The Theory of Shells and Plates.

By

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With 2 Figures.

A. Introduction.[1]

1. Preliminary remarks. A plate and more generally a shell is a special three-dimensional body whose boundary surface has special features. Although we defer defining a shell-like body in precise terms until Sect. 4, for the purpose of these preliminary remarks consider a surface—called a reference surface—and imagine material filaments from above and below surrounding the surface along the normal at each point of the reference surface. Suppose further that the bounding surfaces formed by the end points of the material filaments are equidistant from the reference surface. Such a three-dimensional body is called a shell if the dimension of the body along the normals, called the thickness, is *small*. A shell is said to be *thin* if its thickness is much *smaller* than a certain characteristic length of the reference surface, e.g., the minimum radius of the curvature of the reference surface for initially curved shells.[2]

Interest in the construction of a linear theory for the extensional and flexural deformation of plates (from the three-dimensional equations of linear elasticity) dates back to the early part of the nineteenth century. Following a short period of controversy (especially with regard to the nature of boundary conditions), the complete theory for bending of thin elastic plates, under certain special assumptions, was finally derived in 1850 by Kirchhoff.[3] This theory, now classical (and occasionally referred to as Poisson-Kirchhoff theory for bending of plates), remains virtually unchanged even in its details. The corresponding development for shells from the three-dimensional equations of linear elasticity was given some thirty-eight years later in a pioneering paper by Love (cited below). This paper, containing the first complete linear bending theory for *thin* shells, employs certain special assumptions analogous to Kirchhoff's; in the current literature on

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I should like to express my appreciation also to M. M. Carroll, J. L. Ericksen, R. P. Nordgren and C. Truesdell, who offered helpful comments on the final form of the manuscript.

[2] In the case of initially flat plates, the characteristic length is taken to be the smallest dimension of the reference plane.

[3] Kirchhoff [1850, *1*]. See also the *Historical Introduction* in Love's Treatise [1892, *1*] and [1893, *2*] or its subsequent editions, e.g., [1927, *1*] or [1944, *4*].

shell theory, these assumptions are sometimes referred to as Kirchhoff-Love assumptions. This theory, which has come to be known as Love's first approximation, despite shortcomings, has since occupied a position of prominence.[4] The rather large number of papers in recent years devoted to re-examinations, rederivations, extensions and generalizations of the equations of the linear theory of thin shells, in itself may be a sufficient indication of not only the shortcomings of LOVE'S theory, but also that the foundations of the theory (even linear theory) as derived from the three-dimensional equations is not as yet firmly established. Indeed, there have been continual efforts by a number of investigators (especially during the last two decades) to rigorize and systematize the derivation of the (two-dimensional) equations of the classical linear theory of thin shells; and these efforts have resulted in considerable improvements toward a more satisfactory derivation of an approximate linear theory of shells, particularly the constitutive equations, from the three-dimensional equations. Although these improvements, achieved by a variety of means (e.g., use of variational theorems of classical elasticity, asymptotic expansion technique or consideration of error estimates, etc.), have contributed to our understanding of the subject, *alas* too slowly, a fully satisfactory derivation of an approximate system of constitutive equations along with an appraisal of their accuracy as compared with the three-dimensional equations still remains an open issue. Further remarks on this point are made in Sect. 4 and again in Sect. 20.

Historically, interest in the construction of theories of shells and plates grew from the desire to treat vibrations of plates and shells, aimed at deducing the tones of vibrating bells.[5] However, the motivating factors for the development of a general bending theory of sufficiently thin shells stem mainly from the following considerations: (i) The reduction of an otherwise intractable (at least analytically) three-dimensional problem to one characterized by two-dimensional equations; and (ii) the need to treat shell-like bodies when bending effects are prominent. Such bending effects arise in the open shell which has an edge but they may also be significant in both open and closed shells due to purely geometrical or load discontinuities. Thus, the purpose of a theory of shells (and plates) is to provide appropriate two-dimensional equations applicable to shell-like bodies. Examples of bodies to which shell theory is applicable are numerous and range from machine parts, electronic devices, domes, variety of ship and aerospace structures to components of physiological systems such as arteries, the cornea of the eye and the periodontal membrane, to name only a few.

Because of the considerable difficulties associated with the derivation of shell theory from the three-dimensional equations mentioned above, the possibility of employing a (two-dimensional) *model* for a thin shell presents itself in a natural way. Indeed, such an approach for thin shells was conceived of and dealt with (although at the time with some limitations) by the brothers E. and F. COSSERAT but remained largely unknown or unnoticed until fairly recently.[6] The idea of representing a (three-dimensional) thin shell by a two-dimensional continuum

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[4] LOVE [1888, *1*]. The reference to LOVE's first approximation in contemporary literature is often confusing, since different versions bearing his name can be found. The difference between these and their origin is briefly discussed in Sect. 21 A.

[5] Interestingly enough in the Summary of his [1888, *1*] paper, LOVE wrote: "This paper is really an attempt to construct a theory of the vibrations of bells." For historical background on the subject, see LOVE's [1944, *4*] *Historical Introduction* and TODHUNTER and PEARSON's history of elasticity [1886, *1*], [1893, *3*].

[6] [1908, *1*] and [1909, *1*]. The monograph by E. and F. COSSERAT [1909, *1*] and a number of recent papers bearing on the matter are cited and elaborated upon at some length in Sect. 4.

which would then permit the development of shell theory by direct approach (rather than from the three-dimensional equations) is not as strange as may at first appear to some. The notion of a model for an idealized body, a system or even a universe permeates the structure of classical physics; and is, in fact, the cornerstone of all field theories. Before continuing our discussion of the derivation of shell theory by direct approach, it is desirable to enumerate briefly some examples pertaining to different ways in which the idea of a model is used in the classical (non-polar) continuum mechanics.

To begin with the *continuum* itself is a model representing an idealized body in some sense. Roughly speaking, we may recall that the continuum model (in classical mechanics) is intended to represent phenomena in nature which appear at a scale larger than the interatomic distances. From such intuitive notions and by assigning a (continuous) mass density to the continuum model (corresponding to that of the macroscopic behavior of the general medium in question) we put forth a well-defined model for which the classical field theories of mechanics can be constructed. The plausibility of such an idealized model depends, of course, on its relevance; but this hardly needs emphasis here, in view of the success of the continuum theories and wide and extensive demonstrations of their usefulness and relevance to phenomena in the physical world for nearly two centuries. To continue, the idea of the use of a model in the classical three-dimensional (non-polar) theory is not limited to the concept of the continuum itself and is, in fact, adopted also for a different purpose. To elaborate, we recall that since the field equations in the classical continuum mechanics hold for every medium, it is only the constitutive equations which differ from one medium to another. Now constitutive equations (even those which embrace considerable generality) are always developed with a view toward a particular *model*.[7] For example, constitutive equations which define Newtonian viscous fluids describe the behavior of a class of fluids in all motions. Moreover, the success and the prominence of the theory of Newtonian viscous fluids is simply due to its usefulness for certain purposes and not because it is a valid theory for all fluid media; indeed, for another fluid medium (having a different behavior from Newtonian viscous fluid) we require a different set of constitutive equations. We have come to accept the point of view of describing the behavior of different materials through different sets of constitutive equations which, in turn, represent characterization of a particular model we have in mind.

In the classical theories of the type just referred to, it is the equations representing the material behavior which distinguish one theory from another; i.e., it is the characterization of constitutive equations which is intended to represent the particular behavior of a phenomenon in nature, whenever the classical theory can be assumed to hold. But, as in the case of the continuum itself, we need not confine the idea of a model to the sole purpose of characterizing the behavior of materials by constitutive equations; and, in fact, we may appeal to a suitable model for different purposes whenever such notions are conceptually helpful.[8]

Returning to our earlier discussion regarding the foundations and formulation of the general theory of shells (and plates), we observe here that the two-dimen-

[7] Our use of the term model here is intended to reflect the nature of the behavior of materials and should not be regarded as synonymous with other usages of the term, e.g., with reference to combinations of springs and dashpots.

[8] Indeed it was a point of view of this kind that prompted DUHEM [1893, *1*] to propose the concept of *oriented* media, which was subsequently adopted by the COSSERATS. Additional remarks, in this connection, are made in Sect. 4.

sional field equations for shells often have been derived by direct procedures (rather than from the three-dimensional equations). For example as noted in Sect. 12A, almost from the very early developments in shell theory, the derivation of the (two-dimensional) equations of equilibrium by LOVE and others was accomplished by considering a portion of a reference surface (embedded in a Euclidean 3-space) subjected to load resultants acting on the reference surface and various stress-resultants and stress-couples on the edge curves of the reference surface (corresponding to the middle surface of the shell in a reference configuration).[9] Similarly, in some of the literature on the linear shell theory devoted to derivations from the three-dimensional equations, a (two-dimensional) virtual work principle in terms of two-dimensional variables is stated *ab initio* and is assumed to be valid without any previous appeal to its derivation from the corresponding virtual work principle in the three-dimensional theory. The justification for such an approach (which is not uncommon even in some of the recent or current literature) is of course based on the fact that the two-dimensional principle is *postulated* to be valid on the middle surface of the shell. As another example, consider the membrane theory of elastic shells. Whenever this special theory is regarded to be applicable, it is tacitly assumed that the behavior of the (three-dimensional) shell can be represented by a membrane with only the gradient of the position vector of a material surface (corresponding to, say, the initial middle surface of the shell) in the deformed configuration as its kinematic ingredient. In this connection, it may be recalled that the idea of a *membrane* is not limited to a special theory of elastic shells and in a wider sense pertains to a model which reflects only extensional properties of a thin shell-like body.

The foregoing remarks are intended not only to illustrate the usefulness of the direct approach, but also to remind the reader of the fact that the direct approach has been known and utilized all along in shell theory. Thus, in addition to the definition of the (three-dimensional) shell-like body, we also motivate and introduce in Sect. 4 a (two-dimensional) model which portrays a thin shell. This model [described in detail in Sect. 4, following (4.34)], called a *Cosserat surface*, consists of a surface with a single director (i.e., a deformable vector) assigned to every point of the surface. As will become evident in subsequent chapters, such a *directed* two-dimensional continuum is conceptually simple and provides a fruitful means for characterization and direct development of a general theory for shells. In Chaps. B to D we pursue the construction of a general theory of shells and plates, both by direct approach (based mainly on the concept of a Cosserat surface) and from the three-dimensional equations of the classical (non-polar) continuum mechanics. Such parallel developments are illuminating and provide at the same time a basis of comparison between the various results (especially the field equations) from the direct approach and those emerging from a derivation via the three-dimensional theory. Inasmuch as considerable difficulties remain in the derivation of an *approximate* system of (two-dimensional) equations for thin shells (especially with regard to an approximate set of constitutive equations) from the three-dimensional theory, the alternative development by the direct approach offers a great deal of appeal and is relatively simple. We emphasize the latter approach in Chap. E, which is confined to the linear theory, with the aim of demonstrating the relevance and applicability of the theory of Cosserat surface to shells and plates. The Cosserat surface, as emphasized in Sect. 4, is not a two-dimensional surface alone and can be regarded as represent-

[9] The derivation of the equations of equilibrium for shells from the three-dimensional equations is of a more recent origin. An account of the history of the derivations of the equations of equilibrium for shells is given in Sect. 12A.

ing a model for a thin shell. It will become evident in Chaps. D and E that the director is an effective part of the model reflecting the three-dimensional effects in thin shells and plates.

2. Scope and contents. This monograph is concerned mainly with the foundations of the general theory of shells and plates. Our point of view and approach to the subject are motivated and spelled out in some detail at the beginning of Chap. B (Sect. 4). The preliminary remarks and various definitions in Sect. 4 should be kept in mind in connection with the remaining developments. An effort is made to provide a systematic treatment of the subject in the context of the nonlinear theory, both by direct approach and also from the three-dimensional equations of the classical continuum mechanics. While the equations of the linear theories of shells and plates are included and fully discussed, generally these are obtained by a systematic linearization of the results from the nonlinear theory. On the whole the subject matter is directed toward recent developments, and all aspects of the theory pertaining to the mechanical and thermodynamical foundations of the subject are treated. The problem of stability, however, is not considered. A number of results included here have not appeared or been discussed previously in the literature.

No attempt is made to provide a complete list of works on shells and plates. Our guideline in compiling the bibliography has been to select those which directly bear on our treatment, those which are pertinent to various discussions and provide a source for further related references and a few of the historical papers on the subject.[10] Neither the contents, nor the list of works cited are exhaustive. Nevertheless, it is hoped that the developments presented reflect accurately the state of knowledge in the foundations of the general theory. Some familiarity with the classical three-dimensional (non-polar) continuum mechanics is assumed. In this connection and for background and additional information, we frequently refer to TRUESDELL and TOUPIN and to TRUESDELL and NOLL.[11]

The kinematics of the subject and primitive concepts associated with the basic principles are developed and emphasized only to the extent that they are needed in our treatment of the subject. Moreover, we have made an effort not to take the reader through an extended excursion of the most primitive notions and ideas which are familiar from their counterparts in the three-dimensional theory. For example, the idea of force and couple vectors, each per unit length of a curve on a surface, and associated results are introduced as rapidly as possible (without sacrifice to clarity) but not such details as the notion of the stress vector, the stress tensor, etc., which are discussed in a treatise on continuum mechanics. Similarly, some of the formulae such as those for linearized kinematic measures can be put in a variety of forms by straightforward (although on occasion lengthy) manipulations. Whenever possible such lengthy formulae are conveniently catalogued in separate tables. This should enable a reader not interested in detailed linear kinematic formulae to pass over such special topics easily. A short appendix is included at the end (Chap. F), where for convenience selected formulae from the differential geometry of a surface and related results are collected.

The kinematics of shells and plates, both by direct approach and from the three-dimensional theory, are discussed in Chap. B (Sects. 4–7) at some length. In Sect. 4, we first define a shell-like body and then elaborate in some detail

[10] Our bibliography includes several papers, cited in the text, which became available when this work was almost completed. Consequently, we have not had an opportunity to examine fully these papers which either appeared in recen: periodicals or were kindly sent by their authors in manuscript form.

[11] [1960, *14*] and [1965, *9*].

about the nature of shell theory and motivate the introduction of a Cosserat surface as a continuum portraying a *thin* shell. This background information should be kept in mind in connection with all subsequent developments. Sects. 5 and 6 are concerned with kinematical results by direct approach while their counterparts from the three-dimensional theory are developed in Sect. 7.

Chap. C (Sects. 8–12A) deals with basic principles for shells and plates and shell-like bodies and derivations of the field equations both by direct approach (Sects. 8–10) and from the three-dimensional equations (Sects. 11–12). The basic principles and conservation laws for a Cosserat surface are discussed in detail in Sect. 8 and the local field equations are deduced in Sect. 9, where the field equations appropriate to the linearized theory are also obtained by specialization. Sect. 10 contains a derivation of the field equations for a restricted theory by direct approach, i.e., a theory in which the director is not admitted as an independent primitive kinematic ingredient. The contents of Sects. 11–12 are concerned with the construction of shell theory from the three-dimensional equations of the classical nonlinear continuum mechanics and deal chiefly with the derivation of the local field equations for shell-like bodies. In addition, the nature of the results which may be deduced for an approximate system of field equations for *thin* shells and their relationship with corresponding known results in the classical linear theory of shells and plates are elaborated upon in some detail. Also included in this chapter is a sketch of the history of the derivations of equations of equilibrium for shells (Sect. 12A).

Chap. D (Sects. 13–22) is concerned mainly with constitutive equations for elastic shells and plates, both by direct approach (Sects. 13–16) and from the three-dimensional theory (Sects. 17–21). This chapter also includes a number of related results pertaining to the initial boundary-value problem of the dynamical theory (or the boundary-value problem of the equilibrium theory), special cases of the general theory such as the membrane theory and the inextensional theory (Sect. 14), as well as the complete restricted theory by direct method (Sect. 15) which bears on the classical bending theory of shells. A fairly detailed development of the constitutive equations for an elastic Cosserat surface is given in Sect. 13 and their subsequent linearization is considered in Sect. 16. In the derivation of these results, particular attention is paid to the role played by the director and the manner in which the director (and its gradient) affect the structure of the constitutive equations. The corresponding developments from the three-dimensional theory, including certain approximation schemes for the purpose of obtaining an approximate system of (two-dimensional) equations for thin shells and plates, are discussed in Sects. 17–20. The contents of Sect. 20 (together with part of the remarks made in Sect. 21) provide an account of the classical results and their generalizations, as well as the nature of recent efforts, in the approximate linear theories of shells and plates obtained from the three-dimensional equations. In addition, a sketch of the history of the derivation of linear constitutive equations for elastic shells is provided in Sect. 21A. Also included in Chap. D is a general discussion concerning the relationship and correspondence between the results derived from the three-dimensional theory and those in the theory of Cosserat surface (Sect. 22).

Chap. E (Sects. 23–26) deals exclusively with the linear isothermal theory of elastic plates and shells. It represents a culmination of the point of view expressed in Sect. 4 and begins with a system of equations for the complete linear theory derived by direct method in the previous chapters. Since the constitutive coefficients are arbitrary (in the developments by direct approach) and are not predetermined from an approximate expression for the three-dimensional strain energy density function, a major portion of this chapter (Sects. 24–25) is devoted to the determination of the constitutive coefficients for an isotropic Cosserat surface and the corresponding results in the restricted theory. Chap. E also includes a uniqueness theorem, as well as some remarks on the general theorems in the linear theory of elastic shells.

Apart from the present introductory chapter, the contents of the remaining four Chaps. B to E are so arranged that to a large extent the various sections can be read independently of each other. As should be clear from the table of contents, the developments in each of the three Chaps. B to D are carried out both by direct approach and from the three-dimensional theory. By the time a reader has reached the end of Chap. D, he should be convinced of the following: (1) The development of kinematical results and derivation of field equations can be pursued in a systematic manner from both approaches; (2) the various results for thin shells, in particular the field equations and general aspects of the constitutive equations (in terms of a thermodynamic potential or a strain energy density function), are formally equivalent; and (3) while explicit forms of constitutive equations by direct approach can be dealt with systematically (and free from *ad hoc* assumptions), in general a great deal of difficulty is encountered when explicit constitutive equations are sought from the three-dimensional equations. It is partly here that the theory of a Cosserat surface can be put into fruitful and effective use, as brought out in Chap. E for the linear theory.

3. Notation and a list of symbols used. General convected curvilinear coordinates θ^i $(i=1, 2, 3)$ are used to identify a particle of a body. Similarly, a particle on a surface is identified by convected curvilinear coordinates θ^α $(\alpha=1, 2)$. Throughout this work Latin indices (subscripts or superscripts) have the range 1, 2, 3, Greek indices have the range 1, 2 and the usual summation convention is employed. We use a vertical bar $(|)$ for covariant differentiation with respect to the first fundamental form of a surface, a comma for partial differentiation with respect to surface coordinates θ^α and a superposed dot for material time derivative, i.e., differentiation with respect to time, holding the material coordinates (either θ^α or θ^i) fixed.

Fields and functions which are defined throughout a three-dimensional body are clearly distinguished from the corresponding fields and functions defined over a two-dimensional manifold. Whenever the same symbol is used in both cases, an asterisk is added to the symbol representing a field or a function in a three-dimensional body. Symbols standing for quantities associated with superposed rigid body motions are distinguished by placing a plus sign $(+)$ on the upper right-hand side of the symbol.

While the basic developments are carried out (to a large extent) in an invariant vector notation, the various vector quantities which occur in the field equations and constitutive equations are also expressed in terms of their tensor components. Boldface symbols are employed to designate vector fields or functions defined either throughout a region of a Euclidean 3-space or on a two-dimensional manifold. In general, the boldface symbols represent three-dimensional vector fields or functions; but occasionally they are also used for tangential vector fields defined over a surface. Greek lower case letters are used (although not exclusively) for local thermodynamic scalar functions or variables.

Since the various developments in Chaps. B to D are pursued both by direct approach and also from the three-dimensional theory, often by choice we employ the same symbols in the two developments. This choice of notations for similar quantities is suggestive and will not be confusing, since the two developments are carried out in parallel and entirely separately from one another. Moreover, in order to emphasize the separate nature of these two parallel developments, sometimes formulae of the same type and forms are repeated in a section pertaining to a derivation from the three-dimensional theory even though similar formulae have been already recorded in an earlier section concerned with direct approach.

The notations and formulae given in the Appendix (Chap. F) concerning the geometry of a surface and related results are used throughout and are particularly helpful in early sections. Although all symbols are defined when first introduced, a list of frequently used symbols is provided in a table below. It has not been possible to maintain a complete uniformity in notations and on occasions we have found it necessary to deviate from the scheme of the table and use the same symbols in different senses and in entirely different contexts.

Table of frequently used symbols

Symbol	Name or description	Place of definition or first occurrence
$\boldsymbol{a}_\alpha$, $\boldsymbol{A}_\alpha$	Base vectors of a surface in the present and reference configurations	(4.10)
$\boldsymbol{a}_3$, $\boldsymbol{A}_3$	Unit normal vector to a surface in the present and reference configurations	(4.11)
$a_{\alpha\beta}$, $A_{\alpha\beta}$	First fundamental form of a surface in the present and reference configurations	(4.12)
B	Flexural rigidity	(20.13)
$b_{\alpha\beta}$, $B_{\alpha\beta}$	Second fundamental form of a surface in the present and reference configurations	(4.13)
$\mathscr{B}$	Body	Sect. 4
c	A curve on the surface $\mathfrak{s}$	Sect. 8
$\boldsymbol{c}$, c_i	Acceleration vector for a Cosserat surface	(9.46)
C	Extensional rigidity	(20.13)
$\mathscr{C}$	Cosserat surface	Sect. 4
$\boldsymbol{d}$, $\boldsymbol{D}$	Director of the continuum $\mathscr{C}$ in the present and reference configurations	Sect. 4; $(5.1)_2$ and $(5.2)_2$
D	Component D_3 of $\boldsymbol{D}$	(4.35)
d_i, D_i	Components of director $\boldsymbol{d}$, $\boldsymbol{D}$	(5.25), $(5.34)_2$
$\boldsymbol{d}_N$, $\boldsymbol{D}_N$	Vector fields occuring in a representation of $\boldsymbol{p}$, $\boldsymbol{P}$	(7.1)–(7.2)
d_{Ni}, D_{Ni}	Components of $\boldsymbol{d}_N$, $\boldsymbol{D}_N$	(7.13)
$e_{\alpha\beta}$	A surface kinematic measure	(5.31)
E	Young's modulus of elasticity	(19.3)
$\mathscr{E}$	Internal energy of a Cosserat surface	(8.9)
$\boldsymbol{f}$	Assigned force per unit mass of a surface	(8.6)

Symbol	Name or description	Place of definition or first occurrence
$\bar{\boldsymbol{f}}$	Difference of $\boldsymbol{f}$ and acceleration vector	$(8.19)_1$
$\boldsymbol{f}^*$	External body force density	(8.26)
f^i, F^i	Components of $\boldsymbol{f}$ in the present and reference configurations	(9.38)
$\boldsymbol{g}_i, \boldsymbol{g}^i, g_{ij}, g^{ij}$	Base vectors, metric tensor and conjugate tensor referred to coordinates θ^i in the present configuration	(4.7)
$\boldsymbol{G}_i, \boldsymbol{G}^i, G_{ij}, G^{ij}$	Base vectors, metric tensor and conjugate tensor referred to coordinates θ^i in the reference configuration	(4.23)
$\boldsymbol{G}'_i, G'_{ij}$	Base vectors and metric tensor referred to a normal coordinate system in the reference configuration	(4.29)
h	Initial thickness of (three-dimensional) shell or plate	(4.30)
k, k^N	Scalar functions of position defined by mass density and determinant of the metric tensor	(4.21), (11.28)
$\mathcal{K}$	Kinetic energy of a Cosserat surface	(8.11)
$\boldsymbol{l}$	Assigned director couple per unit mass of a surface	(8.6)
$\grave{\boldsymbol{l}}$	Assigned couple per unit mass of a surface (a tangential vector field)	(10.1)
$\bar{\boldsymbol{l}}$	Difference of $\boldsymbol{l}$ and the inertia term due to director acceleration	$(8.19)_2$
$\boldsymbol{l}^N$ $(N=0, 1, 2, \ldots)$	Body force resultants	(11.29)–(11.30)
$l^i, \bar{l}^i, L^i, \bar{L}^i$	Components of $\boldsymbol{l}$ and $\bar{\boldsymbol{l}}$ in the present and reference configurations	(9.38), (9.57)
$\grave{l}^\alpha, \grave{L}^\alpha$	Components of $\grave{\boldsymbol{l}}$ in the present and reference configurations	(10.20), $(10.27)_2$
$\boldsymbol{m}, {}_R\boldsymbol{m}$	Surface director couple measured per unit area in the present and reference configurations	(8.16), $(8.58)_3$
m^i	Components of the surface director couple $\boldsymbol{m}$	(9.39)
$\boldsymbol{m}^N, m^{Ni}$ $(N=0, 1, 2, \ldots)$	Shear stress- and normal stress-resultants	(11.55), (12.11)
$\boldsymbol{M}, {}_R\boldsymbol{M}$	Contact director couple measured per unit length of a curve on a surface in the present and reference configurations	(8.4), (8.56)
$\grave{\boldsymbol{M}}$	Contact couple per unit length of a curve on a surface (a tangential vector field)	(10.1)

Symbol	Name or description	Place of definition of first occurrence
$\boldsymbol{M}^N$ $(N=0, 1, 2, \ldots)$ $\boldsymbol{M}^{N\alpha}$, $M^{N\alpha i}$ $(N=0, 1, 2, \ldots)$	Stress-resultants and stress-couples	(11.34), (11.36), (12.10)
$M^{\alpha i}$	Components of the contact director couple $\boldsymbol{M}$	(9.42)–(9.43)
$\grave{M}^{\alpha\gamma}$	Components of the contact couple $\grave{\boldsymbol{M}}$	(10.20)
$\boldsymbol{n}$	Outward unit normal vector to a surface in a body	Sect. 8
$\boldsymbol{N}$, ${}_R\boldsymbol{N}$	Contact force measured per unit length of a curve on a surface in the present and reference configurations	(8.4), (8.56)
$N^{\alpha i}$	Components of the contact force $\boldsymbol{N}$	(9.40)–(9.41)
$N'^{\alpha\beta}$	Certain combination of the components $N^{\alpha\beta}$, m^α and $M^{\alpha\beta}$	(9.31), (9.53)
$\grave{N}^{\alpha\beta}$	Certain combination of the components $N^{\alpha\beta}$ and $\grave{M}^{\alpha\beta}$	(10.26)
$\boldsymbol{p}$, $\boldsymbol{P}$	Position vector of the place in a body occupied by the material point in the present and reference configurations	(4.5)–(4.6)
P, ${}_RP$	Scalar quantities representing mechanical power	(9.28), (9.83)
$\mathscr{P}$	Part of a surface	Sect. 4; (4.43)
$\mathscr{P}^*$	Part of a body containing the corresponding part of the surface $\xi=0$	Sect. 11
$\overline{\mathscr{P}}$	Part of a body (not necessarily the same as $\mathscr{P}^*$)	Sect. 11
$\boldsymbol{q}$, ${}_R\boldsymbol{q}$	Heat flux vector per unit time for a Cosserat surface measured per unit length in the present and reference configurations	(9.27), (9.76)
$\boldsymbol{q}^*$, q^{*k}	Heat flux vector per unit area per unit time	(11.11), (11.15)
q_α	Components of $\boldsymbol{q}$ per unit length in the present configuration	(9.26)–(9.27)
${}_Rq_\alpha$	Components of ${}_R\boldsymbol{q}$ per unit length in the reference configuration	$(9.76)_3$
Q_α	Components of $\boldsymbol{q}$ per unit length in the reference configuration	(9.59)
Q^i_j, Q	Proper orthogonal tensor	(4.33), (5.37)
r	Heat supply function per unit mass per unit time for a Cosserat surface	(8.10)
r^*	Heat supply function per unit mass per unit time	$(11.11)_2$

Symbol	Name or description	Place of definition or first occurrence
R	Rate of work by contact and assigned forces and couples for a Cosserat surface	(8.8)
$\boldsymbol{r}$, $\boldsymbol{R}$	Position vector of the place on a surface occupied by the material point in the present and reference configurations	Sect. 4; $(5.1)_1$ and $(5.2)_1$
s, s_α	Relative kinematic measures for a Cosserat surface arising from normal components of director and its gradient	(13.58)
$\mathscr{s}$, $\mathscr{S}$	Surface of the continuum $\mathscr{C}$ in the present and reference configurations	Sect. 4
$\mathfrak{s}$, $\mathfrak{S}$	Surface $\xi = 0$ in a body $\mathscr{B}$	Sect. 4
t	Time	(4.6)
$\boldsymbol{t}$	Stress vector	(8.26)
$\boldsymbol{u}$, u^i	Infinitesimal displacement vector of a surface	(6.1)
$\boldsymbol{u}^*$, u^{*i}	Infinitesimal displacement vector of a shell-like body	(7.50)–(7.51)
$\boldsymbol{v}$, v^i	Velocity vector of a surface at time t	$(5.3)_1$, (5.7)
$\boldsymbol{v}^*$	Velocity vector of a shell-like body at time t	(7.4)–(7.5)
V^α, V^3	Combination of certain components of m^i and $M^{\alpha i}$ for a Cosserat surface	(9.62), (9.67) and (9.72)
$V_{i\alpha}$, $V^i_{.\alpha}$	Components of the gradient of a vector field $\boldsymbol{V}$	(5.5)
$\boldsymbol{w}$, w_i	Director velocity vector of a Cosserat surface at time t	$(5.3)_2$, (5.26)
$\grave{\boldsymbol{w}}$	Angular velocity of the unit normal to a surface at time t	$(5.61)_1$
W_{ki}	Spin tensor (a space tensor)	(5.13)–(5.14)
x_i, x^i	Rectangular Cartesian coordinates	(4.1)
α	Coefficient of director inertia	(8.11)
$\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{13}$	Constitutive coefficients in the linear theory (direct approach)	(16.21), (16.33)
$\beta_1, \beta_2, \beta_5, \beta_6$	Constitutive coefficients in the restricted linear theory (direct approach)	(16.40)
$\boldsymbol{\beta}$, β_α	Infinitesimal kinematic measure arising from rotation of the unit normal to a surface	(6.6)
γ	Specific entropy production for a Cosserat surface	(8.22)
γ_i	A relative surface kinematic measure due to components of director	(5.33)

Symbol	Name or description	Place of definition or first occurrence
γ_{Ni}	A kinematic measure in the three-dimensional theory	(7.19)
γ^*_{ij}	Strain tensor	(7.27)
$\boldsymbol{\Gamma}, \boldsymbol{\Gamma}_{:\alpha}$	Measures of rate of deformation arising from director velocity and director velocity gradient	(5.26), (5.29)
$\boldsymbol{\delta}$, δ_i	Infinitesimal director displacement vector	(6.2)
δ^i_j, δ^α_β	Components of a unit tensor	(4.7), (4.12)
ε	Specific internal energy of a Cosserat surface	(8.9)
ε^*	Specific internal energy	$(11.11)_1$
ε^n $(n=0, 1, 2, \ldots)$	Specific internal energy resultants	(11.43)
ε_{kim}, ε^{kim}; $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$; $\bar{\varepsilon}_{\alpha\beta}$, $\bar{\varepsilon}^{\alpha\beta}$	Absolute permutation symbols	(5.43), (5.63), (6.30)
ζ	Third coordinate in a normal coordinate system $[\theta^\alpha, \zeta]$	(4.25)
η	Specific entropy for a Cosserat surface	(8.21)
η^*	Specific entropy	Sect. 11; (11.12)
η^n $(n=0, 1, 2, \ldots)$	Specific entropy resultants	(11.44)
$\eta_{\alpha\beta}$, $\boldsymbol{\eta}_\alpha$	Surface rate of deformation	(5.15), (5.19)
θ	Temperature field defined on a surface	(8.20)
θ^*	Temperature field in a body	(11.12)
θ^i	Material coordinates in a body	(4.1) and (4.4)
θ^α	Material coordinates on a surface	Sect. 4
$\varkappa_{i\alpha}$	A relative surface kinematic measure due to components of the director gradients	(5.32)
$\bar{\varkappa}_{\beta\alpha}$	A surface kinematic measure (linear theory)	Sect. 6
$\varkappa_{Ni\alpha}$	A kinematic measure in the three-dimensional theory	(7.19)
$\boldsymbol{\lambda}$, λ^α	Unit tangent vector to a curve on a surface in the present configuration	(8.1)
$\lambda_{i\alpha}$, $\Lambda_{i\alpha}$	Components of the director gradient $\boldsymbol{d}_{,\alpha}$ and $\boldsymbol{D}_{,\alpha}$	(5.28) and (5.34)
$\lambda_{Ni\alpha}$, $\Lambda_{Ni\alpha}$	Kinematic measures in the three-dimensional theory	$(7.13)_4$, (7.14), $(7.20)_3$
μ	Shear modulus of elasticity	(20.13)
ν	Poisson's ratio	(19.3)

Symbol	Name or description	Place of definition or first occurrence
$\boldsymbol{\nu}$, ν^{α}	Unit normal to a curve on a surface in the present configuration	(8.2)
${}_0\boldsymbol{\nu}$, ${}_0\nu^{\alpha}$	Unit normal to a curve on a surface in the reference configuration	(8.55)
ξ	Third coordinate in a general convected coordinate system θ^i	(4.4)
ϱ, ϱ_0	Mass densities (per unit area) of a surface	Sect. 4; (4.17), (4.21), (4.38)
ϱ^*, ϱ_0^*	Mass densities of a body	Sect. 4; (4.16), (4.36)
$\varrho_{i\alpha}$, $\bar{\varrho}_{\alpha\beta}$	Surface kinematic measures (linear theory)	(6.24), (20.37), (25.16)
σ, σ_{α}	Certain kinematic variables for a Cosserat surface arising from normal components of director and its gradient	$(13.33)_{1,2}$
σ_{ij}	Cartesian components of the stress tensor	(24.1)
Σ	Strain energy density for a Cosserat surface	(14.3)–(14.4)
τ_{ij}	Symmetric stress tensor of Cauchy in coordinates θ^i	(11.7)
ψ	Specific Helmholtz free energy for a Cosserat surface	(9.34)
ψ^*	Specific Helmholtz free energy	(11.18)
ψ^n $(n=0, 1, 2, \ldots)$	Specific Helmholtz free energy resultants	(11.44)
φ	Specific Gibbs free energy for a Cosserat surface (linear theory)	(16.12)
φ^*	Specific Gibbs free energy	(19.1)
φ_N $(N=0, 1, \ldots)$	Scalar functions in a representation for temperature θ^*	(11.42)
$\boldsymbol{\omega}$, ω^i Ω, Ω_{ki}	Vector and skew-symmetric space tensor representing rigid body angular velocity	(5.39)–(5.42)
${}_0\bar{\boldsymbol{\omega}}$	Infinitesimal rigid body rotation	(6.42)
dv	Element of volume	$(4.19)_1$
$d\sigma$, $d\Sigma$	Element of area	$(4.19)_2$, (4.39)
ds, dS	Line element	(8.4), (8.56)
$(\,)_{,\alpha}$	Partial differentiation with respect to surface coordinates	(4.13)
$(\,)_{\|\alpha}$	Covariant differentiation with respect to first fundamental form of a surface	(4.13)

Symbol	Name or description	Place of definition or first occurrence
$(\ )_{\Vert i}$	Covariant differentiation with respect to the metric tensor g_{ij}	(11.15), (A.1.27)
$(\dot{\ })$	Material time derivative	(5.3)
$[L]$	Physical dimension of length	(4.30)
$[M]$	Physical dimension of mass	(4.37)
$[T]$	Physical dimension of time	(8.5)

B. Kinematics of shells and plates.

This chapter is concerned with the kinematics of shells and initially flat plates both by direct approach and from the three-dimensional theory of classical continuum mechanics. Definition of a shell-like body and motivation for introducing another continuum, namely a Cosserat surface, as a *model* reflecting the main features of a *thin* shell are included in Sect. 4. This preliminary and background material should be kept in mind in relation to the remaining sections of this chapter, as well as most of the subsequent developments in Chaps. C and D.

4. Coordinate systems. Definitions. Preliminary remarks. A shell or a plate is a three-dimensional body whose boundary surface enjoys special features. Before describing such a body in precise terms, it is convenient to introduce suitable coordinate systems. Let the points of a region $\mathscr{R}$ in a Euclidean 3-space be referred to a fixed right-handed rectangular Cartesian coordinate system x_i $(i=1, 2, 3)$ and let $\{\theta^1, \theta^2, \theta^3\}$ be a general *convected* curvilinear system defined by the transformation relations[1]

$$x_i = x_i(\theta^1, \theta^2, \theta^3). \tag{4.1}$$

We assume

$$\det\,(\partial x_i/\partial\theta^j) \neq 0, \tag{4.2}$$

so that (4.1) is nonsingular in $\mathscr{R}$ and has a unique inverse

$$\theta^i = \theta^i(x_1, x_2, x_3). \tag{4.3}$$

In what follows, all Latin indices (subscripts or superscripts) take the values 1, 2, 3 and Greek indices (subscripts or superscripts) take the values 1, 2. Also, for later convenience, we set $\theta^3 = \xi$ and adopt the notation

$$\theta^i = \{\theta^\alpha, \xi\}. \tag{4.4}$$

Consider now a three-dimensional body $\mathscr{B}$, embedded in the region $\mathscr{R}$ of the Euclidean 3-space, and let the particles of $\mathscr{B}$ be identified by a general convected coordinate system (4.3). Let $\boldsymbol{P}$ denote the position vector, relative to a fixed

[1] Often our notation and particularly the choice of convected (or moving) curvilinear coordinates is patterned after GREEN and ZERNA [1954, *1*] or [1968, *9*]. Although the use of a convected coordinate system is by no means essential, it is particularly suited in studies of special bodies (such as shells, plates and rods) and often results in simplification of intermediate steps in the development of the subject.

origin, of a typical particle of $\mathscr{B}$ in a reference configuration. Then,[2]

$$\boldsymbol{P}=\boldsymbol{P}(\theta^\alpha,\xi) \tag{4.5}$$

which can also be expressed as a function of x_i, in view of (4.2). We denote the position vector, relative to the same fixed origin, of a typical particle of $\mathscr{B}$ in the deformed configuration at time t by

$$\boldsymbol{p}=\boldsymbol{p}(\theta^\alpha,\xi,t). \tag{4.6}$$

Thus, (4.5) specifies the place occupied by the material point θ^i in a reference configuration while the place occupied by the material point θ^i in the deformed configuration is given by (4.6). The region of space at time t into which the body is mapped by the vector function $\boldsymbol{p}$ in (4.6) is the region occupied by the body in a given configuration. We assume that the vector function $\boldsymbol{p}$—a 1-parameter family of configurations with t as the real parameter—which describes the motion of the body $\mathscr{B}$, is sufficiently smooth in the sense that it is differentiable with respect to θ^α, ξ and t as many times as may be needed. We recall the formulae[3]

$$\begin{aligned}&\boldsymbol{g}_i=\frac{\partial\boldsymbol{p}}{\partial\theta^i}, \quad g_{ij}=\boldsymbol{g}_i\cdot\boldsymbol{g}_j, \quad g=\det(g_{ij}),\\ &\boldsymbol{g}^i=g^{ij}\boldsymbol{g}_j, \quad \boldsymbol{g}^i\cdot\boldsymbol{g}^j=g^{ij}, \quad \boldsymbol{g}^i\cdot\boldsymbol{g}_j=\delta^i_j,\end{aligned} \tag{4.7}$$

and further assume that

$$g^{\frac{1}{2}}=[\boldsymbol{g}_1\,\boldsymbol{g}_2\,\boldsymbol{g}_3]>0 \tag{4.8}$$

for physically possible motions.[4] In (4.7), $\boldsymbol{g}_i$ and $\boldsymbol{g}^i$ are the covariant and the contravariant base vectors at time t, respectively, g_{ij} is the metric tensor, g^{ij} is its conjugate and δ^i_j is the Kronecker symbol in 3-space. Formulae analogous to those in (4.7), valid in a reference configuration, can be deduced from (4.5) but we postpone recording such results.

A material surface in $\mathscr{B}$ can be defined by the equation $\xi=\xi(\theta^\alpha)$; the equations resulting from (4.5) and (4.6) with $\xi=\xi(\theta^\alpha)$ represent the parametric forms of this surface in the reference and deformed configurations. In particular, with reference to (4.6), $\xi=0$ defines a 1-parameter family of surfaces in space each of which we assume to be smooth and non-intersecting. We refer to the surface $\xi=0$ at time t (i.e., in the deformed configuration) by $\mathfrak{s}$. Any point of this surface is specified by the position vector $\boldsymbol{r}$, relative to the same fixed origin to which $\boldsymbol{p}$ is referred, where

$$\boldsymbol{r}=\boldsymbol{r}(\theta^\alpha,t)=\boldsymbol{p}(\theta^\alpha,0,t). \tag{4.9}$$

Let $\boldsymbol{a}_\alpha$ denote the base vectors along the θ^α-curves on the surface $\mathfrak{s}$. By (4.9) and $(4.7)_1$,

$$\boldsymbol{a}_\alpha=\frac{\partial\boldsymbol{r}}{\partial\theta^\alpha}=\boldsymbol{g}_\alpha(\theta^\gamma,0,t), \tag{4.10}$$

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[2] We may recall that when the particles of the body are referred to a convected (or moving) coordinate system, the numerical values of the coordinates associated with each material point remain the same for all time.

In (4.5) and in most of the developments that follow, the same symbol can be used for a function and its value without confusion. Only on occasions and wherever it serves clarity, the symbol for a function will be distinguished from that of its value. When a function is first introduced we either state or exhibit its arguments; but, subsequently we often employ the symbol designating the function without an explicit indication of the arguments as this will be clear from the context.

[3] For details see Sect. A.1 of the Appendix (Chap. F).

[4] Strictly speaking, for physically possible motions we only need to assume that $g^{\frac{1}{2}}\neq 0$ with the understanding that in any given motion $[\boldsymbol{g}_1\,\boldsymbol{g}_2\,\boldsymbol{g}_3]$ is either >0 or <0. The condition (4.8) also requires that θ^i be a right-handed coordinate system.

and the unit normal $\boldsymbol{a}_3$ to $\mathfrak{s}$ may be defined by

$$\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_3 = 0, \quad \boldsymbol{a}_3 \cdot \boldsymbol{a}_3 = 1, \quad \boldsymbol{a}_3 = \boldsymbol{a}^3, \quad [\boldsymbol{a}_1\, \boldsymbol{a}_2\, \boldsymbol{a}_3] > 0. \tag{4.11}$$

We also recall the formulae[5]

$$\begin{gathered} a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta, \quad a = \det(a_{\alpha\beta}), \\ \boldsymbol{a}^\alpha = a^{\alpha\beta} \boldsymbol{a}_\beta, \quad \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta = a^{\alpha\beta}, \quad a^{\alpha\gamma} a_{\gamma\beta} = \delta^\alpha_\beta, \quad a_{\alpha\gamma} a^{\gamma\beta} = \delta^\beta_\alpha, \end{gathered} \tag{4.12}$$

$$\begin{gathered} b_{\alpha\beta} = b_{\beta\alpha} = -\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_{3,\beta} = \boldsymbol{a}_3 \cdot \boldsymbol{a}_{\alpha,\beta}, \\ \boldsymbol{a}_{\alpha|\beta} = b_{\alpha\beta} \boldsymbol{a}_3, \quad \boldsymbol{a}_{3,\alpha} = -b^\gamma_\alpha \boldsymbol{a}_\gamma, \quad b_{\alpha\beta|\gamma} = b_{\alpha\gamma|\beta}, \end{gathered} \tag{4.13}$$

where $\boldsymbol{a}^\alpha$ denote the reciprocal base vectors of the surface $\mathfrak{s}$, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are its first and its second fundamental forms, a comma denotes partial differentiation with respect to the surface coordinates θ^α, a vertical bar stands for covariant differentiation with respect to $a_{\alpha\beta}$ and δ^α_β is the Kronecker symbol in 2-space.

Let $\partial\mathscr{B}$, the boundary of the body $\mathscr{B}$, be specified by (i) the material surfaces

$$\xi = \alpha(\theta^\alpha), \quad \xi = \beta(\theta^\alpha), \quad \alpha < 0 < \beta, \tag{4.14}$$

with the surface $\xi = 0$ lying entirely between them; and (ii) a material surface

$$f(\theta^1, \theta^2) = 0, \tag{4.15}$$

which is such that $\xi = \text{const.}$ are closed smooth curves on the surface (4.15).[6] Since the convected coordinates (4.4) are identified as material coordinates, the material surfaces $(4.14)_{1,2}$ have the same parametric representations in all configurations; and, in general α and β are functions of the surface coordinates θ^α but in special cases they may be constants. We observe that by virtue of (4.8) and $(4.14)_3$, the material surfaces $(4.14)_{1,2}$ do not intersect themselves, each other, or the surface $\xi = 0$; and in keeping with the designation of the surface $\xi = 0$ at time t by $\mathfrak{s}$, we refer to the surfaces $(4.14)_{1,2}$ in the deformed configuration as $\mathfrak{s}^-$ and $\mathfrak{s}^+$, respectively. The surface $\mathfrak{s}$ is not necessarily midway between the bounding surfaces $\mathfrak{s}^-$ and $\mathfrak{s}^+$ and the middle surface is arbitrarily situated with respect to the boundary surfaces $(4.14)_{1,2}$; however, a reference configuration may be chosen in which the initial middle surface is midway between the surfaces defined by $(4.14)_{1,2}$.

Let $\varrho^*(\theta^\alpha, \xi, t)$ and $\varrho^*_0(\theta^\alpha, \xi)$ be the mass densities of $\mathscr{B}$ in the deformed and reference configurations, respectively. Then, the (local) equation of conservation of mass is

$$\varrho^* g^{\frac{1}{2}} = \varrho^*_0 G^{\frac{1}{2}}, \tag{4.16}$$

where G is the dual of g in a reference configuration. We define a mass per unit area of $\mathfrak{s}$ at time t, namely $\varrho(\theta^\alpha, t)$, by the formula

$$\varrho a^{\frac{1}{2}} = \int_\alpha^\beta \varrho^* g^{\frac{1}{2}}\, d\xi, \tag{4.17}$$

where a and g are given by $(4.12)_2$ and $(4.7)_3$. Since θ^i are convected coordinates and since $\varrho^* g^{\frac{1}{2}}$ is independent of time, it follows from (4.17) that $\varrho a^{\frac{1}{2}}$ is also independent of time, although both ϱ and a may depend on t. The mass of an

[5] See Sects. A.2 and A.3 of the Appendix (Chap. F).

[6] In place of (4.15) we can specify a more general boundary surface of the form $\bar{f}(\theta^\alpha, \xi) = 0$ such that $\xi = \text{const}$ are closed smooth curves on this surface. However, (4.15) will suffice for our present purpose.

arbitrary portion of $\mathscr{B}$ bounded by the surfaces (4.14) and a surface of the form (4.15) may be expressed as

$$\iiint \varrho^* g^{\frac{1}{2}}\, d\theta^1\, d\theta^2\, d\theta^3 = \int \varrho^*\, dv = \iint \varrho\, a^{\frac{1}{2}}\, d\theta^1\, d\theta^2 = \int \varrho\, d\sigma, \tag{4.18}$$

where the ranges of integration in (4.18) are clear and dv and $d\sigma$ given by

$$\begin{aligned} dv &= (\boldsymbol{g}_1 \times \boldsymbol{g}_2 \cdot \boldsymbol{g}_3)\, d\theta^1\, d\theta^2\, d\theta^3 = g^{\frac{1}{2}}\, d\theta^1\, d\theta^2\, d\theta^3, \\ d\sigma &= (\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3)\, d\theta^1\, d\theta^2 = a^{\frac{1}{2}}\, d\theta^1\, d\theta^2 \end{aligned} \tag{4.19}$$

are an element of volume of the body $\mathscr{B}$ in the deformed configuration and an element of area of the surface $\mathfrak{s}$, respectively.

The relation of the surface $\xi = 0$ at time t, i.e., the surface $\mathfrak{s}$, to the bounding surfaces $\mathfrak{s}^-$ and $\mathfrak{s}^+$ can be fixed by imposing the condition[7]

$$\int_\alpha^\beta \varrho^* g^{\frac{1}{2}}\, \xi\, d\xi = \int_\alpha^\beta k\, \xi\, d\xi = 0, \tag{4.20}$$

where we have put for later convenience

$$k = k(\theta^\alpha, \xi) = \varrho^* g^{\frac{1}{2}} = \varrho_0^* G^{\frac{1}{2}}. \tag{4.21}$$

The condition (4.20) is independent of time, so that once the relative position of $\mathfrak{s}$ (i.e., relative to $\mathfrak{s}^-$ and $\mathfrak{s}^+$) is determined by such an equation (in, say, a reference configuration) it remains so determined. This completes the description of a shell-like body, i.e., a three-dimensional continuum bounded by the surfaces (4.14) and (4.15).

In order to continue our preliminary remarks, it is desirable to dispose of some further notation. Henceforth, let the initial configuration of $\mathscr{B}$ be taken as the reference configuration. Then, by (4.5), the initial position vector of the body $\mathscr{B}$, at time $t = 0$, is specified by

$$\boldsymbol{P} = \boldsymbol{P}(\theta^\alpha, \xi) = \boldsymbol{p}(\theta^\alpha, \xi, 0). \tag{4.22}$$

The initial values of $\boldsymbol{g}_i$, $\boldsymbol{g}^i$, g_{ij} will be designated by $\boldsymbol{G}_i$, $\boldsymbol{G}^i$, G_{ij} and we have the formulae

$$\begin{aligned} &\boldsymbol{G}_i = \frac{\partial \boldsymbol{P}}{\partial \theta^i}, \qquad G_{ij} = \boldsymbol{G}_i \cdot \boldsymbol{G}_j, \qquad G = \det(G_{ij}), \\ &\boldsymbol{G}^i = G^{ij}\, \boldsymbol{G}_j, \qquad \boldsymbol{G}^i \cdot \boldsymbol{G}^j = G^{ij}, \qquad \boldsymbol{G}^i \cdot \boldsymbol{G}_j = \delta^i_j, \end{aligned} \tag{4.23}$$

as the duals of those in (4.7). The initial surfaces in the initial configuration of $\mathscr{B}$, which become the surfaces $\mathfrak{s}$, $\mathfrak{s}^-$, $\mathfrak{s}^+$ at time t, will be referred to as $\mathfrak{S}$, $\mathfrak{S}^-$, $\mathfrak{S}^+$, respectively. Similarly, we write the initial position vector of $\mathfrak{S}$ as

$$\boldsymbol{R} = \boldsymbol{R}(\theta^\alpha) = \boldsymbol{P}(\theta^\alpha, 0) = \boldsymbol{r}(\theta^\alpha, 0) \tag{4.24}$$

and designate the initial values of $\boldsymbol{a}_i$, $a_{\alpha\beta}$, $b_{\alpha\beta}$, a by $\boldsymbol{A}_i$, $A_{\alpha\beta}$, $B_{\alpha\beta}$, A, respectively, and we note that formulae of the type (4.10) to (4.13) hold also for the surface $\mathfrak{S}$.

We have already defined a general shell-like body. But, in order to fix ideas, it is desirable to describe a shell in somewhat less general and more familiar terms. For this purpose, consider a surface (embedded in a Euclidean 3-space) which

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[7] This restriction is not severe and imposes only a minor loss of generality. The formula (4.17) defines the mass, per unit area of $\mathfrak{s}$, for an arbitrary portion of $\mathscr{B}$ (above and below the surface $\xi = 0$). The relation (4.20), involving essentially the "moment" of $\varrho^* g^{\frac{1}{2}}$, is in effect a condition on the distribution of mass above and below the surface $\xi = 0$. Condition (4.20) was introduced by GREEN, LAWS and NAGHDI [1968, *4*] in their derivation of two-dimensional shell equations from the three-dimensional equations of classical continuum mechanics.

we temporarily designate by $\mathfrak{S}'$. Let the position vector of any point on this surface be denoted by $\boldsymbol{R}'(y^1, y^2)$ with $\{y^1, y^2\}$ being simply parameters as yet unrelated to the coordinates θ^α. With formulae of the type (4.10) and (4.11) in mind, let $\boldsymbol{A}'_\alpha(y^1, y^2)$ denote the base vectors of $\mathfrak{S}'$ and $\boldsymbol{A}'_3(y^1, y^2)$ its unit normal such that $[\boldsymbol{A}'_1 \boldsymbol{A}'_2 \boldsymbol{A}'_3] > 0$. Assume the existence of a neighborhood $\mathfrak{R}(\mathfrak{S}')$ in which points in space lie along one and only one normal to $\mathfrak{S}'$. Let y^3, a parameter measured to the scale of the (rectangular Cartesian coordinates) x_i along the positive direction of the uniquely defined normal $\boldsymbol{A}'_3(y^1, y^2)$ from $\mathfrak{S}'$, denote the distance to any point in $\mathfrak{R}(\mathfrak{S}')$. Then, any point in $\mathfrak{R}(\mathfrak{S}')$ can be located by $\boldsymbol{R}'(y^1, y^2) + y^3 \boldsymbol{A}'_3(y^1, y^2)$. If we now identify $\mathfrak{S}'$ with $\mathfrak{S}$, the parameters $\{y^1, y^2\}$ with θ^α and put for later convenience $y^3 = \zeta$, then

$$y^i = \{\theta^\alpha, \zeta\} \tag{4.25}$$

may be regarded as a *normal* coordinate system in which y^α coincides with the convected θ^α on $\mathfrak{S}$ and y^3 is normal to $\mathfrak{S}$.

Now in a reference configuration of $\mathscr{B}$, which again we take to be the initial configuration, the convected general coordinates θ^i can always be related to y^i with ζ a specified function of θ^α and ξ. For simplicity and to avoid undue complications, we specify ζ in the form[8]

$$\zeta = \zeta'(\theta^\alpha)\, \xi, \tag{4.26}$$

where ζ' is a function of θ^α only. In the special case that $\zeta'(\theta^\alpha) = 1, \zeta = \xi$, the coordinates (4.4) become coincident with (4.25) in the reference configuration. Thus, with ζ specified by (4.26), the position vector of the body $\mathscr{B}$ in the initial configuration referred to the normal coordinate system y^i is

$$\boldsymbol{P} = \boldsymbol{P}(\theta^\alpha, \xi) = \boldsymbol{P}'(\theta^\alpha, \zeta) = \boldsymbol{R}(\theta^\alpha) + \zeta \boldsymbol{A}_3(\theta^\alpha) \tag{4.27}$$

and (4.27) represents the transformation relations between the normal coordinates (4.25) and x_i, the rectangular Cartesian components of $\boldsymbol{P}$. The existence of a neighborhood $\mathfrak{R}(\mathfrak{S})$ in which every point in the initial configuration of $\mathscr{B}$ is uniquely located by (4.27) may be verified in the manner discussed in Sect. A.3 of the Appendix [between (A.3.14)–(A.3.17) in Chap. F]. Indeed, if R_1 and R_2 are the principal radii of curvature of the surface $\mathfrak{S}$ and since $A = \det(A_{\alpha\beta}) \neq 0$, we need only choose

$$\mathfrak{R}(\mathfrak{S}) = \{(\theta^\alpha, \zeta) : |\zeta| < R\}, \qquad R = \min(|R_1|, |R_2|) \neq 0, \tag{4.28}$$

to ensure that a condition of the form (4.3) between the coordinates y^i and x_i is always satisfied in the (initial) reference configuration. With this choice of $\mathfrak{R}(\mathfrak{S})$, (4.27) is nonsingular and hence the duals of the formulae of the type (4.10)–(4.13) remain valid. Let the base vectors and the metric tensor in the reference configuration referred to the coordinates y^i, with ζ given by (4.26), be denoted by $\boldsymbol{G}'_i(\theta^\alpha, \zeta)$ and $G'_{ij}(\theta^\alpha, \zeta)$. These and other results of the type (4.23) can be calculated using (4.27). In particular, on the surface $\zeta = 0$, the metric tensor G'_{ij}, its conjugate G'^{ij} and the determinant G' reduce to

$$\begin{gathered} G'_{\alpha\beta}(\theta^\gamma, 0) = A_{\alpha\beta}, \qquad G'_{\alpha 3}(\theta^\gamma, 0) = 0, \qquad G'_{33}(\theta^\gamma, 0) = 1, \\ G'^{\alpha\beta}(\theta^\gamma, 0) = A^{\alpha\beta}, \qquad G'^{\alpha 3}(\theta^\gamma, 0) = 0, \qquad G'^{33}(\theta^\gamma, 0) = 1, \\ G'(\theta^\alpha, 0) = A, \end{gathered} \tag{4.29}$$

respectively.

[8] In place of (4.26) we can write $\zeta = \bar{\zeta}(\theta^\alpha, \xi)$, $\bar{\zeta}$ being a function of θ^α and ξ; but this generality is not needed in our subsequent developments.

Consider again the part of $\partial\mathscr{B}$ referred to previously by $\mathfrak{S}^-$ and $\mathfrak{S}^+$ in the reference configuration. Recalling (4.27), let these bounding surfaces be specified by[9]

$$\begin{aligned} &\zeta = h_1(\theta^\alpha), \quad \zeta = h_2(\theta^\alpha), \qquad h_1 < 0 < h_2, \\ &h = h(\theta^\alpha) = h_2(\theta^\alpha) - h_1(\theta^\alpha), \quad \text{phys. dim. } h = [L], \end{aligned} \tag{4.30}$$

where the functions h_1 and h_2, as well as h, have the physical dimension of length designated by $[L]$ and $\zeta = 0$ lies entirely between the surfaces $(4.30)_{1,2}$. Then, with reference to the initial configuration of the body and (4.27)–(4.30), a shell may be defined as a region of space $h_1 < \zeta < h_2$, $\max(|h_1|, |h_2|) < R$ bounded by the two surfaces $(4.30)_{2,1}$, i.e., $\mathfrak{S}^+$ and $\mathfrak{S}^-$ (called the upper and lower surfaces or faces) which are situated above and below a surface $\mathfrak{S}$ (specified by $\zeta = 0$) and a lateral surface [or an edge boundary, i.e., the surface corresponding to (4.15) in the reference configuration], the intersection of which with surfaces $\zeta =$ constant are closed smooth curves. By (4.27) and (4.30) the distance between $\mathfrak{S}^-$ and $\mathfrak{S}^+$, measured along $\boldsymbol{A}_3$ is h and is called the (initial) thickness of the shell. If h is constant, the shell is said to be of uniform thickness, otherwise of variable thickness.

It is clear from either of the above descriptions that a shell (or a plate) is a three-dimensional continuum whose boundary surface has special features, as remarked at the beginning of this section. If *full* information is desired regarding the motion and deformation of such bodies in the context of the classical three-dimensional theory of continuum mechanics, then there would be no point to the present article or the extensive and continuing efforts on the foundations of the theories of shells and plates which have spanned the literature during the twentieth century. In this connection, it is worth recalling the well-known fact that as a problem in the three-dimensional theory, the closed shell is often amenable to analytical treatment at least within the scope of linear elastostatics, whereas an exact analysis of the open shell (i.e., one with an edge boundary) would lead to a formidable task.

Suppose, instead, we are content with only partial information (in some sense) for a sufficiently *thin* shell, i.e., when the thickness h is much smaller than the minimum radius of the curvature R defined in (4.28) or equivalently when[10]

$$\frac{h}{R} \ll 1. \tag{4.31}$$

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[9] Instead of $(4.30)_{1,2}$, it is tempting to specify $\mathfrak{S}^-$ and $\mathfrak{S}^+$ by the symmetric conditions

$$\zeta = \mp \frac{h}{2}$$

in which case $\zeta = 0$ will be the middle surface. Because of (4.20), our specification of $\mathfrak{S}^-$ and $\mathfrak{S}^+$ in the forms $(4.30)_{1,2}$ is simply motivated by the fact that we do not wish to exclude an important class of shell-like bodies whose initial mass density ϱ_0^* is independent of ζ. In general, the symmetric specification

$$\zeta = \mp \frac{h}{2}$$

with $\zeta = 0$ as the middle surface is possible only for initially flat plates; but it will not be consistent with the condition corresponding to (4.20) in the (initial) reference configuration of a curved shell, unless the mass density ϱ_0^* is dependent on ζ. Further remarks on this point, including the consideration of (4.20) in the context of an approximate theory for a shell with its initial mass density independent of ζ, is made in Sect. 7 [Subsect. β) between (7.42)–(7.48)].

[10] The criterion (4.31) is generally used to define a *thin* shell and is also invoked in the development of approximate theories of thin shells from the three-dimensional equation. In the case of an initially flat plate (for which R is infinite), in place of (4.31), it is assumed that h is much smaller than the smallest dimension of its middle plane.

By partial information we mean, for example, information concerning quantities which can be regarded as representing the response of the surface $\zeta = 0$ (or its neighborhood) as a consequence of the (three-dimensional) motion of the body $\mathscr{B}$ or the determination of certain averages of quantities resulting from the (three-dimensional) motion of $\mathscr{B}$. Indeed, the desire for such partial or limited information is the basic motivation for the construction of a two-dimensional theory for a thin shell as defined above, with the aim of providing a simpler theory for the partial or limited information sought. A useful two-dimensional theory of this type is necessarily approximate and is referred to as shell theory, in order to distinguish it from the three-dimensional theory. In fact, the main problem of the general theory of thin shells (and plates) may be stated as follows:

(a) The development of a two-dimensional theory—an approximate theory relative to the three-dimensional theory—so constructed as to be capable of supplying the partial information mentioned above.

(b) The development of a scheme or a systematic procedure for estimating the "error" involved in the use of the (approximate) two-dimensional theory in comparison with that of the full three-dimensional theory. Alternatively, we may ask under what circumstances do the equations of shell theory supply an approximate solution to the three-dimensional equations and how "close" is this approximate solution to the exact solution? (4.32)

With reference to (b) in (4.32), of course, an idea of the range of applicability of shell theory often can be had *a priori* through intuitive reasoning and for certain specific problems (especially in the linear theory) it is possible to support such intuitive reasoning by analysis or similar considerations.[11] In fact, historically speaking, such *a priori* considerations have been partially relied upon by various investigators in developing specialized or more general theories of shells and plates. However, an explicit answer to (b) is not available at present and is not attempted here.[12] Rather, the main developments of this article are concerned with (a) and associated topics. Even here there are considerable difficulties when the complete theory (i.e., all field equations and constitutive equations) is deduced from the full three-dimensional equations. The main difficulties stem chiefly from the fact that at some stage in the development of shell theory, from the three-dimensional equations, approximations must be introduced and that the nature of validity of such approximations probably cannot be entirely divorced from the question posed under (b) in (4.32). Again, with reference to (a) in (4.32) and a general development of the linear theory of thin elastic shells, we may note that there remained unsettled questions for almost three-quarters of a century be-

[11] In addition to intuitive notions, such a priori knowledge may be based on available exact or asymptotic solutions of specific problems obtained from special linear theories of shells such as plate theory, membrane theory or other special cases of the general theory. Also, for certain simple problems (such as pure bending of a rectangular plate, torsion of a rectangular plate and torsion of a circular cylindrical shell) a solution via a general linear theory of shells may be expected to agree (exactly or very nearly) with the corresponding known exact solutions from the three-dimensional theory.

[12] The problem posed under (b) in (4.32) is a formidable one and has not been solved with finality even in the case of the linear theory of bending of elastic plates. Recently, however, some effort in this direction has been made by JOHN [1965, *5*] and [1969, *4*] in connection with the v. Kármán theory of plates.

ginning with the pioneering work of LOVE.[13] From a practical point of view, there seems to be a reasonable agreement among a number of recent and different derivations of an approximate linear bending theory of shells starting from the three-dimensional equations; but still these derivations employ a number of approximations or special assumptions (sometimes in an *ad hoc* manner) which, when taken collectively, are not particularly appealing and often leave something to be desired. Again, the nature of the difficulties here is not unrelated to (b) in (4.32).

With reference to (a) in (4.32), two aspects of the above remarks or observations are worth recapitulating: Firstly, shell theory (being an approximate two-dimensional theory) can neither be expected to, nor can it be capable of, predicting full and exact information (in the sense of three-dimensional theory), except possibly in very special circumstances; and secondly the derivation of a theory for thin shells from the three-dimensional equations involve considerable difficulties which are largely mathematical and have to do with the nature of approximations and special assumptions introduced.[14] It then seems natural to ask if it is possible to replace the continuum characterizing the body $\mathscr{B}$ with another continuum, a *model* which would reflect the main features of a thin shell and which would then permit the development of an exact theory without recourse to the approximations or special assumptions mentioned above? To this end and preliminary to the description of the alternative model, we recall that a motion of a body in classical (three-dimensional) continuum mechanics is said to be *rigid* if and only if the rectangular Cartesian components of the position of every material point θ^i at time t are related to the rectangular Cartesian components $x^i(\theta^k)$ in a reference position by a relation of the form[15]

$$x^{+i}(\theta^k, t) = C^i(t) + Q^i_j(t)\, x^j(\theta^k). \tag{4.33}$$

In (4.33), C^i is some vector-valued function of time and Q^i_j are the components of a proper orthogonal tensor (or matrix) function of time which may be interpreted as the rotation tensor. Consider now a body regarded as consisting not only of material particles identified with material points θ^i, but also of directions associated with the material points. For such a model of *oriented media*, the directions are characterized by deformable vector fields—called the *directors*—which are capable of rotation and stretches independently of the deformation of materials points.[16] In a three-dimensional theory of an oriented medium with a

[13] [1888, *1*].

[14] One of the main obstacles in the development of a general theory of thin shells (from the three-dimensional equations), even in the case of linear theory, lies in the difficulty of rendering the notion of "thinness" precise. What is generally invoked is the criterion (4.31), often supplemented by other special assumptions.

[15] Although subscripts and superscripts are employed in writing (4.33), we recall that no distinction between the character of these indices is necessary in a rectangular Cartesian coordinate system.

[16] Historically the concept of "directed" or "oriented" media was originated by DUHEM [1893, *1*] and a first systematic development of theories of oriented media in one, two and three dimensions (the first two being motivated by rods and shells) was carried out by E. and F. COSSERAT [1909, *1*]. In their work (see also [1968, *3*]—translation of the original [1909, *1*]), the COSSERATS represented the orientation of each point of their continuum by a set of mutually perpendicular rigid vectors. A general development of the kinematics of oriented media, in the presence of n stretchable directors (in n-dimensional space), has been given more recently by ERICKSEN and TRUESDELL [1958, *1*] who also introduced the terminology of *directors*. An exposition of the kinematics of the theory of oriented bodies, together with references to other contributions on the subject prior to 1960, may be found in TRUESDELL and TOUPIN [1960, *14*].

single deformable director[17] $\boldsymbol{d}$ at every material point of the body, the above definition of a rigid motion displayed through (4.33), is supplemented by

$$\bar{d}^{+i}(\theta^k, t) = Q^i_j(t)\, \bar{d}^j(\theta^k), \tag{4.34}$$

where $\bar{d}^{+i}$ and $\bar{d}^i$ are the rectangular Cartesian components of the director field $\boldsymbol{d}$ at x^{+i} and x^i, respectively. The condition (4.34) relates the components $\bar{d}^{+i}$ at every material point θ^i at time t to the reference values $\bar{d}^i$ under rigid motions. It can be readily verified from (4.34) that the director has the property that it remains unaltered in magnitude under rigid motions of the continuum.

Returning to our objective of an alternative model for a thin shell, consider a body $\mathscr{C}$ consisting of a surface embedded in a Euclidean 3-space together with a single deformable director assigned to every point of the surface. Let the particles on the surface of $\mathscr{C}$ be identified by the convected coordinates θ^α $(\alpha = 1, 2)$ and let $\mathscr{S}$ and $\mathfrak{s}$ refer to the surface of $\mathscr{C}$ in the reference and deformed configurations, respectively. Let[18] $\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha, t)$ denote the position vector of $\mathfrak{s}$, relative to a fixed origin (say relative to the same fixed origin used previously), which specifies the place occupied by the material point θ^α in the deformed configuration at time t. Let $\boldsymbol{a}_3 = \boldsymbol{a}_3(\theta^\alpha, t)$ be the unit normal to $\mathfrak{s}$ and let $\boldsymbol{d} = \boldsymbol{d}(\theta^\alpha, t)$ stand for the deformable director assigned to every point of $\mathfrak{s}$. The three-dimensional vector field $\boldsymbol{d}$ at $\boldsymbol{r}$ which we specify to be dimensionless[19] is not necessarily along the normal to $\mathfrak{s}$ and, as already noted, has the property that it remains invariant in magnitude under rigid motions of the continuum. Henceforth (except when noted otherwise) we identify the reference surface $\mathscr{S}$ (i.e., the surface which becomes the surface $\mathfrak{s}$ at time t) with the initial surface and denote the initial values, at time $t = 0$, of the position vector, the unit normal to $\mathscr{S}$ and the initial director as $\boldsymbol{R}$, $\boldsymbol{A}_3$ and $\boldsymbol{D}$, respectively. Also, we denote the base vectors of $\mathfrak{s}$ and $\mathscr{S}$ by $\boldsymbol{a}_\alpha$ and $\boldsymbol{A}_\alpha$, respectively, and note that formulae of the type $(4.10)_1$ and (4.11)–(4.13) and their duals hold also for the surface of $\mathscr{C}$ in the deformed and reference configurations.

The continuum just described is called a *Cosserat surface*.[20] It may be emphasized that this continuum is not just a two-dimensional surface alone; it consists of a surface with a director assigned to every point of the surface. The

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[17] Fairly general nonlinear theories of this type, developed in different contexts, have been given by ERICKSEN [1961, *1*] and GREEN, NAGHDI and RIVLIN [1965, *3*]. Also, TOUPIN [1964, *8*] has discussed, among other developments, a mechanical theory of couple-stress with directors for elastic materials. References to related works on the subject may be found in the above papers and in TRUESDELL and NOLL [1965, *9*].

[18] Although the symbols for the position vector and the unit normal of the surface $\mathfrak{s}$ are the same as those used for the surface $\mathfrak{S}$, this need not give rise to confusion. If desired, the two surfaces may be identified; but we postpone such identifications until later.

[19] Our specification of $\boldsymbol{d}$ as a dimensionless vector field is for later convenience and in anticipation of later interpretations. For other directed media it may be more convenient to regard $\boldsymbol{d}$ as having the dimension of length.

[20] Such a surface is also referred to as a directed or an oriented surface in the recent literature. The idea of representing a (three-dimensional) shell by such a model was initially conceived by E. and F. COSSERAT [1909, *1*]; see also [1968, *3*]. In their work, however, the COSSERATS had considered a surface with a triad of rigid directors assigned to every point of the surface. A development of the kinematics of such directed or oriented surfaces with n deformable directors is contained in the paper of ERICKSEN and TRUESDELL [1958, *1*]. Here, as in the paper of GREEN, NAGHDI and WAINWRIGHT [1965, *4*], we use a single deformable director. A triad of deformable directors assigned to every point of the surface, based on the kinematics of ERICKSEN and TRUESDELL, has been employed by COHEN and DESILVA [1966, *2*] in their study of directed surfaces. With reference to shells and plates, however, at present it appears that the use of a single director should be sufficient to model a shell, as has been remarked also by TOUPIN [1964, *8*].

assigned director is intended to portray the "thickening" about the surface $\xi=0$ (or $\zeta=0$) of the three-dimensional shell and its component along the unit normal to the surface can be regarded as representing the thickness of the three-dimensional shell. More specifically, in the initial configuration of a Cosserat surface, we may specify the initial director $\boldsymbol{D}$ (which is dimensionless) to be directed along $\boldsymbol{A}_3$, i.e., $\boldsymbol{D}=D\boldsymbol{A}_3$; and we may then regard the magnitude of $\boldsymbol{D}$, namely D, as representing the initial thickness of the (three-dimensional) shell:

$$D \propto h \quad \text{or} \quad D=\frac{h}{h_R}, \tag{4.35}$$

where h_R designates a reference value of the thickness h and may be specified by the maximum value of h. For shells and plates of constant thickness, since $h_R=h$, $D=1$.

To complete the specification of the continuum $\mathscr{C}$, we need to specify its mass. Let $\mathscr{P}_{\mathscr{C}}$ refer to an arbitrary region of the material surface of $\mathscr{C}$ which is mapped into a part $\mathscr{P}_0$ of $\mathscr{S}$ in the reference configuration and into a corresponding part $\mathscr{P}$ of $\mathfrak{s}$ in the deformed configuration at time t. Then, the mass $m(\mathscr{P}_{\mathscr{C}})$ for each part $\mathscr{P}_{\mathscr{C}}$ of the Cosserat surface can be defined by a non-negative scalar measure[21]

$$m(\mathscr{P}_{\mathscr{C}})=\int_{\mathscr{P}} \varrho\, d\sigma=\int_{\mathscr{P}_0} \varrho_0\, d\Sigma. \tag{4.36}$$

In (4.36), $\varrho=\varrho(\theta^\alpha, t)$ is the mass density at time t having the physical dimension

$$\text{phys. dim. } \varrho=[M\,L^{-2}], \tag{4.37}$$

with the symbol $[M]$ standing for the physical dimension of mass, and

$$\varrho_0=\varrho_0(\theta^\alpha)=\varrho(\theta^\alpha, 0) \tag{4.38}$$

is the initial mass density. Also, the element of area $d\sigma$ of $\mathfrak{s}$ is given by a formula in the form $(4.19)_2$ and

$$d\Sigma=(\boldsymbol{A}_1\times\boldsymbol{A}_2\cdot\boldsymbol{A}_3)\,d\theta^1\,d\theta^2=A^{\frac{1}{2}}\,d\theta^1\,d\theta^2, \tag{4.39}$$

the dual of $(4.19)_2$ is an element of area of $\mathscr{S}$. It is easily verified from $(4.19)_2$ and (4.39) that the area elements $d\sigma$ and $d\Sigma$ are related by the formula

$$d\sigma=J\,d\Sigma, \tag{4.40}$$

where for later reference we have introduced the notation

$$J=\left(\frac{a}{A}\right)^{\frac{1}{2}}. \tag{4.41}$$

The formula (4.40) relates the elements of area in the deformed and undeformed configurations of the surface. By (4.40), an immediate consequence of (4.36) is the relation

$$\varrho=J^{-1}\varrho_0 \quad \text{or} \quad \varrho_0=J\varrho, \tag{4.42}$$

[21] For background information regarding the concept of mass in continuum mechanics, see Truesdell and Toupin [1960, *14*]. Note also that the physical dimension of ϱ given by (4.37) differs from the physical dimension of the mass density ϱ^* (of the three-dimensional body), but agrees with that defined by (4.17).

as one form of the equation of continuity.[22] Other forms of the continuity equation for the Cosserat surface will be given in Chap. C.

While the mass $m(\mathscr{P}_{\mathscr{C}})$ is a part of the specification of the continuum—here the Cosserat surface $\mathscr{C}$—the mass density depends on the particular configuration which $\mathscr{C}$ will occupy and its value in different configurations is determined by the motion of the continuum. In subsequent developments we need to introduce certain physical entities for a part $\mathscr{P}_{\mathscr{C}}$ of the Cosserat surface which are defined by means of integrals whose range of integration is over $\mathscr{P}$ in the present configuration. To be specific, let $f(\theta^\alpha, t)$ be any scalar-valued or vector-valued function denoting a physical quantity per unit mass ϱ and consider an integral of the type

$$\int_{\mathscr{P}} \varrho f \, d\sigma. \tag{4.43}$$

The above integral is associated with the motion of the continuum and represents a physical entity (say F) defined for a part of the Cosserat surface which occupies the region $\mathscr{P}$ in the present configuration. Strictly speaking an integral of the type (4.43) should be designated as $F(\mathscr{P}_{\mathscr{C}})$; however, in order to avoid cumbersome notations, we shall write $F(\mathscr{P})$ in place of $F(\mathscr{P}_{\mathscr{C}})$ when defining expressions of the type

$$F(\mathscr{P}) = \int_{\mathscr{P}} \varrho f \, d\sigma \tag{4.44}$$

for each part $\mathscr{P}$ of the Cosserat surface in the present configuration.[23] No confusion should arise from the above abbreviated notation, since it will be clear from the particular context [e.g., the presence of $d\sigma$ on the right-hand side of (4.44)] which region of Euclidean space $\mathscr{C}$ occupies. Whenever we need to emphasize in the same equation the distinction between the region of integration over a part of $\mathscr{C}$ in the present configuration and the corresponding part in the reference configuration, we employ the designation $\mathscr{P}_0$ for the latter.

As will become apparent later, a general and exact theory of a Cosserat surface can be constructed systematically and in the same spirit as currently enjoyed by the classical theory of continuum mechanics. This approach, which we call a *direct* approach, will not have the difficulties involving the approximations when shell theory is developed from the three-dimensional equations of the classical theory.[24] On the other hand, the direct approach (via a Cosserat surface) requires additional and sometimes difficult considerations, namely the interpretations of the results of the complete theory (at least in special cases), the identification of the constitutive coefficients and the demonstration of the relevance and the applicability of the theory of a Cosserat surface to shells and plates (regarded as three-dimensional bodies). Either of the two approaches, when carried to completion, has difficulties of its own. Nevertheless, much can be gained by studying both, with the ultimate goal of showing the relationship between them. With this point of view, in this and the next two chapters the subject is discussed both from a direct approach and also from the three-dimensional theory. In each

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[22] This is a two-dimensional analogue of (4.16) which expresses conservation of mass in the three-dimensional theory.

[23] Our abbreviated notation here is similar (but not identical) to that in Sect. 15 of TRUESDELL and NOLL [1965, *9*].

[24] It will become evident by the end of Chap. D that considerable difficulties are associated with the derivation of the constitutive equations from the three-dimensional theory, even in the case of the linear theory of thin elastic shells; and it is partly for this reason that the direct approach is emphasized in Chap. E. However, these remarks and those made earlier at the end of Sect. 2 should not be construed as minimizing or discouraging the efforts directed toward the solution of the problem posed under (b) in (4.32).

chapter the two approaches are considered separately so that no confusion should arise when sometimes the same symbols are used for a Cosserat surface and in the theory developed from the three-dimensional equations.

5. Kinematics of shells: I. Direct approach. We develop in this section the basic kinematical results for a Cosserat surface already defined in Sect. 4. For ease of reference, however, we repeat here once more that a Cosserat surface is a body $\mathscr{C}$ comprising a surface (embedded in a Euclidean 3-space) and a single deformable director attached to every point of the surface. Moreover, the directors which are not necessarily along the unit normals to the surface have, in particular, the property that they remain unaltered in magnitude under superposed rigid body motions.

α) *General kinematical results*. Let the particles (or the material points) of the material surface of $\mathscr{C}$ be identified with the convected coordinates θ^α. In a reference configuration of the Cosserat surface $\mathscr{C}$ which we take to be the initial configuration, let the reference surface be referred to by $\mathscr{S}$, let $\boldsymbol{R}$ be the position vector of $\mathscr{S}$ and $\boldsymbol{D}$ the reference value of the director. Further, let the surface occupied by the material surface of $\mathscr{C}$ in the present configuration at time t be referred to by $\mathscr{s}$, let $\boldsymbol{r}$ be the position vector of $\mathscr{s}$ and $\boldsymbol{d}$ the director at $\boldsymbol{r}$. Then, the motion of a Cosserat surface is defined by

$$\boldsymbol{r}=\boldsymbol{r}(\theta^\alpha, t), \quad \boldsymbol{d}=\boldsymbol{d}(\theta^\alpha, t), \quad [\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{d}]>0, \tag{5.1}$$

where the condition $(5.1)_3$ ensures that $\boldsymbol{d}$ is not tangent to the surface $\mathscr{s}$. Alternatively, instead of $(5.1)_3$, it will suffice to assume that $[\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{d}]\neq 0$ with the understanding that in any given motion $[\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{d}]$ is either >0 or <0. In the reference configuration, the initial position vector $\boldsymbol{R}$ of the surface $\mathscr{S}$ and the initial director $\boldsymbol{D}$ at $\boldsymbol{R}$ are:

$$\boldsymbol{R}=\boldsymbol{R}(\theta^\alpha)=\boldsymbol{r}(\theta^\alpha, 0), \quad \boldsymbol{D}=\boldsymbol{D}(\theta^\alpha)=\boldsymbol{d}(\theta^\alpha, 0). \tag{5.2}$$

At the risk of being repetitious, we observe that $(5.2)_1$ specifies the place occupied by the material point θ^α of the surface of $\mathscr{C}$ in the reference configuration while the place occupied by the material point of the surface of $\mathscr{C}$ in the deformed configuration at time t is given by $(5.1)_1$. We also note that the base vectors $\boldsymbol{a}_\alpha$, the unit normal $\boldsymbol{a}_3$ and the first and the second fundamental forms of $\mathscr{s}$, as well as their duals associated with the reference surface $\mathscr{S}$, satisfy the relations of the forms (4.10)–(4.13).

The vector functions $\boldsymbol{r}$ and $\boldsymbol{d}$, jointly represent a 1-parameter family of configurations which describe the motion of the Cosserat surface. We assume that these vector functions are sufficiently smooth in the sense that they are differentiable with respect to θ^α and t as many times as required. Let $\boldsymbol{v}$ and $\boldsymbol{w}$, each a three-dimensional vector field, denote the velocity of a point of $\mathscr{s}$ and the director velocity at time t. Then

$$\boldsymbol{v}=\frac{d}{dt}\boldsymbol{r}(\theta^\alpha, t)=\dot{\boldsymbol{r}}(\theta^\alpha, t), \quad w=\frac{d}{dt}\boldsymbol{d}(\theta^\alpha, t)=\dot{\boldsymbol{d}}(\theta^\alpha, t), \tag{5.3}$$

where a superposed dot stands for the material time derivative with respect to t, holding θ^α fixed.

In what follows, we frequently encounter the partial derivatives of three-dimensional vector fields or functions with respect to the surface coordinates. In this connection, let $\boldsymbol{V}$ be a three-dimensional vector field defined on $\mathscr{s}$ and let V^i be the components of $\boldsymbol{V}$ referred to the base vectors $\boldsymbol{a}_i=\{\boldsymbol{a}_\alpha, \boldsymbol{a}_3\}$. Then, $\boldsymbol{V}$ can be expressed as

$$\boldsymbol{V}=V^i\,\boldsymbol{a}_i=V^\alpha\,\boldsymbol{a}_\alpha+V^3\,\boldsymbol{a}_3=V_i\,\boldsymbol{a}^i=V_\alpha\,\boldsymbol{a}^\alpha+V_3\,\boldsymbol{a}^3, \tag{5.4}$$

and we recall that the gradient of $\boldsymbol{V}$ and its components are[25]

$$\begin{aligned} &\boldsymbol{V}_{,\alpha} = \boldsymbol{V}_{|\alpha} = V_{i\alpha}\,\boldsymbol{a}^i = V^i_{.\alpha}\,\boldsymbol{a}_i, \\ &V_{i\alpha} = \boldsymbol{a}_i \cdot \boldsymbol{V}_{,\alpha}, \qquad V^i_{.\alpha} = \boldsymbol{a}^i \cdot \boldsymbol{V}_{,\alpha}, \\ &V_{\lambda\alpha} = V_{\lambda|\alpha} - b_{\alpha\lambda} V_3, \quad V_{3\alpha} = V_{3,\alpha} + b^\lambda_\alpha V_\lambda, \\ &V^\lambda_{.\alpha} = V^\lambda{}_{|\alpha} - b^\lambda_\alpha V_3, \quad V^3_{.\alpha} = V^3{}_{,\alpha} + b_{\lambda\alpha} V^\lambda, \end{aligned} \tag{5.5}$$

where a vertical bar stands for covariant differentiation with respect to $a_{\alpha\beta}$ and $V^\alpha \boldsymbol{a}_\alpha$ and $V_\alpha \boldsymbol{a}^\alpha$ are surface vectors with contravariant and covariant components V^α and V_α, respectively. We also note here that since $\boldsymbol{a}_3$ is a unit normal to $\mathscr{s}$ and satisfies $(4.11)_1$, the lowering and raising of superscripts and subscripts of space tensor functions such as V^i in (5.4) and $V_{i\alpha}$ in (5.5) can be accomplished by using a space metric tensor g_{ij} defined by

$$g_{\alpha\beta} = a_{\alpha\beta}, \quad g_{\alpha 3} = 0, \quad g_{33} = 1. \tag{5.6}$$

Consider now the velocity vector $\boldsymbol{v}$ which can be written in the form

$$\boldsymbol{v} = v^i \boldsymbol{a}_i = v^\alpha \boldsymbol{a}_\alpha + v^3 \boldsymbol{a}_3 = v_i \boldsymbol{a}^i. \tag{5.7}$$

Since the coordinate curves on $\mathscr{s}$ are convected, it follows that

$$\dot{\boldsymbol{a}}_\alpha = \boldsymbol{v}_{,\alpha} = \boldsymbol{v}_{|\alpha}, \tag{5.8}$$

and by (5.5) we have

$$\begin{aligned} &\boldsymbol{v}_{|\alpha} = v_{i\alpha}\,\boldsymbol{a}^i, \quad v_{i\alpha} = \boldsymbol{a}_i \cdot \boldsymbol{v}_{,\alpha}, \\ &v_{\lambda\alpha} = \boldsymbol{a}_\lambda \cdot \boldsymbol{v}_{,\alpha} = v_{\lambda|\alpha} - b_{\alpha\lambda} v_3, \\ &v_{3\alpha} = \boldsymbol{a}_3 \cdot \boldsymbol{v}_{,\alpha} = v_{3,\alpha} + b^\lambda_\alpha v_\lambda. \end{aligned} \tag{5.9}$$

From $(4.11)_{1,2}$ and (5.9), it can be easily shown that

$$\dot{\boldsymbol{a}}_3 = -(\boldsymbol{a}_3 \cdot \boldsymbol{v}_{,\alpha})\,\boldsymbol{a}^\alpha = -v_{3\alpha}\,\boldsymbol{a}^\alpha = -(v_{3,\alpha} + b^\lambda_\alpha v_\lambda)\,\boldsymbol{a}^\alpha. \tag{5.10}$$

Since each of the vectors $\dot{\boldsymbol{a}}_i$ $(i = 1, 2, 3)$, may be expressed as a linear combination of $\boldsymbol{a}^i$, we may write

$$\dot{\boldsymbol{a}}_i = c_{ki}\,\boldsymbol{a}^k, \quad c_{ki} = \boldsymbol{a}_k \cdot \dot{\boldsymbol{a}}_i. \tag{5.11}$$

Let $T_{(ik)}$ and $T_{[ik]}$ stand, respectively, for the symmetric and the skew-symmetric parts of a second order tensor T_{ik}, i.e.,

$$T_{ik} = T_{(ik)} + T_{[ik]}, \quad T_{(ik)} = \tfrac{1}{2}(T_{ik} + T_{ki}), \quad T_{[ik]} = \tfrac{1}{2}(T_{ik} - T_{ki}). \tag{5.12}$$

Then, with the notation

$$\eta_{ki} = c_{(ki)}, \quad W_{ki} = c_{[ki]}, \tag{5.13}$$

it follows from (5.11) that

$$\begin{aligned} &c_{ki} = \eta_{ki} + W_{ki}, \\ &2\eta_{ki} = \boldsymbol{a}_k \cdot \dot{\boldsymbol{a}}_i + \boldsymbol{a}_i \cdot \dot{\boldsymbol{a}}_k = \dot{\overline{\boldsymbol{a}_k \cdot \boldsymbol{a}_i}} = \dot{a}_{ki} = 2\eta_{ik}, \\ &2W_{ki} = \boldsymbol{a}_k \cdot \dot{\boldsymbol{a}}_i - \boldsymbol{a}_i \cdot \dot{\boldsymbol{a}}_k = -2W_{ik}. \end{aligned} \tag{5.14}$$

[25] See Eqs. (A.2.54) in Chap. F.

From $(5.14)_{2,3}$, together with (5.5) and $(4.11)_{1,2}$, we obtain

$$\begin{aligned} 2\eta_{\alpha\beta} &= v_{\alpha|\beta} + v_{\beta|\alpha} - 2b_{\alpha\beta}\, v_3, \\ \eta_{3\alpha} &= \eta_{\alpha 3} = 0, \qquad \eta_{33} = 0, \end{aligned} \tag{5.15}$$

$$\begin{aligned} 2W_{\alpha\beta} &= -2W_{\beta\alpha} = v_{\alpha|\beta} - v_{\beta|\alpha}, \\ W_{\alpha 3} &= -W_{3\alpha} = -\boldsymbol{a}_3 \cdot \boldsymbol{v}_{,\alpha} = -(v_{3,\alpha} + b_\alpha^\beta\, v_\beta), \\ W_{33} &= 0. \end{aligned} \tag{5.16}$$

In view of (5.15) and (5.16), the components of the velocity gradient $\boldsymbol{v}_{,\alpha}$ are

$$v_{\lambda\alpha} = \eta_{\lambda\alpha} + W_{\lambda\alpha}, \qquad v_{3\alpha} = W_{3\alpha} \tag{5.17}$$

and we may express $\dot{\boldsymbol{a}}_i$ in the form

$$\dot{\boldsymbol{a}}_i = (\eta_{k\,i} + W_{k\,i})\, \boldsymbol{a}^k. \tag{5.18}$$

It is clear from (5.15)–(5.18) that $\eta_{\alpha\beta}$ and $W_{\alpha\beta}$ (a subtensor of $W_{k\,i}$) are *surface tensors* whereas $W_{k\,i}$ is a *space tensor*. The functions $\eta_{\alpha\beta}$ and $W_{\alpha\beta}$ may be called the *surface rate of deformation* tensor and the *surface spin* tensor, respectively. For later reference, we introduce the notation[26]

$$\boldsymbol{\eta}_\alpha = \eta_{k\alpha}\, \boldsymbol{a}^k \tag{5.19}$$

and also record here the time rate of change of the determinant of $a_{\alpha\beta}$. Thus, from (4.12) and $(5.14)_2$,

$$\begin{aligned} \dot{a} &= \overline{\det(a_{\alpha\beta})}^{\,\boldsymbol{\cdot}} = \frac{\partial}{\partial a_{\lambda\nu}} [\det(a_{\alpha\beta})]\, \dot{a}_{\lambda\nu} \\ &= a\, a^{\alpha\beta}\, \dot{a}_{\alpha\beta} = 2a\, \eta_\alpha^\alpha \\ &= 2a\, (v^\alpha{}_{|\alpha} - b_\alpha^\alpha\, v_3). \end{aligned} \tag{5.20}$$

By (5.20), the time rate of change of J defined by (4.41) is

$$\dot{J} = \tfrac{1}{2} J\, a^{-1}\, \dot{a} = J\, \eta_\alpha^\alpha. \tag{5.21}$$

Before proceeding further, we need suitable expressions for the time rate of change of the reciprocal base vectors. From differentiation of $(4.12)_5$ follows

$$\dot{a}^{\alpha\beta} = -a^{\alpha\lambda}\, a^{\beta\nu}\, \dot{a}_{\lambda\nu}. \tag{5.22}$$

Recalling $(4.12)_3$ and using (5.14), (5.18) and (5.11), we obtain

$$\begin{aligned} \dot{\boldsymbol{a}}^\alpha &= \overline{a^{\alpha\beta}\, \boldsymbol{a}_\beta}^{\,\boldsymbol{\cdot}} \\ &= a^{\alpha\beta} (\eta_{k\beta} + W_{k\beta})\, \boldsymbol{a}^k - 2a^{\alpha\beta}\, a^{\lambda\nu}\, \eta_{\beta\nu}\, \boldsymbol{a}_\lambda \\ &= a^{\alpha\beta} (W_{k\beta} - \eta_{k\beta})\, \boldsymbol{a}^k \end{aligned} \tag{5.23}$$

and from $(4.11)_3$, (5.10) and (5.16), we have

$$\dot{\boldsymbol{a}}^3 = \dot{\boldsymbol{a}}_3 = W_{\alpha 3}\, \boldsymbol{a}^\alpha = -W_{3\alpha}\, \boldsymbol{a}^\alpha = -v_{3\alpha}\, \boldsymbol{a}^\alpha. \tag{5.24}$$

We now introduce additional kinematical results in terms of the director and its derivatives. Let $\boldsymbol{d}$ be referred to the reciprocal base vectors $\boldsymbol{a}^i$. Then,

$$\boldsymbol{d} = d_i\, \boldsymbol{a}^i = d_\alpha\, \boldsymbol{a}^\alpha + d_3\, \boldsymbol{a}^3, \qquad d_i = \boldsymbol{a}_i \cdot \boldsymbol{d} \tag{5.25}$$

[26] The vector function $\boldsymbol{\eta}_\alpha$ as defined in (5.19) is the negative of the corresponding quantity in the paper of GREEN, NAGHDI and WAINWRIGHT [1965, *4*]. We note that $\boldsymbol{\eta}_\alpha$ is a vector tangent to the surface (as $\eta_{3\alpha} = 0$).

and by $(5.3)_2$, (5.23) and (5.24), the director velocity $\boldsymbol{w}$ can be put in the forms

$$\begin{aligned} \boldsymbol{w} &= w_k\,\boldsymbol{a}^k \\ &= \boldsymbol{\Gamma} + d_i\,\dot{\boldsymbol{a}}^i = \boldsymbol{\Gamma} + d^i\,W_{ki}\,\boldsymbol{a}^k - d^\alpha\,\boldsymbol{\eta}_\alpha \\ &= \boldsymbol{\Gamma} + d^\alpha(\boldsymbol{v}_{,\alpha} - 2\boldsymbol{\eta}_\alpha) + d^3\,W_{k3}\,\boldsymbol{a}^k \\ &= [\dot{d}_k + d^i(W_{ki} - \eta_{ki})]\,\boldsymbol{a}^k, \end{aligned} \tag{5.26}$$

where

$$\boldsymbol{\Gamma} = \dot{d}_i\,\boldsymbol{a}^i \tag{5.27}$$

and $\boldsymbol{\eta}_\alpha$ is defined by (5.19). The gradient of the director $\boldsymbol{d}$, with the help of (5.5), can be written as

$$\begin{aligned} &\boldsymbol{d}_{,\alpha} = \lambda_{i\alpha}\,\boldsymbol{a}^i = \lambda^i_{.\alpha}\,\boldsymbol{a}_i, && \lambda_{i\alpha} = \boldsymbol{a}_i\cdot\boldsymbol{d}_{,\alpha}, \\ &\lambda_{\beta\alpha} = d_{\beta|\alpha} - b_{\alpha\beta}\,d_3, && \lambda_{3\alpha} = d_{3,\alpha} + b^\beta_\alpha\,d_\beta, \\ &\lambda^\beta_{.\alpha} = a^{\beta\gamma}\,\lambda_{\gamma\alpha}, && \lambda^3_{.\alpha} = \lambda_{3\alpha}. \end{aligned} \tag{5.28}$$

Also, from (5.26), the gradient of the director velocity is

$$\begin{aligned} \boldsymbol{w}_{,\alpha} &= \boldsymbol{\Gamma}_{:\alpha} + \lambda^i_{.\alpha}\,W_{ki}\,\boldsymbol{a}^k - \lambda^\beta_{.\alpha}\,\boldsymbol{\eta}_\beta \\ &= [\dot{\lambda}_{k\alpha} + \lambda^\beta_{.\alpha}(W_{k\beta} - \eta_{k\beta}) + \lambda^3_{.\alpha}\,W_{k3}]\,\boldsymbol{a}^k, \end{aligned} \tag{5.29}$$

where

$$\dot{\lambda}_{i\alpha} = \boldsymbol{a}_i\cdot\boldsymbol{\Gamma}_{:\alpha}. \tag{5.30}$$

The kinematic quantities introduced above involve mainly $a_{\alpha\beta}$, $\lambda_{i\alpha}$, d_i and their rates.[27] Often, it is convenient to employ the alternative kinematic measures

$$e_{\alpha\beta} = \tfrac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}), \tag{5.31}$$

$$\varkappa_{\gamma\alpha} = \lambda_{\gamma\alpha} - \Lambda_{\gamma\alpha}, \qquad \varkappa_{3\alpha} = \lambda_{3\alpha} - \Lambda_{3\alpha}, \tag{5.32}$$

$$\gamma_\alpha = d_\alpha - D_\alpha, \qquad \gamma_3 = d_3 - D_3, \tag{5.33}$$

where

$$\Lambda_{i\alpha} = \boldsymbol{A}_i\cdot\boldsymbol{D}_{,\alpha}, \qquad D_i = \boldsymbol{A}_i\cdot\boldsymbol{D} \tag{5.34}$$

are the initial values of $\lambda_{i\alpha}$ and d_i. We note that

$$\dot{e}_{\alpha\beta} = \tfrac{1}{2}\,\dot{a}_{\alpha\beta} = \eta_{\alpha\beta}, \qquad \dot{\varkappa}_{i\alpha} = \dot{\lambda}_{i\alpha}, \qquad \dot{\gamma}_i = \dot{d}_i. \tag{5.35}$$

β) *Superposed rigid body motions.* For later considerations, we need to determine whether or not the above kinematical quantities remain invariant under superposed rigid body motions. For this purpose, we consider a motion of the Cosserat surface which differs from the previous motion, defined by (5.1), only by superposed rigid body motions of the whole continuum at different times. Suppose that under such superposed rigid body motions (since the surface $\mathfrak{s}$ now assumes a new orientation in space) the position $\boldsymbol{r}$ and the director $\boldsymbol{d}$ at $\boldsymbol{r}$ are displaced to the position $\boldsymbol{r}^+$ and the director $\boldsymbol{d}^+$ at $\boldsymbol{r}^+$. Then

$$\begin{aligned} \boldsymbol{r}^+ &= \boldsymbol{r}^+(\theta^\alpha, t') = \boldsymbol{r}_0^+(t') + Q(t)\,[\boldsymbol{r}(\theta^\alpha, t) - \boldsymbol{r}_0(t)], \\ \boldsymbol{d}^+ &= \boldsymbol{d}^+(\theta^\alpha, t') = Q(t)\,\boldsymbol{d}(\theta^\alpha, t), \end{aligned} \tag{5.36}$$

[27] These kinematical results were given by GREEN, NAGHDI and WAINWRIGHT [1965, *4*]. Apart from differences in notation, these results can be brought into correspondence with the kinematical results of COHEN and DESILVA [1966, *2*] when their theory is properly specialized to a single director. As already mentioned in Sect. 4, COHEN and DESILVA employ a triad of deformable directors and their analysis is based on ERICKSEN and TRUESDELL's general kinematics of oriented media in n-dimensional space [1958, *1*].

where $\boldsymbol{r}_0^+$ and $\boldsymbol{r}_0$ are vector-valued functions of t' and t, respectively, $t'=t+a'$, a' being an arbitrary constant, and Q is a proper orthogonal tensor-valued function of t. The tensor Q, a second order space tensor, satisfies the conditions

$$QQ^T = Q^T Q = I, \qquad \det(Q) = 1, \tag{5.37}$$

where I stands for the unit tensor and Q^T denotes the transpose of Q. In what follows, we designate the quantities associated with the motion (5.36) by the same symbols to which we also attach a plus sign $(+)$. Thus, let the base vectors of the surface associated with $(5.36)_1$ be denoted by $\boldsymbol{a}_i^+$. Then, from (4.10), (4.11) and (5.36), we have

$$\boldsymbol{a}_\alpha^+ = Q\,\boldsymbol{a}_\alpha, \qquad \boldsymbol{a}_3^+ = Q\,\boldsymbol{a}_3, \qquad \boldsymbol{d}_{,\alpha}^+ = Q\,\boldsymbol{d}_{,\alpha}. \tag{5.38}$$

Let Ω be a second order space tensor-valued function of time defined by

$$\Omega = \Omega(t) = \dot{Q}(t)\,Q(t)^T. \tag{5.39}$$

Then, by $(5.37)_1$,

$$\dot{Q} = \Omega Q, \qquad \Omega = -\Omega^T, \tag{5.40}$$

so that Ω is a skew-symmetric tensor. Hence, there exists a vector-valued function $\boldsymbol{\omega}$ such that for any vector $\boldsymbol{V}$

$$\Omega \boldsymbol{V} = \boldsymbol{\omega} \times \boldsymbol{V}. \tag{5.41}$$

In particular,

$$\begin{aligned} \boldsymbol{\omega} \times \boldsymbol{a}_\alpha &= -\varepsilon_{k\alpha m}\,\omega^m\,\boldsymbol{a}^k = -\Omega_{k\alpha}\,\boldsymbol{a}^k, \\ \omega^m = \boldsymbol{\omega}\cdot\boldsymbol{a}^m, \quad \Omega_{ki} &= \varepsilon_{kim}\,\omega^m = a^{\frac{1}{2}}\,e_{kim}\,\omega^m = -\Omega_{ik}, \end{aligned} \tag{5.42}$$

where the ε-system is related to the permutation symbols e_{kim}, e^{kim} through

$$\varepsilon_{kim} = a^{\frac{1}{2}}\,e_{kim}, \qquad \varepsilon^{kim} = a^{-\frac{1}{2}}\,e^{kim}. \tag{5.43}$$

Using (5.40), it follows from $(5.36)_1$ that

$$\Omega(\boldsymbol{r}^+ - \boldsymbol{r}_0^+) = \dot{Q}\,(\boldsymbol{r} - \boldsymbol{r}_0). \tag{5.44}$$

The velocity vector $\boldsymbol{v}^+$, obtained from $(5.36)_1$, can then be written in the forms

$$\begin{aligned} \boldsymbol{v}^+ = \dot{\boldsymbol{r}}^+ &= \dot{\boldsymbol{r}}_0^+ + Q(\boldsymbol{v} - \dot{\boldsymbol{r}}_0) + \Omega(\boldsymbol{r}^+ - \boldsymbol{r}_0^+) \\ &= [\dot{\boldsymbol{r}}_0^+ - Q\,\dot{\boldsymbol{r}}_0 - \dot{Q}\boldsymbol{r}_0] + Q\,\boldsymbol{v} + \dot{Q}\boldsymbol{r}, \end{aligned} \tag{5.45}$$

where the quantity in the square bracket on the right-hand side of $(5.45)_2$ is a function of time only and we note that the material time derivative operator $\dot{\overline{(\ )}}$ is unaltered under superposed rigid body motions $(5.36)_1$.

Before proceeding further, it is instructive to consider a special case of (5.36) for which the function Ω (and therefore $\boldsymbol{\omega}$) is constant for all time. To this end we consider a motion of the type $(5.36)_1$ such that for a given time the function Q is specified by a special value. For later convenience we take t to be the given time and specify Q by

$$Q(\tau) = \exp[\Omega_0(\tau - t)], \qquad \Omega_0 = -\Omega_0^T = \text{const}, \tag{5.46}$$

τ being real. Then,

$$\dot{Q}(\tau) = \Omega_0\,Q(\tau) \tag{5.47}$$

and

$$Q(t) = I, \qquad \dot{Q}(t) = \Omega_0. \tag{5.48}$$

But, since (5.40) holds for all time, from comparison of $(5.40)_1$ and (5.47) we have $\Omega(\tau)=\Omega_0$. Hence, for the special motion with Q specified by (5.46), Ω is constant for all time and (5.45) can be reduced to

$$\begin{aligned} \boldsymbol{v}^+(\tau) &= \boldsymbol{b}(\tau) + Q(\tau)\,\boldsymbol{v}(\tau) + \Omega_0\, Q(\tau)\,\boldsymbol{r}(\tau), \\ \boldsymbol{b}(\tau) &= \dot{\boldsymbol{r}}_0^+(\tau) - Q(\tau)\,\dot{\boldsymbol{r}}_0(\tau) - \Omega_0\, Q(\tau)\,\boldsymbol{r}_0(\tau), \end{aligned} \tag{5.49}$$

or for time t to

$$\boldsymbol{v}^+(t) = \boldsymbol{v}(t) + [\boldsymbol{b}(t) + \boldsymbol{\omega}_0 \times \boldsymbol{r}(t)]. \tag{5.50}$$

The square bracket on the right-hand side of (5.50) is due to superposed rigid body motion, $\boldsymbol{\omega}_0$ is a uniform rigid body angular velocity and $b(t)$ may be interpreted as a uniform rigid body translatory velocity at time t.

Returning to (5.38), we have

$$\dot{\boldsymbol{a}}_i^+ = Q\,\dot{\boldsymbol{a}}_i + \dot{Q}\,\boldsymbol{a}_i = Q\,[\dot{\boldsymbol{a}}_i + Q^T \Omega\, Q\,\boldsymbol{a}_i] \tag{5.51}$$

which also provides the expression for the velocity gradient $\boldsymbol{v}^+_{,\alpha} = \dot{\boldsymbol{a}}^+_\alpha$. In addition, the director velocity and its gradient associated with the motion (5.36) are

$$\begin{aligned} \boldsymbol{w}^+ &= Q\,\boldsymbol{w} + \dot{Q}\,\boldsymbol{d} = Q[\boldsymbol{w} + Q^T \Omega\, Q\,\boldsymbol{d}], \\ \boldsymbol{w}^+_{,\alpha} &= Q\,\boldsymbol{w}_{,\alpha} + \dot{Q}\,\boldsymbol{d}_{,\alpha} = Q[\boldsymbol{w}_{,\alpha} + Q^T \Omega\, Q\,\boldsymbol{d}_{,\alpha}]. \end{aligned} \tag{5.52}$$

Recalling the expressions for the first and the second fundamental forms of the surface, as well as $(5.28)_2$, from (5.38) and (5.36) and using the relation

$$\boldsymbol{U}\cdot Q\boldsymbol{V} = Q^T\boldsymbol{U}\cdot\boldsymbol{V} \tag{5.53}$$

with $\boldsymbol{U}$ and $\boldsymbol{V}$ being any two vectors, we have

$$\begin{aligned} a^+_{\alpha\beta} &= a_{\alpha\beta}, \qquad & b^+_{\alpha\beta} &= b_{\alpha\beta}, \\ d^+_i &= d_i, \qquad & \lambda^+_{i\alpha} &= \lambda_{i\alpha}, \end{aligned} \tag{5.54}$$

for all proper orthogonal Q. Similarly, with the use of (5.51), it can be readily verified that

$$\begin{aligned} &\eta^+_{ki} = \eta_{ki}, \qquad \dot{a}^+ = \dot{a}, \qquad \boldsymbol{\eta}^+_\alpha = Q\,\boldsymbol{\eta}_\alpha, \\ &W^+_{ki} = W_{ki} + \tfrac{1}{2}[Q^T\dot{Q}\,\boldsymbol{a}_i\cdot\boldsymbol{a}_k - Q^T\dot{Q}\,\boldsymbol{a}_k\cdot\boldsymbol{a}_i] \end{aligned} \tag{5.55}$$

and

$$\begin{aligned} \dot{d}^+_i &= \dot{d}_i, \qquad & \dot{\lambda}^+_{i\alpha} &= \dot{\lambda}_{i\alpha}, \\ \boldsymbol{\Gamma}^+ &= Q\boldsymbol{\Gamma}, \qquad & \boldsymbol{\Gamma}^+_{:\alpha} &= Q\boldsymbol{\Gamma}_{:\alpha}. \end{aligned} \tag{5.56}$$

The foregoing results, except for (5.46)–(5.50), are valid for every proper orthogonal Q and for all t. In the special case of (5.36) in which Q has the value specified by (5.46), some of the formulae simplify and assume a more revealing form. In particular, with $b(\tau)=0$ in $(5.49)_2$ and with Q and $\dot{Q}$ given by (5.48) at time t, the superposed velocity and the superposed velocity gradient at time t become

$$\begin{aligned} \boldsymbol{v}^+ &= \boldsymbol{v} + \boldsymbol{\omega}_0 \times \boldsymbol{r}, \\ \boldsymbol{v}^+_{,\alpha} &= \dot{\boldsymbol{a}}^+_\alpha = \boldsymbol{v}_{,\alpha} + \boldsymbol{\omega}_0 \times \boldsymbol{a}_\alpha = \boldsymbol{v}_{,\alpha} - \Omega^0_{k\alpha}\,\boldsymbol{a}_k \end{aligned} \tag{5.57}$$

and (5.51)–(5.52) reduce to

$$\dot{\boldsymbol{a}}^+_i = \dot{\boldsymbol{a}}_i + \boldsymbol{\omega}_0 \times \boldsymbol{a}_i \tag{5.58}$$

and

$$\begin{aligned} \boldsymbol{w}^+ &= \boldsymbol{w} + \boldsymbol{\omega}_0 \times \boldsymbol{d} = \boldsymbol{w} + d^i\,\Omega^0_{ik}\,\boldsymbol{a}^k, \\ \boldsymbol{w}^+_{,\alpha} &= \boldsymbol{w}_{,\alpha} + \boldsymbol{\omega}_0 \times \boldsymbol{d}_{,\alpha} = \boldsymbol{w}_{,\alpha} + \lambda^i_{.\alpha}\,\Omega^0_{ik}\,\boldsymbol{a}^k, \end{aligned} \tag{5.59}$$

where Ω^0_{ki} are related to the components of $\boldsymbol{\omega}_0$ by (5.42) and we have omitted the argument t [corresponding to the given time in (5.46)] from the various functions in (5.57)–(5.59). Similarly, in view of $(5.46)_1$, most of the remaining expressions have an obvious simplification in this case; in particular $(5.55)_4$ becomes

$$W^+_{ki} = W_{ki} - \Omega^0_{ki}. \tag{5.60}$$

We emphasize that the special results (5.57)–(5.60) are obtained corresponding to (5.46) for a given time t and with Q and $\dot{Q}$ specified by (5.48).

γ) *Additional kinematics.* In the remainder of this section we consider some additional (but unrelated) kinematics which will be used subsequently.

The preceding developments in this section represent kinematic results by direct approach appropriate to the theory of a Cosserat surface. However, other developments by direct approach, in the absence of the director field and less general than the earlier results [in Subsect. α)], are possible; and we discuss now one such possibility suitable for a theory which we call a *restricted theory*. Briefly, consider a material surface and identify the material points of the surface with convected coordinates θ^α. Adopting the previous notation and terminology, we continue to refer to the (initial) reference surface (with position vector $\boldsymbol{R}$) by $\mathscr{S}$ and to the surface in the present configuration at time t (with position vector $\boldsymbol{r}$) by $\mathscr{s}$. Since we do not admit a director, the motion of the surface is simply characterized by $(5.1)_1$ and instead of $(5.1)_3$ we have $(4.11)_4$. The velocity vector $\boldsymbol{v}$ of $\mathscr{s}$ at time t is defined by $(5.3)_1$. In anticipation of results to be derived for the restricted theory (in Sect. 10) and in order to easily contrast these with those of the more general theory of the Cosserat surface (in Sects. 8–9), we introduce the notations

$$\dot{\boldsymbol{w}} = \dot{\boldsymbol{a}}_3, \quad \dot{\boldsymbol{w}}_{,\alpha} = \dot{\boldsymbol{a}}_{3,\alpha}, \quad \lambda_{\alpha\beta} = \lambda_{\beta\alpha} = -b_{\alpha\beta}, \quad \dot{\lambda}_{\alpha\beta} = -\dot{b}_{\alpha\beta}, \tag{5.61}$$

where $\dot{\boldsymbol{w}}$ is the angular velocity of the surface $\mathscr{s}$. Then, by $(4.13)_3$, (5.18) and (5.23), we may write

$$\begin{aligned} \dot{\boldsymbol{w}}_{,\alpha} &= -\overline{(b^\gamma_\alpha \boldsymbol{a}_\gamma)}^{\boldsymbol{\cdot}} = \dot{\lambda}^\gamma_\alpha \boldsymbol{a}_\gamma + \lambda^\gamma_\alpha (\eta_{k\gamma} + W_{k\gamma})\, \boldsymbol{a}^k \\ &= \dot{\lambda}_{\gamma\alpha} \boldsymbol{a}^\gamma + \lambda^\gamma_\alpha (W_{k\gamma} - \eta_{k\gamma})\, \boldsymbol{a}^k. \end{aligned} \tag{5.62}$$

This completes our brief discussion of the kinematics of the restricted theory. We note, however, that earlier formulae of this section which do not involve the director or its gradient (including those under superposed rigid body motions) remain valid in the restricted theory. For later reference, we also recall here the formulae which relate the two-dimensional ε-system $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$ to the two-dimensional permutation symbols:

$$\begin{aligned} &\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta 3} = a^{\frac{1}{2}}\, e_{\alpha\beta}, \qquad &&\varepsilon^{\alpha\beta} = \varepsilon^{\alpha\beta 3} = a^{-\frac{1}{2}}\, e^{\alpha\beta}, \\ &e_{11} = e_{22} = e^{11} = e^{22} = 0, \qquad &&e_{12} = -e_{21} = e^{12} = -e^{21} = 1. \end{aligned} \tag{5.63}$$

Next, we consider the kinematics of a surface integral and deduce an integral formula which will be utilized in the next chapter. Let $\varphi(\theta^\alpha, t)$ stand for a (sufficiently smooth) scalar-valued or vector-valued function of position and time and define the integral

$$\int_{\mathscr{P}} \varphi\, d\sigma \tag{5.64}$$

over $\mathscr{P}$ in the present configuration. Since the above integral is a function of time, its derivative with respect to t can be calculated as follows:

$$\frac{d}{dt}\int_{\mathscr{P}} \varphi\, d\sigma = \frac{d}{dt}\int_{\mathscr{P}_0} J\varphi\, d\Sigma = \int_{\mathscr{P}_0} \overline{J\varphi}^{\boldsymbol{\cdot}}\, d\Sigma = \int_{\mathscr{P}} (\dot{\varphi} + J^{-1}\dot{J}\,\varphi)\, d\sigma, \tag{5.65}$$

where J is defined by (4.41) and the region of integration of the last integral is again over $\mathscr{P}$. But $J^{-1}\dot{J} = \eta^\alpha_\alpha$ by (5.21). Hence, from $(5.65)_3$ we have

$$\frac{d}{dt}\int_{\mathscr{P}} \varphi\, d\sigma = \int_{\mathscr{P}} (\dot{\varphi} + \eta^\alpha_\alpha\, \varphi)\, d\sigma \tag{5.66}$$

which is the desired result. The last formula is essentially the two-dimensional analogue of the transport theorem in the three-dimensional theory.

6. Kinematics of shells continued (linear theory): I. Direct approach. This section is devoted to linearized kinematics for shells and plates by direct approach. In particular, we deduce the linearized kinematic measures for a Cosserat surface with infinitesimal displacements and infinitesimal director displacements as a special case of the general results in Sect. 5.

δ) *Linearized kinematics.* Let

$$\boldsymbol{r} = \boldsymbol{R} + \varepsilon\,\boldsymbol{u}, \quad \boldsymbol{u} = u^i \boldsymbol{A}_i, \quad \boldsymbol{v} = \varepsilon\,\dot{\boldsymbol{u}}, \tag{6.1}$$

$$\boldsymbol{d} = \boldsymbol{D} + \varepsilon\,\boldsymbol{\delta}, \quad \boldsymbol{\delta} = \delta_i \boldsymbol{A}^i, \quad \boldsymbol{w} = \varepsilon\,\dot{\boldsymbol{\delta}}, \tag{6.2}$$

where ε is a non-dimensional parameter. We say the motion of a Cosserat surface characterized by $(6.1)_1$ and $(6.2)_1$ describes infinitesimal deformation if the magnitudes of $\boldsymbol{u}$, $\boldsymbol{\delta}$ and all their derivatives are bounded by 1 and if

$$\varepsilon \ll 1. \tag{6.3}$$

We shall be concerned in the following developments with (scalar, vector or tensor) functions of position and time, determined by $\varepsilon\,\boldsymbol{u}$, $\varepsilon\,\boldsymbol{\delta}$ and their surface and time derivatives. We denote these functions by the customary order symbol $O(\varepsilon^n)$ if there exists a real number C, independent of ε, $\boldsymbol{u}$, $\boldsymbol{\delta}$ and their derivatives, such that

$$|O(\varepsilon^n)| < C\,\varepsilon^n, \tag{6.4}$$

as $\varepsilon \to 0$.

We emphasize that the infinitesimal theory which we wish to obtain as a special case of the results in Sect. 5 and in the sense of (6.3) is such that all kinematical quantities (including the displacement $\boldsymbol{u}$, the director displacement $\boldsymbol{\delta}$ and such measures as $e_{\alpha\beta}$, $\varkappa_{i\alpha}$ and γ_i, as well as their derivatives with respect to the surface coordinates and t) are of $O(\varepsilon)$. Moreover, throughout this section, we again use a vertical bar to denote covariant differentiation; this, however, should not be confusing. The designation of a vertical bar in the present section (or whenever we are concerned with linearized measures) is for covariant differentiation with respect to $A_{\alpha\beta}$ of the undeformed surface, in contrast to the meaning of a vertical bar in Sect. 5 and also in parts of later sections.[28]

From $(6.1)_1$ and $(6.2)_1$, we have

$$\boldsymbol{a}_\alpha = \boldsymbol{A}_\alpha + \varepsilon\,\boldsymbol{u}_{,\alpha}, \quad \boldsymbol{d}_{,\alpha} = \boldsymbol{D}_{,\alpha} + \varepsilon\,\boldsymbol{\delta}_{,\alpha} \tag{6.5}$$

and by $(6.5)_1$ and $(4.11)_{1,2}$ we can show that

$$\boldsymbol{a}_3 = \boldsymbol{A}_3 + \varepsilon\,\boldsymbol{\beta} + O(\varepsilon^2), \quad \boldsymbol{\beta} = \beta^\alpha \boldsymbol{A}_\alpha = \beta_\alpha \boldsymbol{A}^\alpha, \quad \beta_\alpha = \boldsymbol{\beta}\cdot\boldsymbol{A}_\alpha = -\boldsymbol{u}_{,\alpha}\cdot\boldsymbol{A}_3. \tag{6.6}$$

[28] It will be clear from the particular context, in later sections, whenever a vertical bar denotes covariant differentiation with respect to $a_{\alpha\beta}$ or $A_{\alpha\beta}$.

