# INTRO COMMENTS

- "MENTORING A PROJECT" = "SHOWING YOU SOME INTERESTING PHYSICS/MATH UNTIL WE RUN OUT OF TIME"
- BEST APPROACH FOR SIX-DAY FORMAT: DO SOME STATS/DATA ANALYSIS CALCULATIONS, EXPLAIN AS MUCH OF THE PHYSICS AS TIME WILL ALLOW
- BACKGROUND: CALCULUS? SOFTWARE? (LAPTOPS?)

[DISTRIBUTE MATH PROBLEMS HERE]

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STANDARD COSMOLOGICAL MODEL: 6 PARAMETERS

- 3 EXPANSION HISTORY ($\Omega_b, \Omega_m, H_0$)
- 2 PERTURBATIONS ($\Delta_s, n_s$)
- 1 NUISANCE ($\tau$)

## EXPANSION HISTORY

CONTENTS OF UNIVERSE (AS FRACTION OF ENERGY DENSITY $\rho_{tot}$)

69% "DARK ENERGY" (A COSMOLOGICAL CONSTANT $\Lambda \neq 0$?)

26% "COLD DARK MATTER"

4.9% "BARYONIC MATTER" (PROTONS + NEUTRONS + ELECTRONS)

STANDARD NOTATION

$$\left.\begin{aligned} \Omega_b &= \frac{\rho_{bary}}{\rho_{tot}} = 0.049 \\ \Omega_c &= \frac{\rho_{CDM}}{\rho_{tot}} = 0.26 \\ \Omega_\Lambda &= \frac{\rho_{DE}}{\rho_{tot}} = 0.7 \end{aligned}\right\} \text{ONLY 2 INDEPENDENT PARAMETERS SINCE } \sum \Omega_i = 1$$

EXPANSION RATE (OR HUBBLE PARAMETER) TODAY

$$H_0 = 68 \text{ km s}^{-1} \text{ (MEGA PARSEC)}^{-1}$$

WEIRD UNITS! ORIGINAL INTERPRETATION (HUBBLE'S LAW)

$$\begin{bmatrix} \text{RECESSIONAL VELOCITY OF} \\ \text{DISTANT GALAXY} \end{bmatrix} = H_0 \times \begin{bmatrix} \text{DISTANCE} \\ \text{TO GALAXY} \end{bmatrix}$$

MODERN INTERPRETATION: FRACTIONAL EXPANSION OF UNIVERSE PER UNIT TIME

$$H_0 = 0.069 \text{ (GIGA YEAR)}^{-1}$$

THERE IS A SET OF DIFFERENTIAL EQUATIONS WHICH CAN BE SOLVED TO GET THE COMPLETE EXPANSION HISTORY (FRIEDMANN EQS)

$\Rightarrow$ $H_0, \Omega_i$ TODAY DETERMINE $H, \Omega_i$ FOR ALL TIMES

PERTURBATIONS (2 PARAMS)

- AT EARLY TIMES, A FEW SECONDS AFTER THE BIG BANG, THE FLUCTUATIONS IN MATTER DENSITY ARE RANDOM, GAUSSIAN, WITH A POWER LAW SPECTRUM.

WE'LL SPEND THE NEXT 2 DAYS DEFINING THESE TERMS!

FOR NOW WE'LL JUST SAY THAT A COMPLETE STATISTICAL DESCRIPTION OF THE INITIAL CONDITIONS IS GIVEN BY TWO PARAMETERS

$\Delta_s = 4.7 \times 10^{-5}$ TYPICAL FRACTIONAL SIZE OF INITIAL FLUCTUATIONS

$n_s = 0.97$ "SPECTRAL INDEX" OR SCALE DEPENDENCE

- DEFINED SO THAT $n_s = 1$

$\Rightarrow$ SCALE INVARIANT

NUISANCE PARAMETER $\tau$ [OPTICAL DEPTH TO CMB]

BACKGROUND:

- UNIVERSE IS IONIZED (OPAQUE) FOR THE FIRST ~400000 YEARS AFTER THE BIG BANG (UNTIL $z \sim 1100$)
- THEN BECOMES TRANSPARENT, PHOTONS FREESTREAM AND ARE OBSERVED TODAY AS THE CMB
- AT $z \sim 6$, SOMETHING ELSE HAPPENS: HIGH ENERGY PHOTONS FROM THE FIRST STARS RE-IONIZE THE UNIVERSE, NOT PERFECTLY TRANSPARENT ANYMORE

