CHAP 4 2-d Finite Element Formulation.

4.1 3-d Stress-strain relation

등방성 재료에서 if $\underset{\sim}{\varepsilon}^o = 0$

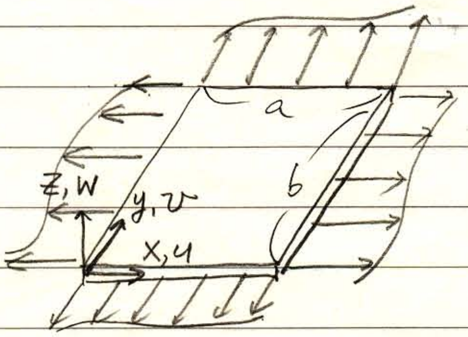
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ & \text{SYM} & & & \frac{1-2\nu}{2(1-\nu)} & 0 \\ & & & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}$$

or

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & \text{SYM} & & & 2(1+\nu) & 0 \\ & & & & & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$$

$$G = \frac{E}{2(1+\nu)}$$

(a) plane stress assumption



두께 $t \ll a, b$

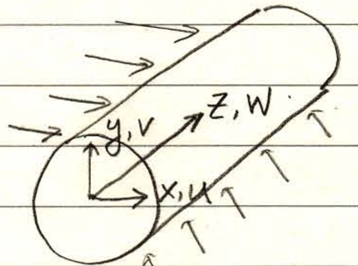
윗면, 아래면에서
$\sigma_{xz}, \sigma_{yz}, \sigma_{zz} = 0$

두께 방향으로

$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} \approx 0$

Then.
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}.$$

(b) plane strain assumption



$x, y \ll z$

$\frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0, \quad w = 0$

$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$

$\varepsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0$

$\varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0$

4.2 2-D Finite Element Formulation

principle of virtual work 에서

$$\delta\Pi = \int_V \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{C}\, \underset{\sim}{\varepsilon}\, dv - \int_V \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{C}\, \underset{\sim}{\varepsilon}^{\circ}\, dv - \int_V \delta\underset{\sim}{u}^T \underset{\sim}{F_B}\, dv$$

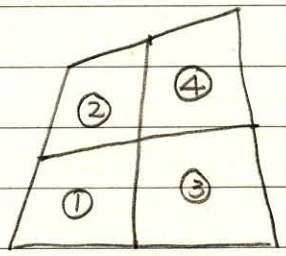
$$- \int_{S_\sigma} \delta\underset{\sim}{u}^T \underset{\sim}{\bar{T}}\, ds \quad = 0$$

and B.C on S_u

$$\underset{\sim}{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}. \qquad \underset{\sim}{F_B} = \begin{Bmatrix} X_B \\ Y_B \end{Bmatrix} \qquad \underset{\sim}{\bar{T}} = \begin{Bmatrix} \bar{T}_x \\ \bar{T}_y \end{Bmatrix}$$

$$\delta\underset{\sim}{u} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}$$

FE discretization을 위해서 여러 요소로 나눈다.

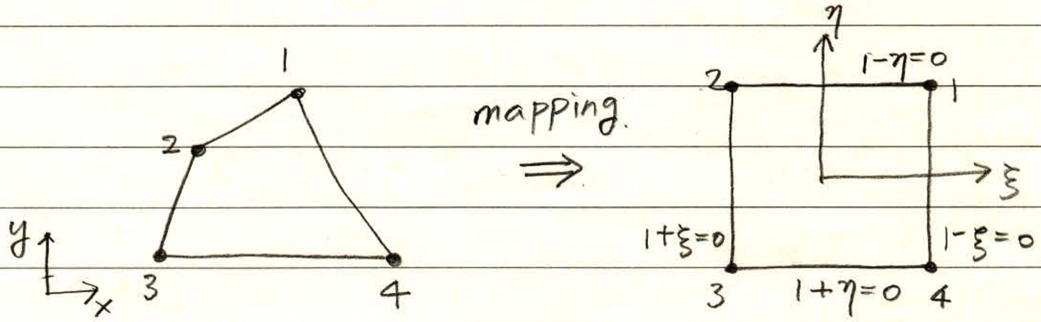


$$\delta\Pi = \sum_{i=1}^{n} \delta\Pi_i$$

$\delta\Pi_i$ = contribution of i-th element to $\delta\Pi$

n = total number of element

4.2.1. Four-node Quadrilateral element



$$N_1 = \frac{1}{4}(1+\xi)(1+\eta)$$: bilinear polynomial

then $N_1 = 1$ at node 1

$N_1 = 0$ at node 2, 3, 4

$$N_2 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$N_3 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_4 = \frac{1}{4}(1+\xi)(1-\eta)$$

Then, 4-node 요소에서

$$\left.\begin{aligned} x &= \sum_{i=1}^{4} N_i x_i \\ y &= \sum_{i=1}^{4} N_i y_i \end{aligned}\right\}$$ exact interpolation.