With the use of (6.5) and $(6.6)_1$ and recalling $(4.12)_1$, (4.13), $(5.25)_2$ and $(5.28)_2$, the expressions for $a_{\alpha\beta}$, $b_{\alpha\beta}$, d_i and $\lambda_{i\alpha}$ can be written as

$$\begin{aligned} a_{\alpha\beta} &= \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = A_{\alpha\beta} + \varepsilon(\boldsymbol{u}_{,\alpha} \cdot \boldsymbol{A}_\beta + \boldsymbol{u}_{,\beta} \cdot \boldsymbol{A}_\alpha) + O(\varepsilon^2), \\ b_{\alpha\beta} &= -\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_{3,\beta} = B_{\alpha\beta} - \varepsilon(\boldsymbol{u}_{,\alpha} \cdot \boldsymbol{A}_{3,\beta} + \boldsymbol{A}_\alpha \cdot \boldsymbol{\beta}_{,\beta}) + O(\varepsilon^2), \end{aligned} \tag{6.7}$$

$$\begin{aligned} d_\alpha &= \boldsymbol{a}_\alpha \cdot \boldsymbol{d} = D_\alpha + \varepsilon(\boldsymbol{A}_\alpha \cdot \boldsymbol{\delta} + \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{D}) + O(\varepsilon^2), \\ d_3 &= \boldsymbol{a}_3 \cdot \boldsymbol{d} = D_3 + \varepsilon(\boldsymbol{A}_3 \cdot \boldsymbol{\delta} + \boldsymbol{\beta} \cdot \boldsymbol{D}) + O(\varepsilon^2), \end{aligned} \tag{6.8}$$

$$\begin{aligned} \lambda_{\beta\alpha} &= \boldsymbol{a}_\beta \cdot \boldsymbol{d}_{,\alpha} = \Lambda_{\beta\alpha} + \varepsilon(\boldsymbol{A}_\beta \cdot \boldsymbol{\delta}_{,\alpha} + \boldsymbol{u}_{,\beta} \cdot \boldsymbol{D}_{,\alpha}) + O(\varepsilon^2), \\ \lambda_{3\alpha} &= \boldsymbol{a}_3 \cdot \boldsymbol{d}_{,\alpha} = \Lambda_{3\alpha} + \varepsilon(\boldsymbol{A}_3 \cdot \boldsymbol{\delta}_{,\alpha} + \boldsymbol{\beta} \cdot \boldsymbol{D}_{,\alpha}) + O(\varepsilon^2), \end{aligned} \tag{6.9}$$

where $A_{\alpha\beta}$, $B_{\alpha\beta}$, D_i and $\Lambda_{i\alpha}$ are the initial reference values of the functions given by (6.7)–(6.9). These results can now be used to obtain the appropriate expressions for $e_{\alpha\beta}$, $\varkappa_{i\alpha}$ and γ_i in (5.29) to (5.31) in terms of $\varepsilon\,\boldsymbol{u}$, $\varepsilon\,\boldsymbol{\delta}$ and their derivatives.

It is desirable at this stage to elaborate on the manner in which the process of linearization may be accomplished. For this purpose, let $\boldsymbol{u}'$ and $\boldsymbol{\delta}'$ be vector functions defined by

$$\begin{aligned} \boldsymbol{u}' &= \varepsilon\,\boldsymbol{u} = O(\varepsilon), \qquad u'^i = \boldsymbol{A}^i \cdot \boldsymbol{u}' = O(\varepsilon), \\ \boldsymbol{\delta}' &= \varepsilon\,\boldsymbol{\delta} = O(\varepsilon), \qquad \delta'_i = \boldsymbol{A}_i \cdot \boldsymbol{\delta}' = O(\varepsilon), \end{aligned} \tag{6.10}$$

which can be used to express all kinematical quantities in terms of $\boldsymbol{u}'$, $\boldsymbol{\delta}'$ and their derivatives. For example, if we define $e'_{\alpha\beta}$ and $\varkappa'_{\beta\alpha}$ by

$$e'_{\alpha\beta} = \tfrac{1}{2}\,(\boldsymbol{u}'_{,\alpha} \cdot \boldsymbol{A}_\beta + \boldsymbol{u}'_{,\beta} \cdot \boldsymbol{A}_\alpha), \qquad \varkappa'_{\beta\alpha} = \boldsymbol{A}_\beta \cdot \boldsymbol{\delta}'_{,\alpha} + \boldsymbol{u}'_{,\beta} \cdot \boldsymbol{D}_{,\alpha}, \tag{6.11}$$

each of which is of $O(\varepsilon)$, we can then write (5.31)–(5.32) and the ratio a/A in (4.41) as

$$\begin{aligned} e_{\alpha\beta} &= e'_{\alpha\beta} + O(\varepsilon^2) = O(\varepsilon), \qquad & \varkappa_{\beta\alpha} &= \varkappa'_{\beta\alpha} + O(\varepsilon^2) = O(\varepsilon), \\ \frac{a}{A} &= 1 + 2A^{\alpha\beta}\, e'_{\alpha\beta} + O(\varepsilon^2), \qquad & \left(\frac{a}{A}\right)^{\frac{1}{2}} &= 1 + e'^{\alpha}_{\alpha} + O(\varepsilon^2) \end{aligned} \tag{6.12}$$

and other kinematical quantities can be expressed similarly. A straightforward procedure is now to retain only terms of $O(\varepsilon)$ in such expressions as (6.12), hence approximate $e_{\alpha\beta}$ and $\varkappa_{\beta\alpha}$ by $e'_{\alpha\beta}$ and $\varkappa'_{\beta\alpha}$, etc., and complete the linearization in this manner. However, in order to avoid the introduction of unnecessary additional notations, we may proceed with the linearization from (6.7)–(6.9) by retaining only terms of $O(\varepsilon)$ and after the approximations, without loss in generality, we set $\varepsilon = 1$. In what follows, we adopt this latter procedure. Thus, the kinematic measures (5.31)–(5.33), after linearization, reduce to

$$\begin{aligned} & e_{\alpha\gamma} = \tfrac{1}{2}(\boldsymbol{u}_{,\alpha} \cdot \boldsymbol{A}_\gamma + \boldsymbol{u}_{,\gamma} \cdot \boldsymbol{A}_\alpha), \\ & \varkappa_{\gamma\alpha} = \boldsymbol{A}_\gamma \cdot \boldsymbol{\delta}_{,\alpha} + \boldsymbol{u}_{,\gamma} \cdot \boldsymbol{D}_{,\alpha}, \qquad \varkappa_{3\alpha} = \boldsymbol{A}_3 \cdot \boldsymbol{\delta}_{,\alpha} + \boldsymbol{\beta} \cdot \boldsymbol{D}_{,\alpha}, \\ & \gamma_\alpha = \boldsymbol{A}_\alpha \cdot \boldsymbol{\delta} + \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{D}, \qquad \gamma_3 = \boldsymbol{A}_3 \cdot \boldsymbol{\delta} + \boldsymbol{\beta} \cdot \boldsymbol{D} \end{aligned} \tag{6.13}$$

and (4.42) becomes

$$\varrho = \varrho_0(1 - e^{\alpha}_{\alpha}), \tag{6.14}$$

in view of (6.7)–(6.9) and $(6.12)_4$. In (6.13), $\boldsymbol{\beta}$ is defined by $(6.6)_3$ and the partial derivatives of various vector functions can be calculated by using a formula of the type (5.5). In particular, we record here the following formulae

$$\begin{aligned} &\boldsymbol{u}_{,\alpha}=u_{i\alpha}\boldsymbol{A}^i=u^i_{.\alpha}\boldsymbol{A}_i, \quad u^\lambda_{.\alpha}=A^{\lambda\nu}u_{\nu\alpha}, \quad u^3_{.\alpha}=u_{3\alpha}, \\ &u_{\gamma\alpha}=\boldsymbol{A}_\gamma\cdot\boldsymbol{u}_{,\alpha}=u_{\gamma|\alpha}-B_{\alpha\gamma}u_3, \quad u_{3\alpha}=\boldsymbol{A}_3\cdot\boldsymbol{u}_{,\alpha}=u_{3,\alpha}+B^\gamma_\alpha u_\gamma \end{aligned} \tag{6.15}$$

and

$$\begin{aligned} &\boldsymbol{A}_\gamma\cdot\boldsymbol{\delta}_{,\alpha}=\delta_{\gamma|\alpha}-B_{\alpha\gamma}\delta_3, \quad \boldsymbol{A}_3\cdot\boldsymbol{\delta}_{,\alpha}=\delta_{3,\alpha}+B^\gamma_\alpha\delta_\gamma, \\ &\boldsymbol{u}_{,\gamma}\cdot\boldsymbol{D}_{,\alpha}=\Lambda_{\nu\alpha}(u^\nu|_\gamma-B^\nu_\gamma u_3)+\Lambda_{3\alpha}(u_{3,\gamma}+B^\nu_\gamma u_\nu), \\ &\boldsymbol{\beta}\cdot\boldsymbol{D}_{,\alpha}=\Lambda_{\nu\alpha}\beta^\nu, \quad \boldsymbol{u}_{,\alpha}\cdot\boldsymbol{D}=D^i u_{i\alpha}=D_i u^i_{.\alpha}, \quad \boldsymbol{\beta}\cdot\boldsymbol{D}=\beta^\alpha D_\alpha, \end{aligned} \tag{6.16}$$

where

$$\Lambda_{\nu\alpha}=\boldsymbol{A}_\nu\cdot\boldsymbol{D}_{,\alpha}=D_{\nu|\alpha}-B_{\alpha\nu}D_3, \quad \Lambda_{3\alpha}=\boldsymbol{A}_3\cdot\boldsymbol{D}_{,\alpha}=D_{3,\alpha}+B^\nu_\alpha D_\nu, \tag{6.17}$$

by $(5.34)_1$. Also, in (6.15)–(6.17) and throughout the present section, a vertical bar denotes covariant differentiation with respect to $A_{\alpha\beta}$ of the initial undeformed surface. Introducing the above results in (6.13), we finally obtain the following expressions for the kinematic measures and their time rates: [29]

$$\begin{aligned} &e_{\alpha\gamma}=\tfrac{1}{2}(u_{\alpha|\gamma}+u_{\gamma|\alpha})-B_{\alpha\gamma}u_3, \\ &\varkappa_{\gamma\alpha}=\delta_{\gamma|\alpha}-B_{\alpha\gamma}\delta_3+\Lambda_{\nu\alpha}u^\nu_{.\gamma}+\Lambda_{3\alpha}u^3_{.\gamma}, \\ &\varkappa_{3\alpha}=\delta_{3,\alpha}+B^\gamma_\alpha\delta_\gamma+\Lambda_{\nu\alpha}\beta^\nu, \quad \beta_\alpha=-u^3_{.\alpha}=-(u_{3,\alpha}+B^\gamma_\alpha u_\gamma), \\ &\gamma_\alpha=\delta_\alpha+D_\lambda u^\lambda_{.\alpha}-D_3\beta_\alpha=\delta_\alpha+D_i u^i_{.\alpha}, \quad \gamma_3=\delta_3+D_\alpha\beta^\alpha \end{aligned} \tag{6.18}$$

and

$$\begin{aligned} &\dot{e}_{\alpha\beta}=\tfrac{1}{2}(v_{\alpha|\gamma}+v_{\gamma|\alpha})-B_{\alpha\gamma}v_3, \\ &\dot{\varkappa}_{\gamma\alpha}=w_{\gamma|\alpha}-B_{\alpha\gamma}w_3+\Lambda_{\nu\alpha}v^\nu_{.\gamma}+\Lambda_{3\alpha}v^3_{.\gamma}, \\ &\dot{\varkappa}_{3\alpha}=w_{3,\alpha}+B^\gamma_\alpha w_\gamma+\Lambda_{\nu\alpha}\dot{\beta}^\nu, \\ &\dot{\gamma}_\alpha=w_\alpha+D_\lambda v^\lambda_{.\alpha}-D_3\dot{\beta}_\alpha, \quad \dot{\gamma}_3=w_3+D_\alpha\dot{\beta}^\alpha, \end{aligned} \tag{6.19}$$

where u_i and δ_i are components of the infinitesimal displacement and the infinitesimal director displacement and where

$$v_i=\dot{u}_i, \quad w_i=\dot{\delta}_i, \quad v^\lambda_{.\alpha}=v^\lambda|_\alpha-B^\lambda_\alpha v_3, \quad v^3_{.\alpha}=-\dot{\beta}_\alpha=v_{3,\alpha}+B^\lambda_\alpha v_\lambda. \tag{6.20}$$

ε) *A catalogue of linear kinematic measures.* The linearized kinematic measures (6.18)–(6.19) are valid for a variable initial director. Apart from (6.14) and $(6.18)_1$ which do not depend on $\boldsymbol{D}$, some of the kinematical results simplify if the initial director $\boldsymbol{D}$ is along the initial unit normal $\boldsymbol{A}_3$, i.e., if

$$\boldsymbol{D}=D\boldsymbol{A}_3, \quad D_\alpha=0, \quad D_3=D, \tag{6.21}$$

where the notation $D_3=D$ is introduced for convenience. Below we collect a catalogue of formulae for linearized kinematic measures when $\boldsymbol{D}$ is of the form (6.21) or a more specialized case in which $\boldsymbol{D}=\boldsymbol{A}_3$. The resulting expressions, which can be expressed in a variety of forms, are of particular interest in connection with kinematic measures in the classical linear theories of shells and plates.

Formulae A

$$\begin{gathered} \boldsymbol{D}=D\boldsymbol{A}_3, \quad \Lambda_{\nu\alpha}=-B_{\nu\alpha}D, \quad \Lambda_{3\alpha}=D_{,\alpha}, \\ e_{\alpha\gamma}=\tfrac{1}{2}(u_{\alpha|\gamma}+u_{\gamma|\alpha})-B_{\alpha\gamma}u_3, \\ \delta_\alpha=\gamma_\alpha+D\beta_\alpha, \quad \delta_3=\gamma_3, \quad \beta_\alpha=-(u_{3,\alpha}+B^\nu_\alpha u_\nu), \\ \varkappa_{\gamma\alpha}=\gamma_{\gamma|\alpha}-B_{\alpha\gamma}\gamma_3-D(B_{\nu\alpha}u^\nu_{.\gamma}-\beta_{\gamma|\alpha})=\varrho_{\gamma\alpha}-B_{\alpha\gamma}\gamma_3, \\ \varrho_{\gamma\alpha}=\gamma_{\gamma|\alpha}-D(B_{\nu\alpha}u^\nu_{.\gamma}-\beta_{\gamma|\alpha}), \\ \varkappa_{3\alpha}=\gamma_{3,\alpha}+B^\nu_\alpha\gamma_\nu=\varrho_{3\alpha}+B^\nu_\alpha\gamma_\nu, \quad \varrho_{3\alpha}=\gamma_{3,\alpha}. \end{gathered} \tag{6.22}$$

[29] GREEN, NAGHDI and WAINWRIGHT [1965, 4].

Formulae B

$$\boldsymbol{D} = D\boldsymbol{A}_3, \quad \Lambda_{\nu\alpha} = -B_{\nu\alpha} D, \quad \Lambda_{3\alpha} = D_{,\alpha},$$
$$\hat{\boldsymbol{\delta}} = \frac{1}{D}\hat{\boldsymbol{\delta}}, \quad \hat{\delta}_i = \frac{1}{D}\delta_i, \quad \hat{\gamma}_i = \frac{1}{D}\gamma_i, \quad \hat{\varkappa}_{i\alpha} = \frac{1}{D}\varkappa_{i\alpha}, \quad \hat{\varrho}_{i\alpha} = \frac{1}{D}\varrho_{i\alpha},$$
$$\begin{gathered} e_{\alpha\gamma} = \tfrac{1}{2}(u_{\alpha|\gamma} + u_{\gamma|\alpha}) - B_{\alpha\gamma} u_3, \\ \hat{\delta}_\alpha = \hat{\gamma}_\alpha + \beta_\alpha, \quad \hat{\delta}_3 = \hat{\gamma}_3, \quad \beta_\alpha = -(u_{3,\alpha} + B^\nu_\alpha u_\nu), \\ \hat{\varkappa}_{\gamma\alpha} = \hat{\varrho}_{\gamma\alpha} - B_{\alpha\gamma}\hat{\gamma}_3, \quad \hat{\varrho}_{\gamma\alpha} = \hat{\gamma}_{\gamma|\alpha} + \frac{D_{,\alpha}}{D}\hat{\gamma}_\gamma - (B_{\nu\alpha} u^\nu_{.\gamma} - \beta_{\gamma|\alpha}), \\ \hat{\varkappa}_{3\alpha} = \hat{\varrho}_{3\alpha} + B^\nu_\alpha \hat{\gamma}_\nu, \quad \hat{\varrho}_{3\alpha} = \hat{\gamma}_{3,\alpha} + \frac{D_{,\alpha}}{D}\hat{\gamma}_3. \end{gathered} \tag{6.23}$$

Formulae C

$$\begin{gathered} \boldsymbol{D} = \boldsymbol{A}_3, \quad \Lambda_{\nu\alpha} = -B_{\nu\alpha}, \quad \Lambda_{3\alpha} = 0, \\ e_{\alpha\gamma} = \tfrac{1}{2}(u_{\alpha|\gamma} + u_{\gamma|\alpha}) - B_{\alpha\gamma} u_3, \\ \delta_\alpha = \gamma_\alpha + \beta_\alpha, \quad \delta_3 = \gamma_3, \quad \beta_\alpha = -(u_{3,\alpha} + B^\nu_\alpha u_\nu), \\ \varkappa_{\gamma\alpha} = \varkappa_{(\gamma\alpha)} + \varkappa_{[\gamma\alpha]} = \varrho_{\gamma\alpha} - B_{\alpha\gamma}\gamma_3, \\ \varkappa_{3\alpha} = \varrho_{3\alpha} + B^\gamma_\alpha \gamma_\gamma, \\ \varrho_{\gamma\alpha} = \gamma_{\gamma|\alpha} - \bar{\varkappa}_{\gamma\alpha}, \quad \varrho_{3\alpha} = \gamma_{3,\alpha}, \\ \bar{\varkappa}_{\gamma\alpha} = \bar{\varkappa}_{\alpha\gamma} = u_{3|\gamma\alpha} + B^\nu_{\gamma|\alpha} u_\nu + B^\nu_\alpha u_{\nu|\gamma} + B^\nu_\gamma u_{\nu|\alpha} - B^\nu_\alpha B_{\nu\gamma} u_3 = (B_{\nu\alpha} u^\nu_{.\gamma} - \beta_{\gamma|\alpha}), \\ \varkappa_{(\gamma\alpha)} = \varrho_{(\gamma\alpha)} - B_{\alpha\gamma}\gamma_3 = \tfrac{1}{2}(\gamma_{\gamma|\alpha} + \gamma_{\alpha|\gamma}) - \bar{\varkappa}_{\alpha\gamma} - B_{\alpha\gamma}\gamma_3, \\ \varkappa_{[\gamma\alpha]} = \varrho_{[\gamma\alpha]} = \tfrac{1}{2}(\gamma_{\gamma|\alpha} - \gamma_{\alpha|\gamma}). \end{gathered} \tag{6.24}$$

Formulae D

$$\begin{gathered} \boldsymbol{D} = \boldsymbol{A}_3, \quad B_{\alpha\beta} = 0, \\ e_{\alpha\gamma} = \tfrac{1}{2}(u_{\alpha|\gamma} + u_{\gamma|\alpha}), \quad \varkappa_{3\alpha} = \varrho_{3\alpha} = \gamma_{3,\alpha}, \quad \gamma_3 = \delta_3, \\ \varkappa_{\gamma\alpha} = \varrho_{\gamma\alpha} = \varkappa_{(\gamma\alpha)} + \varkappa_{[\gamma\alpha]}, \quad \gamma_\alpha = \delta_\alpha - \beta_\alpha = \delta_\alpha + u_{3,\alpha}, \\ \varkappa_{(\gamma\alpha)} = \tfrac{1}{2}(\gamma_{\gamma|\alpha} + \gamma_{\alpha|\gamma}) - u_{3|\alpha\gamma}, \quad \varkappa_{[\gamma\alpha]} = \tfrac{1}{2}(\gamma_{\gamma|\alpha} - \gamma_{\alpha|\gamma}). \end{gathered} \tag{6.25}$$

We briefly elaborate on the nature of the above catalogue of formulae for linearized kinematic measures.[30] The relative simplicity of Formulae A, in

[30] The linearized kinematic measures (6.18)–(6.19) were obtained by GREEN, NAGHDI and WAINWRIGHT [1965, *4*]. The linearized measures in Formulae D and C were employed by GREEN and NAGHDI in a number of studies concerned with linear constitutive equations for a Cosserat surface and their applicability to thin elastic shells and plates: [1967, *4*], [1968, *6*], [1969, *3*]. Formulae B involving the variables $\hat{\gamma}_i$ and $\hat{\varkappa}_{i\alpha}$ have been used by GREEN, NAGHDI and WENNER [1971, *6*] in connection with application of the linear theory of a Cosserat surface (with variable initial director) to plates of variable thickness.

Linearized kinematics for a surface with a single director are given also by GÜNTHER [1961, *4*] who, however, allows the director to undergo infinitesimal rotation without stretch. Thus GÜNTHER's kinematical results are more restrictive than those given by Formulae C. GÜNTHER's paper [1961, *4*] seems to be the first attempt in recent years to construct a complete linear mechanical theory for shells based on the concept of oriented media. However, apart from his somewhat restrictive kinematics, his paper has another undesirable feature: After the development of his kinematics and the equations of motion for the linear theory by direct approach, the rest of his developments are obtained from the three-dimensional equations and he considers the question of constitutive relations on the basis of the generalized Hooke's law (in the three-dimensional theory of non-polar linear elasticity). It is difficult to assess the nature of the final results in his paper which, in addition to an awkward notation for kinematic quantities, involves approximations of the type often used in the development of shell theory from three-dimensional equations.

comparison with (6.18), is mainly due to the simpler expressions for $\Lambda_{\nu\alpha}$ and $\Lambda_{3\alpha}$ in $(6.22)_{2,3}$ which follow from (6.17) when $\boldsymbol{D}$ is specified by (6.21). In Formulae B the infinitesimal director displacement is put in non-dimensional form and an alternative set of kinematic variables are introduced which consist of $e_{\alpha\beta}$ and the new variables $\hat{\gamma}_i$ and $\hat{\varkappa}_{i\alpha}$ in place of γ_i and $\varkappa_{i\alpha}$. The expressions for $\varrho_{\gamma\alpha}$ in (6.22) and $\hat{\varrho}_{\gamma\alpha}$ in (6.23) may also be expressed in terms of $\bar{\varkappa}_{\gamma\alpha}$ in (6.24). Formulae C are special cases of (6.18) in which the initial director is of constant length and coincident with the unit normal to the initial surface. The results in (6.24) can also be obtained as special cases of those in Formulae B, since with $D=1$ the distinctions between $\hat{\delta}_i$ and δ_i, $\hat{\gamma}_i$ and γ_i, $\hat{\varkappa}_{i\alpha}$ and $\varkappa_{i\alpha}$ disappear. Most of the above results simplify in the case of an initially flat Cosserat surface for which

$$B_{\alpha\beta}=0. \tag{6.26}$$

In particular, Formulae D represent the linearized kinematic measures for an initially flat Cosserat surface as special cases of the results in (6.24). Evidently, when (6.26) holds, the kinematic measures separate into two sets given by $(6.25)_{3,4,5}$ and $(6.25)_{6,7}$, respectively. The kinematic variables in the former set, namely $e_{\alpha\gamma}$, $\varkappa_{3\alpha}$ and γ_3 arise from u_γ and δ_3 and characterize the *extensional motion* (or the stretching) of the Cosserat surface. The kinematic variables $\varkappa_{\gamma\alpha}$ and γ_α in the latter set, on the other hand, are specified in terms of u_3 and δ_α which represent the *flexural motion* (or the bending) of the Cosserat surface.

The foregoing kinematic measures in Formulae A to D are appropriate to a linear *direct* theory for the infinitesimal deformation of a Cosserat surface. The primitive relative kinematic measures of this direct theory are the displacement $\boldsymbol{u}$ and the director displacement $\boldsymbol{\delta}$ defined by (6.1)–(6.2)[31] and the resulting kinematic measures (in Formulae A to D) include several features which should be noted. Among these,[32] with reference to Formulae C, we mention the presence of: (i) The component $\delta_3=\gamma_3$ of the director representing extensibility in the direction of the unit normal to $\mathscr{S}$; (ii) the components $\gamma_\alpha=\delta_\alpha-\beta_\alpha$, i.e., the difference in components of rotation of the director relative to the unit normal and the components of the angular displacement β_α, which may be regarded as representing the effect of "transverse shear deformation"; and (iii) the anti-symmetric $\varkappa_{[\gamma\alpha]}$, as well as the components $\varkappa_{3\alpha}$. Parallel observations can be made in the case of other kinematic measures and, in particular, for those in Formulae D.

It is possible to construct by direct approach a *restricted* theory in which the director is not admitted. In a restricted theory of this type (appropriate for a deformable surface embedded in a Euclidean 3-space), the primitive kinematic quantities may be specified by the displacement $\boldsymbol{u}$ and the angular displacement $\boldsymbol{\beta}$ in (6.1) and (6.6). A special set of kinematic measures which emerges for such a restricted theory is summarized below: [33]

---

[31] Inasmuch as the unit normal to the surface is determined by the surface base vectors, the angular displacement $\boldsymbol{\beta}$ in (6.6) is not a primitive kinematic measure.

[32] These features are ordinarily absent or are accounted for only approximately in the existing derivations of the linear theory of shells from the three-dimensional equations.

[33] Although the special set of kinematic measures given by Formulae E resembles that sometimes used in the literature on shell theory, it should not be confused with the set of kinematic measures obtained from the three-dimensional equations under special assumptions such as Kirchhoff-Love hypothesis. In this connection, see the remarks in Sect. 7 preceding (7.76).

Formulae E

$$
\begin{aligned}
&\boldsymbol{u} = u^i \boldsymbol{A}_i = u_i \boldsymbol{A}^i, && \boldsymbol{\beta} = \beta^\gamma \boldsymbol{A}_\gamma = \beta_\gamma \boldsymbol{A}^\gamma,\\
&\boldsymbol{u}_{,\alpha} = u^i_{.\alpha} \boldsymbol{A}_i = u_{\gamma\alpha} \boldsymbol{A}^\gamma - \beta_\alpha \boldsymbol{A}^3, && \boldsymbol{\beta}_{,\alpha} = \beta_{\gamma|\alpha} \boldsymbol{A}^\gamma + B^\gamma_\alpha \beta_\gamma \boldsymbol{A}^3,\\
&e_{\gamma\alpha} = \tfrac{1}{2}(u_{\gamma|\alpha} + u_{\alpha|\gamma}) - B_{\alpha\gamma} u_3, && \beta_\alpha = -u^3_{.\alpha} = -(u_{3,\alpha} + B^\nu_\alpha u_\nu),\\
&\varkappa_{\gamma\alpha} = \varrho_{\gamma\alpha} = B_{\nu\alpha} u^\nu_{.\gamma} - \beta_{\gamma|\alpha} = -\bar{\varkappa}_{\gamma\alpha} = -\bar{\varkappa}_{\alpha\gamma}, && \bar{\varkappa}_{\alpha\gamma} \text{ defined in (6.24)}.
\end{aligned}
\tag{6.27}
$$

The above special kinematic measures can also be obtained as a special case of Formulae C if we put $\gamma_i = 0$. In contrast to certain features of Formulae C noted above, Formulae E contain only a symmetric $\varkappa_{\gamma\alpha} = \varrho_{\gamma\alpha} = -\bar{\varkappa}_{\gamma\alpha}$ and do not contain (i) a measure for the extensibility along the normal to the surface or (ii) a measure representing the "transverse shear deformation."

ζ) *Additional linear kinematic formulae.* Some of the expressions in Formulae C can be expressed in slightly different forms but the interrelations are not always immediately apparent. To facilitate such comparisons and in order to record some additional formulae for later use, we introduce the notations

$$
\begin{gathered}
u_{\lambda|\alpha} = \gamma_{\lambda\alpha} = \gamma_{(\lambda\alpha)} + \gamma_{[\lambda\alpha]},\\
\gamma_{(\lambda\alpha)} = \tfrac{1}{2}(u_{\lambda|\alpha} + u_{\alpha|\lambda}), \qquad \gamma_{[\lambda\alpha]} = \tfrac{1}{2}(u_{\lambda|\alpha} - u_{\alpha|\lambda}) = -\gamma_{[\alpha\lambda]},
\end{gathered}
\tag{6.28}
$$

where $\gamma_{(\lambda\alpha)}$ is the part of the strain measure $e_{\lambda\alpha}$ resulting from the displacement gradient $(6.28)_1$ and $\gamma_{[\lambda\alpha]}$ can be interpreted as the infinitesimal rotation at a point about the unit normal to the surface $\mathscr{S}$. Keeping this interpretation in mind and remembering that $\boldsymbol{\beta}$ (with components $\beta_\alpha = \boldsymbol{\beta} \cdot \boldsymbol{A}_\alpha$) in (6.6) is a measure of the infinitesimal rotation of the unit normal to the surface, we introduce a three-dimensional vector field $\bar{\boldsymbol{\omega}}$ defined by

$$
\begin{gathered}
\bar{\boldsymbol{\omega}} = \bar{\omega}^i \boldsymbol{A}_i = \bar{\omega}_i \boldsymbol{A}^i, \qquad \gamma_{[\lambda\alpha]} = -\bar{\varepsilon}_{\lambda\alpha} \bar{\omega}^3,\\
u_{3\alpha} = -\beta_\alpha = \bar{\varepsilon}_{\lambda\alpha} \bar{\omega}^\lambda, \qquad \bar{\omega}^3 = -\tfrac{1}{2} \bar{\varepsilon}^{\lambda\alpha} \gamma_{[\lambda\alpha]}, \qquad \bar{\omega}^\gamma = -\bar{\varepsilon}^{\gamma\alpha} \beta_\alpha,
\end{gathered}
\tag{6.29}
$$

where $\bar{\varepsilon}_{\alpha\beta}$, $\bar{\varepsilon}^{\alpha\beta}$ are the two-dimensional ε-system for the surface $\mathscr{S}$ defined similarly to those in (5.63) but with $a^{\frac{1}{2}}$ replaced by $A^{\frac{1}{2}}$:

$$
\bar{\varepsilon}_{\alpha\beta} = A^{\frac{1}{2}} e_{\alpha\beta}, \qquad \bar{\varepsilon}^{\alpha\beta} = A^{-\frac{1}{2}} e^{\alpha\beta}. \tag{6.30}
$$

We note here the identity

$$
\bar{\boldsymbol{\omega}} \times \boldsymbol{A}_\alpha = \gamma_{[\lambda\alpha]} \boldsymbol{A}^\lambda + u_{3\alpha} \boldsymbol{A}^3, \tag{6.31}
$$

which may be used to express the displacement gradient $\boldsymbol{u}_{,\alpha}$ in terms of $e_{\lambda\alpha} \boldsymbol{A}^\lambda$ and the rotation $\bar{\boldsymbol{\omega}} \times \boldsymbol{A}_\alpha$.

For later reference, we record below the expression resulting from covariant derivative of $\gamma_{[\lambda\nu]}$ given by $(6.28)_3$. Thus

$$
\begin{aligned}
\gamma_{[\lambda\nu]|\alpha} &= \tfrac{1}{2}(u_{\lambda|\alpha\nu} - u_{\nu|\alpha\lambda}) + \tfrac{1}{2}(R^\gamma_{.\lambda\nu\alpha} - R^\gamma_{.\nu\lambda\alpha}) u_\gamma\\
&= \tfrac{1}{2}(u_{\lambda|\alpha\nu} - u_{\nu|\alpha\lambda}) + \tfrac{1}{2}(u_{\alpha|\lambda\nu} - u_{\alpha|\lambda\nu}) + \tfrac{1}{2}(R^\gamma_{.\lambda\nu\alpha} - R^\gamma_{.\nu\lambda\alpha}) u_\gamma\\
&= \tfrac{1}{2}(u_{\lambda|\alpha\nu} + u_{\alpha|\lambda\nu}) - \tfrac{1}{2}(u_{\nu|\alpha\lambda} + u_{\alpha|\nu\lambda}) + \tfrac{1}{2}[R^\gamma_{.\lambda\nu\alpha} - R^\gamma_{.\nu\lambda\alpha} - R^\gamma_{.\alpha\lambda\nu}] u_\gamma\\
&= (e_{\lambda\alpha} + B_{\lambda\alpha} u_3)_{|\nu} - (e_{\nu\alpha} + B_{\nu\alpha} u_3)_{|\lambda} - R^\gamma_{.\alpha\lambda\nu} u_\gamma\\
&= (e_{\lambda\alpha|\nu} - e_{\nu\alpha|\lambda}) - (B_{\lambda\alpha} \beta_\nu - B_{\nu\alpha} \beta_\lambda),
\end{aligned}
\tag{6.32}
$$

where

$$
R^\gamma_{.\lambda\nu\alpha} = -R^\gamma_{.\lambda\alpha\nu} = B_{\lambda\alpha} B^\gamma_\nu - B_{\lambda\nu} B^\gamma_\alpha \tag{6.33}
$$

is the Riemann-Christoffel surface tensor. We also collect below a list of formulae which hold when $\boldsymbol{D}=\boldsymbol{A}_3$ (as in Formulae C) and which will be useful in our subsequent derivation of the compatibility equations later in this section:

Formulae F

$$
\begin{aligned}
&u_{\lambda|\alpha}-B_{\alpha\lambda}u_3=e_{\lambda\alpha}+\gamma_{[\lambda\alpha]}, \qquad \beta_{\lambda|\alpha}=-\bar{\varkappa}_{\alpha\lambda}+B_\alpha^\nu(e_{\nu\lambda}+\gamma_{[\nu\lambda]}),\\
&B_\beta^\lambda\beta_{\lambda|\alpha}=-B_\beta^\lambda\bar{\varkappa}_{\lambda\alpha}+B_\beta^\lambda B_\alpha^\nu(e_{\nu\lambda}+\gamma_{[\nu\lambda]}),\\
&\beta_{\lambda|\alpha\beta}=-\bar{\varkappa}_{\alpha\lambda|\beta}+B_{\alpha|\beta}^\nu(e_{\nu\lambda}+\gamma_{[\nu\lambda]})+B_\alpha^\nu(e_{\nu\lambda|\beta}+\gamma_{[\nu\lambda]|\beta}),\\
&\boldsymbol{u}_{,\alpha}=(e_{\lambda\alpha}+\gamma_{[\lambda\alpha]})\boldsymbol{A}^\lambda-\beta_\alpha\boldsymbol{A}^3=e_{\lambda\alpha}\boldsymbol{A}^\lambda+\overline{\boldsymbol{\omega}}\times\boldsymbol{A}_\alpha,\\
&\boldsymbol{\delta}_{,\alpha}=[\varkappa_{\lambda\alpha}+B_\alpha^\nu(e_{\nu\lambda}+\gamma_{[\nu\lambda]})]\boldsymbol{A}^\lambda+[\varkappa_{3\alpha}+B_\alpha^\nu\beta_\nu]\boldsymbol{A}^3=\boldsymbol{J}_\alpha-B_\alpha^\nu\overline{\boldsymbol{\omega}}\times\boldsymbol{A}_\nu,\\
&\boldsymbol{J}_\alpha=(\varkappa_{\lambda\alpha}+B_\alpha^\nu e_{\nu\lambda})\boldsymbol{A}^\lambda+\varkappa_{3\alpha}\boldsymbol{A}^3,\\
&\boldsymbol{J}_{\alpha|\beta}=[\varkappa_{\lambda\alpha|\beta}+B_{\alpha|\beta}^\nu e_{\nu\lambda}+B_\alpha^\nu e_{\nu\lambda|\beta}-B_{\lambda\beta}\varkappa_{3\alpha}]\boldsymbol{A}^\lambda\\
&\qquad\qquad+[\varkappa_{3\alpha|\beta}+B_\beta^\lambda\varkappa_{\lambda\alpha}+B_\beta^\lambda B_\alpha^\nu e_{\nu\lambda}]\boldsymbol{A}^3,\\
&\bar{\varepsilon}^{\alpha\beta}\{\boldsymbol{J}_{\alpha|\beta}-B_\beta^\nu\boldsymbol{A}_\nu\times\overline{\boldsymbol{\omega}}_{,\alpha}\}=\bar{\varepsilon}^{\alpha\beta}\{[\varkappa_{\lambda\alpha|\beta}+B_\alpha^\nu e_{\nu\lambda|\beta}-B_{\lambda\beta}\varkappa_{3\alpha}+B_\beta^\nu e_{\lambda\alpha|\nu}-B_\beta^\nu e_{\nu\alpha|\lambda}]\boldsymbol{A}^\lambda\\
&\qquad\qquad+[\varkappa_{3\alpha|\beta}-B_\alpha^\nu\gamma_{\nu|\beta}]\boldsymbol{A}^3\}.
\end{aligned}
\tag{6.34}
$$

Expressions for $\overline{\boldsymbol{\omega}}$ and $\overline{\boldsymbol{\omega}}_{,\alpha}$ are listed in (6.35).

Formulae G

$$
\begin{aligned}
\overline{\boldsymbol{\omega}}&=-\tfrac{1}{2}\bar{\varepsilon}^{\lambda\nu}\gamma_{[\lambda\nu]}\boldsymbol{A}_3-\bar{\varepsilon}^{\lambda\nu}\beta_\nu\boldsymbol{A}_\lambda,\\
\overline{\boldsymbol{\omega}}_{,\alpha}&=[-\tfrac{1}{2}\bar{\varepsilon}^{\lambda\nu}\gamma_{[\lambda\nu]|\alpha}-\bar{\varepsilon}^{\lambda\nu}\beta_\nu B_{\lambda\alpha}]\boldsymbol{A}_3+[+\tfrac{1}{2}\bar{\varepsilon}^{\lambda\nu}\gamma_{[\lambda\nu]}B_\alpha^\gamma-\bar{\varepsilon}^{\gamma\nu}\beta_{\nu|\alpha}]\boldsymbol{A}_\gamma\\
&=\bar{\varepsilon}^{\nu\lambda}e_{\lambda\alpha|\nu}\boldsymbol{A}_3+\bar{\varepsilon}^{\gamma\nu}(\gamma_{\nu|\alpha}-\varrho_{\nu\alpha}-B_\alpha^\sigma e_{\sigma\nu})\boldsymbol{A}_\gamma\\
&=\bar{\varepsilon}^{\nu\lambda}e_{\lambda\alpha|\nu}\boldsymbol{A}_3+\bar{\varepsilon}^{\gamma\nu}(\gamma_{\nu|\alpha}-\varkappa_{\nu\alpha}-B_{\nu\alpha}\gamma_3-B_\alpha^\sigma e_{\sigma\nu})\boldsymbol{A}_\gamma,\\
\bar{\varepsilon}^{\alpha\beta}\overline{\boldsymbol{\omega}}_{|\alpha\beta}&=\bar{\varepsilon}^{\alpha\beta}\{\bar{\varepsilon}^{\nu\lambda}e_{\lambda\alpha|\nu\beta}+B_{\tau\beta}\bar{\varepsilon}^{\tau\nu}(\gamma_{\nu|\alpha}-\varkappa_{\nu\alpha}-B_{\nu\alpha}\gamma_3-B_\alpha^\sigma e_{\sigma\nu})\}\boldsymbol{A}_3\\
&\quad+\bar{\varepsilon}^{\alpha\beta}\{\bar{\varepsilon}^{\tau\nu}[\gamma_{\nu|\alpha\beta}-\varkappa_{\nu\alpha|\beta}-B_{\nu\alpha}\gamma_{3|\beta}-B_\alpha^\sigma e_{\sigma\nu|\beta}]-\bar{\varepsilon}^{\nu\lambda}B_\beta^\tau e_{\lambda\alpha|\nu}\}\boldsymbol{A}_\tau.
\end{aligned}
\tag{6.35}
$$

Before turning our attention to a derivation of compatibility equations, we make one further observation regarding the kinematic measures obtained in this section. The strain measures (6.18) or an equivalent set given by (6.24) when $\boldsymbol{D}=\boldsymbol{A}_3$, should be unaffected by infinitesimal rigid body displacement of the Cosserat surface. To show this, we first introduce the notations

$$
{}_0\boldsymbol{R}=\boldsymbol{R}({}_0\theta^\alpha),\qquad {}_0\boldsymbol{D}=\boldsymbol{D}({}_0\theta^\alpha),\qquad {}_0\boldsymbol{u}=\boldsymbol{u}({}_0\boldsymbol{R}),\qquad {}_0\boldsymbol{\delta}=\boldsymbol{\delta}({}_0\boldsymbol{R}),
\tag{6.36}
$$

where ${}_0\boldsymbol{R}$ and ${}_0\boldsymbol{D}$ stand for the position vector and the director of an arbitrary reference point of $\mathscr{S}$ while $(6.36)_{3,4}$ are abbreviations for the infinitesimal displacement and the infinitesimal director displacement at ${}_0\boldsymbol{R}$. Now, in order to obtain the appropriate expressions for $\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{R})$ and $\boldsymbol{\delta}=\boldsymbol{\delta}(\boldsymbol{R})$ due to purely *infinitesimal rigid body displacements* of the Cosserat surface, we only need to consider a special case of (5.36) in which the reference values of position and director are specified to be $\boldsymbol{R}$ and $\boldsymbol{D}$. With this proviso, it follows from (5.36) that under rigid body displacement alone the position $\boldsymbol{r}$ and the director $\boldsymbol{d}$ can be written as

$$
\boldsymbol{r}=\boldsymbol{C}+Q\boldsymbol{R},\qquad \boldsymbol{d}=Q\boldsymbol{D},
\tag{6.37}
$$

where the contributions corresponding to $\boldsymbol{r}_0^+$ and $\boldsymbol{r}_0$ in (5.36) have been absorbed into $\boldsymbol{C}$. Recalling $(6.1)_1$–$(6.2)_1$ and the earlier linearization procedure adopted in this section, we have to $O(\varepsilon)$:

$$\boldsymbol{u} = \boldsymbol{r} - \boldsymbol{R} = \boldsymbol{C} + {}_0Q\boldsymbol{R}, \qquad \boldsymbol{\delta} = {}_0Q\boldsymbol{D}, \tag{6.38}$$

$$\boldsymbol{C} = O(\varepsilon), \qquad {}_0Q = Q - I = O(\varepsilon). \tag{6.39}$$

Using the notations of $(6.36)_{3,4}$, the expressions (6.38) when evaluated at the reference point ${}_0\boldsymbol{R}$ yield

$${}_0\boldsymbol{u} = \boldsymbol{C} + {}_0Q\,{}_0\boldsymbol{R}, \qquad {}_0\boldsymbol{\delta} = {}_0Q\,{}_0\boldsymbol{D}. \tag{6.40}$$

By subtraction, from (6.38) and (6.40) follow

$$\boldsymbol{u} = {}_0\boldsymbol{u} + {}_0Q(\boldsymbol{R} - {}_0\boldsymbol{R}), \qquad \boldsymbol{\delta} = {}_0\boldsymbol{\delta} + {}_0Q(\boldsymbol{D} - {}_0\boldsymbol{D}), \tag{6.41}$$

as the displacement and the director displacement relative to those at ${}_0\boldsymbol{R}$. In (6.38)–(6.41), the vector $\boldsymbol{C}$ is a uniform infinitesimal rigid body translatory displacement while the second order tensor ${}_0Q$ represents a uniform infinitesimal rotation. In order to write (6.38) and (6.41) in alternative forms, we express the orthogonality condition $(5.37)_1$ in terms of ${}_0Q$ defined by $(6.39)_2$. Since $(I + {}_0Q)^T = I + {}_0Q^T$,

$$QQ^T = (I + {}_0Q)(I + {}_0Q^T) = I + {}_0Q + {}_0Q^T + O(\varepsilon^2) = I.$$

It is easily seen from the last result that $(5.37)_1$ is satisfied to $O(\varepsilon)$ if ${}_0Q$ is a skew-symmetric tensor; hence, there exists an infinitesimal vector-valued function ${}_0\bar{\boldsymbol{\omega}}$ such that for any vector $\boldsymbol{V}$

$${}_0Q\boldsymbol{V} = {}_0\bar{\boldsymbol{\omega}} \times \boldsymbol{V}, \qquad {}_0Q = -{}_0Q^T \tag{6.42}$$

and ${}_0\bar{\boldsymbol{\omega}}$ can be interpreted as a uniform infinitesimal rigid body angular displacement vector. Using $(6.42)_1$, (6.38) and (6.41) can be written in more familiar forms

$$\boldsymbol{u} = \boldsymbol{C} + {}_0\bar{\boldsymbol{\omega}} \times \boldsymbol{R}, \qquad \boldsymbol{\delta} = {}_0\bar{\boldsymbol{\omega}} \times \boldsymbol{D} \tag{6.43}$$

and

$$\boldsymbol{u} = {}_0\boldsymbol{u} + {}_0\bar{\boldsymbol{\omega}} \times (\boldsymbol{R} - {}_0\boldsymbol{R}), \qquad \boldsymbol{\delta} = {}_0\boldsymbol{\delta} + {}_0\bar{\boldsymbol{\omega}} \times (\boldsymbol{D} - {}_0\boldsymbol{D}), \tag{6.44}$$

respectively. Moreover, since ${}_0\boldsymbol{u}$, ${}_0\boldsymbol{\delta}$ and ${}_0\bar{\boldsymbol{\omega}}$ are independent of surface coordinates (but may be functions of time), from (6.43) or (6.44) and (6.6) we have

$$\boldsymbol{u}_{,\alpha} = {}_0\bar{\boldsymbol{\omega}} \times \boldsymbol{A}_\alpha, \qquad \boldsymbol{\delta}_{,\alpha} = {}_0\bar{\boldsymbol{\omega}} \times \boldsymbol{D}_{,\alpha}, \qquad \boldsymbol{\beta} = {}_0\bar{\boldsymbol{\omega}} \times \boldsymbol{A}_3, \tag{6.45}$$

where in obtaining $(6.45)_3$ we have used the identity

$$[(\boldsymbol{V} \times \boldsymbol{A}_\lambda) \cdot \boldsymbol{A}_3]\,\boldsymbol{A}^\lambda = -[(\boldsymbol{V} \times \boldsymbol{A}_3) \cdot \boldsymbol{A}_\lambda]\,\boldsymbol{A}^\lambda = -\boldsymbol{V} \times \boldsymbol{A}_3 \tag{6.46}$$

which holds for any vector $\boldsymbol{V}$. It can be easily verified, with the help of (6.13) and (6.45), that the rigid body displacements (6.44) have no effect on the kinematic measures $e_{\alpha\beta}$, γ_i, $\varkappa_{i\alpha}$ in (6.24).[34]

η) *Compatibility equations.* We include here a relatively simple derivation of compatibility equations which provides both necessary and sufficient conditions for the existence of single-valued displacements $\boldsymbol{u}$ and $\boldsymbol{\delta}$. For the purpose of the

[34] It may be of interest to note here that in the older literature of linear shell theory (developed from the three-dimensional equations) some of the kinematic measures are not invariant under infinitesimal rigid body displacements and this, of course, affects the constitutive relations. Further remark on this is made in Sect. 21 A.

derivation at hand, we assume $\mathscr{S}$ to be a simply connected surface and suppose that the displacement ${}_0\boldsymbol{u}=\boldsymbol{u}({}_0\boldsymbol{R})$, ${}_0\boldsymbol{\delta}=\boldsymbol{\delta}({}_0\boldsymbol{R})$ and the rotation ${}_0\overline{\boldsymbol{\omega}}=\overline{\boldsymbol{\omega}}({}_0\boldsymbol{R})$ are known at some point ${}_0\boldsymbol{R}$ of this surface. We wish to determine the displacement $\boldsymbol{u}$ and the director displacement $\boldsymbol{\delta}$ at any other point $R'=R(\theta'^{\alpha})$ of $\mathscr{S}$ in terms of the known kinematic measures $e_{\alpha\beta}$, γ_i, $\varkappa_{i\alpha}$ (assumed to be at least twice continuously differentiable) by means of the line integrals

$$\boldsymbol{u}(\boldsymbol{R}')={}_0\boldsymbol{u}+\int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'}\frac{\partial\boldsymbol{u}}{\partial\theta^{\alpha}}\,d\theta^{\alpha},\qquad \boldsymbol{\delta}(\boldsymbol{R}')={}_0\boldsymbol{\delta}+\int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'}\frac{\partial\boldsymbol{\delta}}{\partial\theta^{\alpha}}\,d\theta^{\alpha}\tag{6.47}$$

over a continuous curve joining the points ${}_0\boldsymbol{R}$ and $\boldsymbol{R}'$. For simplicity, we carry out the derivation with reference to the kinematic measures in Formulae C which hold when the initial director[35] $\boldsymbol{D}=\boldsymbol{A}_3$.

Consider first $(6.47)_1$ which, with the use of $(6.34)_5$, can be written as

$$\boldsymbol{u}(\boldsymbol{R}')={}_0\boldsymbol{u}+\int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'}e_{\lambda\alpha}\boldsymbol{A}^{\lambda}\,d\theta^{\alpha}+\int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'}\overline{\boldsymbol{\omega}}\times\boldsymbol{A}_{\alpha}\,d\theta^{\alpha}.\tag{6.48}$$

The last integral in (6.48) after an integration by parts gives

$$\begin{aligned}\int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'}\overline{\boldsymbol{\omega}}\times\boldsymbol{A}_{\alpha}\,d\theta^{\alpha}&=\int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'}\overline{\boldsymbol{\omega}}\times d\boldsymbol{R}=\int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'}\overline{\boldsymbol{\omega}}\times d(\boldsymbol{R}-\boldsymbol{R}')\\&={}_0\overline{\boldsymbol{\omega}}\times(\boldsymbol{R}'-{}_0\boldsymbol{R})+\int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'}(\boldsymbol{R}-\boldsymbol{R}')\times\overline{\boldsymbol{\omega}}_{,\alpha}\,d\theta^{\alpha}.\end{aligned}\tag{6.49}$$

Hence, from (6.48)–(6.49),

$$\boldsymbol{u}(\boldsymbol{R}')={}_0\boldsymbol{u}+{}_0\overline{\boldsymbol{\omega}}\times(\boldsymbol{R}'-{}_0\boldsymbol{R})+\int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'}\boldsymbol{U}_{\alpha}\,d\theta^{\alpha}\tag{6.50}$$

and we have put

$$\boldsymbol{U}_{\alpha}=e_{\lambda\alpha}\boldsymbol{A}^{\lambda}+(\boldsymbol{R}-\boldsymbol{R}')\times\overline{\boldsymbol{\omega}}_{,\alpha},\tag{6.51}$$

which is a known function of the kinematic measures [see $(6.35)_2$]. Since the displacement $\boldsymbol{u}$ must be independent of the path of integration for simply connected surfaces, the integrand $\boldsymbol{U}_{\alpha}\,d\theta^{\alpha}$ must be an exact differential. A necessary and sufficient condition that the integrand in (6.50) be an exact differential is $\boldsymbol{U}_{1,2}=\boldsymbol{U}_{2,1}$. The latter condition can equivalently be stated as

$$\bar{\varepsilon}^{\alpha\beta}\,\boldsymbol{U}_{\alpha|\beta}=0,\tag{6.52}$$

since in the expression for the covariant derivative of $\boldsymbol{U}_{\alpha|\beta}$ the Christoffel symbol is symmetric in α, β. Further, since

$$\begin{aligned}\boldsymbol{A}_{\beta}\times\overline{\boldsymbol{\omega}}_{,\alpha}&=(\boldsymbol{A}_{\beta}\times\overline{\boldsymbol{\omega}})_{|\alpha}-B_{\beta\alpha}\boldsymbol{A}_3\times\overline{\boldsymbol{\omega}}\\&=-[\boldsymbol{u}_{,\beta}-e_{\lambda\beta}\boldsymbol{A}^{\lambda}]_{|\alpha}-B_{\beta\alpha}\boldsymbol{A}_3\times\overline{\boldsymbol{\omega}},\end{aligned}$$

[35] Our derivation of the compatibility equations differs from similar previous developments in the literature. A discussion of compatibility equations by direct approach is contained in GÜNTHER's paper [1961, *4*] which, as noted earlier, employs a restrictive kinematic measure. Compatibility equations (in terms of the kinematic measures in Formulae C), as necessary conditions for the existence of single-valued displacements $\boldsymbol{u}$ and $\boldsymbol{\delta}$, have been considered also by CROCHET [1967, *2*] but his development appears to be incomplete.

Compatibility equations are sometimes useful in the formulation of a class of boundary-value problems for shells with infinitesimal deformation. Their utility in the linear theory of elastic shells is somewhat similar to the use of compatibility equations in the formulation of two-dimensional problems (in terms of a stress function) in the classical theory of linear elasticity.

from (6.51) we have

$$\begin{aligned}\boldsymbol{U}_{\alpha|\beta} &= (e_{\lambda\alpha}\boldsymbol{A}^{\lambda})_{|\beta} + \boldsymbol{A}_{\beta}\times\overline{\boldsymbol{\omega}}_{|\alpha} + (\boldsymbol{R}-\boldsymbol{R}')\times\overline{\boldsymbol{\omega}}_{|\alpha\beta} \\ &= (e_{\lambda\alpha}\boldsymbol{A}^{\lambda})_{|\beta} + (e_{\lambda\beta}\boldsymbol{A}^{\lambda})_{|\alpha} - \boldsymbol{u}_{|\beta\alpha} - B_{\beta\alpha}\boldsymbol{A}_3\times\overline{\boldsymbol{\omega}} + (\boldsymbol{R}-\boldsymbol{R}')\times\overline{\boldsymbol{\omega}}_{|\alpha\beta}.\end{aligned}$$

Each of the third and fourth terms in the last equation, as well as the combination of the first and second term, is symmetric in α, β and vanishes identically when multiplied by $\bar{\varepsilon}^{\alpha\beta}$. Hence, application of (6.52) to the above expression for $\boldsymbol{U}_{\alpha|\beta}$ yields

$$(\boldsymbol{R}-\boldsymbol{R}')\times\bar{\varepsilon}^{\alpha\beta}\,\overline{\boldsymbol{\omega}}_{|\alpha\beta} = 0.$$

But, since the last equation must hold for an arbitrary choice of $\boldsymbol{R}'$ and therefore $(\boldsymbol{R}-\boldsymbol{R}')$, it follows that

$$\bar{\varepsilon}^{\alpha\beta}\,\overline{\boldsymbol{\omega}}_{|\alpha\beta} = 0. \tag{6.53}$$

We postpone examining the further implication of (6.53) and now turn our attention to $(6.47)_2$ which, with the use of $(6.34)_4$, can be written in the form

$$\boldsymbol{\delta}(\boldsymbol{R}') = {}_0\boldsymbol{\delta} + \int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'} \boldsymbol{J}_{\alpha}\,d\theta^{\alpha} + \int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'} \overline{\boldsymbol{\omega}}\times\boldsymbol{D}_{,\alpha}\,d\theta^{\alpha}, \tag{6.54}$$

where we have written $\boldsymbol{D}_{,\alpha}$ $(=\boldsymbol{A}_{3,\alpha})$ in place of $-B_{\alpha}^{\nu}\boldsymbol{A}_{\nu}$ for clarity. Again, by an integration by parts of the last integral in (6.54) and using the notation $\boldsymbol{D}'=\boldsymbol{D}(\boldsymbol{R}')$,

$$\begin{aligned}\int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'} \overline{\boldsymbol{\omega}}\times\boldsymbol{D}_{,\alpha}\,d\theta^{\alpha} &= \int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'} \overline{\boldsymbol{\omega}}\times d\boldsymbol{D} = \int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'} \overline{\boldsymbol{\omega}}\times d(\boldsymbol{D}-\boldsymbol{D}') \\ &= {}_0\overline{\boldsymbol{\omega}}\times(\boldsymbol{D}'-{}_0\boldsymbol{D}) + \int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'} (\boldsymbol{D}-\boldsymbol{D}')\times\overline{\boldsymbol{\omega}}_{,\alpha}\,d\theta^{\alpha}\end{aligned} \tag{6.55}$$

and hence (6.54) becomes

$$\boldsymbol{\delta}(\boldsymbol{R}') = {}_0\boldsymbol{\delta} + {}_0\overline{\boldsymbol{\omega}}\times(\boldsymbol{D}'-{}_0\boldsymbol{D}) + \int_{{}_0\boldsymbol{R}}^{\boldsymbol{R}'} \boldsymbol{V}_{\alpha}\,d\theta^{\alpha}, \tag{6.56}$$

where we have put

$$\boldsymbol{V}_{\alpha} = \boldsymbol{J}_{\alpha} + (\boldsymbol{D}-\boldsymbol{D}')\times\overline{\boldsymbol{\omega}}_{,\alpha}. \tag{6.57}$$

Again, since the director displacement must be independent of the path of integration for a simply connected surface, the integrand $\boldsymbol{V}_{\alpha}\,d\theta^{\alpha}$ must be an exact differential. Hence, $\boldsymbol{V}_{1,2}=\boldsymbol{V}_{2,1}$ which is a necessary and sufficient condition for the integrand in (6.56) to be an exact differential. But the latter condition, similar to (6.52), can also be written as

$$\bar{\varepsilon}^{\alpha\beta}\,\boldsymbol{V}_{\alpha|\beta} = 0. \tag{6.58}$$

From (6.57),

$$\boldsymbol{V}_{\alpha|\beta} = \boldsymbol{J}_{\alpha|\beta} - B_{\beta}^{\nu}\boldsymbol{A}_{\nu}\times\overline{\boldsymbol{\omega}}_{,\alpha} + (\boldsymbol{D}-\boldsymbol{D}')\times\overline{\boldsymbol{\omega}}_{|\alpha\beta}.$$

But, in view of (6.53), application of (6.58) to the last result yields

$$\bar{\varepsilon}^{\alpha\beta}\{\boldsymbol{J}_{\alpha|\beta} - B_{\beta}^{\nu}\boldsymbol{A}_{\nu}\times\boldsymbol{\omega}_{,\alpha}\} = 0. \tag{6.59}$$

The conditions (6.53) and (6.59) are primitive forms of equations of compatibility. By taking the scalar products of these two vector equations with $\boldsymbol{A}^{\gamma}$ and $\boldsymbol{A}_3$ and using the appropriate expressions in (6.34)–(6.35), we readily deduce the following equations:

$$\begin{aligned}\bar{\varepsilon}^{\alpha\beta}[\bar{\varepsilon}^{\gamma\nu}(\varkappa_{\nu\alpha|\beta} + B_{\alpha}^{\sigma}\,e_{\sigma\nu|\beta} - \gamma_{\nu|\alpha\beta} + B_{\nu\alpha}\,\gamma_{3|\beta}) + \bar{\varepsilon}^{\nu\lambda}\,B_{\beta}^{\gamma}\,e_{\lambda\alpha|\nu}] &= 0, \\ \bar{\varepsilon}^{\alpha\beta}\,\bar{\varepsilon}^{\lambda\nu}[e_{\alpha\lambda|\nu\beta} - B_{\nu\beta}(\varkappa_{\lambda\alpha} + B_{\alpha}^{\sigma}\,e_{\sigma\lambda} - \gamma_{\lambda|\alpha} + B_{\lambda\alpha}\,\gamma_3 = 0])\end{aligned} \tag{6.60}$$

and

$$\bar{\varepsilon}^{\alpha\beta}[\gamma_{\nu|\alpha\beta} + B_{\nu\alpha} B^{\sigma}_{\beta} \gamma_{\sigma}] = 0, \qquad \bar{\varepsilon}^{\alpha\beta}(\varkappa_{3\alpha|\beta} - B^{\nu}_{\alpha} \gamma_{\nu|\beta}) = 0, \tag{6.61}$$

where $(6.61)_1$ is obtained with the help of $(6.60)_1$ and is equivalent to that resulting from (6.59) and (6.34). The two sets of compatibility equations $(6.60)_{1,2}$ and $(6.61)_{1,2}$ consist of six equations expressed in terms of the kinematic measures $e_{\lambda\alpha}$, $\varkappa_{i\lambda}$, γ_i. Alternatively, they can also be expressed in terms of the measures $e_{\lambda\alpha}$, $\varrho_{i\lambda}$ and γ_i [see (6.24)] but we do not record these.

We examine now the reduction of (6.60)–(6.61) for the restricted theory whose kinematic measures are summarized in (6.27). Thus, if we put $\gamma_i = 0$, the two equations in (6.61) vanish identically and $(6.60)_{1,2}$ reduce to

$$\begin{gathered} \bar{\varepsilon}^{\alpha\beta}[\bar{\varepsilon}^{\gamma\nu}(\varrho_{(\nu\alpha)|\beta} + B^{\sigma}_{\alpha} e_{\sigma\nu|\beta}) + \bar{\varepsilon}^{\nu\lambda} B^{\gamma}_{\nu} e_{\lambda\alpha|\beta}] = 0, \\ \bar{\varepsilon}^{\alpha\beta} \bar{\varepsilon}^{\lambda\nu}[e_{\alpha\lambda|\nu\beta} - B_{\nu\beta}(\varrho_{(\lambda\alpha)} + B^{\sigma}_{\alpha} e_{\sigma\lambda})] = 0. \end{gathered} \tag{6.62}$$

The fact that $\varrho_{\nu\alpha}$ is symmetric in the restricted theory has been explicitly indicated in (6.62).[36] In $(6.62)_1$, we have also used the identity $\bar{\varepsilon}^{\alpha\beta} \bar{\varepsilon}^{\nu\lambda} B^{\gamma}_{\nu} e_{\lambda\alpha|\beta} = \bar{\varepsilon}^{\alpha\beta} \bar{\varepsilon}^{\nu\lambda} B^{\gamma}_{\beta} e_{\lambda\alpha|\nu}$.

The foregoing derivation of the compatibility equations provides both necessary and sufficient conditions for the existence of single-valued displacements $\boldsymbol{u}$ and $\boldsymbol{\delta}$: Given the strain measures $e_{\alpha\gamma}$, γ_i, $\varkappa_{i\alpha}$ satisfying the conditions (6.52) and (6.58) or equivalently (6.60) and (6.61), then (6.50) and (6.56) determine the displacements $\boldsymbol{u}$ and $\boldsymbol{\delta}$ (corresponding to the given strain measures) at any point $\boldsymbol{R}'$ of the surface $\mathscr{S}$ uniquely to within rigid displacements of the forms (6.44). On the other hand, if the functions $e_{\alpha\gamma}$, γ_i, $\varkappa_{i\alpha}$ satisfy the differential equations in u_i and δ_i given by (6.24), then (6.52) and (6.58) or (6.60) and (6.61) are necessary conditions for the existence of single-valued displacements $\boldsymbol{u}$ and $\boldsymbol{\delta}$.

7. Kinematics of shells: II. Developments from the three-dimensional theory. We have already defined a shell-like body in Sect. 4. Here we derive the kinematics of such three-dimensional continua from the three-dimensional theory. Often, we employ (by choice) the same symbols which have been used previously in Sects. 5 and 6; but this need not be confusing, since the contents of this and the two previous sections are developed independently of each other.

α) *General kinematical results.* We begin our development of the kinematical results from the three-dimensional equations by assuming that the position vector $\boldsymbol{p}(\theta^{\alpha}, \xi, t)$ of a material point in the deformed shell is an analytic function of ξ in the region $\alpha < \xi < \beta$. Thus, recalling (4.6) and (4.9), $\boldsymbol{p}$ can be represented as[37]

$$\boldsymbol{p} = \boldsymbol{r}(\theta^{\alpha}, t) + \sum_{N=1}^{\infty} \xi^N \boldsymbol{d}_N(\theta^{\alpha}, t) \tag{7.1}$$

and its dual in a reference configuration is

$$\boldsymbol{P} = \boldsymbol{R}(\theta^{\alpha}) + \sum_{N=1}^{\infty} \xi^N \boldsymbol{D}_N(\theta^{\alpha}), \tag{7.2}$$

where $\boldsymbol{r}$ and $\boldsymbol{R}$, defined by (4.9) and (4.24), are the position vectors of the surface $\xi = 0$ in the deformed and the reference configurations, respectively, $\boldsymbol{d}_N$ are

[36] We may observe that these equations are of the same form as the corresponding compatibility equations in the classical theory of shells derived from the three-dimensional equations. Compare, for example, with Eq. (3.4) in [1963, *7*].

[37] The representation (7.1), along with the interpretation for $\boldsymbol{d}_N$ stated after (7.8), was introduced by GREEN, LAWS and NAGHDI [1968, *4*].

vector functions of θ^α, t and their reference values are denoted by $\boldsymbol{D}_N$, i.e.,

$$\boldsymbol{D}_N(\theta^\alpha) = \boldsymbol{d}_N(\theta^\alpha, 0). \tag{7.3}$$

We assume that the two series (7.1)–(7.2) may be differentiated as many times as required with respect to any of their variables, at least in the open region $\alpha < \xi < \beta$.

The velocity vector $\boldsymbol{v}^*$, of the three-dimensional continuum, at time t is given by

$$\boldsymbol{v}^* = \frac{d\boldsymbol{p}(\theta^\alpha, \xi, t)}{dt} = \dot{\boldsymbol{p}}(\theta^\alpha, \xi, t), \tag{7.4}$$

where a superposed dot denotes the material time derivative, holding $\theta^i = \{\theta^\alpha, \xi\}$ fixed. From (7.4) and (7.1), we have

$$\boldsymbol{v}^* = \boldsymbol{v} + \sum_{N=1}^{\infty} \xi^N \boldsymbol{w}_N, \tag{7.5}$$

where

$$\boldsymbol{v} = \dot{\boldsymbol{r}}, \quad \boldsymbol{w}_N = \dot{\boldsymbol{d}}_N. \tag{7.6}$$

We recall that when the motion of $\mathscr{B}$ differs from (4.6) only by superposed rigid body motions, the position $\boldsymbol{p}^+$ (using a by now familiar notation) has the form

$$\boldsymbol{p}^+ = \boldsymbol{p}^+(\theta^i, t') = \boldsymbol{p}_0^+(t') + Q(t)\left[\boldsymbol{p}(\theta^i, t) - \boldsymbol{p}_0(t)\right], \tag{7.7}$$

where Q is a proper orthogonal tensor function of time which satisfies the conditions (5.37). Since under superposed rigid body motions the position vector $\boldsymbol{r}^+$ of the surface $\mathfrak{s}$ transforms by a formula of the form $(5.36)_1$ and since

$$\boldsymbol{p}^+ - \boldsymbol{r}^+ = \sum_{N=1}^{\infty} \xi^N \boldsymbol{d}_N^+$$

by (7.1), it follows that the vector functions $\boldsymbol{d}_N^+$ must transform according to

$$\boldsymbol{d}_N^+(\theta^\alpha, t) = Q(t)\, \boldsymbol{d}_N(\theta^\alpha, t) \tag{7.8}$$

and hence remain unchanged in magnitude. We may, therefore, call the vectors $\boldsymbol{d}_N$ *directors* and $\boldsymbol{w}_N$ *director velocities*. Also, for reasons that will become apparent later, we introduce the notations

$$\boldsymbol{d} = \boldsymbol{d}_1, \quad \boldsymbol{D} = \boldsymbol{D}_1, \quad \boldsymbol{w} = \boldsymbol{w}_1. \tag{7.9}$$

By $(4.7)_1$ and (7.1), the base vectors $\boldsymbol{g}_i$ can be written as

$$\boldsymbol{g}_\alpha = \boldsymbol{a}_\alpha + \sum_{N=1}^{\infty} \xi^N \frac{\partial \boldsymbol{d}_N}{\partial \theta^\alpha}, \quad \boldsymbol{g}_3 = \sum_{N=1}^{\infty} N\, \xi^{N-1} \boldsymbol{d}_N, \tag{7.10}$$

where $\boldsymbol{a}_\alpha$ are the base vectors of the surface $\mathfrak{s}$ (i.e., $\xi = 0$ in the deformed configuration) defined by (4.10). The unit normal $\boldsymbol{a}_3$ to $\mathfrak{s}$ and the first and the second fundamental forms of $\mathfrak{s}$ are given by (4.11)–(4.13). The base vectors $\boldsymbol{g}_i(\theta^\alpha, \xi, t)$ when evaluated on the surface $\mathfrak{s}$ reduce to

$$\boldsymbol{g}_\alpha(\theta^\gamma, 0, t) = \boldsymbol{a}_\alpha(\theta^\gamma, t), \quad \boldsymbol{g}_3(\theta^\gamma, 0, t) = \boldsymbol{d}(\theta^\gamma, t), \tag{7.11}$$

where $\boldsymbol{d}$ is defined by $(7.9)_1$. The restriction (4.8) holds for all time and all values of θ^i. In particular, it is valid for $\xi = 0$ so that by (7.11) we also have

$$[\boldsymbol{a}_1\, \boldsymbol{a}_2\, \boldsymbol{d}] > 0. \tag{7.12}$$

We now introduce some additional kinematical quantities. Let the three-dimensional vector fields $\boldsymbol{d}_N$ and their partial derivatives with respect to θ^α be referred to the base vectors $\boldsymbol{a}_i$. Then,[38]

$$\begin{aligned}
&\boldsymbol{d}_N = d_{Ni}\,\boldsymbol{a}^i = d_{N\cdot}^{\;i}\,\boldsymbol{a}_i, \quad d_{N\cdot}^{\;\gamma} = a^{\gamma\beta}\, d_{N\beta},\\
&d_{N\cdot}^{\;3} = d_{N3}, \qquad \frac{\partial \boldsymbol{d}_N}{\partial\theta^\alpha} = \lambda_{Ni\alpha}\,\boldsymbol{a}^i \qquad (N \geqq 2),\\
&\boldsymbol{d}_1 = \boldsymbol{d} = d_i\,\boldsymbol{a}^i = d^i\,\boldsymbol{a}_i, \quad d^\gamma = a^{\gamma\beta}\, d_\beta,\\
&d^3 = d_3, \qquad \frac{\partial \boldsymbol{d}}{\partial\theta^\alpha} = \lambda_{1i\alpha}\,\boldsymbol{a}^i = \lambda_{i\alpha}\,\boldsymbol{a}^i.
\end{aligned} \tag{7.13}$$

By application of the general formula (5.5), the components $\lambda_{Ni\alpha}$ are

$$\begin{aligned}
&\lambda_{N\gamma\alpha} = \boldsymbol{a}_\gamma \cdot \boldsymbol{d}_{N,\alpha} = d_{N\gamma|\alpha} - b_{\gamma\alpha}\, d_{N3}, \qquad \lambda_{N\cdot\alpha}^{\;\gamma} = a^{\gamma\beta}\,\lambda_{N\beta\alpha},\\
&\lambda_{N3\alpha} = \boldsymbol{a}_3 \cdot \boldsymbol{d}_{N,\alpha} = d_{N3,\alpha} + b_\alpha^\gamma\, d_{N\gamma}, \qquad \lambda_{N\cdot\alpha}^{\;3} = \lambda_{N3\alpha}
\end{aligned} \tag{7.14}$$

and

$$\begin{aligned}
&\lambda_{\gamma\alpha} = \lambda_{1\gamma\alpha} = d_{\gamma|\alpha} - b_{\gamma\alpha}\, d_3, \qquad \lambda_{\cdot\alpha}^{\gamma} = a^{\gamma\beta}\,\lambda_{\beta\alpha},\\
&\lambda_{3\alpha} = \lambda_{13\alpha} = d_{3,\alpha} + b_\alpha^\gamma\, d_\gamma, \qquad \lambda_{\cdot\alpha}^{3} = \lambda_{3\alpha}
\end{aligned} \tag{7.15}$$

for $N=1$, where a vertical bar denotes covariant differentiation with respect to $a_{\alpha\beta}$. Also, in anticipation of certain results to be obtained presently, we record the expressions

$$\begin{aligned}
&\boldsymbol{d}_N \cdot \boldsymbol{d}_M = d_{N\cdot}^{\;\gamma}\, d_{M\gamma} + \sigma_{NM}, \qquad \sigma_{NM} = d_{N\cdot}^{\;3}\, d_{M3},\\
&\boldsymbol{d}_N \cdot \boldsymbol{d}_{M,\alpha} = d_{N\cdot}^{\;\gamma}\,\lambda_{M\gamma\alpha} + \sigma_{NM\alpha}, \qquad \sigma_{NM\alpha} = d_{N\cdot}^{\;3}\,\lambda_{M3\alpha},\\
&\boldsymbol{d}_{N,\alpha} \cdot \boldsymbol{d}_{M,\beta} = \lambda_{N\cdot\alpha}^{\;\gamma}\,\lambda_{M\gamma\beta} + \sigma_{NM\alpha\beta}, \qquad \sigma_{NM\alpha\beta} = \lambda_{N\cdot\alpha}^{\;3}\,\lambda_{M3\beta}
\end{aligned} \tag{7.16}$$

and

$$\begin{aligned}
&\boldsymbol{d} \cdot \boldsymbol{d} = \boldsymbol{d}_1 \cdot \boldsymbol{d}_1 = d^\gamma\, d_\gamma + \sigma, \qquad \sigma = (d_3)^2,\\
&\boldsymbol{d} \cdot \boldsymbol{d}_{,\alpha} = \boldsymbol{d}_1 \cdot \boldsymbol{d}_{1,\alpha} = d^\gamma\,\lambda_{\gamma\alpha} + \sigma_\alpha, \qquad \sigma_\alpha = d^3\,\lambda_{3\alpha},\\
&\boldsymbol{d}_{,\alpha} \cdot \boldsymbol{d}_{,\beta} = \boldsymbol{d}_{1,\alpha} \cdot \boldsymbol{d}_{1,\beta} = \lambda_{\cdot\alpha}^{\gamma}\,\lambda_{\gamma\beta} + \sigma_{\alpha\beta}, \qquad \sigma_{\alpha\beta} = \lambda_{\cdot\alpha}^{3}\,\lambda_{3\beta}
\end{aligned} \tag{7.17}$$

for $N, M = 1$.

We further introduce the kinematic variables

$$2e_{\alpha\beta} = a_{\alpha\beta} - A_{\alpha\beta}, \tag{7.18}$$

$$\begin{aligned}
&\varkappa_{Ni\alpha} = \lambda_{Ni\alpha} - \Lambda_{Ni\alpha}, \qquad \gamma_{Ni} = d_{Ni} - D_{Ni},\\
&\varkappa_{i\alpha} = \varkappa_{1i\alpha} = \lambda_{i\alpha} - \Lambda_{i\alpha}, \quad \gamma_i = \gamma_{1i} = d_i - D_i,
\end{aligned} \tag{7.19}$$

where D_{Ni} and $\Lambda_{Ni\alpha}$, the reference values of d_{Ni} and $\lambda_{Ni\alpha}$, are given by

$$\begin{aligned}
&D_{Ni} = \boldsymbol{A}_i \cdot \boldsymbol{D}_N, \qquad D_i = D_{1i} = \boldsymbol{A}_i \cdot \boldsymbol{D},\\
&\Lambda_{Ni\alpha} = \boldsymbol{A}_i \cdot \frac{\partial \boldsymbol{D}_N}{\partial\theta^\alpha}, \quad \Lambda_{i\alpha} = \Lambda_{1i\alpha} = \boldsymbol{A}_i \cdot \frac{\partial \boldsymbol{D}}{\partial\theta^\alpha},
\end{aligned} \tag{7.20}$$

$\boldsymbol{D}$ is defined by $(7.9)_2$ and $\boldsymbol{A}_i = \{\boldsymbol{A}_\gamma, \boldsymbol{A}_3\}$ are the base vectors and the unit normal of the surface $\mathfrak{S}$ (i.e., $\xi = 0$ in the reference configuration). We observe that the kinematic quantities d_{Ni} and $\lambda_{Ni\alpha}$, as well as those in (7.18)–(7.19), remain unaltered under superposed rigid body motions. This can be easily verified in a

[38] The kinematic variables (7.13)–(7.15) were employed by GREEN, LAWS and NAGHDI [1968, *4*] and GREEN and NAGHDI [1970, *2*] but the variables (7.16)–(7.17) were not explicitly introduced in these papers. The latter variables, which appear in (7.25)–(7.26), will subsequently bear on the structure of the constitutive equations derived in Sect. 17 from the three-dimensional theory.

manner similar to the invariance conditions expressed in (5.54). We record here the relative measures

$$\begin{aligned}\boldsymbol{d}_N\cdot\boldsymbol{d}_M-\boldsymbol{D}_N\cdot\boldsymbol{D}_M&=d_{N}^{\cdot\gamma}\,d_{M\gamma}-D_{N}^{\cdot\gamma}\,D_{M\gamma}+s_{NM},\\ \boldsymbol{d}_N\cdot\boldsymbol{d}_{M,\alpha}-\boldsymbol{D}_N\cdot\boldsymbol{D}_{M,\alpha}&=d_{N}^{\cdot\gamma}\,\lambda_{M\gamma\alpha}-D_{N}^{\cdot\gamma}\,\Lambda_{M\gamma\alpha}+s_{NM\alpha},\\ \boldsymbol{d}_{N,\alpha}\cdot\boldsymbol{d}_{M,\beta}-\boldsymbol{D}_{N,\alpha}\cdot\boldsymbol{D}_{M,\beta}&=\lambda_{N\cdot\alpha}^{\cdot\gamma}\,\lambda_{M\gamma\beta}-\Lambda_{N\cdot\alpha}^{\cdot\gamma}\,\Lambda_{M\gamma\beta}+s_{NM\alpha\beta},\end{aligned}\tag{7.21}$$

where

$$\begin{gathered}s_{NM}=d_{N}^{\cdot 3}\,d_{M3}-D_{N}^{\cdot 3}\,D_{M3},\qquad s_{NM\alpha}=d_{N}^{\cdot 3}\,\lambda_{M3\alpha}-D_{N}^{\cdot 3}\,\Lambda_{M3\alpha},\\ s_{NM\alpha\beta}=\lambda_{N\cdot\alpha}^{\cdot 3}\,\lambda_{M3\beta}-\Lambda_{N\cdot\alpha}^{\cdot 3}\,\Lambda_{M3\beta}\end{gathered}\tag{7.22}$$

and note that the expressions corresponding to (7.21)–(7.22) for $N, M=1$, in the notations of $(7.9)_1$, are:

$$\begin{aligned}\boldsymbol{d}\cdot\boldsymbol{d}-\boldsymbol{D}\cdot\boldsymbol{D}&=(d^\gamma\,d_\gamma-D^\gamma\,D_\gamma)+s,\\ \boldsymbol{d}\cdot\boldsymbol{d}_{,\alpha}-\boldsymbol{D}\cdot\boldsymbol{D}_{,\alpha}&=(d^\gamma\,\lambda_{\gamma\alpha}-D^\gamma\,\Lambda_{\gamma\alpha})+s_\alpha,\\ \boldsymbol{d}_{,\alpha}\cdot\boldsymbol{d}_{,\beta}-\boldsymbol{D}_{,\alpha}\cdot\boldsymbol{D}_{,\beta}&=(\lambda_{\cdot\alpha}^{\gamma}\,\lambda_{\gamma\beta}-\Lambda_{\cdot\alpha}^{\gamma}\,\Lambda_{\gamma\beta})+s_{\alpha\beta}\end{aligned}\tag{7.23}$$

and

$$s=(d_3)^2-(D_3)^2,\qquad s_\alpha=d^3\,\lambda_{3\alpha}-D^3\,\Lambda_{3\alpha},\qquad s_{\alpha\beta}=\lambda_{\cdot\alpha}^{3}\,\lambda_{3\beta}-\Lambda_{\cdot\alpha}^{3}\,\Lambda_{3\beta}.\tag{7.24}$$

For later reference, it is desirable to calculate the components of a three-dimensional strain measure in terms of the variables (7.13) referred to the base vectors $\boldsymbol{a}^i$. For this purpose, using (7.10), we first record the expressions for the components g_{ij} as follows:

$$\begin{aligned}g_{\alpha\beta}&=\Big(\boldsymbol{a}_\alpha+\sum_{N=1}^{\infty}\xi^N\frac{\partial\boldsymbol{d}_N}{\partial\theta^\alpha}\Big)\cdot\Big(\boldsymbol{a}_\beta+\sum_{M=1}^{\infty}\xi^M\frac{\partial\boldsymbol{d}_M}{\partial\theta^\beta}\Big)\\ &=a_{\alpha\beta}+\sum_{N=1}^{\infty}\xi^N\Big(\boldsymbol{a}_\beta\cdot\frac{\partial\boldsymbol{d}_N}{\partial\theta^\alpha}+\boldsymbol{a}_\alpha\cdot\frac{\partial\boldsymbol{d}_N}{\partial\theta^\beta}\Big)+\sum_{P=2}^{\infty}\xi^P\sum_{M=1}^{P-1}\frac{\partial\boldsymbol{d}_{P-M}}{\partial\theta^\alpha}\cdot\frac{\partial\boldsymbol{d}_M}{\partial\theta^\beta}\\ &=a_{\alpha\beta}+\xi\Big(\boldsymbol{a}_\beta\cdot\frac{\partial\boldsymbol{d}_1}{\partial\theta^\alpha}+\boldsymbol{a}_\alpha\cdot\frac{\partial\boldsymbol{d}_1}{\partial\theta^\beta}\Big)\\ &\quad+\sum_{P=2}^{\infty}\xi^P\Big[\boldsymbol{a}_\beta\cdot\frac{\partial\boldsymbol{d}_P}{\partial\theta^\alpha}+\boldsymbol{a}_\alpha\cdot\frac{\partial\boldsymbol{d}_P}{\partial\theta^\beta}+\sum_{M=1}^{P-1}\Big(\frac{\partial\boldsymbol{d}_{P-M}}{\partial\theta^\alpha}\cdot\frac{\partial\boldsymbol{d}_M}{\partial\theta^\beta}\Big)\Big],\\ g_{\alpha3}&=\Big(\boldsymbol{a}_\alpha+\sum_{N=1}^{\infty}\xi^N\frac{\partial\boldsymbol{d}_N}{\partial\theta^\alpha}\Big)\cdot\Big(\sum_{M=1}^{\infty}M\,\xi^{M-1}\,\boldsymbol{d}_M\Big)\\ &=\boldsymbol{a}_\alpha\cdot\boldsymbol{d}_1+\sum_{P=2}^{\infty}\xi^{P-1}\Big[P\,\boldsymbol{a}_\alpha\cdot\boldsymbol{d}_P+\sum_{M=1}^{P-1}M\,\frac{\partial\boldsymbol{d}_{P-M}}{\partial\theta^\alpha}\cdot\boldsymbol{d}_M\Big],\\ g_{33}&=\Big(\sum_{N=1}^{\infty}N\,\xi^{N-1}\,\boldsymbol{d}_N\Big)\cdot\Big(\sum_{M=1}^{\infty}M\,\xi^{M-1}\,\boldsymbol{d}_M\Big)\\ &=\sum_{P=2}^{\infty}\xi^{P-2}\Big(\sum_{M=1}^{P-1}(P-M)\,M\,\boldsymbol{d}_{P-M}\cdot\boldsymbol{d}_M\Big).\end{aligned}\tag{7.25}$$

A close examination of the series in each of the second expressions of $(7.25)_{1,2}$ reflects a particular form for the dependence of $g_{\alpha\beta}$ and $g_{\alpha3}$ on $\boldsymbol{d}_N$ and $\partial\boldsymbol{d}_N/\partial\theta^\alpha$. To see this more easily, we write the right-hand side of each of $(7.25)_{1,2,3}$ in expanded form indicating explicitly all terms involving only $\boldsymbol{d}_1$ and $\partial\boldsymbol{d}_1/\partial\theta^\alpha$.

Thus, with the notations of $(7.9)_1$, (7.15) and (7.17):

$$\begin{aligned} g_{\alpha\beta} &= a_{\alpha\beta} + \xi\left(\boldsymbol{a}_\beta \cdot \frac{\partial \boldsymbol{d}}{\partial \theta^\alpha} + \boldsymbol{a}_\alpha \cdot \frac{\partial \boldsymbol{d}}{\partial \theta^\beta}\right) + \xi^2 \frac{\partial \boldsymbol{d}}{\partial \theta^\alpha} \cdot \frac{\partial \boldsymbol{d}}{\partial \theta^\beta} + \cdots \\ &= a_{\alpha\beta} + \xi(\lambda_{\beta\alpha} + \lambda_{\alpha\beta}) + \xi^2(\lambda^{\nu}_{.\alpha}\,\lambda_{\nu\beta} + \sigma_{\alpha\beta}) + \cdots, \\ g_{\alpha 3} &= \boldsymbol{a}_\alpha \cdot \boldsymbol{d} + \xi\,\boldsymbol{d} \cdot \frac{\partial \boldsymbol{d}}{\partial \theta^\alpha} + \cdots \\ &= d_\alpha + \xi(d^\nu\,\lambda_{\nu\alpha} + \sigma_\alpha) + \cdots, \\ g_{33} &= \boldsymbol{d} \cdot \boldsymbol{d} + \cdots = d^\gamma\,d_\gamma + \sigma + \cdots, \end{aligned} \tag{7.26}$$

which readily reveal the manner that the components of $\boldsymbol{d} = \boldsymbol{d}_1$ and $\boldsymbol{d}_{,\alpha}$ contribute to the components of g_{ij}. Results similar to (7.25) for the components G_{ij} can be calculated with the help of (7.2). We now recall the formula[39]

$$\gamma^*_{ij} = \tfrac{1}{2}(\boldsymbol{g}_i \cdot \boldsymbol{g}_j - \boldsymbol{G}_i \cdot \boldsymbol{G}_j) = \tfrac{1}{2}(g_{ij} - G_{ij}), \tag{7.27}$$

where γ^*_{ij} are the covariant components of a strain measure in the three-dimensional (nonlinear) theory. Using (7.25) and the corresponding expressions for G_{ij}, the components of γ^*_{ij} may be expressed in the following forms:

$$\begin{aligned} 2\gamma^*_{\alpha\beta} &= 2e_{\alpha\beta} + \xi(\varkappa_{\beta\alpha} + \varkappa_{\alpha\beta}) \\ &\quad + \sum_{P=2}^{\infty} \xi^P \left[\varkappa_{P\beta\alpha} + \varkappa_{P\alpha\beta} + \sum_{M=1}^{P-1} (\boldsymbol{d}_{(P-M),\alpha} \cdot \boldsymbol{d}_{M,\beta} - \boldsymbol{D}_{(P-M),\alpha} \cdot \boldsymbol{D}_{M,\beta})\right], \\ 2\gamma^*_{\alpha 3} &= \gamma_\alpha + \sum_{P=2}^{\infty} \xi^{P-1}\left[P\gamma_{P\alpha} + \sum_{M=1}^{P-1} M(\boldsymbol{d}_{(P-M),\alpha} \cdot \boldsymbol{d}_M - \boldsymbol{D}_{(P-M),\alpha} \cdot \boldsymbol{D}_M)\right], \\ 2\gamma^*_{33} &= \sum_{P=2}^{\infty} \xi^{P-2}\left[\sum_{M=1}^{P-1}(P-M)\,M(\boldsymbol{d}_{(P-M)} \cdot \boldsymbol{d}_M - \boldsymbol{D}_{(P-M)} \cdot \boldsymbol{D}_M)\right]. \end{aligned} \tag{7.28}$$

Again, in order to give an idea regarding the manner of dependence of γ^*_{ij} on the components of $\boldsymbol{d}_N$ and $\partial \boldsymbol{d}_N/\partial\theta^\alpha$, we rewrite (7.28) in expanded form similar to (7.26) indicating explicitly only terms due to $\boldsymbol{d} = \boldsymbol{d}_1$, $\boldsymbol{d}_{1,\alpha} = \boldsymbol{d}_{,\alpha}$ and their reference values. Thus

$$\begin{aligned} 2\gamma^*_{\alpha\beta} &= 2e_{\alpha\beta} + \xi(\varkappa_{\beta\alpha} + \varkappa_{\alpha\beta}) + \xi^2[(\lambda^{\nu}_{.\alpha}\,\lambda_{\nu\beta} - \Lambda^{\nu}_{.\alpha}\,\Lambda_{\nu\beta}) + s_{\alpha\beta}] + \cdots, \\ 2\gamma^*_{\alpha 3} &= \gamma_\alpha + \xi[(d^\nu\,\lambda_{\nu\alpha} - D^\nu \Lambda_{\nu\alpha}) + s_\alpha] + \cdots, \\ 2\gamma^*_{33} &= (d^\gamma\,d_\gamma - D^\gamma D_\gamma) + s + \cdots, \end{aligned} \tag{7.29}$$

where all terms in (7.29) not explicitly recorded arise from $\boldsymbol{d}_N$ and $\boldsymbol{D}_N$ for $N \geqq 2$.

The kinematic variables (7.18)–(7.19) and (7.21)–(7.24) which occur in (7.28) are independent of ξ and thus can be referred to as two-dimensional variables. Given (7.1), the two-dimensional kinematic quantities may be regarded as an exact characterization of the kinematics of shells; however, (7.18)–(7.19) and (7.21)–(7.22) form an infinite set of variables and this is an undesirable feature of such characterizations. It is therefore clear that the introduction of suitable approximations is necessary in order to obtain useful measures of deformation for shells in line with the objective stated under (a) in (4.32). This requires additional elaboration but we postpone further comments and take up the question of suitable approximation later in this section and again in Chap. D.

[39] See, e.g., GREEN and ZERNA [1968, *9*].

β) *Some results valid in a reference configuration.* Before proceeding further, we dispose of some additional results which are independent of linearization, although their utility in a linear theory is particularly significant. First, we observe that the convected coordinates θ^i can always be so chosen in the reference configuration that $\boldsymbol{D}_N = 0$ for $N \geqq 2$. Hence, instead of (7.2), without loss in generality we may write the position vector in the reference configuration of $\mathscr{B}$ as

$$\boldsymbol{P} = \boldsymbol{R}(\theta^\alpha) + \xi\, \boldsymbol{D}(\theta^\alpha), \tag{7.30}$$

with $\boldsymbol{D}$ specified by

$$D_\alpha = 0, \quad D = D_3, \quad \boldsymbol{D} = D\boldsymbol{A}_3. \tag{7.31}$$

From (4.23) and (7.30)–(7.31)$_3$, it follows that the base vectors and the metric tensor in the (initial) reference configuration are

$$\begin{gathered} \boldsymbol{G}_\alpha = \boldsymbol{A}_\alpha + \xi\, \boldsymbol{D}_{,\alpha} = \nu_\alpha^\gamma \boldsymbol{A}_\gamma + \xi\, D_{,\alpha} \boldsymbol{A}_3, \quad \boldsymbol{G}_3 = D\boldsymbol{A}_3, \\ G_{\alpha\beta} = \nu_\alpha^\gamma\, \nu_\beta^\delta\, A_{\gamma\delta} + \xi^2\, D_{,\alpha}\, D_{,\beta}, \quad G_{\alpha 3} = \xi D D_{,\alpha} = \tfrac{1}{2}\,\xi\,(D^2)_{,\alpha}, \quad G_{33} = D^2, \end{gathered} \tag{7.32}$$

where

$$\nu_\alpha^\gamma = \delta_\alpha^\gamma - \xi\, D\, B_\alpha^\gamma. \tag{7.33}$$

We note for later reference that

$$\nu = D \det(\nu_\alpha^\gamma) = \left(\frac{G}{A}\right)^{\frac{1}{2}} = D\,[1 - 2\xi\, DH + \xi^2\, D^2\, K], \tag{7.34}$$

with H and K being the mean curvature and the Gaussian curvature of the surface $\xi = 0$ in the reference configuration defined by

$$\begin{aligned} H &= \tfrac{1}{2} B_\alpha^\alpha, \\ K &= \det(B_\beta^\alpha) = A^{-1} \det(B_{\alpha\beta}) = B_1^1\, B_2^2 - B_2^1\, B_1^2. \end{aligned} \tag{7.35}$$

In view of (7.31) and our choice (7.30), the functions $\Lambda_{Ni\alpha}$ in (7.20) reduce to

$$\Lambda_{\beta\alpha} = \Lambda_{1\beta\alpha} = -D\, B_{\alpha\beta}, \quad \Lambda_{3\alpha} = \Lambda_{13\alpha} = D_{,\alpha}, \quad \Lambda_{Ni\alpha} = 0 \quad (N \geqq 2). \tag{7.36}$$

Since the convected coordinates θ^i may be identified with the normal coordinates (4.25) in the reference configuration of $\mathscr{B}$, the initial position vector can also be taken in the form (4.27) instead of (7.30). Recalling our notations for the base vectors and the metric tensor associated with the coordinates (4.25), we have

$$\begin{aligned} &\boldsymbol{G}'_\alpha = \mu_\alpha^\gamma\, \boldsymbol{A}_\gamma, \qquad \boldsymbol{G}'_3 = \boldsymbol{A}_3, \\ &G'_{\alpha\beta} = \mu_\alpha^\gamma\, \mu_\beta^\delta\, A_{\gamma\delta}, \quad G'_{\alpha 3} = 0, \quad G'_{33} = 1, \end{aligned} \tag{7.37}$$

where (4.27) has been used and where

$$\mu_\alpha^\gamma = \delta_\alpha^\gamma - \zeta\, B_\alpha^\gamma \tag{7.38}$$

and

$$\mu = \det(\mu_\alpha^\gamma) = \left(\frac{G'}{A}\right)^{\frac{1}{2}} = 1 - 2\zeta H + \zeta^2 K. \tag{7.39}$$

From comparison of (4.27) and (7.30) with $\boldsymbol{D}$ specified by (7.31), we have

$$y^\alpha = \theta^\alpha, \quad \zeta = D\,\xi \tag{7.40}$$

as the transformation relations between the coordinate system y^i or (4.25) and θ^i in the reference configuration. Moreover, under the transformations (7.40), from (7.37)–(7.39) and (7.32)–(7.34) follow the relations

$$\mu^\alpha_\beta = \nu^\alpha_\beta, \qquad \mu = \frac{\nu}{D}. \tag{7.41}$$

It may be noted here that the metric tensors G'_{ij} and G_{ij} become identical when evaluated on the surface $\zeta = 0$ (or $\xi = 0$) in the reference configuration and both reduce to (4.29).

With reference to the condition (4.20), recall that k is independent of time and suppose further that k is also independent of ξ. Then, for $k = k(\theta^\alpha)$, (4.20) gives

$$\int_\alpha^\beta k\,\xi\,d\xi = \frac{k}{2}(\beta^2 - \alpha^2) = 0 \Rightarrow \alpha = -\beta, \tag{7.42}$$

since $\alpha < 0$. By (4.21) and (7.34), the mass density in the reference configuration can be written in the form

$$\varrho_0^* = k\,G^{-\frac{1}{2}} = \frac{k}{DA^{\frac{1}{2}}[1 - 2\xi\,DH + \xi^2 D^2 K]}, \qquad k = k(\theta^\alpha). \tag{7.43}$$

In the case of an initially flat plate, for which both H and K vanish, the expression $(7.43)_1$ is independent of ξ and satisfies (4.20) and therefore (7.42) exactly. For initially curved shells, on the other hand, the mass density ϱ_0^* depends on ξ or ζ in view of $(7.40)_2$. In terms of the principal radii of curvature of the surface $\xi = 0$ in the (initial) reference configuration,

$$H = -\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \qquad K = \frac{1}{R_1 R_2} \tag{7.44}$$

and $(7.43)_1$ can be expressed as

$$\varrho_0^* = \frac{k}{DA^{\frac{1}{2}}\left[1 + \left(\frac{\zeta}{R_1} + \frac{\zeta}{R_2}\right) + \left(\frac{\zeta}{R_1}\right)\left(\frac{\zeta}{R_2}\right)\right]}, \tag{7.45}$$

where we have also used $(7.40)_2$. Recalling the notation of $(4.28)_2$, we may invoke (4.31) for sufficiently *thin* shells and neglect ζ/R in comparison with unity in the denominator in (7.45). Thus, if we approximate (7.45) by

$$\varrho_0^* \cong \frac{k}{DA^{\frac{1}{2}}}, \tag{7.46}$$

ϱ_0^* is independent of ζ (or ξ) and the condition (7.42) is also satisfied to the order of approximation used in the sense of (4.31).

With the above approximation for the initial mass density and with $\boldsymbol{P}$ specified by (7.30)–(7.31), it follows from $(7.42)_2$ that the boundary surfaces (4.14) can be regarded as symmetrically situated [approximately in the sense of (4.31)] about the surface $\xi = 0$ in the reference configuration of the initially curved shells. Hence, the limits of integration in such integrals as (4.17) and (4.20) can be replaced by $-\beta, \beta$. Also, in view of (4.29), from $(7.40)_2$ and $(7.42)_2$ follows

$$D\beta = \frac{h}{2}, \tag{7.47}$$

which relates β to D and h. In particular if $D = 1$ so that $(7.31)_3$ becomes $\boldsymbol{D} = \boldsymbol{A}_3$, the distinction between ξ and ζ in (7.30) and (4.27) disappears and we have

$$\begin{gathered} D = 1, \quad \xi = \zeta, \quad \beta = -\alpha = \frac{h}{2}, \\ \nu^\gamma_\alpha = \mu^\gamma_\alpha, \quad \mu = \nu. \end{gathered} \tag{7.48}$$

γ) *Linearized kinematics.* We now proceed to obtain the linearized version of the foregoing kinematical results. Since the linearization procedure is analogous to that employed in Sect. 6, details will not be given; but we note that a vertical bar, in the linearized expressions, will denote covariant differentiation with respect to $A_{\alpha\beta}$. Also, in the remainder of this section, instead of (7.2) we specify the position vector in the reference configuration of $\mathscr{B}$ by (7.30)–(7.31) so that the reference values $(7.20)_{1,3}$ for $N \geqq 2$ become[40]

$$D_{Ni} = 0, \quad \Lambda_{Ni\alpha} = 0 \quad (N \geqq 2). \tag{7.49}$$

Let

$$\boldsymbol{p} = \boldsymbol{P} + \varepsilon\,\boldsymbol{u}^*, \quad \boldsymbol{u}^* = u^{*i}\boldsymbol{A}_i, \quad \boldsymbol{v}^* = \varepsilon\,\dot{\boldsymbol{u}}^* \tag{7.50}$$

and put

$$\boldsymbol{u}^* = \boldsymbol{u}^*(\theta^\alpha, \xi, t) = \boldsymbol{u}(\theta^\alpha, t) + \sum_{N=1}^{\infty} \xi^N \boldsymbol{\delta}_N(\theta^\alpha, t). \tag{7.51}$$

Introduction of (7.51) into (7.50), together with (7.1) and (7.30), results in

$$\begin{aligned} &\boldsymbol{r} = \boldsymbol{R} + \varepsilon\,\boldsymbol{u}, && \boldsymbol{u} = u_i\boldsymbol{A}^i = u^i\boldsymbol{A}_i, && \boldsymbol{v} = \varepsilon\,\dot{\boldsymbol{u}}, \\ &\boldsymbol{d} = \boldsymbol{d}_1 = \boldsymbol{D} + \varepsilon\,\boldsymbol{\delta}, && \boldsymbol{\delta} = \boldsymbol{\delta}_1 = \delta_i\boldsymbol{A}^i, && \boldsymbol{w} = \boldsymbol{w}_1 = \varepsilon\,\dot{\boldsymbol{\delta}}, \\ &\boldsymbol{d}_N = \varepsilon\,\boldsymbol{\delta}_N, && \boldsymbol{\delta}_N = \delta_{Ni}\boldsymbol{A}^i, && \boldsymbol{w}_N = \varepsilon\,\dot{\boldsymbol{\delta}}_N \quad (N \geqq 2). \end{aligned} \tag{7.52}$$

The base vectors $\boldsymbol{G}_i$ are given by $(7.32)_{1,2}$ and the base vectors $\boldsymbol{g}_i$, by $(4.7)_1$, $(4.23)_1$ and (7.52), can be expressed as

$$\begin{aligned} \boldsymbol{g}_i &= \boldsymbol{G}_i + \varepsilon\,\boldsymbol{u}^*_{,i}, \\ \boldsymbol{g}_\alpha &= (\boldsymbol{A}_\alpha + \varepsilon\,\boldsymbol{u}_{,\alpha}) + \xi\frac{\partial}{\partial\theta^\alpha}(\boldsymbol{D} + \varepsilon\,\boldsymbol{\delta}) + \sum_{N=2}^{\infty}\xi^N\frac{\partial}{\partial\theta^\alpha}(\varepsilon\,\boldsymbol{\delta}_N), \\ \boldsymbol{g}_3 &= (\boldsymbol{D} + \varepsilon\,\boldsymbol{\delta}) + \sum_{N=2}^{\infty} N\,\xi^{N-1}(\varepsilon\,\boldsymbol{\delta}_N). \end{aligned} \tag{7.53}$$

The above formulae on the surface $\xi = 0$ become

$$\begin{aligned} \boldsymbol{a}_\alpha = \boldsymbol{g}_\alpha(\theta^\gamma, 0, t) &= \boldsymbol{G}_\alpha(\theta^\gamma, 0) + \varepsilon\,\boldsymbol{u}^*_{,\alpha}(\theta^\gamma, 0, t) \\ &= \boldsymbol{A}_\alpha(\theta^\gamma) + \varepsilon\,\boldsymbol{u}_{,\alpha}(\theta^\gamma, t), \\ \boldsymbol{d} = \boldsymbol{g}_3(\theta^\gamma, 0, t) &= \boldsymbol{G}_3(\theta^\gamma, 0) + \varepsilon\,\boldsymbol{u}^*_{,3}(\theta^\gamma, 0, t) \\ &= \boldsymbol{D}(\theta^\gamma) + \varepsilon\,\boldsymbol{\delta}(\theta^\gamma, t), \end{aligned} \tag{7.54}$$

in view of (4.9) and (7.11).

We say $(7.50)_1$–(7.51) characterize the infinitesimal motion of the three-dimensional continuum $\mathscr{B}$ if the magnitude of $\boldsymbol{u}^*$ and all its derivatives are bounded by 1 and if (6.3) holds. Also, we use the order symbol $O(\varepsilon^n)$ in the sense of (6.4). The components of the vector functions $\boldsymbol{u}$ and $\boldsymbol{\delta}_N$ in (7.52) are referred to the base vectors $\boldsymbol{A}_i$ or $\boldsymbol{A}^i$ and subsequently the components of $\boldsymbol{u}_{,\alpha}$ and $\boldsymbol{\delta}_{N,\alpha}$ will be defined with reference to $\boldsymbol{A}_i$. On the other hand, the components of the vector functions $\boldsymbol{d}_N$ and $\boldsymbol{d}_{N,\alpha}$ are defined with respect to the base vectors $\boldsymbol{a}_i$ or $\boldsymbol{a}^i$ [see (7.13)]. For this reason, in the sequel, we need to have the approximate expressions for $\boldsymbol{a}_i$ to $O(\varepsilon)$. We have already obtained the desired approximation for $\boldsymbol{a}_\alpha$ to $O(\varepsilon)$ in (7.54) and the corresponding expression for the unit normal

[40] As noted already, there is no loss of generality in writing the reference position vector in the form (7.30). Later, in order to simplify some of the calculations, we also take $D = 1$ but the various formulae between (7.50)–(7.57) hold with $\boldsymbol{D}$ specified by (7.31).

$\boldsymbol{a}_3$ to $\mathfrak{s}$ can be deduced from $(7.54)_1$ and $(4.11)_{1,2}$. Thus,

$$\begin{gathered} \boldsymbol{a}_3 = \boldsymbol{A}_3 + \varepsilon\boldsymbol{\beta} + O(\varepsilon^2), \\ \boldsymbol{\beta} = \beta_\alpha \boldsymbol{A}^\alpha = \beta^\alpha \boldsymbol{A}_\alpha, \qquad \beta_\alpha = -(u_{3,\alpha} + B_\alpha^\lambda u_\lambda), \end{gathered} \tag{7.55}$$

which together with $(7.54)_1$ can be used to rewrite (7.18)–(7.19) in terms of $\varepsilon\boldsymbol{u}$, $\varepsilon\boldsymbol{\beta}$, $\varepsilon\boldsymbol{\delta}_N$ and their derivatives. For example, the functions γ_{Ni} in $(7.19)_{2,4}$ can be written as

$$\begin{gathered} \gamma_\alpha = \varepsilon(\boldsymbol{A}_\alpha \cdot \boldsymbol{\delta} + \boldsymbol{u}_{,\alpha} \cdot \boldsymbol{D}) + O(\varepsilon^2) = O(\varepsilon), \\ \gamma_3 = \varepsilon(\boldsymbol{A}_3 \cdot \boldsymbol{\delta} + \boldsymbol{\beta} \cdot \boldsymbol{D}) + O(\varepsilon^2) = O(\varepsilon), \\ \gamma_{Ni} = \varepsilon(\boldsymbol{A}_i \cdot \boldsymbol{\delta}_N) + O(\varepsilon^2) = O(\varepsilon) \quad (N \geqq 2). \end{gathered} \tag{7.56}$$

Also, the variables $\varkappa_{i\alpha}$ and those in (7.24) upon linearization become

$$\begin{aligned} \varkappa_{\beta\alpha} &= \varepsilon(\boldsymbol{u}_{,\beta} \cdot \boldsymbol{D}_{,\alpha} + \boldsymbol{A}_\beta \cdot \boldsymbol{\delta}_{,\alpha}) + O(\varepsilon^2) = O(\varepsilon), \\ \varkappa_{3\alpha} &= \varepsilon(\boldsymbol{A}_3 \cdot \boldsymbol{\delta}_{,\alpha} + \boldsymbol{\beta} \cdot \boldsymbol{D}_{,\alpha}) + O(\varepsilon^2) = O(\varepsilon), \\ s &= 2\varepsilon[D(\boldsymbol{A}_3 \cdot \boldsymbol{\delta} + \boldsymbol{\beta} \cdot \boldsymbol{D})] + O(\varepsilon^2) = O(\varepsilon), \\ s_\alpha &= \varepsilon[D(\boldsymbol{A}_3 \cdot \boldsymbol{\delta}_{,\alpha} + \boldsymbol{\beta} \cdot \boldsymbol{D}_{,\alpha}) + \Lambda_{3\alpha}(\boldsymbol{A}_3 \cdot \boldsymbol{\delta} + \boldsymbol{\beta} \cdot \boldsymbol{D}) + O(\varepsilon^2) = O(\varepsilon), \\ s_{\alpha\beta} &= \varepsilon[\Lambda_{3\alpha}(\boldsymbol{A}_3 \cdot \boldsymbol{\delta}_{,\beta} + \boldsymbol{\beta} \cdot \boldsymbol{D}_{,\beta}) + \Lambda_{3\beta}(\boldsymbol{A}_3 \cdot \boldsymbol{\delta}_{,\alpha} + \boldsymbol{\beta} \cdot \boldsymbol{D}_{,\alpha})] + O(\varepsilon^2) = O(\varepsilon) \end{aligned} \tag{7.57}$$

and the expressions for other quantities in (7.21)–(7.23) may be put in similar forms.

In what follows, in order to achieve some simplification in otherwise lengthy and elaborate calculations, we confine attention to the case corresponding to $D=1$ as in (7.48). Thus, with $\boldsymbol{D} = \boldsymbol{A}_3$, in addition to the reference values (7.49) we now also have

$$D_\alpha = 0, \qquad D = 1, \qquad \Lambda_{\beta\alpha} = -B_{\beta\alpha}, \qquad \Lambda_{3\alpha} = 0. \tag{7.58}$$

The simplification resulting from (7.58) is rather substantial. For example, the first term in $(7.56)_2$ reduces to $\varepsilon(\boldsymbol{A}_3 \cdot \boldsymbol{\delta})$. Similarly, in view of $(7.58)_{3,4}$, the last three of (7.57) now become $s = 2\gamma_3$, $s_\alpha = \varkappa_{3\alpha}$ and $s_{\alpha\beta} = 0$, respectively. Now remembering the remarks made in Sect. 6 [between (6.10) and (6.13)] concerning the linearization process, again we retain only terms of $O(\varepsilon)$; and after the approximations, without loss of generality, we set $\varepsilon = 1$ in order to avoid the introduction of additional notations. In this way, the linearized kinematic measures resulting from (7.18)–(7.19) are

$$\begin{gathered} e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - B_{\alpha\beta} u_3, \\ \gamma_\alpha = \delta_\alpha - \beta_\alpha, \qquad \gamma_3 = \delta_3, \\ \varkappa_{\beta\alpha} = \delta_{\beta|\alpha} - B_{\alpha\beta}\delta_3 - B_\alpha^\nu(u_{\nu|\beta} - B_{\nu\beta} u_3), \\ \varkappa_{3\alpha} = \delta_{3,\alpha} + B_\alpha^\nu(\delta_\nu - \beta_\nu), \\ \gamma_{N\alpha} = \delta_{N\alpha}, \qquad \gamma_{N3} = \delta_{N3} \quad (N \geqq 2), \\ \varkappa_{N\beta\alpha} = \delta_{N\beta|\alpha} - B_{\beta\alpha}\delta_{N3}, \qquad \varkappa_{N3\alpha} = \delta_{N3,\alpha} + B_\alpha^\beta \delta_{N\beta} \quad (N \geqq 2), \end{gathered} \tag{7.59}$$

where the vertical bar now stands for covariant differentiation with respect to $A_{\alpha\beta}$.

The infinitesimal (three-dimensional) strain tensor, resulting from linearization of (7.27), is given by

$$\gamma_{ij}^* = \tfrac{1}{2}(\boldsymbol{G}_i \cdot \boldsymbol{u}_{,j}^* + \boldsymbol{G}_j \cdot \boldsymbol{u}_{,i}^*). \tag{7.60}$$

The relationship between the two-dimensional kinematic variables (7.59) and the infinitesimal (three-dimensional) strain tensor can be found either by linearization of (7.28), remembering also (7.49) and (7.58), or directly from (7.60) and (7.51). Here, we record the final results:

$$
\begin{aligned}
2\gamma^*_{\alpha\beta} &= 2e_{\alpha\beta} + \xi(\varkappa_{\alpha\beta} + \varkappa_{\beta\alpha}) - 2\xi^2 B^\nu_\beta B^\mu_\alpha e_{\mu\nu} + \sum_{N=2}^{\infty} \xi^N (\varkappa_{N\alpha\beta} + \varkappa_{N\beta\alpha}) \\
&\quad - \sum_{N=1}^{\infty} \xi^{N+1} [B^\gamma_\beta \varkappa_{N\gamma\alpha} + B^\gamma_\alpha \varkappa_{N\gamma\beta}], \\
2\gamma^*_{\alpha 3} &= 2\gamma^*_{3\alpha} = \gamma_\alpha + \xi(\varkappa_{3\alpha} - B^\nu_\alpha \gamma_\nu) \\
&\quad + \sum_{N=2}^{\infty} \xi^{N-1} [N\gamma_{N\alpha} + \xi\varkappa_{N3\alpha} - N\xi B^\nu_\alpha \gamma_{N\nu}], \\
\gamma^*_{33} &= \gamma_3 + \sum_{N=2}^{\infty} N\xi^{N-1} \gamma_{N3}.
\end{aligned}
\tag{7.61}
$$

The complexity of the expressions (7.61) for γ^*_{ij} is perhaps suggestive of the degree of the difficulties that can be encountered even in the development of the linear theory. However, prior to consideration of an approximation scheme, it is instructive to specialize (7.59) and (7.61) to the case of an initially flat plate for which (6.26) holds. Thus, for an initially flat plate, the kinematic measures (7.59) reduce to

$$
\begin{gathered}
e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}), \\
\gamma_\alpha = \delta_\alpha - \beta_\alpha, \quad \gamma_3 = \delta_3, \quad \beta_\alpha = -u_{3,\alpha}, \\
\varkappa_{\beta\alpha} = \gamma_{\beta|\alpha} - u_{3|\alpha\beta}, \quad \varkappa_{3\alpha} = \gamma_{3,\alpha}, \\
\gamma_{N\alpha} = \delta_{N\alpha}, \quad \gamma_{N3} = \delta_{N3} \quad (N \geqq 2), \\
\varkappa_{N\beta\alpha} = \gamma_{N\beta|\alpha}, \quad \varkappa_{N3\alpha} = \gamma_{N3,\alpha} \quad (N \geqq 2)
\end{gathered}
\tag{7.62}
$$

and the expressions in (7.61) assume the forms

$$
\begin{aligned}
\gamma^*_{\alpha\beta} &= e_{\alpha\beta} + \xi\,\varkappa_{(\alpha\beta)} + \tfrac{1}{2}\sum_{N=2}^{\infty} \xi^N (\varkappa_{N\alpha\beta} + \varkappa_{N\beta\alpha}), \\
\gamma^*_{\alpha 3} &= \gamma^*_{3\alpha} = \tfrac{1}{2}\left[\gamma_\alpha + \sum_{N=2}^{\infty} N\xi^{N-1}\gamma_{N\alpha} + \sum_{N=1}^{\infty} \xi^N \varkappa_{N3\alpha}\right] \\
&= \tfrac{1}{2}\left[\sum_{N=1}^{\infty} \xi^{N-1}(N\gamma_{N\alpha} + \xi\varkappa_{N3\alpha})\right], \\
\gamma^*_{33} &= \gamma_3 + \sum_{N=2}^{\infty} N\xi^{N-1}\gamma_{N3} = \sum_{N=1}^{\infty} N\xi^{N-1}\gamma_{N3},
\end{aligned}
\tag{7.63}
$$

where the notation $\varkappa_{(\alpha\beta)}$ stands for

$$
\varkappa_{(\alpha\beta)} = \tfrac{1}{2}(\varkappa_{\alpha\beta} + \varkappa_{\beta\alpha}) = \tfrac{1}{2}(\gamma_{\alpha|\beta} + \gamma_{\beta|\alpha}) - u_{3|\alpha\beta}
\tag{7.64}
$$

and conforms to that introduced in (5.12).

The above kinematical results for initially flat plates, although simpler than (7.61), still involve two infinite sets of variables δ_{Ni} and $\varkappa_{Ni\alpha}$. Nevertheless, some observations may be made regarding the structure of γ^*_{ij} in (7.63). The components $\gamma^*_{\alpha\beta}$ are independent of δ_{Ni} and $\varkappa_{N3\alpha}$ while the components γ^*_{i3} are independent of $\varkappa_{N\beta\alpha}$. This uncoupling of the effects of δ_{Ni}, $\varkappa_{N3\alpha}$ and $\varkappa_{N\beta\alpha}$ immediately suggests consideration of an approximate theory for thin plates in which an approximate expression for $\gamma^*_{\alpha\beta}$, obtained from $(7.63)_1$ by ignoring $\varkappa_{N\beta\alpha}(N\geqq 2)$, plays a dominant role.[41] We return to this problem of approximations later.

δ) *Approximate linearized kinematic measures.* The complexity of the foregoing linearized kinematic measures, especially for shells, clearly indicates the need for a *suitable* approximative scheme which would make possible the development of a complete theory which is both useful and manageable. However, in general, the introduction of an approximation for the kinematic quantities alone (and separate from the rest of the theory) is difficult to justify, since any approximation introduced here must be compatible with the entire theory including all field equations and the constitutive relations.[42] Thus, strictly speaking, the introduction of any approximative scheme should be postponed until a complete theory (which includes all field equations and constitutive relations) from the three-dimensional equations is developed. On the other hand, this may require too much patience from the reader. For this reason and in anticipation of certain results, whose range of validity and limitations will be spelled out eventually, we discuss here a set of kinematic measures which are obtained by an approximation from (7.61).

With reference to (7.59) and (7.61), suppose now that γ_{Ni} and $\varkappa_{Ni\alpha}$ vanish for $N\geqq 2$, i.e.,

$$\gamma_{Ni}=0, \quad \varkappa_{Ni\alpha}=0 \quad (N\geqq 2). \tag{7.65}$$

Then, in view of (7.65), by $(7.59)_{6,7}$ $\delta_{Ni}=0$ for $N\geqq 2$. Hence

$$\boldsymbol{\delta}_N=0 \quad (N\geqq 2), \tag{7.66}$$

and (7.51) assumes the simple form

$$\boldsymbol{u}^*=\boldsymbol{u}+\xi\,\boldsymbol{\delta}. \tag{7.67}$$

We expect (7.67) to be a valid approximation for *thin* shells in the region $\alpha<\xi<\beta$ We postpone the justification of (7.65) or the limitations under which it may be assumed; but, given (7.65), the simple expression (7.67) follows without further assumption.[43]

[41] The simple form of such an approximate expression for $\gamma^*_{\alpha\beta}$, corresponding to the first two terms on the right-hand side of $(7.63)_1$, is partially the reason for the success in the construction of an approximate classical theory of plates as compared to that for shells. We postpone further remarks on this until later in this section [Subsect. ε)] and again in Sect. 20. Here, however, we note that the development of the approximate theory just referred to usually begins with a set of displacements (or special assumptions for the three-dimensional strain tensor) which amounts to ignoring γ_{Ni} $(N\geqq 1)$, $\varkappa_{N3\alpha}$ $(N\geqq 1)$ and $\varkappa_{N\beta\alpha}$ $(N\geqq 2)$ or equivalently $\gamma^*_{\alpha 3}$ and $\varkappa_{N\beta\alpha}$ $(N\geqq 2)$ in the kinematical results, but later include (through an additional assumption) the effect of γ^*_{33} in the constitutive relations.

[42] Some of the difficulties and inconsistencies in the past developments of shell theory may be traced directly to the fact that approximations for kinematics were introduced at the outset and without due consideration for their effects in the rest of the theory. To compensate for these difficulties in the derivations of the linear theory of elastic shells, sometimes in recent years use has been made of variational theorems such as those of the Hellinger-Reissner, Hu-Washizu or variants thereof: HELLINGER [1914, *1*], REISSNER [1950, *5*] and [1953, *4*], HU [1955, *4*], WASHIZU [1955, *7*], REISSNER [1964, *7*], NAGHDI [1964, *6*] and REISSNER [1965, *7*]. What these (three-dimensional) variational theorems have in common, apart from the boundary conditions, is one form of the constitutive relations for linear elasticity as part of their Euler equations; they differ from one another mainly in the degree of generality provided by their additional Euler equations for the variational problem. For derivations of an approximate shell theory by means of variational theorems (in the three-dimensional theory) of the type mentioned above, see, e.g., [1963, *6*] and [1964, *6*] which contain additional related references.

[43] Alternatively, we may assume (7.67) together with (7.66) and conclude (7.65) but this appears to be more difficult to justify. An assumption for the displacement vector in the form (7.67) or a more specialized version of it is made in nearly all developments of the linear theory of thin shells at the outset. In this connection, see also Subsect. ε) of this section.

