3.4 THE DUAL SIMPLEX METHOD

In modeling applied problems as linear programming problems, it frequently becomes necessary to add more constraints to the model. These constraints are generally used to make the model more accurately represent the real problem, and their need becomes evident when the researcher compares the solution to the linear programming problem with the situation being modeled. However, adding one or more constraints may cause the existing solution to become infeasible. In this case the dual simplex method, discussed in this section, can be used to restore feasibility without having to resolve the entire new problem.

When we use the simplex algorithm on a primal problem we begin with a feasible but nonoptimal solution. Each iteration of the simplex algorithm finds a feasible solution that is closer to optimality, and this procedure continues until an optimal solution is reached. In the meantime, what is happening to the dual problem? Let us examine the sawmill problem in this context.

EXAMPLE 1. The primal problem in standard form for the model is

$$\begin{aligned}&\text{Maximize} \quad z = 120x + 100y\\&\text{subject to}\\&\qquad 2x + 2y \le 8\\&\qquad 5x + 3y \le 15\\&\qquad x \ge 0, \quad y \ge 0.\end{aligned}$$

The dual problem is

$$\begin{aligned}&\text{Minimize} \quad z' = 8s + 15t\\&\text{subject to}\\&\qquad 2s + 5t \ge 120\\&\qquad 2s + 3t \ge 100\\&\qquad s \ge 0, \quad t \ge 0.\end{aligned}$$

The initial tableau for the primal problem, after adding the necessary slack variables, is as follows.

Tableau 3.21

		120	100	0	0	
$\mathbf{c}_B$		x	y	u	v	$\mathbf{x}_B$
0	u	2	2	1	0	8
0	v	5		0	1	15
		-120	-100	0	0	0

From this tableau we see that

$$\mathbf{c}_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and we may compute from the formula $\mathbf{w}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$ that

$$\mathbf{w}^T = [s \quad t] = [0 \quad 0] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [0 \quad 0].$$

Note that this "solution" to the dual problem satisfies the nonnegativity conditions but neither of the constraints.

We now pivot in Tableau 3.21 and obtain Tableau 3.22.

Tableau 3.22

		120	100	0	0	
$\mathbf{c}_B$		x	y	u	v	$\mathbf{x}_B$
0	u	0	$\frac{4}{5}$	1	$-\frac{2}{5}$	2
120	x	1	$\frac{3}{5}$	0	$\frac{1}{5}$	3
		0	-28	0	24	360

From this tableau we see that

$$\mathbf{c}_B = \begin{bmatrix} 0 \\ 120 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & -\frac{2}{5} \\ 0 & \frac{1}{5} \end{bmatrix}$$

and that

$$\mathbf{w}^T = [s \quad t] = \begin{bmatrix} 0 \\ 120 \end{bmatrix} \begin{bmatrix} 1 & -\frac{2}{5} \\ 0 & \frac{1}{5} \end{bmatrix} = [0 \quad 24].$$

We now have a "solution" to the dual problem that satisfies the nonnegativity conditions and also satisfies the first but not the second constraint of the dual problem. We pivot again, obtaining Tableau 3.23.

Tableau 3.23

		120	100	0	0	
$\mathbf{c}_B$		x	y	u	v	$\mathbf{x}_B$
100	y	0	1	$\frac{5}{4}$	$-\frac{1}{2}$	$\frac{5}{2}$
120	x	1	0	$-\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{2}$
		0	0	35	10	430

From this tableau we see that

$$\mathbf{c}_B = \begin{bmatrix} 100 \\ 120 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^{-1} = \begin{bmatrix} \frac{5}{4} & -\frac{1}{2} \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix}.$$

Thus,

$$\mathbf{w}^{\mathrm{T}} = [s \quad t] = [35 \quad 10].$$

This is a feasible solution to the dual problem: it satisfies the nonnegativity conditions and both of the constraints of the problem. The objective function value for the dual problem using this solution is the same as the objective function value for the primal problem with the corresponding solution. From the Duality Theorem, we have found an optimal solution to the dual problem. △

From this example we have seen that if the primal problem has a solution that is feasible and nonoptimal, then the solution determined for the dual problem is infeasible. As the simplex method progresses, the solutions determined for the dual problem are all infeasible until the optimal solution is attained for the primal problem. The dual solution corresponding to the optimal primal solution is both optimal and feasible. The goal for the primal problem when using the simplex method is to achieve optimality. The goal for a corresponding method for the dual problem is to achieve feasibility, that is, to have both nonnegativity constraints and resource constraints satisfied.

