Chap8. Mixed and Hybrid Formulation

Hellinger - Reissner variational Principle

$\left\{ \begin{array}{l} \text{Equilibrium eqn} \\ \text{constraint equation} \end{array} \right\}$ 연립 방정식.

식 2개 (set)

미지수 2개 (set) $\left\{ \begin{array}{l} \text{assumed displacement field} \\ \text{assumed strain or stress field} \end{array} \right\}$.

Boundary value problem 은 PVW (principle of virtual work) 를 이용하여 다음과 같이 표시.

$$\boxed{\delta\Pi = \int_V \delta\bar{\varepsilon}^T \sigma \, dv - \int_V \delta u^T F_B \, dv - \int_{S_\sigma} \delta u^T T \, ds = 0} \quad (1)$$

equilibrium equation

$$\delta\Pi = \delta U - \delta W = 0 \quad \text{or} \quad \delta U = \delta W$$

$\left\{ \begin{array}{l} \text{virtual strain energy} \\ \text{or internal virtual work} \end{array} \right\}$ = external virtual work

여기서 $\sigma = C\,(\varepsilon^{total} - \varepsilon^{o}) = C\,\varepsilon^{m}$

- C: constitutive matrix
- ε^{total}: total strain
- ε^{o}: initial strain
- ε^{m}: mechanical strain

$\bar{\varepsilon} = B\,q$ = displacement-dependent strain if 3-d strain state,

$$\underset{\sim}{\bar{\varepsilon}} = \lfloor \bar{\varepsilon}_{xx},\ \bar{\varepsilon}_{yy},\ \bar{\varepsilon}_{zz},\ \bar{\varepsilon}_{xy},\ \bar{\varepsilon}_{yz},\ \bar{\varepsilon}_{zx} \rfloor^T$$

$$\underset{\sim}{B} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} = \text{matrix of differential operator}$$

$$\underset{\sim}{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

& Boundary Condition $\begin{Bmatrix} u = \bar{u} \\ v = \bar{v} \\ w = \bar{w} \end{Bmatrix}$ on S_u

stress - strain relation 에서

$$\underset{\sim}{C}(\underset{\sim}{\bar{\varepsilon}} - \underset{\sim}{\varepsilon}^{\circ}) - \underset{\sim}{\sigma} = 0$$

or $\underset{\sim}{\bar{\varepsilon}} - \underset{\sim}{\varepsilon}^{\circ} - \underset{\sim}{S}\,\underset{\sim}{\sigma} = 0.$

$\underset{\sim}{S}$ = flexibility matrix
or Compliance matrix

$= \underset{\sim}{C}^{-1}$

stress - strain relation을 이용하여 다음 적분식 구성

$$\boxed{\delta I = \int_V \delta \underset{\sim}{\sigma}^T (\underset{\sim}{\bar{\varepsilon}} - \underset{\sim}{\varepsilon}^{\circ} - \underset{\sim}{S}\,\underset{\sim}{\sigma})\, dv = 0.} \qquad (2)$$

constraint equation or compatibility equation.

$\underset{\sim}{\sigma}$ = independent stress vector

$\delta \underset{\sim}{\sigma}$ = virtual independent stress vector

(Note) constraint equation은 stress - strain relation을

strain energy의 integral sense 에서 만족

Therefore, $\boxed{\delta \Pi_R = \delta \Pi + \delta I = 0}$

$\delta \Pi_R = 0$ 는 $\left\{ \begin{array}{ll} \delta \Pi = 0 & (1)\text{식} \\ \delta I = 0 & (2)\text{식} \end{array} \right\}$ 을 의미한다.

