Chapter: 4

MEK4560
The Finite Element Method in Solid Mechanics II

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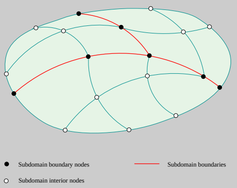
TORGEIR RUSTEN

(E-post:torgeiru@math.uio.no)

Department of Mathematics
University of Oslo



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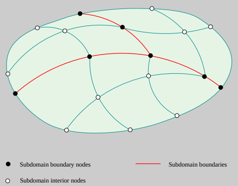

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4. Plates

The topic of the following Chapters are plate theory, both classical thin plate theory and theory for thick plates. While the membrane formulation is characterized by the fact that loads and responses are in the plane, the plate models consider out of plane loads and responses.

Recall that the classical beam models are forth order differential equations, and that C^1 continuity is required for the finite element approximation. This is straightforward to achieve with cubic polynomials.

We will see below a model of thin plates, based on Kirchhoff theory, which also result in a forth order differential equation. Thus, for a conforming element method C^1 continuity is required across inter element boundaries. In 2D this is more difficult to achieve, it turns out that for triangular elements a quintic polynomial is required.

An alternative to thin plates are thick plates based on Mindlin-Reissner theory, similar to the model of beams. Here C^0 continuity is sufficient and the usual finite elements can be used, however the problem with shear locking is also present for plates.

In the sequel we discuss:

1. *Kirchhoff* plates, thin plates where transverse shear strains assumed to vanish.
2. *Mindlin-Reissner* plates, nonzero transverse shear strains.

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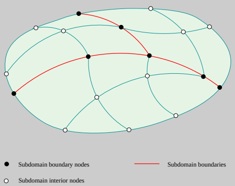

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Note: no plate models are compatible with tree dimensional elasticity. They are useful when the “thickness” is small compared to the size of the plate.

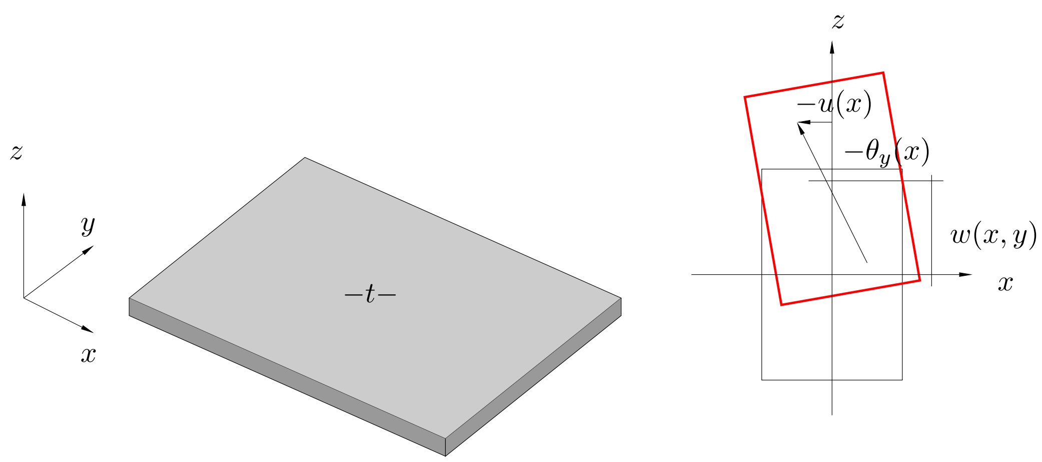

Figure 4.1: Plate geometry. Deformation in a x-z-cross section.

In this chapter thin plate theory is introduced, the basic assumptions, the potential energy functional, the differential equations and finite element methods. The topic of the next two chapters are thick plate models and element methods.

The relevant sections in the text book [Cook et al., 2002]Cook:01 are 15.1, 15.2, 15.3, 15.4 and 15.5.

4.1. Introduction to plates

A plate is:

1. The geometry is plane and the "thickness" t is small compared to the "length" L.

$\frac{t}{L} > \frac{1}{3}$	thick plates, full three dimensional analysis.
$\frac{1}{3} > \frac{t}{L} > \frac{1}{10}$	medium thick plates, analysis using thick plate theory.
$\frac{t}{L} < \frac{1}{10}$	thin plates, thin plate theory.

2. The load is "out of plane".

3. The response in bending; the stress throughout the thickness is not uniform.

Remark 4.1 In order to use a plate model, the thickness can not be to small, eg. kites and hot air balloons can not be analyzed using plate models.

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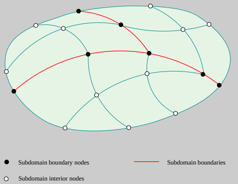

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4.2. Assumptions

The derivation of a model of thin plates are based on the following assumptions:

1. The geometry is given by:

$$V = \left\{ (x, y, x) \in \mathbb{R}^3 | (x, y) \in \mathcal{A} \subset \mathbb{R}^2,\ z \in \left(-\frac{t}{2}, \frac{t}{2}\right) \right\}$$

   where t is the plate thickness and $\mathcal{A}$ is the middle plane.