With the approximation (7.65), the kinematic measures (7.59) can be written as

$$
\begin{aligned}
e_{\alpha\beta} &= \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - B_{\alpha\beta}\, u_3, \\
\gamma_\alpha &= \delta_\alpha - \beta_\alpha, \quad \gamma_3 = \delta_3, \quad \beta_\alpha = -(u_{3,\alpha} + B^\lambda_\alpha\, u_\lambda), \\
\varkappa_{\beta\alpha} &= \varrho_{\beta\alpha} - B_{\alpha\beta}\,\gamma_3, \quad \varkappa_{3\alpha} = \varrho_{3\alpha} + B^\lambda_\alpha\,\gamma_\lambda, \\
\varrho_{\beta\alpha} &= \gamma_{\beta|\alpha} - \bar{\varkappa}_{\beta\alpha}, \quad \varrho_{3\alpha} = \gamma_{3,\alpha}, \\
\bar{\varkappa}_{\beta\alpha} = \bar{\varkappa}_{\alpha\beta} &= [u_{3|\beta\alpha} + B^\nu_{\beta|\alpha}\, u_\nu + B^\nu_\alpha\, u_{\nu|\beta} + B^\nu_\beta\, u_{\nu|\alpha} - B^\nu_\alpha\, B_{\nu\beta}\, u_3] \\
&= -\tfrac{1}{2}(\beta_{\alpha|\beta} + \beta_{\beta|\alpha}) + \tfrac{1}{2}[B^\nu_\alpha(e_{\nu\beta} + \gamma_{[\nu\beta]}) + B^\nu_\beta(e_{\nu\alpha} + \gamma_{[\nu\alpha]})], \\
\gamma_{[\alpha\beta]} &= \tfrac{1}{2}(u_{\alpha|\beta} - u_{\beta|\alpha}),
\end{aligned}
\tag{7.68}
$$

where $\varrho_{i\alpha}$ defined by the last three of (7.68) are introduced for later reference. The components of γ^*_{ij} in (7.61) now simplify considerably and reduce to

$$
\begin{aligned}
\gamma^*_{\alpha\beta} &= e_{\alpha\beta} + \xi\,\varkappa_{(\alpha\beta)} - \xi^2\,\chi_{\alpha\beta}, \\
\gamma^*_{\alpha 3} &= \gamma^*_{3\alpha} = \gamma_\alpha + \xi\,\varrho_{3\alpha}, \quad \gamma^*_{33} = \gamma_3,
\end{aligned}
\tag{7.69}
$$

where $\varkappa_{(\alpha\beta)}$ is the symmetric part of $\varkappa_{\alpha\beta}$ and where we have put

$$
\begin{aligned}
\chi_{\alpha\beta} &= \bar{\chi}_{\alpha\beta} + B^\lambda_\alpha\, B^\nu_\beta\, e_{\lambda\nu} = \chi_{\beta\alpha}, \\
\bar{\chi}_{\alpha\beta} &= \tfrac{1}{2}(B^\gamma_\alpha\,\varkappa_{\gamma\beta} + B^\gamma_\beta\,\varkappa_{\gamma\alpha}).
\end{aligned}
\tag{7.70}
$$

It is not difficult to see that the kinematic measures (7.68) are formally equivalent to those given in Sect. 6 for a Cosserat surface [compare with (6.24)], if the two surfaces $\mathfrak{S}$ and $\mathscr{S}$ are identified. However, we postpone such identifications until later. Also, as might be expected, only the symmetric part of $\varkappa_{\beta\alpha}$ occurs in $(7.69)_1$.

ε) *Other kinematic approximations in the linear theory.* The literature on the linear theory, especially for shells, abounds with a variety of kinematic approximations. Most of these are variants, or a special case, of (7.67) and often constitute the starting point in the development of complete approximate theories.[44] In order to indicate here the nature of these approximations and at the same time call attention to a certain kinematic approximation which is generally adopted for classical theories of plates and shells, it is expedient to consider the case of a flat plate.

Referred to the base vectors $\boldsymbol{A}_i$, the approximate expression (7.67) in component form reads

$$
u^*_\alpha = u_\alpha + \xi\,\delta_\alpha, \quad u^*_3 = u_3 + \xi\,\delta_3. \tag{7.71}
$$

For initially flat plates, in view of (6.26) and the assumptions (7.65), the approximate kinematic measures (7.68) simplify and reduce to the first six relations in (7.62). It is then easily seen that the linearized kinematic variables for initially flat plates separate into two parts, namely

$$
\begin{aligned}
E &= \{u_\alpha,\ \delta_3,\ e_{\alpha\gamma},\ \gamma_3,\ \varkappa_{3\alpha}\}, \\
F &= \{u_3,\ \delta_\alpha,\ \varkappa_{\gamma\alpha},\ \gamma_\alpha\}.
\end{aligned}
\tag{7.72}
$$

The displacements u_α, δ_3 in the former set E characterize the extensional motion while u_3, δ_α in the latter set F represent the displacements associated with the flexural motion of a plate.[45] Moreover, with reference to (7.71), it is clear that the tangential and normal components of $\boldsymbol{u}^*$ are, respectively, even and odd in ξ for the extensional motion while they are odd and even in ξ for the flexural motion.

---

[44] An account of such kinematic approximations can be found in [1963, *6*].

[45] This observation parallels that made in Sect. 6 [following (6.26)] for a Cosserat surface.

Special cases of the kinematic variables listed in (7.72) correspond to those employed in the classical theory of plates.[46] The classical extensional case does not include δ_3 (and hence the components $\gamma_3, \varkappa_{3\alpha}$) and only admits the tangential displacements u_α. Such a kinematic assumption implies, in turn, inextensibility along the normal to the plate. The kinematic variables for the classical flexural (or bending) theory is obtained from the set F in (7.72), if we put $\gamma_\alpha = 0$. Then, the relevant kinematic variables reduce to

$$\delta_\alpha = \beta_\alpha = -u_{3,\alpha}, \qquad \varkappa_{\beta\alpha} = -u_{3|\alpha\beta}. \tag{7.73}$$

The above results for the classical theory of plates is summarized in the displacement assumption

$$u_\alpha^* = u_\alpha + \xi\, \beta_\alpha, \qquad u_3^* = u_3, \tag{7.74}$$

which is known as the Kirchhoff hypothesis.[47] It is clear that the approximate displacements (7.74) imply inextensibility (in the extensional case) along the normal to the plate[48] and also include only the angular displacements β_α, so that the effect of transverse shear deformation is ignored (in the case of bending theory).

The kinematic assumptions in the derivations of the classical theory of shells, beginning with LOVE's paper[49]—known as Love's first approximation—are similar to those in the classical theory of plates. In particular, it is assumed that (i) normals to the undeformed middle surface remain normals and (ii) suffer no extension. From these assumptions, sometimes referred to as Kirchhoff-Love hypothesis (especially in the Russian literature), it follows that the components of displacements must have the form (7.74) or equivalently can be obtained from[50]

$$\boldsymbol{u}^* = \boldsymbol{u} + \xi\, \boldsymbol{\beta}, \tag{7.75}$$

where $\boldsymbol{\beta}$ is defined by $(7.55)_{2,3}$. With the displacement vector given by (7.75), the components $\gamma_{\alpha 3}^*$ and γ_{33}^* (for transverse shear and transverse normal strain) vanish[51] and since now $\gamma_i = 0$ the variables (7.68) reduce to the set

$$e_{\alpha\beta}, \quad \varkappa_{\beta\alpha} = \varrho_{\alpha\beta} = -\bar{\varkappa}_{\alpha\beta}, \quad \delta_\alpha = \beta_\alpha, \tag{7.76}$$

where the expressions for $e_{\alpha\beta}, \bar{\varkappa}_{\alpha\beta}, \beta_\alpha$ in terms of u_α, u_3 and their derivatives are those recorded in (7.68).[52]

The above remarks pertain to kinematic approximations in the classical theories of plates and shells. Other displacement assumptions or numerous other

[46] KIRCHHOFF [1850, *1*].

[47] Despite the implication of (7.74) regarding inextensibility along the normal, the effect of the component of the strain tensor γ_{33}^* is accounted for in the constitutive equations. This seemingly inconsistent set of assumptions, however, leads to the correct linear constitutive relations.

[48] See also Sect. 20.

[49] LOVE [1888, *1*].

[50] Given the kinematic assumptions (i) and (ii) above, the form (7.75) follows even without the limitation to smallness of $\boldsymbol{u}^*$ or its components. See Sect. 4 of [1963, *6*].

[51] Even though (7.75) imply the inextensibility along the normal to the middle surface, the effect of γ_{33}^* is nevertheless included in the constitutive relations. This is similar to the case of plate noted above.

[52] The expressions for these kinematic variables have the same forms as those given by (6.27) for a restricted theory by direct approach.

ad hoc schemes have continuously appeared in the literature but we do not elaborate on these here.[53]

A derivation of compatibility equations, in terms of the kinematic measures resulting from an approximate expression for the displacement $\boldsymbol{u}^*$ in the form (7.67) or the more restrictive assumption (7.75), may be accomplished in a manner similar to the derivation by direct method given in Sect. 6 [Subsect. η)]. We do not include here such a derivation but note that the resulting compatibility equations will be of the same forms as those given by (6.60)–(6.61) and (6.62). In the literature on the classical linear shell theory, compatibility equations [corresponding to those in (6.62)] are generally obtained using the equations of Gauss and Mainardi-Codazzi as the starting point. A derivation of this kind is readily available elsewhere and was first given by GOL'DENVEIZER.[54]

C. Basic principles for shells and plates.

This chapter is devoted to the basic principles, derivations of the appropriate field equations and related results for shells and plates. Again various developments are pursued both by direct approach (Sects. 8–10) and from the three-dimensional theory of (non-polar) classical continuum mechanics (Sects. 11–12).

8. Basic principles for shells: I. Direct approach. This section is concerned with basic principles for a Cosserat surface, including conservation laws and invariance requirements under superposed rigid body motions. Our developments are in the context of a general thermodynamical theory. But a reader who prefers to confine himself to the purely mechanical theory should be able to do so without difficulty, although some adjustment and omission [especially that of Subsect. β)] will then be necessary.

The ideas of force, couple, stress vector, etc., are familiar from the three-dimensional theory. Here, we need to define the corresponding quantities for a Cosserat surface; and, it is not difficult to see that they may be introduced in a manner which parallels the concepts of the external body force and the contact force in the three-dimensional theory. However, prior to a statement of conservation laws, it is more enlightening (and perhaps even economical) in the present development to introduce the notion of force and related quantities for a Cosserat surface through their rate of work expressions.

α) *Conservation laws.* Let $\mathscr{P}$ be a part of $\mathscr{s}$ occupied by the arbitrary part $\mathscr{P}_{\mathscr{C}}$ of the material region of the Cosserat surface $\mathscr{C}$ (defined in Sect. 4) in the deformed configuration at time t. Let c be any curve on $\mathscr{s}$ which may also be taken as the

[53] The nature of LOVE's second approximation in which instead of (7.75) terms involving ξ^2 are also retained in the expansion of $\boldsymbol{u}^*$ was examined and explored by HILDEBRAND, REISSNER and THOMAS [1949, *4*]. A further account of such assumptions for displacements in the linear theory of shells may be found in [1963, *6*]. Similar and other types of kinematic approximations are used in the papers of PARKUS [1950, *4*], RÜDIGER [1959, *5*], DUDDECK [1962, *1*], ZERNA [1962, *8*] and ZERNA [1968, *14*].

[54] GOL'DENVEIZER [1940, *1*]. See also his book [1961, *3*]. The earliest attempt to derive the compatibility equations in shell theory appears to be by ODQVIST [1937, *1*]. A short account of compatibility equations in the context of the classical theory may be found also in the paper of GOL'DENVEIZER and LUR'E [1947, *3*]. A derivation in lines of curvature coordinates is given in NOVOZHILOV's book [1959, *3*]. Another derivation in general coordinates, involving kinematic variables based on (7.75), can be found in [1963, *6*] and its generalization is considered by KOLLMAN [1966, *5*]. Compatibility equations in lines of curvature coordinates and in terms of physical components of the variables (7.76) are given by SANDERS [1959, *6*]. A shorter version in general coordinates is included in [1963, *7*] and another (in coordinate free notation) is contained in a recent paper by STEELE [1971, *9*].

boundary $\partial\mathscr{P}$ of $\mathscr{P}$. As the boundary of $\mathscr{P}$, c will be a closed curve defined for the points $\boldsymbol{r}$ in $\partial\mathscr{P}$ of the deformed configuration at time t. Let $\theta^\alpha = \theta^\alpha(s)$ be the parametric equations of the curve c, with s as the arc parameter; and let $\boldsymbol{\lambda}$ denote the unit tangent vector to the curve c defined for the points $\boldsymbol{r}(\theta^\alpha(s), t)$ on c. Then,

$$\boldsymbol{\lambda} = \frac{\partial \boldsymbol{r}}{\partial s} = \lambda^\alpha \boldsymbol{a}_\alpha, \qquad \lambda^\alpha = \frac{d\theta^\alpha(s)}{ds} \tag{8.1}$$

and the outward unit normal $\boldsymbol{\nu}$ to c lying in the surface is given by

$$\boldsymbol{\nu} = \boldsymbol{\lambda} \times \boldsymbol{a}_3 = \nu^\alpha \boldsymbol{a}_\alpha = \nu_\alpha \boldsymbol{a}^\alpha = \varepsilon_{\alpha\beta} \lambda^\beta \boldsymbol{a}^\alpha, \tag{8.2}$$

where $\varepsilon_{\alpha\beta}$ is defined in (5.63). For later reference, we also note that $\boldsymbol{\lambda}$ can be expressed as

$$\boldsymbol{\lambda} = \boldsymbol{a}_3 \times \boldsymbol{\nu} = \boldsymbol{a}_3 \times \nu_\alpha \boldsymbol{a}^\alpha = \varepsilon^{\alpha\beta} \nu_\alpha \boldsymbol{a}_\beta. \tag{8.3}$$

Let[1] $\boldsymbol{N} = \boldsymbol{N}(\theta^\alpha, t; \boldsymbol{\nu})$ and $\boldsymbol{M} = \boldsymbol{M}(\theta^\alpha, t; \boldsymbol{\nu})$, each a three-dimensional vector field, be defined for points $\boldsymbol{r}$ on the boundary curve c of $\mathscr{P}$ as follows: If for all arbitrary velocity fields $\boldsymbol{v}$, the scalar $\boldsymbol{N} \cdot \boldsymbol{v}$ is a rate of work per unit length of c, then $\boldsymbol{N}$ is called a *contact force* (or a curve force) vector per unit length of c. Similarly, if the scalar $\boldsymbol{M} \cdot \boldsymbol{w}$ is a rate of work per unit length of c for all arbitrary director velocities $\boldsymbol{w}$, then $\boldsymbol{M}$ is called a *contact director couple* (or a curve director couple) per unit length of c. It is clear that the definition for $\boldsymbol{M}$ parallels that for $\boldsymbol{N}$ and can be obtained from the sentence preceding the last if we replace the symbols $\boldsymbol{N}$ and $\boldsymbol{v}$ by $\boldsymbol{M}$ and $\boldsymbol{w}$ and the words velocity and force by director velocity and director couple, respectively. The resultant contact force $\boldsymbol{F}_c(\mathscr{P})$ and the resultant contact director couple $\boldsymbol{G}_c(\mathscr{P})$ exerted on the part $\mathscr{P}$ of the Cosserat surface $\mathscr{C}$ at time t are defined by the line integrals

$$\boldsymbol{F}_c(\mathscr{P}) = \int_{\partial\mathscr{P}} \boldsymbol{N}\, ds, \qquad \boldsymbol{G}_c(\mathscr{P}) = \int_{\partial\mathscr{P}} \boldsymbol{M}\, ds \tag{8.4}$$

over the boundary $\partial\mathscr{P}$ of $\mathscr{P}$ in the present configuration.[2] Moreover, it is clear from the above definitions and the physical dimensions of $\boldsymbol{v}$ and $\boldsymbol{w}$ that $\boldsymbol{N}$ and $\boldsymbol{M}$ have, respectively, the physical dimensions of force and couple per unit length, namely

$$\begin{aligned} \text{phys. dim. } \boldsymbol{N} &= \left[\frac{MLT^{-2}}{L}\right] = [MT^{-2}], \\ \text{phys. dim. } \boldsymbol{M} &= \left[\frac{ML^2T^{-2}}{L}\right] = [MLT^{-2}], \end{aligned} \tag{8.5}$$

where the symbol $[T]$ designates the dimension of time and the symbols $[M]$ and $[L]$ for the dimensions of mass and length were introduced previously.

The vector fields $\boldsymbol{N}$ and $\boldsymbol{M}$ act across any curve on $\mathfrak{s}$. In an analogous fashion we may define $\boldsymbol{f} = \boldsymbol{f}(\theta^\alpha, t)$ and $\boldsymbol{l} = \boldsymbol{l}(\theta^\alpha, t)$, each a three-dimensional vector field per unit mass, for points $\boldsymbol{r}$ on the part $\mathscr{P}$ of $\mathfrak{s}$: If the scalar $\boldsymbol{f} \cdot \boldsymbol{v}$ is a rate of work per unit mass for all arbitrary velocities $\boldsymbol{v}$, then $\boldsymbol{f}$ is called an *assigned force* vector per unit mass of $\mathfrak{s}$. The definition for the *assigned director couple* $\boldsymbol{l}$ parallels that for $\boldsymbol{f}$ and can be obtained from the preceding sentence, if we replace the symbols $\boldsymbol{f}$ and $\boldsymbol{v}$ by $\boldsymbol{l}$ and $\boldsymbol{w}$ and the words velocities and assigned force by director veloc-

[1] To emphasize the dependence of such vector fields as $\boldsymbol{N}$ and $\boldsymbol{M}$ on the unit normal $\boldsymbol{\nu}$, it is customary to write them as $\boldsymbol{N}_{(\boldsymbol{\nu})}$ and $\boldsymbol{M}_{(\boldsymbol{\nu})}$. Here we omit the subscript $(\boldsymbol{\nu})$ from $\boldsymbol{N}_{(\boldsymbol{\nu})}$ and $\boldsymbol{M}_{(\boldsymbol{\nu})}$, except on one or two occasions which may serve clarity.

[2] In (8.4), (8.6) and other definitions in this chapter, the abbreviated notation introduced in Sect. 4 [following (4.43)] is used. The designation of the left-hand sides of $(8.4)_{1,2}$ by $\boldsymbol{F}_c(\mathscr{P})$ and $\boldsymbol{G}_c(\mathscr{P})$, instead of $\boldsymbol{F}_c(\mathscr{P}_\mathscr{C})$ and $\boldsymbol{G}_c(\mathscr{P}_\mathscr{C})$, is in accord with our previous notational agreement.

ities and assigned director couple, respectively. The *resultant assigned force* $\boldsymbol{F}_b(\mathscr{P})$ and the *resultant assigned director couple* $\boldsymbol{G}_b(\mathscr{P})$ acting on the part $\mathscr{P}$ of the Cosserat surface $\mathscr{C}$ at time t are defined by the surface integrals

$$\boldsymbol{F}_b(\mathscr{P}) = \int_{\mathscr{P}} \varrho \boldsymbol{f} \, d\sigma, \qquad \boldsymbol{G}_b(\mathscr{P}) = \int_{\mathscr{P}} \varrho \boldsymbol{l} \, d\sigma \tag{8.6}$$

over $\mathscr{P}$ in the present configuration, where $\varrho = \varrho(\theta^\alpha, t)$ is the mass density of $\mathscr{s}$ defined by (4.36) and the area element $d\sigma$ is given by $(4.19)_2$. In view of (4.37) it is evident that the vector fields $\boldsymbol{f}$ and $\boldsymbol{l}$ have, respectively, the physical dimensions of force and couple per unit mass, namely

$$\text{phys. dim. } \boldsymbol{f} = \left[\frac{MLT^{-2}}{M}\right] = [LT^{-2}], \qquad \text{phys. dim. } \boldsymbol{l} = \left[\frac{ML^2T^{-2}}{M}\right] = [L^2T^{-2}]. \tag{8.7}$$

We assume that all forces and couples are continuously distributed.[3] The assigned force $\boldsymbol{f}$ and the assigned director couple $\boldsymbol{l}$, each per unit mass, act throughout an arbitrary part $\mathscr{P}$ of $\mathscr{C}$ in the present configuration; and the contact force $\boldsymbol{N}$ and the contact director couple $\boldsymbol{M}$, each per unit length, act across the boundary $\partial\mathscr{P}$ of $\mathscr{P}$ in the present configuration at time t. It is convenient to record here the rate of work by these contact and assigned forces and couples in the form

$$\begin{gathered} R(\mathscr{P}) = R_c(\mathscr{P}) + R_b(\mathscr{P}), \\ R_c(\mathscr{P}) = \int_{\partial\mathscr{P}} (\boldsymbol{N}\cdot\boldsymbol{v} + \boldsymbol{M}\cdot\boldsymbol{w})\, ds, \qquad R_b(\mathscr{P}) = \int_{\mathscr{P}} \varrho(\boldsymbol{f}\cdot\boldsymbol{v} + \boldsymbol{l}\cdot\boldsymbol{w})\, d\sigma. \end{gathered} \tag{8.8}$$

We now introduce some additional quantities which we associate with the motion of the Cosserat surface $\mathscr{C}$. We assume the existence of a scalar potential function per unit mass $\varepsilon = \varepsilon(\theta^\alpha, t)$, called the specific internal energy. The surface integral

$$\mathscr{E}(\mathscr{P}) = \int_{\mathscr{P}} \varrho\, \varepsilon\, d\sigma \tag{8.9}$$

defines the *internal energy* for each part $\mathscr{P}$ in the present configuration. We also introduce a scalar field $r = r(\theta^\alpha, t)$ per unit mass per unit time, called the specific heat supply (or heat absorption); and the heat flux, across a curve with the unit normal $\boldsymbol{\nu}$, by the scalar[4] $h = h(\theta^\alpha, t; \boldsymbol{\nu})$ per unit length per unit time. The integral[5]

$$H(\mathscr{P}) = \int_{\mathscr{P}} \varrho\, r\, d\sigma - \int_{\partial\mathscr{P}} h\, ds, \tag{8.10}$$

where $\partial\mathscr{P}$ is the boundary of $\mathscr{P}$, defines the heat per unit time entering the part $\mathscr{P}$ of $\mathscr{C}$ in the present configuration. The first term on the right-hand side of (8.10) represents the heat transmitted into the surface by radiation and the second term the heat entering the surface by conduction.

The kinetic energy $\mathscr{K}(\mathscr{P})$ for each part $\mathscr{P}$ of $\mathscr{C}$ in the present configuration is defined by

$$\mathscr{K}(\mathscr{P}) = \int_{\mathscr{P}} \tfrac{1}{2}\varrho(\boldsymbol{v}\cdot\boldsymbol{v} + \alpha\,\boldsymbol{w}\cdot\boldsymbol{w})\, d\sigma \tag{8.11}$$

which includes a contribution $\frac{1}{2}\alpha\,\boldsymbol{w}\cdot\boldsymbol{w}$ per unit mass due to director velocity $\boldsymbol{w}$, the coefficient $\alpha = \alpha(\theta^\gamma)$ being independent of time but possibly a function of surface coordinates. We also define, for each part $\mathscr{P}$ of $\mathscr{C}$ in the present con-

[3] The terminology of *director force* and *assigned director force* was used in [1965, *4*] for $\boldsymbol{M}$ and $\boldsymbol{l}$, respectively, in place of *director couple* and *assigned director couple*. The latter terminology, adopted here, seems to be more suggestive.

[4] The use of the symbol h here is temporary and need not be confused with that in (4.30).

[5] Alternatively we can introduce the heat flux by a vector field $\boldsymbol{q} = \boldsymbol{q}(\theta^\alpha, t)$ such that $\boldsymbol{q}\cdot\boldsymbol{\nu} = h$. The negative sign in (8.10) is in accord with the usual convention, since $\boldsymbol{q}\cdot\boldsymbol{\nu} < 0$ at points where heat is entering the surface through the boundary curve. Here, our r and $\boldsymbol{q}$ are the two-dimensional counterparts of q and $-\boldsymbol{h}$ (in a three-dimensional theory) employed by some writers; see, e.g., Sect. 79 of TRUESDELL and NOLL [1965, *9*].

figuration, the integrals[6]

$$\begin{aligned} &\text{the linear momentum:} && \mathcal{L}(\mathcal{P}) = \int_{\mathcal{P}} \varrho\, \boldsymbol{v}\, d\sigma, \\ &\text{the director momentum:} && \mathcal{D}(\mathcal{P}) = \int_{\mathcal{P}} \varrho\, \alpha\, \boldsymbol{w}\, d\sigma, \\ &\text{the moment of momentum:} && \mathcal{M}(\mathcal{P}) = \int_{\mathcal{P}} \varrho\, \mathcal{A}\, d\sigma, \end{aligned} \tag{8.12}$$

where

$$\mathcal{A} = \boldsymbol{r} \times \boldsymbol{v} + \boldsymbol{d} \times \alpha\, \boldsymbol{w}. \tag{8.13}$$

Further, let $\boldsymbol{A}(\mathcal{P})$ represent the sum of the resultants of the supply of moment of momentum $\boldsymbol{A}_b(\mathcal{P})$ due to $\boldsymbol{f}$, $\boldsymbol{l}$ and the flux of moment of momentum $\boldsymbol{A}_c(\mathcal{P})$ due to $\boldsymbol{N}$, $\boldsymbol{M}$. Then,

$$\begin{gathered} \boldsymbol{A}(\mathcal{P}) = \boldsymbol{A}_b(\mathcal{P}) + \boldsymbol{A}_c(\mathcal{P}), \\ \boldsymbol{A}_b(\mathcal{P}) = \int_{\mathcal{P}} [\boldsymbol{r} \times \varrho\, \boldsymbol{f} + \boldsymbol{d} \times \varrho\, \boldsymbol{l}]\, d\sigma, \quad \boldsymbol{A}_c(\mathcal{P}) = \int_{\partial\mathcal{P}} (\boldsymbol{r} \times \boldsymbol{N} + \boldsymbol{d} \times \boldsymbol{M})\, ds. \end{gathered} \tag{8.14}$$

We also admit the existence of a vector field[7] $\boldsymbol{m} = \boldsymbol{m}(\theta^\alpha, t)$, an *intrinsic* (or *surface*) *director couple* per unit area of $\mathfrak{s}$ which makes no contribution to the supply of moment of momentum; the physical dimension of $\boldsymbol{m}$ is

$$\text{phys. dim. } \boldsymbol{m} = \left[\frac{ML^2 T^{-2}}{L^2}\right] = [MT^{-2}], \tag{8.15}$$

i.e., a physical dimension of couple per unit area. Recalling $(8.6)_2$, we define the combined resultant director couple due to $\varrho\, \boldsymbol{l}$ and $\boldsymbol{m}$ for each part $\mathcal{P}$ of $\mathcal{C}$ in the present configuration as

$$\boldsymbol{G}_b'(\mathcal{P}) = \int_{\mathcal{P}} (\varrho\, \boldsymbol{l} - \boldsymbol{m})\, d\sigma = \boldsymbol{G}_b(\mathcal{P}) - \int_{\mathcal{P}} \boldsymbol{m}\, d\sigma. \tag{8.16}$$

Having disposed of the foregoing preliminaries, we are now in a position to state the conservation laws (or principles) for a Cosserat surface. With reference to the present configuration, these conservation laws may be stated in the forms[8]

$$\begin{aligned} &\frac{d}{dt}\int_{\mathcal{P}} \varrho\, d\sigma = 0, \\ &\frac{d}{dt}\int_{\mathcal{P}} \varrho\, \boldsymbol{v}\, d\sigma = \boldsymbol{F}_b(\mathcal{P}) + \boldsymbol{F}_c(\mathcal{P}), \\ &\frac{d}{dt}\int_{\mathcal{P}} \varrho\, \alpha\, \boldsymbol{w}\, d\sigma = \boldsymbol{G}_b'(\mathcal{P}) + \boldsymbol{G}_c(\mathcal{P}), \\ &\frac{d}{dt}\int_{\mathcal{P}} \varrho\, \mathcal{A}\, d\sigma = \boldsymbol{A}_b(\mathcal{P}) + \boldsymbol{A}_c(\mathcal{P}), \\ &\frac{d}{dt}\int_{\mathcal{P}} \varrho\left[\varepsilon + \frac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{v} + \alpha\, \boldsymbol{w}\cdot\boldsymbol{w})\right] d\sigma = R(\mathcal{P}) + H(\mathcal{P}). \end{aligned} \tag{8.17}$$

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[6] The director momentum $\mathcal{D}(\mathcal{P})$ in $(8.12)_2$ has the dimension of moment of momentum, since $\boldsymbol{d}$ was specified to be dimensionless. In other theories of directed media, where $\boldsymbol{d}$ more conveniently may be defined to have the dimension of length, the expression corresponding to $\mathcal{D}(\mathcal{P})$ would have the dimension of linear momentum.

[7] In contrast to the contact force $\boldsymbol{N}$ and the contact couple $\boldsymbol{M}$, $\boldsymbol{m}$ does not depend on the unit normal $\boldsymbol{\nu}$.

[8] Apart from minor variations in form, these conservation laws may be regarded as the two-dimensional counterparts of those in the three-dimensional theory of directed media with a single deformable director: ERICKSEN [1961, *1*], GREEN, NAGHDI and RIVLIN [1965, *3*]. See also, in this connection, Sect. 127 of TRUESDELL and NOLL [1965, *9*] which contains an interesting approach to the conservation laws for linear momentum and linear director momentum in the context of ERICKSEN's theory of liquid crystals [1961, *1*].

The first of (8.17) is a mathematical statement of conservation of mass, the second that of the linear momentum principle, the third that of the director momentum, the fourth is the moment of momentum and the fifth represents the balance of energy for a Cosserat surface. The left-hand sides of the last four in (8.17) represent, respectively, the rate of increase of the linear momentum, the director momentum, the moment of momentum (including ordinary momentum and director momentum) and the total energy, i.e., the sum of internal and kinetic energies.

Assuming that ϱ is continuously differentiable and recalling (5.66), from $(8.17)_1$ we obtain

$$\int_{\mathscr{P}} (\dot{\varrho} + \varrho\, \eta^{\alpha}_{\alpha})\, d\sigma = 0,$$

which holds for each part $\mathscr{P}$ in the present configuration. Hence follows

$$\dot{\varrho} + \varrho\, \eta^{\alpha}_{\alpha} = \dot{\varrho} + \varrho\,(v^{\alpha}{}_{|\alpha} - b^{\alpha}_{\alpha}\, v_3) = 0 \tag{8.18}$$

as the spatial form of the continuity equation for the Cosserat surface in contrast to the material form of the continuity equation expressed by (4.42). We postpone the derivations of other local field equations as consequences of the remaining conservation laws in (8.17) and consider first some additional thermodynamical preliminaries. However, in anticipation of certain future results, we introduce here vector fields $\bar{\boldsymbol{f}}$ and $\bar{\boldsymbol{l}}$ defined by

$$\bar{\boldsymbol{f}} = \boldsymbol{f} - \dot{\boldsymbol{v}}, \qquad \bar{\boldsymbol{l}} = \boldsymbol{l} - \alpha\, \dot{\boldsymbol{w}}. \tag{8.19}$$

In (8.19), $\bar{\boldsymbol{f}}$ is the difference of the assigned force $\boldsymbol{f}$ and the acceleration vector $\dot{\boldsymbol{v}}$ of the surface $\mathfrak{s}$ and $\bar{\boldsymbol{l}}$ is the difference of the assigned director couple $\boldsymbol{l}$ and the inertia term due to the director velocity.

β) *Entropy production.* We introduce now further thermodynamic preliminaries. Let the temperature field be denoted by $\theta = \theta(\theta^{\alpha}, t)$ which we assume to have positive values, i.e.,

$$\theta > 0 \tag{8.20}$$

and define a scalar field per unit mass $\eta = \eta(\theta^{\alpha}, t)$, called the *specific entropy*. The surface integral

$$\mathscr{H}(\mathscr{P}) = \int_{\mathscr{P}} \varrho\, \eta\, d\sigma \tag{8.21}$$

defines the entropy of a part $\mathscr{P}$ in the present configuration and we define the *production of entropy* per unit time in a part $\mathscr{P}$ (in the present configuration) at time t by

$$\begin{aligned} \Gamma(\mathscr{P}) &= \frac{d}{dt}\int_{\mathscr{P}} \varrho\, \eta\, d\sigma - \left[\int_{\mathscr{P}} \varrho\, \frac{r}{\theta}\, d\sigma - \int_{\partial\mathscr{P}} \frac{h}{\theta}\, ds\right], \\ \Gamma(\mathscr{P}) &= \int_{\mathscr{P}} \varrho\, \gamma\, d\sigma, \end{aligned} \tag{8.22}$$

where the scalar γ per unit mass is the *specific production of entropy*. The quantity r/θ in $(8.22)_1$ is the entropy due to radiation entering $\mathscr{P}$ and

$$-\frac{h}{\theta}$$

is the flux of entropy due to conduction entering $\mathscr{P}$ through the boundary $\partial\mathscr{P}$.

For a given Cosserat surface $\mathscr{C}$, it is convenient to speak of a *thermodynamic process* or simply a *process* if the set of twelve functions consisting of the vector functions $\boldsymbol{r}$ and $\boldsymbol{d}$ in (5.1), the scalar function η and the mechanical and thermal fields $\mathfrak{M}$, $\mathfrak{T}$, $\mathfrak{B}$, namely

$$\mathfrak{M}=\{\boldsymbol{N},\boldsymbol{M},\boldsymbol{m}\},\quad \mathfrak{T}=\{\varepsilon,\theta,h\},\quad \mathfrak{B}=\{\boldsymbol{f},\boldsymbol{l},r\} \tag{8.23}$$

as functions of θ^α and t and with $\{\boldsymbol{N},\boldsymbol{M},h\}$ being also dependent on $\boldsymbol{\nu}$, satisfy all conservation laws in (8.17). We speak of an *admissible* process if the set of functions $\mathfrak{M}$ and $\mathfrak{T}$ in $(8.23)_{1,2}$ are specified by constitutive equations (characterizing the thermo-mechanical behavior of a given material) which hold at each material point θ^α and at all times. We return to the above terminology in Sect. 9 and amplify the remarks concerning a process and an admissible process.

Recalling $(8.22)_1$, we now postulate the inequality

$$\Gamma\geqq 0 \tag{8.24}$$

which we require to be valid for every admissible process describing the motion and the thermo-mechanical behavior of the Cosserat surface.[9] The entropy production inequality (8.24) is the two-dimensional counterpart of the Clausius-Duhem inequality (in the three-dimensional theory) and is a statement of the second law of thermodynamics.[10] The inequality (8.24) will be viewed as a condition to be satisfied identically by every admissible process which describes the motion and the material behavior of the continuum. Thus, it will narrow the class of all admissible processes and will place restrictions on the functions in $(8.23)_{1,2}$ to be specified by constitutive equations.

γ) *Invariance conditions.* Previously, in Sect. 5, we have considered motions of a Cosserat surface which differ from (5.1) only by superposed rigid body motions and have also obtained the transformation relations for various kinematical quantities under such superposed rigid body motions. We consider now important invariance conditions to be satisfied by various dynamic and thermodynamic quantities when the position vector $\boldsymbol{r}^+$ and the director $\boldsymbol{d}^+$ are specified by (5.36).

Preliminary to our consideration of invariance conditions, we make certain observations regarding $(5.36)_1$ and its characteristic features. It is clear from $(5.36)_1$ that $\{\boldsymbol{r}^+, t^+\}$ and $\{\boldsymbol{r}, t\}$ are related by rigid transformations combined with a time shift. An immediate consequence of $(5.36)_1$ is that for each time t and corresponding to any two material points (say θ^α and ${}_0\theta_\alpha$) of the Cosserat surface $\mathscr{C}$, the magnitude of the relative displacement $|\boldsymbol{r}(\theta^\alpha)-\boldsymbol{r}({}_0\theta^\alpha)|$ remains unaltered:

$$|\boldsymbol{r}^+(\theta^\alpha)-\boldsymbol{r}^+({}_0\theta^\alpha)|=|\boldsymbol{r}(\theta^\alpha)-\boldsymbol{r}({}_0\theta^\alpha)|.$$

Hence, $(5.36)_1$ is distance (or length) preserving and it follows that the element of area of the surface $\mathscr{s}$ and therefore its mass density ϱ remains unaltered under the transformation. We have already seen in Sect. 5 that the relationship $(5.36)_1$

[9] The point of view adopted here is that currently used in the three-dimensional theory of continuum mechanics. For a more detailed treatment of the basic concepts in thermodynamics of deformable media, we refer the reader to Chap. E of TRUESDELL and TOUPIN [1960, *14*] and to Sects. 79–81 of TRUESDELL and NOLL [1965, *9*].

[10] There is some discussion in the current literature regarding the limitation of (8.24) or a more general form for the entropy inequality, but these need not affect our limited use of the inequality (8.24) here. In later sections, we shall appeal to (8.24) only in the context of elastic materials. Moreover, although our subsequent developments are carried out within the framework of a thermodynamic theory, most of the results without much effort can be specialized to the *isothermal* or the purely mechanical theory. Thus a reader who prefers to avoid the use of the inequality (8.24) should be able to do so easily.

induces certain transformations on various kinematic scalar or vector fields and that some of these transform according to formulae of the type

$$V^+ = V, \quad \boldsymbol{V}^+ = Q(t)\,\boldsymbol{V}, \tag{8.25}$$

where V and $\boldsymbol{V}$ stand for a scalar and a vector field and $Q(t)$ is the proper orthogonal tensor appearing in (5.36).[11] However, not all kinematic quantities transform according to (8.25) and this is plainly evident from (5.45), (5.51) and $(5.55)_4$. The foregoing remarks parallel corresponding observations that can be made regarding the characteristic features of superposed rigid body motions in the (non-polar) three-dimensional theory. In particular, since the relationship (7.7) between the position $\boldsymbol{p}^+$ and $\boldsymbol{p}$ is distance preserving, volume elements and therefore the mass density ϱ^* remain unaltered under the transformation (7.7).

In order to motivate the invariance conditions sought, we deviate briefly from our main task and consider the nature of invariance conditions under superposed rigid body motions in the context of the classical three-dimensional continuum mechanics. Confining attention for simplicity to the purely mechanical theory, we recall that the mechanical fields which enter the linear momentum and the moment of momentum principles in the three-dimensional theory are the stress vector $\boldsymbol{t} = \boldsymbol{t}(\theta^i, t; \boldsymbol{n})$ and the body force $\boldsymbol{f}^* = \boldsymbol{f}^*(\theta^i, t)$ per unit mass; the latter is defined throughout a region of space occupied at time t by a corresponding material region of the body $\mathscr{B}$ while the former is defined over the boundary surface of the region in question at time t, with $\boldsymbol{n}$ as its outward unit normal vector. Consider now a second motion of $\mathscr{B}$ which differs from the given one at time t only by superposed rigid body motions. The second motion imparts a change in the orientation of the body, so that the outward unit normal vector to the same material surface (in the present configuration) becomes $\boldsymbol{n}^+$ under superposed rigid body motions; and, by virtue of (7.7), $\boldsymbol{n}^+$ is related to $\boldsymbol{n}$ through $\boldsymbol{n}^+ = Q\,\boldsymbol{n}$. Further, let $\boldsymbol{t}^+ = \boldsymbol{t}^+(\theta^i, t; \boldsymbol{n}^+)$ denote the stress vector in the second motion acting on the surface (in the present configuration) whose unit normal is $\boldsymbol{n}^+$. Because the transformation (7.7) is distance preserving, we expect the stress vector $\boldsymbol{t}^+$ (i) to have the same magnitude as $\boldsymbol{t}$ (which acts on the surface whose unit normal is $\boldsymbol{n}$) and (ii) to have the same orientation relative to $\boldsymbol{n}^+$ as $\boldsymbol{t}$ has relative to $\boldsymbol{n}$. These remarks, in mathematical terms, can be stated as $\boldsymbol{t}^+(\theta^i, t; \boldsymbol{n}^+) = Q(t)\,\boldsymbol{t}(\theta^i, t; \boldsymbol{n})$; and hence, under superposed rigid body motions, the stress vector transforms according to $(8.25)_2$.

To examine the manner in which $\boldsymbol{f}^*$ transforms under superposed rigid body motions, we make use of the linear momentum principle (from the three-dimensional theory) which holds for every motion and for each part $\mathscr{P}_{\mathscr{B}}$ of the body $\mathscr{B}$; and, in particular, compare a statement of this principle for two motions, one specified by $\boldsymbol{p}$ in (4.6) and another by $\boldsymbol{p}^+$ in (7.7) at time t. We observe that these two motions differ by only superposed rigid body motions and that the comparison in question is sought with reference to the same region of space (in the present configuration) at time t. For this purpose let $\mathscr{P}^1_{\mathscr{B}}$ and $\mathscr{P}^2_{\mathscr{B}}$ refer to two different materials rgions of the body $\mathscr{B}$ with at least one material point in common. Further let $\mathscr{P}^1_{\mathscr{B}}$ and $\mathscr{P}^2_{\mathscr{B}}$ occupy the same region of space at time t designated by[12] $\bar{\mathscr{P}}$ with the common material point coincident in the present configuration at

[11] In addition to the mass density noted above, examples of kinematic quantities which transform according to (8.25) under superposed rigid body motions are a, $\dot{a}$, $\boldsymbol{a}_\alpha$ and $\boldsymbol{\eta}_\alpha$. See $(5.54)_1$, $(5.55)_{2,3}$ and $(5.38)_1$.

[12] Our notation here for $\mathscr{P}^1_{\mathscr{B}}$ and $\mathscr{P}^2_{\mathscr{B}}$ is temporary and in line with that adopted in Sect. 4 [following (4.42)].

time t. The linear momentum principle corresponding to the motion (4.6) and for the part $\mathscr{P}^1_{\mathscr{B}}$ in the present configuration may be stated in the form[13]

$$\int_{\bar{\mathscr{P}}} \varrho^* \dot{\boldsymbol{v}}^* \, dv = \int_{\bar{\mathscr{P}}} \varrho^* \boldsymbol{f}^* \, dv + \int_{\partial\bar{\mathscr{P}}} \boldsymbol{t} \, d\sigma. \tag{8.26}$$

Similarly, we may write

$$\int_{\bar{\mathscr{P}}} \varrho^{*+} \dot{\boldsymbol{v}}^{*+} \, dv = \int_{\bar{\mathscr{P}}} \varrho^{*+} \boldsymbol{f}^{*+} \, dv + \int_{\partial\bar{\mathscr{P}}} \boldsymbol{t}^+ \, d\sigma, \tag{8.27}$$

corresponding to the motion (7.7) and for the part $\mathscr{P}^2_{\mathscr{B}}$ in the present configuration. But, since $\boldsymbol{t}^+$ transforms by $(8.25)_2$, the last integral in (8.27) can be written as

$$\int_{\partial\bar{\mathscr{P}}} \boldsymbol{t}^+ \, d\sigma = Q \int_{\partial\bar{\mathscr{P}}} \boldsymbol{t} \, d\sigma = Q \int_{\bar{\mathscr{P}}} \varrho^* (\dot{\boldsymbol{v}}^* - \boldsymbol{f}^*) \, dv, \tag{8.28}$$

where (8.26) has been used. Substitution of (8.28) into (8.27) yields

$$Q \int_{\bar{\mathscr{P}}} \varrho^* (\dot{\boldsymbol{v}}^* - \boldsymbol{f}^*) \, dv = \int_{\bar{\mathscr{P}}} \varrho^{*+} (\dot{\boldsymbol{v}}^{*+} - \boldsymbol{f}^{*+}) \, dv \Rightarrow Q(\dot{\boldsymbol{v}}^* - \boldsymbol{f}^*) = \dot{\boldsymbol{v}}^{*+} - \boldsymbol{f}^{*+}, \tag{8.29}$$

since $(8.29)_1$ holds for each part $\bar{\mathscr{P}}$ and since ϱ^* transforms according to $(8.25)_1$. Thus, $(8.29)_2$ provides the relation between $\boldsymbol{f}^{*+}$ and $\boldsymbol{f}^*$ under the distance preserving transformation (7.7).

Returning to our main objective, we first observe that the mechanical fields (corresponding to the stress vector and the body force per unit mass of the three-dimensional theory) in the theory of a Cosserat surface under consideration are specified by $\{\boldsymbol{N}, \boldsymbol{M}\}$ and $\{\boldsymbol{f}, \boldsymbol{l}\}$. Each of the vector fields $\boldsymbol{N}$ and $\boldsymbol{M}$ depends on the unit normal vector $\boldsymbol{\nu}$ [defined by (8.2)] which transforms by $(8.25)_2$ under superposed rigid body motions. By an argument which entirely parallels that for the stress vector in the paragraph preceding the last, we can motivate the manner in which the curve force vector $\boldsymbol{N}$ and the curve director couple $\boldsymbol{M}$ are affected when a second motion of $\mathscr{C}$ differs from (5.1) only by superposed rigid body motions. In this way, we arrive at the conclusion that both $\boldsymbol{N}$ and $\boldsymbol{M}$ must transform according to $(8.25)_2$ under the transformation $(5.36)_1$. Consider next the conservation law $(8.17)_2$. With the help of (5.66) and after invoking (8.18), $(8.17)_2$ can be written in the form

$$\int_{\mathscr{P}} \varrho \, \dot{\boldsymbol{v}} \, d\sigma = \int_{\mathscr{P}} \varrho \, \boldsymbol{f} \, d\sigma + \int_{\partial\mathscr{P}} \boldsymbol{N} \, ds,$$

which is the analogue of (8.26) for the Cosserat surface. By repeating the steps between (8.26)–(8.29) but now applied to the last equation, we arrive at the conclusion that $\bar{\boldsymbol{f}}$ defined by $(8.19)_1$ must transform according to $(8.25)_2$. In a similar manner, by considering the conservation laws $(8.17)_{3,4}$ and remembering also that the director transforms by $(5.36)_2$, we can motivate the manner in which the vector fields $\boldsymbol{m}$ and $\bar{\boldsymbol{l}}$ defined by $(8.19)_2$ must transform under superposed rigid body motions (5.36). The above background consideration for the invariance requirements can be extended to thermal fields. For example (as in the three-dimensional theory), we expect such scalar fields as the specific internal energy, entropy, temperature and the heat flux h to remain unaffected by superposed rigid body motions.

[13] In this form the conservation of mass is already invoked.

The foregoing discussion was intended to serve as background and motivation for invariance conditions under superposed rigid body motions which we now postulate:

$$\varrho^+ = \varrho, \quad h^+ = h, \tag{8.30}$$

$$\varepsilon^+ = \varepsilon, \quad \eta^+ = \eta, \quad \theta^+ = \theta, \tag{8.31}$$

$$\boldsymbol{N}^+ = Q(t)\, \boldsymbol{N}, \quad \boldsymbol{M}^+ = Q(t)\, \boldsymbol{M}, \quad \boldsymbol{m}^+ = Q(t)\, \boldsymbol{m} \tag{8.32}$$

and

$$r^+ = r, \quad \bar{\boldsymbol{f}}^+ = Q(t)\, \bar{\boldsymbol{f}}, \quad \bar{\boldsymbol{l}}^+ = Q(t)\, \bar{\boldsymbol{l}}. \tag{8.33}$$

These conditions may be viewed as physical requirements—consistent with the conservation laws—imposed on certain physical quantities when the motion of the Cosserat surface differs from (5.1) only by superposed rigid body motions. In subsequent developments, we shall frequently encounter functions and fields [such as those in (8.30)–(8.33)] whose values are scalars and vectors and which obey transformation laws of the type (8.25). For brevity we may refer to such functions and fields as *objective*.[14]

It is clear that the requirements (8.30)–(8.33) impose restrictions on the class of admissible functions characterizing the thermo-mechanical response of the material; however, we have imposed no restrictions relating to any symmetry that the material may possess. Finally, we may adopt the terminology of *equivalent processes* often employed in the three-dimensional theory of continuum mechanics. Thus, recalling that a process is specified here by the motion of the Cosserat surface, the entropy η and the functions $\mathfrak{M}$, $\mathfrak{T}$ and $\mathfrak{B}$ in (8.23), we say two processes are *equivalent* if the motions are related by (5.36) and if the temperature and the various functions in (8.23) are related by the transformations (8.30)–(8.33).

δ) *An alternative statement of the conservation laws.* Starting with the balance of energy $(8.17)_5$ which holds for every motion of the Cosserat surface, we include here a derivation which shows that three of the conservation laws, namely $(8.17)_{1,2,4}$, can be deduced from the energy balance together with the postulated invariance requirements (8.30)–(8.33) under superposed rigid body motions.[15] Such a derivation, apart from any intrinsic value that it may have, is often useful in the construction of a dynamical theory of the type under consideration (or for other generalized continua) and provides some insight into the nature of the conservation laws.

Consider first a special motion of the type (5.36), namely one for which

$$Q(t) = I, \quad \dot{Q}(t) = 0. \tag{8.34}$$

Recalling (5.45) and $(5.52)_1$, we see that the superposed velocity $\boldsymbol{v}^+$ and the superposed director velocity $\boldsymbol{w}^+$ in the above special motion include those which may be obtained from the linear transformations of the forms

$$\begin{aligned} &\boldsymbol{v} \to \boldsymbol{v} + \boldsymbol{b}, \quad \boldsymbol{b} = \text{const}, \\ &\boldsymbol{w} \to \boldsymbol{w}, \end{aligned} \tag{8.35}$$

[14] The use of the term *objective* here is different from the corresponding usage by many who appeal to the *principle of material frame-indifference* and thus allow Q to be an orthogonal tensor and regard $(5.36)_1$ as a change of frame. For a discussion of differences between invariance requirements under superposed rigid body motions and the requirements demanded by the principle of material frame-indifference, see Truesdell and Noll [1965, *9*].

[15] The invariance requirements (8.30)–(8.32) must be utilized in the development of constitutive equations for all materials and are not additional assumptions in any complete theory.

the first of which corresponds to a superposed *uniform* rigid body translational velocity of the continuum while the director velocity $\boldsymbol{w}$ remains unaltered. Moreover, the fields

$$\varrho,\ r,\ h,\ \varepsilon,\ \boldsymbol{N},\ \boldsymbol{M},\ \bar{\boldsymbol{f}},\ \bar{\boldsymbol{l}} \tag{8.36}$$

all remain unchanged under the above transformation, in view of (8.30)–(8.33). On the other hand, the kinetic energy (8.11) and the rate of work expressions $(8.8)_{2,3}$ become[16]

$$\begin{aligned}
&\mathscr{K} \to \mathscr{K} + \boldsymbol{b}\cdot\int_{\mathscr{P}} \varrho\,\boldsymbol{v}\,d\sigma + \tfrac{1}{2}(\boldsymbol{b}\cdot\boldsymbol{b})\int_{\mathscr{P}} \varrho\,d\sigma,\\
&R_b \to R_b + \boldsymbol{b}\cdot\int_{\mathscr{P}} \varrho\,\boldsymbol{f}\,d\sigma,\\
&R_c \to R_c + \boldsymbol{b}\cdot\int_{\partial\mathscr{P}} \boldsymbol{N}\,ds,
\end{aligned} \tag{8.37}$$

since $\boldsymbol{b}$ is a constant vector. Also,

$$\begin{aligned}
&\frac{d\mathscr{E}}{dt} \to \frac{d\mathscr{E}}{dt},\\
&\left(\frac{d\mathscr{K}}{dt} - R_b\right) \to \left(\frac{d\mathscr{K}}{dt} - R_b\right) + \boldsymbol{b}\cdot\left[\frac{d}{dt}\int_{\mathscr{P}} \varrho\,\boldsymbol{v}\,d\sigma - \int_{\mathscr{P}} \varrho\,\boldsymbol{f}\,d\sigma\right]\\
&\qquad + \frac{1}{2}(\boldsymbol{b}\cdot\boldsymbol{b})\frac{d}{dt}\int_{\mathscr{P}} \varrho\,d\sigma.
\end{aligned} \tag{8.38}$$

Thus, if in the balance of energy $(8.17)_5$ we replace $\boldsymbol{v}$ and $\boldsymbol{w}$ by the transformations (8.35) and keep in mind (8.37)–(8.38), then after subtraction we deduce

$$\boldsymbol{b}\cdot\left\{\frac{d}{dt}\int_{\mathscr{P}} \varrho\,\boldsymbol{v}\,d\sigma - \int_{\mathscr{P}} \varrho\,\boldsymbol{f}\,d\sigma - \int_{\partial\mathscr{P}} \boldsymbol{N}\,ds\right\} + \frac{1}{2}(\boldsymbol{b}\cdot\boldsymbol{b})\frac{d}{dt}\int_{\mathscr{P}} \varrho\,d\sigma = 0, \tag{8.39}$$

which holds for all arbitrary $\boldsymbol{b}$. By replacing $\boldsymbol{b}$ by $\beta\,\boldsymbol{b}$, β being an arbitrary scalar, from (8.39) we obtain the Eq. $(8.17)_1$ for conservation of mass and

$$\frac{d}{dt}\int_{\mathscr{P}} \varrho\,\boldsymbol{v}\,d\sigma - \int_{\mathscr{P}} \varrho\,\boldsymbol{f}\,d\sigma - \int_{\partial\mathscr{P}} \boldsymbol{N}\,ds = 0, \tag{8.40}$$

which is the conservation law $(8.17)_2$.

Next, consider a motion of the type (5.36) in which the position $\boldsymbol{r}^+$ is related to $\boldsymbol{r}$ by rotation alone while Ω in (5.39) is a constant tensor, i.e.,

$$\boldsymbol{r}^+ = Q(t)\,\boldsymbol{r}, \quad \boldsymbol{d}^+ = Q(t)\,\boldsymbol{d}, \quad \Omega = \Omega_0 = \text{const}. \tag{8.41}$$

In this case, corresponding to (5.47),

$$\dot{Q}(t) = \Omega_0\,Q(t), \quad \dot{Q}(t)^T = Q(t)^T\,\Omega_0^T = -Q(t)^T\,\Omega_0, \quad \ddot{Q}(t) = \Omega_0^2\,Q(t), \tag{8.42}$$

$$\begin{aligned}
&\boldsymbol{v}^+ = Q\,\boldsymbol{v} + \dot{Q}\,\boldsymbol{r} = Q\,\boldsymbol{v} + \Omega_0\,Q\,\boldsymbol{r},\\
&\dot{\boldsymbol{v}}^+ = Q\,\dot{\boldsymbol{v}} + 2\Omega_0\,Q\,\boldsymbol{v} + \Omega_0^2\,Q\,\boldsymbol{r},
\end{aligned} \tag{8.43}$$

and the kinetic energy per unit mass due to velocity $\boldsymbol{v}^+$ is

$$\begin{aligned}
\tfrac{1}{2}\boldsymbol{v}^+\cdot\boldsymbol{v}^+ &= \tfrac{1}{2}(Q\,\boldsymbol{v} + \Omega_0\,Q\,\boldsymbol{r})\cdot(Q\,\boldsymbol{v} + \Omega_0\,Q\,\boldsymbol{r})\\
&= \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot Q^T\,\Omega_0\,Q\,\boldsymbol{r} + \tfrac{1}{2}(\Omega_0\,Q\,\boldsymbol{r})\cdot(\Omega_0\,Q\,\boldsymbol{r})
\end{aligned} \tag{8.44}$$

[16] In (8.37)–(8.38) and again in (8.52)–(8.53), for brevity we have used the symbols $\mathscr{E}$ and $\mathscr{K}$ in place of the integrals in (8.9) and (8.11).

with similar expressions for $\boldsymbol{w}^+$, $\dot{\boldsymbol{w}}^+$ and $\frac{1}{2}\alpha\,\boldsymbol{w}^+\cdot\boldsymbol{w}^+$ which we do not record here. We now observe that

$$\frac{d}{dt}Q^T\Omega_0 Q=\dot{Q}^T\Omega_0 Q+Q^T\Omega_0\dot{Q}=-Q^T\Omega_0^2 Q+Q^T\Omega_0^2 Q=0. \tag{8.45}$$

Hence the quantity $Q^T\Omega_0 Q$ which occurs in the second term on the right-hand side of (8.44) is independent of time and is equal to its initial value Ω_0 for all time. For later reference, we record here the identities

$$\Omega_0 Q\,\boldsymbol{v}\cdot Q\,\boldsymbol{v}=Q^T\Omega_0 Q\,\boldsymbol{v}\cdot\boldsymbol{v}=\Omega_0\,\boldsymbol{v}\cdot\boldsymbol{v}=0 \tag{8.46}$$

and

$$\begin{aligned}2\Omega_0 Q\,\boldsymbol{v}\cdot\Omega_0 Q\,\boldsymbol{r}+\Omega_0^2 Q\,\boldsymbol{r}\cdot Q\,\boldsymbol{v}&=2\Omega_0 Q\,\boldsymbol{v}\cdot\Omega_0 Q\,\boldsymbol{r}+\Omega_0 Q\,\boldsymbol{r}\cdot\Omega_0^T Q\,\boldsymbol{v}\\&=\Omega_0 Q\,\boldsymbol{v}\cdot\Omega_0 Q\,\boldsymbol{r},\end{aligned} \tag{8.47}$$

where in obtaining (8.46) we have made use of (8.45). By $(8.33)_2$, (8.43) and (8.46)–(8.47),

$$\begin{aligned}\boldsymbol{f}^+&=Q(\boldsymbol{f}-\dot{\boldsymbol{v}})+\dot{\boldsymbol{v}}^+=Q\,\boldsymbol{f}+2\Omega_0 Q\,\boldsymbol{v}+\Omega_0^2 Q\,\boldsymbol{r},\\ \boldsymbol{f}^+\cdot\boldsymbol{v}^+&=\boldsymbol{f}\cdot\boldsymbol{v}+\boldsymbol{f}\cdot Q^T\Omega_0 Q\,\boldsymbol{r}+[(\Omega_0 Q\,\boldsymbol{v})\cdot(\Omega_0 Q\,\boldsymbol{r})+(\Omega_0^2\,Q\,\boldsymbol{r})\cdot(\Omega_0 Q\,\boldsymbol{r})]\end{aligned} \tag{8.48}$$

with similar expressions for $\boldsymbol{l}^+$ and $\boldsymbol{l}^+\cdot\boldsymbol{w}^+$. Also, in view of (8.18) and recalling (5.66), we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathscr{P}}\varrho\left[\frac{1}{2}(\Omega_0 Q\,\boldsymbol{r})\cdot(\Omega_0 Q\,\boldsymbol{r})\right]d\sigma\\&\qquad=\int_{\mathscr{P}}\varrho[(\Omega_0 Q\,\boldsymbol{v})\cdot(\Omega_0 Q\,\boldsymbol{r})+(\Omega_0^2 Q\,\boldsymbol{r})\cdot(\Omega_0 Q\,\boldsymbol{r})]\,d\sigma,\end{aligned} \tag{8.49}$$

where the quantity in the square bracket on the right-hand side of (8.49) is the material time derivative of the third term on the right-hand side of (8.44).

Consider now a special case of the motion (8.41) with Q corresponding to (5.46) so that (5.48) holds. Then, the superposed velocity $\boldsymbol{v}^+$ and the superposed director velocity $\boldsymbol{w}^+$ of the special motion under consideration are given by the linear transformations $(5.57)_1$ and $(5.59)_1$ which correspond to a superposed *uniform* rigid body angular velocity, the continuum occupying the same position at time t. Under these transformations, the fields (8.36) all remain unchanged, in view of (8.30)–(8.33). Keeping the results (8.44)–(8.49) in mind, with the use of identities of the type

$$(\boldsymbol{\omega}_0\times\boldsymbol{r})\cdot\boldsymbol{v}=\boldsymbol{\omega}_0\cdot(\boldsymbol{r}\times\boldsymbol{v}),\qquad \boldsymbol{v}\times\boldsymbol{v}\equiv 0 \tag{8.50}$$

and the temporary notation

$$\begin{aligned}\mathscr{W}=&(\boldsymbol{\omega}_0\times\boldsymbol{v})\cdot(\boldsymbol{\omega}_0\times\boldsymbol{r})+(\boldsymbol{\omega}_0\times\boldsymbol{\omega}_0\times\boldsymbol{r})\cdot(\boldsymbol{\omega}_0\times\boldsymbol{r})\\&+\alpha[(\boldsymbol{\omega}_0\times\boldsymbol{w})\cdot(\boldsymbol{\omega}_0\times\boldsymbol{d})+(\boldsymbol{\omega}_0\times\boldsymbol{\omega}_0\times\boldsymbol{d})\cdot(\boldsymbol{\omega}_0\times\boldsymbol{d}),\end{aligned} \tag{8.51}$$

we see that the rate of the kinetic energy and the expressions for R_b and R_c transform according to

$$\begin{aligned}&\frac{d\mathscr{K}}{dt}\to\frac{d\mathscr{K}}{dt}+\boldsymbol{\omega}_0\cdot\frac{d}{dt}\int_{\mathscr{P}}\varrho\,\mathscr{A}\,d\sigma+\int_{\mathscr{P}}\varrho\,\mathscr{W}\,d\sigma,\\&R_b\to R_b+\boldsymbol{\omega}_0\cdot\int_{\mathscr{P}}\varrho[\boldsymbol{r}\times\boldsymbol{f}+\boldsymbol{d}\times\boldsymbol{l}]\,d\sigma+\int_{\mathscr{P}}\varrho\,\mathscr{W}\,d\sigma,\\&R_c\to R_c+\boldsymbol{\omega}_0\cdot\int_{\partial\mathscr{P}}(\boldsymbol{r}\times\boldsymbol{N}+\boldsymbol{d}\times\boldsymbol{M})\,ds,\end{aligned} \tag{8.52}$$

since $\boldsymbol{\omega}_0$ is a constant vector. Also,

$$\begin{gathered}\frac{d\mathscr{E}}{dt}\to\frac{d\mathscr{E}}{dt},\\ \left(\frac{d\mathscr{K}}{dt}-R_b\right)\to\left(\frac{d\mathscr{K}}{dt}-R_b\right)+\boldsymbol{\omega}_0\cdot\left\{\frac{d}{dt}\int\limits_{\mathscr{P}}\varrho\,\mathscr{A}\,d\sigma-\boldsymbol{A}_b\cdot\right\}.\end{gathered}\tag{8.53}$$

Thus, if in the balance of energy $(8.17)_5$, we replace $\boldsymbol{v}$ and $\boldsymbol{w}$ by the transformations $(5.57)_1$ and $(5.59)_1$ and use (8.52)–(8.53), then after subtraction and since $\boldsymbol{\omega}_0$ is a constant vector we arrive at the conservation law $(8.17)_4$ or equivalently

$$\frac{d}{dt}\int\limits_{\mathscr{P}}\varrho\,\mathscr{A}\,d\sigma-\int\limits_{\mathscr{P}}\varrho[\boldsymbol{r}\times\boldsymbol{f}+\boldsymbol{d}\times\boldsymbol{l}]\,d\sigma-\int\limits_{\partial\mathscr{P}}(\boldsymbol{r}\times\boldsymbol{N}+\boldsymbol{d}\times\boldsymbol{M})\,ds=0,\tag{8.54}$$

where $\mathscr{A}$ is defined by (8.13) and we have used (8.14) in writing the right-hand side of (8.54).

In the foregoing derivation, the conservation of mass in the form $(8.17)_1$, (8.40) and (8.54) have been deduced with the use of the invariance requirements (8.30)–(8.33) and the balance of energy $(8.17)_5$. This clearly shows that the four conservation laws $(8.17)_{1,2,4,5}$ are *equivalent* to the balance of energy and the invariance requirements under superposed rigid body motions.[17]

ε) *Conservation laws in terms of field quantities in a reference state.* The conservation laws (8.17) have been stated with reference to the present configuration and in terms of field quantities which are measured per unit length, per unit area or per unit mass of $\mathfrak{s}$ in the present configuration. For certain purposes, it is more convenient to have available a statement of these conservation laws in terms of field quantities which are measured per unit length, per unit area or per unit mass of the material surface of $\mathscr{C}$ in a reference configuration.

Although the basic structure of the conservation laws (8.17) remains unaltered, certain modifications are necessary in the definitions of some of the field quantities. We assume, in what follows, that a reference configuration of the Cosserat surface is specified by (5.2) and do not insist that it be necessarily an initial configuration. However, in order to simplify the notation as much as possible, we continue (i) to refer to each part $\mathscr{P}_{\mathscr{C}}$ of the Cosserat surface by $\mathscr{P}_0$ as in (4.36), (ii) to identify the material surface of $\mathscr{C}$ in the reference configuration by $\mathscr{S}$ and (iii) to designate the mass density in the reference configuration by ϱ_0. Now, let C be a closed curve on the reference surface $\mathscr{S}$ which becomes a curve c on $\mathfrak{s}$ in the present configuration, let $\theta^\alpha=\theta^\alpha(S)$ be the parametric equations of

[17] The idea of this manner of obtaining the conservation laws from the balance of energy and the invariance requirements under superposed rigid body motions was evidently known to ERICKSEN (see the final remark in his [1961, *1*] paper on liquid crystals). An explicit derivation of this kind in the context of the three-dimensional theory of continuum mechanics was first given by GREEN and RIVLIN [1964, *3*], where Cauchy's equations of motion and the local equation for conservation of mass are derived from the balance of energy and the invariance requirements under superposed rigid body motions. Although the spirit of our development in Subsect. δ) is patterned after that given by GREEN, NAGHDI and WAINWRIGHT [1965, *4*] for a Cosserat surface, the method of derivation differs somewhat from that in [1964, *3*] and [1965, *4*].

The conservation law for the director momentum in the form $(8.17)_3$ cannot be deduced from the balance of energy and the invariance requirements under superposed rigid body motions. In their derivation of the field equations via the energy balance and the invariance requirements, GREEN, NAGHDI and WAINWRIGHT [1965, *4*] did not explicitly state a separate postulate for the integral form of the director momentum principle but assumed a local form of the equations of motion for the director couple which can be deduced from $(8.17)_3$.

the curve C with S as the arc parameter and let

$$_0\boldsymbol{\nu} = {}_0\nu_\alpha\, \boldsymbol{A}^\alpha = {}_0\nu^\alpha\, \boldsymbol{A}_\alpha \tag{8.55}$$

be the outward unit normal to C lying in the surface $\mathscr{S}$. Formulae analogous to (8.1)–(8.3) hold also for $_0\boldsymbol{\nu}$ and the unit tangent vector $_0\boldsymbol{\lambda}$ to C on $\mathscr{S}$.

We introduce here a contact force and a contact director couple, each of which acts across c on $\mathscr{s}$ but is measured per unit length of C on $\mathscr{S}$. Thus, the three-dimensional vector field $_R\boldsymbol{N} = {}_R\boldsymbol{N}(\theta^\alpha, t; {}_0\boldsymbol{\nu})$ defined for points $\boldsymbol{r}$ on c and measured per unit length of C will be called a contact force if the scalar $_R\boldsymbol{N}\cdot\boldsymbol{v}$ is a rate of work per unit length of C for all arbitrary velocity fields $\boldsymbol{v}$. Similarly, the three-dimensional vector field $_R\boldsymbol{M} = {}_R\boldsymbol{M}(\theta^\alpha, t; {}_0\boldsymbol{\nu})$ defined for points $\boldsymbol{r}$ on c and measured per unit length of C will be called a contact director couple if the scalar $_R\boldsymbol{M}\cdot\boldsymbol{w}$ is a rate of work per unit length of C for all arbitrary director velocity $\boldsymbol{w}$. It is clear that the above definitions for $_R\boldsymbol{N}$ and $_R\boldsymbol{M}$ are analogous to that of the Piola-Kirchhoff stress vector in the classical three-dimensional theory. In terms of $_R\boldsymbol{N}$ and $_R\boldsymbol{M}$ and the notation of (4.36), the resultant contact force and the resultant contact director couple exerted on the part $\mathscr{P}_{\mathscr{C}}$ of the Cosserat surface at time t are defined by the line integrals

$$\int_{\partial\mathscr{P}_0} {}_R\boldsymbol{N}\, dS, \quad \int_{\partial\mathscr{P}_0} {}_R\boldsymbol{M}\, dS, \tag{8.56}$$

the integration being over the boundary $\partial\mathscr{P}_0$ of $\mathscr{P}_0$ in the reference configuration. Also, let $_Rh = {}_Rh(\theta^\alpha, t; {}_0\boldsymbol{\nu})$ be the heat flux per unit time, acting across a curve c on $\mathscr{s}$ but measured per unit length of C in the reference configuration. The line integral

$$\int_{\partial\mathscr{P}_0} {}_Rh\, dS, \tag{8.57}$$

where $\partial\mathscr{P}_0$ is the boundary of $\mathscr{P}_0$, defines the heat per unit time entering the surface by conduction.

The expressions (8.56) and (8.57) in terms of $_R\boldsymbol{N}$, $_R\boldsymbol{M}$, $_Rh$ define resultants which parallel the line integrals in (8.4) and (8.10). In a like manner, we introduce now a vector field $_R\boldsymbol{m} = {}_R\boldsymbol{m}(\theta^\alpha, t)$ as an intrinsic (surface) director couple per unit area of $\mathscr{S}$. The remaining field quantities were defined previously [Sect. 9, Subsect. α)] per unit mass and hence require no new definition. The only modification occurs in the surface integrals such as those in (8.6) and $(8.8)_3$, where now $\varrho\, d\sigma$ is replaced by $\varrho_0\, d\Sigma$ and the integration is over $\mathscr{P}_0$ in the reference configuration. With the above background, the conservation laws (8.17) can be easily rewritten and expressed in terms of field quantities in the reference state as follows:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathscr{P}_0} \varrho_0\, d\Sigma = 0, \quad \frac{d}{dt}\int_{\mathscr{P}_0} \varrho_0\, \boldsymbol{v}\, d\Sigma = \int_{\mathscr{P}_0} \varrho_0 \boldsymbol{f}\, d\Sigma + \int_{\partial\mathscr{P}_0} {}_R\boldsymbol{N}\, dS,\\
&\frac{d}{dt}\int_{\mathscr{P}_0} \varrho_0\, \alpha\, \boldsymbol{w}\, d\Sigma = \int_{\mathscr{P}_0} (\varrho_0\, \boldsymbol{l} - {}_R\boldsymbol{m})\, d\Sigma + \int_{\partial\mathscr{P}_0} {}_R\boldsymbol{M}\, dS,\\
&\frac{d}{dt}\int_{\mathscr{P}_0} \varrho_0\, \mathscr{A}\, d\Sigma = \int_{\mathscr{P}_0} (\boldsymbol{r}\times\varrho_0 \boldsymbol{f} + \boldsymbol{d}\times\varrho_0\, \boldsymbol{l})\, d\Sigma + \int_{\partial\mathscr{P}_0} (\boldsymbol{r}\times{}_R\boldsymbol{N} + \boldsymbol{d}\times{}_R\boldsymbol{M})\, dS,\\
&\frac{d}{dt}\int_{\mathscr{P}_0} \varrho_0 \left[\varepsilon + \frac{1}{2}\,(\boldsymbol{v}\cdot\boldsymbol{v} + \alpha\boldsymbol{w}\cdot\boldsymbol{w})\right] d\Sigma = \int_{\mathscr{P}_0} \varrho_0 (\boldsymbol{f}\cdot\boldsymbol{v} + \boldsymbol{l}\cdot\boldsymbol{w} + r)\, d\Sigma\\
&\qquad\qquad + \int_{\partial\mathscr{P}_0} ({}_R\boldsymbol{N}\cdot\boldsymbol{v} + {}_R\boldsymbol{M}\cdot\boldsymbol{w} - {}_Rh)\, dS.
\end{aligned} \tag{8.58}$$

The conservation of mass expressed by $(8.58)_1$ is equivalent to (4.36). The remaining four conservation laws in (8.58), in the order listed, correspond to the last four of (8.17).

For completeness, we also record here the entropy production corresponding to (8.22) in terms of the field quantities in the reference state. Thus, the production of entropy per unit time in a part of $\mathscr{C}$ at time t and measured relative to the reference configuration is defined by

$$\Gamma = \int_{\mathscr{P}_0} \varrho_0\, \gamma\, d\Sigma = \frac{d}{dt}\int_{\mathscr{P}_0} \varrho_0\, \eta\, d\Sigma - \left[\int_{\mathscr{P}_0} \varrho_0 \frac{r}{\theta}\, d\Sigma - \int_{\partial\mathscr{P}_0} \frac{R^h}{\theta}\, dS\right] \tag{8.59}$$

and satisfies the inequality (8.24).

9. Derivation of the basic field equations for shells: I. Direct approach. We derive in this section the basic field equations for a Cosserat surface from the conservation laws of the previous section. As the local form of the conservation of mass has been already given by (8.18), we shall be concerned with the remaining four conservation laws, namely (8.40), (8.54), $(8.17)_3$ and $(8.17)_5$. First we deduce the basic field equations in vector form, using an invariant vector notation, but subsequently we record these equations in terms of tensor components. In these general results, a vertical bar stands for covariant differentiation with respect to the surface metric tensor $a_{\alpha\beta}$; however, in the latter part of this section dealing with linearized field equations and the basic equations in a reference state, a vertical bar denotes covariant differentiation with respect to $A_{\alpha\beta}$ of the reference surface.

α) *General field equations in vector forms.* Consider an arbitrary part of the material region of the surface of $\mathscr{C}$ which is mapped into a part $\mathscr{P}$ of the surface $\mathscr{s}$ in the present configuration at time t. Let $\mathscr{P}$ be divided into two regions $\mathscr{P}_1$, $\mathscr{P}_2$ separated by a curve $\mathscr{p}$ on $\mathscr{s}$ in the present configuration (see Fig. 1). Further let $\partial\mathscr{P}_1$, $\partial\mathscr{P}_2$ refer to the boundaries of $\mathscr{P}_1$, $\mathscr{P}_2$, respectively; and let $\partial\mathscr{P}'$, $\partial\mathscr{P}''$ be the portions of the boundaries of $\mathscr{P}_1$, $\mathscr{P}_2$ such that $\partial\mathscr{P}' = \partial\mathscr{P}_1 \cap \partial\mathscr{P}$, $\partial\mathscr{P}'' = \partial\mathscr{P}_2 \cap \partial\mathscr{P}$. The above description can be summarized as follows:

$$\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2, \quad \partial\mathscr{P} = \partial\mathscr{P}' \cup \partial\mathscr{P}'', \quad \partial\mathscr{P}_1 = \partial\mathscr{P}' \cup \mathscr{p}, \quad \partial\mathscr{P}_2 = \partial\mathscr{P}'' \cup \mathscr{p}. \tag{9.1}$$

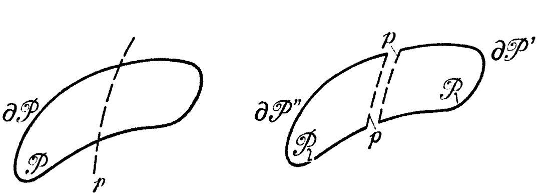

Fig. 1. A part of the surface $\mathscr{s}$ divided into two regions separated by a curve $\mathscr{p}$

Application of the principle of linear momentum (8.40) separately to the parts $\mathscr{P}_1$, $\mathscr{P}_2$ and again to $\mathscr{P}_1 \cup \mathscr{P}_2$ in the present configuration yields

$$\begin{aligned} \frac{d}{dt}\int_{\mathscr{P}_1} \varrho\, \boldsymbol{v}\, d\sigma - \int_{\mathscr{P}_1} \varrho\, \boldsymbol{f}\, d\sigma - \int_{\partial\mathscr{P}_1} \boldsymbol{N}\, ds = 0,\\ \frac{d}{dt}\int_{\mathscr{P}_2} \varrho\, \boldsymbol{v}\, d\sigma - \int_{\mathscr{P}_2} \varrho\, \boldsymbol{f}\, d\sigma - \int_{\partial\mathscr{P}_2} \boldsymbol{N}\, ds = 0 \end{aligned} \tag{9.2}$$

and

$$\frac{d}{dt}\int_{\mathscr{P}_1\cup\mathscr{P}_2} \varrho\, \boldsymbol{v}\, d\sigma - \int_{\mathscr{P}_1\cup\mathscr{P}_2} \varrho \boldsymbol{f}\, d\sigma - \int_{\partial\mathscr{P}'\cup\partial\mathscr{P}''} \boldsymbol{N}\, ds = 0. \tag{9.3}$$

The curve force vector $\boldsymbol{N}$ in $(9.2)_1$ acting over the boundary $\partial\mathscr{P}_1$ (in the present configuration) is due to forces exerted *by* the material on one side of the boundary (exterior to $\mathscr{P}_1$) *on* the material of the other side ($\mathscr{P}_1$). Parallel remarks can be made for the curve force vector in $(9.2)_2$ and (9.3). Let $\boldsymbol{\nu}$ denote the outward unit normal at a point on $\not{p}$ when $\not{p}$ is a portion of $\partial\mathscr{P}_1$. Then, the outward unit normal at the same point on $\not{p}$ when $\not{p}$ is a portion of $\partial\mathscr{P}_2$ is $-\boldsymbol{\nu}$. Keeping this in mind and recalling $(9.1)_{3,4}$, from combination of $(9.2)_{1,2}$ we obtain an equation which must hold also for $\mathscr{P}_1\cup\mathscr{P}_2$. Comparison of the latter result with (9.3) gives[18]

$$\int_{\not{p}} [\boldsymbol{N}_{(\boldsymbol{\nu})} + \boldsymbol{N}_{(-\boldsymbol{\nu})}]\, ds = 0 \tag{9.4}$$

over the arbitrary curve $\not{p}$ of $\mathscr{P}$ in the present configuration. Assuming that $\boldsymbol{N}_{(\boldsymbol{\nu})}$ is a continuous function of position and $\boldsymbol{\nu}$, from (9.4) we conclude that

$$\boldsymbol{N}_{(\boldsymbol{\nu})} = -\boldsymbol{N}_{(-\boldsymbol{\nu})}, \tag{9.5}$$

an analogue of a familiar result in classical continuum mechanics. According to (9.5), the curve force vectors acting on opposite sides of the same curve at a given point are equal in magnitude and opposite in direction.

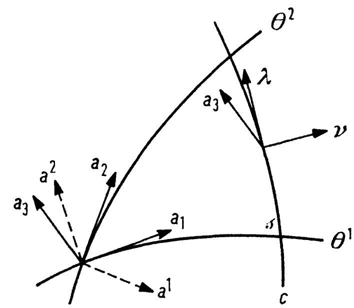

Fig. 2. An elementary curvilinear triangle on $\mathscr{s}$ bounded by coordinate curves and a curve c

Next, consider an elementary curvilinear triangle on $\mathscr{s}$ (see Fig. 2) bounded by the coordinate curves θ^1, θ^2 and a curve c with a unit tangent $\boldsymbol{\lambda}$ and an outward unit normal $\boldsymbol{\nu}$ defined by (8.1)–(8.2). Let ds and ds_α denote, respectively, the element of arc length of c and each of the coordinate curves θ^α. Then,

$$ds_1 = (a_{11})^{\frac{1}{2}}\, d\theta^1, \qquad ds_2 = (a_{22})^{\frac{1}{2}}\, d\theta^2 \tag{9.6}$$

on θ^1 and θ^2 curves, respectively. The unit tangent vectors to θ^1 and θ^2 coordinate curves are[19] $\boldsymbol{\lambda}^{(1)} = \boldsymbol{a}_1 (a_{11})^{-\frac{1}{2}}$ and $\boldsymbol{\lambda}^{(2)} = -\boldsymbol{a}_2 (a_{22})^{-\frac{1}{2}}$. Using (8.2), the outward unit normal vectors to θ^1 and θ^2 curves are given by

$$\boldsymbol{\nu}^{(1)} = -\frac{a^{\frac{1}{2}}\boldsymbol{a}^2}{(a_{11})^{\frac{1}{2}}} = -\frac{\boldsymbol{a}^2}{(a^{22})^{\frac{1}{2}}}, \qquad \boldsymbol{\nu}^{(2)} = -\frac{a^{\frac{1}{2}}\boldsymbol{a}^1}{(a_{22})^{\frac{1}{2}}} = -\frac{\boldsymbol{a}^1}{(a^{11})^{\frac{1}{2}}}, \tag{9.7}$$

---

[18] Temporarily we use $\boldsymbol{N}_{(\boldsymbol{\nu})}$ in place of $\boldsymbol{N}$, in order to emphasize its dependence on $\boldsymbol{\nu}$.

[19] Here the sign of the unit tangent vector is chosen in accord with the convention that if one proceeds along a curve in a closed circuit, the area is always kept to the left. This is consistent with $\{\boldsymbol{\nu}, \boldsymbol{\lambda}, \boldsymbol{a}_3\}$ being a right-handed triad.

respectively (see Fig. 2). Also,

$$\begin{aligned}\boldsymbol{\lambda}\, ds &= -\boldsymbol{a}_1 (a_{11})^{-\frac{1}{2}}\, ds_1 + \boldsymbol{a}_2 (a_{22})^{-\frac{1}{2}}\, ds_2 \\ &= \varepsilon^{\alpha\beta}\, \nu_\alpha\, \boldsymbol{a}_\beta\, ds = \Big[\sum_{\alpha,\beta} e^{\alpha\beta} \frac{\boldsymbol{a}_\beta}{(a_{\beta\beta})^{\frac{1}{2}}}\, \nu_\alpha (a^{\alpha\alpha})^{\frac{1}{2}}\Big]\, ds,\end{aligned} \tag{9.8}$$

where the expression in the square bracket on the right-hand side of $(9.8)_2$ is written with the help of (8.3) and $(5.63)_2$. From $(9.8)_1$ and $(9.8)_2$, we obtain the formulae

$$ds_1 = (a^{22})^{\frac{1}{2}}\, \nu_2\, ds, \qquad ds_2 = (a^{11})^{\frac{1}{2}}\, \nu_1\, ds \tag{9.9}$$

relating ds_α to ds.

We apply (8.40) to the elementary curvilinear triangle described above. Over the curve c with the outward unit normal $\boldsymbol{\nu}$ the physical curve force vector is $\boldsymbol{N}_{(\boldsymbol{\nu})}$. Let $\boldsymbol{n}^{(1)}$ and $\boldsymbol{n}^{(2)}$ denote the physical force vectors acting on the sides of the coordinate curves with the outward unit normals $\boldsymbol{a}^1/(a^{11})^{\frac{1}{2}}$ and $\boldsymbol{a}^2/(a^{22})^{\frac{1}{2}}$, respectively, so that by (9.5) the physical force vectors on the opposite sides of the coordinate curves [corresponding to the outward unit normals (9.7)] are $-\boldsymbol{n}^{(1)}$ and $-\boldsymbol{n}^{(2)}$. Assume that the fields $\varrho\,\dot{\boldsymbol{v}}$ and $\varrho\,\boldsymbol{f}$ are bounded and that $\boldsymbol{N}$ is a continuous function of position and $\boldsymbol{\nu}$. Next, estimate the surface integrals in (8.40) and apply the mean-value theorem to the line integral. Then, in the limit, as the point under consideration (the vertex of the curvilinear triangle in Fig. 2) approaches the boundary curve[20] c:

$$\boldsymbol{N}_{(\boldsymbol{\nu})}\, ds - \boldsymbol{n}^{(2)}\, ds_1 - \boldsymbol{n}^{(1)}\, ds_2 = 0. \tag{9.10}$$

From (9.10), after use of (9.9), follows the relation

$$\begin{aligned}\boldsymbol{N} = \boldsymbol{N}_{(\boldsymbol{\nu})} &= \sum_\alpha \nu_\alpha\, \boldsymbol{n}^{(\alpha)} (a^{\alpha\alpha})^{\frac{1}{2}} = \boldsymbol{N}^\alpha \nu_\alpha, \\ \boldsymbol{N}^\alpha &= \boldsymbol{n}^{(\alpha)} (a^{\alpha\alpha})^{\frac{1}{2}} \quad (\text{no sum on } \alpha),\end{aligned} \tag{9.11}$$

which also indicates that $\boldsymbol{N}^\alpha$ transforms as a contravariant surface vector. In obtaining the above results, the point under consideration was taken to be an interior point on $\mathscr{s}$. But the argument can easily be extended to hold if the point is on the bounding curve with a continuous tangent.

Assume now that $\boldsymbol{N}$ is continuously differentiable, $\varrho\,\dot{\boldsymbol{v}}$ and $\varrho\,\boldsymbol{f}$ are continuous and substitute $(9.11)_1$ into (8.40). Then, recalling (5.66), using (8.18) and applying Stokes' theorem to the line integral, we obtain

$$\int_{\mathscr{P}} [\varrho(\dot{\boldsymbol{v}} - \boldsymbol{f}) - \boldsymbol{N}^\alpha{}_{|\alpha}]\, d\sigma = 0, \tag{9.12}$$

which holds for each part $\mathscr{P}$ of the Cosserat surface. Hence, from the vanishing of the integrand follows the local equations of motion

$$\boldsymbol{N}^\alpha{}_{|\alpha} + \varrho\,\boldsymbol{f} = \varrho\,\dot{\boldsymbol{v}}, \tag{9.13}$$

where a vertical bar stands for covariant differentiation with respect to $a_{\alpha\beta}$.

Next, turning to (8.54), with the help of $(9.11)_1$ we write the line integral containing the term $\boldsymbol{r}\times\boldsymbol{N}$ as

$$\begin{aligned}\int_{\partial\mathscr{P}} (\boldsymbol{r}\times\boldsymbol{N})\, ds &= \int_{\partial\mathscr{P}} (\boldsymbol{r}\times\boldsymbol{N}^\alpha \nu_\alpha)\, ds = \int_{\mathscr{P}} (\boldsymbol{r}\times\boldsymbol{N}^\alpha)_{|\alpha}\, d\sigma \\ &= \int_{\mathscr{P}} (\boldsymbol{r}\times\boldsymbol{N}^\alpha{}_{|\alpha} + \boldsymbol{a}_\alpha\times\boldsymbol{N}^\alpha)\, d\sigma.\end{aligned} \tag{9.14}$$

[20] The argument in obtaining (9.10) parallels that in three-dimensional continuum mechanics. See, for example, Sect. 203 of TRUESDELL and TOUPIN [1960, *14*].

By use of (9.14) and (9.13), as well as (8.18), (8.54) can be reduced to

$$\int_{\mathscr{P}} (\boldsymbol{a}_\alpha \times \boldsymbol{N}^\alpha + \varrho\, \boldsymbol{d} \times \bar{\boldsymbol{l}})\, d\sigma + \int_{\partial\mathscr{P}} \overline{\boldsymbol{M}}\, ds = 0, \tag{9.15}$$

where we have used $(8.19)_2$ and where we have introduced the field $\overline{\boldsymbol{M}}$, which has the physical dimension of couple per unit length:

$$\overline{\boldsymbol{M}} = \boldsymbol{d} \times \boldsymbol{M}, \qquad \text{phys. dim. } \overline{\boldsymbol{M}} = \left[\frac{M L^2 T^{-2}}{L}\right] = [M L T^{-2}]. \tag{9.16}$$

Recalling (9.1), $(8.4)_2$ and (8.16), from the application of the director momentum principle $(8.17)_3$ to the parts $\mathscr{P}_1$, $\mathscr{P}_2$ and again to $\mathscr{P}_1 \cup \mathscr{P}_2$ we can deduce the result [21]

$$\boldsymbol{M}_{(\boldsymbol{v})} = -\boldsymbol{M}_{(-\boldsymbol{v})}. \tag{9.17}$$

Since the contact director couple $\boldsymbol{M}_{(\boldsymbol{v})}$ depends on the unit normal $\boldsymbol{v}$, the vector $\overline{\boldsymbol{M}}$ in $(9.16)_1$ also depends on $\boldsymbol{v}$. Moreover, as the director field $\boldsymbol{d}$ does not depend on $\boldsymbol{v}$, it follows that

$$\overline{\boldsymbol{M}}_{(\boldsymbol{v})} = -\overline{\boldsymbol{M}}_{(-\boldsymbol{v})}. \tag{9.18}$$

Let $\boldsymbol{m}^{(1)}$, $\boldsymbol{m}^{(2)}$ denote the physical director couple vectors (associated with $\boldsymbol{M} = \boldsymbol{M}_{(\boldsymbol{v})}$) acting on the sides of the coordinate curves whose outward unit normal vectors are $\boldsymbol{a}^1/(a^{11})^{\frac{1}{2}}$, $\boldsymbol{a}^2/(a^{22})^{\frac{1}{2}}$; and let $\overline{\boldsymbol{m}}^{(1)} = \boldsymbol{d} \times \boldsymbol{m}^{(1)}$, $\overline{\boldsymbol{m}}^{(2)} = \boldsymbol{d} \times \boldsymbol{m}^{(2)}$ be the corresponding physical quantities associated with $\overline{\boldsymbol{M}} = \overline{\boldsymbol{M}}_{(\boldsymbol{v})}$. Assume $\boldsymbol{d} \times \varrho\, \bar{\boldsymbol{l}}$ is bounded and that $\boldsymbol{d} \times \boldsymbol{M}$ (and therefore $\overline{\boldsymbol{M}}$) is a continuous function of position and $\boldsymbol{v}$. Then, with the help of (9.17) and by an argument similar to that which led to (9.11), the application of $(8.17)_3$ to an elementary curvilinear triangle yields

$$\boldsymbol{M} = \boldsymbol{M}_{(\boldsymbol{v})} = \boldsymbol{M}^\alpha \nu_\alpha, \qquad \boldsymbol{M}^\alpha = \boldsymbol{m}^{(\alpha)} (a^{\alpha\alpha})^{\frac{1}{2}} \quad (\text{no sum on } \alpha), \tag{9.19}$$

and we also conclude that

$$\begin{aligned} &\overline{\boldsymbol{M}} = \overline{\boldsymbol{M}}_{(\boldsymbol{v})} = \overline{\boldsymbol{M}}^\alpha \nu_\alpha = (\boldsymbol{d} \times \boldsymbol{M}^\alpha)\, \nu_\alpha, \\ &\overline{\boldsymbol{M}}^\alpha = \overline{\boldsymbol{m}}^{(\alpha)} (a^{\alpha\alpha})^{\frac{1}{2}} = \boldsymbol{d} \times \boldsymbol{M}^\alpha = \boldsymbol{d} \times \boldsymbol{m}^{(\alpha)} (a^{\alpha\alpha})^{\frac{1}{2}} \quad (\text{no sum on } \alpha), \end{aligned} \tag{9.20}$$

in view of $(9.16)_1$. Assume now that $\boldsymbol{M}$ is continuously differentiable and that $\boldsymbol{d} \times \varrho\, \bar{\boldsymbol{l}}$ is continuous. After introducing (9.20) into (9.15) and transforming the line integral into a surface integral by Stokes' theorem, there follows the equation

$$\boldsymbol{a}_\alpha \times \boldsymbol{N}^\alpha + \boldsymbol{d} \times \varrho\, \bar{\boldsymbol{l}} + (\boldsymbol{d} \times \boldsymbol{M}^\alpha)_{|\alpha} = 0, \tag{9.21}$$

as a consequence of the moment of momentum principle. Similarly, after introducing $(9.19)_1$ in $(8.17)_3$, recalling (5.66) and making suitable continuity assumptions, transforming the line integral into a surface integral and using also (8.18), we deduce in the usual manner the second set of equations of motion:

$$\boldsymbol{M}^\alpha{}_{|\alpha} + \varrho\, \bar{\boldsymbol{l}} = \boldsymbol{m}, \tag{9.22}$$

as a consequence of the director momentum principle. In view of (9.22), (9.21) can also be written as

$$\boldsymbol{a}_\alpha \times \boldsymbol{N}^\alpha + \boldsymbol{d} \times \boldsymbol{m} + \boldsymbol{d}_{,\alpha} \times \boldsymbol{M}^\alpha = 0. \tag{9.23}$$

[21] The argument leading to (9.17) is entirely similar to that used in obtaining (9.5). It should be evident that a statement paralleling that made following (9.5) is applicable also to (9.17) if we replace the word "force" by "director couple".

We consider next the energy balance $(8.17)_5$. Recalling (5.66), using $(9.11)_1$ and $(9.19)_1$, and after invoking (8.18), (9.13), (9.22) and rearranging the result, we reduce the energy balance to

$$\int_{\mathscr{P}} [\varrho r - \varrho\, \dot{\varepsilon} + \boldsymbol{N}^\alpha \cdot \boldsymbol{v}_{,\alpha} + \boldsymbol{m} \cdot \boldsymbol{w} + \boldsymbol{M}^\alpha \cdot \boldsymbol{w}_{,\alpha}]\, d\sigma - \int_{\partial\mathscr{P}} h\, ds = 0. \tag{9.24}$$

Assuming that the integrand of the surface integral is bounded and that h is a continuous function of position and $\boldsymbol{\nu}$, by repeated application of (9.24), first to $\mathscr{P}_1$, $\mathscr{P}_2$ and $\mathscr{P}_1 \cup \mathscr{P}_2$ defined in (9.1) and then to an elementary curvilinear triangle on $\mathfrak{s}$, we get[22]

$$h = h_{(\boldsymbol{\nu})} = -h_{(-\boldsymbol{\nu})}, \tag{9.25}$$

together with

$$h = q^\alpha \nu_\alpha = \boldsymbol{q} \cdot \boldsymbol{\nu}, \qquad q^\alpha = h^{(\alpha)} (a^{\alpha\alpha})^{\frac{1}{2}}, \tag{9.26}$$

where $h^{(1)}$, $h^{(2)}$ are the values of the flux of heat across the coordinate curves whose outward unit normals are $\boldsymbol{a}^1/(a^{11})^{\frac{1}{2}}$, $\boldsymbol{a}^2/(a^{22})^{\frac{1}{2}}$ and $q^\alpha = q^\alpha(\theta^\gamma, t)$ are the contravariant components of the heat flux vector

$$\boldsymbol{q} = q^\alpha \boldsymbol{a}_\alpha. \tag{9.27}$$

Substituting $(9.26)_1$ into (9.24), making the appropriate continuity assumptions and transforming the line integral into a surface integral, we finally obtain

$$\varrho r - q^\alpha{}_{|\alpha} - \varrho\, \dot{\varepsilon} + P = 0, \qquad P = \boldsymbol{N}^\alpha \cdot \boldsymbol{v}_{,\alpha} + \boldsymbol{m} \cdot \boldsymbol{w} + \boldsymbol{M}^\alpha \cdot \boldsymbol{w}_{,\alpha}. \tag{9.28}$$

The scalar P in (9.28) may be called the *mechanical power* and corresponds to the *stress power* in the three-dimensional theory of continuum mechanics.

As the above energy equation is not in invariant form, we consider now the previous argument about *uniform* superposed rigid body angular velocity, the continuum occupying the same position at time t. From $(9.26)_1$–(9.27) and $(8.30)_2$, it follows that under superposed rigid body motions $\boldsymbol{q}$ and q^α transform according to $(8.25)_2$ and $(8.25)_1$, respectively. Keeping this in mind and recalling (8.30)–(8.32), then under superposed rigid body motions at time t in which the values of Q and $\dot{Q}$ are specified by (5.48), we again obtain (9.23) and the local energy equation is st ll given by $(9.28)_1$ but with P in the invariant form

$$\begin{aligned} P &= \boldsymbol{N}^\alpha \cdot \boldsymbol{\eta}_\alpha + \boldsymbol{m} \cdot (\boldsymbol{\Gamma} - d^\alpha \boldsymbol{\eta}_\alpha) + \boldsymbol{M}^\alpha \cdot (\boldsymbol{\Gamma}_{:\alpha} - \lambda^\beta_{\cdot\alpha} \boldsymbol{\eta}_\beta) \\ &= \boldsymbol{N}'^\alpha \cdot \boldsymbol{\eta}_\alpha + \boldsymbol{m} \cdot \boldsymbol{\Gamma} + \boldsymbol{M}^\alpha \cdot \boldsymbol{\Gamma}_{:\alpha}, \end{aligned} \tag{9.29}$$

where various kinematic results of Sect. 5 [including $(5.57)_2$ and (5.59)] have been used and where we have introduced the notation

$$\boldsymbol{N}'^\alpha = \boldsymbol{N}^\alpha - d^\alpha \boldsymbol{m} - \lambda^\alpha_{\cdot\gamma} \boldsymbol{M}^\gamma. \tag{9.30}$$

While the above expression for $\boldsymbol{N}'^\alpha$ is a convenient notation, it should be noted that the only values of (9.30) which occur in $(9.29)_2$, namely in the term

$$\boldsymbol{N}'^\alpha \cdot \boldsymbol{\eta}_\alpha = \boldsymbol{N}'^\alpha \cdot \eta_{\beta\alpha} \boldsymbol{a}^\beta,$$

are the symmetric components

$$N'^{\alpha\beta} = \tfrac{1}{2} (\boldsymbol{N}'^\alpha \cdot \boldsymbol{a}^\beta + \boldsymbol{N}'^\beta \cdot \boldsymbol{a}^\alpha). \tag{9.31}$$

The conservation of mass (8.18), the equations of motion (9.13) and (9.22), the Eq. (9.23) [or equivalently (9.21)] and the energy equation $(9.28)_1$ are the

[22] The arguments here again parallel those used previously in obtaining (9.5) and (9.11).

basic field equations for a Cosserat surface. Each of these five sets of equations is a necessary and sufficient condition for the respective conservation laws stated in (8.17). For completeness, we also obtain here a local entropy production inequality. Recalling (8.24), we introduce $(9.26)_1$ into (8.24) and after transforming the line integral into a surface integral, we obtain the local inequality

$$\varrho\,\theta\,\gamma = \varrho\,\theta\,\dot{\eta} - \varrho\,r + q^{\alpha}{}_{|\alpha} - q^{\alpha}\frac{\theta_{,\alpha}}{\theta} \geqq 0, \tag{9.32}$$

which is deduced with the help of (8.18). From combination of (9.32) and the energy equation $(9.28)_1$ follows the inequality

$$-\varrho\,\dot{\varepsilon} + \varrho\,\theta\,\dot{\eta} + P - q^{\alpha}\frac{\theta_{,\alpha}}{\theta} \geqq 0. \tag{9.33}$$

With reference to the terminology of a process (or a thermodynamic process) introduced in Sect. 8, we now observe that to define a process it is sufficient to prescribe the nine functions $\boldsymbol{r}$, $\boldsymbol{d}$, η and those in $(8.23)_{1,2}$ such that (9.23) is satisfied. The remaining three functions $(8.23)_3$ can then be determined from (9.13), (9.22) and $(9.28)_1$. Also, in order to specify an admissible process it is sufficient to prescribe $\boldsymbol{r}$, $\boldsymbol{d}$ and η (as functions of position and time) and to specify the fields $(8.23)_{1,2}$ by constitutive equations consistent with (9.23). Then, to each such choice of $\boldsymbol{r}$, $\boldsymbol{d}$ and η, there corresponds a unique admissible process since the functions $(8.23)_3$ can be so chosen that the field equations (9.13), (9.22) and $(9.28)_1$ are satisfied. Moreover, since (8.24) was postulated to hold for every admissible process, it follows that the inequality (9.33) will serve to restrict the class of admissible processes or equivalently the constitutive equations which describe the thermo-mechanical behavior of the material. In this connection, we note that (9.23) is regarded also as a restriction on the constitutive equations.

The energy equation $(9.28)_1$, which involves the specific internal energy ε, can be written in terms of an alternative thermodynamic potential, namely the specific *Helmholtz free energy*. For this purpose, we introduce a (two-dimensional) Helmholtz function per unit mass $\psi = \psi(\theta^{\alpha}, t)$ by

$$\psi = \varepsilon - \eta\,\theta. \tag{9.34}$$

By use of (9.34), the energy equation $(9.28)_1$ can be expressed as

$$\varrho\,r - q^{\alpha}{}_{|\alpha} - \varrho(\theta\,\dot{\eta} + \dot{\theta}\,\eta) - \varrho\,\dot{\psi} + P = 0 \tag{9.35}$$

and when this is combined with (9.32), we obtain the inequality

$$-\varrho\,\dot{\psi} - \varrho\,\eta\,\dot{\theta} + P - q^{\alpha}\frac{\theta_{,\alpha}}{\theta} \geqq 0, \tag{9.36}$$

where P is given by $(9.28)_2$ or the invariant form $(9.29)_2$. The remarks made following (9.33) refer to the energy equation $(9.28)_1$ and the entropy inequality in the form (9.33) and when the entropy η is regarded as an independent thermodynamic variable. If, on the other hand, the temperature θ (as a function of position and time) is regarded as the independent thermodynamic variable (in place of η) and the energy equation (9.35) and the inequality (9.36) are employed instead of $(9.28)_1$ and (9.33), the previous remarks [following (9.33)] require only a minor modification. In particular, in order to specify an admissible process, it will be sufficient to prescribe $\boldsymbol{r}$, $\boldsymbol{d}$ and θ (as functions of position and time) and to specify the mechanical fields $(8.23)_1$, as well as ψ, η and the components of heat flux q^α, by constitutive equations consistent with (9.23).[23]

β) *Alternative forms of the field equations.* The elegance and the simplicity of the foregoing derivation does not display the relative complexity of the results. For this reason and for future use, we now deduce the basic field equations in tensor components. The contact force $\boldsymbol{N}$ and the contact couple $\boldsymbol{M}$, when referred to the base vectors $\boldsymbol{a}_i$, can be written as

$$\boldsymbol{N} = N^i \boldsymbol{a}_i = N^\alpha \boldsymbol{a}_\alpha + N^3 \boldsymbol{a}_3, \qquad \boldsymbol{M} = M^i \boldsymbol{a}_i = M^\alpha \boldsymbol{a}_\alpha + M^3 \boldsymbol{a}_3. \tag{9.37}$$

Similarly, the assigned force $\boldsymbol{f}$, the assigned couple $\boldsymbol{l}$ and the surface couple $\boldsymbol{m}$ can be expressed as

$$\boldsymbol{f} = f^i \boldsymbol{a}_i, \qquad \boldsymbol{l} = l^i \boldsymbol{a}_i \tag{9.38}$$

and

$$\boldsymbol{m} = m^i \boldsymbol{a}_i. \tag{9.39}$$

Since $\boldsymbol{N}^\alpha$ in $(9.11)_1$ transforms as a contravariant surface vector, we can put[24]

$$\boldsymbol{N}^\alpha = N^{\alpha i} \boldsymbol{a}_i = N^{\alpha\gamma} \boldsymbol{a}_\gamma + N^{\alpha 3} \boldsymbol{a}_3. \tag{9.40}$$

Also, by $(9.37)_1$,

$$N^i = N^{\alpha\gamma} \nu_\alpha, \qquad N^\gamma = N^{\alpha\gamma} \nu_\alpha, \qquad N^3 = N^{\alpha 3} \nu_\alpha, \tag{9.41}$$

where $N^{\alpha\gamma}$ and $N^{\alpha 3}$ are surface tensors under the transformation of surface coordinates. In a similar manner, since $\boldsymbol{M}^\alpha$ in $(9.19)_1$ transforms as a contravariant surface vector, we can set

$$\boldsymbol{M}^\alpha = M^{\alpha i} \boldsymbol{a}_i = M^{\alpha\gamma} \boldsymbol{a}_\gamma + M^{\alpha 3} \boldsymbol{a}_3 \tag{9.42}$$

and by $(9.37)_2$

$$M^i = M^{\alpha i} \nu_\alpha, \qquad M^\gamma = M^{\alpha\gamma} \nu_\alpha, \qquad M^3 = M^{\alpha 3} \nu_\alpha, \tag{9.43}$$

where $M^{\alpha\gamma}$ and $M^{\alpha 3}$ are surface tensors under the transformation of surface coordinates. It is also clear that $\overline{\boldsymbol{M}}$ defined by (9.16) and $\overline{\boldsymbol{M}}^\alpha$ in $(9.20)_1$ may be expressed as

$$\overline{\boldsymbol{M}} = \overline{M}^i \boldsymbol{a}_i, \qquad \overline{\boldsymbol{M}}^\alpha = \overline{M}^{\alpha i} \boldsymbol{a}_i, \qquad \overline{M}^i = \overline{M}^{\alpha i} \nu_\alpha, \tag{9.44}$$

the components $\overline{M}^{\alpha\gamma}$ and $\overline{M}^{\alpha 3}$ being surface tensors under the transformation of surface coordinates.

To obtain the equations of motion (9.13) in component form, consider the scalar product of (9.13) with $\boldsymbol{a}^\beta$ and again with $\boldsymbol{a}^3$ and deduce

$$\begin{aligned} &(\boldsymbol{a}^\beta \cdot \boldsymbol{N}^\alpha)_{|\alpha} - \boldsymbol{N}^\alpha \cdot \boldsymbol{a}^\beta{}_{|\alpha} + \varrho (\boldsymbol{f} - \dot{\boldsymbol{v}}) \cdot \boldsymbol{a}^\beta = 0, \\ &(\boldsymbol{a}_3 \cdot \boldsymbol{N}^\alpha)_{|\alpha} - \boldsymbol{N}^\alpha \cdot \boldsymbol{a}_{3|\alpha} + \varrho (\boldsymbol{f} - \dot{\boldsymbol{v}}) \cdot \boldsymbol{a}_3 = 0. \end{aligned} \tag{9.45}$$

Introducing c^i for the components of the acceleration vector by

$$\boldsymbol{c} = \dot{\boldsymbol{v}} = c^i \boldsymbol{a}_i \tag{9.46}$$

and using (9.40) and $(4.13)_{2,3}$, from (9.45) we deduce

$$N^{\alpha\beta}{}_{|\alpha} - b^\beta_\alpha N^{\alpha 3} + \varrho f^\beta = \varrho c^\beta, \qquad N^{\alpha 3}{}_{|\alpha} + b_{\alpha\beta} N^{\alpha\beta} + \varrho f^3 = \varrho c^3. \tag{9.47}$$

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[23] The results in Subsect. α) can be easily specialized to the isothermal or the purely mechanical theory. For the isothermal theory with the heat flux $q^\alpha = 0$, r in $(9.28)_1$ or (9.35) is a specified function in order to balance the energy equation.

[24] Our notation for the order of superscripts αi in $N^{\alpha i}$ (and also in $M^{\alpha i}$ to be introduced presently) differs from that used by GREEN, NAGHDI and WAINWRIGHT [1965, *4*]. The above notation also differs from that used by GREEN and NAGHDI [1967, *4*] but is in agreement with a number of subsequent papers, e.g., [1968, *7*] and [1971, *6*].

In an entirely similar manner, using (9.42) and (9.39), we obtain from (9.22) the equations

$$M^{\alpha\beta}{}_{|\alpha} - b^{\beta}_{\alpha} M^{\alpha 3} + \varrho\, \bar{l}^{\beta} = m^{\beta}, \qquad M^{\alpha 3}{}_{|\alpha} + b_{\alpha\beta} M^{\alpha\beta} + \varrho\, \bar{l}^{3} = m^{3}, \tag{9.48}$$

where $\bar{l}^i$ are the components of $\bar{\boldsymbol{l}}$ in $(8.19)_2$ referred to the base vectors $\boldsymbol{a}_i$, i.e.,

$$\bar{\boldsymbol{l}} = \bar{l}^i \boldsymbol{a}_i. \tag{9.49}$$

In an analogous manner, from (9.23) we have

$$\varepsilon_{jim}[\delta^j_\alpha N^{\alpha i} + d^j m^i + \lambda^j_{.\alpha} M^{\alpha i}] = 0, \tag{9.50}$$

or equivalently

$$\begin{gathered} \varepsilon_{\beta\alpha}[N^{\alpha\beta} + m^\beta d^\alpha + M^{\gamma\beta} \lambda^\alpha_{.\gamma}] = 0, \\ N^{\alpha 3} + (m^3 d^\alpha - m^\alpha d^3) + M^{\gamma 3} \lambda^\alpha_{.\gamma} - M^{\gamma\alpha} \lambda^3_{.\gamma} = 0, \end{gathered} \tag{9.51}$$

where ε_{jim} and $\varepsilon_{\beta\alpha}$ are defined by (5.43) and (5.63) and the components of $\lambda^i_{.\alpha}$ are given by (5.28). In order to obtain the energy equation in terms of the tensor components of the various quantities in $(9.28)_1$ or (9.35), it will suffice to record here the expression for P in the form

$$P = N'^{\alpha\beta} \eta_{\alpha\beta} + m^i \dot{d}_i + M^{\alpha i} \dot{\lambda}_{i\alpha}, \tag{9.52}$$

where $N'^{\alpha\beta}$ defined by (9.31) can be written as

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} - m^\alpha d^\beta - M^{\gamma\alpha} \lambda^\beta_{.\gamma}. \tag{9.53}$$

The above expression is equivalent to $(9.51)_1$.

The various field equations derived in this section, including the component forms (9.47)–(9.48) and (9.51) of the equations of motion, were obtained previously.[25] Although GREEN, NAGHDI and WAINWRIGHT did not postulate a director momentum principle corresponding to $(8.17)_3$, in the course of derivation they assumed the local equation (9.22) from which $(9.48)_{1,2}$ result. Without such an assumption, the right-hand sides of $(9.48)_{1,2}$ would be zero and the resulting theory would be too restrictive for a Cosserat surface. Earlier in Sect. 4, we called attention to a paper of ERICKSEN and TRUESDELL which contains a general development of the kinematics of oriented media; this paper also includes a derivation of equilibrium equations for shells by direct approach.[26] In the context of an oriented surface, the equilibrium equations given by ERICKSEN and TRUESDELL are incomplete or restrictive;[27] and this is mainly because their basic principles do not include a director momentum principle corresponding to $(8.17)_3$. However, if their basic principles are supplemented by $(8.17)_3$ [or equivalently (9.22)], then their equilibrium equations can be shown to be of the same form as (9.47)–(9.48) and (9.51).[28]

[25] GREEN, NAGHDI and WAINWRIGHT [1965, *4*]. The equations of equilibrium contained in the paper of COHEN and DE SILVA [1966, *2*], apart from a generality resulting from the fact that they admit a triad of deformable directors at each material point of their directed surface, are similar to (9.47)–(9.48) and (9.51).

[26] Sects. 24–27 of ERICKSEN and TRUESDELL [1958, *1*]. This derivation of equilibrium equations for shells may also be found in Sect. 212 of TRUESDELL and TOUPIN [1960, *14*].

[27] The equilibrium equations of ERICKSEN and TRUESDELL are of the type obtained in Sect. 10 for the restricted theory.

[28] In this connection and with reference to Sect. 26 of the paper by ERICKSEN and TRUESDELL [1958, *1*], we may note that their notations $\boldsymbol{S}$, $\boldsymbol{F}$, $\boldsymbol{M}$, $\boldsymbol{L}$ correspond, respectively, to our $\boldsymbol{N}$, $\varrho\boldsymbol{f}$, $\boldsymbol{d}\times\boldsymbol{M}$, $\boldsymbol{d}\times\varrho\boldsymbol{l}$.

γ) *Linearized field equations.* Previously, with reference to the linearization of the kinematical results for a Cosserat surface (Sect. 6), it was assumed that all kinematic measures such as $e_{\alpha\beta}$, $\varkappa_{i\alpha}$ and γ_i, as well as their derivatives with respect to the surface coordinates and time, are of $O(\varepsilon)$. These must now be supplemented by additional assumptions in a complete infinitesimal theory. Thus, let θ_0 and η_0 refer to a standard temperature and entropy in the (initial) reference configuration and also assume that the vector fields $\boldsymbol{N}^\alpha, \boldsymbol{M}^\alpha, \boldsymbol{m}$ are all zero in the reference configuration. We further assume that $\boldsymbol{N}^\alpha$, $\boldsymbol{M}^\alpha$, $\boldsymbol{m}$ (or their components) when expressed in suitable non-dimensional forms, as well as their derivatives with respect to the surface coordinates, are of $O(\varepsilon)$; and that $(\theta-\theta_0)/\theta_0$ and $(\eta-\eta_0)/\eta_0$ and their derivatives are $O(\varepsilon)$.

Recalling the linearization procedures of Sect. 6, we again avoid introducing additional notations but now regard $N^{\alpha i}$, $M^{\alpha i}$, m^i, etc., as well as θ and η (measured from their reference values), as infinitesimal quantities of $O(\varepsilon)$. As a result of linearization, all tensors are now referred to the initial undeformed surface and covariant differentiation is with respect to $A_{\alpha\beta}$. It follows that in the Eqs. (9.47)–(9.51) each term is $O(\varepsilon)$ and that $b_{\alpha\beta}$, d_i, $\lambda_{i\alpha}$ must be replaced to the order ε by $B_{\alpha\beta}$, D_i, $\Lambda_{i\alpha}$, respectively. We omit the details which involve a straightforward calculation and merely record the linearized versions of (9.47)–(9.51) as follows:

$$N^{\alpha\beta}{}_{|\alpha}-B^\beta_\alpha N^{\alpha 3}+\varrho_0\bar{F}^\beta=0, \qquad N^{\alpha 3}{}_{|\alpha}+B_{\alpha\beta}N^{\alpha\beta}+\varrho_0\bar{F}^3=0, \tag{9.54}$$

$$M^{\alpha\beta}{}_{|\alpha}-B^\beta_\alpha M^{\alpha 3}+\varrho_0\bar{L}^\beta=m^\beta, \qquad M^{\alpha 3}{}_{|\alpha}+B_{\alpha\beta}M^{\alpha\beta}+\varrho_0\bar{L}^3=m^3 \tag{9.55}$$

and

$$\begin{gathered}\bar{\varepsilon}_{\beta\alpha}[N^{\alpha\beta}+m^\beta D^\alpha+M^{\gamma\beta}\Lambda^{\alpha}_{\cdot\gamma}]=0,\\ N^{\alpha 3}+(m^3 D^\alpha-m^\alpha D^3)+M^{\gamma 3}\Lambda^{\alpha}_{\cdot\gamma}-M^{\gamma\alpha}\Lambda^{3}_{\cdot\gamma}=0,\end{gathered} \tag{9.56}$$

where $\bar{\varepsilon}_{\beta\alpha}$ was introduced previously in (6.30), the vertical bar in (9.54)–(9.55) and in the rest of this subsection stands for covariant differentiation with respect to $A_{\alpha\beta}$, all quantities are now referred to the base vectors $\boldsymbol{A}_i$ of the initial Cosserat surface and the notations $\bar{F}^i$ and $\bar{L}^i$ stand for the components of the vector fields $\bar{\boldsymbol{f}}$ and $\bar{\boldsymbol{l}}$ in (8.19) referred to the base vectors $\boldsymbol{A}_i$, i.e.,

$$\bar{F}^i=\bar{\boldsymbol{f}}\cdot\boldsymbol{A}^i, \qquad \bar{L}^i=\bar{\boldsymbol{l}}\cdot\boldsymbol{A}^i. \tag{9.57}$$

Also, upon linearization, (9.53) becomes

$$N'^{\alpha\beta}=N'^{\beta\alpha}=N^{\alpha\beta}-m^\alpha D^\beta-M^{\gamma\alpha}\Lambda^{\beta}_{\cdot\gamma}. \tag{9.58}$$

Before recording the linearized energy equation, we note that the heat flux h in the infinitesimal theory assumes the form

$$h={}_0\nu^\alpha Q_\alpha={}_0\nu_\alpha Q^\alpha, \qquad Q^\alpha=\boldsymbol{q}\cdot\boldsymbol{A}^\alpha, \tag{9.59}$$

where Q_α defined by $(9.59)_2$ are the components of the heat flux vector per unit length (in the reference configuration) per unit time and ${}_0\nu^\alpha$ are the components of the outward unit normal ${}_0\boldsymbol{\nu}={}_0\nu^\alpha\boldsymbol{A}_\alpha$ to the θ^α-curves on the reference surface $\mathscr{S}$. By virtue of the assumptions stated above and in view of (9.52), the energy equation (9.35) becomes

$$\begin{gathered}\varrho_0 r-Q^\alpha{}_{|\alpha}-\varrho_0(\theta\dot{\eta}+\eta\dot{\theta})-\varrho_0\dot{\psi}+P=0,\\ P=N'^{\alpha\beta}\dot{e}_{\alpha\beta}+m^i\dot{\gamma}_i+M^{\alpha i}\dot{\varkappa}_{i\alpha},\end{gathered} \tag{9.60}$$

where $N'^{\alpha\beta}$ is given by (9.58) and the infinitesimal kinematical quantities in $(9.60)_2$ are defined by (6.18). At this point the energy equation $(9.60)_1$ still contains terms of up to $O(\varepsilon^2)$ such as $\eta\dot{\theta}$ and those in P. However, as will become apparent later (in Chap. D), these terms will cancel others resulting from $\dot{\psi}$; and the final linearized (residual) energy equation, after linearization of $\theta\dot{\eta}$ term, will only contain terms of $O(\varepsilon)$ in line with the linearized versions of the other field equations.

It is instructive to discuss briefly special cases of Eqs. (9.54)–(9.58). Consider first the special case in which the initial director is along the unit normal $\boldsymbol{A}_3$ specified by (6.21). In this case, $\Lambda_{i\alpha}$ are given by $(6.22)_{2,3}$ and corresponding to the alternative set of the kinematic measures (6.22)–(6.23), we introduce the variables[29]

$$\hat{N}^{\alpha\beta} = N^{\alpha\beta}, \quad \hat{M}^{\alpha i} = D M^{\alpha i}, \tag{9.61}$$

$$\begin{aligned} V^\alpha &= N^{\alpha 3} = D m^\alpha + D B^\alpha_\gamma M^{\gamma 3} + M^{\beta\alpha} D_{,\beta}, \\ V^3 &= D m^3 - D B_{\alpha\beta} M^{\alpha\beta} + D_{,\alpha} M^{\alpha 3}. \end{aligned} \tag{9.62}$$

The first of (9.62) satisfies $(9.56)_2$ while in terms of the above variables (9.58), which satisfies $(9.56)_1$, becomes

$$N'^{\alpha\beta} = N'^{\beta\alpha} = \hat{N}^{\alpha\beta} + \hat{M}^{\gamma\alpha} B^\beta_\gamma. \tag{9.63}$$

The equations of motion (9.54)–(9.55) can then be written in the more compact forms

$$\hat{N}^{\alpha\beta}{}_{|\alpha} - B^\beta_\alpha V^\alpha + \varrho_0 \bar{F}^\beta = 0, \qquad V^\alpha{}_{|\alpha} + B_{\alpha\beta} \hat{N}^{\alpha\beta} + \varrho_0 \bar{F}^3 = 0, \tag{9.64}$$

$$\hat{M}^{\alpha\beta}{}_{|\alpha} + \varrho_0 \hat{L}^\beta = V^\beta, \qquad \hat{M}^{\alpha 3}{}_{|\alpha} + \varrho_0 \hat{L}^3 = V^3, \tag{9.65}$$

where we have put

$$\hat{L}^i = D L^i. \tag{9.66}$$

We now further specialize the above alternative forms of the equations of motion to the case in which the initial director $\boldsymbol{D}$ is of constant magnitude and coincident with the unit normal $\boldsymbol{A}_3$. With $D=1$, the distinction between $\hat{M}^{\alpha i}$ and $M^{\alpha i}$, $\hat{L}^i$ and L^i disappears while (9.62)–(9.63) and $(9.56)_1$ reduce to

$$\begin{gathered} V^\alpha = N^{\alpha 3} = m^\alpha + B^\alpha_\gamma M^{\gamma 3}, \qquad V^3 = m^3 - B_{\alpha\beta} M^{\alpha\beta}, \\ N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} + M^{\gamma\alpha} B^\beta_\gamma \end{gathered} \tag{9.67}$$

and

$$\bar{\varepsilon}_{\beta\alpha} [N^{\alpha\beta} - B^\alpha_\gamma M^{\gamma\beta}] = 0. \tag{9.68}$$

Then, the equations of motion (9.64)–(9.65) can be expressed as[30]

$$N^{\alpha\beta}{}_{|\alpha} - B^\beta_\alpha V^\alpha + \varrho_0 \bar{F}^\beta = 0, \qquad V^\alpha{}_{|\alpha} + B_{\alpha\beta} N^{\alpha\beta} + \varrho_0 \bar{F}^3 = 0, \tag{9.69}$$

$$M^{\alpha\beta}{}_{|\alpha} + \varrho_0 L^\beta = V^\beta, \qquad M^{\alpha 3}{}_{|\alpha} + \varrho_0 L^3 = V^3. \tag{9.70}$$

Also, with reference to the energy equation (9.60), we observe that the expression for P can be expressed in an alternative form when $\boldsymbol{D} = \boldsymbol{A}_3$. Thus, in terms of the kinematic variables $e_{\alpha\beta}$, $\varrho_{i\alpha}$, γ_i in (6.24), P becomes

$$P = N'^{\alpha\beta} \dot{e}_{\alpha\beta} + M^{\alpha i} \dot{\varrho}_{i\alpha} + V^i \dot{\gamma}_i, \tag{9.71}$$

where we have used the definitions (9.67).

