# Problems: Gaussian random variables and a little bit of linear algebra

ISSYP 2016

1. *Mean and variance for biased coin flips.* Recall the following properties of random variables:

$$\begin{aligned}
\langle X_1 + X_2 \rangle &= \langle X_1 \rangle + \langle X_2 \rangle && \text{always} \\
\langle cX \rangle &= c\langle X \rangle && \text{if } c \text{ is constant (i.e. not a random variable)} \\
\langle X_1 X_2 \rangle &= \langle X_1 \rangle \langle X_2 \rangle && \text{if } X_1, X_2 \text{ are independent random variables}
\end{aligned} \tag{1}$$

(a) Prove that if $X = X_1 + X_2$, where $X_1, X_2$ are independent random variables, then $\mathrm{Var}(X) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$.

(b) Consider a biased coin which is heads with probability $p$, and tails with probability $(1-p)$. Let $H_1$ be the number of heads after a single flip (either 0 or 1). What are the mean and variance of the random variable $H_1$? Let $H_N$ be the number of heads after $N$ flips. What are the mean and variance of $H_N$?

2. *Simulating Gaussian random variables.* A question which sometimes arises is how to simulate a Gaussian random variable in $N$ dimensions, assuming the existence of an algorithm for randomly simulating a one-dimensional Gaussian.[1] In this problem we will study this question in the case $N = 2$.

(a) Consider a 2-component Gaussian random variable $X_i$, with mean $\bar{X}_i = 0$ assumed for simplicity, and covariance matrix

$$C_{ij} = \langle X_i X_j \rangle = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} \tag{2}$$

Suppose we simulate independent one-dimensional Gaussian random numbers $u_1, u_2$ with variance $\sigma^2 = 1$. Show that if we define:

$$\begin{aligned}
X_1 &= \sqrt{C_{11}}\, u_1 \\
X_2 &= \frac{C_{12}}{\sqrt{C_{11}}} u_1 + \sqrt{C_{22} - \frac{C_{12}^2}{C_{11}}}\, u_2
\end{aligned} \tag{3}$$

then $\langle X_i X_j \rangle = C_{ij}$. This is one algorithm for simulating a two-dimensional Gaussian.

(b) More generally, show that a linear transformation of the form $X_i = A_{ij} u_j$ gives covariance matrix $\langle X_i X_j \rangle = C_{ij}$ if and only if $C_{ij} = A_{ik} A_{jk}$.[2]

(c) If you're comfortable with computer programming then try coding this up! For a specific choice of covariance matrix, say $C_{11} = 2$, $C_{12} = 1$, and $C_{22} = 3$, try making random realizations of $X_i$ using Eq. (3), and verify that the averages $\langle X_i X_j \rangle$ converge to $C_{ij}$ in the limit of many realizations.

3. *2-by-2 matrix inversion.* Consider a general 2-by-2 matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{4}$$

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[1] All modern programming lanugages have this built in. For example, in python you can get a Gaussian random number with `numpy.random.standard_normal()`.

[2] If this equation holds then we say that $A$ is a "matrix square root" of $C$. Thus the problem of simulating multivariate Gaussian random numbers is linked to the problem of finding matrix square roots (which turn out to be non-unique). The coefficients given in Eq. (3) are one choice of matrix square root in the 2-by-2 case.

Show that the matrix elements of the inverse matrix

$$A^{-1} = \begin{pmatrix} A^{-1}_{11} & A^{-1}_{12} \\ A^{-1}_{21} & A^{-1}_{22} \end{pmatrix} \tag{5}$$

are given explicitly by

$$\begin{aligned} A^{-1}_{11} &= \frac{A_{22}}{A_{11}A_{22} - A_{12}A_{21}} \\ A^{-1}_{12} &= -\frac{A_{12}}{A_{11}A_{22} - A_{12}A_{21}} \\ A^{-1}_{21} &= -\frac{A_{21}}{A_{11}A_{22} - A_{12}A_{21}} \\ A^{-1}_{22} &= \frac{A_{11}}{A_{11}A_{22} - A_{12}A_{21}} \end{aligned} \tag{6}$$

A brute force approach is to show that the defining condition for the inverse matrix, namely $A_{ij}A^{-1}_{jk} = \delta_{ik}$, holds for all four choices of indices $i, k$. If you have taken some linear algebra, you may know some machinery which is more efficient than the brute force approach – if so then feel free to use it!

4. *Gaussian integrals.* In this problem, you can assume that:

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi} \tag{7}$$

(If you're curious how this is shown, there is a famous trick which is explained in the appendix!)

(a) By change of variable show that

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\pi} a^{-1/2} \tag{8}$$

where $a > 0$ is a real number.

(b) Now show that

$$\int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2} = \frac{\sqrt{\pi}}{2} a^{-3/2} \tag{9}$$

using one of two possible approaches. One way is to group the integrand on the LHS as $(x)(xe^{-ax^2})$ and use integration by parts. The second way is to differentiate both sides of Eq. (8) with respect to $a$.

(c) Using the previous results show that the Gaussian probability distribution in one variable

$$p(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{10}$$

is correctly normalized (i.e. $\int p(x) = 1$) with variance $\sigma^2$ (i.e. $\int x^2 p(x) = \sigma^2$), as implicitly assumed in the lecture.

(d) Can you generalize part (b) to give a formula for $\int x^N e^{-ax^2}$, where $N$ is a positive integer?

## Appendix A: A famous trick for calculating $\int_{-\infty}^{\infty} dx e^{-x^2}$

As far as I know, the following strange trick is the only way of doing the integral! You'll need to have studied a little bit of multivariate calculus, in particular changing variables from Cartesian to polar coordinates in a 2D integral. Define:

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2} \tag{A1}$$

We can write $I^2$ as a 2D integral over the $(x, y)$ plane.

$$I^2 = \left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)\left(\int_{-\infty}^{\infty} dy\, e^{-y^2}\right) = \int\int dx\, dy\, e^{-x^2-y^2} \tag{A2}$$

This doesn't appear to be making progress, but if we now change variables to polar coordinates $(r, \theta)$, then we can do the integral! (In the steps below, we integrate over $\theta$ first and then $r$.)

$$I^2 = \int dr\, d\theta\, r e^{-r^2} = 2\pi \int_0^{\infty} dr\, r e^{-r^2} = 2\pi \left(\frac{1}{2}\right) \tag{A3}$$

Note that the extra factor of $r$ we picked up in the change of variables to polar coordinates is what allows us to do the integral. Taking the square root on both sides we now get $I = \sqrt{\pi}$ as desired.