Optimization of M/M/1 Queue System for Minimal Service Rate and Maximal Arrival Rate

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Objective: To optimize the performance of an M/M/1 queue system by determining the minimal service rate ((μ)) required for various arrival rates ((λ)) and the maximal arrival rate ((λ)) that can be handled for given service rates ((μ)) while maintaining a specified probability threshold ((ε)) for the queue size.

Methodology:

1. **Model Description:**
   - The M/M/1 queue model is a single-server queue where arrivals follow a Poisson process with rate (λ) and service times follow an exponential distribution with rate (μ).
   - The system is analysed to ensure the probability of the queue size exceeding a threshold (k) remains below a specified value (ε).
2. **Functions Implemented:**
   - **find_min_service_rate:** Determines the minimal service rate ((μ)) for a given arrival rate ((λ)) to ensure the probability of exceeding queue size (k) is less than (ε).
   - **find_max_arrival_rate:** Determines the maximal arrival rate ((λ)) for a given service rate ((μ)) to ensure the probability of exceeding queue size (k) is less than (ε).
   - **P_more_than_k:** Calculates the probability of the queue size exceeding (k) for given (λ) and (μ).
3. **Parameters:**
   - ($k = 10$): Queue size threshold.
   - ($\varepsilon = 0.01$): Probability threshold.
   - Arrival rates ((λ)) and service rates ((μ)) are varied to analyse the system.
4. **Script Execution:**
   - The script calculates and displays the minimal service rates for various arrival rates and the maximal arrival rates for various service rates.
   - A plot is generated to show the probability of exceeding the queue size (k) for different (λ/μ) ratios.

Results:

1. **Minimal Service Rate for Various Arrival Rates:**
   - The minimal service rate ((μ)) increases linearly with the arrival rate ((λ)), ensuring the probability of exceeding the queue size (k) remains below the threshold (ε).

2. **Maximal Arrival Rate for Various Service Rates:**
   - The maximal arrival rate ((λ)) is calculated to ensure the system remains stable, with the probability of exceeding the queue size (k) below the threshold (ε).
3. **Graph Interpretation:**
   - The graph shows the probability of exceeding the queue size (k) for different (λ/μ) ratios, which increases exponentially as (λ) approaches (μ).

Explanation of the Model Simulated

The M/M/1 queue model is a fundamental model in queueing theory used to describe systems with a single server. The key characteristics of this model are:

1. **Arrival Process:**
   - Arrivals follow a Poisson process with rate (λ), meaning the time between arrivals is exponentially distributed.
2. **Service Process:**
   - Service times are exponentially distributed with rate (μ), meaning the time to serve each customer is memoryless.
3. **Queue Dynamics:**
   - The system is analysed to ensure stability, which requires (λ < μ). If (λ ≥ μ), the queue would grow indefinitely.
4. **Optimization Goals:**
   - **Minimal Service Rate ((μ)):** For a given arrival rate ((λ)), the minimal service rate is determined to ensure the probability of the queue size exceeding (k) is less than (ε).
   - **Maximal Arrival Rate ((λ)):** For a given service rate ((μ)), the maximal arrival rate is determined to ensure the probability of the queue size exceeding (k) is less than (ε).
5. **Probability Calculation:**
   - The probability of the queue size exceeding (k) is given by ((λ / μ) ^{k+1}). This probability must be less than (ε) to meet the specified threshold.

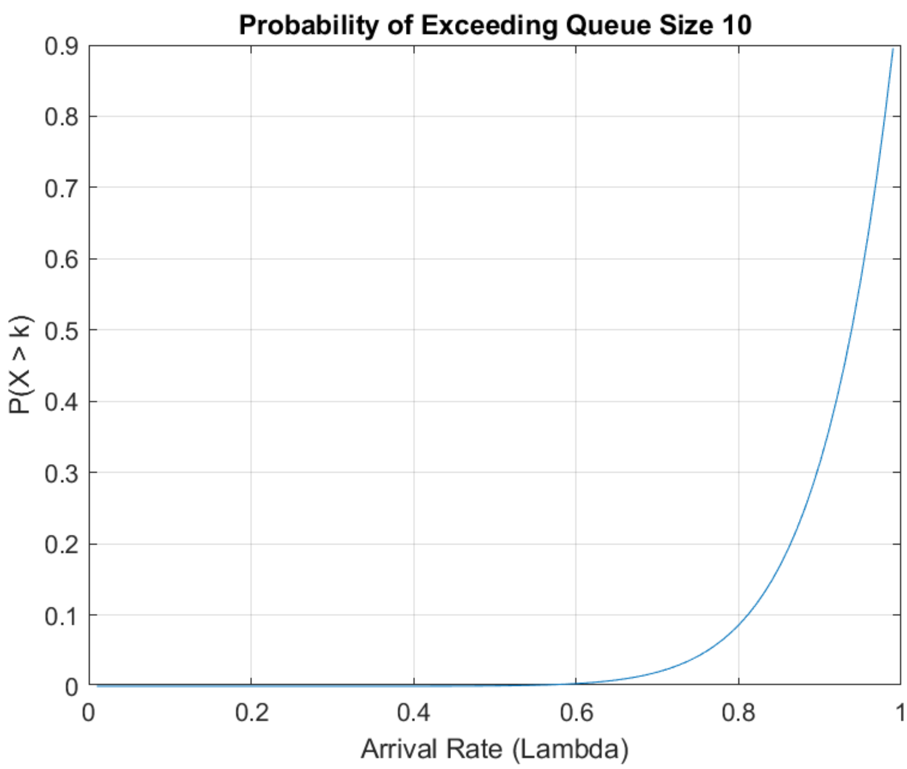
Queue size threshold = 10; ε = 0.01; Probability threshold; arrival rates = [1/10, 1/5, 1/2, 1]; service times = [1/10, 1/5, 1/2, 1].