$\delta \Pi_R = 0$ 는 2개의 연립방정식

→ 미지수 2개 $\underset{\sim}{\sigma}$, $\underset{\sim}{\bar{\varepsilon}}$ (or $\underset{\sim}{q}$)

i^{th} element 에서

(a) Assumed displacement ($\underset{\sim}{q}$)

$\underset{\sim}{u} = \underset{\sim}{N}\,\underset{\sim}{q}_i \longrightarrow \underset{\sim}{\bar{\varepsilon}} = \underset{\sim}{B}\,\underset{\sim}{q}_i$

$\delta \underset{\sim}{u} = \underset{\sim}{N}\,\delta \underset{\sim}{q}_i \longrightarrow \delta \underset{\sim}{\bar{\varepsilon}} = \underset{\sim}{B}\,\delta \underset{\sim}{q}_i$

(b) Assumed independent stress

$\underset{\sim}{\sigma} = \underset{\sim}{P}\, \underset{\sim}{\beta_i}$

$\delta\underset{\sim}{\sigma} = \underset{\sim}{P}\, \delta\underset{\sim}{\beta_i}$

여기서 $\underset{\sim}{P}$ = assumed stress polynomial matrix

$\underset{\sim}{\beta_i}$ = assumed stress parameter

(Note) assumed independent stress $\underset{\sim}{\sigma}$ 는

assumed displacement $\underset{\sim}{u}$ 또는

displacement-dependent strain $\underset{\sim}{\varepsilon}$ 와는 무관한.

별개의 독립 변수.

즉 β_i 는 q_i 와 무관.

Then, 식(1)에서 ith element 에 대해.

$$\int_{V_i} \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, dv - \int_{V_i} \delta\underset{\sim}{u}^T \underset{\sim}{F_B}\, dv - \int_{S_{\sigma_i}} \delta\underset{\sim}{u}^T \underset{\sim}{\bar{T}}\, ds$$

$$= \int_{V_i} \delta\underset{\sim}{q_i}^T \underset{\sim}{B}^T \underset{\sim}{P}\, \underset{\sim}{\beta_i}\, dv - \int_{V_i} \delta\underset{\sim}{q_i}^T \underset{\sim}{N}^T \underset{\sim}{F_B}\, dv$$

$$- \int_{S_{\sigma_i}} \delta\underset{\sim}{q_i}^T \underset{\sim}{L}^T \underset{\sim}{\bar{T}}\, ds$$

$$= \delta q_i^T \underbrace{\left[\int_{V_i} B^T P \, dv\right]}_{= G^T} \beta_i$$

$$- \delta q_i^T \underbrace{\left[\int_{V_i} N^T F_B \, dv + \int_{S_{\sigma i}} L^T \bar{T} \, ds\right]}_{= Q_i^a}$$

$$= \delta q_i^T \left[G^T \beta_i - Q_i^a \right] \qquad (3)$$

여기서 $L = N$ on $S_{\sigma i}$

식 (2) 는 ith element 에서

$$\int_{V_i} \delta \sigma^T \left(\bar{\varepsilon} - \varepsilon^0 - S \sigma \right) dv$$

$$= \int_{V_i} \delta \beta_i^T P^T B \, q_i \, dv - \int_{V_i} \delta \beta_i^T P^T \varepsilon^0 \, dv$$

$$- \int_{V_i} \delta \beta_i^T P^T S P \beta_i \, dv$$

$$= \delta \beta_i^T \underbrace{\left[\int_{V_i} P^T B \, dv\right]}_{= G} q_i - \delta \beta_i^T \underbrace{\left[\int_{V_i} P^T \varepsilon^0 \, dv\right]}_{= G^0}$$

$$- \delta \beta_i^T \underbrace{\left[\int_{V_i} P^T S P \, dv\right]}_{= H} \beta_i$$

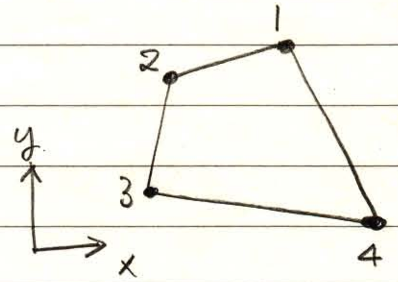
$$= \delta\underset{\sim}{\beta_i}^T \left(\underset{\sim}{G}\, \underset{\sim}{g_i} - \underset{\sim}{G_o} - \underset{\sim}{H}\, \underset{\sim}{\beta_i} \right) \qquad (4)$$

여기서 $\underset{\sim}{H}$ = symmetric ($\because$ $\underset{\sim}{S}$ = symmetric)

8.1 Mixed Formulation I

4-node plane element 에서 $\underset{\sim}{\beta_i}$ 가 ith element 의

nodal stress vector 라면



nodal dof = 5개 $\begin{Bmatrix} \sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \\ u, v \end{Bmatrix}$

element 당 8-DOF 에서 20-DOF 로 증가.

