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ON A STRESS RESULTANT GEOMETRICALLY EXACT SHELL MODEL. PART I: FORMULATION AND OPTIMAL PARAMETRIZATION

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1. Introduction and overview

1.1. Overview

Over the past two decades computational shell analysis has been, to a large extent, dominated by the so-called *degenerated solid approach*, which finds its point of departure in the paper of Ahmad, Irons and Ziekiewicz [3]. The works of Ramm [40], Parish [39], Hughes and Liu [28, 29], Hughes and Carnoy [30], Bathe and Dvorkin [13], Hallquist, Benson and Goudreau [27], Parks and Stanley [38], and Liu, Law, Lam and Belytschko [33], among many others, constitute representative examples of this methodology carried over in its full generality to the nonlinear regime. The thesis of Stanley [48], and the books of Bathe [12], Hughes [21], and Crisfield [20], offer comprehensive overviews of the degenerated solid approach and related methodologies which involve some type of reduction to a *resultant formulation*. An alternative approach to the development of shell elements is found in the pioneering work of Argyris et al. [6–10], which makes use of the classical matrix displacement method with high order interpolations (5th- and 7th-order polynominals) within the context of the author's *natural approach*.

By contrast, the present work, the first part of a series of papers, constitutes a departure from the aforementioned methodology. In a sense, the proposed approach represents a return to the origins of classical nonlinear shell theory, which has its modern point of departure in the pioneering work of the Cosserats [19], subsequently rediscovered by Ericksen and Truesdell [21], and further elaborated upon by a number of authors; notably Green and Laws [23], Green and Zerna [25], or Cohen and DaSilva [18]. We refer to [35] for a compehensive review including many references to the classical literature and historical overviews, to [4, 5] for a careful analysis the mathematical foundations of classical shell theory, and to [44] for a discussion of the underlying Hamiltonian structure.

Although the hypothesis underlying the degenerated solid approach and classical shell theory are essentially the same, the reduction to resultant form is typically carried out numerically in the former, and analytically in the latter. Conceptually, this appears to be the only essential difference between the two approaches. A point frequently made concerning the degenerated approach is that it avoids the mathematical complexities associated with classical shell theory, and hence is better conditioned for numerical implementation. A main thrust of the present work is to demonstrate that classical shell theory, phrased as one-director Cosserat surface, leads itself to an efficient numerical implementation which is free from mathematical complexities and suitable for large scale computation. As an illustration, Figs. 1, 2, and 3 contain simulations involving extremely large displacements and rotations obtained with the formulation described in this paper. In Part II of this paper, it will be shown that the present approach is able to reproduce the exact solutions of standard benchmark linear problems often used to assess the performance of numerical formulations based on the degenerated solid approach.

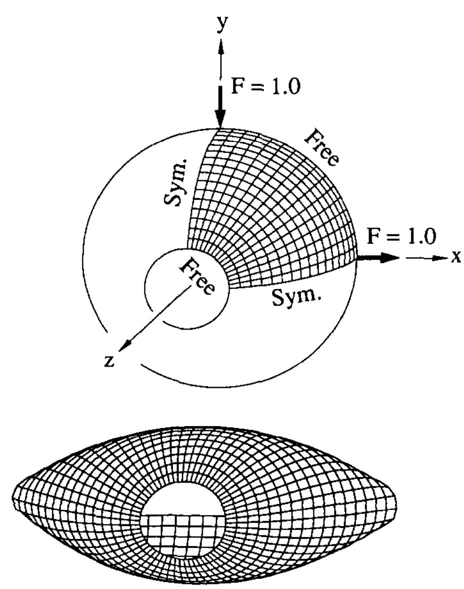

Fig. 1. Large deformation of a pinched hemisphere, a standard benchmark test problem for linear shell elements.

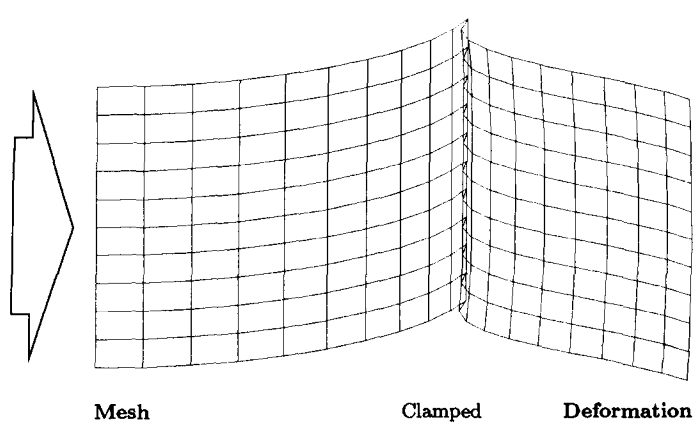

Fig. 2. Large deflection of one quarter of a cylinder shell. The deformation shown (with no magnification) is obtained in one single step loading.

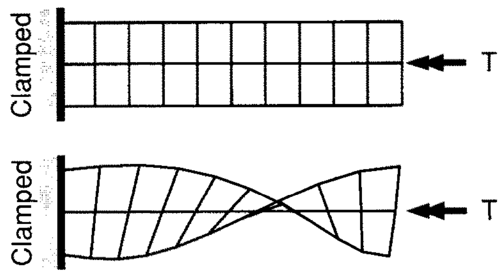

Fig. 3. Numerical simulation involving 180 degree torsion of a thin, initially flat, shell.

To conclude this overview, we observe that most of the existing numerical formulations employing the degenerated approach are inspired by numerical analysis of classical *stress resultant models*; such as the Kirchhoff–Love, Euler, and Timoshenko beam models, and the Poisson–Kirchhoff and the Mindlin–Reissner plate models. Techniques for dealing with phenomena such as *shear and membrane locking* are typically borrowed from the numerical analysis of these models (or related stress-resultant models i.e., Marguere shallow beam and shell equations). As with these classical linear stress-resultant models, we believe that a deeper understanding of the numerical analysis issues involved in finite element shell analysis can be gained only from a careful formulation of the weak form of the momentum equations in resultant form. A case in point is the analysis of the Koiter–Sanders shell model by Bernadou and Boisserie [15].

1.2. Scope of the paper

In this paper, we adopt a classical point of view and regard the shell as an *inextensible* one-director Cosserat surface. The objective is a precise formulation of the local balance laws, the local constitutive equations, and the weak form of the momentum equations in a form suitable for numerical analysis and finite element implementation. Although many expositions of classical theory are available, notably the aforementioned reviews of Naghdi [35, 36] and Antman [50], the present work places special emphasis on the geometric structure underlying stress resultant (Cosserat) shell models. We believe that this geometric point of view is essential in a numerical implementation for the following reasons:

(i) A *singularity-free* parametrization of the *rotation field* is developed by identifying the geometric structure of the configuration space; essentially a differentiable manifold (and not a linear space!) modeled on the unit sphere (S^2) and $\mathbb{R}^3$. This parametrization is *optimal* in the sense that the number of parameters describing the rotation field cannot be further reduced without introducing points of singularity. The situation parallels that found in nonlinear (geometrically exact) rod models; see [39, 41–47]. An important difference, however, concerns the role of *drill rotations* (rotations about the director) which, in sharp contrast with rods, are irrelevant in shells.

(ii) Update procedures for the rotation field, which are exact and preserve objectivity for *any* magnitude of the director rotation increment, can be constructed merely by employing the discrete version of the exponential map in the unit sphere. We address these procedures in detail in Part II of this work.

(iii) Constraints such as inextensibility of the director field are enforced exactly without resorting to penalty methods by exploiting the connection between the unit sphere and a certain subset of the rotation group. In particular, given two vectors in the unit sphere there is a remarkably simple expression for the orthogonal transformation that rotates one vector into the other *without drill*.

(iv) The underlying geometric structure associated with an inextensible one-director Cosserat surface makes the fact that there is no couple stress about the director transparent. This is at variance with some theories based on the use of the "rotation vector," as in [32, 41, 42]; see also the discussion in [24].

Additional noteworthy features emphasized in the present work, which are motivated by computational considerations, are the following:

(v) In classical shell theory, it is customary to resolve the surface displacement and surface director in components relative to a Gaussian (intrinsic) frame. In a computational context, however, it is far more convenient to retain the components of the surface displacement relative to a fixed inertial frame. As for the components of the rotation fields, the most convenient resolution is in terms of a local Cartesian frame, denoted by $\{t_1, t_2, t_3\}$, and obtained by exploiting the correspondence between the unit sphere and the rotation group.

(vi) The weak form of the momentum equations is parametrized in a way that avoids explicit appearance of the Reimannian connection of the mid-surface. In particular, objects such as Christoffel symbols or the second fundamental form, which are not readily accessible in a computational context, *do not* explicitly appear in the formulation.

(vii) The restriction that balance of angular momentum places on the admissible form of the constitutive equations leads to a symmetric form of the consistent tangent operator even away from equilibrium.

(viii) Finally, unnecessary assumptions frequently made in the context of the degenerated approach concerning the form and symmetry of the stress resultants are not made. In particular, only a certain symmetric combination of the stress resultants (known as the *effective stress resultants*) appears in the weak form of the momentum balance equations.

An outline of the remainder of this paper is as follows. First, we develop in detail some geometric preliminaries that play a central role in the formulation of the theory and in its numerical implementation. Next, we focus our attention on the basic kinematics of the model, including the precise (classical) definitions of the resultant linear, angular, and director momentum. We then introduce the local momentum balance equations, formulated in terms of resultants, and make physical definitions of these resultants within the context of the three-dimensional theory. Properly invariant constitutive equations in terms of stress resultants and conjugate strain measures are developed by exploiting the exact expression for the stress power. We conclude our presentation with the formulation of the weak form of the momentum balance equations and the exact expression for the linearized weak form about an arbitrary (not necessarily equilibrated) configuration.

2. Geometric preliminaries. Parametrization

In this section, we summarize some basic properties of the rotation group, SO(3), and the unit sphere, S^2, needed for subsequent developments. For further details we refer to [1, Section 4.1; 2; 17, pp. 181–194].

2.1. The rotation group and its Lie algebra

Following standard notation, we denote by SO(3) the group of orthogonal transformations, i.e.

$$\mathrm{SO}(3) := \{\boldsymbol{\Lambda}\colon \mathbb{R}^3 \to \mathbb{R}^3 \mid \boldsymbol{\Lambda}^{\mathrm{t}} = \boldsymbol{\Lambda}^{-1} \text{ and } \det \boldsymbol{\Lambda} = +1\}\,. \tag{2.1}$$

Any $\boldsymbol{\Lambda} \in \mathrm{SO}(3)$ possesses an eigenvector $\boldsymbol{\psi} \in \mathbb{R}^3$ such that $\boldsymbol{\Lambda\psi} = \boldsymbol{\psi}$. Geometrically, $\boldsymbol{\Lambda}$ represents a rotation about $\boldsymbol{\psi}$. The tangent space to SO(3) at the identity, denoted by $\mathrm{T}_1\mathrm{SO}(3)$, is so(3), the set of skew-symmetric tensors, and is defined as

$$\mathrm{so}(3) = \{\hat{\boldsymbol{\Theta}}\colon \mathbb{R}^3 \to \mathbb{R}^3 \mid \hat{\boldsymbol{\Theta}} + \hat{\boldsymbol{\Theta}}^{\mathrm{t}} = \mathbf{0}\}\,. \tag{2.2}$$

Any $\hat{\boldsymbol{\Theta}} \in \mathrm{so}(3)$ processes an eigenvector $\boldsymbol{\Theta} \in \mathbb{R}^3$ such that $\hat{\boldsymbol{\Theta}}\boldsymbol{\Theta} = \mathbf{0}$. This defines an isomorphism $\hat{}\colon \mathrm{so}(3) \to \mathbb{R}^3$ by the relation

$$\hat{\boldsymbol{\Theta}}\boldsymbol{h} = \boldsymbol{\Theta} \times \boldsymbol{h} \quad \text{for any } \boldsymbol{h} \in \mathbb{R}^3\,. \tag{2.3}$$

In what follows, we denote by $\{\boldsymbol{E}_I\}_{I=1,2,3}$ an *inertial* (fixed) basis, chosen to be the standard basis, i.e.

$$\boldsymbol{E}_1 = \begin{Bmatrix}1\\0\\0\end{Bmatrix}, \quad \boldsymbol{E}_2 = \begin{Bmatrix}0\\1\\0\end{Bmatrix}, \quad \boldsymbol{E}_3 = \begin{Bmatrix}0\\0\\1\end{Bmatrix}. \tag{2.4}$$

Thus, $\boldsymbol{E}_I = \boldsymbol{E}^I$. For any $\hat{\boldsymbol{\Theta}} \in \mathrm{so}(3)$ we have the matrix representation (relative to $\{\boldsymbol{E}_I\}$)

$$[\hat{\Theta}^I_J] = \begin{bmatrix} 0 & -\Theta^3 & \Theta^2 \\ \Theta^3 & 0 & -\Theta^1 \\ -\Theta^2 & \Theta^1 & 0 \end{bmatrix}, \qquad \{\Theta^I\} = \begin{Bmatrix}\Theta^1\\ \Theta^2\\ \Theta^3\end{Bmatrix}. \tag{2.5}$$

Any $\hat{\boldsymbol{\Theta}} \in \mathrm{so}(3)$ represents an *infinitesimal rotation* about $\boldsymbol{\Theta} \in \mathbb{R}^3$. The tangent space at any $\boldsymbol{\Lambda} \in \mathrm{SO}(3)$ is defined (by either left or right translations of so(3)) as[1]

$$\mathrm{T}_{\Lambda}\mathrm{SO}(3) := \{\boldsymbol{\Lambda}\hat{\boldsymbol{\Theta}} \equiv \hat{\boldsymbol{\theta}}\boldsymbol{\Lambda} \mid \hat{\boldsymbol{\Theta}} \in \mathrm{so}(3) \quad \text{or} \quad \hat{\boldsymbol{\theta}} = \boldsymbol{\Lambda}\hat{\boldsymbol{\Theta}}\boldsymbol{\Lambda}^{\mathrm{t}} \in \mathrm{so}(3)\}\,. \tag{2.6}$$

Note that $\boldsymbol{\Lambda}\hat{\boldsymbol{\Theta}} \in \mathrm{T}_{\Lambda}\mathrm{SO}(3)$ can be thought of as a finite rotation superposed onto an infinitesimal rotation. Alternatively, $\hat{\boldsymbol{\theta}}\boldsymbol{\Lambda} \in \mathrm{T}_{\Lambda}\mathrm{SO}(3)$, where $\hat{\boldsymbol{\theta}} = \boldsymbol{\Lambda}\hat{\boldsymbol{\Theta}}\boldsymbol{\Lambda}^{\mathrm{t}} \in \mathrm{so}(3)$ may be thought of as an infinitesimal rotation $\hat{\boldsymbol{\theta}}$ superposed onto a finite rotation.

Finally, the *exponential mapping* exp: $\mathrm{T}_1\mathrm{SO}(3) \to \mathrm{SO}(3)$ maps infinitesimal rotations into finite rotations according to

$$\hat{\boldsymbol{\Theta}} \in \mathrm{so}(3) \ \mapsto\ \exp[\hat{\boldsymbol{\Theta}}] := \sum_{k=0}^{\infty} \frac{1}{k!}\,\hat{\boldsymbol{\Theta}}^k \in \mathrm{SO}(3)\,. \tag{2.7}$$

[1] One speaks of $\hat{\boldsymbol{\theta}}\boldsymbol{\Lambda} \in \mathrm{T}_{\Lambda}\mathrm{SO}(3)$ and $\boldsymbol{\Lambda}\hat{\boldsymbol{\Theta}} \in \mathrm{T}_{\Lambda}\mathrm{SO}(3)$ as the right and left representation of the tangent space of SO(3) at $\boldsymbol{\Lambda}$.

(See Fig. 4 for a geometric illustration of the exponential map.) Remarkably, one has the following closed-form explicit characterization for the exponential map:

$$\exp[\hat{\boldsymbol{\Theta}}] = \mathbf{1} + \frac{\sin\|\boldsymbol{\Theta}\|}{\|\boldsymbol{\Theta}\|}\hat{\boldsymbol{\Theta}} + \tfrac{1}{2}\frac{\sin^2(\frac{1}{2}\|\boldsymbol{\Theta}\|)}{(\frac{1}{2}\|\boldsymbol{\Theta}\|)^2}\hat{\boldsymbol{\Theta}}^2 . \tag{2.8a}$$

Alternatively, since $\hat{\boldsymbol{\Theta}}^2\boldsymbol{h} = \boldsymbol{\Theta}\times(\boldsymbol{\Theta}\times\boldsymbol{h}) = (\boldsymbol{\Theta}\cdot\boldsymbol{h})\boldsymbol{\Theta} - \|\boldsymbol{\Theta}\|^2\boldsymbol{h}$, making use of the half-angle formulae we have the following equivalent expression for the exponential map in SO(3):

$$\exp[\hat{\boldsymbol{\Theta}}] = \cos(\|\boldsymbol{\Theta}\|)\mathbf{1} + \sin\|\boldsymbol{\Theta}\|\hat{\boldsymbol{e}} + [1-\cos(\|\boldsymbol{\Theta}\|)]\boldsymbol{e}\otimes\boldsymbol{e} , \tag{2.8b}$$

where $\boldsymbol{e} := \boldsymbol{\Theta}/\|\boldsymbol{\Theta}\|$ and $\hat{\boldsymbol{e}}$ is the skew-symmetric tensor defined by the isomorphism (2.3).

Formulae (2.8a), (2.8b) are often accredited to Rodrigues (see [22, p. 165; 6] and play a central role in our developments. For a historical review, see [14].

