnitude*. We will denote the eigenvectors by $\mathbf{c}^{(n)}$. They must be normalized to satisfy $\mathbf{c}^{(n)} \cdot \mathbf{c}^{(n)} = 1$
3. Finally, calculate $\mathbf{C}^{1/2} = \sum_{n=1}^{3} \lambda_n \mathbf{c}^{(n)} \otimes \mathbf{c}^{(n)}$, where $\mathbf{c}$ denotes a dyadic product (See Appendix B). In components, this can be written $C_{ij}^{1/2} = \sum_{n=1}^{3} \lambda_n c_i^{(n)} c_j^{(n)}$
4. As an additional bonus, you can quickly compute the inverse square root (which is needed to find **R)** as
$$\mathbf{C}^{-1/2} = \sum_{n=1}^{3} \frac{1}{\lambda_n} \mathbf{c}^{(n)} \otimes \mathbf{c}^{(n)} \quad \text{or} \quad C_{ij}^{-1/2} = \sum_{n=1}^{3} \frac{1}{\lambda_n} c_i^{(n)} c_j^{(n)}$$

To see the physical significance of these tensors, observe that

1. The definition of the rotation tensor shows that
$$\mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1} \Leftrightarrow \mathbf{F} = \mathbf{R} \cdot \mathbf{U}$$
$$\mathbf{R} = \mathbf{V}^{-1} \cdot \mathbf{F} \Leftrightarrow \mathbf{F} = \mathbf{V} \cdot \mathbf{R}$$
2. The multiplicative decomposition of a constant tensor $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ can be regarded as a sequence of two homogeneous deformations – **U**, followed by **R**. Similarly, $\mathbf{F} = \mathbf{V} \cdot \mathbf{R}$ is **R** followed by **V**.
3. **R** is proper orthogonal (it satisfies $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$ and det(**R**)=1), and therefore represents a rotation. To see this, note that **U** is symmetric, and therefore satisfies $\mathbf{U}^{-T} = \mathbf{U}^{-1}$, so that
$$\begin{aligned} \mathbf{R}^T\mathbf{R} &= \left(\mathbf{F} \cdot \mathbf{U}^{-1}\right)^T \cdot \left(\mathbf{F} \cdot \mathbf{U}^{-1}\right) \\ &= \mathbf{U}^{-T} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \mathbf{U}^{-1} \\ &= \mathbf{U}^{-1} \cdot \mathbf{U}^2 \cdot \mathbf{U}^{-1} = \mathbf{I} \end{aligned}$$
and det(**R**)=det(**F**)det($\mathbf{U}^{-1}$)=1
4. **U** can be expressed in the form
$$\mathbf{U} = \lambda_1 \mathbf{u}^{(1)} \otimes \mathbf{u}^{(1)} + \lambda_2 \mathbf{u}^{(2)} \otimes \mathbf{u}^{(2)} + \lambda_3 \mathbf{u}^{(3)} \otimes \mathbf{u}^{(3)}$$
where $\mathbf{u}^{(i)}$ are the three (mutually perpendicular) eigenvectors of **U**. (By construction, these are identical to the eigenvectors of **C**). If we interpret $\mathbf{u}^{(i)}$ as basis vectors, we see that **U** is *diagonal* in this basis, and so corresponds to stretching parallel to each basis vector, as shown in the figure below.



The decompositions

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} \text{ and } \mathbf{F} = \mathbf{V} \cdot \mathbf{R}$$

are known as the *right* and *left* polar decomposition of **F**. (The right and left refer to the positions of **U** and **V**). They show that every homogeneous deformation can be decomposed into a stretch followed by a rigid rotation, or equivalently into a rigid rotation followed by a stretch. The decomposition is discussed in more detail in the next section.

3.18 Principal stretches

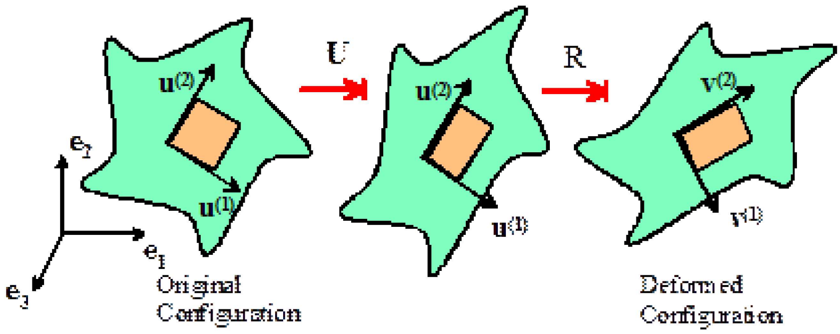
The principal stretches can be calculated from any one of the following (they all give the same answer)

1. The eigenvalues of the right stretch tensor **U**
2. The eigenvalues of the left stretch tensor **V**
3. The square root of the eigenvalues of the right Cauchy-Green tensor **C**
4. The square root of the eigenvalues of the left Cauchy-Green tensor **B**

The principal stretches are also related to the eigenvalues of the Lagrange and Eulerian strains. The details are left as an exercise.

There are two sets of **principal stretch directions**, associated with the undeformed and deformed solids.

1. The principal stretch directions in the **undeformed** solid are the (normalized) eigenvectors of **U** or **C**. Denote these by $\mathbf{u}^{(i)}$.
2. The principal stretch directions in the **deformed** solid are the (normalized) eigenvectors of **V** or **B**. Denote these by $\mathbf{v}^{(i)}$.

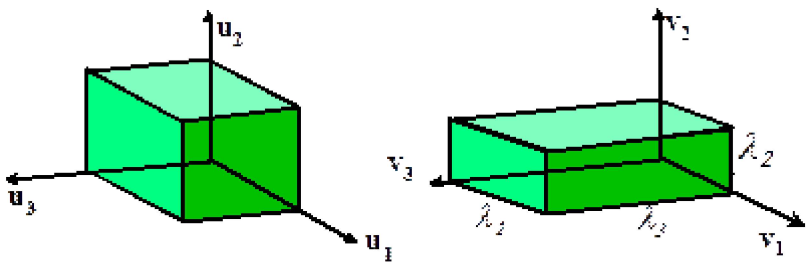


To visualize the physical significance of principal stretches and their directions, note that a deformation can be decomposed as $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ into a sequence of a stretch followed by a rotation.

Note also that

1. The principal directions $\mathbf{u}^{(i)}$ are mutually perpendicular. You could draw a little cube on the undeformed solid with faces perpendicular to these directions, as shown above.
2. The stretch **U** will stretch the cube by an amount λ_i parallel to each $\mathbf{u}^{(i)}$. The faces of the stretched cube remain perpendicular to $\mathbf{u}^{(i)}$.
3. The rotation **R** will rotate the stretched cube so that the directions $\mathbf{u}^{(i)}$ rotate to line up with $\mathbf{v}^{(i)}$.
4. The faces of the deformed cube are perpendicular to $\mathbf{v}^{(i)}$

The decomposition $\mathbf{F} = \mathbf{V} \cdot \mathbf{R}$ can be visualized in much the same way. In this case, the directions $\mathbf{u}^{(i)}$ are first rotated to coincide with $\mathbf{v}^{(i)}$. The cube is then stretched parallel to each $\mathbf{v}^{(i)}$ to produce the same shape change.



We could compare the undeformed and deformed cubes by placing them side by side, with the vectors $\mathbf{v}^{(i)}$ and $\mathbf{u}^{(i)}$ parallel, as shown in the figure.

3.19 Generalized strain measures

The polar decompositions $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ and $\mathbf{F} = \mathbf{V} \cdot \mathbf{R}$ provide a way to define additional strain measures. Let λ_i denote the principal stretches, and let $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$ denote the normalized eigenvectors of **U** and **V**. Then one could define strain tensors through

$$\text{Lagrangian Nominal strain:} \qquad \sum_{\text{i}=1}^{3} (\lambda_i - 1)\mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}$$

$$\text{Lagrangian Logarithmic strain:} \qquad \sum_{\text{i}=1}^{3} \log(\lambda_i)\mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}$$

The correspoinding Eulerian strain measures are

$$\text{Eulerian Nominal strain:} \qquad \sum_{i=1}^{3} (\lambda_i - 1)\mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}$$

$$\text{Eulerian Logarithmic strain:} \qquad \sum_{i=1}^{3} \log(\lambda_i)\mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}$$

Another strain measure can be defined as

$$\text{Green's strain:} \qquad \mathbf{E}_G = \sum_{i=1}^{3} \frac{1}{2}(\lambda_i^2 - 1)\mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}$$

This can be computed directly from the deformation gradient as

$$\mathbf{E}_G = \tfrac{1}{2}\left(\mathbf{F} \cdot \mathbf{F}^T - \mathbf{I}\right)$$

and is very similar to the Lagrangean strain tensor, except that its principal directions are rotated through the rigid rotation **R**.

3.20 Measure of rate of deformation - the velocity gradient

We now list several measures of the *rate* of deformation. The velocity gradient is the basic measure of deformation rate, and is defined as

$$\mathbf{L} = \nabla_{\mathbf{y}} \mathbf{v} \equiv L_{ij} = \frac{\partial v_i}{\partial y_j}$$

It quantifies the relative velocities of two material particles at positions **y** and **y**+*d***y** in the deformed solid, in the sense that

$$dv_i = v_i(\mathbf{y} + d\mathbf{y}) - v_i(\mathbf{y}) = \frac{\partial v_i}{\partial y_j} dy_j$$

The velocity gradient can be expressed in terms of the deformation gradient and its time derivative as

$$\nabla_{\mathbf{y}} \mathbf{v} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \qquad \frac{\partial v_i}{\partial y_j} = \dot{F}_{ik} F_{kj}^{-1}$$

To see this, note that

$$dv_i = \tfrac{d}{dt} dy_i = \tfrac{d}{dt}(F_{ij} dx_j) = \dot{F}_{ij} dx_j$$

and recall that $dy_j = F_{ji} dx_i \Rightarrow dx_j = F_{jk}^{-1} dy_k$, so that

$$dv_i = \dot{F}_{ij} F_{jk}^{-1} dy_k$$

3.21 Stretch rate and spin (vorticity) tensors

The **stretch rate tensor** is defined as $\mathbf{D} = \left(\mathbf{L} + \mathbf{L}^T\right)/2 \qquad D_{ij} = \left(L_{ij} + L_{ji}\right)/2$

The **spin tensor** or **Vorticity tensor** is defined as $\mathbf{W} = \left(\mathbf{L} - \mathbf{L}^T\right)/2 \qquad W_{ij} = \left(L_{ij} - L_{ji}\right)/2$

A general velocity gradient can be decomposed into the sum of stretch rate and spin, as

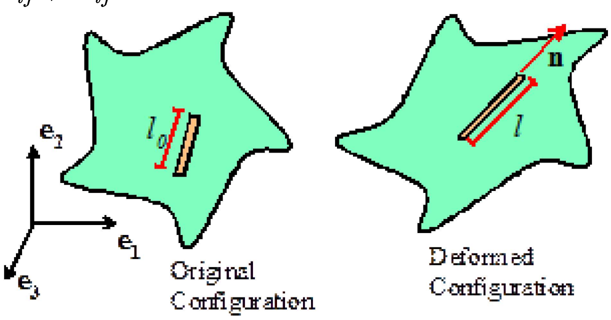
$$\mathbf{L} = \mathbf{D} + \mathbf{W} \qquad L_{ij} = D_{ij} + W_{ij}$$

The stretch rate quantifies the rate of stretching of material fibers in the deformed solid, in the sense that

$$\tfrac{1}{l}\tfrac{dl}{dt} = \mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n} = n_i D_{ij} n_j$$

is the rate of stretching of a material fiber with length l and orientation **n** in the deformed solid. To see this, let $d\mathbf{y} = l\mathbf{n}$, so that

$$\tfrac{d}{dt} d\mathbf{y} = \tfrac{dl}{dt}\mathbf{n} + l\tfrac{d\mathbf{n}}{dt}$$



By definition,

$$\tfrac{d}{dt} d\mathbf{y} = \tfrac{d}{dt}(\mathbf{F} \cdot d\mathbf{x}) = \dot{\mathbf{F}} \cdot d\mathbf{x} = \dot{\mathbf{F}} \cdot \left(\mathbf{F}^{-1} d\mathbf{y}\right) = \dot{\mathbf{F}}\mathbf{F}^{-1} \cdot d\mathbf{y} = \mathbf{L} \cdot d\mathbf{y} = (\mathbf{D} + \mathbf{W}) \cdot l\mathbf{n}$$

Hence

$$(\mathbf{D} + \mathbf{W}) \cdot l\mathbf{n} = \tfrac{dl}{dt}\mathbf{n} + l\tfrac{d\mathbf{n}}{dt}$$

Finally, take the dot product of both sides with **n**, note that since **n** is a unit vector $d\mathbf{n}/dt$ must be perpendicular to **n** and therefore $\mathbf{n} \cdot d\mathbf{n}/dt = 0$. Note also that $\mathbf{n} \cdot \mathbf{W} \cdot \mathbf{n} = 0$, since **W** is skew-symmetric. It is easiest to show this using index notation: $n_i W_{ij} n_j = n_i(L_{ij} - L_{ji})n_j/2 = 0$. Therefore

$$\mathbf{n} \cdot (\mathbf{D} + \mathbf{W}) \cdot l\mathbf{n} = \tfrac{dl}{dt}\mathbf{n} \cdot \mathbf{n} + l\mathbf{n} \cdot \tfrac{d\mathbf{n}}{dt} \Rightarrow \mathbf{n} \cdot \mathbf{D} \cdot l\mathbf{n} = \tfrac{dl}{dt}$$

The spin tensor **W** can be shown to provide a measure of the average angular velocity of all material fibers passing through a material point.

The **vorticity vector** is another measure of the angular velocity. It is defined as

$$\mathbf{w} = curl(\mathbf{v}) \qquad w_i = \in_{ijk} \frac{\partial v_k}{\partial y_j}$$

It is related to the spin tensor as

$$\boldsymbol{\omega} = 2dual(\mathbf{W}) \qquad \omega_i = - \in_{ijk} W_{jk}$$

Where dual **(W)** denotes the dual vector of the skew tensor **W**.

The vorticity vector has the property that, for any vector **g**, $\mathbf{Wg} = \frac{1}{2}\boldsymbol{\omega} \times \mathbf{g} \qquad W_{ji} g_i = \frac{1}{2} \in_{jki} \omega_k g_i$.

A motion satisfying **W**=curl(v)= **0** is said to be *irrotational* – such motions are of interest in fluid mechanics.

