Summation Symbol: Changing the Order

Asked 8 years, 10 months ago Active 4 months ago Viewed 19k times

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I have some questions regarding the order of the summation signs (I have tried things out and also read the wikipedia page, nevertheless some questions remained unanswered):

Original 1. wikipedia says that:

$$\sum_{k=1}^{m} a_k \sum_{k=1}^{n} b_l = \sum_{k=1}^{m} \sum_{l=1}^{n} a_k b_l$$

does **not** necessarily hold. What would be a *concrete* example for that?

Edited 1. wikipedia says that:

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2.As far as I see generally it holds that:

$$\sum_{j=1}^{m} \sum_{i=1}^{n} a_i b_j = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j$$

why is that? It is not due to the property, that multiplication is commutative, is it?

3.What about infinite series, when does:

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_k b_l = \sum_{k=1}^{\infty} a_k \sum_{l=1}^{\infty} b_l$$

hold? And does here too

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_k b_l = \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} a_k b_l$$

hold?

Thanks

real-analysis sequences-and-series summation

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edited Jul 15 '13 at 8:09
NNOX Apps
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asked Mar 26 '13 at 19:41
TestGuest
943 ● 2 ■ 8 ▲ 15

Could you please clarify what you mean in the first double sum? You have summed over k twice on the right hand side, which doesn't make consistent since, and some of the answerers have taken it to mean different things! – Tom Oldfield Mar 26 '13 at 20:19

You are right, so one of the k's could just be replaced with an l, in fact I will edit it. Pardon me – TestGuest Mar 26 '13 at 20:42

Actually I see now that on wikipedia too it is summed twice over k as it was done previously in my post: de.wikipedia.org/wiki/Summe – TestGuest Mar 26 '13 at 23:21

1 Just for clarity, the wikipedia page only sums over k **once** on the left hand side in the article, (the right hand side of your original post) where you were summing twice (and over different ranges). This is what was inconsistent. Using k twice on the right hand side is okay (but arguably poor notation) since the k's do not interact. This inequality comes from the difference between adding over the whole grid, and just adding up the diagonals, if you think about the sum being over a grid, as in my answer. Brian Scott provided an example, but almost any values will show this. – Tom Oldfield Mar 26 '13 at 23:29

4 Answers

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For the *original* first question where $l = k$, let $m = n = 2$, $a_1 = b_1 = 1$, and $a_2 = b_2 = 2$; then

$$\sum_{k=1}^{2} a_k \sum_{k=1}^{2} b_k = \sum_{k=1}^{2} a_k(1+2) = 1\cdot 3 + 2\cdot 3 = 9\ ,$$

but

$$\sum_{k=1}^{2}\sum_{k=1}^{2} a_k b_k = \sum_{k=1}^{2}(1^2+2^2) = 5+5 = 10\ .$$

For the second question, imagine arranging the terms $a_i b_j$ in an $n \times m$ array:

$$\begin{array}{ccccc|c}
a_1b_1 & a_1b_2 & a_1b_3 & \dots & a_1b_m & \sum_{j=1}^m a_1b_j \\
a_2b_1 & a_2b_2 & a_2b_3 & \dots & a_2b_m & \sum_{j=1}^m a_2b_j \\
a_3b_1 & a_3b_2 & a_3b_3 & \dots & a_3b_m & \sum_{j=1}^m a_3b_j \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
a_nb_1 & a_nb_2 & a_nb_3 & \dots & a_nb_m & \sum_{j=1}^m a_nb_j \\ \hline
\sum_{i=1}^n a_ib_1 & \sum_{i=1}^n a_ib_2 & \sum_{i=1}^n a_ib_3 & \dots & \sum_{i=1}^n a_ib_m &
\end{array}$$

For each $j = 1, \dots, m$, $\sum_{i=1}^n a_i b_j$ is the sum of the entries in column j, and for each $i = 1, \dots, n$, $\sum_{j=1}^m a_i b_j$ is the sum of the entries in row i. Thus,

$$\begin{aligned}
\sum_{j=1}^m \sum_{i=1}^n a_i b_j &= \sum_{j=1}^m \text{sum of column } j \\
&= \sum_{i=1}^n \text{sum of row } i \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i b_j \;.
\end{aligned}$$

For infinite double series the situation is a bit more complicated, since an infinite series need not converge. However, it is at least true that if either of

$$\sum_{j=1}^m \sum_{i=1}^n |a_i b_j| \quad \text{and} \quad \sum_{i=1}^n \sum_{j=1}^m |a_i b_j|$$

converges, then the series without the absolute values converge and are equal. [This PDF] has much more information on double sequences and series.