$\tau$ = PROBABILITY THAT A ~~PHOTON~~ CMB PHOTON SCATTERS BETWEEN $z = 1100$ AND NOW

= 0.06 [6% PROBABILITY]

~~NOTES~~

~~END OF~~

WE'VE NOW DESCRIBED THE PARAMETERS OF THE STANDARD COSMOLOGICAL MODEL. A FEW COMMENTS (ABRIDGED IN NOTES, TO BE ELABORATED IN LECTURE)

- NON-PARAMETERS: MANY THINGS WE LOOK FOR BUT DON'T SEE (SPATIAL CURVATURE, EXTRA NEUTRINO SPECIES, GRAVITY WAVES, AND MANY MORE)

- SEVENTH "CLOCK-LIKE" PARAMETER IS REALLY PART OF THE MODEL BUT NOT LISTED EXPLICITLY BY CONVENTION

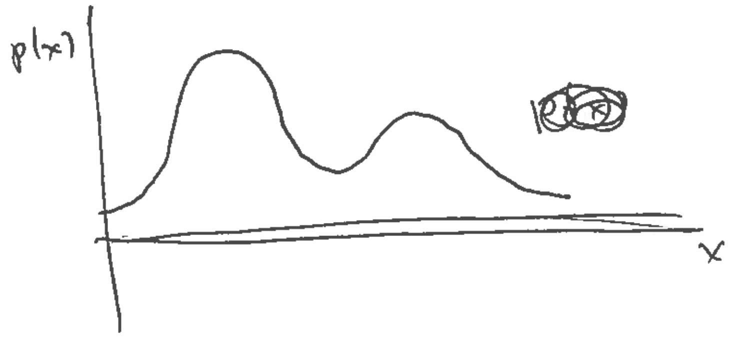
- OPTICAL DEPTH $\tau$ IS STRICTLY SPEAKING NOT INDEPENDENT OF THE OTHERS IN PRINCIPLE, BUT IS AN INDEPENDENT PARAMETER IN PRACTICE

- COSMOLOGY IS A STATISTICAL THEORY, PREDICT PROBABILITY OF OBSERVING A GIVEN CMB REALIZATION GIVEN MODEL PARAMS

- THE FUN PART IS TRYING TO INVERT THIS: GIVEN A SET OF OBSERVATIONS, WHAT ARE THE COSMOLOGICAL PARAMETERS?

- TODAY + TOMORROW WE'LL TALK MOSTLY ABOUT RANDOM VARIABLES AND MATH PRELIMINARIES

RANDOM VARIABLE $X$: ANYTHING THAT HAS A PROBABILITY DISTRIBUTION $p(x)$




$$\text{PROB}(x_0 \leq X \leq x_0 + dx) = p(x)\, dx$$

$$\text{PROB}(x_0 \leq X \leq x_1) = \int_{x_0}^{x_1} dx\, p(x)$$

NOTE THAT $\int_{-\infty}^{\infty} p(x)\, dx = 1$ ALWAYS

DEFINITIONS

$\bar{X} = \langle X \rangle =$ MEAN (EXP. VALUE) OF $X$

$\text{VAR}(X) = \langle (X - \bar{X})^2 \rangle =$ "VARIANCE" OF $X$

[SQUARE OF "TYPICAL" FLUCTUATION AROUND MEAN]

WHEN DOING CALCULATIONS WITH EXPECTATION VALUES, WATCH OUT FOR THIS PITFALL:

$\langle X_1 + X_2 \rangle = \langle X_1 \rangle + \langle X_2 \rangle$ ALWAYS

$\langle cX \rangle = c \langle X \rangle$ ALWAYS, IF $c$ IS A CONSTANT (I.E., NOT A RANDOM VARIABLE)

(*) BUT $\langle X_1 X_2 \rangle = \langle X_1 \rangle \langle X_2 \rangle$ ONLY IF THE RANDOM VARIABLES $X_1, X_2$ ARE INDEPENDENT

RELATED: IF $X = X_1 + \cdots + X_N$ THEN

$\bar{X} = \sum_i \bar{X}_i$ ALWAYS

$\mathrm{VAR}(X) = \sum_i \mathrm{VAR}(X_i)$ IF $X_i$ ARE INDEPENDENT

(NOT OBVIOUS BUT FOLLOWS FROM (★) + SOME ALGEBRA)

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CENTRAL LIMIT THEOREM: EASIEST TO EXPLAIN BY PICTURE, SEE SLIDES!

