

Principle of virtual work and the Finite Element Method

On this subject, there exist a large number of textbooks, many of which are on the shelves of the library. A classic is *The Finite Element Method* by O.C. Zienkiewicz and R.L. Taylor. A text used in ES 128 is *Introduction to Finite Elements in Engineering* by T.R. Candrupatla and A.D. Belegundu.

In this course, we will give only a very brief introduction to FEM, so that the student gains an impression of the basics. Homework, some of which will be carried out using ABAQUS in the computer lab, will help the student gain some practical knowledge.

FEM is perhaps the most significant accomplishment in mechanics in the last century. The method has become such a robust tool that today far more people are users than developers. You do not need sophisticated knowledge of FEM to use a commercial package. However, the method can be extended to solve new problems that are not within the scope of commercial packages. To make such extensions, you may benefit from taking a full course on FEM.

Field equations and boundary conditions

Consider a body of an elastic material. Inside the body, the following equations hold

Force balance:
$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0$$

Strain-displacement relations:
$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Stress-strain relations:
$$\sigma_{ij} = C_{ijpq} \epsilon_{pq} .$$

The surface of the body is divided into two parts, S_u and S_t . On S_u , the displacement vector \mathbf{u} is prescribed. On S_t , the traction vector \mathbf{t} is prescribed, namely,

$$\sigma_{ij} n_j = t_i ,$$

where n_i is the unit vector normal to the surface.

Principle of virtual work

Let $\delta \mathbf{u}$ be a virtual displacement field in the body: the field is unrelated to the real displacement field in the body. The field is arbitrary, except that it vanishes on the part of the boundary where the actual displacement field is prescribed, namely, $\delta \mathbf{u} = \mathbf{0}$ on S_u .

Define the external virtual work by

$$\text{EVW} = \int b_i \delta u_i dV + \int t_i \delta u_i dA.$$

The first integral extends over the volume of the body, and the second integral extends the surface of the body. Note that the traction is known on S_t , and $\delta \mathbf{u} = \mathbf{0}$ on S_u .

Associated with virtual displacement field is a virtual strain field $\delta \varepsilon_{ij}$, defined by

$$\delta \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right).$$

Define the internal virtual work by

$$\text{IVW} = \int \sigma_{ij} \delta \varepsilon_{ij} dV.$$

The integral extends over the volume of the body.

The principle of virtual work (PVW) states that *the stress, body force and traction are in equilibrium if and only if the IVW equals the EVW for every virtual displacement field.*

Proof

Let $f(x_1, x_2, x_3)$ be a function defined in a volume. The divergence theorem states that

$$\int \frac{\partial f}{\partial x_i} dV = \int f n_i dA.$$

The integral on the left hand side extends over the volume, and the integral on the right hand side extends over the surface enclosing the volume. We now apply the divergence theorem to the IVW:

$$\begin{aligned}
IVW &= \int \sigma_{ij} \delta \epsilon_{ij} dV \\
&= \int \sigma_{ij} \frac{\partial \delta u_i}{\partial x_j} dV \\
&= \int \frac{\partial (\sigma_{ij} \delta u_i)}{\partial x_j} dV - \int \frac{\partial \sigma_{ij}}{\partial x_j} \delta u_i dV \\
&= \int \sigma_{ij} n_j \delta u_i dA - \int \frac{\partial \sigma_{ij}}{\partial x_j} \delta u_i dV
\end{aligned}$$

When $IVW = EVW$ for every virtual displacement field, each factor in front of the corresponding virtual displacement must equal. Consequently, the PVW recovers the equilibrium equation in the volume of the body,

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0,$$

and the stress-traction relation on S_t ,

$$\sigma_{ij} n_j = t_i.$$

Disclaimers

The EVW looks like actual work done by the body force and the prescribed traction, except that the displacement field is a fake. Please do not look for any physical or philosophical meaning of EVW. There is none. The same comments apply to IVW.

The virtual displacement field is sometimes considered an arbitrary small variation in the actual displacement field, so that EVW really looks like work. However, in the above proof, the virtual displacement need not be small at all.

Note that PVW mentions the actual displacement field in only one place: the virtual displacement field must vanish on the part of the surface where the actual displacement is prescribed. Otherwise, the PVW does not invoke the actual displacement field. Nor does the PVW invoke actual strain field.

Nor does the PVW invoke any stress-strain relation. The body may as well be a liquid or a plastic solid or whatever.