Optimization Results (Minimal Service Rate for Various Arrival Rates):

Arrival Rate (Λ)	Minimal Service Rate (M)
0.10	1.10
0.20	1.20
0.50	1.50
1.00	2.00

Optimization Results (Maximal Arrival Rate for Various Service Rates):

Service Rate (M)	Maximal Arrival Rate (Λ)
0.10	0.06
0.20	0.13
0.50	0.32
1.00	0.65



the following are the Parameters.

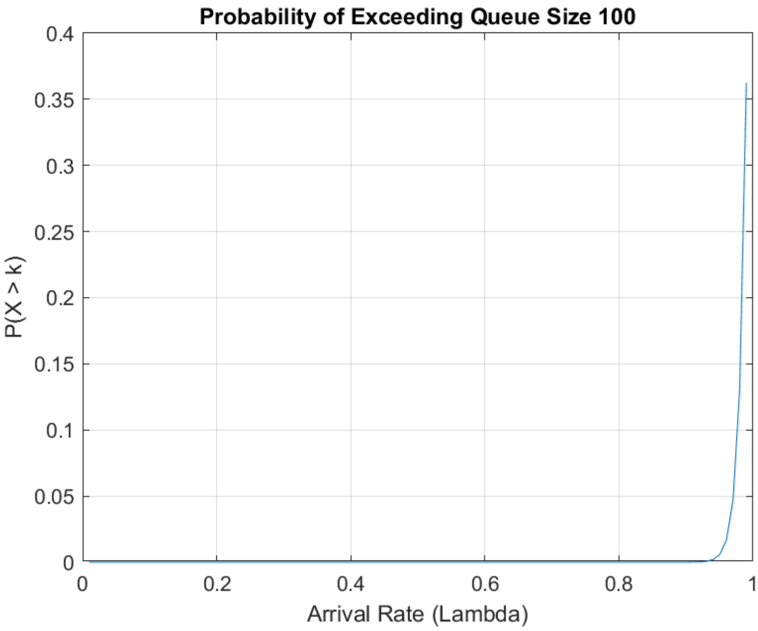
Queue size threshold k = 100; ε Probability threshold=0.01; arrival rates = [2/10, 2/5, 1/4, 1]; service times = [1/10, 1/5, 1/2, 1]

Optimization Results (Minimal Service Rate for Various Arrival Rates):

Arrival Rate (Λ)	Minimal Service Rate (M)
0.20	1.20
0.40	1.40
0.25	1.25
1.00	2.00

Optimization Results (Maximal Arrival Rate for Various Service Rates):

Service Rate (M)	Maximal Arrival Rate (Λ)
0.10	0.09
0.20	0.19
0.50	0.47
1.00	0.95



Based on the result we can infer the following

1. **Minimal Service Rate ((μ)):**
   - For both queue size thresholds ((k = 10) and (k = 100)), the minimal service rate increases linearly with the arrival rate. This indicates that as more jobs

arrive per unit time, the service rate must increase proportionally to maintain the probability of the queue size exceeding (k) below the threshold (ε).

2. **Maximal Arrival Rate ((λ)):**
   - The maximal arrival rate increases with the service rate, but the increase is more pronounced for a larger queue size threshold ((k = 100)). This suggests that a larger queue can handle higher arrival rates before the probability of exceeding the queue size threshold becomes significant.
3. **Impact of Queue Size Threshold ((k)):**
   - A larger queue size threshold ((k = 100)) allows for higher arrival rates and requires slightly higher service rates for the same arrival rates compared to a smaller queue size threshold ((k = 10)). This is because a larger queue can buffer more jobs, reducing the likelihood of exceeding the threshold.
4. **System Stability:**
   - The results confirm that for the system to remain stable ((λ < μ)), the service rate must always be greater than the arrival rate. The calculated minimal service rates and maximal arrival rates ensure that the system operates within this stable region.

Conclusion

The simulation results provide valuable insights into the behaviour of the M/M/1 queue system under different conditions. Key takeaways include:

- **Linear Relationship:** There is a linear relationship between the arrival rate and the minimal service rate required to maintain system stability.
- **Higher Thresholds:** Larger queue size thresholds allow for higher arrival rates and require slightly higher service rates.
- **System Design:** These insights can be used to design and optimize queue systems in various applications, ensuring efficient operation and stability.