FORMAL STATEMENT: IF X IS THE SUM OF MANY <u>INDEPENDENT</u>, <u>IDENTICALLY DISTRIBUTED</u> RANDOM VARIABLES

$$X = X_1 + \cdots + X_N$$

THEN THE PROB. DISTRIBUTION OF X IS WELL-APPROXIMATED BY A UNIVERSAL FORM ("GAUSSIAN DISTRIBUTION")

$$p(x) \approx \frac{1}{\sigma (2\pi)^{1/2}} \exp\left( - \frac{(x-\mu)^2}{2\sigma^2} \right)$$

WHERE $\mu = \langle x \rangle$ AND $\sigma^2 = \mathrm{VAR}(x)$

- PROOF OMITTED (TOO MUCH OF A DIGRESSION)
- GAUSSIAN RANDOM VARIABLES ARE COMMON SINCE COMPLICATED RV'S ARE OFTEN BUILT UP FROM MANY INDEPENDENT CONTRIBUTIONS (EXAMPLES: REPEATED COIN FLIPS, VELOCITY OF A THERMAL PARTICLE)

- ONCE A RANDOM VARIABLE HAS "GAUSSIANIZED", ONLY ITS MEAN $\mu$ AND VARIANCE $\sigma^2$ MATTER (GAUSSIAN PROB. DISTRIBUTION ONLY DEPENDS ON THESE 2 PARAMS)

- FUN FACT: YOU CAN GET "$e$" AND "$\pi$" BY REPEATED COIN FLIPPING

- THE CMB IS GAUSSIAN, IN A STRONG SENSE, BUT NOT BECAUSE OF THE CENTRAL LIMIT THEOREM [ELABORATE IN LECTURE]

# 2D GAUSSIAN RANDOM VARIABLES

NOW LET'S CONSIDER A RANDOM VARIABLE $X$ WITH TWO COMPONENTS: $X = (X_1, X_2)$

E.G. $X_1, X_2$ MIGHT BE CMB TEMPERATURES AT TWO POINTS ON THE SKY WITH SOME FIXED SEPARATION

NOW WE HAVE INDICES, E.G.

RANDOM VARIABLE $X_i$      2-VECTOR

MEAN $\langle X_i \rangle = \bar{X}_i$      2-VECTOR

COVARIANCE $C_{ij} = \langle (X_i - \bar{X}_i)(X_j - \bar{X}_j) \rangle$      2-BY-2 MATRIX

NOTE THAT $C_{ij} = C_{ji}$, I.E. $C$ IS A SYMMETRIC MATRIX

TO GIVE SOME INTUITION:

$C_{ij} = \mathrm{VAR}(X_1)$

$C_{22} = \mathrm{VAR}(X_2)$

$C_{12}$ IS ZERO IF $X_1, X_2$ ARE UNCORRELATED

$= \pm\sqrt{C_{11} C_{22}}$ IF $X_1, X_2$ ARE PERFECTLY (ANTI) CORRELATED

THE CENTRAL LIMIT THEOREM HAS A 2D ANALOGUE

[SHOW SLIDES]

FORM OF THE GAUSSIAN PROB. DISTRIBUTION IN 2D IS (ASSUMING $\bar{X}_i = 0$ FOR SIMPLICITY)

$$p(x_1, x_2) = (\text{CONST.}) \exp\left[\frac{-a x_1^2 - b x_2^2 - c x_1 x_2}{2}\right]$$

~~TO EXPLAIN HOW TO GET VALUES~~

I'LL GENERALLY USE THE FIRST NOTATION (INDICES EXPLICIT BUT SUMMATIONS IMPLICIT) 9

LET'S SOLVE THE SYSTEM, BUT FOR MORE GENERALITY WE'LL LET THE RHS BE A GENERAL VECTOR $Y_i$:

$$2x_1 + x_2 = Y_1$$
$$x_1 - x_2 = Y_2$$

$$\Rightarrow \quad 3x_1 = Y_1 + Y_2 \qquad \text{[SUM]}$$
$$\Rightarrow \quad x_1 = \tfrac{1}{3}Y_1 + \tfrac{1}{3}Y_2$$
$$\Rightarrow \quad x_2 = \tfrac{1}{3}Y_1 - \tfrac{2}{3}Y_2 \qquad \text{[BACKSUBSTITUTE]}$$