[29] [1968, *6*] and [1970, *7*]. The variables (9.61)–(9.62) are of special significance in regard to the classical linear theories of shells and plates. At this stage the forms of these new variables can be easily motivated from the energy equation (9.60) with the help of the alternative kinematic measures (6.22)–(6.23).

[30] The equations of motion (or equilibrium) in the linear theory of a Cosserat surface, either in the general forms (9.69)–(9.70) or their special cases, have been used in a number of recent papers, e.g., [1967, *4*], [1967, *6*] and [1968, *6*]. A special case of the variables (9.61)–(9.62) and the Eqs. (9.64)–(9.65) with $B_{\alpha\beta} = 0$ have been employed in the application of the theory with initial variable directors to plates of variable thickness [1971, *6*].

We close this subsection by making an observation regarding a special case of (9.69)–(9.70) in the linearized theory of an initially flat Cosserat plate for which (6.26) holds. In this case (9.67)–(9.68) reduce to

$$V^\alpha = N^{\alpha 3} = m^\alpha, \qquad V^3 = m^3, \qquad N'^{\alpha\beta} = N^{\alpha\beta} = N^{\beta\alpha}. \tag{9.72}$$

Hence, for a flat Cosserat surface, $N^{\alpha 3}$ and m^α are identical, $N^{\alpha\beta}$ is symmetric and the equations of motion (9.69)–(9.70) become

$$N^{\alpha\beta}{}_{|\alpha} + \varrho_0 \bar{F}^\beta = 0, \qquad M^{\alpha 3}{}_{|\alpha} + \varrho_0 \bar{L}^3 = V^3, \tag{9.73}$$

$$M^{\alpha\beta}{}_{|\alpha} + \varrho_0 \bar{L}^\beta = V^\beta, \qquad V^\alpha{}_{|\alpha} + \varrho_0 \bar{F}^3 = 0. \tag{9.74}$$

Evidently the variables occurring in the above equations separate into two sets, namely $\{N^{\alpha\beta}, V^3, M^{\alpha 3}; \bar{F}^\beta, \bar{L}^3\}$ and $\{M^{\alpha\beta}, V^\beta; \bar{F}^3, \bar{L}^\beta\}$, and this is analogous to the separation of the kinematic variables in (6.25), as discussed following (6.26). In fact, as will become apparent later, the former set governed by the equations of motion (9.73) is associated with the extensional motion while the latter set of variables governed by the equations of motion (9.74) is associated with the flexural motion of the Cosserat plate.

δ) *The basic field equations in terms of a reference state.* The various forms of the field equations in this section deduced from the conservation laws (8.17) involve field quantities which are measured in the present configuration. We now proceed to obtain the counterparts of these results from the conservation laws (8.58) which involve field quantities measured in the reference configuration.

First, we observe that by arguments similar to those used earlier in this section [Subsect. α)], we can readily deduce the results

$$_R\boldsymbol{N} = {}_R\boldsymbol{N}^\alpha \, {}_0\nu_\alpha, \qquad\qquad {}_R\boldsymbol{M} = {}_R\boldsymbol{M}^\alpha \, {}_0\nu_\alpha, \tag{9.75}$$

$$_Rh = {}_Rq^\alpha \, {}_0\nu_\alpha = {}_R\boldsymbol{q} \cdot {}_0\boldsymbol{\nu}, \qquad\qquad {}_R\boldsymbol{q} = {}_Rq^\alpha \boldsymbol{A}_\alpha, \tag{9.76}$$

where ${}_Rq^\alpha = {}_Rq^\alpha(\theta^\gamma; t)$ are the contravariant components of the heat flux vector ${}_R\boldsymbol{q}$ defined by $(9.76)_3$. The relation $(9.75)_1$ is the counterpart of (9.11), while $(9.75)_2$ and (9.76) follow from the conservation laws (8.58) at a stage comparable to that where (9.19) and (9.26) were obtained from those in (8.17). Next, making smoothness assumptions similar to those made in obtaining (9.13) and using $(9.75)_1$, from $(8.58)_2$ we obtain the local equations of motion in terms of ${}_R\boldsymbol{N}^\alpha$:

$$_R\boldsymbol{N}^\alpha{}_{|\alpha} + \varrho_0 \boldsymbol{f} = \varrho_0 \dot{\boldsymbol{v}}, \tag{9.77}$$

where now the vertical bar stands for covariant differentiation with respect to $A_{\alpha\beta}$ of the reference surface. The remaining field equations are obtained in a similar manner from the last three of (8.58). Thus, in place of (9.22), (9.23) and (9.28) we now have

$$_R\boldsymbol{M}^\alpha{}_{|\alpha} + \varrho_0 \boldsymbol{l} = {}_R\boldsymbol{m}, \tag{9.78}$$

$$\boldsymbol{a}_\alpha \times {}_R\boldsymbol{N}^\alpha + \boldsymbol{d} \times {}_R\boldsymbol{m} + \boldsymbol{d}_{,\alpha} \times {}_R\boldsymbol{M}^\alpha = 0, \tag{9.79}$$

$$\varrho_0 r - {}_Rq^\alpha{}_{|\alpha} - \varrho_0 \dot{\varepsilon} + {}_R\boldsymbol{N}^\alpha \cdot \boldsymbol{v}_{,\alpha} + {}_R\boldsymbol{m} \cdot \boldsymbol{w} + {}_R\boldsymbol{M}^\alpha \cdot \boldsymbol{w}_{,\alpha} = 0 \tag{9.80}$$

and we emphasize that the vertical bar throughout the present subsection stands for covariant differentiation with respect to $A_{\alpha\beta}$. Also, from (8.59) and (8.24), the entropy production inequality in terms of the field quantities in the reference state is given by

$$\varrho_0 \theta \gamma = \varrho_0 \theta \dot{\eta} - \varrho_0 r + {}_Rq^\alpha{}_{|\alpha} - {}_Rq^\alpha \frac{\theta_{,\alpha}}{\theta} \geqq 0. \tag{9.81}$$

By use of (9.34), from combination of (9.80) and (9.81) follows the inequality

$$-\varrho_0\,\dot{\psi}-\varrho_0\,\eta\,\dot{\theta}+{}_R P-{}_R q^\alpha\,\frac{\theta_{,\alpha}}{\theta}\geqq 0, \tag{9.82}$$

where

$${}_R P={}_R\boldsymbol{N}^\alpha\cdot\boldsymbol{v}_{,\alpha}+{}_R\boldsymbol{m}\cdot\boldsymbol{w}+{}_R\boldsymbol{M}^\alpha\cdot\boldsymbol{w}_{,\alpha}. \tag{9.83}$$

The inequality (9.82) is the counterpart of (9.36) and the mechanical power ${}_R P$ is the counterpart of $(9.28)_2$ in terms of the reference state.

The structure of (9.77)–(9.80) is entirely analogous to that of the field equations (9.13), (9.22), (9.23) and (9.28). The field equations in terms of a reference state and in tensor components can be obtained from (9.77)–(9.80) but we do not consider these in detail here. We note, however, that if we put

$${}_R\boldsymbol{N}^\alpha={}_R N^{\alpha i}\,\boldsymbol{A}_i,\qquad {}_R\boldsymbol{m}={}_R m^i\,\boldsymbol{A}_i,\qquad {}_R\boldsymbol{M}^\alpha={}_R M^{\alpha i}\,\boldsymbol{A}_i \tag{9.84}$$

and also refer $\boldsymbol{f}$, $\dot{\boldsymbol{v}}$, $\boldsymbol{l}$ to the reference base vectors $\boldsymbol{A}_i$, then the equations of motion in tensor components resulting from (9.77)–(9.78) will be similar in forms to the equations of motion (9.47)–(9.48).

It is of some interest to obtain the relations between the field variables ${}_R\boldsymbol{N}^\alpha$, ${}_R\boldsymbol{M}^\alpha$, ${}_R h$ and $\boldsymbol{N}^\alpha$, $\boldsymbol{M}^\alpha$, h which are measured in the present configuration. For this purpose, recall the expression for the resultant contact force as defined by the line integrals $(8.4)_1$ and $(8.56)_1$. Their integrands, by use of (9.11), $(9.75)_1$ and formulae of the type (8.1)–(8.2), can be expressed as

$$\begin{aligned}\boldsymbol{N}\,ds&=\boldsymbol{N}^\alpha\,\nu_\alpha\,ds=a^{\frac{1}{2}}\,e_{\alpha\beta}\,\boldsymbol{N}^\alpha\,\lambda^\beta\,ds=a^{\frac{1}{2}}\,e_{\alpha\beta}\,\boldsymbol{N}^\alpha\,d\theta^\beta,\\ {}_R\boldsymbol{N}\,dS&={}_R\boldsymbol{N}^\alpha\,{}_0\nu_\alpha\,dS=A^{\frac{1}{2}}\,e_{\alpha\beta}\,{}_R\boldsymbol{N}^\alpha\,{}_0\lambda^\beta\,dS=A^{\frac{1}{2}}\,e_{\alpha\beta}\,{}_R\boldsymbol{N}^\alpha\,d\theta^\beta,\end{aligned} \tag{9.85}$$

where $e_{\alpha\beta}$ is the permutation symbol defined in (5.63) and where ${}_0\lambda^\beta$ are the components of the unit tangent vector to the curve C lying in the reference surface. Since for a given Cosserat surface in the present configuration, the resultants $(8.4)_1$ and $(8.56)_1$ must be the same, it follows that $\boldsymbol{N}\,ds={}_R\boldsymbol{N}\,dS$ and hence by (9.85) and (4.41) we have

$$a^{\frac{1}{2}}\,\boldsymbol{N}^\alpha=A^{\frac{1}{2}}\,{}_R\boldsymbol{N}^\alpha\quad\text{or}\quad {}_R\boldsymbol{N}^\alpha=J\,\boldsymbol{N}^\alpha. \tag{9.86}$$

Similar results hold between $\boldsymbol{M}^\alpha$ and ${}_R\boldsymbol{M}^\alpha$, between q^α and ${}_R q^\alpha$ and between $\boldsymbol{m}$ and ${}_R\boldsymbol{m}$. The latter follows from the fact that $\boldsymbol{m}\,d\sigma={}_R\boldsymbol{m}\,d\Sigma$. For later reference, we summarize the last results as

$$\{{}_R\boldsymbol{N}^\alpha,\,{}_R\boldsymbol{m},\,{}_R\boldsymbol{M}^\alpha,\,{}_R q^\alpha\}=\frac{\varrho_0}{\varrho}\,\{\boldsymbol{N}^\alpha,\,\boldsymbol{m},\,\boldsymbol{M}^\alpha,\,q^\alpha\}, \tag{9.87}$$

where we have also used (4.42).

10. Derivation of the basic field equations of a restricted theory: I. Direct approach. Our derivation of the field equations by a direct approach in Sects. 8–9 is founded on the concept of a Cosserat surface whose basic kinematic variables are the position vector $\boldsymbol{r}$ and the director $\boldsymbol{d}$. As already mentioned in Sect. 5 [Subsect. γ)], other developments by direct approach in which a director is not admitted are possible. For example, we may consider a material surface embedded in a Euclidean 3-space and construct a theory in which the basic kinematic ingredients are the position vector of the surface and suitable first and higher order gradients (with respect to the surface coordinates) of the position vector.[31] Although we do not undertake a general development of this type in this section or even a special case in which the basic kinematic variables are the position vector $\boldsymbol{r}$ and the gradient[32] $\boldsymbol{r}_{,\alpha}$ (giving rise to velocity and velocity gradient $\boldsymbol{v}_{,\alpha}=v_{i\alpha}\boldsymbol{a}^i$), we consider a restricted theory which bears on the classical theory of shells.[33]

In the developments of the restricted theory we retain, of course, the principles $(8.17)_{1,2}$ but replace the remaining conservation laws with a different set. For this purpose, we need to introduce certain quantities not defined previously. Let a (tangential) vector field $\grave{\boldsymbol{M}}=\grave{\boldsymbol{M}}(\theta^\alpha, t; \boldsymbol{v})$ with components $\grave{M}^\gamma=\grave{\boldsymbol{M}}\cdot\boldsymbol{a}^\gamma$ be defined for points $\boldsymbol{r}$ on the boundary curve c of the part $\mathscr{P}$ of $\mathfrak{s}$. If for all arbitrary angular velocity fields $\grave{\boldsymbol{w}}$ [defined in (5.61)], the scalar $\grave{\boldsymbol{M}}\cdot\grave{\boldsymbol{w}}$ is a rate of work per unit length, then $\grave{\boldsymbol{M}}$ is called a *contact couple* (or a *curve couple*) vector per unit length of[34] c. Further, let a (tangential) vector field $\grave{\boldsymbol{l}}=\grave{\boldsymbol{l}}(\theta^\alpha, t)$ per unit mass with components $\grave{l}^\gamma=\grave{\boldsymbol{l}}\cdot\boldsymbol{a}^\gamma$ be defined for points $\boldsymbol{r}$ on the part $\mathscr{P}$ of $\mathfrak{s}$. Then, if the scalar $\grave{\boldsymbol{l}}\cdot\grave{\boldsymbol{w}}$ is a rate of work per unit mass for all arbitrary angular velocities $\grave{\boldsymbol{w}}$, $\grave{\boldsymbol{l}}$ is called an *assigned couple* per unit mass of $\mathfrak{s}$. The *resultant contact couple* $\grave{\boldsymbol{G}}_c(\mathscr{P})$ and the *resultant assigned* couple $\grave{\boldsymbol{G}}_b$ over a part $\mathscr{P}$ in the present configuration are defined by

$$\grave{\boldsymbol{G}}_c=\int_{\partial\mathscr{P}}\grave{\boldsymbol{M}}\,ds, \qquad \grave{\boldsymbol{G}}_b=\int_{\mathscr{P}}\varrho\,\grave{\boldsymbol{l}}\,d\sigma. \tag{10.1}$$

Similarly, corresponding to (8.14), the sum $\grave{\boldsymbol{A}}(\mathscr{P})$ of the supply of moment of momentum $\grave{\boldsymbol{A}}_b(\mathscr{P})$ due to the assigned force and the assigned couple, each per unit mass, and the flux of moment of momentum $\grave{\boldsymbol{A}}_c(\mathscr{P})$ due to the contact force and the contact couple, each per unit length, is defined by

$$\begin{gathered}\grave{\boldsymbol{A}}(\mathscr{P})=\grave{\boldsymbol{A}}_b(\mathscr{P})+\grave{\boldsymbol{A}}_c(\mathscr{P}),\\ \grave{\boldsymbol{A}}_b(\mathscr{P})=\int_{\mathscr{P}}[\boldsymbol{r}\times\varrho\boldsymbol{f}+\boldsymbol{a}_3\times\varrho\,\grave{\boldsymbol{l}}]\,d\sigma, \qquad \grave{\boldsymbol{A}}_c(\mathscr{P})=\int_{\partial\mathscr{P}}(\boldsymbol{r}\times\boldsymbol{N}+\boldsymbol{a}_3\times\grave{\boldsymbol{M}})\,ds.\end{gathered} \tag{10.2}$$

We also record below the expression for the total rate of work by the contact force and the contact couple and by the assigned force and the assigned couple in the form

$$\begin{gathered}\grave{R}(\mathscr{P})=\grave{R}_c(\mathscr{P})+\grave{R}_b(\mathscr{P}),\\ \grave{R}_c(\mathscr{P})=\int_{\partial\mathscr{P}}(\boldsymbol{N}\cdot\boldsymbol{v}+\grave{\boldsymbol{M}}\cdot\grave{\boldsymbol{w}})\,ds, \qquad \grave{R}_b(\mathscr{P})=\int_{\mathscr{P}}\varrho(\boldsymbol{f}\cdot\boldsymbol{v}+\grave{\boldsymbol{l}}\cdot\grave{\boldsymbol{w}})\,d\sigma.\end{gathered} \tag{10.3}$$

[31] A theory of this type, concerned with a deformable surface with simple force multipoles in which $\boldsymbol{r}$ and its first and second gradients $(\boldsymbol{r}_{,\alpha}, \boldsymbol{r}_{,\alpha\beta})$ are taken as the basic kinematic variables, has been developed by BALABAN, GREEN and NAGHDI [1967, *1*]. A similar theory, but less general than that in [1967, *1*], is given by COHEN and DE SILVA [1966, *1*] who have subsequently modified their analysis [1968, *2*]. The work of COHEN and DE SILVA in [1968, 2] may be compared with a special case of the results in [1967, *1*], called the restricted theory of simple force dipoles. Also, SERBIN [1963, *13*] has considered an exact linear (isothermal) theory of an elastic surface by direct approach; but, in early stages of his analysis, he assumes a strain energy function for the linear theory which is too restrictive.

[32] A direct theory in which $\boldsymbol{r}$ and $\boldsymbol{r}_{,\alpha}$ are regarded as the basic kinematic variables will obviously have some overlapping features with that of a Cosserat surface; but, in general the two theories are different in character.

[33] In the existing literature, however, the classical theory is often derived from the linearized three-dimensional equations.

[34] The use of the term contact couple is justified in view of the physical dimension of $\grave{\boldsymbol{w}}$.

It is clear from the above definitions and the physical dimension of $\grave{\boldsymbol{w}}$ that $\grave{\boldsymbol{M}}$ and $\grave{\boldsymbol{l}}$ have, respectively, the physical dimensions of couple per unit length and couple per unit mass, namely

$$\begin{aligned}\text{phys. dim. } \grave{\boldsymbol{M}} &= \left[\frac{ML^2T^{-2}}{L}\right] = [MLT^{-2}],\\ \text{phys. dim. } \grave{\boldsymbol{l}} &= \left[\frac{ML^2T^{-2}}{M}\right] = [L^2T^{-2}].\end{aligned} \tag{10.4}$$

Having disposed of the above preliminaries, we begin the development of the restricted theory under consideration by adopting $(8.17)_{1,2}$ supplemented by

$$\begin{aligned}\frac{d}{dt}\int_{\mathscr{P}} \varrho(\boldsymbol{r}\times\boldsymbol{v})\, d\sigma &= \grave{\boldsymbol{A}}_b(\mathscr{P}) + \grave{\boldsymbol{A}}_c(\mathscr{P}),\\ \frac{d}{dt}\int_{\mathscr{P}} \left(\varepsilon + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}\right) d\sigma &= \grave{R}(\mathscr{P}) + H(\mathscr{P}),\end{aligned} \tag{10.5}$$

which are stated with reference to the present configuration of the surface. The conservation laws $(10.5)_{1,2}$ represent, respectively, the balance of moment of momentum and the balance of energy in the restricted theory.[35] They should be contrasted with $(8.17)_{4,5}$; and we note that $(8.17)_3$ has no counterpart here, since the director is not admitted in the restricted theory.

In place of the invariance condition $(8.32)_2$ we now require that $\grave{\boldsymbol{M}}$ be objective. Similarly, since moment of momentum due to angular velocity is excluded, instead of $(8.33)_3$ we require that $\grave{\boldsymbol{l}}$ be objective. To summarize, it follows that the invariance conditions under superposed rigid body motions in the restricted theory consist of (8.30)–(8.31), together with

$$\boldsymbol{N}^+ = Q(t)\,\boldsymbol{N}, \qquad \grave{\boldsymbol{M}}^+ = Q(t)\,\grave{\boldsymbol{M}} \tag{10.6}$$

and

$$r^+ = r, \qquad \bar{\boldsymbol{f}}^+ = Q(t)\,\bar{\boldsymbol{f}}, \qquad \grave{\boldsymbol{l}}^+ = Q(t)\,\grave{\boldsymbol{l}}. \tag{10.7}$$

By (5.66), (8.18) and (9.13)–(9.14), $(10.5)_1$ can be reduced to

$$\int_{\mathscr{P}} (\boldsymbol{a}_\alpha \times \boldsymbol{N}^\alpha + \varrho\, \boldsymbol{a}_3 \times \grave{\boldsymbol{l}})\, d\sigma + \int_{\partial\mathscr{P}} \boldsymbol{a}_3 \times \grave{\boldsymbol{M}}\, ds = 0. \tag{10.8}$$

Using the temporary notation $\grave{\boldsymbol{M}}_{(\boldsymbol{\nu})}$ in place of $\grave{\boldsymbol{M}}$ (in order to emphasize its dependence on $\boldsymbol{\nu}$), in the same manner that (9.5) and (9.17) were obtained, we can deduce the result $\boldsymbol{a}_3 \times [\grave{\boldsymbol{M}}_{(\boldsymbol{\nu})} + \grave{\boldsymbol{M}}_{(-\boldsymbol{\nu})}] = 0$. But since $\grave{\boldsymbol{M}}_{(\boldsymbol{\nu})}$ is a tangential vector field, it follows that

$$\grave{\boldsymbol{M}}_{(\boldsymbol{\nu})} = -\grave{\boldsymbol{M}}_{(-\boldsymbol{\nu})}. \tag{10.9}$$

Let $\grave{\boldsymbol{m}}^{(1)}$, $\grave{\boldsymbol{m}}^{(2)}$ denote the physical couple vectors acting on the sides of the coordinate curves whose unit normal vectors are $\boldsymbol{a}^1/(a^{11})^{\frac{1}{2}}$, $\boldsymbol{a}^2/(a^{22})^{\frac{1}{2}}$, respectively. Then, with the help of (10.9), application of (10.8) to an elementary curvilinear triangle on $\mathscr{s}$ (Fig. 2) yields

$$\grave{\boldsymbol{M}} = \grave{\boldsymbol{M}}_{(\boldsymbol{\nu})} = \grave{\boldsymbol{M}}^\alpha \nu_\alpha, \qquad \grave{\boldsymbol{M}}^\alpha = \grave{\boldsymbol{m}}^{(\alpha)} (a^{\alpha\alpha})^{\frac{1}{2}} \quad (\text{no sum on } \alpha). \tag{10.10}$$

[35] In writing (10.5), the contributions of the moment of momentum and of the kinetic energy due to the angular velocity have been excluded. These contributions are usually absent in the classical (approximate) theories derived from the three-dimensional equations. Their omission here does not lead to an essential limitation; they can be easily included if desired.

With the use of (10.10) and under suitable continuity assumptions, from (10.8) we obtain the equation

$$\boldsymbol{a}_\alpha \times \boldsymbol{N}^\alpha + \boldsymbol{a}_3 \times \varrho\, \grave{\boldsymbol{l}} + (\boldsymbol{a}_3 \times \grave{\boldsymbol{M}}^\alpha)_{|\alpha} = 0 \tag{10.11}$$

or equivalently

$$\boldsymbol{a}_\alpha \times (\boldsymbol{N}^\alpha - b^\alpha_\gamma \grave{\boldsymbol{M}}^\gamma) + \boldsymbol{a}_3 \times (\grave{\boldsymbol{M}}^\alpha{}_{|\alpha} + \varrho\, \grave{\boldsymbol{l}}) = 0, \tag{10.12}$$

as a consequence of the moment of momentum principle $(10.5)_1$.

We now turn to the energy balance $(10.5)_2$ which, with the help of (8.8) and (9.13), can be reduced to

$$\begin{aligned} \int_{\mathscr{P}} [\varrho\, r - \varrho\, \dot{\varepsilon} + \boldsymbol{N}^\alpha \cdot \boldsymbol{v}_{,\alpha} + \grave{\boldsymbol{M}}^\alpha \cdot \grave{\boldsymbol{w}}_{,\alpha}]\, d\sigma \\ + \int_{\mathscr{P}} (\grave{\boldsymbol{M}}^\alpha{}_{|\alpha} + \varrho\, \grave{\boldsymbol{l}}) \cdot \grave{\boldsymbol{w}}\, d\sigma - \int_{\partial\mathscr{P}} h\, ds = 0. \end{aligned} \tag{10.13}$$

Before proceeding further, we indicate a simplification which can be effected in (10.13). To this end, consider the scalar product of (10.11) with $\boldsymbol{a}_3 \times \grave{\boldsymbol{w}}$ and obtain

$$[(\boldsymbol{a}_3 \times \grave{\boldsymbol{M}}^\alpha)_{|\alpha} + (\boldsymbol{a}_3 \times \varrho\, \grave{\boldsymbol{l}})] \cdot (\boldsymbol{a}_3 \times \grave{\boldsymbol{w}}) = -(\boldsymbol{a}_\alpha \times \boldsymbol{N}^\alpha) \cdot (\boldsymbol{a}_3 \times \grave{\boldsymbol{w}}). \tag{10.14}$$

By use of the formulae for scalar triple product and vector triple product of vectors, namely

$$\boldsymbol{U} \cdot (\boldsymbol{V} \times \boldsymbol{W}) = (\boldsymbol{U} \times \boldsymbol{V}) \cdot \boldsymbol{W}, \quad (\boldsymbol{U} \times \boldsymbol{V}) \times \boldsymbol{W} = \boldsymbol{V}(\boldsymbol{U} \cdot \boldsymbol{W}) - \boldsymbol{U}(\boldsymbol{V} \cdot \boldsymbol{W}), \tag{10.15}$$

the various terms in (10.14) can be reduced as follows:

$$\begin{aligned} (\boldsymbol{a}_3 \times \grave{\boldsymbol{M}}^\alpha)_{|\alpha} \cdot (\boldsymbol{a}_3 \times \grave{\boldsymbol{w}}) &= [(\boldsymbol{a}_{3,\alpha} \times \grave{\boldsymbol{M}}^\alpha + \boldsymbol{a}_3 \times \grave{\boldsymbol{M}}^\alpha{}_{|\alpha}) \times \boldsymbol{a}_3] \cdot \dot{\boldsymbol{a}}_3 \\ &= [\grave{\boldsymbol{M}}^\alpha (\boldsymbol{a}_{3,\alpha} \cdot \boldsymbol{a}_3) - \boldsymbol{a}_{3,\alpha} (\grave{\boldsymbol{M}}^\alpha \cdot \boldsymbol{a}_3) \\ &\quad + \grave{\boldsymbol{M}}^\alpha{}_{|\alpha} (\boldsymbol{a}_3 \cdot \boldsymbol{a}_3) - \boldsymbol{a}_3 (\grave{\boldsymbol{M}}^\alpha{}_{|\alpha} \cdot \boldsymbol{a}_3] \cdot \dot{\boldsymbol{a}}_3 \\ &= \grave{\boldsymbol{M}}^\alpha{}_{|\alpha} \cdot \grave{\boldsymbol{w}}, \end{aligned} \tag{10.16}$$

$$\begin{aligned} (\boldsymbol{a}_3 \times \varrho\, \grave{\boldsymbol{l}}) \cdot (\boldsymbol{a}_3 \times \grave{\boldsymbol{w}}) &= \varrho\, [\grave{\boldsymbol{l}} (\boldsymbol{a}_3 \cdot \boldsymbol{a}_3) - \boldsymbol{a}_3 (\grave{\boldsymbol{l}} \cdot \boldsymbol{a}_3)] \cdot \dot{\boldsymbol{a}}_3 = \varrho\, \grave{\boldsymbol{l}} \cdot \grave{\boldsymbol{w}}, \\ (\boldsymbol{a}_\alpha \times \boldsymbol{N}^\alpha) \cdot (\boldsymbol{a}_3 \times \grave{\boldsymbol{w}}) &= [\boldsymbol{N}^\alpha (\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_3) - \boldsymbol{a}_\alpha (\boldsymbol{N}^\alpha \cdot \boldsymbol{a}_3)] \cdot \dot{\boldsymbol{a}}_3 \\ &= -(\boldsymbol{N}^\alpha \cdot \boldsymbol{a}_3)(\boldsymbol{a}_\alpha \cdot \dot{\boldsymbol{a}}_3) = \boldsymbol{N}^\alpha \cdot (v_{3\alpha}\, \boldsymbol{a}_3), \end{aligned}$$

where temporarily we have recalled (5.61) and we have used $(4.11)_{1,2}$ and (5.10). Substitution of (10.16) into (10.14) results in

$$(\grave{\boldsymbol{M}}^\alpha{}_{|\alpha} + \varrho\, \grave{\boldsymbol{l}}) \cdot \grave{\boldsymbol{w}} = -\boldsymbol{N}^\alpha \cdot (v_{3\alpha}\, \boldsymbol{a}_3). \tag{10.17}$$

The combination of the term $\boldsymbol{N}^\alpha \cdot \boldsymbol{v}_{,\alpha}$ in the first integral of (10.13) and the terms in the second integral, in view of (10.17), yield $\boldsymbol{N}^\alpha \cdot v_{\gamma\alpha}\, \boldsymbol{a}^\gamma$ and the energy balance becomes

$$\int_{\mathscr{P}} [\varrho r - \varrho\, \dot{\varepsilon} + \boldsymbol{N}^\alpha \cdot v_{\gamma\alpha}\, \boldsymbol{a}^\gamma + \grave{\boldsymbol{M}}^\alpha \cdot \grave{\boldsymbol{w}}_{,\alpha}]\, d\sigma - \int_{\partial\mathscr{P}} h\, ds = 0, \tag{10.18}$$

which is the counterpart of (9.24) in the present development. The results (9.25)–(9.27) still hold and, after making the appropriate continuity assumptions, from (10.18) we obtain

$$\varrho r - q^\alpha{}_{|\alpha} - \varrho\, \dot{\varepsilon} + \boldsymbol{N}^\alpha \cdot v_{\gamma\alpha}\, \boldsymbol{a}^\gamma + \grave{\boldsymbol{M}}^\alpha \cdot \grave{\boldsymbol{w}}_{,\alpha} = 0. \tag{10.19}$$

It is convenient at this point to write (10.11) in component form. Since $\grave{\boldsymbol{M}}$ and $\grave{\boldsymbol{l}}$ are tangential vector fields and since $\grave{\boldsymbol{M}}^\alpha$ by (10.10) transforms as a contravariant surface vector, we have

$$\grave{\boldsymbol{M}} = \grave{M}^\gamma \boldsymbol{a}_\gamma, \qquad \grave{\boldsymbol{l}} = \grave{l}^\gamma \boldsymbol{a}_\gamma, \qquad \grave{\boldsymbol{M}}^\alpha = \grave{M}^{\alpha\gamma} \boldsymbol{a}_\gamma, \qquad \grave{M}^\gamma = \grave{M}^{\alpha\gamma} \nu_\alpha. \tag{10.20}$$

Then, recalling (9.40), from (10.11) we obtain the equations of motion

$$\grave{M}^{\gamma\alpha}{}_{|\gamma} - N^{\alpha 3} + \varrho\, \grave{l}^\alpha = 0, \qquad \varepsilon_{\beta\alpha}[N^{\alpha\beta} - \grave{M}^{\gamma\beta} b^\alpha_\gamma] = 0. \tag{10.21}$$

The set of Eqs. (10.21) are the counterparts of $(9.48)_1$ and $(9.51)_1$ in the restricted theory.[36] We also observe here that the differential equations of motion (9.47) and $(10.21)_1$ can be easily written in alternative forms by elimination of $N^{\alpha 3}$. Thus, substituting for $N^{\alpha 3}$ from $(10.21)_1$ into $(9.47)_{1,2}$, we obtain

$$N^{\alpha\beta}{}_{|\alpha} - b^\beta_\alpha \grave{M}^{\gamma\alpha}{}_{|\gamma} + \varrho \bar{f}^\beta = 0, \qquad \grave{M}^{\gamma\alpha}{}_{|\gamma\alpha} + b_{\alpha\beta} N^{\alpha\beta} + \varrho \bar{f}^3 = 0, \tag{10.22}$$

where we have set

$$\varrho \bar{f}^\beta = \varrho (f^\beta - c^\beta + b^\beta_\alpha \grave{l}^\alpha), \qquad \varrho \bar{f}^3 = \varrho (f^3 - c^3) + (\varrho\, \grave{l}^\alpha)_{|\alpha}. \tag{10.23}$$

Returning (10.19), as in Sect. 9, we now use the argument about uniform superposed rigid body angular velocity, the continuum occupying the same position at time t. Recalling that the fields which occur in (8.30)–(8.31) and (10.6)–(10.7) [as well as q^α] are unaltered under such superposed rigid body motions, with the use of $(5.17)_1$ and (5.62), we again obtain $(10.21)_2$ and the local energy equation in the invariant form

$$\begin{aligned} &\varrho r - q^\alpha{}_{|\alpha} - \varrho \dot{\varepsilon} + \grave{P} = 0, \\ &\grave{P} = \grave{\boldsymbol{N}}^\alpha \cdot \boldsymbol{\eta}_\alpha + \grave{\boldsymbol{M}}^\alpha \cdot \dot{\grave{\lambda}}_{\gamma\alpha} \boldsymbol{a}^\lambda \\ &\quad = \grave{N}^{\alpha\beta} \eta_{\alpha\beta} - \grave{M}^{\alpha\gamma} \dot{b}_{\gamma\alpha}, \end{aligned} \tag{10.24}$$

where we have introduced the notation

$$\grave{\boldsymbol{N}}^\alpha = \boldsymbol{N}^\alpha + b^\alpha_\gamma \grave{\boldsymbol{M}}^\gamma \tag{10.25}$$

and where $\grave{N}^{\alpha\beta}$ which satisfies $(10.21)_2$ is given by

$$\begin{aligned} \grave{N}^{\alpha\beta} &= \grave{N}^{\beta\alpha} = \tfrac{1}{2} (\grave{\boldsymbol{N}}^\alpha \cdot \boldsymbol{a}^\beta + \grave{\boldsymbol{N}}^\beta \cdot \boldsymbol{a}^\alpha) \\ &= N^{\alpha\beta} + \grave{M}^{\gamma\alpha} b^\beta_\gamma = N^{\beta\alpha} + \grave{M}^{\gamma\beta} b^\alpha_\gamma. \end{aligned} \tag{10.26}$$

It is clear that the only values of $N^{\alpha\beta} = \boldsymbol{N}^\alpha \cdot \boldsymbol{a}^\beta$ which are present in $(10.24)_2$ are the symmetric components $N^{(\alpha\beta)}$. Also, the components $N^{\alpha 3} = \boldsymbol{N}^\alpha \cdot \boldsymbol{a}_3$ do not occur in the energy equation and will be absent from an inequality resulting from combination of $(10.24)_1$ and (9.32). Moreover, since $b_{\gamma\alpha}$ is symmetric, it follows that only the symmetric part of $\grave{M}^{\alpha\gamma}$ occurs in the energy equation $(10.24)_1$. We may therefore replace $\grave{M}^{\alpha\gamma}$ in (10.24) by $\grave{M}^{(\alpha\gamma)}$. This completes our derivation of the basic field equations of the restricted theory. They consist of the continuity equation (8.18), the equations of motion (9.47) and (10.21) [or equivalently (10.22)

[36] Eqs. $(9.48)_2$ and $(9.51)_2$ have no counterparts in the restricted theory. Had we included the moment of momentum due to the angular velocity in (10.5), instead of $\varrho\, \grave{\boldsymbol{l}}$ in (10.11) we should have $\varrho (\grave{\boldsymbol{l}} - \dot{\grave{\boldsymbol{w}}})$ representing the difference of the assigned couple vector per unit mass and the inertia term due to $\grave{\boldsymbol{w}}$.

and (10.21)] and the energy equation (10.24) which can also be expressed in terms of the specific Helmholtz free energy.[37]

Before closing this section we observe that the linearized version of the above field equations can be obtained in the same manner as the linearized field equations of a Cosserat surface in Sect. 9 [Subsect. γ)]. In particular, all quantities are referred to the base vectors $\boldsymbol{A}_i$, covariant differentiation is with respect to $A_{\alpha\beta}$ and the components of $\grave{\boldsymbol{M}}^\alpha$ and $\grave{\boldsymbol{l}}$ are now defined through

$$M^\gamma = \grave{\boldsymbol{M}} \cdot \boldsymbol{A}_\gamma, \quad \grave{L}^\gamma = \grave{\boldsymbol{l}} \cdot \boldsymbol{A}_\gamma, \quad \grave{\boldsymbol{M}}^\alpha = \grave{M}^{\alpha\gamma} \boldsymbol{A}_\gamma. \tag{10.27}$$

The linearized versions of (10.21) then become

$$\grave{M}^{\alpha\gamma}{}_{|\gamma} - N^{\alpha 3} + \varrho_0 \grave{L}^\alpha = 0, \tag{10.28}$$

$$\bar{\varepsilon}_{\beta\alpha}[N^{\alpha\beta} - \grave{M}^{\gamma\beta} B_\gamma^\alpha] = 0, \tag{10.29}$$

where the notation $\bar{\varepsilon}_{\alpha\beta} = A^{\frac{1}{2}} e_{\alpha\beta}$ was introduced previously in (6.30). Similarly, the linearized equations of motion corresponding to (10.22) can be obtained from (10.28) and (9.54). Also, the energy equation in the restricted linear theory can be written as

$$\varrho_0 r - Q^\alpha{}_{|\alpha} - \varrho_0 \dot{\varepsilon} + \tilde{N}^{\alpha\beta} \dot{e}_{\alpha\beta} + \grave{M}^{\alpha\gamma} \dot{\bar{\varkappa}}_{\gamma\alpha} = 0. \tag{10.30}$$

In (10.30), Q^α is defined in (9.59), $\tilde{N}^{\alpha\beta}$ is of the form (10.26) with b_γ^β replaced by B_γ^β and $\bar{\varkappa}_{\gamma\alpha}$ [which can be obtained from the linearization of $(b_{\alpha\beta} - B_{\alpha\beta})$] is given by the symmetric expression in (6.24). Again, since $\bar{\varkappa}_{\gamma\alpha} = \bar{\varkappa}_{\alpha\gamma}$, we may replace $\grave{M}^{\alpha\gamma}$ in (10.30) by $\grave{M}^{(\alpha\gamma)}$.

11. Basic field equations for shells: II. Derivation from the three-dimensional theory. Prior to our derivation of the basic field equations for shell-like bodies from the three-dimensional equations of the classical continuum mechanics, we need to recall some preliminary results from the three-dimensional theory for non-polar media and also define certain resultants, including stress-resultants and stress-couples. Since our developments are carried out in the context of a thermodynamic theory, we also recall the Clausius-Duhem inequality and obtain its analogue for shells.[38]

α) *Some preliminary results.* A (three-dimensional) shell-like body $\mathscr{B}$ with its boundary surface $\partial\mathscr{B}$ [specified by (4.14)–(4.15)] is defined in Sect. 4, where the relation of the material surface $\xi = 0$ to the bounding surfaces $(4.14)_{1,2}$ is fixed by the condition (4.20). Let[39] $\mathscr{P}_{\mathscr{B}}$ refer to an arbitrary region of the material

[37] These field equations can be brought into correspondence with those of the restricted theory of simple force dipoles; see Sect. 8 of the paper by BALABAN, GREEN and NAGHDI [1967, *1*].

[38] Our developments in this and the next section follow largely GREEN, LAWS and NAGHDI [1968, *4*] and GREEN and NAGHDI [1970, *2*] and to some extent NAGHDI [1964, *4*] and GREEN, NAGHDI and WENNER [1971, *6*]. The thermodynamic results are mainly those obtained in [1964, *4*] and [1970, *2*].

We again remark that there is some discussion in the current literature regarding the limitations of the Clausius-Duhem inequality or a more general form for the entropy inequality, but these need not affect our limited use of the inequality (11.20) here. Later, we shall appeal to (11.20) only in the context of elastic materials. We further remark here that while our subsequent developments are carried out within the framework of a thermodynamic theory, most of the results can be easily specialized to the isothermal or the purely mechanical theory.

[39] Although the symbol $\mathscr{P}$ designating a part of the surface $\xi = 0$ in the present configuration at time t is the same as that used previously [in Sects. 4, 8–9] to designate a part which the Cosserat surface $\mathscr{C}$ occupies at time t, this need not give rise to confusion. If desired, the two surfaces may be identified; but we postpone such identifications until later.

surface $\xi=0$ which is mapped into a part $\mathscr{P}$ in the present configuration and let $\partial\mathscr{P}$ denote the boundary of $\mathscr{P}$. Further let $\mathscr{P}^*_{\mathscr{B}}$ refer to an arbitrary part of $\mathscr{B}$ which is mapped into a part $\mathscr{P}^*$ in the present configuration such that (i) $\mathscr{P}^*$ contains $\mathscr{P}$; (ii) the boundary $\partial\mathscr{P}^*$ of $\mathscr{P}^*$ consists of portions of the surfaces $(4.14)_{1,2}$ and a surface of the form (4.15) at time t; and (iii) the boundary $\partial\mathscr{P}^*$ coincides with $\partial\mathscr{P}$ on $\xi=0$. Also, for later reference, let $\partial\mathscr{P}^*_n$ refer to the part of $\partial\mathscr{P}^*$ specified by a normal surface of the form (4.15) so that $\partial\mathscr{P}^*_n=\partial\mathscr{P}^*=\partial\mathscr{P}$ on $\xi=0$, and let $\partial\mathscr{P}^{*c}_n=\partial\mathscr{P}^*-\partial\mathscr{P}^*_n$ refer to the complement of $\partial\mathscr{P}^*_n$ in $\partial\mathscr{P}^*$.

Recalling that ϱ is a mass per unit area of $\mathfrak{s}$ (the surface $\xi=0$ at time t) given by (4.17), we define the mass of any part $\mathscr{P}_{\mathscr{B}}$ by

$$m(\mathscr{P}_{\mathscr{B}})=\int_{\mathscr{P}}\varrho\, d\sigma, \tag{11.1}$$

where the surface integral is over the part $\mathscr{P}$ of the surface $\xi=0$ at time t. Let $m^*(\mathscr{P}^*_{\mathscr{B}})$ denote the mass of a part $\mathscr{P}^*_{\mathscr{B}}$ of the shell-like body $\mathscr{B}$. The mass density of $\mathscr{B}$ in different configurations is determined by the motion of $\mathscr{B}$; it is a relation between the physical mass $m^*(\mathscr{P}^*_{\mathscr{B}})$ and the volume of the region of the Euclidean space occupied by $\mathscr{P}^*_{\mathscr{B}}$ in a given configuration as defined, for example, by the volume integral in (4.18). From (4.18) and (11.1) follows the relation

$$m^*(\mathscr{P}^*_{\mathscr{B}})=m(\mathscr{P}_{\mathscr{B}}). \tag{11.2}$$

A local form of the (three-dimensional) continuity equation has been stated previously by (4.16). Integration of this equation with respect to ξ, between the limits $\xi=\alpha,\beta$, results in

$$\varrho\, a^{\frac{1}{2}}=\varrho_0\, A^{\frac{1}{2}}. \tag{11.3}$$

The right-hand side of (11.3) is the reference value of (4.17) and

$$\varrho_0=\varrho_0(\theta^\alpha)=\varrho(\theta^\alpha,0)$$

is a mass per unit area of $\mathfrak{S}$ (the surface $\xi=0$ in the initial reference configuration).[40]

Assume that a shell-like body $\mathscr{B}$ and a motion of $\mathscr{B}$ are given. Let $\overline{\mathscr{P}}$, not necessarily the same as $\mathscr{P}^*$, refer to an arbitrary part of $\mathscr{B}$ in the present configuration and let $\partial\overline{\mathscr{P}}$ denote the boundary of $\overline{\mathscr{P}}$. Then, within the scope of non-polar continuum mechanics, the system of forces acting over any part $\overline{\mathscr{P}}_{\mathscr{B}}$ of the body $\mathscr{B}$ in motion consist of the sum of two types of forces $\boldsymbol{F}^*_b$ and $\boldsymbol{F}^*_c$ defined as follows: Let $\boldsymbol{f}^*=\boldsymbol{f}^*(\theta^i,t)$ be a vector field, per unit mass, defined for material points in the region occupied by $\mathscr{B}$ at time t; it is called the *external body force* or simply the *body force*. The *resultant body force* exerted on the part $\overline{\mathscr{P}}$ at time t is defined by the volume integral[41]

$$\boldsymbol{F}^*_b(\overline{\mathscr{P}})=\int_{\overline{\mathscr{P}}}\varrho^*\boldsymbol{f}^*\, dv \tag{11.4}$$

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[40] Despite similarity and our use of the same symbols in (11.3) and the continuity equation (4.42) for the Cosserat surface, we emphasize that these two equations have been deduced on the basis of different concepts. Clearly, they may be brought into one-to-one correspondence, but we postpone such identifications until later.

[41] Strictly speaking, the left-hand sides of (11.4) and (11.6) should be denoted as $\boldsymbol{F}^*_b(\overline{\mathscr{P}}_{\mathscr{B}})$ and $\boldsymbol{F}^*_c(\overline{\mathscr{P}}_{\mathscr{B}})$, respectively; however, in order to avoid cumbersome notation, we adopt the simpler designations on the left-hand sides of (11.4), (11.6) and elsewhere in this section. Our abbreviated notation which is similar to that in (4.44) should not cause confusion, since it will be clear from the particular context [e.g., the presence of dv on the right-hand side of (11.4)] which region of Euclidean space $\mathscr{B}$ occupies.

over $\overline{\mathscr{P}}$ in the present configuration and the element of volume dv is given by $(4.19)_1$. Further, let

$$\boldsymbol{n} = n_i\, \boldsymbol{g}^i = n^i\, \boldsymbol{g}_i \tag{11.5}$$

be the outward unit normal vector at the material point on the boundary $\partial\overline{\mathscr{P}}$ at time t and let $\boldsymbol{t} = \boldsymbol{t}(\theta^i, t; \boldsymbol{n})$ be defined for the material points on the boundary $\partial\overline{\mathscr{P}}$ at time t. The vector field[42] $\boldsymbol{t}$ is called the *contact force* or the *stress vector* acting on the part $\overline{\mathscr{P}}$ of $\mathscr{B}$. The *resultant contact force* exerted on the part $\overline{\mathscr{P}}$ at time t is defined by the surface integral

$$\boldsymbol{F}_c^*(\overline{\mathscr{P}}) = \int_{\partial\overline{\mathscr{P}}} \boldsymbol{t}(\theta^i, t; \boldsymbol{n})\, da \tag{11.6}$$

over the boundary $\partial\overline{\mathscr{P}}$ of $\overline{\mathscr{P}}$ in the present configuration, where da is an element of area whose outward unit normal is $\boldsymbol{n}$.

Under suitable continuity assumptions, the principles of linear momentum and moment of momentum for non-polar media imply[43] (i) the existence of a tensor field $\tau^{ij} = \tau^{ij}(\theta^k, t)$ such that

$$\boldsymbol{t} = \frac{\boldsymbol{T}^i\, n_i}{g^{\frac{1}{2}}} = \tau^{ij}\, n_i\, \boldsymbol{g}_j, \qquad \boldsymbol{T}^i = g^{\frac{1}{2}}\, \tau^{ij}\, \boldsymbol{g}_j = g^{\frac{1}{2}}\, \tau^i_j\, \boldsymbol{g}^j, \tag{11.7}$$

and (ii) the Cauchy equations of motion, which may be expressed as

$$\boldsymbol{T}^i{}_{,i} + \varrho^*\, \boldsymbol{f}^*\, g^{\frac{1}{2}} = \varrho^*\, \dot{\boldsymbol{v}}^*\, g^{\frac{1}{2}}, \qquad \boldsymbol{g}_i \times \boldsymbol{T}^i = 0. \tag{11.8}$$

In (11.7), τ^{ij} and τ^i_j are the contravariant and the mixed components of the stress tensor and it follows from $(11.7)_2$ and $(11.8)_2$ that τ^{ij} is symmetric, i.e.,

$$\tau^{ij} = \tau^{ji}. \tag{11.9}$$

We observe that if $\boldsymbol{n}$ is the outward unit normal to $\partial\mathscr{B}$, then $\boldsymbol{t}(\theta^i, t; \boldsymbol{n})$ reduces to a function of position for points on $\partial\mathscr{B}$, i.e.,

$$\boldsymbol{t} = \bar{\boldsymbol{t}}(\theta^i, t), \tag{11.10}$$

where $\bar{\boldsymbol{t}}$ is the surface traction prescribed on the boundary $\partial\mathscr{B}$.

We introduce the following additional five quantities which we associate with a motion of the body:

The specific *internal energy* $\varepsilon^* = \varepsilon^*(\theta^i, t)$.

The *heat flux* vector $\boldsymbol{q}^* = \boldsymbol{q}^*(\theta^i, t)$.

The *heat supply* or *heat absorption* $r^* = r^*(\theta^i, t)$.

The specific *entropy* $\eta^* = \eta^*(\theta^i, t)$.

The local temperature $\theta^* = \theta^*(\theta^i, t)$ which is assumed to be always positive, i.e., $\theta^* > 0$.

Consider the eight functions consisting of the vector function $\boldsymbol{p}$ in (4.6) which describes the motion of the body, the body force $\boldsymbol{f}^*$, the surface force $\boldsymbol{T}^i$ (or equivalently the stress tensor τ^{ij}) and the five functions defined above. It is convenient to speak of a *thermodynamic process* or simply a *process* if the eight functions are so prescribed that the conservation laws, namely the principles of

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[42] Often $\boldsymbol{t}$ is denoted by $\boldsymbol{t}_{(\boldsymbol{n})}$ in order to emphasize its dependence on the unit normal $\boldsymbol{n}$, but we omit the subscript $\boldsymbol{n}$ here.

[43] GREEN and ZERNA [1968, *9*, Chap. 2]. We may observe here that the *equivalence* of the conservation laws for mass, linear momentum, moment of momentum and energy *and* the balance of energy together with invariance requirements under superposed rigid body motions can be demonstrated in a manner similar to that in Sect. 8.

linear and moment of momentum and balance of energy, are satisfied for every part $\bar{\mathscr{P}}$ of the body $\mathscr{B}$. The internal energy $\mathscr{E}^*(\bar{\mathscr{P}})$ of a part $\bar{\mathscr{P}}$ and the heat $H^*(\bar{\mathscr{P}})$ entering the part $\bar{\mathscr{P}}$ per unit time are defined by[44]

$$\mathscr{E}^*(\bar{\mathscr{P}}) = \int_{\bar{\mathscr{P}}} \varrho^* \varepsilon^* \, dv, \qquad H^*(\bar{\mathscr{P}}) = \int_{\bar{\mathscr{P}}} \varrho^* r^* \, dv - \int_{\partial\bar{\mathscr{P}}} \boldsymbol{q}^* \cdot \boldsymbol{n} \, da. \tag{11.11}$$

Also, the *entropy* $\mathscr{H}^*(\bar{\mathscr{P}})$ of a part $\bar{\mathscr{P}}$ of $\mathscr{B}$ and the *production of entropy* Γ^* per unit time in a part $\bar{\mathscr{P}}$ are defined by

$$\begin{gathered} \mathscr{H}^*(\bar{\mathscr{P}}) = \int_{\bar{\mathscr{P}}} \varrho^* \eta^* \, dv, \\ \Gamma^*(\bar{\mathscr{P}}) = \int_{\bar{\mathscr{P}}} \varrho^* \gamma^* \, dv = \frac{d}{dt} \int_{\bar{\mathscr{P}}} \varrho^* \eta^* \, dv - \left[\int_{\bar{\mathscr{P}}} \varrho^* \frac{r^*}{\theta^*} \, dv - \int_{\partial\bar{\mathscr{P}}} \frac{\boldsymbol{q}^* \cdot \boldsymbol{n}}{\theta^*} \, da \right], \end{gathered} \tag{11.12}$$

where γ^* is the *specific production of entropy*, r^*/θ^* in (11.12) is the entropy due to radiation entering $\bar{\mathscr{P}}$ and $-(\boldsymbol{q}^* \cdot \boldsymbol{n})/\theta^*$ is the flux of entropy due to conduction entering $\bar{\mathscr{P}}$ through the boundary $\partial\bar{\mathscr{P}}$.

For later convenience, we recall the principle of balance of energy (for non-polar media) over any part $\bar{\mathscr{P}}$ in the form

$$\frac{d}{dt} \int_{\bar{\mathscr{P}}} \left(\varepsilon^* + \frac{1}{2} \boldsymbol{v}^* \cdot \boldsymbol{v}^* \right) \varrho^* \, dv = \int_{\bar{\mathscr{P}}} (r^* + \boldsymbol{f}^* \cdot \boldsymbol{v}^*) \, \varrho^* \, dv + \int_{\partial\bar{\mathscr{P}}} (\boldsymbol{t} \cdot \boldsymbol{v}^* - h^*) \, da, \tag{11.13}$$

the left-hand side of which represents the rate of increase of the sum of internal and kinetic energies, $\boldsymbol{t} \cdot \boldsymbol{v}^*$ and $\boldsymbol{f}^* \cdot \boldsymbol{v}^*$ are rate of work by surface and body forces,

$$h^* = \boldsymbol{q}^* \cdot \boldsymbol{n} \tag{11.14}$$

is the flux of heat entering $\bar{\mathscr{P}}$ through the boundary $\partial\bar{\mathscr{P}}$ and all other quantities have been defined previously. Making the appropriate continuity assumptions, recalling (7.27) and using (11.8), we can write the local balance of energy as

$$\varrho^* r^* - \varrho \, \dot{\varepsilon}^* + \tau^{ij} \dot{\gamma}^*_{ij} - q^{*k}{}_{\|k} = 0, \tag{11.15}$$

where q^{*k} and γ_{ij} are defined by

$$\boldsymbol{q}^* = q^{*k} \boldsymbol{g}_k, \qquad \dot{\gamma}^*_{ij} = \tfrac{1}{2} \dot{g}_{ij}, \tag{11.16}$$

and the double vertical line in (11.15) denotes covariant differentiation with respect to the metric tensor g_{ij}. For later use, we record the local energy equation in the alternative form

$$\varrho^* r^* - \varrho^* \dot{\varepsilon}^* + g^{-\frac{1}{2}} [\boldsymbol{T}^i \cdot \boldsymbol{v}^*_{,i} - (q^{*k} g^{\frac{1}{2}})_{,k}] = 0, \tag{11.17}$$

and note that this equation and (11.15) can also be expressed in terms of the (three-dimensional) Helmholtz free energy function defined by

$$\psi^* = \varepsilon^* - \theta^* \eta^*. \tag{11.18}$$

[44] The negative sign in $(11.11)_2$ is in accord with the usual convention, since $\boldsymbol{q}^* \cdot \boldsymbol{n} < 0$ at points where heat is entering the surface. The sign convention in (11.11) agrees with that used by some writers (see, e.g., Sects. 2.7–2.8 of GREEN and ZERNA [1968, 9]) but differs from that employed by others (see, e.g., Sect. 79 of TRUESDELL and NOLL [1965, 9]). Our r^* and $\boldsymbol{q}^*$ correspond to q and $-\boldsymbol{h}$ in [1965, 9].

Before stating the Clausius-Duhem entropy production inequality, we briefly recall certain additional results from the (three-dimensional) theory of the classical continuum mechanics. The invariance conditions under superposed rigid body motions are assumed to be the requirements that physical quantities

$$\varrho^*, r^*, h^* \text{ (or } \boldsymbol{q}^*), \quad \varepsilon^*, \eta^*, \theta^*, \psi^*, \quad \boldsymbol{T}_i, \tau_{ij}, (\boldsymbol{f}^* - \dot{\boldsymbol{v}}^*) \tag{11.19}$$

transform according to (8.25) and are therefore *objective.*[45] With reference to the terminology introduced preceding (11.11), a process is said to be *admissible* if the set of five functions τ^{ij} (or $\boldsymbol{T}^i$,) ε^*, θ^*, η^* and h^* (or $\boldsymbol{q}^*$) are specified by constitutive equations which hold at each material point and for all time. If the vector function $\boldsymbol{p}$ in (4.6) and either θ^* and η^* are specified, then the body force $\boldsymbol{f}^*$ and the heat supply r^* are uniquely determined from $(11.8)_1$ and (11.15), respectively, while $(11.8)_2$ [or the symmetry condition (11.9)] is assumed to be satisfied identically.

The Clausius-Duhem inequality (in the three-dimensional theory) is specified by

$$\Gamma^*(\bar{\mathscr{P}}) \geqq 0, \tag{11.20}$$

which we require to be valid for every admissible process describing the motion and the thermo-mechanical behavior of the body. From (11.20) and $(11.12)_2$, after introducing the appropriate continuity assumptions, follows the local inequality

$$\varrho^* \theta^* \dot{\eta}^* - \varrho^* r^* + \theta^* g^{-\frac{1}{2}} \left(\frac{q^{*k} g^{\frac{1}{2}}}{\theta^*} \right)_{,k} \geqq 0. \tag{11.21}$$

Elimination of r^* and $(q^{*k} g^{\frac{1}{2}})_{,k}$ between (11.15) and (11.21) yields the inequality

$$\varrho(\theta^* \dot{\eta}^* - \dot{\varepsilon}^*) + \tau^{ij} \dot{\gamma}_{ij} - \frac{q^{*k}}{\theta^*} \theta^*_{,k} \geqq 0, \tag{11.22}$$

which in terms of the Helmholtz function ψ^* becomes

$$-\varrho(\eta^* \dot{\theta}^* + \dot{\psi}^*) + \tau^{ij} \dot{\gamma}_{ij} - \frac{q^{*k}}{\theta^*} \theta_{,k} \geqq 0. \tag{11.23}$$

We observe that the left-hand sides of the energy equation (11.15) and the inequality (11.22), as well as the left-hand side of (11.23), remain unaltered under superposed rigid body motions in view of the invariance requirements noted above. Following a current procedure in continuum mechanics, we regard (11.21) as a condition to be satisfied identically by every admissible process describing the motion and the thermo-mechanical behavior of the medium. Thus, (11.22) or (11.23) will narrow the class of all admissible processes and will place restrictions on the constitutive equations to be specified for the five functions τ^{ij}, ε^* (or ψ^*), η^*, θ^* and q^{*k}.

β) *Stress-resultants, stress-couples and other resultants for shells.* We introduce here the definitions of various resultants of the field quantities which occur in the three-dimensional equations by suitable integration with respect to ξ between the limits $\xi = \alpha(\theta^\alpha)$ and $\xi = \beta(\theta^\alpha)$. First, however, we need the expressions for the

[45] See in this connection the earlier remarks made following (8.25). Our use of the term *objective* is different from the corresponding usage by many who appeal to the *principle of material frame-indifference* which is concerned with invariance under a change of frame employing orthogonal transformations. For a discussion of differences between invariance requirements demanded by the principle of material frame-indifference and those under superposed rigid body motions, see TRUESDELL and NOLL [1965, *9*].

element of area of the surfaces $(4.14)_{1,2}$. The outward unit normal vector to the surfaces $(4.14)_{1,2}$ are:

$$\begin{aligned} \boldsymbol{n} &= \mathfrak{f}_{(\alpha)}^{-1}[\alpha_{,1}\boldsymbol{g}^1 + \alpha_{,2}\boldsymbol{g}^2 - \boldsymbol{g}^3] \quad \text{on} \quad \xi = \alpha(\theta^\alpha),\\ \boldsymbol{n} &= \mathfrak{f}_{(\beta)}^{-1}[-\beta_{,1}\boldsymbol{g}^1 - \beta_{,2}\boldsymbol{g}^2 + \boldsymbol{g}^3] \quad \text{on} \quad \xi = \beta(\theta^\alpha), \end{aligned} \tag{11.24}$$

where a comma denotes partial differentiation with respect to surface coordinates and where

$$\mathfrak{f}_{(\alpha)} = [(\alpha_{,1})^2 g^{11} + (\alpha_{,2})^2 g^{22} + g^{33} + 2(\alpha_{,1}\alpha_{,2} g^{12} - \alpha_{,1} g^{13} - \alpha_{,2} g^{23})]^{\frac{1}{2}} \tag{11.25}$$

with an analogous expression for $\mathfrak{f}_{(\beta)}$. The elements of area on the surfaces $(4.14)_{1,2}$ are then given by

$$\begin{aligned} da &= \mathfrak{f}_{(\alpha)}\, g^{\frac{1}{2}}\, d\theta^1\, d\theta^2 \quad \text{on} \quad \xi = \alpha(\theta^\alpha),\\ da &= \mathfrak{f}_{(\beta)}\, g^{\frac{1}{2}}\, d\theta^1\, d\theta^2 \quad \text{on} \quad \xi = \beta(\theta^\alpha). \end{aligned} \tag{11.26}$$

In each of the above expressions, $\boldsymbol{g}^i$, g^{ij}, g are evaluated either at $\xi = \alpha(\theta^\alpha)$ or at $\xi = \beta(\theta^\alpha)$ as indicated. If α and β are constants, then both expressions in (11.26) reduce to

$$da = (g\, g^{33})^{\frac{1}{2}}\, d\theta^1\, d\theta^2 \quad (\text{for } \alpha,\ \beta \text{ constants}). \tag{11.27}$$

Recalling (4.17) and (4.21), we put

$$\varrho\, a^{\frac{1}{2}} = \int_\alpha^\beta k\, d\xi, \qquad \varrho\, k^N a^{\frac{1}{2}} = \int_\alpha^\beta \xi^N k\, d\xi \quad (N \geqq 2) \tag{11.28}$$

and note that α, β are not necessarily constants and an integral corresponding to the right-hand side of $(11.28)_2$ with $N=1$ has been introduced previously in (4.20). We define two-dimensional body forces and a two-dimensional heat supply by

$$\begin{aligned} \varrho\, \boldsymbol{f}\, a^{\frac{1}{2}} = \varrho\, \boldsymbol{l}^0 a^{\frac{1}{2}} &= \int_\alpha^\beta \varrho^* \boldsymbol{f}^* g^{\frac{1}{2}}\, d\xi + [\bar{\boldsymbol{t}}\, g^{\frac{1}{2}}\, \mathfrak{f}_{(\beta)}]_{\xi=\beta} + [\bar{\boldsymbol{t}}\, g^{\frac{1}{2}}\, \mathfrak{f}_{(\alpha)}]_{\xi=\alpha}\\ &= \int_\alpha^\beta \varrho^* \boldsymbol{f}^* g^{\frac{1}{2}}\, d\xi + [-\beta_{,1}\boldsymbol{T}^1 - \beta_{,2}\boldsymbol{T}^2 + \boldsymbol{T}^3]_{\xi=\beta}\\ &\quad - [-\alpha_{,1}\boldsymbol{T}^1 - \alpha_{,2}\boldsymbol{T}^2 + \boldsymbol{T}^3]_{\xi=\alpha}, \end{aligned} \tag{11.29}$$

$$\begin{aligned} \varrho\, \boldsymbol{l}^N a^{\frac{1}{2}} &= \int_\alpha^\beta \varrho^* \boldsymbol{f}^* g^{\frac{1}{2}} \xi^N\, d\xi + [\bar{\boldsymbol{t}}\, \xi^N g^{\frac{1}{2}}\, \mathfrak{f}_{(\beta)}]_{\xi=\beta} + [\bar{\boldsymbol{t}}\, \xi^N g^{\frac{1}{2}}\, \mathfrak{f}_{(\alpha)}]_{\xi=\alpha}\\ &= \int_\alpha^\beta \varrho^* \boldsymbol{f}^* g^{\frac{1}{2}} \xi^N\, d\xi + [\xi^N(-\beta_{,1}\boldsymbol{T}^1 - \beta_{,2}\boldsymbol{T}^2 + \boldsymbol{T}^3)]_{\xi=\beta}\\ &\quad - [\xi^N(-\alpha_{,1}\boldsymbol{T}^1 - \alpha_{,2}\boldsymbol{T}^2 + \boldsymbol{T}^3)]_{\xi=\alpha} \end{aligned} \tag{11.30}$$

and

$$\varrho\, r\, a^{\frac{1}{2}} = \int_\alpha^\beta \varrho^* r^* g^{\frac{1}{2}}\, d\xi - [h^* g^{\frac{1}{2}}\, \mathfrak{f}_{(\beta)}]_{\xi=\beta} - [h^* g^{\frac{1}{2}}\, \mathfrak{f}_{(\alpha)}]_{\xi=\alpha}, \tag{11.31}$$

where $\bar{\boldsymbol{t}}$ in (11.29)–(11.30) is the prescribed surface load on the surfaces $(4.14)_{1,2}$.

Expressions for the stress-resultant and the stress-couples, in terms of the stress tensor τ^{ij} or alternatively the stress vector $\boldsymbol{t}$, can be defined directly using (11.7). First, however, we need certain preliminaries. Remembering the boundary surface (4.15), let $\boldsymbol{\nu}$ be the outward unit normal in the surface $\mathfrak{s}$ (i.e., the surface $\xi = 0$ at time t) to a curve of the form $f(\theta^1, \theta^2) = 0$, $\xi = 0$ and let ds denote the element of arc length of this curve. Then,

$$\boldsymbol{\nu} = \nu_\alpha \boldsymbol{a}^\alpha, \qquad \nu_1\, ds = a^{\frac{1}{2}}\, d\theta^2, \qquad \nu_2\, ds = -a^{\frac{1}{2}}\, d\theta^1. \tag{11.32}$$

Also, an element of surface area in a surface of the form (4.15) with outward unit normal $\boldsymbol{n}=n^i\,\boldsymbol{g}_i=n_i\,\boldsymbol{g}^i$ is given by

$$\begin{gathered} n_1\,da=g^{\frac{1}{2}}\,d\theta^2\,d\theta^3, \quad n_2\,da=-g^{\frac{1}{2}}\,d\theta^1\,d\theta^3, \\ da=(n^1\,d\theta^2-n^2\,d\theta^1)\,g^{\frac{1}{2}}\,d\theta^3. \end{gathered} \tag{11.33}$$

Let $\boldsymbol{N}=\boldsymbol{N}(\theta^\alpha, t; \boldsymbol{\nu})$ and $\boldsymbol{M}^N=\boldsymbol{M}^N(\theta^\alpha, t; \boldsymbol{\nu})$ denote, respectively, the resultant force and the resultant couple vectors of order N ($N=1, 2, \ldots$), each per unit length of a curve c on $\partial\mathscr{P}$ in the deformed configuration. These resultants are defined by the conditions

$$\int_{\partial\mathscr{P}} \boldsymbol{N}\,ds=\int_{\partial\mathscr{P}_n^*} \boldsymbol{t}\,da, \quad \int_{\partial\mathscr{P}} \boldsymbol{M}^N\,ds=\int_{\partial\mathscr{P}_n^*} \boldsymbol{t}\,\xi^N\,da, \tag{11.34}$$

the integration on the right-hand sides of $(11.34)_{1,2}$ being over a surface of the form (4.15) between $\xi=\alpha$ and $\xi=\beta$. The above conditions stipulate that the action of $\boldsymbol{N}$ and $\boldsymbol{M}^N$ on a portion of the curve c is equipollent to the action of the stress vector $\boldsymbol{t}$ upon a corresponding portion of the normal surface $\partial\mathscr{P}_n^*$ (which as defined at the beginning of this section coincides with $\partial\mathscr{P}$ on $\xi=0$). Similarly, the resultant heat flux can be introduced through the surface integral involving h^* in (11.13). Thus, the resultant heat flux $h=h(\theta^\alpha, t; \boldsymbol{\nu})$ per unit time entering $\mathscr{P}$ through the boundary $\partial\mathscr{P}$ is defined by the condition

$$\int_{\partial\mathscr{P}} h\,ds=\int_{\partial\mathscr{P}_n^*} h^*\,da, \tag{11.35}$$

which stipulates that the effect of the resultant heat flux h entering $\mathscr{P}$ is equivalent to the effect of the flux of heat h^* entering through a corresponding portion of the normal surface $\partial\mathscr{P}_n^*$. We also define the resultants $\boldsymbol{N}^\alpha$ and $\boldsymbol{M}^{N\alpha}$ ($N=1, 2, \ldots$), as well as q^α, by

$$\boldsymbol{N}^\alpha\,a^{\frac{1}{2}}=\int_\alpha^\beta \boldsymbol{T}^\alpha\,d\xi, \quad \boldsymbol{M}^{N\alpha}\,a^{\frac{1}{2}}=\int_\alpha^\beta \xi^N\,\boldsymbol{T}^\alpha\,d\xi, \quad \boldsymbol{M}^{0\alpha}=\boldsymbol{N}^\alpha, \tag{11.36}$$

$$q^\alpha\,a^{\frac{1}{2}}=\int_\alpha^\beta q^{*\alpha}\,g^{\frac{1}{2}}\,d\xi. \tag{11.37}$$

Consider now $(11.34)_1$ and substitute for $\boldsymbol{t}$ from $(11.7)_1$ to obtain

$$\begin{aligned} \int_{\partial\mathscr{P}} \boldsymbol{N}\,ds &= \int_{\partial\mathscr{P}_n^*} g^{-\frac{1}{2}}\,\boldsymbol{T}^i\,n_i\,da=\int_{\partial\mathscr{P}}\int_\alpha^\beta (\boldsymbol{T}^1\,d\theta^2-\boldsymbol{T}^2\,d\theta^1)\,d\xi \\ &= \int_{\partial\mathscr{P}} a^{\frac{1}{2}}(\boldsymbol{N}^1\,d\theta^2-\boldsymbol{N}^2\,d\theta^1)=\int_{\partial\mathscr{P}} \boldsymbol{N}^\alpha\,\nu_\alpha\,ds, \end{aligned} \tag{11.38}$$

where (11.33), $(11.36)_1$ and (11.32) have been used. From (11.38) follows the relation[46]

$$\boldsymbol{N}=\boldsymbol{N}^\alpha\,\nu_\alpha. \tag{11.39}$$

In a similar manner, we can deduce the results

$$\boldsymbol{M}^N=\boldsymbol{M}^{N\alpha}\,\nu_\alpha, \quad h=q^\alpha\,\nu_\alpha. \tag{11.40}$$

[46] This corresponds to $(9.11)_1$, but here (11.39) is obtained from the three-dimensional theory through the condition $(11.34)_1$ and the definition of the resultant $\boldsymbol{N}^\alpha$ in $(11.36)_1$.

As in (7.1), we assume that the temperature function $\theta^*(\theta^\alpha, \xi, t)$ at a material point in the deformed shell is an analytic function of ξ in the region $\alpha<\xi<\beta$. Thus, with the notation

$$\varphi_0 = \theta = \theta(\theta^\alpha, t) = \theta^*(\theta^\alpha, 0, t) \tag{11.41}$$

we write[47]

$$\theta^* = \varphi_0 + \sum_{N=1}^{\infty} \xi^N \varphi_N, \tag{11.42}$$

where φ_N are scalar functions of the surface coordinates θ^α and t. We also assume that the series (11.42) may be differentiated as many times as required with respect to any of its variables, at least in the open region $\alpha<\xi<\beta$. We complete our definitions of resultants by introducing the functions $\varepsilon^n = \varepsilon^n(\theta^\alpha, t)$, $(n=0, 1, 2, \ldots)$ as

$$\begin{aligned} \varrho\, \varepsilon\, a^{\frac{1}{2}} &= \int_\alpha^\beta \varepsilon^*\, k\, d\xi, \\ \varrho\, \varepsilon^n\, a^{\frac{1}{2}} &= \int_\alpha^\beta \varepsilon^*\, k\, \xi^n\, d\xi \quad (n=0, 1, 2, \ldots), \qquad \varepsilon^0 = \varepsilon, \end{aligned} \tag{11.43}$$

together with the functions $\eta^n = \eta^n(\theta^\alpha, t)$ and $\psi^n = \psi^n(\theta^\alpha, t)$, $(n=0, 1, 2, \ldots)$, through

$$\begin{aligned} \varrho\, \eta^n\, a^{\frac{1}{2}} &= \int_\alpha^\beta \eta^*\, k\, \xi^n\, d\xi, \qquad \eta^0 = \eta, \\ \varrho\, \psi^n\, a^{\frac{1}{2}} &= \int_\alpha^\beta \psi^*\, k\, \xi^n\, d\xi, \qquad \psi^0 = \psi. \end{aligned} \tag{11.44}$$

The relation

$$\psi^n = \varepsilon^n - \sum_{N=0}^{\infty} \eta^{N+n}\, \varphi_N \tag{11.45}$$

follows from (11.18) and (11.41)–(11.43). Our motivation for definitions (11.43)–(11.45) will become apparent presently.

γ) Developments from the energy equation. Entropy inequalities. First, with the help of the resultants (11.29)–(11.31), (11.34)–(11.35) and $(11.43)_1$, we deduce a two-dimensional energy balance from (11.13). The expression for the kinetic energy of the shell over $\mathscr{P}^*$, using (7.5), (11.28) and (11.2) becomes

$$\tfrac{1}{2}\int_{\mathscr{P}^*} \varrho^*\, \boldsymbol{v}^* \cdot \boldsymbol{v}^*\, dv = \tfrac{1}{2}\int_{\mathscr{P}} \varrho \left[\boldsymbol{v}\cdot\boldsymbol{v} + 2\sum_{N=2}^{\infty} k^N\, \boldsymbol{v}\cdot\boldsymbol{w}_N + \sum_{M,N=1}^{\infty} k^{M+N}\, \boldsymbol{w}_M\cdot\boldsymbol{w}_N\right] d\sigma, \tag{11.46}$$

where k^N is given by $(11.28)_2$ and

$$\varrho\, k^{M+N}\, a^{\frac{1}{2}} = \int_\alpha^\beta \xi^{M+N}\, k\, d\xi \qquad (M \geqq 1,\ N \geqq 1). \tag{11.47}$$

We note that k^N, k^{M+N} and k defined by (4.21) are functions of θ^1, θ^2 but are independent of t. The surface integral in (11.13) with $\partial\overline{\mathscr{P}}$ identified with $\partial\mathscr{P}_n^*$,

[47] This representation and associated definition for η^n in (11.44) were introduced in [1964, *4*].

using (7.5), (11.36)–(11.37) and (11.39)–(11.40), can be reduced as follows:

$$
\begin{aligned}
\int_{\partial\mathscr{P}_n^*} (\boldsymbol{t}\cdot\boldsymbol{v}^* - h^*)\, da &= \int_{\partial\mathscr{P}_n^*} \boldsymbol{t}\cdot\left(\boldsymbol{v} + \sum_{N=1}^{\infty} \xi^N \boldsymbol{w}_N\right) da - \int_{\partial\mathscr{P}_n^*} h^*\, da,\\
&= \iint_{\partial\mathscr{P}_n^*} (\boldsymbol{T}^1\, d\theta^2 - \boldsymbol{T}^2\, d\theta^1)\cdot\left(\boldsymbol{v} + \sum_{N=1}^{\infty} \xi^N \boldsymbol{w}_N\right) d\xi\\
&\quad - \iint_{\partial\mathscr{P}_n^*} h^*(n^1\, d\theta^2 - n^2\, d\theta^1)\, g^{\frac{1}{2}}\, d\xi\\
&= \int_{\partial\mathscr{P}} \left(\boldsymbol{N}^\alpha\cdot\boldsymbol{v} + \sum_{N=1}^{\infty} \boldsymbol{M}^{N\alpha}\cdot\boldsymbol{w}_N\right) \nu_\alpha\, ds - \int_{\partial\mathscr{P}} a^{\frac{1}{2}}(q^1\, d\theta^2 - q^2\, d\theta^1)\\
&= \int_{\partial\mathscr{P}} \left(\boldsymbol{N}\cdot\boldsymbol{v} + \sum_{N=1}^{\infty} \boldsymbol{M}^N\cdot\boldsymbol{w}_N\right) ds - \int_{\partial\mathscr{P}} q^\alpha \nu_\alpha\, ds\\
&= \int_{\partial\mathscr{P}} \left(\boldsymbol{N}\cdot\boldsymbol{v} + \sum_{N=1}^{\infty} \boldsymbol{M}^N\cdot\boldsymbol{w}_N - h\right) ds.
\end{aligned}
\tag{11.48}
$$

Also, with the help of (7.5) and (11.29)–(11.31), it can be readily shown that the volume integral on the right-hand side of (11.13) with $\overline{\mathscr{P}}$ identified with $\mathscr{P}^*$ and contributions due to the surface integral in (11.13) evaluated on the surfaces $(4.14)_{1,2}$ may be written as

$$
\int_{\mathscr{P}^*} \varrho^*(r^* + \boldsymbol{f}^*\cdot\boldsymbol{v}^*)\, dv + \int_{\partial\mathscr{P}_n^{*c}} (\boldsymbol{t}\cdot\boldsymbol{v}^* - h^*)\, da = \int_{\mathscr{P}} \varrho\left[r + \boldsymbol{f}\cdot\boldsymbol{v} + \sum_{N=1}^{\infty} \boldsymbol{l}^N\cdot\boldsymbol{w}_N\right] d\sigma. \tag{11.49}
$$

By use of the results (11.46)–(11.49), as well as $(11.43)_1$, the (three-dimensional) balance of energy (11.13) reduces to

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathscr{P}} \varrho\left(\varepsilon + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + \sum_{N=2}^{\infty} k^N \boldsymbol{w}_N\cdot\boldsymbol{v} + \frac{1}{2}\sum_{M,N=1}^{\infty} k^{M+N} \boldsymbol{w}_M\cdot\boldsymbol{w}_N\right) d\sigma\\
= \int_{\mathscr{P}} \varrho\left(r + \boldsymbol{f}\cdot\boldsymbol{v} + \sum_{N=1}^{\infty} \boldsymbol{l}^N\cdot\boldsymbol{w}_N\right) d\sigma + \int_{\partial\mathscr{P}} \left(\boldsymbol{N}\cdot\boldsymbol{v} + \sum_{N=1}^{\infty} \boldsymbol{M}^N\cdot\boldsymbol{w}_N - h\right) ds
\end{aligned}
\tag{11.50}
$$

over $\mathscr{P}$ in the present configuration.[48]

In what follows, we require generalized forms of the energy equation and the entropy production inequality and these can be obtained from (11.17) and (11.21). Let $\Phi = \Phi(\theta^i)$ be a scalar function of the coordinates θ^i. For example, a useful form of Φ used below in the development of shell theory is $\Phi = \xi^n$ $(n = 0, 1, 2, \ldots)$. Now multiply (11.17) by Φ and integrate over an arbitrary part $\mathscr{P}^*$ (not $\overline{\mathscr{P}}$) in the present configuration. After some straightforward manipulation and using (11.8), we obtain

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathscr{P}^*} \left(\varepsilon^* + \frac{1}{2}\boldsymbol{v}^*\cdot\boldsymbol{v}^*\right) \varrho^*\, \Phi\, dv = &\int_{\mathscr{P}^*} (r^* + \boldsymbol{f}^*\cdot\boldsymbol{v}^*)\, \varrho^*\, \Phi\, dv\\
&+ \int_{\partial\mathscr{P}^*} (\boldsymbol{t}\cdot\boldsymbol{v}^* - \boldsymbol{q}^*\cdot\boldsymbol{n})\, \Phi\, da\\
&- \int_{\mathscr{P}^*} [\boldsymbol{T}^i\cdot\boldsymbol{v}^* - q^{*i} g^{\frac{1}{2}}]\, g^{-\frac{1}{2}} \frac{\partial\Phi}{\partial\theta^i}\, dv.
\end{aligned}
\tag{11.51}
$$

[48] The two-dimensional energy balance (11.50) was obtained in [1968, 4]. A less general result was given previously in [1964, 4]. The derivation in the rest of this section closely follows that of GREEN and NAGHDI [1970, 2].

Again let $\Psi=\Psi(\theta^i)$, a non-negative scalar function of coordinates, be defined over a part $\mathscr{P}^*$ of $\mathscr{B}$:

$$\Psi \geqq 0. \tag{11.52}$$

Multiply (11.21) by Ψ/θ^* and integrate over a part $\mathscr{P}^*$ to obtain

$$\begin{gathered}\frac{d}{dt}\int\limits_{\mathscr{P}^*} \varrho^* \eta^* \Psi\, dv - \int\limits_{\mathscr{P}^*} \varrho^* \frac{r^*}{\theta^*} \Psi\, dv + \int\limits_{\partial\mathscr{P}^*} \frac{\boldsymbol{q}^*\cdot\boldsymbol{n}}{\theta^*} \Psi\, da \\ - \int\limits_{\mathscr{P}^*} \frac{q^{*i}}{\theta^*} \frac{\partial\Psi}{\partial\theta^i}\, dv \geqq 0.\end{gathered} \tag{11.53}$$

It is clear that when Φ and Ψ are constants, (11.51) and (11.53) reduce to the integral forms of balance of energy and the Clausius-Duhem inequality over a part $\mathscr{P}^*$ of $\mathscr{B}$ in the present configuration.

We now substitute $\Phi=\xi^n$ $(n=1, 2, \ldots)$ in (11.51) and by a procedure similar to that used in obtaining (11.50) we deduce

$$\begin{aligned}\frac{d}{dt}\int\limits_{\mathscr{P}} \varrho \left[\varepsilon^n + \frac{1}{2} k^n \boldsymbol{v}\cdot\boldsymbol{v} + \sum_{N=1}^{\infty} k^{N+n} \boldsymbol{w}_N\cdot\boldsymbol{v} + \frac{1}{2}\sum_{M,N=1}^{\infty} k^{M+N+n} \boldsymbol{w}_M\cdot\boldsymbol{w}_N\right] d\sigma \\ = \int\limits_{\mathscr{P}} \cdot\varrho \left[\left(r^n + R^n + \boldsymbol{l}^n\cdot\boldsymbol{v} + \sum_{N=1}^{\infty} \boldsymbol{l}^{N+n}\cdot\boldsymbol{w}_N\right) - \boldsymbol{m}^n\cdot\boldsymbol{v}\right. \\ \left. - \sum_{N=1}^{\infty} \frac{n}{N+n} \boldsymbol{m}^{N+n}\cdot\boldsymbol{w}_N\right] d\sigma \\ + \int\limits_{\partial\mathscr{P}} \left[\boldsymbol{M}^n\cdot\boldsymbol{v} + \sum_{N=1}^{\infty} \boldsymbol{M}^{N+n}\cdot\boldsymbol{w}_N - h^n\right] ds,\end{aligned} \tag{11.54}$$

where we have introduced

$$\boldsymbol{m}^N a^{\frac{1}{2}} = N\int\limits_{\alpha}^{\beta} \xi^{N-1} \boldsymbol{T}^3\, d\xi \qquad (N\geqq 0) \tag{11.55}$$

and

$$\varrho\, R^n a^{\frac{1}{2}} = n\int\limits_{\alpha}^{\beta} q^{*3} \xi^{n-1} g^{\frac{1}{2}}\, d\xi, \tag{11.56}$$

$$q^{n\alpha} a^{\frac{1}{2}} = \int\limits_{\alpha}^{\beta} q^{*\alpha} \xi^n g^{\frac{1}{2}}\, d\xi, \qquad h^n = q^{n\alpha} \nu_\alpha, \tag{11.57}$$

$$\varrho\, r^n a^{\frac{1}{2}} = \int\limits_{\alpha}^{\beta} r^* k\, \xi^n\, d\xi - [\xi^n h^* g^{\frac{1}{2}} \mathfrak{f}_{(\beta)}]_{\xi=\beta} - [\xi^n h^* g^{\frac{1}{2}} \mathfrak{f}_{(\alpha)}]_{\xi=\alpha}, r^0 = r, \tag{11.58}$$

in addition to resultants defined previously. Under suitable continuity assumptions (11.54) can be reduced to

$$\begin{aligned}\varrho\, r^n + \varrho\, R^n - \varrho\, \dot{\varepsilon}^n + (\boldsymbol{M}^{n\alpha}{}_{|\alpha} + \varrho\, \bar{\boldsymbol{l}}^n - \boldsymbol{m}^n)\cdot\boldsymbol{v} + \boldsymbol{M}^{n\alpha}\cdot\boldsymbol{v}_{,\alpha} \\ + \sum_{N=1}^{\infty} \left[\boldsymbol{M}^{(N+n)\alpha}{}_{|\alpha} + \varrho\, \bar{\boldsymbol{l}}^{N+n} - \frac{n}{N+n} \boldsymbol{m}^{N+n}\right]\cdot\boldsymbol{w}_N \\ + \sum_{N=2}^{\infty} \boldsymbol{M}^{(N+n)\alpha}\cdot\boldsymbol{w}_{N,\alpha} - q^{n\alpha}{}_{|\alpha} = 0 \qquad (n=0, 1, 2, \ldots),\end{aligned} \tag{11.59}$$

where we have put

$$\begin{aligned}\bar{\boldsymbol{f}}=\bar{\boldsymbol{l}}^0=\boldsymbol{f}-\dot{\boldsymbol{v}}-\sum_{N=1}^{\infty}k^N\dot{\boldsymbol{w}}_N, \qquad \bar{\boldsymbol{l}}=\bar{\boldsymbol{l}}^1=\boldsymbol{l}-\sum_{M=1}^{\infty}k^{M+1}\dot{\boldsymbol{w}}_M,\\ \bar{\boldsymbol{l}}^N=\boldsymbol{l}^N-k^N\dot{\boldsymbol{v}}-\sum_{M=1}^{\infty}k^{M+N}\dot{\boldsymbol{w}}_M \qquad (N\geqq 2).\end{aligned} \tag{11.60}$$

In (11.59), for clarity, parentheses have been placed around the number $N+n$ which occurs as a superscript in $\boldsymbol{M}^{(N+n)\alpha}$. Henceforth, however, we shall omit the parentheses and will write $\boldsymbol{M}^{(N+n)\alpha}$ as $\boldsymbol{M}^{N+n\alpha}$.

We postpone further consideration of (11.59) and turn our attention to the inequality (11.53). Recalling (11.42) and the fact that $\theta^*>0$, we introduce

$$\Phi^*=\frac{1}{\theta^*}=\sum_{N=0}^{\infty}\xi^N\Phi_N, \qquad \Phi_N=\Phi_N(\theta^1,\theta^2,t), \tag{11.61}$$

so that

$$\varphi_0\,\Phi_0=1, \qquad \sum_{N=0}^{r}\varphi_N\,\Phi_{r-N}=0 \qquad (r=1,2,\ldots), \tag{11.62}$$

in view of (11.42). Also, remembering (11.52), we set

$$\Psi=(-\alpha+\xi)^n \qquad (\alpha\leqq\xi\leqq\beta). \tag{11.63}$$

Substituting (11.61) and (11.63) into (11.53) and making the appropriate continuity assumptions, we obtain the inequalities

$$\begin{aligned}&\varrho\sum_{r=0}^{n}\binom{n}{r}(-\alpha)^{n-r}\left[\dot{\eta}^r-\sum_{N=0}^{\infty}\gamma^{r+N}\,\Phi_N\right]\\ &\quad+\sum_{r=0}^{n}\binom{n}{r}(-\alpha)^{n-r}\sum_{N=0}^{\infty}\left(q^{r+N\alpha}{}_{|\alpha}\,\Phi_N+q^{r+N\alpha}\,\Phi_{N,\alpha}\right)\\ &\quad-\varrho\sum_{r=0}^{n-1}\binom{n-1}{r}(-\alpha)^{n-r-1}\sum_{N=0}^{\infty}\frac{n}{r+N+1}\,R^{r+N+1}\,\Phi_N\geqq 0\\ &(n=0,1,2,\ldots).\end{aligned} \tag{11.64}$$

Since φ_N and Φ_N are not defined for negative values of N, we may write $(11.62)_2$ as

$$\sum_{N=0}^{\infty}\varphi_N\,\Phi_{r-N}=0 \quad (r=1,2,\ldots), \qquad \varphi_M=\Phi_M=0 \quad \text{for} \quad M<0. \tag{11.65}$$

Also, for later reference, we record here the identity

$$\sum_{N=0}^{\infty}\sum_{M=0}^{\infty}\Phi_N\,\dot{\eta}^{M+N+r}\varphi_M=\dot{\eta}^r. \tag{11.66}$$

To verify the last result, let the left-hand side of (11.66) be denoted by $\mathfrak{g}$ and write

$$\begin{aligned}\mathfrak{g}&=\sum_{N=0}^{\infty}\sum_{M=0}^{\infty}\Phi_N\,\dot{\eta}^{M+N+r}\,\varphi_M\\ &=\sum_{n'=N'}^{\infty}\sum_{N'=0}^{\infty}\Phi_{n'-N'}\,\varphi_{N'}\,\dot{\eta}^{n'+r}\\ &=\sum_{n=0}^{\infty}\sum_{N=0}^{\infty}\Phi_{n-N}\,\varphi_N\,\dot{\eta}^{n+r},\end{aligned}$$

where in the last of the above the first summation sign $\sum_{n=N}^{\infty}$ has been replaced with $\sum_{n=0}^{\infty}$ since $\Phi_{n-N}=0$ for $N>n$. Now the last expression can be written as

$$\mathfrak{g}=\sum_{N=0}^{\infty}\Phi_{-N}\,\varphi_N\,\dot{\eta}^r+\sum_{n=1}^{\infty}\left[\sum_{N=0}^{\infty}\Phi_{n-N}\,\varphi_N\right]\dot{\eta}^{n+r}. \tag{11.67}$$

But the term in the square bracket vanishes by virtue of (11.65) and the first term on the right-hand side of (11.67) is $\Phi_0\,\varphi_0\,\dot{\eta}^r=\dot{\eta}^r$ by $(11.62)_1$ and $(11.65)_2$. Hence, (11.67) reduces to (11.66).

12. Basic field equations for shells continued: II. Derivation from the three-dimensional theory. We continue here the development of basic field equations for shells from the three-dimensional theory. After the derivation of the local field equations, several related aspects of the subject are discussed including the linearization of the field equations and their relations to the corresponding results in the classical linear theories of shells and plates.

δ) *General field equations.* The derivation of the equations of motion for shells can be accomplished by direct integration of $(11.8)_{1,2}$ after their multiplication by $\xi^N\,(N\geqq 0)$ and with the use of the resultants $(11.36)_{1,2}$, (11.55) and (11.29)–(11.30). Here, however, we deduce the equations of motion for shells from the energy equations (11.59) together with the invariance requirements under superposed rigid body motions. From the fact that the quantities listed in (11.19) are objective in the sense of (8.25) and the definitions of the various resultants, it follows that

$$\begin{aligned}&\boldsymbol{M}^{n\alpha},\ \boldsymbol{m}^n,\ \boldsymbol{l}^n\quad (n\geqq 0),\\ &\varepsilon^n,\ \eta^n,\ \varphi^n,\ \psi^n\quad (n\geqq 0),\\ &r^n,\ R^n,\ q^{n\alpha}\qquad (n\geqq 0)\end{aligned} \tag{12.1}$$

are also objective and hence transform according to (8.25) under superposed rigid body motions.

Consider now two special motions corresponding to those under superposed rigid body motions when the body occupies the same position at time t. Then, with Q and $\dot{Q}$ given by (5.48) at time t, the superposed velocities are given, respectively, by the linear transformations

$$\boldsymbol{v}^*\to\boldsymbol{v}^*+\boldsymbol{b},\qquad \boldsymbol{b}=\text{const.}, \tag{12.2}$$

$$\boldsymbol{v}^*\to\boldsymbol{v}^*+\boldsymbol{\omega}_0\times\boldsymbol{p},\qquad \boldsymbol{\omega}_0=\text{const.}, \tag{12.3}$$

and these, in turn, imply the transformations

$$\boldsymbol{v}\to\boldsymbol{v}+\boldsymbol{b},\qquad \boldsymbol{w}_N\to\boldsymbol{w}_N, \tag{12.4}$$

$$\boldsymbol{v}\to\boldsymbol{v}+\boldsymbol{\omega}_0\times\boldsymbol{r},\qquad \boldsymbol{w}_N\to\boldsymbol{w}_N+\boldsymbol{\omega}_0\times\boldsymbol{d}_N, \tag{12.5}$$

in view of (7.1) and (7.5). The linear transformations (12.2) or (12.4) correspond to superposed uniform rigid body translational velocity while (12.3) or (12.5) represent uniform rigid body angular velocity, the (three-dimensional) continuum occupying the same position at time t.

Since the energy equations (11.59) hold for every motion of the shell, by a familiar argument and using (12.4)–(12.5), the equations of motion can quickly be deduced from (11.59). Thus, if we replace $\boldsymbol{v}$ and $\boldsymbol{w}_N$ in (11.59) by the transformations (12.4) and keep in mind that the functions listed in (12.1) remain unaltered under superposed rigid body motions, then after subtraction we obtain

$$\boldsymbol{N}^{\alpha}{}_{|\alpha} + \varrho \bar{\boldsymbol{f}} = 0, \quad \boldsymbol{M}^{n\alpha}{}_{|\alpha} + \varrho \bar{\boldsymbol{l}}^{n} - \boldsymbol{m}^{n} = 0 \quad (n = 1, 2, \ldots). \tag{12.6}$$

With the use of (12.6) and recalling the remark (concerning notation) following (11.60), the energy equations (11.59) can be reduced to

$$\begin{aligned} \varrho r^{n} + \varrho R^{n} - \varrho \dot{\varepsilon}^{n} + \boldsymbol{M}^{n\alpha} \cdot \boldsymbol{v}_{,\alpha} + \sum_{N=1}^{\infty} \frac{N}{N+n} \boldsymbol{m}^{N+n} \cdot \boldsymbol{w}_{N} \\ + \sum_{N=1}^{\infty} \boldsymbol{M}^{N+n\alpha} \cdot \boldsymbol{w}_{N,\alpha} - q^{n\alpha}{}_{|\alpha} = 0 \quad (n = 0, 1, 2, \ldots). \end{aligned} \tag{12.7}$$

Next, considering a motion of the shell corresponding to (12.5) and again keeping in mind that the functions listed in (12.1) remain unaltered under such superposed rigid body motions, from (12.7) we deduce the equations

$$\boldsymbol{M}^{n\alpha} \times \boldsymbol{a}_{\alpha} + \sum_{N=1}^{\infty} \left(\frac{N}{N+n} \boldsymbol{m}^{N+n} \times \boldsymbol{d}_{N} + \boldsymbol{M}^{N+n\alpha} \times \boldsymbol{d}_{N,\alpha} \right) = 0 \quad (n = 0, 1, 2, \ldots). \tag{12.8}$$

For $n = 0$, (12.8) can be written in a more transparent form, namely

$$\boldsymbol{N}^{\alpha} \times \boldsymbol{a}_{\alpha} + \sum_{N=1}^{\infty} (\boldsymbol{m}^{N} \times \boldsymbol{d}_{N} + \boldsymbol{M}^{N\alpha} \times \boldsymbol{d}_{N,\alpha}) = 0. \tag{12.9}$$

As noted earlier, the equations of motion (12.6) can also be obtained directly from $(11.8)_1$ while (12.8) can be deduced from $(11.8)_2$ and is therefore a consequence of the symmetry of the stress tensor.[49]

The Eqs. (12.6) and (12.8) are the equations of motion for shells in vector form. For later reference, it is desirable to record these equations also in tensor components. By (11.39)–$(11.40)_1$, the resultants $\boldsymbol{N}^{\alpha}$ and $\boldsymbol{M}^{N\alpha}$ transform as contravariant surface vectors. When referred to the base vectors $\boldsymbol{a}_i$, these vector fields and the resultants $\boldsymbol{m}^{N}$ can be written as[50]

$$\boldsymbol{N}^{\alpha} = N^{\alpha i} \boldsymbol{a}_i, \quad \boldsymbol{M}^{N\alpha} = M^{N\alpha i} \boldsymbol{a}_i \tag{12.10}$$

and

$$\boldsymbol{m}^{N} = m^{Ni} \boldsymbol{a}_i. \tag{12.11}$$

Similarly, the resultants in (11.60) can be expressed as

$$\bar{\boldsymbol{f}} = \bar{\boldsymbol{l}}^{0} = \bar{f}^{i} \boldsymbol{a}_i, \quad \bar{\boldsymbol{l}} = \bar{\boldsymbol{l}}^{1} = \bar{l}^{i} \boldsymbol{a}_i, \quad \bar{\boldsymbol{l}}^{N} = \bar{l}^{Ni} \boldsymbol{a}_i \quad (N \geqq 2). \tag{12.12}$$

---

[49] The equations of motion (12.6) and (12.8) in vector forms were derived by GREEN and NAGHDI [1970, *2*] and the component forms (12.13)–(12.15) were obtained earlier by GREEN, LAWS and NAGHDI [1968, *4*]. Two-dimensional equations of motion (or equilibrium) of this type, limited to the linear theory of flat plates, can be found in the papers of TIFFEN and LOWE [1963, *15*] and [1965, *8*]. We also mention here an extensive work of CHIEN [1944, *2*], where his equations of equilibrium (corresponding to the equations of the restricted theory in Sect. 10) are those which were derived in [1941, *2*] by direct approach; however, in the remainder of the series of papers [1944, *2*], the stress tensor and the body force are expanded in powers of the thickness coordinate. This procedure is not the same as that in which resultants of the type (11.29)–(11.30) and (11.36) are employed.

[50] The order of indices αi in $N^{\alpha i}$ and $M^{N\alpha i}$ differs from those in [1968, *4*] and [1970, *2*] but agrees with that in [1971, *6*] and corresponds to the usual notation in shell theory.

With the help of (12.10)–(12.12), the equations of motion (12.6) in component form are[51]

$$N^{\alpha\beta}{}_{|\alpha} - b^\alpha_\beta N^{\alpha 3} + \varrho \bar{f}^\beta = 0, \qquad N^{\alpha 3}{}_{|\alpha} + b_{\alpha\beta} N^{\alpha\beta} + \varrho \bar{f}^3 = 0, \tag{12.13}$$

$$\left.\begin{aligned} &M^{N\alpha\beta}{}_{|\alpha} - b^\beta_\alpha M^{N\alpha 3} + \varrho \bar{l}^{N\beta} = m^{N\beta},\\ &M^{N\alpha 3}{}_{|\alpha} + b_{\alpha\beta} M^{N\alpha\beta} + \varrho \bar{l}^{N3} = m^{N3} \qquad (N = 1, 2, \ldots). \end{aligned}\right\} \tag{12.14}$$

In a similar manner and by recalling the kinematical results (7.13)–(7.15), from (12.8) we deduce

$$\left.\begin{aligned} &N'^{\beta\alpha} = N'^{\alpha\beta} = N^{\alpha\beta} - \sum_{N=1}^{\infty} (m^{N\alpha} d_{N\cdot}^{\beta} + M^{N\gamma\alpha} \lambda_{N\cdot\gamma}^{\beta}),\\ &N^{\alpha 3} + \sum_{N=1}^{\infty} (m^{N3} d_{N\cdot}^{\alpha} - m^{N\alpha} d_{N\cdot}^{3}) + \sum_{N=1}^{\infty} (M^{N\gamma 3} \lambda_{N\cdot\gamma}^{\alpha} - M^{N\gamma\alpha} \lambda_{N\cdot\gamma}^{3}) = 0 \end{aligned}\right\} \tag{12.15}$$

and

$$\left.\begin{aligned} &M'^{n\alpha\beta} = M'^{n\beta\alpha} = M^{n\alpha\beta} - \sum_{N=1}^{\infty} \left(\frac{N}{N+n} m^{N+n\alpha} d_{N\cdot}^{\beta} + M^{N+n\gamma\alpha} \lambda_{N\cdot\gamma}^{\beta}\right) \quad (n = 1, 2, \ldots),\\ &M^{n\alpha 3} + \sum_{N=1}^{\infty} \frac{N}{N+n} (m^{N+n3} d_{N\cdot}^{\alpha} - m^{N+n\alpha} d_{N\cdot}^{3})\\ &\qquad + \sum_{N=1}^{\infty} (M^{N+n\gamma 3} \lambda_{N\cdot\gamma}^{\alpha} - M^{N+n\gamma\alpha} \lambda_{N\cdot\gamma}^{3}) = 0 \qquad (n = 1, 2, \ldots). \end{aligned}\right\} \tag{12.16}$$

Eqs. (12.15) are the component form of (12.9) or (12.8) for $n = 0$. It is clear that the conditions (12.15)–(12.16), which result from the symmetry of the stress tensor, are identities in the exact theory.

Recalling the kinematic variables (7.13)–(7.16) and using (12.8), the energy equations (12.7) reduce to

$$\varrho r^n + \varrho R^n - \varrho \dot{\varepsilon}^n + P^n - q^{n\alpha}{}_{|\alpha} = 0, \tag{12.17}$$

where

$$P^n = M'^{n\alpha\beta} \eta_{\alpha\beta} + \sum_{N=1}^{\infty} \left(\frac{N}{N+n} m^{N+ni} \dot{d}_{Ni} + M^{N+n\alpha i} \dot{\lambda}_{Ni\alpha}\right), \tag{12.18}$$

$M'^{n\alpha\beta}$ is given by $(12.16)_1$ and we have introduced the notation

$$\eta_{\alpha\beta} = \dot{e}_{\alpha\beta} = \tfrac{1}{2} \dot{a}_{\alpha\beta} \tag{12.19}$$

for the time rate of the kinematic measure (7.18). In terms of the resultants ψ^n defined by $(11.44)_3$, the energy equations (12.17) can be expressed as

$$\varrho r^n + \varrho R^n - \varrho \left[\dot{\psi}^n + \sum_{N=0}^{\infty} (\dot{\eta}^{N+n} \varphi_N + \eta^{N+n} \dot{\varphi}_N)\right] + P^n - q^{n\alpha}{}_{|\alpha} = 0. \tag{12.20}$$

For completeness, we also record here the inequalities which can be deduced from the combination of (11.64) and (12.20). In view of (11.66), we have

$$\begin{aligned} &\dot{\eta}^r - \sum_{N=0}^{\infty} \Phi_N \left[\dot{\psi}^{r+N} + \sum_{M=0}^{\infty} \overline{\eta^{M+N+r} \varphi_M}^{\,\boldsymbol{\cdot}}\right]\\ &\quad = \dot{\eta}^r - \sum_{N=0}^{\infty} \Phi_N \left[\dot{\psi}^{r+N} + \sum_{M=0}^{\infty} \eta^{M+N+r} \dot{\varphi}_M\right] - \sum_{n=0}^{\infty} \sum_{N=0}^{\infty} \dot{\eta}^{n+r} \Phi_{n-N} \varphi_N\\ &\quad = -\sum_{N=0}^{\infty} \Phi_N \left[\dot{\psi}^{r+N} + \sum_{M=0}^{\infty} \eta^{M+N+r} \dot{\varphi}_M\right]. \end{aligned} \tag{12.21}$$

[51] These are deduced in a manner similar to (9.42) by considering the scalar product of each of (12.6) with $\boldsymbol{a}^\beta$ and again with $\boldsymbol{a}^3$.

Substituting for r^n from (12.20) into (11.64) and using (12.21), we finally obtain the inequalities[52]

$$
\begin{aligned}
\sum_{r=0}^{n} \binom{n}{r} (-\alpha)^{n-r} \Bigg\{ \sum_{N=0}^{\infty} \Phi_N \Bigg[-\varrho \left(\dot{\psi}^{r+N} + \sum_{M=0}^{\infty} \eta^{M+N+r}\, \dot{\varphi}_M \right) + M'^{r+N\alpha\beta}\, \eta_{\alpha\beta} \\
+ \sum_{M=1}^{\infty} \frac{M}{M+N+r}\, m^{M+N+ri}\, \dot{d}_{Mi} + \sum_{M=1}^{\infty} M^{M+N+r\alpha i}\, \dot{\lambda}_{Mi\alpha} + \frac{N}{N+r}\, \varrho\, R^{N+r} \Bigg] \qquad (12.22) \\
+ \sum_{N=0}^{\infty} q^{N+r\alpha}\, \Phi_{N,\alpha} \Bigg\} \geqq 0 \qquad (n=0,1,2,\ldots).
\end{aligned}
$$

ε) *An approximate system of equations of motion.* Although the equations of motion (12.6) and (12.8) or (12.13)–(12.16) are exact, they consist in an infinite system of equations in an infinite number of unknowns. The complexity of this system and a need for a *suitable* approximation are evident. As remarked in Sect. 7, the introduction of an approximative scheme is premature at this stage and strictly speaking should be postponed until the complete theory (including the constitutive equations) has been developed. However, in order to give an indication of the nature of the field equations of an approximate theory (derived from the three-dimensional equations), we consider briefly a system of approximate equations of motion but postpone its justification.[53]

Suppose that[54]

$$M^{N\alpha i} = 0, \qquad m^{Ni} = 0 \qquad (N \geqq 2). \tag{12.23}$$

Then, the equations of motion (12.14) for $N \geqq 2$ are satisfied if we specify

$$\bar{l}^{Ni} = 0 \qquad (N \geqq 2). \tag{12.24}$$

Eqs. (12.24) are usually satisfied approximately, since the quantities $\bar{l}^{Ni}$ $(N \geqq 2)$ involve moments (of order greater than one across the thickness of the shell) of body forces, applied surface loads and inertia terms. Recalling (11.60) and the notation in (7.9), consistent with the above approximation, we also take

$$\bar{\boldsymbol{f}} = \boldsymbol{f} - \dot{\boldsymbol{v}}, \qquad \bar{\boldsymbol{l}} = \bar{\boldsymbol{l}}^1 = \boldsymbol{l} - k^{11}\, \dot{\boldsymbol{w}}, \tag{12.25}$$

where

$$\varrho\, k^{11}\, a^{\frac{1}{2}} = \int_{\alpha}^{\beta} \varrho^*\, g^{\frac{1}{2}}\, \xi^2\, d\xi. \tag{12.26}$$

With the notations

$$M^{1\alpha i} = M^{\alpha i}, \qquad m^{1i} = m^i, \tag{12.27}$$

as a consequence of the above approximation specified by (12.23)–(12.25), the equations of motion (12.13)–(12.14) reduce to

$$N^{\alpha\beta}{}_{|\alpha} - b^{\beta}_{\alpha}\, N^{\alpha 3} + \varrho\, \bar{f}^{\beta} = 0, \qquad N^{\alpha 3}{}_{|\alpha} + b_{\alpha\beta}\, N^{\alpha\beta} + \varrho\, \bar{f}^3 = 0, \tag{12.28}$$

$$M^{\alpha\beta}{}_{|\alpha} - b^{\beta}_{\alpha}\, M^{\alpha 3} + \varrho\, \bar{l}^{\beta} = m^{\beta}, \qquad M^{\alpha 3}{}_{|\alpha} + b_{\alpha\beta}\, M^{\alpha\beta} + \varrho\, \bar{l}^3 = m^3, \tag{12.29}$$

[52] GREEN and NAGHDI [1970, *2*].

[53] The approximations under which the system of approximate equations of motion are obtained were employed by GREEN, LAWS and NAGHDI [1968, *4*] and GREEN and NAGHDI [1970, *2*].

[54] In view of the generality of our development leading to (12.13)–(12.16), conditions (12.23) represent an approximation. On the other hand, in most of the existing literature on shell theory confined to a less general development, the introduction of the resultants $M^{N\alpha i}$ and m^{Ni} for $N \geqq 2$ is avoided and thereby the question of approximation (12.23) does not arise.

while Eqs. $(12.15)_{1,2}$ become

$$\begin{aligned} N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} - m^{\alpha} d^{\beta} - M^{\gamma\alpha} \lambda^{\beta}_{.\gamma}, \\ N^{\alpha 3} + m^3 d^{\alpha} - m^{\alpha} d^3 + M^{\gamma 3} \lambda^{\alpha}_{.\gamma} - M^{\gamma\alpha} \lambda^{3}_{.\gamma} = 0. \end{aligned} \tag{12.30}$$

Also, the Eqs. $(12.16)_{1,2}$ reduce to

$$\begin{aligned} M'^{1\alpha\beta} = M^{\alpha\beta} = M^{\beta\alpha}, \\ M^{1\alpha 3} = M^{\alpha 3} = 0. \end{aligned} \tag{12.31}$$

Although $(12.16)_{1,2}$ are identities in an exact theory, in general we cannot expect that they be satisfied in an approximate theory. Indeed, from the restrictive nature of (12.31), it can be inferred that they could be satisfied only in special cases of an approximate theory. The identities $(12.16)_{1,2}$ and the question of their subsequent approximations do not arise in most of the existing literature on the classical theory of shells developed from the three-dimensional equations. This is simply due to the fact that the introduction of certain resultants such as $M^{N\alpha i}$ and m^{Ni} for $N \geqq 2$ is avoided *ab initio*.

As our purpose of describing the above approximation at this stage is merely to give an indication of the nature of a system of approximate equations of motion, we do not dwell on the corresponding approximation for the temperature functions in (11.61) and their subsequent effect, as well as those in (12.23), on the energy equations (12.20) and the inequalities (12.22). However, it is worth observing that if the surfaces $\mathfrak{s}$ and $\mathscr{s}$ are identified, the set of Eqs. (12.28)–(12.30) are formally the same as the equations of motion for a Cosserat surface derived in Sect. 9.

ζ) *Linearized field equations.* Previously in Sect. 7, where the linearization of the kinematic quantities was considered, the (three-dimensional) infinitesimal strain tensor was expressed in terms of linearized two-dimensional kinematic measures. In carrying out the linearization procedure, it was assumed that all (two-dimensional) kinematic measures such as those in (7.59), as well as their derivatives with respect to the surface coordinates and time, are of $O(\varepsilon)$ in the sense of (6.4). We now suppose that the continuum is initially stress-free and let θ_0^* and η_0^* refer to a standard temperature and entropy in the (initial) reference configuration. We further recall that in a complete linearized (three-dimensional) theory, $\boldsymbol{T}^i$ or the stress tensor τ^{ij} when expressed in suitable non-dimensional form, as well as its derivative, is of $O(\varepsilon)$; $(\theta^* - \theta_0^*)/\theta_0^*$ and $(\eta^* - \eta_0^*)/\eta_0^*$ and their derivatives are of $O(\varepsilon)$; and that the internal energy ε^* includes terms of $O(\varepsilon)$ and $O(\varepsilon^2)$ while the free energy ψ^* is of $O(\varepsilon^2)$. Keeping this in mind, the order of magnitude of the various resultants defined in Sect. 11 is clear in a linearized theory. In particular, the vector fields $\boldsymbol{N}^{\alpha}$, $\boldsymbol{M}^{N\alpha}$, $\boldsymbol{m}^{N}$ $(N \geqq 1)$ or their tensor components when expressed in suitable non-dimensional forms, as well as their derivatives with respect to the surface coordinates, are of $O(\varepsilon)$.

Thus, in obtaining the linearized version of the basic field equations, all tensors in these equations are referred to the initial undeformed surface $\mathfrak{S}$, covariant differentiation is with respect to $A_{\alpha\beta}$ and in Eqs. (12.13)–(12.20), as well as the inequality (12.22), $b_{\alpha\beta}$, d_{Ni}, $\lambda_{Ni\alpha}$, ϱ must be replaced to order ε by their initial values $B_{\alpha\beta}$, D_{Ni}, $\Lambda_{Ni\alpha}$, ϱ_0, respectively. Similarly, in (12.10)–(12.11) and in the definitions of the various resultants [e.g., (11.36)–(11.37), (11.55), (11.43)], $\boldsymbol{g}_i$, $\boldsymbol{a}_i$, g and a may now be replaced with their initial values. We omit the details of the linearization of all field equations but in the rest of this subsection confine our attention to the linearized versions of the approximate equations of motion (12.28)–(12.30). Let $\bar{F}^i$ and $\bar{L}^i$ denote the components of $\bar{\boldsymbol{f}}$ and $\bar{\boldsymbol{l}}$ in (12.25) referred to the base vectors $\boldsymbol{A}_i$, i.e.,

$$\bar{F}^i = \bar{\boldsymbol{f}} \cdot \boldsymbol{A}^i, \quad \bar{L}^i = \bar{\boldsymbol{l}} \cdot \boldsymbol{A}^i. \tag{12.32}$$

Then, upon linearization, (12.28)–(12.30) become

$$N^{\alpha\beta}|_\alpha - B^\beta_\alpha N^{\alpha 3} + \varrho_0 \bar{F}^\beta = 0, \qquad N^{\alpha 3}|_\alpha + B_{\alpha\beta} N^{\alpha\beta} + \varrho_0 \bar{F}^3 = 0, \tag{12.33}$$

$$M^{\alpha\beta}|_\alpha - B^\beta_\alpha M^{\alpha 3} + \varrho_0 \bar{L}^\beta = m^\beta, \qquad M^{\alpha 3}|_\alpha + B_{\alpha\beta} M^{\alpha\beta} + \varrho_0 \bar{L}^3 = m^3, \tag{12.34}$$

and

$$\begin{aligned} N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} - m^\alpha D^\beta - M^{\gamma\alpha} \Lambda^\beta_{.\gamma}, \\ N^{\alpha 3} + m^3 D^\alpha - m^\alpha D^3 + M^{\gamma 3} \Lambda^\alpha_{.\gamma} - M^{\gamma\alpha} \Lambda^3_{.\gamma} = 0. \end{aligned} \tag{12.35}$$

The above equations are the linearized equations of motion, obtained under the linearized version of the approximations (12.23)–(12.25).[55] The Eqs. (12.33)–(12.35) are formally equivalent to the linearized equations of motion for a Cosserat surface given by (9.47)–(9.51).

η) *Relationship with results in the classical linear theory of thin shells and plates.* In our description of a shell-like body in Sect. 4 it was emphasized that the surface $\xi = 0$ is not necessarily midway between the surfaces $(4.14)_{1,2}$, although a reference configuration could be chosen in which $\xi = 0$ is the initial middle surface. The various resultants which occur in the field equations of this section, e.g., (12.13)–(12.16) or (12.28)–(12.30), are defined between the limits α, β and in terms of $\boldsymbol{T}^i$ (or equivalently the symmetric stress tensor τ^{ij}) in the present configuration.

In what follows, we confine our attention to the field equations of the linearized theory and assume that the position vector in the initial reference configuration is given by (7.30) with $\boldsymbol{D}$ specified by $(7.31)_3$. For sufficiently *thin* shells, we may also adopt the criterion (4.31) which is generally assumed in the classical linear theories of shells developed from the three-dimensional equations. Then, the mass density is given by the approximate expression (7.46) and the limits of integration in the various resultants may be replaced by[56] $-\beta, \beta$. Moreover, in the special case in which $\boldsymbol{D} = \boldsymbol{A}_3$, by (7.48) the limits of integration may be replaced by

$$-\frac{h}{2}, \frac{h}{2}.$$

As noted above, within the scope of the linear theory, $\boldsymbol{g}_i$, $\boldsymbol{a}_i$, g and a may be replaced by their initial values in the definitions of the various resultants. For example, the resultants which occur in the linearized equations (12.33)–(12.35) can be written in the forms

$$\begin{aligned} N^{\alpha\beta} &= \int_{-\beta}^{\beta} \nu\, \tau^{\alpha\gamma} \nu^\beta_\gamma \, d\xi, & N^{\alpha 3} &= \int_{-\beta}^{\beta} \nu (\xi D_{,\beta}\, \tau^{\alpha\beta} + D \tau^{\alpha 3})\, d\xi, \\ M^{\alpha\beta} &= \int_{-\beta}^{\beta} \nu\, \tau^{\alpha\gamma} \nu^\beta_\gamma \xi \, d\xi, & M^{\alpha 3} &= \int_{-\beta}^{\beta} \nu (\xi D_{,\beta}\, \tau^{\alpha\beta} + D \tau^{\alpha 3})\, \xi \, d\xi, \\ m^\alpha &= \int_{-\beta}^{\beta} \nu\, \tau^{3\gamma} \nu^\alpha_\gamma \, d\xi, & m^3 &= \int_{-\beta}^{\beta} \nu (\xi D_{,\alpha}\, \tau^{3\alpha} + D \tau^{33})\, d\xi. \end{aligned} \tag{12.36}$$

[55] The equations of motion (12.33)–(12.35) are obtained by ignoring $(12.31)_{1,2}$ which result from $(12.16)_{1,2}$ by approximation. As was noted earlier, the latter conditions do not arise if (for thin shells) resultants of the type $M^{N\alpha i}$ and m^{Ni} for $N \geqq 2$ are not admitted. A further remark on this point will be made later in this section.

[56] In this connection, recall the remarks made following (7.46).

The above expressions for thin shells have been obtained from the definitions (11.36) and (11.55) with $N=1$ referred to the base vectors $\boldsymbol{G}_i$ given by $(7.32)_{1,2}$.

The resultants (12.36) are defined in terms of the stress tensor τ^{ij} referred to the convected coordinates θ^i. Let $\hat{\tau}^{ij}$ denote the contravariant stress tensor referred to the normal coordinates y^i defined by the transformation relations (7.40). The relationships between the components of τ^{ij} and $\hat{\tau}^{ij}$ can be readily found from (7.40) and the transformation law between two second order tensors. Thus

$$\begin{gathered}\hat{\tau}^{\alpha\beta}=\tau^{\alpha\beta}, \qquad \hat{\tau}^{\alpha 3}=\xi D_{,\beta}\,\tau^{\alpha\beta}+D\,\tau^{\alpha 3},\\ \hat{\tau}^{33}=\xi^2 D_{,\alpha} D_{,\beta}\,\tau^{\alpha\beta}+2\xi D D_{,\alpha}\,\tau^{\alpha 3}+D^2\,\tau^{33}.\end{gathered}\tag{12.37}$$

We define a new set of stress-resultants and stress-couples in terms of $\hat{\tau}^{ij}$ by

$$\begin{gathered}\hat{N}^{\alpha\beta}=\int_{-h/2}^{h/2}\mu\,\hat{\tau}^{\alpha\gamma}\mu_\gamma^\beta\,d\zeta, \qquad V^\alpha=\int_{-h/2}^{h/2}\mu\,\hat{\tau}^{\alpha 3}\,d\zeta,\\ \hat{M}^{\alpha\beta}=\int_{-h/2}^{h/2}\mu\,\hat{\tau}^{\alpha\gamma}\mu_\gamma^\beta\,\zeta\,d\zeta, \qquad \hat{M}^{\alpha 3}=\int_{-h/2}^{h/2}\mu\,\hat{\tau}^{\alpha 3}\,\zeta\,d\zeta,\\ V^3=\int_{-h/2}^{h/2}\mu\,(\hat{\tau}^{33}-B_{\alpha\beta}\,\hat{\tau}^{\alpha\gamma}\mu_\gamma^\beta\,\zeta)\,d\zeta,\end{gathered}\tag{12.38}$$

where μ_γ^β and μ are given by (7.38)–(7.39). Using (7.41), (7.47) and (12.37) in (12.38) and recalling (12.36), we obtain

$$\begin{aligned}&\hat{N}^{\alpha\beta}=N^{\alpha\beta}, && V^\alpha=N^{\alpha 3}=D\,m^\alpha+D\,B_\gamma^\alpha\,M^{\gamma 3}+M^{\beta\alpha}\,D_{,\beta},\\ &\hat{M}^{\alpha i}=D\,M^{\alpha i}, && V^3=D\,m^3-D\,B_{\alpha\beta}\,M^{\alpha\beta}+D_{,\alpha}\,M^{\alpha 3},\end{aligned}\tag{12.39}$$

which relate the definitions (12.36) and (12.38). The definitions (12.38) are those usually employed in the classical theories of shells.[57] The corresponding expressions for initially flat plates are obtained by putting $B_{\alpha\beta}=0$, $\mu=1$ and $\mu_\beta^\alpha=\delta_\beta^\alpha$. It is clear from (7.48) that with $D=1$ (corresponding to $\boldsymbol{D}=\boldsymbol{A}_3$), the distinction between $M^{\alpha i}$ and $\hat{M}^{\alpha i}$ disappears and that the two sets of resultants are equivalent, apart from the expressions which involve τ^{33} or $\hat{\tau}^{33}$.

