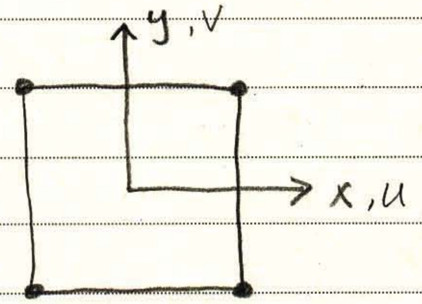
Chap 7. Spurious Kinematic mode & Locking effect

7.1 Spurious Kinematic mode

작은 하중에서 비현실적으로 큰 변위 발생.



4-node plane element에서

$$u = a_1 + a_2 x + a_3 y + a_4 xy$$

$$v = b_1 + b_2 x + b_3 y + b_4 xy$$

Then

$$\varepsilon_{xx} = a_2 + a_4 y$$

$$\varepsilon_{yy} = b_3 + b_4 x$$

$$\varepsilon_{xy} = a_3 + a_4 x + b_2 + b_4 y$$

$$\delta\varepsilon_{xx} = \delta a_2 + \delta a_4 y$$

$$\delta\varepsilon_{yy} = \delta b_3 + \delta b_4 x$$

$$\delta \varepsilon_{xy} = \delta a_3 + \delta a_4 x + \delta b_2 + \delta b_4 y.$$

plane stress 에서.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}.$$

$\therefore\ \sigma_{xx} \sim 1,\ x,\ y$

$\sigma_{yy} \sim 1,\ x,\ y$

$\sigma_{xy} \sim 1,\ x,\ y$

$$\delta U = \int_V \left(\delta\varepsilon_{xx}\sigma_{xx} + \delta\varepsilon_{yy}\sigma_{yy} + \delta\varepsilon_{xy}\sigma_{xy} \right) dV$$

$\delta\varepsilon_{xx}\sigma_{xx} \sim 1,\ x,\ y,\ xy,\ y^2.$

$\delta\varepsilon_{yy}\sigma_{yy} \sim 1,\ x,\ y,\ xy,\ x^2$

$\delta\varepsilon_{xy}\sigma_{xy} \sim 1,\ x,\ y,\ xy,\ x^2,\ y^2.$

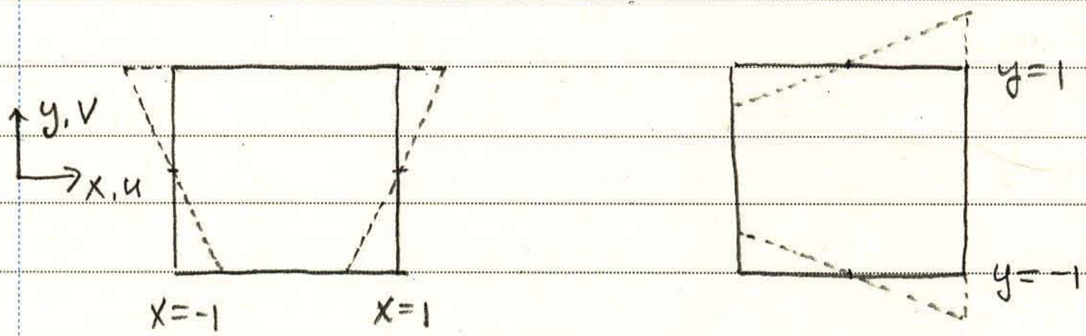
4 HOT

if full integration (2x2 point), $(2n-1) = 3$차 항까지 정확히 적분. → 모든 항을 적분

if reduced integration (1 point), $(2n-1) = 1$차 항까지 정확히 적분 → 4개의 HOT는 정확히 적분 못함.

strain energy 에서 4 HOT 는

$$\left\{ \begin{aligned} u &= a_4 xy \\ v &= b_4 xy \end{aligned} \right\}$$ 에 해당 (spurious mode)



$u = a_4 xy$ $\qquad$ $v = b_4 xy$

4-node plane element 에서 reduced integration 하면

2개의 spurious mode 발생.

(NOTE) disp field 는 $u = a_4 xy$, $v = b_4 xy$ 표시 가능

그러나 δU 계산할때 2개의 mode 에 해당되는 strain

항을 적분 못함

즉, zero strain energy 에서도 2개의 mode 발생.

spurious mode or zero strain energy mode 는

strain energy 없이 발생 가능.