From every virtual displacement field to some virtual displacement fields

To maintain $IVW = EVW$ for *every* virtual displacement field is asking for a lot. Instead, we will only maintain $IVW = EVW$ for a *subset* of virtual displacement fields. This will lead to an approximate solution of the actual displacement field. In this sense, the PVW is a **weak statement** of the equilibrium conditions.

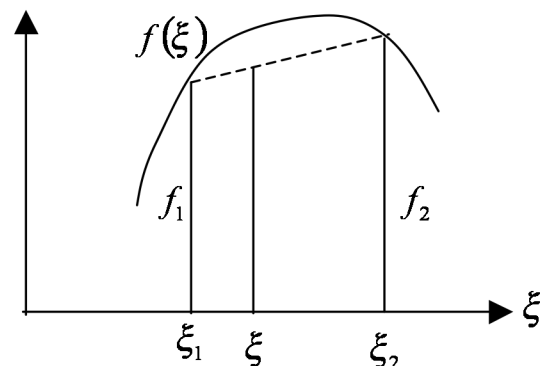
The finite element method is a particular way to construct this subset of virtual displacement fields, as outlined below.

1. Divide the body into elements, such as quadrilaterals. The basic variables are the displacement vectors at all the nodes, called the nodal displacements.
2. Interpolate the displacement field by the nodal displacements. Also interpolate the virtual displacement field by the virtual nodal displacements.
3. Use the strain-displacement relations to express the strain field in terms of the nodal displacements. Use the same procedure to express the virtual strain field in terms of the virtual nodal displacements.
4. Use the stress-strain relations to express the stress field in terms of the nodal displacements.
5. Require that $IVW = EVW$ for every set of virtual nodal displacements. This leads to a set of algebraic equations for the nodal displacements.

When the elements become very small, the nodal displacements approach the actual displacement field. In practice, the element sizes are finite, not infinitesimal; hence the name finite element method. The next few paragraphs walk you through the steps to implement this idea, using plane-strain problems and four-node elements.

1. Interpolate a function of one variable

Let us first look at interpolation of a one-variable function $f(\xi)$. The two nodes are ξ_1 and ξ_2 . The element is the segment between the two nodes. The values of the function at the two nodes are f_1 and f_2 . We can approximate the function at an arbitrary point in the element by:



$$f(\xi) = \frac{\xi_2 - \xi}{\xi_2 - \xi_1} f_1 + \frac{\xi - \xi_1}{\xi_2 - \xi_1} f_2.$$

The interpolation is linear in ξ . We can write the interpolation in the form

$$f(\xi) = N_1(\xi) f_1 + N_2(\xi) f_2.$$

The coefficients N_1, N_2 , are functions of ξ , and are known as the **shape functions**. They must satisfy the following requirements:

$$N_1(\xi_1) = 1, N_1(\xi_2) = 0, N_2(\xi_2) = 1, N_2(\xi_1) = 0.$$

If, in addition, we stipulate that the shape functions are linear in ξ , we can uniquely determine the shape functions, as given above.

Map a square to a quadrilateral

To solve a plane-strain problem, we divide the cross section of the body, on the (x, y) plane, into quadrilateral elements. To represent a body of a general shape, we must consider quadrilaterals of arbitrary shapes. To ease integration in evaluating the virtual work, we map a unit square on a different plane, (ξ, η) , to the quadrilateral in the (x, y) plane.

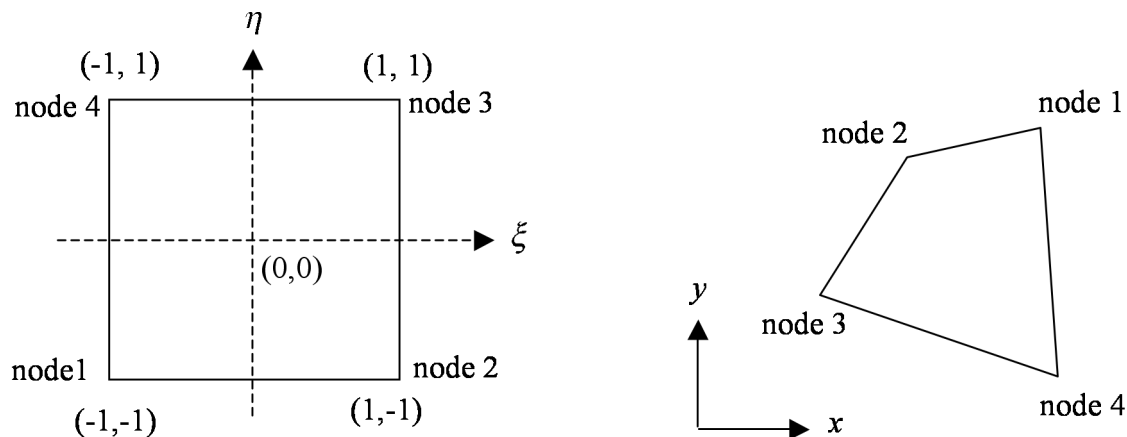
Now consider a quadrilateral element. Label its four nodes, *counterclockwise*, as 1, 2, 3, 4. On the (x, y) plane, the four nodes have coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$. The following functions map a point in the (ξ, η) plane to a point in the (x, y) plane:

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

The functions are linear combinations of the nodal coordinates. The coefficients N_1, N_2, N_3, N_4 are functions of ξ and η . They are the shape functions, so constructed that the four corners of the square on the (ξ, η) plane map to the four nodes of the quadrilateral on the (x, y) plane.

Let us determine the shape functions. For example, on the (ξ, η) plane, N_1 should be 1 at node 1, and vanish at the other three nodes. The simplest function that fulfills the second requirement is $N_1(\xi, \eta) = c(1 - \xi)(1 - \eta)$, where c is a constant. To fulfill the first requirement, $N_1(-1, -1) = 1$, we set the constant to be $c = 1/4$. Thus,



$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)$$

We can similarly write out the other three shape functions:

$$N_2 = \frac{1}{4}(1 + \xi)(1 - \eta)$$

$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta)$$

$$N_4 = \frac{1}{4}(1 - \xi)(1 + \eta)$$

Interpolate displacement field

Let the displacements at the four nodes of the quadrilateral be (u_1, v_1) , (u_2, v_2) , (u_3, v_3) , (u_4, v_4) . Interpolate the displacement (u, v) of a point inside the quadrilateral as

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4,$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4.$$

We have used the same shape functions to interpolate the coordinates and the displacements. This is known as **isoparametric interpolation**. List the displacements of the four nodes of the quadrilateral in a column matrix:

$$\mathbf{q} = [u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4]^T.$$

Write

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}.$$

In the matrix form, the displacement vector at a point in the element is an interpolation of the nodal displacements:

$$\mathbf{u} = \mathbf{N}\mathbf{q}.$$

Let the virtual nodal displacements be $\delta\mathbf{q}$, the virtual displacement field in the elements is

$$\delta\mathbf{u} = \mathbf{N}\delta\mathbf{q}.$$

Express strains in terms of nodal displacements

This step contains tedious algebra, but the final result is easy to understand: We can represent the strain field in terms of the nodal displacements, $\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q}$, where \mathbf{B} is a matrix depending on (ξ, η) . To gain a general impression of the finite element method, a beginning student may as well skip the derivation.

To calculate the strains, we need to calculate displacement gradients, such as $\partial u / \partial x$. However, in the above interpolations, the displacement is given as a function $u(\xi, \eta)$. The displacement gradient on the (ξ, η) plane is readily calculated:

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}.$$

We need to convert the gradient on the (ξ, η) to that on the (x, y) plane. Using the chain rule of differentiation, we obtain that

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}.$$

This equation relates the gradients in the two planes. The two by two matrix is the Jacobian matrix \mathbf{J} of the map $(\xi, \eta) \rightarrow (x, y)$. The elements of the Jacobian matrix are readily calculated:

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{1}{4}(1-\eta)(-x_1 + x_2) + \frac{1}{4}(1+\eta)(x_3 - x_4),$$

$$J_{12} = \frac{\partial y}{\partial \xi} = \frac{1}{4}(1-\eta)(-y_1 + y_2) + \frac{1}{4}(1+\eta)(y_3 - y_4),$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{1}{4}(1-\xi)(-x_1 + x_4) + \frac{1}{4}(1+\xi)(-x_2 + x_3),$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \frac{1}{4}(1-\xi)(-y_1 + y_4) + \frac{1}{4}(1+\xi)(-y_2 + y_3).$$