3.22 Spatial (Eulerian) description of acceleration

The acceleration of a material particle is, by definition

$$\mathbf{a} = \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{x}=const}$$

In fluid mechanics, it is often convenient to use a *spatial* description of velocity and acceleration – that is to say the velocity field is expressed as a function of position **y** in the deformed solid as $\mathbf{v}(\mathbf{y},t)$. The acceleration of the material particle with instantaneous position **y** in the deformed solid can be expressed as

$$\begin{aligned} a_i &= \frac{\partial v_i}{\partial y_k}\frac{\partial y_k}{\partial t} + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const} = L_{ik} v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const} \\ &= (D_{ik} + W_{ik}) v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const} = (D_{ik} + W_{ik}) v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const} \end{aligned}$$

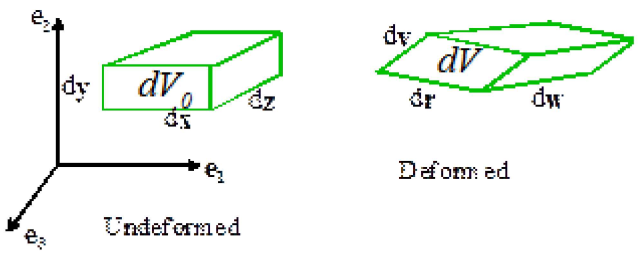
3.23 Acceleration - spin – vorticity relations

In fluid mechanics, equations relating the acceleration to the spatial velocity field are useful. In particular, it can be shown that

- $a_i = \left.\frac{\partial v_i}{\partial t}\right|_{x_k=const} = \left.\frac{\partial v_i}{\partial t}\right|_{y_k=const} + \frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k) + 2W_{ik} v_k$
- $a_i = \left.\frac{\partial v_i}{\partial t}\right|_{x_k=const} = \left.\frac{\partial v_i}{\partial t}\right|_{y_k=const} + \frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k) + \in_{ijk} \omega_j v_k$
- $\in_{ijk} \frac{\partial a_k}{\partial y_j} = \left.\frac{\partial \omega_i}{\partial t}\right|_{\mathbf{x}=const} - D_{ij}\omega_j + \frac{\partial v_k}{\partial y_k}\omega_i$

Deriving these relations is left as an exercise.

3.24 Rate of change of volume



We have seen that

$$J = \det(\mathbf{F})$$

quantifies the volume change associated with a deformation, in that

$$J dV = dV_0$$

In rate form:

$$\frac{dJ}{dt} = \frac{dJ}{dF_{ij}}\frac{dF_{ij}}{dt} = J F_{ji}^{-1} \dot{F}_{ij} = J L_{ii} = J\frac{\partial v_i}{\partial y_i} = J D_{ii}.$$

The trace of **D,** trace of **L** or the trace of grad(**v**) are therefore measures of rate of change of volume.

3.25 Infinitesimal strain rate and rotation rate

For *small strains* the rate of deformation tensor is approximately equal to the infinitesimal strain rate, while the spin can be approximated by the time derivative of the infinitesimal rotation tensor

$$\frac{d}{dt}\boldsymbol{\varepsilon} = \frac{d}{dt}\frac{1}{2}\left(\mathbf{u}\otimes\nabla + (\mathbf{u}\otimes\nabla)^T\right) \approx \mathbf{D} \quad \text{or} \quad \dot{\varepsilon}_{ij} \approx D_{ij}$$

$$\frac{d}{dt}\mathbf{w} = \frac{d}{dt}\frac{1}{2}\left(\mathbf{u}\otimes\nabla - (\mathbf{u}\otimes\nabla)^T\right) \approx \mathbf{W} \quad \text{or} \quad \dot{w}_{ij} \approx W_{ij}$$

The approximation is because the infinitesimal strain and rotation involve derivatives with respect to position in the reference configuration, while the stretch rate and spin tensors are defined in terms of spatial derivatives. Similarly, you can show that

$$\frac{d}{dt}\frac{\partial u_i}{\partial x_j} = \dot{F}_{ij} = \dot{\varepsilon}_{ij} + \dot{w}_{ij} \approx L_{ij}$$

3.26 Other deformation rate measures

The rate of deformation tensor can be related to time derivatives of other strain measures. For example the time derivative of the Lagrange strain tensor can be shown to be

$$\frac{d\mathbf{E}}{dt} = \mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F} \qquad \dot{E}_{ij} = F_{ki} D_{kl} F_{lj}$$

Other useful results are

- For a pure rotation $\dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \dot{\mathbf{R}}^T = \mathbf{0}$, or equivalently $\dot{\mathbf{R}} \cdot \mathbf{R}^T = -\left(\dot{\mathbf{R}} \cdot \mathbf{R}^T\right)^T$. To see this, recall that $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ and evaluate the time derivative.
- If the deformation gradient is decomposed into a stretch followed by a rotation as $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ then $\mathbf{D} = \mathbf{R} \cdot \left(\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} + \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}\right) \cdot \mathbf{R}^T / 2$ and $\mathbf{W} = \dot{\mathbf{R}} \cdot \mathbf{R}^T + \mathbf{R} \cdot \left(\dot{\mathbf{U}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1} \cdot \dot{\mathbf{U}}\right) \cdot \mathbf{R}^T / 2$

For *small strains* the rate of change of Lagrangian strain $\mathbf{E}$ is approximately equal to the rate of change of infinitesimal strain ε:

$$\frac{d\mathbf{E}}{dt} \approx \frac{d}{dt}\boldsymbol{\varepsilon} \qquad \dot{E}_{ij} \approx \frac{d}{dt}\varepsilon_{ij}$$

3.27 Path lines, streamlines, and vortex lines

Path lines, streamlines, and vortex lines are useful concepts in fluid mechanics.

- A **path line** is the curve traced by a material particle as it moves through space. If the curve is described in parametric form by $y_i(\lambda)$, with λ a scalar, then the curve satisfies

$$\frac{dy_i(t)}{dt} = v_i(\mathbf{X}, t)$$

- A **stream line** is a curve that is everywhere tangent to the spatial velocity vector. In general, streamlines may be functions of time. If $y_i(\lambda, t)$ is the parametric representation of the curve, at time t, then $y_i(\lambda, t)$ is a member of the family of solutions to the differential equation

$$\frac{dy_i(\lambda,t)}{d\lambda} = v_i\left(\mathbf{y}(\lambda), t\right)$$

For the particular case of a *steady flow*, the spatial velocity field is (by definition) independent of time, and therefore the curves are fixed in space.

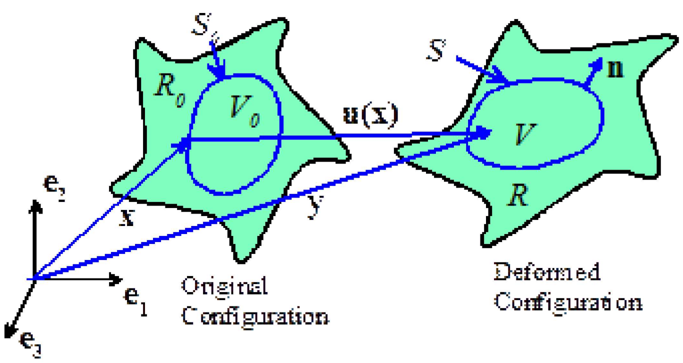
- A **vortex line** is a curve that is everywhere tangent to the vorticity vector. These curves satisfy the differential equation

$$\frac{dy_i(\lambda,t)}{d\lambda} = \omega_i\left(\mathbf{y}(\lambda), t\right)$$

Again, for the special case of a *steady flow* the vortex lines are independent of time.

3.27 Reynolds Transport Relation

The Reynolds transport theorem is a useful way to calculate the rate of change of a quantity inside a volume that deforms with a solid (e.g. the total mass of a volume). Let $\phi(\mathbf{y}, t)$ be any scalar valued property of a material particle at position $\mathbf{y}$ in the deformed solid. The Reynolds transport relation states that rate of change of the total value of this property within a volume V of a deformed solid can be calculated as



$$\frac{d}{dt}\int_V \phi dV = \int_V \left(\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} + \phi\frac{\partial v_i}{\partial y_i}\right) dV = \int_V \left(\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} + \phi D_{kk}\right) dV = \int_V \left(\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{y}=const}\right) dV + \int_S \left(\phi v_k n_k\right) dA$$

Note that the material volume V and surface S convect with the deforming solid – they are not control volumes.

To see this, note that we can't take the time derivative inside the integral because the volume changes with time as the solid deforms. But we can map the integral back to the reference configuration, which is time independent – the derivative can then be taken inside the integral.

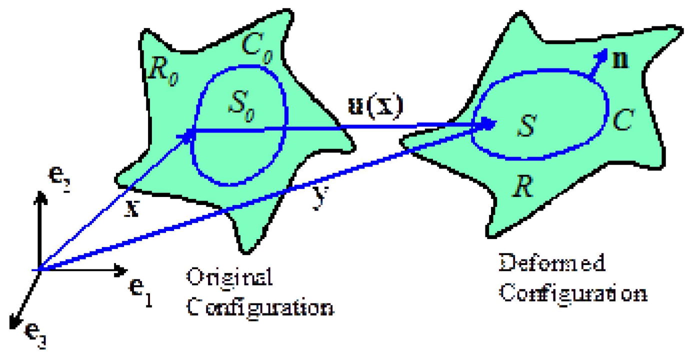
$$\frac{d}{dt}\int_V \phi dV = \frac{d}{dt}\int_{V_0} \phi J dV = \int_{V_0}\left(J\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} + \phi\frac{\partial J}{\partial t}\right) dV$$

$$= \int_{V_0}\left(\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} + \phi D_{kk}\right) J dV = \int_{V}\left(\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} + \phi D_{kk}\right) dV$$

The last result follows by noting that $\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} = \left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \phi}{\partial y_i}v_i$. Then note that $\frac{\partial \phi}{\partial y_i}v_i + \phi\frac{\partial v_i}{\partial y_i} = \frac{\partial(\phi v_i)}{\partial y_i}$ and apply the divergence theorem to this term.

3.28 Transport Relations for material curves and surfaces

Similar transport relations can be derived for material curves and surfaces which convect with a deformable solid or fluid.



Let C be a material curve in a deformable solid; and let S be an interior surface with normal vector $\mathbf{n}$. Let $\phi(\mathbf{y},t)$ be any scalar valued property of a material particle at position $\mathbf{y}$ in the deformed solid. Then

1. $\frac{d}{dt}\int_C \phi\tau_i ds = \int_C\left(\delta_{ij}\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} + \phi\frac{\partial v_i}{\partial y_j}\right)\tau_j ds$
2. $\frac{d}{dt}\int_S \phi n_i dA = \int_S\left(\delta_{ij}\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} + \delta_{ij}\phi\frac{\partial v_k}{\partial y_k} - \phi\frac{\partial v_j}{\partial y_i}\right) n_j dA$

To show the first result, start by mapping the integral to the reference configuration, then take the time derivative, and map back to the current configuration, as follows

$$\frac{d}{dt}\int_C \phi\tau_i ds = \frac{d}{dt}\int_{C_0}\phi F_{ij}\tau_j^0 ds_0 = \int_{C_0}\left(\left.\frac{d\phi}{dt}\right|_{\mathbf{x}} F_{ij} + \phi\frac{dF_{ij}}{dt}\right)\tau_j^0 ds_0$$

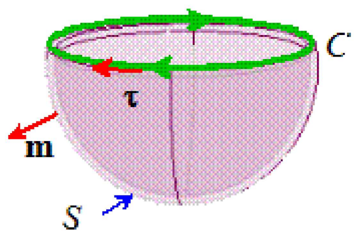
$$= \int_C\left(\left.\frac{d\phi}{dt}\right|_{\mathbf{x}}\delta_{ik} + \phi\frac{dF_{ij}}{dt}F_{jk}^{-1}\right)\tau_k ds = \int_C\left(\delta_{ij}\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{x}=const} + \phi\frac{\partial v_i}{\partial y_j}\right)\tau_j ds$$

To show the second, apply the same process to the surface integral. The details are left as an exercise…

3.28 Circulation and the circulation transport relation

The *circulation* of the velocity field around a closed curve C is defined as

$$I_C = \int_C \mathbf{v}\cdot\boldsymbol{\tau} ds,$$



where $\boldsymbol{\tau}$ is a unit vector tangent to the curve. If C is a reducible curve (i.e. if there is a regular, open surface S bounded by C that lies within the configuration) then Stokes theorem shows that

$$I_C = \int_C \mathbf{v}\cdot\boldsymbol{\tau} ds = \int_S curl(\mathbf{v})\cdot\mathbf{m} dA$$

The circulation transport relation states that

$$\left.\frac{\partial I_C}{\partial t}\right|_{\mathbf{x}=const} = \int_C \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{x}=const}\cdot\boldsymbol{\tau} ds$$

for any material curve (i.e. a curve that convects with material particles within a body). To see this recall the transport relation for a material curve, and set $\phi = v_i$

$$\frac{d}{dt}\int_C v_i\tau_i ds = \int_C\left(\left.\frac{\partial v_j}{\partial t}\right|_{\mathbf{x}=const} + v_i\frac{\partial v_i}{\partial y_j}\right)\tau_j ds$$

Note that

$$v_i\frac{\partial v_i}{\partial y_j} = \frac{1}{2}\frac{\partial(v_i v_i)}{\partial y_j}$$

and hence

$$\int_C \left(v_i \frac{\partial v_i}{\partial y_j} \right) \tau_j ds = \int_C \frac{d(v_i v_i)}{ds} ds = 0$$

because C is a closed curve.

Kelvin's circulation theorem is a direct consequence of this result. The theorem states that if the acceleration is the gradient of a potential, then the circulation around any closed material curve remains constant. To see this, let

$$\left. \frac{\partial v_i}{\partial t} \right|_{\mathbf{x}=const} = \frac{\partial \phi}{\partial y_i}$$

$$\left. \frac{\partial I_C}{\partial t} \right|_{\mathbf{x}=const} = \int_C \frac{\partial \phi}{\partial y_i} \cdot \tau_i ds = \int_C \frac{\partial \phi}{\partial s} ds = 0$$

4. Kinetics

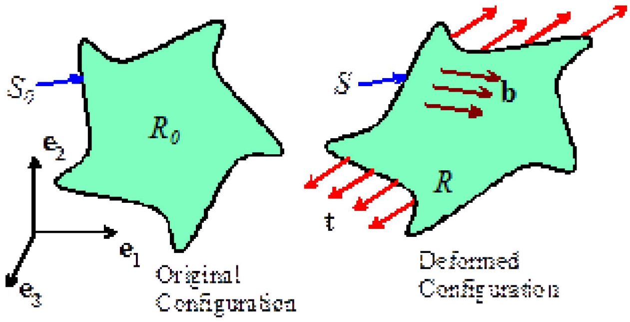
Our next objective is to outline the mathematical formulas that describe internal and external forces acting on a solid. Just as there are many different strain measures, there are several different definitions of internal force. We shall see that internal forces can be described as a second order tensor, which must be symmetric. Thus, internal forces can always be quantified by a set of six numbers, and the various different definitions are all equivalent.