Assume $$\left.\begin{aligned} u &= \sum_{i=1}^{4} N_i u_i \\ v &= \sum_{i=1}^{4} N_i v_i \end{aligned}\right\}$$ u 와 v 를 가정

⇒ isoparametric assumption.

다시 쓰면.

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \underbrace{\begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix}}_{\underset{\sim}{N}} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}}_{\underset{\sim}{q_i}} = \underset{\sim}{N}\,\underset{\sim}{q_i}$$

$$\underset{\sim}{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

u, v 는 ξ, η 의 함수 ($N_i(\xi, \eta)$).

$\underset{\sim}{\varepsilon}$ 을 구하려면 $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ 와 $\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}$ 의 관계식 필요.

chain rule 에서

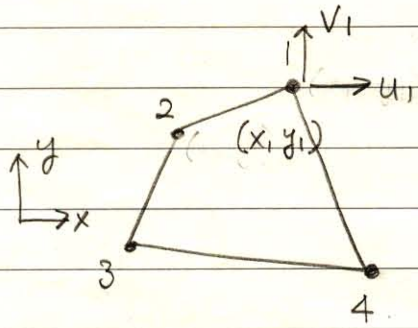
$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta}$$

or $$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\underset{\sim}{J} = \text{Jacobian}} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \underset{\sim}{J} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$

or
$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = \underset{\sim}{J}^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}.$$

$|J|$ = determinant of $\underset{\sim}{J}$



Since $x = \sum N_i x_i$

$y = \sum N_i y_i$

$$\frac{\partial x}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} x_i \qquad \frac{\partial x}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} x_i$$

$$\frac{\partial y}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} y_i \qquad \frac{\partial y}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} y_i$$

Since
$$\frac{\partial}{\partial x} = \frac{1}{|J|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}.$$

$$\frac{\partial}{\partial y} = \frac{1}{|J|} \begin{bmatrix} -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix}.$$

$$\underset{\sim}{\varepsilon} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

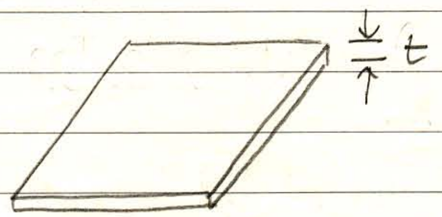
$$= \underbrace{\frac{1}{|J|}\begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} & 0 & 0 \\ 0 & 0 & -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \end{bmatrix}}_{= \underset{\sim}{D}} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

$$= \underbrace{\underset{\sim}{D} \cdot \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & 0 & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix}}_{= \underset{\sim}{B}} \underbrace{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}}_{= \underset{\sim}{q}_i}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{B}\, \underset{\sim}{q}$$

$$\delta \underset{\sim}{\varepsilon} = \underset{\sim}{B}\, \delta \underset{\sim}{q}$$

얇은 평판에서



유한요소 지배방정식을 적용하면

$$\int_{V_i} \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{C}\, \underset{\sim}{\varepsilon}\, dV = \int_{A_i} \delta \underset{\sim}{q}_i^T \underset{\sim}{B}^T \underset{\sim}{C}\, \underset{\sim}{B}\, \underset{\sim}{q}_i\, t\, dA$$

$$= \delta \underset{\sim}{q}_i^T \Big[\underbrace{\int_{A_i} \underset{\sim}{B}^T \underset{\sim}{C}\, \underset{\sim}{B}\, t\, dA}_{\parallel} \Big] \underset{\sim}{q}_i$$

$\underset{\sim}{K}^i$ = element stiffness matrix

$$= \delta \underset{\sim}{q}_i^T \underset{\sim}{K}^i \underset{\sim}{q}_i$$

$$\int_{V_i} \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{C}\, \underset{\sim}{\varepsilon}^0\, dV = \int_{A_i} \delta \underset{\sim}{q}_i^T \underset{\sim}{B}^T \underset{\sim}{C}\, \underset{\sim}{\varepsilon}^0\, t\, dA$$

$$= \delta \underset{\sim}{q}_i^T \underbrace{\int_{A_i} \underset{\sim}{B}^T \underset{\sim}{C}\, \underset{\sim}{\varepsilon}^0\, t\, dA}_{\parallel}$$

initial strain에 의한 element load vector

$$\int_{V_i} \delta \underset{\sim}{u}^T \underset{\sim}{F}_B\, dV = \int_{A_i} \delta \underset{\sim}{q}_i^T \underset{\sim}{N}^T \underset{\sim}{F}_B\, t\, dA$$