2. Transverse share strains are zero

$$\gamma_{xz} = \gamma_{yz} = 0$$

3. The stress normal to the plate midlplane are negligible

$$\sigma_{zz} = 0$$

4. Small rotations and small displacements:

$$w(x, y) << t, \qquad \sin\alpha = \alpha = \frac{\partial w}{\partial x}$$

   similarly for the rotations around the x-axis

5. No stress in the middle plane:

$$\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$$

   The boundary conditions must be compatible to this assumption.

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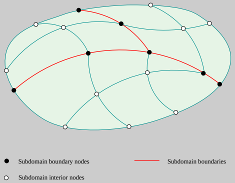

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Remark 4.2 In general the thickness can be a function of x and y, $t = t(x, y)$. However, the variation must be "sufficiently" slow in order to avoid three dimensional effects.

Remark 4.3 The term *inextensional bending* is used when the middle-plane has zero stress. It is also called *plate bending*. If the the middle-plane has "sufficiently" large deformations it is called *extensional bending* and a shell model is appropriate.

4.3. Kinematics

Displacements: The in plane displacements are related to the normal displacements as follows:

$$u(x, y, z) = -z\frac{\partial w}{\partial x}, \qquad v(x, y, z) = -z\frac{\partial w}{\partial y} \qquad \text{and} \qquad w(x, y, z) = w(x, y)$$

Thus, just as for the *Euler-Bernoulli beam*:

Plane cross sections remains plane and normal to the middle-plane during deformations.

Based on the above assumptions on the displacement field we can find:

- strain, stress, potential energy and equilibrium.

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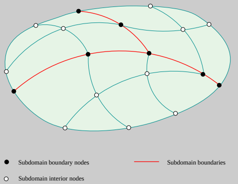

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Strains: The strain vector is composed of in-plane strains:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} -z\frac{\partial w}{\partial x} \\ -z\frac{\partial w}{\partial y} \end{pmatrix} = -z \begin{pmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{pmatrix} = -z\boldsymbol{\kappa}(w)$$

where the bending vector, $\boldsymbol{\kappa}(w)$, is introduced:

$$\boldsymbol{\kappa}(w) = \begin{pmatrix} \kappa_{xx}(w) \\ \kappa_{yy}(w) \\ \kappa_{xy}(w) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{pmatrix}$$

Remark 4.4 Using the kinematic assumptions

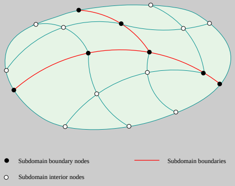
$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0, \qquad 2\varepsilon_{xz} = \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0$$

$$\text{and} \qquad 2\varepsilon_{yz} = \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} = 0$$

thus the requirement of vanishing transverse shear follows from the kinematic assumptions.

Remark 4.5 In the derivation of the beam theory a certain inconsistency existed. A similar inconsistency exist for plate theory.

Note that the plate theory also has certain inconsistencies, just as the beam models. In particular, the shear strains are zero. For an isotropic material this implies that the shear



stresses, $\sigma_{xz} = \sigma_{yz} = 0$, also vanish. As for the beam the shear forces are in the equilibrium equations.

The normal strains, ε_{zz}, are also zeros, i.e. we have a plane strain condition. On the other hand, we also have $\sigma_{zz} = 0$, which more natural physical assumption. For an isotropic material the conditions of plane stress and plane strain implies that $\nu = 0$.

4.4. Moment-bending relations

Stresses: The stress strain relations are found from the material law:

$$\sigma = \mathbf{E}\varepsilon = -z\mathbf{E}\kappa$$

In plate theory is is useful to introduce shear forces, stresses integrated across the plate thickness. They result in moments:

$$\boldsymbol{M} = \begin{pmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{pmatrix} = \int_{-\frac{t}{2}}^{-\frac{t}{2}} -\boldsymbol{\sigma} z \, dz = \int_{-\frac{t}{2}}^{-\frac{t}{2}} \boldsymbol{E} z^2 \, dz \boldsymbol{\kappa} = \boldsymbol{D}\boldsymbol{\kappa}$$

E int the expression above can be a general material law, however in the following we make the assumption of an isotropic material.

Remark 4.6 Note that the reference plane is the middle plane, thus bending is not coupled to axial deformations in the middle plane.

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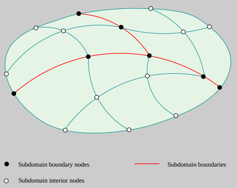

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4.5. Potential energy

Using the above in the potential energy functional for for three dimensional elasticity, we obtain the potential energy functional for plate bending.