2.2. The unit sphere S^2*. Relation to* SO(3)

In the development of shell theory as an inextensible one-director Cosserat surface, the unit sphere, denoted by S^2, plays a central role. We set

$$S^2 := \{\boldsymbol{t}\in\mathbb{R}^3 \mid \|\boldsymbol{t}\| = 1\} . \tag{2.9}$$

The tangent space at $\boldsymbol{t}\in S^2$ is the linear space of vectors tangent to curves $\varepsilon \mapsto \boldsymbol{t}_\varepsilon \in S^2$, the $\boldsymbol{t}_\varepsilon|_{\varepsilon=0} = \boldsymbol{t}$, and is obtained by noting that

$$\boldsymbol{t}_\varepsilon\cdot\boldsymbol{t}_\varepsilon = 1 \Rightarrow \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\|\boldsymbol{t}_\varepsilon\|^2 = \boldsymbol{t}\cdot\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}\boldsymbol{t}_\varepsilon = 0 ; \tag{2.10}$$

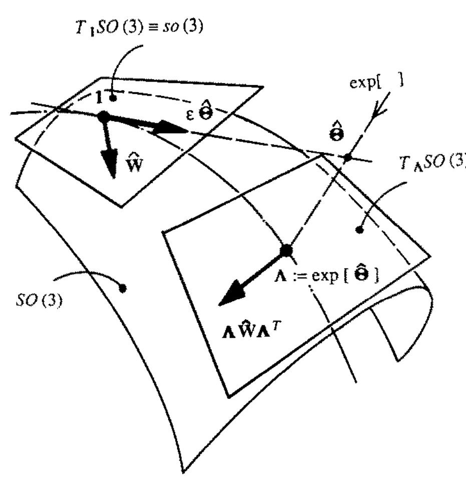

Fig. 4. The special orthogonal group SO(3) and its tangent spaces. The exponential mapping exp: $T_1SO(3)\to SO(3)$ maps skew-symmetric tensors into orthogonal tensors.

consequently, $\mathrm{T}_t\mathrm{S}^2$ is defined as

$$\mathrm{T}_t\mathrm{S}^2 := \{\bar{\boldsymbol{w}} \in \mathbb{R}^3 \mid \bar{\boldsymbol{w}} \cdot \boldsymbol{t} = 0\} . \tag{2.11}$$

The unit sphere S^2 is in a one-to-one correspondence with a subset $\mathrm{S}^2_E \subset \mathrm{SO}(3)$ defined as follows. Let $\boldsymbol{E} \in \mathbb{R}^3$ be a *fixed* but otherwise arbitrary vector. Set

$$\mathrm{S}^2_E := \{\boldsymbol{\Lambda} \in \mathrm{SO}(3) \mid \text{where } \boldsymbol{\psi} \in \mathbb{R}^3, \text{ such that } \boldsymbol{\Lambda}\boldsymbol{\psi} = \boldsymbol{\psi}, \text{ satisfies } \boldsymbol{\psi} \cdot \boldsymbol{E} = 0\} ; \tag{2.12}$$

that is, S^2_E is the set of (finite) rotations whose rotation axis is perpendicular to $\boldsymbol{E}$. The tangent space $\mathrm{T}_1\mathrm{S}^2_E$ at the identity $\mathbf{1} \in \mathrm{S}^2_E$ is then given by

$$\mathrm{T}_1\mathrm{S}^2_E := \{\hat{\boldsymbol{\Theta}} \in \mathrm{so}(3) \mid \boldsymbol{\Theta} \cdot \boldsymbol{E} = 0\} . \tag{2.13}$$

The tangent space at any arbitrary element $\boldsymbol{\Lambda} \in \mathrm{S}^2_E$ is defined by either left or right translations of the tangent space at the identity, $\mathrm{T}_1\mathrm{S}^2_E$, by the expression

$$\mathrm{T}_\Lambda\mathrm{S}^2_E := \{\boldsymbol{\Lambda}\hat{\boldsymbol{\Theta}} \equiv \hat{\boldsymbol{\theta}}\boldsymbol{\Lambda} \mid \hat{\boldsymbol{\Theta}}, \hat{\boldsymbol{\theta}} \in \mathrm{so}(3) \text{ and } \boldsymbol{\theta} \cdot \boldsymbol{\Lambda}\boldsymbol{E} = \boldsymbol{\Theta} \cdot \boldsymbol{E} = 0\} . \tag{2.14}$$

Next, we prove a fundamental result concerning the relation between S^2 and S^2_E.

PROPOSITION 2.1. Given any two vectors $\boldsymbol{E}, \boldsymbol{t} \in \mathrm{S}^2$, with $\boldsymbol{t} \neq -\boldsymbol{E}$ there exists a unique $\boldsymbol{\Lambda} \in \mathrm{S}^2_E$ such that

$$\boldsymbol{t} = \boldsymbol{\Lambda}\boldsymbol{E} , \tag{2.15a}$$

where

$$\boldsymbol{\Lambda} := (\boldsymbol{t} \cdot \boldsymbol{E})\mathbf{1} + [\widehat{\boldsymbol{E} \times \boldsymbol{t}}] + \frac{1}{1 + \boldsymbol{t} \cdot \boldsymbol{E}} (\boldsymbol{E} \times \boldsymbol{t}) \otimes (\boldsymbol{E} \times \boldsymbol{t}) . \tag{2.15b}$$

PROOF. Let $\hat{\boldsymbol{\Theta}} \in \mathrm{T}_1\mathrm{S}^2_E$, so that $\boldsymbol{\Theta} \cdot \boldsymbol{E} = 0$. We show that $\hat{\boldsymbol{\Theta}} \in \mathrm{T}_1\mathrm{S}^2_E$ is uniquely determined from $\boldsymbol{t} \in \mathrm{S}^2$ (and $\boldsymbol{E} \in \mathrm{S}^2$) and that (2.15b) follows from (2.8b). Let $\boldsymbol{t} = \boldsymbol{\Lambda}\boldsymbol{E}$. Since $\boldsymbol{e} \cdot \boldsymbol{E} = 0$, where $\boldsymbol{e} := \boldsymbol{\Theta}/\Theta$ and $\Theta := \|\boldsymbol{\Theta}\|$, from (2.8b) we let $\boldsymbol{\Lambda} = \exp[\hat{\boldsymbol{\Theta}}]$. Thus, we have

$$\cos \Theta = \boldsymbol{t} \cdot \boldsymbol{E} \;\Rightarrow\; \Theta = \cos^{-1}(\boldsymbol{t} \cdot \boldsymbol{E}) . \tag{2.16}$$

It also follows from (2.8b) that $\boldsymbol{t} = \boldsymbol{\Lambda}\boldsymbol{E} = \cos \Theta\boldsymbol{E} + \sin \Theta\boldsymbol{e} \times \boldsymbol{E}$. Consequently, we have

$$\boldsymbol{E} \times \boldsymbol{t} = \sin \Theta\boldsymbol{E} \times (\boldsymbol{e} \times \boldsymbol{E}) = \sin \Theta[\boldsymbol{e} - (\boldsymbol{E} \cdot \boldsymbol{e})\boldsymbol{E}] = \sin \Theta\boldsymbol{e} , \tag{2.17}$$

From $\boldsymbol{e}$ and Θ, $\boldsymbol{\Theta}$, or equivalently $\hat{\boldsymbol{\Theta}}$, is determined and the result follows by substituting (2.16) and (2.17) into (2.8b). Uniqueness of $\boldsymbol{\Lambda}$ follows at once from the construction. □

The following correspondence between tangent fields in the unit sphere, S^2, and tangent fields in S^2_E also plays an important role in subsequent developments of the shell theory.

COROLLARY 2.2. $\mathrm{T}_\Lambda \mathrm{S}^2_E$ *and* $\mathrm{T}_t\mathrm{S}^2$ *are in one-to-one correspondence through the isomorphism*

$$\hat{\boldsymbol{\omega}}\boldsymbol{\Lambda} \in \mathrm{T}_\Lambda \mathrm{S}^2_E \mapsto \bar{\boldsymbol{\omega}} \in \mathrm{T}_t\mathrm{S}^2 \quad \textit{with } \bar{\boldsymbol{\omega}} = \boldsymbol{\omega} \times \boldsymbol{t} . \tag{2.18}$$

PROOF. (*i*) Let $\hat{\boldsymbol{\omega}}\boldsymbol{\Lambda} \in \mathrm{T}_\Lambda \mathrm{S}^2_E$. Hence $\boldsymbol{\omega} \cdot \boldsymbol{\Lambda E} = 0$. Define $\bar{\boldsymbol{\omega}} \in \mathbb{R}^3$ through the relation $\bar{\boldsymbol{\omega}} := \boldsymbol{\omega} \times \boldsymbol{t}$, where $\boldsymbol{t} = \boldsymbol{\Lambda E}$. Then $\boldsymbol{t} \cdot \bar{\boldsymbol{\omega}} = 0$ and $\bar{\boldsymbol{\omega}} \in \mathrm{T}_t\mathrm{S}^2$.

(ii) Conversely, let $\bar{\boldsymbol{\omega}} \in \mathrm{T}_t\mathrm{S}^2$, so that $\bar{\boldsymbol{\omega}} \cdot \boldsymbol{t} = 0$. Define $\boldsymbol{\omega} \in \mathbb{R}^3$ through the relation $\boldsymbol{\omega} := \boldsymbol{t} \times \bar{\boldsymbol{\omega}}$. Since

$$\boldsymbol{\omega} \cdot \boldsymbol{\Lambda E} = \boldsymbol{\omega} \cdot \boldsymbol{t} = (\boldsymbol{t} \times \bar{\boldsymbol{\omega}}) \cdot \boldsymbol{t} = 0 \Rightarrow \hat{\boldsymbol{\omega}}\boldsymbol{\Lambda} \in \mathrm{T}_\Lambda \mathrm{S}^2_E , \tag{2.19}$$

the result follows. □

REMARK 2.3. By particularizing (2.18) to the identity, we obtain the one-to-one correspondence between $\mathrm{T}_1\mathrm{S}^2_E$ and $\mathrm{T}_E\mathrm{S}^2$

$$\hat{\boldsymbol{\Omega}} \in \mathrm{T}_1\mathrm{S}^2_E \mapsto \bar{\boldsymbol{\Omega}} \in \mathrm{T}_E\mathrm{S}^2 \quad \text{with } \bar{\boldsymbol{\Omega}} = \boldsymbol{\Omega} \times \boldsymbol{E} , \tag{2.20}$$

where $\boldsymbol{\Omega} \cdot \boldsymbol{E} = \bar{\boldsymbol{\Omega}} \cdot \boldsymbol{E} = 0$. □

A geometric illustration of the preceding concepts is contained in Fig. 5.

2.3. *The exponential map in S^2*

Making use of the correspondence established in Proposition 2.1 between the unit sphere S^2 and the subset S^2_t (or S^2_E) $\subset \mathrm{SO}(3)$, we derive below the *closed-form* expression for the exponential map in S^2.

Let $\boldsymbol{t} \in \mathrm{S}^2$ be any fixed (but otherwise arbitrary) vector in S^2. We have the following.

PROPOSITION 2.4. The following closed-form expression for the exponential map $\exp_t : \mathrm{T}_t\mathrm{S}^2 \to \mathrm{S}^2$ *holds:*

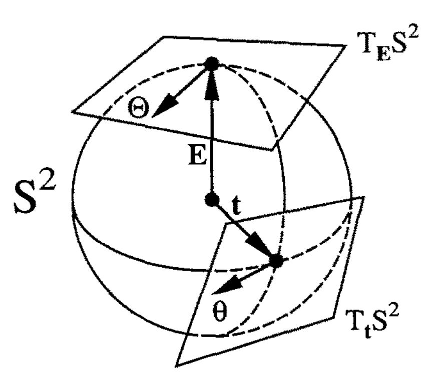

Fig. 5. The S^2-sphere and its tangent spaces. $\boldsymbol{\Lambda} \in \mathrm{S}^2_E$ is the orthogonal transformation mapping $\boldsymbol{E} \mapsto \boldsymbol{t} = \boldsymbol{\Lambda E} \in \mathrm{S}^2$.

$$\exp_t[\bar{\boldsymbol{\theta}}] = \cos\|\bar{\boldsymbol{\theta}}\| \boldsymbol{t} + \frac{\sin\|\bar{\boldsymbol{\theta}}\|}{\|\bar{\boldsymbol{\theta}}\|} \bar{\boldsymbol{\theta}} , \tag{2.21}$$

where $\bar{\boldsymbol{\theta}} \in \mathrm{T}_t\mathrm{S}^2$.

PROOF. By Corollary 2.2, $\boldsymbol{\theta} := \boldsymbol{t} \times \bar{\boldsymbol{\theta}}$ is such that $\hat{\boldsymbol{\theta}} \in \mathrm{T}_1\mathrm{S}_t^2$. Consequently, $\boldsymbol{\Lambda} := \exp[\hat{\boldsymbol{\theta}}]$ is in S_t^2. By Proposition 2.1, this defines a unique $\boldsymbol{t}' \in \mathrm{S}^2$ through the relation $\boldsymbol{t}' = \boldsymbol{\Lambda t}$. Thus set

$$\boldsymbol{t}' \equiv \exp[\hat{\boldsymbol{\theta}}]\boldsymbol{t} =: \exp_t[\bar{\boldsymbol{\theta}}] . \tag{2.22}$$

By Rodrigues' formula (2.8b), since $\boldsymbol{\theta} \cdot \boldsymbol{t} = \bar{\boldsymbol{\theta}} \cdot \boldsymbol{t} = 0$, we have

$$\exp_t[\bar{\boldsymbol{\theta}}] = \cos\|\boldsymbol{\theta}\| \boldsymbol{t} + \frac{\sin\|\boldsymbol{\theta}\|}{\|\boldsymbol{\theta}\|} \boldsymbol{\theta} \times \boldsymbol{t} . \tag{2.23}$$

The result then follows by noting that $\|\boldsymbol{\theta}\| = \|\bar{\boldsymbol{\theta}}\|$ along with $\bar{\boldsymbol{\theta}} = \boldsymbol{\theta} \times \boldsymbol{t}$. □

REMARK 2.5. In particular, Proposition 2.4 may be used to parameterize the unit sphere as follows. Choose $\{\boldsymbol{E}_I\}_{I=1,2,3}$ such that $\boldsymbol{E}_3 = \boldsymbol{E}$ defines the north pole. We then have the following characterization of $\exp_E$: $\mathrm{T}_E\mathrm{S}^2 \rightarrow \mathrm{S}^2$, for $\boldsymbol{E} \in \mathrm{S}^2$,

$$\bar{\boldsymbol{\Theta}} \in \mathrm{T}_E\mathrm{S}^2 \;\mapsto\; \exp_E[\bar{\boldsymbol{\Theta}}] := \cos\|\bar{\boldsymbol{\Theta}}\| \boldsymbol{E} + \frac{\sin\|\bar{\boldsymbol{\Theta}}\|}{\|\bar{\boldsymbol{\Theta}}\|} \bar{\boldsymbol{\Theta}} \in \mathrm{S}^2 , \tag{2.24}$$

where $\boldsymbol{\Theta} = \boldsymbol{E} \times \bar{\boldsymbol{\Theta}} = \Theta^1\boldsymbol{E}_1 + \Theta^2\boldsymbol{E}_2$ with $\hat{\boldsymbol{\Theta}} \in \mathrm{T}_1\mathrm{S}_E^2$. □

The geometric notions in this section are key to the development of a well-conditioned singularity-free parametrization of the director field $\boldsymbol{t} \in \mathrm{S}^2$ and its associated orthogonal transformation $\boldsymbol{\Lambda}$. We shall address these issues in Part II of this work where computational aspects are examined in detail.

3. Kinematic description of the shell

In this section we consider the basic kinematic results underlying the shell model.

3.1. The basic kinematic assumption. Configurations

To state precisely the basic kinematic assumption, we define the set $\mathscr{C}$ (a differentiable manifold) as

$$\mathscr{C} := \{(\boldsymbol{\varphi}, \boldsymbol{t}) \colon \mathscr{A} \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \times \mathrm{S}^2\} . \tag{3.1}$$

Here, $\mathscr{A} \subset \mathbb{R}^2$ is an open set with smooth boundary $\partial\mathscr{A}$, compact closure $\bar{\mathscr{A}}$, and points denoted by $\boldsymbol{\xi} \in \mathcal{A}$. We set

$$\boldsymbol{\xi} = \xi^1 \boldsymbol{E}_1 + \xi^2 \boldsymbol{E}_2\,, \quad (\xi^1, \xi^2) \in \mathbb{R}^2\,. \tag{3.2}$$

The basic kinematic assumption is that of an *inextensible one-director Cosserat surface.* Accordingly, any configuration of the shell is described by a pair $(\boldsymbol{\varphi}, \boldsymbol{t}) \in \mathscr{C}$, where:

(i) The map $\boldsymbol{\varphi}$: $\mathcal{A} \to \mathbb{R}^3$ defines the position of the mid-surface of the shell.

(ii) The map $\boldsymbol{t}$: $\mathcal{A} \to S^2$ defines a unit vector field at each point of the surface, referred to as the director (or fiber) field.

One is then led to the following kinematic hypothesis.

BASIC ASSUMPTION. Any configuration of the shell $\mathscr{S} \subset \mathbb{R}^3$ is assumed to be defined as

$$\mathscr{S} := \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x} = \boldsymbol{\varphi} + \xi \boldsymbol{t} \text{ where } (\boldsymbol{\varphi}, \boldsymbol{t}) \in \mathscr{C} \text{ and } \xi \in [h^-, h^+]\}\,. \tag{3.3}$$

In particular, the *reference configuration* is exactly described as[2]

$$\mathscr{B} := \{\boldsymbol{x}_0 \in \mathbb{R}^3 \mid \boldsymbol{x}_0 = \boldsymbol{\varphi}_0 + \xi \boldsymbol{t}_0 \text{ with } (\boldsymbol{\varphi}_0, \boldsymbol{t}_0) \in \mathscr{C} \text{ and } \xi \in [h^-, h^+]\}\,. \tag{3.4}$$

Here, $[h^-, h^+] \subset \mathbb{R}$, with $h^+ > h^-$ and $h = h^+ - h^-$ is the thickness of the shell. □

We shall often use the notation

$$\boldsymbol{x} = \boldsymbol{\Phi}(\xi^1, \xi^2, \xi) := \boldsymbol{\varphi}(\xi^1, \xi^2) + \xi \boldsymbol{t}(\xi^1, \xi^2)\,. \tag{3.5}$$

It follows that $\boldsymbol{\Phi}$: $\mathcal{A} \times [h^-, h^+] \to \mathbb{R}^3$. A *deformation* of the shell, then, is a mapping

$$\boldsymbol{\chi}: \mathscr{B} \to \mathscr{S}\,, \qquad \boldsymbol{\chi} := \boldsymbol{\Phi} \circ \boldsymbol{\Phi}_0^{-1}\,. \tag{3.6}$$

An illustration of the basic kinematics of the shell model is contained in Fig. 6.

3.2. The tangent map at a configuration. Deformation gradient

Given a configuration $\boldsymbol{\Phi}$: $\mathcal{A} \times [h^-, h^+] \to \mathbb{R}^3$ the *tangent map* is the Frechet derivative, denoted by $\nabla \boldsymbol{\Phi}$. Relative to $\{\boldsymbol{E}_I\}_{I=1,2,3}$ one has (with $\xi^3 \equiv \xi$)

$$\nabla \boldsymbol{\Phi} := \frac{\partial \boldsymbol{\Phi}}{\partial \xi^I} \otimes \boldsymbol{E}^I \equiv \boldsymbol{g}_I \otimes \boldsymbol{E}^I\,. \tag{3.7}$$

One refers to $\boldsymbol{g}_I = \partial \boldsymbol{\Phi} / \partial \xi^I$ as the *convected* basis. By the chain rule, the *deformation gradient* associated with a deformation $\boldsymbol{\chi}$: $\mathscr{B} \to \mathscr{S}$ is

$$\boldsymbol{F} \equiv T\boldsymbol{\chi} := \nabla \boldsymbol{\Phi} \circ (\nabla \boldsymbol{\Phi}_0)^{-1}\,. \tag{3.8}$$

An explicit expression for $\nabla \boldsymbol{\Phi}$ is contained in the following.

[2]This parameterization is often termed *normal coordinate chart*; i.e. see [30].

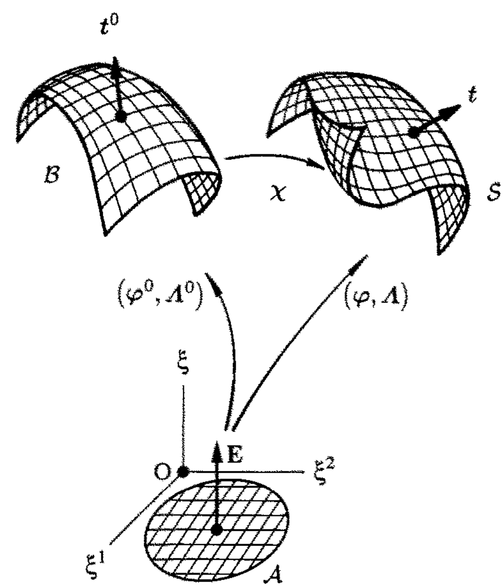

Fig. 6. Geometry of an *inextensible* one-director *Cosserat surface*.

