Appendix:

Adaptation of Numerical Solution Method for NorSand

Provided as ***Supplementary Information*** to paper in **Geotechnical Research**

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A .1 Overview of Algorithm

The numerical solution method used for integrating NorSand in the simple shear test, implemented in the VBA code, is an adaptation of the *elastic predictor-plastic corrector* (EP-PC) method. This method is conceptually similar to the widely used *initial stress* solution technique of the finite element method. Forward Euler integration is used, the analysis stepping forward in time with the new converged stress state $\overline{\sigma}^C$ related to the converged stress state at the previous load step ($\overline{\sigma}^O$) by:

$$\overline{\sigma}^C = \overline{\sigma}^O + \dot{\overline{\sigma}} \quad \text{[A.1]}$$

The incremental load is expressed using the elastic-plastic strain decomposition:

$$\dot{\overline{\sigma}} = D\left(\dot{\varepsilon} - \dot{\varepsilon}^p\right) \quad \text{[A.2]}$$

…where D is the elastic load-displacement matrix. If we are inside the yield surface the plastic strains are zero and [A.2] reduces to elasticity. For yielding, the plastic strains are given by:

$$\dot{\varepsilon}^p = \lambda^s \frac{\partial Q}{\partial \sigma} \quad \text{[A.3]}$$

Or, on substituting [A.3] in [A.2]:

$$\dot{\overline{\sigma}} = D\left(\dot{\varepsilon} - \lambda^s \frac{\partial Q}{\partial \sigma}\right) \quad \text{[A.4]}$$

…where Q is the plastic potential and λ^s is the plastic multiplier, discussed later. Note that λ^s is not the slope of the CSL with its usual semi-log idealization; although both uses of λ are long-standing conventions, the same Greek letter (confusingly) stands for two very different things in numerical implementations of constitutive models with a CSL. We follow the convention that superscript ‘s’ applied to λ denotes the plastic multiplier while the subscript ‘e’ denotes the soil property.

The change in stress in [A.4] can be viewed as two parts: i) an elastic stress increment is added to the converged stress state at the previous “old” load step ($\overline{\sigma}^0$) to give the initial “elastic” estimate of the new stress state ($\overline{\sigma}^E$) assuming that all strains are elastic; ii) the elastic estimate is then corrected for the plastic strains in the current loadstep to give the new “converged” stress state $\overline{\sigma}^C$ that is on the hardened or softened yield surface. The finite element literature refers to this as the 'initial stress' method using a reduced elasto-plastic *D* matrix, but in this implementation [A.4] is used directly.

Importantly, note that this algorithm assumes stress increments are co-axial with the stress state at the start of the loadstep. This means that the current principal stress direction α is used to map stress increments, with α then being updated for the consequent changes in the stress state at the next loadstep (i.e. principal stress rotation lags stress changes by one step).

A .2 Incremental Loads with NorSand

The NorSand equations are most simply derived using principal stresses, but these relate to a *1,2,3* coordinate frame that rotates. Conversely, stress analysis (e.g. finite element codes) normally adopts a spatially fixed *x,y,z* frame and this is also easiest for the simple shear test. Assuming that σ_z is the intermediate principal stress (and σ_2 transposes with σ_3 if not), the converged stress state in the *x,y,z* frame in terms of the principal stresses increments is:

$$\bar{\sigma}_x^C = \bar{\sigma}_x^O + \left(\Delta\dot{\bar{\sigma}}_1 \cos^2(90+\alpha) + \Delta\dot{\bar{\sigma}}_3 \sin^2(90+\alpha)\right) \quad \text{[A.5a]}$$

$$\bar{\sigma}_y^C = \bar{\sigma}_y^O + \left(\Delta\dot{\bar{\sigma}}_1 \cos^2\alpha + \Delta\dot{\bar{\sigma}}_3 \sin^2\alpha\right) \quad \text{[A.5b]}$$

$$\bar{\sigma}_z^C = \bar{\sigma}_z^O + \Delta\dot{\bar{\sigma}}_2 \quad \text{[A.5c]}$$

$$\bar{\sigma}_{xy}^C = \bar{\sigma}_{xy}^O + 0.5\left(\Delta\dot{\bar{\sigma}}_1 - \Delta\dot{\bar{\sigma}}_3\right)\sin 2\alpha \quad \text{[A.5d]}$$