The strain energy becomes:

$$U(w) = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\, dV = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^T \boldsymbol{E}\boldsymbol{\varepsilon}\, dV = \frac{1}{2}\int_A \int_{-\frac{t}{2}}^{\frac{t}{2}} z^2 \boldsymbol{\kappa}^T \boldsymbol{E}\boldsymbol{\kappa}\, dz\, dA = \frac{1}{2}\int_A \boldsymbol{\kappa}(w)^T \boldsymbol{D}\boldsymbol{\kappa}(w)\, dA$$

$$\text{or} \qquad U(w) = \frac{1}{2}\int_A \boldsymbol{M}^T \boldsymbol{\kappa}(w)\, dA$$

For an isotropic and homogeneous material the internal energy becomes:

$$U(w) = \frac{Et^3}{24(1-\nu^2)}\int_A \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dA \qquad (4.1)$$

In case the loading consist of transversal loads the load potential becomes

$$W(w) = \int_V \boldsymbol{F}^T \boldsymbol{u}\, dV = \int_A q w\, dA$$

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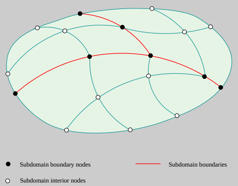

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4.6. Equilibrium I

The equilibrium conditions can be derived by minimizing the potential energy functional. The minimum, or stationary value, is taken for a function w satisfying the *Euler-Lagrange* equations. The functional above is of the form:

$$\Pi = \int_V F(x, y, w, w_{,x}, w_{,y}, w_{,xx}, w_{,xy}, w_{,yy})\, dV$$

The Euler-Lagrange equations, see [Gelfand and Fomin, 1963][2], are:

$$\frac{\partial F}{\partial w} - \frac{\partial}{\partial x}\frac{\partial F}{\partial w_{,x}} - \frac{\partial}{\partial y}\frac{\partial F}{\partial w_{,y}} + \frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial w_{,xx}} + \frac{\partial^2}{\partial x \partial y}\frac{\partial F}{\partial w_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial w_{,yy}} = 0$$

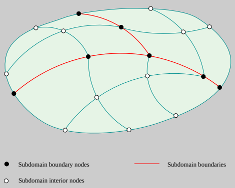
For a isotropic and homogeneous material the associated differential equations becomes

$$\begin{aligned} q + \frac{Et^3}{24(1-\nu)}\left[\frac{\partial^2}{\partial x^2}(2w_{,xx} + 2\nu w_{,yy}) + \frac{\partial^2}{\partial x \partial y}(4(1-\nu)w_{,xy}) + \frac{\partial^2}{\partial y^2}(2w_{,yy} + 2\nu w_{,xx})\right] \\ = q + \frac{Et^3}{24(1-\nu)}\left[2w_{,xxxx} + 2\nu w_{xxyy} + 4(1-\nu)w_{,xxyy} + 2w_{,yyyy} + 2\nu w_{xxyy}\right] \\ = q + D\left[w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}\right] = 0 \end{aligned}$$

Thus the thin plate equation is:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{q}{D}$$

[2] I. M. Gelfand and S. V. Fomin. *Calculus of Variations*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963.



$$\Delta\Delta w = -\frac{q}{D} \quad \text{or} \quad \Delta^2 w = -\frac{q}{D}$$

or, with a slight abuse of notation since ∇^2 is taken to mean $\nabla \cdot \nabla = \nabla^T \nabla$,

$$\nabla^2 \nabla^2 w = -\frac{q}{D} \quad \text{or} \quad \nabla^4 w = -\frac{q}{D} \tag{4.2}$$

where

$$D = \frac{Et^3}{12(1-\nu^2)}$$

4.7. Equilibrium II

Above we used a result from calculus of variations to derive the Euler-Lagrange equations for the thin plate model. Here we show how to find the minimum of the potential energy functionals arising in linearized elasticity.

First, consider the function

$$\Pi(V) = \frac{1}{2} V^T K V - V^T F \tag{4.3}$$

where V and F are vectors of dimension n and K is an n by n matrix. Think of K as the stiffness matrix, F as a load vector and V the degrees of freedom of a finite element model. Thus (4.3) is a finite element approximation of the potential energy functional.

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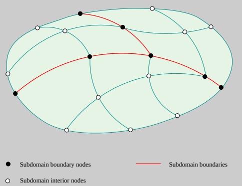

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In order to find the minimum one could compute the gradient and find the vector U where the gradient is zero. I.e. set all the partial derivatives to zeros, or

$$\frac{\partial \Pi}{\partial V_i} = 0 \tag{4.4}$$

for $i = 1, \ldots, n$.

Note that

$$\frac{\partial \Pi}{\partial V_i} = \left[\Pi'(V + te_i)\right]_{t=0} \tag{4.5}$$

where e_i is an n vector with 1 in location i and zeroes elsewhere. Consequently, if

$$\left[\Pi'(U + tV)\right]_{t=0} = 0 \tag{4.6}$$

for all vectors V it follows that at the point U all the partial derivatives are zero.