PROPOSITION 3.1. The tangent map $\nabla\boldsymbol{\Phi}$ *associated with* $\boldsymbol{\Phi}$: $\mathcal{A} \times [h^-, h^+] \rightarrow \mathbb{R}^3$ *is given by*

$$\begin{aligned}\nabla\boldsymbol{\Phi} &= (\boldsymbol{\varphi}_{,\alpha} + \xi\boldsymbol{t}_{,\alpha}) \otimes \boldsymbol{E}^\alpha + \boldsymbol{t} \otimes \boldsymbol{E} \\ &= \boldsymbol{\Lambda}[(\boldsymbol{\Gamma}_\alpha + \xi\bar{\boldsymbol{K}}_\alpha) \otimes \boldsymbol{E}^\alpha + \boldsymbol{E}_3 \otimes \boldsymbol{E}^3]\,,\end{aligned} \tag{3.9}$$

where

$$\boldsymbol{\Gamma}_\alpha := \boldsymbol{\Lambda}^{\mathrm{t}}\boldsymbol{\varphi}_{,\alpha} \quad \text{and} \quad \bar{\boldsymbol{K}}_\alpha = \boldsymbol{\Lambda}^{\mathrm{t}}\boldsymbol{t}_{,\alpha} \tag{3.10}$$

are referred to as the rotated strains, and $\boldsymbol{\Lambda}$: $\mathcal{A} \rightarrow \mathrm{S}^2_E$ *is such that* $\boldsymbol{t} = \boldsymbol{\Lambda}\boldsymbol{E}$.

PROOF. From (3.5) and (3.7) one has immediately

$$\nabla\boldsymbol{\Phi} = (\boldsymbol{\varphi}_{,\alpha} + \xi\boldsymbol{t}_{,\alpha}) \otimes \boldsymbol{E}^\alpha + \boldsymbol{t} \otimes \boldsymbol{E}\,. \tag{3.11}$$

Since by Proposition 2.1 we have $\boldsymbol{t} = \boldsymbol{\Lambda}\boldsymbol{E}$, where $\boldsymbol{E}_3 = \boldsymbol{E}^3 \equiv \boldsymbol{E}$, substitution of (3.10) into (3.11) yields the result. □

REMARKS 3.2. (1) In the assumption of "small strains" and large rotations one has $\nabla\boldsymbol{\Phi} \simeq \boldsymbol{\Lambda}$, i.e. an orthogonal transformation. Such an assumption will not be made here.

(2) In addition to the *convected basis* $\{\boldsymbol{g}_I\}_{I=1,2,3}$ (where $\boldsymbol{g}_3 \equiv \boldsymbol{t}$) one defines the *reciprocal basis* $\{\boldsymbol{g}^I\}_{I=1,2,3}$ by the standard relation $\boldsymbol{g}_I \cdot \boldsymbol{g}^J = \delta^J_I$. Thus

$$\boldsymbol{g}_I = \nabla\boldsymbol{\Phi}\boldsymbol{E}_I \quad \text{and} \quad \boldsymbol{g}^I = \nabla\boldsymbol{\Phi}^{-\mathrm{t}}\boldsymbol{E}^I\,. \tag{3.12}$$

(3) We shall use the following notation:

$$j := \det[\nabla\boldsymbol{\Phi}]\,, \qquad j_0 := \det[\nabla\boldsymbol{\Phi}_0]\,, \qquad J = j/j_0\,, \tag{3.13}$$

where $J := \det[\boldsymbol{F}]$ is the Jacobian of the deformation gradient which is given by (3.8).

(4) Observe that while $\boldsymbol{g}_3 = \boldsymbol{t}$, $\boldsymbol{g}^3$ is given by

$$\boldsymbol{g}^3 = \frac{1}{j}\,\boldsymbol{g}_1 \times \boldsymbol{g}_2\,. \tag{3.14}$$

Thus, $\|\boldsymbol{t}\| \equiv \|\boldsymbol{g}_3\| = 1$, and $j = [\boldsymbol{g}_1 \times \boldsymbol{g}_2] \cdot \boldsymbol{g}_3$, but $\|\boldsymbol{g}^3\| \neq 1$. □

3.3. Reference frames on the mid-surface

In addition to the fixed inertial frame $\{\boldsymbol{E}_I\}_{I=1,2,3}$, we define two reference frames on the mid-surface that play an important role in the subsequent developments.

(i) *Surface convected frame*. Denote by $\{\boldsymbol{a}_I\}_{I=1,2,3}$ the surface convected frame defined as $\boldsymbol{a}_I = \boldsymbol{g}_I|_{\xi=0}$. Note that $\{\boldsymbol{a}_\alpha\}_{\alpha=1,2}$ span the tangent space to the mid-surface. It follows from (3.5) and (3.7) that

$$\boldsymbol{a}_\alpha = \boldsymbol{\varphi}_{,\alpha} \quad \text{and} \quad \boldsymbol{a}_3 = \boldsymbol{g}_3 = \boldsymbol{t}\,. \tag{3.15}$$

The *metric tensor* (first fundamental form) on the reference surface is then

$$\boldsymbol{a} = \alpha_{\alpha\beta}\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta\,, \qquad a_{\alpha\beta} = \boldsymbol{\varphi}_{,\alpha} \cdot \boldsymbol{\varphi}_{,\beta}\,, \tag{3.16}$$

where $\{\boldsymbol{a}^I\}_{I=1,2,3}$ denotes the *dual surface convected* basis defined by the standard relation $\boldsymbol{a}_I \cdot \boldsymbol{a}^J = \delta^J_I$.

(ii) *Director orthogonal frame*. Denote by $\{\boldsymbol{t}_I\}_{I=1,2,3}$ the director orthogonal frame defined through the orthogonal transformation $\boldsymbol{\Lambda}$: $\mathscr{A} \subset \mathbb{R}^2 \to \mathrm{S}^2_E$ as

$$\boldsymbol{t}_\alpha = \boldsymbol{\Lambda}\boldsymbol{E}_\alpha\,, \qquad \boldsymbol{t}_3 = \boldsymbol{\Lambda}\boldsymbol{E}_3 \equiv \boldsymbol{t}\,. \tag{3.17}$$

Note that $\{\boldsymbol{t}_I\}_{I=1,2,3}$ is the *orthonormal basis* which, by virtue of (3.9), is the "closest" to $\{\boldsymbol{a}_I\}_{I=1,2,3}$. In addition, $\{\boldsymbol{t}_1, \boldsymbol{t}_2\}$ span $\mathrm{T}_t\mathrm{S}^2$, the tangent space normal to $\boldsymbol{t}$, since $\boldsymbol{t} \cdot \boldsymbol{t}_\alpha = 0$.

REMARK 3.3. Our definition of the orthogonal basis $\{\boldsymbol{t}_I\}_{I=1,2,3}$ is intrinsic, and is motivated by expression (3.9) for $\nabla\boldsymbol{\Phi}$. This definition bypasses ad-hoc constructions often made in the computational literature; see [31, Ch. 6]. □

These frames are illustrated in Fig. 7. Finally, we recall that the element of mid-surface area is given by the differential (two-form)

$$\mathrm{d}\mathscr{A} := \boldsymbol{a}_1 \times \boldsymbol{a}_2\, \mathrm{d}\xi^1\, \mathrm{d}\xi^2\,. \tag{3.18a}$$

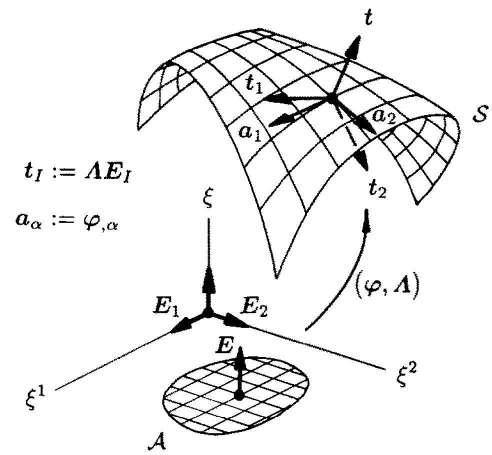

Fig. 7. Reference frames on the mid-surface.

We use the notation

$$\bar{j}_0 := \|\boldsymbol{a}_{01} \times \boldsymbol{a}_{02}\| , \qquad \bar{j} := \|\boldsymbol{a}_1 \times \boldsymbol{a}_2\| \quad \text{and} \quad \bar{J} := \bar{j}/\bar{j}_0 , \tag{3.18b}$$

to designate the *mid-surface Jacobians* in $\mathscr{B}$ and $\mathscr{S}$, and the relative mid-surface Jacobian, respectively.

3.4. Resultant linear, angular, and director momentum

We conclude this section with a derivation of the expressions for resultant linear, angular, and director momentum.

A *motion* is a curve of configurations; that is, a mapping $\boldsymbol{\Phi}_t$: $\mathscr{A} \times [h^-, h^+] \times \mathbb{R}_+ \to \mathbb{R}^3$. Associated with a motion, we have the mapping $t \mapsto (\varphi_t, \boldsymbol{t}_t) \in \mathscr{C}$, which defines the motion of the mid-surface and the director.

3.4.1. Angular velocity of the director field

By Proposition 2.1, there is a unique $\boldsymbol{\Lambda}_t$: $\mathscr{A} \times \mathbb{R}_+ \to \mathrm{S}^2_E$ such that $\boldsymbol{t}_t = \boldsymbol{\Lambda}_t \boldsymbol{E}$. Time differentiation yields

$$\dot{\boldsymbol{t}}_t = \dot{\boldsymbol{\Lambda}}_t \boldsymbol{E} = \boldsymbol{\Lambda}_t \hat{\boldsymbol{W}}_t \boldsymbol{E} = \hat{\boldsymbol{w}}_t \boldsymbol{t} , \tag{3.19}$$

where

$$\hat{\boldsymbol{W}}_t = \boldsymbol{\Lambda}_t^{\mathrm{t}} \dot{\boldsymbol{\Lambda}}_t \quad \text{and} \quad \hat{\boldsymbol{w}}_t := \dot{\boldsymbol{\Lambda}}_t \boldsymbol{\Lambda}_t^{\mathrm{t}} \tag{3.20}$$

are skew-symmetric tensors. Observe that $\hat{\boldsymbol{w}}_t \boldsymbol{\Lambda}_t \in \mathrm{T}_\Lambda \mathrm{S}^2_E$ and $\hat{\boldsymbol{W}}_t \in \mathrm{T}_1 \mathrm{S}^2_E$, whereas $\dot{\boldsymbol{t}} \in \mathrm{T}_t \mathrm{S}^2$ since $\boldsymbol{t}_t \cdot \dot{\boldsymbol{t}}_t = 0$. Consequently

$$\dot{\boldsymbol{t}}_t = \boldsymbol{w}_t \times \boldsymbol{t}_t = \boldsymbol{\Lambda}_t[\boldsymbol{W}_t \times \boldsymbol{E}] , \tag{3.21a}$$

with

$$\boldsymbol{w}_t \cdot \boldsymbol{t}_t = 0 \quad \text{and} \quad \boldsymbol{W}_t \cdot \boldsymbol{E} = 0 . \tag{3.21b}$$

One refers to $\boldsymbol{w}_t$ and $\boldsymbol{W}_t$ as the *spatial* and *rotated* velocity of the director field, respectively.

3.4.2. Density and inertia

Let $\rho_0(\xi^1, \xi^2, \xi)$ and $\rho(\xi^1, \xi^2, \xi, t)$ be the mass density in the reference and current configurations, $\mathscr{B}$ and $\mathscr{S}$, respectively. Recall that conservation of mass requires $\rho_0 = J\rho$, where $J = \det \boldsymbol{F}_t \equiv j/j_0$ is the Jacobian of the deformation gradient $\boldsymbol{F}_t = [\nabla \boldsymbol{\Phi}_t][\nabla \boldsymbol{\Phi}_0]^{-1}$ associated with the motion $\boldsymbol{\chi}_t = \boldsymbol{\Phi}_t \circ \boldsymbol{\Phi}_0^{-1}$: $\mathscr{B} \to \mathscr{S}$. We select the mid-surface $\boldsymbol{\varphi}_0$: $\mathscr{A} \to \mathbb{R}^3$ (which *does not* necessarily correspond to the geometric center of the cross-section) such that

$$\int_{h^-}^{h^+} \xi j_0 \rho_0 \, \mathrm{d}\xi \equiv \int_{h^-}^{h^+} \xi j \rho \, \mathrm{d}\xi = 0 \,. \tag{3.22}$$

Define the surface densities $\bar{\rho}_0$: $\mathscr{A} \to \mathbb{R}$ and $\bar{\rho}$: $\mathscr{A} \to \mathbb{R}$ by the expressions

$$\bar{\rho}_0 := \frac{1}{\bar{j}_0} \int_{h^-}^{h^+} \rho_0 j_0 \, \mathrm{d}\xi \quad \text{and} \quad \bar{\rho} := \frac{1}{\bar{j}} \int_{h^-}^{h^+} \rho j \, \mathrm{d}\xi \,. \tag{3.23}$$

In addition, define the surface inertias $\bar{I}_{0\rho}$: $\mathscr{A} \to \mathbb{R}$ and $\bar{I}_\rho := \mathscr{A} \to \mathbb{R}$ as

$$\bar{I}_{0\rho} := \frac{1}{\bar{j}_0} \int_{h^-}^{h^+} j_0 \rho_0 \xi^2 \, \mathrm{d}\xi \quad \text{and} \quad \bar{I}_\rho := \frac{1}{\bar{j}} \int_{h^-}^{h^+} j \rho \xi^2 \, \mathrm{d}\xi \,. \tag{3.24}$$

The significance of these definitions will become apparent in what follows. By (3.23), (3.24) and the conservation of mass relation $\rho_0 = J\rho$, we have

$$\bar{\rho}_0 = \bar{J}\bar{\rho} \quad \text{and} \quad \bar{I}_{0\rho} = \bar{J}\bar{I}_\rho \,, \tag{3.25}$$

which constitutes the statement of conservation of mass for shells.

3.4.3. Resultant linear, angular, and director momentum

For a motion $\boldsymbol{\Phi}_t$: $\mathscr{A} \times [h^-, h^+] \times \mathbb{R}_+ \to \mathbb{R}^3$, time differentiation of expression (3.5) yields

$$\dot{\boldsymbol{\Phi}}_t = \dot{\boldsymbol{\varphi}}_t + \xi \dot{\boldsymbol{t}}_t = \dot{\boldsymbol{\varphi}}_t + \xi \boldsymbol{w}_t \times \boldsymbol{t}_t \,. \tag{3.26}$$

Define the *resultant linear momentum* $\boldsymbol{p}_t$, and the *resultant angular momentum* $\boldsymbol{\pi}_t$ by the expressions

$$\boldsymbol{p}_t := \frac{1}{\bar{j}} \int_{h^-}^{h^+} \rho \dot{\boldsymbol{\Phi}}_t j \, \mathrm{d}\xi \,, \qquad \boldsymbol{\pi}_t := \frac{1}{\bar{j}} \int_{h^-}^{h^+} (\boldsymbol{\Phi}_t - \boldsymbol{\varphi}_t) \times \rho \dot{\boldsymbol{\Phi}}_t j \, \mathrm{d}\xi \,. \tag{3.27}$$

Making use of (3.22), (3.26), the kinematic hypothesis (3.5) and introducing definitions (3.23) and (3.24) we have

$$\boldsymbol{p}_t = \bar{\rho} \dot{\boldsymbol{\varphi}}_t \quad \text{and} \quad \boldsymbol{\pi}_t = \bar{I}_\rho \boldsymbol{w}_t \,, \tag{3.28}$$

where the relation $\boldsymbol{w}_t \cdot \boldsymbol{t}_t = 0$ was employed.

Since $\boldsymbol{\pi}_t \cdot \boldsymbol{t}_t = 0$, by Corollary 2.2, there is a *unique* $\bar{\boldsymbol{\pi}}_t \in \mathrm{T}_t\mathrm{S}^2$ such that $\bar{\boldsymbol{\pi}}_t = \boldsymbol{\pi}_t \times \boldsymbol{t}_t$; in fact

$$\bar{\boldsymbol{\pi}}_t := \boldsymbol{\pi}_t \times \boldsymbol{t}_t = \bar{I}_p \boldsymbol{w}_t \times \boldsymbol{t}_t \equiv \bar{I}_p \dot{\boldsymbol{t}}_t \,. \tag{3.29}$$

$\bar{\boldsymbol{\pi}}_t$ is referred to as the *resultant director momentum.*

4. Stress resultants and stress couples. Local balance laws

In this section we define stress resultants and stress couple resultants from the three-dimensional theory and develop the balance laws in terms of these resultants.

Given a *motion* $\boldsymbol{\chi}_t$: $\mathscr{B} \times \mathbb{R}_+ \to \mathscr{S}$, where $\boldsymbol{\chi}_t = \boldsymbol{\Phi}_t \circ \boldsymbol{\Phi}_0^{-}$, we denote by $\boldsymbol{\sigma}$ the symmetric Cauchy stress tensor in $\mathscr{S}$, and by $\boldsymbol{P}$ the nonsymmetric (first) Piola–Kirchhoff stress tensor relative to $\mathscr{B}$ and $\mathscr{S}$.

4.1. *Definitions for the three-dimensional theory*

Consider sections in the current configuration $\mathscr{S} \subset \mathbb{R}^3$ defined as

$$\mathscr{S}^\alpha := \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x} = \boldsymbol{\Phi}|_{\xi^\alpha = \mathrm{const.}}\} \,, \quad \alpha = 1, 2 \,. \tag{4.1}$$

The (one-form) field normal to $\mathscr{S}^1$ is given by

$$\mathrm{d}\mathscr{S}^1 := j[\nabla\boldsymbol{\Phi}]^{-\mathrm{t}}\boldsymbol{E}^1 \,\mathrm{d}\xi^2 \,\mathrm{d}\xi \equiv j\boldsymbol{g}^1 \,\mathrm{d}\xi^2 \,\mathrm{d}\xi \,, \tag{4.2}$$

and the analogous expression holds for $\mathrm{d}\mathscr{S}^2$. Consequently, the force acting on $\mathscr{S}^1$ per unit of coordinate length ξ^2 is

$$\boldsymbol{R}^1 := \int_{h^-}^{h^+} \boldsymbol{\sigma} \frac{\mathrm{d}\mathscr{S}^1}{\mathrm{d}\xi^2} = \int_{h^-}^{h^+} \boldsymbol{\sigma}\boldsymbol{g}^1 j \,\mathrm{d}\xi \,. \tag{4.3}$$

Similarly, the torque acting on $\mathscr{S}^1$ per unit of coordinate length ξ^2 is

$$\boldsymbol{T}^1 := \int_{h^-}^{h^+} (\boldsymbol{x} - \boldsymbol{\varphi}) \times \boldsymbol{\sigma} \frac{\mathrm{d}\mathscr{S}^1}{\mathrm{d}\xi^2} = \int_{h^-}^{h^+} (\boldsymbol{x} - \boldsymbol{\varphi}) \times \boldsymbol{\sigma}\boldsymbol{g}^1 j \,\mathrm{d}\xi \,. \tag{4.4}$$

We define the *stress resultant* $\boldsymbol{n}^\alpha$, and the *stress couple* $\boldsymbol{m}^\alpha$, by normalizing $\boldsymbol{R}^\alpha$ and $\boldsymbol{T}^\alpha$ with the surface Jacobian $\bar{j} = \|\boldsymbol{a}_1 \times \boldsymbol{a}_2\|$. Accordingly, we set

$$\boldsymbol{n}^\alpha := \frac{1}{\bar{j}} \int_{h^-}^{h^+} \boldsymbol{\sigma}\boldsymbol{g}^\alpha j \,\mathrm{d}\xi \,, \tag{4.5a}$$

$$\boldsymbol{m}^\alpha := \frac{1}{\bar{j}} \int_{h^-}^{h^+} (\boldsymbol{x} - \boldsymbol{\varphi}) \times \boldsymbol{\sigma}\boldsymbol{g}^\alpha j \,\mathrm{d}\xi \,, \tag{4.5b}$$

where $\boldsymbol{x} = \boldsymbol{\Phi}(\xi^1, \xi^2, \xi)$. Finally, one defines the *across-the-thickness stress resultant*, denoted by $\boldsymbol{l}$, by the expression

$$\boldsymbol{l} = \frac{1}{\bar{j}} \int_{h^-}^{h^+} \boldsymbol{\sigma} \boldsymbol{g}^3 j \, \mathrm{d}\xi \equiv \frac{1}{\bar{j}} \int_{h^-}^{h^+} \boldsymbol{\sigma} \boldsymbol{\nu} \| \boldsymbol{g}_1 \times \boldsymbol{g}_2 \| \, \mathrm{d}\xi \,, \tag{4.6}$$

where $\boldsymbol{\nu} = (\boldsymbol{g}_1 \times \boldsymbol{g}_2)/\|\boldsymbol{g}_1 \times \boldsymbol{g}_2\| = (j\boldsymbol{g}^3)/\|\boldsymbol{g}_1 \times \boldsymbol{g}_2\|$ is the (one-form) normal to the "laminae" surface.