Writing the elastic strain increments in terms of total and plastic increments, using the standard plastic strain decomposition $\varepsilon^p = \varepsilon - \varepsilon^e$, the increment of principal stresses are given in terms of the strain increments by:

$$\dot{\bar{\sigma}}_1 = A\left(\dot{\varepsilon}_1 - \dot{\varepsilon}_1^p\right) + B\left(\dot{\varepsilon}_2 - \dot{\varepsilon}_2^p\right) + B\left(\dot{\varepsilon}_3 - \dot{\varepsilon}_3^p\right)$$

$$\Rightarrow \dot{\bar{\sigma}}_1 = A\dot{\varepsilon}_1 + B\left(\dot{\varepsilon}_2 + \dot{\varepsilon}_3\right) - \left(A\dot{\varepsilon}_1^p + B\dot{\varepsilon}_2^p + B\dot{\varepsilon}_3^p\right) \quad \text{[A.6]}$$

…where *A, B* being two elastic coefficients :

$$A = K + 4G/3 \quad \text{[A.7a]}$$

$$B = K - 2G/3 \quad \text{[A.7b]}$$

The other two stress increments follow similarly (using cyclic substitution of subscripts).

Defining the ratios of plastic principal strain rates as z_2, z_3:

$$z_2 = \dot{\varepsilon}_2^p / \dot{\varepsilon}_1^p \qquad \text{[A.8a]}$$

$$z_3 = \dot{\varepsilon}_3^p / \dot{\varepsilon}_1^p \qquad \text{[A.8b]}$$

The strain increments ratios z_2 and z_3 are unique functions of the converged stress state $\bar{\sigma}^0$ at the previous loadstep as defined in Table 1 of the Paper. The ratio z_3 uses a cosine function of the Lode angle in the π plane; this is an empirical function to the detailed test data on the plane strain behaviour of Brasted sand – still the best data set available for looking outside the triaxial situation. Intriguingly, the cosine function used for z_3 is similar in form to that used for M and delivers a close approximation of normality in the π-plane. The second ratio z_2 ensures that the plastic strains remain consistent with the idealized plastic work dissipation. This approach is needed because, while NorSand is associated under triaxial conditions, no work is done in the π plane by the Lode angle; the work-based considerations underlying normality only have contributions from the σ_q, σ_m stress invariants (see Jefferies & Shuttle, 2002, for longer discussion of this point) and which then leaves an ambiguity about normality in the π-plane. Using the z_2, z_3, re-write [A.6] as:

$$\dot{\bar{\sigma}}_1 = A\dot{\varepsilon}_1 + B(\dot{\varepsilon}_2 + \dot{\varepsilon}_3) - \dot{\varepsilon}_1^p(A + Bz_2 + Bz_3) \qquad \text{[A.9a]}$$

$$\dot{\bar{\sigma}}_2 = A\dot{\varepsilon}_2 + B(\dot{\varepsilon}_3 + \dot{\varepsilon}_1) - \dot{\varepsilon}_1^p(Az_2 + Bz_3 + B) \qquad \text{[A.9b]}$$

$$\dot{\bar{\sigma}}_3 = A\dot{\varepsilon}_3 + B(\dot{\varepsilon}_1 + \dot{\varepsilon}_2) - \dot{\varepsilon}_1^p(Az_3 + B + Bz_2) \qquad \text{[A.9c]}$$

