BACHELOR OF SCIENCE (HONS) IN
- APPLIED COMPUTING
- COMPUTER FORENSICS & SECURITY

EXAMINATION:

DISCRETE MATHEMATICS (COMMON MODULE)

SEMESTER 1 - YEAR 1

DECEMBER 2024 **DURATION: 2 HOURS**

INTERNAL EXAMINERS:	**DR DENIS FLYNN**	**DATE:**	**16 DEC 2024**
	DR KIERAN MURPHY	**TIME:**	**11.45 AM**
		VENUE:	**MAIN HALL**
EXTERNAL EXAMINER:	**DR JULIE CROWLEY**		

INSTRUCTIONS TO CANDIDATES

1. ANSWER ALL QUESTIONS.
2. TOTAL MARKS = 100.
3. EXAM PAPER (5 PAGES EXCLUDING THIS COVER PAGE), FORMULA AND PYTHON SHEETS (2 PAGES)

MATERIALS REQUIRED

1. NEW MATHEMATICS TABLES.
2. GRAPH PAPER

SOUTH EAST TECHNOLOGICAL UNIVERSITY

Question 1

(a)

Partial marks if failed to make all propositions atomic.

We have atomic propositions

- $I =$ "Internet connection is stable"
- $S =$ "Server is running"
- $W =$ "Website will be accessible"

Then the given sentence as propositional logic expression is

$$(I \wedge S) \to W$$

(b)

(i)
- **Reflexive**: A relation is reflexive if for every element $a \in A$, $(a, a) \in R$. Since $(1,1), (2,2), (3,3) \in R$, the relation is **reflexive**.
- **Symmetric**: A relation is symmetric if $(a, b) \in R$ implies $(b, a) \in R$. Since $(1,2) \in R$ but $(2,1) \notin R$, the relation is **not symmetric**.
- **Transitive**: A relation is transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$. Since there are no such pairs in R, the relation is **transitive**.

(ii) Since R is not symmetric it cannot be an equivalence relation.

(iii) **Antisymmetric**: A relation is antisymmetric if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$. Since $(1, 2) \in R$ but $(2, 1) \notin R$, the relation is **antisymmetric**.

6 marks $= 3 \times 2$

(c)

(i) *Start with the sub-string 1101.*

No constraints on remaining 8 bits, so $2^8 = 256$.

(ii) *Have weight 6 (i.e., contain exactly six 1's) and start with the sub-string 1101.*

In the remaining 8 bits, three of which must be 1, so $\binom{8}{3} = 56$

(iii) *Either start with* 1101 *or end with* 1010 *(or both).*

Start with 1101: No constraints on remaining 8 bits, so $2^8 = 256$.

End with 1010: No constraints on preceding 4 bits, so $2^8 = 256$.

Start with 1101 and end with 1010: No constraints on middle 4 bits, so $2^4 = 16$.

Ans (remove double counting): $2^8 + 2^8 - 2^4 = 256 + 256 - 16 = 496$

(iv) *Have weight 6, start with* 1101, *and end with* 1010.

The middle 4 digits must have weight 1 so that the entire string has weight 6. Hence have 4 possibilities.

6 marks $= 1 + 1 + 2 + 2$**, justification required**

(d)

Partial marks for correct parsing of expression, demonstrating ability to compute logical expression, implication of satisfiability definition.

$$\underbrace{\underbrace{(\neg P \vee Q)}_{1} \rightarrow \underbrace{\underbrace{(P \wedge \neg R)}_{2} \vee \underbrace{(Q \wedge R)}_{3}}_{4}}_{E}$$

P	Q	R	$\neg P$	$\neg Q$	$\neg R$	$\overbrace{(\neg P \vee Q)}^{1}$	$\overbrace{(P \wedge \neg R)}^{2}$	$\overbrace{(Q \wedge R)}^{3}$	$\overbrace{(P \wedge \neg R) \vee (Q \wedge R)}^{4}$	E
F	F	F	**T**	**T**	**T**	**T**	F	F	F	F
F	F	**T**	**T**	**T**	F	**T**	F	F	F	F
F	**T**	F	**T**	F	**T**	**T**	F	F	F	F
F	**T**	**T**	**T**	F	F	**T**	F	**T**	**T**	**T**
T	F	F	F	**T**	**T**	F	**T**	F	**T**	**T**
T	F	**T**	F	**T**	F	F	F	F	F	**T**
T	**T**	F	F	F	**T**	**T**	**T**	F	**T**	**T**
T	**T**	**T**	F	F	F	**T**	F	**T**	**T**	**T**

The expression is satisfiable but is not a tautology.