REMARKS 4.1. (1) For the stress resultant $\boldsymbol{n}^\alpha$ we have the equivalent expressions

$$\boldsymbol{n}^\alpha = \frac{1}{\bar{j}} \int_{h^-}^{h^+} \boldsymbol{\sigma} \boldsymbol{g}^\alpha j \, \mathrm{d}\xi \equiv \frac{1}{\bar{j}} \int_{h^-}^{h^+} \boldsymbol{P} \boldsymbol{g}_0^\alpha j_0 \, \mathrm{d}\xi \,, \tag{4.7}$$

the second of which follows by recalling the relation $\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-\mathrm{t}}$, where $\boldsymbol{F} = \nabla\boldsymbol{\Phi}[\nabla\boldsymbol{\Phi}_0]^{-1}$ is the deformation gradient and $J := \det \boldsymbol{F} = j/j_0$, and noting that

$$j\boldsymbol{\sigma}\nabla\boldsymbol{\Phi}^{-\mathrm{t}} = j_0 \boldsymbol{P} \nabla\boldsymbol{\Phi}_0^{-\mathrm{t}} \,. \tag{4.8}$$

(2) Similarly, noting that $(\boldsymbol{x} - \boldsymbol{\varphi}) = \xi \boldsymbol{t}$, for the stress couple $\boldsymbol{m}^\alpha$ we have the alternative expressions in terms of the Piola–Kirchhoff stress

$$\boldsymbol{m}^\alpha = \boldsymbol{t} \times \frac{1}{\bar{j}} \int_{h^-}^{h^+} \xi \boldsymbol{\sigma} \boldsymbol{g}^\alpha j \, \mathrm{d}\xi \equiv \boldsymbol{t} \times \frac{1}{\bar{j}} \int_{h^-}^{h^+} \xi \boldsymbol{P} \boldsymbol{g}_0^\alpha j_0 \, \mathrm{d}\xi \,. \tag{4.9}$$

(3) In terms of the Piola–Kirchhoff stress, the across-the-thickness resultant can be written

$$\boldsymbol{l} = \frac{1}{\bar{j}} \int_{h^-}^{h^+} \boldsymbol{\sigma} \boldsymbol{g}^3 j \, \mathrm{d}\xi \equiv \frac{1}{\bar{j}} \int_{h^-}^{h^+} \boldsymbol{P} \boldsymbol{g}_0^3 j_0 \, \mathrm{d}\xi \,. \tag{4.10}$$

(4) Observe that $\boldsymbol{m}^\alpha \cdot \boldsymbol{t} = 0$. Hence, *there is no component of the stress couple along the director* $\boldsymbol{t}$. This is at variance with some formulations of shell theory employing the rotation vector, as in [32]. Alternatively, one may define the director stress couple, $\tilde{\boldsymbol{m}}^\alpha$, according to the expression

$$\boldsymbol{m}^\alpha = \boldsymbol{t} \times \tilde{\boldsymbol{m}}^\alpha \;\Rightarrow\; \tilde{\boldsymbol{m}}^\alpha := \frac{1}{\bar{j}} \int_{h^-}^{h^+} \xi \boldsymbol{\sigma} \boldsymbol{g}^\alpha j \, \mathrm{d}\xi = \frac{1}{\bar{j}} \int_{h^-}^{h^+} \xi \boldsymbol{P} \boldsymbol{g}_0^\alpha j_0 \, \mathrm{d}\xi \,. \tag{4.11}$$

It should be noted that $\tilde{\boldsymbol{m}}^\alpha \cdot \boldsymbol{t} \neq 0$. However, because of the constraint $\boldsymbol{t} \in \mathrm{S}^2$, the component of $\tilde{\boldsymbol{m}}^\alpha$ along $\boldsymbol{t}$ does not enter explicitly in the subsequent developments. In fact, this component could be eliminated completely by defining $\bar{\boldsymbol{m}}^\alpha := \tilde{\boldsymbol{m}}^\alpha - (\tilde{\boldsymbol{m}}^\alpha \cdot \boldsymbol{t})\boldsymbol{t}$, so that $\bar{\boldsymbol{m}}^\alpha \cdot \boldsymbol{t} = 0$, and (4.11) would be of the form $\boldsymbol{m}^\alpha = \boldsymbol{t} \times \bar{\boldsymbol{m}}^\alpha$. □

4.2. The momentum equations

Starting with the momentum balance equations of the three-dimensional theory, it can be shown (see Appendix A) that the resultant local form of the momentum balance equations take the following form:

$$\frac{1}{\bar{j}}(\bar{j}\boldsymbol{n}^{\alpha})_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}\ddot{\boldsymbol{\varphi}}_t\,, \tag{4.12a}$$

$$\frac{1}{\bar{j}}(\bar{j}\boldsymbol{m}^{\alpha})_{,\alpha} + \boldsymbol{\varphi}_{,\alpha} \times \boldsymbol{n}^{\alpha} + \bar{\boldsymbol{m}} = \bar{I}_{\rho}\dot{\boldsymbol{w}}_t, \tag{4.12b}$$

where $\bar{\boldsymbol{n}}$ and $\bar{\boldsymbol{m}}$ are the applied resultant force and applied resultant coupler per unit length as defined in Appendix A. We note that

$$\boldsymbol{m}^{\alpha} \cdot \boldsymbol{t} = 0 \quad \text{and} \quad \bar{\boldsymbol{m}} \cdot \boldsymbol{t} = 0\,. \tag{4.13}$$

Equations (4.12) are in the form considered by several authors; see Green and Zerna [25, pp. 379–380] or Libai and Simmonds [32]. These authors, however, immediately proceed to derive component equations relative to the Gauss frame $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{t}\}$ leading, inevitably, to the explicit appearance of the Christoffel symbols associated with the Riemannian connection of the mid-surface. In this regard, see also [35]. We show below that the direct use of the vector equations (4.12) is all that is needed in the weak formulation of the equations. The Riemannian connection of the mid-surface does not enter *explicitly* in the formulation.

4.3. Constitutive restriction. Alternative form of angular momentum

The balance of angular momentum of the three-dimensional theory is expressed by the symmetry of the Cauchy stress tensor, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$. In what follows, this balance law is interpreted as a restriction balance of angular momentum places on the admissible form of the constitutive equations for the resultants $\boldsymbol{n}^{\alpha}$, $\boldsymbol{m}^{\alpha}$ (or equivalently $\tilde{\boldsymbol{m}}^{\alpha}$), and $\boldsymbol{l}$.

4.3.1. Constitutive restriction

Expressing $\boldsymbol{\sigma}$ in components relative to the convected basis $\boldsymbol{g}_I$ we have $\boldsymbol{\sigma} = \sigma^{IJ}\boldsymbol{g}_I \otimes \boldsymbol{g}_J$. The symmetry condition $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$ implies $\boldsymbol{g}_I \times \boldsymbol{g}_J\sigma^{IJ} = \boldsymbol{g}_I \times \boldsymbol{\sigma}\boldsymbol{g}^I = 0$. Integration of the latter relation over $[h^-, h^+] \subset \mathbb{R}$ and use of the expressions $\boldsymbol{g}_{\alpha} = \boldsymbol{a}_{\alpha} + \xi\boldsymbol{t}_{,\alpha}$ and $\boldsymbol{g}_3 = \boldsymbol{t}$ yields

$$\boldsymbol{\varphi}_{,\alpha} \times \int_{h^-}^{h^+} \boldsymbol{\sigma}\boldsymbol{g}^{\alpha} j \,\mathrm{d}\xi + \boldsymbol{t}_{,\alpha} \times \int_{h^-}^{h^+} \xi\boldsymbol{\sigma}\boldsymbol{g}^{\alpha} j \,\mathrm{d}\xi + \boldsymbol{t} \times \int_{h^-}^{h^+} \boldsymbol{\sigma}\boldsymbol{g}^{3} j \,\mathrm{d}\xi = 0\,. \tag{4.14}$$

Introducing definitions (4.5a), (4.6) and (4.11), (4.14) reduces to

$$\boldsymbol{\varphi}_{,\alpha} \times \boldsymbol{n}^{\alpha} + \boldsymbol{t}_{,\alpha} \times \tilde{\boldsymbol{m}}^{\alpha} + \boldsymbol{t} \times \boldsymbol{l} = \boldsymbol{0}\,. \tag{4.15}$$

Equation (4.15) is the restriction that balance of angular momentum places on the admissible form of the constitutive equations.

4.3.2. Alternative form of the momentum equations

With the help of (4.15), the angular momentum equation (4.12b) can be recast in the following form. From (4.11) we have

$$\frac{1}{\bar{j}}(\bar{j}\boldsymbol{m}^{\alpha})_{,\alpha} = \boldsymbol{t}_{,\alpha} \times \tilde{\boldsymbol{m}}^{\alpha} + \boldsymbol{t} \times \frac{1}{\bar{j}}(\bar{j}\tilde{\boldsymbol{m}}^{\alpha})_{,\alpha} . \tag{4.16}$$

Recalling that $\dot{\boldsymbol{w}} = \boldsymbol{t} \times \ddot{\boldsymbol{t}}$, substitution of (4.15) and (4.16) into (4.12b) yields

$$\boldsymbol{t} \times \left[\frac{1}{\bar{j}}(\bar{j}\tilde{\boldsymbol{m}}^{\alpha})_{,\alpha} - \boldsymbol{l} + \tilde{\bar{\boldsymbol{m}}} - \bar{I}_{\rho}\ddot{\boldsymbol{t}}\right] = \boldsymbol{0} , \tag{4.17}$$

where $\tilde{\bar{\boldsymbol{m}}}$ is the applied director couple per unit length which satisfies

$$\tilde{\bar{\boldsymbol{m}}} \cdot \boldsymbol{t} = 0 \;\Rightarrow\; \tilde{\bar{\boldsymbol{m}}} = \bar{\boldsymbol{m}} \times \boldsymbol{t} . \tag{4.18}$$

Consequently, we obtain the equivalent system of resultant local momentum balance equations

$$\frac{1}{\bar{j}}(\bar{j}\boldsymbol{n}^{\alpha})_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}\ddot{\boldsymbol{\varphi}}_{t} , \qquad \frac{1}{\bar{j}}(\bar{j}\tilde{\boldsymbol{m}}^{\alpha})_{,\alpha} - \bar{\boldsymbol{l}} + \tilde{\bar{\boldsymbol{m}}} = \bar{I}_{\rho}\ddot{\boldsymbol{t}}_{t} , \tag{4.19}$$

where $\bar{\boldsymbol{l}} = \boldsymbol{l} + \bar{\lambda}\boldsymbol{t}$, and $\bar{\lambda}: \mathscr{A} \to \mathbb{R}$ is an undetermined director "pressure," whose significance is analogous to the hydrostatic pressure in incompressible elasticity. Further elaboration on the significance of $\bar{\boldsymbol{l}}$ is given below.

4.4. Further reduction. The across-the-thickness stress resultant

By making use of the constraint condition $\|\boldsymbol{t}\| = 1$, we derive an explicit expression for the across-the-thickness stress resultant and obtain a further reduction of the constitutive restriction (4.15). These expressions play a fundamental role in the variational formulation of the momentum equations considered in Section 6.

4.4.1. Further reduction of the constitutive restriction

We now consider component expressions relative to the surface convected basis. We start the development by setting

$$\boldsymbol{t}_{,\alpha} = \lambda_{\alpha}^{\mu}\boldsymbol{\varphi}_{,\mu} + \lambda_{\alpha}^{3}\boldsymbol{t} . \tag{4.20}$$

Using the constraint condition $\|\boldsymbol{t}\| = 1$, λ_{α}^{3} is determined in terms of λ_{α}^{μ} from the condition $\boldsymbol{t} \cdot \boldsymbol{t}_{,\alpha} = 0$ as

$$\lambda_{\alpha}^{3} = -\lambda_{\alpha}^{\mu}\boldsymbol{\varphi}_{,\mu} \cdot \boldsymbol{t} \equiv -\lambda_{\alpha}^{\mu}\gamma_{\mu} . \tag{4.21}$$

Next, we resolve the resultants $\boldsymbol{n}^{\alpha}$ and $\tilde{\boldsymbol{m}}^{\alpha}$ into components along $\{\boldsymbol{\varphi}_{,\alpha}, \boldsymbol{t}\}$ as

$$\boldsymbol{n}^{\alpha} = n^{\beta\alpha}\boldsymbol{\varphi}_{,\beta} + q^{\alpha}\boldsymbol{t}\,, \qquad \tilde{\boldsymbol{m}}^{\alpha} = \tilde{m}^{\beta\alpha}\boldsymbol{\varphi}_{,\beta} + \tilde{m}^{3\alpha}\boldsymbol{t}\,. \tag{4.22}$$

Restriction (4.15) then becomes

$$(n^{\beta\alpha} - \lambda^{\beta}_{\mu}\tilde{m}^{\alpha\mu})\boldsymbol{\varphi}_{,\beta} \times \boldsymbol{\varphi}_{,\alpha} + \boldsymbol{t} \times [\boldsymbol{l} - (q^{\alpha} + \lambda^{\alpha}_{\mu}\tilde{m}^{3\mu} - \lambda^{3}_{\mu}\tilde{m}^{\alpha\mu})\boldsymbol{\varphi}_{,\alpha}] = \boldsymbol{0}\,. \tag{4.23}$$

By taking the dot product of (4.23) with $\boldsymbol{t}$ we obtain

$$\bar{j}e_{\alpha\beta}(n^{\beta\alpha} - \lambda^{\beta}_{\mu}\tilde{m}^{\alpha\mu}) = 0\,, \tag{4.24}$$

where $e_{\alpha\beta}$ is the surface alternator tensor. Consequently, we define the *symmetric* resultant

$$\tilde{n}^{\beta\alpha} := n^{\beta\alpha} - \lambda^{\beta}_{\mu}\tilde{m}^{\alpha\mu} \equiv \tilde{n}^{\alpha\beta}\,. \tag{4.25}$$

In addition, from (4.23) and (4.25) one obtains the explicit expression for $\boldsymbol{l}$

$$\boldsymbol{l} = \lambda\boldsymbol{t} + (q^{\alpha} + \lambda^{\alpha}_{\mu}\tilde{m}^{3\mu} - \lambda^{3}_{\mu}\tilde{m}^{\alpha\mu})\boldsymbol{\varphi}_{,\alpha}\,. \tag{4.26}$$

Introducing the definition

$$\tilde{q}^{\alpha} := q^{\alpha} - \lambda^{3}_{\mu}\tilde{m}^{\alpha\mu} \equiv q^{\alpha} + \lambda^{\beta}_{\mu}\gamma_{\beta}\tilde{m}^{\alpha\mu}\,, \tag{4.27}$$

expression (4.26) now becomes

$$\boldsymbol{l} = \lambda\boldsymbol{t} + \tilde{q}^{\alpha}\boldsymbol{\varphi}_{,\alpha} + \lambda^{\alpha}_{\mu}\tilde{m}^{3\mu}\boldsymbol{\varphi}_{,\alpha}\,. \tag{4.28}$$

We shall refer to $\tilde{n}^{\beta\alpha}$ and $\tilde{q}^{\alpha}$ as the *effective membrane* and *effective shear* stress resultants. The significance of these definitions will become apparent in the following section.

5. Local (elastic) constitutive equations

In this section, we derive properly invariant elastic constitutive equations for the effective stress resultants $\tilde{n}^{\beta\alpha}$ and $\tilde{q}^{\alpha}$ and for the resultant stress couple $\tilde{\boldsymbol{m}}^{\alpha}$. To this end, we first obtain appropriate conjugate strain measures by reducing the general expression for the stress power of the three-dimensional theory by means of the basic kinematic assumption (3.5).

5.1. Reduced stress power. Conjugate strain measures

The main result to be exploited in the formulation of constitutive equations is contained in the following.

PROPOSITION 5.1. By making use of the basic kinematic assumption (3.5), *the stress power of the three-dimensional theory is expressed in the form*

$$\mathcal{W} := \int_{\mathcal{B}} \boldsymbol{P} : \dot{\boldsymbol{F}} \, \mathrm{d}\mathcal{V} = \int_{\bar{\mathcal{A}}} [\boldsymbol{n}^{\alpha} \cdot \dot{\boldsymbol{\varphi}}_{,\alpha} + \tilde{\boldsymbol{m}}^{\alpha} \cdot \dot{\boldsymbol{t}}_{,\alpha} + \boldsymbol{l} \cdot \dot{\boldsymbol{t}}] \, \mathrm{d}\mu \,, \tag{5.1}$$

where $\mathrm{d}\mu = \bar{j}\, \mathrm{d}\xi^1 \, \mathrm{d}\xi^2$ *is the current surface area measure,* $\boldsymbol{P}$ *is the first Piola–Kirchhoff stress tensor, and* $\boldsymbol{F}$ *is the deformation gradient given by* (3.8).

PROOF. Using (3.8), along with Proposition 3.1, time differentiation yields

$$\begin{aligned} \dot{\boldsymbol{F}} &= [\nabla \dot{\boldsymbol{\Phi}}] \nabla \boldsymbol{\Phi}_0^{-1} \,, \\ \dot{\boldsymbol{F}} &= [(\dot{\boldsymbol{\varphi}}_{,\alpha} + \xi \dot{\boldsymbol{t}}_{,\alpha}) \otimes \boldsymbol{E}^{\alpha} + \dot{\boldsymbol{t}} \otimes \boldsymbol{E}^3] \nabla \boldsymbol{\Phi}_0^{-1} \,. \end{aligned} \tag{5.2}$$

We can now write

$$\boldsymbol{P} : \dot{\boldsymbol{F}} = \boldsymbol{P} \nabla \boldsymbol{\Phi}_0^{-1} \boldsymbol{E}^{\alpha} \cdot (\dot{\boldsymbol{\varphi}}_{,\alpha} + \xi \dot{\boldsymbol{t}}_{,\alpha}) + \boldsymbol{P} \nabla \boldsymbol{\Phi}_0^{-1} \boldsymbol{E}^3 \cdot \dot{\boldsymbol{t}} \,. \tag{5.3}$$

Thus the stress power relation is expressed

$$\mathcal{W} = \int_{\bar{\mathcal{A}}} \left[\int_{h^-}^{h^+} \boldsymbol{P} \boldsymbol{g}_0^{\alpha} j \, \mathrm{d}\xi \cdot \dot{\boldsymbol{\varphi}}_{,\alpha} + \int_{h^-}^{h^+} \xi \boldsymbol{P} \boldsymbol{g}_0^{\alpha} j \, \mathrm{d}\xi \cdot \dot{\boldsymbol{t}}_{,\alpha} + \int_{h^-}^{h^+} \boldsymbol{P} \boldsymbol{g}_0^{3} j \, \mathrm{d}\xi \cdot \dot{\boldsymbol{t}} \right] \mathrm{d}\xi^1 \, \mathrm{d}\xi^2 \,. \tag{5.4}$$

The result (5.1) then follows by recalling definitions (4.7), (4.10), and (4.11) of the stress and stress couple resultants in terms of the Piola–Kirchhoff stress tensor. □

An alternative form of the stress-power relation (5.1) is obtained by introducing the effective stress resultants (4.25) and (4.27). This result is summarized in Corollary 5.2. First, we make the following definitions.