4.1 Surface traction and internal body force

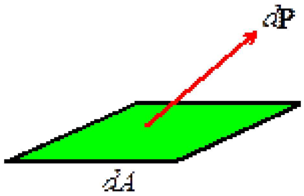
Forces can be applied to a solid body in two ways.

(i) A force can be applied to its boundary: examples include fluid pressure, wind loading, or forces arising from contact with another solid.

(ii) The solid can be subjected to *body forces*, which act on the interior of the solid. Examples include gravitational loading, or electromagnetic forces.



● The ***surface traction vector*** **t** at a point on the surface represents the force acting on the surface per unit area of the deformed solid.



Formally, let dA be an element of area on a surface. Suppose that dA is subjected to a force $d\mathbf{P}$. Then

$$\mathbf{t} = \lim_{dA \to 0} \frac{d\mathbf{P}}{dA}$$

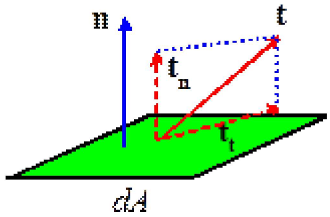
The resultant force acting on any portion S of the surface of the deformed solid is

$$\mathbf{P} = \int_S \mathbf{t} \, dA$$

Surface traction, like `true stress,' should be thought of as acting on the deformed solid.

● **Normal and shear tractions**

The traction vector is often resolved into components acting normal and tangential to a surface, as shown in the picture.

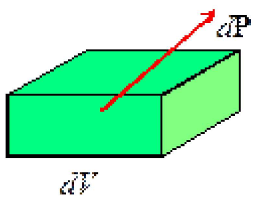


The normal component is referred to as the **normal traction**, and the tangential component is known as the **shear traction**.

Formally, let **n** denote a unit vector normal to the surface. Then

$$\mathbf{t}_\mathbf{n} = (\mathbf{t} \cdot \mathbf{n})\, \mathbf{n} \qquad \mathbf{t}_\mathbf{t} = \mathbf{t} - \mathbf{t}_\mathbf{n}$$

● The ***body force vector*** denotes the external force acting on the interior of a solid, per unit mass.



Formally, let dV denote an infinitesimal volume element within the deformed solid, and let ρ denote the mass density (mass per unit deformed volume). Suppose that the element is subjected to a force $d\mathbf{P}$. Then

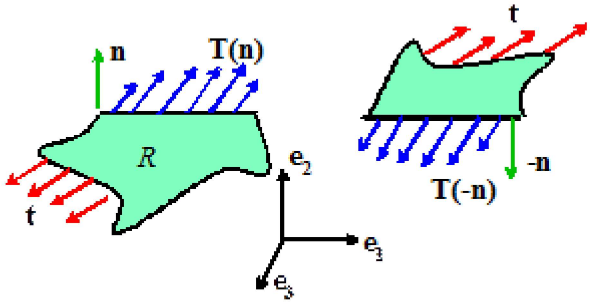
$$\mathbf{b} = \frac{1}{\rho} \lim_{dV \to 0} \frac{d\mathbf{P}}{dV}$$

The resultant body force acting on any volume V within the **deformed** solid is

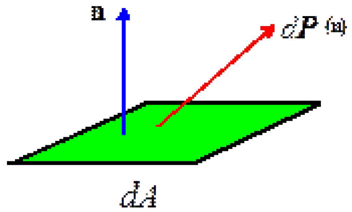
$$\mathbf{P} = \int_V \rho \, \mathbf{b} \, dV$$

4.2 Traction acting on planes within a solid

Every plane in the interior of a solid is subjected to a distribution of traction. To see this, consider a loaded, solid, body in static equilibrium. Imagine cutting the solid in two. The two parts of the solid must each be in static equilibrium. This is possible only if forces act on the planes that were created by the cut.



● The **internal traction vector T(n)** represents the force per unit area acting on a section of the deformed body across a plane with outer normal vector **n**.



Formally, let dA be an element of area in the interior of the solid, with normal **n**. Suppose that the material on the underside of dA is subjected to a force $d\mathbf{P}^{(\mathbf{n})}$ across the plane dA. Then

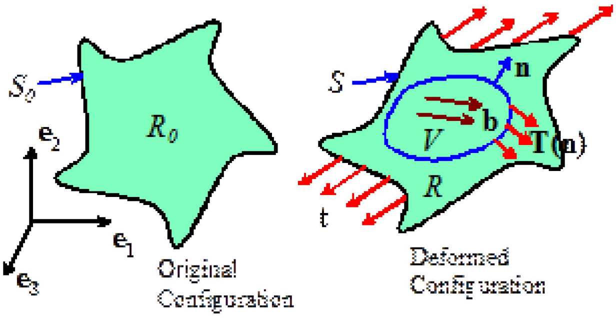
$$\mathbf{T}(\mathbf{n}) = \lim_{dA\to 0} \frac{d\mathbf{P}^{(\mathbf{n})}}{dA}$$

Note that internal traction is the *force per unit area of the deformed solid*, like `true stress'

● **The resultant force acting on any internal volume V with boundary surface A within a deformed solid is**

$$\mathbf{P} = \int_A \mathbf{T}(\mathbf{n})dA + \int_V \rho\,\mathbf{b}\,dV$$

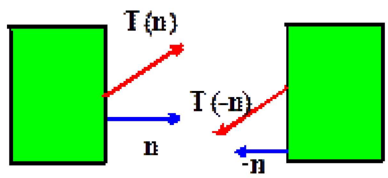
The first term is the resultant force acting on the internal surface A, the second term is the resultant body force acting on the interior V.



● **Newton's third law (every action has an equal and opposite reaction) requires that**

$$\mathbf{T}(-\mathbf{n}) = -\mathbf{T}(\mathbf{n})$$

To see this, note that the forces acting on planes separating two adjacent volume elements in a solid must be equal and opposite.

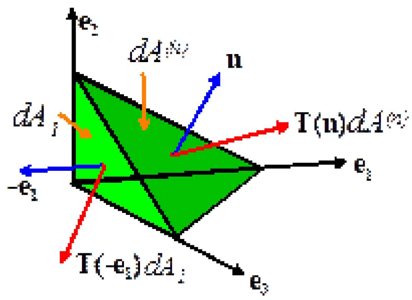


● **Traction acting on different planes passing through the same point are related, in order to satisfy Newton's second law (F=ma).**

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis. Let $T_i(\mathbf{e}_1)$, $T_i(\mathbf{e}_2)$, $T_i(\mathbf{e}_3)$ denote the components of traction acting on planes with normal vectors in the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ directions, respectively. Then, the traction components $T_i(\mathbf{n})$ acting on a surface with normal **n** are given by

$$T_i(\mathbf{n}) = T_i(\mathbf{e}_1)n_1 + T_i(\mathbf{e}_2)n_2 + T_i(\mathbf{e}_3)n_3$$

where n_i are the components of **n**.



To see this, consider the forces acting on the infinitesimal tetrahedron shown in the figure. The base and sides of the tetrahedron have normals in the $-\mathbf{e}_2$, $-\mathbf{e}_1$ and $-\mathbf{e}_3$ directions. The fourth face has normal **n**. Suppose the volume of the tetrahedron is dV, and let $dA_1,\ dA_2,\ dA_3,\ dA_n$ denote the areas of the faces. Assume that the material within the tetrahedron has mass density ρ and is subjected to a body force **b**. Let **a** denote the acceleration of the center of mass of the tetrahedron. Then, **F**=m**a** for the tetrahedron requires that

$$\mathbf{T}(\mathbf{n})dA^{(n)} + \mathbf{T}(-\mathbf{e}_1)dA_1 + \mathbf{T}(-\mathbf{e}_2)dA_2 + \mathbf{T}(-\mathbf{e}_3)dA_3 + \rho\mathbf{b}dV = \rho dV\mathbf{a}$$

Recall that $\mathbf{T}(-\mathbf{e}_i) = -\mathbf{T}(\mathbf{e}_i)$ and divide through by dA_n:

$$\mathbf{T}(\mathbf{n}) - \mathbf{T}(\mathbf{e}_1)\frac{dA_1}{dA^{(n)}} - \mathbf{T}(\mathbf{e}_2)\frac{dA_2}{dA^{(n)}} - \mathbf{T}(\mathbf{e}_3)\frac{dA_3}{dA^{(n)}} + \rho\mathbf{b}\frac{dV}{dA^{(n)}} = \rho\frac{dV}{dA^{(n)}}\mathbf{a}$$

Finally, let $dA_n \to 0$. We can show that

$$\frac{dA_1}{dA^{(n)}} = n_1 \qquad \frac{dA_2}{dA^{(n)}} = n_2 \qquad \frac{dA_3}{dA^{(n)}} = n_3 \qquad \lim_{dA^{(n)}\to 0}\frac{dV}{dA^{(n)}} = 0$$

so

$$\mathbf{T}(\mathbf{n}) = \mathbf{T}(\mathbf{e}_1)n_1 - \mathbf{T}(\mathbf{e}_2)n_2 - \mathbf{T}(\mathbf{e}_3)n_3$$

or, using index notation

$$T_i(\mathbf{n}) = T_i(\mathbf{e}_1)n_1 + T_i(\mathbf{e}_2)n_2 + T_i(\mathbf{e}_3)n_3$$

The significance of this result is that the tractions acting on planes with normals in the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ directions completely characterize the internal forces that act at a point. Given these tractions, we can deduce the tractions acting on any other plane. This leads directly to the definition of the Cauchy stress tensor in the next section.

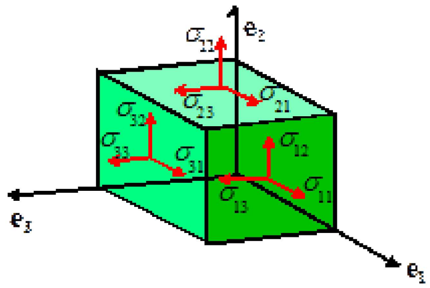
4.3 The Cauchy (true) stress tensor

Consider a solid which deforms under external loading. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis. Let $T_i(\mathbf{e}_1)$, $T_i(\mathbf{e}_2)$, $T_i(\mathbf{e}_3)$ denote the components of traction acting on planes with normals in the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ directions, respectively, as outlined in the preceding section

- **Define the components of the Cauchy stress tensor σ_{ij} by**

$$\sigma_{ij} = T_j(\mathbf{e}_i)$$

$$\equiv \begin{cases} \sigma_{11} = T_1(\mathbf{e}_1) & \sigma_{12} = T_2(\mathbf{e}_1) & \sigma_{13} = T_3(\mathbf{e}_1) \\ \sigma_{21} = T_1(\mathbf{e}_2) & \sigma_{22} = T_2(\mathbf{e}_2) & \sigma_{23} = T_3(\mathbf{e}_2) \\ \sigma_{31} = T_1(\mathbf{e}_3) & \sigma_{32} = T_2(\mathbf{e}_3) & \sigma_{33} = T_3(\mathbf{e}_3) \end{cases}$$



Then, the traction $T_i(\mathbf{n})$ acting on any plane with normal $\mathbf{n}$ follows as

$$\mathbf{T}(\mathbf{n}) = \mathbf{n} \cdot \boldsymbol{\sigma} \qquad \text{or} \qquad T_i(\mathbf{n}) = n_j \sigma_{ji}$$

To see this, recall the last result from the preceding section

$$T_i(\mathbf{n}) = T_i(\mathbf{e}_1)n_1 + T_i(\mathbf{e}_2)n_2 + T_i(\mathbf{e}_3)n_3$$

and substitute for $T_i(\mathbf{e}_j)$ in terms of the components of the Cauchy stress tensor

$$T_i(\mathbf{n}) = \sigma_{1i}n_1 + \sigma_{2i}n_2 + \sigma_{3i}n_3 = n_j \sigma_{ji}$$

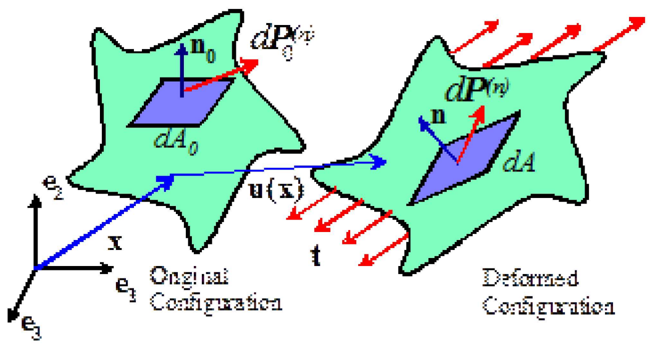
The Cauchy stress tensor completely characterizes the internal forces acting in a deformed solid. The physical significance of the components of the stress tensor is illustrated in the figure: σ_{ji} represents the ith component of traction acting on a plane with normal in the $\mathbf{e}_j$ direction.

Note the Cauchy stress represents force per unit area of the *deformed* solid. In elementary strength of materials courses it is called `true stress,' for this reason.

HEALTH WARNING: Some texts define stress as the *transpose* of the definition used here, so that $\mathbf{T}(\mathbf{n}) = \boldsymbol{\sigma} \cdot \mathbf{n}$ or $T_i(\mathbf{n}) = \sigma_{ij} n_j$. In this case the first index for each stress component denotes the direction of traction, while the second denotes the normal to the plane. We will see later that Cauchy stress is always symmetric, so there is no confusion if you use the wrong definition. But some stress measures are *not* symmetric (see below) and in this case you need to be careful to check which convention the author has chosen.

4.4 Other stress measures – Kirchhoff, Nominal and Material stress tensors

Cauchy stress σ_{ij} (the actual force per unit area acting on an actual, deformed solid) is the most physical measure of internal force. Other definitions of stress often appear in constitutive equations, however.