assembly 와 boundary condition 적용.

$$\delta \underset{\sim}{q}^T \left[(\underset{\sim}{G}^*)^T \underset{\sim}{\beta} - \underset{\sim}{Q}^* \right] = 0$$

$$\delta \underset{\sim}{\beta}^T \left[\underset{\sim}{G}^* \underset{\sim}{q} - \underset{\sim}{G_0}^* - \underset{\sim}{H}^* \underset{\sim}{\beta} \right] = 0$$

여기서 $\underset{\sim}{G}^*$, $\underset{\sim}{H}^*$, $\underset{\sim}{G_0}^*$, $\underset{\sim}{q}$, $\underset{\sim}{\beta}$ = global quantity

임의의 $\delta \underset{\sim}{q}$ 와 $\delta \underset{\sim}{\beta}$ 에서

$$\underset{\sim}{G}^{*T} \underset{\sim}{\beta} - \underset{\sim}{Q}^* = 0$$

$$\underset{\sim}{G}^* \underset{\sim}{q} - \underset{\sim}{G_0}^* - \underset{\sim}{H}^* \underset{\sim}{\beta} = 0$$

or $$\begin{bmatrix} \underset{\sim}{H}^{*} & -\underset{\sim}{G}^{*} \\ -\underset{\sim}{G}^{*T} & \underset{\sim}{0} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\beta} \\ \underset{\sim}{q} \end{Bmatrix} = \begin{Bmatrix} -\underset{\sim}{G_0}^{*} \\ -\underset{\sim}{Q}^{*} \end{Bmatrix} \qquad (5)$$

(Note) Mixed Formulation I is not practical

DOF 증가가 너무 크다.

8.2 Mixed Formulation II

If, stress 가 element 사이에서 연속이라는 조건이 없으면, stress parameter β_i는 ith element의 independent parameter 이다.

→ β_i를 global 에서 계산하지 않고, 각 element 에서 계산

→ β_i는 element level 에서 소거

식 (4)에서 $\delta\beta_i$는 임의 값이므로

$$\underset{\sim}{G}\,\underset{\sim}{q}_i - \underset{\sim}{G}_0 - \underset{\sim}{H}\,\underset{\sim}{\beta}_i = 0$$

$$\rightarrow \underset{\sim}{\beta}_i = \underset{\sim}{H}^{-1}\underset{\sim}{G}\,\underset{\sim}{q}_i - \underset{\sim}{H}^{-1}\underset{\sim}{G}_0 \qquad (6)$$

식 (6)을 식 (3)에 대입.

$$\delta\underset{\sim}{q}_i^T\left[\underset{\sim}{G}^T(\underset{\sim}{H}^{-1}\underset{\sim}{G}\,\underset{\sim}{q}_i - \underset{\sim}{H}^{-1}\underset{\sim}{G}_0) - \underset{\sim}{Q}_i^a\right]$$

$$= \delta\underset{\sim}{q}_i^T\left[\underbrace{\underset{\sim}{G}^T\underset{\sim}{H}^{-1}\underset{\sim}{G}}_{\underset{\sim}{K}_i}\,\underset{\sim}{q}_i - \underbrace{(\underset{\sim}{G}^T\underset{\sim}{H}^{-1}\underset{\sim}{G}_0 + \underset{\sim}{Q}_i^a)}_{\underset{\sim}{Q}_i}\right]$$

$$= \delta\underset{\sim}{q}_i^T(\underset{\sim}{K}_i\,\underset{\sim}{q}_i - \underset{\sim}{Q}_i)$$

여기서 $\underset{\sim}{K_i} = \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G}$

$$= \left[\int_{V_i} \underset{\sim}{B}^T \underset{\sim}{P}\, dv\right]\left[\int_{V_i} \underset{\sim}{P}^T \underset{\sim}{S}\, \underset{\sim}{P}\, dv\right]^{-1}\left[\int_{V_i} \underset{\sim}{P}^T \underset{\sim}{B}\, dv\right]$$

