Mathematics for Computing

Topic 01 : Sets

Lecture 01 : Introduction to Sets

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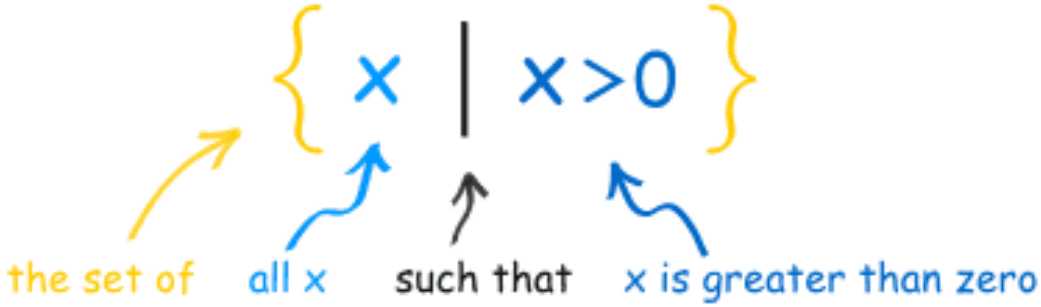
Outline

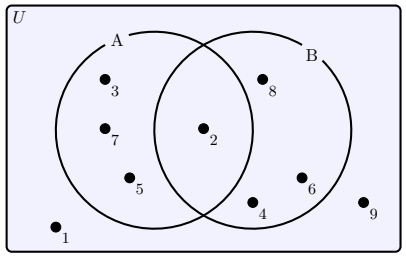
- Representing sets using enumeration, set builder notation and Venn diagrams.
- Parts of a set (subsets and proper subsets).
- Java — constructing a set, adding/removing elements, is element test, is subset test.
- **Book** §1.1

Introduction to Sets — Summary

1. Review of Fundamental Concepts 3

- A set is an unordered well-defined collection of unique objects.
- Represent sets using a list of elements, set builder notation, and Venn diagrams

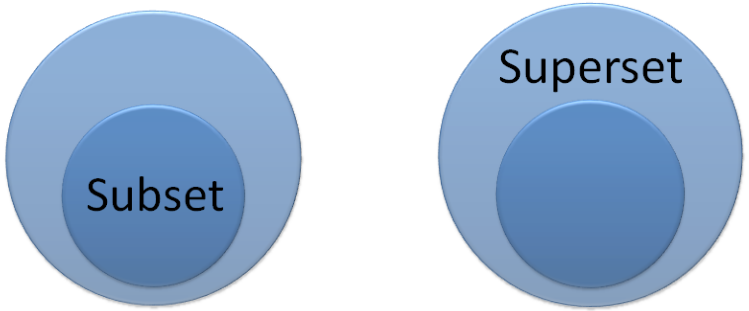




- Representing sets in Java.

2. Subsets 15

- A subset is a part or whole of another set.



- A proper subset is a part (but not whole) of another set.
- Subset testing in Java.

Sets

Sets are the most basic, fundamental data structures in math and computer science:

Definition (Set, element)

A set is a unordered, well defined, collection of distinct objects, called elements.

Examples

1. Vowels in the English language:

$$\{a, e, i, o, u\}$$

2. Set of planets in our solar system

$$\{\text{Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune}\}$$

Non-Examples

1. The set of famous actors — not a set because it is not well defined.
2. $\{m, o, o, n\}$ — not a set because the elements are not unique.

Set Notation

- Sets are usually denoted by capital letters and its elements by lower case letters.
- Given set A, we write $a \in A$ to denote that a is an element of A.
- The notation $a \notin A$ denotes that a is not an element of A.
- The special set that has no elements is called the empty set or the null set and is denoted by $\emptyset$ or by $\{\}$.

Examples

1. Let P represent the set of planets in our solar system, then

$$\text{Mercury} \in P, \text{Venus} \in P, \ldots, \text{Neptune} \in P$$

but

$$\text{Pluto} \notin P$$

2. Let A represent the set of atheist Roman Catholic popes*, then

$$A = \{\}$$

*Here I am making a 'reasonable' assumption !

Set Cardinality — How big is a set?

Definition (Set Cardinality)

Given set A, if there are n distinct elements in A then n is called the cardinality of A and is denoted by $|A|$ or $\#A$.

Example

1. Let D be the set of hexadecimal digits, that is

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$$

then $|D| = 16$.

Properties

- The cardinality of a set is a non-negative integer.
- The empty set, $\emptyset$, has cardinality zero, i.e., $|\emptyset| = 0$.

Special Sets

Definition ($\mathbb{N}$ and $\mathbb{Z}$)

- The set
$$\mathbb{N} = 0, 1, 2, 3, 4, ...$$
is the set of natural numbers.
- The set
$$\mathbb{Z} = \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$$
is the set of integers.