The dual simplex method handles problems for which it is easy to obtain an initial basic solution that is infeasible but satisfies the optimality criterion. That is, the initial tableau has nonnegative entries in the objective row but negative entries in the right-hand column. The following example will be used as we present our description of the dual simplex algorithm.

EXAMPLE 2. Consider the following linear programming problem:

$$\text{Maximize} \quad z = -x_1 - 2x_2$$

subject to

$$\begin{aligned} x_1 - 2x_2 + x_3 &\geq 4 \\ 2x_1 + x_2 - x_3 &\geq 6 \end{aligned}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.$$

We change each constraint to an $\leq$ inequality and then introduce slack variables x_4 and x_5. The result is a problem in canonical form:

$$\text{Maximize} \quad z = -x_1 - 2x_2$$

subject to

$$\begin{aligned} -x_1 + 2x_2 - x_3 + x_4 &= -4 \\ -2x_1 - x_2 + x_3 + x_5 &= -6 \end{aligned}$$

$$x_j \geq 0, \quad j = 1, 2, \ldots, 5.$$

The initial tableau for the simplex algorithm is given in Tableau 3.24. It has x_4 and x_5 as the initial basic variables. The solution that this tableau represents is

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = -4, \quad x_5 = -6.$$

Tableau 3.24

		-1	-2	0	0	0	
$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	$\mathbf{x}_B$
0	x_4	-1	2	-1	1	0	-4
0	x_5	-2	-1	1	0	1	-6
		1	2	0	0	0	0

This solution is not feasible, and here $z = 0$. The entries in the objective row show that the optimality criterion is satisfied. △

The dual simplex method consists of two parts: a feasibility criterion that tells us whether the current solution (which satisfies the optimality criterion) is feasible, and a procedure for getting a new solution that removes some of the infeasibilities of the current solution and consequently drives the current solution toward a feasible solution. The dual simplex method consists of the following steps.

1. Find an initial basic solution such that all entries in the objective row are nonnegative and at least one basic variable has a negative value. (Tableau 3.24 represents this step for our example.)
2. Select a departing variable by examining the basic variables and choosing the most negative one. This is the departing variable and the row it labels is the pivotal row.
3. Select an entering variable. This selection depends on the ratios of the objective row entries to the corresponding pivotal row entries. The ratios are formed only for those entries of the pivotal row that are negative. If all entries in the pivotal row are nonnegative, the problem has no feasible solution. Among all the ratios (which must all be nonpositive), select the maximum ratio. The column for which this ratio occurred is the pivotal column and the corresponding variable is the entering variable. In case of ties among the ratios, choose one column arbitrarily.
4. Perform pivoting to obtain a new tableau. The objective row can be computed as $z_j - c_j = \mathbf{c}_B^T \mathbf{t}_j - c_j$, where $\mathbf{t}_j$ is the jth column of the new tableau.
5. The process stops when a basic solution that is feasible (all variables ≥ 0) is obtained.

A flowchart for the dual simplex method is given in Figure 3.4 and a structure diagram is given in Figure 3.5.

EXAMPLE 2 (CONTINUED). Continuing with our example, we perform Step 2 of the dual simplex algorithm. We see that $x_5 = -6$ is the most

Input is a tableau which satisfies the optimality criterion.

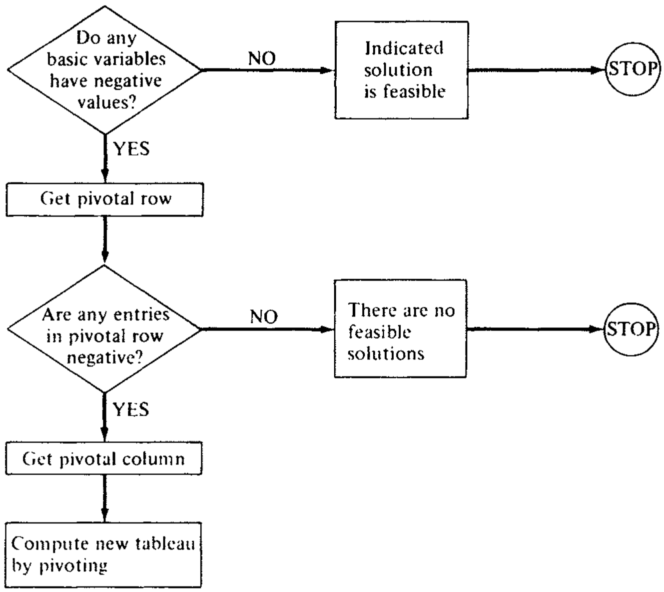

FIGURE 3.4 Flowchart for the dual simplex algorithm.