The determinant of the Jacobian matrix is

$$\det \mathbf{J} = J_{11}J_{22} - J_{12}J_{21}$$

We invert the matrix equation by using Cramer's rule, and obtain that

$$\frac{\partial u}{\partial x} = \frac{1}{\det \mathbf{J}} \begin{vmatrix} \frac{\partial u}{\partial \xi} & J_{12} \\ \frac{\partial u}{\partial \eta} & J_{22} \end{vmatrix} = \frac{1}{\det \mathbf{J}} \left(J_{22} \frac{\partial u}{\partial \xi} - J_{12} \frac{\partial u}{\partial \eta} \right),$$

$$\frac{\partial u}{\partial y} = \frac{1}{\det \mathbf{J}} \begin{vmatrix} J_{11} & \frac{\partial u}{\partial \xi} \\ J_{21} & \frac{\partial u}{\partial \eta} \end{vmatrix} = \frac{1}{\det \mathbf{J}} \left(J_{11} \frac{\partial u}{\partial \eta} - J_{21} \frac{\partial u}{\partial \xi} \right).$$

These express the displacement gradients as linear combinations of the nodal displacements:

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{-J_{22}(1-\eta) + J_{12}(1-\xi)}{4 \det \mathbf{J}} & \frac{J_{22}(1-\eta) + J_{12}(1+\xi)}{4 \det \mathbf{J}} & \frac{J_{22}(1+\eta) - J_{12}(1+\xi)}{4 \det \mathbf{J}} & \frac{-J_{22}(1+\eta) - J_{12}(1-\xi)}{4 \det \mathbf{J}} \\ \frac{-J_{11}(1-\xi) + J_{21}(1-\eta)}{4 \det \mathbf{J}} & \frac{-J_{11}(1+\xi) - J_{21}(1-\eta)}{4 \det \mathbf{J}} & \frac{J_{11}(1+\xi) - J_{21}(1+\eta)}{4 \det \mathbf{J}} & \frac{J_{11}(1-\xi) + J_{21}(1+\eta)}{4 \det \mathbf{J}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

We can also write similar expressions for $\partial v / \partial x$ and $\partial v / \partial y$.

Recall that the strain column is

$$\boldsymbol{\varepsilon} = \left[\left(\frac{\partial u}{\partial x} \right), \left(\frac{\partial v}{\partial y} \right), \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right]^T.$$

Consequently, the strain column of a point in the element is linear in the nodal displacement column. We write

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{q}.$$

The entries to the matrix \mathbf{B} can be worked out readily by combining the above relations. For example,

$$B_{11} = \frac{1}{4 \det \mathbf{J}} [-J_{22}(1-\eta) + J_{12}(1-\xi)].$$

Similarly, the virtual strain field is

$$\delta \boldsymbol{\varepsilon} = \mathbf{B} \delta \mathbf{q}.$$

Express the stress field in terms of the nodal displacements

List the in-plane stresses by a column: $\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \sigma_{xy}]^T$. The stress column relates to the strain column as

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}.$$

Written explicitly, the stress-strain relation under the plane strain conditions is

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}.$$

Recall that $\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{q}$. The stress column is also linear in the displacement column:

$$\boldsymbol{\sigma} = \mathbf{D}\mathbf{B}\mathbf{q}.$$

Assemble equilibrium equations

The internal virtual work is

$$IVW = \int \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV .$$

Inserting the above expressions into the expression for the internal virtual work, we obtain that

$$IVW = \sum \delta \mathbf{q}^T \mathbf{k} \mathbf{q} .$$

The sum is taken over all the elements. The element stiffness matrix is an integral over the volume of the element:

$$\mathbf{k} = \int \mathbf{B}^T \mathbf{D} \mathbf{B} dV .$$

Write the internal virtual work in terms of the column of the global degrees of freedom:

$$IVW = \delta \mathbf{Q}^T \mathbf{K} \mathbf{Q} .$$

The global stiffness matrix \mathbf{K} sums over the contributions from all the elements.

The external virtual work is

$$EVW = \int \delta \mathbf{u}^T \mathbf{b} dV + \int \delta \mathbf{u}^T \mathbf{t} dA .$$

The first integral is over the volume of the body. The second integral is over the surface of the body, where the traction is prescribed. Replace the displacement variation by $\delta \mathbf{u} = \mathbf{N} \delta \mathbf{q}$, and we obtain that

$$EVW = \sum \delta \mathbf{q}^T \mathbf{f}$$

The sum is over all the elements in the body. The force column \mathbf{f} is the sum of two contributions. The body force term integrates over the volume of an element, giving

$$\mathbf{f}_b = \int \mathbf{N}^T \mathbf{b} dV .$$

The surface traction term integrates over the surface area of an element on which traction is prescribed, giving

$$\mathbf{f}_t = \int \mathbf{N}^T \mathbf{t} dA .$$

List all the nodal displacements in the body by the column matrix \mathbf{Q} and write the EVW in terms of the global columns:

$$\text{EVW} = \delta \mathbf{Q}^T \mathbf{F} .$$

The global load column \mathbf{F} sums over the contribution of the body force from all elements, and the traction from all surface elements.