= element stiffness matrix

(Note) displacement formulation 에서

$$\underset{\sim}{K_i} = \int_{V_i} \underset{\sim}{B}^T \underset{\sim}{C}\, \underset{\sim}{B}\, dv.$$

$$\underset{\sim}{Q_i} = \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G_o} + \underset{\sim}{Q_i}^a$$

$$= \left[\int_{V_i} \underset{\sim}{B}^T \underset{\sim}{P}\, dv\right]\left[\int_{V_i} \underset{\sim}{P}^T \underset{\sim}{S}\, \underset{\sim}{P}\, dv\right]^{-1}\left[\int_{V_i} \underset{\sim}{P}^T \underset{\sim}{\varepsilon}^o\, dv\right]$$

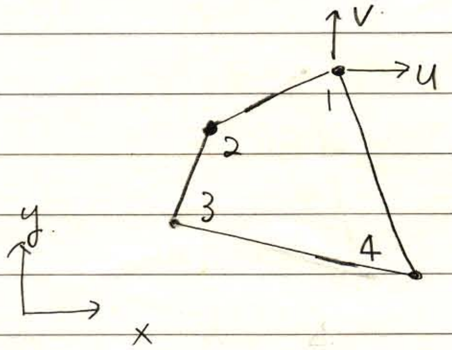
$$+ \left[\int_{V_i} \underset{\sim}{N}^T \underset{\sim}{F_B}\, dv + \int_{S_{\sigma i}} \underset{\sim}{L}^T \underset{\sim}{\bar{T}}\, ds\right]$$

= element nodal load vector

(Note) displacement formulation 에서

$$\underset{\sim}{Q_i} = \int_{V_i} \underset{\sim}{B}^T \underset{\sim}{C}\, \underset{\sim}{\varepsilon}^o\, dv + \int_{V_i} \underset{\sim}{N}^T \underset{\sim}{F_B}\, dv + \int_{S_{\sigma i}} \underset{\sim}{L}^T \underset{\sim}{\bar{T}}\, ds.$$

(example) Four-node plane element 에서



4-node

8-dof

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

assumed stress를 다음 같이 가정.

$$\left.\begin{aligned} \sigma_{xx} &= \beta_1 + \beta_2 x + \beta_3 y \\ \sigma_{yy} &= \beta_4 + \beta_5 x + \beta_6 y \\ \sigma_{xy} &= \beta_7 + \beta_8 x + \beta_9 y \end{aligned}\right\}$$

assumed displacement에서 계산한 stress와 동일한 polynomial

$$\underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & x & y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y \end{bmatrix}}_{= \underset{\sim}{P}} \underbrace{\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_9 \end{Bmatrix}}_{= \underset{\sim}{\beta}}$$

$$= \underset{\sim}{P} \underset{\sim}{\beta}$$

Assembly 와 boundary condition 적용

$$\delta \underset{\sim}{q}^T (\underset{\sim}{K}\, \underset{\sim}{q} - \underset{\sim}{F}) = 0$$

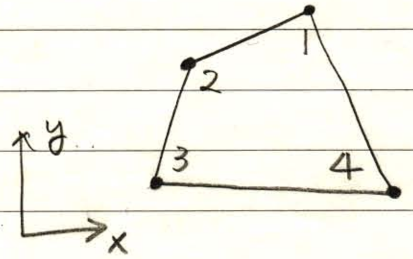
or $\underset{\sim}{K}\, \underset{\sim}{q} = \underset{\sim}{F}$

8.3 Hybrid stress formulation
(or Assumed stress hybrid formulation)

각 element 의 assumed stress 를 선택할 때

stress equilibrium equation 을 만족시키도록 선택한다.

(example) 4-node plane element
body force 가 존재하지 않는 2-d prob.



set assumed stress

$$\sigma_{xx} = \beta_1 + \beta_4 x + \beta_5 y$$

$$\sigma_{yy} = \beta_2 + \beta_6 y + \beta_7 x$$

$$\sigma_{xy} = \beta_3 - \beta_4 y - \beta_6 x$$

Then, stress equilibrium equation

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= \beta_4 - \beta_4 = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= -\beta_6 + \beta_6 = 0 \end{aligned}\right\} \text{만족}$$

$9\beta - 2\beta = 7\beta$ 사용

$$\underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x & y & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & y & x \\ 0 & 0 & 1 & -y & 0 & -x & 0 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_7 \end{Bmatrix}$$

$$= \underset{\sim}{P} \; \underset{\sim}{\beta}.$$

8.4 Assumed strain Mixed Formulation.

Hellinger - Reissner principle 에서 assumed stress 대신

assumed strain 사용.