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edited Sep 11 '21 at 6:11

answered Mar 26 '13 at 20:13

Brian M. Scott

583k ● 51 ■ 689 ▲ 1134

The pdf link is no longer avaiable :(– drewdles Sep 30 '15 at 16:11

@Anant: I found the new address and fixed the link. – Brian M. Scott Sep 30 '15 at 16:38

@BrianM.Scott it has been removed from that link as well – stateless Apr 7 '21 at 16:35

1 @stateless: I’ve replaced that link with one that works. I’m not positive that it’s the same PDF, but I think that it is, and it does at least appear to be a workable substitute. – Brian M. Scott Apr 7 '21 at 17:38

@BrianM.Scott thanks! – stateless Apr 7 '21 at 19:46

@BrianM.Scott thanks! – stateless Apr 7 '21 at 19:46

11

First of all, by the distributivity of multiplication over addition, the following *is* true:

$$\left(\sum_{l=1}^{m} a_l\right)\left(\sum_{k=1}^{n} b_k\right) = \sum_{l=1}^{m}\left(a_l \sum_{k=1}^{n} b_k\right) = \sum_{l=1}^{m}\sum_{k=1}^{n} a_l b_k$$

This can be seen by writing out the sums explicitly.

This is also true:

$$\sum_{j=1}^{m}\sum_{i=1}^{n} a_i b_j = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j$$

Commutativity is not necessarily involved because each pair of numbers being multiplied together are also done so in the same order. One reason equality holds is because of the commutitivity of addition. Think of an $n \times m$ grid in the xy plane. If the point with co-ordinate (i, j) has the number $a_i b_j$ written on it, the sum of all the numbers on the grid is the

same if we add along the rows first (the left hand sum) or if we add along the columns first (the right hand sum).

When it comes to infinite series, things get a lot more complicated. One thing that is true is that if

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |a_k b_l|$$

converges, then:

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_k b_l = \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} a_k b_l$$

You can follow this link:

http://www.math.ubc.ca/~feldman/m321/twosum.pdf

To see an example of where changing the order **does** matter.

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edited Nov 29 '17 at 11:38
dopexxx
180 8

answered Mar 26 '13 at 20:17
Tom Oldfield
12.2k 1 33 71

Thank you very much, I have another question actually too: Is $\sum_{k,l}^{m} a_k b_l$ equal to $\sum_{k}^{m}\sum_{l}^{m} a_k b_l$? – TestGuest Mar 26 '13 at 22:46

1 @TestGuest Yes, it's just another way of writing the same thing (although if you don't know that k and l both go up to m the left hand version might be a bit vague. Happy to help! – Tom Oldfield Mar 26 '13 at 22:51

shouldn't the first one be:

1

$$\sum_{k=1}^{m} a_k \sum_{k=1}^{n} b_k = \sum_{k=1}^{m} \sum_{l=1}^{n} a_k b_l$$

?

anyway,

$$\sum_{k=1}^{m} a_k = a_1 + \ldots + a_m$$

$$\sum_{k=1}^{n} b_k = b_1 + \ldots + b_n$$

$$\sum_{k=1}^{m} a_k \sum_{k=1}^{n} b_k = (a_1 + \ldots + a_m)(b_1 + \ldots + b_n) =$$

$$(1) = a_1 b_1 + \ldots + a_1 b_n + \ldots + a_m b_1 + \ldots a_m b_n$$

$$\sum_{k=1}^{m} \sum_{l=1}^{n} a_k b_l = \sum_{k=1}^{m} (a_k b_1 + \ldots + a_k b_n) =$$

$$(2) = a_1 b_1 + \ldots + a_1 b_n + \ldots + a_m b_1 + \ldots a_m b_n$$

(1) and (2) looks the same to me

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edited Mar 26 '13 at 20:00

answered Mar 26 '13 at 19:49

Kamil Mikolajczyk
147 3

Hi, welcome to the website! This isn't really an answer I'm afraid, things like this are better off in the comments section underneath the original question. – Tom Oldfield Mar 26 '13 at 19:54

Hi, yeah, I totally agree with you, but I can't find such possibility to put a comment under original question – Kamil Mikolajczyk Mar 26 '13 at 20:01

@Tom: Users with less than 50 reputation points cannot comment on posts they don't own. – hmakholm left over Monica Mar 26 '13 at 21:52

Here is a proof by induction for (2).

0

base case: $n = 1$

$$\sum_{j=1}^{m}\sum_{i=1}^{1} a_i \times b_j = \sum_{j=1}^{m} a_1 \times b_j = \sum_{i=1}^{1}\sum_{j=1}^{m} a_i \times b_j$$

Assume the property holds for $n = k$, and now prove if for $n = k + 1$:

$$\sum_{j=1}^{m}\sum_{i=1}^{(k+1)} a_i \times b_j = \sum_{j=1}^{m}(\sum_{i=1}^{k}(a_i \times b_j) + a_{k+1} \times b_j) = \sum_{j=1}^{m}\sum_{i=1}^{k} a_i \times b_j + \sum_{j=1}^{m} a_{k+1} \times b_j$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{m} a_i \times b_j + \sum_{j=1}^{m} a_{k+1} b_j$$

$$= \sum_{i=1}^{k+1}\sum_{j=1}^{m}(a_i \times b_j)$$

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edited Aug 8 '17 at 22:43

answered Aug 8 '17 at 22:36

Tony

944 7 14