The PVW then requires that $\text{IVW} = \text{EVW}$, namely,

$$\delta \mathbf{Q}^T \mathbf{K} \mathbf{Q} = \delta \mathbf{Q}^T \mathbf{F}$$

for virtual nodal displacement column, $\delta \mathbf{Q}$. Consequently, the displacement column \mathbf{Q} satisfies

$$\mathbf{K} \mathbf{Q} = \mathbf{F} .$$

After enforcing the prescribed displacements, the computer solves the linear algebraic equation.

Differential volume and area

To calculate the virtual work, we will need to integrate over the volume of the body. The body is now divided into elements. So, we will integrate over each element, and then sum over all elements. For plane problems, the differential volume dV is the differential area dA on the (x, y) plane times the thickness h of the element. We have expressed various fields as functions of (ξ, η) . We need to carry out the integral on the (ξ, η) plane.

Consider a rectangular infinitesimal element on the (ξ, η) plane, defined by four points (ξ, η) , $(\xi + d\xi, \eta)$, $(\xi, \eta + d\eta)$ and $(\xi + d\xi, \eta + d\eta)$. This infinitesimal element maps to a quadrilateral infinitesimal element on the (x, y) plane. The point (ξ, η) maps to a vector $\mathbf{x}(\xi, \eta)$ on the (x, y) plane. The point $(\xi + d\xi, \eta)$ maps to point $\mathbf{x}(\xi + d\xi, \eta)$. Consequently, one side of the quadrilateral is the vector

$$d\mathbf{x}_\xi = \mathbf{x}(\xi + d\xi, \eta) - \mathbf{x}(\xi, \eta) = \frac{\partial \mathbf{x}}{\partial \xi} d\xi .$$

Another side of the quadrilateral is the vector

$$d\mathbf{x}_\eta = \mathbf{x}(\xi, \eta + d\eta) - \mathbf{x}(\xi, \eta) = \frac{\partial \mathbf{x}}{\partial \eta} d\eta .$$

Recall that the cross product of any two vectors is a vector, whose magnitude is the area of the parallelogram defined by the two vectors. Consequently, the area of the quadrilateral infinitesimal element is

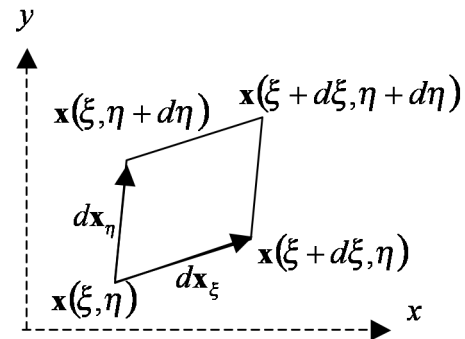
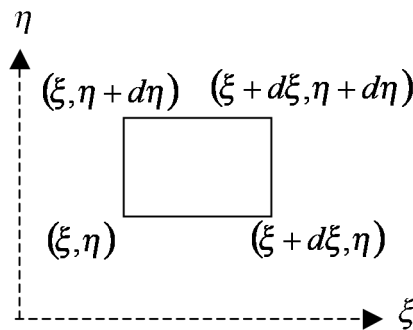
$$dA = |d\mathbf{x}_\xi \times d\mathbf{x}_\eta|.$$

Calculate the cross product, and we obtain that

$$d\mathbf{x}_\xi \times d\mathbf{x}_\eta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial \xi} d\xi & \frac{\partial y}{\partial \xi} d\xi & 0 \\ \frac{\partial x}{\partial \eta} d\eta & \frac{\partial y}{\partial \eta} d\eta & 0 \end{vmatrix} = \mathbf{k} \left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} \right) d\xi d\eta.$$

Consequently, the area of the differential element is

$$dA = \det \mathbf{J} d\xi d\eta.$$



In calculating the EVW due to the traction, we will integrate over the surface area of the body. For plane elasticity problems, the differential surface area becomes the differential line length dL times the element thickness h . Say the line is the edge of a quadrilateral element between node 2 and 3, namely, $\xi = 1$. The differential line length is

$$dL = |d\mathbf{x}_\eta| = \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2} d\eta,$$

or

$$dL = \frac{l_{23}}{2} d\eta,$$

where l_{23} is the distance between node 2 and 3. Thus, the volume integrals become

$$\mathbf{k} = \int_{-1}^1 \int_{-1}^1 h \mathbf{B}^T \mathbf{D} \mathbf{B} \det \mathbf{J} d\xi d\eta$$

$$\mathbf{f}_b = \int_{-1}^1 \int_{-1}^1 h \mathbf{N}^T \mathbf{b} \det \mathbf{J} d\xi d\eta$$