The linearized field equations and the linearized version of (12.22) can be expressed in different forms depending on the choice of the definitions of the type (12.36) or (12.38). We illustrate this with reference to the approximate equations of motion (12.33)–(12.35) in which, along with the definitions (12.36) and (12.38), corresponding expressions for the resultants F^i, L^i (and $\bar{F}^i$, $\bar{L}^i$) with the limits of integration

$$-\beta,\ \beta \text{ and } \frac{h}{2},\ \frac{h}{2}$$

are used.[58] Below a catalogue of formulae is provided which, in particular, are of interest in connection with the definitions of resultants and the equations of motion in the classical linear theories of shells and plates. With reference to the equations of motion (12.28)–(12.31), it was remarked earlier in this section that the question of approximation does not arise and that these equations are exact

[57] The expressions corresponding to $(12.38)_{4,5}$ are not defined in most of the literature on the linear theory of shells. The results (12.39) relating the two sets of definitions (12.36) and (12.38) were noted by Green, Naghdi and Wenner [1971, *6*].

[58] These expressions are not listed here since they can be easily obtained from (11.29), (11.30) with $N=1$ after linearization and use of (12.32).

if the resultants corresponding to $M^{N\alpha i}$ and m^{Ni} for $N \geqq 2$ are not admitted in the development of the theory. We now observe specifically that if (as in the classical shell theory) only the resultants of the type (12.38) are admitted and $\boldsymbol{D}$ is identified with $\boldsymbol{A}_3$, then the resulting equations of motion recorded in Formulae C below are exact.[59]

Formulae A

$$\boldsymbol{D} = D\boldsymbol{A}_3, \quad \Lambda_{\nu\alpha} = -B_{\nu\alpha} D, \quad \Lambda_{3\alpha} = D_{,\alpha}.$$

Stress-resultants and stress-couples defined by (12.36),

$$\begin{gathered}
N^{\alpha\beta}{}_{|\alpha} - B^{\beta}_{\alpha} N^{\alpha 3} + \varrho_0 \overline{F}^{\beta} = 0, \qquad N^{\alpha 3}{}_{|\alpha} + B_{\alpha\beta} N^{\alpha\beta} + \varrho_0 \overline{F}^{3} = 0, \\
M^{\alpha\beta}{}_{|\alpha} - B^{\beta}_{\alpha} M^{\alpha 3} + \varrho_0 \overline{L}^{\beta} = m^{\beta}, \qquad M^{\alpha 3}{}_{|\alpha} + B_{\alpha\beta} M^{\alpha\beta} + \varrho_0 \overline{L}^{3} = m^{3}, \\
N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} + D\, M^{\gamma\alpha} B^{\beta}_{\gamma}, \\
N^{\alpha 3} - D\, m^{\alpha} - D\, M^{\gamma 3} B^{\alpha}_{\gamma} - M^{\gamma\alpha} D_{,\gamma} = 0.
\end{gathered} \tag{12.40}$$

Formulae B

$$\boldsymbol{D} = D\boldsymbol{A}_3, \quad \Lambda_{\nu\alpha} = -B_{\nu\alpha} D, \quad \Lambda_{3\alpha} = D_{,\alpha}.$$

Stress-resultants and stress-couples defined by (12.38),

$$\begin{gathered}
\hat{N}^{\alpha\beta}{}_{|\alpha} - B^{\beta}_{\alpha} V^{\alpha} + \varrho_0 \overline{F}^{\beta} = 0, \qquad V^{\alpha}{}_{|\alpha} + B_{\alpha\beta} \hat{N}^{\alpha\beta} + \varrho_0 \overline{F}^{3} = 0, \\
\hat{M}^{\alpha\beta}{}_{|\alpha} + \varrho_0 \hat{\overline{L}}^{\beta} = V^{\beta}, \qquad \hat{M}^{\alpha 3}{}_{|\alpha} + \varrho_0 \overline{L}^{3} = V^{3}, \\
N'^{\alpha\beta} = N'^{\beta\alpha} = \hat{N}^{\alpha\beta} + \hat{M}^{\gamma\alpha} B^{\beta}_{\gamma}, \\
V^{\alpha} = D\, m^{\alpha} + B^{\alpha}_{\gamma} \hat{M}^{\gamma 3} + \hat{M}^{\beta\alpha} \frac{D_{,\beta}}{D}, \qquad V^{3} = D\, m^{3} - B_{\alpha\beta} \hat{M}^{\alpha\beta} + \hat{M}^{\alpha 3} \frac{D_{,\alpha}}{D}, \\
\hat{\overline{L}}^{\beta} = D\, \overline{L}^{\beta}, \quad \hat{\overline{L}}^{3} = D\, \overline{L}^{3}.
\end{gathered} \tag{12.41}$$

Formulae C

$$\begin{gathered}
\boldsymbol{D} = \boldsymbol{A}_3, \quad \Lambda_{\beta\alpha} = -B_{\alpha\beta}, \quad \Lambda_{3\alpha} = 0, \\
N^{\alpha\beta} = \int_{-h/2}^{h/2} \mu\, \tau^{\alpha\gamma} \mu^{\beta}_{\gamma}\, d\zeta, \qquad V^{\alpha} = \int_{-h/2}^{h/2} \mu\, \tau^{\alpha 3}\, d\zeta, \\
M^{\alpha\beta} = \int_{-h/2}^{h/2} \mu\, \tau^{\alpha\gamma} \mu^{\beta}_{\gamma} \zeta\, d\zeta, \qquad M^{\alpha 3} = \int_{-h/2}^{h/2} \mu\, \tau^{\alpha 3} \zeta\, d\zeta, \\
V^{3} = \int_{-h/2}^{h/2} \mu\, (\tau^{33} - B_{\alpha\beta} \tau^{\alpha\gamma} \mu^{\beta}_{\gamma} \zeta)\, d\zeta, \\
N^{\alpha\beta}{}_{|\alpha} - B^{\beta}_{\alpha} V^{\alpha} + \varrho_0 \overline{F}^{\beta} = 0, \qquad V^{\alpha}{}_{|\alpha} + B_{\alpha\beta} N^{\alpha\beta} + \varrho_0 \overline{F}^{3} = 0, \\
M^{\alpha\beta}{}_{|\alpha} + \varrho_0 \overline{L}^{\beta} = V^{\beta}, \qquad M^{\alpha 3}{}_{|\alpha} + \varrho_0 \overline{L}^{3} = V^{3}, \\
N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} + M^{\gamma\alpha} B^{\beta}_{\gamma}, \\
V^{\alpha} = N^{\alpha 3} = m^{\alpha} + B^{\alpha}_{\gamma} M^{\gamma 3}, \qquad V^{3} = m^{3} - B_{\alpha\beta} M^{\alpha\beta}.
\end{gathered} \tag{12.42}$$

[59] A derivation leading to the equations of motion in (12.42) from the three-dimensional stress equations of motion and with the use of the resultants $(12.38)_{1,2,3}$ is given in Sect. 5 of [1963, *6*].

Formulae D

$$\boldsymbol{D}=\boldsymbol{A}_3, \qquad B_{\alpha\beta}=0,$$

$$N^{\alpha\beta}=\int_{-h/2}^{h/2}\tau^{\alpha\beta}\,d\zeta, \qquad V^{\alpha}=\int_{-h/2}^{h/2}\tau^{\alpha 3}\,d\zeta,$$

$$M^{\alpha\beta}=\int_{-h/2}^{h/2}\tau^{\alpha\beta}\zeta\,d\zeta, \qquad M^{\alpha 3}=\int_{-h/2}^{h/2}\tau^{\alpha 3}\zeta\,d\zeta,$$

$$V^{3}=\int_{-h/2}^{h/2}\tau^{33}\,d\zeta, \tag{12.43}$$

$$N^{\alpha\beta}{}_{|\alpha}+\varrho_0\bar{F}^{\beta}=0, \qquad M^{\alpha 3}{}_{|\alpha}+\varrho_0\bar{L}^{3}=V^{3},$$

$$M^{\alpha\beta}{}_{|\alpha}+\varrho_0\bar{L}^{\beta}=V^{\beta}, \qquad V^{\alpha}{}_{|\alpha}+\varrho_0\bar{F}^{3}=0,$$

$$N'^{\alpha\beta}=N^{\alpha\beta}, \qquad V^{\alpha}=N^{\alpha 3}=m^{\alpha}, \qquad V^{3}=m^{3}.$$

12A. Appendix on the history of derivations of the equations of equilibrium for shells. A method of derivation of equations of equilibrium for shells which is only partly direct was originated by LOVE; it has long received acceptance and has been widely practiced and reproduced in books on the subject. Briefly, this method consists in two parts: (i) First the stress-resultants and the stress-couples are defined by integrals of the type in (12.42), together with similar definitions for load resultants [see (12.32) and (11.29)–(11.30)]; and then (ii) the equilibrium equations for shells are derived not from the three-dimensional equations but by consideration of the equilibrium of an element of the curved shell (or effectively its middle surface) under the action of the stress-resultants and the stress-couples (each per unit length of curves on the middle surface), as well as load resultant (per unit area of the middle surface). As a whole, this method is neither direct nor one in which the equations of equilibrium are derived fully from the three-dimensional equations. It is the second part of the procedure which is direct; and, hence this manner of obtaining the equilibrium equations for shells may properly be regarded as a derivation by direct approach.

The derivation of the equilibrium equations for shells via the direct method (and in lines of curvature coordinates) was given by LOVE.[60] In effect, this derivation is equivalent to that of our restricted theory when the equilibrium equations corresponding to those in Sect. 10 are specialized to lines of curvature coordinates and are also expressed in terms of physical components. A neat vectorial treatment of LOVE's derivation (again in lines of curvature coordinates) was given by REISSNER.[61] The corresponding derivation in general coordinates was supplied by SYNGE and CHIEN.[62] A derivation of equilibrium equations by direct method was also considered in 1958 by ERICKSEN and TRUESDELL to which reference was made in Sect. 9 [Subsect. β)]. This derivation of ERICKSEN and TRUESDELL is shorter and more appealing than that of SYNGE and CHIEN or other earlier

[60] Sect. 340 of LOVE [1893, *2*] and subsequent editions of his treatise, e.g., Sect. 331 of [1944, *4*]. Although LOVE defines the stress-resultants and the stress-couples, these are not used *per se* in his subsequent derivation of equilibrium equations according to the procedure mentioned above. With the help of formulae of the type (A.4.11)–(A.4.12) in Chap. F, the equilibrium equations in lines of curvature coordinates can easily be obtained from (12.28)–(12.30) or from (9.47)–(9.48) and (9.51). The corresponding results for the restricted theory can be obtained from (9.47) and (10.21). See also Eqs. (A.4.13)–(A.4.14) in Chap. F.

[61] REISSNER [1941, *1*].

[62] SYNGE and CHIEN [1941, *2*]. Derivations of this type by direct approach and in terms of tensor components are contained also in a paper by ZERNA [1949, *6*] and Chap. 10 of GREEN and ZERNA [1954, *1*].

derivations by direct approach. References to other recent direct derivations of equations of motion or equilibrium (after 1958) have already been cited in Sects. 9–10 and need not be repeated here.

The definitions of stress-resultants and stress-couples in terms of the three-dimensional stress tensor, corresponding to the resultants in (12.42) but in lines of curvature coordinates and for physical components of $N^{\alpha\beta}$, $M^{\alpha\beta}$, V^{α}, are due to LOVE.[63] The definitions of the resultants in general coordinates corresponding to $N^{\alpha\beta}$, $M^{\alpha\beta}$, V^{α} in (12.42) were given by ZERNA.[64] The earliest derivation which can be regarded as fully derived from the three-dimensional equations appears to be due to NOVOZHILOV and to NOVOZHILOV and FINKELSTEIN.[65] This derivation, carried out in lines of curvature coordinates and in terms of physical components of the resultants, is accomplished by integration of the (three-dimensional) differential equations of equilibrium across the thickness of the shell; the derivation is independent of any kinematic assumption and no approximation is involved. An exposition of the derivation of NOVOZHILOV and of NOVOZHILOV and FINKELSTEIN (again in lines of curvature coordinates) is given by TRUESDELL and TOUPIN.[66] A derivation in general coordinates (from the three-dimensional theory), resulting in the equilibrium equations in (12.42) which involve $N^{\alpha\beta}$, $M^{\alpha\beta}$, V^{α}, was given by NAGHDI.[67] References to more recent derivations of equations of motion for shells (from the three-dimensional equations) in terms of more general definitions for resultants [such as those in (11.36)] are cited in Sects. 11–12.

D. Elastic shells.

This chapter is concerned mainly with the development of constitutive equations for elastic shells, both by direct approach and from the three-dimensional equations.[1] Nonlinear and linearized constitutive relations are discussed and the complete theory is recapitulated. While we confine our attention here to elastic shells, it may be noted that the previous developments (in Chaps. B and C) are not limited to elastic materials.[2]

13. Constitutive equations for elastic shells (nonlinear theory): I. Direct approach. In this section, we introduce nonlinear constitutive equations for thermo-

[63] Sect. 339 of LOVE [1893, *2*] and subsequent editions of his treatise, e.g., Sect. 328 of [1944, *4*]. The main ingredient for the definitions of the resultants of the type in (12.42), namely the presence of the curvature factor, was noted by LAMB [1890, *2*] and BASSET [1890, *1*]. In his paper of [1888, *1*], LOVE defines the resultants approximately (without the curvature factor) and obtains his equations of motion by integration of the (three-dimensional) virtual work principle and in terms of displacements of the middle surface.

[64] ZERNA [1949, *6*]. See also GREEN and ZERNA [1954, *1*] and [1968, *9*].

[65] NOVOZHILOV [1943, *1*] and NOVOZHILOV and FINKELSTEIN [1943, *2*]. A less general derivation of this kind from the three-dimensional equations was given independently by TRUESDELL [1945, *3*] in a paper which deals chiefly with the membrane theory of shells of revolution.

[66] Sect. 213 of TRUESDELL and TOUPIN [1960, *14*] which also contains some historical remarks concerning the definitions of resultants and derivation of equilibrium equations for shells from the three-dimensional equations.

[67] NAGHDI [1963, *6*]. See also Chap. 10 (in 2nd edition) of GREEN and ZERNA [1968, *9*], where the equilibrium equations in general coordinates are derived from the three-dimensional equations.

[1] What we regard here as *elastic materials* are often called *hyperelastic materials* in the recent literature on continuum mechanics. For an extensive account of the three-dimensional (non-polar) theory of hyperelastic materials, together with thermodynamic aspects of the subject, see TRUESDELL and NOLL [1965, *9*].

[2] With reference to elastic-plastic shells and plates, mention may be made of a paper by GREEN, NAGHDI and OSBORN [1968, *8*] which deals with an elastic-plastic Cosserat surface.

mechanical behavior of an elastic Cosserat surface and then deduce the restrictions placed on them by the invariance requirements under superposed rigid body motions, by the entropy inequality and by material symmetries.

α) *General considerations. Thermodynamical results.* We recall that a material is defined by a *constitutive assumption* which characterizes the thermo-mechanical behavior of the medium; the constitutive assumption places a restriction on the processes which are admissible in a body—here the Cosserat surface $\mathscr{C}$. Preliminary to the introduction of the constitutive assumption for an elastic Cosserat surface, we recall the invariance requirements under superposed rigid body motions and examine their implications regarding the tensor components of the various vector fields in (8.30)–(8.32). Consider $(8.32)_1$ according to which $\boldsymbol{N}$ transforms as

$$\boldsymbol{N} \to Q\boldsymbol{N},$$

where Q is a proper orthogonal tensor defined by (5.37). By (9.11) and $(5.38)_1$ and since $\boldsymbol{v}$ transforms as $\boldsymbol{v} \to Q\,\boldsymbol{v}$, $v_\alpha \to v_\alpha$, we readily deduce the transformations

$$\boldsymbol{N}^\alpha \to Q\boldsymbol{N}^\alpha, \qquad N^{\alpha i} \to N^{\alpha i}, \tag{13.1}$$

for $\boldsymbol{N}^\alpha$ and $N^{\alpha i}$ under superposed rigid body motions.[3] Similar objective transformations can be obtained for $\boldsymbol{m}, \boldsymbol{M}^\alpha, \boldsymbol{q}$ or their components $m^i, M^{\alpha i}, q^\alpha$, as well as the temperature gradient $\theta_{,\alpha}$, all of which follow from the invariance conditions (8.30)–(8.32). Collecting these results, in addition to (13.1), we have[4]

$$\boldsymbol{m} \to Q\,\boldsymbol{m}, \qquad \boldsymbol{M}^\alpha \to Q\boldsymbol{M}^\alpha, \qquad \boldsymbol{N}'^\alpha \to Q\boldsymbol{N}'^\alpha, \qquad \boldsymbol{q} \to Q\boldsymbol{q}, \tag{13.2}$$

or

$$m^i \to m^i, \qquad M^{\alpha i} \to M^{\alpha i}, \qquad N'^{\alpha\beta} \to N'^{\alpha\beta}, \qquad q^\alpha \to q^\alpha, \tag{13.3}$$

as well as

$$\theta_{,\alpha} \to \theta_{,\alpha}. \tag{13.4}$$

An elastic Cosserat surface is defined by a set of response functions which depend on appropriate kinematic and thermal variables. For example, if in addition to the kinematic variables, the temperature and the temperature gradient are also taken as the independent variables, then the response functions consist of the six functions[5]

$$\bar{\boldsymbol{N}}^\alpha \text{ (or } \bar{\boldsymbol{N}}'^\alpha\text{)}, \ \bar{\boldsymbol{m}}, \ \bar{\boldsymbol{M}}^\alpha, \ \bar{\psi}, \ \bar{\eta}, \ \bar{\boldsymbol{q}}, \tag{13.5}$$

or an equivalent set

$$\bar{N}^{\alpha i}, \ \bar{m}^i, \ \bar{M}^{\alpha i}, \ \bar{\psi}, \ \bar{\eta}, \ \bar{q}^\alpha. \tag{13.6}$$

We introduce constitutive equations which must hold at each material point and for all t in terms of the response functions (13.5).[6] In this connection, we

[3] While $\boldsymbol{N}^\alpha$ transforms as a vector under superposed rigid body motions, $N^{\alpha i}$ (with six components) transform as six scalars; this is because of our use of convected coordinates.

[4] Recalling the definitions (9.30)–(9.31), the transformations for $\boldsymbol{N}'^\alpha$ and $N'^{\alpha\beta}$ follow from (13.1), $(13.2)_{1,2}$, $(13.3)_{1,2}$ and (5.54).

[5] The overbar in (13.5) is introduced temporarily for added clarity and in order to distinguish a function from its value. The notation $\bar{\boldsymbol{M}}^\alpha$ in (13.5) should not be confused with the previous use of the same symbol in (9.20) and (9.44).

[6] Often, in the three-dimensional theory of (non-polar) continuum mechanics, the material point in a body is identified with its position in a reference configuration and the constitutive equations are then introduced relative to the reference configuration. Although such a procedure is also possible here, it is simpler to introduce the constitutive equations at each material point of $\mathscr{C}$ relative to a material coordinate system, rather than in terms of the response functions relative to a reference configuration. Thus, our developments in the first part of this section [between (13.8)–(13.57)] are independent of any reference configuration; and, until further notice in this section, we disregard the fact that the coordinates representing the particles (or the material points) of $\mathscr{C}$ were identified previously (Sect. 4) with the convected coordinates on the surface in the (initial) reference configuration.

recall that the displacement function $\boldsymbol{r}$ in $(5.1)_1$ is the place occupied by the material point θ^μ (representing a typical surface particle of $\mathscr{C}$) in the present configuration; and, similarly, the function $\boldsymbol{d}$ in $(5.1)_2$ is the director at the material point in the present configuration. Thus, apart from temperature and its gradient, the local state of an elastic Cosserat surface $\mathscr{C}$ can be defined by the functions in $(5.1)_{1,2}$ and their gradients at each material point in the present configuration, namely

$$\boldsymbol{r},\ \boldsymbol{r}_{,\alpha},\ \boldsymbol{d},\ \boldsymbol{d}_{,\gamma}.$$

But, since the response functions must remain unaltered under superposed rigid body translational displacement, dependence of the functions (13.5) on $\boldsymbol{r}$ must be excluded. Hence, the last three in the above set, or equivalently

$$\boldsymbol{a}_\alpha,\ \boldsymbol{d},\ \boldsymbol{d}_{,\gamma}, \tag{13.7}$$

can be regarded as the primitive kinematic ingredients which define the local state and which occur in the constitutive equations. Keeping this in mind, we assume that the constitutive equations for an elastic Cosserat surface depend on the kinematic variables (13.7), as well as θ and $\theta_{,\alpha}$. We therefore write[7]

$$\begin{aligned}
\psi &= \bar{\psi}(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}, \theta, \theta_{,\alpha};\ \theta^\mu),\\
\eta &= \bar{\eta}(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}, \theta, \theta_{,\alpha};\ \theta^\mu),\\
\boldsymbol{N}^\alpha &= \bar{\boldsymbol{N}}^\alpha(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}, \theta, \theta_{,\alpha};\ \theta^\mu),\\
\boldsymbol{m} &= \bar{\boldsymbol{m}}(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}, \theta, \theta_{,\alpha};\ \theta^\mu),\\
\boldsymbol{M}^\alpha &= \bar{\boldsymbol{M}}^\alpha(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}, \theta, \theta_{,\alpha};\ \theta^\mu),
\end{aligned} \tag{13.8}$$

and

$$\boldsymbol{q} = \bar{\boldsymbol{q}}(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}, \theta, \theta_{,\alpha};\ \theta^\mu). \tag{13.9}$$

The above constitutive equations, which characterize the thermo-mechanical response of the medium, are assumed to hold at each particle θ^μ of the Cosserat surface and for all times t. Any dependence of the response functions on inhomogeneity, anisotropy or a (physically) preferred reference state is indicated through the argument θ^μ. Moreover, the constitutive equations (13.8)–(13.9) represent the response of the medium relative to a material coordinate system and would be different relative to another material coordinate system. This latter has not been explicitly exhibited in (13.8)–(13.9); and, strictly speaking, a constitutive equation such as $(13.8)_1$ should be recorded as

$$\psi = \bar{\psi}_{(\theta^\mu)}(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}, \theta, \theta_{,\alpha};\theta^\mu),$$

where the subscript (θ^μ) attached to $\bar{\psi}$ indicates the choice of coordinates in contrast to the argument θ^μ which signifies the choice of particle. To elaborate, let θ'^μ be any other material coordinate system related to θ^μ by

$$\theta'^\mu = \bar{\theta}'^\mu(\theta^\nu) \quad \text{and} \quad \theta^\mu = \bar{\theta}^\mu(\theta'^\nu),$$

where the function $\bar{\theta}^\mu$ is the inverse of $\bar{\theta}'^\mu$. Further, temporarily let the surface base vectors and the gradients of the director and the temperature relative to the

[7] These constitutive equations satisfy *equipresence*, which appears to be viewed by some writers as a physical principle. Here, we regard the notion of equipresence as a convenient mathematical procedure. A statement of equipresence, as currently understood, is as follows: An independent variable present in one constitutive equation should be so present in all, unless its presence is contradicted by the conservation laws, the entropy inequality or a rule of invariance. For further background information on equipresence, see Sect. 96 of TRUESDELL and NOLL [1965, *9*].

primed coordinate system be written as

$$\frac{\partial \boldsymbol{r}}{\partial \theta'^{\alpha}}, \quad \frac{\partial \boldsymbol{d}}{\partial \theta'^{\gamma}}, \quad \frac{\partial \theta}{\partial \theta'^{\alpha}},$$

respectively. Then, under the coordinate transformation, the response function for the free energy relative to the primed coordinate system is related to $\overline{\psi}_{(\theta^{\mu})}$ by

$$\begin{aligned}\overline{\psi}_{(\theta'^{\mu})}&\left(\frac{\partial \boldsymbol{r}}{\partial \theta'^{\alpha}}, \boldsymbol{d}, \frac{\partial \boldsymbol{d}}{\partial \theta'^{\gamma}}, \theta, \frac{\partial \theta}{\partial \theta'^{\alpha}};\ \theta'^{\mu}\right)\\ &=\overline{\psi}_{(\theta^{\mu})}\left(\frac{\partial \boldsymbol{r}}{\partial \theta'^{\mu}}\,\frac{\partial \theta'^{\mu}}{\partial \theta^{\alpha}}, \boldsymbol{d}, \frac{\partial \boldsymbol{d}}{\partial \theta'^{\mu}}\,\frac{\partial \theta'^{\mu}}{\partial \theta^{\gamma}}, \theta, \frac{\partial \theta}{\partial \theta'^{\mu}}\,\frac{\partial \theta'^{\mu}}{\partial \theta^{\alpha}};\ \theta^{\mu}\right),\end{aligned}$$

where the argument θ^{μ} refers to the particle identified by $\theta^{\mu}=\bar{\theta}^{\mu}(\theta'^{\nu})$. According to the last relation, from the knowledge of the function $\psi_{(\theta^{\mu})}$ and the coordinate transformation we can readily calculate the response function in any other coordinate system. Having discussed the nature of the dependence of the response functions on the material coordinates, in order to avoid cumbersome notation in what follows we do not explicitly display this dependence, and we continue to write the response functions, as in (13.8)–(13.9), without the subscript (θ^{μ}) attached to them.

Before proceeding further, we first effect a certain simplification in our constitutive assumptions (13.8). We had previously postulated that the inequality (8.24) must hold for every admissible process. We now introduce the constitutive equations (13.8)–(13.9) into the inequality (9.36) which [since it is obtained from (8.24)] must hold for every admissible process. From $(13.8)_1$, we have

$$\begin{aligned}\dot{\overline{\psi}}&=\frac{\partial\overline{\psi}}{\partial \boldsymbol{a}_{\alpha}}\cdot\dot{\boldsymbol{a}}_{\alpha}+\frac{\partial\overline{\psi}}{\partial \boldsymbol{d}}\cdot\dot{\boldsymbol{d}}+\frac{\partial\overline{\psi}}{\partial \boldsymbol{d}_{,\gamma}}\cdot\dot{\boldsymbol{d}}_{,\gamma}+\frac{\partial\overline{\psi}}{\partial\theta}\dot{\theta}+\frac{\partial\overline{\psi}}{\partial\theta_{,\alpha}}\dot{\theta}_{,\alpha}\\ &=\left(\frac{\partial\overline{\psi}}{\partial \boldsymbol{a}_{\alpha}}-d^{\alpha}\frac{\partial\overline{\psi}}{\partial \boldsymbol{d}}-\lambda^{\alpha}_{\cdot\gamma}\frac{\partial\overline{\psi}}{\partial \boldsymbol{d}_{,\gamma}}\right)\cdot\boldsymbol{\eta}_{\alpha}\\ &\quad+\frac{\partial\overline{\psi}}{\partial \boldsymbol{d}}\cdot\boldsymbol{\Gamma}+\frac{\partial\overline{\psi}}{\partial \boldsymbol{d}_{,\gamma}}\cdot\boldsymbol{\Gamma}_{:\gamma}+\frac{\partial\overline{\psi}}{\partial\theta}\dot{\theta}+\frac{\partial\overline{\psi}}{\partial\theta_{,\alpha}}\dot{\theta}_{,\alpha}\\ &\quad+W_{ki}\,\boldsymbol{a}^{k}\cdot\left[\delta^{i}_{\alpha}\frac{\partial\overline{\psi}}{\partial \boldsymbol{a}_{\alpha}}+d^{i}\frac{\partial\overline{\psi}}{\partial \boldsymbol{d}}+\lambda^{i}_{\cdot\gamma}\frac{\partial\overline{\psi}}{\partial \boldsymbol{d}_{,\gamma}}\right],\end{aligned}\tag{13.10}$$

where (5.19), (5.26) and (5.29) have been used. After introducing the constitutive assumption into (9.36) with P given by $(9.29)_2$, we obtain an inequality which must hold for all arbitrary values of[8]

$$\dot{\boldsymbol{a}}_{\alpha},\ \dot{\boldsymbol{d}},\ \dot{\boldsymbol{d}}_{,\alpha},\ \dot{\theta},\ \dot{\theta}_{,\alpha}.\tag{13.11}$$

The coefficients of the above rate quantities in the inequality are independent of the variables (13.11) and, as functions of time, can be chosen arbitrarily. It follows that for a given time (say at the present time t), there exist admissible

[8] For convenience we postpone the use of invariance requirements under superposed rigid body motions which places a restriction on the function $\overline{\psi}$ in (13.8) and therefore $\dot{\overline{\psi}}$ which appears in the inequality. Operators of the form $\partial f/\partial \boldsymbol{x}$, where f is a scalar function whose arguments include the vector $\boldsymbol{x}$, occur in (13.10) and elsewhere in this chapter. By an operator of this type, whenever it exists, we mean the partial derivative with respect to $\boldsymbol{x}$, which satisfies

$$\lim_{\beta\to 0}\frac{f(\boldsymbol{x}+\beta\boldsymbol{V})-f(\boldsymbol{x})}{\beta}=\frac{\partial f}{\partial \boldsymbol{x}}\cdot\boldsymbol{V}$$

for all values of the arbitrary vector $\boldsymbol{V}$.

processes such that all coefficient functions in the inequality and all rate quantities in (13.11) can be assigned arbitrarily. Consider now a process in which all kinematic and thermal variables in the argument of the response functions in (13.8)–(13.9) at time t, as well as all rate quantities in (13.11) at time t, except $\dot{\theta}_{,\alpha}$ are prescribed. Then, all terms in the resulting inequality are fixed except $\dot{\theta}_{,\alpha}$. In order that the inequality holds for all $\dot{\theta}_{,\alpha}$, the coefficient of the term before last in $(13.10)_2$ must vanish:

$$\frac{\partial \overline{\psi}}{\partial \theta_{,\alpha}} = 0. \tag{13.12}$$

Hence $\overline{\psi}$ must be independent of $\theta_{,\alpha}$ and $(13.8)_1$ reduces to[9]

$$\psi = \overline{\psi}(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}, \theta;\ \theta^\mu), \tag{13.13}$$

where $\overline{\psi}$ is now a different function from that in $(13.8)_1$.

Having shown that the constitutive equation $(13.8)_1$ can be reduced to (13.13), with the use of the kinematic results in Sect. 5 and the remaining constitutive equations in (13.8)–(13.9), we write the inequality (9.36) in the form

$$\begin{aligned} -\varrho\left(\overline{\eta} + \frac{\partial \overline{\psi}}{\partial \theta}\right)\dot{\theta} + \left[\overline{\boldsymbol{N}}'^\alpha - \varrho\left(\frac{\partial \overline{\psi}}{\partial \boldsymbol{a}_\alpha} - d^\alpha \frac{\partial \overline{\psi}}{\partial \boldsymbol{d}} - \lambda^\alpha_{\cdot\gamma} \frac{\partial \overline{\psi}}{\partial \boldsymbol{d}_{,\gamma}}\right)\right] \cdot \boldsymbol{\eta}_\alpha \\ + \left[\overline{\boldsymbol{m}} - \varrho \frac{\partial \overline{\psi}}{\partial \boldsymbol{d}}\right] \cdot \boldsymbol{\Gamma} + \left[\overline{\boldsymbol{M}}^\gamma - \varrho \frac{\partial \overline{\psi}}{\partial \boldsymbol{d}_{,\gamma}}\right] \cdot \boldsymbol{\Gamma}_{:\gamma} - (\overline{\boldsymbol{q}} \cdot \boldsymbol{a}^\alpha) \frac{\theta_{,\alpha}}{\theta} \geqq 0, \end{aligned} \tag{13.14}$$

where the function $\overline{\psi}$ defined in (13.13) must satisfy

$$\varepsilon_{jkm}\, \boldsymbol{a}^k \cdot \left[\delta^j_\alpha \frac{\partial \overline{\psi}}{\partial \boldsymbol{a}_\alpha} + d^j \frac{\partial \overline{\psi}}{\partial \boldsymbol{d}} + \lambda^j_{\cdot\gamma} \frac{\partial \overline{\psi}}{\partial \boldsymbol{d}_{,\gamma}}\right] = 0. \tag{13.15}$$

The condition (13.15) arises from the fact that the inequality (9.36) must remain unaffected by superposed rigid body motions. It is not difficult to see that it is deduced from the vanishing of the square bracket in $(13.10)_2$ if, in the notation of (5.42), the skew-symmetric W_{ki} is expressed as $\varepsilon_{kim}\, \omega^m$. As will become apparent later in Subsect. β), (13.15) will be satisfied identically after restriction arising from invariance conditions under superposed rigid body motions is placed on the free energy ψ.

The inequality (13.14) holds for all admissible processes and, in particular, for all arbitrary values of $\dot{\theta}$, $\boldsymbol{\eta}_\alpha$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Gamma}_{:\alpha}$. Consider now an admissible process such that all kinematic and thermal variables which occur in the response functions (including $\theta_{,\alpha}$ at time t), as well as $\boldsymbol{\eta}_\alpha$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Gamma}_{:\alpha}$ at time t, are prescribed. Then, all quantities in (13.14) are fixed except $\dot{\theta}$, which may assume arbitrary values. Hence, in order that (13.14) hold for all $\dot{\theta}$, its coefficient must vanish and we must have

$$\overline{\eta} = -\frac{\partial \overline{\psi}}{\partial \theta}. \tag{13.16}$$

Next, consider an admissible process at time t corresponding to which (13.7), the thermal variables θ and $\theta_{,\alpha}$ and all rate quantities, except $\boldsymbol{\eta}_\alpha$, are prescribed; but, since $\boldsymbol{a}_\beta$ (and therefore $\boldsymbol{a}^\beta$) are tangent vectors at each point of $\mathscr{s}$, $\boldsymbol{\eta}_\alpha = \eta_{\beta\alpha}\, \boldsymbol{a}^\beta$

[9] The argument here and those leading to (13.16)–(13.19) parallels similar arguments by COLEMAN and NOLL [1963, *2*] in the (three-dimensional) theory of nonlinear elastic materials with heat conduction and viscosity. We recall that the justification for such arguments necessarily rests on the notion that the functions $(8.23)_3$ may be arbitrarily chosen and are not assigned *a priori*.

can only be varied arbitrarily in the tangent plane and only the components $\eta_{\beta\alpha}$ may assume arbitrary values. Keeping this in mind and by repeating similar arguments, from (13.14) we obtain the further relations

$$\begin{aligned}\overline{N}'^{\alpha\beta} = \overline{N}'^{\beta\alpha} &= \frac{1}{2}\,(\overline{\boldsymbol{N}}'^{\alpha}\cdot\boldsymbol{a}^{\beta} + \overline{\boldsymbol{N}}'^{\beta}\cdot\boldsymbol{a}^{\alpha})\\ &= \frac{1}{2}\,\varrho\left\{\left(\frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{\alpha}} - d^{\alpha}\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}} - \lambda^{\alpha}_{\cdot\gamma}\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}_{,\gamma}}\right)\cdot\boldsymbol{a}^{\beta}\right.\\ &\quad\left.+\left(\frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{\beta}} - d^{\beta}\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}} - \lambda^{\beta}_{\cdot\gamma}\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}_{,\gamma}}\right)\cdot\boldsymbol{a}^{\alpha}\right\},\end{aligned} \tag{13.17}$$

$$\overline{\boldsymbol{m}} = \varrho\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}}, \qquad \overline{\boldsymbol{M}}^{\gamma} = \varrho\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}_{,\gamma}} \tag{13.18}$$

and the inequality

$$-\overline{q}^{\alpha}\,\theta_{,\alpha} \geqq 0, \tag{13.19}$$

since $\theta > 0$. Also, in view of (9.30), from (13.17)–(13.18) follows the expression for the symmetric part of $\overline{\boldsymbol{N}}^{\alpha}\cdot\boldsymbol{a}^{\beta}$:

$$\overline{N}^{(\alpha\beta)} = \frac{1}{2}\,(\overline{N}^{\alpha\beta} + \overline{N}^{\beta\alpha}) = \frac{1}{2}\,\varrho\left(\frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{\alpha}}\cdot\boldsymbol{a}^{\beta} + \frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{\beta}}\cdot\boldsymbol{a}^{\alpha}\right). \tag{13.20}$$

It is clear that the relations (13.20) and (13.18) are equivalent to the set (13.17)–(13.18).

Previously, we had shown that $\overline{\psi}$ must be independent of the temperature gradient $\theta_{,\alpha}$, but the remaining response functions in (13.8) are still dependent on $\theta_{,\alpha}$. The relations (13.16)–(13.18) now show that the response functions $\overline{\eta}$, $\overline{N}'^{\alpha\beta}$, $\overline{\boldsymbol{m}}$, $\overline{\boldsymbol{M}}^{\gamma}$, as well as $\overline{N}^{(\alpha\beta)}$, which are determined by the partial derivatives of $\overline{\psi}$ are also independent of $\theta_{,\alpha}$. Hence, $\overline{\boldsymbol{q}}$ (or $\overline{q}^{\alpha}$) is the only response function in (13.9) which remains dependent on the temperature gradient. It follows from these results and (13.12) that the five constitutive equations in (13.8) have been reduced to (13.13) and

$$\eta = -\frac{\partial\overline{\psi}}{\partial\theta}, \tag{13.21}$$

$$\begin{aligned}N'^{\alpha\beta} = N'^{\beta\alpha} &= \frac{1}{2}\,\varrho\left\{\left(\frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{\alpha}} - d^{\alpha}\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}} - \lambda^{\alpha}_{\cdot\gamma}\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}_{,\gamma}}\right)\cdot\boldsymbol{a}^{\beta}\right.\\ &\quad\left.+\left(\frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{\beta}} - d^{\beta}\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}} - \lambda^{\beta}_{\cdot\gamma}\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}_{,\gamma}}\right)\cdot\boldsymbol{a}^{\alpha}\right\},\end{aligned} \tag{13.22}$$

$$\boldsymbol{m} = \varrho\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}}, \qquad \boldsymbol{M}^{\gamma} = \varrho\frac{\partial\overline{\psi}}{\partial\boldsymbol{d}_{,\gamma}}. \tag{13.23}$$

Also, the constitutive relation for $N^{(\alpha\beta)}$ is now given by the right-hand side of (13.20):

$$N^{(\alpha\beta)} = \frac{1}{2}\,\varrho\left(\frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{\alpha}}\cdot\boldsymbol{a}^{\beta} + \frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{\beta}}\cdot\boldsymbol{a}^{\alpha}\right).$$

The foregoing results have been deduced from (13.14) or equivalently from the entropy inequality (9.36) with the mechanical power P given by $(9.29)_2$. Alternatively, we could have used the inequality (9.36) with P defined by $(9.28)_2$. If the latter inequality is employed, then with the help of $\dot{\overline{\psi}}$ given by $(13.10)_1$ we should obtain slightly different results at this stage of our derivation. In particular, with the use of (9.36) with P given by $(9.28)_2$, an expression corresponding to (13.15) does not arise and while the results (13.21), (13.23) and (13.19) are again recovered, instead of (13.22) we have

$$\boldsymbol{N}^{\alpha}=\varrho\,\frac{\partial\overline{\psi}}{\partial \boldsymbol{a}_{\alpha}}\,. \tag{13.24}$$

Recalling (9.40), from (13.24) we can readily calculate the expressions for $N^{\alpha 3}=\boldsymbol{N}^{\alpha}\cdot\boldsymbol{a}^{3}$ and $N^{\alpha\beta}=\boldsymbol{N}^{\alpha}\cdot\boldsymbol{a}^{\beta}$. The symmetric part of the latter is easily seen to be the expression for $N^{(\alpha\beta)}$ noted above and at this point it appears that we also have constitutive equations for $N^{\alpha 3}$ and the skew-symmetric part of $N^{\alpha\beta}$. However, after allowance is made for the restriction placed on the free energy ψ by the invariance conditions under superposed rigid body motions, it will become evident in Subsect. β) [see (13.38)–(13.39)] that the expressions for $N^{\alpha 3}$ and $N^{[\alpha\beta]}$ calculated from (13.24) are identical with those resulting from (9.51). In this connection, it is worth observing that substitution of (13.23) and (13.24) into (13.15) at once yields (9.50). We further observe that in obtaining the relations (13.16)–(13.18) or (13.21)–(13.24), as well as the inequality (13.19), we have employed the procedure that the entropy inequality be identically satisfied for every admissible process defined by the constitutive equations. Thus (13.16)–(13.19) are necessary conditions for the validity of the inequality (9.36) and it is easily seen that they are also sufficient.

A different set of constitutive equations can also be obtained in terms of the specific internal energy ε. Through (13.21), θ can be expressed as a function of η and the set of the kinematic variables (13.7). Recalling (9.34) and (13.13), it follows that ε has the form

$$\varepsilon=\overline{\varepsilon}\,(\boldsymbol{a}_{\alpha},\boldsymbol{d},\boldsymbol{d}_{,\gamma},\eta;\;\theta^{\mu}) \tag{13.25}$$

and instead of (13.21)–(13.23) we obtain

$$\theta=\frac{\partial\overline{\varepsilon}}{\partial\eta}\,, \tag{13.26}$$

$$\begin{aligned} N'^{\alpha\beta}=N'^{\beta\alpha}=\frac{1}{2}\,\varrho\,\Big\{&\Big(\frac{\partial\overline{\varepsilon}}{\partial\boldsymbol{a}_{\alpha}}-d^{\alpha}\,\frac{\partial\overline{\varepsilon}}{\partial\boldsymbol{d}}-\lambda^{\alpha}_{\cdot\gamma}\,\frac{\partial\overline{\varepsilon}}{\partial\boldsymbol{d}_{,\gamma}}\Big)\cdot\boldsymbol{a}^{\beta}\\ &+\Big(\frac{\partial\overline{\varepsilon}}{\partial\boldsymbol{a}_{\beta}}-d^{\beta}\,\frac{\partial\overline{\varepsilon}}{\partial\boldsymbol{d}}-\lambda^{\beta}_{\cdot\gamma}\,\frac{\partial\overline{\varepsilon}}{\partial\boldsymbol{d}_{,\gamma}}\Big)\cdot\boldsymbol{a}^{\alpha}\Big\}\,,\\ \boldsymbol{m}=\varrho\,\frac{\partial\overline{\varepsilon}}{\partial\boldsymbol{d}}\,,\qquad &\boldsymbol{M}^{\gamma}=\varrho\,\frac{\partial\overline{\varepsilon}}{\partial\boldsymbol{d}_{,\gamma}}\,. \end{aligned} \tag{13.27}$$

The results (13.26)–(13.27) with $\overline{\varepsilon}$ given by the response function in (13.25), together with the inequality (13.19), may be obtained directly from the inequality (9.33) by a procedure similar to that used earlier in this section and after introducing a set of constitutive assumptions similar to those in (13.8)–(13.9).

β) *Reduction of the constitutive equations under superposed rigid body motions.* As remarked previously, the local state of the Cosserat surface is defined by the kinematic variables (13.7), together with θ and $\theta_{,\alpha}$. These kinematic and thermal variables which occur in the function $\overline{\psi}$ and other response functions are objective, in view of $(5.36)_2$, $(5.38)_{1,3}$ and (13.4). Moreover, by virtue of (8.30)–(8.32) and (13.1)–(13.3), η, θ, $\theta_{,\alpha}$, ε (and therefore ψ), $\boldsymbol{N}^{\alpha}$, $\boldsymbol{m}$, $\boldsymbol{M}^{\alpha}$, $\boldsymbol{N}'^{\alpha}$, $\boldsymbol{q}$ or their components, all are objective under superposed rigid body motions.

In what follows, let a typical constitutive equation such as (13.13), which is an objective scalar, be written in the form

$$f=\overline{f}\,(\boldsymbol{a}_{\alpha},\boldsymbol{d},\boldsymbol{d}_{,\gamma})\,, \tag{13.28}$$

where for brevity dependence on θ and the material point θ^μ is temporarily suppressed. Now a constitutive equation in the form (13.28) which holds for an admissible process must also hold for a motion differing from the given one only by superposed rigid body motions. This requirement is fulfilled if and only if the response function $\bar{f}$ satisfies the identity

$$\bar{f}(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}) = \bar{f}(Q\boldsymbol{a}_\alpha, Q\boldsymbol{d}, Q\boldsymbol{d}_{,\gamma}) \tag{13.29}$$

for all values of the arguments in the domain of $\bar{f}$ and for all proper orthogonal Q. From Cauchy's representation theorem on isotropic functions, $\bar{f}$ may be expressed as a (different) function of the inner products and the scalar triple products of its arguments.[10] The inner products of the arguments of $\bar{f}$ in (13.28) are the set

$$\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = a_{\alpha\beta}, \quad \boldsymbol{a}_\alpha \cdot \boldsymbol{d} = d_\alpha, \quad \boldsymbol{a}_\gamma \cdot \boldsymbol{d}_{,\alpha} = \lambda_{\gamma\alpha}, \tag{13.30}$$

and

$$\boldsymbol{d} \cdot \boldsymbol{d}, \quad \boldsymbol{d} \cdot \boldsymbol{d}_{,\gamma}, \quad \boldsymbol{d}_{,\alpha} \cdot \boldsymbol{d}_{,\gamma}, \tag{13.31}$$

while the independent scalar triple products in question are formed from the vectors $\boldsymbol{a}_\alpha$, $\boldsymbol{d}$ and $\boldsymbol{d}_{,\gamma}$. However, before considering these scalar triple products, it is expedient to examine further the inner products (13.30)–(13.31).

Let each of the two vectors $\boldsymbol{d}$ and $\boldsymbol{d}_{,\alpha}$ be resolved into their respective tangential and normal components. Thus, we write[11]

$$\begin{aligned} &\boldsymbol{d} = \boldsymbol{d}^T + \boldsymbol{d}^N, && \boldsymbol{d}^T = d^\alpha \boldsymbol{a}_\alpha, && \boldsymbol{d}^N = d^3 \boldsymbol{a}_3, \\ &\boldsymbol{d}_{,\alpha} = (\boldsymbol{d}_{,\alpha})^T + (\boldsymbol{d}_{,\alpha})^N, && (\boldsymbol{d}_{,\alpha})^T = \lambda^{\gamma}_{.\alpha} \boldsymbol{a}_\gamma, && (\boldsymbol{d}_{,\alpha})^N = \lambda^3_{.\alpha} \boldsymbol{a}_3, \end{aligned} \tag{13.32}$$

where $\boldsymbol{d}^T$ and $\boldsymbol{d}^N$ designate the tangential and normal components of $\boldsymbol{d}$ while $(\boldsymbol{d}_{,\alpha})^T$ and $(\boldsymbol{d}_{,\alpha})^N$ stand for the tangential and normal components of $\boldsymbol{d}_{,\alpha}$. According to (13.28), f is determined by a response function which depends on the variables characterizing the local state of the Cosserat surface; and we have noted that $\bar{f}$ is expressible as a different function of (13.30)–(13.31) and the scalar triple products. But the latter function already depends on the tangential components $\boldsymbol{d}^T$ and $(\boldsymbol{d}_{,\alpha})^T$, as is evident from $(13.30)_{2,3}$. Hence, without loss of generality, the set (13.31) may be replaced by the set of variables $\sigma, \sigma_\alpha, \sigma_{\alpha\gamma}$ defined by

$$\begin{gathered} \sigma = \boldsymbol{d}^N \cdot \boldsymbol{d}^N = (d^3)^2, \quad \sigma_\alpha = \boldsymbol{d}^N \cdot (\boldsymbol{d}_{,\alpha})^N = d^3 \lambda_{3\alpha}, \\ \sigma_{\alpha\gamma} = (\boldsymbol{d}_{,\alpha})^N \cdot (\boldsymbol{d}_{,\gamma})^N = \lambda_{3\alpha} \lambda_{3\gamma}. \end{gathered} \tag{13.33}$$

Before proceeding further, we note that $(5.25)_1$ can be solved for

$$\boldsymbol{a}_3 = \frac{\boldsymbol{d} - d^\alpha \boldsymbol{a}_\alpha}{d^3}$$

and then $(5.28)_1$ can be expressed in the form

$$\boldsymbol{d}_{,\gamma} = \left(\lambda^{\nu}_{.\gamma} - \frac{d^\nu \sigma_\gamma}{\sigma}\right) \boldsymbol{a}_\nu + \frac{\sigma_\gamma}{\sigma} \boldsymbol{d}. \tag{13.34}$$

As noted above [following (13.31)], the independent scalar triple products are formed from the appropriate combination of the vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{d}$, $\boldsymbol{d}_{,1}$ and $\boldsymbol{d}_{,2}$.

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[10] For a proof of CAUCHY's representation theorem, see TRUESDELL and NOLL [1965, *9*]. The scalar triple products must be included since the identity (13.29) is required to hold only for proper orthogonal Q.

[11] The temporary notations for tangential and normal components in (13.32)–(13.33) should not be confused, respectively, with the use of superscript T for transpose of a second order tensor in Sect. 5 and the use of the index N in formulae of Sect. 7.

One of these, namely $[\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{d}]$ can be expressed in terms of $(13.30)_{1,2}$, $(13.33)_1$ and τ defined by

$$\tau = \operatorname{sgn}\,[\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{d}]. \tag{13.35}$$

Clearly the magnitude of $[\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{d}]$, but not its sign, is determined by $(13.30)_{1,2}$ and $(13.33)_1$. With the help of (13.34), it can be easily verified that the remaining scalar triple products can be expressed in terms of (13.30), $(13.33)_{1,2}$ and (13.35). In addition, it is easily seen from (13.34) that the scalar $\boldsymbol{d}_{,\alpha}\cdot\boldsymbol{d}_{,\gamma}$ can be expressed in terms of the arguments[12] (13.30) and $(13.33)_{1,2}$; hence $\sigma_{\alpha\gamma}$ may be suppressed from the arguments of the response function $\bar{f}$. Thus, from the results between (13.29)–(13.35), it follows that $\bar{f}$ may be expressed as a function of (13.35) and the set of variables

$$\begin{aligned}\mathscr{V}:\ & a_{\alpha\beta},\ d_\alpha,\ \lambda_{\gamma\alpha},\\ & \sigma,\quad \sigma_\alpha.\end{aligned} \tag{13.36}$$

However, since we have considered only motions of a Cosserat surface which are consistent with $(5.1)_3$, τ will always have the positive value 1. With this restriction, (13.35) may be eliminated from the domain of the function $\bar{f}$ which can be replaced by a (different) function of (13.36):[13]

$$\bar{f}(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}) = \tilde{f}(\mathscr{V}). \tag{13.37}$$

By application of the result (13.37) to the function $\bar{\psi}$, the constitutive equation (13.13), apart from its dependence on the material point θ^μ and temperature θ, can alternatively be expressed in terms of a different function of (13.36):

$$\begin{aligned}\psi &= \bar{\psi}(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}, \theta;\ \theta^\mu)\\ &= \tilde{\psi}(a_{\alpha\beta}, d_\alpha, \lambda_{\beta\alpha}, \sigma, \sigma_\alpha, \theta;\ \theta^\mu).\end{aligned} \tag{13.38}$$

In order to obtain an alternative form for the constitutive equations (13.21)–(13.23) in tensor components and in terms of the function $\tilde{\psi}$, we first calculate the partial derivatives which occur in (13.22)–(13.23) in terms of the partial derivatives of $\tilde{\psi}$. Thus, by chain rule differentiation,

$$\begin{aligned}\frac{\partial\bar{\psi}}{\partial\boldsymbol{a}_\beta} &= \left(\frac{\partial\tilde{\psi}}{\partial a_{\beta\gamma}}\boldsymbol{a}_\gamma + \frac{\partial\tilde{\psi}}{\partial a_{\gamma\beta}}\boldsymbol{a}_\gamma\right) + \frac{\partial\tilde{\psi}}{\partial d_\beta}\boldsymbol{d} + \frac{\partial\tilde{\psi}}{\partial\lambda_{\beta\gamma}}\boldsymbol{d}_{,\gamma}\\ &\quad - \left[2d^3 d^\beta\frac{\partial\tilde{\psi}}{\partial\sigma} + (d^\beta\lambda^3_{.\alpha} + d^3\lambda^\beta_{.\alpha})\frac{\partial\tilde{\psi}}{\partial\sigma_\alpha}\right]\boldsymbol{a}_3,\\ \frac{\partial\bar{\psi}}{\partial\boldsymbol{d}} &= \frac{\partial\tilde{\psi}}{\partial d_\gamma}\boldsymbol{a}_\gamma + 2\frac{\partial\tilde{\psi}}{\partial\sigma}d^3\boldsymbol{a}_3 + \frac{\partial\tilde{\psi}}{\partial\sigma_\alpha}\lambda^3_{.\alpha}\boldsymbol{a}_3,\\ \frac{\partial\bar{\psi}}{\partial\boldsymbol{d}_{,\gamma}} &= \frac{\partial\tilde{\psi}}{\partial\lambda_{\alpha\gamma}}\boldsymbol{a}_\alpha + \frac{\partial\tilde{\psi}}{\partial\sigma_\gamma}d^3\boldsymbol{a}_3.\end{aligned} \tag{13.39}$$

Then, with the help of (13.39) and recalling $(5.25)_2$, $(5.28)_2$, (9.39), (9.40) and (9.42), we can express the constitutive relations (13.21)–(13.23) in tensor compo-

[12] This further reduction resulting in the suppression of $\boldsymbol{d}_{,\alpha}\cdot\boldsymbol{d}_{,\gamma}$ was noted by Dr. S. L. PASSMAN. I express here my appreciation to him for bringing it to my attention.

[13] Such a reduction would result eventually for any material symmetry restriction characterized by an improper symmetry transformation, i.e., one in which (13.36) remain invariant and $\tau\to-\tau$, even if we had assumed $[\boldsymbol{a}_1\,\boldsymbol{a}_2\,\boldsymbol{d}]\neq 0$ instead of $(5.1)_3$.

nents as follows:[14]

$$\eta = -\frac{\partial \tilde{\psi}}{\partial \theta} \tag{13.40}$$

and

$$N'^{\alpha\beta} = N'^{\beta\alpha} = 2\varrho \frac{\partial \tilde{\psi}}{\partial a_{\alpha\beta}}, \tag{13.41}$$

$$\begin{aligned} m^\alpha &= \varrho \frac{\partial \tilde{\psi}}{\partial d_\alpha}, & m^3 &= \varrho \left(2d^3 \frac{\partial \tilde{\psi}}{\partial \sigma} + \lambda^3_{\cdot\,\alpha} \frac{\partial \tilde{\psi}}{\partial \sigma_\alpha}\right), \\ M^{\alpha\gamma} &= \varrho \frac{\partial \tilde{\psi}}{\partial \lambda_{\gamma\alpha}}, & M^{\alpha 3} &= \varrho d^3 \frac{\partial \tilde{\psi}}{\partial \sigma_\alpha}. \end{aligned} \tag{13.42}$$

It is clear from the above that only the symmetric part of $N^{\alpha\beta}$ or equivalently $N'^{\alpha\beta}$ is determined by a constitutive equation in terms of $\tilde{\psi}$. Also, using (13.39), (13.24) and (13.41)–(13.42), the expressions for $N^{\alpha 3}$ and the skew-symmetric part of $N^{\alpha\beta}$ are found from the restriction (13.15). These expressions are the same as those provided by (9.51); and this is in line with earlier observations in this section and the remarks in Sect. 9 [Subsect. α)] that (9.23) and therefore (9.51) is regarded as a restriction on the constitutive equations.

For completeness, we also record here a reduced form of the constitutive relation for the components q^α of the heat flux vector. Recalling the transformations $(13.3)_4$–(13.4) under superposed rigid body motions, we see easily that the conclusion (13.37) is also applicable to the response functions $\bar{q}^\alpha = \bar{\boldsymbol{q}} \cdot \boldsymbol{a}^\alpha$ in (13.9). Hence, the constitutive relation (13.9) reduces to

$$q^\alpha = \tilde{q}^\alpha (a_{\gamma\delta}, d_\gamma, \lambda_{\delta\gamma}, \sigma, \sigma_\gamma, \theta, \theta_{,\gamma};\ \theta^\mu), \tag{13.43}$$

and is subject to the restriction (13.19) with $\bar{q}^\alpha$ replaced by $\tilde{q}^\alpha$.

The foregoing nonlinear constitutive equations, namely (13.38) and (13.40)–(13.43), are valid for an elastic Cosserat surface which may be anisotropic with reference to preferred directions associated with the material points of $\mathscr{C}$. Since the response functions for η, $N'^{\alpha\beta}$, m^i and $M^{\alpha i}$ are fully determined from the knowledge of the response functions for the specific Helmholtz free energy, in any discussion of the effect of material symmetry it will suffice to consider only the response functions for the specific free energy and the heat flux vector. In subsequent developments, however, we shall be largely concerned with the isothermal case; and, in the discussion of material symmetries which serve to restrict the form of the constitutive equations, we shall mainly consider the symmetries associated with the function $\bar{\psi}$ or $\tilde{\psi}$.

γ) *Material symmetry restrictions.* According to a primitive notion of *material symmetry*, the isotropy group (also called the symmetry group) $\mathscr{G}$ of a material is the group of density-preserving transformations of the material coordinates which leave the response of the material unaltered.[15] Before describing the

[14] Although we often write the partial derivative of a function with respect to a symmetric tensor such as $a_{\alpha\beta}$ in the form indicated in (13.41), the partial derivative $\partial\tilde{\psi}/\partial a_{\alpha\beta}$ is understood to have the symmetric form

$$\frac{1}{2}\left(\frac{\partial \tilde{\psi}}{\partial a_{\alpha\beta}} + \frac{\partial \tilde{\psi}}{\partial a_{\beta\alpha}}\right).$$

A parallel remark applies to all similar partial derivatives with respect to symmetric tensors elsewhere in this chapter.

[15] In general, the isotropy group consists of density-preserving transformations; but, since we are concerned with elastic solids, we only need to consider length-preserving transformations. A clear development of the basic concepts of material symmetry in the three-dimensional theory can be found in a paper by NOLL [1958, *4*]. See also Sect. 293 of TRUESDELL and TOUPIN [1960, *14*].

consequences of such restrictions in mathematical terms, for background information, it is desirable to elaborate briefly (in descriptive terms) on the primitive notion of material symmetry with reference to the Cosserat surface $\mathscr{C}$. Consider the tangent plane and the normal at each material point of $\mathscr{C}$, which point may be regarded as the origin of a Cartesian coordinate system x^m. We associate with x^m a set of orthonormal basis vectors $\boldsymbol{E}_m$ such that $\boldsymbol{E}_\alpha$ are in the tangent plane and $\boldsymbol{E}_3$ is directed along the normal. If a typical vector $\boldsymbol{V}$ is decomposed in such a coordinate system along the unit vectors $\boldsymbol{E}_m$ into tangential components and a normal component, then under a change from one coordinate system in the tangent plane to another the tangential components transform like the components of a 2-vector under the orthogonal group and the third component remains invariant. The symmetry of the material, defined by preferred directions in the body manifold (i.e., the Cosserat surface) is then characterized by $\boldsymbol{E}_\alpha$ and the appropriate group of (two-dimensional) transformations which specify the equivalent positions of the vectors $\boldsymbol{E}_\alpha$ from one system to another and the constitutive relations must then be form-invariant under each transformation of this group. We consider here symmetries for which the associated group of transformations is a subgroup of the full orthogonal group.

Since the body manifold—here the Cosserat surface $\mathscr{C}$—has no metric property, we begin our discussion of material symmetry by assigning a metric tensor to the material surface of $\mathscr{C}$; this is conveniently realized by assigning the (reference) metric tensor $A_{\alpha\beta}$ at each material point[16] θ^μ. Next, we introduce another coordinate system $\bar{\theta}^\mu = \{\bar{\theta}^1, \bar{\theta}^2\}$ on the material surface of $\mathscr{C}$ by the length-preserving transformation

$$\theta^\alpha = \theta^\alpha(\bar{\theta}^1, \bar{\theta}^2), \tag{13.44}$$

such that

$$\begin{aligned} A_{\gamma\delta} &= H^\alpha_\gamma H^\beta_\delta A_{\alpha\beta}, \\ H^\alpha_\gamma &= \frac{\partial\theta^\alpha}{\partial\bar{\theta}^\gamma}. \end{aligned} \tag{13.45}$$

In other words, (13.44) is so restricted that at a given material point the components H^α_γ satisfy $(13.45)_1$.

The length-preserving coordinate transformation (13.44) relates two curvilinear coordinate systems on the material surface of $\mathscr{C}$; and it is not necessary, for our present purpose, to be more specific than this. However, often it is simpler in applications to introduce the length-preserving transformation by means of two coordinate systems which are locally Cartesian at the material point in question of the body manifold. Thus, let x^α and $\bar{x}^\gamma$ refer to two locally rectangular Cartesian coordinate systems and let the origin of the former be identified with the material point in question. Then, in line with earlier remarks (in this subsection) and instead of (13.44)–(13.45), we may introduce the length-preserving transformation by

$$x^\alpha = \bar{H}^\alpha_\gamma \bar{x}^\gamma + \bar{H}^\alpha, \qquad \bar{H}\,\bar{H}^T = \bar{H}^T\,\bar{H} = I, \tag{13.46}$$

such that $\boldsymbol{E}_\alpha$ are the basis in the x^α coordinates and $\bar{H}$ (with components $\bar{H}^\alpha_\gamma$) is an orthogonal matrix which satisfies $(13.46)_2$.

[16] Since the assigned metric tensor is that of the reference configuration, in reality we are considering symmetries relative to the reference configuration of $\mathscr{C}$. As we are concerned with material symmetries possessed by the body manifold relative to a *local* reference configuration, we have assigned the values of $A_{\alpha\beta}$ to each material point θ^μ in order to render the notion of distance on the material surface of $\mathscr{C}$ meaningful.

For convenience, in what follows, we again employ (13.28) as a typical constitutive equation and study first the effect of material symmetry on the response functions $\bar{f}$ and $\tilde{f}$ in (13.37). If a response function for an elastic Cosserat surface [such as $\bar{f}$ in (13.28)] is form-invariant under a subgroup of the distance-preserving transformations (13.44), then the material is said to have the symmetry represented by that subgroup. Hence, for each element in the symmetry group of the material, we must have[17]

$$\bar{f}\left(\frac{\partial \boldsymbol{r}}{\partial \theta^\alpha}, \boldsymbol{d}, \frac{\partial \boldsymbol{d}}{\partial \theta^\gamma}; \theta^\mu\right) = \bar{f}\left(\frac{\partial \boldsymbol{r}}{\partial \bar{\theta}^\alpha}, \boldsymbol{d}, \frac{\partial \boldsymbol{d}}{\partial \bar{\theta}^\gamma}; \theta^\mu\right), \tag{13.47}$$

where $\partial \boldsymbol{r}/\partial \bar{\theta}^\alpha$ and $\partial \boldsymbol{d}/\partial \bar{\theta}^\alpha$ by (13.44)–(13.45) are:

$$\frac{\partial \boldsymbol{r}}{\partial \bar{\theta}^\alpha} = H_\alpha^\gamma \boldsymbol{a}_\gamma, \qquad \frac{\partial \boldsymbol{d}}{\partial \bar{\theta}^\alpha} = H_\alpha^\gamma \boldsymbol{d}_{,\gamma}. \tag{13.48}$$

Evidently, under the transformation (13.44), a scalar V or a vector field $\boldsymbol{V}$ remains unaffected but tangential derivatives (such as $\partial V/\partial \bar{\theta}^\alpha$, $\partial \boldsymbol{V}/\partial \bar{\theta}^\alpha$) transform according to formulae of the type (13.48).

Under the symmetry transformation (13.48), (13.47) becomes

$$\bar{f}(\boldsymbol{a}_\alpha, \boldsymbol{d}, \boldsymbol{d}_{,\beta}; \theta^\mu) = \bar{f}(H_\alpha^\gamma \boldsymbol{a}_\gamma, \boldsymbol{d}, H_\beta^\delta \boldsymbol{d}_{,\delta}; \theta^\mu). \tag{13.49}$$

Similarly, recalling (13.37) and the variables $\mathscr{V}$ in (13.36), we see that the material symmetry group of the response function $\tilde{f}$ is the set of all length-preserving transformations such that

$$\tilde{f}(\mathscr{V}; \theta^\mu) = \tilde{f}(\overline{\mathscr{V}}; \theta^\mu), \tag{13.50}$$

where $\overline{\mathscr{V}}$ stands for the set of variables

$$\begin{aligned} \overline{\mathscr{V}}: H_\alpha^\tau H_\gamma^\nu a_{\tau\nu}, \; H_\alpha^\tau d_\tau, \; H_\gamma^\nu H_\alpha^\tau \lambda_{\nu\tau}, \\ \sigma, \; H_\alpha^\tau \sigma_\tau \end{aligned} \tag{13.51}$$

resulting from the set $\mathscr{V}$ under the symmetry transformations (13.48).

We can now apply the result (13.49) to $\bar{\psi}$ or (13.50) to $\tilde{\psi}$ in (13.38). Considering only the latter, we have

$$\tilde{\psi}(\mathscr{V}, \theta; \theta^\mu) = \tilde{\psi}(\overline{\mathscr{V}}, \theta; \theta^\mu), \tag{13.52}$$

which must be satisfied by the response function $\tilde{\psi}$ for all H_γ^α in $\mathscr{G}$. It should be apparent that a result similar to (13.52) can also be recorded for the response function $\tilde{q}^\alpha$ in (13.43), except that $H_\alpha^\tau \theta_{,\tau}$ should be added to the variables $\overline{\mathscr{V}}$. Given a group of symmetry transformations (13.44) for an elastic material, the identity (13.52) serves to restrict the response function $\tilde{\psi}$ and hence the constitutive relations $(13.38)_2$ and (13.40)–(13.42). In particular, if the symmetry group $\mathscr{G}$ consists of the set of all length-preserving transformations (13.44), the elastic material is said to be isotropic with a center of symmetry. This completes our discussion of material symmetry restrictions. However, independently of material symmetries, a further restriction may be imposed and this is discussed next.

The material symmetries discussed above pertain to a given Cosserat surface $\mathscr{C}$ and are valid irrespective of the choice of direction associated with the director

[17] Previously in (13.28) the dependence of $\bar{f}$ on the material point θ^μ was suppressed; it is included in (13.47) for clarity.

at each material point of $\mathscr{C}$. Suppose we require that the response functions be also independent of the particular orientation of the director at each material point of the surface which, in turn, implies that the constitutive equations remain unaltered also under the reflection[18]

$$\boldsymbol{d} \to -\boldsymbol{d} \tag{13.53}$$

and hence

$$\boldsymbol{d}_{,\alpha} \to -\boldsymbol{d}_{,\alpha}. \tag{13.54}$$

Then, for a given Cosserat surface, $\boldsymbol{a}_\alpha$ and therefore $\boldsymbol{a}_3$ remain unaltered but the kinematic variables d_i and $\lambda_{i\alpha}$ transform according to

$$\begin{aligned} d_\alpha &\to -d_\alpha, \qquad d_3 \to -d_3, \\ \lambda_{\beta\alpha} &\to -\lambda_{\beta\alpha}, \quad \lambda_{3\alpha} \to -\lambda_{3\alpha}. \end{aligned} \tag{13.55}$$

It is noteworthy that the variables σ and σ_α defined by $(13.33)_{1,2}$ remain unaffected by the results (13.55). To summarize, under the added restriction (13.53), the set of variables $\mathscr{V}$ in (13.36) becomes

$$\begin{gathered} \mathscr{V}': \; a_{\alpha\gamma}, \; -d_\alpha, \; -\lambda_{\gamma\alpha}, \\ \sigma, \; \sigma_\alpha \end{gathered} \tag{13.56}$$

and hence

$$\tilde{\psi}(\mathscr{V}, \theta; \theta^\mu) = \tilde{\psi}(\mathscr{V}', \theta; \theta^\mu). \tag{13.57}$$

The restriction imposed by (13.57) requires that $\tilde{\psi}$ be an even function of the ordered pair $(\lambda_{\beta\alpha}, d_\gamma)$. This must be kept in mind, if the restriction (13.53) is introduced prior to considerations of material symmetry.

δ) *Alternative forms of the constitutive equations.* It is sometimes desirable to express the response functions in terms of *relative* kinematic measures such as those in (5.31)–(5.33), which are obtained relative to a fixed reference configuration. To this end and in view of the dependence of $\tilde{\psi}$ and q^α in (13.38) and (13.43) on the last two of the set (13.36), we introduce the relative measures

$$\begin{aligned} s &= \boldsymbol{d}^N \cdot \boldsymbol{d}^N - \boldsymbol{D}^N \cdot \boldsymbol{D}^N = (d^3)^2 - (D^3)^2, \\ s_\alpha &= \boldsymbol{d}^N \cdot (\boldsymbol{d}_{,\alpha})^N - \boldsymbol{D}^N \cdot (\boldsymbol{D}_{,\alpha})^N = d^3 \lambda_{3\alpha} - D^3 \Lambda_{3\alpha}, \end{aligned} \tag{13.58}$$

where in line with (13.32)–(13.33) we have used the notations

$$\boldsymbol{D}^N = D^3 \boldsymbol{A}_3, \quad (\boldsymbol{D}_{,\alpha})^N = \Lambda^3_{.\alpha} \boldsymbol{A}_3. \tag{13.59}$$

The variable $e_{\alpha\beta}$ defined by (5.31) is a kinematic measure in which $a_{\alpha\beta}$ is computed relative to its reference value $A_{\alpha\beta}$. Hence, a function of $a_{\alpha\beta}$ can be expressed as a different function of $e_{\alpha\beta}$ and $A_{\alpha\beta}$; but it should be kept in mind that a constitutive equation in terms of the latter function is now one that holds relative to a reference configuration, with the material point identified with its position in the reference configuration. Parallel remarks apply to more general response functions.

Henceforth, we identify the material point with its position in the reference configuration and also restrict attention to materials which are homogeneous in the reference configuration. It follows from the above remarks that a response

[18] To give an interpretation of (13.53), we first recall the definition of the three-dimensional shell and the observation in Sect. 4 that the magnitude of the director (say directed along the normal to the material surface prior to any deformation) can be regarded as representing the thickness of the (three-dimensional) shell. Then, the condition (13.53) can be interpreted as reflecting the fact that a material filament above and below the surface $\xi = 0$ of the (three-dimensional) shell possesses no intrinsic positive or negative direction.

function such as $\tilde{\psi}$ in (13.38) may be replaced by a (different) function of temperature, the variables

$$\begin{aligned}\mathscr{U}\colon\ & e_{\alpha\beta},\ \gamma_\alpha,\ \varkappa_{\beta\alpha},\\ & s,\ s_\alpha\end{aligned} \tag{13.60}$$

and the reference values

$$\begin{aligned}\mathscr{U}_R\colon\ & A_{\alpha\beta},\ D_\alpha,\ \Lambda_{\beta\alpha},\\ & (D^3)^2,\ D^3\Lambda_{3\alpha}.\end{aligned} \tag{13.61}$$

In particular, the constitutive equation $(13.38)_2$ for the specific free energy may be replaced by

$$\psi=\hat{\psi}(e_{\alpha\beta},\gamma_\alpha,\varkappa_{\beta\alpha},s,s_\alpha,\theta;\mathscr{U}_R). \tag{13.62}$$

The above constitutive equation holds relative to a fixed homogeneous reference configuration which may be taken to be the initial configuration. It is now a straightforward matter to obtain the constitutive equations in tensor components and in terms of the function $\hat{\psi}$ from (13.40)–(13.42). These are given by[19]

$$\eta=-\frac{\partial\hat{\psi}}{\partial\theta} \tag{13.63}$$

and

$$N'^{\alpha\beta}=N'^{\beta\alpha}=\varrho\frac{\partial\hat{\psi}}{\partial e_{\alpha\beta}}, \tag{13.64}$$

$$\begin{aligned} m^\alpha &=\varrho\frac{\partial\hat{\psi}}{\partial\gamma_\alpha}, & m^3 &=\varrho\left(2d^3\frac{\partial\hat{\psi}}{\partial s}+\lambda^3_{\cdot\alpha}\frac{\partial\hat{\psi}}{\partial s_\alpha}\right),\\ M^{\alpha\gamma} &=\varrho\frac{\partial\hat{\psi}}{\partial\varkappa_{\gamma\alpha}}, & M^{\alpha3} &=\varrho\, d^3\frac{\partial\hat{\psi}}{\partial s_\alpha}.\end{aligned} \tag{13.65}$$

Also, corresponding to (13.43), we have

$$q^\alpha=\hat{q}^\alpha(e_{\alpha\beta},\gamma_\alpha,\varkappa_{\beta\alpha},s,s_\alpha,\theta,\theta_{,\alpha};\mathscr{U}_R). \tag{13.66}$$

Both sets of constitutive equations $(13.38)_2$ and (13.40)–(13.43) and (13.62)–(13.66) hold at each material point of the Cosserat surface $\mathscr{C}$; however, in contrast to the former set, the constitutive equations (13.62)–(13.66) involve kinematic variables which are measured relative to a fixed reference configuration. Moreover, the response functions in (13.62)–(13.66) depend on the choice of the reference configuration as indicated explicitly by the presence of $\mathscr{U}_R$ in the arguments of $\hat{\psi}$ and $\hat{q}^\alpha$. We also observe that the response function $\hat{q}^\alpha$ must satisfy an inequality of the form (13.19) but with $\bar{q}^\alpha$ replaced by $\hat{q}^\alpha$. Also, with the help of (13.62)–(13.65), the residual energy equation can be obtained from (9.35) and (9.52).

The material symmetry restrictions discussed above [Subsect. γ)] may be readily extended to the response functions $\hat{\psi}$ and $\hat{q}^\alpha$. Recalling that the material point is now identified with its position in the reference configuration, it can be shown that $\partial\boldsymbol{R}/\partial\bar{\theta}^\gamma$ and $\partial\boldsymbol{D}/\partial\bar{\theta}^\gamma$ also transform as in (13.48) under the length-preserving transformation (13.44). Moreover, the material isotropy group $\mathscr{G}$ now becomes identical to the isotropy group relative to the reference configuration.[20]

[19] The partial derivative $\partial\hat{\psi}/\partial e_{\alpha\beta}$ in (13.64) is understood to have the symmetric form

$$\frac{1}{2}\left(\frac{\partial\hat{\psi}}{\partial e_{\alpha\beta}}+\frac{\partial\hat{\psi}}{\partial e_{\beta\alpha}}\right).$$

[20] We may recall here that according to NOLL [1958, *4*], the isotropy groups corresponding to two different reference configurations of the same material can be brought into one-to-one correspondence preserving the group structure or that, in more precise terms, they are conjugate and hence isomorphic.

It then follows that the response function $\hat{\psi}$ must satisfy the identity

$$\hat{\psi}(\mathscr{U}, \theta; \mathscr{U}_R) = \hat{\psi}(\overline{\mathscr{U}}, \theta; \overline{\mathscr{U}}_R), \tag{13.67}$$

where $\overline{\mathscr{U}}$ and $\overline{\mathscr{U}}_R$ stand for

$$\begin{gathered} \overline{\mathscr{U}}\colon\ H_\alpha^\tau H_\gamma^\nu e_{\tau\nu},\ H_\alpha^\tau \gamma_\tau,\ H_\gamma^\nu H_\alpha^\tau \varkappa_{\nu\tau}, \\ s,\ H_\alpha^\tau s_\tau \end{gathered} \tag{13.68}$$

and

$$\begin{gathered} \overline{\mathscr{U}}_R\colon\ H_\alpha^\tau H_\gamma^\nu A_{\tau\nu},\ H_\alpha^\tau D_\tau,\ H_\gamma^\nu H_\alpha^\tau \Lambda_{\nu\tau}, \\ (D^3)^2,\ H_\alpha^\tau(D^3 \Lambda_{3\tau}). \end{gathered} \tag{13.69}$$

An identity similar to (13.67) holds also for $\hat{q}^\alpha$. The symmetry restriction (13.67) serves to restrict the response function $\hat{\psi}$ and hence the constitutive equation (13.62). The previous remarks concerning the symmetry group $\mathscr{G}$ made with reference to (13.52) apply also to (13.67).

Now let the director in the (initial) reference configuration be specified along the unit normal to $\mathscr{S}$ so that

$$\begin{gathered} \boldsymbol{D} = D\,\boldsymbol{A}_3, \quad D_\alpha = 0, \quad D_3 = D, \\ \Lambda_{\beta\alpha} = -D\,B_{\alpha\beta}, \quad \Lambda_{3\alpha} = D_{,\alpha}, \end{gathered} \tag{13.70}$$

where the notation $(13.70)_3$ is introduced for convenience. Further, if $\boldsymbol{D}$ is of constant magnitude and coincident with the unit normal $\boldsymbol{A}_3$, we have

$$\begin{gathered} \boldsymbol{D} = \boldsymbol{A}_3, \quad D_\alpha = 0, \quad D_3 = D = 1, \\ \Lambda_{\beta\alpha} = -B_{\alpha\beta}, \quad \Lambda_{3\alpha} = 0. \end{gathered} \tag{13.71}$$

Then, corresponding to (13.71), the reference values (13.61) become simply

$$\mathscr{U}_R'\colon\ A_{\alpha\beta},\ -B_{\alpha\beta} \tag{13.72}$$

and the constitutive equation (13.62) reduces to

$$\psi = \hat{\psi}(e_{\alpha\beta}, \gamma_\alpha, \varkappa_{\beta\alpha}, s, s_\alpha, \theta; \mathscr{U}_R'), \tag{13.73}$$

where $\hat{\psi}$ is a different function from that in (13.62). In our further developments of the constitutive equations (both nonlinear and linear) by direct approach, we shall restrict attention mainly to a Cosserat surface whose (initial) reference director is specified by (13.71).

Previously [in Subsect. γ)], with reference to $(13.38)_2$ and (13.40)–(13.43), we also examined the consequences of an added restriction (13.53). This restriction, which is separate from material symmetry restrictions, requires that the response of the Cosserat surface be independent of the particular orientation of the director at each point of the surface. Since the constitutive equations (13.62)–(13.66) hold relative to the (initial) reference configuration, the added restriction requires (13.53)–(13.54) together with

$$\boldsymbol{D} \to -\boldsymbol{D} \tag{13.74}$$

and hence

$$\boldsymbol{D}_{,\alpha} \to -\boldsymbol{D}_{,\alpha}. \tag{13.75}$$

The conditions (13.74)–(13.75), in turn, imply

$$\begin{gathered} D_\alpha \to -D_\alpha, \quad D_3 \to -D_3, \\ \Lambda_{\beta\alpha} \to -\Lambda_{\beta\alpha}, \quad \Lambda_{3\alpha} \to -\Lambda_{3\alpha}. \end{gathered} \tag{13.76}$$

Under the additional restriction (13.53) and (13.74), the set of variables $\mathcal{U}$ and $\mathcal{U}_R$ in (13.60)–(13.61) become

$$\begin{gathered}\mathcal{U}'': e_{\alpha\gamma}, \ -\gamma_\alpha, \ -\varkappa_{\gamma\alpha}, \\ s, \ s_\alpha\end{gathered} \tag{13.77}$$

and

$$\begin{gathered}\mathcal{U}''_R: A_{\alpha\beta}, \ -D_\alpha, \ -\Lambda_{\gamma\alpha}, \\ (D^3)^2, \ D^3 \Lambda_{3\alpha}\end{gathered} \tag{13.78}$$

and hence the response function $\hat{\psi}$ must also satisfy

$$\hat{\psi}(\mathcal{U}, \theta; \mathcal{U}_R) = \hat{\psi}(\mathcal{U}'', \theta; \mathcal{U}''_R), \tag{13.79}$$

in addition to (13.67) resulting from material symmetry restrictions. It should be clear that the restriction specified by (13.53) and (13.74) could have been introduced prior to any consideration of material symmetries.

The constitutive relations for $N'^{\alpha\beta}$, m^i and $M^{\alpha i}$ can be expressed in somewhat simpler forms than those in (13.64)–(13.65); but, this can be accomplished at the expense of a loss in the representation for the function $\tilde{\psi}$ or $\hat{\psi}$. To elaborate, consider the constitutive relations given by (13.62)–(13.65) which reflect, in particular, the manner in which $\hat{\psi}$ depends on the last two variables in each of (13.60)–(13.61). Suppose, instead of (13.62), the constitutive equation for ψ is assumed to depend on the temperature, the kinematic variables (5.31)–(5.33) and the reference values $A_{\alpha\beta}, D_i, \Lambda_{i\alpha}$:

$$\psi = \psi'(e_{\alpha\beta}, \gamma_i, \varkappa_{i\alpha}, \theta; A_{\alpha\beta}, D_i, \Lambda_{i\alpha}). \tag{13.80}$$

Then, from (9.36) with P given by (9.52) and by a procedure similar to that used earlier, we can deduce the expressions[21]

$$\eta = -\frac{\partial \psi'}{\partial \theta}, \tag{13.81}$$

$$N'^{\alpha\beta} = N'^{\beta\alpha} = \varrho \frac{\partial \psi'}{\partial e_{\alpha\beta}}, \quad m^i = \varrho \frac{\partial \psi'}{\partial \gamma_i}, \quad M^{\alpha i} = \varrho \frac{\partial \psi'}{\partial \varkappa_{i\alpha}}. \tag{13.82}$$

Mathematically, the constitutive equations (13.80)–(13.82) and a similar constitutive equation for q^α are as general as those in (13.62)–(13.66). But the simpler forms (13.82) are gained at a loss in the representation for the specific free energy, since the function ψ' is not endowed with the property that its arguments $\gamma_3, \varkappa_{3\alpha}$ and their reference values appear according to the forms of the last two variables in each of (13.60)–(13.61). Of course, the constitutive equations (13.80)–(13.82) can be supplemented by separate conditions reflecting the effects of (13.58) and their initial values in the response functions.[22]

We close this section with a remark concerning the nature of constitutive equations in an alternative formulation of the theory of a Cosserat surface in terms of a reference state [Sect. 8, Subsect. ε) and Sect. 9, Subsect. δ)]. We recall that the basic field equations in terms of a reference state involve, in particular, the

[21] The constitutive equations in the form (13.80)–(13.82) were derived by GREEN, NAGHDI and WAINWRIGHT [1965, *4*]. For a purely mechanical theory, results similar in forms to (13.82) and in terms of a strain energy function were also obtained by COHEN and DE SILVA [1966, *2*] who admit three independent directors at each point of their directed surfaces.

[22] In the context of the linear theory and on the basis of a different argument, a restriction corresponding to (13.79) was introduced by GREEN and NAGHDI [1967, *4*], [1968, *6*] and imposed by them on the linearized versions of the constitutive equations (13.80)–(13.82). In this connection, see Sect. 16 [Subsect. γ)].

variables

$$\{{}_R\boldsymbol{N}^\alpha, {}_R\boldsymbol{m}, {}_R\boldsymbol{M}^\alpha, {}_Rq^\alpha\}, \tag{13.83}$$

which are measured relative to a reference configuration. The constitutive relations and related thermodynamical results for an elastic Cosserat surface in terms of the variables (13.83) can be obtained from (9.82) in the same manner that various results in this section [Subsect. α)] were deduced in terms of the variables $\{\boldsymbol{N}^\alpha, \boldsymbol{m}, \boldsymbol{M}^\alpha, q^\alpha\}$ from the inequality (9.36). However, any additional discussion of this kind is unnecessary, in view of the relations (9.87) between the two sets of variables. In fact, apart from a factor of ϱ_0/ϱ, it is clear that the constitutive equations for the variables (13.83) are also given by (13.24), (13.23) and (13.9).

14. The complete theory. Special results: I. Direct approach. We recapitulate here the complete thermodynamical theory of an elastic Cosserat surface and then separately discuss the constitutive equations of the purely mechanical theory of an elastic Cosserat surface. Also, we briefly indicate the nature of some special theories and some available special results and solutions within the scope of the general isothermal (or mechanical) theory.

α) *The boundary-value problem in the general theory.* The basic field equations of the nonlinear theory consist in the equations of motion (9.47)–(9.48), (9.51) and the energy equation $(9.28)_1$ with P given by $(9.28)_2$ or (9.52). The constitutive equations for an elastic Cosserat surface are specified by $(13.38)_2$, (13.40)–(13.43) or any of the alternative forms discussed in the previous section. Also the heat flux q^α, whose constitutive equation has the form (13.43), must satisfy an inequality of the form (13.19).

The above field equations and constitutive relations characterize the initial boundary-value problem in the nonlinear theory of an elastic Cosserat surface. The nature of the boundary conditions in this theory is clear from the rate of work expression $R_c(\mathscr{P})$ in (8.8) and the line integral in (8.10). In particular, the force and the director couple boundary conditions are given by (9.41) and (9.43) and these hold pointwise on the boundary c of the Cosserat surface with outward unit normal ν_α.

The rather difficult question of existence and nonuniqueness for elastostatic boundary-value problems has been considered very recently by ANTMAN, whose analysis is confined to axisymmetric deformations in the isothermal theory of a Cosserat surface.[23] Employing the direct methods of the calculus of variations and imposing fairly mild restrictions on the material response, ANTMAN's paper deals with existence, regularity and nonuniqueness of solutions for equilibrium problems of shells of revolution (under a wide class of boundary conditions) subjected to hydrostatic pressure and distributed axial load.

β) *Constitutive equations in a mechanical theory.* It is desirable to indicate briefly one way in which the constitutive relations of the purely mechanical theory can be expressed in terms of an *elastic potential.* For this purpose, we first recall the expression for the rate of work by contact force and contact director couple for each part $\mathscr{P}$ of $\mathscr{C}$ in the present configuration. Thus, after introducing (9.11) and (9.20) into $(8.8)_2$ and transforming the line integral into a surface integral, the expression for $R_c(\mathscr{P})$ becomes

$$\begin{aligned} R_c(\mathscr{P}) &= \int_{\mathscr{P}} [(\boldsymbol{N}^\alpha \cdot \boldsymbol{v})_{|\alpha} + (\boldsymbol{M}^\alpha \cdot \boldsymbol{w})_{|\alpha}]\, d\sigma \\ &= -\int_{\mathscr{P}} \varrho[\bar{\boldsymbol{f}} \cdot \boldsymbol{v} + \bar{\boldsymbol{l}} \cdot \boldsymbol{w}]\, d\sigma + \int_{\mathscr{P}} [\boldsymbol{N}^\alpha \cdot \boldsymbol{v}_{,\alpha} + \boldsymbol{m} \cdot \boldsymbol{w} + \boldsymbol{M}^\alpha \cdot \boldsymbol{w}_{,\alpha}]\, d\sigma, \end{aligned} \tag{14.1}$$

[23] ANTMAN [1971, *1*].

where in obtaining $(14.1)_2$ use has been made of (9.13) and (9.22). Consider next the reduction of the rate of work of the contact and the assigned forces and couples in $(8.8)_1$. Combining (14.1) and $(8.8)_3$, recalling the kinematical results (5.18)–(5.19), (5.26), (5.29) and using (9.23), we finally obtain

$$\begin{aligned} R(\mathscr{P}) &= \frac{d}{dt}\mathscr{K}(\mathscr{P}) + \int_{\mathscr{P}} [\boldsymbol{N}'^{\alpha}\cdot\boldsymbol{\eta}_{\alpha} + \boldsymbol{m}\cdot\boldsymbol{\Gamma} + \boldsymbol{M}^{\alpha}\cdot\boldsymbol{\Gamma}_{:\alpha}]\,d\sigma \\ &= \frac{d}{dt}\mathscr{K}(\mathscr{P}) + \int_{\mathscr{P}} [N'^{\alpha\beta}\eta_{\alpha\beta} + m^i\dot{d}_i + M^{\alpha i}\dot{\lambda}_{i\alpha}]\,d\sigma, \end{aligned} \tag{14.2}$$

where the kinetic energy $\mathscr{K}(\mathscr{P})$ is defined by (8.11) and the integrand in the second surface integral is the component form of the integrand in the first integral.

We assume the existence of a *strain energy* or a *stored energy* per unit mass $\Sigma(\theta^{\alpha}, t)$ such that

$$\boldsymbol{N}'^{\alpha}\cdot\boldsymbol{\eta}_{\alpha} + \boldsymbol{m}\cdot\boldsymbol{\Gamma} + \boldsymbol{M}^{\alpha}\cdot\boldsymbol{\Gamma}_{:\alpha} = N'^{\alpha\beta}\eta_{\alpha\beta} + m^i\dot{d}_i + M^{\alpha i}\dot{\lambda}_{i\alpha} = \varrho\dot{\Sigma} \tag{14.3}$$

and we define the strain energy for each part $\mathscr{P}$ in the present configuration at time t by the surface integral

$$U(\mathscr{P}) = \int_{\mathscr{P}} \varrho\,\Sigma\,d\sigma. \tag{14.4}$$

From (14.2)–(14.4), (5.66) and (8.18) follows

$$R(\mathscr{P}) = \frac{d}{dt}[\mathscr{K}(\mathscr{P}) + U(\mathscr{P})], \tag{14.5}$$

which is the analogue of a familiar result in the three-dimensional theory. According to (14.5) the rate of work by the contact and the assigned forces and couples $R(\mathscr{P})$ is equal to the sum of the rate of the kinetic energy $\mathscr{K}(\mathscr{P})$ and the rate of the strain energy $U(\mathscr{P})$.

Returning to (14.3), in parallel with the developments of Sect. 13 [Subsect. α)], we assume that for an elastic Cosserat surface the strain energy density Σ at each material point of $\mathscr{C}$ and for all t is specified by a response function which depends on the kinematic variables (13.7):

$$\Sigma = \bar{\Sigma}(\boldsymbol{a}_{\alpha}, \boldsymbol{d}, \boldsymbol{d}_{,\gamma}; \theta^{\mu}). \tag{14.6}$$

Since

$$\dot{\Sigma} = \frac{\partial\bar{\Sigma}}{\partial\boldsymbol{a}_{\alpha}}\cdot\dot{\boldsymbol{a}}_{\alpha} + \frac{\partial\bar{\Sigma}}{\partial\boldsymbol{d}}\cdot\dot{\boldsymbol{d}} + \frac{\partial\bar{\Sigma}}{\partial\boldsymbol{d}_{,\gamma}}\cdot\dot{\boldsymbol{d}}_{,\gamma}$$

and since (14.3) is defined to hold for all arbitrary values of $(13.11)_{1,2,3}$ or equivalently $\eta_{\alpha\beta}$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Gamma}_{:\alpha}$, from (14.3) we can deduce a condition of the form (13.15) together with constitutive equations of the forms (13.22)–(13.23) but with $\bar{\psi}$ replaced by $\bar{\Sigma}$. Further, by an argument entirely similar to that which led to (13.38) and (13.41)–(13.42), the constitutive equations can be reduced to

$$\Sigma = \tilde{\Sigma}(a_{\alpha\beta}, d_{\alpha}, \lambda_{\beta\alpha}, \sigma, \sigma_{\alpha}; \theta^{\mu}) \tag{14.7}$$

and

$$N'^{\alpha\beta} = N'^{\beta\alpha} = 2\varrho\frac{\partial\tilde{\Sigma}}{\partial a_{\alpha\beta}}, \tag{14.8}$$

$$\begin{aligned} m^{\alpha} &= \varrho\frac{\partial\tilde{\Sigma}}{\partial d_{\alpha}}, \qquad & m^3 &= \varrho\left(2d^3\frac{\partial\tilde{\Sigma}}{\partial\sigma} + \lambda^3_{.\alpha}\frac{\partial\tilde{\Sigma}}{\partial\sigma_{\alpha}}\right), \\ M^{\alpha\gamma} &= \varrho\frac{\partial\tilde{\Sigma}}{\partial\lambda_{\gamma\alpha}}, \qquad & M^{\alpha 3} &= \varrho\,d^3\frac{\partial\tilde{\Sigma}}{\partial\sigma_{\alpha}}. \end{aligned} \tag{14.9}$$

Alternative forms of the constitutive equations in which the strain energy density depends on the relative kinematic measures (13.60) and their reference values (13.61), the material symmetry, as well as the restriction imposed by the condition (13.53), can be discussed as in Sect. 13.

The above results, in terms of the response function $\tilde{\Sigma}$, represent the constitutive equations of the purely mechanical theory of an elastic Cosserat surface. They can be regarded as a special case of the formulae (13.38) and (13.41)–(13.42) when the temperature is constant. Alternatively, they can be regarded as a special case of those resulting from (13.25)–(13.27) when the entropy is constant. We note that the constitutive relations (14.8)–(14.9) may also be deduced from a virtual work principle.

γ) *Some special results.* We include here some remarks concerning certain special results, including a class of large deformation solutions. Based on the above theory and when the constitutive equations are restricted to correspond to an elastic Cosserat surface which is initially isotropic with a center of symmetry, a class of static and dynamic solutions for finite isothermal deformation has been given by CROCHET and NAGHDI.[24] The problems considered by CROCHET and NAGHDI are concerned mainly with a sector of a circular cylindrical surface, problems of a closed circular cylindrical surface and spherically symmetric deformation of a spherical surface. The solutions are obtained without specializing the form of the specific free energy (or the strain energy density) function and are such that in all cases (i) the initial director is taken to be coincident with the unit normal to the initial undeformed surface and (ii) the displacement gradients and the director displacements are independent of surface coordinates, but are either constants or functions of time. These solutions may be regarded as the surface counterpart of a number of existing exact solutions in the three-dimensional (non-polar) finite elasticity, often called controllable solutions in the recent literature.[25] We also note here that the theory of infinitesimal deformation superimposed upon a given deformation of an elastic Cosserat surface can be constructed by employing concepts similar to the corresponding developments in the three-dimensional theory.[26]

Within the scope of the purely mechanical theory of an elastic Cosserat surface and with the limitation to elastostatic problems, recently ERICKSEN has posed the question of what is meant by a uniform state for shells and has provided conditions for the existence of uniform states.[27] In essence, ERICKSEN defines a local state by the kinematic variables (13.7) and proceeds to obtain an equivalence relation between local states at different points of the same surface at time t. His motivation is based on the idea that easily analyzed problems in shell theory will involve such uniform states.

δ) *Special theories.* The theory of a Cosserat surface summarized above [Sect. 14, Subsect. α)] includes, or is more general than, a number of special or restrictive theories of shells. Notable among these is the restricted theory whose developments began in Sect. 10 and will be completed in the next section. Here, we discuss two well-known special theories or special cases of the general theory, namely the *membrane theory* and the *inextensional theory*.

[24] CROCHET and NAGHDI [1969, *1*].

[25] A detailed account of such exact solutions in the three-dimensional theory can be found in GREEN and ADKINS [1960, *5*] and in TRUESDELL and NOLL [1965, *9*].

[26] General results concerning superposed small deformations on a large deformation of an elastic Cosserat surface are contained in a paper by GREEN and NAGHDI [1971, *5*].

[27] ERICKSEN [1970, *1*].

The membrane theory of shells can be obtained by returning to the general principles of Sect. 8 and omitting the vector $\boldsymbol{d}$. Alternatively, the membrane theory follows from the general theory of an elastic Cosserat surface if we set $\bar{\boldsymbol{l}}=0$ and assume that the free energy function in (13.38) is independent of $\boldsymbol{d}$ and $\boldsymbol{d}_{,\gamma}$. Then, $\boldsymbol{M}^\alpha$ and $\boldsymbol{m}$ vanish by (13.23) and from (9.51) and (9.53) we obtain

$$N^{\alpha\beta}=N^{\beta\alpha}=N'^{\alpha\beta}, \qquad N^{\alpha 3}=0. \tag{14.10}$$

Hence, in the membrane theory, $N^{\alpha\beta}$ is a symmetric tensor and the equations of motion (9.47) reduce to

$$N^{\alpha\beta}{}_{|\alpha}+\varrho f^\beta=\varrho c^\beta, \qquad b_{\alpha\beta} N^{\alpha\beta}+\varrho f^3=\varrho c^3, \qquad \varepsilon_{\beta\alpha} N^{\alpha\beta}=0. \tag{14.11}$$

The energy equation and the entropy inequality are still of the forms (9.35)–(9.36) but with $P=N^{\alpha\beta}\eta_{\alpha\beta}$. For completeness we record below the constitutive equations for an elastic membrane which follow from (13.62)–(13.66):

$$\psi=\hat{\psi}(e_{\alpha\beta},\theta;A_{\alpha\beta}), \qquad \eta=-\frac{\partial\hat{\psi}}{\partial\theta}, \qquad N^{\alpha\beta}=N^{\beta\alpha}=\varrho\frac{\partial\hat{\psi}}{\partial e_{\alpha\beta}} \tag{14.12}$$

and

$$q^\alpha=\hat{q}^\alpha(e_{\gamma\delta},\theta,\theta_{,\gamma};A_{\gamma\delta}), \tag{14.13}$$

where the response functions $\hat{\psi}$, $\hat{q}^\alpha$ are now different functions from those in (13.62) and (13.66) and $\hat{q}^\alpha$ must satisfy an inequality of the form (13.19). Alternatively, the constitutive equations for an elastic membrane may be obtained from specialization of (13.38) and (13.40)–(13.43), in terms of the kinematic measure $a_{\alpha\beta}$ rather than $e_{\alpha\beta}$, but we do not record these here. The material symmetry restrictions discussed in Sect. 13 [Subsects. γ) and δ)] with an obvious modification (in the absence of $\boldsymbol{d}$, $\boldsymbol{d}_{,\gamma}$) apply also to the response functions in (14.12)–(14.13). In this connection, we observe that the appropriate material symmetries could be used in application to thin shells reinforced with cords.[28]

It is perhaps worth observing here that such phenomenon as *surface tension* can be described as a special case of the foregoing membrane theory. The classical theory of surface tension may be summarized by the statement that the total free energy is proportional to the surface area. In the context of the three-dimensional theory, for a *thin* film, the reference (or middle) surface has about the same area as the lower and upper surfaces so that adding their contributions results in an energy proportional to the area of the reference surface. Within the framework of the above membrane theory, the free energy can contain such a contribution. Indeed, in view of (4.40), if the response function $\tilde{\psi}$ is assumed to depend on $a^{\frac{1}{2}}$ only or equivalently if $\psi=\tilde{\psi}(a^{\frac{1}{2}})$, then the expressions for $N^{\alpha\beta}$ appropriate to surface tension can be calculated from $(14.10)_1$ and (13.41). In the special case that $\tilde{\psi}$ is quadratic in $a^{\frac{1}{2}}$; i.e., $\tilde{\psi}=\beta a$, where β is a constant, the results in lines of curvature coordinates will correspond to those usually given in the context of the classical theory of surface tension.

We consider now a special theory, called the inextensional theory, wherein the length of each element of the surface $\mathscr{C}$ is assumed to remain constant throughout all motions. This requirement can be expressed by the three equations

$$a_{\alpha\beta}=A_{\alpha\beta}=\text{const.}, \tag{14.14}$$

[28] In the context of the purely mechanical theory, RIVLIN [1959, *4*] has discussed the deformation of a net of inextensible cords regarded as a membrane of fabric with two directions of inextensibility at each point of the surface. Related problems of sheets reinforced with perfectly flexible but inextensible cords can be found in Chap. 7 of GREEN and ADKINS [1960, *5*].

which by (5.31) is equivalent to $e_{\alpha\beta}=0$. The condition (14.14) implies that the surface area is "preserved" for all times and it follows that $\varrho=\varrho_0$, in view of the continuity equation (4.42).[29] It is clear that (14.14) is a restriction on the class of possible motions and hence represents an internal constraint. We can therefore deduce the inextensional theory from the general theory of a Cosserat surface only by the introduction of additional assumptions appropriate to a theory with internal constraints. Although we do not consider here a general development of internal constraints for shells, the development of the theory in the presence of mechanical constraints of the type (14.14) is fairly straightforward and may be patterned after analogous treatments of internal constraints in the three-dimensional (non-polar) continuum mechanics.[30]

From differentiation of (14.14) with respect to t follow the three constraint equations $\dot{a}_{\alpha\beta}=0$. These constraint equations remain unaltered under superposed rigid body motions, and by (5.14) may be written as

$$\eta_{\alpha\beta}=0. \tag{14.15}$$

From the energy equation (9.29) and the conditions (14.15), it is at once apparent that $\boldsymbol{N}'^{\alpha}$ can no longer be determined entirely by constitutive equations in the sense that a part of $\boldsymbol{N}'^{\alpha}$ which occurs in

$$\boldsymbol{N}'^{\alpha}\cdot\boldsymbol{\eta}_{\alpha}=(\boldsymbol{N}'^{\alpha}\cdot\boldsymbol{a}^{\beta})\,\eta_{\alpha\beta}$$

is *workless*. It is evident that the indeterminate part of $\boldsymbol{N}'^{\alpha}$, i.e., the part not specified by constitutive equations, pertains to the tangential components $N'^{\alpha\beta}$ defined by (9.31) and not to the normal component $\boldsymbol{N}'^{\alpha}\cdot\boldsymbol{a}^3$. The indeterminate part of $\boldsymbol{N}'^{\alpha}$ may be thought of as the "forces required to maintain the constraint" and for brevity may be referred to as the *constraint response*.[31]

Keeping the above in mind, in the development of the inextensional theory under consideration, we make the following assumptions: (i) The vector functions $\boldsymbol{N}'^{\alpha}$ are determined to within an additive constraint response in the form

$$\boldsymbol{N}'^{\alpha}=\mathcal{N}^{\alpha}+{}_E\boldsymbol{N}'^{\alpha}, \tag{14.16}$$

where ${}_E\boldsymbol{N}'^{\alpha}$ are specified by constitutive equations, while the constraint response $\mathcal{N}^{\alpha}$ are functions of position and time but are independent of the rate quantities (13.11); (ii) the indeterminate forces $\mathcal{N}^{\alpha}$ are workless in all motions consistent with (14.15), i.e.,[32]

$$\mathcal{N}^{\alpha}\cdot\boldsymbol{\eta}_{\alpha}=0, \tag{14.17}$$

where $\boldsymbol{\eta}_{\alpha}$ is defined by (5.19) and (5.15). Now by any of a number of standard procedures, from (14.15) and (14.17) we may conclude that $\mathcal{N}^{\alpha}\cdot\boldsymbol{a}^{\beta}=\lambda^{\alpha\beta}$ or equivalently

$$\mathcal{N}^{\alpha}=\lambda^{\alpha\beta}\,\boldsymbol{a}_{\beta}, \qquad \lambda^{\alpha\beta}=\lambda^{\beta\alpha}, \tag{14.18}$$

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[29] The inextensional deformation specified by (14.14) implies $\varrho=$ const (and hence "surface incompressibility"), but the converse is not true.

[30] For an account of mechanical constraints in the three-dimensional theory, see Sect. 30 of TRUESDELL and NOLL [1965, *9*]. A thermodynamical theory of a continuum in the presence of thermo-mechanical constraints has been given recently by GREEN, NAGHDI and TRAPP [1970, *3*]. A development of the inextensional theory by direct approach, based on the theory of a Cosserat surface subject to the conditions (14.14), is contained in a paper by CROCHET [1971, *3*].

[31] This terminology is used in [1970, *3*].

[32] In the context of the thermodynamical treatment of constraints in [1970, *3*], the requirement (14.17) is equivalent to the assumption that the local production of entropy due to $\mathcal{N}^{\alpha}$ is zero.

where $\lambda^{\alpha\beta}$ is an arbitrary symmetric tensor function of position and time.[33] It may be noted that since only the tangential components of $\boldsymbol{N}'^{\alpha}$ are affected by the constraint conditions (14.15), in the case of the inextensional theory we could absorb the tangential components of ${}_E\boldsymbol{N}'^{\alpha}$ into $\mathcal{N}^{\alpha}$ and replace (14.16) by $N'^{\alpha\beta} = N'^{\beta\alpha} = \mathcal{N}^{\alpha\beta}$. In this way we would again arrive at the conclusion $\mathcal{N}^{\alpha\beta} = \lambda^{\alpha\beta}$.

We summarize now the field equations and the constitutive equations for an inextensible Cosserat surface. The symmetric tensor $\lambda^{\alpha\beta}$ in (14.18) is governed by the equations of motion (9.13) or equivalently[34] $(9.47)_{1,2}$. The equations of motion of the inextensional theory are given by (9.48) and the components $N^{[\alpha\beta]}$ and $N^{\alpha 3}$ are still determined from (9.51). The energy equation and the entropy inequality are still of the forms (9.35)–(9.36) but instead of P in $(9.29)_2$ or (9.52) we now have

$$\begin{aligned} P &= \boldsymbol{m} \cdot \boldsymbol{\Gamma} + \boldsymbol{M}^{\alpha} \cdot \boldsymbol{\Gamma}_{:\alpha} \\ &= m^i \dot{d}_i + M^{\alpha i} \dot{\lambda}_{i\alpha}. \end{aligned} \tag{14.19}$$

The developments of the constitutive equations for an elastic inextensional Cosserat surface now parallels that given in Sect. 13. Thus, except for $\boldsymbol{N}^{\alpha}$, we may begin by introducing constitutive assumptions similar to those in (13.8) in terms of response functions which, aside from their dependence on the material point θ^{μ}, are functions of

$$\boldsymbol{d},\ \boldsymbol{d}_{,\gamma},\ \theta \tag{14.20}$$

and the temperature gradient $\theta_{,\alpha}$. Moreover, because of the constraint conditions (14.15), instead of the inequality (13.14) we now have

$$-\varrho\left(\bar{\eta} + \frac{\partial\bar{\psi}}{\partial\theta}\right)\dot{\theta} + \left(\overline{\boldsymbol{m}} - \varrho\frac{\partial\bar{\psi}}{\partial\boldsymbol{d}}\right)\cdot\boldsymbol{\Gamma} + \left(\overline{\boldsymbol{M}}^{\gamma} - \varrho\frac{\partial\bar{\psi}}{\partial\boldsymbol{d}_{,\gamma}}\right)\cdot\boldsymbol{\Gamma}_{:\gamma} - (\overline{\boldsymbol{q}}\cdot\boldsymbol{a}^{\alpha})\frac{\theta_{,\alpha}}{\theta} \geqq 0, \tag{14.21}$$

where $\bar{\psi}$ is a different function from that in (13.14) and depends on the variables (14.20). The rest of the developments, with obvious modifications, are entirely similar to those in Sect. 13.[35]

15. The complete restricted theory: I. Direct approach. Previously, the field equations of the restricted theory in which the motion is characterized by $(5.1)_1$ alone were derived in Sect. 10. Here we complete this theory for an elastic surface in a manner similar to the more general developments of Sect. 13. Preliminary to the introduction of the constitutive assumption appropriate to the restricted theory, we observe that under superposed rigid body motions the results (13.1) still hold but instead of (13.2) we now have the transformations

$$\check{\boldsymbol{M}}^{\alpha} \to Q\,\check{\boldsymbol{M}}^{\alpha}, \qquad \check{\boldsymbol{N}}^{\alpha} \to Q\,\check{\boldsymbol{N}}^{\alpha}, \qquad \boldsymbol{q} \to Q\,\boldsymbol{q} \tag{15.1}$$

under superposed rigid body motions. It follows that the transformation (13.4) still holds but those in (13.3) are now replaced by

$$\check{M}^{\alpha\beta} \to \check{M}^{\alpha\beta}, \qquad \check{N}^{\alpha\beta} \to \check{N}^{\alpha\beta}, \qquad q^{\alpha} \to q^{\alpha}, \tag{15.2}$$

where $\check{N}^{\alpha\beta}$ and $\check{M}^{\alpha\beta}$ are defined by (10.26) and $(10.20)_3$.

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[33] It is clear that $\lambda^{\alpha\beta}$ play the role of Lagrange multipliers.

[34] These equations may not always uniquely determine $\lambda^{\alpha\beta}$ throughout the surface of $\mathscr{C}$.

[35] Inasmuch as the variables $\boldsymbol{a}_{\alpha}$ are absent from the argument of the response functions, the inequality (14.21) is unaffected by the constraint equations (14.15). In the presence of more general constraints of the type $\boldsymbol{\gamma}^{\alpha} \cdot \dot{\boldsymbol{a}}_{\beta} = 0$, where $\boldsymbol{\gamma}^{\alpha}$ are vector functions of $\boldsymbol{a}_{\nu}$ and the variables (14.20), the inequality corresponding to (14.21) will be subject to the constraint equations and this must also be taken into account when obtaining results similar to (13.18). See, in this connection, Sect. 4 of [1970, *3*] for a discussion of constitutive equations for an elastic continuum subject to a general thermo-mechanical constraint.

Recalling the general remarks concerning constitutive equations at the beginning of Sect. 13 [between (13.4) and (13.8)], within the scope of the restricted theory, we define an elastic surface by the following five response functions:

$$\grave{\overline{\boldsymbol{N}}}^{\alpha}(\text{or } \overline{\boldsymbol{N}}^{\alpha}),\ \grave{\overline{\boldsymbol{M}}}^{\alpha},\ \overline{\psi},\ \overline{\eta},\ \overline{q}^{\alpha}. \tag{15.3}$$

We introduce constitutive equations, in terms of the response functions (15.3), at each material point θ^{μ} of the surface in the present configuration; and we assume that the response functions depend on the kinematic variables

$$\boldsymbol{a}_{\alpha},\ \boldsymbol{a}_{3,\gamma}, \tag{15.4}$$

as well as the temperature and the temperature gradient.[36] For example, the constitutive equation for the specific Helmholtz free energy will have the form

$$\psi = \overline{\psi}(\boldsymbol{a}_{\alpha}, \boldsymbol{a}_{3,\gamma}, \theta, \theta_{,\alpha}; \theta^{\mu}) \tag{15.5}$$

with similar constitutive equations for $\grave{\boldsymbol{N}}^{\alpha}$, $\grave{\boldsymbol{M}}^{\alpha}$, η and q^{α} in terms of the response functions in (15.3). These constitutive equations are assumed to hold at each material point θ^{μ} and for all times t.

From combination of the entropy inequality (9.32) and the energy equation $(10.24)_1$ in terms of ψ in (9.34), follows the inequality

$$-\varrho\,\dot{\psi} - \varrho\,\eta\,\dot{\theta} + \grave{P} - q^{\alpha}\frac{\theta_{,\alpha}}{\theta} \geqq 0, \tag{15.6}$$

which must hold for every admissible process and where $\grave{P}$ is given by $(10.24)_2$. After introducing the above constitutive assumptions for the restricted theory in (15.6) and following the procedure of Sect. 13, we can show that $\overline{\psi}$ in (15.5) must be independent of $\theta_{,\alpha}$. Hence, ψ can be expressed in terms of a (different) response function of θ and the variables (15.4):

$$\psi = \overline{\psi}(\boldsymbol{a}_{\alpha}, \boldsymbol{a}_{3,\gamma}, \theta; \theta^{\mu}). \tag{15.7}$$

Moreover, the inequality (15.6) can then be written in the form

$$\begin{aligned} -\varrho\left(\overline{\eta} + \frac{\partial\overline{\psi}}{\partial\theta}\right)\dot{\theta} + \left[\grave{\overline{\boldsymbol{N}}}^{\alpha} - \varrho\left(\frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{\alpha}} + b_{\gamma}^{\alpha}\frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{3,\gamma}}\right)\right]\cdot\boldsymbol{\eta}_{\alpha} \\ + \left[\grave{\overline{\boldsymbol{M}}}^{\gamma} - \varrho\frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{3,\gamma}}\right]\cdot\dot{\grave{\lambda}}_{\alpha\gamma}\,\boldsymbol{a}^{\alpha} - \overline{q}^{\alpha}\frac{\theta_{,\alpha}}{\theta} \geqq 0, \end{aligned} \tag{15.8}$$

where the function $\overline{\psi}$ in (15.7) must satisfy

$$\varepsilon_{\gamma\alpha}\,\boldsymbol{a}^{\gamma}\cdot\left[\frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{\alpha}} - b_{\lambda}^{\alpha}\frac{\partial\overline{\psi}}{\partial\boldsymbol{a}_{3,\lambda}}\right] = 0 \tag{15.9}$$

and $\grave{\lambda}_{\alpha\gamma}$ is defined by $(5.61)_3$. The condition (15.9) arises from the fact that the inequality (15.6) must remain unaffected by superposed rigid body motions.[37] Using arguments similar to those in Sect. 13 [Subsect. α)] and recalling our

[36] Apart from the temperature and its gradient, the variables (15.4) define the local state in the restricted theory. They may be contrasted with the kinematic variables (13.7) which define the local state in the theory of an elastic Cosserat surface. In fact, since the unit normal to a surface is determined by the surface base vectors, the variables (15.4) representing the local state in the restricted theory follow from (13.7) with $\boldsymbol{d} = \boldsymbol{a}_3$.

[37] The inequality (15.8) and the condition (15.9) are, respectively, the counterparts of (13.14) and (13.15) in the restricted theory.

constitutive assumptions, from (15.8) we deduce the relations

$$\eta = -\frac{\partial \bar{\psi}}{\partial \theta}, \tag{15.10}$$

$$\begin{aligned} \overset{\prime}{N}^{\alpha\beta} = \overset{\prime}{N}^{\beta\alpha} &= \frac{1}{2}\varrho\left\{\left(\frac{\partial\bar{\psi}}{\partial \boldsymbol{a}_\alpha} + b^\alpha_\gamma \frac{\partial \bar{\psi}}{\partial \boldsymbol{a}_{3,\gamma}}\right)\cdot \boldsymbol{a}^\beta + \left(\frac{\partial\bar{\psi}}{\partial \boldsymbol{a}_\beta} + b^\beta_\gamma \frac{\partial \bar{\psi}}{\partial \boldsymbol{a}_{3,\gamma}}\right)\cdot \boldsymbol{a}^\alpha\right\}, \\ \overset{\prime}{\boldsymbol{M}}^{(\gamma\alpha)} &= \frac{1}{2}\varrho\left\{\frac{\partial\bar{\psi}}{\partial \boldsymbol{a}_{3,\gamma}}\cdot \boldsymbol{a}^\alpha + \frac{\partial\bar{\psi}}{\partial \boldsymbol{a}_{3,\alpha}}\cdot \boldsymbol{a}^\gamma\right\} \end{aligned} \tag{15.11}$$

and an inequality in the form (13.19) but with $\bar{q}^\alpha$ now a function of θ, $\theta_{,\alpha}$ and the variables (15.4).

Next, we consider the reduction of (15.7) under superposed rigid body motions. As in Sect. 13 [Subsect. β)], we observe that a constitutive equation in the form (15.7) which holds for an admissible process must also hold for a motion which differs from $(5.1)_1$ only by superposed rigid body motions. This requirement is fulfilled if and only if a response function such as $\bar{\psi}$ satisfies the identity

$$\bar{\psi}(\boldsymbol{a}_\alpha, \boldsymbol{a}_{3,\gamma}, \theta; \theta^\mu) = \bar{\psi}(Q\,\boldsymbol{a}_\alpha, Q\,\boldsymbol{a}_{3,\gamma}, \theta; \theta^\mu), \tag{15.12}$$

for all proper orthogonal Q. An identity similar to (15.12) holds for $\bar{q}^\alpha$. We now use Cauchy's representation theorem for isotropic functions and express $\bar{\psi}$ as a (different) function in the form

$$\psi = \tilde{\psi}(a_{\alpha\beta}, \grave{\lambda}_{\alpha\beta}, \theta; \theta^\mu), \tag{15.13}$$

where $(4.12)_1$, $(4.13)_1$ and the definition $(5.61)_3$ for $\grave{\lambda}_{\alpha\beta}$ have been used. With the help of (15.7) and (15.13) and using the chain rule for differentiation, the constitutive equations (15.10)–(15.11) can be reduced to[38]

$$\eta = -\frac{\partial\tilde{\psi}}{\partial\theta}, \tag{15.14}$$

$$\overset{\prime}{N}^{\alpha\beta} = \overset{\prime}{N}^{\beta\alpha} = 2\varrho\frac{\partial\tilde{\psi}}{\partial a_{\alpha\beta}}, \qquad \overset{\prime}{\boldsymbol{M}}^{(\alpha\gamma)} = \varrho\frac{\partial\tilde{\psi}}{\partial\grave{\lambda}_{\alpha\beta}}. \tag{15.15}$$

Also, the constitutive equation for the heat flux can be expressed as

$$q^\alpha = \tilde{q}^\alpha(a_{\gamma\delta}, \grave{\lambda}_{\gamma\delta}, \theta, \theta_{,\gamma}; \theta^\mu). \tag{15.16}$$

The discussion of material symmetry restrictions and other considerations for constitutive equations can be pursued in a manner similar to that in Sect. 13. We can also write the constitutive equations in terms of relative kinematic measures and the reference values (13.72) but we do not record these.

The components $N^{\alpha 3}$ are absent from the energy equation and also from the inequality (15.6), as was noted in Sect. 10. Hence, the components $N^{\alpha 3}$ cannot be specified by constitutive equations and may be determined from the equations of motion $(10.21)_1$. Moreover, as is evident from (15.15), the restricted theory provides constitutive equations for only the symmetric $\overset{\prime}{N}^{\alpha\beta}$ and the symmetric part of $\overset{\prime}{\boldsymbol{M}}^{\alpha\beta}$ or equivalently for $N^{(\alpha\beta)}$ and $\overset{\prime}{\boldsymbol{M}}^{(\alpha\beta)}$. The skew-symmetric part of $N^{\alpha\beta}$ is determined from $(10.21)_2$ but the latter also involves $\overset{\prime}{\boldsymbol{M}}^{\alpha\beta}$. Thus, in order to obtain a determinate theory, we now assume that

$$\overset{\prime}{\boldsymbol{M}}^{[\alpha\beta]} = 0. \tag{15.17}$$

[38] Once more we recall that in evaluating partial derivatives with respect to symmetric tensors such as $\partial\tilde{\psi}/\partial\grave{\lambda}_{\alpha\beta}$, the tensor $\grave{\lambda}_{\alpha\beta}$ is understood to stand for $\frac{1}{2}(\grave{\lambda}_{\alpha\beta} + \grave{\lambda}_{\beta\alpha})$.