But

$$\Pi(U + tV) = \frac{1}{2}(U^T KU + 2tV^T KU + t^2 V^T KV) - (U^T F + tU^T F) \tag{4.7}$$

Thus

$$\Pi'(U + tV) = V^T KU + tV^T KV - V^T F \tag{4.8}$$

and evaluating this at $t = 0$ we have shown that

$$V^T(KU - F) = 0 \quad \text{for all } V \tag{4.9}$$

Since it holds for all V, it holds for all the unit vectors e_i, thus the vector U satisfies

$$KU = F \tag{4.10}$$

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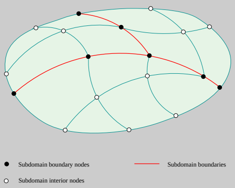

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Then, we consider the continuous plate problem. In order to simplify the writing

$$K(w,v) = \int_A \kappa(w) D \kappa(v)\, dA \quad \text{and} \quad F(v) = \int_A qv\, dA \tag{4.11}$$

Note that $K(w,v) = K(v,w)$ and that $K(v,v) \geq 0$

Then the potential energy functional for the thin plate formulation can be written

$$\Pi(v) = \frac{1}{2} K(v,v) - F(v) \tag{4.12}$$

As above the function w that minimize the potential energy functional satisfy

$$\Pi'(w + tv) = 0 \tag{4.13}$$

for all v and $t = 0$.

Note that

$$F(w + tv) = F(w) + tF(v) \tag{4.14}$$

and

$$K(w + tv, w + tv) = K(w,w) + 2tK(w,v) + t^2 K(v,v) \tag{4.15}$$

Thus

$$\Pi'(w + tv) = K(w,v) + tK(v,v) - F(v) \tag{4.16}$$

and the function w minimizing $\Pi(v)$ satisfies

$$K(w,v) = F(v) \quad \text{for all } v \tag{4.17}$$

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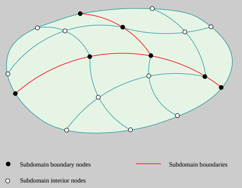

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It is beyond the scope this course to discuss the existence of a function w.

Note that (4.17) correspond to the *principle of virtual work*, and is called a *weak formulation*.

Note that the derivation is general and with suitable definition of $K(w,v)$ and $F(v)$ all formulations of linearized elasticity, e.g. three dimensional, membrane, beam, etc. is covered.

4.8. Field equations I

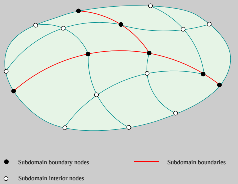
In order to derive the differential equation, note that the weak form of the thin plate problem is: Find w satisfying

$$\int_A M_{xx}(w)v_{,xx} + M_{yy}(w)v_{,yy} + 2M_{xy}(w)v_{,xy}\,dA = \int_A q\,v\,dA \quad \text{for all } v \tag{4.18}$$

Now, the Greens formula is used to move derivatives from the test function v to the moments $\mathbf{M}$.

First

$$\begin{aligned}\int_A M_{xx}v_{,xx}\,dA &= \int_{\partial A} M_{xx}v_{,x}n_x\,dS - \int_A M_{xx,x}v_{,x}\,dA \\ &= \int_{\partial A} M_{xx}v_{,x}n_x\,dS - \int_{\partial A} M_{xx,x}vn_x\,dS + \int_A M_{xx,xx}v\,dA \end{aligned}\tag{4.19}$$



where n_x is the x component of the unit outward normal vector. Then

$$\begin{aligned}\int_A M_{yy} v_{,yy}\, dA &= \int_{\partial A} M_{yy} v_{,y} n_y\, dS - \int_A M_{yy,y} v_{,y}\, dA \\ &= \int_{\partial A} M_{yy} v_{,y} n_y\, dS - \int_{\partial A} M_{yy,y} v n_y\, dS + \int_A M_{yy,yy} v\, dA \qquad (4.20)\end{aligned}$$

where n_y is the y component of the unit outward normal vector.

The $2M_{xy}v_{,xy}$ is handled similarly, for one term the Green formula is used for x first and then for y, for the other term it is used for y first and then for x.

Collecting the terms we obtain: Find w satisfying

$$\begin{aligned}\int_A (M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy}) v\, dA + \int_{\partial A} v_{,x} M_{xx} n_x + v_{,y} M_{yy} n_y + v_{,x} M_{xy} n_y + v_{,y} M_{xy} n_x\, dS \\ - \int_{\partial A} v (M_{xx,x} n_x + M_{yy,y} n_y + M_{xy,y} n_x + M_{xy,x} n_y)\, dS = \int_A q\, v\, dA \quad \text{for all } v \qquad (4.21)\end{aligned}$$

In order to discuss the boundary conditions the boundary integrals is rewritten. The outward normal on the boundary is n and the unit tangent vector is s. Since we are in two dimensions $s = (-n_y, n_x)$. First

$$v_{,x} M_{xx} n_x + v_{,y} M_{yy} n_y + v_{,x} M_{xy} n_y + v_{,y} M_{xy} n_x = (\nabla v)^T \begin{pmatrix} M_{xx} n_x + M_{xy} n_y \\ M_{xy} n_x + M_{yy} n_y \end{pmatrix} \qquad (4.22)$$