- The natural number set[†] allow us to count objects.
- The positive integer set $\mathbb{P} = \mathbb{N}^+ = \{1, 2, 3, \ldots\}$
- Beyond Z, we have $\mathbb{Q}$ the set of rationals, then $\mathbb{P}$ [illegible] numbers, then $\mathbb{C}$ the set of [illegible] the set of hyper-[illegible]

(important in mathematics, less so in computer science)

[†]Some authors do not include the number 0 in the set $\mathbb{N}$. It is no big deal either way as long as everyone knows which definition is used.

Representing Sets using a List of Elements (Enumeration)

- For small sets, the simplest ways to represent a set is to list all the elements in the set.
- Braces "{" and "}" are used with the elements listed (in any order, since order is not important) between them.

Examples

1. The set, D, of the digits in base 16 (hex) can be written:

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$$

2. The set, A, of all of the lower case letters can be written as:

$$A = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$$

3. The set, E, of all even integers greater than 10 and less than 20 can be written as:

$$E = \{12, 14, 16, 18\}$$

Representing Sets using Set Builder Notation

In set builder notation the elements of the set are not listed but are specified by stating the property a object must possess to be an element of the set.

The symbol "$|$" should be read as "such that".

Examples

- The set E of all integers greater than 10 and less than 20 can be written:

$$E = \{x | x \text{ is an integer greater than 10 and less than 20}\}$$

or using the more concise mathematical notation

$$E = \{x \in \mathbb{N} | 10 < x < 20\}$$

- The set of planets in our solar system could be defined as

$$\left\{ x \in \{\text{all objects orbiting the sun}\} \;\middle|\; \begin{array}{l} x \text{ big enough to be spherical,} \\ \text{and sweeps its orbit clear of} \\ \text{other objects.} \end{array} \right\}$$

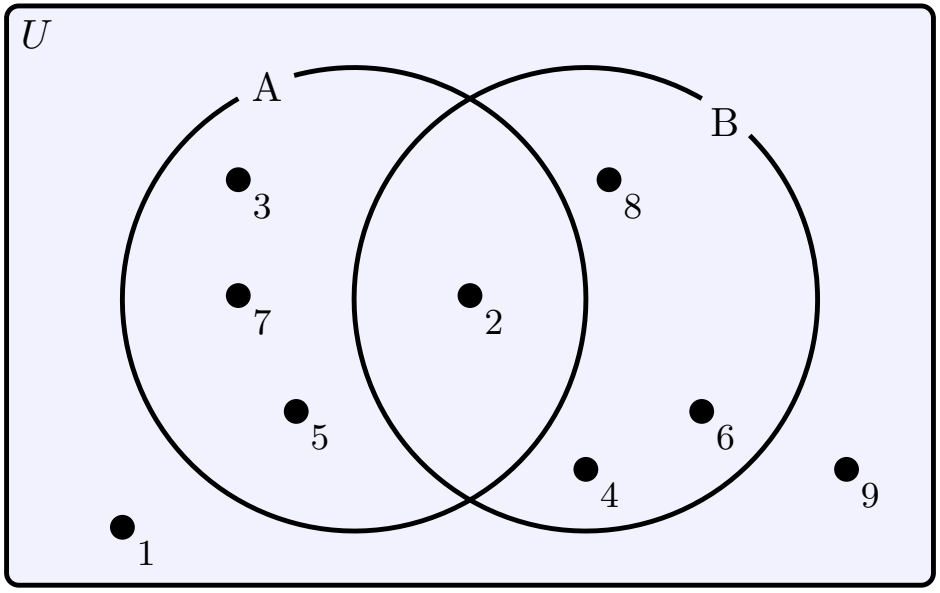
Representing Sets using Venn Diagrams

Set can be represented visually using Venn diagrams where:

- Sets are represented by ovals.
- Elements within sets are represented by points within the ovals.
- The universal set, typically labelled U, is indicated by a rectangle. (If the universal set is not relevant, then it is usually not drawn.).

Example

Consider the universal set of positive integers less than 10. Then, a Venn diagram representing the sets $A = \{2, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$ is:



Java Implementation I

Java has multiple implements of sets (e.g., `HashSet` and `TreeSet`) that allow you to create, access and modify sets similar to the way we do in mathematics — the syntax is different due to the rules of the Java language, coding style or for performance reasons.

Mathematics ↔ Java

- To construct a set, called `setA`, in Java use the code:

```
HashSet<String> setA = new HashSet<String>();
```

- To add an element to the set `setA`, use the `add` method:

```
setA.add("Apple");   // add an element to the set
```

- To remove an element from a set, use the `remove` method.
- To check if something is an element of a set, use the `contains` method.
- To determine the set cardinality, i.e., number of elements in the set, use the `size` method.