Then, from $(10.21)_2$, the skew-symmetric part of $N^{\alpha\beta}$ is given by

$$N^{[\alpha\beta]} = \tfrac{1}{2}\{b^{\alpha}_{\gamma}\,\overset{*}{M}{}^{(\gamma\beta)} - b^{\beta}_{\gamma}\,\overset{*}{M}{}^{(\gamma\alpha)}\}. \tag{15.18}$$

Further, with the help of (15.18), the differential equations of motion (10.22) can be written in the alternative form

$$\begin{aligned} N^{(\alpha\beta)}{}_{|\alpha} - \tfrac{1}{2}[b^{\beta}_{\gamma}\,\overset{*}{M}{}^{(\gamma\alpha)}]_{|\alpha} + \tfrac{1}{2}[b^{\alpha}_{\gamma}\,\overset{*}{M}{}^{(\gamma\beta)}]_{|\alpha} - b^{\beta}_{\alpha}\,\overset{*}{M}{}^{(\gamma\alpha)}{}_{|\gamma} + \varrho\,\bar{f}^{\beta} &= 0,\\ \overset{*}{M}{}^{(\alpha\beta)}{}_{|\alpha\beta} + b_{\alpha\beta}\,N^{(\alpha\beta)} + \varrho\,\bar{f}^{3} &= 0. \end{aligned} \tag{15.19}$$

The above differential equations involve only the symmetric $N^{(\alpha\beta)}$ and $\overset{*}{M}{}^{(\alpha\beta)}$. They can also be expressed in terms of $\overset{*}{N}{}^{\alpha\beta}$ and $\overset{*}{M}{}^{(\alpha\beta)}$ as follows:

$$\begin{aligned} \overset{*}{N}{}^{\alpha\beta}{}_{|\alpha} - b^{\beta}_{\gamma|\alpha}\,\overset{*}{M}{}^{(\gamma\alpha)} - 2b^{\beta}_{\alpha}\,\overset{*}{M}{}^{(\gamma\alpha)}{}_{|\gamma} + \varrho\,\bar{f}^{\beta} &= 0,\\ \overset{*}{M}{}^{(\alpha\beta)}{}_{|\alpha\beta} + b_{\alpha\beta}\,\overset{*}{N}{}^{\alpha\beta} - b_{\alpha\beta}\,b^{\beta}_{\gamma}\,\overset{*}{M}{}^{(\gamma\alpha)} + \varrho\,\bar{f}^{3} &= 0. \end{aligned} \tag{15.20}$$

To summarize, the basic field equations and the constitutive relations for an elastic surface in the restricted theory consist of the equations of motion (9.47) and (10.21) together with (15.17) [or any of the alternative sets recorded above], the energy equation (10.24) and the constitutive equations (15.13)–(15.16). Also the response function $\tilde{q}^{\alpha}$ must satisfy an inequality of the form (13.19).[39] It remains to consider the nature of the boundary conditions in the restricted theory. Recalling the expression for the rate of work of contact force and couple in (10.3) and remembering $(10.20)_3$, (15.17), $(5.61)_1$ and (5.24), we have

$$\begin{aligned} \overset{*}{R}_c(\mathscr{P}) &= \int_{\partial\mathscr{P}} (\boldsymbol{N}\cdot\boldsymbol{v} + \overset{*}{\boldsymbol{M}}\cdot\dot{\boldsymbol{w}})\,ds\\ &= \int_{\partial\mathscr{P}} \nu_{\alpha}\{N^{\alpha i}\,v_i - \overset{*}{M}{}^{(\alpha\gamma)}\,b^{\beta}_{\gamma}\,v_{\beta} - \overset{*}{M}{}^{(\alpha\gamma)}\,v_{3,\gamma}\}\,ds. \end{aligned} \tag{15.21}$$

Let $\partial/\partial\nu$ stand for the directional derivative along the unit normal $\boldsymbol{\nu}$ to the boundary curve c of the surface s in the present configuration. Then

$$v_{3,\gamma} = \nu_{\gamma}\frac{\partial v_3}{\partial\nu} - \varepsilon_{\gamma\alpha}\,\nu^{\alpha}\frac{\partial v_3}{\partial s}, \tag{15.22}$$

where $\partial/\partial s$ is the directional derivative along the tangent to c introduced in (8.1). Provided the quantities in (15.21) are single-valued on a (sufficiently smooth) closed curve c, with the use of (15.22) and an integration by parts, (15.21) can be reduced to

$$R_c(\mathscr{P}) = \int_{\partial\mathscr{P}} \left\{P^{\beta}_{\nu}\,v_{\beta} + P^{3}\,v_3 - G\frac{\partial v_3}{\partial\nu}\right\}ds, \tag{15.23}$$

[39] These results, including the constitutive relations of the restricted theory for an elastic surface, can be brought into correspondence with those of the restricted theory of simple force dipole contained in the paper of BALABAN, GREEN and NAGHDI [1967, *1*] and referred to earlier in Sect. 10.

In the context of the purely mechanical theory, there exist in the literature derivations (with various degrees of approximation) from the three-dimensional equations which are aimed at the types of results supplied by the restricted theory. Such developments from the three-dimensional theory which employ special or restrictive kinematic assumptions (e.g., NAGHDI and NORDGREN [1963, *8*]) are necessarily approximate in character; and although they often contain formulae similar in form to those of the restricted theory [including those corresponding to (15.15) with ψ regarded as the strain energy density], they should not be confused with the latter. The essential point here to be born in mind is that although the two sets of formulae are similar, the constitutive coefficients in (15.15) are arbitrary and are not predetermined from an (approximate) expression of the three-dimensional specific free energy or strain energy density.

where we have put

$$P^\beta = \nu_\alpha [N^{\alpha\beta} - \grave{M}^{(\alpha\gamma)} b^\beta_\gamma], \qquad G = \grave{M}^{(\alpha\gamma)} \nu_\alpha \nu_\gamma, \tag{15.24}$$

$$P^3 = \nu_\alpha [\grave{M}^{(\beta\alpha)}|_\beta + \varrho\, \grave{l}^\alpha] - \frac{\partial}{\partial s} [\varepsilon_{\beta\gamma} \grave{M}^{(\alpha\beta)} \nu_\alpha \nu^\gamma], \tag{15.25}$$

and in obtaining (15.25) use has been made of $(10.21)_1$. The nature of the boundary conditions in the restricted theory is now clear from (15.23). In particular, the modified force and couple boundary conditions which hold pointwise on c are given by (15.24)–(15.25).[40]

As remarked earlier, the above restricted theory bears on the classical theory of shells.[41] This will become apparent in Sect. 25, where the complete restricted linear theory is discussed. We note here that the membrane theory follows also from the above restricted theory by setting $\grave{l}^\alpha = 0$ and assuming that $\tilde{\psi}$ in (5.13) is independent of $\grave{\lambda}_{\alpha\beta}$. Similarly, by introducing additional assumptions, an inextensional theory—less general than that given in Sect. 14 [Subsect. δ)]—can be derived from the restricted theory subject to the constraint equations (14.15).

16. Linear constitutive equations: I. Direct approach. Previously in Sects. 6 and 9, we considered linearization of the kinematic variables and the field equations in a systematic manner. Here, we complete the linearization procedure in a complete theory of an elastic Cosserat surface and obtain explicit expressions for the constitutive equations.

α) *General considerations.* We begin our general considerations of constitutive equations of the linear theory with the thermodynamical results (13.62)–(13.66). However, as in $(13.70)_1$, we take the initial director $\boldsymbol{D}$ to be along the normal to the reference surface $\mathscr{S}$ so that the set of variables $\mathscr{U}_R$ are still given by (13.61) but with values D_i and $\Lambda_{i\alpha}$ in (13.70).

Since the kinematic variables $\gamma_3, \varkappa_{3\alpha}$ and their reference values occur in the arguments of the response functions according to the last two variables in each of (13.60)–(13.61), in addition to the linear kinematic variables of Sect. 6, we need also the expressions resulting from linearization of (13.58). Thus, by the procedure of Sect. 6 and recalling $(6.8)_2$, $(6.9)_2$ and (6.13), we readily find

$$\begin{aligned} s &= 2\varepsilon (D^3 \gamma_3) + O(\varepsilon^2) = O(\varepsilon), \\ s_\alpha &= \varepsilon (D^3 \varkappa_{3\alpha} + \Lambda^3_{.\alpha} \gamma_3) + O(\varepsilon^2) = O(\varepsilon). \end{aligned} \tag{16.1}$$

Recalling the earlier remarks concerning the linear theory [Sect. 9, Subsect. γ)], the response function $\hat{\psi}$ in (13.62) now becomes a quadratic function of the appropriate kinematic variables either in the forms (6.22) or (6.23) together with

$$s = 2D^3 \gamma_3, \qquad s_\alpha = D^3 \varkappa_{3\alpha} + \Lambda^3_{.\alpha} \gamma_3, \tag{16.2}$$

which are obtained from (16.1) in accordance with the linearization procedure of Sect. 6. Moreover, since the left-hand side of each constitutive equation in (13.63)–(13.66) is now of $O(\varepsilon)$, it follows that on the right-hand sides $\varrho, d^3, \lambda^3_{.\alpha}$ must be replaced to the order ε by $\varrho_0, D^3, \Lambda^3_{.\alpha}$, respectively. It is easily seen that the linearized constitutive equations for $\eta, N'^{\alpha\beta}, m^\alpha, M^{\alpha\gamma}$ will have the same

[40] These results were given in [1968, *7*]. They correspond to similar formulae obtained in the literature on the classical linear theory of shells; see, e.g., [1963, *7*].

[41] A special case of the theory of Cosserat surface resulting in a system of equations analogous to that of the restricted theory has been discussed by GREEN and NAGHDI [1968, *5*]. The constitutive equations used in [1968, *5*] are, however, those in the form (13.80)–(13.82).

forms as those in (13.63)–(13.65) but the linear constitutive equations for m^3 and $M^{\alpha 3}$ are

$$m^3 = \varrho_0 \left(2D^3 \frac{\partial \hat{\psi}}{\partial s} + \Lambda^3_{.\alpha} \frac{\partial \hat{\psi}}{\partial s_\alpha} \right), \quad M^{\alpha 3} = \varrho_0 D^3 \frac{\partial \hat{\psi}}{\partial s_\alpha}, \tag{16.3}$$

where $\hat{\psi}$ is a different function from that in (13.62) and is quadratic in the arguments $e_{\alpha\gamma}, \gamma_\alpha, \varkappa_{\gamma\alpha}, s, s_\alpha$ and θ.

The foregoing results are obtained from (13.62)–(13.65) for a Cosserat surface whose (initial) reference director is specified by $(13.70)_1$. Consider now a Cosserat surface with its (initial) reference director coincident with the unit normal to $\mathscr{S}$ given by $(13.71)_1$. For a Cosserat surface $\mathscr{C}$ so specified in its reference configuration, in view of $(13.71)_{3,5}$, the linearized expressions (16.2) to $O(\varepsilon)$ become

$$s = 2\gamma_3, \qquad s_\alpha = \varkappa_{3\alpha}. \tag{16.4}$$

Then, the right-hand sides of $(16.3)_{1,2}$ reduce to

$$\varrho_0 \frac{\partial \hat{\psi}}{\partial \gamma_3} \quad \text{and} \quad \varrho_0 \frac{\partial \hat{\psi}}{\partial \varkappa_{3\alpha}},$$

respectively; and the constitutive equations of the linear theory may be expressed as[42]

$$\psi = \psi(e_{\alpha\beta}, \gamma_i, \varkappa_{i\alpha}, \theta; \mathscr{U}'_R) \tag{16.5}$$

and

$$\eta = -\frac{\partial \psi}{\partial \theta}, \tag{16.6}$$

$$N'^{\alpha\beta} = N'^{\beta\alpha} = \varrho_0 \frac{\partial \psi}{\partial e_{\alpha\beta}}, \quad m^i = \varrho_0 \frac{\partial \psi}{\partial \gamma_i}, \quad M^{\alpha i} = \varrho_0 \frac{\partial \psi}{\partial \varkappa_{i\alpha}}. \tag{16.7}$$

In (16.5)–(16.7) θ and η are now the temperature difference and the entropy difference, each of $O(\varepsilon)$ measured from a standard temperature θ_0 and entropy η_0 in the initial undeformed surface; and the response function ψ is quadratic in the temperature θ and the kinematic variables $e_{\alpha\beta}, \gamma_i, \varkappa_{i\alpha}$. Also, instead of (13.66), we now have a constitutive equation for the linearized heat flux Q^α defined by $(9.59)_2$. The response function for the heat flux, apart from its dependence on $\mathscr{U}'_R$, is now a linear function of the remaining arguments in (16.5) in addition to $\theta_{,\alpha}$. Moreover, instead of (13.19), in the linear theory we have the inequality

$$-Q^\alpha \theta_{,\alpha} \geqq 0, \tag{16.8}$$

which places restriction on the response function Q^α. It then follows that Q^α must be linear in the temperature gradient and can depend in addition only on the reference values $\mathscr{U}'_R$. We defer recording an explicit constitutive equation for Q^α but note here that the residual energy equation in the linear theory becomes

$$\varrho_0 r - \varrho_0 \theta_0 \dot{\eta} - Q^\alpha{}_{|\alpha} = 0, \tag{16.9}$$

where the vertical bar denotes covariant differentiation with respect to $A_{\alpha\beta}$. The above residual energy equation is obtained from $(9.60)_1$ with the help of (16.5)–(16.7). Each term in (16.9) is of $O(\varepsilon)$ consistent with our linearization procedure and in accord with the remarks made in Sect. 9 [Subsect. γ)].

In the remainder of this section and in our further considerations of the linear theory by direct approach, we confine attention to an elastic Cosserat surface

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[42] Here and in most of the developments that follow we return to our earlier notation and often employ the same symbol for a function and its value without confusion.

with its (initial) reference director specified by $(13.71)_1$. The constitutive relations are then given by (16.5)–(16.7), apart from an equation for heat flux.[43] Moreover, we observe here that the results (16.5)–(16.7) can also be deduced by linearization of (13.80)–(13.82) with the reference values (13.71). It is rather striking that the distinction between the two sets of constitutive equations (13.62)–(13.65) and (13.80)–(13.82) disappears for the linear theory and with $\boldsymbol{D}$ specified by $(13.71)_1$.

If we recall the energy equation $(9.60)_1$ with P in the form (9.71), it becomes evident that thermodynamical results corresponding to (16.7) can be deduced also for $N'^{\alpha\beta}$, $M^{\alpha i}$ and V^i. Thus, in terms of the kinematic variables $e_{\alpha\beta}$, $\varrho_{i\alpha}$ and γ_i, the specific free energy in the linear theory can alternatively be expressed as[44]

$$\psi = \overline{\psi}(e_{\alpha\beta}, \gamma_i, \varrho_{i\alpha}, \theta; \mathscr{U}'_R) \tag{16.10}$$

and we can deduce the results

$$N'^{\alpha\beta} = N'^{\beta\alpha} = \varrho_0 \frac{\partial \overline{\psi}}{\partial e_{\alpha\beta}}, \qquad V^i = \varrho_0 \frac{\partial \overline{\psi}}{\partial \gamma_i}, \qquad M^{\alpha i} = \varrho_0 \frac{\partial \overline{\psi}}{\partial \varrho_{i\alpha}} \tag{16.11}$$

in place of (16.7). The relation for entropy is still of the form (16.6) but with the function ψ replaced by $\overline{\psi}$.

The constitutive equations of the linear theory may also be expressed in terms of the specific internal energy. Such results can be deduced in a manner similar to those in Sect. 13 [between (13.25)–(13.27)] but will not be recorded here. Instead we consider next a different set of constitutive equations for the linear theory in terms of another thermodynamic potential, namely the specific *Gibbs free energy function* φ. Occasionally, it is useful to be able to express the kinematic variables such as $e_{\alpha\beta}$, γ_i, $\varkappa_{i\alpha}$ in terms of θ and $N'^{\alpha\beta}$, m^i, $M^{\alpha i}$, regarded as the independent variables. This can be achieved by introduction of the specific Gibbs function φ defined by

$$\varphi = \varphi(N'^{\alpha\beta}, m^i, M^{\alpha i}, \theta; \mathscr{U}'_R) = \psi - \frac{1}{\varrho_0}\left[N'^{\alpha\beta} e_{\alpha\beta} + m^i \gamma_i + M^{\alpha i} \varkappa_{i\alpha}\right], \tag{16.12}$$

where the specific Helmholtz free energy function ψ is of the form (16.5). Then, we have

$$\eta = -\frac{\partial \varphi}{\partial \theta}, \tag{16.13}$$

$$e_{\alpha\beta} = -\varrho_0 \frac{\partial \varphi}{\partial N'^{\alpha\beta}}, \qquad \gamma_i = -\varrho_0 \frac{\partial \varphi}{\partial m^i}, \qquad \varkappa_{i\alpha} = -\varrho_0 \frac{\partial \varphi}{\partial M^{\alpha i}}, \tag{16.14}$$

instead of the relations (16.6)–(16.7).

β) *Explicit results for linear constitutive equations.* Let the surface $\mathscr{S}$ in the initial configuration of the Cosserat surface $\mathscr{C}$ be homogeneous, free from curve force and director couple and in the state of rest at a constant temperature θ_0 and entropy η_0. Then, to the order of approximation considered, it is sufficient to express ψ in (16.5) as a quadratic function of $e_{\alpha\beta}$, γ_i, $\varkappa_{i\alpha}$ and θ. Thus, if we put

---

[43] The constitutive relations (16.5)–(16.7), being valid for a Cosserat surface whose initial director $\boldsymbol{D}$ is of constant magnitude, are intended for shells and plates of uniform thickness.

[44] The use of the symbol $\overline{\psi}$ here should not be confused with a similar notation in Sect. 13. In (16.10), $\overline{\psi}$ is a quadratic function of temperature and the kinematic variables $e_{\alpha\beta}$, γ_i, $\varrho_{i\alpha}$.

$\eta_0 = 0$, we have[45]

$$
\begin{aligned}
\varrho_0 \psi = {}& {}_1C^{\alpha\beta\gamma\delta} e_{\alpha\beta} e_{\gamma\delta} + {}_2C^{\alpha\beta\gamma\delta} \varkappa_{\alpha\beta} \varkappa_{\gamma\delta} + {}_3C^{\alpha\beta\gamma\delta} e_{\alpha\beta} \varkappa_{\gamma\delta} \\
& + {}_1C^{\alpha\beta\gamma} \varkappa_{3\alpha} \varkappa_{\beta\gamma} + {}_2C^{\alpha\beta\gamma} e_{\alpha\beta} \gamma_\gamma \\
& + {}_3C^{\alpha\beta\gamma} e_{\alpha\beta} \varkappa_{3\gamma} + {}_4C^{\alpha\beta\gamma} \gamma_\alpha \varkappa_{\beta\gamma} \\
& + {}_1C^{\alpha\beta} \gamma_\alpha \gamma_\beta + {}_2C^{\alpha\beta} \varkappa_{3\alpha} \varkappa_{3\beta} + {}_3C^{\alpha\beta} \gamma_\alpha \varkappa_{3\beta} \\
& + {}_4C^{\alpha\beta} e_{\alpha\beta} \gamma_3 + {}_5C^{\alpha\beta} \varkappa_{\alpha\beta} \gamma_3 + {}_4C'^{\alpha\beta} e_{\alpha\beta}\, \theta + {}_5C'^{\alpha\beta} \varkappa_{\alpha\beta}\, \theta \\
& + {}_1C^{\alpha} \gamma_\alpha \gamma_3 + {}_2C^{\alpha} \varkappa_{3\alpha} \gamma_3 + {}_1C'^{\alpha} \gamma_\alpha\, \theta + {}_2C'^{\alpha} \varkappa_{3\alpha}\, \theta \\
& + C(\gamma_3)^2 + C'\, \theta^2 + C''\, \gamma_3\, \theta,
\end{aligned}
\tag{16.15}
$$

where some of the coefficients satisfy certain symmetry conditions, e.g.,

$$
\begin{gathered}
{}_1C^{\alpha\beta\gamma\delta} = {}_1C^{\beta\alpha\gamma\delta} = {}_1C^{\alpha\beta\delta\gamma} = {}_1C^{\gamma\delta\alpha\beta}, \\
{}_2C^{\alpha\beta\gamma\delta} = {}_2C^{\gamma\delta\alpha\beta}, \quad {}_3C^{\alpha\beta\gamma\delta} = {}_3C^{\beta\alpha\gamma\delta}, \quad {}_2C^{\alpha\beta\gamma} = {}_2C^{\beta\alpha\gamma}.
\end{gathered}
\tag{16.16}
$$

From (16.15), we can readily calculate explicit relations for η, $N'^{\alpha\beta}$, m^i, $M^{\alpha i}$ which would be valid for an anisotropic Cosserat surface, but we do not record these here. For completeness we also record an explicit constitutive relation for Q^α. Recalling the remark made following (16.8), we write[46]

$$
Q^\alpha = -\varkappa^{\alpha\beta}\, \theta_{,\beta}, \qquad \varkappa^{\alpha\beta} = \varkappa^{\beta\alpha}, \tag{16.17}
$$

where the conductivity coefficients $\varkappa^{\alpha\beta}$ (assumed to be symmetric) depend on the reference values $\mathscr{U}'_R$ and satisfy the restrictions

$$
\varkappa^{11} \geqq 0, \qquad \varkappa^{22} \geqq 0, \qquad \varkappa^{11} \varkappa^{22} - \varkappa^{12} \varkappa^{21} \geqq 0 \tag{16.18}
$$

imposed by the inequality (16.8). We emphasize here that the free energy function (16.15) [and also the response function $(16.17)_1$] is recorded for a Cosserat surface with its (initial) reference director specified by $(13.71)_1$; and hence, in line with the remark made in Sect. 4 [following (4.35)], is appropriate for a shell which is of uniform thickness in the initial reference configuration. For a shell which is of variable thickness in the reference configuration, the free energy is still of the form (16.15) except that the various coefficients are now functions of the reference values (13.61) [instead of (13.72)] and also the kinematic variables γ_3 and $\varkappa_{3\alpha}$ in (16.15) must be replaced with s and s_α, respectively. Moreover, in this case, the constitutive relations for m^3 and $M^{\alpha 3}$ are found from (16.3) instead of the corresponding expressions in (16.7).

Henceforth, with reference to (16.15) and (16.17), we restrict our attention to an elastic Cosserat surface possessing holohedral isotropy (i.e., isotropy with a center of symmetry). In this case, a tensor basis is given by $A^{\alpha\beta}$ and since there are no holohedral isotropic tensors of odd order, it follows that all odd order coefficients in (16.15) must vanish:

$$
{}_1C^{\alpha\beta\gamma} = {}_2C^{\alpha\beta\gamma} = {}_3C^{\alpha\beta\gamma} = {}_4C^{\alpha\beta\gamma} = 0, \qquad {}_1C^{\alpha} = {}_2C^{\alpha} = {}_1C'^{\alpha} = {}_2C'^{\alpha} = 0. \tag{16.19}
$$

---

[45] Our results between (16.15)–(16.25) follow GREEN, NAGHDI and WAINWRIGHT [1965, *4*]. The various coefficients in (16.15) are functions of the reference values $\mathscr{U}'_R$ in (13.72).

Since for simplicity we have specified the initial director to be $\boldsymbol{D} = \boldsymbol{A}_3$ (with $D^3 = 1$), the dependence of the various coefficients in (16.15) on $(D^3)^2$ and therefore the magnitude of $\boldsymbol{D}$ is not explicit. [Recall the reference values (13.61).] Indeed, instead of specifying $D^3 = 1$, we could leave $(D^3)^2$ as an arbitrary parameter in the various coefficients of (16.15).

[46] This is simply the two-dimensional analogue of the Fourier law in the linear three-dimensional theory.

Moreover, the remaining coefficients in (16.15) must be homogeneous linear functions of products of $A^{\alpha\beta}$ so that, for example, ${}_1C^{\alpha\beta\gamma\delta}$ may be written as

$$2\,{}_1C^{\alpha\beta\gamma\delta} = \alpha_1 A^{\alpha\beta} A^{\gamma\delta} + \alpha_2 A^{\alpha\gamma} A^{\beta\delta} + \alpha_3 A^{\alpha\delta} A^{\beta\gamma}. \tag{16.20}$$

But by $(16.16)_1$, $\alpha_2 = \alpha_3$ and the same conclusions can be reached by $(16.16)_2$. Similar arguments can be applied to other coefficients in (16.15). In particular, it can be shown that ${}_3C^{\alpha\beta\gamma\delta}$, as ${}_1C^{\alpha\beta\gamma\delta}$, has only two independent scalar coefficients but ${}_2C^{\alpha\beta\gamma\delta}$ involves three independent scalar coefficients.

Keeping the above in mind, we can finally write the specific free energy function ψ in the form

$$\begin{aligned}
\varrho_0 \psi = {} & \tfrac{1}{2}[\alpha_1 A^{\alpha\beta} A^{\gamma\delta} + \alpha_2 (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\, e_{\alpha\beta}\, e_{\gamma\delta} \\
& + \tfrac{1}{2}\alpha_3 A^{\alpha\beta} \gamma_\alpha \gamma_\beta + \tfrac{1}{2}\alpha_4 (\gamma_3)^2 + \alpha_4' \gamma_3\, \theta + \tfrac{1}{2}\alpha_4'' \theta^2 \\
& + \tfrac{1}{2}[\alpha_5 A^{\alpha\beta} A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma} A^{\beta\delta} + \alpha_7 A^{\alpha\delta} A^{\beta\gamma}]\, \varkappa_{\alpha\beta}\, \varkappa_{\gamma\delta} \\
& + \tfrac{1}{2}\alpha_8 A^{\alpha\beta} \varkappa_{3\alpha} \varkappa_{3\beta} + \alpha_9 A^{\alpha\beta} e_{\alpha\beta} \gamma_3 + \alpha_9' A^{\alpha\beta} e_{\alpha\beta}\, \theta \\
& + [\alpha_{10} A^{\alpha\beta} A^{\gamma\delta} + \alpha_{11} (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\, e_{\alpha\beta}\, \varkappa_{\gamma\delta} \\
& + \alpha_{12} A^{\alpha\beta} \varkappa_{\alpha\beta} \gamma_3 + \alpha_{12}' A^{\alpha\beta} \varkappa_{\alpha\beta}\, \theta \\
& + \alpha_{13} A^{\alpha\beta} \gamma_\alpha \varkappa_{3\beta},
\end{aligned} \tag{16.21}$$

and by (16.7) we also have

$$\begin{aligned}
N'^{\alpha\beta} = {} & [\alpha_1 A^{\alpha\beta} A^{\gamma\delta} + \alpha_2 (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\, e_{\gamma\delta} \\
& + [\alpha_{10} A^{\alpha\beta} A^{\gamma\delta} + \alpha_{11} (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\, \varkappa_{\gamma\delta} \\
& + \alpha_9 A^{\alpha\beta} \gamma_3 + \alpha_9' A^{\alpha\beta}\, \theta,
\end{aligned} \tag{16.22}$$

$$\begin{aligned}
m^\alpha &= \alpha_3 A^{\alpha\gamma} \gamma_\gamma + \alpha_{13} A^{\alpha\gamma} \varkappa_{3\gamma}, \\
m^3 &= \alpha_4 \gamma_3 + \alpha_4' \theta + \alpha_9 A^{\alpha\beta} e_{\alpha\beta} + \alpha_{12} A^{\alpha\beta} \varkappa_{\alpha\beta},
\end{aligned} \tag{16.23}$$

and

$$\begin{aligned}
M^{\beta\alpha} = {} & [\alpha_5 A^{\alpha\beta} A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma} A^{\beta\delta} + \alpha_7 A^{\alpha\delta} A^{\beta\gamma}]\, \varkappa_{\gamma\delta} \\
& + [\alpha_{10} A^{\alpha\beta} A^{\gamma\delta} + \alpha_{11} (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\, e_{\gamma\delta} \\
& + \alpha_{12} A^{\alpha\beta} \gamma_3 + \alpha_{12}' A^{\alpha\beta}\, \theta, \\
M^{\alpha 3} = {} & \alpha_8 A^{\alpha\gamma} \varkappa_{3\gamma} + \alpha_{13} A^{\alpha\gamma} \gamma_\gamma,
\end{aligned} \tag{16.24}$$

where the coefficients $\alpha_1, \ldots, \alpha_{13}, \alpha_4', \alpha_4'', \alpha_9'$ and α_{12}' depend on the initial values $A_{\alpha\beta}$, $B_{\alpha\beta}$. We also note that for an isotropic material with a center of symmetry the constitutive relation for Q^α in (16.17) reduces to

$$Q^\alpha = -\varkappa A^{\alpha\beta} \theta_{,\beta}, \qquad \varkappa \geqq 0. \tag{16.25}$$

Apart from dependence of the coefficients $\alpha_1, \ldots, \alpha_{13}, \alpha_4', \ldots, \alpha_{12}', \alpha_4''$ on the reference values $\mathscr{U}_R'$ in (13.72), the foregoing results for an elastic Cosserat surface which is initially homogeneous and possesses holohedral isotropy are not particularly simple even in the case of an initially flat Cosserat surface for which (6.26) holds. For this reason and for later reference we consider next an additional restriction corresponding to (13.53) and (13.74) and examine its consequences.

γ) *A restricted form of the constitutive equations for an isotropic material.* We consider now a further restriction on the constitutive equations introduced previously in Sect. 13. According to this restriction, which is separate from material symmetries, for a given Cosserat surface the constitutive equations are required to remain unaltered under the reflection (13.53) and hence (13.54) or equivalently

$$\boldsymbol{\delta} \to -\boldsymbol{\delta}, \qquad \boldsymbol{\delta}_{,\alpha} \to -\boldsymbol{\delta}_{,\alpha}, \tag{16.26}$$

together with (13.74)–(13.75).[47] In order to obtain the appropriate transformation relations for the kinematic variables (which occur in the constitutive equations) under the restriction $(16.26)_1$, we should return to (16.2) and examine the effect of $(16.26)_1$ prior to the specification of $\boldsymbol{D} = \boldsymbol{A}_3$. For this purpose and with reference to the constitutive equations resulting from linearization of (13.63)–(13.65) with $\boldsymbol{D}$ along the normal to $\mathscr{S}$ [such as the constitutive equation (16.3)], suppose that the initial director along the normal is of constant magnitude specified by $\boldsymbol{D} = \bar{D}\boldsymbol{A}_3$ with $\bar{D}$ being constant. Then, $D^3 = \bar{D} = \text{const}$, $\Lambda_{3\alpha} = 0$ and the linearized variables (16.2) reduce to

$$s = 2\bar{D}\,\gamma_3, \qquad s_\alpha = \bar{D}\,\varkappa_{3\alpha}. \tag{16.27}$$

Keeping the above in mind, we see that the kinematic variables $\gamma_\alpha, \varkappa_{\gamma\alpha}, s, s_\alpha$ which occur in the linear constitutive equations (with $\boldsymbol{D} = \bar{D}\boldsymbol{A}_3$) transform according to

$$\begin{aligned} &\gamma_\alpha \to -\gamma_\alpha, \qquad \varkappa_{\gamma\alpha} \to -\varkappa_{\gamma\alpha}, \\ &s \to s, \qquad\quad\; s_\alpha \to s_\alpha, \end{aligned} \tag{16.28}$$

under the conditions (16.26). Now, without loss in generality, we put $\bar{D} = 1$ (or $\boldsymbol{D} = \boldsymbol{A}_3$) and conclude that under the restriction $(16.26)_1$ the kinematic variables $\gamma_i, \varkappa_{i\alpha}$ [which occur in the constitutive equations (16.5)–(16.7)] must transform according to

$$\begin{aligned} &\gamma_\alpha \to -\gamma_\alpha, \qquad \varkappa_{\gamma\alpha} \to -\varkappa_{\gamma\alpha}, \\ &\gamma_3 \to \gamma_3, \qquad\;\; \varkappa_{3\alpha} \to \varkappa_{3\alpha}, \end{aligned} \tag{16.29}$$

in view of the relations (16.4) between[48] s, γ_3 and $s_\alpha, \varkappa_{3\alpha}$.

We restrict attention in the remainder of this section to the isothermal theory, although this is by no means essential, and we return to our objective of obtaining a restricted form of the constitutive equations (16.21)–(16.24). It is instructive to consider first the case of an initially flat Cosserat plate for which the specific free energy (or the strain energy density, since $\theta = \text{const.}$) takes the form[49]

$$\psi = \psi(e_{\alpha\beta}, \varkappa_{i\alpha}, \gamma_i, A_{\alpha\beta}), \tag{16.30}$$

where the kinematic variables in the argument of ψ are given by (6.25). It is clear that the constitutive equations obtained from (16.30) and (16.7), for an isotropic material with a center of symmetry, will have the same form as (16.21)–(16.24) except that the coefficients $\alpha_1, \ldots, \alpha_{13}$ are now constants. We now introduce the further restriction that ψ be invariant under the reflection $(16.26)_1$ which, in

[47] See also the remarks made preceding both (13.53) and (13.74). The interpretation which may be associated with $(16.26)_1$ parallels that given in Sect. 13 with reference to (13.53).

[48] Note that with $\bar{D} = 1$, the expressions (16.27) reduce to (16.4) and the previous conclusion regarding the reduction of the constitutive equations (16.3) to corresponding expressions in (16.7) holds.

[49] This follows from (16.5) for an isothermal theory of an initially flat Cosserat plate for which (6.26) holds.

view of (16.29), can be stated as[50]

$$\psi(e_{\alpha\beta}, \varkappa_{\gamma\alpha}, \varkappa_{3\alpha}, \gamma_\alpha, \gamma_3, A_{\alpha\beta}) = \psi(e_{\alpha\beta}, -\varkappa_{\gamma\alpha}, \varkappa_{3\alpha}, -\gamma_\alpha, \gamma_3, A_{\alpha\beta}). \tag{16.31}$$

The condition (16.31) is a further restriction on the free energy function ψ, separate from that arising from holohedral isotropy (or isotropy with a center of symmetry), and results in simpler expressions for ψ, $N'^{\alpha\beta}$, m^i, $M^{\alpha i}$. These expressions can be obtained as special cases of (16.21)–(16.24), apart from the terms involving θ, if we put

$$\alpha_{10} = \alpha_{11} = \alpha_{12} = \alpha_{13} = 0. \tag{16.32}$$

We postpone recording the explicit form of these constitutive relations, but note that the simplification leading to (16.32) can also be achieved in the case of (16.10)–(16.11) by specializing (16.10) for the isothermal theory of an initially flat Cosserat plate. For later convenience, we record the resulting expression for $\overline{\psi}$ in the form

$$\begin{aligned}
\overline{\psi} &= \overline{\psi}_e + \overline{\psi}_b, \\
2\varrho_0 \overline{\psi}_e &= [\alpha_1 A^{\alpha\beta} A^{\gamma\delta} + \alpha_2 (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\, e_{\alpha\beta}\, e_{\gamma\delta} \\
&\quad + \alpha_4 (\gamma_3)^2 + \alpha_8 A^{\alpha\beta} \varrho_{3\alpha} \varrho_{3\beta} + 2\alpha_9 A^{\alpha\beta} e_{\alpha\beta} \gamma_3, \\
2\varrho_0 \overline{\psi}_b &= [\alpha_5 A^{\alpha\beta} A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma} A^{\beta\delta} + \alpha_7 A^{\alpha\delta} A^{\beta\gamma}]\, \varrho_{\alpha\beta}\, \varrho_{\gamma\delta} \\
&\quad + \alpha_3 A^{\alpha\beta} \gamma_\alpha \gamma_\beta.
\end{aligned} \tag{16.33}$$

We may note that the variable $\varrho_{\gamma\alpha}$ in (16.33) may be replaced with $\varkappa_{\gamma\alpha}$, in view of $(6.25)_6$.

We now turn to the more general case (in which $B_{\alpha\beta} \neq 0$) but discuss the additional restriction with reference to the constitutive relations (16.10)–(16.11). Again, limiting the discussion to the isothermal theory, we see that the free energy as a function of the kinematic variables (6.24) takes the form

$$\psi = \overline{\psi}(e_{\alpha\beta}, \varrho_{i\alpha}, \gamma_i; \mathscr{U}'_R), \tag{16.34}$$

where $\overline{\psi}$ is now a function different from that in (16.10). The additional restriction that $\overline{\psi}$ remain unaltered under the transformations $(16.26)_1$ and (13.74), in view of (16.29) and (13.78) with $\boldsymbol{D} = \boldsymbol{A}_3$, implies that[51]

$$\begin{aligned}
&\overline{\psi}(e_{\alpha\beta}, \varrho_{\alpha\beta}, \varrho_{3\alpha}, \gamma_\alpha, \gamma_3; A_{\alpha\beta}, -B_{\alpha\beta}) \\
&\quad = \overline{\psi}(e_{\alpha\beta}, -\varrho_{\alpha\beta}, \varrho_{3\alpha}, -\gamma_\alpha, \gamma_3; A_{\alpha\beta}, B_{\alpha\beta}).
\end{aligned} \tag{16.35}$$

The restriction (16.35), together with that due to isotropy with a center of symmetry, enables us to write down the complete form for $\overline{\psi}$ which is quadratic in $e_{\alpha\beta}, \varrho_{i\alpha}, \gamma_i$. The resulting expression will contain terms of the type in (16.21) with $\varrho_{i\alpha}$ defined in (6.24), in addition to terms involving $B_{\alpha\beta}$. As the latter terms (which vanish in the case of an initially flat surface) are rather unwieldy, we may regard the free energy to have the form (16.33) but with the kinematic variables defined by (6.24). This is equivalent to the specification that the response function

---

[50] The restriction (16.31) was introduced by GREEN and NAGHDI [1967, *4*], [1968, *6*] on the basis of a different argument than that given here, but the results and the conclusions remain the same.

[51] The restriction (16.35) was introduced by GREEN and NAGHDI [1968, *6*], [1969, *3*] on the basis of an argument different from that given here, but the results and conclusions remain the same.

$\bar{\psi}$ be independent of $B_{\alpha\beta}$. It should be emphasized that this is *not* an approximation but merely a particular choice for $\bar{\psi}$ as a special case of the general theory.[52]

δ) *Constitutive equations of the restricted linear theory.* The results in Subsect. γ) for a restricted form of constitutive equations of an elastic Cosserat surface should not be confused with the corresponding set of linear constitutive equations appropriate to the restricted theory (Sects. 10 and 15). It is not difficult to see that the latter set (with different constitutive coefficients) may be obtained in a similar manner by linearization of the nonlinear constitutive equations of the restricted theory in Sect. 15. In particular, with the limitation to isothermal deformations, the specific free energy in the restricted linear theory is a quadratic function of the kinematic variables $e_{\alpha\beta}$, $\varrho_{(\beta\alpha)}$ defined in (6.27). Hence, in parallel with (16.34), we may write

$$\psi = \bar{\psi}(e_{\alpha\beta}, \varrho_{(\beta\alpha)}; \mathcal{U}'_R), \tag{16.36}$$

where $\bar{\psi}$ is a different function from that in (16.34).

We may also impose a condition similar to (16.35) on the response function $\bar{\psi}$ in (16.36) by requiring that the function $\bar{\psi}$ remain unaltered under the reflection of the unit normal and hence its gradient:[53]

$$\boldsymbol{A}_3 \to -\boldsymbol{A}_3, \quad \boldsymbol{A}_{3,\alpha} \to -\boldsymbol{A}_{3,\alpha}. \tag{16.37}$$

Under the transformations (16.37), the linearized kinematic formulae of the restricted theory [collected in (6.27)] transform according to

$$\begin{gathered} u_\alpha \to u_\alpha, \quad u_3 \to -u_3, \quad B_{\alpha\beta} \to -B_{\alpha\beta}, \\ e_{\alpha\beta} \to e_{\alpha\beta}, \quad \varrho_{(\beta\alpha)} \to -\varrho_{(\beta\alpha)}. \end{gathered} \tag{16.38}$$

Hence, the added requirement that $\bar{\psi}$ in (16.36) remain unaltered under the reflection (16.37) implies that

$$\bar{\psi}(e_{\alpha\beta}, \varrho_{(\beta\alpha)}; A_{\alpha\beta}, -B_{\alpha\beta}) = \bar{\psi}(e_{\alpha\beta}, -\varrho_{(\beta\alpha)}; A_{\alpha\beta}, B_{\alpha\beta}). \tag{16.39}$$

In line with the remarks made following (16.35), we now regard the response function $\bar{\psi}$ in (16.39) to be independent of $B_{\alpha\beta}$ but the kinematic variables in its argument being those in (6.27). It then follows that for an isotropic material with a center of symmetry, the constitutive equation for ψ [in parallel to (16.33)] can be written in the form

$$\begin{aligned} \psi &= \bar{\psi}_e + \bar{\psi}_b, \\ 2\varrho_0 \bar{\psi}_e &= [\beta_1 A^{\alpha\beta} A^{\gamma\delta} + \beta_2 (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\, e_{\alpha\beta}\, e_{\gamma\delta}, \\ 2\varrho_0 \bar{\psi}_b &= [\beta_5 A^{\alpha\beta} A^{\gamma\delta} + \beta_6 (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\, \varrho_{(\alpha\beta)}\, \varrho_{(\gamma\delta)}, \end{aligned} \tag{16.40}$$

where $\beta_1, \beta_2, \beta_5$ and β_6 are arbitrary constants and do not necessarily have the same values as the corresponding coefficients in (16.33). We postpone recording

[52] The ultimate plausibility for such a special choice of the specific free energy (or the strain energy) function depends, of course, on its usefulness and a demonstration of its relevance to shell theory. Indeed, as remarked by GREEN and NAGHDI [1968, *6*], the problem of choice of the strain energy function here parallels the corresponding problem for a suitable choice of the strain energy function in the three-dimensional theory of nonlinear or linear elasticity and is a common feature of all general theories. Moreover, it is worth noting that in all existing developments for a linear shell theory from the three-dimensional equations (beginning with the paper of LOVE [1888, *1*]), an assumption that the strain energy density (after an integration with respect to the thickness coordinate) has the same form as the (integrated) strain energy density for a flat plate is implicit or is implied by other assumptions. This corresponds to the above special choice for $\bar{\psi}$.

[53] The conditions (16.37) are the counterparts of (13.74)–(13.75) in the restricted theory. We emphasize that (16.37) are conditions which should be imposed only on the arguments of the response functions.

the constitutive equations for $\tilde{N}^{\alpha\beta}$ and $\tilde{M}^{(\alpha\beta)}$ which can be obtained from (16.40) and the linearized version of (15.15).

17. The complete theory for thermoelastic shells: II. Derivation from the three-dimensional theory. On the basis of the three-dimensional equations of the classical continuum mechanics, two-dimensional kinematical results, field equations, entropy inequalities and related aspects of the subject were developed above for shell-like bodies in Sects. 7 and 11–12. Here we first obtain the corresponding two-dimensional constitutive relations and thermodynamical results for elastic materials and then briefly summarize the basic equations of the complete theory for thermo-elastic shells.

α) *Constitutive equations in terms of two-dimensional variables. Thermodynamical results.* We recall that in the three-dimensional theory of (non-polar) elastic materials the constitutive relations for the specific free energy, the entropy and the stress tensor can be expressed in the forms[54]

$$\psi^* = \hat{\psi}^*(\gamma_{ij}^*, \theta^*), \tag{17.1}$$

$$\eta^* = -\frac{\partial \hat{\psi}^*}{\partial \theta^*}, \tag{17.2}$$

$$\tau^{ij} = \varrho^* \frac{\partial \hat{\psi}^*}{\partial \gamma_{ij}^*}. \tag{17.3}$$

In addition, the constitutive equation for the heat flux vector has the form

$$q^{*k} = \hat{q}^{*k}(\gamma_{ij}^*, \theta^*, \theta^*{}_{,m}) \tag{17.4}$$

and the response function $\hat{q}^{*k}$ is restricted by the inequality

$$-\hat{q}^{*k}\,\theta^*{}_{,k} \geqq 0. \tag{17.5}$$

The results (17.2)–(17.3) and (17.5) are deduced from (11.23) and hold for every admissible process.[55] Also, with the help of (17.2)–(17.3), the residual energy equation is obtained from (11.15) and (11.18):

$$\varrho^* r^* - q^{*k}{}_{\|k} - \varrho^*\,\theta^*\,\dot{\eta}^* = 0. \tag{17.6}$$

We now proceed to deduce the two-dimensional counterparts of the above results in terms of variables defined in Sects. 11–12. To indicate the nature of the reduction, using the chain rule we first observe the relations:

$$\frac{\partial \psi^*}{\partial \boldsymbol{g}_k} = \frac{\partial \hat{\psi}^*}{\partial \gamma_{ij}^*}\frac{\partial \gamma_{ij}^*}{\partial \boldsymbol{g}_k} = \frac{\partial \hat{\psi}^*}{\partial \gamma_{ij}^*}\left[\frac{1}{2}\left(\delta_i^k\,\boldsymbol{g}_j + \delta_j^k\,\boldsymbol{g}_i\right)\right] = \frac{\partial \hat{\psi}^*}{\partial \gamma_{kj}^*}\boldsymbol{g}_j \tag{17.7}$$

and

$$\begin{gathered}\frac{\partial \psi^*}{\partial \boldsymbol{a}_\alpha} = \frac{\partial \hat{\psi}^*}{\partial \gamma_{ij}^*}\frac{\partial \gamma_{ij}^*}{\partial \boldsymbol{g}_k}\frac{\partial \boldsymbol{g}_k}{\partial \boldsymbol{a}_\alpha} = \frac{\partial \hat{\psi}^*}{\partial \gamma_{kj}^*}\boldsymbol{g}_j\,\delta_\alpha^k = \frac{\partial \hat{\psi}^*}{\partial \gamma_{\alpha j}^*}\boldsymbol{g}_j,\\ \frac{\partial \psi^*}{\partial \boldsymbol{d}_{N,\alpha}} = \frac{\partial \hat{\psi}^*}{\partial \gamma_{\alpha j}^*}\boldsymbol{g}_j\,\xi^N, \qquad \frac{\partial \psi^*}{\partial \boldsymbol{d}_N} = \frac{\partial \hat{\psi}^*}{\partial \gamma_{3j}^*}\boldsymbol{g}_j\,(N\,\xi^{N-1}),\\ \frac{\partial \psi^*}{\partial \varphi_M} = \frac{\partial \hat{\psi}^*}{\partial \theta^*}\xi^M.\end{gathered} \tag{17.8}$$

[54] The functions $\hat{\psi}^*$ in (17.1) and $\hat{q}^{*k}$ in (17.4) depend also on the reference values G_{ij}, although this is not exhibited in (17.1), (17.4) and elsewhere in this section. The partial derivative of a function with respect to a symmetric tensor such as that in (17.3) is understood to have the symmetric form

$$\frac{1}{2}\left(\frac{\partial \hat{\psi}^*}{\partial \gamma_{ij}^*} + \frac{\partial \hat{\psi}^*}{\partial \gamma_{ji}^*}\right).$$

[55] See Sects. 79–82 of TRUESDELL and NOLL [1965, 9].

The relation (17.7) follows directly from (17.1) when the argument γ_{ij}^* is expressed in terms of the base vectors $\boldsymbol{g}_i$ and $\boldsymbol{G}_i$ as in $(7.27)_1$ while those in (17.8) are obtained with the help of (7.10) and (11.42). Consider now the constitutive equations for the components $\tau^{\alpha i}$ in (17.3). Multiply both sides by $g^{\frac{1}{2}}\boldsymbol{g}_j$ and integrate the resulting expression with respect to ξ between the limits α, β to obtain

$$\int_\alpha^\beta \boldsymbol{T}^\alpha \, d\xi = \int_\alpha^\beta \varrho^* g^{\frac{1}{2}} \frac{\partial \hat{\psi}^*}{\partial \gamma_{\alpha j}^*} \boldsymbol{g}_j \, d\xi, \tag{17.9}$$

where by $(11.7)_2$ we have used $\boldsymbol{T}^\alpha = g^{\frac{1}{2}} \tau^{\alpha j} \boldsymbol{g}_j$. By $(17.8)_1$, (4.21) and $(11.44)_{3,4}$, the right-hand side of (17.9) can be reduced as follows:

$$\int_\alpha^\beta \varrho^* g^{\frac{1}{2}} \frac{\partial \hat{\psi}^*}{\partial \gamma_{\alpha j}^*} \boldsymbol{g}_j \, d\xi = \int_\alpha^\beta k \frac{\partial \hat{\psi}^*}{\partial \boldsymbol{a}_\alpha} d\xi = \frac{\partial}{\partial \boldsymbol{a}_\alpha} \int_\alpha^\beta k \, \hat{\psi}^* \, d\xi = \varrho a^{\frac{1}{2}} \frac{\partial \overline{\psi}}{\partial \boldsymbol{a}_\alpha}. \tag{17.10}$$

From $(11.36)_1$ and (17.9)–(17.10), it is easily seen that the stress-resultants $\boldsymbol{N}^\alpha$ are related to the partial derivatives $\partial \hat{\psi}^* / \partial \boldsymbol{a}_\alpha$ and similar relations can be deduced for other resultants. The relation

$$\overline{\psi} = \frac{1}{\varrho(\theta^\alpha)\, a^{\frac{1}{2}}(\theta^\alpha)} \int_\alpha^\beta k(\theta^\alpha, \xi)\, \hat{\psi}^*(\gamma_{\alpha\beta}^*, \gamma_{\alpha 3}^*, \gamma_{33}^*, \theta^*)\, d\xi, \tag{17.11}$$

where the arguments of $\hat{\psi}^*$ have the structure indicated by (7.28) and (11.42), defines the function $\overline{\psi}$ which occurs in (17.10).[56] Clearly, with the help of $(11.44)_{3,4}$, (17.11) may also be obtained from (17.1) and $\overline{\psi}$ can be regarded as a function of the variables

$$\boldsymbol{a}_\alpha, \ \boldsymbol{d}_N, \ \boldsymbol{d}_{N,\alpha}, \ \varphi_M \qquad (N = 1, 2, \ldots; M = 0, 1, 2, \ldots).$$

In a manner similar to that indicated above, from (7.28), (11.42), (11.28), (11.36), (11.44), (11.55), (17.7)–(17.8) and (17.1)–(17.3), by direct calculations we deduce the following two-dimensional constitutive equations:[57]

$$\psi = \overline{\psi}(\boldsymbol{a}_\alpha, \boldsymbol{d}_N, \boldsymbol{d}_{N,\gamma}, \varphi_M) \qquad (N = 1, 2, \ldots; M = 0, 1, 2, \ldots), \tag{17.12}$$

$$\eta^M = -\frac{\partial \overline{\psi}}{\partial \varphi_M} \qquad (M = 0, 1, 2, \ldots), \tag{17.13}$$

$$\left.\begin{aligned} \boldsymbol{N}^\alpha &= \varrho \frac{\partial \overline{\psi}}{\partial \boldsymbol{a}_\alpha}, \\ \boldsymbol{m}^N = \varrho \frac{\partial \overline{\psi}}{\partial \boldsymbol{d}_N}, \quad \boldsymbol{M}^{N\alpha} &= \varrho \frac{\partial \overline{\psi}}{\partial \boldsymbol{d}_{N,\alpha}} \qquad (N = 1, 2, \ldots). \end{aligned}\right\} \tag{17.14}$$

The above constitutive equations involve only the specific free energy ψ corresponding to $n = 0$ in the definition $(11.44)_3$ and an explanation is necessary as to why we have not used the expressions involving ψ^n $(n = 1, 2, \ldots)$. To elaborate, consider for example $(17.14)_3$ which is obtained from the constitutive relation for $\tau^{\alpha i}$ in (17.3) after multiplication of both sides of the equation by $\xi^N g^{\frac{1}{2}} \boldsymbol{g}_j$. However, if instead of $\xi^N g^{\frac{1}{2}} \boldsymbol{g}_j$ both sides of the equation are multiplied by $\xi^{N+n} g^{\frac{1}{2}} \boldsymbol{g}_j$ prior to integration, then

$$\boldsymbol{M}^{(N+n)\alpha} = \varrho \frac{\partial \overline{\psi}^n}{\partial \boldsymbol{d}_{N,\alpha}} \qquad (N = 1, 2, \ldots; n = 0, 1, 2, \ldots)$$

[56] The function $\hat{\psi}^*$ in (17.11) depends also on the reference values $G_{\alpha\beta}, G_{\alpha 3}, G_{33}$.

[57] The function $\overline{\psi}$ in (17.12) depends also on the reference values $\boldsymbol{A}_\alpha$, $\boldsymbol{D}_N$, $\boldsymbol{D}_{N,\gamma}$. These reference values, although not exhibited in (17.12), arise from the duals of (7.25) for the reference values $G_{\alpha\beta}, G_{\alpha 3}, G_{33}$ in the arguments of $\hat{\psi}^*$ in (17.11).

will result instead of $(17.14)_3$, where $\overline{\psi}^n$ depends on the variables which occur on the right-hand side of (17.12) and where for clarity we have again used the notation $M^{(N+n)\alpha}$ [as in (11.59)] instead of $M^{N+n\alpha}$ in (12.7). Similar results can be obtained for

$$\eta^{(n+M)}, \quad \frac{N}{n+N}\,\boldsymbol{m}^{(N+n)\alpha}, \quad \boldsymbol{N}^{n\alpha} \qquad \begin{pmatrix} n=0, 1, 2, \ldots \\ N=1, 2, \ldots; M=0, 1, 2, \ldots \end{pmatrix}$$

in terms of $\overline{\psi}^n$. It is clear that these constitutive relations for $n \geqq 1$ are redundant since those in (17.13)–(17.14) already hold for all integer values of N and M. Hence, it will suffice to consider only the set (17.12)–(17.14) as the appropriate two-dimensional constitutive equations.

The constitutive equations (17.1)–(17.4) are expressed in terms of response functions which depend, in particular, on the relative kinematic measures γ_{ij}^* and are specially useful when we obtain their linearized counterparts later in this chapter. Alternatively, the constitutive equations in nonlinear thermo-elasticity may be expressed as functions of g_{ij} (rather than γ_{ij}^*), apart from their dependence on thermal variables and on the reference values G_{ij}. Using a response function for the specific free energy which depends on g_{ij} (instead of γ_{ij}^*), we can readily record a relation similar to (17.11). In this latter relation, the arguments of $\hat{\psi}^*$ (now a different function) have the structure indicated by (7.25) and (11.42); and the response function for the two-dimensional specific free energy can be regarded as a different function of φ_M ($M=0, 1, 2, \ldots$) and the variables

$$a_{\alpha\beta}, \; d_{N\alpha}, \; \lambda_{N\beta\alpha}, \; \sigma_{NK}, \; \sigma_{NK\alpha}, \; \sigma_{NK\alpha\beta}, \qquad (N, K=1, 2, \ldots), \tag{17.15}$$

where the last five of the above are defined in (7.13)–(7.17). Introducing the notations

$$\sigma_N = \sigma_{1N} = d^3\, d_{N3}, \qquad \sigma_{N\alpha} = \sigma_{1N\alpha} = d^3\, \lambda_{N3\alpha} \qquad (N=1, 2, \ldots) \tag{17.16}$$

and using the expression for $\boldsymbol{a}_3$ in terms of $\boldsymbol{d}(=\boldsymbol{d}_1)$ and its components [see the equation preceding (13.34)], we can write $\boldsymbol{d}_N$ and its partial derivatives as

$$\boldsymbol{d}_N = d_{N\cdot}^{\;\alpha}\, \boldsymbol{a}_\alpha + d_{N\cdot}^{\;3}\, \boldsymbol{a}_3 = \left(d_{N\cdot}^{\;\nu} - \frac{d^\nu \sigma_N}{\sigma}\right)\boldsymbol{a}_\nu + \frac{\sigma_N}{\sigma}\,\boldsymbol{d},$$

$$\frac{\partial \boldsymbol{d}_N}{\partial \theta^\alpha} = \lambda_{N\cdot\alpha}^{\;\nu}\, \boldsymbol{a}_\nu + \lambda_{N\cdot\alpha}^{\;3}\, \boldsymbol{a}_3 = \left(\lambda_{N\cdot\alpha}^{\;\nu} - \frac{d^\nu \sigma_{N\alpha}}{\sigma}\right)\boldsymbol{a}_\nu + \frac{\sigma_{N\alpha}}{\sigma}\,\boldsymbol{d},$$

where $\sigma(=\sigma_1)$ is defined by $(7.17)_2$. We now make an observation parallel to that in Sect. 13 [between (13.32)–(13.33)] regarding the tangential components of $\boldsymbol{d}_N$ and $\boldsymbol{d}_{N,\alpha}$; and also, from the last two expressions, we note that the last three sets of variables in (17.15) can be expressed in terms of (17.16) and the first three of (17.15). It follows that σ_{NK} (for $N \geqq 2$), $\sigma_{NK\alpha}$ (for $N \geqq 2$) and $\sigma_{KN\alpha\beta}$ (for $N, K \geqq 1$) may be suppressed from the arguments of the (two-dimensional free energy) response function and that, instead of (17.15), the relevant variables are

$$\mathscr{V}_N\colon \; a_{\alpha\beta}, \; d_{N\alpha}, \; \lambda_{N\beta\alpha}, \; \sigma_N, \; \sigma_{N\alpha} \qquad (N=1, 2, \ldots). \tag{17.17}$$

Hence, we write[58]

$$\begin{aligned} \psi &= \overline{\psi}(\boldsymbol{a}_\alpha, \boldsymbol{d}_N, \boldsymbol{d}_{N,\gamma}, \varphi_M) \\ &= \tilde{\psi}(\mathscr{V}_N, \varphi_M) \qquad (N=1, 2, \ldots; M=0, 1, 2, \ldots), \end{aligned} \tag{17.18}$$

where $\mathscr{V}_N$ stand for the set of kinematic variables (17.17). As in (13.40)–(13.42), using the chain rule for differentiation, the constitutive equations (17.13)–

[58] As already noted with reference to (17.12), the function $\overline{\psi}$ depends also on the reference values $\boldsymbol{A}_\alpha$, $\boldsymbol{D}_N$, $\boldsymbol{D}_{N,\gamma}$. Similarly, the function $\tilde{\psi}$ in $(17.18)_2$ depends also on the reference values which are the duals of those in (17.17).

(17.14) can be reduced to a set in terms of $\tilde{\psi}$ and the tensor components of the resultants defined by (12.10)–(12.11) and (12.15).

Before proceeding further, we make an observation regarding the representation of the two-dimensional specific free energy function. As long as no kinematic approximation is introduced in the argument of $\hat{\psi}^*$ in (17.11), the two-dimensional function $\bar{\psi}$ (and therefore $\tilde{\psi}$) is exact. However, if instead of the exact expressions (7.28) the kinematic variables in the argument of the response function $\hat{\psi}^*$ are replaced by a set of approximate expressions, then neither $\bar{\psi}$ nor $\tilde{\psi}$ will necessarily reflect the structure of the original three-dimensional response function.[59] Because of this, if an approximation is contemplated, it is equally acceptable to obtain an approximate two-dimensional specific free energy from a function of the kinematic variables $a_{\alpha\beta}, d_{Ni}, \lambda_{Ni\alpha}$ in the form[60]

$$\psi = \psi'(a_{\alpha\beta}, d_{Ni}, \lambda_{Ni\alpha}, \varphi_M) \qquad (N=1, 2, \ldots; M=0, 1, 2, \ldots). \tag{17.19}$$

However, even though an approximation procedure will be considered in the next section, in what follows we retain the constitutive equation for ψ in the form (17.18) since the variables $\mathscr{V}_N$ in the argument of $\tilde{\psi}$ do reflect the structure of the leading terms in the expansion of g_{ij} and hence γ^*_{ij} [see (7.25)–(7.26) and (7.28)–(7.29)].

Returning to our main task in this section, instead of recording the two-dimensional constitutive equations in terms of $\tilde{\psi}$, for later convenience we express the two-dimensional specific free energy as a function of the relative kinematic measures

$$\mathscr{U}_N:\ e_{\alpha\beta}, \gamma_{N\alpha}, \varkappa_{N\beta\alpha}, s_N, s_{N\alpha} \qquad (N=1, 2, \ldots) \tag{17.20}$$

and the reference values

$$\begin{aligned} {}_R\mathscr{U}_N:\ & A_{\alpha\beta}, D_{N\alpha}, \Lambda_{N\beta\alpha}, \\ & D^3 D_{N3}, D^3 \Lambda_{N3\alpha} \qquad (N=1, 2, \ldots). \end{aligned} \tag{17.21}$$

The last two relative measures in (17.20) correspond to those in (17.16) and are given by

$$s_N = s_{1N} = \sigma_N - D^3 D_{N3}, \qquad s_{N\alpha} = s_{1N\alpha} = \sigma_{N\alpha} - D^3 \Lambda_{N3\alpha},$$

where $s_{1N}, s_{1N\alpha}$ and $s_1 (=s)$, $s_{1\alpha} (=s_\alpha)$ are defined by $(7.22)_{1,2}$ and $(7.24)_{1,2}$. Thus, with

$$\psi = \hat{\psi}(\mathscr{U}_N, \varphi_M; {}_R\mathscr{U}_N) \qquad (N=1, 2, \ldots; M=0, 1, 2, \ldots), \tag{17.22}$$

the constitutive equations corresponding to (17.13)–(17.14) are:

$$\eta^M = -\frac{\partial\psi}{\partial\varphi_M} \qquad (M=0, 1, 2, \ldots), \tag{17.23}$$

$$N'^{\alpha\beta} = N'^{\beta\alpha} = \varrho \frac{\partial\hat{\psi}}{\partial e_{\alpha\beta}}, \tag{17.24}$$

$$\begin{aligned} m^{N\alpha} &= \varrho \frac{\partial\hat{\psi}}{\partial\gamma_{N\alpha}}, \qquad m^{N3} = \varrho\left(d^3 \frac{\partial\hat{\psi}}{\partial s_N} + \delta^N_1 \sum_{M=1}^{\infty} \left\{\frac{\partial\hat{\psi}}{\partial s_M} d_M{}^3_{\cdot} + \frac{\partial\hat{\psi}}{\partial s_{M\alpha}} \lambda_M{}^3_{\cdot\alpha}\right\}\right), \\ M^{N\alpha\beta} &= \varrho \frac{\partial\hat{\psi}}{\partial\varkappa_{N\beta\alpha}}, \quad M^{N\alpha 3} = \varrho d^3 \frac{\partial\hat{\psi}}{\partial s_{N\alpha}} \qquad (N=1, 2, \ldots). \end{aligned} \tag{17.25}$$

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[59] In effect, the approximation will partially mask the manner of dependence of the two-dimensional free energy function on the detailed structure of γ^*_{ij} in (7.28).

[60] The form (17.19) is used in [1970, *2*]. Two-dimensional constitutive equations of the type (17.13)–(17.14), either in terms of $\tilde{\psi}$ or ψ', may also be obtained from combination of (11.59) and (11.64) or from (12.22). This manner of obtaining the two-dimensional constitutive equations will also include those in terms of $\tilde{\psi}^n$ or ψ'^n ($n \geqq 1$); but, as noted above, the expressions involving $\tilde{\psi}^n$ or ψ'^n ($n \geqq 1$) are redundant.

The results (17.23)–(17.25) are obtained from (17.22) and (17.18) by using the chain rule and the tensor components of the resultants defined by (12.10)–(12.11) and (12.15).

To complete our development, we now proceed to obtain the two-dimensional counterparts of the inequality (17.5) and the residual energy equation (17.6). Use of (11.42) and (11.56)–(11.57) followed by integration of (17.5) yields

$$-\sum_{r=0}^{n}\binom{n}{r}(-\alpha)^{n-r}\sum_{N=0}^{\infty}\left[\frac{N}{r+N}\,\varrho\,R^{r+N}\,\varphi_N+q^{r+N\alpha}\,\varphi_{N,\alpha}\right]\geqq 0 \qquad (n=0,1,2,\ldots), \tag{17.26}$$

where R^n and $q^{n\alpha}$, which require constitutive equations, depend on the variables in the argument of the function $\hat{\psi}$ in (17.22) or $\tilde{\psi}$ in (17.18) and $\varphi_{N,\alpha}$. Alternatively, observing that the inequality (17.5) is equivalent to

$$q^{*k}\,\Phi^{*}{}_{,k}\geqq 0,$$

instead of (17.26), we can deduce the equivalent inequalities

$$\sum_{r=0}^{n}\binom{n}{r}(-\alpha)^{n-r}\sum_{N=0}^{\infty}\left[\frac{N}{r+N}\,\varrho\,R^{r+N}\,\Phi_N+q^{r+N\alpha}\,\Phi_{N,\alpha}\right]\geqq 0 \qquad (n=0,1,2,\ldots). \tag{17.27}$$

In order to obtain the two-dimensional residual energy equations, we first multiply (17.6) by ξ^n and then integrate over a part $\mathscr{P}^*$. The resulting equation, after use of (11.42), $(11.44)_1$ and (11.56)–(11.58), becomes

$$\int_{\mathscr{P}}\varrho\,(r^n+R^n)\,d\sigma-\int_{\mathscr{P}}\varrho\sum_{M=0}^{\infty}\dot{\eta}^{n+M}\,\varphi_M\,d\sigma-\int_{\partial\mathscr{P}}h^n\,ds=0, \tag{17.28}$$

from which follow the residual energy equations

$$\varrho\,(r^n+R^n)-\varrho\sum_{M=0}^{\infty}\varphi_M\,\dot{\eta}^{n+M}-q^{n\alpha}{}_{|\alpha}=0 \qquad (n=0,1,2,\ldots). \tag{17.29}$$

The foregoing two-dimensional results are obtained by direct integration of (17.1)–(17.6) and with the use of various resultants defined in Sects. 11–12. Alternatively, by introducing suitable constitutive assumptions, the results (17.22)–(17.25), (17.27) and (17.29) can be deduced from the inequalities (12.22) and the energy equations (12.20).

β) *Summary of the basic equations in a complete theory.* In addition to the constitutive relations and thermodynamical results (17.22)–(17.29) or an equivalent set of results, the complete theory includes the equations of motion (12.13)–(12.14), Eqs. $(12.15)_{1,2}$ which give values for $N^{\beta\alpha}$ and $N^{\alpha 3}$ and the set of Eqs.[61] $(12.16)_{1,2}$. Earlier we observed that constitutive equations can also be deduced in terms of the functions $\bar{\psi}^n$ or $\tilde{\psi}^n$ $(n\geqq 1)$ [and therefore also in terms of $\hat{\psi}^n$ $(n\geqq 1)$], obtained from $(11.44)_3$; however, such results were not recorded since they are redundant. Keeping this in mind, we see that the set of Eqs. (12.16) is satisfied identically. As a result, there is some redundancy in the system of two-dimensional equations obtained for shell-like bodies and we summarize below the essential results.

The equations of motion are given by (12.13)–(12.15). The constitutive equations in terms of the function $\hat{\psi}$ have the forms (17.22)–(17.25). The constitutive

[61] Eqs. (12.15)–(12.16) arise from the symmetry of the stress tensor.

equations for $q^{0\alpha}$ and $q^{n\alpha}$ and R^n, $(n=1, 2, \ldots)$, depend on $\varphi_{M,\alpha}$ and the variables in the argument of the function $\hat{\psi}$ in (17.22). The functions R^n and $q^{n\alpha}$ satisfy the inequalities (17.26) and the residual energy equations are given by (17.29).

Special cases of the above development can be discussed in a manner somewhat analogous to the special theory considered in Sect. 15 [Subsect. α)]. In particular, the membrane theory can be obtained by assuming that the function $\hat{\psi}$ in (17.22) depends only on $e_{\alpha\beta}$ and φ_0 [or θ in the notation of (11.41)] and by introducing other appropriate specializations. The resulting theory, under isothermal conditions, will be of the same form as the nonlinear theory of elastic membranes discussed by GREEN and ADKINS.[62]

18. Approximation for thin shells: II. Developments from the three-dimensional theory. While the two-dimensional equations summarized in Sect. 17 [Subsect. β)] have been obtained systematically from the corresponding three-dimensional equations of thermoelasticity, they consist in an infinite set of equations for an infinite number of unknowns. The desirability of an approximation procedure for *thin* shells is, therefore, self-evident. Indeed, the need for a suitable approximative scheme in conjunction with the exact two-dimensional results deduced from the three-dimensional equations is already indicated in Sect. 7 [Subsects. α), δ)] and in Sect. 12 [Subsect. ε)]. Here, we first outline an approximation procedure suggested by GREEN and NAGHDI[63] and then make some remarks pertaining to the resulting approximate theory and other approximations.

α) *An approximation procedure.* Before introducing an approximation procedure, we recall that the initial position vector (without loss in generality) can always be taken in the form (7.30)–(7.31) so that, as indicated in (7.49), D_{Ni} and $\Lambda_{Ni\alpha}$ vanish for $N \geqq 2$. In addition, henceforth we specify $D_3 = D = 1$. Then, (7.48) and (7.58) hold and the reference values (17.21) reduce to

$$A_{\alpha\beta}, \quad -B_{\alpha\beta}. \tag{18.1}$$

We now assume that the free energy function $\hat{\psi}$ in (17.22) for sufficiently *thin* shells, can be represented by an approximate expression in terms of the kinematic variables

$$\begin{gathered} e_{\alpha\beta}, \ \gamma_\alpha, \ \varkappa_{(\beta\alpha)}, \\ s, \ s_\alpha, \end{gathered} \tag{18.2}$$

the reference values (18.1) and φ_N $(N=0, 1, 2, \ldots)$ only.[64] Thus, we set[65]

$$\psi = \psi(e_{\alpha\beta}, \gamma_\alpha, \varkappa_{(\beta\alpha)}, s, s_\alpha, \varphi_N) \qquad (N=0, 1, 2, \ldots) \tag{18.3}$$

approximately. The problem of how to determine the approximate form of ψ in (18.3) from $\hat{\psi}$ in (17.22) and therefore from (17.11) is not considered here.[66]

[62] GREEN and ADKINS [1960, *5*].

[63] [1970, *2*].

[64] At this stage it is not necessary to introduce an approximation for the temperature, which can be considered separately. The kinematic variables (18.2) correspond to those in (17.20) for $N=1$ only; however, in introducing the approximation, we also write $\varkappa_{(\beta\alpha)}$ in place of $\varkappa_{\beta\alpha} = \varkappa_{1\beta\alpha}$. The latter is motivated by the fact that the leading terms of $\gamma^*_{\alpha\beta}$ in the argument of $\hat{\psi}^*$ in (17.11) contain only the symmetric $\varkappa_{(\beta\alpha)}$.

[65] In (18.3) and throughout this section, we again use the same symbol for a function and its value. Although not explicitly exhibited, the dependence of the function ψ in (18.3) on the (initial) reference values (18.1) is understood. In view of (7.58), the variables γ_α and s_α in the argument of ψ can be replaced, respectively, by d_α and σ_α defined in (7.17). But we retain the set of variables (18.2).

[66] Except possibly in very special cases, it appears to be exceedingly difficult to calculate ψ in (18.3) from the free energy function $\hat{\psi}^*$ or $\hat{\psi}$ in (17.22).

It follows from (18.3) and (17.25) that

$$M^{N\alpha i}=0, \quad m^{Ni}=0 \quad (N\geqq 2),$$

approximately.[67] The equations of motion (12.14) for $N\geqq 2$ are then satisfied if we also specify (12.24). The remaining equations of motion are given by (12.28)–(12.30). The constitutive relation for η^N is still of the form (17.23) but with the function $\hat{\psi}$ replaced by that in (18.3). However, in place of (17.24)–(17.25) we now have

$$N'^{\alpha\beta}=N'^{\beta\alpha}=\varrho\frac{\partial\psi}{\partial e_{\alpha\beta}}, \quad m^{\alpha}=\varrho\frac{\partial\psi}{\partial\gamma_{\alpha}}, \quad M^{\alpha\beta}=M^{(\alpha\beta)}=\varrho\frac{\partial\psi}{\partial\varkappa_{(\beta\alpha)}}, \tag{18.4}$$

$$m^3=\varrho\left(2d^3\frac{\partial\psi}{\partial s}+\lambda^3_{.\alpha}\frac{\partial\psi}{\partial s_\alpha}\right), \quad M^{\alpha 3}=\varrho d^3\frac{\partial\psi}{\partial s_\alpha}, \tag{18.5}$$

where the notations of (12.27) have been used. As we have made no approximation so far about the temperature, the residual energy equations are still given by (17.29) and constitutive equations are required for $q^{n\alpha}$ and R^n. These constitutive equations must satisfy the inequalities (17.26).

Consider now the question of approximation for the temperature and the remaining thermal variables. In view of (11.41)–(11.42), one possibility is to make the approximation

$$\theta^*(\theta^\alpha,\xi,t)=\varphi_0(\theta^\alpha,t)+\xi\,\varphi_1(\theta^\alpha,t), \tag{18.6}$$

or adopt the more specialized approximation[68]

$$\theta^*(\theta^\alpha,\xi,t)=\theta(\theta^\alpha,t)=\varphi_0(\theta^\alpha,t). \tag{18.7}$$

In the former case we also assume that $q^{n\alpha}=0$, $R^n=0$ for $n\geqq 2$. Then, the approximate expression for the free energy ψ in (18.3) will depend only on φ_0 and φ_1 and the complete theory will involve only two energy equations and two entropy inequalities corresponding to $n=0, 1$ in (17.29) and (17.26). On the other hand, if the approximation (18.7) is adopted, we also assume that $q^{n\alpha}=0$, $R^n=0$ for $n\geqq 1$. The approximate expression for the free energy will now depend only on θ (in place of φ_N) and it follows from a relation of the form (17.23) that $\eta^N=0$ for $N\geqq 1$ and

$$\eta=-\frac{\partial\psi}{\partial\theta}, \tag{18.8}$$

where we use the notation of $(11.44)_2$. Moreover, in this case $r^n=0$ for $n\geqq 1$ and we have the single residual energy equation

$$\varrho\,r-\varrho\,\theta\,\dot{\eta}-q^{\alpha}{}_{|\alpha}=0 \tag{18.9}$$

for the determination of temperature and the single inequality

$$-q^\alpha\,\theta_{,\alpha}\geqq 0, \tag{18.10}$$

in place of (17.29) and (17.26). It is worth observing that if the approximation (18.7) was adopted at an earlier stage in place of (11.42), then there would have been no need to introduce the thermal resultants $\varepsilon^n, \eta^n, \psi^n, R^n$ for $n>0$; and, as a result, we would obtain a single energy equation and a single entropy in-

[67] These results, which were stated in (12.23), hold approximately, since they are obtained with the use of (18.3).

[68] The approximation (18.6) allows for temperature variation through the thickness of the shell while the approximation (18.7) accounts only for temperature variation on the surface $\xi=0$.

equality corresponding to (11.59) and (11.64) for $n=0$ only.[69] A similar remark can be made if (18.6) was adopted earlier in place of (11.42).

Apart from the approximation for the temperature, the constitutive equations of the above approximate theory are functions of the kinematic variables (18.2) and the reference values (18.1). We now adopt the approximation (18.7) for the temperature; and, for later reference and subsequent linearization, record the two-dimensional free energy as

$$\psi=\psi(e_{\alpha\beta}, \gamma_\alpha, \varkappa_{(\beta\alpha)}, s, s_\alpha, \theta), \tag{18.11}$$

where ψ in (18.11) is a different function from that in (18.3) and depends also on the reference values (18.1). The constitutive equations for $\eta, N'^{\alpha\beta}, m^\alpha, M^{(\alpha\beta)}, m^3, M^{\alpha 3}$ are still of the forms in (18.4)–(18.5) and (18.8) but with ψ now given by that in (18.11). Also the constitutive equation for q^α depends on $\theta, \theta_{,\alpha}$, the variables (18.2) and the reference values (18.1).[70] One further remark should be made regarding the nature of the equation of motion in the above approximation procedure, either with ψ given by the approximate expression (18.3) or (18.11). In considering the approximate equations of motion [Sect. 12, Subsect. ε)], it was observed that while $(12.16)_{1,2}$ are identities in an exact theory, they cannot in general be satisfied in an approximate theory. In addition to the remarks made following (12.31), we note that since the expression (18.3) [or (18.11)] for the free energy is no longer exact, we expect that some of the equations in $(12.16)_{1,2}$ may not be satisfied by the approximation procedure used in obtaining (12.28)–(12.30), (18.4)–(18.5) and related results. In fact, in view of (12.23) which follow from the constitutive relations and the approximate expression for the free energy, only equations corresponding to $n=1$ in $(12.16)_2$ are violated and these reduce to $(12.31)_2$.

β) *Approximation in the linear theory.* We summarize here the main results of the linear theory for thermoelastic shells based on the above approximation procedure but confine our attention to the case in which the free energy is specified by (18.11). Some aspects of linearized kinematics and the forms of the linearized field equations were discussed previously in Sect. 7 [Subsect. γ)] and Sect. 12 [Subsect. ζ)]. In particular, we recall that when the reference values $D_i, \Lambda_{i\alpha}$ are specified by (7.58), the variables s and s_α to $O(\varepsilon)$ become $2\gamma_3$ and $\varkappa_{3\alpha}$, respectively, as was noted in Sect. 7 [see the remarks following (7.58)]. Thus, when the components of $\boldsymbol{D}$ are specified by $(7.58)_{1,2}$, the distinction between the linearized kinematic variables corresponding to those in the argument of ψ in (18.11) and the set defined by (7.68), namely

$$e_{\alpha\beta}, \gamma_i, \varkappa_{(\beta\alpha)}, \varkappa_{3\alpha} \tag{18.12}$$

disappears. Also, the temperature θ is now measured from the reference temperature θ_0^* and is of $O(\varepsilon)$ [see Sect. 12, Subsect. ζ)].

The field equations and constitutive relations of the approximate theory can be linearized in the same way as their counterparts in the exact theory. In particular, the specific free energy of the approximate linear theory has the form[71]

$$\psi=\psi(e_{\alpha\beta}, \gamma_i, \varkappa_{i\alpha}, \theta), \tag{18.13}$$

[69] The approximation (18.7) for the temperature is employed by GREEN, LAWS and NAGHDI [1968, *4*] prior to consideration of constitutive equations.

[70] These constitutive equations of the approximate theory with the free energy specified by (18.11) are of the same form as those in (13.62)–(13.66), apart from an extra generality for the skew-symmetric part of $M^{\alpha\gamma}$ in $(13.65)_3$.

[71] In (18.13) and subsequent results for the approximate linear theory, we no longer emphasize the symmetry of $\varkappa_{\beta\alpha}$ and, for convenience, write $\varkappa_{i\alpha}$ for $\{\varkappa_{\beta\alpha}, \varkappa_{3\alpha}\}$. However, it is understood that $\varkappa_{\beta\alpha}$, as well as $M^{\alpha\beta}$ in the constitutive relation $(18.4)_3$ are symmetric.

where ψ is quadratic in the variables (18.12) and the temperature θ (measured from the reference temperature) and depends also on the reference values (18.1). It can be readily seen that the remaining constitutive equations upon linearization of (18.8), (18.4)–(18.5) and by virtue of the reduction of s, s_α mentioned above reduce to

$$\eta = -\frac{\partial \psi}{\partial \theta}, \tag{18.14}$$

$$N'^{\alpha\beta} = N'^{\beta\alpha} = \varrho_0 \frac{\partial \psi}{\partial e_{\alpha\beta}}, \quad m^i = \varrho_0 \frac{\partial \psi}{\partial \gamma_i}, \quad M^{\alpha i} = \varrho_0 \frac{\partial \psi}{\partial \varkappa_{i\alpha}}, \tag{18.15}$$

where η in (18.14) is measured from the reference entropy η_0^* and is of $O(\varepsilon)$ [see Sect. 12, Subsect. ζ)]. The approximate linear constitutive equations (18.13)–(18.15) have the same forms as (16.5)–(16.7), apart from an extra generality for the skew-symmetric part of $M^{\alpha\beta}$ in $(16.7)_3$.

Recalling (7.37)–(7.39) and the linearized version of (11.37), the definition of the heat flux resultant is now given by

$$Q^\alpha = \boldsymbol{q} \cdot \boldsymbol{A}^\alpha, \quad Q^\alpha = \int_{-h/2}^{h/2} \mu\, q^{*\alpha}\, d\zeta, \tag{18.16}$$

where Q^α are the contravariant components of the heat flux resultant $\boldsymbol{q}$ and are measured per unit length (in the reference configuration) per unit time. The stress-resultants, the stress couples and the equations of motion of the approximate linear theory are given by (12.42). The residual energy equation (18.9) now reduces to

$$\varrho_0\, r - \varrho_0\, \theta_0^*\, \dot{\eta} - Q^\alpha|_\alpha = 0, \tag{18.17}$$

where the vertical bar denotes covariant differentiation with respect to $A_{\alpha\beta}$. The constitutive relation for Q^α will be of a form similar to that stated in Sect. 16 and its coefficients will be restricted by an inequality in the form (18.10) but with q^α replaced by Q^α. Our above discussion of the approximate linear theory utilizes the approximation (18.7) for the temperature. A more general development can be pursued in the presence of the approximation (18.6) and has been given in some detail by GREEN and NAGHDI.[72]

19. An alternative approximation procedure in the linear theory: II. Developments from the three-dimensional theory. The central point in the approximation procedure outlined in Sect. 18 is the assumption that the specific free energy can be approximated by an expression of the form (18.11) or in the case of the linear theory by a function which is quadratic in the infinitesimal temperature θ and the variables (18.12). However, even in the linear theory an explicit form of the function ψ in (18.13) was not obtained from the full three-dimensional expression for the free energy so that the constitutive coefficients (of the linear theory) still remain arbitrary and unrelated to the elastic constants in the three-dimensional

[72] GREEN and NAGHDI [1970, *2*]. In the older literature on the linear theories of elastic shells and plates, thermal effects were generally confined to thermal stresses without full thermodynamical considerations. Within the scope of the classical plate theory and in the presence of steady state temperature distribution of the form (18.6), thermal stresses in plates have been discussed by MARGUERRE [1935, *1*] and by MELAN and PARKUS [1953, *1*]. A more general derivation of heat conduction equations in the linear theory of shells is given by BOLOTIN [1960, *1*] and is used in [1962, *6*] in connection with a formulation of non-isothermal elastokinetic problems of shallow shells; BOLOTIN's approximate equations do not account for thermo-mechanical coupling effects. The results given in [1970, *2*] with the use of approximation (18.6) for the temperature and upon the neglect of thermo-mechanical coupling effects reduce to those obtained by BOLOTIN [1960, *1*].

theory. In order to provide explicit constitutive relations (for the approximate linear theory) in which the coefficients are related to the elastic constants in the three-dimensional theory, we outline in this section an approximation procedure in terms of the specific *Gibbs free energy function*. This approximation procedure will be helpful in effecting an explicit derivation of the constitutive relations for thin plates and shells from the three-dimensional equations.[73]

Our developments in this and the next section are carried out in the context of elastostatic theories of shells and plates and are also confined to isothermal deformation and to isotropic materials. However, the latter limitations are not essential and are adopted in order to focus attention on the main features of the approximation procedure.

We recall that the constitutive equations of the isothermal linear theory of elasticity may be expressed in terms of the (three-dimensional) specific Gibbs free energy function φ^* in the form[74]

$$\gamma_{ij}^* = -\varrho_0^* \frac{\partial \varphi^*}{\partial \tau^{ij}}, \tag{19.1}$$

where the infinitesimal strain γ_{ij}^* is given by (7.60),

$$\varphi^* = \varphi^*(\tau^{ij}) = \psi^*(\gamma_{ij}^*) - \frac{1}{\varrho_0^*} \tau^{ij} \gamma_{ij}^* \tag{19.2}$$

and φ^* and ψ^* are quadratic functions of their arguments and both also depend upon the reference values G_{ij}. The Gibbs function φ^* (or the complementary energy function in the isothermal theory) for an initially homogeneous and isotropic material can be expressed as

$$\varrho_0^* \varphi^* = \left[-\frac{1+\nu}{2E} G_{im} G_{jn} + \frac{\nu}{2E} G_{ij} G_{mn} \right] \tau^{ij} \tau^{mn}, \tag{19.3}$$

where G_{ij} is the initial metric tensor defined in (4.23), E is Young's modulus of elasticity and ν is Poisson's ratio.[75]

Within the scope of the linear theory and remembering the remarks made following (7.46), corresponding to the resultant (11.43) and $(11.44)_3$ with $n=1$, we define

$$\varrho_0 \varphi A^{\frac{1}{2}} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \varrho_0^* \varphi^* G^{\frac{1}{2}} d\xi, \tag{19.4}$$

where φ is a two-dimensional Gibbs free energy (or a complementary energy in the isothermal theory). With the use of the linearized versions of $(11.7)_2$, (11.36), (11.55), (12.10)–(12.11), as well as (19.1)–(19.2) and (19.4), in a manner similar to the development in Sect. 17 we can show that the constitutive equations of

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[73] The approximation procedure in terms of the Gibbs function is proposed by GREEN, NAGHDI and WENNER [1971, *6*] and is used by them to obtain an approximate theory for plates of variable thickness from the three-dimensional equations. This procedure has some features in common with an approximation procedure employing a variational theorem in which assumptions are admitted simultaneously for both stresses and displacements.

[74] The partial derivative $\partial\varphi^*/\partial\tau^{ij}$ is understood to have the symmetric form

$$\frac{1}{2}\left(\frac{\partial \varphi^*}{\partial \tau^{ij}} + \frac{\partial \varphi^*}{\partial \tau^{ji}}\right).$$

[75] The use of the letter ν in (19.3) and other constitutive relations should not be confused with that in (7.34).

the linear isothermal theory may also be expressed in the forms

$$\begin{aligned}\varrho_0\,\varphi &= \varrho_0\,\overline{\varphi}(N'^{\alpha\beta}, M^{N\alpha i}, m^{Ni}) \\ &= \varrho_0\,\overline{\psi} - \left\{N'^{\alpha\beta}\,e_{\alpha\beta} + \sum_{N=1}^{\infty}(M^{N\alpha i}\,\varkappa_{Ni\alpha} + m^{Ni}\,\gamma_{Ni})\right\},\end{aligned} \tag{19.5}$$

$$e_{\alpha\beta} = -\varrho_0\frac{\partial\overline{\varphi}}{\partial N'^{\alpha\beta}}, \quad \varkappa_{Ni\alpha} = -\varrho_0\frac{\partial\overline{\varphi}}{\partial M^{N\alpha i}}, \quad \gamma_{Ni} = -\varrho_0\frac{\partial\overline{\varphi}}{\partial m^{Ni}}. \tag{19.6}$$

In (19.5), $\overline{\varphi}$ is a quadratic function of its arguments, $\overline{\psi}$ is a quadratic function of $e_{\alpha\beta}, \varkappa_{Ni\alpha}, \gamma_{Ni}$ and both $\overline{\psi}$ and $\overline{\psi}$ depend also on the reference values (18.1).

The constitutive equations (19.5)–(19.6), together with the linearized version of the equations of motion (12.13)–(12.16), form a system of infinite equations in an infinite number of unknowns. In parallel to the approximation procedure of Sect. 18, we now assume that the Gibbs free energy function $\overline{\varphi}$ can be represented by an approximate expression which is independent of $M^{N\alpha i}$, m^{Ni} for $N \geqq 2$. Thus, using the notations of (12.27), we set

$$\varphi = \tilde{\varphi}(N'^{\alpha\beta}, M^{\alpha i}, m^i) \tag{19.7}$$

approximately, where $\tilde{\varphi}$ is a different function from that in (19.5). Using (19.7) it follows from $(19.6)_{2,3}$ that

$$\gamma_{Ni} = 0, \qquad \varkappa_{Ni\alpha} = 0 \qquad (N \geqq 2),$$

approximately.[76] We also assume that in the linearized equations of equilibrium, $M^{N\alpha i}$, m^{Ni} and L^{Ni} (for $N \geqq 2$) can be neglected so that these equations reduce to (12.33)–(12.35) with D_i, $\Lambda_{i\alpha}$ given by (7.58) and with F^i, L^i in place of $\overline{F}^i$, $\overline{L}^i$, respectively. Moreover, as in the approximation procedure of Sect. 18, the linearized versions of the conditions (12.16) are satisfied approximately, except those for $n = 1$. These reduce to the forms (12.31), which may be violated in an approximate theory and about which we have already remarked in Sect. 18 and in Sect. 12 following (12.31). By means of the above procedure, the constitutive relations of the approximate theory are easily seen to be[77]

$$e_{\alpha\beta} = -\varrho_0\frac{\partial\tilde{\varphi}}{\partial N'^{\alpha\beta}}, \quad \varkappa_{i\alpha} = -\varrho_0\frac{\partial\tilde{\varphi}}{\partial M^{\alpha i}}, \quad \gamma_i = -\varrho_0\frac{\partial\tilde{\varphi}}{\partial m^i}, \tag{19.8}$$

where $\tilde{\varphi}$ is the function in (19.7). Alternatively, the constitutive equations (19.8) can be expressed in terms of the variables $e_{\alpha\beta}, \varrho_{(\beta\alpha)}, \varrho_{3\alpha}, \gamma_i$ [see (7.68)] and those in (12.42). Thus, writing $\varrho_{i\alpha}$ for $\{\varrho_{(\beta\alpha)}, \varrho_{3\alpha}\}$, we have

$$e_{\alpha\beta} = -\varrho_0\frac{\partial\hat{\varphi}}{\partial N'^{\alpha\beta}}, \quad \varrho_{i\alpha} = -\varrho_0\frac{\partial\hat{\varphi}}{\partial M^{\alpha i}}, \quad \gamma_i = -\varrho_0\frac{\partial\hat{\varphi}}{\partial V^i}, \tag{19.9}$$

where $\hat{\varphi}$ is a function of $N'^{\alpha\beta}$, $M^{\alpha i}$, V^i [and also depends on the reference values (18.1)]:

$$\varphi = \hat{\varphi}(N'^{\alpha\beta}, M^{\alpha i}, V^i). \tag{19.10}$$

Further conclusions may be obtained from the above procedure without additional assumptions. We recall again that the initial position vector can always be taken

[76] These results which were stated in (7.65) hold approximately, since they are obtained with the use of (19.7).

[77] Although in (19.7)–(19.8) and elsewhere in this section we have written $\varkappa_{i\alpha}$ for $\{\varkappa_{\beta\alpha}, \varkappa_{3\alpha}\}$ and $M^{\alpha i}$ for $\{M^{\alpha\beta}, M^{\alpha 3}\}$, it is understood that $\varkappa_{\beta\alpha}$ and $M^{\alpha\beta}$ in the constitutive relations $(19.8)_2$ and (19.7) are symmetric tensors. This is analogous to the notation adopted in (18.13) and (18.15).

in the form (7.30), with the initial values of D_i and $\Lambda_{i\alpha}$ specified by (7.58). Then, as discussed in Sect. 12 [Subsect. δ)], it follows from (7.65) and (7.59) that consistently with the above approximation procedure the displacement $\boldsymbol{u}^*$ has the form (7.67) and by (7.68) the relevant (two-dimensional) kinematic measures are either the set $e_{\alpha\beta}, \varkappa_{i\alpha}, \gamma_i$ or $e_{\alpha\beta}, \varrho_{i\alpha}, \gamma_i$.

20. Explicit constitutive equations for approximate linear theories of plates and shells: II. Developments from the three-dimensional theory. Using the approximation procedure of Sect. 19, in this section we obtain explicit forms for constitutive equations in approximate linear theories of thin plates and shells of uniform thickness h. Although our derivations are carried out in the context of elastostatic theories, some remarks regarding dynamical problems are also included. Moreover, we confine attention to isotropic materials but the extension to anisotropic materials will be essentially similar.

α) *Approximate constitutive equations for plates.* Prior to the calculation of an explicit form for the approximate expression (19.10), we need to dispose of certain preliminaries. For this purpose, recall (7.48) and let the surface loads on the bounding surfaces $\zeta = \pm h/2$ of the initially flat plate be specified by

$$\begin{aligned} \boldsymbol{P}'' &= p''^{\alpha} \boldsymbol{A}_\alpha - p'' \boldsymbol{A}_3 \quad \text{on} \quad \zeta = \frac{h}{2}, \\ \boldsymbol{P}' &= p'^{\alpha} \boldsymbol{A}_\alpha + p' \boldsymbol{A}_3 \quad \text{on} \quad \zeta = -\frac{h}{2}. \end{aligned} \tag{20.1}$$

Then, in the absence of three-dimensional body force $\boldsymbol{f}^*$, from resultants of the type (11.29)–(11.30) we find

$$\begin{aligned} &\varrho_0 F^\alpha = p''^\alpha + p'^\alpha, &&\varrho_0 F^3 = -p'' + p', \\ &\varrho_0 L^\alpha = \frac{h}{2}(p''^\alpha - p'^\alpha), &&\varrho_0 L^3 = -\frac{h}{2}(p'' + p'). \end{aligned} \tag{20.2}$$

The outward unit normals to the surfaces $\zeta = \pm h/2$ have the components $\{n_\alpha = 0, n_3 = \pm 1\}$. Hence, by (11.10) and $(11.7)_1$, the surface tractions on $\zeta = \pm h/2$ are

$$\begin{aligned} \tau^{3\alpha} &= \begin{Bmatrix} p''^\alpha \\ -p'^\alpha \end{Bmatrix} \quad \text{on} \quad \zeta = \pm \frac{h}{2}, \\ \tau^{33} &= \begin{Bmatrix} -p'' \\ -p' \end{Bmatrix} \quad \text{on} \quad \zeta = \pm \frac{h}{2}. \end{aligned} \tag{20.3}$$

In order to calculate an approximate expression for $\hat{\varphi}$ in the form (19.10), we need to introduce *suitable* assumptions for the stresses τ^{ij} in terms of the resultants $N^{\alpha\beta}$, $M^{\alpha i}$, V^i defined in (12.43). Also, in the calculation of $\hat{\varphi}$, we shall assume that the effect of surface loads is negligible and attempt to satisfy the following conditions as closely as possible: The stresses must satisfy (i) the definitions of the resultants in (12.43) and (ii) the boundary conditions (20.3) with $p', p'', p'^\alpha, p''^\alpha$ all zero.[78]

Since the two-dimensional equations governing the behavior of isotropic plates separate into those for bending and extensional theories, it is instructive

[78] We could delete (ii) and satisfy the boundary conditions in the presence of $p', p'', p'_\alpha, p''_\alpha$; but then the effect of the surface loads will appear in the constitutive relations. It may be noted that the effect of the surface loads is sometimes retained in the constitutive equations of the approximate linear theories of shells and plates. See, e.g., Sect. 7.7 of GREEN and ZERNA [1968, *9*].

to carry out the calculation for $\hat{\varphi}$ in (19.10) in two parts. Thus, employing again the same symbol for a function and its value (in line with our earlier notation), we write

$$\varphi = \varphi_e + \varphi_b, \tag{20.4}$$

where φ_e is associated with the extensional theory and φ_b with the bending theory, and calculate separately the approximate expressions for $\varrho_0\,\varphi_e$ and $\varrho_0\,\varphi_b$.

Consider first the case of bending and introduce the following approximate expressions for stresses:

$$\tau^{\alpha\beta} = \frac{6\,M^{\alpha\beta}}{h^2}\,\frac{\zeta}{h/2}, \qquad \tau^{\alpha 3} = \tau^{3\alpha} = \frac{3}{2h}\,V^\alpha\left[1 - \left(\frac{\zeta}{h/2}\right)^2\right], \qquad \tau^{33} = 0. \tag{20.5}$$

The expressions (20.5) meet the conditions (i) and (ii) stated above.[79] Introducing (20.5) into (19.3) and using the resulting expression in (19.4), we obtain

$$\begin{aligned}\varrho_0\,\varphi_b = {} & \frac{6}{E h^3}\left[\nu\,A_{\alpha\beta}\,A_{\gamma\delta} - \frac{1+\nu}{2}\,(A_{\alpha\gamma}\,A_{\beta\delta} + A_{\alpha\delta}\,A_{\beta\gamma})\right] M^{(\alpha\beta)}\,M^{(\gamma\delta)} \\ & - \frac{6\,(1+\nu)}{5Eh}\,A_{\alpha\beta}\,V^\alpha\,V^\beta,\end{aligned} \tag{20.6}$$

where we have written $M^{(\alpha\beta)}$ for $M^{\alpha\beta}$ in order to emphasize that $M^{\alpha\beta}$ in (20.6) is symmetric. The constitutive equations for $\varrho_{(\alpha\beta)}$ and γ_α can be obtained from (19.9) and (20.6). Thus,

$$\begin{aligned}\varrho_{(\alpha\beta)} &= \frac{12}{E h^3}\,[-\nu A_{\alpha\beta} A_{\gamma\delta} + (1+\nu)\,A_{\alpha\gamma}\,A_{\beta\delta}]\,M^{(\gamma\delta)}, \\ \gamma_\alpha &= \frac{12\,(1+\nu)}{5Eh}\,A_{\alpha\beta}\,V^\beta\end{aligned} \tag{20.7}$$

and there is no constitutive relation for $\varrho_{[\alpha\beta]}$ since the symmetric $M^{(\alpha\beta)}$ occurs in (20.6).

For the extensional theory of plates, we again assume that the effect of surface loads (in the constitutive relations) are negligible and write the following approximate expressions for stresses:

$$\tau^{\alpha\beta} = \frac{N^{\alpha\beta}}{h}, \qquad \tau^{\alpha 3} = \tau^{3\alpha} = \frac{15}{h^2}\,M^{\alpha 3}\left[\frac{\zeta}{h/2} - \left(\frac{\zeta}{h/2}\right)^3\right], \qquad \tau^{33} = \frac{V^3}{h}. \tag{20.8}$$

The expressions (20.8) meet the condition (i) above and also satisfy $(20.3)_1$ with $p'^\alpha = p''^\alpha = 0$. The approximate expression for φ_e can now be obtained from (19.3)–(19.4) and (20.8). Thus,

$$\begin{aligned}\varrho_0\,\varphi_e = {} & \frac{1}{2Eh}\left[\nu\,A_{\alpha\beta}\,A_{\gamma\delta} - \frac{1+\nu}{2}\,(A_{\alpha\gamma}\,A_{\beta\delta} + A_{\alpha\delta}\,A_{\beta\gamma})\right] N^{\alpha\beta}\,N^{\gamma\delta} \\ & + \frac{\nu}{Eh}\,A_{\alpha\beta}\,N^{\alpha\beta}\,V^3 - \frac{1}{2Eh}\,(V^3)^2 \\ & - \frac{120}{7}\,\frac{1+\nu}{E h^3}\,A_{\alpha\beta}\,M^{\alpha 3}\,M^{\beta 3}\end{aligned} \tag{20.9}$$

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[79] Our assumption for τ^{33} in (20.5) could be easily modified so as to satisfy $(20.3)_2$. But, as already mentioned, this would result in the effect of the surface loads appearing in the constitutive relations. We also note here that $(20.5)_{1,2}$ satisfy the local forms of the first two of the three-dimensional equations of equilibrium $\boldsymbol{T}^i{}_{,i} = 0$, rather than merely the integrated forms of equilibrium equations.

and the constitutive relations for $e_{\alpha\beta}$, $\varrho_{3\alpha}$, γ_3 are found from (19.9) and (20.9):

$$\begin{aligned} e_{\alpha\beta} &= \frac{1}{Eh}\left[-\nu\, A_{\alpha\beta}\, A_{\gamma\delta} + (1+\nu)\, A_{\alpha\gamma}\, A_{\beta\delta}\right] N^{\gamma\delta} - \frac{\nu}{Eh} A_{\alpha\beta}\, V^3, \\ \gamma_3 &= \frac{1}{Eh} V^3 - \frac{\nu}{Eh} A_{\alpha\beta}\, N^{\alpha\beta}, \\ \varrho_{3\alpha} &= \frac{240}{7}\,\frac{1+\nu}{Eh^3} A_{\alpha\beta}\, M^{\beta 3}. \end{aligned} \tag{20.10}$$

The constitutive equations (20.10) and (20.7) with the help of the dual of $(4.12)_6$, i.e., $A_{\alpha\gamma}\, A^{\gamma\beta} = \delta_\alpha^\beta$, can be easily inverted to give the relations

$$\begin{aligned} N^{\alpha\beta} &= (1-\nu)\, C\left\{\left[\frac{\nu}{1-2\nu} A^{\alpha\beta} A^{\gamma\delta} + A^{\alpha\gamma} A^{\beta\delta}\right] e_{\gamma\delta} + \frac{\nu}{1-2\nu} A^{\alpha\beta}\,\gamma_3\right\}, \\ V^3 &= \frac{1-\nu}{1-2\nu}\, C\left[(1-\nu)\,\gamma_3 + \nu\, A^{\alpha\beta}\, e_{\alpha\beta}\right], \\ M^{\alpha 3} &= \frac{7}{20}(1-\nu)\, B\, A^{\alpha\beta}\,\varrho_{3\beta} \end{aligned} \tag{20.11}$$

for the extensional theory and the relations

$$\begin{aligned} M^{(\alpha\beta)} &= B\left[\nu\, A^{\alpha\beta} A^{\gamma\delta} + (1-\nu)\, A^{\alpha\gamma} A^{\beta\delta}\right] \varrho_{(\gamma\delta)}, \\ V^\alpha &= \frac{5}{6}\mu\, h\, A^{\alpha\beta}\,\gamma_\beta \end{aligned} \tag{20.12}$$

for the bending theory. In (20.11)–(20.12), the shear modulus of elasticity μ and the coefficients C and B are given by

$$\mu = \frac{E}{2(1+\nu)}, \qquad C = \frac{Eh}{1-\nu^2}, \qquad B = \frac{Eh^3}{12(1-\nu^2)}. \tag{20.13}$$

Also, the approximate Helmholtz free energy function corresponding to $\hat{\varphi}$ in (19.10) can be calculated with the help of (19.5), (7.65), (20.9), (20.6) and (20.11)–(20.12). Thus

$$\begin{aligned} \psi &= \psi_e + \psi_b, \\ \varrho_0\,\psi_e &= \frac{1-\nu}{2}\, C\left\{\left[\frac{\nu}{1-2\nu} A^{\alpha\beta} A^{\gamma\delta} + \frac{1}{2}\left(A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma}\right)\right] e_{\alpha\beta}\, e_{\gamma\delta}\right. \\ &\quad \left. + \frac{1-\nu}{1-2\nu}(\gamma_3)^2 + \frac{2\nu}{1-2\nu} A^{\alpha\beta}\, e_{\alpha\beta}\,\gamma_3\right\} + \frac{7}{40}(1-\nu)\, B\, A^{\alpha\beta}\,\varrho_{3\alpha}\,\varrho_{3\beta}, \\ \varrho_0\,\psi_b &= \frac{1}{2} B\left[\nu\, A^{\alpha\beta} A^{\gamma\delta} + \frac{1-\nu}{2}\left(A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma}\right)\right] \varrho_{(\alpha\beta)}\,\varrho_{(\gamma\delta)} \\ &\quad + \frac{5}{12}\mu\, h A^{\alpha\beta}\,\gamma_\alpha\,\gamma_\beta. \end{aligned} \tag{20.14}$$

The constitutive relations (20.11), apart from the presence of V^3 and $M^{\alpha 3}$, correspond to the classical result for the extensional theory of isotropic plates.[80] The constitutive Eqs. (20.12) for equilibrium problems in the bending theory of

[80] The relation between $(20.11)_1$ and the constitutive equation for $N^{\alpha\beta}$ in the classical extensional theory (i.e., for generalized plane stress) will become apparent below in Subsect. β). The value of $\frac{7}{20}(1-\nu)\,B$ for the coefficient of $(20.11)_3$ is a consequence of our approximation (20.8) and was also obtained in [1971, *6*]. An explicit relation of the form $(20.11)_3$ has not been discussed previously in the literature and there is no evidence available at present regarding the reasonableness of $(20.11)_3$ as an approximate relation for $M^{\alpha 3}$.

plates were first derived by REISSNER, using a variational theorem.[81] This theory includes the effect of transverse shear deformation γ_α in the constitutive equations and requires the specification of three boundary conditions at each edge of the plate. These are either the displacement conditions[82]

$$\gamma_\alpha,\ u_3 \text{ on } \partial\mathfrak{S} \tag{20.15}$$

or the boundary conditions for the resultants (in the bending theory)

$${}_0\nu_\alpha\, M^{\alpha\beta},\ {}_0\nu_\alpha\, V^\alpha \quad \text{on } \partial\mathfrak{S}, \tag{20.16}$$

where $\partial\mathfrak{S}$ refers to the boundary $\mathfrak{S}$ of the middle surface $\xi=0$ of the plate and ${}_0\nu_\alpha$ are the components of the outward unit normal ${}_0\boldsymbol{\nu}$ to $\partial\mathfrak{S}$. This is in contrast to the classical bending theory of plates which, as will be seen below, requires only two boundary conditions at each edge of the plate.

A theory similar to REISSNER'S but for flexural vibrations of plates is developed by MINDLIN.[83] In MINDLIN'S development, the constitutive equation for the shear-stress resultant corresponding to $(20.12)_2$ is obtained as follows: In the early stage of his derivation he assumes a constitutive relation in the form

$$V^\alpha = \mu' \, h A^{\alpha\beta} \gamma_\beta, \qquad \mu' = \varkappa^2 \mu, \tag{20.17}$$

where $\varkappa^2$ (and therefore μ') is a constant but unspecified. Then, at the completion of his derivation by comparing the solution for the circular frequency of the thickness-shear vibration with the corresponding exact three-dimensional solution due to LAMB,[84] MINDLIN makes the identification $\varkappa^2 = \pi^2/12$ [a value which is very close to REISSNER'S $5/6$ in $(20.12)_2$].

β) *The classical plate theory. Additional remarks.* We recall that the constitutive relations in the three-dimensional theory of elasticity, referred to normal coordinates (4.25) with metric tensor components of the type in (7.37), may be expressed in the form

$$\begin{aligned}
\tau_{\alpha\beta} &= 2\mu \left[\frac{\nu}{1-\nu}\, \gamma^{*\lambda}_{\ \lambda}\, G_{\alpha\beta} + \gamma^*_{\alpha\beta}\right] + \frac{\nu}{1-\nu}\, \tau_{33}\, G_{\alpha\beta}, \\
\tau_{\alpha 3} &= 2\mu\, \gamma^*_{\alpha 3}, \\
\gamma^*_{33} &= -\frac{\nu}{1-\nu}\, \gamma^{*\lambda}_{\ \lambda} + \frac{1-2\nu}{1-\nu}\, \frac{1}{2\mu}\, \tau_{33}.
\end{aligned} \tag{20.18}$$

The constitutive equations of the classical theory of plates is generally obtained from integration of (20.18) or from the counterparts of (17.1) and (17.3) in the linear isothermal theory with $\hat{\psi}^*$ being a quadratic function of the linearized γ^*_{ij},

---

[81] REISSNER [1945, *2*]; see also [1944, *5*], [1947, *5*] and [1950, *5*]. A derivation of REISSNER'S theory of bending of plates from the three-dimensional equations of classical elasticity is given by GREEN [1949, *2*] and may also be found in GREEN and ZERNA [1968, *9*]. Some related additional references on the subject include the papers by BOLLE [1947, *1*], SCHÄFER [1952, *3*], NAGHDI and ROWLEY [1953, *2*] and ESSENBURG and NAGHDI [1958, *2*]. The last contains a derivation of constitutive equations [corresponding to (20.12)] for plates of variable thickness.

[82] The nature of the boundary conditions for the bending of plates in REISSNER'S theory, as well as those for the extensional theory, is clear from an examination of the linearized balance of energy or a corresponding virtual work principle.

[83] MINDLIN [1951, *1*]. A previous development for flexural vibrations of plates was given by UFLYAND [1948, *1*]. Related aspects of the subject are discussed in several papers by MINDLIN: E.g., [1951, *2*], [1955, *6*] and [1960, *8*]. For extensional vibrations of plates at moderate or high frequencies, reference may be made to the papers by MINDLIN [1960, *8*], [1961, *8*], [1963, *5*] where additional references are cited.

[84] LAMB [1917, *1*].

after introduction of the kinematic approximation (7.74) and certain additional assumptions. These assumptions pertain to (a) the neglect of the effect of "transverse shear deformation" due to $\gamma^*_{\alpha 3}$ and (b) the neglect of transverse normal stress τ_{33} in the constitutive equations. From the displacement approximation (7.74) and the assumption (a) follows the expression $(7.73)_1$ for β_α. In view of assumption (a) and $(20.18)_2$, it is clear that there will be no constitutive equation for the shear stress-resultant V^α defined in (12.43). Further, by assumption (b) and the definitions for $N^{\alpha\beta}$ and $M^{\alpha\beta}$ in (12.43), constitutive equations of the classical theory of plates can easily be obtained from integration of $(20.18)_1$.

Alternatively, the constitutive equations of the classical theory for $N^{\alpha\beta}$ and $M^{\alpha\beta}$ in (12.43) may be deduced in the following manner: The expression for the three-dimensional response function appropriate to the linear theory (mentioned above) can be calculated from $\varrho_0^* \hat{\psi}^*(\gamma^*_{ij}) = \frac{1}{2}\tau^{ij}\gamma^*_{ij}$, where $\hat{\psi}^*$ is a quadratic function of the infinitesimal strain tensor. Substitute for τ_{33} from $(20.18)_3$ into $(20.18)_1$ and introduce the resulting expression along with $(20.18)_2$ on the right-hand side of $\hat{\psi}^*$. Then, after invoking the assumptions (a) and (b) and using the displacement approximation (7.74), integration of $\hat{\psi}^*$ by means of the linearized version of a formula of the type (17.11) yields a two-dimensional function $\bar{\psi}$ which is quadratic in $e_{\alpha\beta}$ and $\varrho_{(\alpha\beta)}$ calculated from (7.74). Remembering the procedure by which the approximate constitutive equations of the linear theory were obtained in Sect. 18 and observing that the resultant V^α will not occur in the energy equation [as a consequence of assumption (a)], we are led to constitutive equations of the type (18.15) with ψ replaced by $\bar{\psi}$ for $N'^{\alpha\beta} = N^{\alpha\beta}$ and $M^{(\alpha\beta)}$.

The discussion in the preceding two paragraphs should indicate the manner in which the constitutive equations of the classical theory of plates are usually derived. Here, in keeping with our development in Subsect. α), we obtain the constitutive equations of the classical plate theory by the approximation method of Sect. 19. For this purpose, however, we must return to (20.5), (20.8) and modify the assumptions for stresses. With reference to constitutive equations of the type (19.9) in terms of a two-dimensional Gibbs free energy function (or a complementary energy function in the isothermal theory) and in view of the remarks made following (7.74), since $\gamma_i = 0$ in the classical theory, it follows from $(19.9)_{2,3}$ that $\hat{\varphi}$ in (19.10) cannot depend on V^α, $M^{\alpha 3}$ and V^3. Hence, consistently with this observation, we introduce the following assumptions for stresses:[85]

$$\begin{aligned} \tau^{\alpha\beta} &= \frac{N^{\alpha\beta}}{h} + \frac{6M^{\alpha\beta}}{h^2}\frac{\zeta}{h/2}, \\ \tau^{\alpha 3} &= 0, \qquad \tau^{33} = 0. \end{aligned} \tag{20.19}$$

Using the assumptions (20.19), from (19.4) we calculate the approximate expressions

$$\begin{aligned} \varphi &= \varphi_e + \varphi_b, \\ \varrho_0\,\varphi_e &= \frac{1}{2Eh}\left[\nu A_{\alpha\beta}A_{\gamma\delta} - \frac{1+\nu}{2}(A_{\alpha\gamma}A_{\beta\delta} + A_{\alpha\delta}A_{\beta\gamma})\right] N^{\alpha\beta}N^{\gamma\delta}, \\ \varrho_0\,\varphi_b &= \frac{6}{Eh^3}\left[\nu A_{\alpha\beta}A_{\gamma\delta} - \frac{1+\nu}{2}(A_{\alpha\gamma}A_{\beta\delta} + A_{\alpha\delta}A_{\beta\gamma})\right] M^{(\alpha\beta)}M^{(\gamma\delta)} \end{aligned} \tag{20.20}$$

[85] The approximate expression $(20.19)_1$ represents combined contribution to both stretching and bending of a plate. The first and the second term on the right-hand side of $(20.19)_1$ are the same as $(20.8)_1$ and $(20.5)_1$, respectively.

and also obtain

$$
\begin{aligned}
e_{\alpha\beta} &= \frac{1}{E h} \left[-\nu A_{\alpha\beta} A_{\gamma\delta} + (1+\nu) A_{\alpha\gamma} A_{\beta\delta} \right] N^{\gamma\delta}, \\
\varrho_{(\alpha\beta)} &= \frac{12}{E h^3} \left[-\nu A_{\alpha\beta} A_{\gamma\delta} + (1+\nu) A_{\alpha\gamma} A_{\beta\delta} \right] M^{(\gamma\delta)}.
\end{aligned} \tag{20.21}
$$

The above results can be easily inverted to yield

$$
\begin{aligned}
N^{\alpha\beta} &= C \left[\nu A^{\alpha\beta} A^{\gamma\delta} + (1-\nu) A^{\alpha\gamma} A^{\beta\delta} \right] e_{\gamma\delta}, \\
M^{(\alpha\beta)} &= B \left[\nu A^{\alpha\beta} A^{\gamma\delta} + (1-\nu) A^{\alpha\gamma} A^{\beta\delta} \right] \varrho_{(\gamma\delta)}.
\end{aligned} \tag{20.22}
$$

Similarly, the approximate expression for the Helmholtz free energy function in the classical theory can be calculated from (19.5) and in a manner similar to that in (20.14). Thus, for the classical theory,

$$
\begin{aligned}
\psi &= \psi_e + \psi_b, \\
\varrho_0 \psi_e &= \frac{1}{2} C \left[\nu A^{\alpha\beta} A^{\gamma\delta} + \frac{1-\nu}{2} (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma}) \right] e_{\alpha\beta} e_{\gamma\delta}, \\
\varrho_0 \psi_b &= \frac{1}{2} B \left[\nu A^{\alpha\beta} A^{\gamma\delta} + \frac{1-\nu}{2} (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma}) \right] \varrho_{(\alpha\beta)} \varrho_{(\gamma\delta)}.
\end{aligned} \tag{20.23}
$$

The constitutive equations $(20.21)_{1,2}$ or equivalently $(20.22)_{1,2}$ are those for the classical extensional theory (corresponding to generalized plane stress) and the classical bending theory of plates. The boundary conditions in the classical theory are four at each edge of the plate, two for the extensional theory and two for the bending theory. For the extensional theory they are specified by[86]

$$
u_\alpha \text{ or } {}_0\nu_\alpha N^{\alpha\beta} \quad \text{on } \partial\mathfrak{S} \tag{20.24}
$$

and for the bending theory by

$$
\begin{gathered}
u_3, \ \frac{\partial u_3}{\partial \nu_0} \\
\text{or} \\
M^{(\alpha\gamma)} {}_0\nu_\alpha {}_0\nu_\gamma, \ {}_0\nu_\alpha M^{(\beta\alpha)}{}_{|\beta} - \frac{\partial}{\partial s_0} \left[\bar{\varepsilon}_{\beta\gamma} M^{(\alpha\beta)} {}_0\nu_\alpha {}_0\nu^\gamma \right] \quad \text{on } \partial\mathfrak{S},
\end{gathered} \tag{20.25}
$$

where $\partial\mathfrak{S}$ refers to the boundary of the middle surface $\xi = 0$ of the plate and $\partial/\partial \nu_0$, $\partial/\partial s_0$ denote the directional derivatives along the normal and the tangent to the boundary curve $\partial\mathfrak{S}$, respectively. The Kirchhoff boundary conditions $(20.25)_{3,4}$ should be compared with the boundary conditions (20.16) in REISSNER'S theory.[87] In this connection, we may recall that the differential equations of the classical bending theory (which requires the specification of two boundary conditions at each edge of the plate) can be reduced to the fourth-order partial differential equation

$$
\nabla^4 u_3 = \frac{p}{B}, \tag{20.26}
$$

[86] These boundary conditions may be derived in a manner similar to those for the restricted theory in Sect. 15.

[87] The classical theory of plates, prior to KIRCHHOFF'S paper [1850, *1*], was considered by POISSON (1828) and by CAUCHY (1828). References to these as well as some remarks on the history of the subject may be found in LOVE'S [1944, *4*] *Historical Introduction*. An account of the controversy over POISSON'S boundary conditions, KIRCHHOFF'S derivation of plate theory with correct boundary conditions and KELVIN and TAIT'S interpretation (1867) of KIRCHHOFF'S boundary conditions can be found in TODHUNTER and PEARSON [1886, *1*], Sects. 488–489 and [1893, *3*], Sects. 1238–1239.

where ∇^2 is the two-dimensional Laplacian and where we have used the notation $p = \varrho_0 F^3$ in place of that in $(20.2)_2$.[88] In contrast to (20.26), the differential equations of REISSNER's theory (which requires the specification of three boundary conditions at each edge of the plate) are equivalent to a system of sixth-order partial differential equations.

With a view toward an assessment of the effect of transverse shear deformation, we note that the system of differential equations in REISSNER's theory of bending of plates is characterized by the appropriate equilibrium equations in (12.43) and the constitutive equations (20.12). While this system of equations after elimination of V^α includes as a special case (with $\gamma_\alpha = 0$) the corresponding system of differential equations in KIRCHHOFF's classical theory or equivalently the fourth-order differential equation (20.26), the practical significance of the former should not be exaggerated. Indeed, it is well known that for most purposes the solution of the simpler Eq. (20.26) compares favorably with the solution for the displacement u_3 of the corresponding equilibrium boundary-value problem in REISSNER's theory. On the other hand, the additional ingredients in REISSNER's theory (including the presence of three boundary conditions instead of two) may be significant in certain circumstances. Roughly speaking, the effect of transverse shear deformation may give rise to a noticeable contribution, if the equilibrium boundary-value problem in question involves a geometrical parameter which has the dimension of length and which is smaller than any other characteristic length on the middle plane of the plate but not so small as to be of the same order of magnitude as the plate thickness. Such a parameter may arise due to the presence of a hole in an infinite plate, from a distributed load acting over a small area on the middle plane of the plate or when a small portion of the surface is constrained against normal displacement. Some idea along these lines can be gained from a number of existing solutions in the literature;[89] and, in particular, from REISSNER's solution of the "stress concentration" (or rather stress-couple concentration) problem of an infinite plate with a circular hole subjected to a uniform state of plain bending (and twist) at infinity.[90]

The above discussion regarding the effect of transverse shear deformation is limited to elastostatic flexural problems. For dynamical problems of plates, where a parameter corresponding to wavelength is always present, the significance of the effect of transverse shear deformation (and rotatory inertia) for moderate frequencies has been brought out in extensive studies by MINDLIN and others.[91]

γ) *Approximate constitutive relations for thin shells.* We obtain here a system of constitutive relations for thin elastic shells using the approximation procedure

[88] An account of the classical bending theory of plates can be found in LOVE's treatise [1944, *4*] or in the books by NADAI [1925, *1*] and TIMOSHENKO and WOINOWSKY-KRIEGER [1959, *7*].

[89] Among such solutions we mention those in [1945, *2*], [1953, *2*], [1956, *1*] and [1962, *2*].

[90] REISSNER [1945, *2*]. For a ratio of hole radius to plate thickness $a/h < 5$, the results (especially the stress-couple concentration factor) differ substantially from the solution of the corresponding problem according to the classical theory by GOODIER [1936, *1*]. In this connection, the discussions of the solution in [1945, *2*] by DRUCKER [1946, *1*] and by GOODIER [1946, *2*] are particularly interesting. The solution for the *stress-couple concentration* problem in [1945, *2*] should not be confused with the solution of the corresponding *stress concentration* problem within the scope of the three-dimensional theory. The latter three-dimensional problem has been considered by ALBLAS [1957, *1*] on the basis of an exact formulation of plate theory given by GREEN [1949, *3*]. The work of ALBLAS also includes a comparison with REISSNER's solution.