Input is a tableau which satisfies the optimality criterion.

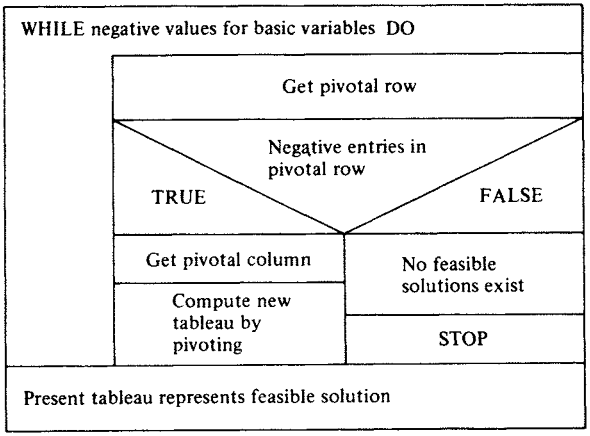

FIGURE 3.5 Structure diagram for the dual simplex algorithm.

negative basic variable, so that x_5 is the departing variable. The ratios of the entries in the objective row to corresponding negative entries in the pivotal row are

$$\text{Column 1:} \quad -\tfrac{1}{2}$$
$$\text{Column 2:} \quad -2.$$

The maximum ratio is $-\frac{1}{2}$, so that x_1 is the entering variable. We repeat Tableau 3.24 with the entering and departing variables and the pivot labeled (Tableau 3.24a).

Tableau 3.24a

			↓					
			-1	-2	0	0	0	
	$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	$\mathbf{x}_B$
	0	x_4	-1	2	-1	1	0	-4
←	0	x_5	-2	-1	1	0	1	-6
			1	2	0	0	0	0

We now perform a pivotal elimination to get Tableau 3.25. The basic solution given by Tableau 3.25 is

$$x_1 = 3, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = -1, \quad x_5 = 0.$$

This solution is still optimal (the objective row has nonnegative entries) but is infeasible. However, it is less infeasible in that only one variable is negative.

Tableau 3.25

					↓			
			-1	-2	0	0	0	
	$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	$\mathbf{x}_B$
←	0	x_4	0	$\frac{5}{2}$	$-\frac{3}{2}$	1	$-\frac{1}{2}$	-1
	-1	x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	3
			0	$\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	-3

For the next iteration of the dual simplex algorithm, x_4 is the departing variable, since it is the only negative basic variable. Forming the ratios of the entries in the objective row to the corresponding negative entries of the pivotal row, we have

$$\text{Column 3:} \quad \tfrac{1}{2}/-\tfrac{3}{2} = -\tfrac{1}{3}$$
$$\text{Column 5:} \quad \tfrac{1}{2}/-\tfrac{1}{2} = -1.$$

The maximum ratio is $-\frac{1}{3}$, so that x_3 is the entering variable. Pivoting, we now obtain Tableau 3.26.

Tableau 3.26

		-1	-2	0	0	0	
$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	$\mathbf{x}_B$
0	x_3	0	$-\frac{5}{3}$	1	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
-1	x_1	1	$-\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{10}{3}$
		0	$\frac{7}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{10}{3}$

The basic solution represented by Tableau 3.26 is

$$x_1 = \tfrac{10}{3}, \quad x_2 = 0, \quad x_3 = \tfrac{2}{3}, \quad x_4 = 0, \quad x_5 = 0.$$

This solution satisfies the optimality criterion, *and* it is feasible since all the variables have nonnegative values. △

While using the dual simplex method in Example 2, we were fortunate in finding an initial basic solution to the given problem that satisfied the optimality criterion but was not feasible. In general, it is difficult to find such a starting point for the dual simplex method. However, the principal use of this method is to restore feasibility when additional constraints are included in a linear programming problem whose solution is known. The following example illustrates this situation.

EXAMPLE 3. There are three typical kinds of dog food that Paws eats each day: dry food, canned wet food, and dog biscuits. The nutritional analysis of his favorite brands is given in the following table in percent by weight.