It is helpful to work in terms of the plastic shear strain invariant to implement NorSand, rather than ε_1, readily done by noting:

$$\dot{\varepsilon}_v^p = \dot{\varepsilon}_1^p + \dot{\varepsilon}_2^p + \dot{\varepsilon}_3^p \;\Rightarrow \dot{\varepsilon}_1^p = \frac{\dot{\varepsilon}_v^p}{1+z_2+z_3} \Rightarrow \dot{\varepsilon}_1^p = \frac{D^p\dot{\varepsilon}_q^p}{1+z_2+z_3} \qquad \text{[A.10]}$$

On substituting [A.10] in [A.9], the principal stress increments become:

$$\dot{\bar{\sigma}}_1 = A\dot{\varepsilon}_1 + B(\dot{\varepsilon}_2 + \dot{\varepsilon}_3) - \left(\frac{A + Bz_2 + Bz_3}{1+z_2+z_3}\right) D^p \ \dot{\varepsilon}_q^p \tag{A.11a}$$

$$\dot{\bar{\sigma}}_2 = A\dot{\varepsilon}_2 + B(\dot{\varepsilon}_3 + \dot{\varepsilon}_1) - \left(\frac{Az_2 + Bz_3 + B}{1+z_2+z_3}\right) D^p \ \dot{\varepsilon}_q^p \tag{A.11b}$$

$$\dot{\bar{\sigma}}_3 = A\dot{\varepsilon}_3 + B(\dot{\varepsilon}_1 + \dot{\varepsilon}_2) - \left(\frac{Az_3 + B + Bz_2}{1+z_2+z_3}\right) D^p \ \dot{\varepsilon}_q^p \tag{A.11c}$$

Notice in [A.11] that *A, B* are elastic constants for any location in the domain of interest and at any converged stress state (although they can vary with stress level when stepping the solution forward). There is now only a single unknown, the plastic shear strain increment that drives the hardening law, to be solved for at the integration point being considered. As usual, this unknown is approached by the plastic multiplier λ^s. With that strain solved, substituting [A.11] in [A.5] immediately gives the new, converged stress state.

A .3 Stress Increments for plastic strain in NorSand

Plastic strain increments are normal to the *plastic potential* (denoted as Q), and their magnitude is an unknown to be found as part of the solution. However, the plastic strain increments are proportional to each other. Thus, the standard plastic 'flowrule' is:

$$\dot{\varepsilon}^p = \lambda^s \frac{\partial Q}{\partial \sigma} \tag{A.4 bis}$$

… where λ^s is an unknown scalar. As a work-based idealization, NorSand is 'associated' for the bullet-shaped part of the yield surface with the plastic potential function Q identical to the yield surface function F. It is convenient to re-write the NorSand yield surface from dimensionless ratios into standard "*F=0*" form with units of stress:

$$\frac{\eta}{M_i} = 1 - \ln\left(\frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}}\right) \Leftrightarrow F = \bar{\sigma}_q - M_i\bar{\sigma}_m + M_i\bar{\sigma}_m \ln(\bar{\sigma}_m) - M_i\bar{\sigma}_m \ln(\bar{\sigma}_{m,i}) \tag{A.12}$$

… with F = 0 indicating plastic yielding and F<0 elastic states.

Derivation of plastic strains has simple form if written using the invariants underlying NorSand, that is:

$$\dot{\varepsilon}_v^p = \lambda^s \frac{\partial F}{\partial \overline{\sigma}_m} \tag{A.13a}$$

$$\dot{\varepsilon}_q^p = \lambda^s \frac{\partial F}{\partial \overline{\sigma}_q} \tag{A.13b}$$

Taking the partial differential of [A.12] with respect to mean stress:

$$\frac{\partial F}{\partial \overline{\sigma}_m} = -M_i + M_i\left(\ln(\overline{\sigma}_m) - \ln(\overline{\sigma}_{m,i})\right) + M_i \overline{\sigma}_m \frac{1}{\overline{\sigma}_m}$$

$$\Rightarrow \frac{\partial F}{\partial \overline{\sigma}_m} = M_i\left(\ln(\overline{\sigma}_m) - \ln(\overline{\sigma}_{m,i})\right)$$