[91] For example, MINDLIN [1951, *1*], [1960, *8*], MINDLIN and ONOE [1957, *3*] and MIKLOWITZ [1960, *7*]. Additional references may be found in [1960, *8*] and in TIERSTEN's monograph [1969, *5*] on vibrations of piezoelectric plates.

of Sect. 19. Again our derivation is limited to elastostatic problems and, for simplicity, attention is confined to isotropic materials. As in Subsect. α), let the surface loads on

$$\zeta = \pm \frac{h}{2}$$

be specified by (20.1). Then, F^i, L^i are given by (20.2) and the boundary conditions on

$$\zeta = \pm \frac{h}{2}$$

are specified by (20.3).

Again, we calculate an approximate expression for φ in the form (19.10) and this requires the introduction of *suitable* assumptions for the stresses τ^{ij} in terms of the resultants $N^{\alpha\beta}$, $M^{\alpha i}$, V^i in (12.42). In this connection, we assume that the effect of surface load is negligible in the evaluation of $\hat{\varphi}$ and attempt to satisfy the conditions (i) and (ii) stated above [in Subsect. α)] as closely as possible. With this background and remembering the definitions of the stress-resultants and the stress-couples in (12.42), we write the following approximate expressions for the stresses:

$$\begin{aligned}
\mu\,\tau^{\alpha\gamma}\,\mu_\gamma^\beta &= \frac{N^{\alpha\beta}}{h} + \frac{6M^{\alpha\beta}}{h^2}\,\frac{\zeta}{h/2},\\
\mu\,\tau^{\alpha 3} = \mu\,\tau^{3\alpha} &= \frac{3V^\alpha}{2h}\left[1-\left(\frac{\zeta}{h/2}\right)^2\right] + \frac{15}{h^2}M^{\alpha 3}\left[\frac{\zeta}{h/2}-\left(\frac{\zeta}{h/2}\right)^3\right],\\
\mu\,\tau^{33} &= \frac{V^3}{h} + \zeta\,B_{\alpha\beta}\left(\frac{N^{\alpha\beta}}{h} + \frac{6M^{\alpha\beta}}{h^2}\,\frac{\zeta}{h/2}\right).
\end{aligned} \tag{20.27}$$

The expressions (20.27) meet the condition (i) above, and they satisfy the boundary conditions $(20.3)_1$ with $p'^\alpha = p''^\alpha = 0$. However, as in the case of flat plates, the boundary conditions $(20.3)_2$ with $p' = p'' = 0$ are not satisfied.[92] In view of the expression for $N'^{\alpha\beta}$ in (12.42), $(20.27)_1$ becomes

$$\mu\,\tau^{\alpha\gamma}\,\mu_\gamma^\beta = \frac{N'^{\alpha\beta}}{h} - \frac{M^{\gamma\alpha}}{h}\,B_\gamma^\beta + \frac{6M^{\alpha\beta}}{h^2}\,\frac{\zeta}{h/2}. \tag{20.28}$$

Since $D=1$ in the present development, $\zeta = \xi$ by (7.40). Hence, use of (7.37)–(7.39) in (19.3) yields

$$\begin{aligned}
\mu\,\varrho_0^*\,\varphi^* = &-\frac{1+\nu}{2E}\,\mu^{-1}\big[A_{\delta\nu}\,A_{\theta\varrho}\,\mu_\alpha^\delta\,\mu_\beta^\nu(\mu\,\tau^{\beta\gamma}\,\mu_\gamma^\varrho)\,(\mu\,\tau^{\alpha\lambda}\,\mu_\lambda^\theta)\\
&+2A_{\delta\nu}\,\mu_\alpha^\delta\,\mu_\beta^\nu(\mu\,\tau^{\alpha 3})\,(\mu\,\tau^{\beta 3}) + (\mu\,\tau^{33})^2\big]\\
&+\frac{\nu}{2E}\,\mu^{-1}\big[A_{\delta\nu}\,\mu_\alpha^\delta(\mu\,\tau^{\alpha\beta}\,\mu_\beta^\nu) + \mu\,\tau^{33}\big]^2.
\end{aligned} \tag{20.29}$$

In writing (20.29) no assumption is introduced beyond those already made in (20.27). However, before calculating φ from (19.4), we invoke the condition (4.31) and also make the following approximations in (20.29):

$$\begin{aligned}
&\frac{1}{\mu} \cong 1, \quad \mu_\beta^\alpha \cong \delta_\beta^\alpha,\\
&\mu\,\tau^{\alpha\gamma}\,\mu_\gamma^\beta \cong \frac{N'^{\alpha\beta}}{h} + \frac{6M^{(\alpha\beta)}}{h^2}\,\frac{\zeta}{h/2},\\
&\mu\,\tau^{33} \cong \frac{V^3}{h},
\end{aligned} \tag{20.30}$$

[92] We can modify $(20.27)_3$ so as to satisfy the boundary conditions $(20.3)_2$ but then the effect of surface loads will appear in the constitutive relations. See [1957, 4].

where in obtaining $(20.30)_3$ the term

$$\frac{M^{\gamma\alpha}}{h}\, B^\beta_\gamma, \text{ which is } O[(h\, B^\gamma_\beta)] \text{ times } \frac{6M^{\alpha\beta}}{h^2}\frac{\zeta}{h/2},$$

has been neglected in view of (4.31). Also, since by $(20.30)_{1,2}$ the left-hand side of $(20.30)_3$ is now $\mu\,\tau^{\alpha\gamma}\,\mu^\beta_\gamma \cong \tau^{\alpha\beta} = \tau^{\beta\alpha}$, we have replaced $M^{\alpha\beta}$ in (20.30) by its symmetric part so that

$$M^{[\alpha\beta]} = 0. \tag{20.31}$$

The approximation (20.30) need not be made at this stage and could be postponed: Since μ^{-1} can be expressed as a convergent series in[93] ζ, (20.29) can be integrated without the introduction of the approximation (20.30). This is however a lengthy procedure and since approximations are eventually introduced after the integration, the development is tantamount to adopting (20.30).[94]

Using (20.29)–(20.30) and $(20.27)_2$ in (19.4) yields the approximate expression

$$\begin{aligned} 2\varrho_0\,\varphi = {} & \frac{1}{Eh}\left[\nu A_{\alpha\beta}\,A_{\gamma\delta} - (1+\nu)\,A_{\alpha\delta}\,A_{\beta\gamma}\right] N'^{\alpha\beta}\,N'^{\gamma\delta} \\ & + \frac{2\nu}{Eh}\,A_{\alpha\beta}\,N'^{\alpha\beta}\,V^3 - \frac{1}{Eh}\,(V^3)^2 - \frac{240(1+\nu)}{7Eh^3}\,A_{\alpha\beta}\,M^{\alpha3}\,M^{\beta3} \\ & + \frac{12}{Eh^3}\left[\nu A_{\alpha\beta}\,A_{\gamma\delta} - (1+\nu)\,A_{\alpha\delta}\,A_{\beta\gamma}\right] M^{(\alpha\beta)}\,M^{(\gamma\delta)} - \frac{6}{5\mu h}\,A_{\alpha\beta}\,V^\alpha\,V^\beta. \end{aligned} \tag{20.32}$$

The remaining constitutive relations, obtained by direct calculation from (19.9) and (20.32), are:

$$\begin{aligned} e_{\alpha\beta} &= \frac{1}{Eh}\{[-\nu A_{\alpha\beta}\,A_{\gamma\delta} + (1+\nu)\,A_{\alpha\gamma}\,A_{\beta\delta}]\,N'^{\gamma\delta} - \nu A_{\alpha\beta}\,V^3\}, \\ \gamma_\alpha &= \frac{6}{5\mu h}\,A_{\alpha\beta}\,V^\beta, \quad \gamma_3 = \frac{1}{Eh}\,[V^3 - \nu A_{\alpha\beta}\,N'^{\alpha\beta}], \\ \varrho_{(\alpha\beta)} &= \frac{12}{Eh^3}\,[-\nu A_{\alpha\beta}\,A_{\gamma\delta} + (1+\nu)\,A_{\alpha\gamma}\,A_{\beta\delta}]\,M^{(\gamma\delta)}, \\ \varrho_{3\alpha} &= \frac{240(1+\nu)}{7Eh^3}\,A_{\alpha\beta}\,M^{\beta3}. \end{aligned} \tag{20.33}$$

From comparison of (20.32)–(20.33) with corresponding results in the case of flat plates, it is easily seen that the approximate expression for φ in (20.32) is of the same form as the combination of φ_e and φ_b in (20.9) and (20.6) and that the constitutive equations in (20.33) are of the same forms as those in (20.10) and (20.7). It follows that the inverted constitutive relations (20.33) will be of the same form as those in (20.11)–(20.12) but with $N^{\alpha\beta}$ replaced by $N'^{\alpha\beta}$. Also, the approximate expression for the Helmholtz free energy ψ obtained from (19.5) and (20.32) will be of the same form as (20.14). We postpone further remarks on the constitutive relations (20.33) or their inverted forms and consider next a derivation of the constitutive equations of the classical theory.

δ) *Classical shell theory. Additional remarks.* It was indicated in Sect. 7 [Subsect. ε)] that the displacement vector usually employed in the derivations of the

[93] Such convergent series representation for μ^{-1} and $\mu^{-1}{}^\alpha_\beta$ are given by NAGHDI [1963, *6*].

[94] In essence the approximation (20.30) will yield an energy function without coupling terms, i.e., one which has the same form as that for a flat plate. If the lengthier procedure is adopted and if the approximation of the type (20.30) is adopted after the integration, then the consequences of the two derivations will be the same.

classical theory (under Kirchhoff-Love assumptions) has the form (7.75). In order to obtain the constitutive relations of the classical theory by the procedure of this section, we must return to (19.4) and introduce a different set of approximate expressions for stresses. Since the classical theory ignores the effect of transverse shear deformation and the transverse normal stress in the constitutive equations, instead of (20.27) we now write

$$\begin{aligned} \mu\, \tau^{\alpha\gamma}\, \mu_\gamma^\beta &= \frac{N^{\alpha\beta}}{h} + \frac{6M^{\alpha\beta}}{h^2}\,\frac{\zeta}{h/2}, \\ \tau^{\alpha 3} &= \tau^{33} = 0. \end{aligned} \tag{20.34}$$

We can then calculate the expression for $\hat{\varphi}$ in a manner analogous to that in Subsect. β): First $\mu\,\varrho_0\,\varphi^*$ can be calculated using (19.3), (7.37)–(7.39) and (7.48). Next we introduce the assumptions (20.34) and use the approximation $(20.30)_{1,2}$ to obtain the function φ for the classical theory. This function will have the same form as that in (20.20) but with $N^{\alpha\beta}$ replaced by $N'^{\alpha\beta}$ and $\varrho_{(\alpha\beta)}$ by that for shells in (7.68). For ease of reference, we record below these constitutive relations in the form

$$\begin{aligned} N'^{\alpha\beta} &= C\,[\nu A^{\alpha\beta} A^{\gamma\delta} + (1-\nu)\, A^{\alpha\gamma} A^{\beta\delta}]\, e_{\gamma\delta}, \\ M^{(\alpha\beta)} &= B\,[\nu A^{\alpha\beta} A^{\gamma\delta} + (1-\nu)\, A^{\alpha\gamma} A^{\beta\delta}]\, \varrho_{(\gamma\delta)}, \\ e_{\gamma\delta} &= \tfrac{1}{2}(u_{\gamma|\delta} + u_{\delta|\gamma}) - B_{\gamma\delta}\, u_3, \qquad \varrho_{(\gamma\delta)} = -\bar{\varkappa}_{\gamma\delta}, \end{aligned} \tag{20.35}$$

where $\bar{\varkappa}_{\gamma\delta}$ is given by $(7.68)_9$.

The above linear constitutive equations of the classical theory involve the kinematic variables $e_{\alpha\beta}$ and $\varrho_{(\beta\alpha)}$. They can also be deduced from expressions of the type (18.15), namely

$$N'^{\alpha\beta} = N'^{\beta\alpha} = \varrho_0 \frac{\partial \psi}{\partial e_{\alpha\beta}}, \qquad M^{(\alpha\beta)} = \varrho_0 \frac{\partial \psi}{\partial \varrho_{(\beta\alpha)}}, \tag{20.36}$$

where ψ is a function different from that in (18.15). In fact, as in (20.23), ψ in (20.36) is a quadratic function of $e_{\alpha\beta}$ and $\varrho_{(\beta\alpha)}$. We consider now an alternative form of the constitutive equations of the classical theory employing a slightly different set of kinematic variables. For this purpose, put

$$\bar{\varrho}_{(\alpha\beta)} = \varrho_{(\alpha\beta)} + \tfrac{1}{2}(B_\alpha^\nu\, e_{\nu\beta} + B_\beta^\nu\, e_{\nu\alpha}). \tag{20.37}$$

Assuming now for ψ the form

$$\psi = \bar{\psi}(e_{\alpha\beta}, \bar{\varrho}_{(\beta\alpha)}), \tag{20.38}$$

we can then deduce the results

$$N^{(\alpha\beta)} = \varrho_0 \frac{\partial \bar{\psi}}{\partial e_{\alpha\beta}}, \qquad M^{(\alpha\beta)} = \varrho_0 \frac{\partial \bar{\psi}}{\partial \bar{\varrho}_{(\beta\alpha)}}, \tag{20.39}$$

where

$$\begin{aligned} N^{\alpha\beta} &= N^{(\alpha\beta)} + N^{[\alpha\beta]}, \\ N^{(\alpha\beta)} &= N'^{\alpha\beta} - \tfrac{1}{2}(M^{\lambda\alpha} B_\lambda^\beta + M^{\lambda\beta} B_\lambda^\alpha), \\ N^{[\alpha\beta]} &= \tfrac{1}{2}[B_\lambda^\alpha M^{\lambda\beta} - B_\lambda^\beta M^{\lambda\alpha}]. \end{aligned} \tag{20.40}$$

Because of the approximations of the type (20.30) already used in the calculation from the three-dimensional expressions for φ^*, i.e., the neglect of all terms of $O(h/R)$ or smaller compared with those in $\bar{\psi}$ [or a function φ in the form (20.20)], to the same order of approximation we can neglect the second and the third terms on the right-hand side of $(20.40)_2$ and write

$$N^{(\alpha\beta)} = N'^{\alpha\beta}. \tag{20.41}$$

It should be evident from the above development [between (20.34)–(20.41)] that the two sets of constitutive relations (20.36) and (20.39) are equivalent. These equations are effectively of the type originally given by LOVE (in lines of curvature coordinates) but with added recent improvements.[95]

To summarize the system of differential equations which characterize the (classical) bending theory of shells with the approximation (7.75) for the displacement, we first recall that resultants corresponding to $(12.38)_{4,5}$ are not admitted in the classical theory. Moreover, in line with the approximations used in the derivation of the constitutive equations and in order to render the theory determinate, the skew-symmetric part of $M^{\alpha\beta}$ is specified by (20.31). Then, $M^{\alpha\beta}$ can be replaced by $M^{(\alpha\beta)}$ also in the equations of equilibrium (or motion) while the skew-symmetric part of $N^{\alpha\beta}$, calculated from $(12.35)_1$ or the corresponding expression in (12.42), is given by

$$N^{[\alpha\beta]} = \tfrac{1}{2}\{B^\alpha_\gamma M^{(\gamma\beta)} - B^\beta_\gamma M^{(\gamma\alpha)}\}. \tag{20.42}$$

It follows that the equilibrium equations in the classical bending theory are given by the first three differential equations in (12.42) in the absence of the inertia terms and with $L^i = 0$. These equations involve the resultants V^α, $N^{\alpha\beta}$ and the symmetric $M^{(\alpha\beta)}$ by virtue of (20.31). The shear stress-resultant V^α may be eliminated from the equilibrium equations in (12.42) in a manner similar to that in Sect. 15 [see (15.19)–(15.20)]. Thus, from the three equilibrium equations in (12.42), we obtain either the set[96]

$$\begin{aligned} N'^{\alpha\beta}{}_{|\alpha} - B^\beta_{\gamma|\alpha} M^{(\gamma\alpha)} - 2B^\beta_\gamma M^{(\gamma\alpha)}{}_{|\alpha} + \varrho_0 F^\beta &= 0, \\ M^{(\alpha\beta)}{}_{|\alpha\beta} + B_{\alpha\beta} N'^{\alpha\beta} - B_{\alpha\beta} B^\beta_\gamma M^{(\gamma\alpha)} + \varrho_0 F^3 &= 0 \end{aligned} \tag{20.43}$$

or the set

$$\begin{aligned} N^{(\alpha\beta)}{}_{|\alpha} - \tfrac{1}{2}[B^\beta_\gamma M^{(\gamma\alpha)}]_{|\alpha} + \tfrac{1}{2}[B^\alpha_\gamma M^{(\gamma\beta)}]_{|\alpha} - B^\beta_\gamma M^{(\gamma\alpha)}{}_{|\alpha} + \varrho_0 F^\beta &= 0, \\ M^{(\alpha\beta)}{}_{|\alpha\beta} + B_{\alpha\beta} N^{(\alpha\beta)} + \varrho_0 F^3 &= 0. \end{aligned} \tag{20.44}$$

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[95] The constitutive equations (20.35) or (20.36), as well as (20.39), possess a certain symmetry in structure and the kinematic variables $\varrho_{(\alpha\beta)}$ and $\bar{\varrho}_{(\alpha\beta)}$ are unaffected by infinitesimal rigid body displacements. This is in contrast to corresponding kinematic quantities in various versions of LOVE's first approximation (see Sect. 21 A). The set of constitutive equations (20.39) when expressed in a form similar to (20.35) and specialized to lines of curvature coordinates have features which are essentially those contained in a derivation first given by NOVOZHILOV [1946, *3*]. (For further details, see Sect. 21 A.) The kinematic measure $\bar{\varrho}_{\alpha\beta}$ was introduced by SANDERS [1959, *6*] in lines of curvature coordinates and by KOITER [1960, *6*], [1961, *5*] in a form which can be expressed as in (20.37). The choice of $\bar{\varrho}_{(\alpha\beta)}$ by SANDERS is motivated mainly from his consideration of a (two-dimensional) virtual work principle for shells. KOITER [1960, *6*], who does not introduce a displacement assumption such as (7.75), obtains his expression for $\bar{\varrho}_{\alpha\beta}$ by approximation and begins his calculation of a strain energy function by assuming that the state of stress as in (20.34) is approximately plane so that transverse shear stress and transverse normal stress may be neglected in the (three-dimensional) strain energy density; and he subsequently defines a modified stress-resultant and obtains his equilibrium equations by a variational method. An account of SANDER's and KOITER's contributions may be found in [1963, *6*]. Our derivation of (20.35) or the relations that follow from (20.39) is patterned after that given by NAGHDI [1963, *7*] and [1964, *5*], who used a virtual work principle for shells derived from the corresponding principle in the three-dimensional theory. See also NAGHDI [1966, *7*].

[96] The Eqs. (20.43) can be deduced from the original differential equations in (12.42) even without the specification of (20.31). See Eqs. (3.8) in [1963, *7*].

Both sets of equations are remarkably free from the skew-symmetric $N^{[\alpha\beta]}$. The former involves only the symmetric $N'^{\alpha\beta}$ and $M^{(\alpha\beta)}$ while the latter is expressed in terms of the symmetric part of $N^{\alpha\beta}$ and $M^{(\alpha\beta)}$.

It is now clear that the classical theory may be characterized by two entirely equivalent systems of differential equations: Either by the system of Eqs. (20.43) and (20.36) or by (20.44) and (20.39).[97] The boundary conditions appropriate to the classical theory may be obtained by considering the linearized (three-dimensional) rate of work expression. After integrating with respect to ζ between the limits

$$-\frac{h}{2},\ \frac{h}{2}$$

and using the definitions of the resultants in (12.42), the procedure to be followed is similar to that in Sect. 15 [between (15.21)–(15.25)]. For equilibrium problems, these boundary conditions are either[98]

$$u_\beta,\ u_3,\ \frac{\partial u_3}{\partial \nu_0} \quad \text{on } \partial\mathfrak{S} \tag{20.45}$$

or

$$\begin{gathered} {}_0\nu_\alpha\left[N'^{\alpha\beta} - 2B^\beta_\gamma M^{(\alpha\gamma)}\right],\ M^{(\alpha\gamma)}\,{}_0\nu_\alpha\,{}_0\nu_\gamma, \\ {}_0\nu_\alpha M^{(\beta\alpha)}{}_{|\beta} - \frac{\partial}{\partial s_0}\left(\bar{\varepsilon}_{\beta\gamma}\, M^{(\alpha\beta)}\,{}_0\nu_\alpha\,{}_0\nu^\gamma\right) \quad \text{on } \partial\mathfrak{S} \end{gathered} \tag{20.46}$$

and hold pointwise on the boundary $\partial\mathfrak{S}$. The boundary condition $(20.46)_1$ can also be expressed in terms of $N^{(\alpha\beta)}$.

The derivation of the approximate constitutive equations in Subsect. γ) is based on the assumption (20.27) and leads to (20.33) or their inverted forms, including a constitutive relation for $M^{\alpha 3}$. These constitutive equations are appropriate for a bending theory whose equilibrium equations are given by those which can be obtained from (12.42) in the absence of inertia terms. Also, it is clear that the number of the independent boundary conditions in this theory are six at each edge of the shell. This is in contrast to the four boundary conditions in the classical theory given by (20.45)–(20.46).

We consider now briefly a (slightly less general) system of equations of the linear theory in which the resultant $M^{\alpha 3}$ is not admitted. The equations of equilibrium are then given by those associated with the first three differential equations in (12.42) and $V^3 = \varrho_0 L^3$. The derivation of constitutive equations in this theory can be accomplished in just the way that led to (20.33) but with a modification in the initial assumption for stresses. In the present development, we retain $(20.27)_{1,3}$ but (since $M^{\alpha 3}$ is not admitted) replace $(20.27)_2$ by

$$\mu\,\tau^{\alpha 3} = \mu\,\tau^{3\alpha} = \frac{3\,V^\alpha}{2h}\left[1 - \left(\frac{\zeta}{h/2}\right)^2\right].$$

The above expression will still meet the boundary conditions $(20.3)_1$ with $p'^\alpha = p''^\alpha = 0$. The rest of the development parallels that in Subsect. γ) with obvious modifications, in view of the absence of $M^{\alpha 3}$. In this manner we finally obtain a system of constitutive equations which are the same as the first four of

[97] In the recent literature, sometimes preference is indicated in favor of the constitutive equations (20.36) or (20.39) together with (20.41) [or a variant of (20.40)] and with $\bar{\varrho}_{(\alpha\beta)}$ expressed in terms of the components of rotation about the normal to the middle surface. In view of the equivalence of the two systems of equations indicated above, such a preference can only be argued on the basis of convenience or a possible simplification resulting with reference to a specific problem. BUDIANSKY and SANDERS [1963, *1*] have argued in favor of a set of constitutive equations corresponding to (20.39); the label of "the 'best' first order linear shell theory" adopted by them and repeated by others is misleading.

[98] The boundary conditions (20.46) can also be derived (from the three-dimensional theory) even without the specification of (20.31). See Eqs. (4.20) in [1963, *7*].

(20.33). For later reference, we record below the inverted forms of these constitutive equations:

$$\begin{aligned}
N'^{\alpha\beta} &= (1-\nu)\, C\left\{\left[\frac{\nu}{1-2\nu} A^{\alpha\beta} A^{\gamma\delta} + A^{\alpha\gamma} A^{\beta\delta}\right] e_{\gamma\delta} + \frac{\nu}{1-2\nu} A^{\alpha\beta}\gamma_3\right\},\\
M^{(\alpha\beta)} &= B\left[\nu A^{\alpha\beta} A^{\gamma\delta} + (1-\nu) A^{\alpha\gamma} A^{\beta\delta}\right] \varrho_{(\gamma\delta)},\\
V^{\alpha} &= \frac{5}{6}\mu\, h A^{\alpha\beta}\gamma_\beta,\\
V^{3} &= \frac{1-\nu}{1-2\nu} C\left[(1-\nu)\gamma_3 + \nu A^{\alpha\beta} e_{\alpha\beta}\right].
\end{aligned} \tag{20.47}$$

The above constitutive equations include the effect of transverse shear deformation and transverse normal stress.[99] The boundary conditions in the above linear theory with constitutive equations (20.47) are either the displacement boundary conditions

$$u_\alpha,\ u_3,\ \gamma_\alpha \quad \text{on } \partial\mathfrak{S} \tag{20.48}$$

or the resultant boundary conditions

$${}_0\nu_\alpha N^{\alpha\beta},\ {}_0\nu_\alpha M^{\alpha\beta},\ {}_0\nu_\alpha V^{\alpha} \quad \text{on } \partial\mathfrak{S}. \tag{20.49}$$

The boundary-value problem of the linear bending theory characterized by the appropriate equations of equilibrium (noted earlier in this paragraph), as well as (20.47)–(20.49), include certain features in common with REISSNER'S plate theory. Remarks similar to those made at the end of Subsect. β) can also be made here.

Before closing this section some additional remarks should be made regarding the nature of the foregoing derivations of constitutive equations for shells and plates from the three-dimensional theory. Although an effort was made to be as systematic as possible, the shortcomings of the derivations associated with the approximations are self-evident. Moreover, no attempt has been made here to provide clear-cut justifications (which can be supported by analysis) for the approximations or to obtain an estimate of the "error" involved in the use of the approximate constitutive equations. As remarked previously (Sect. 4), the problem posed under (b) in (4.32) regarding a scheme for estimating the "error" involved in the use of the (approximate) two-dimensional equations of the classical theory of shells or more general linear theories, as compared to the three-dimensional equations of linear elasticity, has not been solved with finality even in the case of the linear bending theory of plates.[100]

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[99] Constitutive equations of this type with various degrees of generality and from different points of view have been given previously: HILDEBRAND, REISSNER and THOMAS [1949, *4*], GREEN and ZERNA [1950, *2*], REISSNER [1952, *2*], [1964, *7*] and NAGHDI [1957, *4*], [1963, *6*], [1964, *6*]. These papers (except for [1964, *6*]), however, employ a kinematic measure different from $\varrho_{(\gamma\delta)}$ in $(20.47)_2$.

[100] With reference to the classical bending theory of plates, however, MORGENSTERN [1959, *2*] has shown that the stresses and strains obtained from a solution in plate theory converge in a mean square sense to a solution in elasticity theory as the plate thickness approaches zero. A recent related paper by NORDGREN [1971, *7*] contains an explicit estimate of the mean square error for the stresses obtained from a solution in plate theory with respect to the exact solution of the corresponding problem in the three-dimensional theory. By constructing a somewhat more elaborate three-dimensional displacement field than that in [1971, *7*], an improved estimate for the error in the classical bending theory of plates has been obtained by SIMMONDS [1971, *8*]. A more ambitious effort toward the problem posed under (b) in (4.32) was undertaken by JOHN ([1965, *5*] and [1969, *4*]) in the context of a nonlinear theory, as was noted earlier. A comparison of a linear shell theory with the exact three-dimensional linear theory, based to some extent on the work of JOHN [1965, *5*], is given by SENSENIG [1968, *11*]. The question of "error" estimate in the linear theory of shells has been considered recently by KOITER [1970, *4*]. KOITER'S analysis is carried out in the spirit of earlier works of MORGENSTERN [1959, *2*] for the bending of flat plates and MORGENSTERN and SZABO [1961, *9*] for generalized plane stress.

21. Further remarks on the approximate linear and nonlinear theories developed from the three-dimensional equations. In this section, we first supplement the previous remarks on the classical linear theories of shells and plates and then briefly discuss the nature of some of the approximate nonlinear theories obtained from the three-dimensional equations. Preliminary to the discussion which follows, we recall that certain mathematical rules or procedures (such as those pertaining to dimensional invariance, coordinate invariance, consistency, equipresence, etc.) are observed or employed directly in the construction of the constitutive equations.[101] In particular, we recall here that the constitutive equations must fulfill the following requirements: (a) They must be consistent with all conservation laws and equations resulting therefrom—hence they must be consistent with the equations of motion, the energy equation and all energetic theorems and related results; (b) they must remain unaffected by rigid body motions;[102] and (c) they must remain invariant under the transformation of coordinate systems or, in the present context, under the transformation of the middle surface coordinates.[103]

Ordinarily, most of the rules of invariance or procedures referred to above are not explicitly stated; and often, especially in the case of linear theories of continuum mechanics, it is taken for granted that the derived constitutive equations meet all of the above requirements. Yet, as the history of the subject shows,[104] the above requirements have not always been satisfied in the case of the constitutive equations of the linear theory of elastic shells. These shortcomings stem largely from the trend in which the bending theory of elastic shells has been historically developed—often piecemeal and in an *ad hoc* fashion, in contrast to the developments of both the three-dimensional and two-dimensional (plane) linear elasticity theories—but also from the difficult nature of the topic, as well as the use of special coordinates (e.g., lines of curvature coordinates) or special shapes (e.g., cylindrical and spherical shells).[105]

The constitutive equations derived in Sect. 20 [Subsects. γ) and δ)] meet all invariance requirements mentioned above. However, some of the linear constitutive equations employed in the current literature on shell theory and even recent books on the subject violate one or more of the invariance requirements (see Sect. 21 A). Although the extent to which the latter constitutive equations violate the requirements of the type (a) to (c) may be insignificant in most practical applications, such shortcomings are nevertheless undesirable from a theoretical standpoint.[106]

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[101] For an account of these rules, see Sect. 293 of TRUESDELL and TOUPIN [1960, *14*].

[102] In the case of linear theory, this requirement implies that the constitutive equations must remain unaffected by infinitesimal rigid body motions.

[103] This requirement guarantees that the response of the material is unaltered by a different coordinate description; in the case of shell theory, it is easily fulfilled by deriving the equations in tensorial or similar general forms.

[104] A brief history of the derivation of the constitutive equations of the linear theory of elastic shells, beginning with the work of LOVE [1888, *1*], is given in Sect. 21 A.

[105] Some of the trends have surprisingly persisted throughout the years; even in very recent and current literature, it is not uncommon to find the use of lines of curvature coordinates in discussions intended to reveal features of a general theory.

[106] The foregoing remarks are not intended to detract from most of the specific results and solutions in the literature which have been obtained in the past with the use of somewhat simpler sets of constitutive equations [such as the set which includes (21 A.7) and described in Sect. 21 A], despite their shortcomings. As noted by KOITER [1960, *6*], if the nonvanishing physical component of the rotation [corresponding to that in (21 A.8)] is *small* or of the same order of magnitude when compared with the physical components of $e_{\alpha\beta}$, then from a practical point of view the use of the simpler set of constitutive equations may be justified. On the other hand, as brought out by KNOWLES and REISSNER [1958, *3*] and by COHEN'S [1960, *2*] analysis of a helicoidal shell, if the physical component of rotation is *large* in comparison with components of $e_{\alpha\beta}$, then the use of the simpler set of constitutive equations [e.g., the set which includes (21 A.7)] can lead to serious errors.

Having disposed of the foregoing preliminaries, we return once more to the discussion of an approximate linear theory of elastic shells obtained from the three-dimensional equations. Our derivation of linear constitutive equations in Sect. 20 is carried out in the spirit of earlier developments (from the three-dimensional equations) in Chap. D. Alternatively (and with some adjustments in the initial assumptions), the same results can be deduced from the generalized Hooke's law or with the use of the (three-dimensional) virtual work principle or any of the variational theorems of three-dimensional linear elasticity. The latter approach has been popular over the years beginning with a paper of TREFFTZ.[107]

An entirely different approach to the linear theories of plates and shells, based on the use of asymptotic expansion techniques, has attracted considerable attention in recent years. This approach which begins with the three-dimensional equations of linear elasticity, roughly speaking, may be described as follows: With the initial position vector of a material point specified by (4.27), first *suitable* scaling of the coordinates and, in turn of the displacements u_i^* and stresses τ^{ij} are introduced. This scaling is such that it gives distinction between the tangential components u_α^*, $\tau^{\alpha\gamma}$ and the components u_3^*, τ^{i3} along the normal to the middle surface of the shell. Next, with a view toward separate consideration of the "interior" problem (i.e., for the region away from the edge of the shell or plate) and that of the boundary-layer region, all equations are expanded in powers of a *small* parameter (say $\varepsilon = h/R$)[108] and successive approximations to the three-dimensional equations are deduced for the interior and the boundary-layer regions. Inasmuch as a fairly detailed and recent account of the asymptotic expansion procedure for shells is available elsewhere (Chap. 16 of GREEN and ZERNA),[109] we confine ourselves here to only a few remarks in order to give an indication of the nature of such developments.

In the case of flat plates, the earliest derivation by an asymptotic procedure is included in a paper by GOODIER.[110] The subject was considered anew by FRIEDRICHS and DRESSLER with particular reference to the boundary-layer equations for elastic plates.[111] In essence, the lowest order approximation to the "interior" problem leads to the classical Kirchhoff theories for stretching and bending of plates. However, the interior equations are not uniformly valid in the immediate region of the edge surface of the plate in the sense that it is not possible to satisfy arbitrary specified boundary conditions on the edge surface with the interior equations and the boundary-layer problem must be considered. From the latter and by a method first used by GREEN,[112] the usual Kirchhoff boundary conditions

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[107] TREFFTZ [1935, *2*]. The use of variational theorems of a virtual work principle are prominent in a number of papers cited in Sect. 20 and are also employed in most books on the subject: E.g., NOVOZHILOV [1959, *3*], GOL'DENVEIZER [1961, *3*] and AMBARTSUMIAN [1964, *1*].

[108] It is usual in the case of shells to introduce two parameters, one given by ε and another associated with the deformation pattern on the middle surface which implies dependence on the nature of the solutions of the differential equations sought. However, this is not essential and can be avoided as indicated by GREEN and NAGHDI [1965, *2*].

[109] GREEN and ZERNA [1968, *9*].

[110] GOODIER [1938, *1*].

[111] FRIEDRICHS and DRESSLER [1961, *2*]. For related papers on asymptotic expansions and boundary-layer equations for elastic plates, see REISS and LOCKE [1961, *11*], GOL'DENVEIZER [1962, *3*], FOX [1964, *2*], GOL'DENVEIZER and KOLOS [1965, *1*] and LAWS [1966, *6*].

The above papers are all concerned with derivations of classical plate theory by asymptotic expansions of the equations of (non-polar) linear elasticity. GREEN and NAGHDI [1967, *5*] have shown that the equations of the linear theory of an elastic Cosserat plate can be obtained as a first approximation to an asymptotic expansion of an exact linearized three-dimensional theory of a generalized continuum which admits a director. See also a related paper by ERINGEN [1967, *3*] concerning the derivation of plate equations from a theory of generalized continua.

[112] GREEN [1962, *5*]; see also GREEN and LAWS [1966, *3*].

can then be deduced and these must be applied to the major terms of the interior stresses.[113]

As might be expected, the derivations by asymptotic methods for elastic shells are both more complex and laborious than that for flat plates, but the essence of the ideas (though not the details of the method and results) are similar. Asymptotic derivations of this type have been discussed by JOHNSON and REISSNER, REISS, GREEN and others.[114] Roughly speaking, four sets of approximate system of equations for shells are obtained by the asymptotic procedure and these correspond to (1) membrane approximation, (2) inextensional approximation, (3) a simplified system of equations for bending and (4) boundary-layer equations.[115] The four sets of Eqs. (1)–(4) do not constitute a complete system of shell equations in the sense of the classical shell theory obtained in Sect. 20 [Subsect. δ)], but they may be regarded as a basis for obtaining approximate solutions to shell problems. In fact, in the context of the asymptotic procedure used, the Eqs. (20.39) of the classical shell theory contain a significant contribution of an order higher than the first as was noted by GREEN and NAGHDI.[116] In any case, the derivations by asymptotic expansion techniques for shells (and plates) do not appear to be conclusive, especially with regard to the problem posed under (b) in Sect. 4.[117] The derivations by asymptotic expansion techniques, at first sight, may appear to be free from *ad hoc* assumptions; but, in fact, this is not the case. The scaling of stresses and displacement is tantamount to *a priori* special assumptions regarding the transverse components u_3^* and τ^{i3}, although subsequent developments are carried out systematically. Moreover, no proof is available that the expansions obtained are asymptotic and of uniform validity over the entire region of the shell or plate.

We close this section with some remarks concerning the nature of the existing approximate nonlinear theories of elastic plates and shells derived from the three-dimensional equations. In nearly all of these approximate theories, the underlying kinematic assumptions are such that the strains are small while the rotation may be *large* or moderately large[118] and linear constitutive equations are assumed to be

[113] Boundary conditions for plates have been discussed by FRIEDRICHS [1949, *1*], [1950, *1*], FRIEDRICHS and DRESSLER [1961, *2*], REISSNER [1963, *9*] and others. Some of these papers use a "matching" procedure between the interior and the boundary-layer stresses, but the simpler method of GREEN [1962, *5*] is used by LAWS [1966, *6*] to obtain the classical Kirchhoff boundary conditions.

[114] The earliest papers are by JOHNSON and REISSNER [1959, *1*] and REISS [1960, *12*] who confined themselves to symmetric deformations of cylindrical shells. Subsequently, the case of axisymmetric deformation of shells of revolution was considered by REISSNER [1960, *13*] and the unsymmetric deformation of cylindrical shells was dealt with by REISS [1962, *7*]. General derivations of the interior and the boundary-layer equations for shells were given by GREEN [1962, *4*], [1962, *5*]; see also GREEN and LAWS [1966, *3*]. Some simplifications in the procedure, as well as extension of the results to dynamical problems, were noted by GREEN and NAGHDI [1965, *2*]. For other related references, see the papers by GOL'DENVEIZER [1963, *3*], JOHNSON [1963, *4*] and REISSNER [1963, *10*].

[115] The system of equations in each of the categories (1)–(4) represents an approximation to the three-dimensional equations. Sometimes the approximations (1) and (2) are referred to as the membrane theory and the inextensional theory, respectively. But the latter terminologies may possibly detract from clarity here, since what is sought in an asymptotic development are approximations to the three-dimensional equations. While the approximate system of equations in each of the categories (1)–(4) may violate one or more of the requirements stated under (a) to (c) above, it should be noted that this is simply a consequence of the particular approximation used.

[116] [1965, *2*].

[117] This is perhaps partly evident from GOL'DENVEIZER's recent effort [1969, *2*] with regard to the interaction of boundary layer and interior stresses for thin shells.

[118] A discussion of such kinematic assumptions is given in Sects. 48–49 of NOVOZHILOV [1953, *3*]. See also the review of [1953, *3*] by TRUESDELL [1953, *5*].

valid. In general, however, a systematic development of such approximate theories is not available; and even those few contributions which have striven toward a more satisfactory derivation either employ assumptions which are too special or else still contain a number of *ad hoc* approximations. Nevertheless, it should be noted that these approximate developments have been largely motivated by applications and have served a useful purpose over the years. The origin of such approximate theories for initially flat plates goes back to papers of FÖPPL (in which bending effects are ignored) and VON KÁRMÁN.[119] The latter contains the well-known von Kármán equations for nonlinear bending of thin plates characterized by a system of coupled nonlinear partial differential equations in terms of a stress function and the normal displacement.[120]

With reference to initially curved shells, a general derivation of the nonlinear membrane theory (in the context of nonlinear elasticity) is given in GREEN and ADKINS.[121] This derivation does not contain approximations for the tangential components of strain measures and employs nonlinear constitutive equations. A more special nonlinear membrane theory in which the strains (but not the displacements) are assumed to be small was given earlier by BROMBERG and STOKER.[122] A number of approximate nonlinear theories have been developed with reference to special geometries. Notable among these are the papers by DONNELL for cylindrical shells,[123] MARGUERRE for shallow shells[124] and by REISSNER for axisymmetric deformation of shells of revolution.[125] Finally, we note that more general approximate nonlinear theories have been obtained in recent years with different aims and varying degrees of generality.[126]

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[119] FÖPPL [1907, *1*], VON KÁRMÁN [1910, *1*].

[120] An account of VON KÁRMÁN's equations can be found in STOKER's [1968, *12*] monograph and in TIMOSHENKO and WOINOWSKY-KREIGER [1959, *7*]. Related and more general derivations are contained in the papers by REISSNER [1953, *4*] and ERINGEN [1955, *1*]. Large inextensional deformations of plates are discussed by FUNG and WITRICK [1955, *2*], MANSFIELD [1955, *5*] and ASHWELL [1957, *2*].

[121] GREEN and ADKINS [1960, *5*].

[122] BROMBERG and STOKER [1945, *1*]. Related results pertaining to nonlinear membrane theory are contained in the papers of STOKER [1963, *14*], ERINGEN [1952, *1*] and BUDIANSKY [1968, *1*].

[123] DONNELL [1933, *1*]. A linearized version of DONNELL's equations, because of their relative simplicity, has been extensively employed in applications but not without concern regarding their accuracy. In this connection, see a paper by HOFF [1955, *3*] which contains a discussion of the accuracy of DONNELL's equations as compared to those of FLÜGGE [1932, *1*].

[124] MARGUERRE [1938, *2*]. The papers of MARGUERRE and DONNELL are also significant because of the linearized versions of their developments in [1938, *2*] and [1933, *1*]. In fact, MARGUERRE's paper contains, as a special case, the first satisfactory derivation of the linear theory of shallow shells.

Equations equivalent to those given by DONNELL [1933, *1*] and MARGUERRE [1938, *2*] or variants thereof were independently discovered and utilized by others, e.g., VLASOV and MUSHTARI. For references, see VLASOV [1958, *5*], NOVOZHILOV [1959, *3*], ONIASHVILI [1960, *11*] and MUSHTARI and GALIMOV [1961, *10*]. The effect of nonlinearity in [1933, *1*] and [1938, *2*] is accounted for in a manner similar to that in VON KÁRMÁN's equations. A linearized version of DONNELL's equations may also be deduced as a special case of the differential equations for bending of circular cylindrical shells in GREEN and ZERNA [1968, *9*]. Similarly, a linearized version of MARGUERRE's theory of shallow shells may be found in GREEN and ZERNA [1968, *9*].

[125] REISSNER [1950, *6*], [1960, *13*] and [1963, *11*].

[126] We mention, in particular, the papers by LEONARD [1961, *6*], RÜDIGER [1961, *12*], NAGHDI and NORDGREN [1963, *8*], SANDERS [1963, *12*], KOITER [1966, *4*] and BIRICIKOGLU and KALNINS [1971, *2*]. All of these papers invoke the Kirchhoff-Love hypothesis or a similar equivalent set of assumptions and most (but not all) of them employ linear constitutive equations. However, a nonlinear theory in which the strains are small but the constitutive equations (through their coefficients) contain nonlinear effects has been also discussed by ZERNA [1960, *15*] and WAINWRIGHT [1963, *16*].

21 A. Appendix on the history of the derivation of linear constitutive equations for thin elastic shells. We begin our account of the history of the derivation of linear constitutive equations for elastic shells with that of LOVE's first approximation (LOVE's theory of 1888). However, as remarked in Sect. 1, the reference to LOVE's first approximation in the contemporary literature is often confusing, as there are at least three different versions which bear his name.[127] In order to distinguish between these and also provide a basis for their comparison with our results of the classical shell theory (Sect. 20), it is expedient to record the constitutive equations of the classical theory corresponding to one of the alternative forms (20.36) or (20.39) in lines of curvature coordinates and in terms of physical components. Here we choose the latter, since the constitutive equations of LOVE's first approximation involve resultants which correspond to $N^{(\alpha\beta)}$ and $M^{(\alpha\beta)}$ instead of $N'^{\alpha\beta}$ and $M^{(\alpha\beta)}$.

Thus, we refer the surface coordinates in the reference configuration to lines of curvature coordinates so that $A_{12}=B_{12}=0$ and also introduce the notations

$$\begin{gathered}(A_1)^2=A_{11}=|\boldsymbol{A}_1|^2, \quad (A_2)^2=A_{22}=|\boldsymbol{A}_2|^2,\\ B_1^1=-\frac{1}{R_1}, \quad B_2^2=-\frac{1}{R_2}, \quad B_{11}=-\frac{(A_1)^2}{R_1}, \quad B_{22}=-\frac{(A_2)^2}{R_2},\end{gathered} \tag{21 A.1}$$

where R_1 and R_2 are the principal radii of curvature of the initial reference surface and the notations A_1, A_2 are introduced for convenience. Further, let the physical components of tensors of the type $e_{\alpha\beta}$, $N^{\alpha\beta}$ be designated as $e_{\langle\alpha\beta\rangle}$, $N_{\langle\alpha\beta\rangle}$. Then[128]

$$\begin{aligned}e_{\langle\alpha\beta\rangle}&=\frac{e_{\alpha\beta}}{A_\alpha A_\beta} &&\text{(no summation over } \alpha, \beta),\\ e_{\alpha\beta}&=A_\alpha A_\beta\, e_{\langle\alpha\beta\rangle} &&\text{(no summation over } \alpha, \beta),\\ N_{\langle\alpha\beta\rangle}&=A_\alpha A_\beta N^{\alpha\beta} &&\text{(no summation over } \alpha, \beta),\\ N^{\alpha\beta}&=\frac{N_{\langle\alpha\beta\rangle}}{A_\alpha A_\beta} &&\text{(no summation over } \alpha, \beta).\end{aligned} \tag{21 A.2}$$

We record below the physical components of the symmetric strain measures $e_{\alpha\beta}$ and $\bar{\varrho}_{\alpha\beta}$, namely

$$\begin{aligned}e_{\langle 11\rangle}&=\frac{1}{A_1}\left[u_{\langle 1\rangle,1}+\frac{A_{1,2}}{A_2}u_{\langle 2\rangle}\right]+\frac{u_3}{R_1},\\ e_{\langle 22\rangle}&=\frac{1}{A_2}\left[u_{\langle 2\rangle,2}+\frac{A_{2,1}}{A_1}u_{\langle 1\rangle}\right]+\frac{u_3}{R_2},\\ e_{\langle 12\rangle}&=e_{\langle 21\rangle}=\frac{1}{2}\left\{\frac{A_1}{A_2}\left[\frac{u_{\langle 1\rangle}}{A_1}\right]_{,2}+\frac{A_2}{A_1}\left[\frac{u_{\langle 2\rangle}}{A_2}\right]_{,1}\right\}\end{aligned} \tag{21 A.3}$$

and

$$\begin{aligned}\bar{\varrho}_{\langle 11\rangle}&=-\frac{1}{A_1}\left[\frac{u_{3,1}}{A_1}-\frac{u_{\langle 1\rangle}}{R_1}\right]_{,1}-\frac{A_{1,2}}{A_1A_2}\left[\frac{u_{3,2}}{A_2}-\frac{u_{\langle 2\rangle}}{R_1}\right],\\ \bar{\varrho}_{\langle 22\rangle}&=-\frac{1}{A_2}\left[\frac{u_{3,2}}{A_2}-\frac{u_{\langle 2\rangle}}{R_2}\right]_{,2}-\frac{A_{2,1}}{A_1A_2}\left[\frac{u_{3,1}}{A_1}-\frac{u_{\langle 1\rangle}}{R_2}\right],\\ 2\bar{\varrho}_{\langle 12\rangle}&=2\bar{\varrho}_{\langle 21\rangle}=\left\{\frac{1}{A_1A_2}\left[(A_1)^2\left(\frac{u_{\langle 1\rangle}}{A_1R_1}\right)_{,2}+(A_2)^2\left(\frac{u_{\langle 2\rangle}}{A_2R_2}\right)_{,1}\right]\right.\\ &\quad\left.+\frac{2}{A_1A_2}\left[-u_{3,12}+\frac{A_{1,2}}{A_1}u_{3,1}+\frac{A_{2,1}}{A_2}u_{3,2}\right]\right\}\\ &\quad+\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\left[\frac{1}{2A_1A_2}\left(\frac{1}{(A_1)^2}\{(A_1)^3u_{\langle 1\rangle}\}_{,2}-\frac{1}{(A_2)^2}\{(A_2)^3u_{\langle 2\rangle}\}_{,1}\right)\right].\end{aligned} \tag{21 A.4}$$

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[127] LOVE's original derivations [1888, *1*] and [1893, *2*], as well as related subsequent developments, were carried out in lines of curvature coordinates and in terms of physical components of the stress-resultants and the stress-couples.

[128] See Sect. A.4. of the Appendix (Chap. F).

The physical constitutive equations for $N^{(\alpha\beta)}$ and $M^{(\alpha\beta)}$ which follow from (20.39) in the case of initially homogeneous and isotropic elastic materials are then given by relations of the type

$$\begin{aligned} N_{\langle 11\rangle} &= C\left[e_{\langle 11\rangle}+\nu\, e_{\langle 22\rangle}\right], \\ N_{\langle(12)\rangle} &= N_{\langle(21)\rangle}=(1-\nu)\, C\, e_{\langle 12\rangle}, \\ M_{\langle 11\rangle} &= B\left[\bar{\varrho}_{\langle 11\rangle}+\nu\, \bar{\varrho}_{\langle 22\rangle}\right], \\ M_{\langle 12\rangle} &= M_{\langle 21\rangle}=(1-\nu)\, B\, \bar{\varrho}_{\langle 12\rangle}, \end{aligned} \tag{21A.5}$$

where the kinematic measures are those in (21 A.3)–(21 A.4). Also, the physical equation for the skew-symmetric part of $N^{\alpha\beta}$ becomes

$$N_{\langle 12\rangle}-N_{\langle 21\rangle}=\left(\frac{1}{R_2}-\frac{1}{R_1}\right) M_{\langle 12\rangle}. \tag{21 A.6}$$

Returning to LOVE's first approximation, we first observe that a derivation of LOVE's constitutive equations in lines of curvature coordinates is contained in a paper by REISSNER.[129] This formulation of LOVE's first approximation, as will become apparent presently, is different from that in LOVE's treatise or the version originally given by LOVE in 1888. The essential difference between the physical constitutive equations (21 A.5) and the version of LOVE's equations as derived by REISSNER is in the expression for $M_{\langle 12\rangle}$, namely[130]

$$\begin{aligned} M_{\langle 12\rangle} &= M_{\langle 21\rangle}=\frac{1-\nu}{2} B\, \tau, \\ \tau &= \left(\frac{A_2}{A_1}\right)\left[\frac{\beta_{\langle 2\rangle}}{A_2}\right]_{,1}+\left(\frac{A_1}{A_2}\right)\left[\frac{\beta_{\langle 1\rangle}}{A_1}\right]_{,2}, \end{aligned} \tag{21 A.7}$$

where $\beta_{\langle 1\rangle}, \beta_{\langle 2\rangle}$ are the physical components of β_α given by $(7.68)_4$. The remaining constitutive equations of LOVE's theory (as derived by REISSNER) are the same as the corresponding equations in (21 A.5). Thus, the difference between the two sets of constitutive equations lies in the kinematic measures which occur in (21 A.7) and $(21\,\text{A}.5)_4$, respectively. An easy comparison between τ and $\bar{\varrho}_{\langle 12\rangle}$ can be made if, with the use of $(20.35)_4$ and the second expression for $\bar{\varkappa}_{\alpha\beta}$ in (7.68), we write (20.37) in the form

$$\bar{\varrho}_{\alpha\beta}=\tfrac{1}{2}\left(\beta_{\alpha|\beta}+\beta_{\beta|\alpha}\right)-\tfrac{1}{2}\left(B_\alpha^\nu\, \gamma_{[\nu\beta]}+B_\beta^\nu\, \gamma_{[\nu\alpha]}\right). \tag{21 A.8}$$

It is then easily seen that the expression for τ in $(21\,\text{A}.7)_2$ arises only from the first parenthesis on the right-hand side of (21 A.8). As noted above, the version of LOVE's theory as given in his treatise differs from that which includes (21 A.7) and has also been used extensively in the literature (at least until the early 1950's). The version of the constitutive equations in LOVE's treatise differs from the set derived by REISSNER only in the relation for $M_{\langle 12\rangle}$, where the expression corresponding to the kinematic measure τ is not symmetric in $u_{\langle 1\rangle}$ and $u_{\langle 2\rangle}$; and hence, in this sense, the constitutive equations in LOVE's treatise are considered unsatisfactory.[131]

[129] REISSNER [1941, *1*]. A similar derivation is included in [1949, *4*].

[130] The difference of the factor of $\frac{1}{2}$ in the coefficients of $(21\,\text{A}.5)_4$ and $(21\,\text{A}.7)_1$ is due to the definition of τ in $(21\,\text{A}.7)_2$.

[131] See Sect. 329 of LOVE [1944, *4*] or the corresponding sections of earlier editions of his treatise. Still, shortly after his [1941, *1*] paper, another version was proposed by REISSNER [1942, *1*]. The constitutive equations in [1942, *1*] contain higher-order terms in h/R in comparison with the set of equations which include (21 A.7); and, in particular, those corresponding to $M_{\langle 12\rangle}$ and $M_{\langle 21\rangle}$ are not the same.

Almost immediately after its appearance, LOVE'S work of 1888 drew criticism from Lord RAYLEIGH[132] who, favoring his own inextensional theory, objected to the application of LOVE'S equations to extensional vibrations of shells.[133] Shortly thereafter, LAMB[134] published a paper in which he derived by another approach equations essentially equivalent to those of LOVE. But despite LAMB'S praise of LOVE'S work, at the time LOVE appears to have been pessimistic about his own 1888 derivation of the general bending theory of thin shells.[135] Responding to RAYLEIGH'S criticism,[136] LOVE after again pointing out that the inextensional theory violates the boundary conditions and agreeing that the extensional theory will not predict the lowest frequency of free vibrations,[137] also notes that it may be necessary to retain higher-order terms in h/R as was done by BASSETT in special cases.[138] In this connection, it may be noted that LOVE'S second approximation[139] does contain some higher order terms in h/R.

During the past two or three decades, when speaking of LOVE'S first approximation, some authors have in mind the version of LOVE'S theory with $M_{\langle 12\rangle} = M_{\langle 21\rangle}$ given by (21 A.7) while others (especially the Russian investigators) have reference to the equations originally supplied by LOVE or those in his treatise mentioned above. Neither of the two sets of constitutive equations of LOVE'S first approximation (i) satisfies the equilibrium equation arising from the symmetry of the stress tensor or equivalently (21 A.6). Nor are they (ii) invariant under infinitesimal rigid body displacement.[140] Objections to LOVE'S first approximation (in the form given originally by LOVE) were first raised by VLASOV[141] who, in addition to the above shortcomings (i) and (ii), also pointed out that LOVE'S equations (iii) violate a reciprocity theorem, i.e., a two-dimensional analogue of the (BETTI-RAYLEIGH) reciprocity theorem in three-dimensional linear elasticity. The version of LOVE'S first approximation as derived by REISSNER also has the shortcomings (i) and (ii) but not (iii); and, moreover, REISSNER'S version can be put in a tensorially invariant form, i.e., in a form which remains unaltered under the transformation of the middle surface coordinates.[142]

Subsequent attempts to remedy the unsatisfactory state of the subject that existed around 1940 were largely confined to derivations within the framework

[132] RAYLEIGH [1888, *2*].

[133] In his paper of [1888, *1*], after deriving the equations of the bending theory, LOVE considers the special cases of the extensional vibrations of cylindrical and spherical shells.

[134] LAMB [1890, 2].

[135] LOVE [1891, *1*].

[136] In his response to RAYLEIGH'S criticism and with reference to his own theory of [1888, *1*], LOVE writes ([1891, *1*]) "it would most probably be sufficiently exact for the application of a method of approximation."

[137] However, the argument given by Lord RAYLEIGH [1888, *2*] is faulty since the inextensional and the extensional displacements both violate the boundary conditions; and, therefore, RAYLEIGH'S principle as used in the argument does not apply.

[138] BASSETT [1890, *1*].

[139] See Sect. 330 of [1944, *4*]. A discussion of LOVE'S second approximation is included also in [1949, *4*].

[140] In the case of (21 A.7), this can be easily verified with the use of expressions of the type $(6.44)_1$ and $(6.45)_3$ for infinitesimal rigid body displacements of the shell. See also Sect. 6 of [1963, *6*].

[141] VLASOV [1944, *6*]. See also his book [1958, *5*].

[142] With reference to REISSNER'S version of LOVE'S first approximation [1941, *1*] the shortcomings (i) and (ii) above were also mentioned by KNOWLES and REISSNER [1958, *3*]. The fact that this version admits a reciprocity theorem was noted in [1960, *10*] and the observation that it can be put in tensorially invariant form was made in [1963, *6*].

of the Kirchhoff-Love assumption.[143] In this connection, we now indicate the nature of a system of constitutive equations, derived under the Kirchhoff-Love assumption [see Sect. 7, Subsect. ε)], where higher-order terms in the thickness coordinate (i.e., in h/R) are retained. For this purpose, we introduce the notations

$$\begin{aligned}\gamma_1^0 &= \frac{1}{A_1}\left[u_{\langle 2\rangle,1} - \frac{u_{\langle 1\rangle}}{A_2} A_{1,2}\right], & \gamma_2^0 &= \frac{1}{A_2}\left[u_{\langle 1\rangle,2} - \frac{u_{\langle 2\rangle}}{A_1} A_{2,1}\right],\\ \tau_1 &= \frac{1}{A_1}\left[\beta_{\langle 2\rangle,1} - \frac{\beta_{\langle 1\rangle}}{A_2} A_{1,2}\right], & \tau_2 &= \frac{1}{A_2}\left[\beta_{\langle 1\rangle,2} - \frac{\beta_{\langle 2\rangle}}{A_1} A_{2,1}\right]\end{aligned} \tag{21 A.9}$$

and also observe that the above quantities are such that $\gamma_1^0 + \gamma_2^0 = 2e_{\langle 12\rangle}$ and $\tau_1 + \tau_2 = \tau$. Then, referred to lines of curvature coordinates and in terms of physical components, the constitutive equations under consideration can be written in the form[144]

$$\begin{aligned}N_{\langle 11\rangle} &= C\left\{[e_{\langle 11\rangle} + \nu\, e_{\langle 22\rangle}] + \frac{h^2}{12}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{e_{\langle 11\rangle}}{R_1} - \bar{\varrho}_{\langle 11\rangle}\right)\right\},\\ N_{\langle 22\rangle} &= C\left\{[e_{\langle 22\rangle} + \nu\, e_{\langle 11\rangle}] + \frac{h^2}{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(\frac{e_{\langle 22\rangle}}{R_2} - \bar{\varrho}_{\langle 22\rangle}\right)\right\},\\ N_{\langle 12\rangle} &= \frac{1-\nu}{2}\, C\left\{2\, e_{\langle 12\rangle} + \frac{h^2}{12}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\left(\frac{\gamma_1^0}{R_1} - \tau_1\right)\right\},\\ N_{\langle 21\rangle} &= \frac{1-\nu}{2}\, C\left\{2\, e_{\langle 12\rangle} + \frac{h^2}{12}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(\frac{\gamma_2^0}{R_2} - \tau_2\right)\right\}\end{aligned} \tag{21 A.10}$$

and

$$\begin{aligned}M_{\langle 11\rangle} &= B\left\{[\bar{\varrho}_{\langle 11\rangle} + \nu\, \bar{\varrho}_{\langle 22\rangle}] - \left(\frac{1}{R_1} - \frac{1}{R_2}\right) e_{\langle 11\rangle}\right\},\\ M_{\langle 22\rangle} &= B\left\{[\bar{\varrho}_{\langle 22\rangle} + \nu\, \bar{\varrho}_{\langle 11\rangle}] - \left(\frac{1}{R_2} - \frac{1}{R_1}\right) e_{\langle 22\rangle}\right\},\\ M_{\langle 12\rangle} &= \frac{1-\nu}{2}\, B\left\{\tau - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\gamma_1^0\right\},\\ M_{\langle 21\rangle} &= \frac{1-\nu}{2}\, B\left\{\tau - \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\gamma_2^0\right\}.\end{aligned} \tag{21 A.11}$$

[143] An exception is the work of CHIEN [1944, *2*] which, although quite general in scope, does not appear to have directed itself toward the resolution of the issues that existed at the time. Criticisms of CHIEN's work can be found in the papers of GOL'DENVEIZER and LUR'E [1947, *3*] and GREEN and ZERNA [1950, *2*].

[144] In the literature on shell theory, (21 A.10)–(21 A.11) are often referred to as the Flügge-Lur'e-Byrne equations. These constitutive relations were originally derived in the case of cylindrical shells by FLÜGGE [1932, *1*] and in general form by LUR'E [1940, *2*]. Similar developments were constructed independently in general form by BYRNE (1941) and for shells of revolution by TRUESDELL (1943). The paper by LUR'E [1940, *2*] did not become known until after World War II and the works of BYRNE and TRUESDELL were evidently delayed in publication (BYRNE [1944, 1] and TRUESDELL [1945, *3*]); see also, in this regard, the remarks by TRUESDELL in the last paragraph of [1953, *5*], where references to the original works of BYRNE (1941) and TRUESDELL (1943) are cited. In view of this background, the relations (21 A. 10)–(21 A. 11) may be referred to as the FLÜGGE-LUR'E-BYRNE-TRUESDELL equations. For accounts of constitutive equations (21 A.10)–(21 A.11) see also FLÜGGE's books [1934, *1*], [1960, *4*] and "Notes to Chapter I" at the end of the book by LUR'E [1947, *4*]. A general derivation of (21 A.10)–(21 A.11) can be found in [1963, *6*].

In recording the constitutive equations (21 A.10)–(21 A.11), so far as possible, we have employed the same notations as in (21 A.5). As should be apparent from (21 A.8) when specialized to lines of curvature coordinates, the measures $\bar{\varrho}_{\langle 11\rangle}$ and $\bar{\varrho}_{\langle 22\rangle}$ in (21 A.10)–(21 A.11) are equivalent to the physical components of $\beta_{\alpha|\beta}$ for values of $\alpha = \beta$. It is the latter variables that occur in the constitutive equations (21 A.10)–(21 A.11) as used in the literature on shell theory.

If the second set of terms on the right-hand sides of the above equations, i.e., those involving

$$\pm\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

are neglected, then we recover the version of LOVE's first approximation which includes (21 A.7). The constitutive equations (21 A.10)–(21 A.11) do not have any of the shortcomings associated with different versions of LOVE's first approximation and, in particular, are invariant under infinitesimal rigid body displacement and under transformation of the middle surface coordinates.[145] However, it is quite clear that the additional terms in (21 A.10)–(21 A.11) which involve

$$\pm\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

are of $O(h/R)$ and thus these equations represent more than a first approximation in the linear theory. Mention should be made here of several related derivations which include the effect of transverse shear deformation and sometimes also the effect of transverse normal stress.[146] Most of these results are such that upon the neglect of the latter effects, they reduce to (21 A.10)–(21 A.11) or variants thereof.

A noteworthy contribution toward a satisfactory derivation of the constitutive equations of a first approximation theory was published in 1946 by NOVOZHILOV.[147] This derivation, which is carried out in lines of curvature coordinates, remained virtually unknown in the Western hemisphere until the early 1960's. NOVOZHILOV's constitutive equations have certain features which (in lines of curvature coordinates) appear to be close to those in (21 A.5).[148] Whereas NOVOZHILOV's equations remain invariant under infinitesimal rigid body displacement, satisfy the equilibrium equation resulting from the symmetry of the stress tensor and also admit two-dimensional energetic theorems (including a reciprocity theorem), nevertheless they are not entirely satisfactory. For example, these equations do not attain a simple form for spherical shells and NOVOZHILOV himself recommends the use of the constitutive equations (21 A.10)–(21 A.11) in this case.[149] Moreover, NOVOZHILOV's equations are not invariant under the transformation of the middle surface coordinates. This shortcoming no doubt is because approximations were made in lines of curvature coordinates rather than in a derivation carried out in general coordinates; however, if this deficiency is remedied and the set of equations given by NOVOZHILOV is put in an invariant form, then part of the simplicity is lost and some of the equations will contain additional terms of $O(h/R)$ similar to those in (21 A.10)–(21 A.11).[150] References to other and more recent contributions concerning the derivation of constitutive equations of a first approximation theory are cited in Sects. 20–21 and need not be repeated here.

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[145] For details on these and related discussions concerning the constitutive equations (21 A.10)–(21 A.11), see [1963, *6*] and [1966, *8*].

[146] Among these we mention the derivations by HILDEBRAND, REISSNER and THOMAS [1949, *4*], GREEN and ZERNA [1950, *2*], REISSNER [1952, *2*] and NAGHDI [1957, *4*], [1963, *6*]. A criticism by GOL'DENVEIZER (cited in [1960, *9*]) to the effect that the constitutive equations in [1957, *4*] are not invariant under infinitesimal rigid body displacements has no basis. This was acknowledged by him later, as was noted in [1960, *9*].

[147] NOVOZHILOV [1946, *3*]. See also the English translation of his book [1959, *3*]. According to NOVOZHILOV [1959, *3*, p. 54] similar approximate constitutive equations were obtained independently by L. I. BALABUKH. The reference to the work of BALABUKH (1946) is cited on p. 83 of [1961, *3*].

[148] See NOVOZHILOV's equations (10.16) in [1959, *3*].

[149] See the remark on p. 54 of [1959, *3*].

[150] A more detailed discussion of NOVOZHILOV's equations can be found in Sects. 6.3 and 6.4 of [1963, *6*].

22. Relationship of results from the three-dimensional theory and the theory of Cosserat surface. We return to the field equations and the constitutive relations of the approximate nonlinear theory (derived from the three-dimensional equations) and consider their comparison with the corresponding results derived by direct approach (and summarized in Sect. 14). A close examination of the field equations (9.47)–(9.48) and (9.51) readily reveals that these equations are of the same form as (12.28)–(12.30). Moreover, the set of constitutive equations (13.64)–(13.65) for the Cosserat surface has the same form as (18.4)–(18.5), apart from an extra generality in Sect. 18 for the temperature. If we also adopt the approximation (18.7) for the temperature, identify the surface $\hat{s}$ of the (three-dimensional) shell-like body with s and (11.1) with (4.36), then the field equations and the constitutive equations for elastic shells in the two developments are formally equivalent.

In the theory of a Cosserat surface, the skew-symmetric part of $M^{\alpha\beta}$ is specified by a constitutive equation which has no counterpart in the approximate development from the three-dimensional equations. The latter is due to the nature of the approximation adopted for the (two-dimensional) specific free energy ψ in the derivation from the three-dimensional theory. Our reasons for this choice of an approximate expression for ψ were discussed in Sects. 17–18. Alternatively, a specification of a different (and equally acceptable) approximate expression for ψ could result in a constitutive equation for $M^{\alpha\beta}$ rather than $M^{(\alpha\beta)}$.

Although the constitutive equations in Sect. 18 are obtained from the three-dimensional equations, they involve an approximate function for the free energy and, as already remarked, in general it is difficult to evaluate this approximate expression from the full three-dimensional expression for the free energy. On the other hand, given the Cosserat surface $\mathscr{C}$ as a model for a thin shell and the balance principles stated in Sect. 8, the resulting theory is exact; but, it requires the additional (and sometimes difficult) considerations regarding the interpretations of the results and identification of the constitutive coefficients.

The contact force $\boldsymbol{N}$ and the contact director couple $\boldsymbol{M}$ in the theory of a Cosserat surface, as well as $\boldsymbol{N}^\alpha$ in (9.11) and $\boldsymbol{M}^\alpha$ in (9.19), have respectively the physical dimensions of force per unit length and couple per unit length, as indicated in (8.5). Moreover, these vector fields have the same physical dimensions as the stress-resultants and the stress-couples defined by (11.36). Hence, in view of the formal equivalence of the two developments noted above, we may identify $\boldsymbol{N}^\alpha$ and $\boldsymbol{M}^\alpha$ (in the theory of a Cosserat surface) with the corresponding stress-resultants and stress-couples. Similar identifications can be made for other quantities, including the identification of the assigned force $\boldsymbol{f}$ and the assigned director couple $\boldsymbol{l}$ with the corresponding load resultants in (11.30) with $N=0, 1$. Further, recalling (8.19), we may identify $f^i - c^i = \bar{f}^i$ and $\bar{l}^i$ in (9.47)–(9.48) with the corresponding quantities in (12.28)–(12.29). The identification of $\bar{l}^i$ in the two sets of equations entails also that we put the director inertia coefficient α in $(8.19)_2$ equal to k^{11} in $(12.25)_2$, i.e.,

$$\alpha = k^{11}. \tag{22.1}$$

With (22.1) the identification is complete and the equations of motion (9.47)–(9.48) correspond exactly to (12.28)–(12.29).

While a one-to-one correspondence can be established between the various quantities in our two developments, namely that by direct approach and from the approximate three-dimensional theory, the relationship of the latter to the results in the classical theory of shells needs further elaboration. In the classical (approximate) linear theory of shells developed from the three-dimensional equations, where the initial position vector is of the form (4.27), the stress-couple resultant is defined by a tangential vector field[151]

$$\begin{aligned}\mathcal{M}^\alpha &= \int_{-h/2}^{h/2} (\boldsymbol{A}_3 \times \boldsymbol{T}^\alpha)\, \zeta\, d\zeta, \\ &= M^{\alpha\beta}\, \boldsymbol{A}_3 \times \boldsymbol{A}_\beta.\end{aligned} \tag{22.2}$$

On the other hand, in the linearized version of the field equations as derived (from the three-dimensional equations) in Sects. 11–12 and with the initial position vector specified by (7.30)–(7.31), it is the quantity $\boldsymbol{D} \times \boldsymbol{M}^\alpha$ or

$$\begin{aligned}\boldsymbol{D} \times \boldsymbol{M}^\alpha &= D \boldsymbol{A}_3 \times M^{\alpha i} \boldsymbol{A}_i, \\ &= D M^{\alpha\beta} \boldsymbol{A}_3 \times \boldsymbol{A}_\beta,\end{aligned} \tag{22.3}$$

which corresponds to (22.2). In view of earlier remarks, a similar comparison also holds between $\boldsymbol{D} \times \boldsymbol{M}^\alpha$ (in the theory of a Cosserat surface) and $\mathcal{M}^\alpha$ as defined by (22.2). It is therefore clear that in the comparison with the classical theory (where $M^{\alpha 3}$ is not defined) it is the quantity $D M^{\alpha\beta}$ [or $\boldsymbol{D} \times \boldsymbol{M}^\alpha$] which must be identified with $M^{\alpha\beta}$ in (22.2). This is simply due to the fact that in the classical theory of shells $M^{\alpha\beta}$ is defined through (22.2), or through the stress-couples in $(12.42)_6$, and *not* by $\mathcal{M}^\alpha = M^{\alpha i} \boldsymbol{A}_i$. In this connection, see also the remarks made in Sect. 12 [Subsect. η)].

Before closing this section, we calculate explicitly the value of the inertia coefficient k^{11} in (22.1) appropriate to the linear theory and when the position vector of the shell-like body in the initial reference configuration is given by (7.30) with $\boldsymbol{D} = \boldsymbol{A}_3$. Recalling (4.21), (7.46) and $(7.48)_{1,3}$, from (12.26) we readily obtain

$$\varrho\, k^{11}\, a^{\frac{1}{2}} = \varrho_0^*\, A^{\frac{1}{2}} \frac{h^3}{12}. \tag{22.4}$$

But $\varrho_0 = \varrho_0^*\, h$ by (4.17) and (7.46) and $\varrho\, a^{\frac{1}{2}} = \varrho_0\, A^{\frac{1}{2}}$ by (4.41)–(4.42). Using these results, in (22.4) yields $k^{11} = h^2/12$. Hence, the director inertia coefficient α which occurs in $\bar{L}^i$ of the linearized equations of motion (9.55) is given by

$$\alpha = \frac{h^2}{12}, \tag{22.5}$$

in view of the identification (22.1).

E. Linear theory of elastic plates and shells.

This concluding chapter, though limited only to the linear theory, is a culmination of the point of view adopted earlier in Sect. 4 regarding shells and plates. Our starting point here is a system of equations for the linear theory derived in the previous chapters by direct approach. After providing for ease of reference a brief summary of the field equations and the constitutive relations, the remainder of the chapter is devoted to the detailed considerations of the linear theory. Special attention is given to the determination of the constitutive coefficients, proof of a uniqueness theorem and certain results pertaining to the classical linear theory of shells and plates.

In order to concentrate attention on the main aspects of the subject, especially in regard to the identification of the constitutive coefficients, we limit ourselves to the isothermal theory. We note, however, that this limitation is not essential. Moreover, it should be clear that the developments of this chapter can be regarded

[151] See for example Chap. 10 of Green and Zerna [1968, *9*].

as appropriate for the purely mechanical theory in line with the remarks made in Sect. 14 [Subsect. β)].

23. The boundary-value problem in the linear theory. In this section, we summarize the field equations and a system of constitutive relations in the linear isothermal theory of a Cosserat surface which characterize the initial boundary-value problem of elastic shells and plates. However, the case of a flat plate is considered separately from that of the initially curved shell, as this will be more illuminating.

α) *Elastic plates.* It is convenient to recall here the basic equations for elastic plates with reference to rectangular Cartesian coordinates. Thus, let the surface coordinates on the initial surface $\mathscr{S}$ be identified with the rectangular Cartesian coordinates x_α. Then, referred to rectangular Cartesian coordinates, the equations of motion (9.73)–(9.74) can be written as

$$N_{\alpha\beta,\alpha} + \varrho_0 \bar{F}_\beta = 0, \quad M_{\alpha 3,\alpha} + \varrho_0 \bar{L}_3 = V_3, \tag{23.1}$$

$$M_{\alpha\beta,\alpha} + \varrho_0 \bar{L}_\beta = V_\beta, \quad V_{\alpha,\alpha} + \varrho_0 \bar{F}_3 = 0, \tag{23.2}$$

where a comma denotes partial differentiation with respect to x_α and all quantities in (23.1)–(23.2) are now referred to rectangular Cartesian coordinates so that no distinction between the position of the indices (subscripts and superscripts) in such quantities as $N^{\alpha\beta}$, $M^{\alpha i}$, L^i is necessary. The above equations of motion include the effect of inertia due to both displacement $\boldsymbol{u}$ and director displacement $\boldsymbol{\delta}$. The latter, which corresponds to "rotatory inertia", occurs through $\bar{L}_i$ defined by $(9.57)_2$ and $(8.19)_2$ with the director inertia coefficient α given by (22.5).

The constitutive relations of the linear theory considered here are those for an isotropic Cosserat plate obtained under the restriction (16.31). Recalling (16.33) and (9.72), from (16.11) referred to rectangular Cartesian coordinates, we have the following constitutive relations for initially flat elastic Cosserat plates:

$$\begin{aligned} N_{\alpha\beta} = N_{\beta\alpha} &= \alpha_1 \delta_{\alpha\beta} e_{\gamma\gamma} + 2\alpha_2 e_{\alpha\beta} + \alpha_9 \delta_{\alpha\beta} \gamma_3, \\ V_3 &= \alpha_4 \gamma_3 + \alpha_9 e_{\gamma\gamma}, \quad M_{\alpha 3} = \alpha_8 \varkappa_{3\alpha} \end{aligned} \tag{23.3}$$

and

$$M_{\alpha\beta} = \alpha_5 \delta_{\alpha\beta} \varkappa_{\gamma\gamma} + \alpha_6 \varkappa_{\beta\alpha} + \alpha_7 \varkappa_{\alpha\beta}, \quad V_\alpha = \alpha_3 \gamma_\alpha. \tag{23.4}$$

The kinematic measures in (23.3)–(23.4) are those in (6.25) referred to rectangular Cartesian coordinates so that now covariant differentiation in (6.25) is replaced with partial differentiation.

An examination of the equations of motion (23.1)–(23.2) and the constitutive equations (23.3)–(23.4) readily reveals that the differential equations of the linear theory of an isotropic Cosserat plate separate into two systems of uncoupled equations: One system, consisting of (23.1), (23.3) and $(6.25)_{3,4,5}$, represents the stretching (or the extensional motion) of the plate while the other, given by (23.2), (23.4) and $(6.25)_{6,7}$, characterizes the bending (or the flexural motion) of the plate. Among the features of the extensional theory, we note the presence of five constitutive coefficients in (23.3), a constitutive equation for the normal force V_3 (corresponding to that for the normal stress-resultant) and a constitutive equation for $M_{\alpha 3}$ (corresponding to that for the shear stress-couple). Similarly, the bending theory[1] contains four constitutive coefficients in (23.4) and includes a constitutive

[1] Apart from the presence of a constitutive equation for $M_{[\alpha\beta]}$, the equations of the bending theory are of the same form as those in REISSNER's plate theory discussed in Sect. 20.

equation for V^α (corresponding to that for the shear stress-resultant) in terms of the kinematic variable γ_α; the latter can be regarded as representing the effect of "transverse shear deformation".

The nature of the boundary conditions in the above linear theory is clear from the linearized version of the rate of work expression (8.8). In particular, the force and the director couple boundary conditions are given by[2]

$$N_\gamma = N_{\alpha\gamma}\, {}_0\nu_\alpha, \quad M_3 = M_{\alpha 3}\, {}_0\nu_\alpha, \quad N_3 = V_\alpha\, {}_0\nu_\alpha, \quad M_\gamma = M_{\alpha\gamma}\, {}_0\nu_\alpha, \tag{23.5}$$

where ${}_0\nu_\alpha$ in (23.5) are the components of the outward unit normal to the boundary curve of the initial surface $\mathscr{S}$ referred to rectangular Cartesian coordinates. The boundary conditions (23.5) hold pointwise on the boundary $\partial\mathscr{S}$.

β) *Elastic shells.* We recall the equations of motion for shells in terms of the variables $N^{\alpha\beta}$, $M^{\alpha i}$, V^i and in the forms (9.68)–(9.70) which correspond to those in (12.42). The equations of motion (9.70) include the inertia terms in L^i arising from the director displacement. These terms, with director inertia coefficient α given by (22.5), represent the effect of "rotatory inertia".

The constitutive equations considered here are those for an isotropic material which are obtained under the restriction (16.35) with the further stipulation that $\bar{\psi}$ be independent of $B_{\alpha\beta}$. Thus, in line with the discussion following (16.35), we assume that $\bar{\psi}$ is specified by (16.33) and from (16.11) obtain the desired constitutive relations for an elastic Cosserat surface. These constitutive relations, which are similar to those in (16.22)–(16.24), can be expressed in the form

$$\begin{aligned} N'^{\alpha\beta} &= [\alpha_1 A^{\alpha\beta} A^{\gamma\delta} + \alpha_2 (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\, e_{\gamma\delta} + \alpha_9 A^{\alpha\beta} \gamma_3, \\ M^{\beta\alpha} &= [\alpha_5 A^{\alpha\beta} A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma} A^{\beta\delta} + \alpha_7 A^{\alpha\delta} A^{\beta\gamma}]\, \varrho_{\gamma\delta} \end{aligned} \tag{23.6}$$

$$V^\alpha = \alpha_3 A^{\alpha\gamma} \gamma_\gamma, \qquad V^3 = \alpha_4 \gamma_3 + \alpha_9 A^{\alpha\beta} e_{\alpha\beta}, \tag{23.7}$$

$$M^{\alpha 3} = \alpha_8 A^{\alpha\gamma} \varrho_{3\gamma}, \tag{23.8}$$

where the kinematic measures are defined in (6.24). The above constitutive relations have been expressed in terms of the variables $N'^{\alpha\beta}$, $M^{\alpha i}$, V^i and the kinematic measures $e_{\alpha\beta}$, $\varrho_{i\alpha}$, γ_i. Alternatively, we could adopt the set $e_{\alpha\beta}$, $\varkappa_{i\alpha}$, γ_i as the kinematic variables for the linear theory and record the constitutive relations corresponding to (23.6) in terms of $N'^{\alpha\beta}$, $M^{\alpha i}$, m^i. We note that upon specialization and with $B_{\alpha\beta}=0$, (23.6)–(23.8) reduce to (23.3)–(23.4), since for flat plates $\varrho_{i\alpha} = \varkappa_{i\alpha}$.

We again observe that the nature of the boundary conditions is clear from the linearized version of the rate of work expression (8.8). In particular, with reference to the above forms of the equations of motion and constitutive equations, the force and the director couple boundary conditions for shells are given by

$$N^\gamma = N^{\alpha\gamma}\, {}_0\nu_\alpha, \qquad N^3 = V^\alpha\, {}_0\nu_\alpha, \qquad M^i = M^{\alpha i}\, {}_0\nu_\alpha, \tag{23.9}$$

where ${}_0\nu_\alpha$ are the components of the outward unit normal ${}_0\boldsymbol{\nu}$ to the boundary curve $\partial\mathscr{S}$ of the initial surface $\mathscr{S}$ defined previously in Sect. 9 [Subsect. γ)]. It should be noted that the components $N^{\alpha\gamma}$ in $(23.9)_1$ involve both the symmetric components $N^{(\alpha\gamma)}$ and the skew-symmetric part $N^{[\alpha\gamma]}$. The former is calculated from $(23.6)_1$ and $(9.67)_3$ while the latter is given by[3]

$$N^{[\alpha\gamma]} = \tfrac{1}{2} (M^{\beta\gamma} B^{\alpha}_{\beta} - M^{\beta\alpha} B^{\gamma}_{\beta}). \tag{23.10}$$

[2] These follow from linearized versions of (9.41) and (9.43), as well as (9.72).

[3] Recall that only the symmetric part of $N^{\alpha\beta}$ has a constitutive equation. The expression (23.10) follows from (9.68).

Before closing this section, we briefly indicate the manner in which a (slightly less general) system of equations for the linear theory can be obtained as a special case of those stated above.[4] Let

$$\alpha_8 = 0 \tag{23.11}$$

and hence[5]

$$M^{\alpha 3} = 0. \tag{23.12}$$

Then, the differential equations of motion are given by (9.69)–$(9.70)_1$ and in place of $(9.70)_2$ we have

$$V^3 = \varrho_0 \bar{L}^3. \tag{23.13}$$

The constitutive equations are given by (23.6)–(23.7) and the specification of five (instead of six) boundary conditions are required at each edge of the shell. These boundary conditions are either the displacement boundary conditions

$$u_\alpha,\ u_3,\ \delta_\alpha\ (\text{or } \gamma_\alpha) \tag{23.14}$$

or the force boundary conditions $(23.9)_{1,2}$ and

$$M^\gamma = M^{\alpha\gamma}\,{}_0\nu_\alpha. \tag{23.15}$$

Among other features of this theory we note the presence of: (i) a constitutive equation for V^α representing the effect of "transverse shear deformation"; (ii) a constitutive equation for the normal force V^3 per unit length (corresponding to a normal stress-resultant); (iii) a constitutive equation for the skew-symmetric director couple $M^{[\beta\alpha]}$; and (iv) eight constitutive coefficients in (23.6)–(23.7).

24. Determination of the constitutive coefficients. The relationship and the correspondence between the theory of a Cosserat surface and the theory of shells and plates obtained from the general three-dimensional equations have been already brought out. Moreover, in view of the observations made in Sect. 22, we may identify $N^{\alpha\beta}$ and $N^{\alpha 3}$ (or V^α) as stress-resultants and shear stress-resultants, $M^{\alpha\beta}$ and $M^{\alpha 3}$ as stress-couples and shear stress-couples and V^3 may be called a normal stress-resultant. In the constitutive relations (23.6)–(23.8), as well as in (23.3)–(23.4), the coefficients $\alpha_1, \ldots, \alpha_9$ are so far arbitrary; however, as will become evident presently, most of these coefficients can be identified by comparison of certain simple solutions with corresponding exact solutions in the three-dimensional theory of linear elasticity. In what follows, we first consider the determination of the constitutive coefficients for initially flat plates and then discuss the constitutive coefficients for shells separately.

α) *The constitutive coefficients for plates.* Before proceeding with the identification of the constitutive coefficients, we elaborate once more on the relationship between the variables $N^{\alpha\beta}$, $M^{\alpha i}$, V^i and the stress-resultants and the stress-couples defined in the course of our derivation from the three-dimensional equations. For this purpose, consider a (three-dimensional) plate of uniform thickness h. Let the plate, referred to a system of rectangular Cartesian coordinates x_i, be defined by its middle plane $x_3 = 0$ and the region

$$-\frac{h}{2} \leq x_3 \leq \frac{h}{2}.$$

[4] The system of equations discussed below [between (23.11)–(23.15)] correspond to those in Sect. 20 with constitutive equations (20.47).

[5] It should be noted in obtaining (23.12) no restriction is placed on δ_3 or $\varrho_{3\gamma}$.

Further, let σ_{ij} denote the Cartesian components of the symmetric stress tensor and consider the resultants

$$\int_{-h/2}^{h/2} \sigma_{\alpha\beta}\, dx_3, \quad \int_{-h/2}^{h/2} \sigma_{\alpha 3}\, dx_3, \quad \int_{-h/2}^{h/2} \sigma_{33}\, dx_3, \quad \int_{-h/2}^{h/2} \sigma_{\alpha\beta}\, x_3\, dx_3, \quad \int_{-h/2}^{h/2} \sigma_{\alpha 3}\, x_3\, dx_3. \tag{24.1}$$

It follows from (12.43) and the remarks made in Sect. 22 that the resultants (24.1), in the order listed, may be identified with $N_{\alpha\beta}$, V_α, V_3, $M_{(\alpha\beta)}$, $M_{\alpha 3}$ in the linear theory of a Cosserat plate.[6]

We now proceed to identify the constitutive coefficients in (23.3)–(23.4). We consider, in particular, three simple elastostatic problems, namely pure bending of a plate, extensional deformation in the plane of the plate and a plate under a uniform hydrostatic pressure. From a comparison of solutions of these simple examples with the corresponding exact solutions in linear three-dimensional elasticity, we determine seven of the nine coefficients in (23.3)–(23.4). In addition, we also make some remarks pertaining to the specification of the remaining coefficients. In this connection and for later reference, we recall here the three-dimensional linear constitutive relations for transversely isotropic elastic materials. These constitutive relations which involve five independent coefficients are given by[7]

$$\begin{aligned} \sigma_{11} &= c_{11}\, e^*_{11} + c_{12}\, e^*_{22} + c_{13}\, e^*_{33}, & \sigma_{23} &= 2 c_{44}\, e^*_{23}, \\ \sigma_{22} &= c_{12}\, e^*_{11} + c_{11}\, e^*_{22} + c_{13}\, e^*_{33}, & \sigma_{31} &= 2 c_{44}\, e^*_{31}, \\ \sigma_{33} &= c_{13} (e^*_{11} + e^*_{22}) + c_{33}\, e^*_{33}, & \sigma_{12} &= (c_{11} - c_{12})\, e^*_{12}, \end{aligned} \tag{24.2}$$

where e^*_{ij} are the Cartesian components of the three-dimensional strain tensor defined by (7.60) which is now referred to rectangular Cartesian coordinates and the coefficients $c_{11}, \ldots, c_{44}$ in (24.2) are the elasticity constants. In the case of an isotropic material, the coefficients $c_{11}, \ldots, c_{44}$ assume the values

$$c_{11} = c_{33} = \lambda + 2\mu, \qquad c_{12} = c_{13} = \lambda, \qquad c_{44} = \mu, \tag{24.3}$$

where λ and μ are the Lamé constants.

(i) *Pure bending of a plate.* Consider a rectangular Cosserat plate in equilibrium, bounded by the lines (or edges) $x_1 = a_1, a_2$, $x_2 = b_1, b_2$, and subjected to uniform couples of constant magnitude M_1 and M_2 along the edges $x_1 =$ const. and $x_2 =$const., respectively. The appropriate boundary conditions in this case are[8]

$$\begin{aligned} M_{11} &= M_1, \quad M_{12} = V_1 = 0 \quad \text{on} \quad x_1 = a_1, a_2, \\ M_{22} &= M_2, \quad M_{21} = V_2 = 0 \quad \text{on} \quad x_2 = b_1, b_2. \end{aligned} \tag{24.4}$$

The differential equations for elastostatic bending problems of a Cosserat plate subjected to edge tractions alone are given by (23.2) and (23.4) in the absence of $\overline{L}_\beta$ and $\overline{F}_3$. We seek a solution of these equations for which

$$\begin{gathered} \gamma_\alpha = 0, \\ \varkappa_{11} = -u_{3,11} = \text{const.}, \qquad \varkappa_{22} = -u_{3,22} = \text{const.}, \qquad \varkappa_{12} = -u_{3,12} = 0. \end{gathered} \tag{24.5}$$

[6] It should be recalled that in the linear theory of a Cosserat plate characterized by (23.1)–(23.4), the initial director is of constant length and coincident with the unit normal to the initial surface (i.e., $D_\alpha = 0$, $D_3 = D = 1$). Also, since $\sigma_{\alpha\beta}$ is symmetric, only the symmetric part of $M_{\alpha\beta}$, namely $M_{(\alpha\beta)}$, can be identified with $(24.1)_4$.

[7] See Green and Zerna [1968, *9*, p. 178].

[8] Since we are concerned with bending of an isotropic plate, it is not necessary to consider the equations of the extensional theory. However, for the example under consideration, we note that a zero solution for $\{N_{\alpha\beta}, V_3, M_{\alpha 3}\}$ identically satisfies (23.1) and (23.3) in the absence of $\overline{F}_\beta$ and $\overline{L}_3$.

It then follows that all equilibrium equations in (23.2) with $L_\beta = \bar{F}_3 = 0$ are identically satisfied and the only non-identically vanishing constitutive relations are those for M_{11} and M_{22} from which we obtain

$$-u_{3,11} = \varkappa_{11} = \frac{M_1 - \beta M_2}{\alpha(1-\beta^2)}, \qquad -u_{3,22} = \varkappa_{22} = \frac{M_2 - \beta M_1}{\alpha(1-\beta^2)}, \tag{24.6}$$

where we have put

$$\alpha = \alpha_5 + \alpha_6 + \alpha_7, \qquad \beta = \frac{\alpha_5}{\alpha}. \tag{24.7}$$

The solution of the system of differential Eqs. $(24.5)_4$ and (24.6) is

$$u_3 = -\tfrac{1}{2}(\varkappa_{11}\, x_1^2 + \varkappa_{22}\, x_2^2), \tag{24.8}$$

where the constants $\varkappa_{11}$ and $\varkappa_{22}$ are given by the right-hand sides of each of the expressions in $(24.6)_{1,2}$ and where the arbitrary constants of integration have been set equal to zero without loss in generality. The solution (24.8) includes those for bending of a Cosserat plate into a spherical surface ($\varkappa_{11} = \varkappa_{22}$, $M_{11} = M_{22}$) or into an anticlastic surface ($\varkappa_{11} = -\varkappa_{22}$, $M_{11} = -M_{22}$).

Consider now a three-dimensional rectangular plate of uniform thickness h which initially is transversely isotropic with respect to the normals of the plate and let the plate be subjected to uniform bending couples of magnitude M_1 and M_2, each per unit length, along the edges parallel to x_2 and x_1 directions, respectively. From an elementary result in three-dimensional elasticity, the solution of this example for $\varkappa_{11}$ and $\varkappa_{22}$ has the same form as those given by the right-hand members in each of the expressions (24.6) and the solution for u_3 (on the middle plane only) is of the same form as (24.8).[9] Hence, by comparison, we set[10]

$$\begin{gathered} \alpha_6 + \alpha_7 = \frac{h^3}{12}(c_{11} - c_{12}), \qquad \alpha_6 - \alpha_7 = 0, \\ \alpha_5 = \frac{h^3}{12}\,\frac{c_{12}c_{33} - c_{13}^2}{c_{33}}, \end{gathered} \tag{24.9}$$

where $c_{11}, c_{12}, c_{13}, c_{33}$ are the elastic coefficients for a plate which is transversely isotropic with respect to its normals. In the case of an isotropic plate, these coefficients are given by (24.3) and the coefficients (24.9) reduce to

$$\alpha_5 = \nu B, \qquad \alpha_6 = \alpha_7 = \tfrac{1}{2}(1-\nu)\,B, \tag{24.10}$$

where the flexural rigidity B is defined by $(20.13)_3$ and ν is Poisson's ratio.

(ii) *Extensional deformation of a plate.* Consider a rectangular Cosserat plate in equilibrium, bounded by the lines $x_1 = a_1, a_2$, $x_2 = b_1, b_2$ and subjected to uniform forces per unit length (in the plane of the plate) of magnitude N_1 and N_2 along the edges $x_1 = \text{const.}$ and $x_2 = \text{const.}$, respectively. The appropriate boundary conditions in this case are[11]

$$\begin{aligned} N_{11} &= N_1, \quad N_{12} = M_{13} = 0 \quad \text{on} \quad x_1 = a_1, a_2, \\ N_{22} &= N_2, \quad N_{21} = M_{23} = 0 \quad \text{on} \quad x_2 = b_1, b_2. \end{aligned} \tag{24.11}$$

[9] In the case of transversely isotropic materials for which the components σ_{i3} of the stress tensor vanish, the results can be easily deduced using the equations given in GREEN and ZERNA [1968, *9*, p. 178]. The corresponding solution for an isotropic plate may be found in LOVE [1944, *4*, p. 132], or in TIMOSHENKO and GOODIER [1951, *3*, p. 255].

[10] In making such comparisons we put $\alpha_6 - \alpha_7 = 0$, since the coefficient $(\alpha_6 - \alpha_7)$ has no counterpart in (non-polar) three-dimensional linear elasticity. The results (24.9) and (24.10) were given previously by GREEN and NAGHDI [1967, *4*] through consideration of pure bending of a plate as discussed here.

[11] Since we are concerned with extensional deformation of an isotropic plate, it is not necessary to consider the equations of the bending theory. However, for the example under consideration, we note that a zero solution for $\{M_{\alpha\beta}, V_\alpha\}$ identically satisfies (23.2) and (23.4) in the absence of $\bar{L}_\beta$ and $\bar{F}_3$.

The differential equations for elastostatic extensional problems of a Cosserat plate subjected to edge tractions alone are given by (23.1) and (23.3) in the absence of $\bar{F}_\beta$ and $\bar{L}_3$. Here we seek a solution of these equations in the form

$$\begin{gathered}\gamma_3 = \text{const.},\\ e_{11} = u_{1,1} = \text{const.}, \qquad e_{22} = u_{2,2} = \text{const.}, \qquad e_{12} = u_{1,2} = 0.\end{gathered} \tag{24.12}$$

From $(23.3)_3$ we have $M_{\alpha 3} = 0$ and, in the absence of $\bar{L}_3$, $(23.1)_2$ yields $V_3 = 0$ which along with $(23.3)_2$ gives

$$\gamma_3 = -\frac{\alpha_9}{\alpha_4} e_{\gamma\gamma}. \tag{24.13}$$

The equilibrium equation $(23.1)_1$ with $\bar{F}_\beta = 0$ is identically satisfied and the only remaining non-identically vanishing constitutive relations are those for N_{11} and N_{22} from which we obtain

$$\begin{aligned} u_{1,1} &= \frac{\bar{\beta} N_2 - \bar{\alpha} N_1}{\bar{\beta}^2 - \bar{\alpha}^2} = \text{const.} = \lambda_1 \text{ (say)},\\ u_{2,2} &= \frac{\bar{\beta} N_1 - \bar{\alpha} N_2}{\bar{\beta}^2 - \bar{\alpha}^2} = \text{const.} = \lambda_2 \text{ (say)}, \end{aligned} \tag{24.14}$$

where

$$\bar{\alpha} = 2\alpha_2 + \bar{\beta}, \quad \bar{\beta} = \alpha_1 - \frac{\alpha_9^2}{\alpha_4} = \alpha_1 - \alpha_9 \bar{\gamma}, \quad \bar{\gamma} = \frac{\alpha_9}{\alpha_4} \tag{24.15}$$

and we have introduced $(24.15)_3$ for later convenience. The solution of the system of differential equations (24.14) and $(24.12)_4$ is

$$u_1 = \lambda_1 x_1, \qquad u_2 = \lambda_2 x_2, \tag{24.16}$$

where the constants λ_1 and λ_2 are those in $(24.14)_{1,2}$. Also, by (24.13) and (24.16), γ_3 can be written as

$$\gamma_3 = -\bar{\gamma}(\lambda_1 + \lambda_2). \tag{24.17}$$

Consider now a three-dimensional rectangular plate of uniform thickness h, as defined earlier in this section. The plate is initially transversely isotropic with respect to its normals and is subjected to uniform tractions N_1 and N_2, each per unit length, along the edges parallel to the x_1 and x_2 directions. The boundary conditions in this case can be written as

$$\begin{aligned} &\sigma_{11} = \frac{N_1}{h}, \quad \sigma_{12} = \sigma_{13} = 0 \quad \text{on} \quad x_1 = a_1, a_2,\\ &\sigma_{22} = \frac{N_2}{h}, \quad \sigma_{21} = \sigma_{23} = 0 \quad \text{on} \quad x_2 = b_1, b_2,\\ &\qquad\qquad \sigma_{3i} = 0 \quad \text{on} \quad x_3 = \pm\frac{h}{2}. \end{aligned} \tag{24.18}$$

From an elementary result in three-dimensional elasticity for extensional deformation of a plate, the (three-dimensional) displacements are

$$u_1^* = \lambda_1 x_1, \qquad u_2^* = \lambda_2 x_2, \qquad u_3^* = \lambda_3 x_3. \tag{24.19}$$

With the use of (24.19), (24.2) and the boundary conditions (24.18), the solutions for e_{11}^*, e_{22}^* and e_{33}^* are easily found to have the same form as those given by the right-hand members in each of (24.14) and (24.17). Hence, by comparison, we may set

$$\bar{\alpha} = \alpha_1 + 2\alpha_2 - \frac{\alpha_9^2}{\alpha_4} = h\left(c_{11} - \frac{c_{13}^2}{c_{33}}\right), \quad \bar{\beta} = \alpha_1 - \frac{\alpha_9^2}{\alpha_4} = h\left(c_{12} - \frac{c_{13}^2}{c_{33}}\right),$$
$$\bar{\gamma} = \frac{\alpha_9}{\alpha_4} = \frac{c_{13}}{c_{33}}. \tag{24.20}$$

The results in (24.20) provide only three relations involving the four coefficients $\alpha_1, \alpha_2, \alpha_4, \alpha_9$; and, before the coefficients $\alpha_1, \ldots, \alpha_9$ can be solved in terms of the elastic constants $c_{11}, c_{12}, c_{13}, c_{33}$, we need a fourth relation which is obtained next.

(iii) *A plate under a uniform hydrostatic pressure.* Consider a plate of uniform thickness h subjected to a uniform hydrostatic pressure and recall first its solution as an elastostatic problem in the three-dimensional theory. The solution for the stresses is simply

$$\sigma_{ij} = -p\,\delta_{ij}, \tag{24.21}$$

where p is a positive constant. The above solution satisfies all stress boundary conditions and the (three-dimensional) equilibrium equations in the absence of body forces. Using (24.21), the constitutive equations (24.2) yield

$$e_{11}^* = e_{22}^* = -\left[\frac{c_{13} - c_{33}}{2c_{13}^2 - (c_{11} + c_{12})\,c_{33}}\right] p = \lambda \quad \text{(say)},$$
$$e_{33}^* = -\left[\frac{2c_{13} - (c_{11} + c_{12})}{2c_{13}^2 - (c_{11} + c_{12})\,c_{33}}\right] p = \bar{\lambda} \quad \text{(say)}. \tag{24.22}$$
$$\text{All other } e_{ij}^* = 0.$$

From (24.22) and the strain-displacements relations, we readily obtain the displacements

$$u_1^* = \lambda\,x_1, \quad u_2^* = \lambda\,x_2, \quad u_3^* = \bar{\lambda}\,x_3, \tag{24.23}$$

for a body under a uniform hydrostatic pressure.

Consider now the same example within the scope of the theory of Cosserat surface. Thus, for a Cosserat plate in equilibrium bounded by the lines (or edges) $x_1 = a_1, a_2$, $x_2 = b_1, b_2$, we specify the boundary conditions by

$$N_{11} = -h\,p, \quad N_{12} = M_{13} = 0 \quad \text{on} \quad x_1 = a_1, a_2,$$
$$N_{22} = -h\,p, \quad N_{21} = M_{23} = 0 \quad \text{on} \quad x_2 = b_1, b_2, \tag{24.24}$$

and put[12]

$$\varrho_0\,L_3 = -h\,p. \tag{24.25}$$

The relevant differential equations in this case are given by (23.3) and (23.1) in the absence of $\bar{F}_\beta$ and we seek a solution of these equations[13] in the form

$$\gamma_3 = \text{const.}, \quad e_{11} = e_{22} = \text{const.}, \quad e_{12} = 0. \tag{24.26}$$

Then, from (23.3), we have $M_{\alpha 3} = 0$, $N_{12} = 0$ and we also conclude that the components N_{11}, N_{22}, V_3 given by

$$N_{11} = N_{22} = 2(\alpha_1 + \alpha_2)\,e_{11} + \alpha_9\,\gamma_3, \quad V_3 = \alpha_4\,\gamma_3 + 2\alpha_9\,e_{11} \tag{24.27}$$

[12] The specification of (24.25) is suggested by the definitions $(12.32)_2$ and (11.30) for $N = 1$.

[13] If one considers the remaining Eqs. (23.4) and (23.2), it is easily seen that these are identically satisfied with $\bar{L}_\beta = 0$, $\bar{F}_3 = 0$.

are constants. It follows that $(23.1)_1$ is identically satisfied and $(23.1)_2$ yields

$$V_3 = \varrho_0 L_3. \tag{24.28}$$

From (24.27)–(24.28) and (24.24)–(24.25) we obtain

$$2(\alpha_1+\alpha_2)\,e_{11}+\alpha_9\,\gamma_3 = -h\,p, \qquad \alpha_4\,\gamma_3+2\alpha_9\,e_{11} = -h\,p. \tag{24.29}$$

Introducing the notation

$$\bar{\delta} = \frac{2(\alpha_1+\alpha_2)}{\alpha_4} \tag{24.30}$$

and solving for e_{11} and γ_3 from (24.29) we finally obtain the expressions

$$e_{11} = e_{22} = -\frac{1-\bar{\gamma}}{\bar{\alpha}+\bar{\beta}}\,h\,p, \qquad \gamma_3 = -\frac{\bar{\delta}-2\bar{\gamma}}{\bar{\alpha}+\bar{\beta}}\,h\,p, \tag{24.31}$$

where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are defined by (24.15). By comparison of (24.31) and (24.22), we may set

$$\frac{h(1-\bar{\gamma})}{\bar{\alpha}+\bar{\beta}} = \frac{c_{13}-c_{33}}{2c_{13}^2-(c_{11}+c_{12})\,c_{33}}, \qquad \frac{h(\bar{\delta}-2\bar{\gamma})}{\bar{\alpha}+\bar{\beta}} = \frac{2c_{13}-(c_{11}+c_{12})}{2c_{13}^2-(c_{11}+c_{12})\,c_{33}}. \tag{24.32}$$

The previous expressions for $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ given by (24.20) identically satisfy $(24.32)_1$ and when used in $(24.32)_2$ result in

$$\bar{\delta} = \frac{2(\alpha_1+\alpha_2)}{\alpha_4} = \frac{c_{11}+c_{12}}{c_{33}}. \tag{24.33}$$

The four relations in (24.20) and (24.33) can be solved for α_1, α_2, α_4, α_9 as follows:

$$\alpha_1 = h\,c_{12}, \quad 2\alpha_2 = h(c_{11}-c_{12}), \quad \alpha_4 = h\,c_{33}, \quad \alpha_9 = h\,c_{13}. \tag{24.34}$$

For an isotropic plate, using (24.3), the above coefficients become

$$\alpha_1 = \frac{\nu(1-\nu)}{1-2\nu}\,C, \quad \alpha_2 = \frac{1-\nu}{2}\,C, \quad \alpha_4 = \frac{(1-\nu)^2}{1-2\nu}\,C, \quad \alpha_9 = \frac{\nu(1-\nu)}{1-2\nu}\,C, \tag{24.35}$$

where C is defined by $(20.13)_2$ and ν is Poisson's ratio.

This completes the determination of seven of the constitutive coefficients in (23.3)–(23.4) for initially flat Cosserat plates.[14] In the theory of a Cosserat plate characterized by (23.1)–(23.4), the coefficients α_6 and α_7 or equivalently $(\alpha_6+\alpha_7)$ and $(\alpha_6-\alpha_7)$ are arbitrary and there is a constitutive relation for $M_{[\alpha\beta]}$ which involves the coefficient $(\alpha_6-\alpha_7)$. However, since we have undertaken to determine the coefficients by direct comparison with the exact three-dimensional solutions, then we should put $\alpha_6=\alpha_7$ in view of the symmetry of the stress-tensor in the three-dimensional theory.

Two other coefficients, namely α_3 and α_8 which occur in $(23.4)_2$ and $(23.3)_3$, remain arbitrary and we note that these coefficients have the orders of magnitude

$$\alpha_3 = O(C), \qquad \alpha_8 = O(B). \tag{24.36}$$

The coefficients α_1, α_2, α_4, α_5, α_6, α_7 and α_9 in (24.10) and (24.35), apart from their dependence on the thickness h, are expressed in terms of material properties which are constants in the three-dimensional theory. By contrast, a little reflection will reveal that the coefficients α_3 and α_8 in (24.36) cannot be determined as

[14] The determination of the constitutive coefficients (24.10) and (24.35) has been discussed in a number of papers: [1967, *4*], [1967, *6*], [1968, *6*], [1969, *3*] and [1970, *2*]. However, the identification (24.35) was achieved by different procedures in these papers.

constants from comparison with the three-dimensional solutions.[15] Even for a slightly less general system of equations [discussed following (23.10)], where the coefficient $\alpha_8=0$ and $M_{\alpha 3}=0$, we still have α_3 unspecified. Instead of assigning a definite approximate value to α_3, it seems preferable to allow α_3 (and also α_8 when a constitutive equation for $M_{\alpha 3}$ is present) to have different possible values depending on the particular context in which the theory of Cosserat surface is used. This requires elaboration but first we consider below a further example which will be helpful in our subsequent discussion.

(iv) *Torsion of a rectangular plate.* Consider a rectangular Cosserat plate in equilibrium and let the boundary lines (or edges)

$$x_1=\pm\frac{a}{2}$$

be traction free while the boundaries $x_2=\pm l$ rotate and are free from normal tractions. These boundary conditions can be stated as

$$M_{11}=M_{12}=V_1=0 \quad \text{on} \quad x_1=\pm\frac{a}{2}, \tag{24.37}$$

$$M_{22}=0, \quad \delta_1=\mp\bar{\theta}\,l, \quad u_3=\pm\bar{\theta}\,x_1\,l \quad \text{on} \quad x_2=\pm l, \tag{24.38}$$

where $\bar{\theta}$ is the angle of twist per unit length. We assume $M_{\alpha\beta}$, V_α to be independent of x_2 so that the kinematic variables $\varrho_{\alpha\beta}$ and γ_α are also functions of x_1 only. In the absence of L_β and $\bar{F}_3$, Eqs. (23.2) reduce to

$$M_{11,1}=V_1, \quad M_{12,1}=V_2, \quad V_{1,1}=0. \tag{24.39}$$

Keeping the boundary conditions $(24.38)_{2,3}$ in mind, we assume the displacement solutions in the form

$$u_3=\bar{\theta}\,x_1\,x_2, \quad \delta_1=-\bar{\theta}\,x_2, \quad \delta_2=\delta_2(x_1). \tag{24.40}$$

From $(24.39)_{1,3}$, (23.4) and the kinematic results (6.25), as well as the boundary conditions $(24.37)_{1,3}$ and $(24.38)_1$, it follows that

$$\begin{gathered}\gamma_1=0, \quad \gamma_2=\gamma_2(x_1)=\delta_2+\bar{\theta}\,x_1,\\ \varrho_{11}=\varrho_{22}=0, \quad \varrho_{12}=-\bar{\theta}, \quad \varrho_{21}=\gamma_{2,1}-\bar{\theta},\\ M_{11}=0, \quad M_{22}=0, \quad V_1=0.\end{gathered} \tag{24.41}$$

By use of the expressions for γ_2 and ϱ_{12} in (24.41), the remaining constitutive equations in (23.4) become[16]

$$M_{(12)}=\alpha_6(\gamma_{2,1}-2\,\bar{\theta}), \quad V_2=\alpha_3\,\gamma_2. \tag{24.42}$$

A differential equation for γ_2 can be obtained by substituting (24.42) into $(24.39)_2$. From the solution of this equation for γ_2 (which we expect to be an odd function of x_1) and the use of the boundary condition $(24.37)_2$, we readily deduce

$$\gamma_2=A\,\operatorname{Sinh}\frac{x_1}{\lambda}, \quad \lambda^2=\frac{\alpha_6}{\alpha_3}, \quad A=\frac{2\lambda\,\bar{\theta}}{\operatorname{Cosh}\frac{a}{2\lambda}}. \tag{24.43}$$

[15] This will become evident also in the example (iv) considered below.

[16] Since we have already determined the coefficients α_6 and α_7 according to (24.10), only the symmetric part of $M_{\alpha\beta}$ is given by a constitutive equation and $M_{[\alpha\beta]}=0$.

We do not record here the final expressions for $M_{(12)}$ and V_2 but note that the resultant torque is given by

$$T = \int_{-a/2}^{a/2} [x_1 V_2 - M_{(12)}]\, dx_1 = -2 \int_{-a/2}^{a/2} M_{(12)}\, dx_1, \tag{24.44}$$

in view of $(24.39)_2$ and the edge conditions (24.37). For a rectangular strip of breadth a and thickness h, the expression for the torsional rigidity resulting from (24.44) is easily found to be

$$\mu \frac{h^3 a}{3} \left[1 - \frac{2\lambda}{a} \tanh \frac{a}{2\lambda}\right], \tag{24.45}$$

where μ is defined by $(20.13)_1$ and we have also used the value of α_6 given by $(24.10)_2$. The above expression, for a wide range of a/h, will be in remarkably close agreement with the prediction of the corresponding result in the Saint-Venant theory of torsion if we choose $\lambda^2 = h^2/10$ or equivalently

$$\alpha_3 = \tfrac{5}{6} \mu h. \tag{24.46}$$

This completes our consideration of torsion of a rectangular plate.[17]

We return now to our previous discussion concerning the coefficients α_3, α_8 and also recall the remarks made following (24.36). As far as $(24.36)_2$ is concerned, at present we have no definite evidence regarding a suitable approximate value for[18] α_8. However, in a theory which is only slightly less general than that characterized by (23.1)–(23.4), $M_{\alpha 3}$ is absent and the coefficient $\alpha_8 = 0$ or does not arise.[19]

With reference to $(24.36)_1$, in view of the conclusion reached in example (iv) above, we may specify α_3 by the approximate value (24.46) for elastostatic problems. On the other hand, for dynamical problems, we can determine α_3 by comparison of an appropriate elastodynamic solution of the system of equations of the bending theory [i.e., the system of Eqs. (23.2) and (23.4) with values (24.10)] with the prediction of the corresponding exact (three-dimensional) solution due to LAMB[20] for the circular frequency of the first anti-symmetric mode of thickness-shear

[17] The approximate value for α_3 is the same as that in REISSNER's plate theory discussed in Sect. 20. Our above solution for torsion of a rectangular plate [between (24.37)–(24.44)] parallels that contained in REISSNER's [1945, *2*] paper. It is perhaps of interest to indicate here the nature of a corresponding result for torsion of a rectangular plate of variable thickness, with the thickness being dependent on one coordinate (say x_1), as given by ESSENBURG and NAGHDI [1958, *2*]. Using constitutive equations appropriate to plates of variable thickness (derived from the three-dimensional equations), they have shown that when the cross-section is an ellipse or an equilateral triangle the results agree exactly with the corresponding solutions in the Saint-Venant theory of torsion. A solution parallel to that in [1958, *2*] is included in the paper of GREEN, NAGHDI and WENNER [1971, *6*] who employ the theory of a Cosserat surface with the initial director in the form (6.21) and with D as a function of x_1.

[18] A value for α_8 is suggested by the coefficient in $(20.11)_3$. This value is a consequence of the approximation $(20.8)_2$ and there is no evidence in the existing literature as to its being a reasonable approximate value, as remarked also following (20.14). However, it is possible to specify an approximate value for α_8 by a suitable comparison of an extensional solution of (23.1) and (23.3) with the corresponding solution of a simple two-dimensional problem in elasticity theory.

[19] The nature of this linear theory, in the absence of $M_{\alpha 3}$, was discussed following (23.10). This type of linear theory already includes all existing linear theories of shells and plates currently employed in the literature.

[20] LAMB [1917, *1*].

vibration in a plate. Such a comparison leads to the value[21]

$$\alpha_3 = \frac{\pi^2}{12} \mu h,$$

which is very close to (24.46). In subsequent developments in this chapter, we shall not record (or utilize) the constitutive equations for V_α and $M_{\alpha 3}$ with specific values assigned to the coefficients (24.36); but the foregoing discussion regarding the specification of α_8 and α_3 will be understood.

β) The constitutive coefficients for shells. It is clear from the developments of Sect. 16 that the constitutive coefficients in the linear theory of shells depend, in general, on $B_{\alpha\beta}$. However, the constitutive relations (23.6)–(23.8) are deduced by assuming a special form for the Helmholtz free energy function which does not depend explicitly on[22] $B_{\alpha\beta}$. Moreover, since these constitutive relations must reduce to those appropriate for flat plates, we adopt the values of the coefficients (24.10) and (24.35). Hence, the constitutive relations (23.6)–(23.8) can be written as

$$\begin{aligned}
N'^{\alpha\beta} &= N'^{\beta\alpha} = C H^{\alpha\beta\gamma\delta} e_{\gamma\delta} + \frac{\nu}{1-\nu} A^{\alpha\beta} V^3, \\
M^{\alpha\beta} &= M^{\beta\alpha} = B H^{\alpha\beta\gamma\delta} \varrho_{\gamma\delta}, \\
V^\alpha &= N^{\alpha 3} = \alpha_3 A^{\alpha\beta} \gamma_\beta, \\
V^3 &= C \frac{(1-\nu)^2}{1-2\nu} \gamma_3 + C \frac{\nu(1-\nu)}{1-2\nu} A^{\alpha\beta} e_{\alpha\beta}, \\
M^{\alpha 3} &= \alpha_8 A^{\alpha\gamma} \varrho_{3\gamma},
\end{aligned} \tag{24.47}$$

where

$$\begin{aligned}
H^{\alpha\beta\gamma\delta} &= \tfrac{1}{2}\{A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma} + \nu[2A^{\alpha\beta} A^{\gamma\delta} - A^{\alpha\gamma} A^{\beta\delta} - A^{\alpha\delta} A^{\beta\gamma}]\} \\
&= \tfrac{1}{2}\{A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma} + \nu(\bar{\varepsilon}^{\alpha\gamma} \bar{\varepsilon}^{\beta\delta} + \bar{\varepsilon}^{\alpha\delta} \bar{\varepsilon}^{\beta\gamma})\}
\end{aligned} \tag{24.48}$$

and $\bar{\varepsilon}^{\alpha\beta}$ is an ε-symbol defined previously in (6.30).

For reasons stated earlier in this section, we have left the coefficients α_3 and α_8 unspecified in (24.47). However, we emphasize that the discussion at the end of Subsect. α) is also pertinent to the constitutive coefficients in $(24.47)_{3,5}$. In particular, the values suggested for α_3 can also be used for shells. We further observe that some of the remaining coefficients in (24.47) can be determined by means of solutions of special examples in the theory of Cosserat surface with $B_{\alpha\beta} \neq 0$ (rather than those for initially flat plates). For example, the coefficients α_2, α_6 and α_7 with values the same as those in (24.35) and (24.10) can be deter-

[21] This manner of determining the value of α_3 for dynamical problems is similar to that used by MINDLIN [1951, *1*] and mentioned previously in Sect. 20 [Subsect. α)]. As noted with reference to (20.17), in the early stage of his derivation of plate equations for dynamical problems (from the three-dimensional equations), MINDLIN assumes the coefficient corresponding to α_3 to be unspecified; but subsequently he determines this coefficient by a comparison with LAMB's solution [1917, *1*]. This manner of determining the constitutive coefficients is in accord with our point of view in this chapter; in fact, the use of the equations of the Cosserat surface (whose constitutive coefficients are not predetermined) supplies a justification for MINDLIN's procedure.

[22] As already remarked in Sect. 16, the ultimate plausibility for such a special choice of the free energy (or the strain energy) function depends, of course, on its usefulness. In the context of the three-dimensional theory, it is clear from comparison of (16.33) and (20.14) that the relations (23.6)–(23.8) correspond to the neglect of terms of $O(h/R)$ or smaller compared with those in the special free energy function (16.33).

mined from a comparison of the solution for torsion of a cylindrical Cosserat surface with the corresponding results in the Saint-Venant theory of torsion.[23]

25. The boundary-value problem of the restricted linear theory. For convenience, we first summarize below the linearized field equations of the restricted isothermal theory. Thus, it follows from the linearized versions of the results in Sects. 10 and 15 that the equations of motion are given by (9.69) and

$$\grave{M}^{(\gamma\alpha)}{}_{|\gamma} = V^{\alpha}, \qquad \bar{\varepsilon}_{\alpha\beta}[N^{\alpha\beta} - \grave{M}^{(\gamma\beta)} B^{\alpha}_{\gamma}] = 0. \tag{25.1}$$

The above equations are the same as those in (10.28)–(10.29), together with (15.17), except that in (25.1) we have used V^{α} in place of $N^{\alpha 3}$ for ready comparison with $(9.70)_1$ and we have also put[24] $\grave{L}^{\alpha} = 0$. We recall here that in the restricted theory only $\grave{M}^{(\gamma\alpha)}$ and the linearized expression corresponding to (10.26), namely the symmetric

$$\grave{N}^{\alpha\beta} = \grave{N}^{\beta\alpha} = N^{\alpha\beta} + \grave{M}^{(\gamma\alpha)} B^{\beta}_{\gamma}, \tag{25.2}$$

are specified by constitutive equations; and that V^{α}, which is not specified by a constitutive equation, is determined from $(25.1)_1$. Remembering the brief remarks included in Sect. 16 [Subsect. δ)] concerning the linear constitutive equations of the restricted isothermal theory, from (16.40) we obtain the following constitutive equation for an isotropic elastic material:

$$\begin{aligned} \grave{N}^{\alpha\beta} &= [\beta_1 A^{\alpha\beta} A^{\gamma\delta} + \beta_2 (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\, e_{\gamma\delta}, \\ \grave{M}^{(\alpha\beta)} &= [\beta_5 A^{\alpha\beta} A^{\gamma\delta} + \beta_6 (A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\, \varrho_{\gamma\delta}. \end{aligned} \tag{25.3}$$

The kinematic measures in (25.3) are defined in (6.27) and the coefficients β_1, β_2, β_5 and β_6 are arbitrary constants and do not necessarily have the values obtained previously in Sect. 24 for the corresponding coefficients in (23.6).