Question 2

(a)

(i) $T \land S \to A$

(ii) $\neg T \lor \neg S \to \neg A$

(iii) $T \to A$

(iv) $\neg S \to \not A$

(b)

(i) *How many distinct paths are there from $(0,0)$ to $(7,4)$?*

The total number of distinct paths from $(0,0)$ to $(7,4)$ is given by the binomial coefficient:

$$\binom{11}{4} = \frac{11!}{4!(11-4)!} = 330$$

(ii) *How many paths pass through $(4,2)$?*

First, calculate the number of paths from $(0,0)$ to $(4,2)$:

$$\binom{6}{2} = 15$$

Then, calculate the number of paths from $(4,2)$ to $(7,4)$:

$$\binom{5}{2} = 10$$

Therefore, the total number of paths passing through $(4,2)$ is:

$$\text{Paths through } (4,2) = \binom{6}{2} \times \binom{5}{2} = 15 \times 10 = 150$$

(iii) *How many paths pass through either $(3,1)$ or $(5,3)$?*

We use the inclusion-exclusion principle.

- Paths through $(3,1)$ are given by:

$$\binom{4}{1} \times \binom{7}{3} = 4 \times 35 = 140$$

- Paths through $(5,3)$ are:

$$\binom{8}{3} = 56$$

- Paths through both $(3,1)$ and $(5,3)$ are:

$$\binom{4}{1} \times \binom{4}{2} \times \binom{3}{1} = 4 \times 6 \times 3 = 72$$

Using inclusion-exclusion, the total number of paths passing through either $(3,1)$ or $(5,3)$ is:

$$140 + 56 - 72 = 124$$

9 marks $= 2 + 3 + 4$

(c)

The GP is $(a = 10,\ r = 0.5)$

$$\underbrace{10}_{a_0}, \underbrace{5}_{a_1}, \underbrace{2.5}_{a_2}, \underbrace{1.25}_{a_3}, \underbrace{0.625}_{a_4}, \underbrace{0.3125}_{a_5}, \ldots, \underbrace{ar^{n-1}}_{a_n}, \ldots,$$

So 0.5 is not an element in this progression. Or can use the formula for a general term

$$10 \times 0.5^{n-1} = 0.5 \implies 0.5^{n-1} = 0.05 \implies n = 5.32 \notin \mathbb{N}$$

Question 3

(a)

(i) *Explain why f has a inverse.*

$f(x)$ is a linear function with a non-zero slope, so it is bijective (one-to-one and onto), and an inverse exists.

(ii) *Determine $f^{-1}(x)$.* To determine f^{-1} solve $f(x) = 3x - 5 = y$ for x

$$y = 3x - 5 \quad \Longrightarrow \quad x = \frac{y+5}{3}$$

So $f^{-1}(x) = \frac{x+5}{3}$.

(iii) *Verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.*

$$f^{-1}(f(x)) = f^{-1}(3x-5) = \frac{[3x-5]+5}{3} = x$$

And

$$f(f^{-1}(x)) = f\left(\frac{x+5}{3}\right) = 3\left[\frac{x+5}{3}\right] - 5 = x$$

5 marks $= 1 + 2 + 2$

(b)

(i) *Express the* **number of pixels** *at each step as a geometric sequence.*

The number of pixels in the image at each step forms a geometric sequence. If a_0 is the initial number of pixels, then each step reduces the number of pixels by a factor of $\frac{1}{4}$ (since both width and height are halved).

$$a_0 = 1024 \times 768 = 786,432$$

The geometric sequence representing the number of pixels after n applications of the reductions step is:

$$a_n = 786432 \times \left(\frac{1}{4}\right)^n$$

(ii) *Find the total number of pixels at the 4th reduction step.*

After three applications of the reduction step we have

$$a_3 = 786,432 \times \left(\frac{1}{4}\right)^3 = 12,288$$

(iii) *How many reductions are needed until the number of pixels drops below 10,000?*

We want to find n such that $a_n < 10000$

$$786432 \times \left(\frac{1}{4}\right)^3 < 1000 \quad \Longrightarrow \quad n \approx 4.8 \Longrightarrow n = 5$$

7 marks $= 2 + 2 + 3$

(c)

Converting to mathematical notation ...

$$\sum_{k=1}^{2} \prod_{j=0}^{k-1} (j+k)^2$$

Evaluating (by expanding outer loop first) ...

