Problems: Python, and integrating the expansion history of the universe

ISSYP 2016

I. PYTHON LOOPS

1. Write a program which prints the values of $(1/n^2)$ for $n = 1, \cdots, 20$.
2. There is a math identity (this is not supposed to be obvious, the proof is hard!) which says that the infinite series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \tag{1}$$

is equal to $(\pi^2/6)$. Let's check this result numerically as follows. Write a loop which calculates the sum to some maximum value of n, and evaluate the result for $n_{\rm max}$ equal to 10, 100, 1000, and 10000. You should be able to see emperically that the result is converging to $\pi^2/^6$.

3. In one of the problems from yesterday (number 3), we figured out how to compute the total density of the universe $\rho_{\rm tot}$ at a specific value of a (namely $a = 0.43$). Write a program which prints the density of the universe at $a = 0.1, 0.2, \cdots, 0.9, 1$.
4. (Hard) The Fibonacci numbers $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, $F_5 = 8$, $\cdots$ are defined by the recursion relation $F_n = F_{n-1} + F_{n-2}$. Write a python program to print the value of F_{50}. Can you modify your program to also print the value of $F_0 + F_1 + \cdots + F_{50}$?

II. PYTHON LISTS

5. Make a length-50 list whose n-th element (for $0 \le n < 50$) is equal to $(1.1)^n + 5n^2$.
6. Make a list of 100 numbers t_i uniformly spaced between 0 and (2π). Make another length-100 list containing the values of $\cos(t_i)$. By calling `matplotlib.pyplot.plt()` with these lists as arguments, make a plot of t vs $\cos(t)$.

III. INTEGRATING FRIEDMANN'S EQUATION WITH PYTHON

Warning: everything in this section is supposed to be a challenge! Don't worry if you find it tough at first.

Friedmann's equation is the differential equation:

$$\frac{da}{dt} = aH(a) \tag{2}$$

where $H(a)$ has a somewhat messy algebraic form that you derived in question 2 above.

For numerically integrating this equation I suggest making time be the *dependent variable* and using $\log a$ (not a!) as the independent variable. In other words, your numerical integration would take equal steps in $\log a$, and evolve the value of t (the elapsed time since the big bang).

Something which isn't supposed to be obvious, but which you can show if you've taken some calculus, is that the form of Friedmann's equations in these variables is:

$$\frac{dt}{d\log a} = \frac{1}{H(a)} \tag{3}$$

or equivalently, defining $u = \log(a)$ for notational clarity

$$\frac{dt}{du} = \frac{1}{H(e^u)} \tag{4}$$

Write a python program to integrate this differential equation and tabulate t as a function of u. Based on your tabulated values, answer the following questions:

7. What is the age of the universe?
8. Consider the redshifted galaxy from question 3 above. How long ago was the light from the galaxy emitted?
9. How different would the age of the universe be if there were no dark energy, but the matter and radiation densities ρ_m and ρ_{rad} were the same?
10. Determine whether the expansion of the universe is accelerating or decelerating today, by determining whether da/dt is increasing or decreasing with the exapansion. Would the answer be different if there were no dark energy?
11. Warning: this one is particularly difficult, I'll give some hints on Monday! What is the comoving size of the universe? I.e. if we consider the furthest point we can look back to, how far away is that point today?