	Fat	*Protein*	*Fiber*	*Moisture*	*Cost (¢/oz)*
Dry food	8.0	14.0	5.5	12.0	4.1
Wet food	6.0	9.0	1.5	78.0	2.5
Biscuit	8.0	21.0	4.5	12.0	7.3

The veterinarian has suggested that Paws get at least 5 oz of protein and at most 1 oz of fiber each day. His owner has set up the following linear programming problem to model the dietary requirements and minimize the cost, where x_1 is the amount of dry food measured in ounces offered to Paws. Similarly x_2 denotes the amount of wet food and x_3 denotes the amount of biscuits.

$$\begin{aligned}&\text{Minimize} \quad z = 4.1x_1 + 2.5x_2 + 7.3x_3\\&\text{subject to}\\&\quad 0.14x_1 + 0.09x_2 + 0.21x_3 \geq 5.0\\&\quad 0.055x_1 + 0.015x_2 + 0.045x_3 \leq 1.0\\&\qquad x_j \geq 0, \quad j = 1,2,3\end{aligned}$$

We first transform this problem to standard form and then to canonical form introducing the slack variables x_4 and x_5:

$$\begin{aligned}&\text{Maximize} \quad z = -4.1x_1 - 2.5x_2 - 7.3x_3\\&\text{subject to}\\&\quad -0.14x_1 - 0.09x_2 - 0.21x_3 + x_4 = -5.0\\&\quad 0.055x_1 + 0.015x_2 + 0.045x_3 + x_5 = 1.0\\&\qquad x_j \geq 0, \quad j = 1,2,\ldots,5.\end{aligned}$$

Solving this linear programming problem using the dual simplex method we obtain (verify) Tableau 3.27.

Tableau 3.27

		-4.1	-2.5	-7.3	0	0	
$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	$\mathbf{x}_B$
-2.5	x_2	1.556	1	2.333	-11.111	0	55.556
0	x_5	0.032	0	0.010	0.167	1	0.167
		0.211	0	1.467	27.778	0	-138.889

From this tableau we see that the minimum-cost diet for Paws consists of 55.556 ounces of wet food per day (no dry food and no biscuits, poor Paws) at a cost of \$1.39 per day.

At the most recent visit to the veterinarian, she also suggested that Paws' fat intake be limited to 2.5 oz per day. The new constraint,

$$0.08x_1 + 0.06x_2 + 0.08x_3 \leq 2.5,$$

expresses this limitation. Introducing the slack variable x_6 the new constraint becomes

$$0.08x_1 + 0.06x_2 + 0.08x_3 + x_6 = 2.5.$$

We now include this constraint into Tableau 3.27 and form Tableau 3.28. Observe that the new slack variable x_6 does not appear in the other two constraints nor in the objective function, so that its coefficient in these two constraints and in the objective row will be zero.

Tableau 3.28

		-4.1	-2.5	-7.3	0	0	0	
$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{x}_B$
-2.5	x_2	1.556	1	2.333	-11.111	0	0	55.556
0	x_5	0.032	0	0.010	0.167	1	0	0.167
0	x_6	0.08	0.06	0.08	0	0	1	2.5
		0.211	0	1.467	27.778	0	0	-138.889

Moreover, the new slack variable will be a basic variable. Since x_2 is a basic variable, all the entries in the column labeled by x_2 must be zero except for the entry in the row labeled by x_2. Hence, we add (-0.06) times the first row to the third row and obtain Tableau 3.29.

Tableau 3.29

			$\downarrow$						
			-4.1	-2.5	-7.3	0	0	0	
	$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{x}_B$
	-2.5	x_2	1.556	1	2.333	-11.111	0	0	55.556
	0	x_5	0.032	0	0.010	0.167	1	0	0.167
$\leftarrow$	0	x_6	-0.013	0	-0.06	0.667	0	1	-0.833
			0.211	0	1.467	27.778	0	0	-138.889

This tableau represents an infeasible solution that satisfies the optimality criterion.

Using the dual simplex method, we may restore feasibility. We see that x_6 is the departing variable and that x_1 is the entering variable for this step. Completing several iterations of the dual simplex method we obtain Tableau 3.30 (verify).

Tableau 3.30

		-4.1	-2.5	-7.3	0	0	0	
$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{x}_B$
-2.5	x_2	0	1	0	5.031	-35.220	33.019	22.170
-4.1	x_1	1	0	0	9.434	33.962	5.660	0.943
-7.3	x_3	0	0	1	-13.208	-7.547	-17.925	13.679
		0	0	0	45.157	3.899	25.094	-159.151

We conclude that after his fat intake was restricted, the minimum-cost diet for Paws is 0.943 oz of dry food, 22.17 oz of wet food, and 13.679 oz of dog biscuits per day at a total cost of \$1.59. $\triangle$

Much of our discussion of the simplex method centered on finding an initial basic feasible solution to a linear programming problem in arbitrary form. We developed the concept of artificial variables to provide a method for constructing a starting point for such a problem. The situation with the dual simplex method is different. In this book we will use the dual simplex method to restore feasibility in a tableau that represents a solution that is infeasible but satisfies the optimality criterion. Thus, we will not need any procedures for finding an initial basic feasible solution when using the dual simplex method.