H-R 에서

$$\delta\Pi = \int_V \delta\bar{\underset{\sim}{\varepsilon}}^T \underset{\sim}{\sigma}\, dv - \int_V \delta\underset{\sim}{u}^T \underset{\sim}{F_B}\, dv - \int_{S_\sigma} \delta\underset{\sim}{u}^T \underset{\sim}{\bar{T}}\, ds = 0 \quad (1)$$

$$\delta I = \int_V \delta\underset{\sim}{\sigma}^T (\bar{\underset{\sim}{\varepsilon}} - \underset{\sim}{\varepsilon}^\circ - \underset{\sim}{S}\,\underset{\sim}{\sigma})\, dv = 0 \quad (2)$$

여기서 $\bar{\underset{\sim}{\varepsilon}} = \underset{\sim}{B}\,\underset{\sim}{q}$ = displacement-dependent strain

$\underset{\sim}{\sigma} = \underset{\sim}{P}\,\underset{\sim}{\beta}$ = independent assumed stress

$\underset{\sim}{\sigma} = \underset{\sim}{C}\,\underset{\sim}{\varepsilon}$ 이므로

$$\delta\Pi = \int_V \delta\bar{\underset{\sim}{\varepsilon}}^T \underset{\sim}{C}\,\underset{\sim}{\varepsilon}\, dv - \int_V \delta\underset{\sim}{u}^T \underset{\sim}{F_B}\, dv - \int_{S_\sigma} \delta\underset{\sim}{u}^T \underset{\sim}{\bar{T}}\, ds = 0 \quad (3)$$

$$\delta I = \int_V \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{C}\, (\bar{\underset{\sim}{\varepsilon}} - \underset{\sim}{\varepsilon}^\circ - \underset{\sim}{\varepsilon})\, dv = 0 \quad (4)$$

여기서 $\bar{\underset{\sim}{\varepsilon}} = \underset{\sim}{B}\,\underset{\sim}{q}$, $\delta\bar{\underset{\sim}{\varepsilon}} = \underset{\sim}{B}\,\delta\underset{\sim}{q}$

$\underset{\sim}{\varepsilon} = \underset{\sim}{P}\,\underset{\sim}{\alpha}$, $\delta\underset{\sim}{\varepsilon} = \underset{\sim}{P}\,\delta\underset{\sim}{\alpha}$

$\underset{\sim}{P}$ = assumed strain polynomial matrix

$\underset{\sim}{\alpha}$ = assumed strain parameter

Then, 식 (3)은

$$\int_V \delta \tilde{\varepsilon}^T \underset{\sim}{C}\, \underset{\sim}{\varepsilon}\, dv - \int_V \delta \underset{\sim}{u}^T \underset{\sim}{F_B}\, dv - \int_{S_\sigma} \delta \underset{\sim}{u}^T \underset{\sim}{T}\, ds.$$

$$= \delta \underset{\sim}{q}^T \underbrace{\left[\int_V \underset{\sim}{B}^T \underset{\sim}{C}\, \underset{\sim}{P}\, dv\right]}_{\underset{\sim}{G}^T} \underset{\sim}{\alpha} - \delta \underset{\sim}{q}^T \underbrace{\left[\int_V \underset{\sim}{N}^T \underset{\sim}{F_B}\, dv + \int_{S_\sigma} \underset{\sim}{L}^T \underset{\sim}{T}\, ds\right]}_{\underset{\sim}{Q}^a}$$

$$= \delta \underset{\sim}{q}^T \left[\underset{\sim}{G}^T \underset{\sim}{\alpha} - \underset{\sim}{Q}^a\right] = 0 \qquad (5)$$