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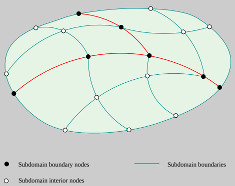

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The gradient can be decomposed into

$$\nabla v = (\nabla v)^T n\, n + (\nabla v)^T s\, s = \frac{\partial v}{\partial n} n + \frac{\partial v}{\partial s} s \tag{4.23}$$

We also introduce the notation

$$M_{nn} = n^T \begin{pmatrix} M_{xx} n_x + M_{xy} n_y \\ M_{xy} n_x + M_{yy} n_y \end{pmatrix} \quad \text{and} \quad M_{ns} = s^T \begin{pmatrix} M_{xx} n_x + M_{xy} n_y \\ M_{xy} n_x + M_{yy} n_y \end{pmatrix} \tag{4.24}$$

Thus

$$v_{,x} M_{xx} n_x + v_{,y} M_{yy} n_y + v_{,x} M_{xy} n_y + v_{,y} M_{xy} n_x = \frac{\partial v}{\partial n} M_{nn} + \frac{\partial v}{\partial s} M_{ns} \tag{4.25}$$

For the second term we introduce the shear forces Q_x and Q_y and obtain

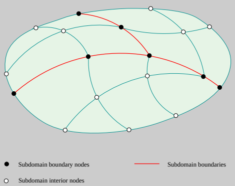
$$M_{xx,x} n_x + M_{yy,y} n_y + M_{xy,y} n_x + M_{xy,x} n_y = Q_x n_x + Q_y n_y = Q_n \tag{4.26}$$

The weak form takes the form: Find w satisfying

$$\begin{aligned} &\int_A (M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy}) v \, dA \\ &\qquad + \int_{\partial A} \frac{\partial v}{\partial n} M_{nn} + \frac{\partial v}{\partial s} M_{ns} \, dS - \int_{\partial A} v\, Q_n \, dS = \int_A q\, v \, dA \quad \text{for all } v \end{aligned} \tag{4.27}$$

Now, use test functions v such that $v = \frac{\partial v}{\partial n} = 0$, then

$$M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} = q \tag{4.28}$$



Otherwise it would be possible to find a basis function that violates the equality in the weak form.

Then, assume that $v = 0$ on the boundary, but the normal derivative is nonzero. Since $v = 0$, $\frac{\partial v}{\partial s} = 0$ and

$$M_{nn} = 0 \tag{4.29}$$

To treat the final terms on the boundary we assume that the boundary is smooth and use integration by parts

$$\int_{\partial A} \frac{\partial v}{\partial s} M_{ns} \, dS = - \int_{\partial A} v \frac{\partial M_{ns}}{\partial s} \, dS \tag{4.30}$$

Combining this with the shear force term

$$\int_{\partial A} v \left(Q_n - \frac{\partial M_{ns}}{\partial s} \right) dS = 0 \tag{4.31}$$

Thus

$$V_n = Q_n - \frac{\partial M_{ns}}{\partial s} = 0 \tag{4.32}$$

The combinations of boundary conditions are summarized below. If the boundary is not smooth corner forces enters at the corner points, see below.

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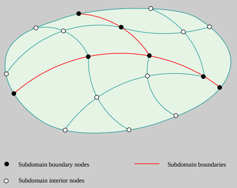

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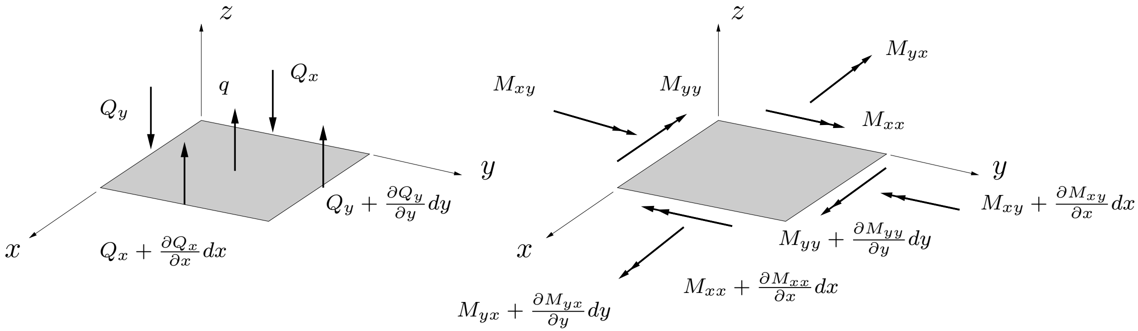
4.9. Field equations II

Equilibrium: The equilibrium equation can also be derived by deriving the equilibrium of a sub square of the plate. In the derivation shear forces enters, since we neglected shear forces they where not part of the potential energy functional above. The figure below show the direction of the shear forces and moments.