The surface integral becomes

$$\mathbf{f}_t = \frac{l_{23}}{2} \int_{-1}^1 h \mathbf{N}^T \mathbf{t} d\eta$$

Only nodes on the edge of the element, $\xi = 1$, contribute to the force.

Gaussian quadrature

Consider the integral

$$I = \int_{-1}^1 f(\xi) d\xi.$$

A generic way to construct a numerical integration method is as follows. Select a set of n points $\xi_1, \xi_2, \dots, \xi_n$ in the interval $(-1, 1)$. Evaluate the function at these points: $f(\xi_1), f(\xi_2), \dots, f(\xi_n)$.

The integral is a linear combination of the function values:

$$I = \int_{-1}^1 f(\xi) d\xi = w_1 f(\xi_1) + w_2 f(\xi_2) + \dots + w_n f(\xi_n).$$

The coefficients are known as the weights. The weights are selected so that the above formula gives exact results for some known functions.

Evaluating function values is costly. The Gaussian quadrature selects the points $\xi_1, \xi_2, \dots, \xi_n$ and the weights such that the above formula is exact for polynomials of as large a degree as possible.

For n points and n weights, the formula is exact for polynomial of degree $2n-1$, for it has $2n$ terms.

As an example, consider the two-point Gaussian quadrature:

$$I = \int_{-1}^1 f(\xi) d\xi \approx w_1 f(\xi_1) + w_2 f(\xi_2).$$

We need to determine four numbers: ξ_1, ξ_2, w_1, w_2 . A polynomial of degree 3 is a linear combination of monomials $1, x, x^2$ and x^3 . The integral I is a number linear in the function f . To ensure the integral is exact for *every* polynomial of degree up to 3, all we need to do is to ensure that the formula is exact for the four monomials. Thus,

$$2 = w_1 + w_2$$

$$0 = w_1 \xi_1 + w_2 \xi_2$$

$$\frac{2}{3} = w_1 \xi_1^2 + w_2 \xi_2^2$$

$$0 = w_1 \xi_1^3 + w_2 \xi_2^3$$

This is a set of nonlinear algebraic equations. The solution is

$$w_1 = 1, \xi_1 = -1/\sqrt{3}$$

$$w_2 = 1, \xi_2 = +1/\sqrt{3}$$

Many mathematics handbooks list integration points and weights for Gaussian quadrature.

A two dimensional integral is evaluated as

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = \int_{-1}^1 \sum_{i=1}^n w_i f(\xi_i, \eta) d\eta = \sum_{j=1}^n \sum_{i=1}^n w_i w_j f(\xi_i, \eta_j).$$

For example, the 2-point quadrature in one dimension becomes 4-point quadrature in two dimensions:

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

Additional comments

We have walked through the steps of implementing the finite element method. The following extensions are straightforward and have little educational value. These options are available in ABAQUS, and are important in applications. We simply list them, with brief comments.

Element type

You do not have to divide the plane into quadrilaterals. For example, triangular elements may be more versatile in modeling odd-shaped bodies. You do not have to restrict the nodes to the corners of an element. Some elements have nodes on the sides of the elements, inside the elements, as well as at the corners of the elements. If an element contains more nodes, the displacements in the element vary as a higher order polynomial of the coordinates. This represents a larger family of displacement fields. To divide a body into elements, you can increase the number of nodes by using either a large number of low order elements, or a small number of high order elements.

Axisymmetric problems

In practice, you would like to avoid as much as possible three-dimensional problems. Such problems dramatically increase the size of computation job. Even if the computer is not tired of the job size, you will, for it will take a long time for you to go through the output to extract useful information. When the deformation field is axisymmetric, all the fields are function of the axial coordinate z and the radial coordinate r . Consequently, both input and output are made in the plane (r, z) . The amount of work is comparable to solving the plane elasticity problems.

Three-dimensional problems

If you have to solve a three-dimensional model, ABAQUS provides such an option.

Other fields

The finite element method has been developed for many other fields, such as temperature and electrostatic fields. ABAQUS provides many options.