CONCLUSION: IF $A_{ij}x_j = Y_j$, THEN $x_i = B_{ij}Y_j$ WHERE $B$ IS THE MATRIX

$$B = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

THE MATRICES $A$ AND $B$ ARE "INVERSES" AND WE WOULD NORMALLY WRITE $B_{ij} = A^{-1}_{ij}$.

$$A_{ij}x_j = Y_j \quad \Leftrightarrow \quad x_i = A^{-1}_{ij}Y_j$$

MULTIPLYING BY $A^{-1}_{ij}$ "UNDOES" MULTIPLICATION BY $A_{ij}$

QUESTION: HOW ARE THE VALUES OF $\{a, b, c\}$ RELATED TO THE COVARIANCE MATRIX $C_{ij}$?

TO ANSWER THIS QUESTION WE'LL NEED A LINEAR ALGEBRA DIGRESSION

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LINEAR ALGEBRA DIGRESSION

CONSIDER A 2-BY-2 SYSTEM OF LINEAR EQS, E.G.

$$2x_1 + x_2 = 6$$
$$x_1 - x_2 = -3$$

LET'S WRITE THIS IN "INDEX" FORM

$$\sum_{j=1}^{2} A_{ij} x_j = y_i \qquad (\star)$$

WHERE $y_j = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ $\quad A_{ij} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$

AN EQUATION LIKE ($\star$) WITH ONE "FREE" (I.E. UNSUMMED) INDEX ON EACH SIDE ($i$) STANDS FOR TWO EQUATIONS, ONE FOR EACH VALUE OF $i$.

ALTERNATE NOTATIONS FOR ($\star$)

$A_{ij} x_j = y_i$ EINSTEIN SUMMATION CONVENTION: REPEATED INDICES (E.G. $j$) IMPLICITLY SUMMED

$\underbrace{Ax}_{\text{MATRIX-VECTOR PRODUCT}} = y$ MATRIX NOTATION: ALL INDICES AND SUMMATIONS IMPLICIT

MORE FUN WITH MATRICES: THERE IS AN "IDENTITY MATRIX" $\delta_{ij}$ (SOMETIMES WRITTEN $I_{ij}$) DEFINED BY

$$\delta_{ij} = \begin{cases} 0 & \text{IF } i \neq j \\ 1 & \text{IF } i = j \end{cases}$$

IT HAS THE PROPERTY THAT

$$\delta_{ij} x_j = x_i \qquad \delta_{ij} A_{jk} = A_{ik} \qquad \text{ETC.}$$

ONE DEFINITION OF THE INVERSE MATRIX:

$$A^{-1}_{ij} A_{jk} = \delta_{ik} \qquad \text{[DEFINES } A^{-1} \text{ GIVEN } A\text{]}$$

END OF THE LINEAR ALGEBRA DIGRESSION -- BUT THERE WILL BE A LOT MORE LINEAR ALGEBRA IN THE NEXT FEW DAYS!

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NOW WE CAN EXPLAIN HOW THE COEFFS $(a, b, c)$ IN THE 2D GAUSSIAN PROB. DISTRIBUTION

$$p(x_1, x_2) = (\text{CONST.}) \exp\left[\frac{-ax_1^2 - bx_2^2 - cx_1 x_2}{2}\right] \qquad (*)$$

ARE RELATED TO THE 2-BY-2 COVARIANCE MATRIX $C_{ij}$.

THE GENERAL STATEMENT IS THAT

$$p(x_1, x_2) = (\text{CONST}) \exp\left[-\frac{1}{2} x_i C^{-1}_{ij} x_j\right] \qquad (**)$$

IF WE WRITE THE INVERSE COVARIANCE MATRIX EXPLICITLY AS

$$C^{-1}_{ij} = \begin{pmatrix} a & c/2 \\ c/2 & b \end{pmatrix}$$

THEN WE GET (*). HOWEVER IT'S PROBABLY EASIER TO THINK "IN MATRICES" AND USE THE ~~[illegible]~~ ALGEBRAIC FORM (**)

WE'LL GIVE A DERIVATION OF (**) LATER!