Define the following spatial tensors:

$$\begin{aligned} \tilde{\boldsymbol{n}} &:= \tilde{n}^{\beta\alpha} \boldsymbol{a}_{\beta} \otimes \boldsymbol{a}_{\alpha} \equiv \tilde{\boldsymbol{n}}^{\mathrm{t}} \,, \\ \tilde{\boldsymbol{q}} &:= \tilde{q}^{\alpha} \boldsymbol{a}_{\alpha} \,, \\ \tilde{\boldsymbol{m}} &:= \tilde{m}^{\beta\alpha} \boldsymbol{a}_{\beta} \otimes \boldsymbol{a}_{\alpha} \,. \end{aligned} \tag{5.5}$$

Define kinematic variables as follows:

$$a_{\alpha\beta} := \boldsymbol{\varphi}_{,\alpha} \cdot \boldsymbol{\varphi}_{,\beta} \,, \qquad a_{0\alpha\beta} = \boldsymbol{\varphi}_{0,\alpha} \cdot \boldsymbol{\varphi}_{0,\beta} \,, \tag{5.6a}$$

$$\gamma_{\alpha} := \boldsymbol{\varphi}_{,\alpha} \cdot \boldsymbol{t} \,, \qquad \gamma_{0\alpha} = \boldsymbol{\varphi}_{0,\alpha} \cdot \boldsymbol{t}_0 \,, \tag{5.6b}$$

$$\kappa_{\alpha\beta} := \boldsymbol{\varphi}_{,\alpha} \cdot \boldsymbol{t}_{,\beta} \,, \qquad \kappa_{0\alpha\beta} = \boldsymbol{\varphi}_{0,\alpha} \cdot \boldsymbol{t}_{0,\beta} \,. \tag{5.6c}$$

The corresponding relative strain measures are defined relative to the dual spatial surface basis as

$$\boldsymbol{\varepsilon} := \tfrac{1}{2}(a_{\alpha\beta} - a_{0\alpha\beta})\boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}^{\beta} ,$$

$$\boldsymbol{\delta} := (\gamma_{\alpha} - \gamma_{0\alpha})\boldsymbol{a}^{\alpha} , \tag{5.7}$$

$$\boldsymbol{\rho} := (\kappa_{\alpha\beta} - \kappa_{0\alpha\beta})\boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}^{\beta} .$$

With this notation at hand, we present the following corollary to Proposition 5.1.

COROLLARY 5.2. The stress power of the three-dimensional theory may be expressed in the equivalent form

$$\mathcal{W} := \int_{\mathcal{B}} \boldsymbol{P} : \dot{\boldsymbol{F}} \, \mathrm{d}\mathcal{V} = \int_{\bar{\mathcal{A}}} [\tilde{n}^{\beta\alpha} \tfrac{1}{2} \dot{a}_{\beta\alpha} + \tilde{q}^{\alpha} \dot{\gamma}_{\alpha} + \tilde{m}^{\beta\alpha} \dot{\kappa}_{\beta\alpha}] \, \mathrm{d}\mu \tag{5.8a}$$

$$= \int_{\bar{\mathcal{A}}} [\tilde{\boldsymbol{n}} : L_v \boldsymbol{\varepsilon} + \tilde{\boldsymbol{q}} \cdot L_v \boldsymbol{\delta} + \tilde{\boldsymbol{m}} : L_v \boldsymbol{\rho}] \, \mathrm{d}\mu , \tag{5.8b}$$

where L_v represents the convected time derivative.

PROOF. First, we use the component expressions (4.22) to rewrite (5.1) as

$$\mathcal{W} = \int_{\bar{\mathcal{A}}} [n^{\beta\alpha} \boldsymbol{\varphi}_{,\beta} \cdot \dot{\boldsymbol{\varphi}}_{,\alpha} + q^{\alpha} \boldsymbol{t} \cdot \dot{\boldsymbol{\varphi}}_{,\alpha} + \tilde{m}^{\beta\alpha} \boldsymbol{\varphi}_{,\beta} \cdot \dot{\boldsymbol{t}}_{,\alpha} + \tilde{m}^{3\alpha} \boldsymbol{t} \cdot \dot{\boldsymbol{t}}_{,\alpha} + \boldsymbol{l} \cdot \dot{\boldsymbol{t}}] \, \mathrm{d}\mu . \tag{5.9}$$

From (5.6c) and the constraint $\|\boldsymbol{t}\| = 1$,

$$\dot{\kappa}_{\beta\alpha} = \dot{\boldsymbol{\varphi}}_{,\beta} \cdot \boldsymbol{t}_{,\alpha} + \boldsymbol{\varphi}_{,\beta} \cdot \dot{\boldsymbol{t}}_{,\alpha} ,$$

$$\boldsymbol{t} \cdot \boldsymbol{t}_{,\alpha} = 0 \;\Rightarrow\; \boldsymbol{t} \cdot \dot{\boldsymbol{t}}_{,\alpha} = -\dot{\boldsymbol{t}} \cdot \boldsymbol{t}_{,\alpha} . \tag{5.10}$$

Recalling the expression (4.20) for $\boldsymbol{t}_{,\alpha}$ in components and the expression (4.28) for $\boldsymbol{l}$, we substitute (5.10) into (5.9) to find that terms involving $\tilde{m}^{3\alpha}$ cancel, to leave

$$\mathcal{W} = \int_{\bar{\mathcal{A}}} [(n^{\beta\alpha} - \lambda^{\beta}_{\mu} \tilde{m}^{\alpha\mu}) \boldsymbol{\varphi}_{,\beta} \cdot \dot{\boldsymbol{\varphi}}_{,\alpha} + \tilde{m}^{\beta\alpha} \dot{\kappa}_{\beta\alpha} + (q^{\alpha} - \lambda^{3}_{\mu} \tilde{m}^{\alpha\mu}) \dot{\boldsymbol{\varphi}}_{,\alpha} \cdot \boldsymbol{t} + \tilde{q}^{\alpha} \boldsymbol{\varphi}_{,\alpha} \cdot \dot{\boldsymbol{t}}] \, \mathrm{d}\mu . \tag{5.11}$$

Result (5.8a) follows immediately from the symmetry of the effective membrane stress resultant (4.25), the definition of the effective shear stress resultant (4.27), and time differentiation of γ_{α}. Result (5.8b) follows from the definition of the convected time derivative

$$L_v \boldsymbol{\varepsilon} = \tfrac{1}{2} \dot{a}_{\alpha\beta} \boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}^{\beta} ,$$

$$L_v \boldsymbol{\delta} = \dot{\gamma}_{\alpha} \boldsymbol{a}^{\alpha} , \tag{5.12}$$

$$L_v \boldsymbol{\rho} = \dot{\kappa}_{\alpha\beta} \boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}^{\beta} . \qquad \square$$

REMARK 5.3. As defined, the convected time derivative is a particular form of the Lie derivative.[3] □

5.2. General elastic constitutive equations

Within the context of the purely mechanical theory, hyperelastic constitutive equations may be formulated by postulating the existence of a stored energy function of the form $\hat{\Psi}(\boldsymbol{\xi}, \boldsymbol{a}, \gamma_\alpha, \boldsymbol{t}_{,\alpha}; \boldsymbol{a}_0, \gamma_{0\alpha}, \boldsymbol{t}_{0,\alpha})$ or equivalently $\Psi(\boldsymbol{\xi}, \boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\rho}; \boldsymbol{a}_0, \gamma_{0\alpha}, \boldsymbol{t}_{0,\alpha})$. We now reduce the form of Ψ by exploiting standard invariance requirements under superposed rigid-body motions. Given any superposed rigid-body rotation $t \mapsto \boldsymbol{Q}(t) \in \mathrm{SO}(3)$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\delta}$, $\boldsymbol{\rho}$, ψ, $\tilde{\boldsymbol{n}}$, $\tilde{\boldsymbol{q}}$, $\tilde{\boldsymbol{m}}$, and $\bar{\rho}$ must transform according to

$$\boldsymbol{\varepsilon}^+ = \boldsymbol{Q}(t)\boldsymbol{\varepsilon}\boldsymbol{Q}^{\mathrm{t}}(t)\,, \qquad \tilde{\boldsymbol{n}}^+ = \boldsymbol{Q}(t)\tilde{\boldsymbol{n}}\boldsymbol{Q}^{\mathrm{t}}(t)\,, \tag{5.13a}$$

$$\boldsymbol{\delta}^+ = \boldsymbol{Q}(t)\boldsymbol{\delta}\,, \qquad \tilde{\boldsymbol{q}}^+ = \boldsymbol{Q}(t)\tilde{\boldsymbol{q}}\,, \tag{5.13b}$$

$$\boldsymbol{\rho}^+ = \boldsymbol{Q}(t)\boldsymbol{\rho}\boldsymbol{Q}^{\mathrm{t}}(t)\,, \qquad \tilde{\boldsymbol{m}}^+ = \boldsymbol{Q}(t)\tilde{\boldsymbol{m}}\boldsymbol{Q}^{\mathrm{t}}(t)\,, \tag{5.13c}$$

$$\psi^+ = \psi\,, \qquad \bar{\rho}^+ = \bar{\rho}\,. \tag{5.13d}$$

Since the base vectors $\boldsymbol{a}_\alpha$ and the dual vectors $\boldsymbol{a}^\alpha$ must transform according to

$$\boldsymbol{a}_\alpha^+ = \boldsymbol{Q}(t)\boldsymbol{a}_\alpha\,, \qquad \boldsymbol{a}^{\alpha+} = \boldsymbol{Q}(t)\boldsymbol{a}^\alpha\,, \tag{5.14}$$

it follows from (5.13a) and (5.14) that

$$\begin{aligned}\boldsymbol{\varepsilon}^+ &= \overset{+}{\varepsilon}_{\alpha\beta}\boldsymbol{a}^{\alpha+} \otimes \boldsymbol{a}^{\beta+} \\ &= \boldsymbol{Q}(t)(\overset{+}{\varepsilon}_{\alpha\beta}\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta)\boldsymbol{Q}^{\mathrm{t}}(t) \\ &= \boldsymbol{Q}(t)(\varepsilon_{\alpha\beta}\boldsymbol{a}^\alpha \otimes \boldsymbol{a}^\beta)\boldsymbol{Q}^{\mathrm{t}}(t)\,.\end{aligned} \tag{5.15}$$

Result (5.15) implies $\overset{+}{\varepsilon}_{\alpha\beta} = \varepsilon_{\alpha\beta}$. Following the same argument for $\boldsymbol{\delta}$ and $\boldsymbol{\rho}$ we conclude that

$$\overset{+}{\delta}_\alpha = \delta_\alpha \quad \text{and} \quad \overset{+}{\rho}_{\alpha\beta} = \rho_{\alpha\beta}\,. \tag{5.16}$$

Thus Ψ is expressed in invariant form as

$$\Psi(\boldsymbol{\xi}, \boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\rho}; \boldsymbol{a}_0, \gamma_{0\alpha}, \boldsymbol{t}_{0,\alpha}) = \psi(\boldsymbol{\xi}, \varepsilon_{\alpha\beta}, \delta_\alpha, \rho_{\alpha\beta}; a_{0\alpha\beta}, \gamma_{0\alpha}, \lambda^{\mu}_{0\alpha}, \overset{3}{\lambda}_{0\alpha})\,. \tag{5.17}$$

REMARKS 5.4. (1) ψ depends on the *relative* measures $\varepsilon_{\alpha\beta}$, δ_α and $\rho_{\alpha\beta}$.

(2) The dependence of ψ on the reference quantities $a_{0\alpha\beta}$, $\lambda^{\mu}_{0\alpha}$ and $\overset{3}{\lambda}_{0\alpha}$ is discussed in detail in [16].

(3) The dependence of ψ on $\overset{3}{\lambda}_{0\alpha}$ could be eliminated as $\overset{3}{\lambda}_{0\alpha} = -\gamma_{0\mu}\lambda^{\mu}_{0\alpha}$. □

[3] For a further discussion of Lie derivatives see [29, Section 1.6].

Using the Clausius–Duhem inequality and following standard arguments (see [34, Section 2.5] and references cited there), we write the properly invariant hyperelastic constitutive relations

$$\tilde{n}^{\beta\alpha} = \bar{\rho}\, \frac{\partial \psi}{\partial \varepsilon_{\beta\alpha}} , \qquad \tilde{q}^{\alpha} = \bar{\rho}\, \frac{\partial \psi}{\partial \delta_{\alpha}} , \qquad \tilde{m}^{\beta\alpha} = \bar{\rho}\, \frac{\partial \psi}{\partial \rho_{\beta\alpha}} . \tag{5.18}$$

5.3. *Example*: *Isotropic constitutive relations*

The simplest properly invariant isotropic constitutive relations for the effective membrane and shear stress resultants $\tilde{n}^{\beta\alpha}$ and $\tilde{q}^{\alpha}$, and for the stress couple resultant $\tilde{m}^{\beta\alpha}$ is given by

$$\begin{aligned} \tilde{n}^{\beta\alpha} &= \frac{\rho E h}{1-\nu^2}\, H^{\beta\alpha\gamma\delta} \varepsilon_{\gamma\delta} , \\ \tilde{m}^{(\beta\alpha)} &= \frac{\rho E h^3}{12(1-\nu^2)}\, H^{\beta\alpha\gamma\delta} \rho_{(\gamma\delta)} , \\ \tilde{q}^{\alpha} &= \kappa \rho G h a_0^{\alpha\beta} \delta_{\beta} , \end{aligned} \tag{5.19}$$

where E is Young's modulus, ν is Poisson's ratio, κ is the shear reduction coefficient, $\rho = \bar{\rho}/h$ is the three dimensional mass density taken to be independent of the through-the-thickness

Table 1
Basic kinematics

- Definition of the kinematics

$$\begin{aligned} &\varphi: \mathscr{A} \subset \mathbb{R}^2 \to \mathbb{R}^3 , && \text{mid-surface} \\ &t: \mathscr{A} \subset \mathbb{R}^2 \to S^2 , && \text{director field} \\ &\{E_1, E_2, E_3\} , && \text{inertial frame} \end{aligned}$$

- Orthogonal transformation such that $t' = \Lambda t (t' \neq -t)$

$$\Lambda = (t \cdot t')\mathbf{1} + [\widehat{t \times t'}] + \frac{1}{1 + t \cdot t'} (t \times t') \otimes (t \times t')$$

- Exact relation between director and rotation vector $\boldsymbol{\theta}$ (*exponential map*)

$$t' = \cos\|\bar{\boldsymbol{\theta}}\| t + \frac{\sin\|\bar{\boldsymbol{\theta}}\|}{\|\bar{\boldsymbol{\theta}}\|} \bar{\boldsymbol{\theta}} , \quad \text{where } \boldsymbol{\theta} \cdot t = 0 , \ \bar{\boldsymbol{\theta}} = \boldsymbol{\theta} \times t$$

- Surface convected frame and surface Jacobian

$$\begin{aligned} &a_{0\alpha} = \varphi_{0,\alpha} , && a_{03} = t_0 , && \bar{j}_0 = \|a_{01} \times a_{02}\| \\ &a_{\alpha} = \varphi_{,\alpha} , && a_3 = t , && \bar{j} = \|a_1 \times a_2\| \end{aligned}$$

- Relative strain measures

$$\begin{aligned} &a_{\alpha\beta} = a_{\alpha} \cdot a_{\beta} , && \gamma_{\alpha} = a_{\alpha} \cdot t , && \kappa_{\alpha\beta} = a_{\alpha} \cdot t_{,\beta} \\ &\varepsilon_{\alpha\beta} = \tfrac{1}{2}(a_{\alpha\beta} - a_{0\alpha\beta}) , && \delta_{\alpha} = \gamma_{\alpha} - \gamma_{0\alpha} , && \rho_{\alpha\beta} = \kappa_{\alpha\beta} - \kappa_{0\alpha\beta} \end{aligned}$$

variable, and

$$H^{\beta\alpha\gamma\delta} = \{\nu a_0^{\beta\alpha} a_0^{\gamma\delta} + \tfrac{1}{2}(1-\nu)(a_0^{\beta\gamma} a_0^{\alpha\delta} + a_0^{\beta\delta} a_0^{\alpha\gamma})\} . \tag{5.20}$$

REMARK 5.5. (1) The constitutive equation for the resultant couple given above is in terms of the *symmetric part* of $\tilde{m}^{\beta\alpha}$. It is assumed here and in the following developments that the skew-symmetric part of $\tilde{m}^{\beta\alpha}$ is zero, i.e.

$$\tilde{m}^{[\alpha\beta]} = 0 . \tag{5.21}$$

(2) Constitutive equations (5.19) can be justified on the basis of an asymptotic expansion of the three-dimensional Saint Venant–Kirchhoff model that employs definitions (AB4c) and (AB5) from Appendix B and retains terms up to order $O(h/R)$, where $h = h^+ - h^-$ is the thickness of the shell and R is the radius of curvature. Since terms involving products of $a^{\alpha\beta}$, $\lambda^\alpha_{0\beta}$ and λ^α_β are of higher order, in classical shell theory the resulting constitutive equations are typically assumed to depend on the refence surface only though the first fundamental form (i.e., $a_0^{\alpha\beta}$); see [25; 35, p. 606; 37]. The general dependence of the constitutive equations on the reference surface is discussed in detail in [16]. □

For convenience, the basic theory presented in Sections 1–5 is summarized in Tables 1 and 2.

Table 2
Local momentum equations. Constitutive relations

- Balance of linear and director momentum

$$\frac{1}{\bar{j}}(\bar{j}\boldsymbol{n}^\alpha)_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}\ddot{\boldsymbol{\varphi}}$$

$$\frac{1}{\bar{j}}(\bar{j}\tilde{\boldsymbol{m}}^\alpha)_{,\alpha} - \bar{\boldsymbol{l}} + \bar{\tilde{\boldsymbol{m}}} = \bar{I}_\rho \ddot{\boldsymbol{t}}$$

- Stress resultants and stress couples

$$\boldsymbol{n}^\alpha = n^{\beta\alpha}\boldsymbol{a}_\beta + q^\alpha \boldsymbol{t}$$

$$\tilde{\boldsymbol{m}}^\alpha = \tilde{m}^{\beta\alpha}\boldsymbol{a}_\beta + \tilde{m}^{3\alpha}\boldsymbol{t}$$

$$\bar{\boldsymbol{l}} = \bar{\lambda}\boldsymbol{t} + \boldsymbol{l}$$

$$\boldsymbol{l} = \lambda\boldsymbol{t} + \tilde{q}^\alpha \boldsymbol{a}_\alpha + \lambda^\alpha_\mu \tilde{m}^{3\mu}\boldsymbol{a}_\alpha$$

- Effective stress resultants

$$\tilde{n}^{\beta\alpha} = n^{\beta\alpha} - \lambda^\beta_\mu \tilde{m}^{\alpha\mu}$$

$$\tilde{q}^\alpha = q^\alpha - \lambda^3_\mu \tilde{m}^{\alpha\mu}$$

$$= q^\alpha + \lambda^\beta_\mu \gamma_\beta \tilde{m}^{\alpha\mu}$$

- Constitutive equations: $\Psi = \psi(\boldsymbol{\xi}, \varepsilon_{\alpha\beta}, \delta_\alpha, \rho_{\alpha\beta}; a_{0\alpha\beta}, \gamma_{0\alpha}, \lambda^\mu_{0\alpha}, \lambda^3_{0\alpha})$

$$\tilde{n}^{\beta\alpha} = \bar{\rho}\,\frac{\partial\psi}{\partial\varepsilon_{\beta\alpha}}, \quad \tilde{q}^\alpha = \bar{\rho}\,\frac{\partial\psi}{\partial\delta_\alpha}, \qquad \tilde{m}^{\beta\alpha} = \bar{\rho}\,\frac{\partial\psi}{\partial\rho_{\beta\alpha}}$$

6. The (weak) variational formulation

In this section we construct the weak form of the field equations summarized in Table 2. This variational formulation is of paramount importance in our subsequent numerical treatment considered in Part II of this work.