The other stress measures regard forces as acting on the *undeformed* solid. Consequently, to define them we must know not only what the deformed solid looks like, but also what it looked like before deformation. The deformation is described by a displacement vector $\mathbf{u}(\mathbf{x})$ and the associated deformation gradient

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u} \qquad F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$$

as outlined in Section 2.1. In addition, let $J = \det(\mathbf{F})$

We then define the following stress measures

- **Kirchhoff stress** $\boldsymbol{\tau} = J\boldsymbol{\sigma} \quad \tau_{ij} = J\sigma_{ij}$

- **Nominal (First Piola-Kirchhoff) stress** $\mathbf{S} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma}$ $\quad S_{ij} = JF_{ik}^{-1}\sigma_{kj}$
- **Material (Second Piola-Kirchhoff) stress** $\boldsymbol{\Sigma} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T}$ $\quad \Sigma_{ij} = JF_{ik}^{-1}\sigma_{kl}F_{jl}^{-1}$

The inverse relations are also useful – the one for Kirchhoff stress is obvious – the others are

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F} \cdot \mathbf{S} \quad \sigma_{ij} = \frac{1}{J}F_{ik}S_{kj} \qquad\qquad \boldsymbol{\sigma} = \frac{1}{J}\mathbf{F} \cdot \boldsymbol{\Sigma} \cdot \mathbf{F}^{T} \quad \sigma_{ij} = \frac{1}{J}F_{ik}\Sigma_{kl}F_{jl}$$

The **Kirchoff stress** has no obvious physical significance.

The **nominal stress tensor** can be regarded as the internal force per unit undeformed area acting within a solid, as follows

1. Visualize an element of area dA in the deformed solid, with normal **n**, which is subjected to a force $d\mathbf{P}^{(n)}$ by the internal traction in the solid;
2. Suppose that the element of area dA has started out as an element of area dA_0 with normal $\mathbf{n}_0$ in the undeformed solid, as shown in the figure;
3. Then, the force $d\mathbf{P}^{(n)}$ is related to the nominal stress by $dP_j^{(\mathbf{n})} = dA_0 n_i^0 S_{ij}$

To see this, note that one can show that

$$dA\mathbf{n} = J\mathbf{F}^{-T} \cdot dA_0\mathbf{n}_0 \qquad dAn_i = JF_{ki}^{-1}n_k^0 dA_0$$

Recall that the Cauchy stress is defined so that

$$dP_i^{(\mathbf{n})} = dAn_j\sigma_{ji}$$

Substituting for dAn_j and rearranging shows that

$$dP_i^{(\mathbf{n})} = JdA_0 n_k^0 \left(F_{kj}^{-1}\sigma_{ji}\right) = dA_0 n_k^0 S_{ki}$$

The **material stress tensor** can also be visualized as force per unit undeformed area, except that the forces are regarded as acting within the undeformed solid, rather than on the deformed solid. Specifically

1. The infinitesimal force $d\mathbf{P}^{(\mathbf{n})}$ is assumed to behave like an infinitesimal material fiber in the solid, in the sense that it is stretched and rotated just like an small vector d**x** in the solid
2. This means that we can define a (fictitious) force in the reference configuration $d\mathbf{P}^{(\mathbf{n}0)}$ that is related to $d\mathbf{P}^{(\mathbf{n})}$ by $\mathbf{F} \cdot d\mathbf{P}^{(\mathbf{n}0)} = d\mathbf{P}^{(\mathbf{n})}$ or $F_{ij}dP_j{}^{(\mathbf{n}0)} = dP_i{}^{(\mathbf{n})}$.
3. This fictitious force is related to material stress by $dP_i^{(\mathbf{n}0)} = dA_0 n_j^0 \Sigma_{ji}$

To see this, substitute into the expression relating $d\mathbf{P}^{(n)}$ to nominal stress to see that

$$F_{ik}dP_k^{(\mathbf{n}0)} = dA_0 n_j^0 S_{ji}$$

Finally multiply through by F_{li}^{-1}, note $F_{li}^{-1}F_{ik} = \delta_{lk}$, and rearrange to see that

$$dP_l^{(\mathbf{n}0)} = dA_0 n_j^0 S_{ji}F_{li}^{-1} = dA_0 n_j^0 \Sigma_{jl}$$

where we have noted that $\Sigma_{jl} = S_{ji}F_{li}^{-1}$

In practice, it is best not to try to attach too much physical significance to these stress measures. Cauchy stress is the best physical measure of internal force – it is the force per unit area acting inside the deformed solid. The other stress measures are best regarded as *generalized forces* (in the sense of Lagrangian mechanics), which are work-conjugate to particular strain measures. This means that the stress measure multiplied by the time derivative of the strain measure tells you the rate of work done by the forces. When setting up any mechanics problem, we always work with conjugate measures of motion and forces.

Specifically, we shall show later that the rate of work $\dot{W}$ done by stresses acting on a small material element with volume dV_0 in the undeformed solid (and volume dV in the deformed solid) can be computed as

$$\dot{W} = D_{ij}\sigma_{ij}dV = D_{ij}\tau_{ji}dV_0 = \dot{F}_{ij}S_{ji}dV_0 = \dot{E}_{ij}\Sigma_{ji}dV_0$$

where D_{ij} is the stretch rate tensor, $\dot{F}_{ij}$ is the rate of change of deformation gradient, and $\dot{E}_{ij}$ is the rate of change of Lagrange strain tensor. Note that Cauchy stress (and also Kirchhoff stress) is not conjugate to any convenient strain measure – this is the main reason that nominal and material stresses need to be defined. The nominal stress is conjugate to the deformation gradient, while the material stress is conjugate to the Lagrange strain tensor.

4.5 Stress measures for infinitesimal deformations

For a problem involving *infinitesimal deformation* (where shape changes are characterized by the infinitesimal strain tensor and rotation tensor) all the stress measures defined in the preceding section are approximately equal.

$$\sigma_{ij} \approx \tau_{ij} \approx S_{ij} \approx \Sigma_{ij}$$

To see this, write the deformation gradient as $F_{ij} = \delta_{ij} + \partial u_i / \partial x_j$; recall that $J = \det(\mathbf{F}) \approx 1 + \partial u_k / \partial x_k$, and finally assume that for infinitesimal motions $\partial u_i / \partial x_j << 1$. Substituting into the formulas relating Cauchy stress, Nominal stress and Material stress, we see that

$$\sigma_{ij} = \frac{1}{J} F_{ik} S_{kj} \approx \frac{1}{1+\partial u_p/\partial x_p}\left(\delta_{ip} + \frac{\partial u_i}{\partial x_p}\right) S_{pj} = S_{pj} + \ldots \approx S_{pj}$$

The same procedure will show that material stress and Cauchy stress are approximately equal, to within a term of order $\partial u_i / \partial x_j << 1$

4.6 Principal Stresses and directions

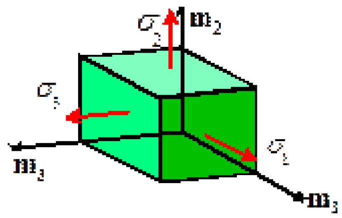
For any stress measure, the *principal stresses* σ_i *and their directions* $\mathbf{n}^{(i)}$, with i=1..3 are defined such that

$$\mathbf{n}^{(i)} \cdot \boldsymbol{\sigma} = \sigma_i \mathbf{n}^{(i)} \quad \text{or} \quad n_j^{(i)} \sigma_{jk} = \sigma_i n_k^{(i)} \quad \text{(no sum on } i\text{)}$$

Clearly,

1. The **principal stresses** are the (left) eigenvalues of the stress tensor
2. The **principal stress directions** are the (left) eigenvectors of the stress tensor

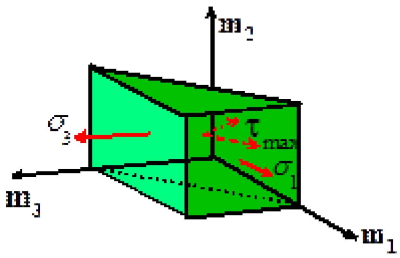
The term `left' eigenvector and eigenvalue indicates that the vector multiplies the tensor on the left. We will see later that Cauchy stress and material stress are both symmetric. For a symmetric tensor the left and right eigenvalues and vectors are the same.



Note that the eigenvectors of a symmetric tensor are orthogonal. Consequently, the principal Cauchy or material stresses can be visualized as tractions acting normal to the faces of a cube. The principal directions specify the orientation of this special cube.

One can also show that if $\sigma_1 > \sigma_2 > \sigma_3$, then σ_1 is the largest normal traction acting on any plane passing through the point of interest, while σ_3 is the lowest. This is helpful in defining damage criteria for brittle materials, which fail when the stress acting normal to a material plane reaches a critical magnitude.

In the same vein, the largest shear stress can be shown to act on the plane with unit normal vector $\mathbf{m}_{\text{shear}} = -(\mathbf{m}_1 + \mathbf{m}_3)/\sqrt{2}$ (at 45° to the $\mathbf{m}_1$ and $\mathbf{m}_3$ axes), and its magnitude is $\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3)$. This observation is useful for defining yield criteria for metal polycrystals, which begin to deform plastically when the shear stress acting on a material plane reaches a critical value.



4.7 Hydrostatic and Deviatoric Stress; von Mises effective stress

Given the Cauchy stress tensor σ, the following may be defined:

The **Hydrostatic stress** is defined as $\sigma_h = \text{trace}(\boldsymbol{\sigma})/3 \equiv \sigma_{kk}/3$
The **Deviatoric stress tensor** is defined as $\sigma'_{ij} = \sigma_{ij} - \sigma_h \delta_{ij}$
The **Von-Mises effective stress** is defined as $\sigma_e = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}}$

The *hydrostatic stress* is a measure of the pressure exerted by a state of stress. Pressure acts so as to change the volume of a material element.

The *deviatoric stress* is a measure of the shearing exerted by a state of stress. Shear stress tends to distort a solid, without changing its volume.

The *Von-Mises effective stress* can be regarded as a uniaxial equivalent of a multi-axial stress state. It is used in many failure or yield criteria. Thus, if a material is known to fail in a uniaxial tensile test (with σ_{11} the only nonzero stress component) when $\sigma_{11} = \sigma_{crit}$, it will fail when $\sigma_e = \sigma_{crit}$ under multi-axial loading (with several $\sigma_{ij} \neq 0$)

The hydrostatic stress and von Mises stress can also be expressed in terms of principal stresses as

$$\sigma_h = (\sigma_1 + \sigma_2 + \sigma_3)/3$$

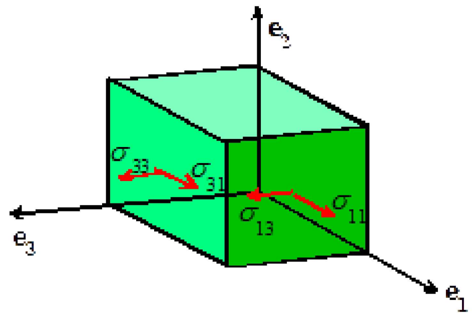
$$\sigma_e = \sqrt{\frac{1}{2}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right\}}$$

The hydrostatic and von Mises stresses are *invariants* of the stress tensor – they have the same value regardless of the basis chosen to define the stress components.

4.8 Stresses near an external surface or edge – boundary conditions on stresses

Note that *at an external surface at which tractions are prescribed, some components of stress are known*. Specifically, let **n** denote a unit vector normal to the surface, and let **t** denote the traction (force per unit area) acting on the surface. Then the Cauchy stress at the surface must satisfy

$$n_i \sigma_{ij} = t_j$$



For example, suppose that a surface with normal in the $\mathbf{e}_2$ direction is subjected to *no* loading. Then (noting that $n_i = \delta_{i2}$) it follows that $\sigma_{2i} = 0$. In addition, two of the principal stress directions must be parallel to the surface; the third (with zero stress) must be perpendicular to the surface.

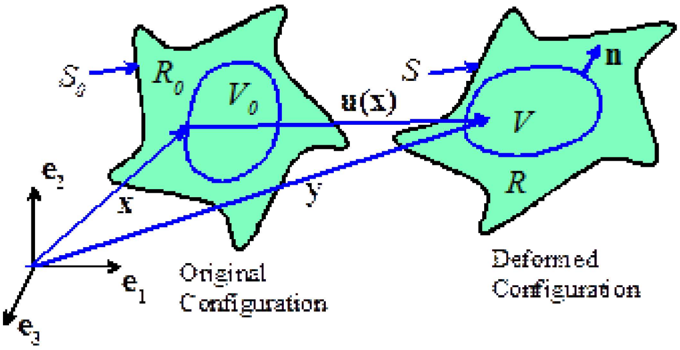
The stress state at an edge is even simpler. Suppose that surfaces with normals in the $\mathbf{e}_2$ and $\mathbf{e}_1$ are traction free. Then $\sigma_{1i} = \sigma_{2i} = 0$, so that 6 stress components are known to be zero.

5. Conservation Laws for Continua

In this section, we generalize Newton's laws of motion (conservation of linear and angular momentum); mass conservation; and the laws of thermodynamics for a continuum.

5.1 Mass Conservation

The total mass of any subregion within a deformable solid must be conserved. We can write express this condition as a constraint in several different ways:



In integral form:

$$\frac{d}{dt}\int_{V_0} \rho(\mathbf{X})dV = \frac{d}{dt}\int_{V} \rho(\mathbf{y},t)dV = 0$$

Or, (using Reynolds transport relation) we can write a *local* mass conservation equation

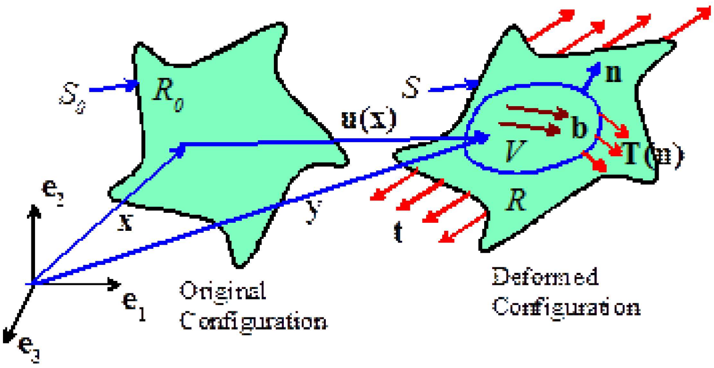
$$\int_V \left(\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho\frac{\partial v_i}{\partial y_i} \right) dV = 0 \Rightarrow \left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho\frac{\partial v_i}{\partial y_i} = 0$$

Alternatively, in spatial form

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho v_i}{\partial y_i} = 0$$

5.2 Linear momentum balance in terms of Cauchy stress

Let σ_{ij} denote the Cauchy stress distribution within a deformed solid. Assume that the solid is subjected to a body force b_i, and let u_i, v_i and a_i denote the displacement, velocity and acceleration of a material particle at position y_i in the deformed solid.