→ singular stiffness matrix

or rank deficiency

or Rows are dependent

check.

eigenvalue analysis.

number of zero eigenvalue for $\underset{\sim}{K}$ (no BC)

= # of rigid body mode

\+ # of spurious mode

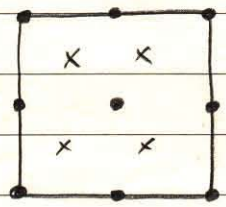
if $\underset{\sim}{K}$에 BC 적용하고, eigenvalue check

\# of zero eigenvalue

= # of spurious mode

spurious mode의 종류

$$\begin{cases} \text{compatible mode} \\ \text{incompatible mode} \end{cases}$$



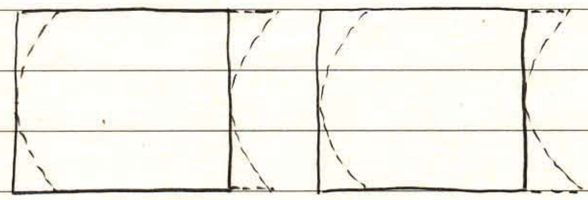
9-node element에서 reduced integration (2×2 적분)

$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^2 y + a_8 xy^2 + a_9 x^2 y^2$$

$$\varepsilon_{xx} = a_2 + 2a_4 x + a_5 y + 2a_7 xy + \underbrace{\boxed{a_8 y^2 + 2a_9 xy^2}}_{\text{제외}}$$

① compatible spurious mode

if $u = a_9 x^2 y^2$ 이 spurious mode.

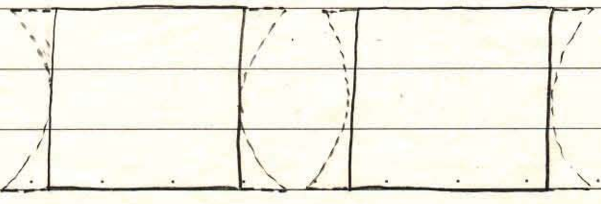


옆의 element와 assemble 해도 계속 존재

→ should be suppressed

② incompatible spurious mode

if $u = a_8 xy^2$ 이 spurious mode.



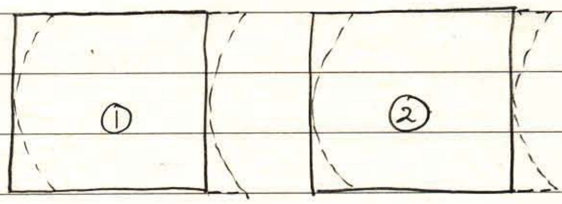
옆의 element와 assemble하면 cancel out

→ practical f.e. mesh에서 suppress할 필요 없음

(참고) y 방향 (변위 v)으로도 2종류의 spurious mode가 발생할 수 있다.

<u>Alternate interpretation</u>

Compatible mode는 증폭되고, incompatible mode는 cancel out 되는 이유



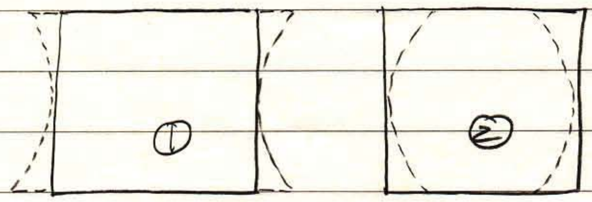
$u = a_9 x^2 y^2$.

두 element의 경계가 일치해야 하므로 (conforming)

element ①, ②의 a_9은 동일

No area change or volume change

→ zero energy mode 가능



$u = a_8 x y^2$.

두 element의 경계가 일치하려면. element ①, ②의

a_8은 크기는 같고 부호는 반대.

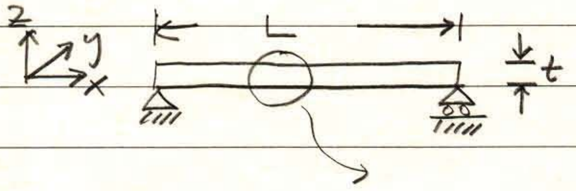
area와 volume change → energy 필요.

zero energy mode 불가능.

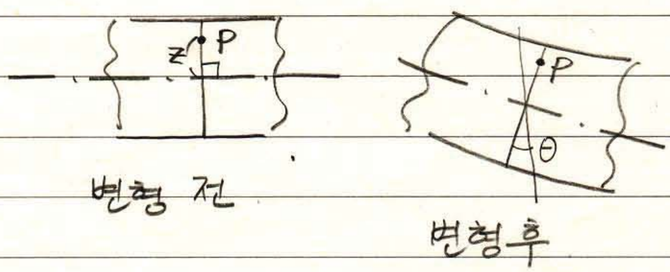
7.2. Locking

(a) Transverse shear locking

bending with transverse shear deformation



E, t = constant



$$u = z\theta$$
$$w = w_0$$

$$\begin{cases} \varepsilon_{xx} = \dfrac{\partial u}{\partial x} = z\dfrac{\partial \theta}{\partial x} \\[2ex] \varepsilon_{xz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} = \theta + \dfrac{\partial w}{\partial x} \neq 0 \end{cases}$$

θ and w are independent variable

$$\sigma_{xx} = E\,\varepsilon_{xx} = E \cdot z \frac{\partial \theta}{\partial x}$$

$$\sigma_{xz} = G\,\varepsilon_{xz} = G\left(\theta + \frac{\partial w}{\partial x}\right)$$