Equilibrium in the z-direction result in equilibrium of the shear forces:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q$$

Then, we consider the equilibrium of moments relative to the three axes. Equilibrium relative to the x and y axes result in the two differential equations for moments:

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -Q_x \qquad \text{and} \qquad \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} = -Q_y$$

while equilibrium relative to the z-axis result in

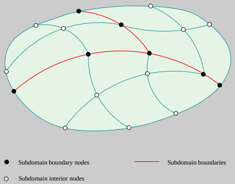
$$M_{xy} = M_{yx}$$

This can be derived from the stresses.

Eliminating Q_x, Q_y and M_{yx} result in

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = q$$

Using the material law for an isotropic material, and eliminating eliminate the moments and curvatures, the biharmonic plate equation, Equation 4.2, can be established.



4.10. Boundary conditions

Before introducing the boundary conditions it is appropriate to discuss *Cauchys law* for shear forces and moments. It is illustrated below.

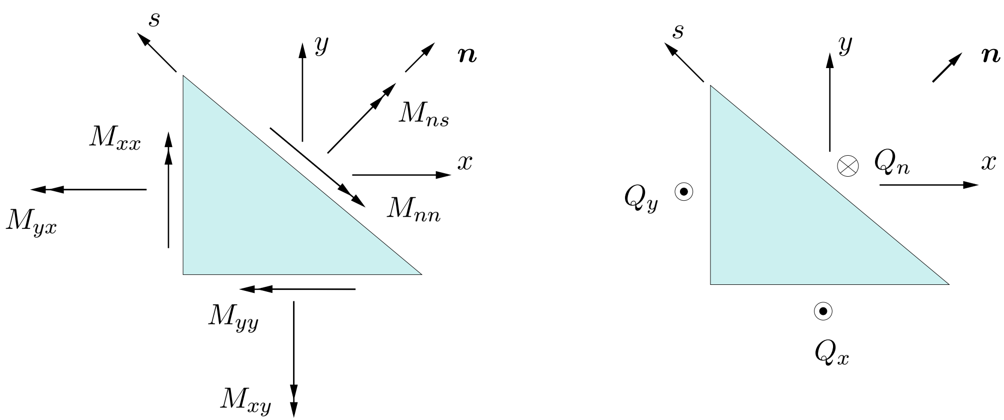


Figure 4.2: Cauchys law for a plate.

For the moments

$$M_{nn} = M_{xx}c^2 + M_{yy}s^2 + 2M_{xy}sc \qquad \text{and} \qquad M_{ns} = (M_{yy} - M_{xx})sc + M_{xy}(c^2 - s^2). \quad (4.33)$$

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where $s = \sin\phi$ and $c = \cos\phi$. For the shear forces

$$Q_n = Q_x c + Q_y s \tag{4.34}$$

In the derivation of suitable boundary conditions *Poisson's paradox* was encountered, it was solved by *Kirchhoff*:

- The plate equation is a fourth order partial differential equation.
- Hence, two boundary conditions must be specified at each point on the boundary.
- It appears that we have three conjugate quantities on the boundary, normal moment, rotational moment and shear force, see Equation 4.33 and Equation 4.34.

We assume that the local axis are oriented as shown in Figure 4.2. The kinematic quantities are given by

$$w, \qquad \frac{\partial w}{\partial n} = -\theta_s \qquad \text{and} \qquad \frac{\partial w}{\partial s} = \theta_n.$$

The *conjugate* quantities at the boundary are

$$Q_n, \qquad M_{nn} \qquad \text{and} \qquad M_{ns}$$

The work on the boundary are

$$W_B = \int_S \left(Q_n w + M_{nn} \frac{\partial w}{\partial n} + M_{ns} \frac{\partial w}{\partial s} \right) ds \tag{4.35}$$

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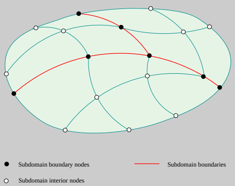

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It seem like we should specify three values at each point of the boundary:

$$\text{Simply supported:} \qquad w = 0, \qquad \frac{\partial w}{\partial s} = 0 \qquad \text{and} \qquad M_{nn} = 0.$$

$$\text{Free:} \qquad Q_n = 0, \qquad M_{nn} = 0 \qquad \text{and} \qquad M_{ns} = 0.$$

However, if $w = 0$ it follows that $\frac{\partial w}{\partial s} = 0$, thus only two quantities are relevant in the first case.