Newton's third law of motion (**F**=m**a**) can be expressed as

$$\nabla_{\mathbf{y}} \cdot \boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a} \quad \text{or} \quad \frac{\partial \sigma_{ij}}{\partial y_i} + \rho\, b_j = \rho\, a_j$$

Written out in full

$$\frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial \sigma_{21}}{\partial y_2} + \frac{\partial \sigma_{31}}{\partial y_3} + \rho b_1 = \rho a_1$$
$$\frac{\partial \sigma_{12}}{\partial y_1} + \frac{\partial \sigma_{22}}{\partial y_2} + \frac{\partial \sigma_{32}}{\partial y_3} + \rho b_2 = \rho a_2$$
$$\frac{\partial \sigma_{13}}{\partial y_1} + \frac{\partial \sigma_{23}}{\partial y_2} + \frac{\partial \sigma_{33}}{\partial y_3} + \rho b_3 = \rho a_3$$

Note that the derivative is taken with respect to position in the actual, deformed solid. For the special (but rather common) case of a solid in static equilibrium in the absence of body forces

$$\frac{\partial \sigma_{ij}}{\partial y_i} = 0$$

Derivation: Recall that the resultant force acting on an arbitrary volume of material V within a solid is

$$P_i = \int_A T_i(\mathbf{n})dA + \int_V \rho\, b_i\, dV$$

where **T(n)** is the internal traction acting on the surface A with normal **n** that bounds V.

The linear momentum of the volume V is

$$\Lambda_i = \int_V \rho\, v_i dV$$

where **v** is the velocity vector of a material particle in the deformed solid. Express **T** in terms of σ_{ij} and set $P_i = d\Lambda_i/dt$

$$\int_A \sigma_{ji} n_j dA + \int_V \rho\, b_i\, dV = \frac{d}{dt}\left\{ \int_V \rho\, v_i dV \right\}$$

Apply the divergence theorem to convert the first integral into a volume integral, and note that the Reynolds transport equation implies that

$$\frac{d}{dt}\left\{ \int_V \rho\, v_i dV \right\} = \int_V \rho a_i dV$$

so

$$\int_V \frac{\partial \sigma_{ji}}{\partial y_j} dV + \int_V \rho\, b_i\, dV = \int_V \rho a_i dV \Rightarrow \int_V \left(\frac{\partial \sigma_{ji}}{\partial y_j} + \rho\, b_i - \rho\, a_i \right)\, dV = 0$$

Since this must hold for every volume of material within a solid, it follows that

$$\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho a_i$$

as stated.

We can also write this in spatial form by recalling that

$$a_i = \frac{\partial v_i}{\partial y_k} v_k + \left. \frac{\partial v_i}{\partial t} \right|_{y_i = const}$$

so that

$$\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho \left(\frac{\partial v_i}{\partial y_k} v_k + \left. \frac{\partial v_i}{\partial t} \right|_{y_i = const} \right)$$

5.3 Angular momentum balance in terms of Cauchy stress

Conservation of angular momentum for a continuum requires that the Cauchy stress satisfy

$$\sigma_{ji} = \sigma_{ij}$$

i.e. the stress tensor must be symmetric.

Derivation: write down the equation for balance of angular momentum for the region V within the deformed solid

$$\int_A \mathbf{y} \times \mathbf{T}\, dA + \int_V \mathbf{y} \times \rho\, \mathbf{b} dV = \frac{d}{dt}\left\{ \int_V \mathbf{y} \times \rho\, \mathbf{v} dV \right\}$$

Here, the left hand side is the resultant moment (about the origin) exerted by tractions and body forces acting on a general region within a solid. The right hand side is the total angular momentum of the solid about the origin.

We can write the same expression using index notation

$$\int_A \in_{ijk}\, y_j T_k dA + \int_V \in_{ijk}\, y_j b_k \rho\, dV = \frac{d}{dt}\left\{ \int_V \in_{ijk}\, y_j v_k \rho\, dV \right\}$$

Express **T** in terms of σ_{ij} and re-write the first integral as a volume integral using the divergence theorem

$$\int_A \in_{ijk}\, y_j T_k dA = \int_A \in_{ijk}\, y_j \sigma_{mk} n_m dA = \int_V \frac{\partial}{\partial y_m} \left(\in_{ijk}\, y_j \sigma_{mk} \right) dV$$

$$= \int_V \in_{ijk} \left(\delta_{jm} \sigma_{mk} + y_j \frac{\partial \sigma_{mk}}{\partial x_m} \right) dV$$

We may also show that

$$\frac{d}{dt}\left\{ \int_V \in_{ijk}\, y_j v_k \rho\, dV \right\} = \int_V \in_{ijk}\, y_j a_k \rho\, dV$$

Substitute the last two results into the angular momentum balance equation to see that

$$\int_V \in_{ijk} \left(\delta_{jm}\sigma_{mk} + y_j \frac{\partial \sigma_{mk}}{\partial x_m} \right) dV + \int_V \in_{ijk} \, y_j b_k \rho \, dV = \int_V \in_{ijk} \, y_j a_k \rho \, dV$$

$$\Rightarrow \int_V \in_{ijk} \, \delta_{jm}\sigma_{mk} = -\int_V \in_{ijk} \, y_j \left(\frac{\partial \sigma_{mk}}{\partial y_m} + \rho b_k - \rho a_k \right) dV$$

The integral on the right hand side of this expression is zero, because the stresses must satisfy the linear momentum balance equation. Since this holds for any volume *V*, we conclude that

$$\begin{aligned} &\in_{ijk} \, \delta_{jm}\sigma_{mk} = \in_{ijk} \, \sigma_{jk} = 0 \\ &\Rightarrow \in_{imn} \in_{ijk} \, \sigma_{jk} = 0 \\ &\Rightarrow \left(\delta_{jm}\delta_{kn} - \delta_{mk}\delta_{nj} \right) \sigma_{jk} = 0 \\ &\Rightarrow \sigma_{mn} - \sigma_{nm} = 0 \end{aligned}$$

which is the result we wanted.

5.4 Equations of motion in terms of other stress measures

In terms of nominal and material stress the balance of linear momentum is

$$\nabla \cdot S + \rho_0 b = \rho_0 \mathbf{a} \qquad \frac{S_{ij}}{\partial x_i} + \rho_0 b_j = \rho_0 a_j$$

$$\nabla \cdot \left[\Sigma \cdot F^T \right] + \rho_0 b = \rho_0 \mathbf{a} \qquad \frac{\partial (\Sigma_{ik} F_{jk})}{\partial x_i} + \rho_0 b_j = \rho_0 a_j$$

Note that the derivatives are taken with respect to position in the *undeformed* solid.

The angular momentum balance equation is

$$\begin{aligned} &F \cdot S = [F \cdot S]^T \\ &\Sigma = \Sigma^T \end{aligned}$$

To derive these results, you can start with the integral form of the linear momentum balance in terms of Cauchy stress

$$\int_A \sigma_{ji} n_j dA + \int_V \rho \, b_i \, dV = \frac{d}{dt} \left\{ \int_V \rho \, v_i dV \right\}$$

Recall that area elements in the deformed and undeformed solids are related through

$$dA n_i = J F_{ki}^{-1} n_k^0 dA_0$$

In addition, volume elements are related by $dV = J dV_0$. We can use these results to re-write the integrals as integrals over a volume in the *undeformed solid* as

$$\int_{A0} \sigma_{ji} J F_{kj}^{-1} n_k^0 dA_0 + \int_{V0} \rho \, b_i \, J dV_0 = \frac{d}{dt} \left\{ \int_{V0} \rho \, v_i \, J dV_0 \right\}$$

Finally, recall that $S_{ij} = J F_{ik}^{-1} \sigma_{kj}$ and that $J\rho = \rho_0$ to see that

$$\int_{A0} S_{ki} n_k^0 dA_0 + \int_{V0} \rho_0 b_i \, dV_0 = \frac{d}{dt} \left\{ \int_{V0} \rho_0 v_i dV_0 \right\}$$

Apply the divergence theorem to the first term and rearrange

$$\int_V \left(\frac{\partial S_{ji}}{\partial x_j} + \rho_0 \, b_i - \rho_0 \frac{dv_i}{dt} \right) dV_0 = 0$$

Once again, since this must hold for any material volume, we conclude that

$$\frac{S_{ij}}{\partial x_i} + \rho_0 b_j = \rho_0 a_j$$

The linear momentum balance equation in terms of material stress follows directly, by substituting into this equation for S_{ij} in terms of Σ_{ij}

The angular momentum balance equation can be derived simply by substituting into the momentum balance equation in terms of Cauchy stress $\sigma_{ij} = \sigma_{ji}$

5.5 Work done by Cauchy stresses

Consider a solid with mass density ρ_0 in its initial configuration, and density ρ in the deformed solid. Let σ_{ij} denote the Cauchy stress distribution within the solid. Assume that the solid is subjected to a body force b_i (per unit mass), and let u_i, v_i and a_i denote the displacement, velocity and acceleration of a material particle at position y_i in the deformed solid. In addition, let

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right)$$

denote the stretch rate in the solid.

The **rate of work done by Cauchy stresses per unit deformed volume** is then $\sigma_{ij}D_{ij}$. This energy is either dissipated as heat or stored as internal energy in the solid, depending on the material behavior.

We shall show that the rate of work done by internal forces acting on any sub-volume V bounded by a surface A in the deformed solid can be calculated from

$$\dot{r} = \int_A T_i^{(\mathbf{n})} v_i dA + \int_V \rho b_i v_i dV = \int_V \sigma_{ij} D_{ij} dV + \frac{d}{dt}\left\{\int_V \frac{1}{2}\rho v_i v_i dV\right\}$$

Here, the two terms on the left hand side represent the rate of work done by tractions and body forces acting on the solid (work done = force x velocity). The first term on the right-hand side can be interpreted as the work done by Cauchy stresses; the second term is the rate of change of kinetic energy.

Derivation: Substitute for $T_i^{(\mathbf{n})}$ in terms of Cauchy stress to see that

$$\dot{r} = \int_A T_i^{(\mathbf{n})} v_i dA + \int_V \rho b_i v_i dV = \int_A n_j \sigma_{ji} v_i dA + \int_V \rho b_i v_i dV$$

Now, apply the divergence theorem to the first term on the right hand side

$$\dot{r} = \int_V \frac{\partial}{\partial y_j}\left(\sigma_{ji} v_i\right) dV + \int_V \rho b_i v_i dV$$

Evaluate the derivative and collect together the terms involving body force and stress divergence

$$\dot{r} = \int_V \left\{\sigma_{ji}\frac{\partial v_i}{\partial y_j} + \left(\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i\right) v_i\right\} dV$$

Recall the equation of motion

$$\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho a_i$$

and note that since the stress is symmetric $\sigma_{ij} = \sigma_{ji}$

$$\sigma_{ji}\frac{\partial v_i}{\partial y_j} = \frac{1}{2}\left(\sigma_{ij} + \sigma_{ji}\right)\frac{\partial v_i}{\partial y_j} = \frac{1}{2}\sigma_{ij}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right) = \sigma_{ij}D_{ij}$$

to see that

$$\dot{r} = \int_V \left\{\sigma_{ij}D_{ij} + \rho a_i v_i\right\} dV$$

Finally, note that

$$\int_V \rho a_i v_i dV = \int_{V_0} \rho_o \frac{dv_i}{dt} v_i dV_0 = \int_{V_0} \rho_o \frac{1}{2}\frac{d}{dt}\left(v_i v_i\right) dV_0$$

$$= \frac{d}{dt}\left(\int_{V_0} \frac{1}{2}\rho_0\left(v_i v_i\right) dV_0\right) = \frac{d}{dt}\left(\int_{V_0} \frac{1}{2}\rho_0\left(v_i v_i\right) dV_0\right) = \frac{d}{dt}\int_V \frac{1}{2}\rho\left(v_i v_i\right) dV$$

Finally, substitution leads to

$$\dot{r} = \int_A T_i^{(\mathbf{n})} v_i dA + \int_V \rho b_i v_i dV = \int_V \sigma_{ij} D_{ij} dV + \frac{d}{dt}\left\{\int_V \frac{1}{2}\rho v_i v_i dV\right\}$$

as required.

5.6 Rate of mechanical work in terms of other stress measures

- The **rate of work done per unit undeformed volume by Kirchhoff stress** is $\tau_{ij}D_{ij}$
- The **rate of work done per unit undeformed volume by Nominal stress** is $S_{ij}\dot{F}_{ji}$
- The **rate of work done per unit undeformed volume by Material stress** is $\Sigma_{ij}\dot{E}_{ij}$

This shows that nominal stress and deformation gradient are *work conjugate*, as are material stress and Lagrange strain.

In addition, the rate of work done on a volume V_0 of the undeformed solid can be expressed as

$$\dot{r} = \int_A T_i^{(\mathbf{n})} v_i dA + \int_V \rho b_i v_i dV = \int_{V_0} \tau_{ij} D_{ij} dV_0 + \frac{d}{dt}\left\{ \int_{V_0} \frac{1}{2} \rho_0 v_i v_i dV_0 \right\}$$

$$\dot{r} = \int_A T_i^{(\mathbf{n})} v_i dA + \int_V \rho b_i v_i dV = \int_{V_0} S_{ij} \dot{F}_{ji} dV_0 + \frac{d}{dt}\left\{ \int_{V_0} \frac{1}{2} \rho_0 v_i v_i dV_0 \right\}$$

$$\dot{r} = \int_A T_i^{(\mathbf{n})} v_i dA + \int_V \rho b_i v_i dV = \int_{V_0} \Sigma_{ij} \dot{E}_{ij} dV_0 + \frac{d}{dt}\left\{ \int_{V_0} \frac{1}{2} \rho_0 v_i v_i dV_0 \right\}$$

Derivations: The proof of the first result (and the stress power of Kirchhoff stress) is straightforward and is left as an exercise. To show the second result, note that $T_i^{(\mathbf{n})} dA = n_j^0 S_{ji} dA_0$ and $dV = J dV_0$ to re-write the integrals over the undeformed solid; then and apply the divergence theorem to see that

$$\dot{r} = \int_{V_0} \frac{\partial}{\partial x_j} \left(S_{ji} v_i \right) dV_0 + \int_{V0} \rho b_i v_i J dV_0$$

Evaluate the derivative, recall that $J\rho = \rho_0$ and use the equation of motion

$$\frac{S_{ij}}{\partial x_i} + \rho_0 b_j = \rho_0 \frac{dv_j}{dt}$$

to see that

$$\dot{r} = \int_{V_0} S_{ji} \frac{\partial v_i}{\partial x_j} dV_0 + \int_{V0} \rho_0 \frac{dv_i}{dt} v_i dV_0$$

Finally, note that $\partial v_i / \partial x_j = (\partial \dot{u}_i / \partial x_j) = \dot{F}_{ij}$ and re-write the second integral as a kinetic energy term as before to obtain the required result.