SimpleSetExample.java

```
3  import static java.lang.System.out;
4  import java.util.HashSet; // fastest type of set (but not sorted)
5
6  public class SimpleSetExample {
7
8      public static void main(String[] args) {
9
10         // create a set - here we want a set of strings
11         HashSet<String> setA = new HashSet<String>();
12
13         setA.add("Apple");    // add an element to the set
14         setA.add("Grape");    // . . .
15         setA.add("Pear");    // . . .
16         setA.add("Pear");    // add an element twice !!!
17         setA.remove("Apple");  // remove an element
18         setA.remove("Apple");  // remove an element twice !!!
19
20         out.println ("Set␣A␣=␣" + setA);          // output the set
21         out.println ("|A|␣=␣" + setA.size());     // output size
22
23         out.println ("Is␣Apple␣in␣set␣?␣" + setA.contains("Apple"));
24         out.println ("Is␣Pear␣in␣set␣?␣" + setA.contains("Pear"));
25     }
26 }
```

```
Set A = [Pear, Grape]
|A| = 2
Is Apple in set ? false
Is Pear in set ? true
```

Set Equality

Two sets, A and B, are equal iff they have the same elements

- If A and B are equal, we write $A = B$; otherwise we write $A \neq B$.

Examples

1. The sets

   $$A = \{\text{Mercury}, \text{Venus}, \text{Earth}\} \qquad \text{and} \qquad B = \{\text{Earth}, \text{Mercury}, \text{Venus}\}$$

   are equal. (Note order does not matter.)

2. The sets

   $$A = \{2x + 1 | x \in \mathbb{N}\} \qquad \text{and} \qquad B = \{2, 4, 6, 8, \ldots\}$$

   are equal. (What is important is the element that both sets contain — not how the sets are written.)

Java Implementation

SetEqualityExample.java

```
3  import java.util.HashSet;   // fastest type of set (but not sorted)
4
5  public class SetEqualityExample {
6
7      public static void main(String[] args) {
8
9          HashSet<String> setA = new HashSet<String>();
10         setA.add("Apple");
11         setA.add("Pear");
12         System.out.println ("Set␣A␣=␣" + setA);
13
14         HashSet<String> setB = new HashSet<String>();
15         setB.add("Pear");
16         setB.add("Apple");
17         System.out.println ("Set␣B␣=␣" + setB);
18
19         System.out.println("Does␣A=B␣?␣␣" + setA.equals(setB));
20     }
21 }
```

```
Set A = [Pear, Apple]
Set B = [Pear, Apple]
Does A=B ?  true
```

Review Exercises 1 (Fundamental Concepts)

Question 1.1
Which of the following are valid sets? Explain your answer.

(a) $\{5, 4, 3, 1\}$ **(b)** $\{2, 5, 7, 2\}$

(c) $\{\{2, 5, 1\}, \{2, 5\}, \{2\}\}$ **(d)** $\{\{\{5\}\}\}$

(e) $\{\text{Apple}, 4\}$

Question 1.2
Which of the following statements are true, and which are false?

(a) $2 \in \{2, 5, 7\}$ **(b)** $3 \notin \mathbb{Z}$

(c) $9.4 \notin \mathbb{Z}$ **(d)** $\{2\} \in \{2, 5, 7\}$

(e) $3 \in \{\{1\}, \{2\}, \{3\}\}$ **(f)** $\{2, 5\} \notin \{\{6, 7\}, \{2, 5, 1\}\}$

(g) $\emptyset \in \{5, 7, 0\}$

Question 1.3
How many elements in the following set

$$A = \{2x | x \in \mathbb{N}, x < 10\}$$

Sets and Their Subsets

Definition (subset)

A set, A, is said to be a subset of set B if every element of A is also an element of B.

We use the notation $A \subseteq B$ to indicate that A is a subset of B.

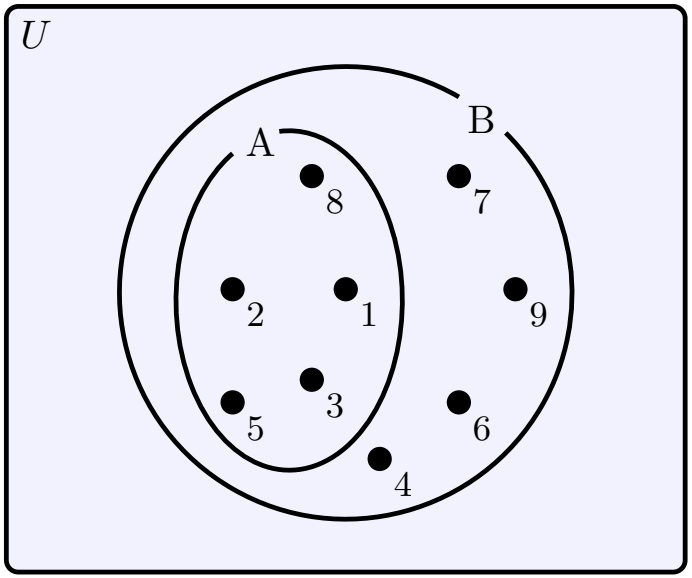
Example

Let B be the set of positive integers less than 10, i.e.,

$$B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

and let A be the set[‡] $\{1, 2, 3, 5, 8\}$.