3.4 EXERCISES

In Exercises 1–5 the given tableau represents a solution to a linear programming problem that satisfies the optimality criterion, but is infeasible. Use the dual simplex method to restore feasibility.

1.

		5	6	0	0	0	
$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	$\mathbf{x}_B$
5	x_1	1	0	$\frac{1}{8}$	$-\frac{1}{8}$	0	$\frac{17}{4}$
6	x_2	0	1	$-\frac{1}{12}$	$\frac{5}{12}$	0	$\frac{19}{6}$
0	x_5	0	0	$-\frac{1}{8}$	$-\frac{7}{8}$	1	$-\frac{1}{4}$
		0	0	$\frac{1}{8}$	$\frac{15}{8}$	0	$\frac{161}{4}$

2.

		5	6	0	0	0	0	
$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{x}_B$
5	x_1	1	0	0	-1	1	0	4
6	x_2	0	1	0	1	$-\frac{2}{3}$	0	$\frac{10}{3}$
0	x_3	0	0	1	7	-8	0	2
0	x_6	0	0	0	0	$-\frac{1}{3}$	1	$-\frac{1}{3}$
		0	0	0	1	1	0	40

3.

		4	5	3	0	0	0	0	
$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\mathbf{x}_B$
3	x_3	$\frac{3}{4}$	0	1	$\frac{1}{4}$	0	0	0	$\frac{5}{2}$
0	x_5	$\frac{11}{16}$	0	0	$-\frac{3}{16}$	1	$-\frac{1}{4}$	0	$\frac{17}{8}$
5	x_2	$\frac{9}{16}$	1	0	$-\frac{1}{16}$	0	$\frac{1}{4}$	0	$\frac{19}{8}$
0	x_7	$-\frac{1}{4}$	0	0	$-\frac{1}{4}$	0	0	1	$-\frac{1}{2}$
		$\frac{17}{16}$	0	0	$\frac{7}{16}$	0	$\frac{5}{4}$	0	$\frac{155}{8}$

4.

$\mathbf{c}_B$		5	6	0	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{x}_B$
5	x_1	1	0	0	0	1	0	4
6	x_2	0	1	0	$\frac{1}{3}$	$-\frac{2}{3}$	0	$\frac{10}{3}$
0	x_3	0	0	1	-1	-8	0	2
0	x_6	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$-\frac{1}{3}$
		0	0	0	2	1	0	40

5.

$\mathbf{c}_B$		7	3	0	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{x}_B$
0	x_3	0	0	1	$-\frac{1}{4}$	$-\frac{17}{4}$	0	$\frac{19}{2}$
7	x_1	1	0	0	$\frac{1}{8}$	$-\frac{3}{8}$	0	$\frac{1}{4}$
3	x_2	0	1	0	0	1	0	2
0	x_6	0	0	0	$\frac{1}{8}$	$-\frac{3}{8}$	1	$-\frac{23}{4}$
		0	0	0	$\frac{7}{8}$	$\frac{3}{8}$	0	$\frac{31}{4}$

6. Use the dual simplex method to verify that Tableau 3.27 is correct,
7. For Example 3, verify using the dual simplex method that the final tableau (Tableau 3.30) is correct. Note that your answer may differ slightly from the text due to round-off error.
8. Use the dual simplex method to find a solution to the linear programming problem formed by adding the constraint
$$3x_1 + 5x_3 \geq 15$$
to the problem in Example 2.
9. Example 3 showed that adding a constraint may change the solution to a linear programming problem (i.e., the new solution has different basic variables and the basic variables have different values). There are two other possibilities that may occur when a constraint is added. Describe them.
10. **Computing project**. Compare the structure diagrams for the simplex algorithm and the dual simplex algorithm. How is duality exemplified by these diagrams?

3.5 THE REVISED SIMPLEX METHOD

The revised simplex method makes use of some of the notation and ideas we developed in Section 3.3 to obtain efficiency in computation and storage of intermediate results. The simplex method, as we described it,