6.1. Admissible variations. Tangent space

Let $\boldsymbol{\chi} := \boldsymbol{\Phi} \circ \boldsymbol{\Phi}_0^{-1}: \mathscr{B} \to \mathscr{S}$ be a deformation with $\boldsymbol{\Phi}_0: \mathscr{A} \to \mathscr{B}$ and $\boldsymbol{\Phi}: \mathscr{A} \to \mathscr{S}$ being the reference and current configurations, respectively. Recall from (3.3) that

$$\boldsymbol{\Phi} = \boldsymbol{\varphi} + \xi \boldsymbol{t}, \qquad \xi \in [h^-, h^+] \quad \text{and} \quad (\boldsymbol{\varphi}, \boldsymbol{t}) \in \mathscr{C}. \tag{6.1}$$

In what follows, with a slight abuse in notation, we write $\boldsymbol{\Phi} = (\boldsymbol{\varphi}, \boldsymbol{t}) \in \mathscr{C}$, where $\boldsymbol{\varphi}: \mathscr{A} \to \mathbb{R}^3$ defines the position of the mid-surface, and $\boldsymbol{t}: \mathscr{A} \to \mathrm{S}^2$ defines the director field. We assume displacement boundary conditions of the form

$$\boldsymbol{\varphi} - \boldsymbol{\varphi}_0 = \tilde{\boldsymbol{u}} \quad \text{for } \boldsymbol{\xi} \in \partial_\varphi \mathscr{A}, \qquad \partial_\varphi \mathscr{A} \subset \partial \mathscr{A}, \tag{6.2}$$

$$\boldsymbol{t} - \boldsymbol{t}_0 = \tilde{\boldsymbol{\psi}} \quad \text{for } \boldsymbol{\xi} \in \partial_t \mathscr{A}, \qquad \partial_t \mathscr{A} \subset \partial \mathscr{A}, \tag{6.3}$$

where $\partial\mathscr{A}$ is the boundary of $\mathscr{A} \subset \mathbb{R}^2$. The tangent space at $\boldsymbol{\Phi} = (\boldsymbol{\varphi}, \boldsymbol{t})$, denoted by $\mathrm{T}_{\boldsymbol{\Phi}}\mathscr{C}$, is the linear space of admissible variations defined as

$$\mathrm{T}_{\boldsymbol{\Phi}}\mathscr{C} = \{(\delta\boldsymbol{\varphi}, \delta\boldsymbol{t}) =: \delta\boldsymbol{\Phi}: \mathscr{A} \to \mathbb{R}^3 \times \mathrm{T}_t\mathrm{S}^2 \,|\, \delta\boldsymbol{\varphi}|_{\partial_\varphi \mathscr{A}} = \boldsymbol{0} \text{ and } \delta\boldsymbol{t}|_{\partial_t \mathscr{A}} = \boldsymbol{0}\}. \tag{6.4}$$

Since $\delta\boldsymbol{t}(\boldsymbol{\xi}) \in \mathrm{T}_t\mathrm{S}^2$ for each $\boldsymbol{\xi} \in \mathscr{A}$, we have the following alternative useful characterization of the variations $\delta\boldsymbol{t}: \mathscr{A} \to \mathrm{T}_t\mathrm{S}^2$ associated with the director field:

(i) *Spatial description*. Recalling the one-to-one correspondence between $\mathrm{T}_t\mathrm{S}^2$ and $\mathrm{T}_\Lambda\mathrm{S}_E^2$; we can write

$$\delta\boldsymbol{t} = \delta\boldsymbol{\theta} \times \boldsymbol{t}, \qquad \delta\boldsymbol{\theta} \cdot \boldsymbol{t} = 0, \tag{6.5a}$$

where $\delta\boldsymbol{\theta}: \mathscr{A} \to \mathbb{R}^3$ is the axial vector associated with $\delta\hat{\boldsymbol{\theta}}: \mathscr{A} \to \mathrm{so}(3)$ with $\delta\hat{\boldsymbol{\theta}} \in \mathrm{T}_\Lambda\mathrm{S}_E^2$. (See definition (2.14) of $\mathrm{T}_\Lambda\mathrm{S}_E^2$.)

(ii) *Material (rotated) description*. Since $\delta\boldsymbol{\theta} = \boldsymbol{\Lambda}\,\delta\boldsymbol{\Theta}$, where $\boldsymbol{\Lambda}: \mathscr{A} \to \mathrm{S}_E^2$ is the orthogonal transformation such that $\boldsymbol{t} = \boldsymbol{\Lambda}\boldsymbol{E}$, and $\delta\boldsymbol{\Theta}: \mathscr{A} \to \mathbb{R}^3$ satisfies $\delta\boldsymbol{\Theta} \cdot \boldsymbol{E} = 0$ (see definition (2.13) of $\mathrm{T}_1\mathrm{S}_E^2$), we can write

$$\delta\boldsymbol{t} = \boldsymbol{\Lambda}(\delta\boldsymbol{\Theta} \times \boldsymbol{E}) =: \boldsymbol{\Lambda}\,\delta\boldsymbol{T}, \tag{6.5b}$$

where $\delta\boldsymbol{T}: \mathscr{A} \to \mathbb{R}^3$ also satisfies $\delta\boldsymbol{T} \cdot \boldsymbol{E} = 0$.

REMARK 6.1. The rotated representation (6.5b) is particularly useful in a computational context. The reason for this lies in the relation $\delta\boldsymbol{T} \cdot \boldsymbol{E} = 0$. Following Remark 2.5 we can write

$$\delta \boldsymbol{T} \cdot \boldsymbol{E} = 0 \;\Rightarrow\; \delta \boldsymbol{T} = \delta T^1 \boldsymbol{E}_1 + \delta T^2 \boldsymbol{E}_2 \,. \tag{6.6}$$

Consequently, *only two components* (relative to the inertial frame $\{\boldsymbol{E}_I\}_{I=1,2,3}$) need to be considered. Thus, this representation automatically excludes the "drill" degree of freedom along $\boldsymbol{E}_3 = \boldsymbol{E}$. In the spatial description (6.5a), however, this constraint is only implicit in the condition $\delta \boldsymbol{t} \cdot \boldsymbol{t} = 0$. □

6.2. *Weak form of momentum balance*

By considering arbitrary variations $\delta \boldsymbol{\Phi} = (\delta \boldsymbol{\varphi}, \delta \boldsymbol{t}) \in \mathrm{T}_{\boldsymbol{\Phi}} \mathscr{C}$, and integrating over the current surface $\boldsymbol{\varphi}\colon \bar{\mathscr{A}} \to \mathbb{R}^3$, the weak form of the momentum equations in Table 2 becomes

$$G_{\text{dyn}}(\boldsymbol{\Phi}, \delta \boldsymbol{\Phi}) := G(\boldsymbol{\Phi}, \delta \boldsymbol{\Phi}) + \int_{\bar{\mathscr{A}}} [\bar{\rho} \ddot{\boldsymbol{\varphi}} \cdot \delta \boldsymbol{\varphi} + \bar{I}_\rho \ddot{\boldsymbol{t}} \cdot \delta \boldsymbol{t}] \, \mathrm{d}\mu \,, \tag{6.7}$$

where $\mathrm{d}\mu = \bar{j} \, \mathrm{d}\xi^1 \, \mathrm{d}\xi^2$ is the (two-form) current surface measure and $G(\boldsymbol{\Phi}, \delta \boldsymbol{\Phi})$ is the static weak form or virtual work expression defined in the following subsection.

6.2.1. *Static weak form*

Using the divergence theorem for surfaces (integration by parts), a straightforward manipulation gives the following expression:

$$G(\boldsymbol{\Phi}, \delta \boldsymbol{\Phi}) = \int_{\mathscr{A}} [\boldsymbol{n}^\alpha \cdot \delta \boldsymbol{\varphi}_{,\alpha} + \tilde{\boldsymbol{m}}^\alpha \cdot \delta \boldsymbol{t}_{,\alpha} + \boldsymbol{l} \cdot \delta \boldsymbol{t}] \, \mathrm{d}\mu - G_{\text{ext}}(\delta \boldsymbol{\Phi}) \,, \tag{6.8}$$

where $G_{\text{ext}}(\delta \boldsymbol{\Phi})$ is the virtual work of the external loading given by

$$G_{\text{ext}}(\delta \boldsymbol{\Phi}) = \int_{\mathscr{A}} [\bar{\boldsymbol{n}} \cdot \delta \boldsymbol{\varphi} + \bar{\boldsymbol{m}} \cdot \delta \boldsymbol{t}] \, \mathrm{d}\mu + \int_{\partial_n \mathscr{A}} \bar{\bar{\boldsymbol{n}}} \cdot \delta \boldsymbol{\varphi} \bar{j} \, \mathrm{d}s + \int_{\partial_m \mathscr{A}} \bar{\bar{\boldsymbol{m}}} \cdot \delta \boldsymbol{t} \bar{j} \, \mathrm{d}s \,. \tag{6.9}$$

Here $\bar{\bar{\boldsymbol{n}}}$ and $\bar{\bar{\boldsymbol{m}}}$ are the prescribed force and torque on the boundary of the shell; i.e.

$$\bar{\bar{\boldsymbol{n}}} = \boldsymbol{n}^\alpha \nu_\alpha \quad \text{on } \partial_n \mathscr{A} \quad \text{and} \quad \bar{\bar{\boldsymbol{m}}} = \tilde{\boldsymbol{m}}^\alpha \nu_\alpha \quad \text{on } \partial_m \mathscr{A} \,, \tag{6.10}$$

where $\boldsymbol{\nu} = \nu_\alpha \boldsymbol{a}^\alpha$ is the (one-form) field normal to the boundary $\boldsymbol{\varphi}(\partial \mathscr{A})$. Of course, $\partial_n \mathscr{A} \cap \partial_\varphi \mathscr{A} = \emptyset$ and $\partial_m \mathscr{A} \cap \partial_t \mathscr{A} = \emptyset$.

6.2.2. *Component expression*

We now introduce the explicit expression for $\boldsymbol{l}$ and the component relations (4.20) and (4.22) into expression (6.8). Performing manipulations identical to those in the proof of Corollary 5.2, it can be shown that the weak form (6.8) is expressed in components as

$$G(\boldsymbol{\Phi}, \delta \boldsymbol{\Phi}) = \int_{\mathscr{A}} [\tilde{n}^{\beta\alpha} \tfrac{1}{2} \delta a_{\beta\alpha} + \tilde{m}^{\beta\alpha} \delta \kappa_{\beta\alpha} + \tilde{q}^\alpha \delta \gamma_\alpha] \, \mathrm{d}\mu - G_{\text{ext}}(\delta \boldsymbol{\Phi}) \,. \tag{6.11}$$

6.3. Linearization. Linearized strain measures

Next, we consider the linearization of the weak form (6.11) at a configuration $\boldsymbol{\Phi} := (\boldsymbol{\varphi}, \boldsymbol{t}) \in \mathscr{C}$ in the direction of an incremental tangent field $\Delta\boldsymbol{\Phi} := (\Delta\boldsymbol{\varphi}, \Delta\boldsymbol{t}) \in \mathrm{T}_{\boldsymbol{\Phi}}\mathscr{C}$. The resulting linearized weak form plays a central role in the numerical solution of boundary value problems considered in Part II of this work.

To construct the linearization of (6.11), one considers a one-parameter family of configurations $\varepsilon \mapsto \boldsymbol{\Phi}_\varepsilon = (\boldsymbol{\varphi}_\varepsilon, \boldsymbol{t}_\varepsilon) \in \mathscr{C}$ with the property that

$$\boldsymbol{\Phi}_\varepsilon\big|_{\varepsilon=0} = \boldsymbol{\Phi} \tag{6.12a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \boldsymbol{\Phi}_\varepsilon = \Delta\boldsymbol{\Phi}\,. \tag{6.12b}$$

A systematic procedure for constructing $\boldsymbol{\Phi}_\varepsilon$ is to make use of the exponential map, as follows.

6.3.1. Tangent configurations and exponential map

Let $(\Delta\boldsymbol{\varphi}, \Delta\boldsymbol{t}) \in \mathrm{T}_{\boldsymbol{\Phi}}\mathscr{C}$. Define the one-parameter family of configurations $(\boldsymbol{\varphi}_\varepsilon, \boldsymbol{t}_\varepsilon) \in \mathscr{C}$ by setting

$$\boldsymbol{\varphi}_\varepsilon := \boldsymbol{\varphi} + \varepsilon\,\Delta\boldsymbol{\varphi} \quad \text{and} \quad \boldsymbol{t}_\varepsilon := \exp_{\boldsymbol{t}}[\varepsilon\,\Delta\boldsymbol{t}]\,, \tag{6.13}$$

where $\exp_{\boldsymbol{t}}$: $\mathrm{T}_{\boldsymbol{t}}\mathrm{S}^2 \to \mathrm{S}^2$ is the exponential map in the S^2 sphere (see Proposition 2.4). Recall that one has the explicit expression

$$\exp_{\boldsymbol{t}}[\varepsilon\,\Delta\boldsymbol{t}] := \cos\|\varepsilon\,\Delta\boldsymbol{t}\|\boldsymbol{t} + \frac{\sin\|\varepsilon\,\Delta\boldsymbol{t}\|}{\|\varepsilon\,\Delta\boldsymbol{t}\|}\,\varepsilon\,\Delta\boldsymbol{t}\,, \tag{6.14}$$

where $\Delta\boldsymbol{t} \in \mathrm{T}_1\mathrm{S}^2$. We now verify that (6.13) meets the required properties of (6.12). To this end, first observe that

$$\boldsymbol{\varphi}_\varepsilon\big|_{\varepsilon=0} = \boldsymbol{\varphi} \quad \text{and} \quad \exp_{\boldsymbol{t}}[\varepsilon\,\Delta\boldsymbol{t}]\big|_{\varepsilon=0} = \boldsymbol{t}\,, \tag{6.15}$$

so that (6.12a) holds. Next, by making use of the directional derivative formula we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \boldsymbol{\varphi}_\varepsilon = \Delta\boldsymbol{\varphi}\,,\\ &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \boldsymbol{t}_\varepsilon = -\|\Delta\boldsymbol{t}\|\sin(\|\varepsilon\,\Delta\boldsymbol{t}\|)\boldsymbol{t} + \|\Delta\boldsymbol{t}\|\cos(\|\varepsilon\,\Delta\boldsymbol{t}\|)\,\frac{\Delta\boldsymbol{t}}{\|\Delta\boldsymbol{t}\|}\bigg|_{\varepsilon=0} \equiv \Delta\boldsymbol{t}\,, \end{aligned} \tag{6.16}$$

so that (6.12b) also holds. Finally, we record a relation needed in the next subsection. By differentiating (6.16) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \boldsymbol{\varphi}_{\varepsilon,\alpha} = \Delta\boldsymbol{\varphi}_{,\alpha} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \boldsymbol{t}_{\varepsilon,\alpha} = \Delta\boldsymbol{t}_{,\alpha}\,. \tag{6.17}$$

With these preliminary results at hand, we proceed to the linearization of the strain measures followed by the linearization of the weak form.

6.3.2. Linearized strain measures

Let $(a^{\varepsilon}_{\alpha\beta}, \gamma^{\varepsilon}_{\alpha}, \kappa^{\varepsilon}_{\alpha\beta})$ be the kinematic quantities associated with the one-parameter family of configurations $\varepsilon \mapsto \boldsymbol{\Phi}_{\varepsilon} = (\boldsymbol{\varphi}_{\varepsilon}, \boldsymbol{t}_{\varepsilon})$ defined by (6.13). For convenience, we introduce the notation

$$(\Delta a_{\alpha\beta}, \Delta\gamma_{\alpha}, \Delta\kappa_{\alpha\beta}) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} (a^{\varepsilon}_{\alpha\beta}, \gamma^{\varepsilon}_{\alpha}, \kappa^{\varepsilon}_{\alpha\beta})\,. \tag{6.18}$$

By making use of the results in Section 6.3.1, we readily obtain the expressions

$$\begin{aligned} &\Delta a_{\alpha\beta} := (\Delta\boldsymbol{\varphi}_{,\alpha}\cdot\boldsymbol{\varphi}_{,\beta} + \boldsymbol{\varphi}_{,\alpha}\cdot\Delta\boldsymbol{\varphi}_{,\beta})\,,\\ &\Delta\gamma_{\alpha} := (\boldsymbol{t}\cdot\Delta\boldsymbol{\varphi}_{,\alpha} + \Delta\boldsymbol{t}\cdot\boldsymbol{\varphi}_{,\alpha})\,,\\ &\Delta\kappa_{\alpha\beta} := \Delta\boldsymbol{\varphi}_{,\alpha}\cdot\boldsymbol{t}_{,\beta} + \boldsymbol{\varphi}_{,\alpha}\cdot\Delta\boldsymbol{t}_{,\beta}\,. \end{aligned} \tag{6.19}$$

We now proceed to the linearization of the weak form.

6.4. Linearized weak form. Symmetry

The conceptual procedure is standard; see e.g. [34, Ch. 6]. One substitutes the one-parameter family of configurations (6.13) into (6.11) and then one makes use of the relation between the Frechet directional derivative given by the standard formula

$$\mathrm{DG}\cdot\Delta\boldsymbol{\Phi} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \mathrm{G}(\boldsymbol{\Phi}_{\varepsilon}, \delta\boldsymbol{\Phi})\,. \tag{6.20}$$

The linearization of (6.20) can be broken into two parts, $\mathrm{D_M G}\cdot\Delta\boldsymbol{\Phi}$ and $\mathrm{D_G G}\cdot\Delta\boldsymbol{\Phi}$, each of which possesses classical interpretations.