… and on substitution of [A.12] to eliminate the log terms:

$$\frac{\partial F}{\partial \overline{\sigma}_m} = M_i - \eta = D^p \tag{A.14}$$

Next, taking the partial differential of [A.12] with respect to $\overline{\sigma}_q$, on inspection:

$$\frac{\partial F}{\partial \overline{\sigma}_q} = 1 \tag{A.15}$$

Substituting [A.14] and [A.15] in [A.13] gives the pleasingly simple expressions for the NorSand plastic strain rates:

$$\dot{\varepsilon}_v^p = \lambda^s D^p \tag{A.16a}$$

$$\dot{\varepsilon}_q^p = \lambda^s \tag{A.16b}$$

The next step is to write the stress increments in terms of the unknown plastic multiplier λ^s. Because the shear strain measure ε_q is linear, as is ε_v, the strain invariants can be written as direct expressions of the plastic strain decomposition:

$$\dot{\varepsilon}_q^e = \dot{\varepsilon}_q - \dot{\varepsilon}_q^p \tag{A.17a}$$

$$\dot{\varepsilon}_v^e = \dot{\varepsilon}_v - \dot{\varepsilon}_v^p \tag{A.17b}$$

The stress increment corresponds to the elastic strain increment through the elastic modulus, so on introduction of the shear and bulk stiffness into [A.17]:

$$\dot{\bar{\sigma}}_q = 3G\dot{\varepsilon}_q^e = 3G\dot{\varepsilon}_q - 3G\dot{\varepsilon}_q^p \tag{A.18a}$$

$$\dot{\bar{\sigma}}_m = K\dot{\varepsilon}_v^e = K\dot{\varepsilon}_v - K\dot{\varepsilon}_v^p \tag{A.18b}$$

Combining [A.18] with the plastic strain rates through normality, [A.16], the increments of the stress invariants are:

$$\dot{\bar{\sigma}}_q = 3G\dot{\varepsilon}_q - 3G\lambda^s \quad = 3G\left(\dot{\varepsilon}_q - \lambda^s\right) \tag{A.19a}$$

$$\dot{\bar{\sigma}}_m = K\dot{\varepsilon}_v - K\lambda^s D^p = K\left(\dot{\varepsilon}_v - \lambda^s D^p\right) \tag{A.19b}$$

Equation [19] allows the new stress state to be easily computed once the single unknown, the plastic multiplier λ^s, has been determined – a further equation is needed to do that.

A .4 Plastic Multiplier from Consistency Condition

The plastic multiplier λ^s is found through the consistency condition requiring that plastic strains developing during loading must leave the stress state still on the yield surface: $\dot{F} = 0$. Thus the new stress state depends on the combination of how the yield surface changes size during the load increment (the hardening law) and how the stress state moves across the yield surface ('neutral' loading). The equations implementing the consistency condition are different for outward loading of the bullet-like yield surface to the inward loading on the cap.

In the case of the bullet-like yield surface, our interest here for simple shear, taking the total differential of the yield surface [A.12] and setting to zero, gives the consistency condition:

$$\dot{F} = 0 = \frac{\partial F}{\partial \bar{\sigma}_m}\dot{\bar{\sigma}}_m + \frac{\partial F}{\partial \bar{\sigma}_q}\dot{\bar{\sigma}}_q + \frac{\partial F}{\partial \bar{\sigma}_{m,i}}\dot{\bar{\sigma}}_{m,i} + \frac{\partial F}{\partial M_i}\dot{M}_i \tag{A.20}$$

The partial differentials with respect to stress were established in the previous section:

$$\frac{\partial F}{\partial \bar{\sigma}_m} = D^p \tag{A.14 bis}$$

$$\frac{\partial F}{\partial \bar{\sigma}_q} = 1 \tag{A.15 bis}$$

Taking the partial differential of [A.12] with respect to the image mean stress:

$$\frac{\partial F}{\partial \bar{\sigma}_{m,i}} = -M_i \frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} \qquad \text{[A.21]}$$