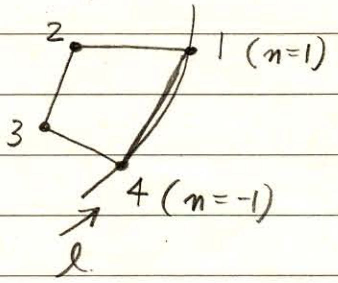
$$= \delta \underset{\sim}{q}_i^T \underbrace{\int \underset{\sim}{N}^T \underset{\sim}{F}_B\, t\, dA}$$

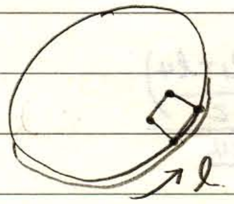
body force에 의한 element load vector

Applied traction term

2-d, 얇은 평판에서

$$\int_{S_{\sigma i}} \delta \underset{\sim}{u}^T \underset{\sim}{T}\, ds = \int_{l_i} \delta \underset{\sim}{u}^T \underset{\sim}{T}\, t\, dl$$





l = 경계선을 따르는 좌표계

곡면 (1-4 면)을 직선면으로 가정 → geometric approximation

면 1-4를 따라서 변위는 linear

$$u = N_1 u_1 + N_4 u_4 = \frac{1}{2}(1+\eta)u_1 + \frac{1}{2}(1-\eta)u_4$$

$$v = N_1 v_1 + N_4 v_4 = \frac{1}{2}(1+\eta)v_1 + \frac{1}{2}(1-\eta)v_4$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2}(1+\eta) & \frac{1}{2}(1-\eta) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1+\eta) & \frac{1}{2}(1-\eta) \end{bmatrix}}_{=\ \underset{\sim}{L}} \begin{Bmatrix} u_1 \\ u_4 \\ v_1 \\ v_4 \end{Bmatrix}$$

$\underset{\sim}{L}$ = shape function on 1-4 면

$$\begin{Bmatrix} \bar{T}_x \\ \bar{T}_y \end{Bmatrix} = \underset{\sim}{L} \begin{Bmatrix} \bar{T}_{x1} \\ \bar{T}_{x4} \\ \bar{T}_{y1} \\ \bar{T}_{y4} \end{Bmatrix}$$

$$\int_{l_i} \delta \underset{\sim}{u}^T \underset{\sim}{\bar{T}} \, t \, dl = \int_{l_i} \lfloor \delta u \;\; \delta v \rfloor \begin{Bmatrix} \bar{T}_x \\ \bar{T}_y \end{Bmatrix} t \, dl$$

$$= \lfloor \delta u_1 \;\; \delta u_4 \;\; \delta v_1 \;\; \delta v_4 \rfloor \left(\int_{-1}^{1} \underset{\sim}{L}^T \underset{\sim}{L} \, t \, \frac{(l_1 - l_4)}{2} \, d\eta \right) \begin{Bmatrix} \bar{T}_{x1} \\ \bar{T}_{x4} \\ \bar{T}_{y1} \\ \bar{T}_{y4} \end{Bmatrix}$$

$$= \lfloor \delta u_1 \;\; \delta u_4 \;\; \delta v_1 \;\; \delta v_4 \rfloor \begin{Bmatrix} Q^*_{x1} \\ Q^*_{x4} \\ Q^*_{y1} \\ Q^*_{y4} \end{Bmatrix}$$

$$= \lfloor \delta u_1 \; \delta u_2 \; \delta u_3 \; \delta u_4 \; \delta v_1 \; \delta v_2 \; \delta v_3 \; \delta v_4 \rfloor \begin{Bmatrix} Q^*_{x1} \\ 0 \\ 0 \\ Q^*_{x4} \\ Q^*_{y1} \\ 0 \\ 0 \\ Q^*_{y4} \end{Bmatrix} = \delta \underset{\sim}{q}_i^T \underset{\sim}{Q}_i^*$$

$\uparrow$ traction에 의한 nodal load vector.

$$\delta \Pi_i = \delta \underset{\sim}{q}_i^T \underset{\sim}{K}^i \underset{\sim}{q}_i - \delta \underset{\sim}{q}_i^T \underset{\sim}{Q}_i = \delta \underset{\sim}{q}_i^T (\underset{\sim}{K}^i \underset{\sim}{q}_i - \underset{\sim}{Q}_i) = 0$$

$\underset{\sim}{Q}_i$ = nodal load vector $\left\{ \begin{array}{l} \text{initial strain} \\ \text{body force} \\ \text{traction} \end{array} \right\}$