The third result follows by straightforward algebraic manipulations – note that by definition

$$S_{ij} \dot{F}_{ji} = \Sigma_{ik} F_{jk} \dot{F}_{ji}$$

Since Σ_{ij} is symmetric it follows that

$$\Sigma_{ik} F_{jk} \dot{F}_{ji} = \tfrac{1}{2} \left(\Sigma_{ik} + \Sigma_{ki} \right) F_{jk} \dot{F}_{ji} = \Sigma_{ik} \tfrac{1}{2} \left(F_{jk} \dot{F}_{ji} + F_{ji} \dot{F}_{jk} \right) = \Sigma_{ik} \dot{E}_{ik}$$

5.7 Rate of mechanical work for infinitesimal deformations

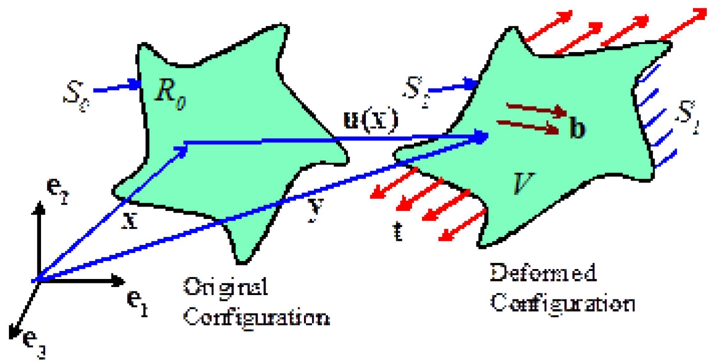
For infintesimal motions all stress measures are equal; and all strain rate measures can be approximated by the infinitesimal strain tensor $\boldsymbol{\varepsilon}$. The rate of work done by stresses per unit volume of either deformed or undeformed solid (the difference is neglected) can be expressed as $\sigma_{ij} \dot{\varepsilon}_{ij}$, and the work done on a volume V_0 of the solid is

$$\dot{r} = \int_A T_i^{(\mathbf{n})} v_i dA + \int_V \rho b_i v_i dV = \int_{V_0} \sigma_{ij} \dot{\varepsilon}_{ij} dV_0 + \frac{d}{dt}\left\{ \int_{V_0} \frac{1}{2} \rho_0 v_i v_i dV_0 \right\}$$

5.8 The principle of Virtual Work

The principle of virtual work forms the basis for the finite element method in the mechanics of solids and so will be discussed in detail in this section.

Suppose that a deformable solid is subjected to loading that induces a displacement field $\mathbf{u}(\mathbf{x})$, and a velocity field $\mathbf{v}(\mathbf{x})$. The loading consists of a prescribed displacement on part of the boundary (denoted by S_1), together with a traction $\mathbf{t}$ (which may be zero in places) applied to the rest of the boundary (denoted by S_2). The loading induces a Cauchy stress σ_{ij}. The stress field satisfies the angular momentum balance equation $\sigma_{ij} = \sigma_{ji}$.



The principle of virtual work is a different way of re-writing partial differential equation for linear moment balance

$$\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho \frac{dv_i}{dt}$$

in an equivalent *integral* form, which is much better suited for computer solution.

To express the principle, we define a **kinematically admissible virtual velocity field** $\delta \mathbf{v}(\mathbf{y})$, satisfying $\delta \mathbf{v} = 0$ on S_1. You can visualize this field as a small change in the velocity of the solid, if you like, but it is really just an arbitrary differentiable vector field. The term `kinematically admissible' is just a complicated way of saying that the field is continuous, differentiable, and satisfies $\delta \mathbf{v}=0$ on S_1 - that is to say, if you perturb the velocity by $\delta \mathbf{v}(\mathbf{y})$, the boundary conditions on displacement are still satisfied.

In addition, we define an associated *virtual velocity gradient, and virtual stretch rate* as

$$\delta L_{ij}=\frac{\partial \delta v_i}{\partial y_j} \qquad \delta D_{ij}=\frac{1}{2}\left(\frac{\partial \delta v_i}{\partial y_j}+\frac{\partial \delta v_j}{\partial y_i}\right)$$

The principal of virtual work may be stated in two ways.

First version of the principle of virtual work

The first is not very interesting, but we will state it anyway. Suppose that the Cauchy stress satisfies:

1. The boundary condition $n_i\sigma_{ij}=t_j$ on S_2
2. The linear momentum balance equation

$$\frac{\partial \sigma_{ji}}{\partial y_j}+\rho b_i=\rho\frac{dv_i}{dt}$$

Then the ***virtual work equation***

$$\int_V \sigma_{ij}\delta D_{ij}\,dV+\int_V \rho\frac{dv_i}{dt}\delta v_i dV-\int_V \rho b_i\delta v_i dV-\int_{S_2} t_i\delta v_i dA=0$$

is satisfied for *all virtual velocity fields*.

Proof: Observe that since the Cauchy stress is symmetric

$$\sigma_{ij}\delta D_{ij}=\frac{1}{2}\sigma_{ij}\left(\frac{\partial \delta v_i}{\partial y_j}+\frac{\partial \delta v_j}{\partial y_i}\right)=\frac{1}{2}\left(\sigma_{ji}\frac{\partial \delta v_i}{\partial y_j}+\sigma_{ij}\frac{\partial \delta v_j}{\partial y_i}\right)=\sigma_{ji}\frac{\partial \delta v_i}{\partial y_j}$$

Next, note that

$$\sigma_{ji}\frac{\partial v_i}{\partial y_j}=\frac{\partial}{\partial y_j}\left(\sigma_{ji}\delta v_i\right)-\frac{\partial \sigma_{ji}}{\partial y_j}\delta v_i$$

Finally, substituting the latter identity into the virtual work equation, applying the divergence theorem, using the linear momentum balance equation and boundary conditions on $\boldsymbol{\sigma}$ and $\delta \mathbf{v}(\mathbf{y})$ we obtain the required result.

Second version of the principle of virtual work

The converse of this statement is much more interesting and useful. Suppose that σ_{ij} satisfies the virtual work equation

$$\int_V \sigma_{ij}\delta D_{ij}\,dV+\int_V \rho\frac{dv_i}{dt}\delta v_i dV-\int_V \rho b_i\delta v_i dV-\int_{S_2} t_i\delta v_i dA=0$$

for all virtual velocity fields $\delta \mathbf{v}(\mathbf{y})$. Then the stress field must satisfy

3. The boundary condition $n_i\sigma_{ij}=t_j$ on S_2
4. The linear momentum balance equation

$$\frac{\partial \sigma_{ji}}{\partial y_j}+\rho b_i=\rho\frac{dv_i}{dt}$$

The significance of this result is that it gives us an alternative way to solve for a stress field that satisfies the linear momentum balance equation, *which avoids having to differentiate the stress*. It is not easy to differentiate functions accurately in the computer, but it is easy to integrate them. The virtual work statement is the starting point for any finite element solution involving deformable solids.

Proof: Follow the same preliminary steps as before, i..e.

$$\sigma_{ij}\delta D_{ij}=\frac{1}{2}\sigma_{ij}\left(\frac{\partial \delta v_i}{\partial y_j}+\frac{\partial \delta v_j}{\partial y_i}\right)=\frac{1}{2}\left(\sigma_{ji}\frac{\partial \delta v_i}{\partial y_j}+\sigma_{ij}\frac{\partial \delta v_j}{\partial y_i}\right)=\sigma_{ji}\frac{\partial \delta v_i}{\partial y_j}$$

$$\sigma_{ji}\frac{\partial v_i}{\partial y_j}=\frac{\partial}{\partial y_j}\left(\sigma_{ji}\delta v_i\right)-\frac{\partial \sigma_{ji}}{\partial y_j}\delta v_i$$

and substitute into the virtual work equation

$$\int_V\left\{\frac{\partial}{\partial y_j}\left(\sigma_{ji}\delta v_i\right)-\frac{\partial \sigma_{ji}}{\partial y_j}\delta v_i\right\}dV+\int_V \rho\frac{dv_i}{dt}\delta v_i dV-\int_V \rho b_i\delta v_i dV-\int_{S_2} t_i\delta v_i dA=0$$

Apply the divergence theorem to the first term in the first integral, and recall that $\delta \mathbf{v}=0$ on S_1, we see that

$$-\int_V\left\{\frac{\partial \sigma_{ji}}{\partial y_j}+\rho b_i-\rho\frac{dv_i}{dt}\right\}\delta v_i dV+\int_{S_2}\left(\sigma_{ji}n_j-t_i\right)\delta v_i dA=0$$

Since this must hold for all virtual velocity fields we could choose

$$\delta v_i=f(\mathbf{y})\left\{\frac{\partial \sigma_{ji}}{\partial y_j}+\rho b_i-\rho\frac{dv_i}{dt}\right\}$$

where $f(\mathbf{y}) = 0$ is an arbitrary function that is positive everywhere inside the solid, but is equal to zero on S. For this choice, the virtual work equation reduces to

$$-\int_V f(\mathbf{y}) \left\{ \frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i - \rho \frac{dv_i}{dt} \right\} \left\{ \frac{\partial \sigma_{ki}}{\partial y_k} + \rho b_i - \rho \frac{dv_i}{dt} \right\} dV = 0$$

and since the integrand is positive everywhere the only way the equation can be satisfied is if

$$\frac{\partial \sigma_{ji}}{\partial y_j} + \rho b_i = \rho \frac{dv_i}{dt}$$

Given this, we can next choose a virtual velocity field that satisfies

$$\delta v_i = (\sigma_{ji} n_j - t_i)$$

on S_2. For this choice (and noting that the volume integral is zero) the virtual work equation reduces to

$$+\int_{S_2} (\sigma_{ji} n_j - t_i)(\sigma_{ki} n_k - t_i)\, dA = 0$$

Again, the integrand is positive everywhere (it is a perfect square) and so can vanish only if

$$\sigma_{ji} n_j = t_i$$

as stated.

5.9 The Virtual Work equation in terms of other stress measures.

It is often convenient to implement the virtual work equation in a finite element code using different stress measures.

To do so, we define

1. The actual deformation gradient in the solid $F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$
2. The virtual rate of change of deformation gradient $\delta \dot{F}_{ij} = \frac{\partial \delta v_i}{\partial y_k} F_{kj} = \frac{\partial \delta v_i}{\partial x_j}$
3. The virtual rate of change of Lagrange strain $\delta \dot{E}_{ij} = \frac{1}{2}\left(F_{ki} \delta \dot{F}_{kj} + \delta \dot{F}_{ki} F_{kj} \right)$

In addition, we define (in the usual way)

1. Kirchhoff stress $\tau_{ij} = J\sigma_{ij}$
2. Nominal (First Piola-Kirchhoff) stress $S_{ij} = JF_{ik}^{-1}\sigma_{kj}$
3. Material (Second Piola-Kirchhoff) stress $\Sigma_{ij} = JF_{ik}^{-1}\sigma_{kl}F_{jl}^{-1}$

In terms of these quantities, the virtual work equation may be expressed as

$$\int_{V0} \tau_{ij} \delta D_{ij}\, dV_0 + \int_{V_0} \rho_0 \frac{dv_i}{dt} \delta v_i dV_0 - \int_{V0} \rho_0 b_i \delta v_i dV_0 - \int_{S_2} t_i \delta v_i dA = 0$$

$$\int_{V0} S_{ij} \delta \dot{F}_{ji}\, dV_0 + \int_{V_0} \rho_0 \frac{dv_i}{dt} \delta v_i dV_0 - \int_{V0} \rho_0 b_i \delta v_i dV_0 - \int_{S_2} t_i \delta v_i dA = 0$$

$$\int_{V0} \Sigma_{ij} \delta \dot{E}_{ij}\, dV_0 + \int_{V_0} \rho_0 \frac{dv_i}{dt} \delta v_i dV_0 - \int_{V0} \rho_0 b_i \delta v_i dV_0 - \int_{S_2} t_i \delta v_i dA = 0$$

Note that all the volume integrals are now taken over the *undeformed solid* – this is convenient for computer applications, because the shape of the undeformed solid is known. The area integral is evaluated over the *deformed* solid, unfortunately. It can be expressed as an equivalent integral over the undeformed solid, but the result is messy and will be deferred until we actually need to do it.

5.10 The Virtual Work equation for infinitesimal deformations.

For infintesimal motions, the Cauchy, Nominal, and Material stress tensors are equal; and the virtual stretch rate can be replaced by the virtual infinitesimal strain rate

$$\delta \dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \delta v_i}{\partial x_j} + \frac{\partial \delta v_j}{\partial x_i} \right)$$

There is no need to distinguish between the volume or surface area of the deformed and undeformed solid. The virtual work equation can thus be expressed as

$$\int_{V0} \sigma_{ij} \delta \dot{\varepsilon}_{ij}\, dV_0 + \int_{V_0} \rho_0 \frac{dv_i}{dt} \delta v_i dV_0 - \int_{V0} \rho_0 b_i \delta v_i dV_0 - \int_{S_2} t_i \delta v_i dA_0 = 0$$

for all kinematically admissible velocity fields.