Then $A \subseteq B$ since every element of A is also an element of B as the diagram[§] illustrates.



[‡] The set A is the set of the first 5 Fibonacci numbers.

[§] We could have skipped drawing the universal set here.

Sets and Their Subsets II

Properties

1. For any set, A we have $\emptyset \subseteq A$. This is true since $\emptyset$ contains no elements and hence every element in $\emptyset$ is also in A. In fact,

   $\emptyset$ is a subset of every set.

2. Let U be the universal set, and A be any arbitrary set. Then

   $$\emptyset \subseteq A \subseteq U.$$

   and hence

   $$0 \leq |A| \leq |U|.$$

You need to take great care when using a symbolic language since similar symbols[¶] may have significantly different meanings, e.g., $0 \neq \emptyset$

[¶] In one of my exam papers last summer I used symbols x, X, and $\mathcal{X}$ all of which represent specific and different things.

Sets and Their Subsets III

Properties (cont.)

1. If A is a subset of B, then A cannot contain more elements than B, i.e.,

$$\text{If } A \subseteq B \text{ then } |A| \leq |B|$$

2. When it is desirable to emphasise that A is a subset of B, but $A \neq B$ we write $A \subset B$. In this case A is said to be a proper subset of B.
3. If A is a proper subset of B, then A must contain fewer elements than B, i.e.,

$$\text{If } A \subset B \text{ then } |A| < |B|$$

4. The statements $A \supseteq B$ and $A \supset B$ denote that the set A is a superset of B and a proper superset of B respectively.
5. If A is a subset of B and B is a subset of A, then A and B are equal:

$$\text{If } A \subseteq B \text{ and } B \subseteq A \text{ then } A = B.$$

Java Implementation

Mathematics $\leftrightarrow$ Java

- To test whether set `setB` is a subset of `setA` use the `containsAll` method.

"$\in$" and "$\subseteq$" are two separate tests

- The "is element of" test checks whether something is a element of a set.
- The "is subset of" test checks whether a set is a part or whole of another set.

Example

Let $A = \{a, b, c\}$. Then

- $a \in A$, but $\{a\} \notin A$. However $\{a\} \subseteq A$.
- $\{a, b\} \subseteq A$, but $\{a, b\} \notin A$.

$\Longrightarrow$ Java methods `contains` and `containsAll` are distinct methods

```
Set A = [b, c, a]
Set B = [b, a]
Is A a subset of B ? false
Is B a subset of A ? true
Is A an element of B ? false
Is B an element of A ? false
```

```
3  import static java.lang.System.out;
4  import java.util.HashSet;
5
6  public class SubsetExample {
7
8      public static void main(String[] args) {
9          HashSet<String> setA = new HashSet<String> ();
10         setA.add("a");
11         setA.add("b");
12         setA.add("c");
13         System.out.println("Set A = " + setA);
14
15         HashSet<String> setB = new HashSet<String> ();
16         setB.add("a");
17         setB.add("b");
18         System.out.println("Set B = " + setB);
19
20         // subset tests
21         out.println("Is A a subset of B ? " + setB.containsAll(setA));
22         out.println("Is B a subset of A ? " + setA.containsAll(setB));
23
24         // element tests
25         out.println("Is A an element of B ? " + setB.contains(setA));
26         out.println("Is B an element of A ? " + setA.contains(setB));
27     }
28 }
```

Review Exercises 2 (Subsets)

Question 2.1

Let $A = \{2, 3\}$, $B = \{0, 2, 3\}$, and $C = \{1, 5, 9\}$. Determine which of the following statements are true. Give reasons for your answers.

(a) $2 \in A$ **(b)** $\{2\} \in A$ **(c)** $\{2\} \subseteq A$

(d) $\{0\} \in B$ **(e)** $\{\} \in B$ **(f)** $\{\} \subseteq B$

(g) $\{2, 3\} \subseteq A$ **(h)** $\{2, 3\} \subseteq B$ **(i)** $\{2, 3\} \subseteq C$

(j) $\{2, 3\} \subset A$ **(k)** $\{2, 3\} \subset B$ **(l)** $\{2, 3\} \subset C$