We proceed to identify the constitutive coefficients in (25.3) in a manner entirely similar to that employed in Sect. 24. On this occasion we only need to consider two simple elastostatic problems, namely those for flexural and extensional deformations of a plate, discussed in examples (i) and (ii) in Sect. 24. Thus, by comparisons of these solutions with corresponding exact solutions in the three-dimensional theory, we determine all four coefficients in (25.3) as follows:

$$\beta_1 = \nu\, C, \quad \beta_2 = \frac{1-\nu}{2}\, C, \quad \beta_5 = \nu\, B, \quad \beta_6 = \frac{1-\nu}{2}\, B. \tag{25.4}$$

With the above coefficients, the constitutive equations (25.3) can be expressed as

$$\grave{N}^{\alpha\beta} = C\, H^{\alpha\beta\gamma\delta} e_{\gamma\delta}, \qquad \grave{M}^{(\alpha\beta)} = B\, H^{\alpha\beta\gamma\delta} \varrho_{\gamma\delta}, \tag{25.5}$$

where $H^{\alpha\beta\gamma\delta}$ is given by (24.48).

The boundary conditions appropriate to the above restricted linear theory can be obtained from the linearized version of the corresponding results in Sect. 15.

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[23] The determination of α_2, α_6 and α_7 in this manner can be found in [1967, *6*]. The value of α_1 appearing in Eq. (3.20) of this paper contains an error and should conform to that in (24.35); however, this does not affect the remaining discussion in [1967, *6*] concerning torsion of a circular cylindrical Cosserat surface. Related results for torsion of a cylindrical Cosserat surface, in which the edge curve perpendicular to the generator is not necessarily circular, are given by Wenner [1968, *13*].

[24] This restriction is not essential. It is introduced here simply because a quantity corresponding to $\grave{L}^{\alpha}$ is usually absent or is neglected in the derivations of the classical theory of shells and plates from the three-dimensional equations.

These conditions require the specification of either the displacement boundary conditions

$$u_\gamma,\quad u_3,\quad \frac{\partial u_3}{\partial \nu_0} \tag{25.6}$$

or the force and couple boundary conditions

$$\begin{aligned}{}_0P^\beta &= {}_0\nu_\alpha\,[N^{\alpha\beta} - \grave{M}^{(\alpha\gamma)}\,B^\beta_\gamma], \qquad {}_0G = \grave{M}^{(\alpha\gamma)}\,{}_0\nu_\alpha\,{}_0\nu_\gamma,\\ {}_0P^3 &= {}_0\nu_\alpha\,(\grave{M}^{(\beta\alpha)}{}_{|\beta}) - \frac{\partial}{\partial s_0}\,[\bar{\varepsilon}_{\beta\gamma}\,\grave{M}^{(\alpha\beta)}\,{}_0\nu_\alpha\,{}_0\nu^\gamma],\end{aligned} \tag{25.7}$$

where ${}_0\nu_\alpha$ are the components of the outward unit normal ${}_0\boldsymbol{\nu}$ to the boundary curve on the reference surface $\mathscr{S}$ and $\partial/\partial\nu_0$, $\partial/\partial s_0$ denote the directional derivatives along the normal and the tangent to the boundary curve on $\mathscr{S}$.

The system of equations and boundary conditions in the above restricted theory correspond to those of the classical theory of shells and plates obtained (from the three-dimensional equations) under the Kirchhoff (or the Kirchhoff-Love) hypothesis or any other equivalent set of assumptions. The derivation of the restricted theory (by direct approach) is, however, free of the inconsistencies present in the usual derivations of the classical theory.[25] In this connection, we emphasize that the constitutive coefficients in (25.3) are arbitrary and can be assigned the specified values (25.4) without violating any of the kinematic assumptions appropriate to the restricted theory. This is in contrast to the usual derivations of the classical theory in which an additional assumption is introduced (beyond that of the kinematic assumptions), in order to compensate for the fact that the constitutive coefficients in an approximate expression for the three-dimensional strain energy density are predetermined.

The system of equations of the restricted theory can also be obtained as a special case of those in the linear theory of a Cosserat surface [Sect. 23, Subsect. β)] under suitable constraints. To see this, with reference to (23.6)–(23.8), let

$$\alpha_8 = 0,\qquad \frac{\alpha_3}{\alpha_5}\to\infty,\qquad \gamma_\alpha\to 0, \tag{25.8}$$

in such a manner that

$$V^\alpha \to \text{finite limit}. \tag{25.9}$$

Then, $M^{\alpha 3}=0$ and V^α is not determined by a constitutive equation. In addition, under the condition $(25.8)_3$, the kinematic measure $\varrho_{\gamma\alpha}$ in (6.24) becomes symmetric in the indices γ,α [see (6.27)] and hence the skew-symmetric part of $M^{\beta\alpha}$ in $(23.6)_2$ vanishes, i.e., $M^{[\beta\alpha]}=0$. We also assume that the assigned director couple and the inertia terms due to director velocity are either zero or negligible so that $L^\beta=0$, $L^3=0$. Then, from (9.70) and (25.9), we have

$$M^{(\gamma\alpha)}{}_{|\gamma} = V^\alpha,\qquad V^3 = 0, \tag{25.10}$$

while the equations of motion (9.69) remain unchanged. In view of $(25.10)_2$, the constitutive relation $(24.47)_1$ reduces to

$$N'^{\alpha\beta} = N'^{\beta\alpha} = C\,H^{\alpha\beta\gamma\delta}\,e_{\gamma\delta} \tag{25.11}$$

and $(24.47)_4$ yields

$$\gamma_3 = -\frac{\nu}{1-\nu}\,A^{\alpha\beta}\,e_{\alpha\beta}. \tag{25.12}$$

[25] These inconsistencies refer, on the one hand, to the kinematic assumptions in the classical theory of shells and plates [discussed in Sect. 7, Subsect. ε)]; and, on the other hand, to the manner in which the constitutive equations with appropriate constitutive coefficients are derived.

Further, by (25.8) and in view of the expressions for the symmetric $\bar{\varkappa}_{\gamma\delta}$ in (6.24), $\varrho_{\gamma\delta}$ is now given by the last expression in (6.27) and $(24.47)_2$ becomes

$$M^{(\alpha\beta)} = B H^{\alpha\beta\gamma\delta} \varrho_{\gamma\delta}, \qquad \varrho_{\gamma\delta} = -\bar{\varkappa}_{\gamma\delta}. \tag{25.13}$$

Also, the remaining Eq. (9.68) takes the form

$$N'^{\alpha\beta} = N'^{\beta\alpha} = N^{\alpha\beta} + M^{(\gamma\alpha)} B^{\beta}_{\gamma}. \tag{25.14}$$

The boundary conditions for the above special theory can be obtained from the more general boundary conditions using (25.8)–(25.9) and other results between (25.10)–(25.14). These reduced boundary conditions will be similar in form to those in (25.6)–(25.7) and hence will not be recorded. It should be clear that the system of equations of the restricted theory characterized by (9.69), (25.1)–(25.2) and (25.5) are formally equivalent to those of the special theory [discussed between (25.8)–(25.14)] if we identify $\grave{M}^{(\alpha\beta)}$ with $M^{(\alpha\beta)}$ and $\grave{N}^{\alpha\beta}$ with $N'^{\alpha\beta}$, apart from (25.12) in the special theory.[26]

Returning to the restricted theory, we recall from Sect. 15 that the equations of motion (after elimination of V^{α}) can be expressed in either one of the alternative forms resulting from linearization of (15.19) and (15.20). The linearized equations of motion in terms of $\grave{N}^{\alpha\beta}$ and $\grave{M}^{(\alpha\beta)}$ obtained by linearization of $(15.20)_{1,2}$, together with the constitutive equations (25.3) in terms of the kinematic variables $e_{\alpha\beta}$ and $\varrho_{\alpha\beta}$, constitute a determinate system. However, in obtaining solutions to boundary-value problems (or initial boundary-value problems), it is also possible to employ the equations of motion which result from the linearization of $(15.19)_{1,2}$, namely

$$\begin{gathered} N^{(\alpha\beta)}{}_{|\alpha} - \tfrac{1}{2}[B^{\beta}_{\gamma} \grave{M}^{(\gamma\alpha)}]_{|\alpha} + \tfrac{1}{2}[B^{\alpha}_{\gamma} \grave{M}^{(\gamma\beta)}]_{|\alpha} - B^{\beta}_{\alpha} \grave{M}^{(\gamma\alpha)}{}_{|\gamma} + \varrho_0 \bar{F}^{\beta} = 0, \\ \grave{M}^{(\alpha\beta)}{}_{|\alpha\beta} + B_{\alpha\beta} N^{(\alpha\beta)} + \varrho_0 \bar{F}^3 = 0, \end{gathered} \tag{25.15}$$

where the components $\bar{F}^i$ are defined by $(9.57)_1$ instead of the linearized version of (10.23) since we have put $\grave{L}^{\alpha} = 0$. Although $\grave{M}^{(\alpha\beta)}$ in (25.15) can be identified with $M^{(\alpha\beta)}$ in view of the equivalence of the restricted theory and a special case of the general theory noted above, we continue our use of the notations $\grave{N}^{\alpha\beta}$ and $\grave{M}^{(\alpha\beta)}$ in order to distinguish between the restricted theory and the general theory.

The equations of motion (25.15) involve $N^{(\alpha\beta)}$, $\grave{M}^{(\alpha\beta)}$ rather than $\grave{N}^{\alpha\beta}$, $\grave{M}^{(\alpha\beta)}$. Hence, it is more convenient to express the constitutive equations also in terms of the former set whenever (25.15) is employed. To this end, we introduce the symmetric kinematic variable

$$\bar{\varrho}_{\alpha\beta} = \varrho_{\alpha\beta} + \tfrac{1}{2}(B^{\nu}_{\alpha} e_{\nu\beta} + B^{\nu}_{\beta} e_{\nu\alpha}), \tag{25.16}$$

where $\varrho_{\alpha\beta}$, $e_{\alpha\beta}$ are defined in (6.27).[27] Assuming now for the specific free energy (or the strain energy density) the form

$$\psi = \grave{\psi}(e_{\alpha\beta}, \bar{\varrho}_{\alpha\beta}; A_{\alpha\beta}, -B_{\alpha\beta}), \tag{25.17}$$

we can then deduce the results[28]

$$N^{(\alpha\beta)} = \varrho_0 \frac{\partial \grave{\psi}}{\partial e_{\alpha\beta}}, \qquad \grave{M}^{(\alpha\beta)} = \varrho_0 \frac{\partial \grave{\psi}}{\partial \bar{\varrho}_{\alpha\beta}}, \tag{25.18}$$

[26] The reduction of the general linear theory to the special theory as discussed above is contained in a paper by Green and Naghdi [1968, 6] and is patterned after that discussed by them [1968, *5*] in a more general context.

[27] The kinematic variable in (25.16) corresponds to that defined by (20.37) in an alternative formulation of the classical shell theory [Sect. 20, Subsect. δ)].

[28] The expressions (25.18) may be compared with (20.39) in the classical linear theory of shells, obtained from the three-dimensional equations by approximation.

where $\dot{\psi}$ is a quadratic function of $e_{\alpha\beta}$ and $\bar{\varrho}_{\alpha\beta}$. Further, by imposing a restriction on the function $\dot{\psi}$ similar to (16.39) and for an isotropic material with a center of symmetry, we can write the constitutive equation for ψ in a form similar to (16.40) but in terms of $e_{\alpha\beta}$ and $\bar{\varrho}_{\alpha\beta}$. The linear constitutive equations for $N^{(\alpha\beta)}$ and $\check{M}^{(\alpha\beta)}$ in terms of $e_{\alpha\beta}$ and $\bar{\varrho}_{\alpha\beta}$ then follow but we do not pursue the matter further. We note, however, the relationship between $N^{(\alpha\beta)}$ and $\check{N}^{\alpha\beta}$, namely

$$N^{(\alpha\beta)} = \check{N}^{\alpha\beta} - \tfrac{1}{2}(B^{\alpha}_{\lambda}\check{M}^{(\lambda\beta)} + B^{\beta}_{\lambda}\check{M}^{(\lambda\alpha)}), \tag{25.19}$$

which is obtained from (25.2). Also, from (10.29), the skew-symmetric part of $N^{\alpha\beta}$ is given by

$$N^{[\alpha\beta]} = \tfrac{1}{2}[B^{\alpha}_{\lambda}\check{M}^{(\lambda\beta)} - B^{\beta}_{\lambda}\check{M}^{(\lambda\alpha)}]. \tag{25.20}$$

26. A uniqueness theorem. Remarks on the general theorems. This concluding section is concerned mainly with a uniqueness theorem for solutions of the initial (isothermal) boundary-value problems of elastic shells and plates. In addition, some remarks are included here concerning the nature of the general theorems which can be obtained in the linear theory. Although we establish a uniqueness theorem below for the initial mixed boundary-value problems of the dynamical theory, a parallel development can also be given for equilibrium mixed boundary-value problems. We first obtain the conditions *sufficient* for uniqueness, using the constitutive equations (16.11) with a quadratic specific free energy (or the strain energy density) function $\bar{\psi}$ in the form (16.10);[29] and then we further examine the restrictions on the constitutive coefficients required by uniqueness when $\bar{\psi}$ has the special form (16.33).

Let $\mathscr{S}$ be a bounded regular region of two-dimensional space occupied by the initial elastic Cosserat surface. (By a regular region we mean one to which the divergence theorem is applicable.) Let $\partial\mathscr{S}$ be the boundary and let $\mathscr{S}^0$ denote the interior of $\mathscr{S}$ and introduce the regions of space-time by

$$\begin{aligned} \mathscr{R} &= \{(\theta^\alpha, t) : \theta^\alpha \in \mathscr{S}, t \geqq 0\}, \\ \mathscr{R}^0 &= \{(\theta^\alpha, t) : \theta^\alpha \in \mathscr{S}^0, t \geqq 0\}. \end{aligned} \tag{26.1}$$

Let ${}_0\boldsymbol{\nu} = {}_0\nu_\alpha \boldsymbol{A}^\alpha$ be the outward unit normal to $\partial\mathscr{S}$ and $\partial\mathscr{S}_1$, $\partial\mathscr{S}_2$ be arbitrary disjoint subsets of $\partial\mathscr{S}$ such that $\partial\mathscr{S}_1 \cup \partial\mathscr{S}_2 = \partial\mathscr{S}$. Then, the initial boundary-value problem of the isothermal linear theory of elastic shells is characterized by the equations of motion (9.68)–(9.70) and the constitutive relations (16.11) with the function $\bar{\psi}$ given by that in (16.10). We require that the free energy (or the strain energy density) function $\bar{\psi}$ be nonnegative, i.e.,

$$\bar{\psi} \geqq 0 \quad \text{for} \quad t \geqq 0. \tag{26.2}$$

To the above field equations and constitutive relations, we supplement the boundary conditions

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{v}', & \boldsymbol{w} &= \boldsymbol{w}' & &\text{on } \partial\mathscr{S}_1 \quad (\text{for } t \geqq 0), \\ \boldsymbol{N} &= \boldsymbol{N}', & \boldsymbol{M} &= \boldsymbol{M}' & &\text{on } \partial\mathscr{S}_2 \quad (\text{for } t \geqq 0) \end{aligned} \tag{26.3}$$

and the initial conditions

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}^0, & \boldsymbol{\delta} &= \boldsymbol{\delta}^0 & &\text{on } \mathscr{S} \quad (\text{for } t = 0), \\ \boldsymbol{v} &= \boldsymbol{v}^0, & \boldsymbol{w} &= \boldsymbol{w}^0 & &\text{on } \mathscr{S} \quad (\text{for } t = 0). \end{aligned} \tag{26.4}$$

[29] Since we are concerned with the isothermal theory, θ in the argument of $\bar{\psi}$ in (16.10) is now a constant.

Here $\boldsymbol{u}^0$, $\boldsymbol{\delta}^0$, $\boldsymbol{v}^0$, $\boldsymbol{w}^0$, $\boldsymbol{v}'$, $\boldsymbol{w}'$, $\boldsymbol{N}'$, $\boldsymbol{M}'$ are prescribed functions on the appropriate domains and the mass density ϱ_0 is a strictly positive constant.

We now state the following uniqueness theorem: Let $\boldsymbol{u}$, $\boldsymbol{\delta}$ be the displacement vector and the director displacement vector which satisfy the above mentioned field equations on $\mathscr{R}$; and let the assigned surface force $\boldsymbol{f}$ and the assigned director couple $\boldsymbol{l}$ be prescribed. Then, provided (26.2) holds and that the director inertia coefficient $\alpha > 0$, there exists at most one set of functions $\boldsymbol{u}$, $\boldsymbol{\delta}$ satisfying (26.3)–(26.4) such that u_α, δ_i are of class C^1 and u_3 is of class C^2 on $\mathscr{R}$ while u_α, δ_i are of class C^2 and u_3 is of class C^3 on $\mathscr{R}^0$.

Our method of proof is similar to the classical uniqueness proofs in three-dimensional linear elasticity.[30] In essence, we consider the difference of two possible solutions and make use of an equation for the mechanical balance of energy, i.e., an equation in which the rate of increase of the sum of kinetic energy and internal energy is equal to the rate of work by the contact and the assigned forces and couples. Thus, assume that there are two sets of solutions to the initial boundary-value problem under consideration, namely

$$\begin{gathered} {}^{(1)}u_i,\ {}^{(1)}\delta_i,\ {}^{(1)}N'^{\alpha\beta},\ {}^{(1)}M^{\alpha i},\ {}^{(1)}V^\alpha,\\ {}^{(2)}u_i,\ {}^{(2)}\delta_i,\ {}^{(2)}N'^{\alpha\beta},\ {}^{(2)}M^{\alpha i},\ {}^{(2)}V^\alpha, \end{gathered} \tag{26.5}$$

and denote the difference of the two sets of solutions by

$$\begin{gathered} \bar{u}_i = {}^{(1)}u_i - {}^{(2)}u_i, \qquad \bar{\delta}_i = {}^{(1)}\delta_i - {}^{(2)}\delta_i,\\ \bar{N}'^{\alpha\beta} = {}^{(1)}N'^{\alpha\beta} - {}^{(2)}N'^{\alpha\beta}, \quad \bar{M}^{\alpha i} = {}^{(1)}M^{\alpha i} - {}^{(2)}M^{\alpha i}, \quad \bar{V}^\alpha = {}^{(1)}V^\alpha - {}^{(2)}V^\alpha. \end{gathered} \tag{26.6}$$

Because of the linear character of all field equations and constitutive relations, it is clear that the set of functions $\bar{u}_i$, $\bar{\delta}_i$, etc., defined by (26.6) satisfy equations of the forms (9.68)–(9.70) in the absence of the assigned force and the assigned director couple F^i, L^i [defined as in (9.57)] and also equations of the forms (16.11) with $\bar{\psi}$ given by that in (16.10) as a quadratic function of its arguments. Moreover, since each of the two sets of solutions in (26.5) satisfies the boundary conditions, we have

$$\begin{aligned} \bar{\boldsymbol{v}} &= 0, & \bar{\boldsymbol{w}} &= 0 \quad \text{on } \partial\mathscr{S}_1 \quad (\text{for } t \geqq 0),\\ \bar{\boldsymbol{N}} &= 0, & \bar{\boldsymbol{M}} &= 0 \quad \text{on } \partial\mathscr{S}_2 \quad (\text{for } t \geqq 0). \end{aligned} \tag{26.7}$$

We keep these in mind in what follows and for simplicity delete the overbar from the difference functions in (26.6).

Since F_i and L_i are prescribed, from the linearized versions of (8.8) and (14.2) applied to the difference of the two solutions, we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathscr{S}} \frac{1}{2}\varrho_0(\boldsymbol{v}\cdot\boldsymbol{v} + \alpha\,\boldsymbol{w}\cdot\boldsymbol{w})\,d\Sigma + \int_{\mathscr{S}} [N'^{\alpha\beta}\dot{e}_{\alpha\beta} + V^i\dot{\gamma}_i + M^{\alpha i}\dot{\varrho}_{i\alpha}]\,d\Sigma&\\ = \int_{\partial\mathscr{S}} (\boldsymbol{N}\cdot\boldsymbol{v} + \boldsymbol{M}\cdot\boldsymbol{w})\,dS&, \end{aligned} \tag{26.8}$$

where the element of area $d\Sigma$ is defined by (4.39), dS is an element of length of the boundary curve with outward unit normal ${}_0\boldsymbol{\nu}$ and we have omitted the overbar from the difference of the two sets of functions. Using (16.11) and the bound-

[30] For the classical uniqueness theorems in three-dimensional elasticity, see Love [1944, *4*, p. 170 and p. 176] or Sokolnikoff [1956, *2*, p. 86]. The requirement that the free energy function $\bar{\psi}$ be nonnegative can probably be relaxed as in the paper of Knops and Payne [1968, *10*], which is concerned with uniqueness in three-dimensional elastodynamics.

ary conditions (26.7), (26.8) leads to

$$\frac{d}{dt}\int_{\mathscr{S}}\left\{\frac{1}{2}\varrho_0(\boldsymbol{v}\cdot\boldsymbol{v}+\alpha\boldsymbol{w}\cdot\boldsymbol{w})+\varrho_0\overline{\psi}\right\}d\Sigma=0. \tag{26.9}$$

Since $\overline{\psi}$ is nonnegative and $\alpha>0$, it follows from (26.9) that the difference of the functions for velocity and director velocity must vanish:

$$\boldsymbol{v}=0, \qquad \boldsymbol{w}=0. \tag{26.10}$$

Moreover, in view of the initial conditions (26.4), we also have

$$\boldsymbol{u}=0, \qquad \boldsymbol{\delta}=0, \tag{26.11}$$

which proves uniqueness.

The above uniqueness proof is valid for a general linear theory of shells and is not limited to isotropic materials. Obviously, it is valid also for the initial mixed boundary-value problem characterized by the equations of the restricted linear theory [Sect. 16, Subsect. δ) and Sect. 25].[31] It is straightforward to obtain corresponding sufficient conditions for uniqueness of the equilibrium mixed boundary-value problem in which case instead of (26.2) we require that $\overline{\psi}$ be positive definite, i.e.,

$$\overline{\psi}>0 \tag{26.12}$$

for all non-zero values of the kinematic variables; however, in view of the close similarity of the proof with that given above, it will not be included here.

The above theorem provides sufficient conditions for uniqueness without detailed specification of the quadratic function $\overline{\psi}$ in (16.10). We now briefly examine the restrictions on the constitutive coefficients (required by our uniqueness proof) when $\overline{\psi}$ is specified by (16.33)[32] for an isotropic material and further limit ourselves to the case of flat plates. Since the differential equations characterizing the streching of the plate and those for the bending of the plate separate into two distinct sets (Sect. 23), the nature of the restrictions on the constitutive coefficients can be examined separately. Consider first the extensional case and introduce $(16.33)_2$ in the condition (26.2). It then follows that

$$\alpha_1+\alpha_2\geqq 0, \quad \alpha_2\geqq 0, \quad \alpha_4\geqq 0, \quad (\alpha_1+\alpha_2)\alpha_4\geqq\alpha_9^2 \tag{26.13}$$

and

$$\alpha_8\geqq 0. \tag{26.14}$$

Similarly, for the flexure of the plate, the condition (26.2) with $\overline{\psi}$ given by $(16.33)_3$ yields

$$\alpha_6\geqq 0, \quad \alpha_6\pm\alpha_7\geqq 0, \quad 2\alpha_5+\alpha_6+\alpha_7\geqq 0 \tag{26.15}$$

and

$$\alpha_3\geqq 0. \tag{26.16}$$

[31] In the case of the classical theory of plates (corresponding to equations for flat plates resulting from the restricted theory), the uniqueness proof was given by KIRCHHOFF [1850, *1*]. The earliest proof of uniqueness for shells, based on the constitutive relations (21 A.10)–(21 A.11), is contained in a paper by BYRNE [1944, *1*]. A uniqueness theorem is also given by GOL'DENVEIZER [1944, *3*]; see also, GOL'DENVEIZER and LUR'E [1947, *3*]. Uniqueness of solution in the context of thermoelastic shells (using a system of differential equations which include those recorded in Sect. 23), along with some related results pertaining to the nature of the restrictions placed on the constitutive coefficients, is discussed in [1971, *4*].

[32] Recall that the constitutive equations for shells and plates in Sect. 23 are those which are obtained from $\overline{\psi}$ in (16.33).

If we now make use of the values (24.10) and (24.35), the inequalities (26.13) imply

$$\mu \geqq 0, \quad -1 \leqq \nu < \tfrac{1}{2} \tag{26.17}$$

while those in (26.15) give

$$\mu \geqq 0, \quad -1 \leqq \nu < 1. \tag{26.18}$$

The conditions (26.17) are the same as those found (by energy arguments) for the initial boundary-value problem in the three-dimensional theory of elasticity for isotropic materials. This is in contrast to the results of the classical theory of generalized plane stress for the stretching of the plate, where the corresponding conditions deduced by energy arguments are $\mu \geqq 0$, $-1 \leqq \nu < 1$. These latter conditions can be obtained by putting

$$\alpha_4 \gamma_3 + 2\alpha_9 A^{\alpha\beta} e_{\alpha\beta} = 0$$

in $(16.33)_2$ before using $(26.2)_1$. Returning to (26.18), we note that these conditions are less restrictive than the corresponding conditions (26.17) for the stretching of the plate. The conditions (26.18) are the same as those found for the bending of the plate according to the classical thin plate theory [Sect. 20, Subsect. β)] using energy arguments. This completes our discussion of the restrictions on the constitutive coefficients required by the above uniqueness proof.

Before closing this section, we make a few remarks concerning the nature of the general theorems for shells. It is not difficult to see that most of the general theorems in the three-dimensional linear theory of elasticity have analogues in shell theory. We have already used an analogue of a three-dimensional result (in the above uniqueness proof), namely the linearized versions of (8.8) and (14.2) according to which the rate of kinetic energy and internal energy is equal to the rate of work by $\boldsymbol{N}$, $\boldsymbol{M}$ and $\boldsymbol{f}$, $\boldsymbol{l}$. From this result and the structure of the linear constitutive equations for shells (Sect. 23), it should be apparent that one can easily obtain analogues of all variational theorems of classical elasticity (including the minimum potential energy and complementary energy theorems), the analogue of BETTI'S (also known as the BETTI-RAYLEIGH) reciprocity theorem, as well as a number of related representation theorems.[33]

All of the general theorems mentioned above may be established using the constitutive equations (16.11), with a quadratic function $\overline{\psi}$ of the form (16.10) specialized for isothermal theory, and thus their validity is not limited to isotropic materials. Some of these theorems are analogues of those in the three-dimensional dynamical theory and others represent the counterparts of the corresponding theorems in the three-dimensional elastostatic theory. There exists, however, a correspondence theorem for elastostatic problems of shells which has no direct counterpart in the classical three-dimensional theory. This correspondence theorem, known as *static-geometric analogy*, was introduced simultaneously and independently by GOL'DENVEIZER and LUR'E in the context of the classical theory of shells.[34] According to this analogy, in the absence of the assigned surface

[33] Theorems of this kind, over the years, have been obtained for shells with the use of a variety of constitutive equations which are less general than those in Sect. 23. In particular, an analogue of Betti's reciprocity theorem was first derived by GOL'DENVEIZER [1944, *3*]. For this and analogues of related results, see also [1947, *3*], [1960, *10*] and [1963, *6*].

[34] The static-geometric analogy was introduced by GOL'DENVEIZER [1940, *1*] and LUR'E [1940, *2*], within the scope of the classical shell theory under the Kirchhoff-Love hypothesis. Since the classical theory does not provide a constitutive equation for the shear-stress resultants V^{α}, the analogy in its most convenient form makes use of the equations of equilibrium corresponding to (20.43) or (20.44). Related to this is the idea of reducing the number of the stress-resultants and the stress-couples on the one hand, and the number of the independent strain measures on the other hand, which was apparently first suggested by LUR'E [1950, *3*].

forces $\{F^\beta, F^3\}$, a one-to-one correspondence exists between the equations of equilibrium and the compatibility equations.[35] Although the static-geometric analogy can be discussed with reference to the linear theory summarized in Sect. 23, we confine attention here to the restricted theory since the analogy is particularly useful in the latter framework.

In order to show (within the scope of the restricted theory) the truth of the assertion concerning the static-geometric analogy stated above, it will suffice to consider one of the alternative forms of the equilibrium equations resulting from the linearization of (15.19) or (15.20). We consider here the former or equivalently the equilibrium equations associated with (25.15). For this purpose, recall the compatibility equations $(6.62)_{1,2}$ for the restricted theory and write these in terms of the kinematic variable $\bar{\varrho}_{\alpha\beta}$ defined by (25.16). Thus

$$\begin{aligned} \bar{\varepsilon}^{\alpha\beta}\{\bar{\varepsilon}^{\gamma\nu}[\bar{\varrho}_{\nu\alpha|\beta} + \tfrac{1}{2}(B^\lambda_\alpha e_{\lambda\nu})_{|\beta} - \tfrac{1}{2}(B^\lambda_\nu e_{\lambda\alpha})_{|\beta}] + \bar{\varepsilon}^{\nu\lambda} B^\gamma_\nu e_{\lambda\alpha|\beta}\} = 0, \\ \bar{\varepsilon}^{\alpha\beta}\bar{\varepsilon}^{\lambda\nu}[e_{\alpha\lambda|\nu\beta} - B_{\nu\beta}\bar{\varrho}_{\lambda\alpha}] = 0, \end{aligned} \tag{26.19}$$

where $(4.13)_4$ has been used. Introducing now the correspondence

$$N^{(\alpha\beta)} \Leftrightarrow -\bar{\varepsilon}^{\alpha\sigma}\bar{\varepsilon}^{\beta\nu}\bar{\varrho}_{\sigma\nu}, \qquad \tilde{M}^{(\alpha\beta)} \Leftrightarrow \bar{\varepsilon}^{\alpha\sigma}\bar{\varepsilon}^{\beta\nu} e_{\sigma\nu}, \tag{26.20}$$

we can verify easily that substitution of (26.20) into the equilibrium equations (25.15) with $F^\beta = F^3 = 0$ yields the compatibility equations (26.19) and this establishes the correspondence sought.[36] Moreover, by virtue of the correspondence (26.20), the equilibrium and the compatibility equations can be combined into a single system of complex differential equations. For this purpose, put

$$P^{\alpha\beta} = N^{(\alpha\beta)} - i K \bar{\varepsilon}^{\alpha\sigma}\bar{\varepsilon}^{\beta\nu}\bar{\varrho}_{\sigma\nu}, \qquad Q^{\alpha\beta} = \tilde{M}^{(\alpha\beta)} + i K \bar{\varepsilon}^{\alpha\sigma}\bar{\varepsilon}^{\beta\nu} e_{\sigma\nu}, \tag{26.21}$$

where $i = (-1)^{\frac{1}{2}}$ and K is an arbitrary (real) constant. Using (26.21), by a suitable combination of (26.19) and (25.15) with $\bar{F}^i$ replaced with F^i we obtain

$$\begin{aligned} P^{\alpha\beta}{}_{|\beta} - \tfrac{1}{2}[B^\alpha_\lambda Q^{\beta\lambda}]_{|\beta} + \tfrac{1}{2}[B^\beta_\lambda Q^{\lambda\alpha}]_{|\beta} - B^\alpha_\lambda Q^{\beta\lambda}{}_{|\beta} + \varrho_0 F^\alpha = 0, \\ Q^{\alpha\beta}{}_{|\alpha\beta} + B_{\alpha\beta} P^{\alpha\beta} + \varrho_0 F^3 = 0, \end{aligned} \tag{26.22}$$

the real and imaginary parts of which yield the equations of equilibrium and compatibility, respectively. The complex differential equations (26.22) can be further reduced in the case of isotropic shells but we do not pursue the matter here.[37]

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[35] The static-geometric analogy, in lines of curvature coordinates, is discussed in the books by NOVOZHILOV [1959, *3*] and GOL'DENVEIZER [1961, *3*] and was also noted independently by SANDERS [1959, *6*]. A related analogy has been discussed more recently by LUR'E [1961, *7*]. An account of these results can be found in [1963, *6*]. Within the scope of the classical shell theory and in terms of the variables employed in Sect. 20 [Subsect. δ)], different versions of the analogy can be deduced. See, in this connection, [1963, *7*], [1964, *5*] and [1966, *7*].

[36] As noted earlier, the correspondence theorem has no direct counterpart in the classical (non-polar) three-dimensional theory. However, a *static-geometric analogue* can be shown to hold in the three-dimensional linear theory with couple stresses. An observation of this kind is made in [1965, *6*], where it is also shown that the analogy in the non-polar case merely degenerates to the representation of the stress tensor in terms of a Beltrami-Gwyther-Finzi stress function; for the latter, see TRUESDELL and TOUPIN [1960, *14*].

[37] The complex differential equations (26.22) and their further reduction in the case of isotropic shells were obtained by NAGHDI [1966, *7*]. Similar complex differential equations, together with their reduction for isotropic shells, were previously discussed by NOVOZHILOV [1959, *3*] in lines of curvature coordinates. However, in effecting the reduction, NOVOZHILOV makes use of a different set of constitutive equations along with a further approximation which renders his derivation rigorous only if Poisson's ratio is zero. The complex differential equations (26.22) and the results in [1966, *7*] have been utilized in a recent study by STEELE [1971, *9*].

F. Appendix: Geometry of a surface and related results.

This Appendix contains various formulae from tensor calculus and selected results from the differential geometry of a surface which are essential in our development of the theory of shells and plates (Chaps. B to E). Some familiarity with tensor calculus and elementary differential geometry is assumed. Our short exposition is intended to serve mainly as background and to facilitate and shorten some of the developments in Chaps. B to E. More elaborate accounts and detailed proofs may be found in standard books on tensor calculus and differential geometry.[1]

Our notations in this Appendix generally correspond to that of the main text; in particular, we use the same symbol for a function and its value without confusion. Throughout the Appendix, all Latin indices (subscripts or superscripts) take the values 1, 2, 3, Greek indices (subscripts or superscripts) have the range 1, 2, and the usual summation convention over a repeated index (one subscript and one superscript) is employed. We use a comma for partial differentiation with respect to coordinates, a single vertical line ($|$) for covariant differentiation with respect to the metric tensor of the (two-dimensional) surface and double vertical lines ($\|$) to designate covariant differentiation with respect to the metric tensor of a Euclidean 3-space. Also, in general, whenever the same letter is used for a quantity defined on a (two-dimensional) surface and a corresponding quantity in a Euclidean 3-space, for clarity the latter is distinguished by an added asterisk.

The contents of the Appendix are arranged in four sections as follows: In Sect. A.1, we have summarized the essential results and formulae from tensor calculus for a Euclidean 3-space with real coordinates. The restriction to three-dimensional space is not essential and can be easily dropped. Although the results in Sect. A.1 are collected with reference to Euclidean space, many of the formulae hold in fact in Riemannian space. Sect. A.2 contains selected results from differential geometry pertaining to *intrinsic* and *non-intrinsic* properties of a (two-dimensional) surface embedded in a Euclidean 3-space. In contrast to the developments in Sect. A.2, we consider in Sect. A.3 the geometrical properties of a surface embedded in a Euclidean 3-space covered by *normal coordinates*. We also indicate in Sect. A.3 the relationships between the components of space tensors and their surface counterparts. Finally, in Sect. A.4 we briefly discuss the nature of physical components of surface tensors and their tensor derivatives referred to lines of curvature coordinates on a surface.

A.1. Geometry of Euclidean space. Let x^i $(i=1, 2, 3)$ refer to a fixed right-handed orthogonal Cartesian coordinate system in a Euclidean 3-space and let θ^i denote an arbitrary (real) curvilinear coordinate system defined by the transformation

$$x^i = x^i(\theta^1, \theta^2, \theta^3), \qquad \det\left(\frac{\partial x^i}{\partial \theta^j}\right) \neq 0. \tag{A.1.1}$$

The condition $(\text{A.1.1})_2$ ensures the existence of a unique inverse of $(\text{A.1.1})_1$ so that

$$\theta^i = \theta^i(x^1, x^2, x^3). \tag{A.1.2}$$

Let $\bar{\theta}^i$ be a new set of curvilinear coordinates and consider the invertible transformation

$$\bar{\theta}^i = \bar{\theta}^i(\theta^1, \theta^2, \theta^3) \tag{A.1.3}$$

[1] A large number of books and monographs pertaining to the subject are available: For example, McConnell [1931, *1*], Eisenhart [1947, *2*], Synge and Schild [1949, *5*], Ch. 1 of Green and Zerna [1954, *1*] (see also [1968, *9*]), Willmore [1959, *8*] and Ericksen [1960, *3*].

and its inverse defined by

$$\theta^i = \theta^i(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3). \tag{A.1.4}$$

We assume that the functions describing the coordinate transformations (A.1.3)–(A.1.4) are single-valued and possess as many partial derivatives as required. From (A.1.3)–(A.1.4), the transformation of differentials $d\theta^i$, $d\bar{\theta}^i$ are:

$$d\theta^i = \frac{\partial\theta^i}{\partial\bar{\theta}^j}\, d\bar{\theta}^j, \qquad d\bar{\theta}^i = \frac{\partial\bar{\theta}^i}{\partial\theta^j}\, d\theta^j. \tag{A.1.5}$$

It is clear from $(\text{A.1.5})_1$ that (A.1.4) induces a homogeneous linear transformation on the differentials $d\theta^i$ and $d\bar{\theta}^i$, the coefficients of transformation being functions of the coordinates.

Let $T^i = \{T^1, T^2, T^3\}$ and $T_i = \{T_1, T_2, T_3\}$ be two distinct sets of 3 real numbers associated with a point P of the Euclidean 3-space with coordinates θ^i. Similarly, let there be associated with the same point of space two sets of real numbers $\bar{T}^i = \{\bar{T}^1, \bar{T}^2, \bar{T}^3\}$ and $\bar{T}_i = \{\bar{T}_1, \bar{T}_2, \bar{T}_3\}$ defined with respect to a coordinate system $\bar{\theta}^i$ obtained from (A.1.3). Then, $T^i\,(T_i)$ are said to be components of a contravariant (covariant) tensor of order 1 (in the coordinates θ^i) or simply contravariant (covariant) components of a vector at P if the numbers T^i and $\bar{T}^i$ (T_i and $\bar{T}_i$) satisfy the relations

$$\bar{T}^i = \frac{\partial\bar{\theta}^i}{\partial\theta^j}\, T^j, \qquad T^i = \frac{\partial\theta^i}{\partial\bar{\theta}^j}\, \bar{T}^j, \tag{A.1.6}$$

$$\bar{T}_i = \frac{\partial\theta^j}{\partial\bar{\theta}^i}\, T_j, \qquad T_i = \frac{\partial\bar{\theta}^j}{\partial\theta^i}\, \bar{T}_j. \tag{A.1.7}$$

Consider next a system of 3^2 real numbers, associated with the point P of the Euclidean 3-space, which we designate by T^{ij} and $\bar{T}^{ij}\,(i, j = 1, 2, 3)$ in the coordinates θ^i and $\bar{\theta}^i$, respectively. These numbers are called the components of a contravariant tensor of order 2 (in their respective coordinates) if they are related by the transformation laws

$$\begin{aligned} T^{ij} &= \frac{\partial\theta^i}{\partial\bar{\theta}^k}\, \frac{\partial\theta^j}{\partial\bar{\theta}^l}\, \bar{T}^{kl}, \\ \bar{T}^{ij} &= \frac{\partial\bar{\theta}^i}{\partial\theta^k}\, \frac{\partial\bar{\theta}^j}{\partial\theta^l}\, T^{kl}. \end{aligned} \tag{A.1.8}$$

The set of numbers T^{ij} are the components of the contravariant tensor at P in the coordinates θ^i and similarly $\bar{T}^{ij}$ are the components of the same tensor at P in the coordinates $\bar{\theta}^i$. The transformation laws for the components of a covariant tensor of order 2 can be defined analogously. Let T_{ij} and $\bar{T}_{ij}$ denote a system of 3^2 real numbers (associated with the point P) in the coordinates θ^i and $\bar{\theta}^i$, respectively. These numbers are called the components of a covariant tensor of order 2 if, under transformation of coordinates, T_{ij} and $\bar{T}_{ij}$ are related by

$$\begin{aligned} T_{ij} &= \frac{\partial\bar{\theta}^k}{\partial\theta^i}\, \frac{\partial\bar{\theta}^l}{\partial\theta^j}\, \bar{T}_{kl}, \\ \bar{T}_{ij} &= \frac{\partial\theta^k}{\partial\bar{\theta}^i}\, \frac{\partial\theta^l}{\partial\bar{\theta}^j}\, T_{kl}. \end{aligned} \tag{A.1.9}$$

More generally, let $T_{i_1 \dots i_r}{}^{j_1 \dots j_s}$ and $\bar{T}_{i_1 \dots i_r}{}^{j_1 \dots j_s}\,(i_1, \dots, i_r, j_1, \dots, j_s = 1, 2, 3)$ be two systems of 3^{r+s} real numbers associated with a point P in the coordinates

θ^i and $\bar{\theta}^i$, respectively. We say these numbers (in their respective coordinates) represent the components of a mixed tensor of order $r+s$ at P, covariant of order r and contravariant of order s, if they transform according to the laws

$$\begin{aligned} T_{i_1\ldots i_r}{}^{j_1\ldots j_s}_{\cdot\;\;\;\cdot} &= \frac{\partial\bar{\theta}^{k_1}}{\partial\theta^{i_1}}\cdots\frac{\partial\bar{\theta}^{k_r}}{\partial\theta^{i_r}}\frac{\partial\theta^{j_1}}{\partial\bar{\theta}^{l_1}}\cdots\frac{\partial\theta^{j_s}}{\partial\bar{\theta}^{l_s}}\bar{T}_{k_1\ldots k_r}{}^{l_1\ldots l_s}_{\cdot\;\;\;\cdot}, \\ \bar{T}_{i_1\ldots i_r}{}^{j_1\ldots j_s}_{\cdot\;\;\;\cdot} &= \frac{\partial\theta^{k_1}}{\partial\bar{\theta}^{i_1}}\cdots\frac{\partial\theta^{k_r}}{\partial\bar{\theta}^{i_r}}\frac{\partial\bar{\theta}^{j_1}}{\partial\theta^{l_1}}\cdots\frac{\partial\bar{\theta}^{j_s}}{\partial\theta^{l_s}}T_{k_1\ldots k_r}{}^{l_1\ldots l_s}_{\cdot\;\;\;\cdot}. \end{aligned} \tag{A.1.10}$$

For reasons that will become apparent later, in writing a mixed tensor such as $T_i{}^j{}_{\cdot k}$ or those in (A.1.10), we refrain from placing two indices above each other (along the same vertical line). This is easily effected by reserving a vacant space indicated by a dot.

The above definitions and transformation laws are those appropriate for absolute tensors and require modifications in the case of the so-called relative tensors. Definitions (A.1.6)–(A.1.9) may be regarded as special cases of (A.1.10). A scalar, i.e., a quantity which remains invariant with respect to any coordinate transformation, is called a tensor of order zero while a tensor of order 1 is also known as a vector. A tensor is called *covariant* or *contravariant* if it has only covariant indices (subscripts) or contravariant indices (superscripts), respectively; otherwise the tensor is called *mixed*. If the relative order of two indices—either both subscripts or both superscripts—of a tensor is immaterial, the tensor is called *symmetric with respect to these indices*. A tensor, either covariant or contravariant, is said to be *symmetric* if it is symmetric with respect to all pairs of indices. If two components of a given tensor (either covariant or contravariant) can be obtained from one another by the interchange of two particular indices (either both subscripts or both superscripts) and a change in sign, the tensor is said to be *skew-symmetric (or alternating) with respect to these indices*. A tensor, either covariant or contravariant, is said to be *skew-symmetric* if it is skew-symmetric with respect to all pairs of indices. Any covariant or contravariant tensor of order 2 can be represented uniquely as a sum of symmetric and a skew-symmetric tensor. Thus

$$\begin{gathered} T_{ij} = T_{(ij)} + T_{[ij]}, \\ T_{(ij)} = \tfrac{1}{2}(T_{ij}+T_{ji}), \quad T_{[ij]} = \tfrac{1}{2}(T_{ij}-T_{ji}) = -T_{[ji]}, \end{gathered} \tag{A.1.11}$$

where the notations $T_{(ij)}$, $T_{[ij]}$ stand, respectively, for the symmetric and the skew-symmetric part of T_{ij}. Results analogous to (A.1.11) can be recorded also for T^{ij}.

Two tensors are said to be *of the same type* if they are covariant of the same order and contravariant of the same order. By adding (at the same point in space) the corresponding components of two tensors of the same type (say in the coordinates θ^i), we obtain a third tensor of the same type (in the coordinates θ^i). If we multiply every component of a tensor by a scalar, a tensor of the same type will result. From this multiplication and the addition of two tensors of the same type, it follows that the totality of tensors of the same type form a vector space. If every component of a tensor such as $A_{ij}{}^{kl}_{\cdot\cdot}$ is multiplied by every component of another arbitrary tensor such as $B_{mn}{}^{p}_{\cdot}$, there results

$$T_{ijmn}{}^{klp}_{\cdot\cdot\cdot} = A_{ij}{}^{kl}_{\cdot\cdot}\, B_{mn}{}^{p}_{\cdot}, \tag{A.1.12}$$

called the *outer product* of the tensor A and B. By (A.1.10), the outer product of any two tensors is a tensor whose character is indicated by the position of its indices; the number of covariant and contravariant indices of the resulting tensor T is equal to the sum of the numbers of covariant or contravariant indices of the tensors A and B. Thus, the left-hand side of (A.1.12) are components of a mixed tensor, covariant of order 4 and contravariant of order 3. If a subscript and a superscript of a mixed tensor are identified, so that the same index is repeated with the implied summation, the process is called *contraction*. For example, contraction of the indices j and k of the tensor $T_{ijmn\ldots}^{\quad klp}$ in (A.1.12) results in the mixed tensor $T_{ijmn\ldots}^{\quad jlp}$ which is of order 1 less than the original tensor in both covariant and contravariant indices. In particular, for a mixed tensor of order 2 such as $T^i_{.j}$, contraction results in a scalar $T^i_{.i}$. Multiplication of two tensors accompanied by a contraction results in a tensor called an *inner product*. Contraction may be applied more than once and the number of inner products formed from any two tensors depends on the number of indices involved. For example, from the tensors $A_{i.k}^{\ j}$ and $B^m_{.n}$ different inner products $A_{i.k}^{\ j} B^k_{.n}$, $A_{i.k}^{\ j} B^m_{.j}$, $A_{i.k}^{\ j} B^k_{.j}$, etc., can be obtained.

Let

$$\boldsymbol{p} = \boldsymbol{p}(\theta^i) \tag{A.1.13}$$

denote the position vector of a typical point (with coordinates θ^i) in a region $\mathscr{R}$ of the Euclidean space. Then, the square of a line element is given by

$$ds^{*2} = d\boldsymbol{p}\cdot d\boldsymbol{p} = g_{ij}\, d\theta^i\, d\theta^j, \tag{A.1.14}$$

where

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{p}}{\partial \theta^i}, \qquad g_{ij} = \boldsymbol{g}_i\cdot\boldsymbol{g}_j \tag{A.1.15}$$

are, respectively, the (covariant) base vectors and the metric tensor while a comma denotes partial differentiation with respect to θ^i. The reciprocal (contravariant) base vectors $\boldsymbol{g}^i$ and the conjugate tensor g^{ij} are defined by

$$\begin{gathered} \boldsymbol{g}^i = g^{ij}\,\boldsymbol{g}_j, \qquad g^{ij} = g^{ji} = \boldsymbol{g}^i\cdot\boldsymbol{g}^j = \frac{\mathscr{G}^{ij}}{g}, \\ g = \det(g_{ij}), \end{gathered} \tag{A.1.16}$$

where $\mathscr{G}^{ij}$ are the cofactors of g_{ij} in the expansion of the determinant g. The base vectors as well as the metric and the conjugate tensors, in addition to those indicated by $(\text{A.1.15})_2$ and $(\text{A.1.16})_2$, also satisfy the relations

$$\boldsymbol{g}^i\cdot\boldsymbol{g}_j = \delta^i_j, \qquad g^{ik}\, g_{kj} = g_{jk}\, g^{ki} = \delta^i_j. \tag{A.1.17}$$

The vector products of the base vectors yield

$$\begin{aligned} \boldsymbol{g}_i \times \boldsymbol{g}_j &= \varepsilon_{ijk}\,\boldsymbol{g}^k, \\ \boldsymbol{g}^i \times \boldsymbol{g}^j &= \varepsilon^{ijk}\,\boldsymbol{g}_k, \end{aligned} \tag{A.1.18}$$

where δ^i_j stands for Kronecker delta and the ε-system is related to the permutation symbols e_{ijk}, e^{ijk}, through

$$\varepsilon_{ijk} = g^{\frac{1}{2}}\, e_{ijk}, \qquad \varepsilon^{ijk} = g^{-\frac{1}{2}}\, e^{ijk}. \tag{A.1.19}$$

In contrast to ε_{ijk}, ε^{ijk} which are absolute tensors and transform according to (A.1.10), the permutation symbols are relative tensors and have different transformation laws. We do not elaborate here in detail on relative tensors but note that the permutation symbols e_{ijk} and e^{ijk} are relative tensors of weights -1 and

$+1$, respectively, which transform according to

$$\begin{aligned} e_{ijk} &= J \frac{\partial \bar{\theta}^m}{\partial \theta^i} \frac{\partial \bar{\theta}^n}{\partial \theta^j} \frac{\partial \bar{\theta}^p}{\partial \theta^k} \bar{e}_{mnp}, \\ e^{ijk} &= J^{-1} \frac{\partial \theta^i}{\partial \bar{\theta}^m} \frac{\partial \theta^j}{\partial \bar{\theta}^n} \frac{\partial \theta^k}{\partial \bar{\theta}^p} \bar{e}^{mnp}, \end{aligned} \tag{A.1.20}$$

where $J = \det\left(\frac{\partial \theta^i}{\partial \bar{\theta}^j}\right)$ is the Jacobian of the transformation.

An index of a tensor can be raised or lowered through inner multiplication by g^{ij} or g_{ij}. To illustrate this process, consider a tensor T_{hjk} which may be generated from $T^i_{.jk}$ through inner multiplication by g_{hi}:

$$T_{hjk} = g_{hi} T^i_{.jk}. \tag{A.1.21}$$

It is clear from the right-hand side of (A.1.21) that through inner multiplication by g_{hi}, the superscript i is lowered into the vacant space indicated by a dot in $T^i_{.jk}$. Consider next the raising of an index by inner multiplication of g^{ih} and T_{hjk}. Thus, using (A.1.21) and $(\text{A.1.17})_2$, we have

$$\begin{aligned} g^{ih} T_{hjk} &= g^{ih} g_{hm} T^m_{.jk} \\ &= \delta^i_m T^m_{.jk} = T^i_{.jk}, \end{aligned} \tag{A.1.22}$$

which also shows that the process of raising and lowering indices is reversible. However, in general, the expressions

$$g^{ki} T_{ij} = T^k_{.j}, \qquad g^{ki} T_{ji} = T_j^{.k} \tag{A1.23}$$

are different. They become identical if $T_{ij} = T_{ji}$.

The Christoffel symbols of the first and second kinds are defined by

$$\overset{*}{\Gamma}_{ijk} = g_{mk} \overset{*}{\Gamma}{}^m_{ij} = \tfrac{1}{2}[g_{ik,j} + g_{jk,i} - g_{ij,k}], \quad \overset{*}{\Gamma}_{ijk} = \overset{*}{\Gamma}_{jik}, \quad \overset{*}{\Gamma}{}^k_{ij} = \overset{*}{\Gamma}{}^k_{ji}. \tag{A.1.24}$$

In view of (A.1.15) and since $\boldsymbol{g}_{i,j} = \boldsymbol{g}_{j,i}$, they can also be expressed as

$$\overset{*}{\Gamma}_{ijk} = \overset{*}{\Gamma}_{jik} = \boldsymbol{g}_k \cdot \boldsymbol{g}_{i,j}, \quad \overset{*}{\Gamma}{}^k_{ij} = \overset{*}{\Gamma}{}^k_{ji} = \boldsymbol{g}^k \cdot \boldsymbol{g}_{i,j} = -\boldsymbol{g}_i \cdot \boldsymbol{g}^k{}_{,j} \tag{A.1.25}$$

and hence

$$\boldsymbol{g}_{i,j} = \overset{*}{\Gamma}{}^k_{ij} \boldsymbol{g}_k, \qquad \boldsymbol{g}^i{}_{,j} = -\overset{*}{\Gamma}{}^i_{jk} \boldsymbol{g}^k. \tag{A.1.26}$$

To indicate the nature of covariant differentiation of a tensor function, we record below the expressions for covariant derivatives of a covariant tensor T_i, a contravariant tensor T^i, a mixed second order tensor $T^i_{.j}$ and a contravariant second order tensor T^{ij}. Thus, designating covariant differentiation with respect to g_{ij} by double vertical lines ($\|$), we have:

$$\begin{aligned} T_{i\|k} &= T_{i,k} - \overset{*}{\Gamma}{}^l_{ik} T_l, \\ T^i{}_{\|k} &= T^i{}_{,k} + \overset{*}{\Gamma}{}^i_{lk} T^l, \end{aligned} \tag{A.1.27}$$

$$\begin{aligned} T^i_{.j\|k} &= T^i_{.j,k} + \overset{*}{\Gamma}{}^i_{lk} T^l_{.j} - \overset{*}{\Gamma}{}^l_{jk} T^i_{.l}, \\ T^{ij}{}_{\|k} &= T^{ij}{}_{,k} + \overset{*}{\Gamma}{}^i_{lk} T^{lj} + \overset{*}{\Gamma}{}^j_{lk} T^{il}. \end{aligned} \tag{A.1.28}$$

The covariant derivative $T^i{}_{\|k}$ (and also $T_{i\|k}$) has a simple geometrical meaning. To see this, let T^i and T_i be the contravariant and the covariant components of a vector $\boldsymbol{T}$ so that

$$\boldsymbol{T} = T^i \boldsymbol{g}_i = T_i \boldsymbol{g}^i. \tag{A.1.29}$$

Then, with the help of (A.1.25)–(A.1.26), from $(\text{A.1.29})_1$ we can readily obtain the expression

$$\frac{\partial \boldsymbol{T}}{\partial \theta^k} = \frac{\partial T^i}{\partial \theta^k} \boldsymbol{g}_i + \frac{\partial \boldsymbol{g}_i}{\partial \theta^k} T^i = \{T^i{}_{,k} + \overset{*}{\Gamma}{}^i_{lk} T^l\} \boldsymbol{g}_i = T^i{}_{\|k} \boldsymbol{g}_i. \tag{A.1.30}$$

Similarly, from $(\text{A.1.29})_2$, we obtain

$$\frac{\partial \boldsymbol{T}}{\partial \theta^k} = T_{i\|k} \boldsymbol{g}^i. \tag{A.1.31}$$

The expressions (A.1.30)–(A.1.31) show that the covariant derivatives $T^i{}_{\|k} (T_{i\|k})$ are components of the partial derivative $\partial \boldsymbol{T}/\partial \theta^k$ referred to the base vectors $\boldsymbol{g}_i(\boldsymbol{g}^i)$. We note here that the base vectors $\boldsymbol{g}_i$, $\boldsymbol{g}^i$, the metric tensor g_{ij} and its conjugate g^{ij}, as well as the tensor derivatives defined by (A.1.27)–(A.1.28), are tensors whose covariant and contravariant characters are indicated by the position of their indices. The Christoffel symbols do not transform according to (A.1.10) and hence are not tensors.

We examine now the nature of two successive covariant derivatives of tensor functions. For this purpose, it will suffice to consider the tensors T_i and T_{ij} whose characters are indicated by the position of their respective indices. Then

$$T_{i\|jk} - T_{i\|kj} = \overset{*}{R}{}^m{}_{.ijk} T_m, \qquad T_{ij\|kl} - T_{ij\|lk} = \overset{*}{R}{}^m{}_{.ikl} T_{mj} + \overset{*}{R}{}^m{}_{.jkl} T_{im}, \tag{A.1.32}$$

where the mixed curvature tensor (also known as the mixed Riemann-Christoffel tensor) is given by

$$\overset{*}{R}{}^m{}_{.ijk} = \overset{*}{\Gamma}{}^m_{ik,j} - \overset{*}{\Gamma}{}^m_{ij,k} + \overset{*}{\Gamma}{}^p_{ik} \overset{*}{\Gamma}{}^m_{pj} - \overset{*}{\Gamma}{}^p_{ij} \overset{*}{\Gamma}{}^m_{pk}. \tag{A.1.33}$$

According to the relations (A.1.32), the order of covariant differentiation does not commute unless $\overset{*}{R}{}^m{}_{.ijk} = 0$ which is both a necessary and sufficient condition for the space to be Euclidean. Hence, in a Euclidean space, the order of covariant differentiation is immaterial.

The covariant curvature tensor, namely

$$\overset{*}{R}_{mijk} = g_{mp} \overset{*}{R}{}^p{}_{.ijk}$$

satisfies the relations

$$\overset{*}{R}_{pijk} = -\overset{*}{R}_{ipjk} = -\overset{*}{R}_{pikj} = \overset{*}{R}_{jkpi}, \tag{A.1.34}$$

$$\overset{*}{R}_{mijk} + \overset{*}{R}_{mjki} + \overset{*}{R}_{mkij} = 0, \tag{A.1.35}$$

and the number of independent components of $\overset{*}{R}_{mijk}$ which depends on the dimension of the space, with the aid of (A.1.34)–(A.1.35), may be shown to be[2]

$$\tfrac{1}{12} N^2 (N^2 - 1), \tag{A.1.36}$$

N being the dimension of the space. For a 2-space, the covariant curvature tensor has only one independent component as all of its components either vanish or are

[2] See, e.g., p. 87 of SYNGE and SCHILD [1949, 5].

expressible in terms of $\overset{*}{R}_{1212}$. For $N=3$, it follows from (A.1.36) that the number of independent components of the covariant curvature tensor is six, which can be shown to be

$$\overset{*}{R}_{1212},\ \overset{*}{R}_{3112},\ \overset{*}{R}_{3221},\ \overset{*}{R}_{1313},\ \overset{*}{R}_{2323},\ \overset{*}{R}_{1323}. \tag{A.1.37}$$

A.2. Some results from the differential geometry of a surface. This section contains a heterogeneous collection of certain results and formulae pertaining to (local) intrinsic and non-intrinsic properties of a surface $\mathscr{s}$ embedded in a Euclidean 3-space.

α) *Definition of a surface. Preliminaries.* A surface $\mathscr{s}$ may be specified in terms of parametric equations of the form

$$x_i = f_i(\theta^1, \theta^2), \tag{A.2.1}$$

where $x_i (i=1, 2, 3)$ are rectangular Cartesian coordinates of a point P on the surface $\mathscr{s}$ and $\theta^\alpha (\alpha=1, 2)$ are parameters, as yet unrelated to the curvilinear coordinates θ^i utilized in Sect. A.1. We assume that the functions f_i are single-valued and continuous and also require that they possess as many derivatives as may be required. The position vector $\boldsymbol{r}$ (with rectangular Cartesian components x_i) at P, in view of (A.2.1), may be expressed as a function of the parameters[3] θ^α. Thus, we write

$$\boldsymbol{r} = \boldsymbol{r}(\theta^1, \theta^2), \tag{A.2.2}$$

where the function $\boldsymbol{r}$ in (A.2.2) has the same continuity and differentiability properties as f_i. With any point P of the surface, we can associate a set of base vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ defined by

$$\boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha}, \qquad \boldsymbol{a}_1 \times \boldsymbol{a}_2 \neq 0, \tag{A.2.3}$$

where a comma denotes partial differentiation with respect to θ^α-coordinates. As will become evident presently, the condition $(\text{A.2.3})_2$ on the vector product of the base vectors ensures the existence of a unique normal at each *ordinary point*[4] on $\mathscr{s}$.

We introduce new coordinates $\bar{\theta}^1$, $\bar{\theta}^2$ on the surface $\mathscr{s}$ by means of the coordinate transformation

$$\theta^\alpha = \theta^\alpha(\bar{\theta}^1, \bar{\theta}^2), \qquad \det\left(\frac{\partial\theta^\alpha}{\partial\bar{\theta}^\gamma}\right) \neq 0, \tag{A.2.4}$$

so that the inverse transformation

$$\bar{\theta}^\alpha = \bar{\theta}^\alpha(\theta^1, \theta^2) \tag{A.2.5}$$

exists. From (A.2.4)–(A.2.5), the transformation of the differentials $d\theta^\alpha$ and $d\bar{\theta}^\alpha$ are:

$$d\theta^\alpha = \frac{\partial\theta^\alpha}{\partial\bar{\theta}^\gamma}\, d\bar{\theta}^\gamma, \qquad d\bar{\theta}^\alpha = \frac{\partial\bar{\theta}^\alpha}{\partial\theta^\gamma}\, d\theta^\gamma. \tag{A.2.6}$$

In what follows an overbar will be placed on quantities at P associated with coordinates $\bar{\theta}^\alpha$ in order to distinguish these from the same quantities at P associated with coordinates θ^α. Thus, for example the base vectors at P associated

[3] These parameters are usually called curvilinear coordinates on the surface.

[4] An ordinary point is defined by the condition $(\text{A.2.3})_2$. A point which is not ordinary, such as the vertex of a cone, is called a *singularity*. Here we restrict the domain of θ^α such that every point of $\mathscr{s}$ is ordinary.

with $\bar{\theta}^\alpha$ will be designated as $\bar{\boldsymbol{a}}_\alpha$. It can be readily verified that the property which defines an ordinary point is unaltered under the coordinate transformation (A.2.4), so that

$$\bar{\boldsymbol{a}}_1 \times \bar{\boldsymbol{a}}_2 = \frac{\partial \boldsymbol{r}}{\partial \bar{\theta}^1} \times \frac{\partial \boldsymbol{r}}{\partial \bar{\theta}^2} \neq 0$$

[$\boldsymbol{r}$ being now a different function from that in (A.2.2)].

Definitions of covariant, contravariant and mixed tensors at a point P on $\mathscr{s}$ are similar to those given in Sect. A.1. In fact, with an obvious modification, various tensor transformation laws between (A.1.6)–(A.1.10) and the subsequent discussion until the end of the paragraph containing (A.1.12) hold for the Riemannian space under consideration. For example, let a system of 2^{r+s} real numbers $T_{\alpha_1\ldots\alpha_r\cdot}{}^{\gamma_1\ldots\gamma_s}$ $(\alpha_1, \ldots, \alpha_r, \gamma_1, \ldots, \gamma_s = 1, 2)$ be associated with a point P on $\mathscr{s}$ (a Riemannian 2-space) with coordinates θ^α and let $\bar{T}_{\alpha_1\ldots\alpha_r\cdot}{}^{\gamma_1\ldots\gamma_s}$ be the corresponding 2^{r+s} real numbers at P in the coordinates $\bar{\theta}^\alpha$ defined by (A.2.5). We say that these numbers represent a mixed tensor of order $r+s$ at P, covariant of order r and contravariant of order s, if they are related by the transformation laws

$$\begin{aligned} T_{\alpha_1\ldots\alpha_r\cdot}{}^{\gamma_1\ldots\gamma_s} &= \frac{\partial \bar{\theta}^{\beta_1}}{\partial \theta^{\alpha_1}} \cdots \frac{\partial \bar{\theta}^{\beta_r}}{\partial \theta^{\alpha_r}} \frac{\partial \theta^{\gamma_1}}{\partial \bar{\theta}^{\delta_1}} \cdots \frac{\partial \theta^{\gamma_s}}{\partial \bar{\theta}^{\delta_s}} \bar{T}_{\beta_1\ldots\beta_r\cdot}{}^{\delta_1\ldots\delta_s}, \\ \bar{T}_{\alpha_1\ldots\alpha_r\cdot}{}^{\gamma_1\ldots\gamma_s} &= \frac{\partial \theta^{\beta_1}}{\partial \bar{\theta}^{\alpha_1}} \cdots \frac{\partial \theta^{\beta_r}}{\partial \bar{\theta}^{\alpha_r}} \frac{\partial \bar{\theta}^{\gamma_1}}{\partial \theta^{\delta_1}} \cdots \frac{d \bar{\theta}^{\gamma_s}}{d \theta^{\delta_s}} T_{\beta_1\ldots\beta_r\cdot}{}^{\delta_1\ldots\delta_s}. \end{aligned} \tag{A.2.7}$$

A curve on a surface $\mathscr{s}$ can be determined by a sufficiently smooth parametric representation $\theta^\alpha = \theta^\alpha(u)$, where u is a real variable. The direction of the tangent to a curve on $\mathscr{s}$ is determined by the vector

$$\frac{d\boldsymbol{r}}{du} = \frac{\partial \boldsymbol{r}}{\partial \theta^\alpha} \frac{d\theta^\alpha}{du} = \boldsymbol{a}_\alpha \frac{d\theta^\alpha}{du}, \tag{A.2.8}$$

which, in general, depends on u. Since $\boldsymbol{a}_\alpha$ (which are tangent to the coordinate curves) are non-zero and independent, the tangent to any curve on $\mathscr{s}$ through a point P lies in a plane which contains the two vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ at P; this plane is the *tangent plane* at P. A vector $\boldsymbol{U}$ (in the Euclidean 3-space at P) which lies in the tangent plane to $\mathscr{s}$ at P is called a *tangent vector* or a *vector in the surface.* Any such vector (since it lies in the tangent plane at P) can be expressed as a linear combination of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$:

$$\boldsymbol{U} = U^\alpha \boldsymbol{a}_\alpha. \tag{A.2.9}$$

The base vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ defined by (A.2.3) are linearly independent, are tangent to the θ^α-curves through a point P on $\mathscr{s}$ and span the tangent plane to $\mathscr{s}$ at P. The normal to $\mathscr{s}$ at P is the normal to the tangent plane at P and is therefore perpendicular to $\boldsymbol{a}_1$, $\boldsymbol{a}_2$. Let $\boldsymbol{a}_3$ denote the unit normal to $\mathscr{s}$ at P, the sense of the unit normal being fixed by the convention that $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ form a right-handed triad. It then follows that

$$\boldsymbol{a}_3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}, \qquad |\boldsymbol{a}_1 \times \boldsymbol{a}_2| \neq 0 \tag{A.2.10}$$

and

$$\boldsymbol{a}_3 \cdot \boldsymbol{a}_3 = 1, \qquad \boldsymbol{a}_3 \cdot \boldsymbol{a}_\alpha = 0, \qquad \boldsymbol{a}_3 \cdot \boldsymbol{a}_{3,\alpha} = 0. \tag{A.2.11}$$

The notation $|\boldsymbol{V}|$ in (A.2.10) stands for the magnitude of $\boldsymbol{V}$ and the last of (A.2.11) follows from differentiation of $(\text{A.2.11})_1$.

β) *First and second fundamental forms.* The square of a line element of the surface is given by

$$d s^2 = d\boldsymbol{r}\cdot d\boldsymbol{r} = a_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta, \tag{A.2.12}$$

where

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \tag{A.2.13}$$

and the base vectors $\boldsymbol{a}_\alpha$ are defined by $(\text{A.2.3})_1$. The quadratic form (A.2.12) is called the *first fundamental form* of the surface. The reciprocals of $(\text{A.2.3})_1$ and (A.2.13) are denoted by $\boldsymbol{a}^\alpha$ and $a^{\alpha\beta}$, respectively. They are defined for all ordinary points of the surface, i.e., points for which[5]

$$a = \det(a_{\alpha\beta}) \neq 0, \tag{A.2.14}$$

and are given by

$$\boldsymbol{a}^\alpha = a^{\alpha\beta}\,\boldsymbol{a}_\beta, \qquad a^{\alpha\beta} = a^{\beta\alpha} = \boldsymbol{a}^\alpha\cdot\boldsymbol{a}^\beta = \frac{\mathscr{A}^{\alpha\beta}}{a}, \tag{A.2.15}$$

where $\mathscr{A}^{\alpha\beta}$ are the cofactors of $a_{\alpha\beta}$ in the expansion of the determinant a. The (covariant) base vectors $\boldsymbol{a}_\alpha$, the (contravariant) reciprocal base vectors $\boldsymbol{a}^\alpha$, the symmetric tensor $a_{\alpha\beta}$ and its conjugate $a^{\alpha\beta}$ all satisfy the appropriate laws for tensor transformations of the type (A.2.7). These vectors and tensors also satisfy the relations

$$\boldsymbol{a}^\alpha\cdot\boldsymbol{a}_\beta = \delta^\alpha_\beta, \qquad a^{\alpha\lambda}\, a_{\lambda\beta} = a_{\beta\lambda}\, a^{\lambda\alpha} = \delta^\alpha_\beta. \tag{A.2.16}$$

The raising and the lowering of indices is accomplished here with the use of $a^{\alpha\beta}$ and $a_{\alpha\beta}$; the process is similar to that discussed in Sect. A.1 [between (A.1.21)–(A.1.22)].

The vector products involving $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are

$$\begin{aligned} \boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta &= \varepsilon_{\alpha\beta}\,\boldsymbol{a}_3, & \boldsymbol{a}^\alpha \times \boldsymbol{a}^\beta &= \varepsilon^{\alpha\beta}\,\boldsymbol{a}_3,\\ \boldsymbol{a}_3 \times \boldsymbol{a}_\beta &= \varepsilon_{\beta\lambda}\,\boldsymbol{a}^\lambda, & \boldsymbol{a}_3 \times \boldsymbol{a}^\beta &= \varepsilon^{\beta\lambda}\,\boldsymbol{a}_\lambda, \end{aligned} \tag{A.2.17}$$

where $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$ are the ε-system for the surface defined by

$$\varepsilon_{\alpha\beta} = a^{\frac{1}{2}}\, e_{\alpha\beta}, \qquad \varepsilon^{\alpha\beta} = a^{-\frac{1}{2}}\, e^{\alpha\beta} \tag{A.2.18}$$

and

$$e_{11} = e_{22} = e^{11} = e^{22} = 0, \qquad e_{12} = e^{12} = 1, \qquad e_{21} = e^{21} = -1. \tag{A.2.19}$$

Let $\theta^\alpha = \theta^\alpha(s)$ be the parametric equations of a curve c, where the parameter s is the arc length; and let $\boldsymbol{\lambda}$ denote the unit tangent vector to c defined for points $\boldsymbol{r}(\theta^\alpha(s))$ on c. Then,

$$\boldsymbol{\lambda} = \frac{d\boldsymbol{r}}{ds} = \lambda^\alpha\,\boldsymbol{a}_\alpha, \qquad \lambda^\alpha = \frac{d\theta^\alpha(s)}{ds}, \tag{A.2.20}$$

which follows also from (A.2.8) if the parameter u is identified with the arc length s. The outward unit normal $\boldsymbol{\nu}$ to a curve c on $\mathfrak{s}$, through a point P, is a tangent vector whose sense is fixed by the convention that $\boldsymbol{\nu}$, $\boldsymbol{\lambda}$, $\boldsymbol{a}_3$ form a right-handed triad. Thus, using (A.2.17),

$$\begin{aligned} \boldsymbol{\nu} &= \boldsymbol{\lambda}\times\boldsymbol{a}_3 = \nu^\alpha\,\boldsymbol{a}_\alpha = \nu_\alpha\,\boldsymbol{a}^\alpha = \varepsilon_{\alpha\beta}\,\lambda^\beta\,\boldsymbol{a}^\alpha,\\ \boldsymbol{\lambda} &= \boldsymbol{a}_3\times\boldsymbol{\nu} = \varepsilon^{\alpha\beta}\,\nu_\alpha\,\boldsymbol{a}_\beta. \end{aligned} \tag{A.2.21}$$

We define the generalized Kronecker delta for the surface by

$$\varepsilon^{\alpha\beta}\,\varepsilon_{\lambda\mu} = \delta^{\alpha\beta}_{\lambda\mu} \tag{A.2.22}$$

[5] The condition (A.2.14) is equivalent to $(\text{A.2.3})_2$.

and note that

$$\delta^{\alpha\beta}_{\lambda\beta} = \varepsilon^{\alpha\beta}\,\varepsilon_{\lambda\beta} = a_{\lambda\mu}\,a_{\beta\nu}\,\varepsilon^{\alpha\beta}\,\varepsilon^{\mu\nu} = a^{\alpha\mu}\,a_{\lambda\mu} = \delta^{\alpha}_{\lambda},\qquad \delta^{\alpha\beta}_{\alpha\beta} = \delta^{\alpha}_{\alpha} = 2. \tag{A.2.23}$$

Let $T^{\alpha\beta}$ (not necessarily in the surface) denote the components of a second-order tensor defined at a point on the surface s. Then,

$$T^{\alpha\beta} - T^{\beta\alpha} = \delta^{\alpha\beta}_{\lambda\mu}\,T^{\lambda\mu}. \tag{A.2.24}$$

Similarly, with the help of (A.2.22) and (A.2.19), it can be easily verified that

$$\delta^{\alpha\eta}_{\lambda\nu}\,T^{\lambda}_{\alpha} = \delta^{\eta}_{\nu}\,T^{\alpha}_{\alpha} - T^{\eta}_{\nu}. \tag{A.2.25}$$

The magnitude or the length of a vector in the surface such as $\boldsymbol{U}$ in (A.2.9) is easily seen to be $|\boldsymbol{U}| = [a_{\alpha\beta}\,U^{\alpha}\,U^{\beta}]^{\frac{1}{2}} = [a^{\alpha\beta}\,U_{\alpha}\,U_{\beta}]^{\frac{1}{2}}$. We recall that the element of area of the surface (by definition) is $d\sigma = a^{\frac{1}{2}}\,d\theta^1\,d\theta^2$ and that the expression for the angle between two vectors in the surface involves $a_{\alpha\beta}$. It is, therefore, clear that the coefficients of the first fundamental form determine lengths, angles and areas on a surface. This justifies the use of the term *metric tensor of the surface* for $a_{\alpha\beta}$.

We now turn to the *second fundamental form* of the surface, defined by the scalar product

$$-d\boldsymbol{r}\cdot d\boldsymbol{a}_3 = b_{\alpha\beta}\,d\theta^{\alpha}\,d\theta^{\beta}, \tag{A.2.26}$$

where by $(\text{A.2.3})_1$ and $(\text{A.2.11})_2$,

$$b_{\alpha\beta} = b_{\beta\alpha} = -\boldsymbol{a}_{\alpha}\cdot\boldsymbol{a}_{3,\beta} = \boldsymbol{a}_3\cdot\boldsymbol{a}_{\alpha,\beta}. \tag{A.2.27}$$

The *mean curvature* of the surface is defined by

$$H = \tfrac{1}{2} b^{\alpha}_{\alpha} \tag{A.2.28}$$

and the *Gaussian curvature* of the surface by

$$K = \det(b^{\alpha}_{\beta}) = a^{-1}\det(b_{\alpha\beta}) = b^1_1\,b^2_2 - b^1_2\,b^2_1. \tag{A.2.29}$$

Recalling the well-known formula for the expansion of a determinant, namely

$$\varepsilon^{\lambda\mu}\det(b^{\alpha}_{\beta}) = \varepsilon^{\alpha\beta}\,b^{\lambda}_{\alpha}\,b^{\mu}_{\beta}, \tag{A.2.30}$$

then after multiplication by $\varepsilon_{\lambda\mu}$ and using (A.2.22) and $(\text{A.2.23})_5$, we see that (A.2.29) may alternatively be written in the form

$$K = \tfrac{1}{2}\,\delta^{\alpha\beta}_{\lambda\mu}\,b^{\lambda}_{\alpha}\,b^{\mu}_{\beta}. \tag{A.2.31}$$

It is clear from (A.2.28) and (A.2.31) that both H and K are surface invariants which depend only on the coefficients of the second fundamental form $b_{\alpha\beta}$ of the surface. However, as will be seen presently, the Gaussian curvature can also be expressed entirely in terms of the coefficients of the first fundamental form $a_{\alpha\beta}$ and its derivatives[6].

γ) Covariant derivatives. The curvature tensor. The operator representing the covariant derivative with respect to the metric tensor of the surface will be designated by a vertical line (|). In order to indicate the nature of covariant differentiation of a tensor function defined on the surface, we record below the

[6] See Eq. (A.2.52).

expressions for covariant derivatives, with respect to $a_{\alpha\beta}$, of a covariant tensor T_α, a contravariant tensor T^α, a mixed second order tensor $T^\alpha_{\cdot\beta}$ and a contravariant second order tensor $T^{\alpha\beta}$:

$$\begin{aligned} T_{\alpha|\gamma} &= T_{\alpha,\gamma} - \Gamma^\lambda_{\alpha\gamma} T_\lambda, \qquad T^\alpha{}_{|\gamma} = T^\alpha{}_{,\gamma} + \Gamma^\alpha_{\lambda\gamma} T^\lambda, \\ T^\alpha_{\cdot\beta|\gamma} &= T^\alpha_{\cdot\beta,\gamma} + \Gamma^\alpha_{\lambda\gamma} T^\lambda_{\cdot\beta} - \Gamma^\lambda_{\beta\gamma} T^\alpha_{\cdot\lambda}, \\ T^{\alpha\beta}{}_{|\gamma} &= T^{\alpha\beta}{}_{,\gamma} + \Gamma^\alpha_{\lambda\gamma} T^{\lambda\beta} + \Gamma^\beta_{\lambda\gamma} T^{\alpha\lambda}, \end{aligned} \tag{A.2.32}$$

where the surface Christoffel symbols are given by

$$\begin{aligned} \Gamma_{\alpha\beta\gamma} &= a_{\mu\gamma} \Gamma^\mu_{\alpha\beta} = \tfrac{1}{2}(a_{\alpha\gamma,\beta} + a_{\beta\gamma,\alpha} - a_{\alpha\beta,\gamma}), \\ \Gamma_{\alpha\beta\gamma} &= \Gamma_{\beta\alpha\gamma} = \boldsymbol{a}_\gamma \cdot \boldsymbol{a}_{\alpha,\beta}, \\ \Gamma^\gamma_{\alpha\beta} &= \Gamma^\gamma_{\beta\alpha} = \boldsymbol{a}^\gamma \cdot \boldsymbol{a}_{\alpha,\beta} = -\boldsymbol{a}_\alpha \cdot \boldsymbol{a}^\gamma{}_{,\beta}. \end{aligned} \tag{A.2.33}$$

We note that application of (A.2.32) to (A.2.13), (A.2.15) and (A.2.18) results in

$$a_{\alpha\beta|\gamma} = a^{\alpha\beta}{}_{|\gamma} = 0, \qquad \varepsilon_{\alpha\beta|\gamma} = \varepsilon^{\alpha\beta}{}_{|\gamma} = 0. \tag{A.2.34}$$

By considering the second covariant derivatives of surface vectors and tensors, we find the surface analogues of the expressions (A.1.32). For example

$$\begin{aligned} T_{\alpha|\beta\gamma} - T_{\alpha|\gamma\beta} &= R^\lambda_{\cdot\alpha\beta\gamma} T_\lambda, \\ T_{\alpha\beta|\gamma\delta} - T_{\alpha\beta|\delta\gamma} &= R^\lambda_{\cdot\alpha\gamma\delta} T_{\lambda\beta} + R^\lambda_{\cdot\beta\gamma\delta} T_{\alpha\lambda}, \\ T^{\alpha\beta}{}_{|\gamma\delta} - T^{\alpha\beta}{}_{|\delta\gamma} &= -R^\alpha_{\cdot\lambda\gamma\delta} T^{\lambda\beta} - R^\beta_{\cdot\lambda\gamma\delta} T^{\alpha\lambda}, \end{aligned} \tag{A.2.35}$$

where the curvature tensor (also called the Riemann-Christoffel tensor) for the surface is defined analogously to (A.1.33):

$$R^\lambda_{\cdot\alpha\beta\gamma} = \Gamma^\lambda_{\alpha\gamma,\beta} - \Gamma^\lambda_{\alpha\beta,\gamma} + \Gamma^\mu_{\alpha\gamma} \Gamma^\lambda_{\mu\beta} - \Gamma^\mu_{\alpha\beta} \Gamma^\lambda_{\mu\gamma}. \tag{A.2.36}$$

The covariant curvature tensor for the surface is given by

$$R_{\lambda\alpha\beta\gamma} = a_{\lambda\mu} R^\mu_{\cdot\alpha\beta\gamma} \tag{A.2.37}$$

and satisfies the identities

$$\begin{gathered} R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}, \qquad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}, \qquad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}, \\ R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0. \end{gathered} \tag{A.2.38}$$

It follows from (A.2.38) that in a Riemannian 2-space all components of $R_{\lambda\alpha\beta\gamma}$ either vanish or are expressible in terms of R_{1212}. We return to this later in this section. It is clear from (A.2.35) that the successive covariant differentiations do not commute in a Riemannian space. However, we note here the useful identity

$$T^{\alpha\beta}{}_{|\alpha\beta} = T^{\alpha\beta}{}_{|\beta\alpha}, \tag{A.2.39}$$

where $T^{\alpha\beta}$ is not necessarily symmetric.

δ) Formulae of Weingarten and Gauss. Integrability conditions. At every point P of the surface $\mathscr{s}$, we have a set of base vectors $\boldsymbol{a}_i = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ defined by $(\text{A.2.3})_1$ and (A.2.10). We examine now the partial derivatives, with respect to θ^α, of these vectors which can be represented as a linear combination of $\boldsymbol{a}_i$.

Consider first the partial derivatives $\boldsymbol{a}_{3,1}$ and $\boldsymbol{a}_{3,2}$ which by (A.2.11) lie in the tangent plane to the surface spanned by $\boldsymbol{a}_1, \boldsymbol{a}_2$. Hence, we put

$$\boldsymbol{a}_{3,\alpha} = k^\lambda_\alpha \boldsymbol{a}_\lambda. \tag{A.2.40}$$

The coefficient k_α^λ can be determined by taking the scalar product of both sides of (A.2.40) with $\boldsymbol{a}_\beta$:

$$\boldsymbol{a}_{3,\alpha} \cdot \boldsymbol{a}_\beta = k_\alpha^\lambda \, \boldsymbol{a}_\lambda \cdot \boldsymbol{a}_\beta = a_{\lambda\beta} \, k_\alpha^\lambda .$$

It follows from the last expression and (A.2.27) that $k_\alpha^\lambda = -b_\alpha^\lambda$ and we have

$$\boldsymbol{a}_{3,\alpha} = \boldsymbol{a}_{3|\alpha} = -b_\alpha^\gamma \, \boldsymbol{a}_\gamma . \tag{A.2.41}$$

The relations (A.2.41) are known as the *formulae of Weingarten.*

Next, we write the partial derivatives of $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ as linear combinations of $\boldsymbol{a}_i$ in the form

$$\boldsymbol{a}_{\alpha,\beta} = c_{\alpha\beta}^\gamma \, \boldsymbol{a}_\gamma + c_{\alpha\beta} \, \boldsymbol{a}_3 \tag{A.2.42}$$

and determine the coefficients $c_{\alpha\beta}^\gamma$, $c_{\alpha\beta}$ by taking the scalar product of both sides of (A.2.42) first with $\boldsymbol{a}_\lambda$ and then with $\boldsymbol{a}_3$. In this way, using (A.2.27) and (A.2.33), we obtain

$$c_{\alpha\beta}^\gamma \, a_{\gamma\lambda} = c_{\alpha\beta\lambda} = \boldsymbol{a}_\lambda \cdot \boldsymbol{a}_{\alpha,\beta} = \Gamma_{\alpha\beta\lambda} ,$$
$$c_{\alpha\beta} \, \boldsymbol{a}_3 \cdot \boldsymbol{a}_3 = c_{\alpha\beta} = \boldsymbol{a}_{\alpha,\beta} \cdot \boldsymbol{a}_3 = b_{\alpha\beta} .$$

Hence, the relations (A.2.42) become

$$\boldsymbol{a}_{\alpha,\beta} = \Gamma_{\alpha\beta}^\gamma \, \boldsymbol{a}_\gamma + b_{\alpha\beta} \, \boldsymbol{a}_3 \tag{A.2.43}$$

and are known as the *formulae of Gauss.* In view of $(\text{A.2.32})_1$, the last result may also be written in the useful form

$$\boldsymbol{a}_{\alpha|\beta} = b_{\alpha\beta} \, \boldsymbol{a}_3 . \tag{A.2.44}$$

Weingarten's formula may be used to define the *third fundamental form* of the surface, namely

$$d\boldsymbol{a}_3 \cdot d\boldsymbol{a}_3 = b_\alpha^\lambda \, b_{\lambda\beta} \, d\theta^\alpha \, d\theta^\beta . \tag{A.2.45}$$

We now proceed to obtain certain integrability conditions which ensure that the partial differential equations (A.2.41) and (A.2.43) [or equivalently (A.2.44)] always have solutions. The conditions we derive are necessary but they can also be shown to be sufficient for the existence of solutions of (A.2.41) and (A.2.43).[7] Assuming that (A.2.2) is at least of class C^3, we must have

$$\frac{\partial \boldsymbol{a}_{\alpha,\beta}}{\partial \theta^\lambda} = \frac{\partial \boldsymbol{a}_{\alpha,\lambda}}{\partial \theta^\beta} . \tag{A.2.46}$$

After calculating the partial derivatives of (A.2.43) and substituting the results in (A.2.46), we may rewrite the latter in the form

$$[R_{\delta\alpha\beta\gamma} - (b_{\alpha\gamma} \, b_{\beta\delta} - b_{\alpha\beta} \, b_{\gamma\delta})] \, \boldsymbol{a}^\delta - (b_{\alpha\beta|\gamma} - b_{\alpha\gamma|\beta}) \, \boldsymbol{a}^3 = 0 , \tag{A.2.47}$$

where $R_{\delta\alpha\beta\gamma}$ is defined by (A.2.37).[8] The scalar product of (A.2.47) with $\boldsymbol{a}_3$ yields

$$b_{\alpha\beta|\gamma} = b_{\alpha\gamma|\beta} , \tag{A.2.48}$$

which are known as the *Mainardi-Codazzi equations.* For $\beta = \gamma$ these equations are identically satisfied, while for $\beta \neq \gamma$ they give the relations

$$b_{\alpha 1|2} = b_{\alpha 2|1} . \tag{A.2.49}$$

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[7] See, in this connection, p. 119 of WILLMORE [1959, *8*].

[8] The result (A.2.47) can also be obtained from application of $(\text{A.2.35})_1$ to $\boldsymbol{a}_\alpha$ and the use of (A.2.44).

Also, from the scalar product of (A.2.47) with $\boldsymbol{a}_\mu$, we obtain *the equations of Gauss*:

$$R_{\delta\alpha\beta\gamma} = b_{\alpha\gamma}\, b_{\beta\delta} - b_{\alpha\beta}\, b_{\gamma\delta}. \tag{A.2.50}$$

It follows from the symmetry conditions (A.2.38) that the non-zero components of $R_{\delta\alpha\beta\gamma}$ are those for which $\delta \neq \alpha$ and $\beta \neq \gamma$. Hence, there are only four non-zero components of the curvature tensor for the surface and these are given by

$$\begin{aligned} &R_{2112} = R_{1221} = -R_{1212} = -R_{2121}, \\ &R_{1212} = b_{22}\, b_{11} - (b_{12})^2 = \det(b_{\alpha\beta}). \end{aligned} \tag{A.2.51}$$

The Mainardi-Codazzi equations (A.2.48) and the expression (A.2.50) constitute the integrability conditions for the formulae of Weingarten and Gauss. From $(\text{A.2.51})_5$ and the definition (A.2.29) follows the well-known theorem of Gauss, namely

$$K = \frac{R_{1212}}{a}, \tag{A.2.52}$$

according to which the Gaussian curvature of a surface is independent of the coefficients of the second fundamental form but depends only on the coefficients $a_{\alpha\beta}$ of the first fundamental form and their first and second partial derivatives. It is clear from (A.2.35) and (A.2.52) that, in general, the interchange of the order of covariant differentiation is not permissible unless the Gaussian curvature vanishes. In the special case when the surface (A.2.2) is a *plane*, the unit vector $\boldsymbol{a}_3$ is a constant and $\boldsymbol{a}_{3,\alpha} = 0$. Hence, $b_{\alpha\beta} = 0$, $R_{1212} = 0$ and the order of covariant differentiation is immaterial in the case of a plane.

The formulae of Weingarten and Gauss can be used to calculate the partial derivatives of a vector which is not necessarily in the surface. To show this, consider a three-dimensional vector field $\boldsymbol{V}$ defined on $\mathscr{s}$. Referred to the base vectors $\boldsymbol{a}_i$, $\boldsymbol{V}$ may be expressed in the form

$$\boldsymbol{V} = V^\alpha\, \boldsymbol{a}_\alpha + V^3\, \boldsymbol{a}_3 = V_\alpha\, \boldsymbol{a}^\alpha + V_3\, \boldsymbol{a}^3. \tag{A.2.53}$$

Using (A.2.41) and (A.2.44), the partial derivatives of $\boldsymbol{V}$ are found to be

$$\begin{aligned} \boldsymbol{V}_{,\alpha} \equiv \boldsymbol{V}_{|\alpha} &= (V^\lambda{}_{|\alpha} - b^\lambda_\alpha\, V^3)\, \boldsymbol{a}_\lambda + (V^3{}_{,\alpha} + b_{\lambda\alpha}\, V^\lambda)\, \boldsymbol{a}_3 \\ &= (V_{\lambda|\alpha} - b_{\lambda\alpha}\, V_3)\, \boldsymbol{a}^\lambda + (V_{3,\alpha} + b^\lambda_\alpha\, V_\lambda)\, \boldsymbol{a}^3. \end{aligned} \tag{A.2.54}$$

ε) *Principal curvatures. Lines of curvature.* The unit tangent vector $\boldsymbol{\lambda}$ to a curve on the surface at the point $r = r(\theta^\alpha(s))$ is given by $(\text{A.2.21})_2$. Let $\varkappa(s)$, where s is the arc parameter, be the *curvature* of the curve and let $\boldsymbol{\mu}$ denote the (space) unit *principal normal* to the curve at the point $\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha(s))$. Then, the *curvature vector* $\boldsymbol{\kappa}$ defined by

$$\boldsymbol{\kappa} = \varkappa\, \boldsymbol{\mu} = \frac{d\boldsymbol{\lambda}}{ds} \tag{A.2.55}$$

can, with the help of (A.2.44), be written as

$$\boldsymbol{\kappa} = \left(\frac{d\lambda^\mu}{ds} + \Gamma^\mu_{\alpha\beta}\, \lambda^\alpha \lambda^\beta\right) \boldsymbol{a}_\mu + b_{\alpha\beta}\, \lambda^\alpha\, \lambda^\beta\, \boldsymbol{a}_3. \tag{A.2.56}$$

It should be recalled that the unit principal normal vector $\boldsymbol{\mu}$ is independent of the orientation of a curve while the sense of the unit tangent vector $\boldsymbol{\lambda}$ depends on the orientation of the curve.

The *normal curvature* of a curve at the point $\boldsymbol{r}=\boldsymbol{r}(\theta^\alpha(s))$ is obtained from (A.2.56) through

$$\varkappa_{(n)}=\boldsymbol{\kappa}\cdot\boldsymbol{a}_3=b_{\alpha\beta}\,\lambda^\alpha\,\lambda^\beta, \tag{A.2.57}$$

which, in general, assumes an infinity of values corresponding to an infinity of directions at each point of the surface. Except at points where $\varkappa_{(n)}$ is constant, i.e., at *umbilics*, the directions for which the normal curvature $\varkappa_{(n)}$ assumes an extremal value may be determined by the method of Lagrangian multiplier. This leads to the following set of homogeneous equations:[9]

$$(b_{\alpha\beta}-\varkappa_{(n)}\,a_{\alpha\beta})\,\lambda^\beta=0, \tag{A.2.58}$$

which has a nontrivial solution if and only if the values of $\varkappa_{(n)}$ are the roots of

$$\det(b_{\alpha\beta}-\varphi\,a_{\alpha\beta})=0, \tag{A.2.59}$$

or equivalently

$$\varphi^2-2H\varphi+K=0, \tag{A.2.60}$$

where H and K are given by (A.2.28) and (A.2.29). The roots $\varkappa_{(1)}=\varkappa_1$, $\varkappa_{(2)}=\varkappa_2$ are the *principal curvatures* of the surface at the point in question and the corresponding directions are called the *principal directions*. It can be shown that the roots of (A.2.60) are always real and that at any point which is not umbilic the principal directions $\lambda^\alpha_{(1)}$, $\lambda^\alpha_{(2)}$ are orthogonal. The principal directions $\lambda^\alpha_{(\nu)}$ can be determined in the usual manner from (A.2.58). At an umbilic, since $\varkappa_{(n)}=\text{const}$, $\varkappa_1=\varkappa_2$ and every direction is a principal direction.

We define the principal radii of curvature r_1 and r_2 by

$$r_1=-\frac{1}{\varkappa_1},\qquad r_2=-\frac{1}{\varkappa_2}, \tag{A.2.61}$$

where

$$\varkappa_1+\varkappa_2=2H,\qquad \varkappa_1\,\varkappa_2=K. \tag{A.2.62}$$

For later reference, we note that (A.2.61) are the solutions of

$$\det(\delta^\alpha_\beta-\varphi^{-1}\,b^\alpha_\beta)=0. \tag{A.2.63}$$

The sign convention in (A.2.61) is in accord with the rule that r_1 and r_2 are positive (negative) if the unit normal vector to the surface is directed away from (toward) the center of curvature.[10]

A curve on a surface whose tangent at each point is along a principal direction is called a *line of curvature*. If $\varkappa_1=\varkappa_2$ at every point, then all coordinate lines are lines of curvature (the only cases being the spherical surface and the plane). For $\varkappa_1\neq\varkappa_2$ it can be shown that the conditions necessary and sufficient for the coordinate curves on s to be lines of curvature are

$$a_{12}=b_{12}=0. \tag{A.2.64}$$

A. 3. Geometry of a surface in a Euclidean space covered by normal coordinates. We begin the development of this section by identifying the curvilinear coordinates θ^i of Sect. A.1 with a system of normal coordinates as follows:[11] Consider a

[9] See, e.g., p. 210 of McCONNELL [1931, *1*] or p. 224 of EISENHART [1947, *2*].

[10] It should be noted that the sign convention in (A.2.61) is opposite to that usually employed in differential geometry, but is in accord with that adopted in most investigations dealing with shell theory. Also some authors prefer to define the second fundamental form as negative of that in (A.2.26) and this in turn influences the signs of $\varkappa_1$ and $\varkappa_2$.

[11] For a general discussion of normal coordinate system, see p. 62 of SYNGE and SCHILD [1949, *5*].

singly infinite family of (two-dimensional) surfaces in the Euclidean 3-space such that θ^3, regarded as a parameter, is a constant over each surface. Let $\theta^\alpha (\alpha = 1, 2)$ be the coordinate system on one of the surfaces, say the surface $\theta^3 = 0$ which we can call the *reference surface*. Further, let the position vector and the unit normal to the reference surface ($\theta^3 = 0$) be denoted by $\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha)$ and $\boldsymbol{a}_3 = \boldsymbol{a}_3(\theta^\alpha)$, respectively.[12] Then, adopting the notation $\theta^3 = \zeta$, we may represent the position vector of any point of the space as

$$\boldsymbol{p} = \boldsymbol{r} + \zeta\, \boldsymbol{a}_3, \tag{A.3.1}$$

where $\boldsymbol{r}$ and $\boldsymbol{a}_3$ are functions of θ^1, θ^2 only and satisfy the restriction

$$\boldsymbol{a}_\alpha \cdot \boldsymbol{a}_3 = 0, \qquad \boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha}. \tag{A.3.2}$$

It should be clear that various formulae of the previous section such as (A.2.3), (A.2.11), (A.2.27), (A.2.41), and (A.2.43) which involve $\boldsymbol{r}$, $\boldsymbol{a}_3$ and their partial derivatives hold also here for the reference surface $\boldsymbol{r} = \boldsymbol{p}(\theta^\alpha, 0)$.

In the remainder of this section we consider the geometrical properties of a surface, i.e., the reference surface ($\zeta = 0$) defined above, embedded in the space (A.3.1) and also briefly discuss the relationships between the components of the space tensors and their surface counterparts defined on $\zeta = 0$. Thus, in view of (A.1.15), the covariant base vectors and the metric tensor for the space (A.3.1) are

$$\boldsymbol{g}_\alpha = \boldsymbol{a}_\alpha + \zeta\, \boldsymbol{a}_{3,\alpha} = \mu_\alpha^\gamma\, \boldsymbol{a}_\gamma, \qquad \boldsymbol{g}_3 = \boldsymbol{a}_3 \tag{A.3.3}$$

and

$$g_{\alpha\beta} = \mu_\alpha^\nu\, \mu_\beta^\lambda\, a_{\nu\lambda}, \qquad g_{\alpha 3} = 0, \qquad g_{33} = 1, \tag{A.3.4}$$

where $a_{\nu\lambda} = \boldsymbol{a}_\nu \cdot \boldsymbol{a}_\lambda$ is the metric tensor of the reference surface and where

$$\mu_\alpha^\gamma = \delta_\alpha^\gamma - \zeta\, b_\alpha^\gamma, \tag{A.3.5}$$

b_α^γ being the coefficients of the second fundamental form. The value of $g_{\alpha 3}$ given by $(\text{A.3.4})_2$ is the characteristic property of normal coordinates and is a consequence of $(\text{A.3.2})_1$. The nature of the reduction of the above formulae when evaluated on the reference surface ($\zeta = 0$) is evident. In particular, we note that

$$\boldsymbol{g}_\alpha(\theta^\gamma, 0) = \boldsymbol{a}_\alpha, \qquad g_{\alpha\beta}(\theta^\gamma, 0) = a_{\alpha\beta} \tag{A.3.6}$$

and

$$g(\theta^\alpha, 0) = \det(g_{\alpha\beta})\big|_{\zeta=0} = \det(a_{\alpha\beta}) = a. \tag{A.3.7}$$

Let $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$ denote the ε-system for the reference surface. Then, the components $\varepsilon_{\alpha\beta 3}$, $\varepsilon^{\alpha\beta 3}$ of the space ε-system in (A.1.19) when evaluated on $\zeta = 0$ become

$$\begin{aligned} \varepsilon_{\alpha\beta} &= \varepsilon_{\alpha\beta 3}\big|_{\zeta=0} = a^{\frac{1}{2}}\, e_{\alpha\beta}, \\ \varepsilon^{\alpha\beta} &= \varepsilon^{\alpha\beta 3}\big|_{\zeta=0} = a^{-\frac{1}{2}}\, e^{\alpha\beta}, \end{aligned} \tag{A.3.8}$$

where we have set $e_{\alpha\beta 3} = e_{\alpha\beta}$, $e^{\alpha\beta 3} = e^{\alpha\beta}$ and the components of the permutation symbols $e_{\alpha\beta}$, $e^{\alpha\beta}$ are defined in (A.2.19). We observe that

$$\varepsilon_{\alpha\beta 3} = \left(\frac{g}{a}\right)^{\frac{1}{2}} \varepsilon_{\alpha\beta}, \qquad \varepsilon^{\alpha\beta 3} = \left(\frac{a}{g}\right)^{\frac{1}{2}} \varepsilon^{\alpha\beta}, \tag{A.3.9}$$

which follow from the comparison of (A.1.19), (A.3.8) and (A.2.19). Also, from (A.3.5),

$$\mu = \det(\mu_\beta^\alpha) = \left(\frac{g}{a}\right)^{\frac{1}{2}} = 1 - 2\zeta H + \zeta^2 K, \tag{A.3.10}$$

[12] The designation of the position vector and the unit normal of the reference surface by the same symbols as those used for a surface in Sect. A.2 is made in anticipation of later identification of the two surfaces.

where H and K are defined by (A.2.28)–(A.2.29). Using (A.3.10), we may rewrite (A.3.9) in the form

$$\varepsilon_{\alpha\beta 3} = \mu_\alpha^\nu \mu_\beta^\lambda \varepsilon_{\nu\lambda} = \mu \varepsilon_{\alpha\beta} \tag{A.3.11}$$

and a similar expression holds for $\varepsilon^{\alpha\beta 3}$.

The Christoffel symbols with respect to the reference surface are found by evaluating (A.1.24) on $\zeta=0$ and by using (A.3.3) and (A.1.25):

$$\begin{aligned}
&\overset{*}{\Gamma}_{\alpha\beta\gamma}\big|_{\zeta=0} = \Gamma_{\alpha\beta\gamma}, \quad \overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}\big|_{\zeta=0} = \Gamma^{\alpha}_{\beta\gamma} = a^{\alpha\lambda}\Gamma_{\beta\gamma\lambda},\\
&\overset{*}{\Gamma}{}^{\alpha}_{\beta 3}\big|_{\zeta=0} = \boldsymbol{a}^{\alpha}\cdot\boldsymbol{a}_{3,\beta} = -\boldsymbol{a}_3\cdot\boldsymbol{a}^{\alpha}{}_{,\beta} = -b^{\alpha}_{\beta},\\
&\overset{*}{\Gamma}{}^{3}_{\alpha\beta}\big|_{\zeta=0} = \boldsymbol{a}^{3}\cdot\boldsymbol{a}_{\alpha,\beta} = -\boldsymbol{a}_\beta\cdot\boldsymbol{a}^{3}{}_{,\alpha} = b_{\alpha\beta},\\
&\overset{*}{\Gamma}{}^{\alpha}_{33}\big|_{\zeta=0} = 0, \quad \overset{*}{\Gamma}{}^{3}_{33}\big|_{\zeta=0} = 0, \quad \overset{*}{\Gamma}{}^{3}_{3\alpha}\big|_{\zeta=0}
\end{aligned} \tag{A.3.12}$$

where $\Gamma_{\alpha\beta\gamma}$ is defined by (A.2.33). Using the values (A.3.12), the components of the Riemann-Christoffel tensor (A.1.33) when evaluated on the surface $\zeta=0$ can be put in the forms

$$\overset{*}{R}{}^{\lambda}_{\cdot\alpha\beta\gamma}\big|_{\zeta=0} = R^{\lambda}_{\cdot\alpha\beta\gamma} - b_{\alpha\gamma}\, b^{\lambda}_{\beta} + b_{\alpha\beta}\, b^{\lambda}_{\gamma}, \tag{A.3.13}$$

$$\overset{*}{R}{}^{3}_{\cdot\alpha\beta\gamma}\big|_{\zeta=0} = b_{\alpha\gamma,\beta} - b_{\alpha\beta,\gamma} + \Gamma^{\lambda}_{\alpha\gamma}\, b_{\lambda\beta} - \Gamma^{\lambda}_{\alpha\beta}\, b_{\lambda\gamma}, \tag{A.3.14}$$

where $R^{\lambda}{}_{\alpha\beta\gamma}$ is defined by (A.2.36). Since $\overset{*}{R}{}^{m}_{\cdot ijk}$ vanishes in a Euclidean space, the left-hand sides of each of the last two equations are zero and we again obtain formulae of the forms (A.2.50) and (A.2.48) for the reference surface $\zeta=0$.

In terms of general coordinates θ^i of Sect. A.1, the volume element (by definition) is $dv = g^{\frac{1}{2}}\, d\theta^1\, d\theta^2\, d\theta^3$. In a region of space covered by normal coordinates, the volume element can be expressed in terms of the element of area $d\sigma = a^{\frac{1}{2}}\, d\theta^1\, d\theta^2$ of the reference surface ($\zeta=0$) in the form $dv = \mu\, d\sigma\, d\zeta$ which also involves the determinant μ defined by (A.3.10). In this connection, we ask what restriction must be placed on the space (A.3.1) in order to ensure the existence of nonvanishing μ for all ζ. For this puspose, we set

$$\mu = \det(\delta^{\alpha}_{\beta} - \zeta\, b^{\alpha}_{\beta}) = 0 \tag{A.3.15}$$

and observe that this equation has the same structure as (A.2.63). It follows from (A.2.59)–(A.2.62) and (A.3.10) that the solutions of (A.3.15) in terms of the principal radii of curvature of the reference surface are

$$\zeta = \{r_1, r_2\}. \tag{A.3.16}$$

Evidently, in view of (A.3.16), for $\mu \neq 0$ it is sufficient to require that[13]

$$|\zeta| < r, \qquad r = \min(|r_1|, |r_2|) \neq 0. \tag{A.3.17}$$

The restriction $(\text{A.3.17})_1$ also ensures that the tensor μ^{γ}_{α} is nonsingular and, therefore, possesses a unique inverse $(\mu^{-1})^{\gamma}_{\alpha}$ such that

$$\mu^{\nu}_{\alpha}(\mu^{-1})^{\gamma}_{\nu} = \delta^{\gamma}_{\alpha}. \tag{A.3.18}$$

[13] Of course the restriction $(\text{A.3.17})_1$ is not a necessary condition for the existence of nonvanishing μ. We remark here that (A.3.17) is indeed a weak restriction from the point of view of shell theory.

To obtain an expression for the inverse $(\mu^{-1})^\gamma_\alpha$, multiply $(A.3.11)_2$ by $\varepsilon^{\gamma\beta}$, use (A.2.22) and put the resulting equation in the form

$$\mu^\nu_\alpha\left[\frac{1}{\mu}\,\delta^{\gamma\beta}_{\nu\lambda}\,\mu^\lambda_\beta\right]=\delta^\gamma_\alpha.$$

Then, comparison of this last result and (A.3.18) yields the expression

$$(\mu^{-1})^\gamma_\alpha=\frac{1}{\mu}\,\delta^{\gamma\beta}_{\alpha\lambda}\,\mu^\lambda_\beta. \tag{A.3.19}$$

In a region of space covered by normal coordinates, the relationships between the components of space tensors and their surface counterparts involve the determinant μ, as well as μ^γ_α and its inverse. This is partly evident from (A.3.3)–(A.3.4) and (A.3.11) which involve μ^γ_α and μ. Other transformation relations between space tensors and their surface counterparts, such as those between $\boldsymbol{g}^\alpha$ and $\boldsymbol{a}^\gamma$ and between $g^{\alpha\beta}$ and $a^{\gamma\lambda}$, can be shown to involve the inverse $(\mu^{-1})^\gamma_\alpha$ but we do not elaborate on these here.[14]

A.4. Physical components of surface tensors in lines of curvature coordinates. We consider here briefly the physical components of tensors defined over a surface and referred to lines of curvature coordinates, i.e., when the conditions (A.2.64) hold. We also obtain explicit forms of the physical components of certain tensor derivatives which will be useful in the discussion of the equations of shell theory in lines of curvature coordinates.

From (A.2.58), the nonvanishing components of b^α_β in lines of curvature coordinates are

$$\varkappa_1=b^1_1=\frac{b_{11}}{(a_1)^2}=-\frac{1}{r_1},\qquad \varkappa_2=b^2_2=\frac{b_{22}}{(a_2)^2}=-\frac{1}{r_2}, \tag{A.4.1}$$

where r_1 and r_2 are the principal radii of curvature defined by (A.2.61) and where for convenience we have introduced the notations

$$(a_1)^2=a_{11}=|\boldsymbol{a}_1|^2,\qquad (a_2)^2=a_{22}=|\boldsymbol{a}_2|^2. \tag{A.4.2}$$

When the coordinate curves on $\mathfrak{s}$ are orthogonal ($a_{12}=0$), the components of the Christoffel symbols in (A.2.33) are given by

$$\begin{aligned}
\Gamma^\alpha_{\beta\beta}&=-\frac{1}{2a_{\alpha\alpha}}\frac{\partial a_{\beta\beta}}{\partial\theta^\alpha}=-\frac{a_{\beta\beta}}{a_{\alpha\alpha}}\frac{\partial}{\partial\theta^\alpha}\log\sqrt{a_{\beta\beta}}=-\frac{a_\beta}{(a_\alpha)^2}\,a_{\beta,\alpha}\\
&\qquad\qquad(\alpha\neq\beta,\ \text{no sum over }\alpha,\beta),\\
\Gamma^\alpha_{\alpha\beta}&=\Gamma^\alpha_{\beta\alpha}=\frac{1}{2a_{\alpha\alpha}}\frac{\partial a_{\alpha\alpha}}{\partial\theta^\beta}=\frac{\partial}{\partial\theta^\beta}\log\sqrt{a_{\alpha\alpha}}=\frac{1}{a_\alpha}\,a_{\alpha,\beta}\\
&\qquad\qquad(\text{no sum over }\alpha),
\end{aligned} \tag{A.4.3}$$

where we have recorded the formulae both in terms of the metric tensor $a_{\alpha\beta}$ and also in terms of a_α defined by (A.4.2).

The physical components of tensors in an orthogonal basis are defined by[15]

$$\begin{aligned}
T_{\langle\alpha\ldots\gamma\beta\ldots\delta\rangle}&=\left[\frac{a_{\alpha\alpha}\cdots a_{\gamma\gamma}}{a_{\beta\beta}\cdots a_{\delta\delta}}\right]^{\frac12}T^{\alpha\ldots\gamma}{}_{\beta\ldots\delta}\\
&=(a_{\alpha\alpha}\cdots a_{\delta\delta})^{\frac12}\,T^{\alpha\ldots\delta}=(a_{\alpha\alpha}\cdots a_{\delta\delta})^{-\frac12}\,T_{\alpha\ldots\delta},
\end{aligned} \tag{A.4.4}$$

[14] A general discussion of transformation relations between the components of space tensors and their surface counterparts is given in Sect. 3 of [1963, *6*].

[15] A discussion of the physical components of tensors in orthogonal curvilinear coordinates can be found in McConnell [1931, *1*]. For a more general account of physical components of tensors, see the papers by Truesdell [1953, *6*], [1954, *2*] and Sects. 11–14 of Ericksen [1960, *3*].

where $T_{\langle\alpha\ldots\gamma\beta\ldots\delta\rangle}$ is the symbol designating the physical components. In particular, using the notation of (A.4.2), the physical components of the first and second-order tensors are

$$\begin{aligned} T_{\langle\alpha\rangle} &= \frac{T_\alpha}{a_\alpha} = a_\alpha\, T^\alpha && \text{(no summation over } \alpha\text{)},\\ T_{\langle\alpha\beta\rangle} &= \frac{T_{\alpha\beta}}{a_\alpha a_\beta} = a_\alpha\, a_\beta\, T^{\alpha\beta} && \text{(no summation over } \alpha, \beta\text{)}, \end{aligned} \tag{A.4.5}$$

and

$$\begin{aligned} T_\alpha &= a_\alpha\, T_{\langle\alpha\rangle}, \quad T^\alpha = \frac{T_{\langle\alpha\rangle}}{a_\alpha} && \text{(no summation over } \alpha\text{)},\\ T_{\alpha\beta} &= a_\alpha\, a_\beta\, T_{\langle\alpha\beta\rangle}, \quad T^{\alpha\beta} = \frac{T_{\langle\alpha\beta\rangle}}{a_\alpha\, a_\beta} && \text{(no summation over } \alpha, \beta\text{)}. \end{aligned} \tag{A.4.6}$$

Next, introducing the notations

$$\Gamma^{\langle\alpha\beta\gamma\rangle} = \frac{\sqrt{a_{\gamma\gamma}}}{\sqrt{a_{\alpha\alpha}}\sqrt{a_{\beta\beta}}}\,\Gamma^\gamma_{\alpha\beta} = \frac{a_\gamma}{a_\alpha a_\beta}\,\Gamma^\gamma_{\alpha\beta} \tag{A.4.7}$$

and

$$\frac{\partial}{\partial s_\beta} = \frac{1}{\sqrt{a_{\beta\beta}}}\,\frac{\partial}{\partial\theta^\beta} = \frac{1}{a_\beta}\,\frac{\partial}{\partial\theta^\beta} \quad \text{(no sum over } \beta\text{)}, \tag{A.4.8}$$

from (A.4.3) we obtain

$$\begin{aligned} \Gamma^{\langle\beta\beta\alpha\rangle} &= -\frac{\partial}{\partial s_\alpha}\log\sqrt{a_{\beta\beta}} = -\frac{\partial}{\partial s_\alpha}\log a_\beta && (\alpha \neq \beta, \text{ no sum over } \beta),\\ \Gamma^{\langle\alpha\beta\alpha\rangle} &= \frac{\partial}{\partial s_\beta}\log\sqrt{a_{\alpha\alpha}} = \frac{\partial}{\partial s_\beta}\log a_\alpha && (\text{no sum over } \alpha). \end{aligned} \tag{A.4.9}$$

The physical components of tensor derivatives can be readily obtained by direct substitution from the definitions (A.4.4) and (A.4.7)–(A.4.9) into expressions of the type (A.2.32). Thus, the physical components of $T^{\alpha\cdots\gamma}{}_{.\beta\ldots\delta|\sigma}$ can be calculated from[16]

$$\begin{aligned} T_{\langle\alpha\ldots\gamma\beta\ldots\delta|\sigma\rangle} = {} & \frac{\partial}{\partial s_\sigma} T_{\langle\alpha\ldots\gamma\beta\ldots\delta\rangle} - T_{\langle\alpha\ldots\gamma\beta\ldots\delta\rangle}\frac{\partial}{\partial s_\sigma}\log\frac{a_\alpha\cdots a_\gamma}{a_\beta\cdots a_\delta}\\ & + \Gamma^{\langle\lambda\sigma\alpha\rangle} T_{\langle\lambda\ldots\gamma\beta\ldots\delta\rangle} + \cdots + \Gamma^{\langle\lambda\sigma\gamma\rangle} T_{\langle\alpha\ldots\lambda\beta\ldots\delta\rangle}\\ & - \Gamma^{\langle\beta\sigma\lambda\rangle} T_{\langle\alpha\ldots\gamma\lambda\ldots\delta\rangle} - \cdots - \Gamma^{\langle\delta\sigma\lambda\rangle} T_{\langle\alpha\ldots\gamma\beta\ldots\lambda\rangle}, \end{aligned} \tag{A.4.10}$$

where use has been made of the fact that

$$\frac{a_\beta\cdots a_\delta}{a_\alpha\cdots a_\gamma}\,\frac{\partial}{\partial s_\sigma}\left[\frac{a_\alpha\cdots a_\gamma}{a_\beta\cdots a_\delta}\right] = \frac{\partial}{\partial s_\sigma}\log\frac{a_\alpha\cdots a_\gamma}{a_\beta\cdots a_\delta}.$$

We record below explicit expressions for the physical components of two tensor derivatives, namely $T^\alpha{}_{|\alpha}$ and $T^{\alpha\beta}{}_{|\alpha}$, which frequently occur in the field equations of shell theory. Thus, for the physical components of $T^\alpha{}_{|\alpha}$ we have

$$\begin{aligned} T_{\langle 1|1\rangle} &= \frac{1}{a_1}\left[T_{\langle 1\rangle,1} + \frac{a_{1,2}}{a_2}\,T_{\langle 2\rangle}\right],\\ T_{\langle 2|2\rangle} &= \frac{1}{a_2}\left[T_{\langle 2\rangle,2} + \frac{a_{2,1}}{a_1}\,T_{\langle 1\rangle}\right]. \end{aligned} \tag{A.4.11}$$

[16] The rule (A.4.10) is a special case of a more general formula given by TRUESDELL [1953, 6]; see Sect. 12 of his paper.

Similarly, the physical components of $T^{\alpha\beta}{}_{|\alpha}$ are given by

$$
\begin{aligned}
T_{\langle 11|1\rangle} &= \frac{1}{a_1}\left[T_{\langle 11\rangle,1} + \frac{a_{1,2}}{a_2}\left(T_{\langle 12\rangle} + T_{\langle 21\rangle}\right)\right], \\
T_{\langle 22|2\rangle} &= \frac{1}{a_2}\left[T_{\langle 22\rangle,2} + \frac{a_{2,1}}{a_1}\left(T_{\langle 12\rangle} + T_{\langle 21\rangle}\right)\right], \\
T_{\langle 21|2\rangle} &= \frac{1}{a_2}\left[T_{\langle 21\rangle,2} + \frac{a_{2,1}}{a_1}\left(T_{\langle 11\rangle} - T_{\langle 22\rangle}\right)\right], \\
T_{\langle 12|1\rangle} &= \frac{1}{a_1}\left[T_{\langle 12\rangle,1} + \frac{a_{1,2}}{a_2}\left(T_{\langle 22\rangle} - T_{\langle 11\rangle}\right)\right].
\end{aligned}
\tag{A.4.12}
$$

In obtaining (A.4.4) to (A.4.12), only the first of the two conditions in (A.2.64), namely $a_{12}=0$, has been invoked. With the use of (A.4.11)–(A.4.12) and (A.4.6), as well as (A.4.1), it is now a simple matter to write the various field equations, kinematical results and the constitutive relations of shell theory in lines of curvature coordinates and in terms of physical components. Thus, for example, the equations of equilibrium obtained from $(9.47)_{1,2}$ in lines of curvature coordinates and in terms of physical components can be written as

$$
\begin{aligned}
N_{\langle 11|1\rangle} + N_{\langle 21|2\rangle} + \frac{N_{\langle 13\rangle}}{r_1} + \varrho f_{\langle 1\rangle} &= 0, \\
N_{\langle 12|1\rangle} + N_{\langle 22|2\rangle} + \frac{N_{\langle 23\rangle}}{r_2} + \varrho f_{\langle 2\rangle} &= 0, \\
N_{\langle 13|1\rangle} + N_{\langle 23|2\rangle} - \left(\frac{N_{\langle 11\rangle}}{r_1} + \frac{N_{\langle 22\rangle}}{r_2}\right) + \varrho f_3 &= 0.
\end{aligned}
\tag{A.4.13}
$$

Here, r_1 and r_2 are the principal radii of curvature of the surface $(5.1)_1$ and the components of $N_{\langle\alpha 3|\alpha\rangle}$ are given by expressions of the type (A.4.11). If we substitute formulae of the forms (A.4.11)–(A.4.12) into (A.4.13), then after a slight rearrangement more familiar forms of the equations of equilibrium in lines of curvature coordinates follow:

$$
\begin{aligned}
&\frac{1}{a_1 a_2}\left\{(a_2 N_{\langle 11\rangle})_{,1} + (a_1 N_{\langle 21\rangle})_{,2} + a_{1,2} N_{\langle 12\rangle} - a_{2,1} N_{\langle 22\rangle}\right\} + \frac{N_{\langle 13\rangle}}{r_1} + \varrho f_{\langle 1\rangle} = 0, \\
&\frac{1}{a_1 a_2}\left\{(a_2 N_{\langle 12\rangle})_{,1} + (a_1 N_{\langle 22\rangle})_{,2} - a_{1,2} N_{\langle 11\rangle} + a_{2,1} N_{\langle 21\rangle}\right\} \\
&\qquad\qquad + \frac{N_{\langle 23\rangle}}{r_2} + \varrho f_{\langle 2\rangle} = 0, \\
&\frac{1}{a_1 a_2}\left\{(a_2 N_{\langle 13\rangle})_{,1} + (a_1 N_{\langle 23\rangle})_{,2}\right\} - \left(\frac{N_{\langle 11\rangle}}{r_1} + \frac{N_{\langle 22\rangle}}{r_2}\right) + \varrho f_3 = 0.
\end{aligned}
\tag{A.4.14}
$$

Results similar to (A.4.13) and (A.4.14) hold also for the equations of equilibrium obtained from $(9.48)_{1,2}$ which have essentially the same structure as those in $(9.47)_{1,2}$.

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