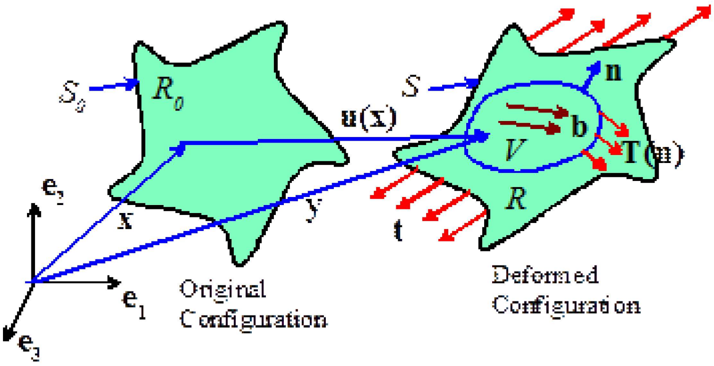
As a special case, this expression can be applied to a quasi-static state with $v_i = 0$. Then, for a stress state σ_{ij} satisfying the static equilibrium equation $\sigma_{ij}/dx_i + \rho_0 b_j = 0$ and boundary conditions $\sigma_{ij} n_j = t_i$ on S_2, the virtual work equation reduces to

$$\int_{V0} \sigma_{ij}\delta\varepsilon_{ij}\, dV_0 = \int_{V0} \rho_0 b_i \delta u_i dV_0 + \int_{S_2} t_i \delta u_i dA$$

In which δu_i are kinematically admissible displacements components ($\delta u_i = 0$ on S_2) and $\delta\varepsilon_{ij} = (\partial\delta u_i/x_j + \partial\delta u_j/x_i)/2$.

Conversely, if the stress state σ_{ij} satisfies $\int_{V0} \sigma_{ij}\delta\varepsilon_{ij}\, dV_0 = \int_{V0} \rho_0 b_i \delta u_i dV_0 + \int_{S_2} t_i \delta u_i dA$ for every set of kinematically admissible virtual displacements, then the stress state σ_{ij} satisfies the static equilibrium equation $\sigma_{ij}/dx_i + \rho_0 b_j = 0$ and boundary conditions $\sigma_{ij} n_j = t_i$ on S_2.

5.10 The first and second laws of thermodynamics for continua



Consider a sub-region V of a deformed solid with surface A. Define:

- The heat flux vector $\mathbf{q}$, which is defined so that $dQ = \mathbf{q}\cdot\mathbf{n}dA$ is the heat flux crossing an internal surface with area dA and normal $\mathbf{n}$ in the deformed solid;
- The heat supply q, defined so that $dQ= qdV$ is the heat supplied from an external source into a volume element dV in the deformed solid;
- The net heat flux into the solid $Q = \int_V q dV - \int_A \mathbf{q}\cdot\mathbf{n}dA$
- The net rate of mechanical work done on the solid $W = \int_V \mathbf{b}\cdot\mathbf{v}dV + \int_A \mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{v}dA$
- The total kinetic energy $KE = \int_V \frac{1}{2}\rho(v_i v_i)dV$
- The total internal energy $E = \int_V \rho\varepsilon dV$ where ε is the specific internal energy (internal energy per unit mass)
- The total entropy $S = \int_V \rho s dV$, where s is the specific entropy (entropy per unit mass)
- The temperature of the solid θ.
- The net external entropy supplied to the volume $\frac{dH}{dt} = \int_A -\frac{\mathbf{q}}{\theta}\cdot\mathbf{n}dA + \int_V \frac{q}{\theta}dV$
- The specific free energy $\psi = \varepsilon - \theta s$
- The total free energy $\Psi = \int_V \rho\psi dV$

The ***first law of thermodynamics*** then requires that

$$\frac{d}{dt}(E + KE) = Q + W$$

for any volume V.

This condition can also be expressed as

$$\rho \left.\frac{\partial\varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \sigma_{ij}D_{ij} - \frac{\partial q_i}{\partial y_i} + q$$

To see this,

1. recall that

$$W = \int_V b_i v_i dV + \int_A \sigma_{ij} n_i v_j dA = \int_V \sigma_{ij} D_{ij} dV + \frac{d}{dt}\left\{\int_V \frac{1}{2}\rho v_i v_i dV\right\} = \int_V \sigma_{ij} D_{ij} dV + \frac{d}{dt}(KE)$$

2. the divergence theorem gives

$$Q = \int_V q dV - \int_A \mathbf{q}\cdot\mathbf{n}dA = \int_V \left(q - \frac{\partial q_i}{\partial y_i}\right)dV$$

3. Therefore

$$\frac{d}{dt}\left(\int\limits_V \rho\varepsilon dV + KE\right) = \int\limits_V \left(q - \frac{\partial q_i}{\partial y_i}\right) dV + \int\limits_V \sigma_{ij} D_{ij} dV + \frac{d}{dt}(KE)$$

4. Note also that
$$\frac{d}{dt}\int\limits_V \rho\varepsilon dV = \frac{d}{dt}\int\limits_{V_0} \rho_0\varepsilon dV = \int\limits_{V_0} \rho_0 \frac{d\varepsilon}{dt} dV = \int\limits_V \rho\frac{d\varepsilon}{dt} dV$$
where $\rho_0 = J\rho$ is the mass density per unit reference volume. Finally
$$\int\limits_V \rho\frac{d\varepsilon}{dt} dV = \int\limits_V \left(q - \frac{\partial q_i}{\partial y_i}\right) dV + \int\limits_V \sigma_{ij} D_{ij} dV$$
This must hold for all V, giving the required result.

The ***second law of thermodynamics*** specifies that the net entropy production within V must be non-negative, i.e.
$$\frac{dS}{dt} - \frac{dH}{dt} \geq 0$$
This can also be expressed as
$$\rho\frac{\partial s}{\partial t} + \frac{\partial(q_i/\theta)}{\partial y_i} - \frac{q}{\theta} \geq 0$$
(this condition is know as the **Clausius-Duhem** inequality).

To see this, simply substitute the definitions and use the divergence theorem.

The first and second laws can be combined to yield the ***free energy imbalance***
$$\sigma_{ij} D_{ij} - \frac{1}{\theta} q_i \frac{\partial\theta}{\partial y_i} - \rho\left(\frac{\partial\psi}{\partial t} + s\frac{\partial\theta}{\partial t}\right) \geq 0$$
This can also be expressed as
$$W - \frac{d(KE)}{dt} - \frac{d\Psi}{dt} - \int\limits_V \left(\rho s\frac{\partial\theta}{\partial t} + \frac{1}{\theta} q_i \frac{\partial\theta}{\partial y_i}\right) dV \geq 0$$

To see the first result,

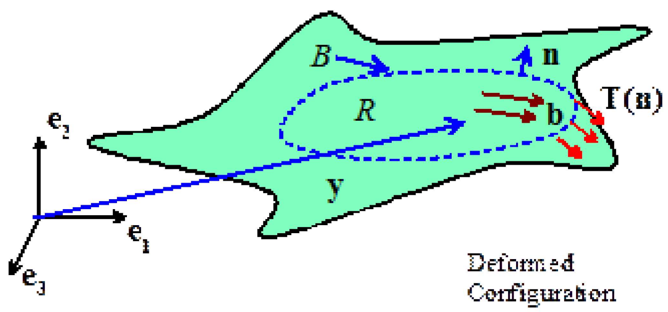
1. note that
$$\rho\frac{\partial s}{\partial t} + \frac{\partial(q_i/\theta)}{\partial y_i} - \frac{q}{\theta} = \rho\frac{\partial s}{\partial t} + \frac{1}{\theta}\frac{\partial q_i}{\partial y_i} - \frac{q}{\theta} - \frac{1}{\theta^2} q_i \frac{\partial\theta}{\partial y_i}$$
2. Use the first law to see that
$$\rho\frac{\partial s}{\partial t} + \frac{1}{\theta}\frac{\partial q_i}{\partial y_i} - \frac{q}{\theta} - \frac{1}{\theta^2} q_i \frac{\partial\theta}{\partial y_i} = \rho\frac{\partial s}{\partial t} + \frac{1}{\theta}\left(-\rho\frac{\partial\varepsilon}{\partial t}\Big|_{\mathbf{x}=const} + \sigma_{ij} D_{ij}\right) - \frac{1}{\theta^2} q_i \frac{\partial\theta}{\partial y_i} \geq 0$$
$$\Rightarrow \sigma_{ij} D_{ij} - \rho\frac{\partial}{\partial t}(\psi + \theta s) + \theta\rho\frac{\partial s}{\partial t} - \frac{1}{\theta} q_i \frac{\partial\theta}{\partial y_i} \geq 0$$
where we have noted that temperature is always positive. This yields the solution.

The second result follows by integrating the local form and using the stress-power work expression.

5.11 Conservation laws for a control volume

To model solids, it is usually convenient to write the equations of motion for a volume that moves with the solid. When modeling fluids, it is often preferable to consider a fixed spatial volume, through which the fluid moves with time. To this end,

- We consider a fixed region in space R, bounded by a surface B.
- Material flows through the region with velocity field $\mathbf{v}(\mathbf{y},t)$.
- The solid has mass density per unit deformed volume (in the spatial configuration) ρ; and is subjected to a body force $\mathbf{b}$ per unit mass.



● A heat flux $\mathbf{q}$ flows through the solid; while an external source injects heat flux Q per unit deformed volume.

The conservation laws can be expressed in terms of integrals over the fixed spatial region (which does not move with the solid) as follows

Mass Conservation: $\frac{d}{dt}\int\limits_R \rho dV + \int\limits_B \rho \mathbf{v}\cdot\mathbf{n}dA = 0$

Linear Momentum Balance $\int\limits_B \mathbf{n}\cdot\boldsymbol{\sigma}dA + \int\limits_R \rho\mathbf{b}dV = \frac{d}{dt}\int\limits_R \rho\mathbf{v}dV + \int\limits_B (\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}dA$

Angular Momentum Balance $\int\limits_B \mathbf{y}\times(\mathbf{n}\cdot\boldsymbol{\sigma})dA + \int\limits_B \mathbf{y}\times(\rho\mathbf{b})dA = \frac{d}{dt}\int\limits_R \mathbf{y}\times\rho\mathbf{v}dV + \int\limits_B (\mathbf{y}\times\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}dA$

Mechanical Power Balance $\int\limits_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v}dA + \int\limits_R \rho\mathbf{b}\cdot\mathbf{v}dV = \int\limits_R \boldsymbol{\sigma}:\mathbf{D}dV + \frac{d}{dt}\int\limits_R \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v})dV + \int\limits_B \frac{1}{2}\rho(\mathbf{v}\cdot\mathbf{v})\mathbf{v}\cdot\mathbf{n}dA$

First law of thermodynamics

$$\int\limits_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v}dA + \int\limits_R \rho\mathbf{b}\cdot\mathbf{v}dV - \int\limits_B \mathbf{q}\cdot\mathbf{n}dA + \int\limits_V qdV = \frac{d}{dt}\int\limits_R \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)dV + \int\limits_B \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\cdot\mathbf{n}dA$$

Second law of thermodynamics

$$\frac{d}{dt}\int\limits_R \rho s dV + \int\limits_B \rho s(\mathbf{v}\cdot\mathbf{n})dA + \int\limits_B \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}dA - \int\limits_R \frac{q}{\theta}dV \geq 0$$

These results can all be derived from the conservation laws for a material volume, using a similar approach. Consider the mass conservation equation as a special case. Start with the local form

$$\left.\frac{\partial\rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho v_i}{\partial y_i} = 0$$

Integrate over a fixed spatial volume

$$\int\limits_R \left.\frac{\partial\rho}{\partial t}\right|_{\mathbf{y}=const} dV + \int\limits_R \frac{\partial \rho v_i}{\partial y_i}dV = 0$$

Note the R is independent of time, and use the divergence theorem

$$\frac{d}{dt}\int\limits_R \rho(\mathbf{y},t)dV + \int\limits_B \rho v_i n_i dA = 0$$

As a second example, for the linear momentum balance equation, start with the local form

$$\frac{\partial\sigma_{ji}}{\partial y_j} + \rho b_i = \rho\left(\frac{\partial v_i}{\partial y_k}v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const}\right)$$

Note that, using mass conservation

$$\frac{\partial}{\partial y_k}(\rho v_i v_k) = \rho v_k\frac{\partial v_i}{\partial y_k} + \frac{\partial(\rho v_k)}{\partial y_k}v_i = \rho v_k\frac{\partial v_i}{\partial y_k} - \frac{\partial\rho}{\partial t}v_i$$

$$\Rightarrow \rho v_k\frac{\partial v_i}{\partial y_k} = \frac{\partial}{\partial y_k}(\rho v_i v_k) + \frac{\partial\rho}{\partial t}v_i$$

Therefore

$$\frac{\partial\sigma_{ji}}{\partial y_j} + \rho b_i = \frac{\partial}{\partial y_k}(\rho v_i v_k) + v_i\left.\frac{\partial\rho}{\partial t}\right|_{y_i=const} + \rho\left.\frac{\partial v_i}{\partial t}\right|_{y_i=const} = \frac{\partial}{\partial y_k}(\rho v_i v_k) + \left.\frac{\partial(\rho v_i)}{\partial t}\right|_{y_i=const}$$

Integrating this expression over the fixed control volume and using the divergence theorem gives the stated answer.

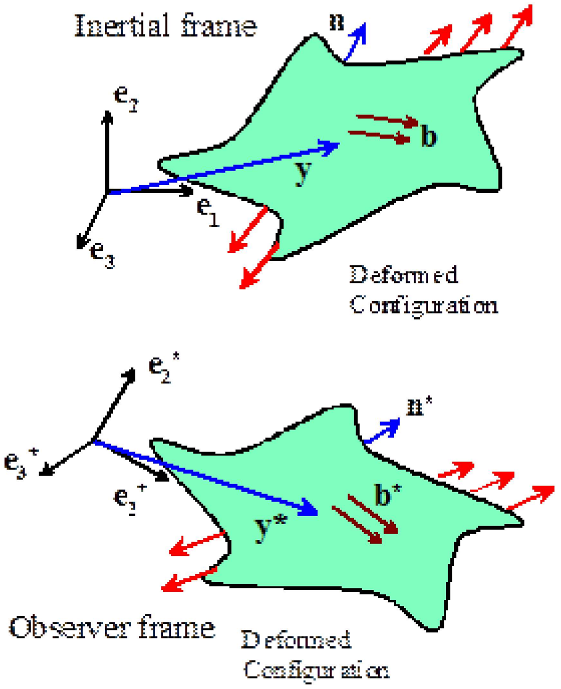
A similar approach can be used to obtain the remaining results – details are left as an exercise.