6.4.1. Operator expressions

For subsequent developments, it proves convenient to introduce a more compact notation. To this end, we define the following resultant stress and stress couple vectors. As is stated in Remark 5.5, we assume $\tilde{m}^{[\beta\alpha]} = 0$, i.e. $\tilde{m}^{\beta\alpha} = \tilde{m}^{(\beta\alpha)}$. We thus have

$$\tilde{\boldsymbol{N}} = \bar{J}\begin{Bmatrix}\tilde{n}^{11}\\ \tilde{n}^{22}\\ \tilde{n}^{12}\end{Bmatrix}, \qquad \tilde{\boldsymbol{Q}} = \bar{J}\begin{Bmatrix}\tilde{q}^{1}\\ \tilde{q}^{2}\end{Bmatrix}, \qquad \tilde{\boldsymbol{M}} = \bar{J}\begin{Bmatrix}\tilde{m}^{11}\\ \tilde{m}^{22}\\ \tilde{m}^{12}\end{Bmatrix}. \tag{6.21}$$

Furthermore, utilizing expression (6.5b) for the director variation field, set

$$\boldsymbol{\varphi} = \begin{Bmatrix}\varphi^{1}\\ \varphi^{2}\\ \varphi^{3}\end{Bmatrix}, \qquad \boldsymbol{t} = \begin{Bmatrix}t^{1}\\ t^{2}\\ t^{3}\end{Bmatrix}, \qquad \Delta\boldsymbol{\varphi} = \begin{Bmatrix}\Delta\varphi^{1}\\ \Delta\varphi^{2}\\ \Delta\varphi^{3}\end{Bmatrix}, \qquad \Delta\boldsymbol{T} = \begin{Bmatrix}\Delta T^{1}\\ \Delta T^{2}\end{Bmatrix}. \tag{6.22}$$

We define the matrix differential operators

$$\mathbb{B}_{\mathrm{m}} = \begin{bmatrix} \boldsymbol{\varphi}^{\mathrm{t}}_{,1} \dfrac{\partial}{\partial \xi^1} \\ \boldsymbol{\varphi}^{\mathrm{t}}_{,2} \dfrac{\partial}{\partial \xi^2} \\ \boldsymbol{\varphi}^{\mathrm{t}}_{,1} \dfrac{\partial}{\partial \xi^2} + \boldsymbol{\varphi}^{\mathrm{t}}_{,2} \dfrac{\partial}{\partial \xi^1} \end{bmatrix}_{3\times 3}, \tag{6.23a}$$

$$\mathbb{B}_{\mathrm{sm}} = \begin{bmatrix} \boldsymbol{t}^{\mathrm{t}} \dfrac{\partial}{\partial \xi^1} \\ \boldsymbol{t}^{\mathrm{t}} \dfrac{\partial}{\partial \xi^2} \end{bmatrix}_{2\times 3}, \qquad \mathbb{B}_{\mathrm{sb}} = \begin{bmatrix} \boldsymbol{\varphi}^{\mathrm{t}}_{,1} \\ \boldsymbol{\varphi}^{\mathrm{t}}_{,2} \end{bmatrix}_{2\times 3} \bar{\boldsymbol{\Lambda}}_{3\times 2}, \tag{6.23b}$$

$$\mathbb{B}_{\mathrm{bm}} = \begin{bmatrix} \boldsymbol{t}^{\mathrm{t}}_{,1} \dfrac{\partial}{\partial \xi^1} \\ \boldsymbol{t}^{\mathrm{t}}_{,2} \dfrac{\partial}{\partial \xi^2} \\ \boldsymbol{t}^{\mathrm{t}}_{,1} \dfrac{\partial}{\partial \xi^2} + \boldsymbol{t}^{\mathrm{t}}_{,2} \dfrac{\partial}{\partial \xi^1} \end{bmatrix}_{3\times 3}, \qquad \mathbb{B}_{\mathrm{bb}} = \begin{bmatrix} \boldsymbol{\varphi}^{\mathrm{t}}_{,1} \dfrac{\partial}{\partial \xi^1} \\ \boldsymbol{\varphi}^{\mathrm{t}}_{,2} \dfrac{\partial}{\partial \xi^2} \\ \boldsymbol{\varphi}^{\mathrm{t}}_{,1} \dfrac{\partial}{\partial \xi^2} + \boldsymbol{\varphi}^{\mathrm{t}}_{,2} \dfrac{\partial}{\partial \xi^1} \end{bmatrix} \bar{\boldsymbol{\Lambda}}_{3\times 2}, \tag{6.23c}$$

where $\bar{\boldsymbol{\Lambda}}$ is the (3×2) matrix obtain by deleting the third column of $\boldsymbol{\Lambda}$, i.e.

$$\bar{\boldsymbol{\Lambda}}_{3\times 2} = [\boldsymbol{t}_1 \boldsymbol{t}_2] = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \\ \Lambda_{31} & \Lambda_{32} \end{bmatrix}. \tag{6.24}$$

With this notation at hand, we can rewrite the weak form (6.11) as

$$G(\boldsymbol{\Phi}, \delta\boldsymbol{\Phi}) = \int_{\mathscr{A}} \{[\mathbb{B}_{\mathrm{m}}\, \delta\boldsymbol{\varphi}]^{\mathrm{t}} \tilde{\boldsymbol{N}} + [\mathbb{B}_{\mathrm{sm}}\, \delta\boldsymbol{\varphi} + \mathbb{B}_{\mathrm{sb}}\, \delta\boldsymbol{T}]^{\mathrm{t}} \tilde{\boldsymbol{Q}} + [\mathbb{B}_{\mathrm{bm}}\, \delta\boldsymbol{\varphi} + \mathbb{B}_{\mathrm{bb}}\, \delta\boldsymbol{T}]^{\mathrm{t}} \tilde{\boldsymbol{M}}\}\, \mathrm{d}\mu_0 - G_{\mathrm{ext}}(\delta\boldsymbol{\Phi}), \tag{6.25}$$

where $\mathrm{d}\mu_0 = j_0\, \mathrm{d}\xi^1\, \mathrm{d}\xi^2$ is the reference surface measure.

Alternatively, defining the total resultant stress vector as

$$\boldsymbol{R} = \begin{bmatrix} \tilde{\boldsymbol{N}} \\ \tilde{\boldsymbol{Q}} \\ \tilde{\boldsymbol{M}} \end{bmatrix}_{8\times 1}, \tag{6.26}$$

and the differential operator

$$\mathbb{B} = \begin{bmatrix} \mathbb{B}_{\mathrm{m}} & \boldsymbol{0}_{3\times 2} \\ \mathbb{B}_{\mathrm{sm}} & \mathbb{B}_{\mathrm{sb}} \\ \mathbb{B}_{\mathrm{bm}} & \mathbb{B}_{\mathrm{bb}} \end{bmatrix}_{8\times 5}, \tag{6.27}$$

the weak form (6.25) can be written as

$$G(\boldsymbol{\Phi}, \delta\boldsymbol{\Phi}) := \int_{\mathscr{A}} \mathbb{B} \begin{Bmatrix} \delta\boldsymbol{\varphi} \\ \delta\boldsymbol{T} \end{Bmatrix} \cdot \boldsymbol{R}\, \mathrm{d}\mu_0 - G_{\mathrm{ext}}(\delta\boldsymbol{\Phi}). \tag{6.28}$$

We emphasized once more that by employing the material representation of the director field variation, the weak form (6.28) is in a "five degree of freedom" format with the constraint condition $\boldsymbol{t}\cdot\delta\boldsymbol{t}=0$ automatically satisfied through the material relation $\delta\boldsymbol{T}\cdot\boldsymbol{E}_3=0$.

REMARK 6.2. For the case of hyperelastic response with properly invariant stored energy function $\psi(\boldsymbol{\xi},\ \varepsilon_{\alpha\beta},\ \delta_\alpha,\ \rho_{\alpha\beta};\ a_{0\alpha\beta},\ \gamma_{0\alpha},\ \lambda^{\mu}_{0\alpha},\ \lambda^{3}_{0\alpha})$, $G(\boldsymbol{\Phi},\delta\boldsymbol{\Phi})$, as given by (6.28), is the first variation of the functional

$$\Pi := \int_{\bar{\mathscr{A}}} \bar{\rho}\psi(\boldsymbol{\xi},\ \varepsilon_{\alpha\beta},\ \delta_\alpha,\ \rho_{\alpha\beta};\ a_{0\alpha\beta},\ \gamma_{0\alpha},\ \lambda^{\mu}_{0\alpha},\ \lambda^{3}_{0\alpha})\,\mathrm{d}\mu + \Pi_{\mathrm{ext}}\,, \tag{6.29}$$

where

$$\Pi_{\mathrm{ext}} := \int_{\bar{\mathscr{A}}} [\tilde{\boldsymbol{n}}\cdot\boldsymbol{\varphi} + \bar{\tilde{\boldsymbol{m}}}\cdot\boldsymbol{t}]\,\mathrm{d}\mu - \int_{\partial_n\mathscr{A}} \bar{\bar{\boldsymbol{n}}}\cdot\boldsymbol{\varphi}\bar{j}\,\mathrm{d}s - \int_{\partial_m\mathscr{A}} \bar{\bar{\boldsymbol{m}}}\cdot\boldsymbol{t}\bar{j}\,\mathrm{d}s \tag{6.30}$$

is the potential energy of the external loading (we assume "dead loading"). □

We now proceed to the linearization of the weak form (6.28).

6.4.2. Material part

The material part, $\mathrm{D_M G}\cdot\Delta\boldsymbol{\Phi}$, of the tangent operator arises as a result of linearizing the constitutive equation at fixed geometry. For illustrative purposes, we consider elastic response. Accordingly,

$$\mathrm{D}[\boldsymbol{R}]\cdot\Delta\boldsymbol{\Phi} = \mathbb{C}\mathbb{B}\begin{Bmatrix}\Delta\boldsymbol{\varphi}\\ \Delta\boldsymbol{T}\end{Bmatrix}, \tag{6.31}$$

where

$$\mathbb{C} = \bar{\rho}_0\begin{bmatrix}
\frac{\partial^2\psi}{\partial\varepsilon_{11}\partial\varepsilon_{11}} & \frac{\partial^2\psi}{\partial\varepsilon_{11}\partial\varepsilon_{22}} & \frac{\partial^2\psi}{\partial\varepsilon_{11}\partial\varepsilon_{12}} & \frac{\partial^2\psi}{\partial\varepsilon_{11}\partial\delta_1} & \frac{\partial^2\psi}{\partial\varepsilon_{11}\partial\delta_2} & \frac{\partial^2\psi}{\partial\varepsilon_{11}\partial\rho_{11}} & \frac{\partial^2\psi}{\partial\varepsilon_{11}\partial\rho_{22}} & \frac{\partial^2\psi}{\partial\varepsilon_{11}\partial\rho_{12}}\\
\frac{\partial^2\psi}{\partial\varepsilon_{22}\partial\varepsilon_{11}} & \frac{\partial^2\psi}{\partial\varepsilon_{22}\partial\varepsilon_{22}} & \frac{\partial^2\psi}{\partial\varepsilon_{22}\partial\varepsilon_{12}} & \frac{\partial^2\psi}{\partial\varepsilon_{22}\partial\delta_1} & \frac{\partial^2\psi}{\partial\varepsilon_{22}\partial\delta_2} & \frac{\partial^2\psi}{\partial\varepsilon_{22}\partial\rho_{11}} & \frac{\partial^2\psi}{\partial\varepsilon_{22}\partial\rho_{22}} & \frac{\partial^2\psi}{\partial\varepsilon_{22}\partial\rho_{12}}\\
\frac{\partial^2\psi}{\partial\varepsilon_{12}\partial\varepsilon_{11}} & \frac{\partial^2\psi}{\partial\varepsilon_{12}\partial\varepsilon_{22}} & \frac{\partial^2\psi}{\partial\varepsilon_{12}\partial\varepsilon_{12}} & \frac{\partial^2\psi}{\partial\varepsilon_{12}\partial\delta_1} & \frac{\partial^2\psi}{\partial\varepsilon_{12}\partial\delta_2} & \frac{\partial^2\psi}{\partial\varepsilon_{12}\partial\rho_{11}} & \frac{\partial^2\psi}{\partial\varepsilon_{12}\partial\rho_{22}} & \frac{\partial^2\psi}{\partial\varepsilon_{12}\partial\rho_{12}}\\
\frac{\partial^2\psi}{\partial\delta_1\partial\varepsilon_{11}} & \frac{\partial^2\psi}{\partial\delta_1\partial\varepsilon_{22}} & \frac{\partial^2\psi}{\partial\delta_1\partial\varepsilon_{12}} & \frac{\partial^2\psi}{\partial\delta_1\partial\delta_1} & \frac{\partial^2\psi}{\partial\delta_1\partial\delta_2} & \frac{\partial^2\psi}{\partial\delta_1\partial\rho_{11}} & \frac{\partial^2\psi}{\partial\delta_1\partial\rho_{22}} & \frac{\partial^2\psi}{\partial\delta_1\partial\rho_{12}}\\
\frac{\partial^2\psi}{\partial\delta_2\partial\varepsilon_{11}} & \frac{\partial^2\psi}{\partial\delta_2\partial\varepsilon_{22}} & \frac{\partial^2\psi}{\partial\delta_2\partial\varepsilon_{12}} & \frac{\partial^2\psi}{\partial\delta_2\partial\delta_1} & \frac{\partial^2\psi}{\partial\delta_2\partial\delta_2} & \frac{\partial^2\psi}{\partial\delta_2\partial\rho_{11}} & \frac{\partial^2\psi}{\partial\delta_2\partial\rho_{22}} & \frac{\partial^2\psi}{\partial\delta_2\partial\rho_{12}}\\
\frac{\partial^2\psi}{\partial\rho_{11}\partial\varepsilon_{11}} & \frac{\partial^2\psi}{\partial\rho_{11}\partial\varepsilon_{22}} & \frac{\partial^2\psi}{\partial\rho_{11}\partial\varepsilon_{12}} & \frac{\partial^2\psi}{\partial\rho_{11}\partial\delta_1} & \frac{\partial^2\psi}{\partial\rho_{11}\partial\delta_2} & \frac{\partial^2\psi}{\partial\rho_{11}\partial\rho_{11}} & \frac{\partial^2\psi}{\partial\rho_{11}\partial\rho_{22}} & \frac{\partial^2\psi}{\partial\rho_{11}\partial\rho_{12}}\\
\frac{\partial^2\psi}{\partial\rho_{22}\partial\varepsilon_{11}} & \frac{\partial^2\psi}{\partial\rho_{22}\partial\varepsilon_{22}} & \frac{\partial^2\psi}{\partial\rho_{22}\partial\varepsilon_{12}} & \frac{\partial^2\psi}{\partial\rho_{22}\partial\delta_1} & \frac{\partial^2\psi}{\partial\rho_{22}\partial\delta_2} & \frac{\partial^2\psi}{\partial\rho_{22}\partial\rho_{11}} & \frac{\partial^2\psi}{\partial\rho_{22}\partial\rho_{22}} & \frac{\partial^2\psi}{\partial\rho_{22}\partial\rho_{12}}\\
\frac{\partial^2\psi}{\partial\rho_{12}\partial\varepsilon_{11}} & \frac{\partial^2\psi}{\partial\rho_{12}\partial\varepsilon_{22}} & \frac{\partial^2\psi}{\partial\rho_{12}\partial\varepsilon_{12}} & \frac{\partial^2\psi}{\partial\rho_{12}\partial\delta_1} & \frac{\partial^2\psi}{\partial\rho_{12}\partial\delta_2} & \frac{\partial^2\psi}{\partial\rho_{12}\partial\rho_{11}} & \frac{\partial^2\psi}{\partial\rho_{12}\partial\rho_{22}} & \frac{\partial^2\psi}{\partial\rho_{12}\partial\rho_{12}}
\end{bmatrix}$$

is the symmetric tangent elasticity tensor. (6.32)

6.4.3. Geometric part

The geometric part, $\mathrm{D_G G}\cdot\Delta\boldsymbol{\Phi}$, of the tangent operator arises from linearizing the geometric part of the static weak form with the material stress resultants held constant. With straight forward manipulations, it can be shown that

$$\left\{\mathrm{D}\mathbb{B}\begin{Bmatrix}\delta\boldsymbol{\varphi}\\ \delta\boldsymbol{T}\end{Bmatrix}\cdot\Delta\boldsymbol{\Phi}\right\}\cdot\boldsymbol{R}=\boldsymbol{Y}\begin{Bmatrix}\delta\boldsymbol{\varphi}\\ \delta\boldsymbol{T}\end{Bmatrix}\cdot\boldsymbol{K}_{\mathrm{G}}\boldsymbol{Y}\begin{Bmatrix}\Delta\boldsymbol{\varphi}\\ \Delta\boldsymbol{T}\end{Bmatrix}\bar{j}\,, \tag{6.33}$$

where

$$\boldsymbol{K}_{\mathrm{G}}=\begin{bmatrix}\tilde{n}^{11}\mathbf{1}_3 & \tilde{n}^{12}\mathbf{1}_3 & \tilde{q}^{1}\mathbf{1}_3 & \tilde{m}^{11}\mathbf{1}_3 & \tilde{m}^{12}\mathbf{1}_3\\ \tilde{n}^{12}\mathbf{1}_3 & \tilde{n}^{22}\mathbf{1}_3 & \tilde{q}^{2}\mathbf{1}_3 & \tilde{m}^{12}\mathbf{1}_3 & \tilde{m}^{22}\mathbf{1}_3\\ \tilde{q}^{1}\mathbf{1}_3 & \tilde{q}^{2}\mathbf{1}_3 & -(\tilde{m}^{\alpha\beta}\kappa_{\alpha\beta}+\tilde{q}^{\alpha}\gamma_\alpha)\mathbf{1}_3 & -\tilde{m}^{1\alpha}\gamma_\alpha\mathbf{1}_3 & -\tilde{m}^{2\alpha}\gamma_\alpha\mathbf{1}_3\\ \tilde{m}^{11}\mathbf{1}_3 & \tilde{m}^{12}\mathbf{1}_3 & -\tilde{m}^{1\alpha}\gamma_\alpha\mathbf{1}_3 & \mathbf{0}_3 & \mathbf{0}_3\\ \tilde{m}^{12}\mathbf{1}_3 & \tilde{m}^{22}\mathbf{1}_3 & -\tilde{m}^{2\alpha}\gamma_\alpha\mathbf{1}_3 & \mathbf{0}_3 & \mathbf{0}_3\end{bmatrix}_{15\times 15} \tag{6.34}$$

and

$$\boldsymbol{Y}=\begin{bmatrix}\mathbf{1}_3\dfrac{\partial}{\partial\xi^1} & \mathbf{0}_3\\ \mathbf{1}_3\dfrac{\partial}{\partial\xi^2} & \mathbf{0}_3\\ \mathbf{0}_3 & \mathbf{1}_3\\ \mathbf{0}_3 & \mathbf{1}_3\dfrac{\partial}{\partial\xi^1}\\ \mathbf{0}_3 & \mathbf{1}_3\dfrac{\partial}{\partial\xi^2}\end{bmatrix}\begin{bmatrix}\mathbf{1}_3 & \mathbf{0}_{3\times 2}\\ \mathbf{0}_3 & \boldsymbol{\Lambda}_{3\times 2}\end{bmatrix}. \tag{6.35}$$

Note that $\boldsymbol{K}_{\mathrm{G}}$, the geometric tangent stiffness, is symmetric.

REMARK 6.3. The primary variables in the linearization process (i.e. the variations to be held constant in the linearization process) are $(\delta\boldsymbol{\varphi},\delta\boldsymbol{\theta})$, where $\delta\boldsymbol{t}=\delta\boldsymbol{\theta}\times\boldsymbol{t}$. Thus $\Delta(\delta\boldsymbol{\theta})\equiv\mathbf{0}$, where $\Delta(\delta\boldsymbol{t})\neq\mathbf{0}$. In fact,

$$\begin{aligned}\Delta(\delta\boldsymbol{t})&=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}(\delta\boldsymbol{\theta}\times\boldsymbol{t}_\varepsilon)\\ &=\delta\boldsymbol{\theta}\times\Delta\boldsymbol{t}\\ &=-(\delta\boldsymbol{t}\cdot\Delta\boldsymbol{t})\boldsymbol{t}\,.\qquad\square\end{aligned} \tag{6.36}$$

The tangent operator is now expressed as

$$\begin{aligned}\mathrm{DG}(\boldsymbol{\Phi},\delta\boldsymbol{\Phi})\cdot\Delta\boldsymbol{\Phi}&=\mathrm{D_M G}(\boldsymbol{\Phi},\delta\boldsymbol{\Phi})\cdot\Delta\boldsymbol{\Phi}+\mathrm{D_G G}(\boldsymbol{\Phi},\delta\boldsymbol{\Phi})\cdot\Delta\boldsymbol{\Phi}\\ &=\int_{\mathscr{A}}\mathbb{B}\begin{Bmatrix}\delta\boldsymbol{\varphi}\\ \delta\boldsymbol{T}\end{Bmatrix}\cdot\mathbb{C}\mathbb{B}\begin{Bmatrix}\Delta\boldsymbol{\varphi}\\ \Delta\boldsymbol{T}\end{Bmatrix}\mathrm{d}\mu_0\\ &\quad+\int_{\mathscr{A}}\boldsymbol{Y}\begin{Bmatrix}\delta\boldsymbol{\varphi}\\ \delta\boldsymbol{T}\end{Bmatrix}\cdot\boldsymbol{K}_{\mathrm{G}}\boldsymbol{Y}\begin{Bmatrix}\Delta\boldsymbol{\varphi}\\ \Delta\boldsymbol{T}\end{Bmatrix}\bar{j}\,\mathrm{d}\mu_0\,.\end{aligned} \tag{6.37}$$

7. Closure and overview of subsequent research

We have presented a careful development of the continuum basis of a geometrically exact shell theory obtained by reduction of the three-dimensional theory by means of the *one inextensible director* kinematic assumption. Our subsequent work will be concerned with numerical analysis and computational aspects involved in the implementation of the present formulation within the framework of the finite element method. In particular, we plan to focus on the following aspects:

(a) *The linearized theory*. The relevant field equations are obtained simply by particularizing the results developed above to the reference configuration. The linear theory provides an ideal framework to address, in detail, issues concerned with interpolation and convergence of the finite element procedure. We provide a detailed account of the proposed interpolation, develop explicit matrix expressions, and assess the performance of the model in standard benchmark problems, as well as new test problems. Our results indicate that the present theory exactly matches results obtained with the degenerated solid approach.