The reduction to two independent variables on the boundary is based on integration by parts of the middle term of Equation 4.35 along a line segment AB

$$W_B|_A^B = \int_{[A,B]} \left[\left(Q_n - \frac{\partial M_{ns}}{\partial s} \right) w + M_{nn} \frac{\partial w}{\partial n} \right] ds + M_{ns} w|_A^B$$

Upon introducing the modified shear force

$$V_n = Q_n - \frac{\partial M_{ns}}{\partial s}$$

the load potential can be written

$$W_B|_A^B = \int_S \left(V_n w + M_{nn} \frac{\partial w}{\partial n} \right) ds + M_{ns} w|_A^B \tag{4.36}$$

This result in the correct number of conjugated quantities at the boundary:

$$w,\ V_n \qquad \text{and} \qquad \frac{\partial w}{\partial n} = -\theta_s,\ M_{nn}$$

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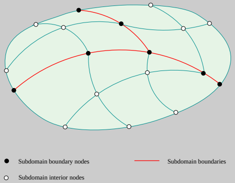


Homogeneous boundary conditions: The homogeneous boundary conditions for a thin plate is:

Clamped:	$w = 0$	and	$\dfrac{\partial w}{\partial n} = 0.$
Simply supported:	$w = 0$	and	$M_{nn} = 0.$
Free:	$V_n = 0$	and	$M_{nn} = 0.$
Symmetric about s:	$V_n = 0$	and	$\dfrac{\partial w}{\partial n} = 0.$

Corner forces: The last term in Equation 4.36 needs to be explained.

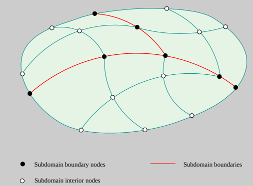
If the boundary is smooth, M_{ns} is continuous on the boundary S and that the boundary is smooth such that $A = B$

$$M_{ns}w|_A^B = 0.$$

In case the plate has corners, the integration is done "edge by edge". In an corder C the moment M_{ns} has a jump, say from M_{ns}^+ to M_{ns}^-. At the corners the jump term becomes

$$M_{ns}w|_A^B = R_c w = (M_{ns}^+ - M_{ns}^-)w$$

since w is continuous. This jump is called *corner forces*.



4.11. Summary

In summary:

The highest derivative m in the PMPE	2	
Highest derivative $2m$ (in the differential equation)	4	
Kinematic boundary conditions	$(0, 1)$	$(w, w_{,n} = -\theta_s)$
Natural boundary condition	$(2, 3)$	(M_{nn}, V_n)
Continuity requirement $m - 1$	1	$(w, w_{,x}, w_{,y})$

The load potential can be added to the potential energy functional:

$$\Pi(w) = \frac{1}{2} \int_A \boldsymbol{\kappa}^T \boldsymbol{D} \boldsymbol{\kappa} \, dA - \int_A q w \, dA - \int_S \left(V_n w + M_{nn} \frac{\partial w}{\partial n} \right) ds - M_{ns} w|_A^B$$

Thus, distributed loads, boundary moments and boundary shear forces can be specified.

Since second derivatives are present in the formulation, C^1 continuity is required for conforming finite element methods. Unfortunately, this i more difficult in two dimensions, for a triangular element quintic polynomials are required.

Kinematic boundary conditions are prescribed for displacements and rotations. Natural boundary conditions for moments and shear forces.

Before discussing finite elements for thin plates we will derive equations for thick plates.

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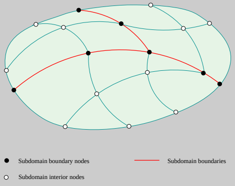

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Remark 4.7 Other sources for information on plate theory is Professor Carlos Felippa, University of Colorado, Boulder. The notes to *Advanced Finite Element Method* has two chapters on plates.

A classical reference to plates is [Timoshenko and Woinowsky-Krieger, 1970][3].

[3] Stephen P. Timoshenko and S. Woinowsky-Krieger. *Theory of plates and shells.* Mc Graw Hill, second edition, 1970.

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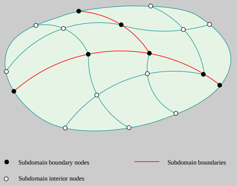

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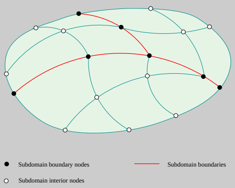

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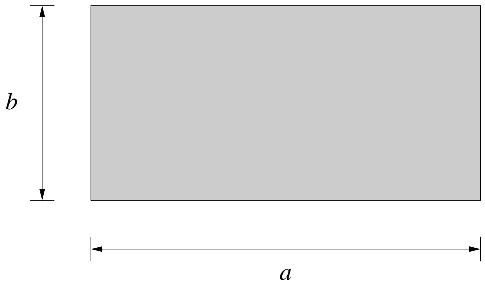
Øving 4.1

Figuren under viser en rektangulær plate.



Anta at platen er fritt opplagt, består av et isotropt materiale og er belastet med en fordelt last gitt ved

$$q(x, y) = q_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Videre så antar vi at løsningen av platens differensialligning er gitt ved

$$w(x, y) = C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

a) Tilfredstiller løsningen, $w(x, y)$, platens randkrav?

b) Bestem C_{mn} slik at platens differensialligning blir tilfredsstilt.

c) Bestem momentene M_{xx}, M_{yy} og M_{xy}.

d) Plot M_{xy} f.eks. i `MATLAB`. Kan du ut i fra figuren og definisjonen av M_{ns} avgjøre om vi får hjørnekrefter i platen?