Finally, taking the partial differential of [A.12] with respect to the operating critical friction ratio M_i:

$$\frac{\partial F}{\partial M_i} = -\bar{\sigma}_m + \bar{\sigma}_m \left(\ln(\bar{\sigma}_m) - \ln(\bar{\sigma}_{m,i}) \right)$$

… and again using [A.12] to eliminate the log terms:

$$\Rightarrow \frac{\partial F}{\partial M_i} = -\bar{\sigma}_m + \bar{\sigma}_m \left(1 - \frac{\eta}{M_i} \right) = -\frac{\bar{\sigma}_q}{M_i} \qquad \text{[A.22]}$$

On collecting [A.14], [A.15], [A.21] and [A.22], the consistency condition [A.20] can be written as:

$$\dot{F} = 0 = D^p \dot{\bar{\sigma}}_m + \dot{\bar{\sigma}}_q - M_i \frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} \dot{\bar{\sigma}}_{m,i} - \frac{\bar{\sigma}_q}{M_i} \dot{M}_i \qquad \text{[A.23]}$$

The last term in [A.23] involving M_i is a tad tedious, and also loses generality as it depends on the chosen idealization for the CSL. The M_i term also varies in the π-plane as the Lode angle changes. In numerical implementations it is both conceptually and practically simplest to use a trailing measure (backward difference) for M_i as it changes quite slowly.

The NorSand hardening rule is (see Table 1 of the Paper):

$$\dot{\bar{\sigma}}_{m,i} = H \frac{M_i}{M_{i,tc}} \left(\frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} \right) \left[\bar{\sigma}_{mx} - \bar{\sigma}_{m,i} \right] \dot{\varepsilon}_q^p + S \frac{\eta}{\eta_L} \dot{\bar{\sigma}}_{mx}$$

…where S is a flag to toggle the additional softening on/off, $\eta_L = M_i \left(1 - \chi_i \psi_i / M_{i.tc} \right)$, and σ_{mx} is the current maximum image mean stress for the yield surface (see Table 1 of the Paper).

This additional softening term is found necessary to deal with rapidly changing mean effective stress during undrained tests, see Appendix D of Jefferies & Been (2006). Strictly, the additional softening term has the status of a "necessary kludge" to fit test data as, at present, it has not been derived from first principles about the nature of particulate behaviour.

The hardening rule can be written as comprising four terms. The first two (X_H and X_C) are multipliers of the current unknown plastic strain increment as $\dot{\varepsilon}_q^p$ and $\dot{\bar{\sigma}}_m$ are related to λ^s through [A.16] and [A.19] respectively. The remaining two (X_D and X_E) take values computed at the previous loadstep. In this arrangement the hardening rule is:

$$\dot{\bar{\sigma}}_{m,i} = X_H \dot{\varepsilon}_q^p + X_C \dot{\bar{\sigma}}_m + X_D \dot{M}_{i.tc} - X_E \dot{\psi}_i = X_H \lambda^s + X_C \dot{\bar{\sigma}}_m + X_D \dot{M}_{i,tc} - X_E \dot{\psi}_i \qquad \text{[A.24]}$$

… where :

$$X_H = H \frac{M_i}{M_{i,tc}} \left(\frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} \right) \left[\bar{\sigma}_{mx} - \bar{\sigma}_{m,i} \right] \qquad \text{[A.25a]}$$

$$X_C = Z \frac{\bar{\sigma}_{mx}}{\bar{\sigma}_m} \frac{\eta}{\eta_L} \qquad \text{[A.25b]}$$

$$X_D = Z \bar{\sigma}_{mx} \frac{\eta}{\eta_L} \frac{\psi_i \chi_i}{M_{i,tc}^2} \qquad \text{[A.25c]}$$

$$X_E = Z \bar{\sigma}_{mx} \frac{\eta}{\eta_L} \frac{\chi_i}{M_{i,tc}} \qquad \text{[A.25d]}$$