(b) *Computational aspects of the fully nonlinear theory*. The central topic to be addressed is the development of discrete update procedure for the director field which is *exact* and *singularity free* regardless of the magnitude of the rotational increments. In contrast with current procedures, often only second-order accurate, the proposed approach is exact, and preserves objectivity for arbitrary large displacement and rotation increments. Computationally, the need for using quaternion parameters to avoid singularities is bypassed. Detailed account of the interpolation procedure, and expressions for the residual and consistent tangent matrix are given. The formulation is assessed through numerical simulations involving instability and bifurcation phenomena.

(c) *Enhanced models including through the thickness stretch*. The model considered in this paper can be easily extended to accommodate thickness changes in the shell by means of a one *extensible* director kinematic assumption. It is shown that this additional effect adds very little complexity to the model while allowing for an entirely new class of problems to be addressed. The advantages of such a model include: exact recovery of the plane stress constitutive equations in the thin shell limit, explicit account of the through-the-thickness-stress for problems dominated by behavior such as delamination and avoidance of numerical problems associated with the undetermined "pressure-like" inextensible constraint.

(d) *Inelasticity in stress resultants at finite strains*. Elastoplastic constitutive models at finite strains that incorporate classical kinematic and isotropic hardening rules are formulated entirely in stress resultants. The formulation completely bypasses the need for the costly integration-through-the-thickness procedure in the traditional degenerated solid formulation. The integration of the elastoplastic equations is performed by means of recently proposed general return mapping algorithms that can accommodate non-smooth yield surfaces.

(e) *Nonlinear dynamics. Computational aspects*. The present geometric setting, and the associated Hamiltonian structure considered in [39] can be exploited to implement time stepping algorithms that exactly preserve the constants of motions. Recent work indicates that one can develop time stepping algorithms which are, in fact, *discrete canonical transformations*, thus preserving the discrete Hamiltonian structure of the model. We plan to address these and related issues in our subsequent work.

Appendix A. Balance equations

For completeness, we include in this appendix a derivation of the resultant form of the balance of linear and angular momentum from the three-dimensional integral balance laws. For a detailed discussion in the general context of Cosserat surfaces, see [35, 36]. An alternative derivation is given in [32].

A.1. Linear momentum

The three-dimensional integral equation of balance of linear momentum is written

$$\iint_{\partial\mathcal{V}} \boldsymbol{\sigma}\hat{\boldsymbol{n}}\,\mathrm{d}\mathcal{S} + \iiint_{\mathcal{V}} \boldsymbol{B}\rho\,\mathrm{d}\mathcal{V} = \iiint_{\mathcal{V}} \rho\ddot{\boldsymbol{\Phi}}\,\mathrm{d}v\,, \tag{A.1}$$

where $\hat{\boldsymbol{n}}$ is the unit normal to the arbitrary spatial volume $\mathcal{V}$ with boundary $\partial\mathcal{V}$ and $\boldsymbol{B}$ is the body force per unit mass. By using the kinematic assumption (3.3), the normal map relation $\hat{\boldsymbol{n}}\,\mathrm{d}\mathcal{S} = j\nabla\boldsymbol{\Phi}^{-\mathrm{t}}\hat{\boldsymbol{N}}\,\mathrm{d}\Gamma$ ($\hat{\boldsymbol{N}}$ is the normal to $\bar{\mathcal{A}}\times[h^-, h^+]$, with surface element $\mathrm{d}\Gamma$), and the change of variables relation, (A.1) becomes

$$\iiint_{\partial(\mathcal{A}\times[h^-,h^+])} j\boldsymbol{\sigma}\nabla\boldsymbol{\Phi}^{-\mathrm{t}}\hat{\boldsymbol{N}}\,\mathrm{d}\Gamma + \iiint_{\bar{\mathcal{A}}\times[h^-,h^+]} j\rho\boldsymbol{B}\,\mathrm{d}\mathcal{A}\,\mathrm{d}\xi = \iiint_{\bar{\mathcal{A}}\times[h^-,h^+]} j\rho[\ddot{\boldsymbol{\varphi}} + \xi\ddot{\boldsymbol{t}}]\,\mathrm{d}\mathcal{A}\,\mathrm{d}\xi\,. \tag{A.2}$$

We now decompose the surface integral into an integral over the lateral surfaces ($\hat{\boldsymbol{N}} = \nu_\alpha \boldsymbol{E}^\alpha$) and an integral over the top and bottom surfaces ($\hat{\boldsymbol{N}} = \pm\boldsymbol{E}^3$). Using the definitions of the reciprocal basis (3.12) and the mid-surface (3.22) we have

$$\begin{aligned}&\int_{\partial\mathcal{A}} \left[\int_{h^-}^{h^+} j\boldsymbol{\sigma}\boldsymbol{g}^\alpha\,\mathrm{d}\xi\right]\nu_\alpha\,\mathrm{d}\gamma + \iint_{\bar{\mathcal{A}}} [(j\boldsymbol{\sigma}\boldsymbol{g}^3)|_{\xi=h^-}^{\xi=h^+}]\,\mathrm{d}\mathcal{A} \\ &\quad + \iint_{\bar{\mathcal{A}}} \left[\int_{h^-}^{h^+} j\rho\boldsymbol{B}\,\mathrm{d}\xi\right]\mathrm{d}\mathcal{A} = \iint_{\bar{\mathcal{A}}} \left[\int_{h^-}^{h^+} j\rho\,\mathrm{d}\xi\right]\ddot{\boldsymbol{\varphi}}\,\mathrm{d}\mathcal{A}\,.\end{aligned} \tag{A.3}$$

Recalling the definition of the stress resultant (4.5) and by using the divergence theorem for surfaces, the first term in (A.3) can be written

$$\int_{\partial\mathcal{A}} \left[\int_{h^-}^{h^+} j\boldsymbol{\sigma}\boldsymbol{g}^\alpha\,\mathrm{d}\xi\right]\nu_\alpha\,\mathrm{d}\gamma = \iint_{\bar{\mathcal{A}}} (\bar{j}\boldsymbol{n}^\alpha)_{,\alpha}\,\mathrm{d}\mathcal{A}\,. \tag{A.4}$$

Defining the surface loading term

$$\bar{\boldsymbol{n}} = \frac{1}{\bar{j}}\left[(j\boldsymbol{\sigma}\boldsymbol{g}^3)|_{\xi=h^-}^{\xi=h^+} + \int_{h^-}^{h^+} j\rho\boldsymbol{B}\,\mathrm{d}\xi\right] \tag{A.5}$$

and using definiton (3.23), we have

$$\iint_{\bar{\mathscr{A}}} \left[\frac{1}{\bar{j}}(\bar{j}\boldsymbol{n}^{\alpha})_{,\alpha} + \bar{\boldsymbol{n}}\right]\bar{j}\,\mathrm{d}\mathscr{A} = \iint_{\bar{\mathscr{A}}} \bar{\rho}\ddot{\boldsymbol{\varphi}}\bar{j}\,\mathrm{d}\mathscr{A}\,. \tag{A.6}$$

Since the choice of $\mathscr{V}$ (or equivalently $\mathscr{A}$) is arbitrary, assuming enough smoothness so that the standard localization result holds; see [26, p. 38], we obtain the resultant form of the local balance of linear momentum (4.12a)

$$\frac{1}{\bar{j}}(\bar{j}\boldsymbol{n}^{\alpha})_{,\alpha} + \bar{\boldsymbol{n}} = \bar{\rho}\ddot{\boldsymbol{\varphi}}\,. \tag{A.7}$$

A.2. Angular momentum

The three-dimensional integral equation of balance of angular momentum is written

$$\iint_{\partial\mathscr{V}} \boldsymbol{\Phi}\times\boldsymbol{\sigma}\hat{\boldsymbol{n}}\,\mathrm{d}\mathscr{S} + \iiint_{\mathscr{V}} \boldsymbol{\Phi}\times\rho\boldsymbol{B}\,\mathrm{d}\mathscr{V} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathscr{V}} \boldsymbol{\Phi}\times\rho\dot{\boldsymbol{\Phi}}\,\mathrm{d}\mathscr{V}\,, \tag{A.8}$$

where the definitions in (A.1) apply. Again employing the kinematic assumption (3.3), the normal map relation $\hat{\boldsymbol{n}}\,\mathrm{d}\mathscr{S} = j\nabla\boldsymbol{\Phi}^{-\mathrm{t}}\hat{\boldsymbol{N}}\,\mathrm{d}\Gamma$, and the change of variables relation, (A.8) becomes

$$\begin{aligned}&\iint_{\partial(\mathscr{A}\times[h^-,h^+])} (\boldsymbol{\varphi}+\xi\boldsymbol{t})\times j\boldsymbol{\sigma}\nabla\boldsymbol{\Phi}^{-\mathrm{t}}\hat{\boldsymbol{N}}\,\mathrm{d}\Gamma + \iiint_{\bar{\mathscr{A}}\times[h^-,h^+]} (\boldsymbol{\varphi}+\xi\boldsymbol{t})\times j\rho\boldsymbol{B}\,\mathrm{d}\xi\,\mathrm{d}\mathscr{A}\\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\bar{\mathscr{A}}\times[h^-,h^+]} (\boldsymbol{\varphi}+\xi\boldsymbol{t})\times j\rho(\dot{\boldsymbol{\varphi}}+\xi\dot{\boldsymbol{t}})\,\mathrm{d}\xi\,\mathrm{d}\mathscr{A}\,.\end{aligned} \tag{A.9}$$

Recall that the normals to the lateral, top, and bottom surfaces are given by $\hat{\boldsymbol{N}} = \nu_{\alpha}\boldsymbol{E}^{\alpha}$, $\hat{\boldsymbol{N}} = \boldsymbol{E}^3$, and $\hat{\boldsymbol{N}} = -\boldsymbol{E}^3$, respectively. Thus, with the definition of the reciprocal basis (3.12) and the definition of the mid-surface (3.22) we can write

$$\begin{aligned}&\int_{\partial\mathscr{A}} \boldsymbol{\varphi}\times\left[\int_{h^-}^{h^+} j\boldsymbol{\sigma}\boldsymbol{g}^{\alpha}\,\mathrm{d}\xi\right]\nu_{\alpha}\,\mathrm{d}\gamma + \int_{\partial\mathscr{A}} \boldsymbol{t}\times\left[\int_{h^-}^{h^+} \xi j\boldsymbol{\sigma}\boldsymbol{g}^{\alpha}\,\mathrm{d}\xi\right]\nu_{\alpha}\,\mathrm{d}\gamma\\ &\quad + \iint_{\bar{\mathscr{A}}} [\boldsymbol{\varphi}\times(j\boldsymbol{\sigma}\boldsymbol{g}^3)|_{\xi=h^-}^{\xi=h^+} + \boldsymbol{t}\times(\xi j\boldsymbol{\sigma}\boldsymbol{g}^3)|_{\xi=h^-}^{\xi=h^+}]\,\mathrm{d}\mathscr{A} + \iint_{\bar{\mathscr{A}}} \left[\boldsymbol{\varphi}\times\left(\int_{h^-}^{h^+} j\rho\boldsymbol{B}\,\mathrm{d}\xi\right)\right]\mathrm{d}\mathscr{A}\\ &\quad + \iint_{\bar{\mathscr{A}}} \boldsymbol{t}\times\left[\left(\int_{h^-}^{h^+} \xi j\rho\boldsymbol{B}\,\mathrm{d}\xi\right)\right]\mathrm{d}\mathscr{A} = \iint_{\bar{\mathscr{A}}} \left[\left(\int_{h^-}^{h^+} j\rho\,\mathrm{d}\xi\right)\boldsymbol{\varphi}\times\ddot{\boldsymbol{\varphi}}\right.\\ &\quad \left. + \left(\int_{h^-}^{h^+} \xi^2 j\rho\,\mathrm{d}\xi\right)\boldsymbol{t}\times\ddot{\boldsymbol{t}}\right]\mathrm{d}\mathscr{A}\,.\end{aligned} \tag{A.10}$$

Using the stress and stress couple resultant definitions (4.5), (A.5), (3.23), and (3.24) and defining the external couple as

$$\bar{\boldsymbol{m}} = \frac{1}{\bar{j}}\, \boldsymbol{t} \times \left[(\xi j \boldsymbol{\sigma} \boldsymbol{g}^3)\big|_{\xi=h^-}^{\xi=h^+} + \int_{h^-}^{h^+} \xi j \rho \boldsymbol{B}\, \mathrm{d}\xi \right], \tag{A.11}$$

we can rephrase (A.10) as

$$\iint_{\bar{\mathscr{A}}} \left[\frac{1}{\bar{j}} (\bar{j} \boldsymbol{m}^\alpha)_{,\alpha} + \boldsymbol{\varphi}_{,\alpha} \times \boldsymbol{n}^\alpha + \bar{\boldsymbol{m}} \right] \bar{j}\, \mathrm{d}\mathscr{A} = \iint_{\bar{\mathscr{A}}} I_\rho \dot{\boldsymbol{w}}_t \bar{j}\, \mathrm{d}\mathscr{A}\ , \tag{A.12}$$

where the divergence theorem, balance of linear momentum (A.7), and the relation $\dot{\boldsymbol{w}}_t = \boldsymbol{t} \times \ddot{\boldsymbol{t}}$ were used. Since $\mathscr{V}$ (or equivalently $\mathscr{A}$) can be chosen arbitrarily, assuming enough smoothness we obtain the local resultant form of the balance of angular momentum (4.12b) as

$$\frac{1}{\bar{j}} (\bar{j} \boldsymbol{m}^\alpha)_{,\alpha} + \boldsymbol{\varphi}_{,\alpha} \times \boldsymbol{n}^\alpha + \bar{\boldsymbol{m}} = I_\rho \dot{\boldsymbol{w}}_t\,. \tag{A.13}$$

Appendix B. Stress resultant components

In this appendix we derive explicit component expressions for the stress and stress couple resultants from the three-dimensional theory. First, we present some preliminary results that are helpful in the following development:

$$\begin{aligned} &\boldsymbol{g}_I = \boldsymbol{\Phi}_{,I} \Rightarrow \boldsymbol{g}_\alpha = \boldsymbol{a}_\alpha + \xi \boldsymbol{t}_{,\alpha}\,, \qquad \boldsymbol{g}_3 = \boldsymbol{t}\,, \\ &\boldsymbol{t}_{,\alpha} = \lambda^\mu_\alpha \boldsymbol{a}_\mu + \lambda^3_\alpha \boldsymbol{t}\,, \qquad \boldsymbol{\sigma} \boldsymbol{g}^\alpha = \sigma^{\beta\alpha} \boldsymbol{g}_\beta + \sigma^{3\alpha} \boldsymbol{t}\,. \end{aligned} \tag{B.1}$$

From definitions (4.7) and (4.11) we recall that

$$\boldsymbol{n}^\alpha = \frac{1}{\bar{j}} \int_{h^-}^{h^+} \boldsymbol{\sigma} \boldsymbol{g}^\alpha j\, \mathrm{d}\xi\ , \qquad \tilde{\boldsymbol{m}}^\alpha = \frac{1}{\bar{j}} \int_{h^-}^{h^+} \xi \boldsymbol{\sigma} \boldsymbol{g}^\alpha j\, \mathrm{d}\xi\ . \tag{B.2}$$

Substitution of (B.1) into (B.2) yields

$$\begin{aligned} \boldsymbol{n}^\alpha &= \frac{1}{\bar{j}} \int_{h^-}^{h^+} \sigma^{\beta\alpha} (\boldsymbol{a}_\beta + \xi \lambda^\mu_\beta \boldsymbol{a}_\mu) j\, \mathrm{d}\xi + \frac{1}{\bar{j}} \int_{h^-}^{h^+} (\sigma^{3\alpha} + \xi \lambda^3_\beta \sigma^{\beta\alpha}) \boldsymbol{t} j\, \mathrm{d}\xi\ , \\ \tilde{\boldsymbol{m}}^\alpha &= \frac{1}{\bar{j}} \int_{h^-}^{h^+} \xi \sigma^{\beta\alpha} (\boldsymbol{a}_\beta + \xi \lambda^\mu_\beta \boldsymbol{a}_\mu) j\, \mathrm{d}\xi + \frac{1}{\bar{j}} \int_{h^-}^{h^+} \xi (\sigma^{3\alpha} + \xi \sigma^{\beta\alpha} \lambda^3_\beta) \boldsymbol{t} j\, \mathrm{d}\xi\ . \end{aligned} \tag{B.3}$$

From component expressions (4.22) relative to the surface convected basis $\{\boldsymbol{a}_\alpha, \boldsymbol{t}\}$ we obtain

$$n^{\beta\alpha} = \frac{1}{\bar{j}} \int_{h^-}^{h^+} (\sigma^{\beta\alpha} + \xi \lambda^\beta_\mu \sigma^{\mu\alpha}) j\, \mathrm{d}\xi\ , \tag{B.4a}$$

$$q^\alpha = \frac{1}{\bar{j}} \int_{h^-}^{h^+} (\sigma^{3\alpha} + \xi \lambda^3_\beta \sigma^{\beta\alpha}) j\, \mathrm{d}\xi\ , \tag{B.4b}$$

$$\tilde{m}^{\beta\alpha} = \frac{1}{\bar{j}} \int_{h^-}^{h^+} (\xi\sigma^{\beta\alpha} + \xi^2 \lambda^{\beta}_{\mu} \sigma^{\mu\alpha}) j \, d\xi \,, \tag{B.4c}$$

$$\tilde{m}^{3\alpha} = \frac{1}{\bar{j}} \int_{h^-}^{h^+} (\xi\sigma^{3\alpha} + \xi^2 \sigma^{\mu\alpha} \lambda^{3}_{\mu}) j \, d\xi \,. \tag{B.4d}$$

From (4.25) and (4.27) we now obtain component relations for the effective membrane and shear stress resultants as

$$\begin{aligned} \tilde{n}^{\beta\alpha} &= n^{\beta\alpha} - \lambda^{\beta}_{\mu} \tilde{m}^{\alpha\mu} \\ &= \frac{1}{\bar{j}} \int_{h^-}^{h^+} (\sigma^{\beta\alpha} - \xi^2 \lambda^{\beta}_{\mu} \sigma^{\mu\gamma} \lambda^{\alpha}_{\gamma}) j \, d\xi \,, \\ \tilde{q}^{\alpha} &= q^{\alpha} - \lambda^{3}_{\mu} \tilde{m}^{\alpha\mu} \\ &= \frac{1}{\bar{j}} \int_{h^-}^{h^+} (\sigma^{3\alpha} + \xi^2 \gamma_{\lambda} \lambda^{\lambda}_{\mu} \sigma^{\mu\gamma} \lambda^{\alpha}_{\gamma}) j \, d\xi \,. \end{aligned} \tag{B.5}$$

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