5.12 Transformation of kinematic and kinetic variables under changes of reference frame

Since physical laws must be constructed so as to be independent of the choice of reference frame, the behavior of kinematic and kinetic variables, the field equations, and constitutive equations under a change of reference frame is of interest. The concept of a reference frame, and the various relations involved in changing reference frames, are both rather obscure concepts. There are several reasons for this:

1. One source of confusion arises because Newtonian mechanics relies on the concept of an *inertial* frame, and Newton's law **F**=m**a** only holds in this frame. The statement 'the laws of physics are independent of reference frame' does not mean that **F**=m**a** in all reference frames – it means that all observers must describe Newton's laws with respect to the same inertial frame, and must do so in a consistent manner. Frame indifference is not the theory of relativity…
2. A second source of confusion stems from the use of a *reference configuration* to quantify shape changes of a solid. We nearly always use the undeformed solid as reference, which gives the impression that the reference configuration, like the deformed configuration, is associated with the inertial frame. In fact, the reference configuration is completely arbitrary,

and even though all observers might choose the same initial configuration of a solid as reference, *they will all assume that the reference configuration is fixed*. The reference configuration is *not* attached to the inertial frame. Moreover, since the reference configuration is arbitrary, two observers could choose different reference configurations, and still devise equations that describe the same physical process. Of course the exact form of the governing equations will change with the choice of reference configuration. There are no restrictions governing transformation of reference configuration between observers, beyond the fact that two reference configurations must be related by an invertible 1:1 mapping.



To make the concept of a change in reference frame in classical continuum mechanics precise, we first introduce the intertial frame. As in all preceding discussions, we assume that the inertial frame is a three-dimensional Euclidean space, and let $\mathbf{y}(t)$ denote a point in the inertial frame. We then define Newtonian measures of velocity and acceleration vectors in the usual way as

$$\mathbf{v} = \frac{\partial \mathbf{y}}{\partial t} \qquad \mathbf{a} = \frac{\partial^2 \mathbf{y}}{\partial t^2}$$

This inertial frame could be viewed by an observer who rotates and translates with respect to the inertial frame. To this observer, all physical quantities associated with the inertial frame would appear to be translated and rotated in the opposite sense. We describe this apparent transformation of space with respect to the observer as a rigid rotation and translation – thus, the position vector of a point seen by the observer is related to position in the inertial frame by

$$\mathbf{y}^* = \mathbf{y}_0^*(t) + \mathbf{Q}(t)(\mathbf{y} - \mathbf{y}_0)$$

where $\mathbf{y}_0$ is an arbitrary fixed point in the inertial frame, $\mathbf{y}_0^*$ is an arbitrary vector, and $\mathbf{Q}(t)$ is a proper orthogonal tensor, representing a rigid rotation. It is convenient to introduce the spin associated with the relative rotation of the inertial frame and the observer's frame

$$\mathbf{\Omega} = \frac{d\mathbf{Q}}{dt}\mathbf{Q}^T$$

We will denote quantities in the observer's reference frame with a star superscript; -for example mass density, body force, Cauchy stress $\rho^*, \mathbf{b}^*, \boldsymbol{\sigma}^*$; those without superscripts will be assumed to be defined in the inertial frame.

All observable physical quantities must transform in particular ways under a change of observer. Specifically:

- Scalar quantities, such as density or temperature are *invariant* – they have the same value in all reference frames.
- Quantities such as body force, a line element in the deformed solid; the normal to a surface; the velocity vector, the acceleration vector, and so on, are physical vectors defined in the inertial reference frame. They can be regarded as connecting two points in the inertial frame, and must transform with the line connecting these two points under a change of reference frame. Thus, a normal vector to a deformed surface, body force, velocity, acceleration vectors must transform as
$$\mathbf{b}^* = \mathbf{Q}\mathbf{b} \;\; \mathbf{n}^* = \mathbf{Q}\mathbf{n} \;\; \mathbf{v}^* = \mathbf{Q}\mathbf{v} \quad \mathbf{a}^* = \mathbf{Q}\mathbf{a}$$
Vectors that transform in this way are said to be **frame indifferent**, or **objective**. Note (1) frame indifference does not mean that vectors are invariant – quite the opposite, in fact – it means that all observers must describe the same physical quantity; (2) vector quantities we make frequent use of in solid mechanics need not necessarily be frame indifferent. For example, the normal to the reference configuration of a solid would *not* be frame indifferent; nor would a material fiber within the reference configuration.
- Tensor quantities that map a frame indifferent vector onto another frame independent vector are similarly said to be **frame indifferent,** or **objective.** Examples include the stretch rate tensor (which specifies the relative velocity of two ends of an infinitesimal material fiber in the spatial configuration); or Cauchy stress (which maps the normal to a surface in the spatial configuration to the physical traction vector. A frame indifferent tensor must transform as
$$\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T \qquad \mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^T$$
Again, not all tensors are frame indifferent. The deformation gradient; the spin tensor; the Lagrange strain tensor are *not* frame indifferent.

Frame indifference can also be looked at as follows. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an inertial basis. Under a change of observer, the basis vectors transform as $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\} = \{\mathbf{Q}\mathbf{e}_1, \mathbf{Q}\mathbf{e}_2, \mathbf{Q}\mathbf{e}_3\}$ (relative to the observer, they appear to rotate with the observed frame. It is important to note that $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*\}$ are time dependent). Now, we can compute components in either basis

$$b_i^* = \mathbf{e}_i^* \cdot \mathbf{b}^* = \mathbf{Q}\mathbf{e}_i \cdot \mathbf{Q}\mathbf{b} = \mathbf{e}_i \cdot \mathbf{Q}^T\mathbf{Q}\,\mathbf{b} = \mathbf{e}_i \cdot \mathbf{b} = b_i$$

$$\sigma_{ij}^* = \mathbf{e}_i^* \cdot \boldsymbol{\sigma}^* \cdot \mathbf{e}_j^* = \mathbf{Q}\mathbf{e}_i \cdot \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T\mathbf{Q}\mathbf{e}_j = \sigma_{ij}$$

The components of a vector with respect to a basis that rotates with an inertial frame is independent of the observer. This is one interpretation of what we mean by a physical process being independent of the observer.

We now examine how several kinematic and kinetic variables commonly used in continuum mechanics transform under a change of observer. To describe deformations, a reference configuration must be selected. A material particle in the reference configuration is identified by a time independent vector $\mathbf{X}$ in *reference space*. The choice of reference space is arbitrary; and there is no reason why different observers will necessarily adopt the same reference space. Discussions are greatly simplified, however, if we choose to assume that all observers use the same space for the reference configuration (behavior under a change of reference configuration can be treated separately).

- **The deformation mapping** transforms as $\mathbf{y}^*(\mathbf{X},t) = \mathbf{y}_0^*(t) + \mathbf{Q}(t)\left(\mathbf{y}(\mathbf{X},t) - \mathbf{y}_0\right)$
- **The deformation gradient** transforms as $\mathbf{F}^* = \frac{\partial \mathbf{y}^*}{\partial \mathbf{X}} = \mathbf{Q}\frac{\partial \mathbf{y}}{\partial \mathbf{X}} = \mathbf{QF}$
- **The right Cauchy Green strain Lagrange strain, the right stretch tensor are invariant**

$$\mathbf{C}^* = \mathbf{F}^{*T}\mathbf{F}^* = \mathbf{F}^T\mathbf{Q}^T\mathbf{QF} = \mathbf{C} \qquad \mathbf{E}^* = \mathbf{E} \quad \mathbf{U}^* = \mathbf{U}$$

- **The left Cauchy Green strain, Eulerian strain, left stretch tensor are frame indifferent**

$$\mathbf{B}^* = \mathbf{F}^*\mathbf{F}^{*T} = \mathbf{QFF}^T\mathbf{Q}^T = \mathbf{QCQ}^T \qquad \mathbf{V}^* = \mathbf{QVQ}^T$$

- **The velocity gradient and spin tensor** transform as

$$\mathbf{L}^* = \dot{\mathbf{F}}^*\mathbf{F}^{*-1} = \left(\dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\dot{\mathbf{F}}\right)\mathbf{F}^{-1}\mathbf{Q}^T = \mathbf{QLQ}^T + \mathbf{\Omega}$$

$$\mathbf{W}^* = (\mathbf{L}^* - \mathbf{L}^{*T})/2 = \mathbf{QWQ}^T + \mathbf{\Omega}$$

- **The velocity and acceleration vectors** transform as

$$\mathbf{v}^* = \mathbf{Qv} = \mathbf{Q}\frac{d\mathbf{y}}{dt} = \mathbf{Q}\frac{d}{dt}\mathbf{Q}^T(\mathbf{y}^* - \mathbf{y}_0^*(t)) = \frac{d\mathbf{y}^*}{dt} - \frac{d\mathbf{y}_0^*}{dt} - \mathbf{\Omega}(\mathbf{y}^* - \mathbf{y}_0^*(t))$$

$$\mathbf{a}^* = \mathbf{Qa} = \mathbf{Q}\frac{d^2\mathbf{y}}{dt^2} = \mathbf{Q}\frac{d^2}{dt^2}\mathbf{Q}^T(\mathbf{y}^* - \mathbf{y}_0^*(t)) = \frac{d^2\mathbf{y}^*}{dt^2} - \frac{d^2\mathbf{y}_0^*}{dt^2} + \left(\mathbf{\Omega}^2 - \frac{d\mathbf{\Omega}}{dt}\right)(\mathbf{y}^* - \mathbf{y}_0^*(t)) - 2\mathbf{\Omega}\left(\frac{d\mathbf{y}^*}{dt} - \frac{d\mathbf{y}_0^*(t)}{dt}\right)$$

(the additional terms in the acceleration can be interpreted as the centripetal and coriolis accelerations)

- **The Cauchy stress is frame indifferent** $\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T$ (you can see this from the formal definition, or use the fact that the virtual power must be invariant under a frame change)
- **The material stress is frame invariant** $\mathbf{\Sigma}^* = \mathbf{\Sigma}$
- **The nominal stress transforms as** $\mathbf{S}^* = J(\mathbf{QF})^{-1}\cdot\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T = J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}\mathbf{Q}^T = \mathbf{SQ}^T$ (note that this transformation rule will differ if the nominal stress is defined as the transpose of the measure used here…)

By this time you are probably asking yourself why anyone could possibly care about all this. This is a fair question – issues of frame indifference arise rather rarely in practice. They do come up, however, when we define *constitutive equations* for a material – which must relate deformation measures to internal force measures. A new constitutive equation is a new physical law – and it is important to make sure that the new law behaves correctly under a change of observer. Most modern constitutive equations try to describe the underlying microscopic processes that govern its response, and if this is done properly, the law will be frame indifferent. But some constitutive laws are just curve-fits – some mathematical relationship between a deformation measure and a force measure – and not all possible relationships will transform correctly.

Problems arise most commonly in trying to develop *rate forms* of constitutive equations, which are intended to relate some measure of strain rate to stress rate. This is because, even if a vector or tensor is itself frame indifferent, its time derivative is generally not. For example, although position vector satisfies $\mathbf{y}^* = \mathbf{Qy}$ and is frame indifferent, this does not mean that $\frac{d\mathbf{y}^*}{dt} = \mathbf{Q}\frac{d\mathbf{y}}{dt}$. Similarly, for the rate of change of Cauchy stress is not frame indifferent, because

$$\frac{d\boldsymbol{\sigma}^*}{dt} = \frac{d\mathbf{Q}}{dt}\boldsymbol{\sigma}\mathbf{Q}^T + \mathbf{Q}\frac{d\boldsymbol{\sigma}}{dt}\mathbf{Q}^T + \mathbf{Q}\boldsymbol{\sigma}\frac{d\mathbf{Q}}{dt}^T = \mathbf{Q}\frac{d\boldsymbol{\sigma}}{dt}\mathbf{Q}^T + \mathbf{\Omega}\boldsymbol{\sigma}^* - \boldsymbol{\sigma}^*\mathbf{\Omega} \neq \mathbf{Q}\frac{d\boldsymbol{\sigma}}{dt}\mathbf{Q}^T$$

In fact, only quantities that are *invariant* under a change of observer can be differentiated safely with respect to time – their time derivatives remain invariant.

This means that if constitutive equations are expressed in rate form – for example something that looks at first glance like the rate form of an elastic constitutive equation

$$\frac{d\boldsymbol{\sigma}}{dt} = \widetilde{\mathbf{C}}\mathbf{D}$$

(here $\mathbf{C}$ is a fourth order constant tensor) – the constitutive equation will *not* be frame indifferent.

There are various fixes for this – the constitutive law can be written in terms of invariant quantities (eg by relating the rate of change of material stress to Lagrange strain rate); they can be derived from physical principles, in which case frame indifferent measures usually emerge naturally from the treatment; or frame indifferent measures of time derivatives can be specially constructed.

As a specific example, one way to construct a frame indifferent stress rate is to use the rate of change of stress components with respect to a basis that rotates with the solid (this is what an observer rotating with the material would actually see). This sounds easy – we just choose some basis vectors $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ with each basis vector parallel to a particular material fiber. But this doesn't quite work, because of course the basis vectors won't generally remain orthogonal under an arbitrary deformation. So rather than attach $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ to particular material fibers, we simply suppose that they rotate with the *average* angular velocity of all material fibers passing through a particular point. This means that

$$\frac{d\mathbf{m}_i}{dt} = \mathbf{W}\mathbf{m}_i$$

where $\mathbf{W}$ is the spin tensor. Now, the time derivative of stress can be written as

$$\frac{d\boldsymbol{\sigma}}{dt} = \frac{d}{dt}(\sigma_{ij}\mathbf{m}_i \otimes \mathbf{m}_j) = \frac{d\sigma_{ij}}{dt}\mathbf{m}_i \otimes \mathbf{m}_j + \sigma_{ij}\mathbf{W}\mathbf{m}_i \otimes \mathbf{m}_j + \sigma_{ij}\mathbf{m}_i \otimes \mathbf{W}\mathbf{m}_j = \overset{\nabla}{\boldsymbol{\sigma}} + \mathbf{W}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{W}$$

Here, the first term can be interpreted as the stress rate seen by an observer rotating with the embedded basis; the second and third are the rates of change of stress arising from the rotation of the material. The first term is of particular interest, and is called the ***Jaumann stress rate.*** It is defined as

$$\overset{\nabla}{\boldsymbol{\sigma}} = \frac{d\boldsymbol{\sigma}}{dt} - \mathbf{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W}$$

It is easy to show that $\overset{\nabla}{\boldsymbol{\sigma}}$ is frame indifferent. Many constitutive equations assume that material stretch rate is proportional to this special stress rate. For example, we could write

$$\overset{\nabla}{\boldsymbol{\sigma}} = \widetilde{\mathbf{C}}\mathbf{D}$$

Provided that $\widetilde{\mathbf{C}}$ is a frame indifferent fourth-order tensor, this constitutive equation would be frame indifferent.

This raises another question, of course. What does it mean for a fourth-order tensor to be frame indifferent? Hopefully you can answer this question for yourself!