Anta at lasten nå kan beskrives ved

$$q(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

For en jevnt fordelt last, q_0, så har vi at

$$q_{mn} = \frac{16 q_0}{\pi^2 mn}.$$

På samme måte så antar vi at forskyvningen er bestemt ved

$$w(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

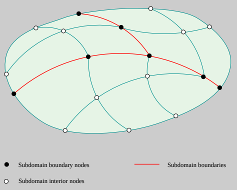
e) Benytt løsningen funnet for C_{mn} i b) over til å bestemme hver av koeffisientene C_{mn}.

f) Bestem hvor mange ledd i summen en trenger for å få fire siffers nøyaktighet.

g) Gjør det samme for momentene. Konvergerer disse like fort som forskyvningen?

Øving 4.2

I denne oppgaven skal vi modellere platen i oppgave 4.1 i ANSYS. Benytt $q_0 = 1$, $a = b = 10$, $E = 10.92$, $\nu = 0.3$ og $t = 0.1$.



- Modeller hele platen ved å benytte `SHELL63` (`KEYOPT(1)` $= 2$, slår av membran effekter).
- Sammenlign resultatene for w_{midt} for elementinndelingene 2×2, 4×4, 8×8 og 16×16 med eksakt løsning.
- Se på reaksjonskreftene for 16×16 elementnettet. Kan vi se om vi har hjørnekrefter (slik som teorien sier).
- Benytt samme element men lag et elementnett bestående av trekanter. Sammenlign trekant- og firkantløsningen.
- Benytt symmetriegenskapene til å modeller en kvart modell. Sammenlign trekantløsningen med trekantløsningen fra en full modell.

Øving 4.3

Tynne plater er ikke tilgjengelig i *COMSOL Multiphysics Structural Module*. Vi kan benytte Mindlin-Reissner formuleringen og sørge for at platen er tynn og så se om vi får resultater som samsvarer med tynnplateløsningen.

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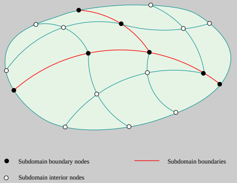

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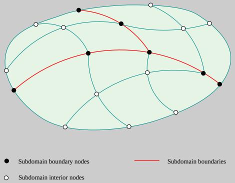

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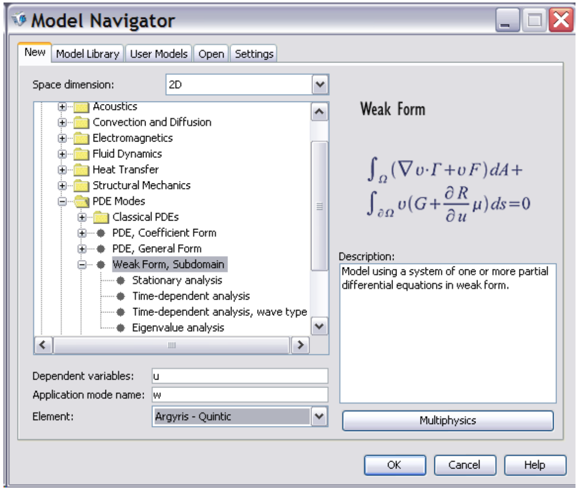
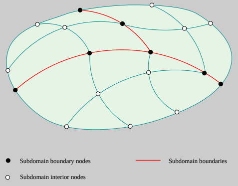
Et alternativ er å benytte *COMSOL Multiphysics* **PDF Modes**. Vi trenger da å modellere de bi-harmoniske ligningene fra grunnen av og vi må velge *Argyris* — *Quintic* under *Element*.



I denne oppgaven skal vi benytte samme data som i 4.2.

- Sett opp de bi-harmoniske ligningene for en tynn plate med isotropt homogent materiale på en form som passer i *COMSOL Multiphysics* (svak form, *Weak Form, Subdomain*).
- Modeller dette problemet i *COMSOL Multiphysics* som en generell svak form for problemet i 4.2.
- Sammenlign resultatene med eksakt løsning.
- I brukermanualen til *COMSOL Multiphysics* er det knyttet kommentarer til dette elementet. Er det noe å ta hensyn til i problemet vi løser her?

- Sammenlign med resultatene fra ANSYS 10.0.
- Benytt også svak formulering for de kinematiske randkravene (*Lagrange* — *Linear* interpolasjon for de kontinuerlige Lagrange multpilikatorene). Hvordan ser disse Lagrange-multiplikatorene ut i hjørene? Tegn randverdiene.



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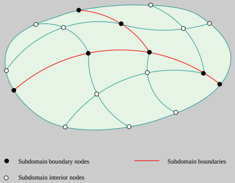

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