식 (4)는

$$\int_{V_i} \delta \tilde{\varepsilon}^T \underset{\sim}{C} \left(\tilde{\varepsilon} - \underset{\sim}{\varepsilon}^\circ - \underset{\sim}{\varepsilon}\right) dv$$

$$= \delta \underset{\sim}{\alpha}^T \underbrace{\left[\int_V \underset{\sim}{P}^T \underset{\sim}{C}\, \underset{\sim}{B}\, dv\right]}_{\underset{\sim}{G}} \underset{\sim}{q} - \delta \underset{\sim}{\alpha}^T \underbrace{\left[\int_V \underset{\sim}{P}^T \underset{\sim}{C}\, \underset{\sim}{\varepsilon}^\circ\, dv\right]}_{\underset{\sim}{G_0}}$$

$$- \delta \underset{\sim}{\alpha}^T \underbrace{\left[\int_V \underset{\sim}{P}^T \underset{\sim}{C}\, \underset{\sim}{P}\, dv\right]}_{\underset{\sim}{H}} \underset{\sim}{\alpha}$$

$$= \delta \underset{\sim}{\alpha}^T \left(\underset{\sim}{G}\, \underset{\sim}{q} - \underset{\sim}{G_0} - \underset{\sim}{H}\, \underset{\sim}{\alpha}\right) = 0 \qquad (6)$$

Since 식 (6) 에서 $\delta \underset{\sim}{\alpha}$ 는 arbitrary

$$\underset{\sim}{G}\, \underset{\sim}{q} - \underset{\sim}{G_0} - \underset{\sim}{H}\, \underset{\sim}{\alpha} = 0$$

$$\rightarrow \underset{\sim}{\alpha} = \underset{\sim}{H}^{-1} \underset{\sim}{G}\, \underset{\sim}{q} - \underset{\sim}{H}^{-1} \underset{\sim}{G_0} \qquad (7)$$

식 (7)을 식 (5)에 대입.

$$\delta \underset{\sim}{q}^T \left[\underset{\sim}{G}^T \left(\underset{\sim}{H}^{-1} \underset{\sim}{G}\, \underset{\sim}{q} - \underset{\sim}{H}^{-1} \underset{\sim}{G}_0 \right) - \underset{\sim}{Q}^a \right]$$

$$= \delta \underset{\sim}{q}^T \left[\underbrace{\underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G}\, \underset{\sim}{q}}_{= \underset{\sim}{K}} - \underbrace{\left(\underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G}_0 + \underset{\sim}{Q}^a \right)}_{= \underset{\sim}{Q}} \right]$$

$$= \delta \underset{\sim}{q}^T \left(\underset{\sim}{K}\, \underset{\sim}{q} - \underset{\sim}{Q} \right)$$

여기서 $\underset{\sim}{K} = \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G}$

$$= \left[\int_V \underset{\sim}{B}^T \underset{\sim}{C}\, \underset{\sim}{P}\, dv \right] \left[\int_V \underset{\sim}{P}^T \underset{\sim}{C}\, \underset{\sim}{P}\, dv \right]^{-1} \left[\int_V \underset{\sim}{P}^T \underset{\sim}{C}\, \underset{\sim}{B}\, dv \right]$$

= element stiffness matrix with assumed strain

$$\underset{\sim}{Q} = \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G}_0 + \underset{\sim}{Q}^a$$

$$= \left[\int_V \underset{\sim}{B}^T \underset{\sim}{C}\, \underset{\sim}{P}\, dv \right] \left[\int_V \underset{\sim}{P}^T \underset{\sim}{C}\, \underset{\sim}{P}\, dv \right]^{-1} \left[\int_V \underset{\sim}{P}^T \underset{\sim}{C}\, \underset{\sim}{\varepsilon}^{\circ}\, dv \right]$$

$$+ \left[\int_V \underset{\sim}{N}^T \underset{\sim}{F}_B\, dv + \int_{S_{\sigma_i}} \underset{\sim}{L}^T \underset{\sim}{T}\, ds \right]$$

= element nodal load vector with assumed strain

Assumed stress formulation	Assumed strain formulation
$\underset{\sim}{\sigma} = \underset{\sim}{P}_{\sigma} \underset{\sim}{\beta}$	$\underset{\sim}{\varepsilon} = \underset{\sim}{P}_{\varepsilon} \underset{\sim}{\alpha}$
Kinematic mode를 조정하기 힘들다.	Kinematic mode 조정 가능
stress equilibrium을 만족시킬 수 있다.	stress equilibrium을 만족시키기 힘들다.