$$\sum_{k=1}^{2} \prod_{j=0}^{k-1} (j+k)^2 = \underbrace{\left[\prod_{j=0}^{0} (j+1)^2\right]}_{k=1} + \underbrace{\left[\prod_{j=0}^{1} (j+2)^2\right]}_{k=2}$$

$$= \left[\underbrace{\left[(0+1)^2\right]}_{j=0}\right] + \left[\underbrace{\left[(0+2)^2\right]}_{j=0} \cdot \underbrace{\left[(1+2)^2\right]}_{j=1}\right] = 1 + 4 \cdot 9 = 37$$

(d)

The AP is $(a = 2,\ d = 4)$

$$\underbrace{2}_{a_0}, \underbrace{6}_{a_1}, \underbrace{10}_{a_2}, \ldots, \underbrace{a + d(n-1)}_{a_n}, \ldots,$$

So the 10th element is $2 + 4(10 - 1) = 38$.

Question 4

(a)

(i) $A \cup B = \{0, 1, 2, 3, 4, 5, 6\}$

(ii) $A \setminus B = \{0, 1, 2\}$

(iii) $B \cap A = \{3, 4\}$

(iv) $A \oplus B = \{0, 1, 2, 5, 6\}$

(v) $(A \cup B) \cap C = \{1, 4, 5, 6\}$

(vi) $\bar{A} = \{5, 6, 7, 8, 9\}$

6 marks$=6 \times 1$

(b)

(i) *How many subsets are there of cardinality 3?*

$\binom{7}{3} = 35$ subsets.

(ii) *How many subsets of S are there? That is, find $|\mathcal{P}(S)|$.*

$2^7 = 128$ subsets.

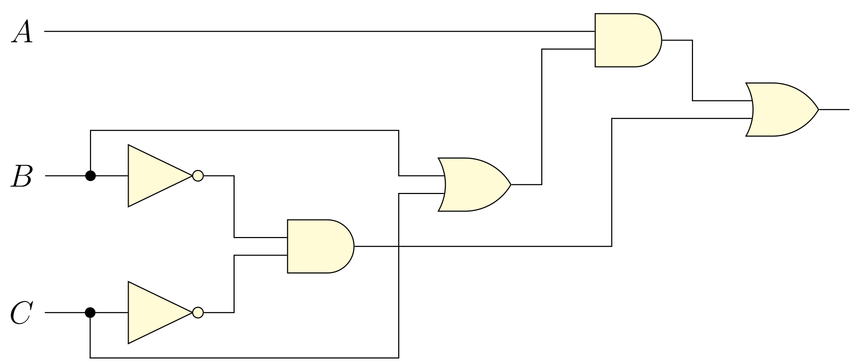
(iii) *How many subsets of S are there where the sum of the elements equals 15?*

$$|\{\{2, 13\}, \{3, 5, 7\}\}$$

4 marks=1+1+2

(c)

A possible layout is



(d)

Function computes whether $A \subseteq B$ — will return **True** iff all elements of A are elements of B.

Question 5

(a)

No + reason

(b)

(i) *Can this function be injective (one-to-one)? Justify your answer*

The function cannot be injective **if** more than one employee works in any on the departments, as then multiple employees will map to the same department.

Therefore, f is not injective.

(ii) *Can this function be surjective (onto)? Justify your answer.*

The function is surjective if every department has at least one employee.

If there exists a department with no employees, the function is not surjective. This is very unlikely but does happen.

(c)

(i) $f(4) = 2 \cdot 4 + 3 = 8 + 3 = 11$

(ii) $g(2) = 2^3 - 1 = 8 - 1 = 7$

(iii) $f(5) + g(2) - h(4) = 2 \cdot 5 + 3 + 7 - \frac{5}{4} = 10 + 3 + 7 - \frac{5}{4} = 20 - \frac{5}{4} = \frac{75}{4} = 18.75$

(iv) $f(g(1)) = f(1^3 - 1) = f(0) = 2 \cdot 0 + 3 = 3$

(v) $g(h(2)) = g(\frac{5}{2}) = (\frac{5}{2})^3 - 1 = \frac{125}{8} - 1 = \frac{117}{8} = 14.63$

(vi) $h(f(3)) = h(2 \cdot 3 + 3) = h(9) = \frac{5}{9} = 0.556$

6 marks$= 6 \times 1$

(d)

Partial marks for identifying graph. 4×2 **marks**

(i) K_9
Complete graph so girth is 3.

(ii) $K_{5,7}$
Complete bipartite graph, so girth is 4.

(iii) C_8
Cycle graph has girth is 8.

(iv) W_8
Wheel graph has girth of 3.