Strain energy U for a beam with rectangular cross-section

$$U = \text{bending energy} + \text{transverse shear energy}$$

$$= \frac{1}{2}\int_V \sigma_{xx}\varepsilon_{xx}\,dV + \frac{1}{2}\int_V \sigma_{xz}\varepsilon_{xz}\,dV$$

$$= \frac{1}{2}\int E z^2 \left(\frac{d\theta}{dx}\right)^2 dx\,dy\,dz + \frac{1}{2}\int G\left(\theta + \frac{\partial w}{\partial x}\right)^2 dx\,dy\,dz$$

$$= \frac{1}{2}\int_0^L EI\left(\frac{\partial\theta}{\partial x}\right)^2 dx + \frac{1}{2}\int GA\left(\theta + \frac{\partial w}{\partial x}\right)^2 dx$$

I = moment of inertia

$= \frac{1}{12}bt^3$

A = 단면적 $= bt$

무차원 x' and w'

$x' = \frac{x}{L}$, $w' = \frac{w}{L}$

$x = Lx'$ $w = Lw'$

$dx = L\,dx'$

$$U = \frac{1}{2}E\left(\frac{1}{12}bt^3\right)\int_0^1 \frac{1}{L^2}\left(\frac{\partial\theta}{\partial x'}\right)^2 L\,dx'$$

$$+ \frac{Gtb}{2}\int_0^1 \left(\theta + \frac{\partial w'}{\partial x'}\right)^2 L\,dx'$$

$$= \frac{1}{2}\frac{Et^3b}{12L}\int_0^1 \left(\frac{\partial\theta}{\partial x'}\right)^2 dx' + \frac{1}{2}GtbL\int_0^1 \left(\theta + \frac{\partial w'}{\partial x'}\right)^2 dx'$$

$$= \frac{1}{2}\frac{Et^3b}{L}\left[\int_0^1 \frac{1}{12}\left(\frac{\partial\theta}{\partial x'}\right)^2 dx' + \frac{G}{E}\left(\frac{L}{t}\right)^2\int_0^1 \left(\theta + \frac{\partial w'}{\partial x'}\right)^2 dx'\right]$$

두께비 $\left(\frac{L}{t}\right)$ 가 커질수록 (thin beam)

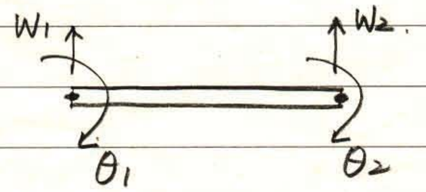
① $\left(\frac{L}{t}\right)^2$ 은 더욱 커지고

② shear strain energy는 점점 작아진다.

$\left(\frac{L}{t}\right)$ 가 아주커지면 $(10^3 \sim 10^5)$, shear strain energy는 very small or almost zero.

$$\left(\theta + \frac{\partial w'}{\partial x'}\right) \to 0 \qquad (1)$$

(Example) 2-node, 4-dof, 2-d beam 요소



$x' = 0$ $\qquad$ $x' = 1.$

$$\theta = a_1 + a_2 x'$$
$$w' = b_1 + b_2 x' \qquad (2)$$

식 (2)를 (1)에 대입

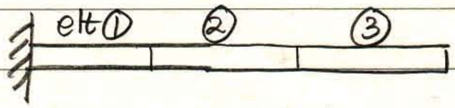
$$a_1 + a_2 x' + b_2 \to 0 \qquad (3)$$

If 2-point 적분 (full integration)

$$\left.\begin{aligned} \therefore\ & a_1 + b_2 \to 0 \\ & a_2 \to 0 \end{aligned}\right\} \text{ 2 constraints among 4 DOF}$$

(NOTE) number of constraints
= number of integration point

clamped B.C. 에서



$x'=0$

at $x'=0$ $\quad w'=0 \rightarrow b_1=0.$

$\theta=0 \rightarrow a_1=0.$

$$\left.\begin{aligned} \therefore\ & a_1=b_1=0 \\ & a_2\rightarrow 0,\ b_2\rightarrow 0 \end{aligned}\right\} \rightarrow \begin{aligned} &\theta\rightarrow 0 \\ &w'\rightarrow 0 \end{aligned}$$

element ① almost does not deform.

같은 방법으로. element ②, ③, ··· almost does not deform

→ transverse shear locking

식(3) 에서 if 1-point 적분 (reduced integration)

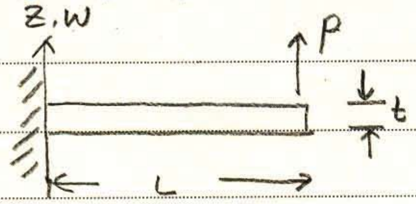
$$\left. a_1 + a_2\frac{1}{2} + b_2 = 0 \right\} \text{ 1 constraint}$$

clamped B.C 에서

at $x'=0$, $w'=0$, $\theta=0$ → $b_1=0$, $a_1=0$

$\therefore$ $a_1=b_1=0$

$\frac{1}{2}a_2+b_2=0$ → allows deformation.



$\frac{EI w_{max}}{PL^3}$	$L/t = 10$	100
Exact with transverse shear strain	0.3353	0.3333
FEM, 2-node, 20-elt, full integ.	0.3037	0.0292
FEM, 2-node, 5-elt, reduced integ.	0.3320	0.3300