Using the new parameters X_H, X_C, X_D and X_E the consistency condition [A.20] becomes:

$$\begin{aligned} \dot{F} = 0 = D^p \dot{\bar{\sigma}}_m + \dot{\bar{\sigma}}_q - M_i \frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} X_H \lambda^s - M_i \frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} X_C \dot{\bar{\sigma}}_m \\ - M_i \frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} X_D \dot{M}_{i,tc} + M_i \frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} X_E \dot{\psi}_i - \frac{\bar{\sigma}_q}{M_i} \dot{M}_i \end{aligned} \qquad \text{[A.26]}$$

Substituting the stress increments in terms of the unknown plastic multiplier λ^s, derived earlier as [A.19], allows the consistency condition [A.20] to be written as:

$$\begin{aligned} & D^p K \left(\dot{\varepsilon}_v - \lambda^s D^p \right) + 3G \left(\dot{\varepsilon}_q - \lambda^s \right) - M_i \frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} X_H \lambda^s - M_i \frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} X_C \dot{\bar{\sigma}}_m \\ & - M_i \frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} X_D \dot{M}_{i,tc} + M_i \frac{\bar{\sigma}_m}{\bar{\sigma}_{m,i}} X_E \dot{\psi}_i - \frac{\bar{\sigma}_q}{M_i} \dot{M}_i = 0 \end{aligned} \qquad \text{[A.27]}$$

On collecting terms in [A.27] and re-arranging…

$$\lambda^s = \frac{3G\dot{\varepsilon}_q + D^p K\dot{\varepsilon}_v - \dfrac{\overline{\sigma}_q}{M_i}\dot{M}_i - M_i\dfrac{\overline{\sigma}_m}{\overline{\sigma}_{m,i}}X_C\dot{\overline{\sigma}}_m - M_i\dfrac{\overline{\sigma}_m}{\overline{\sigma}_{m,i}}X_D\dot{M}_{i,tc} + M_i\dfrac{\overline{\sigma}_m}{\overline{\sigma}_{m,i}}X_E\dot{\psi}_i}{3G + D^p K D^p + M_i\dfrac{\overline{\sigma}_m}{\overline{\sigma}_{m,i}}X_H} \qquad [A.28]$$

Now setting:

$$X_m = M_i\frac{\overline{\sigma}_m}{\overline{\sigma}_{m,i}} \qquad [A.29]$$

… we obtain the simplest explicit form for the plastic multiplier:

$$\lambda^s = \frac{3G\dot{\varepsilon}_q + D^p K\dot{\varepsilon}_v - \dfrac{\overline{\sigma}_q}{M_i}\dot{M}_i + X_m\left(X_E\dot{\psi}_i - X_C\dot{\overline{\sigma}}_m - X_D\dot{M}_{i,tc}\right)}{3G + K\left(D^p\right)^2 + X_m X_H} \qquad [A.30]$$

A .5 Converged Stress State

The converged stress increments are calculated by substituting [A.16b] in [A.11] giving:

$$\dot{\overline{\sigma}}_1 = A\dot{\varepsilon}_1 + B\left(\dot{\varepsilon}_2 + \dot{\varepsilon}_3\right) - \left(\frac{A + Bz_2 + Bz_3}{1 + z_2 + z_3}\right)D^p\ \lambda^s \qquad [A.31a]$$

$$\dot{\overline{\sigma}}_2 = A\dot{\varepsilon}_2 + B\left(\dot{\varepsilon}_3 + \dot{\varepsilon}_1\right) - \left(\frac{Az_2 + Bz_3 + B}{1 + z_2 + z_3}\right)D^p\ \lambda^s \qquad [A.31b]$$

$$\dot{\overline{\sigma}}_3 = A\dot{\varepsilon}_3 + B\left(\dot{\varepsilon}_1 + \dot{\varepsilon}_2\right) - \left(\frac{Az_3 + B + Bz_2}{1 + z_2 + z_3}\right)D^p\ \lambda^s \qquad [A.31c]$$

… where all the terms on the right hand side of [A.31] are now known using [A.30].